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# A Fourier transform for all generalized functions

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# Contents

# Chapter 1

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# Chapter 2

# Abstract

Using the existence of an infinite number $k$ in the non-Archimedean ring of Robinson-Colombeau, we define the hyperfinite Fourier transform (HFT) by considering integration extended to $[-k,k]^n$ instead of $(-\infty,\infty)^n$. In order to realize this idea, the space of generalized functions we consider is that of *generalized smooth functions* (GSF), an extension of classical distribution theory sharing many nonlinear properties with ordinary smooth functions, like the closure with respect to composition, a good integration theory, and several classical theorems of calculus. Even though the final transform depends on $k$, we obtain a new notion that applies to all GSF, in particular to all Schwartz's distributions and to all Colombeau generalized functions, without growth restrictions. We prove that this FT generalizes several classical properties of the ordinary FT, and in this way we also overcome the difficulties of FT in Colombeau's settings. Differences in some formulas, such as in the transform of derivatives, reveal to be meaningful since allow to obtain also global unique non-tempered solutions of differential equations. Before dealing with HFT, we need the correct notion of a limit to interchange with the integral sign. In fact, it is well-known that the notion of limit in the sharp topology of sequences of Colombeau generalized numbers $\widetilde{\mathbb{R}}$ does not generalize classical results. E.g. the sequence $\frac{1}{n} \not\to 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ converges if and only if $x_{n+1}-x_n \to 0$. This has several deep consequences, e.g. in the study of series, analytic generalized functions, or sigma-additivity and classical limit theorems in integration of generalized functions. The lacking of these results is also connected to the fact that $\widetilde{\mathbb{R}}$ is necessarily not a complete ordered set, e.g. the set of all the infinitesimals has neither supremum nor infimum. We first present a solution of these problems with the introduction of the notions of hypernatural number, hypersequence, close supremum and infimum. In this way, we can generalize all the classical theorems for the hyperlimit of a hypersequence.

# Chapter 3

# Kurzfassung

Unter Verwendung der Existenz einer unendlichen Zahl $k$ im nicht-archimedischen Ring von Robinson-Colombeau definieren wir die hyperfinite Fourier-Transformation (HFT), indem wir über $[-k,k]^n$ anstatt $(-\infty,\infty)^n$ integrieren. Um diese Idee zu realisieren, betrachten wir einen bestimmten Raum verallgemeinerter Funktionen, nämlich den der *verallgemeinerten glatten Funktionen* (GSF); dies ist eine Erweiterung der klassischen Distributionstheorie, die mit gewöhnlichen glatten Funktionen viele nichtlineare Eigenschaften gemeinsam hat, wie z.B. Abschluss unter Komposition, eine zufriedenstellende Integrationstheorie und mehrere klassische Theoreme der Infinitesimalrechnung. Obwohl die Transformation letztendlich von k abhängt, erhalten wir einen neuen Begriff, der auf alle GSF, insbesondere auf alle Schwartz-Distributionen und alle verallgemeinerten Funktionen im Sinne von Colombeau, ohne Wachstumsbeschränkungen angewendet werden kann. Wir beweisen, dass diese FT mehrere klassische Eigenschaften der gewöhnlichen FT verallgemeinert, und überwinden auf diese Weise auch die Schwierigkeiten der FT im Colombeau-Setting. Unterschiede in einigen Formeln, wie zum Beispiel bei der Transformation von Ableitungen, erweisen sich als sinnvoll, weil dadurch auch globale eindeutige nichttemperierte Lösungen von Differentialgleichungen erhalten werden können. Bevor wir uns mit der HFT befassen, benötigen wir einen passenden Begriff des Grenzwertes, der mit dem Integralzeichen vertauscht werden kann. Tatsächlich ist bekannt, dass der Begriff des Grenzwertes in der scharfen Topologie von Folgen von Colombeau-verallgemeinerten Zahlen $\widetilde{\mathbb{R}}$ ungeeignet ist zur Verallgemeinerung klassischer Resultate. Z.B. gilt $\frac{1}{n} \not\to 0$, und eine Folge $(x_n)_{n\in\mathbb{N}}$ konvergiert dann, und nur dann, wenn $x_{n+1} - x_n \to 0$. Dies hat mehrere tiefgreifende Konsequenzen, z.B. bei der Untersuchung von Reihen, von analytischen verallgemeinerten Funktionen, oder von Sigma-Additivität und klassischen Grenzwertsätzen in der Integration verallgemeinerter Funktionen. Das Nichtvorhandensein dieser Resultate hängt auch damit zusammen, dass $\widetilde{\mathbb{R}}$ zwangsläufig keine vollständige geordnete Menge ist, denn z.B. besitzt die Menge aller infinitesimalen Elemente weder ein Supremum noch ein Infimum. Wir präsentieren zunächst eine Lösung für diese Probleme durch die Einführung der Begriffe „hypernatürliche Zahl“,

„Hyperfolge“, „enges Supremum bzw. Infimum“. Auf diese Weise können wir alle klassischen Sätze auf den Hypergrenzwert einer Hyperfolge verallgemeinern.

# Chapter 4

# Introduction

Fourier transform (FT) and generalized functions (GF) are naturally interwoven, since the former naturally leads to suitable spaces of the latter. This already occurs even in trivial cases, such as transforming a simple sound wave $f(t) = A\sin(2\pi\omega_0 t)$, whose spectrum must be, in some way, concentrated at the frequencies $\pm\omega_0$. Even the link between constants and delta-like functions was already conceived by Fourier (see e.g. [45]). Although different theories of generalized functions arise for different motivations, from distribution theory of Sobolev, Schwartz [62, 66] up to Hairer's regularity structures [38], almost all these theories are usually augmented with a corresponding calculus of FT, which can be applied to an appropriate subspace of generalized functions. Since the beginning of distribution theory, it was hence natural to try to extend the domain of the FT with less or even with no growth restrictions imposed. In fact, e.g., as a consequence of these restrictions, the only solution of the trivial ODE $y' = y$ we can achieve using tempered distributions is the trivial one. We can hence cite in [25, 26] the definition of the FT as the limit of a sequence of functions integrated on a finite domain, or [74] for a two-sided Laplace transform defined on a space larger than that of tempered distributions, and similarly in [6] for the directional short-time Fourier transform of exponential-type distributions. In the same direction we can inscribe the works [5, 12, 18, 40, 57, 68, 65, 21, 22] on ultradistributions, hyperfunctions and thick distributions.

On the other hand, problems originating from physics, such as singularities and point-source fields, also suggest us to consider alternative modeling, ranging from non-smooth functions as test functions in the theory of distributions (see e.g. [72] and references therein) to non-Archimedean analysis (i.e. mathematical analysis over a ring extending the real field and containing infinitesimal and/or infinite numbers, see [37, 23]). In general, a key concept of non-Archimedean analysis is that extending the real field $\mathbb{R}$ into a ring containing infinitesimals and infinite numbers could eventually lead to the solution of non trivial problems. This is the case, e.g., of Colombeau theory, where nonlinear generalized functions can be viewed as set-theoretical maps on domains consisting of generalized points of the non-Archimedean ring $\widetilde{\mathbb{R}}$. This orientation has become increasingly important in recent years and hence it has led to the study of preliminary notions of $\widetilde{\mathbb{R}}$ (cf., e.g., [54, 4, 2, 55, 3, 7, 71, 35, 49]; see below for a self-contained introduction to the ring of Colombeau generalized numbers $\widetilde{\mathbb{R}}$). In the interplay between mathematics and physics, it is well-known that heuristically manipulating non-linear pointwise equalities such as $H^2 = H$ ($H$ being the Heaviside function) can easily lead to contradictions (see e.g. [11, 37]). This can make particularly difficult to realize the strategy of [47], where the authors search for a metaplectic representation from symplectic maps to symplectic relations. According to A. Weinstein (personal communication, May 2019), this would require an algebra of generalized functions extending the usual algebra of smooth functions and a FT acting on them with the usual inversion formula and transforming the Dirac delta into a constant 1. As we will see more diffusely in the chapter 6, this is not possible in the classical approach to Colombeau's algebra, see [14, 16, 52, 39]. In fact, although the notion of FT in the Colombeau setting shares several properties with the classical one, it lacks e.g. the Fourier inversion theorem, which holds only at the level of equality in the sense of generalized tempered distributions (g.t.d.) [14, 16, 52], see also (6.7.3). See also [67] for a Paley-Wiener like theorem. In other words, we only have e.g. $\mathcal{F}_{\hat{\varphi}}(\partial^\alpha u) =_{\text{g.t.d.}} i^{|\alpha|}\omega^\alpha \mathcal{F}_{\hat{\varphi}}(u)$, $i^{|\alpha|}\mathcal{F}^*{}_{\hat{\varphi}}(\partial^\alpha u) =_{\text{g.t.d.}} x^\alpha \mathcal{F}^*{}_{\hat{\varphi}}(u)$, $\mathcal{F}_{\hat{\varphi}}\mathcal{F}^*{}_{\hat{\varphi}}u =_{\text{g.t.d.}} \mathcal{F}^*{}_{\hat{\varphi}}\mathcal{F}_{\hat{\varphi}}u$, where $\mathcal{F}_{\hat{\varphi}}(u)$ denotes the Fourier transform with respect to the damping measure. Moreover $\langle \iota_{\mathbb{R}}(\hat{T}), \psi\rangle \approx \langle \mathcal{F}_{\hat{\varphi}}\iota_{\mathbb{R}}(T), \psi\rangle$ for all $T \in \mathcal{S}'(\mathbb{R})$ and all $\psi \in \mathcal{S}(\mathbb{R})$, where $\iota_{\mathbb{R}}(T)$ is the embedding of Schwartz distributions as Colombeau generalized functions. The only known possibility to obtain a strict Fourier inversion theorem in Colombeau's theory, is the approach used by [53], where smoothing kernels are used as index set (instead of the simpler $\varepsilon \in I$) and therefore the knowledge of infinite dimensional calculus in convenient vector spaces is needed. Unfortunately, the latter approach is not widely known, even in the community of CGF, and it can be considered as technically involved.

To overcome this type of problems, we are going to use the category of *generalized smooth functions* (GSF), see [29, 30, 46, 28, 31]. This theory seems to be a good candidate, since it is an extension of classical distribution theory which allows to model nonlinear singular problems, while at the same time sharing many nonlinear properties with ordinary smooth functions, like the closure with respect to composition (thereby, they form an algebra extending the algebra of *smooth* functions with pointwise product) and several non trivial classical theorems of the calculus. One could describe GSF as a methodological restoration of Cauchy-Dirac's original conception of generalized function, see [19, 44, 41]. In essence, the idea of Cauchy and Dirac (but also of Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside) was to view generalized functions as suitable types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. For example, the density of a Cauchy-Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties which nowadays are attributed to the Dirac delta, cf. [41].

The basic idea to define a very general FT in this setting is the following: Since GSF form a non-Archimedean framework, we can consider a positive infinite generalized number $k$ (i.e. $k > r$ for all $r \in \mathbb{R}_{>0}$) and define the FT with the usual formula, but integrating over the $n$-dimensional interval $[-k, k]^n$. Although $k$ is an infinite number (hence, $[-k, k]^n \supseteq \mathbb{R}^n$), this interval behaves like a compact set for GSF, so that, e.g., on these domains we always have an extreme value theorem and integrals always exist. Clearly, this leads to a FT, called *hyperfinite* FT, that depends on the parameter $k$, but, on the other hand, where we can transform *all* the GSF defined on this interval and these include all tempered Schwartz distributions, all tempered Colombeau GF, but also a large class of non-tempered GF, such as the exponential functions, or non-linear examples like $\delta^a \circ \delta^b$, $\delta^a \circ H^b$, $a$, $b \in \mathbb{N}$, etc. Not all the properties of the classical FT remain unchanged for this more general transform, but the final formalism still retains the useful properties of the FT in dealing with differential equations. Even more, the new formula for the transform of derivatives leads to discover also exponential solutions of the aforementioned ODE $y' = y$. Since [17] proves that ultradistributions and periodic hyperfunctions can be embedded in Colombeau type algebra (and hence as GSF), this give strong hints to conjecture that the hyperfinite FT is very general, and it justifies the title of this doctoral thesis.

One of the most important results one aims to achieve in developing a FT theory is clearly the Fourier inversion theorem. Trying to generalize the classical proof, a pivotal step is the possibility to interchange limits with integration. This necessarily leaded us, in chapter 5, to firstly develop the correct notion of limit (called *hyperlimit*) for the most useful topology for GSF, i.e. the sharp topology. In fact, the sharp topology on $\widetilde{\mathbb{R}}$ (cf., e.g., [27, 60, 61] and below) is the appropriate notion to deal with continuity of this class of generalized functions and for a suitable concept of well-posedness. This topology necessarily has to deal with balls having infinitesimal radius $r \in \widetilde{\mathbb{R}}$, and thus $\frac{1}{n} \not\to 0$ if $n \to +\infty$, $n \in \mathbb{N}$, because we never have $\mathbb{R}_{>0} \ni \frac{1}{n} < r$ if $r$ is infinitesimal. Another unusual property related to the sharp topology can be derived from the following inequalities (where $m \in \mathbb{N}$, $n \in \mathbb{N}_{\le m}$, $r \in \widetilde{\mathbb{R}}_{>0}$ is an infinitesimal number, and $|x_{k+1} - x_k| \le r^2$)

$$|x_m - x_n| \le |x_m - x_{m-1}| + \ldots + |x_{n+1} - x_n| \le (m - n)r^2 < r,$$

which imply that $(x_n)_{n\in\mathbb{N}} \in \widetilde{\mathbb{R}}^{\mathbb{N}}$ is a Cauchy sequence if and only if $|x_{n+1} - x_n| \to 0$ (actually, this is a well-known property of every ultrametric space, cf., e.g., [42, 60]). Naturally, this has several counter-intuitive consequences (arising from differences with the classical theory) when we have to deal with the study of series, analytic generalized functions, or sigma-additivity and classical limit theorems in integration of generalized functions (cf., e.g., [58, 70, 31]). In order to settle this problem, it is important to generalize the role of the net $(\varepsilon)$, as used in Colombeau theory, into a more general $\rho = (\rho_\varepsilon) \to 0$ (which is called a *gauge*), and hence to generalize $\widetilde{\mathbb{R}}$ into some ${}^\rho\widetilde{\mathbb{R}}$ (see Def. 1). We then introduce the set of *hypernatural numbers* as

$$^{\rho}\widetilde{\mathbb{N}} := \left\{[n_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall\varepsilon\right\},$$

so that it is natural to expect that $\frac{1}{n} \to 0$ in the sharp topology if $n \to +\infty$ with $n \in {}^{\rho}\widetilde{\mathbb{N}}$, because now $n$ can also take infinite values. The notion of sequence is therefore substituted with that of *hypersequence*, as a map $(x_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$, where $\sigma$ is, generally speaking, another gauge. As we will see, (cf. example 37) only in this way we are able to prove e.g. that $\frac{1}{\log n} \to 0$ in ${}^{\rho}\widetilde{\mathbb{R}}$ as $n \in {}^{\sigma}\widetilde{\mathbb{N}}$ but only for a suitable gauge $\sigma$ (depending on $\rho$), whereas this limit does not exist if $\sigma = \rho$.

The notions of supremum and infimum are naturally linked to the notion of limit of a monotonic (hyper)sequence. Being an ordered set, ${}^{\rho}\widetilde{\mathbb{R}}$ already has a definition of, let us say, supremum as least upper bound. However, as already preliminary studied and proved by [24], this definition does not fit well with topological properties of ${}^{\rho}\widetilde{\mathbb{R}}$ because generalized numbers $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ can actually jump as $\varepsilon \to 0^+$ (see Sec. 5.3.1). It is well known that in $\mathbb{R}$ we have $m = \sup(S)$ if and only if $m$ is an upper bound of $S$ and

$$\forall r \in \mathbb{R}_{>0}\, \exists s \in S: \; m - r \leq s. \tag{4.0.1}$$

This could be generalized into the notion of *close supremum* in ${}^{\rho}\widetilde{\mathbb{R}}$, generalizing [24], that results into better topological properties, see Sec. 5.3. The ideas presented in this doctoral thesis, can surely be useful to explore similar ideas in other non-Archimedean settings, such as [10, 9, 64, 56, 42].

Moreover, since our non-Archimedean scalars form a non-totally ordered ring, also the notion of *subpoint* revealed to be very useful with its invertibility on subpoints (see Lem. 10 and its trichotomy (and quadrichotomy!) laws (see Lem. 11 and Lem. 12).

The structure of the current PhD thesis is as follows. In chapter 5, we develop the theory of hyperlimits and its natural links with close supremum and infimum (see [24]) of monotonic hypersequences.

Next, in chapter 6, we start with an introduction into the setting of GSF and give basic notions concerning GSF and its calculus that are needed for a first study of the hyperfinite FT (Sec. 6.1). we then define the hyperfinite FT in Sec. 6.4 and the convolution of compactly supported GSF in Sec. 6.3. In Sec. 6.5, we show how the elementary properties of FT change for the hyperfinite FT. In Sec. 6.6 and Sec. 6.1.2, we respectively prove the inversion theorem and give general condition to guarantee that the embedding of Sobolev-Schwartz tempered distributions preserves their FT, i.e. that the hyperfinite FT commutes with the embedding of Schwartz functions and tempered distributions. In this section, we also recall the problems of FT in the Colombeau's setting and how we overcome them. Finally, in Sec. 6.8 we give classical examples of differential and convolution equations which underscore the new possibility to transform non-tempered generalized functions. We hence prove global existence and uniqueness results of classical differential equations that includes as special cases the usual tempered solutions, but also comprise non-tempered solutions and initial conditions.

Finally, in chapter 7, we introduce the concept of a space of rapidly decreasing GSF, as well as the proper notion of FT in this space. We prove that FT in the space of a rapidly decreasing GSF is a continuous mapping from the space of a rapidly decreasing GSF into itself. We also prove that every compactly supported GSF is rapidly decreasing GSF (and vice versa). Finally, we formulate the properties of FT in the space of rapidly decreasing GSFs and prove the corresponding inversion theorem.

Some conclusions we can summarize at the closure of this thesis are the following:

1. In chapter 5, we showed how to deal with several deficiencies of the ring of Robinson-Colombeau generalized numbers $\widetilde{^\rho\mathbb{R}}$: trichotomy law for the order relations $\le$ and $<$, existence of supremum and infimum and limits of sequences with a topology generated by infinitesimal radii. In each case, we obtain a faithful generalization of the classical case of real numbers. We think that some of the ideas we presented in this article can inspire similar works in other non-Archimedean settings such as (constructive) nonstandard analysis, p-adic analysis, the Levi-Civita field, surreal numbers, etc (see e.g. [10, 9, 64, 56, 42]). Clearly, the notions introduced here open the possibility to extend classical proofs in dealing with series, analytic generalized functions, sigma-additivity in integration of generalized functions, non-Archimedean functional analysis, just to mention a few.

2. The power of a non-Archimedean language permeates the whole thesis since the beginning (e.g. by defining GF as set-theoretical maps with infinite values derivatives or in the use of sharp continuity). This power turned out to be important also for the hyperfinite FT: see the heuristic motivation of the FT in Sec. 6.4.1, Example 103 about application of the uncertainty principle to a delta distribution, or the hyperfinite FT of exponential functions in Example 96 and in Sec. 6.8.

3. The results presented here are deeply founded on a strong and flexible theory of multidimensional integration of GSF on functionally compact sets as developed in [31]: as we mentioned above, the possibility to exchange hyperlimits and integration is an important step in the proof of the Fourier inversion theorem; the possibility to compute $\varepsilon$-wise integrals on intervals is another feature used in several theorems and a key step in defining integration of compactly supported GSF.

4. It can also be worth explicitly mentioning that the definition of hyperfinite FT is based on the classical formulas used only for rapidly decreasing smooth functions and not on duality pairing. In our opinion, this is a strong simplification that even more underscores the strict analogies between ordinary smooth functions and GSF. All this in spite of the fact that the ring of scalars $\widetilde{^\rho\mathbb{R}}$ is not a field and is not totally ordered.

5. Important differences with respect to the classical theory result from the Riemann-Lebesgue Lem. 93 and the differentiation formula (6.5.1). In the former case, we explained these differences as a general consequence of integration by part formula, i.e. of the non-linear framework we are working in, see Thm. 95. For example, the FT $\mathbb{1} := \mathcal{F}(\delta)$ of Dirac's delta equals 1 at all finite points but it is necessarily compactly supported, as a consequence of the Riemann-Lebesgue lemma and the integration by parts formula, i.e. because of the non-linear setting. On the other hand, the compact support of the hyperfinite FT $\mathbb{1}$ of Dirac's delta reveals to be very important in stating and proving the preservation properties of hyperfinite FT, see Sec. 6.7. Surprisingly (the classical formula dates back at least to 1822), in Sec. 6.8 we showed that the new differentiation formula is very important to get out of the constrained world of tempered solutions.

6. Finally, Example 103 of application of the uncertainty principle, further suggests that the space ${}^{\rho}\mathcal{GC}^{\infty}(K)$ may be a useful framework for quantum mechanics, so as to have both GF and smooth functions in a space sharing several properties with the classical $L^2(\mathbb{R}^n)$ (but which, on the other hand, is a *graded* Hilbert space).

# Chapter 5

# Supremum, infimum and hyperlimits in the non-Archimedean ring of Colombeau generalized numbers

## 5.1 The Ring of Robinson Colombeau and the hypernatural numbers

### 5.1.1 The new ring of scalars

In the sequel, $I$ denotes the interval $(0,1] \subseteq \mathbb{R}$ and we will always use the variable $\varepsilon$ for elements of $I$; we also denote $\varepsilon$-dependent nets $x \in \mathbb{R}^I$ simply by $(x_\varepsilon)$. By $\mathbb{N}$ we denote the set of natural numbers, including zero.

We start by introducing a new simple non-Archimedean ring of scalars that extends the real field $\mathbb{R}$. The entire theory is constructive to a high degree, e.g. neither ultrafilter nor non-standard method are used. For all the proofs of results in this section, see [28, 29, 31, 30]. As we mentioned above, in order to accomplish the theory of hyperlimits, it is important to generalize Colombeau generalized numbers by taking an arbitrary asymptotic scale instead of the usual $\rho_\varepsilon = \varepsilon$:

**Definition 1.** Let $\rho = (\rho_\varepsilon) \in (0,1]^I$ be a net such that $(\rho_\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (in the following, such a net will be called a *gauge*), then

1. $\mathcal{I}(\rho) := \{(\rho_\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0}\}$ is called the *asymptotic gauge* generated by $\rho$.

2. If $\mathcal{P}(\varepsilon)$ is a property of $\varepsilon \in I$, we use the notation $\forall^0\varepsilon : \mathcal{P}(\varepsilon)$ to denote $\exists\varepsilon_0 \in I\, \forall\varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$. We can read $\forall^0\varepsilon$ as *for $\varepsilon$ small.*

3. We say that a net $(x_\varepsilon) \in \mathbb{R}^I$ *is $\rho$-moderate*, and we write $(x_\varepsilon) \in \mathbb{R}_\rho$ if

$$\exists (J_\varepsilon) \in \mathcal{I}(\rho) : \ x_\varepsilon = O(J_\varepsilon) \text{ as } \varepsilon \to 0^+,$$

i.e., if

$$\exists N \in \mathbb{N}\, \forall^0\varepsilon : \ |x_\varepsilon| \le \rho_\varepsilon^{-N}.$$

4. Let $(x_\varepsilon)$, $(y_\varepsilon) \in \mathbb{R}^I$, then we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if

$$\forall (J_\varepsilon) \in \mathcal{I}(\rho) : \ x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1}) \text{ as } \varepsilon \to 0^+,$$

that is if

$$\forall n \in \mathbb{N}\, \forall^0\varepsilon : \ |x_\varepsilon - y_\varepsilon| \le \rho_\varepsilon^n. \tag{5.1.1}$$

This is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations, and we can hence define

$${}^\rho\widetilde{\mathbb{R}} := \mathbb{R}_\rho / \sim_\rho,$$

which we call *Robinson-Colombeau ring of generalized numbers*. This name is justified by [59, 13]: Indeed, in [59] A. Robinson introduced the notion of moderate and negligible nets depending on an arbitrary fixed infinitesimal $\rho$ (in the framework of nonstandard analysis); independently, J.F. Colombeau, cf. e.g. [13] and references therein, studied the same concepts without using nonstandard analysis, but considering only the particular gauge $\rho_\varepsilon = \varepsilon$.

We will also use other directed sets instead of $I$: e.g. $J \subseteq I$ such that 0 is a closure point of $J$, or $I \times \mathbb{N}$. The reader can easily check that all our constructions can be repeated in these cases. We can also define an order relation on ${}^\rho\widetilde{\mathbb{R}}$ by saying that $[x_\varepsilon] \le [y_\varepsilon]$ if there exists $(z_\varepsilon) \in \mathbb{R}^I$ such that $(z_\varepsilon) \sim_\rho 0$ (we then say that $(z_\varepsilon)$ is *$\rho$-negligible*) and $x_\varepsilon \le y_\varepsilon + z_\varepsilon$ for $\varepsilon$ small. Equivalently, we have that $x \le y$ if and only if there exist representatives $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$ such that $x_\varepsilon \le y_\varepsilon$ for all $\varepsilon$. Although the order $\le$ is not total, we still have the possibility to define the infimum $[x_\varepsilon] \wedge [y_\varepsilon] := [\min(x_\varepsilon, y_\varepsilon)]$, the supremum $[x_\varepsilon] \vee [y_\varepsilon] := [\max(x_\varepsilon, y_\varepsilon)]$ of a finite number of generalized numbers. See [50] for a complete study of supremum and infimum in ${}^\rho\widetilde{\mathbb{R}}$. Henceforth, we will also use the customary notation ${}^\rho\widetilde{\mathbb{R}}^*$ for the set of invertible generalized numbers, and we write $x < y$ to say that $x \le y$ and $x - y \in {}^\rho\widetilde{\mathbb{R}}^*$. Our notations for intervals are: $[a, b] := \{x \in {}^\rho\widetilde{\mathbb{R}} \mid a \le x \le b\}$, $[a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}$, and analogously for segments $[x, y] := \{x + r \cdot (y - x) \mid r \in [0, 1]\} \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $[x, y]_{\mathbb{R}^n} = [x, y] \cap \mathbb{R}^n$. We also set $\mathbb{C}_\rho := \mathbb{R}_\rho + i \cdot \mathbb{R}_\rho$ and ${}^\rho\widetilde{\mathbb{C}} := {}^\rho\widetilde{\mathbb{R}} + i \cdot {}^\rho\widetilde{\mathbb{R}}$, where $i = \sqrt{-1}$. On the ${}^\rho\widetilde{\mathbb{R}}$-module ${}^\rho\widetilde{\mathbb{R}}^n$ we can consider the natural extension of the Euclidean norm, i.e. $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$, where $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$.

As in every non-Archimedean ring, we have the following

**Definition 2.** Let $x \in {}^\rho\widetilde{\mathbb{R}}^n$ be a generalized number, then

1. $x$ is *infinitesimal* if $|x| \le r$ for all $r \in \mathbb{R}_{>0}$. If $x = [x_\varepsilon]$, this is equivalent to $\lim_{\varepsilon \to 0^+} |x_\varepsilon| = 0$. We write $x \approx y$ if $x - y$ is infinitesimal.

2. $x$ is *finite* if $|x| \le r$ for some $r \in \mathbb{R}_{>0}$.

3. $x$ is *infinite* if $|x| \ge r$ for all $r \in \mathbb{R}_{>0}$. If $x = [x_\varepsilon]$, this is equivalent to $\lim_{\varepsilon \to 0^+} |x_\varepsilon| = +\infty$.

For example, setting $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, we have that $\mathrm{d}\rho^n \in {}^\rho\widetilde{\mathbb{R}}$, $n \in \mathbb{N}_{>0}$, is an invertible infinitesimal, whose reciprocal is $\mathrm{d}\rho^{-n} = [\rho_\varepsilon^{-n}]$, which is necessarily a positive infinite number. Of course, in the ring ${}^\rho\widetilde{\mathbb{R}}$ there exist generalized numbers which are not in any of the three classes of Def. 2, like e.g. $x_\varepsilon = \frac{1}{\varepsilon}\sin\left(\frac{1}{\varepsilon}\right)$.

**Definition 3.** We say that $x$ is a *strong infinite number* if $|x| \ge \mathrm{d}\rho^{-r}$ for some $r \in \mathbb{R}_{>0}$, whereas we say that $x$ is a *weak infinite number* if $|x| \le \mathrm{d}\rho^{-r}$ for all $r \in \mathbb{R}_{>0}$. For example, $x = -N \log \mathrm{d}\rho$, $N \in \mathbb{N}$, is a weak infinite number, whereas if $x_\varepsilon = \rho_\varepsilon^{-1}$ for $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}_{>0}$, and $x_\varepsilon = -\log \rho_\varepsilon$ otherwise, then $x$ is neither a strong nor a weak infinite number.

The following result is useful to deal with positive and invertible generalized numbers. For its proof, see e.g. [37].

**Lemma 4.** *Let $x \in {}^\rho\widetilde{\mathbb{R}}$. Then the following are equivalent:*

1. *$x$ is invertible and $x \ge 0$, i.e. $x > 0$.*

2. *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of $x$ we have $\forall^0\varepsilon :\ x_\varepsilon > 0$.*

3. *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of $x$ we have $\exists m \in \mathbb{N}\, \forall^0\varepsilon :\ x_\varepsilon > \rho_\varepsilon^m$.*

4. *There exists a representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of $x$ such that $\exists m \in \mathbb{N}\, \forall^0\varepsilon :\ x_\varepsilon > \rho_\varepsilon^m$.*

### 5.1.2 Topologies on ${}^\rho\widetilde{\mathbb{R}}^n$

As we mentioned above, on the ${}^\rho\widetilde{\mathbb{R}}$-module ${}^\rho\widetilde{\mathbb{R}}^n$ we defined $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$, where $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$. Even if this generalized norm takes values in ${}^\rho\widetilde{\mathbb{R}}$, it shares some essential properties with classical norms:

$$
\begin{aligned}
&|x| = x \vee (-x) \\
&|x| \ge 0 \\
&|x| = 0 \Rightarrow x = 0 \\
&|y \cdot x| = |y| \cdot |x| \\
&|x + y| \le |x| + |y| \\
&||x| - |y|| \le |x - y|.
\end{aligned}
$$

It is therefore natural to consider on ${}^\rho\widetilde{\mathbb{R}}^n$ a topology generated by balls defined by this generalized norm and the set of radii ${}^\rho\widetilde{\mathbb{R}}_{>0}$ of positive invertible numbers:

**Definition 5.** Let $c \in {}^{\rho}\widetilde{\mathbb{R}}^n$ then:

1. $B_r(c) := \left\{x \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid |x - c| < r\right\}$ for each $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$.

2. $B_r^{\mathrm{E}}(c) := \{x \in \mathbb{R}^n \mid |x - c| < r\}$, for each $r \in \mathbb{R}_{>0}$, denotes an ordinary Euclidean ball in $\mathbb{R}^n$ if $c \in \mathbb{R}^n$.

The relation $<$ has better topological properties as compared to the usual strict order relation $a \leq b$ and $a \neq b$ (that we will *never* use) because the set of balls $\left\{B_r(c) \mid r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0},\ c \in {}^{\rho}\widetilde{\mathbb{R}}^n\right\}$ is a base for a topology on ${}^{\rho}\widetilde{\mathbb{R}}^n$ called *sharp topology*. We will call *sharply open set* any open set in the sharp topology. The existence of infinitesimal neighborhoods (e.g. $r = \mathrm{d}\rho$) implies that the sharp topology induces the discrete topology on $\mathbb{R}$. This is a necessary result when one has to deal with continuous generalized functions which have infinite derivatives. In fact, if $f'(x_0)$ is infinite, we have $f(x) \approx f(x_0)$ only for $x \approx x_0$ , see [28]. Also open intervals are defined using the relation $<$, i.e. $(a, b) := \{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid a < x < b\}$.

**Lemma 6.** *Let $\mathfrak{R}$ be a set of radii and $x$, $y$, $z \in {}^{\rho}\widetilde{\mathbb{R}}$, then*

1. $\neg\,(x <_{\mathfrak{R}} x)$.

2. $x <_{\mathfrak{R}} y$ *and* $y <_{\mathfrak{R}} z$ *imply* $x <_{\mathfrak{R}} z$.

3. $\forall r \in \mathfrak{R}:\ 0 <_{\mathfrak{R}} r$.

The relation $<_{\mathfrak{R}}$ has better topological properties as compared to the usual strict order relation $x \leq y$ and $x \neq y$ (a relation that we will therefore never use) because of the following result:

**Theorem 7.** *The set of balls $\left\{B_r^{\mathfrak{R}}(x) \mid r \in \mathfrak{R},\ x \in {}^{\rho}\widetilde{\mathbb{R}}^n\right\}$ generated by a set of radii $\mathfrak{R}$ is a base for a topology on ${}^{\rho}\widetilde{\mathbb{R}}^n$.*

Henceforth, we will only consider the sets of radii ${}^{\rho}\widetilde{\mathbb{R}}^*_{\geq 0} = {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $\mathbb{R}_{>0}$ and will use the simplified notation $B_r(x) := B_r^{\mathfrak{R}}(x)$ if $\mathfrak{R} = {}^{\rho}\widetilde{\mathbb{R}}_{>0}$. The topology generated in the former case is called *sharp topology*, whereas the latter is called *Fermat topology*. We will call *sharply open set* any open set in the sharp topology, and *large open set* any open set in the Fermat topology; clearly, the latter is coarser than the former. It is well-known (see e.g. [3, 4, 27, 34, 31] and references therein) that this is an equivalent way to define the sharp topology usually considered in the ring of Colombeau generalized numbers. We therefore recall that the sharp topology on ${}^{\rho}\widetilde{\mathbb{R}}^n$ is Hausdorff and Cauchy complete, see e.g. [3, 34].

### 5.1.3 The language of subpoints

The following simple language allows us to simplify some proofs using steps that recall the classical real field $\mathbb{R}$, see [50]. We first introduce the notion of *subpoint*:

**Definition 8.** For subsets $J$, $K \subseteq I$ we write $K \subseteq_0 J$ if 0 is an accumulation point of $K$ and $K \subseteq J$ (we read it as: $K$ *is co-final in* $J$). Note that for any $J \subseteq_0 I$, the constructions introduced so far in Def. 1 can be repeated using nets $(x_\varepsilon)_{\varepsilon \in J}$. We indicate the resulting ring with the symbol ${}^\rho\widetilde{\mathbb{R}}^n|_J$. More generally, no peculiar property of $I = (0, 1]$ will ever be used in the following, and hence all the presented results can be easily generalized considering any other directed set. If $K \subseteq_0 J$, $x \in {}^\rho\widetilde{\mathbb{R}}^n|_J$ and $x' \in {}^\rho\widetilde{\mathbb{R}}^n|_K$, then $x'$ is called a *subpoint* of $x$, denoted as $x' \subseteq x$, if there exist representatives $(x_\varepsilon)_{\varepsilon \in J}$, $(x'_\varepsilon)_{\varepsilon \in K}$ of $x$, $x'$ such that $x'_\varepsilon = x_\varepsilon$ for all $\varepsilon \in K$. In this case we write $x' = x|_K$, $\mathrm{dom}(x') := K$, and the restriction $(-)|_K : {}^\rho\widetilde{\mathbb{R}}^n \longrightarrow {}^\rho\widetilde{\mathbb{R}}^n|_K$ is a well defined operation. In general, for $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ we set $X|_J := \{x|_J \in {}^\rho\widetilde{\mathbb{R}}^n|_J \mid x \in X\}$.

In the next definition, we introduce binary relations that hold only *on subpoints*. Clearly, this idea is inherited from nonstandard analysis, where co-final subsets are always taken in a fixed ultrafilter.

**Definition 9.** Let $x$, $y \in {}^\rho\widetilde{\mathbb{R}}$, $L \subseteq_0 I$, then we say

1. $x <_L y \ :\iff \ x|_L < y|_L$ (the latter inequality has to be meant in the ordered ring ${}^\rho\widetilde{\mathbb{R}}|_L$). We read $x <_L y$ as "$x$ *is less than* $y$ *on* $L$".

2. $x <_{\mathrm{s}} y \ :\iff \ \exists L \subseteq_0 I : \ x <_L y$. We read $x <_{\mathrm{s}} y$ as "*$x$ is less than $y$ on subpoints*".

Analogously, we can define other relations holding only on subpoints such as e.g.: $=_L$, $\in_L$, $\in_{\mathrm{s}}$, $\le_{\mathrm{s}}$, $=_{\mathrm{s}}$, $\subseteq_{\mathrm{s}}$, etc.

For example, we have

$$\begin{aligned} x \le y &\iff \forall L \subseteq_0 I : \ x \le_L y \\ x < y &\iff \forall L \subseteq_0 I : \ x <_L y, \end{aligned}$$

the former following from the definition of $\le$, whereas the latter following from Lem. 4. Moreover, if $\mathcal{P}\{x_\varepsilon\}$ is an arbitrary property of $x_\varepsilon$, then

$$\neg\left(\forall^0 \varepsilon : \ \mathcal{P}\{x_\varepsilon\}\right) \iff \exists L \subseteq_0 I \, \forall \varepsilon \in L : \ \neg \mathcal{P}\{x_\varepsilon\}. \tag{5.1.2}$$

Note explicitly that, generally speaking, relations on subpoints, such as $\le_{\mathrm{s}}$ or $=_{\mathrm{s}}$, do not inherit the same properties of the corresponding relations for points. So, e.g., both $=_{\mathrm{s}}$ and $\le_{\mathrm{s}}$ are not transitive relations.

The next result clarifies how to equivalently write a negation of an inequality or of an equality using the language of subpoints.

**Lemma 10.** *Let $x$, $y \in {}^\rho\widetilde{\mathbb{R}}$, then*

1. $x \not\le y \iff x >_{\mathrm{s}} y$
2. $x \not< y \iff x \ge_{\mathrm{s}} y$
3. $x \ne y \iff x >_{\mathrm{s}} y$ *or* $x <_{\mathrm{s}} y$

Using the language of subpoints, we can write different forms of dichotomy or trichotomy laws for inequality.

**Lemma 11.** *Let $x, y \in {}^{\rho}\widetilde{\mathbb{R}}$, then*

1. $x \le y$ *or* $x >_{\mathrm{s}} y$
2. $\neg(x >_{\mathrm{s}} y$ *and* $x \le y)$
3. $x = y$ *or* $x <_{\mathrm{s}} y$ *or* $x >_{\mathrm{s}} y$
4. $x \le y \ \Rightarrow \ x <_{\mathrm{s}} y$ *or* $x = y$
5. $x \le_{\mathrm{s}} y \iff x <_{\mathrm{s}} y$ *or* $x =_{\mathrm{s}} y$.

*Proof.* 1 and 2 follows directly from Lem. 10. To prove 3, we can consider that $x >_{\mathrm{s}} y$ or $x \not>_{\mathrm{s}} y$. In the second case, Lem. 10 implies $x \le y$. If $y \le x$ then $x = y$; otherwise, once again by Lem. 10, we get $x <_{\mathrm{s}} y$. To prove 4, assume that $x \le y$ but $x \not<_{\mathrm{s}} y$, then $x \ge y$ by Lem. 10.1 and hence $x = y$. The implication $\Leftarrow$ of 5 is trivial. On the other hand, if $x \le_{\mathrm{s}} y$ and $x \not<_{\mathrm{s}} y$, then $y \le x$ from Lem. 10.1, and hence $x =_{\mathrm{s}} y$. □

As usual, we note that these results can also be trivially repeated for the ring ${}^{\rho}\widetilde{\mathbb{R}}|_L$. So, e.g., we have $x \not\le_L y$ if and only if $\exists J \subseteq_0 L : \ x >_J y$, which is the analog of Lem. 10.1 for the ring ${}^{\rho}\widetilde{\mathbb{R}}|_L$.

The second form of trichotomy (which for ${}^{\rho}\widetilde{\mathbb{R}}$ can be more correctly named as *quadrichotomy*) *is* stated as follows:

**Lemma 12.** *Let $x = [x_\varepsilon]$, $y = [y_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$, then*

1. $x \le y$ *or* $x \ge y$ *or* $\exists L \subseteq_0 I : \ L^c \subseteq_0 I, \ x \ge_L y$ *and* $x \le_{L^c} y$
2. *If for all $L \subseteq_0 I$ the following implication holds*
$$x \le_L y, \ \text{or } x \ge_L y \ \Rightarrow \ \forall^0 \varepsilon \in L : \ \mathcal{P}\{x_\varepsilon, y_\varepsilon\}, \tag{5.1.3}$$
*then* $\forall^0 \varepsilon : \ \mathcal{P}\{x_\varepsilon, y_\varepsilon\}$.
3. *If for all $L \subseteq_0 I$ the following implication holds*
$$x <_L y, \ \text{or } x >_L y \text{ or } x =_L y \ \Rightarrow \ \forall^0 \varepsilon \in L : \ \mathcal{P}\{x_\varepsilon, y_\varepsilon\}, \tag{5.1.4}$$
*then* $\forall^0 \varepsilon : \ \mathcal{P}\{x_\varepsilon, y_\varepsilon\}$.

*Proof.* 1: if $x \not\le y$, then $x >_{\mathrm{s}} y$ from Lem. 10.1. Let $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$ be two representatives, and set $L := \{\varepsilon \in I \mid x_\varepsilon \ge y_\varepsilon\}$. The relation $x >_{\mathrm{s}} y$ implies that $L \subseteq_0 I$. Clearly, $x \ge_L y$ (but note that in general we cannot prove $x >_L y$). If $L^c \not\subseteq_0 I$, then $(0, \varepsilon_o] \subseteq L$ for some $\varepsilon_0$, i.e. $x \ge y$. On the contrary, if $L^c \subseteq_0 I$, then $x \le_{L^c} y$.

2: Property 1 states that we have three cases. If $x_\varepsilon \le y_\varepsilon$ for all $\varepsilon \le \varepsilon_0$, then it suffices to set $L := (0, \varepsilon_0]$ in (5.1.3) to get the claim. Similarly, we can proceed if $x \geq y$. Finally, if $x \geq_L y$ and $x \leq_{L^c} y$, then we can apply (5.1.3) both with $L$ and $L^c$ to obtain

$$\forall^0 \varepsilon \in L: \ \mathcal{P}\left\{x_\varepsilon, y_\varepsilon\right\}$$
$$\forall^0 \varepsilon \in L^c: \ \mathcal{P}\left\{x_\varepsilon, y_\varepsilon\right\},$$

from which the claim directly follows.

3: By contradiction, assume

$$\forall \varepsilon \in L: \ \neg\mathcal{P}\left\{x_\varepsilon, y_\varepsilon\right\}, \tag{5.1.5}$$

for some $L \subseteq_0 I$. We apply 1 to the ring ${}^\rho\widetilde{\mathbb{R}}|_L$ to obtain the following three cases:

$$x \leq_L y \text{ or } x \geq_L y \text{ or } \exists J \subseteq_0 L: \ J^c \subseteq_0 L, \ x \geq_J y \text{ and } x \leq_{J^c} y. \tag{5.1.6}$$

If $x \leq_L y$, by Lem. 11.4 for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$, this case splits into two sub-cases: $x =_L y$ or $\exists K \subseteq_0 L: \ x <_K y$. If the former holds, using (5.1.4) we get $\mathcal{P}\left\{x_\varepsilon, y_\varepsilon\right\} \ \forall^0 \varepsilon \in L$, which contradicts (5.1.5). If $x <_K y$, then $K \subseteq_0 I$ and we can apply (5.1.5) with $K$ to get $\mathcal{P}\left\{x_\varepsilon, y_\varepsilon\right\} \ \forall^0 \varepsilon \in K$, which again contradicts (5.1.5) because $K \subseteq_0 L$. Similarly we can proceed with the other three cases stated in (5.1.6). □

Property Lem. 12.2 represents a typical replacement of the usual dichotomy law in $\mathbb{R}$: for arbitrary $L \subseteq_0 I$, we can assume to have two cases: either $x \leq_L y$ or $x \geq_L y$. If in both cases we are able to prove $\mathcal{P}\{x_\varepsilon, y_\varepsilon\}$ for $\varepsilon \in L$ small, then we always get that this property holds for all $\varepsilon$ small. Similarly, we can use the strict trichotomy law stated in 3.

### 5.1.4 Open, closed and bounded sets generated by nets

A natural way to obtain sharply open, closed and bounded sets in ${}^\rho\widetilde{\mathbb{R}}^n$ is by using a net $(A_\varepsilon)$ of subsets $A_\varepsilon \subseteq \mathbb{R}^n$. We have two ways of extending the membership relation $x_\varepsilon \in A_\varepsilon$ to generalized points $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$ (cf. [55, 29]).

**Definition 13.** Let $(A_\varepsilon)$ be a net of subsets of $\mathbb{R}^n$, then

1. $[A_\varepsilon] := \left\{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid \forall^0 \varepsilon: \ x_\varepsilon \in A_\varepsilon\right\}$ is called the *internal set* generated by the net $(A_\varepsilon)$.

2. Let $(x_\varepsilon)$ be a net of points of $\mathbb{R}^n$, then we say that $x_\varepsilon \in_\varepsilon A_\varepsilon$, and we read it as $(x_\varepsilon)$ *strongly belongs to* $(A_\varepsilon)$, if

   (a) $\forall^0 \varepsilon: \ x_\varepsilon \in A_\varepsilon$.

   (b) If $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$, then also $x'_\varepsilon \in A_\varepsilon$ for $\varepsilon$ small.

   Moreover, we set $\langle A_\varepsilon \rangle := \left\{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_\varepsilon A_\varepsilon\right\}$, and we call it the *strongly internal set* generated by the net $(A_\varepsilon)$.

3. We say that the internal set $K = [A_\varepsilon]$ is *sharply bounded* if there exists $M \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $K \subseteq B_M(0)$.

4. Finally, we say that the $(A_\varepsilon)$ is a *sharply bounded net* if there exists $N \in \mathbb{R}_{>0}$ such that $\forall^0\varepsilon\,\forall x \in A_\varepsilon : \ |x| \le \rho_\varepsilon^{-N}$.

Therefore, $x \in [A_\varepsilon]$ if there exists a representative $[x_\varepsilon] = x$ such that $x_\varepsilon \in A_\varepsilon$ for $\varepsilon$ small, whereas this membership is independent from the chosen representative in case of strongly internal sets. An internal set generated by a constant net $A_\varepsilon = A \subseteq \mathbb{R}^n$ will simply be denoted by $[A]$.

The following theorem (cf. [55, 29, 31]) shows that internal and strongly internal sets have dual topological properties:

**Theorem 14.** *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x_\varepsilon \in \mathbb{R}^n$. Then we have*

1. *$[x_\varepsilon] \in [A_\varepsilon]$ if and only if $\forall q \in \mathbb{R}_{>0}\,\forall^0\varepsilon : \ d(x_\varepsilon, A_\varepsilon) \le \rho_\varepsilon^q$. Therefore $[x_\varepsilon] \in [A_\varepsilon]$ if and only if $[d(x_\varepsilon, A_\varepsilon)] = 0 \in {}^\rho\widetilde{\mathbb{R}}$.*

2. *$[x_\varepsilon] \in \langle A_\varepsilon \rangle$ if and only if $\exists q \in \mathbb{R}_{>0}\,\forall^0\varepsilon : \ d(x_\varepsilon, A_\varepsilon^c) > \rho_\varepsilon^q$, where $A_\varepsilon^c := \mathbb{R}^n \setminus A_\varepsilon$. Therefore, if $(d(x_\varepsilon, A_\varepsilon^c)) \in \mathbb{R}_\rho$, then $[x_\varepsilon] \in \langle A_\varepsilon \rangle$ if and only if $[d(x_\varepsilon, A_\varepsilon^c)] > 0$.*

3. *$[A_\varepsilon]$ is sharply closed.*

4. *$\langle A_\varepsilon \rangle$ is sharply open.*

5. *$[A_\varepsilon] = [\mathrm{cl}\,(A_\varepsilon)]$, where $\mathrm{cl}\,(S)$ is the closure of $S \subseteq \mathbb{R}^n$.*

6. *$\langle A_\varepsilon \rangle = \langle \mathrm{int}(A_\varepsilon) \rangle$, where $\mathrm{int}\,(S)$ is the interior of $S \subseteq \mathbb{R}^n$.*

For example, it is not hard to show that the closure in the sharp topology of a ball of center $c = [c_\varepsilon]$ and radius $r = [r_\varepsilon] > 0$ is

$$\overline{B_r(c)} = \left\{ x \in {}^\rho\widetilde{\mathbb{R}}^d \mid |x - c| \le r \right\} = \left[ \overline{B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon)} \right], \tag{5.1.7}$$

whereas

$$B_r(c) = \left\{ x \in {}^\rho\widetilde{\mathbb{R}}^d \mid |x - c| < r \right\} = \langle B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon) \rangle.$$

Using internal sets and adopting ideas similar to those used in proving Lem. 12, we also have the following form of dichotomy law:

**Lemma 15.** *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$. Then we have:*

1. *$x \in [A_\varepsilon]$ or $x \in [A_\varepsilon^c]$ or $\exists L \subseteq_0 I : \ L^c \subseteq_0 I, \ x \in_L [A_\varepsilon], \ x \in_{L^c} [A_\varepsilon^c]$*

2. *If for all $L \subseteq_0 I$ the following implication holds*

$$x \in_L [A_\varepsilon] \text{ or } x \in_L [A_\varepsilon^c] \ \Rightarrow \ \forall^0\varepsilon \in L : \ \mathcal{P}\{x_\varepsilon\},$$

*then $\forall^0\varepsilon : \ \mathcal{P}\{x_\varepsilon\}$.*

*Proof.* 1: If $x \notin [A_\varepsilon^c]$, then $x_\varepsilon \in A_\varepsilon$ for all $\varepsilon \in K$ and for some $K \subseteq_0 I$. Set $L := \{\varepsilon \in I \mid x_\varepsilon \in A_\varepsilon\}$, so that $K \subseteq L \subseteq_0 I$. We have $x \in_L [A_\varepsilon]$. If $L^c \not\subseteq_0 I$, then $(0, \varepsilon_0] \subseteq L$ for some $\varepsilon_0$, i.e. $x \in [A_\varepsilon]$. On the contrary, if $L^c \subseteq_0 I$, then $x \in_{L^c} [A_\varepsilon^c]$.

2: We can proceed as in the proof of Lem. 12.2 using 1. $\square$

## 5.2 Hypernatural numbers

We start by defining the set of hypernatural numbers in ${}^{\rho}\widetilde{\mathbb{R}}$ and the set of $\rho$-moderate nets of natural numbers:

**Definition 16.** We set

1. ${}^{\rho}\widetilde{\mathbb{N}} := \left\{[n_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall\varepsilon\right\}$
2. $\mathbb{N}_\rho := \{(n_\varepsilon) \in \mathbb{R}_\rho \mid n_\varepsilon \in \mathbb{N} \quad \forall\varepsilon\}$.

Therefore, $n \in {}^{\rho}\widetilde{\mathbb{N}}$ if and only if there exists $(x_\varepsilon) \in \mathbb{R}_\rho$ such that $n = [\mathrm{int}(|x_\varepsilon|)]$. Clearly, $\mathbb{N} \subset {}^{\rho}\widetilde{\mathbb{N}}$. Note that the integer part function $\mathrm{int}(-)$ is not well-defined on ${}^{\rho}\widetilde{\mathbb{R}}$. In fact, if $x = 1 = \left[1 - \rho_\varepsilon^{1/\varepsilon}\right] = \left[1 + \rho_\varepsilon^{1/\varepsilon}\right]$, then $\mathrm{int}\left(1 - \rho_\varepsilon^{1/\varepsilon}\right) = 0$ whereas $\mathrm{int}\left(1 + \rho_\varepsilon^{1/\varepsilon}\right) = 1$, for $\varepsilon$ sufficiently small. Similar counter examples can be set for floor and ceiling functions. However, the nearest integer function is well defined on ${}^{\rho}\widetilde{\mathbb{N}}$, as proved in the following

**Lemma 17.** *Let $(n_\varepsilon) \in \mathbb{N}_\rho$ and $(x_\varepsilon) \in \mathbb{R}_\rho$ be such that $[n_\varepsilon] = [x_\varepsilon]$. Let $\mathrm{rpi} : \mathbb{R} \longrightarrow \mathbb{N}$ be the function rounding to the nearest integer with tie breaking towards positive infinity, i.e. $rpi(x) = \lfloor x + \frac{1}{2} \rfloor$. Then $\mathrm{rpi}(x_\varepsilon) = n_\varepsilon$ for $\varepsilon$ small. The same result holds using $\mathrm{rni} : \mathbb{R} \longrightarrow \mathbb{N}$, the function rounding half towards $-\infty$.*

*Proof.* We have $\mathrm{rpi}(x) = \lfloor x + \frac{1}{2} \rfloor$, where $\lfloor - \rfloor$ is the floor function. For $\varepsilon$ small, $\rho_\varepsilon < \frac{1}{2}$ and, since $[n_\varepsilon] = [x_\varepsilon]$, always for $\varepsilon$ small, we also have $n_\varepsilon - \rho_\varepsilon + \frac{1}{2} < x_\varepsilon + \frac{1}{2} < n_\varepsilon + \rho_\varepsilon + \frac{1}{2}$. But $n_\varepsilon \le n_\varepsilon - \rho_\varepsilon + \frac{1}{2}$ and $n_\varepsilon + \rho_\varepsilon + \frac{1}{2} < n_\varepsilon + 1$. Therefore $\lfloor x_\varepsilon + \frac{1}{2} \rfloor = n_\varepsilon$. An analogous argument can be applied to $\mathrm{rni}(-)$. $\square$

Actually, this lemma does not allow us to define a *nearest integer* function $\mathrm{ni} : {}^{\rho}\widetilde{\mathbb{N}} \longrightarrow \mathbb{N}_\rho$ as $\mathrm{ni}([x_\varepsilon]) := \mathrm{rpi}(x_\varepsilon)$ because if $[x_\varepsilon] = [n_\varepsilon]$, the equality $n_\varepsilon = \mathrm{rpi}(x_\varepsilon)$ holds only for $\varepsilon$ small. A simpler approach is to choose a representative $(n_\varepsilon) \in \mathbb{N}_\rho$ for each $x \in {}^{\rho}\widetilde{\mathbb{N}}$ and to define $\mathrm{ni}(x) := (n_\varepsilon)$. Clearly, we must consider the net $(\mathrm{ni}(x)_\varepsilon)$ only for $\varepsilon$ small, such as in equalities of the form $x = [\mathrm{ni}(x)_\varepsilon]$. This is what we do in the following

**Definition 18.** The *nearest integer function* $\mathrm{ni}(-)$ is defined by:

1. $\mathrm{ni} : {}^{\rho}\widetilde{\mathbb{N}} :\longrightarrow \mathbb{N}_\rho$
2. If $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{N}}$ and $\mathrm{ni}([x_\varepsilon]) = (n_\varepsilon)$ then $\forall^0\varepsilon :\ n_\varepsilon = \mathrm{rpi}(x_\varepsilon)$.

In other words, if $x \in {}^{\rho}\widetilde{\mathbb{N}}$, then $x = [\mathrm{ni}(x)_\varepsilon]$ and $\mathrm{ni}(x)_\varepsilon \in \mathbb{N}$ for all $\varepsilon$. Another possibility is to formulate Lem. 17 as

$$[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{N}} \quad \iff \quad [x_\varepsilon] = [\mathrm{rpi}(x_\varepsilon)].$$

Therefore, without loss of generality we may always suppose that $x_\varepsilon \in \mathbb{N}$ whenever $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{N}}$.

*Remark* 19.

1. ${}^{\sigma}\widetilde{\mathbb{N}}$, with the order $\le$ induced by ${}^{\sigma}\widetilde{\mathbb{R}}$, is a directed set; it is closed with respect to sum and product although recursive definitions using ${}^{\sigma}\widetilde{\mathbb{N}}$ are not possible.

2. In ${}^{\sigma}\widetilde{\mathbb{N}}$ we can find several chains (totally ordered subsets) such as: $\mathbb{N}$, $\mathbb{N}\cdot[\mathrm{int}(\rho_\varepsilon^{-k})]$ for a fixed $k\in\mathbb{N}$, $\{[\mathrm{int}(\rho_\varepsilon^{-k})]\mid k\in\mathbb{N}\}$.

3. Generally speaking, if $m$, $n\in{}^{\rho}\widetilde{\mathbb{N}}$, $m^n\notin{}^{\rho}\widetilde{\mathbb{N}}$ because the net $(m_\varepsilon^{n_\varepsilon})$ can grow faster than any power $(\rho_\varepsilon^{-K})$. However, if we take two gauges $\sigma$, $\rho$ satisfying $\sigma\le\rho$, using the net $\left(\sigma_\varepsilon^{-1}\right)$ we can measure infinite nets that grow faster than $(\rho_\varepsilon^{-K})$ because $\sigma_\varepsilon^{-1}\ge\rho_\varepsilon^{-1}$ for $\varepsilon$ small. Therefore, we can take $m$, $n\in{}^{\sigma}\widetilde{\mathbb{N}}$ such that $(\mathrm{ni}(m)_\varepsilon)$, $(\mathrm{ni}(n)_\varepsilon)\in\mathbb{R}_\rho$; we think at $m$, $n$ as $\sigma$-hypernatural numbers growing at most polynomially with respect to $\rho$. Then, it is not hard to prove that if $\rho$ is an arbitrary gauge, and we consider the auxiliary gauge $\sigma_\varepsilon:=\rho_\varepsilon^{e^{1/\rho_\varepsilon}}$. then $m^n\in{}^{\sigma}\widetilde{\mathbb{N}}$.

4. If $m\in{}^{\rho}\widetilde{\mathbb{N}}$, then $1^m:=[(1+z_\varepsilon)^{m_\varepsilon}]$, where $(z_\varepsilon)$ is $\rho$-negligible, is well defined and $1^m=1$. In fact, $\log(1+z_\varepsilon)^{m_\varepsilon}$ is asymptotically equal to $m_\varepsilon z_\varepsilon\to0$, and this shows that $((1+z_\varepsilon)^{m_\varepsilon})$ is moderate. Finally, $|(1+z_\varepsilon)^{m_\varepsilon}-1|\le|z_\varepsilon|\,m_\varepsilon(1+z_\varepsilon)^{m_\varepsilon-1}$ by the mean value theorem.

## 5.3 Supremum and Infimum in ${}^{\rho}\widetilde{\mathbb{R}}$

To solve the problems we explained in the introduction of this article, it is important to generalize at least two main existence theorems for limits: the Cauchy criterion and the existence of a limit of a bounded monotone sequence. The latter is clearly related to the existence of supremum and infimum, which cannot be always guaranteed in the non-Archimedean ring ${}^{\rho}\widetilde{\mathbb{R}}$. As we will see more clearly later (see also [24]), to arrive at these existence theorems, the notion of supremum, i.e. the least upper bound, is not the correct one. More appropriately, we can associate a notion of close supremum (and close infimum) to every topology generated by a set of radii (see Def. 5).

**Definition 20.** Let $\mathfrak{R}$ be a set of radii and let $\tau$ be the topology on ${}^{\rho}\widetilde{\mathbb{R}}$ generated by $\mathfrak{R}$. Let $P\subseteq{}^{\rho}\widetilde{\mathbb{R}}$, then we say that $\tau$ *separates points of* $P$ if

$$\forall p,q\in P:\ p\neq q\ \Rightarrow\ \exists A,B\in\tau:\ p\in A,\ q\in B,\ A\cap B=\emptyset,$$

i.e. if $P$ with the topology induced by $\tau$ is Hausdorff.

**Definition 21.** Let $\tau$ be a topology on ${}^{\rho}\widetilde{\mathbb{R}}$ generated by a set of radii $\mathfrak{R}$ that separates points of $P\subseteq{}^{\rho}\widetilde{\mathbb{R}}$ and let $S\subseteq{}^{\rho}\widetilde{\mathbb{R}}$. Then, we say that $\sigma$ is $(\tau,P)$-*supremum of* $S$ if

1. $\sigma\in P$;

2. $\forall s \in S: \ s \le \sigma$;

3. $\sigma$ is a point of closure of $S$ in the topology $\tau$, i.e. if $\forall A \in \tau: \ \sigma \in A \ \Rightarrow \exists \bar{s} \in S \cap A$.

Similarly, we say that $\iota$ is $(\tau, P)$-*infimum of* $S$ if

1. $\iota \in P$;

2. $\forall s \in S: \ \iota \le s$;

3. $\iota$ is a point of closure of $S$ in the topology $\tau$, i.e. if $\forall A \in \tau: \ \iota \in A \ \Rightarrow \exists \bar{s} \in S \cap A$.

In particular, if $\tau$ is the sharp topology and $P = {}^{\rho}\widetilde{\mathbb{R}}$, then following [24], we simply call the $(\tau, P)$-supremum, the *close supremum* (the adjective close will be omitted if it will be clear from the context) or the *sharp supremum* if we want to underline the dependency on the topology. Analogously, if $\tau$ is the Fermat topology and $P = \mathbb{R}$, then we call the $(\tau, P)$-supremum the *Fermat supremum*. Note that 3 implies that if $\sigma$ is $(\tau, P)$-supremum of $S$, then necessarily $S \ne \emptyset$.

*Remark* 22.

1. Let $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, then from Def. 5 and Thm. 7 we can prove that $\sigma$ is the $(\tau, P)$-supremum of $S$ if and only if

   (a) $\forall s \in S: \ s \le \sigma$;

   (b) $\forall r \in \mathfrak{R} \, \exists \bar{s} \in S: \ \sigma - r \le \bar{s}$.

   In particular, for the sharp supremum, 1b is equivalent to

   $$\forall q \in \mathbb{N} \, \exists \bar{s} \in S: \ \sigma - \mathrm{d}\rho^{q} \le \bar{s}. \tag{5.3.1}$$

   In the following of this article, we will also mainly consider the sharp topology and the corresponding notions of sharp supremum and infimum.

2. If there exists the sharp supremum $\sigma$ of $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ and $\sigma \notin S$, then from (5.3.1) it follows that $S$ is necessarily an infinite set. In fact, applying (5.3.1) with $q_1 := 1$ we get the existence of $\bar{s}_1 \in S$ such that $\sigma - \mathrm{d}\rho^{q_1} < \bar{s}_1$. We have $\bar{s}_1 \ne \sigma$ because $\sigma \notin S$. Hence, Lem. 10.3 and Def. 21.2 yield that $\bar{s}_1 <_{\mathrm{s}} \sigma$. Therefore, $\sigma - \bar{s}_1 \ge_{\mathrm{s}} \mathrm{d}\rho^{q_2}$ for some $q_2 > q_1$. Applying again (5.3.1) we get $\sigma - \mathrm{d}\rho^{q_2} < \bar{s}_2$ for some $\bar{s}_2 \in S \setminus \{\bar{s}_1\}$. Recursively, this process proves that $S$ is infinite. On the other hand, if $S = \{s_1, \ldots, s_n\}$ and $s_i = [s_{i\varepsilon}]$, then $\sup\left([\{s_{1\varepsilon}, \ldots, s_{n\varepsilon}\}]\right) = s_1 \vee \ldots \vee s_n$. In fact, $s_1 \vee \ldots \vee s_n = [\max_{i=1,\ldots,n} s_{n\varepsilon}] \in [\{s_{1\varepsilon}, \ldots, s_{n\varepsilon}\}]$.

3. If $\exists \sup(S) = \sigma$, then there also exists the $\sup(\mathrm{interl}(S)) = \sigma$, where (see [55]) we recall that

   $$\mathrm{interl}(S) := \left\{ \sum_{j=1}^{m} e_{S_j} s_j \mid m \in \mathbb{N}, \ S_j \subseteq_0 I, \ s_j \in S \ \forall j \right\}, \ e_S := [1_S] \in {}^{\rho}\widetilde{\mathbb{R}}$$

($1_S$ is the characteristic function of $S \subseteq I$). This follows from $S \subseteq \text{interl}(S)$. Vice versa, if $\exists \sup(\text{interl}(S)) = \sigma$ and $\text{interl}(S) \subseteq S$ (e.g. if $S$ is an internal or strongly internal set), then also $\exists \sup(S) = \sigma$.

**Theorem 23.** *There is at most one sharp supremum of $S$, which is denoted by* $\sup(S)$.

*Proof.* Assume that $\sigma_1$ and $\sigma_2$ are supremum of $S$. That is Def. 21.2 and (5.3.1) hold both for $\sigma_1$, $\sigma_2$. Then, for all fixed $q \in \mathbb{N}$, there exists $\bar{s}_2 \in S$ such that $\sigma_2 - \mathrm{d}\rho^q \le \bar{s}_2$. Hence $\bar{s}_2 \le \sigma_1$ because $\bar{s}_2 \in S$. Analogously, we have that $\sigma_1 - \mathrm{d}\rho^q \le \bar{s}_1 \le \sigma_2$ for some $\bar{s}_1 \in S$. Therefore, $\sigma_2 - \mathrm{d}\rho^q \le \sigma_1 \le \sigma_2 + \mathrm{d}\rho^q$, and this implies $\sigma_1 = \sigma_2$ since $q \in \mathbb{N}$ is arbitrary. ☐

In [24], the notation $\overline{\sup}(S)$ is used for the close supremum. On the other hand, we will *never* use the notion of supremum as least upper bound. For these reasons, we prefer to use the simpler notation $\sup(S)$. Similarly, we use the notation $\inf(S)$ for the close (or sharp) infimum. From Rem. 22.1a and 1b it follows that

$$\inf(S) = -\sup(-S) \tag{5.3.2}$$

in the sense that the former exists if and only if the latter exists and in that case they are equal. For this reason, in the following we only study the supremum.

**Example 24.**

1. Let $K = [K_\varepsilon] \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}$ be a functionally compact set (see Def. 69), i.e. $K \subseteq B_M(0)$ for some $M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $K_\varepsilon \Subset \mathbb{R}$ for all $\varepsilon$. We can then define $\sigma_\varepsilon := \sup(K_\varepsilon) \in K_\varepsilon$. From $K \subseteq B_M(0)$, we get $\sigma := [\sigma_\varepsilon] \in K$. It is not hard to prove that $\sigma = \sup(K) = \max(K)$. Analogously, we can prove the existence of the sharp minimum of $K$.

2. If $S = (a, b)$, where $a$, $b \in {}^{\rho}\widetilde{\mathbb{R}}$ and $a \le b$, then $\sup(S) = b$ and $\inf(S) = a$.

3. If $S = \left\{ \frac{1}{n} \mid n \in {}^{\rho}\widetilde{\mathbb{N}} \right\}$, then $\inf(S) = 0$.

4. Like in several other non-Archimedean rings, both sharp supremum and infimum of the set $D_\infty$ of all infinitesimals do not exist. In fact, by contradiction, if $\sigma$ were the sharp supremum of $D_\infty$, then from (5.3.1) for $q = 1$ we would get the existence of $\bar{h} \in D_\infty$ such that $\sigma \le \bar{h} + \mathrm{d}\rho$. But then $\sigma \in D_\infty$, so also $2\sigma \in D_\infty$. Therefore, we get $2\sigma \le \sigma$ because $\sigma$ is an upper bound of $D_\infty$, and hence $\sigma = 0 \ge \mathrm{d}\rho$, a contradiction. Similarly, one can prove that there does not exist the infimum of this set.

5. Let $S = (0,1)_{\mathbb{R}} = \{x \in \mathbb{R} \mid 0 < x < 1\}$, then clearly $\sigma = 1$ is the Fermat supremum of $S$ whereas there does not exist the sharp supremum of $S$. Indeed, if $\sigma = \sup(S)$, then $s \le \sigma \le \bar{s} + \mathrm{d}\rho$ for all $s \in S$ and for some $\bar{s} \in S$. Taking any $s \in (\bar{s}, 1)_{\mathbb{R}} \subseteq S$ we get $s \le \sigma \le \bar{s} + \mathrm{d}\rho$, which, for $\varepsilon \to 0$, implies $s \le \bar{s}$ because $s$, $\bar{s} \in \mathbb{R}$. This contradicts $s \in (\bar{s}, 1)$. In particular, 1 is not the sharp supremum. This example shows the importance of

Def. 21, i.e. that the best notion of supremum in a non-Archimedean setting depends on a fixed topology.

6. Let $S = (0,1) \cup \{\hat{s}\}$ where $\hat{s}|_L = 2$, $\hat{s}|_{L^c} = \frac{1}{2}$, $L \subseteq_0 I$, $L^c \subseteq_0 I$, then $\nexists \sup(S)$. In fact, if $\exists \sigma := \sup(S)$, then $\sigma|_L \geq \hat{s}|_L = 2$ and $\sigma|_{L^c} = 1$. Assume that $\exists \bar{s} \in S : \ \sigma - \mathrm{d}\rho \leq \bar{s}$, then $2 - \mathrm{d}\rho|_L \leq \sigma|_L - \mathrm{d}\rho|_L \leq \bar{s}|_L$. Thereby, $\bar{s}|_L > \frac{3}{2}$ and hence $\bar{s} \notin (0,1)$ and $\bar{s} = \hat{s}$. We hence get $\sigma|_{L^c} - \mathrm{d}\rho|_{L^c} \leq \hat{s}|_{L^c}$, i.e. $1 - \mathrm{d}\rho|_{L^c} \leq \frac{1}{2}$, which is impossible. We can intuitively say that the subpoint $\hat{s}|_L$ creates a "$\varepsilon$-hole" (i.e. a "hole" only for some $\varepsilon$) on the right of $S$ and hence $S$ is not "an $\varepsilon$-continuum" on this side. Finally note that the point $u|_L := 2$ and $u|_{L^c} := 1$ is the least upper bound of $S$.

**Lemma 25.** *Let $A$, $B \subseteq {}^\rho\widetilde{\mathbb{R}}$, then*

1. $\forall \lambda \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ \sup(\lambda A) = \lambda \sup(A)$, *in the sense that one supremum exists if and only if the other one exists, and in that case they coincide;*

2. $\forall \lambda \in {}^\rho\widetilde{\mathbb{R}}_{<0} : \ \sup(\lambda A) = \lambda \inf(A)$, *in the sense that one supremum/infimum exists if and only if the other one exists, and in that case they coincide;*

*Moreover, if* $\exists \sup(A)$, $\sup(B)$, *then:*

3. *If* $A \subseteq B$, *then* $\sup(A) \leq \sup(B)$*;*

4. $\sup(A + B) = \sup(A) + \sup(B)$*;*

5. *If* $A$, $B \subseteq {}^\rho\widetilde{\mathbb{R}}_{\geq 0}$, *then* $\sup(A \cdot B) = \sup(A) \cdot \sup(B)$.

*Proof.* 1: If $\exists \sup(\lambda A)$, then we have $a \leq \frac{1}{\lambda} \sup(\lambda A)$ for all $a \in A$. For all $q \in \mathbb{N}$, we can find $\bar{a} \in A$ such that $\sup(\lambda A) - \lambda \bar{a} \leq \mathrm{d}\rho^q$. Thereby, $\frac{1}{\lambda} \sup(\lambda A) - \bar{a} \leq \frac{1}{\lambda} \mathrm{d}\rho^q \to 0$ as $q \to +\infty$ because $\lambda$ is moderate. This proves that $\exists \sup(A) = \frac{1}{\lambda} \sup(\lambda A)$. Similarly, we can prove the opposite implication.

2: From 1 and (5.3.2) we get: $\sup(\lambda A) = \sup(-\lambda(-A)) = -\lambda \sup(-A) = \lambda \inf(A)$.

3: By contradiction, using Lem. 10.1, if $\sup(A) >_L \sup(B)$ for some $L \subseteq_0 I$, then $\sup(A) - \sup(B) >_L \mathrm{d}\rho^q$ for some $q \in \mathbb{N}$ by Lem. 4 for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$. Property (5.3.1) yields $\sup(A) - \mathrm{d}\rho^q \leq \bar{a}$ for some $\bar{a} \in A$, and $\bar{a} \leq \sup(B)$ because $A \subseteq B$. Thereby, $\sup(A) - \sup(B) \leq \mathrm{d}\rho^q$, which implies $\mathrm{d}\rho^q <_L \mathrm{d}\rho^q$, a contradiction.

4 and 5 follow easily from Def. 21.2 and (5.3.1). $\square$

In the next section, we introduce in the non-Archimedean framework ${}^\rho\widetilde{\mathbb{R}}$ how to approximate $\sup(S)$ of $S \subseteq {}^\rho\widetilde{\mathbb{R}}$ using points of $S$ and upper bounds, and the non-Archimedean analogous of the notion of upper bound.

### 5.3.1 Approximations of *Sup*, completeness from above and Archimedean upper bounds

In the real field, we have the following peculiar properties:

1. The notion of least upper bound coincides with that of close supremum, i.e. it satisfies property (4.0.1). We can hence question when these two notions coincide also in ${}^{\rho}\widetilde{\mathbb{R}}$. Example 24.6 shows that the answer is not trivial. A first solution of this problem is already contained in [24, Prop. 1.4], where it is shown that the close supremum, assuming that it exists, coincides with the least upper bound.

2. The notion of upper bound in $\mathbb{R}$ is very useful because it entails the existence of the supremum. Clearly, since there are infinite upper bounds but only one supremum, the notion of upper bound results to be really useful in estimates with inequalities. Moreover, in the ring ${}^{\rho}\widetilde{\mathbb{R}}$, the presence of infinite numbers (of different magnitudes) allows one to have trivial upper bounds, such as in the case $S = (0, 1)$ and $M = \mathrm{d}\rho^{-1}$, or $S = (0, \mathrm{d}\rho^{-1})$ and $M = \mathrm{d}\rho^{-2}$. Therefore, we can also investigate whether we can consider non trivial upper bounds, i.e. numbers which are, intuitively, of the same order of magnitude of the elements of $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$. On the other hand, example 24.6 shows that with respect to any reasonable definition of "same order of magnitude", the upper bound $m = 3$ must be of the same order of any point in $S$, although $\nexists \sup(S)$. We will solve this problem by introducing the definition of *Archimedean upper bound*.

3. If $\emptyset \neq S \subseteq \mathbb{R}$ admits an upper bound, then $\sup(S)$ can be arbitrarily approximated using upper bounds and points of $S$. When is this possible if $\emptyset \neq S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$?

Example 24.6 shows that these problems cannot be solved in general, and we are hence searching for a useful sufficient condition on $S$. As we will see more clearly below, we could also say that we are searching for a practical notion or procedure "at the $\varepsilon$-level" (i.e. working on representatives) to determine whether a set has the supremum or the least upper bound. However, we are actually far from a real solution of this non trivial problem, and the present section presents only preliminary steps in this direction.

We first prove the following useful characterization of the existence of $\sup(S)$, which also solves problem 3:

**Theorem 26.** *Let $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, and let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ denote the set of upper bounds of $S$. Then $S$ has supremum if and only if*

$$\forall q \in \mathbb{N}\, \exists u_q \in U\, \exists s_q \in S : \ u_q - s_q \leq \mathrm{d}\rho^q. \tag{5.3.3}$$

*Proof.* If $\sigma = \sup(S)$, then (5.3.3) simply follows by setting $u_q := \sigma$ and $s_q \in S$ from (5.3.1). Vice versa, if (5.3.3) holds, then

$$-\mathrm{d}\rho^q \leq s_q - u_q \leq u_{q+1} - u_q \leq u_{q+1} - s_{q+1} \leq \mathrm{d}\rho^{q+1} \quad \forall q \in \mathbb{N}.$$

Thereby, $-(p-q)\mathrm{d}\rho^q \le u_p - u_q \le (p-q)\mathrm{d}\rho^p$ for all $p > q$, and hence $-\mathrm{d}\rho^{\min(p,q)-1} \le u_p - u_q \le \mathrm{d}\rho^{\min(p,q)-1}$ for all $p, q \in \mathbb{N}_{>0}$. This shows that $(u_q)_{q\in\mathbb{N}}$ is a Cauchy sequence which thus converges to some $\sigma \in {}^\rho\widetilde{\mathbb{R}}$. Property (5.3.3) yields that also $(s_q)_{q\in\mathbb{N}} \to \sigma$, and this implies condition (5.3.1). Since each $u_q$ is an upper bound, for all $s \in S$ we have $s \le u_q$, which gives $s \le \sigma$ for $q \to +\infty$. $\square$

To solve problem 2, assume that $u \in {}^\rho\widetilde{\mathbb{R}}$ is an upper bound of a non empty $S \subseteq {}^\rho\widetilde{\mathbb{R}}$. Let $[u_\varepsilon] = u$, and for all $s \in S$ choose a representative $[s_\varepsilon(u)] = s$ such that

$$\forall \varepsilon \in I: \ s_\varepsilon(u) \le u_\varepsilon. \tag{5.3.4}$$

We first note that setting

$$\sigma_\varepsilon := \sup\{s_\varepsilon(u) \mid s \in S\} \quad \forall \varepsilon \in I \tag{5.3.5}$$

does not work to define a representative of the supremum, e.g. if $S = (0,1)$. Assume, e.g., that $u_\varepsilon = 3$ and take any sequence $(s_n)_{n\in\mathbb{N}}$ of different points of $S$: $s_i \ne s_j$ if $i \ne j$. Change representatives of $s_n = [s_{n\varepsilon}]$ satisfying (5.3.4) by setting $\bar{s}_{n\varepsilon} := s_{n\varepsilon}(u) = s_{n\varepsilon}(3)$ if $\varepsilon \ne \frac{1}{n}$ and $\bar{s}_{n,\frac{1}{n}} := 3$. These new representatives still satisfy (5.3.4), but defining $\sigma_\varepsilon$ with them as in (5.3.5), we would get $\sigma_{\frac{1}{n}} \ge \sup\left\{\bar{s}_{n,\frac{1}{n}} \mid n \in \mathbb{N}_{>0}\right\} = 3$, and hence $[\sigma_\varepsilon] \ne 1 = \sup(S)$. We want to refine this idea by considering suitable representatives $[s_\varepsilon(u)] = s$ satisfying (5.3.4), and setting

$$\sigma_\varepsilon(S) := \sigma_\varepsilon := \inf\{\sup\{s_\varepsilon(u) \mid s \in S\} \mid u \ge S\} \quad \forall \varepsilon \in I, \tag{5.3.6}$$

$$(\sigma_\varepsilon(S)) \in \mathbb{R}_\rho \ \Rightarrow \ \sigma(S) := [\sigma_\varepsilon(S)] \in {}^\rho\widetilde{\mathbb{R}}, \tag{5.3.7}$$

where $u \ge S$ means that $u$ is an upper bound of $S$, and where the representatives are chosen as follows: set $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, and for all $(u_\varepsilon) \in \mathbb{R}_\infty^I$ and $s \in S$:

$$\begin{cases} s \le [u_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \ \Rightarrow \ \exists\, [s_\varepsilon(u)] = s \, \forall \varepsilon \in I: \ s_\varepsilon(u) \le u_\varepsilon \\ (u_\varepsilon) \notin \mathbb{R}_\rho \text{ or } s \not\le [u_\varepsilon] \ \Rightarrow \ [s_\varepsilon(u)] = s \text{ is any representative of } s. \end{cases} \tag{5.3.8}$$

Note that definition (5.3.6) depends on the chosen representatives $(s_\varepsilon(u))$ for $s \in S$ and $(u_\varepsilon)$ for $u \ge S$; trivially, if $(\bar{\sigma}_\varepsilon(S))$ is defined using different representatives $(\bar{s}_\varepsilon(u))$ and $(\bar{u}_\varepsilon)$, and both $(\bar{\sigma}_\varepsilon(S))$ and $(\sigma_\varepsilon(S))$ well-define the supremum $\sup(S)$ (or the least upper bound $\mathrm{lub}(S)$) of $S$, then $[\bar{\sigma}_\varepsilon(S)] = [\sigma_\varepsilon(S)]$. On the other hand, if we calculate $(\sigma_\varepsilon(S))$ using a certain choice of representatives, and we notice that $(\sigma_\varepsilon(S))$ is not an upper bound of $S$, we do not know whether another choice of representatives can give an upper bound or not. This is one of the weaknesses of the present solution. To highlight this dependence, we will also sometimes use the following notations for our choice functions (their existence depends on the axiom of choice):

$$\begin{aligned} \mathrm{e}(s,u,\varepsilon) &:= s_\varepsilon(u) \quad \forall s \in S \, \forall u \ge S \\ \mathrm{b}(u,\varepsilon) &:= u_\varepsilon \quad \forall u \ge S. \end{aligned} \tag{5.3.9}$$

We first observe that, for all $\varepsilon \in I$:

$$\begin{aligned} \nexists u \geq S \;&\Rightarrow\; \sigma_\varepsilon = +\infty \\ \exists u \geq S \;&\Rightarrow\; \sigma_\varepsilon \leq \sup\left\{s_\varepsilon(u) \mid s \in S\right\} \leq u_\varepsilon \\ S = \emptyset \;&\Rightarrow\; \sigma_\varepsilon = \sup\left\{s_\varepsilon(u) \mid s \in S\right\} = -\infty. \end{aligned} \tag{5.3.10}$$

We therefore have:

**Lemma 27.** *Assume that $S \subseteq {}^\rho\widetilde{\mathbb{R}}$, $(\sigma_\varepsilon(S)) \in \mathbb{R}_\rho$ and $\sigma(S) \geq S$. Then the following properties hold:*

1. *$\sigma(S) = \mathrm{lub}(S)$.*
2. *If $\mathrm{b}(\sigma(S), \varepsilon) = \sigma_\varepsilon(S)$, then $\sigma_\varepsilon(S) = \sup\left\{s_\varepsilon(\sigma(S)) \mid s \in S\right\} = \inf\left\{u_\varepsilon \mid u \geq S\right\}$ for all $\varepsilon \in I$.*

*Proof.* If $\sigma := \sigma(S) \geq S$, inequality (5.3.10) shows that $\sigma$ is the least upper bound of $S$. From (5.3.6) and (5.3.10), we have $\sigma_\varepsilon \leq \sup\left\{s_\varepsilon(\sigma) \mid s \in S\right\} \leq \sigma_\varepsilon$ because $\sigma \geq S$ and $\mathrm{b}(\sigma, \varepsilon) = \sigma_\varepsilon$ (i.e. the chosen representative $(u_\varepsilon)$ for the upper bound $\sigma \geq S$ is exactly $(\sigma_\varepsilon)$ as defined in (5.3.6)). Finally, the inequality $\sigma_\varepsilon \leq \inf\left\{u_\varepsilon \mid u \geq S\right\}$ follows from (5.3.10). The other inequality follows from $\sigma = \sigma(S) \geq S$ and from $\mathrm{b}(\sigma, \varepsilon) = \sigma_\varepsilon$. □

In general, the net $(\sigma_\varepsilon(S))$ is not $\rho$-moderate. In fact, if $(u_n)_{n\in\mathbb{N}}$ is a sequence of different upper bounds and we set $s_{n,\frac{1}{n}}(u_n) = -\rho_{\frac{1}{n}}^{-1/n}$, this yields $\sigma_{\frac{1}{n}} \leq -\rho_{\frac{1}{n}}^{-1/n}$. On the other hand, we have:

**Lemma 28.** *Let $u \in {}^\rho\widetilde{\mathbb{R}}$, $S \subseteq {}^\rho\widetilde{\mathbb{R}}$ with $S \leq u$. Assume that for some $\bar{s} \in S$ we have*

$$\forall^0\varepsilon : \; \sigma_\varepsilon(S) \geq \bar{s}_\varepsilon(u). \tag{5.3.11}$$

*Then $(\sigma_\varepsilon(S)) \in \mathbb{R}_\rho$ and $\bar{s} \leq \sigma(S) \leq u$.*

*Proof.* From (5.3.10), we get $\sigma_\varepsilon \leq u_\varepsilon$. The conclusion thus follows from (5.3.11) and $\bar{s}$, $u \in {}^\rho\widetilde{\mathbb{R}}$. □

Since the set of all infinitesimals $S = D_\infty$ has no least upper bound, the previous two results imply that $\sigma(D_\infty) \ngeq D_\infty$. Using Lem. 28 with $l = -r$, $u = r \in \mathbb{R}_{>0}$, we have that $\sigma(D_\infty)$ is always an infinitesimal (that actually depends on the chosen representatives $(s_\varepsilon(u))$ and $(u_\varepsilon)$).

The following condition solves problem 2:

**Definition 29.** Let $S \subseteq {}^\rho\widetilde{\mathbb{R}}$ and for simplicity use $\sigma_\varepsilon = \sigma_\varepsilon(S)$, then we say that *$S$ is complete from above* if the following conditions hold:

1. $\forall s \in S \, \exists [s_\varepsilon] = s \, \forall^0\varepsilon : \; s_\varepsilon \leq \sigma_\varepsilon$.

2. If $(s^e)_{e\in I}$ is a family of $S$ which satisfies:
$$\exists[u_\varepsilon]\in {}^\rho\widetilde{\mathbb{R}}\,\forall e\in I\,\forall^0\varepsilon:\ s^e_\varepsilon(\sigma)\le u_\varepsilon \tag{5.3.12}$$
then
$$\exists[\bar{s}_\varepsilon]\in\overline{S}\,\forall^0\varepsilon:\ s^\varepsilon_\varepsilon(\sigma)\le\bar{s}_\varepsilon, \tag{5.3.13}$$
where $\overline{S}$ is the closure of $S$ in the sharp topology.

Moreover, if $\exists s\in S:\ s>0$, then we say that $M$ *is an Archimedean upper bound (AUB) of* $S$ if

1. $M\in{}^\rho\widetilde{\mathbb{R}}$ and $\forall s\in S:\ s\le M$;
2. $\exists n\in\mathbb{N}\,\exists\bar{s}\in S:\ M<n\bar{s}$. The minimum $n\in\mathbb{N}$ that satisfies this property is called the *order of* $M$ (clearly, $n\ge 2$). Note that this condition, using an Archimedean-like property, formalizes the idea that $M$ and $\bar{s}$ are of the same order of magnitude.

Dually, we can define the notion of *completeness from below* by reverting all the inequalities in 1 and 2. If $\exists s\in S:\ s<0$, then $N$ is an *Archimedean lower bound (ALB) of* $S$ if it is a lower bound such that $\exists n\in\mathbb{N}\,\exists\bar{s}\in S:\ \bar{s}n<N$.

Note that $\sigma=\sup(S)$ is always an AUB of order 2. In fact, from the existence of $s\in S_{>0}$, we have $s>\mathrm{d}\rho^q$ for some $q\in\mathbb{N}$ and the existence of $\bar{s}\in S$ with $\bar{s}\ge\sigma-\mathrm{d}\rho^{q+1}$. Thereby, $\bar{s}\ge s-\mathrm{d}\rho^{q+1}>\mathrm{d}\rho^q-\mathrm{d}\rho^{q+1}>\mathrm{d}\rho^{q+1}$ and thus $\sigma\le\bar{s}+\mathrm{d}\rho^{q+1}<2\bar{s}$. We also note that $S={}^\rho\widetilde{\mathbb{R}}$ is trivially complete from above (because $\sigma_\varepsilon=+\infty$ from (5.3.10), and by setting $\bar{s}_\varepsilon=u_\varepsilon$) but $\nexists\sup({}^\rho\widetilde{\mathbb{R}})$. Looking at Lem. 28, in the case of a non empty subset $S\subseteq{}^\rho\widetilde{\mathbb{R}}$ bounded from above, the condition of being complete from above can be intuitively described as follows:

1. Choose representatives $[u_\varepsilon]=u$ for each $u\ge S$ and $[s_\varepsilon(u)]=s$ for each $s\in S$ satisfying (5.3.8);
2. Define $\sigma_\varepsilon(S)=:\sigma_\varepsilon\in\mathbb{R}_\infty=\mathbb{R}\cup\{+\infty\}$ as in 5.3.6.
3. Check if the inequality $s_\varepsilon(\sigma)\le\sigma_\varepsilon$ holds (in this case, for the chosen representatives satisfying (5.3.8), without loss of generality, we can assume that $\mathrm{b}(\sigma,\varepsilon)=\sigma_\varepsilon$ for all $\varepsilon\in I$);
4. From *any* family $(s^e)_{e\in I}$ of $S$ (which is therefore bounded from above, so that (5.3.12) always holds) pick the diagonal net $(s^\varepsilon_\varepsilon(\sigma))$ from its representatives (depending on $\sigma\ge S$) and check if $s^\varepsilon_\varepsilon(\sigma)\le\bar{s}_\varepsilon$ for some $\bar{s}$ in the sharp closure $\overline{S}$.
5. If any of the two previous steps do not hold, consider a different set of representatives in the first step 1.

We therefore have the following simplified case:

**Lemma 30.** *Assume that $\emptyset \neq S \subseteq {}^\rho\widetilde{\mathbb{R}}$ is sharply bounded from above, then $S$ is complete from above if and only if the following condition holds*

1. *$\sigma(S) =: \sigma \geq S$*

2. *If $(s^e)_{e \in I}$ is a family of $S$, then $\exists [\bar{s}_\varepsilon] \in \overline{S}\, \forall^0 \varepsilon : \ s^\varepsilon_\varepsilon(\sigma) \leq \bar{s}_\varepsilon$.*

Note that example 24.6 satisfies the first one of these conditions (so that $\sigma(S)$ is its least upper bound) but not the second one because it does not admit supremum (see the following theorem). Cases which remain excluded from the previous lemma are e.g. intervals $(a, +\infty)$, with $-\infty \leq a \in {}^\rho\widetilde{\mathbb{R}}$ which are complete from above even if they do not admit supremum nor least upper bound. The following results solve the remaining problems 1 and 2 we set at the beginning of this section.

**Theorem 31.** *Assume that $\emptyset \neq S \subseteq {}^\rho\widetilde{\mathbb{R}}$, then*

1. *If $S$ is complete and bounded from above and $\mathrm{b}(\sigma(S), \varepsilon) = \sigma_\varepsilon(S)$, then $\exists \sup(S) = \sigma(S)$;*

*Let $(s_q)_{q \in \mathbb{N}}$ and $(u_q)_{q \in \mathbb{N}}$ be two sequences as in Thm. 26, then*

2. *If $\exists s \in S : \ s > 0$, and if there exists $C \in \mathbb{R}_{>0}$ such that $s_q \geq C \mathrm{d}\rho^q$ for all $q \in \mathbb{N}$ large, then $u_q$ is an AUB of $S$ for all $q$ sufficiently large;*

3. *If $\exists s \in S : \ s > 0$, then $u_q$ is an AUB of $S$ of order $2$ for all $q$ sufficiently large.*

*Proof.* 1: From Lem. 28 we get that $\sigma(S) =: \sigma$ is well-defined because $\sigma \geq S$ by definition of completeness from above, i.e. Def. 29.1. Therefore, Lem. 27 and the assumption $\mathrm{b}(\sigma(S), \varepsilon) = \sigma_\varepsilon(S)$, yield that $\sigma_\varepsilon = \sup\{s_\varepsilon(\sigma) \mid s \in S\}$ for all $\varepsilon$. For arbitrary $q \in \mathbb{N}$ and $e \in I$, this yields

$$\sigma_e - \rho_e^{q+2} < s^e_e(\sigma) =: s^e_e \tag{5.3.14}$$

for some $s^e \in S$ (that depends on both $q$ and $e$). By definition of completeness from above, we get the existence of $\bar{s} = [\bar{s}_\varepsilon] \in \overline{S}$ such that $s^\varepsilon_\varepsilon \leq \bar{s}_\varepsilon$ for $\varepsilon$ mall. Setting $e = \varepsilon$ in (5.3.14), we get $\sigma_\varepsilon - \rho_\varepsilon^{q+2} < s^\varepsilon_\varepsilon \leq \bar{s}_\varepsilon$ for $\varepsilon$ small, i.e. $\sigma - \mathrm{d}\rho^{q+2} < \bar{s}$. Since $\bar{s} \in \overline{S}$, there exists $s \in S \cap (\bar{s} - \mathrm{d}\rho^{q+1}, \bar{s} + \mathrm{d}\rho^{q+1})$. Thereby, $\sigma - \mathrm{d}\rho^q + \mathrm{d}\rho^{q+1} < \sigma - \mathrm{d}\rho^{q+2} < \bar{s}$, and hence $\sigma - \mathrm{d}\rho^q < \bar{s} - \mathrm{d}\rho^{q+1} < s$, which proves our claim 1.

Now, assume that $s_q \geq C \mathrm{d}\rho^q$ for some $C \in \mathbb{R}_{>0}$ and for all $q \in \mathbb{N}$ sufficiently large. Then, for these $q$ we have $\frac{s_q + \mathrm{d}\rho^q}{s_q} \leq 1 + \frac{1}{C} \leq \left\lceil 1 + \frac{1}{C} \right\rceil =: n \in \mathbb{N}$. This yields $u_q < s_q + \mathrm{d}\rho^q < n s_q$, i.e. $u_q$ is an AUB of $S$. Finally, from the existence of at least one $s \in S_{>0}$, we get the existence of $p \in \mathbb{N}$ such that $s > \mathrm{d}\rho^p$. Therefore, also $\mathrm{d}\rho^p < s \leq \sigma$. From 1, we hence get that for $q \in \mathbb{N}$ sufficiently large $\mathrm{d}\rho^p < s_q \leq \sigma$, i.e. $\frac{1}{s_q} < \mathrm{d}\rho^{-p}$ and $\frac{s_q + \mathrm{d}\rho^q}{s_q} \leq 1 + \mathrm{d}\rho^{q-p} \leq 2$ for all $q > p$. Proceeding as above we can prove the claim. $\square$

Example 24.6 shows the necessity of the assumption of completeness from above in this theorem.

Directly from Thm. 31.1, we obtain:

**Corollary 32.** *Let $\emptyset \ne S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$. Assume that $S$ is complete from above and $\mathrm{b}(\sigma(S), \varepsilon) = \sigma_{\varepsilon}(S)$, then $\exists \sup(S)$ if and only if $S$ admits an upper bound.*

Now, we can also complete the relationships between close supremum and least upper bound (see also [24, Prop. 1.4]) and study what happens if we consider only the upper bounds $u$ lower than a fixed upper bound $\bar{u}$ in (5.3.6).

**Corollary 33.** *Let $\emptyset \ne S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, then the following properties hold:*

1. *If $\exists \sup(S) = \sigma$, then $\exists \mathrm{lub}(S) = \sigma$.*

2. *If $S$ is complete and bounded from above, then*

$$\exists \sup(S) = \sigma \iff \exists \mathrm{lub}(S) = \sigma.$$

3. *Assume that $\bar{u} \ge S$ and define $\bar{\sigma}_{\varepsilon}(S) := \inf \{\sup \{s_{\varepsilon}(u) \mid s \in S\} \mid \bar{u} \ge u \ge S\}$. Then $\bar{\sigma}(S) := [\bar{\sigma}_{\varepsilon}(S)]$ is well-defined and $\bar{\sigma}(S) \ge \sigma(S)$. If $\sigma(S) \ge S$, then $\bar{\sigma}(S) = \sigma(S)$. If $\bar{\sigma}(S) \ge S$, then $\bar{\sigma}(S)$ is the least upper bound of $S$, thus $\bar{\sigma}(S) = \sigma(S)$ if $S$ is complete from above.*

4. *Assume that $\exists\, \sigma(S) \ge S$, $\mathrm{b}(\sigma(S), \varepsilon) = \sigma_{\varepsilon}(S)$ and $\exists \sup(S)$. Then $S$ is complete from above.*

*Proof.* 1 and 2: Assume that $\exists \sup(S) = \sigma$, and let $u$ be an upper bound of $S$; by condition (5.3.1) we get $\sigma - \mathrm{d}\rho^{q} \le s_{q} \le u$ for all $q \in \mathbb{N}$ and for some $s_{q} \in S$. For $q \to +\infty$, we get $\sigma \le u$. Vice versa, if $S \ne \emptyset$ is complete from above and $\sigma$ is the least upper bound of $S$, then the conclusion follows from Cor. 32.

3: If $\bar{s} \in S \le u$, we can prove that $(\bar{\sigma}_{\varepsilon}(S)) \in \mathbb{R}_{\rho}$ and $\bar{s} \le \bar{\sigma}(S) \le u$ as in the proof of Lem. 28. We always have that $\sigma(S) \le \bar{u}$ because $\bar{u} \ge S$. Therefore, if $\sigma(S) \ge S$, then $\bar{u} \ge \sigma(S) \ge S$ and hence $\bar{\sigma}(S) \le \sigma(S) \le \bar{\sigma}(S)$. Finally, if we assume that $\bar{\sigma}(S) \ge S$ and we consider an arbitrary upper bound $u \ge S$, then either $u \ge \bar{u}$ or $u <_{L} \bar{u}$ for some $L \subseteq_{0} I$. Thereby, $\bar{\sigma}(S) \le u$ or $\bar{\sigma}(S) \le_{L} u$, and hence $\bar{\sigma}(S) \le u$. Therefore, $\bar{\sigma}(S)$ is the least upper bound of $S$, and the final claim follows from 2.

4: From Lem. 27, we have $\sigma(S) = \mathrm{lub}(S) =: \sigma$ and hence $\sup(S) = \sigma \in \overline{S}$ from 1. From $b(\sigma(S), \varepsilon) = \sigma_{\varepsilon}(S)$ and (5.3.8) we have $s_{\varepsilon}(\sigma) \le \sigma_{\varepsilon}$ for all $s \in S$ and all $\varepsilon \in I$. In particular, if $(s^{e})_{e \in I}$ is a family of $S$, we have $s^{e}_{\varepsilon}(\sigma) \le \sigma_{\varepsilon}$ for all $e \in I$ and all $\varepsilon \in I$. Taking $e = \varepsilon$, we get that Def. 29.2 holds. □

**Example 34.**

1. Example 24.6 shows that the assumption of being complete from above is necessary in Cor. 33. On the other hand, using the notation of this example, one can prove that $\sigma_{\varepsilon}(S) = 2$ if $\varepsilon \in L$ and $\sigma_{\varepsilon}(S) = 1$ if $\varepsilon \in L^{c}$. From Lem. 27 it follows that $\sigma(S)$ is the least upper bound of $S$.

This underscores the differences between the order theoretical definition of supremum as least upper bound and the topological definition of closed supremum.

2. Any set having a maximum is trivially complete from above: set $[\bar{s}_\varepsilon] := \max(S)$ in (5.3.13) and consider that $\sigma(S) = \max(S)$.

3. $S = (0, 1)$ is complete from above for $[\bar{s}_\varepsilon] = 1$ and because $\sigma_\varepsilon(S) =: \sigma_\varepsilon = 1$. In fact, $\sigma_\varepsilon \le 1$ from (5.3.10). Now, take any $u = [u_\varepsilon] \ge S$, so that $u_\varepsilon \ge 1$ for all $\varepsilon \ge \varepsilon_0$. For $\varepsilon \ge \varepsilon_0$, by contradiction assume that $1 > \sup\{s_\varepsilon(u) \mid s \in S\}$, and hence $1 > r > \sup\{s_\varepsilon(u) \mid s \in S\}$ for some $r \in (0, 1)_{\mathbb{R}} \subseteq S$. Take $r_\varepsilon = r$ as representative of $r$ in (5.3.8); we have two cases: If $r_\varepsilon(u) = u_\varepsilon \ge 1$, then $1 > \sup\{s_\varepsilon(u) \mid s \in S\} \ge r_\varepsilon(u) = u_\varepsilon \ge 1$; if $r_\varepsilon(u) = r$, then $\sup\{s_\varepsilon(u) \mid s \in S\} \ge r_\varepsilon(u) = r > \sup\{s_\varepsilon(u) \mid s \in S\}$. In any case, we get a contradiction, and this proves that $1 \le \sup\{s_\varepsilon(u) \mid s \in S\}$ for all $\varepsilon \le \varepsilon_0$, and hence $\sigma_\varepsilon \ge 1$.

4. There do not exist neither the supremum nor the least upper bound of $S = 1 + D_\infty$. On the other hand, 2 is an AUB of $S$ and hence $S$ is not complete from above.

5. $D_\infty$ has neither AUB nor ALB; ${}^\rho\widetilde{\mathbb{R}}$ has neither AUB nor ALB; $\{\mathrm{d}\rho^r \mid r \in \mathbb{R}_{>0}\}$ has no supremum and no AUB and hence it is not complete from above.

6. Assume that there does not exist and upper bound of $S$. This means that

$$\forall u \in {}^\rho\widetilde{\mathbb{R}}\, \exists s \in S :\ s >_{\mathrm{s}} u.$$

Thereby, there exists a sequence $(s_q)_{q\in\mathbb{N}}$ of $S$ such that $s_q >_{\mathrm{s}} \mathrm{d}\rho^{-q}$. Based on this, we could set $\sup(S) := +\infty$.

## 5.4 The hyperlimit of a hypersequence

### 5.4.1 Definition and examples

**Definition 35.** A map $x : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$, whose domain is the set of hypernatural numbers ${}^\sigma\widetilde{\mathbb{N}}$ is called a $(\sigma-)$ *hypersequence* (of elements of ${}^\rho\widetilde{\mathbb{R}}$). The values $x(n) \in {}^\rho\widetilde{\mathbb{R}}$ at $n \in {}^\sigma\widetilde{\mathbb{N}}$ of the function $x$ are called *terms* of the hypersequence and, as usual, denoted using an index as argument: $x_n = x(n)$. The hypersequence itself is denoted by $(x_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$, or simply $(x_n)_n$ if the gauge on the domain is clear from the context. Let $\sigma$, $\rho$ be two gauges, $x : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be a hypersequence and $l \in {}^\rho\widetilde{\mathbb{R}}$. We say that $l$ is *hyperlimit* of $(x_n)_n$ as $n \to \infty$ and $n \in {}^\sigma\widetilde{\mathbb{N}}$, if

$$\forall q \in N\, \exists M \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge M} :\ |x_n - l| < \mathrm{d}\rho^q.$$

In the following, if not differently stated, $\rho$ and $\sigma$ will always denote two gauges and $(x_n)_n$ a $\sigma$-hypersequence of elements of ${}^\rho\widetilde{\mathbb{R}}$. Finally, if $\sigma_\varepsilon \ge \rho_\varepsilon$, at least for all $\varepsilon$ small, we simply write $\sigma \ge \rho$.

*Remark* 36. In the assumption of Def. 35, let $k \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, $N \in \mathbb{N}$, then the following are equivalent:

1. $l \in {}^{\rho}\widetilde{\mathbb{R}}$ is the hyperlimit of $(x_n)_n$ as $n \in {}^{\sigma}\widetilde{\mathbb{N}}$.
2. $\forall \eta \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M} :\ |x_n - l| < \eta$.
3. Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ be a sharply open set, if $l \in U$ then $\exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M}$ : $x_n \in U$.
4. $\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M} :\ |x_n - l| < k \cdot \mathrm{d}\rho^q$.
5. $\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M} :\ |x_n - l| < \mathrm{d}\rho^{q-N}$.

Directly by the inequality $|l_1 - l_2| \leq |l_1 - x_n| + |l_2 - x_n| \leq 2\mathrm{d}\rho^{q+1} < \mathrm{d}\rho^q$ (or by using that the sharp topology on ${}^{\rho}\widetilde{\mathbb{R}}$ is Hausdorff) it follows that there exists at most one hyperlimit, so that we can use the notation

$$ {}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} x_n \quad := \quad l. $$

As usual, a hypersequence (not) having a hyperlimit is said to be (non-)convergent. We can also similarly say that $(x_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ is divergent to $+\infty$ $(-\infty)$ if

$$ \forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M} :\ x_n > \mathrm{d}\rho^{-q} \quad (x < -\mathrm{d}\rho^{-q}). $$

**Example 37.**

1. If $\sigma \leq \rho^R$ for some $R \in \mathbb{R}_{>0}$, we have ${}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{1}{n} = 0$. In fact, $\frac{1}{n} < \mathrm{d}\rho^q$ holds e.g. if $n > [\mathrm{int}\,(\rho_\varepsilon^{-q}) + 1] \in {}^{\sigma}\widetilde{\mathbb{N}}$ because $\rho_\varepsilon^{-q} \leq \sigma_\varepsilon^{-q/R}$ for $\varepsilon$ small.
2. Let $\rho$ be a gauge and set $\sigma_\varepsilon := \exp\left(-\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right)$, so that $\sigma$ is also a gauge. We have
$$ {}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{1}{\log n} = 0 \in {}^{\rho}\widetilde{\mathbb{R}} \text{ whereas } \nexists\, {}^{\rho}\lim_{n \in {}^{\rho}\widetilde{\mathbb{N}}} \frac{1}{\log n} $$
In fact, if $n > 1$, we have $0 < \frac{1}{\log n} < \mathrm{d}\rho^q$ if and only if $\log n > \mathrm{d}\rho^{-q}$, i.e. $n > e^{\mathrm{d}\rho^{-q}}$ (in ${}^{\rho}\widetilde{\mathbb{R}}$). We can thus take $M := \left[\mathrm{int}\left(e^{\rho_\varepsilon^{-q}}\right) + 1\right] \in {}^{\sigma}\widetilde{\mathbb{N}}$ because $e^{\rho_\varepsilon^{-q}} < \exp\left(\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right) = \sigma_\varepsilon^{-1}$ for $\varepsilon$ small. Vice versa, by contradiction, if $\exists\, {}^{\rho}\lim_{n \in {}^{\rho}\widetilde{\mathbb{N}}} \frac{1}{\log n} =: l \in {}^{\rho}\widetilde{\mathbb{R}}$, then by the definition of hyperlimit from ${}^{\rho}\widetilde{\mathbb{N}}$ to ${}^{\rho}\widetilde{\mathbb{R}}$, we would get the existence of $M \in {}^{\rho}\widetilde{\mathbb{N}}$ such that
$$ \forall n \in {}^{\rho}\widetilde{\mathbb{N}} : n \geq M \ \Rightarrow\ \frac{1}{\log n} - \mathrm{d}\rho < l < \frac{1}{\log n} + \mathrm{d}\rho \tag{5.4.1} $$

We have to explore two possibilities: if $l$ is not invertible, then $l_{\varepsilon_k} = 0$ for some sequence $(\varepsilon_k) \downarrow 0$ and some representative $[l_\varepsilon] = l$. Therefore from 35, we get

$$\frac{1}{\log M_{\varepsilon_k}} < l_{\varepsilon_k} + \rho_{\varepsilon_k} = \rho_{\varepsilon_k}$$

hence $M_{\varepsilon_k} > e^{-\frac{1}{\rho_{\varepsilon_k}}}$ $\forall k \in \mathbb{N}$, in contradiction with $M \in {}^\rho\widetilde{\mathbb{R}}$. If $l$ is invertible, then $\mathrm{d}\rho^p < |l|$ for some $p \in \mathbb{N}$. Setting $q := \min\{p \in \mathbb{N} \mid \mathrm{d}\rho^p < |l|\}+1$, we get that $l_{\bar{\varepsilon}_k} < \rho^q_{\bar{\varepsilon}_k}$ for some sequence $(\bar{\varepsilon}_k)_k \downarrow 0$. Therefore

$$\frac{1}{\log M_{\bar{\varepsilon}_k}} < l_{\bar{\varepsilon}_k} + \rho_{\bar{\varepsilon}_k} \le |l_{\bar{\varepsilon}_k}| + \rho_{\bar{\varepsilon}_k} < \rho^q_{\bar{\varepsilon}_k} + \rho_{\bar{\varepsilon}_k}$$

and hence $M_{\bar{\varepsilon}_k} > \exp\left(\frac{1}{\rho^q_{\bar{\varepsilon}_k} + \rho_{\varepsilon_k}}\right)$ for all $k \in \mathbb{N}$, which is in contradiction with $M \in {}^\rho\widetilde{\mathbb{R}}$ because $q \ge 1$.
Analogously, we can prove that ${}^\rho\lim_{n \in {}^\sigma\widetilde{\mathbb{N}}} \frac{1}{\log(\log n)} = 0$ if $\sigma = [\sigma_\epsilon] = \left[e^{-e^{\rho_\epsilon^{-\frac{1}{\rho_\epsilon}}}}\right]$ whereas $\nexists\, {}^\rho\lim_{n \in {}^\rho\widetilde{\mathbb{N}}} \frac{1}{\log(\log n)}$ (and similarly using $\log(\log(\overset{k}{\ldots\ldots}(\log n)\ldots)$.

3. Set $x_n := \mathrm{d}\rho^{-n}$ if $n \in \mathbb{N}$, and $x_n := \frac{1}{n}$ if $n \in {}^\rho\widetilde{\mathbb{N}} \setminus \mathbb{N}$, then $\{x_n \mid n \in {}^\rho\widetilde{\mathbb{N}}\}$ is unbounded in ${}^\rho\widetilde{\mathbb{R}}$ even if ${}^\rho\lim_{n \in {}^\rho\widetilde{\mathbb{N}}} x_n = 0$. Similarly, if $x_n := \mathrm{d}\rho^n$ if $n \in \mathbb{N}$ and $x_n := \sin(n)$ otherwise, then $\lim_{\substack{n\to+\infty \\ n\in\mathbb{N}}} x_n = 0$ whereas $\nexists\, {}^\rho\lim_{n \in {}^\rho\widetilde{\mathbb{N}}} x_n$. In general, we can hence only state that convergent hypersequence are eventually bounded:

$$\exists\, {}^\rho\lim_{n \in {}^\sigma\widetilde{\mathbb{N}}} x_n \ \Rightarrow\ \exists M \in {}^\rho\widetilde{\mathbb{R}}\, \exists N \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N} :\ |x_n| \le M.$$

4. If $k <_{\mathrm{s}} 1$ and $k >_{\mathrm{s}} 1$, then ${}^\rho\lim_{n \in {}^\rho\widetilde{\mathbb{N}}} k^n =_{\mathrm{s}} 0$ and ${}^\rho\lim_{n \in {}^\rho\widetilde{\mathbb{N}}} k^n =_{\mathrm{s}} +\infty$, hence $\nexists {}^\rho\lim_{n \in {}^\rho\widetilde{\mathbb{N}}} k^n$.

5. Since for $n \in \mathbb{N}$ we have $(1-\mathrm{d}\rho)^n = 1 - n\mathrm{d}\rho + O_n(\mathrm{d}\rho^2)$, it is not hard to prove that $((1-\mathrm{d}\rho)^n)_{n\in\mathbb{N}}$ is not a Cauchy sequence. Therefore, $\nexists \lim_{n\in\mathbb{N}}(1-\mathrm{d}\rho)^n$, whereas ${}^\rho\lim_{n \in {}^\rho\widetilde{\mathbb{N}}}(1-\mathrm{d}\rho)^n = 0$.

A sufficient condition to extend an ordinary sequence $(a_n)_{n\in\mathbb{N}} : \mathbb{N} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ of $\rho$-generalized numbers to the whole ${}^\sigma\widetilde{\mathbb{N}}$ is

$$\forall n \in {}^\sigma\widetilde{\mathbb{N}} :\ \left(a_{\mathrm{ni}(n)_\varepsilon}\right) \in \mathbb{R}_\rho. \tag{5.4.2}$$

In fact, in this way $a_n := \left[a_{\mathrm{ni}(n)_\varepsilon}\right] \in {}^\rho\widetilde{\mathbb{R}}$ for all $n \in {}^\sigma\widetilde{\mathbb{N}}$, is well-defined because of Lem. 17; on the other hand, we have defined an extension of the old sequence $(a_n)_{n\in\mathbb{N}}$ because if $n \in \mathbb{N}$, then $\mathrm{ni}(n)_\varepsilon = n$ for $\varepsilon$ small and hence $a_n = [a_n]$. For example, the sequence of infinities $a_n = \frac{1}{n} + \mathrm{d}\rho^{-1}$ for all $n \in \mathbb{N}$ can be extended to any ${}^{\sigma}\widetilde{\mathbb{N}}$, whereas $a_n = \mathrm{d}\sigma^{-n}$ can be extended as $a : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ only for some gauges $\rho$, e.g. if the gauges satisfy

$$\exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\, \forall^0\varepsilon : \ \sigma_\varepsilon^n \geq \rho_\varepsilon^N, \tag{5.4.3}$$

(e.g. $\sigma_\varepsilon = \varepsilon$ and $\rho_\varepsilon = \varepsilon^{1/\varepsilon}$).

The following result allows us to obtain hyperlimits by proceeding $\varepsilon$-wise

**Theorem 38.** *Let $(a_{n,\varepsilon})_{n,\varepsilon} : \mathbb{N} \times I \longrightarrow \mathbb{R}$. Assume that for all $\varepsilon$*

$$\exists \lim_{n\to+\infty} a_{n,\varepsilon} =: l_\varepsilon, \tag{5.4.4}$$

*and that $l := [l_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$. Then there exists a gauge $\sigma$ (not necessarily a monotonic one) such that*

1. *There exists $M \in {}^{\sigma}\widetilde{\mathbb{N}}$ and a hypersequence $(a_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ such that $a_n = [a_{ni(n)_\varepsilon,\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}$ for all $n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M}$;*
2. *$l = {}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$.*

*Proof.* From (5.4.4), we have

$$\forall\varepsilon\, \forall q\, \exists M_{\varepsilon q} \in \mathbb{N}_{>0}\, \forall n \geq M_{\varepsilon q} : \ \rho_\varepsilon^q - l_\varepsilon < a_{n,\varepsilon} < \rho_\varepsilon^q + l_\varepsilon. \tag{5.4.5}$$

Without loss of generality, we can assume to have recursively chosen $M_{\varepsilon q}$ so that

$$M_{\varepsilon q} \leq M_{\varepsilon,q+1} \quad \forall\varepsilon\, \forall q. \tag{5.4.6}$$

Set $\bar{M}_\varepsilon := M_{\varepsilon,\lceil\frac{1}{\varepsilon}\rceil} > 0$; since $\forall q \in \mathbb{N}\, \forall^0\varepsilon : \ q \leq \lceil\frac{1}{\varepsilon}\rceil$, (5.4.6) implies

$$\forall q \in \mathbb{N}\, \forall^0\varepsilon : \ \bar{M}_\varepsilon \geq M_{\varepsilon q}. \tag{5.4.7}$$

If the net $(\bar{M}_\varepsilon)$ is $\rho$-moderate, set $\sigma := \rho$, otherwise set $\sigma_\varepsilon := \min\left(\rho_\varepsilon, \bar{M}_\varepsilon^{-1}\right) \in (0,1]$. Thereby, the net $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0^+$ (note that not necessarily $\sigma$ is non-decreasing, e.g. if $\lim_{\varepsilon\to\frac{1}{k}} \bar{M}_\varepsilon = +\infty$ for all $k \in \mathbb{N}_{>0}$ and $\bar{M}_\varepsilon \geq \rho_\varepsilon^{-1}$), i.e. it is a gauge. Now set $\bar{M} := [\bar{M}_\varepsilon] \in {}^{\sigma}\widetilde{\mathbb{N}}$ because our definition of $\sigma$ yields $\bar{M}_\varepsilon \leq \sigma_\varepsilon^{-1}$, $M_q := [M_{\varepsilon q}] \in {}^{\sigma}\widetilde{\mathbb{N}}$ because of (5.4.7), and

$$a_n := \begin{cases} [a_{\mathrm{ni}(n)_\varepsilon,\varepsilon}] & \text{if } n \geq M_1 \text{ in } {}^{\sigma}\widetilde{\mathbb{N}} \\ 1 & \text{otherwise} \end{cases} \qquad \forall n \in {}^{\sigma}\widetilde{\mathbb{N}}. \tag{5.4.8}$$

We have to prove that this well-defines a hypersequence $(a_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$. First of all, the sequence is well-defined with respect to the equality in ${}^{\sigma}\widetilde{\mathbb{N}}$ because of Lem. 17. Moreover, setting $q = 1$ in (5.4.5), we get $\rho_\varepsilon - l_\varepsilon < a_{n,\varepsilon} < \rho_\varepsilon + l_\varepsilon$ for all $\varepsilon$ and for all $n \geq M_{\varepsilon 1}$. If $n \geq M_1$ in ${}^{\sigma}\widetilde{\mathbb{N}}$, then $\mathrm{ni}(n)_\varepsilon \geq M_{\varepsilon 1}$ for $\varepsilon$ small, and hence $\rho_\varepsilon - l_\varepsilon < a_{\mathrm{ni}(n)_\varepsilon,\varepsilon} < \rho_\varepsilon + l_\varepsilon$. This shows that $a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ because we assumed that $l = [l_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$. Finally, (5.4.5) and (5.4.6) yield that if $n \geq M_q$ then $n \geq M_1$ and hence $|a_n - l| < \mathrm{d}\rho^q$. $\square$

From the proof it also follows, more generally, that if $(M_{\varepsilon q})_{\varepsilon,q}$ satisfies (5.4.5) and if

$$\exists(q_\varepsilon) \to +\infty : \ (M_{\varepsilon,q_\varepsilon}) \in \mathbb{R}_\rho,$$

then we can repeat the proof with $q_\varepsilon$ instead of $\lceil \frac{1}{\varepsilon} \rceil$ and setting $\sigma := \rho$.

### 5.4.2 Operations with hyperlimits and inequalities

Thanks to Def. 5 of sharp topology and our notation for $x < y$ (and of the consequent Lem. 4), some results about hyperlimits can be proved by trivially generalizing classical proofs. For example, if $(x_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ and $(y_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ are two convergent hypersequences then their sum $(x_n + y_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$, product $(x_n \cdot y_n)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ and quotient $\left(\frac{x_n}{y_n}\right)_{n\in{}^\sigma\widetilde{\mathbb{N}}}$ (the last one being defined only when $y_n$ is invertible for all $n\in {}^\sigma\widetilde{\mathbb{N}}$) are convergent hypersequences and the corresponding hyperlimits are sum, product and quotient of the corresponding hyperlimits.

The following results generalize the classical relations between limits and inequalities.

**Theorem 39.** *Let $x$, $y$, $z : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be hypersequences, then we have:*

1. *If ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n < {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} y_n$,* then $\exists M\in {}^\sigma\widetilde{\mathbb{N}}$ such that $x_n < y_n$ for all $n \ge M$, $n \in {}^\sigma\widetilde{\mathbb{N}}$.
2. *If $x_n \le y_n \le z_n$ for all $n \in {}^\sigma\widetilde{\mathbb{N}}$ and ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n = {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} z_n =: l$,* then $\exists\, {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} y_n = l$,

*Proof.* 1 follows from Lem. 4 and the Def. 35 of hyperlimit. For 2, the proof is analogous to the classical one. In fact, since ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n = {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} z_n =: l$ given $q \in \mathbb{N}$, there exist $M'$, $M'' \in {}^\sigma\widetilde{\mathbb{N}}$ such that $l - \mathrm{d}\rho^q < x_n$ and $z_n < l + \mathrm{d}\rho^q$ for all $n > M'$, $n > M''$, $n \in {}^\sigma\widetilde{\mathbb{N}}$, then for $n > M := M' \vee M''$, we have $l - \mathrm{d}\rho^q < x_n \le y_n \le z_n < l + \mathrm{d}\rho^q$. □

**Theorem 40.** *Assume that $C$ is a sharply closed subset of ${}^\rho\widetilde{\mathbb{R}}$, that $\exists\, {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n =: l$ and that $x_n$ eventually lies in $C$, i.e. $\exists N \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N} : \ x_n \in C$. Then also $l \in C$. In particular, if $(y_n)_n$ is another hypersequence such that $\exists\, {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} y_n =: k$, then $\exists N \in {}^\sigma\widetilde{\mathbb{N}}\, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N} : \ x_n \ge y_n$ implies $l \ge k$.*

*Proof.* A reformulation of the usual proof applies. In fact, let us suppose that $l \in {}^\rho\widetilde{\mathbb{R}}\setminus C$. Since ${}^\rho\widetilde{\mathbb{R}}\setminus C$ is sharply open, there is an $\eta > 0$, for which $B_\eta(l) \subseteq {}^\rho\widetilde{\mathbb{R}}\setminus C$. Let $\bar{n} \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N}$ be such that $|x_n - l| < \eta$ when $n > \bar{n}$. Then we have $x_n \in C$ and $x_n \in B_\eta(l) \subseteq {}^\rho\widetilde{\mathbb{R}} \setminus C$, a contradiction. □

The following result applies to all generalized smooth functions (and hence to all Colombeau generalized functions, see e.g. [35, 31]; see also [2] for a more general class of functions) because of their continuity in the sharp topology.

**Theorem 41.** *Suppose that $f : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$. Then $f$ is sharply continuous function at $x = c$ if and only if it is hyper-sequentially continuous, i.e. for any hypersequence $(x_n)_n$ in $U$ converging to $c$, the hypersequence $(f(x_n))_n$ converges to $f(c)$, i.e. $f\left({}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n\right) = {}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} f(x_n)$.*

*Proof.* We only prove that the hyper-sequential continuity is a sufficient condition, because the other implication is a trivial generalization of the classical one. By contradiction, assume that for some $Q \in \mathbb{N}$

$$\forall n \in \mathbb{N}\, \exists x_n \in U : \ |x_n - c| < \mathrm{d}\rho^n, \ |f(x_n) - f(c)| >_{\mathrm{s}} \mathrm{d}\rho^Q. \tag{5.4.9}$$

For $n \in \mathbb{N}$ set $\omega_n := n$ and for $n \in {}^\rho\widetilde{\mathbb{N}} \setminus \mathbb{N}$ set $\omega_n := \min\left\{N \in \mathbb{N} \mid n \le \mathrm{d}\rho^{-N}\right\}$ and $x_n := x_{\omega_n}$. Then for all $n \in {}^\rho\widetilde{\mathbb{N}}$, from (5.4.9) we get $|x_n - c| < \mathrm{d}\rho^{\omega_n} \to 0$ because $\omega_n \to +\infty$ as $n \to +\infty$ in $n \in {}^\rho\widetilde{\mathbb{N}}$. Therefore, $(x_n)_n$ is an hypersequence of $U$ that converges to $c$, which yields $f(x_n) \to f(c)$, in contradiction with (5.4.9). $\square$

**Example 42.** Let $\sigma \le \rho^R$ for some $R \in \mathbb{R}_{>0}$. The following inequalities hold for all generalized numbers because they also hold for all real numbers:

$$\begin{aligned} \ln(x) &\le x \\ e\left(\frac{n}{e}\right)^n &\le n! \le en\left(\frac{n}{e}\right)^n. \end{aligned} \tag{5.4.10}$$

From the first one it follows $0 \le \frac{\ln(n)}{n} = \frac{2\ln\sqrt{n}}{n} \le \frac{2\sqrt{n}}{n}$, so that ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{\ln(n)}{n} := 0$ from Thm. 39 and ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} n^{1/n} = 1$ from Thm. 41 and hence ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} (n!)^{1/n} = +\infty$ by (5.4.10). Similarly, we have ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \left(1+\frac{1}{n}\right)^n = e$ because $n\log\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \to 1$ and because of Thm. 41.

A little more involved proof concerns L'Hôpital rule for generalized smooth functions. For the sake of completeness, here we only recall the equivalent definition:

**Definition 43.** Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. We say that $f : X \longrightarrow Y$ is a *generalized smooth function* (GSF) if

1. $f : X \longrightarrow Y$ is a set-theoretical function.
2. There exists a net $(f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^{(0,1]}$ such that for all $[x_\varepsilon] \in X$:
   (a) $f(x) = [f_\varepsilon(x_\varepsilon)]$
   (b) $\forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon))$ is $\rho-$moderate.

For generalized smooth functions lots of results hold: closure with respect to composition, embedding of Schwartz's distributions, differential calculus, one-dimensional integral calculus using primitives, classical theorems (intermediate value, mean value, Taylor, extreme value, inverse and implicit function), multidimensional integration, Banach fixed point theorem, a Picard-Lindelöf theorem for both ODE and PDE, several results of calculus of variations, etc.

In particular, we have the following (see also [27] for the particular case of Colombeau generalized functions):

**Theorem 44.** *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ be a sharply open set and let $f : U \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be a GSF defined by the net of smooth functions $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. Then*

1. *There exists an open neighbourhood $T$ of $U \times \{0\}$ and a GSF $R_f : T \to {}^{\rho}\widetilde{\mathbb{R}}$, called the* generalized incremental ratio *of $f$, such that*
$$f(x+h) = f(x) + h \cdot R_f(x,h) \qquad \forall (x,h) \in T. \tag{5.4.11}$$
*Moreover $R_f(x,0) = [f'_\varepsilon(x_\varepsilon)] = f'(x)$ is another GSF and we can hence recursively define $f^{(k)}(x)$.*

2. *Any two generalized incremental ratios of $f$ coincide on the intersection of their domains.*

3. *More generally, for all $k \in \mathbb{N}_{>0}$ there exists an open neighbourhood $T$ of $U \times \{0\}$ and a GSF $R^k_f : T \to {}^{\rho}\widetilde{\mathbb{R}}$, called $k$-*th order Taylor ratio *of $f$, such that*
$$f(x+h) = \sum_{j=0}^{k-1} \frac{f^{(j)}(x)}{j!} h^j + R^k_f(x,h) \cdot h^k \qquad \forall (x,h) \in T. \tag{5.4.12}$$
*Any two ratios of $f$ of the same order coincide on the intersection of their domains.*

We can now prove the following generalization of one of L'Hôpital rule:

**Theorem 45.** *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ be a sharply open set $(x_n)_n$, $(y_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow U$ be hypersequences converging to $l \in U$ and $m \in U$ respectively and such that*
$$ {}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{x_n - l}{y_n - m} =: C \in {}^{\rho}\widetilde{\mathbb{R}}.$$
*Let $k \in \mathbb{N}_{>0}$ and $f$, $g : U \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be GSF such that for all $n \in {}^{\sigma}\widetilde{\mathbb{N}}$ and all $j = 0, \ldots, k-1$*
$$\begin{aligned} &g^{(j)}(y_n) \in {}^{\rho}\widetilde{\mathbb{R}}^* \\ &f^{(j)}(l) = g^{(j)}(m) = 0 \\ &g^{(k)}(m) \in {}^{\rho}\widetilde{\mathbb{R}}^* \end{aligned} \tag{5.4.13}$$
*Then for all $j = 0, \ldots, k-1$*
$$\exists\, {}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{f^{(j)}(x_n)}{g^{(j)}(y_n)} = C^k \cdot {}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{f^{(k)}(x_n)}{g^{(k)}(y_n)}.$$

*Proof.* Using (5.4.12) and (5.4.13), we can write
$$\frac{f(x_n)}{g(y_n)} = \frac{\sum_{j=0}^{k-1} \frac{f^{(j)}(l)}{j!}(x_n - l)^j + (x_n - l)^k R^k_f(l, x_n - l)}{\sum_{j=0}^{k-1} \frac{g^{(j)}(m)}{j!}(y_n - m)^j + (y_n - m)^k R^k_g(m, y_n - m)} =$$
$$= \left(\frac{x_n - l}{y_n - m}\right)^k \cdot \frac{R^k_f(l, x_n - l)}{R^k_g(m, y_n - m)}.$$

Since $R_f^k$ and $R_g^k$ are GSF, they are sharply continuous. Therefore, the right hand side of the previous equality tends to $C^k \cdot \frac{R_f^k(l,0)}{R_g^k(m,0)} = C^k \cdot \frac{f^{(k)}(l)}{g^{(k)}(m)}$. At the same limit converges the quotient $C^k \frac{f^{(k)}(x_n)}{g^{(k)}(y_n)}$ because $f^{(k)}$ and $g^{(k)}$ are also GSF and hence they are sharply continuous. The claim for $j = 1, \ldots, k-1$ follows by applying the conclusion for $j = 0$ with $f^{(j)}$ and $g^{(j)}$ instead of $f$ and $g$. $\square$

Note that for $x_n = y_n$, $l = m$, we have $C = 1$ and we get the usual L'Hôpital rule (formulated using hypersequences). Note that a similar theorem can also be proved without hypersequences and using the same Taylor expansion argument as in the previous proof.

### 5.4.3 Cauchy criterion and monotonic hypersequences.

In this section, we deal with classical criteria implying the existence of a hyperlimit.

**Definition 46.** We say that $(x_n)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}$ is a *Cauchy hypersequence* if

$$\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n, m \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M} : \ |x_n - x_m| < \mathrm{d}\rho^q.$$

**Theorem 47.** *A hypersequence converges if and only if it is a Cauchy hypersequence*

*Proof.* To prove that the Cauchy criterion is a necessary condition it suffices to consider the inequalities:

$$|x_n - x_m| \leq |x_n - l| + |x_m - l| \leq \mathrm{d}\rho^{q+1} + \mathrm{d}\rho^{q+1} < \mathrm{d}\rho^q$$

Vice versa, assume that

$$\forall q \in \mathbb{N}\, \exists M_q \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n, m \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M_q} : \ |x_n - x_m| < \mathrm{d}\rho^q. \tag{5.4.14}$$

The idea is to use Cauchy completeness of ${}^{\rho}\widetilde{\mathbb{R}}$. In fact, set $h_1 := M_1$ and $h_{q+1} := M_{q+1} \vee h_q$. We claim that $(x_{h_q})_{q\in\mathbb{N}}$ is a standard Cauchy sequence converging to the same limit of $(x_n)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}$. From (5.4.14) it follows that $(x_{h_q})_{q\in\mathbb{N}}$ is a standard Cauchy sequence (in the sharp topology). Therefore, there exists $\bar{x} \in {}^{\rho}\widetilde{\mathbb{R}}$ such that $\lim_{q\to+\infty} x_{h_q} = \bar{x}$. Now, fix $q \in \mathbb{N}$ and pick any $m \geq q+1$ such that

$$|x_{h_m} - \bar{x}| < \mathrm{d}\rho^{q+1}. \tag{5.4.15}$$

Then for all $N \geq M_{q+1}$ we have:

$$|x_N - \bar{x}| \leq |x_N - x_{h_m}| + |x_{h_m} - \bar{x}| < 2\mathrm{d}\rho^{q+1} < \mathrm{d}\rho^q$$

because $h_m \geq h_{q+1} \geq M_{q+1}$ so that we can apply (5.4.14) and (5.4.15). $\square$

**Theorem 48.** *A hypersequence converges if and only if*

$$\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n, m \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq M} : m \geq n \ \Rightarrow \ |x_n - x_m| < \mathrm{d}\rho^q.$$

*Proof.* It suffices to apply the inequality $|x_n - x_m| \le |x_n - x_{n\vee m}| + |x_{n\vee m} - x_m|$. $\square$

The second classical criterion for the existence of a hyperlimit is related to the notion of monotonic hypersequence. The existence of several chains in ${}^{\sigma}\widetilde{\mathbb{N}}$ does not allow to arrive at any $M \in {}^{\sigma}\widetilde{\mathbb{N}}$ starting from any other lower $N \in {}^{\sigma}\widetilde{\mathbb{N}}$ and using the successor operation only a finite number of times. For this reason, the following is the most natural notion of monotonic hypersequence:

**Definition 49.** We say that $(x_n)_{N\in{}^{\sigma}\widetilde{\mathbb{N}}}$ is a *non-decreasing* (or *increasing*) hypersequence if

$$\forall n, m \in {}^{\sigma}\widetilde{\mathbb{N}} : \ n \ge m \ \Rightarrow \ x_n \ge x_m.$$

Similarly, we can define the notion of non-increasing (decreasing) hypersequence.

**Theorem 50.** *Let $(x_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be a non-decreasing hypersequence. Then*

$$\exists\, {}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n \iff \exists \sup \left\{ x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}} \right\},$$

*and in that case they are equal. In particular, if $\left\{ x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}} \right\}$ is complete from above for all the upper bounds, then*

$$\exists\, {}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n \iff \exists U \in {}^{\rho}\widetilde{\mathbb{R}}\, \forall n \in {}^{\sigma}\widetilde{\mathbb{N}} : \ x_n \le U.$$

*Proof.* Assume that $(x_n)_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}$ converges to $l$ and set $S := \{x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}}\}$, we will show that $l = \sup(S)$. Now, using Def. 35, we have that $\forall n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge N} : \ x_n < l + \mathrm{d}\rho^q$ for some $N \in {}^{\sigma}\widetilde{\mathbb{N}}$. But from Def. 49 $\forall n \in {}^{\sigma}\widetilde{\mathbb{N}} : \ x_n \le x_{n\vee N} < l + \mathrm{d}\rho^q$. Therefore $x_n \le l + \mathrm{d}\rho^q$ for all $n \in {}^{\sigma}\widetilde{\mathbb{N}}$, and the conclusion $x_n \le l$ follows since $q \in \mathbb{N}$ is arbitrary. Finally, from Def. 35 of hyperlimit, for all $q \in \mathbb{N}$ we have the existence of $L \in {}^{\sigma}\widetilde{\mathbb{N}}$ such that $l - \mathrm{d}\rho^q < x_L \in S$ which completes the necessity part of the proof. Now, assume that $\exists \sup(S) =: l$. We have to prove that ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n = l$. In fact, using Rem. 1, we get

$$\forall q \in \mathbb{N}\, \exists x_N \in S : \ l - \mathrm{d}\rho^q < x_N,$$

and $x_N \le x_n \le l < l + \mathrm{d}\rho^q$ for all $n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge N}$ by Def. 49 of monotonicity. That is, $|l - x_n| = x_n - l < \mathrm{d}\rho^q$. $\square$

**Example 51.** The hypersequence $x_n := \mathrm{d}\rho^{1/n}$ is non-decreasing. Assume that $(x_n)_n$ converges to $l$ and that $\sigma \le \rho^R$ for some $R \in \mathbb{R}_{>0}$. Since $x_n \ge \mathrm{d}\rho$, by Thm. 40, we get $l \ge \mathrm{d}\rho$. Therefore, applying the logarithm and the exponential functions, from Thm. 41 we obtain that $l = 1$ because from $\sigma \le \rho^R$ it follows that ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{\log(\mathrm{d}\rho)}{n} = 0$. But this is impossible since $1 \approx 1 - \mathrm{d}\rho \not\le \mathrm{d}\rho^{1/n}$. Thereby, $\nexists \sup \left\{ \mathrm{d}\rho^{1/n} \mid n \in {}^{\sigma}\widetilde{\mathbb{N}}_{>0} \right\}$ and this set is also not complete from above.

## 5.5 Limit superior and inferior

We have two possibilities to define the notions of limit superior and inferior in a non-Archimedean setting such as ${}^{\rho}\widetilde{\mathbb{R}}$: the first one is to assume that both $\alpha_m := \sup\{x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq m}\}$ and $\inf\{\alpha_m \mid m \in {}^{\sigma}\widetilde{\mathbb{N}}\}$ exist (the former for all $m \in {}^{\sigma}\widetilde{\mathbb{N}}$); the second possibility is to use inequalities to avoid the use of supremum and infimum. In fact, in the real case we have $\iota \leq \sup_{n\geq m} x_n \leq \iota+\varepsilon$ if and only if

$$\begin{aligned}
&\forall n \geq m: \ x_n \leq \iota+\varepsilon\\
&\forall \varepsilon\,\exists \bar{n} \geq m: \ \iota-\varepsilon \leq x_{\bar{n}}.
\end{aligned}$$

**Definition 52.** Let $(x_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be an hypersequence, then we say that $\iota \in {}^{\rho}\widetilde{\mathbb{R}}$ is the *limit superior* of $(x_n)_n$ if

1. $\forall q \in \mathbb{N}\,\exists N \in {}^{\sigma}\widetilde{\mathbb{N}}\,\forall n \geq N: \ x_n \leq \iota + \mathrm{d}\rho^q$;
2. $\forall q \in \mathbb{N}\,\forall N \in {}^{\sigma}\widetilde{\mathbb{N}}\,\exists \bar{n} \geq N: \ \iota - \mathrm{d}\rho^q \leq x_{\bar{n}}$.

Similarly, we say that $\sigma \in {}^{\rho}\widetilde{\mathbb{R}}$ is the *limit inferior* of $(x_n)_n$ if

3. $\forall q \in \mathbb{N}\,\exists N \in {}^{\sigma}\widetilde{\mathbb{N}}\,\forall n \geq N: \ x_n \geq \sigma - \mathrm{d}\rho^q$;
4. $\forall q \in \mathbb{N}\,\forall N \in {}^{\sigma}\widetilde{\mathbb{N}}\,\exists \bar{n} \geq N: \ \sigma + \mathrm{d}\rho^q \geq x_{\bar{n}}$.

We have the following results (clearly, dual results hold for the limit inferior):

**Theorem 53.** *Let $(x_n)_n$, $(y_n)_n : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be hypersequences, then*

1. *There exists at most one limit superior and at most one limit inferior. They are denoted with ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ and ${}^{\rho}\liminf_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$.*
2. *If $\exists \sup\left\{x_n \mid n \in {}^{\sigma}\widetilde{\mathbb{N}}_{\geq m}\right\} =: \alpha_m$ for all $m \in {}^{\sigma}\widetilde{\mathbb{N}}$, then $\exists\, {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ if and only if $\exists \inf\left\{\alpha_m \mid m \in {}^{\sigma}\widetilde{\mathbb{N}}\right\}$, and in that case*
$$ {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n = {}^{\rho}\lim_{m\in{}^{\sigma}\widetilde{\mathbb{N}}} \alpha_m = \inf\left\{\alpha_m \mid m \in {}^{\sigma}\widetilde{\mathbb{N}}\right\}.$$
3. *${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}(-x_n) = -{}^{\rho}\liminf_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ in the sense that if one of them exists, then also the other one exists and in that case they are equal.*
4. *$\exists^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ if and only if $\exists^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n = {}^{\rho}\liminf_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$.*
5. *If $\exists^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$, ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} y_n$, ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}(x_n+y_n)$, then*
$$ {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}(x_n+y_n) \leq {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n + {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} y_n.$$
*In particular, if $\forall N \in {}^{\sigma}\widetilde{\mathbb{N}}\,\forall \bar{n}, \hat{n} \geq N\,\exists n \geq N: \ x_{\bar{n}} + y_{\hat{n}} \leq x_n + y_n$, then the existence of the single limit superiors implies the existence of the limit superior of the sum.*

6. *If $x_n$, $y_n \ge 0$ for all $n \in {}^\sigma\widetilde{\mathbb{N}}$ and if $\exists^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n$, ${}^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}} y_n$, ${}^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}} (x_n \cdot y_n)$, then*

$$ {}^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}}(x_n \cdot y_n) \le {}^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n \cdot {}^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}} y_n. $$

   *In particular, if $\forall N \in {}^\sigma\widetilde{\mathbb{N}}\,\forall \bar{n}, \hat{n} \ge N\,\exists n \ge N :\ x_{\bar{n}} \cdot y_{\hat{n}} \le x_n \cdot y_n$, then the existence of the single limit superiors implies the existence of the limit superior of the product.*

7. *If $\exists\,{}^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n =: \iota$, then there exists a sequence $(\bar{n}_q)_{q\in\mathbb{N}}$ of ${}^\sigma\widetilde{\mathbb{N}}$ such that*

   (a) *$\bar{n}_{q+1} > \bar{n}_q$ for all $q \in \mathbb{N}$;*

   (b) *$\lim_{q\to+\infty} \bar{n}_q = +\infty$ in ${}^\sigma\widetilde{\mathbb{R}}$;*

   (c) *$\exists \lim_{q\to+\infty} x_{\bar{n}_q} = \iota$.*

8. *Assume to have a sequence $(\bar{n}_q)_{q\in\mathbb{N}}$ satisfying the previous conditions 7a, 7b, 7c and*

$$ \forall n \in {}^\sigma\widetilde{\mathbb{N}}\,\exists p \in \mathbb{N} :\ \bar{n}_p \ge n,\ x_n \le x_{\bar{n}_p}. \tag{5.5.1} $$

   *Then $\exists\,{}^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_n =: \iota$.*

*Proof.* 1: Let $\iota_1$, $\iota_2$ be both limit superior of $(x_n)_n$. Based on Lem. 11.3, without loss of generality we can assume that $\iota_1 <_{\mathrm{s}} \iota_2$. According to Lem. 4, there exists $m \in \mathbb{N}$ such that $\iota_1 + \mathrm{d}\rho^m <_{\mathrm{s}} \iota_2$. Take $q_1$, $q_2$ large enough so that $\mathrm{d}\rho^{q_1} + \mathrm{d}\rho^{q_2} < \mathrm{d}\rho^m$. Using the last two inequalities, we obtain

$$ \iota_1 + \mathrm{d}\rho^{q_1} <_{\mathrm{s}} \iota_2 - \mathrm{d}\rho^{q_2}. \tag{5.5.2} $$

Using Def. 52.1, we can find $N_1 \in {}^\sigma\widetilde{\mathbb{N}}$ such that

$$ \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N_1} :\ x_n \le \iota_1 + \mathrm{d}\rho^{q_1}. \tag{5.5.3} $$

Using Def. 52.2 with $q = q_1$ and $N = N_1$, we get

$$ \exists \bar{n} \in {}^\sigma\widetilde{\mathbb{N}}_{\ge N_1} :\ \iota_2 - \mathrm{d}\rho^{q_2} \le x_{\bar{n}}. \tag{5.5.4} $$

We now use (5.5.2), (5.5.4) and (5.5.3) for $n = \bar{n}$ and we obtain $x_{\bar{n}} \le \iota_1 + \mathrm{d}\rho^{q_1} <_{\mathrm{s}} \iota_2 - \mathrm{d}\rho^{q_2} \le x_{\bar{n}}$, which is a contradiction.

2: Lem. 25.3 implies that $(\alpha_m)_m$ is non-increasing. Therefore, we have ${}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \alpha_m = \inf\left\{\alpha_m \mid m \in {}^\sigma\widetilde{\mathbb{N}}\right\}$ if these terms exist from Thm. 50. But Cor. 33 and Def. 52.1 imply $\alpha_m \le \iota + \mathrm{d}\rho^q$. Finally, Def. 52.2 yields $\iota - \mathrm{d}\rho^q \le x_{\bar{n}} \le \alpha_m$, which proves that $\exists\,{}^\rho\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \alpha_m = {}^\rho\limsup_{n\in{}^\sigma\widetilde{\mathbb{N}}} x_m = \iota$.

3: Directly from Def. 52.

4: Assume that hyperlimit superior and inferior exist and are equal to $l$. From Def. 52.1 and Def. 52.3 we get $l - \mathrm{d}\rho^q \le x_n \le l + \mathrm{d}\rho^q$ for all $n \ge N$. Vice versa, assume that the hyperlimit exists and equals $l$, so that $l - \mathrm{d}\rho^q \le x_n \le$

$l+\mathrm{d}\rho^q$ for all $n \geq N$. Then both Def. 52.1 and Def. 52.3 trivially hold. Finally, Def. 52.2 and Def. 52.4 hold taking e.g. $\bar{n} = N$.

5: Setting

$$\begin{aligned}\iota &:= {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n\\ j &:= {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} y_n\\ l &:= {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} (x_n + y_n)\,,\end{aligned}$$

from Def. 52 we get $l - \mathrm{d}\rho^q \leq x_{\bar{n}} + y_{\bar{n}} \leq \iota + j + 2\mathrm{d}\rho^q$, which implies $l \leq \iota + j$ for $q \to +\infty$. Adding Def. 52.2 we obtain $\iota + j - 2\mathrm{d}\rho^q \leq x_{\bar{n}} + y_{\hat{n}}$ for some $\bar{n}$, $\hat{n} \geq N \in {}^{\sigma}\widetilde{\mathbb{N}}$. Therefore, if $x_{\bar{n}} + y_{\hat{n}} \leq x_n + y_n$ for some $n \geq N$, this yields the second claim. Similarly, one can prove 6.

7: From Def. 52.1, choose an $N_q = N$ for each $q \in \mathbb{N}$, i.e.

$$\forall q \in \mathbb{N}\, \exists N_q \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n \geq N_q :\ x_n \leq \iota + \mathrm{d}\rho^q. \tag{5.5.5}$$

Applying Def. 52.2 with $q > 0$ and $N = N_q \vee (\bar{n}_{q-1} + 1) \vee [\mathrm{int}(\sigma_\varepsilon^{-q})] \in {}^{\sigma}\widetilde{\mathbb{N}}$, we get the existence of $\bar{n}_q \geq N_q$ such that both 7a and 7b hold and $\iota - \mathrm{d}\rho^q \leq x_{\bar{n}_q}$. Thereby, from (5.5.5) we also get 7c.

8: Write 7c as

$$\forall q \in \mathbb{N}\, \exists Q_q \in \mathbb{N}\, \forall p \in \mathbb{N}_{\geq Q_q} :\ \iota - \mathrm{d}\rho^p \leq x_{\bar{n}_p} \leq \iota + \mathrm{d}\rho^p. \tag{5.5.6}$$

Set $N := \bar{n}_{Q_q} \in {}^{\sigma}\widetilde{\mathbb{N}}$. For $n \geq N$, from (5.5.1) we get the existence of $p \in \mathbb{N}$ such that $\bar{n}_p \geq n$ and $x_n \leq x_{\bar{n}_p}$. Thereby, $\bar{n}_p \geq \bar{n}_{Q_q}$ and hence $p \geq Q_q$ because of 7a and thus $x_n \leq x_{\bar{n}_p} \leq \iota + \mathrm{d}\rho^q$. Finally, condition 2 of Def. 52 follows from (5.5.6) and 7b. □

It remains an open problem to show an example that proves as necessary the assumption of Thm. 53.2, i.e. that that the previous definition of limit superior and inferior is strictly more general than the simple transposition of the classical one.

**Example 54.**

1. Directly from Def. 52, we have that

$$ {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}(-1)^n = 1,\ {}^{\rho}\liminf_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}(-1)^n = -1$$

2. Let $\mu \in {}^{\rho}\widetilde{\mathbb{R}}$ be such that $\mu|_L = 1$ and $\mu|_{L^c} = -1$, where $L$, $L^c \subseteq_0 I$. Then $\mu^n \leq 1$ and $1 - \mathrm{d}\rho^q \leq \mu^{\bar{n}}$ if $\mathrm{ni}(\bar{n})_\varepsilon$ is even for all $\varepsilon$ small. Therefore ${}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \mu^n = 1$, $\sup_{n\geq m} \mu^n = 1$, whereas $\nexists\, {}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \mu^n$.

3. From 7 and 8 of Thm. 53 it follows that for an increasing hypersequence $(x_n)_n$, $\exists {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$ if and only if $\exists {}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} x_n$. Therefore, example 51 implies that $\nexists\, {}^{\rho}\limsup_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \mathrm{d}\rho^{1/n}$.

## 5.6 Summary of the chapter 5

To sum up, this chapter can be viewed as ancillary to the following main chapters of the doctoral dissertation. Above, we formulated how to deal with several deficiencies of the ring of Robinson-Colombeau generalized numbers ${}^{\rho}\widetilde{\mathbb{R}}$: trichotomy law for the order relations $\le$ and $<$, existence of supremum and infimum and limits of sequences with a topology generated by infinitesimal radii. In each case, we obtain a faithful generalization of the classical case of real numbers. We think that some of the ideas we presented in this chapter can inspire similar works in other non-Archimedean settings such as (constructive) nonstandard analysis, p-adic analysis, the Levi-Civita field, surreal numbers, etc. Clearly, the notions introduced above open the possibility to extend classical proofs in dealing with series, analytic generalized functions, sigma-additivity in integration of generalized functions, non-Archimedean functional analysis, just to mention a few. In the next chapter, we consider the core results of this doctoral dissertation.

# Chapter 6

# A Fourier transform for all generalized functions

## 6.1 Basic notions

### 6.1.1 Generalized smooth functions and their calculus

Using the ring ${}^{\rho}\widetilde{\mathbb{R}}$, it is easy to consider a Gaussian with an infinitesimal standard deviation. If we denote this probability density by $f(x,\sigma)$, and if we set $\sigma = [\sigma_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, where $\sigma \approx 0$, we obtain the net of smooth functions $(f(-,\sigma_\varepsilon))_{\varepsilon\in I}$. This is the basic idea we are going to develop in the following

**Definition 55.** Let $(\Omega_\varepsilon)$ be a net of open subsets of $\mathbb{R}^n$. Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Then we say that

$$f : X \longrightarrow Y \text{ is a } \textit{generalized smooth function}$$

if there exists a net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ defining the map $f : X \longrightarrow Y$ in the sense that

1. $X \subseteq \langle \Omega_\varepsilon \rangle$,
2. $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \in Y$ for all $x = [x_\varepsilon] \in X$,
3. $(\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_\rho$ for all $x = [x_\varepsilon] \in X$ and all $\alpha \in \mathbb{N}^n$.

The space of generalized smooth functions (GSF) from $X$ to $Y$ is denoted by ${}^{\rho}\mathcal{GC}^\infty(X, Y)$.

Let us note explicitly that this definition states minimal logical conditions to obtain a set-theoretical map from $X$ into $Y$ and defined by a net of smooth functions of which we can take arbitrary derivatives still remaining in the space of $\rho$-moderate nets. In particular, the following Thm. 56 states that the equality $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)]$ is meaningful, i.e. that we have independence from the representatives for all derivatives $[x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in {}^{\rho}\widetilde{\mathbb{R}}^d$, $\alpha \in \mathbb{N}^n$.

**Theorem 56.** *Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Let $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ be a net of smooth functions that defines a generalized smooth map of the type $X \longrightarrow Y$, then*

1. $\forall \alpha \in \mathbb{N}^n \, \forall (x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}^n_\rho : \, [x_\varepsilon] = [x'_\varepsilon] \in X \, \Rightarrow \, (\partial^\alpha f_\varepsilon(x_\varepsilon)) \sim_\rho (\partial^\alpha f_\varepsilon(x'_\varepsilon))$.
2. *Each $f \in {}^{\rho}\mathcal{GC}^\infty(X, Y)$ is continuous with respect to the sharp topologies induced on $X$, $Y$.*
3. *$f : X \longrightarrow Y$ is a GSF if and only if there exists a net $v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ defining a generalized smooth map of type $X \longrightarrow Y$ such that $f = [v_\varepsilon(-)]|_X$.*
4. *GSF are closed with respect to composition, i.e. subsets $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}^s$ with the trace of the sharp topology, and GSF as arrows form a subcategory of the category of topological spaces. We will call this category ${}^{\rho}\mathcal{GC}^\infty$, the* category of GSF. *Therefore, with pointwise sum and product, any space ${}^{\rho}\mathcal{GC}^\infty(X, {}^{\rho}\widetilde{\mathbb{R}})$ is an algebra.*

The differential calculus for GSF can be introduced by showing existence and uniqueness of another GSF serving as incremental ratio (sometimes this is called *derivative á la Carathéodory*, see e.g. [43]).

**Theorem 57** (Fermat-Reyes theorem for GSF)**.** *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be a sharply open set, let $v = [v_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n$, and let $f \in {}^{\rho}\mathcal{GC}^\infty(U, {}^{\rho}\widetilde{\mathbb{R}})$ be a GSF generated by the net of smooth functions $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$. Then*

1. *There exists a sharp neighborhood $T$ of $U \times \{0\}$ and a generalized smooth map $r \in {}^{\rho}\mathcal{GC}^\infty(T, {}^{\rho}\widetilde{\mathbb{R}})$, called the* generalized incremental ratio *of $f$* along *$v$, such that*
$$\forall (x, h) \in T : \, f(x + hv) = f(x) + h \cdot r(x, h).$$
2. *Any two generalized incremental ratios coincide on a sharp neighborhood of $U \times \{0\}$, so that we can use the notation $f[x; h] := r(x, h)$ if $(x, h)$ are sufficiently small.*
3. *We have $f[x; 0] = \left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon)\right]$ for every $x \in U$ and we can thus define $Df(x) \cdot v := \frac{\partial f}{\partial v}(x) := f[x; 0]$, so that $\frac{\partial f}{\partial v} \in {}^{\rho}\mathcal{GC}^\infty(U, {}^{\rho}\widetilde{\mathbb{R}})$.*

Note that this result permits us to consider the partial derivative of $f$ with respect to an arbitrary generalized vector $v \in {}^{\rho}\widetilde{\mathbb{R}}^n$ which can be, e.g., infinitesimal or infinite. Using recursively this result, we can also define subsequent differentials $D^j f(x)$ as $j-$multilinear maps, and we set $D^j f(x) \cdot h^j := D^j f(x)(h, \overset{j}{\ldots}, h)$. The set of all the $j-$multilinear maps $\left({}^{\rho}\widetilde{\mathbb{R}}^n\right)^j \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}^d$ over the ring ${}^{\rho}\widetilde{\mathbb{R}}$ will be denoted by $L^j({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}\widetilde{\mathbb{R}}^d)$. For $A = [A_\varepsilon(-)] \in L^j({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}\widetilde{\mathbb{R}}^d)$, we set $\|A\| := [|A_\varepsilon|]$, the generalized number defined by the operator norms of the multilinear maps $A_\varepsilon \in L^j(\mathbb{R}^n, \mathbb{R}^d)$.

The following result follows from the analogous properties for the nets of smooth functions defining $f$ and $g$.

**Theorem 58.** *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be an open subset in the sharp topology, let $v \in {}^{\rho}\widetilde{\mathbb{R}}^n$ and $f$, $g : U \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be generalized smooth maps. Then*

1. $\frac{\partial(f+g)}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial v}$
2. $\frac{\partial(r \cdot f)}{\partial v} = r \cdot \frac{\partial f}{\partial v} \quad \forall r \in {}^{\rho}\widetilde{\mathbb{R}}$
3. $\frac{\partial(f \cdot g)}{\partial v} = \frac{\partial f}{\partial v} \cdot g + f \cdot \frac{\partial g}{\partial v}$
4. *For each $x \in U$, the map $\mathrm{d}f(x).v := \frac{\partial f}{\partial v}(x) \in {}^{\rho}\widetilde{\mathbb{R}}$ is ${}^{\rho}\widetilde{\mathbb{R}}$-linear in $v \in {}^{\rho}\widetilde{\mathbb{R}}^n$*
5. *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $V \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ be open subsets in the sharp topology and $g \in {}^{\rho}\mathcal{GC}^{\infty}(V, U)$, $f \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{\rho}\widetilde{\mathbb{R}})$ be generalized smooth maps. Then for all $x \in V$ and all $v \in {}^{\rho}\widetilde{\mathbb{R}}^d$, we have $\frac{\partial(f \circ g)}{\partial v}(x) = \mathrm{d}f\left(g(x)\right) . \frac{\partial g}{\partial v}(x)$.*

One dimensional integral calculus of GSF is based on the following

**Theorem 59.** *Let $f \in {}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}})$ be a GSF defined in the interval $[a, b] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, where $a < b$. Let $c \in [a, b]$. Then, there exists one and only one GSF $F \in {}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}})$ such that $F(c) = 0$ and $F'(x) = f(x)$ for all $x \in [a, b]$. Moreover, if $f$ is defined by the net $f_\varepsilon \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R})$ and $c = [c_\varepsilon]$, then $F(x) = \left[\int_{c_\varepsilon}^{x_\varepsilon} f_\varepsilon(s)\mathrm{d}s\right]$ for all $x = [x_\varepsilon] \in [a, b]$.*

We can thus define

**Definition 60.** Under the assumptions of Theorem 59, we denote by $\int_c^{(-)} f := \int_c^{(-)} f(s)\,\mathrm{d}s \in {}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}})$ the unique GSF such that:

1. $\int_c^c f = 0$
2. $\left(\int_u^{(-)} f\right)'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_u^x f(s)\,\mathrm{d}s = f(x)$ for all $x \in [a, b]$.

All the classical rules of integral calculus hold in this setting:

**Theorem 61.** *Let $f \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{\rho}\widetilde{\mathbb{R}})$ and $g \in {}^{\rho}\mathcal{GC}^{\infty}(V, {}^{\rho}\widetilde{\mathbb{R}})$ be two GSF defined on sharply open domains in ${}^{\rho}\widetilde{\mathbb{R}}$. Let $a$, $b \in {}^{\rho}\widetilde{\mathbb{R}}$ with $a < b$ and $c$, $d \in [a, b] \subseteq U \cap V$, then*

1. $\int_c^d (f + g) = \int_c^d f + \int_c^d g$
2. $\int_c^d \lambda f = \lambda \int_c^d f \quad \forall \lambda \in {}^{\rho}\widetilde{\mathbb{R}}$
3. $\int_c^d f = \int_c^e f + \int_e^d f$ *for all* $e \in [a, b]$
4. $\int_c^d f = -\int_d^c f$
5. $\int_c^d f' = f(d) - f(c)$
6. $\int_c^d f' \cdot g = [f \cdot g]_c^d - \int_c^d f \cdot g'$

7. *If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$.*

8. *Let $a$, $b$, $c$, $d \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$ and $c < d$, and $f \in {}^\rho\mathcal{GC}^\infty([a,b]\times[c,d], {}^\rho\widetilde{\mathbb{R}}^d)$, then*
$$\frac{\mathrm{d}}{\mathrm{d}s}\int_a^b f(\tau,s)\,\mathrm{d}\tau = \int_a^b \frac{\partial}{\partial s} f(\tau,s)\,\mathrm{d}\tau \quad \forall s \in [c,d].$$

**Theorem 62.** *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ and $\varphi \in {}^\rho\mathcal{GC}^\infty(V,U)$ be GSF defined on sharply open domains in ${}^\rho\widetilde{\mathbb{R}}$. Let $a$, $b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $[a,b] \subseteq V$, $\varphi(a) < \varphi(b)$, $[\varphi(a), \varphi(b)] \subseteq U$. Finally, assume that $\varphi([a,b]) \subseteq [\varphi(a),\varphi(b)]$. Then*
$$\int_{\varphi(a)}^{\varphi(b)} f(t)\mathrm{d}t = \int_a^b f\left[\varphi(s)\right]\cdot\varphi'(s)\mathrm{d}s.$$

We also have a generalization of Taylor formula:

**Theorem 63.** *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. Let $a$, $b \in {}^\rho\widetilde{\mathbb{R}}^d$ such that the line segment $[a,b] \subseteq U$, and set $h := b - a$. Then, for all $n \in \mathbb{N}$ we have*

1. *$\exists \xi \in [a,b] :\ f(a+h) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!}\cdot h^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!}\cdot h^{n+1}$.*

2. *$f(a+h) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!}\cdot h^j + \frac{1}{n!}\cdot\int_0^1 (1-t)^n\,\mathrm{d}^{n+1} f(a+th)\cdot h^{n+1}\,\mathrm{d}t$.*

*Moreover, there exists some $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that*
$$\forall k \in B_R(0)\,\exists\xi \in [a,a+k] :\ f(a+k) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!}\cdot k^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!}\cdot k^{n+1} \quad (6.1.1)$$
$$\frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!}\cdot k^{n+1} = \frac{1}{n!}\cdot\int_0^1 (1-t)^n\,\mathrm{d}^{n+1} f(a+tk)\cdot k^{n+1}\,\mathrm{d}t \approx 0. \quad (6.1.2)$$

Formulas 1 and 2 correspond to a plain generalization of Taylor's theorem for ordinary smooth functions with Lagrange and integral remainder, respectively. Dealing with generalized functions, it is important to note that this direct statement also includes the possibility that the differential $\mathrm{d}^{n+1} f(\xi)$ may be an infinite number at some point. For this reason, in (6.1.1) and (6.1.2), considering a sufficiently small increment $k$, we get more classical infinitesimal remainders $\mathrm{d}^{n+1} f(\xi)\cdot k^{n+1} \approx 0$. We can also define right and left derivatives as e.g. $f'(a) := f'_+(a) := \lim_{\substack{t\to a\\ a<t}} f'(t)$, which always exist if $f \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d)$.

### 6.1.2 Embedding of Sobolev-Schwartz distributions and Colombeau functions

We finally recall two results that give a certain flexibility in constructing embeddings of Schwartz distributions. Note that both the infinitesimal $\rho$ and the

embedding of Schwartz distributions have to be chosen depending on the problem we aim to solve. A trivial example in this direction is the ODE $y' = y/\mathrm{d}\varepsilon$, which cannot be solved for $\rho = (\varepsilon)$, but it has a solution for $\rho = (e^{-1/\varepsilon})$. As another simple example, if we need the property $H(0) = 1/2$, where $H$ is the Heaviside function, then we have to choose the embedding of distributions accordingly. In other words, both the gauges and the particular embedding we choose have to be thought of elements of the mathematical structure we are considering to deal with the particular problem we want to solve. See also [32, 49] for further details in this direction.
If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $r \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}^n$, we use the notations $r \odot \varphi$ for the function $x \in \mathbb{R}^n \mapsto \frac{1}{r^n} \cdot \varphi\left(\frac{x}{r}\right) \in \mathbb{R}$ and $x \oplus \varphi$ for the function $y \in \mathbb{R}^n \mapsto \varphi(y-x) \in \mathbb{R}$. These notations permit us to highlight that $\odot$ is a free action of the multiplicative group $(\mathbb{R}_{>0}, \cdot, 1)$ on $\mathcal{D}(\mathbb{R}^n)$ and $\oplus$ is a free action of the additive group $(\mathbb{R}_{>0}, +, 0)$ on $\mathcal{D}(\mathbb{R}^n)$. We also have the distributive property $r \odot (x \oplus \varphi) = rx \oplus r \odot \varphi$.

**Lemma 64.** *Let $b \in {}^\rho\widetilde{\mathbb{R}}$ be a net such that $\lim_{\varepsilon\to 0^+} b_\varepsilon = +\infty$. Let $d \in (0,1)_{\mathbb{R}}$, there exists a net $(\psi_\varepsilon)_{\varepsilon\in I}$ of $\mathcal{D}(\mathbb{R}^n)$ with the properties:*

1. *supp$(\psi_\varepsilon) \subseteq B_1(0)$ and $\psi_\varepsilon$ is even for all $\varepsilon \in I$.*
2. *Let $\omega_n$ denote the surface area of $S^{n-1}$ and set $c_n := \frac{2n}{\omega_n}$ for $n > 1$ and $c_1 := 1$, then $\psi_\varepsilon(0) = c_n$ for all $\varepsilon \in I$.*
3. *$\int \psi_\varepsilon = 1$ for all $\varepsilon \in I$.*
4. *$\forall \alpha \in \mathbb{N}^n : \ \sup_{x\in\mathbb{R}^n} |\partial^\alpha \psi_\varepsilon(x)| = O(b_\varepsilon^{2+|\alpha|})$ as $\varepsilon \to 0^+$.*
5. *$\forall j \in \mathbb{N}\, \forall^0\varepsilon : \ 1 \le |\alpha| \le j \Rightarrow \int x^\alpha \cdot \psi_\varepsilon(x)\,\mathrm{d}x = 0$.*
6. *$\forall \eta \in \mathbb{R}_{>0}\, \forall^0\varepsilon : \ \int |\psi_\varepsilon| \le 1 + \eta$.*
7. *If $n = 1$, then the net $(\psi_\varepsilon)_{\varepsilon\in I}$ can be chosen so that $\int_{-\infty}^0 \psi_\varepsilon = d$.*

*In particular $\psi_\varepsilon^b := b_\varepsilon^{-1} \odot \psi_\varepsilon$ satisfies 3 - 6.*

Concerning embeddings of Schwartz distributions, we have the following result, where $\mathrm{c}(\Omega) := \{[x_\varepsilon] \in [\Omega] \mid \exists K \Subset \Omega\, \forall^0\varepsilon : \ x_\varepsilon \in K\}$ is called the set of *compactly supported points in* $\Omega \subseteq \mathbb{R}^n$. Note that $\mathrm{c}(\Omega) = \{x \in [\Omega] \mid x \text{ is finite}\}$ (see Def. 2).

**Theorem 65.** *Under the assumptions of Lemma 64, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $(\psi_\varepsilon^b)$ be the net defined in 64. Then the mapping*

$$\iota_\Omega^b : T \in \mathcal{E}'(\Omega) \mapsto \left[\left(T * \psi_\varepsilon^b\right)(-)\right] \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}}) \tag{6.1.3}$$

*uniquely extends to a sheaf morphism of real vector spaces*

$$\iota^b : \mathcal{D}' \longrightarrow {}^\rho\mathcal{GC}^\infty(\mathrm{c}(-), {}^\rho\widetilde{\mathbb{R}}),$$

*and satisfies the following properties:*

1. *If $b \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ is a strong infinite number, then $\iota^b|_{\mathcal{C}^\infty(-)} : \mathcal{C}^\infty(-) \longrightarrow {}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(-), {}^{\rho}\widetilde{\mathbb{R}})$ is a sheaf morphism of algebras and $\iota^b_\Omega(f)(x) = f(x)$ for all smooth functions $f \in \mathcal{C}^\infty(\Omega)$ and all $x \in \Omega$;*

2. *If $T \in \mathcal{E}'(\Omega)$ then $\mathrm{supp}(T) = \mathrm{stsupp}(\iota^b_\Omega(T))$, where*
$$\mathrm{stsupp}(f) := \left(\bigcup \{\Omega' \subseteq \Omega \mid \Omega' \text{ open},\ f|_{\Omega'} = 0\}\right)^c \tag{6.1.4}$$
*for all $f \in {}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}})$.*

3. *Let $b \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ be a strong infinite number. Then $\left[\int_\Omega \iota^b_\Omega(T)_\varepsilon(x) \cdot \varphi(x)\,\mathrm{d}x\right] = \langle T, \varphi\rangle$ for all $\varphi \in \mathcal{D}(\Omega)$ and all $T \in \mathcal{D}'(\Omega)$;*

4. *$\iota^b$ commutes with partial derivatives, i.e. $\partial^\alpha\left(\iota^b_\Omega(T)\right) = \iota^b_\Omega(\partial^\alpha T)$ for each $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}$.*

5. *Similar results also hold for the embedding of tempered distributions:*
$$\iota^b_\Omega : T \in \mathcal{S}'(\Omega) \mapsto \left[\left(T * \psi^b_\varepsilon\right)(-)\right] \in {}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}}).$$

Concerning the embedding of Colombeau generalized functions (CGF), we recall that the special Colombeau algebra on $\Omega$ is defined as the quotient $\mathcal{G}^s(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}^s(\Omega)$ of *moderate nets* over *negligible nets*, where the former is

$$\mathcal{E}_M(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\}$$

and the latter is

$$\mathcal{N}^s(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^m)\}.$$

Using $\rho = (\varepsilon)$, we have the following compatibility result:

**Theorem 66.** *A Colombeau generalized function $u = (u_\varepsilon) + \mathcal{N}^s(\Omega)^d \in \mathcal{G}^s(\Omega)^d$ defines a GSF $u : [x_\varepsilon] \in \mathrm{c}(\Omega) \longrightarrow [u_\varepsilon(x_\varepsilon)] \in \widetilde{\mathbb{R}}^d$. This assignment provides a bijection of $\mathcal{G}^s(\Omega)^d$ onto ${}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}}^d)$ for every open set $\Omega \subseteq \mathbb{R}^n$.*

**Example 67.**

1. Let $\delta \in {}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(\mathbb{R}^n), {}^{\rho}\widetilde{\mathbb{R}})$ and $H \in {}^{\rho}\mathcal{GC}^\infty(\mathrm{c}(\mathbb{R}), {}^{\rho}\widetilde{\mathbb{R}})$ be the $\iota^b$-embeddings of the Dirac delta and of the Heaviside function. Then $\delta(x) = b^n \cdot \psi(b \cdot x)$, where $\psi(x) := [\psi_\varepsilon(x_\varepsilon)]$ is called *n-dimensional Colombeau mollifier*. Note that $\delta$ is an even function because of Lem. 64.1. We have that $\delta(0) = c_n b^n$ is a strong infinite number and $\delta(x) = 0$ if $|x| > r$ for some $r \in \mathbb{R}_{>0}$ because of Lem. 64.1 (see Lem. 64.2 for the definition of $c_n \in \mathbb{R}_{>0}$). If $n = 1$, by the intermediate value theorem (see [31]), $\delta$ takes any value in the interval $[0, b] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$. Similar properties can be stated e.g. for $\delta^2(x) = b^2 \cdot \psi(b \cdot x)^2$. Using these formulas, we can simply consider $\delta \in {}^{\rho}\mathcal{GC}^\infty({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}\widetilde{\mathbb{R}})$ and $H \in {}^{\rho}\mathcal{GC}^\infty({}^{\rho}\widetilde{\mathbb{R}}, {}^{\rho}\widetilde{\mathbb{R}})$.

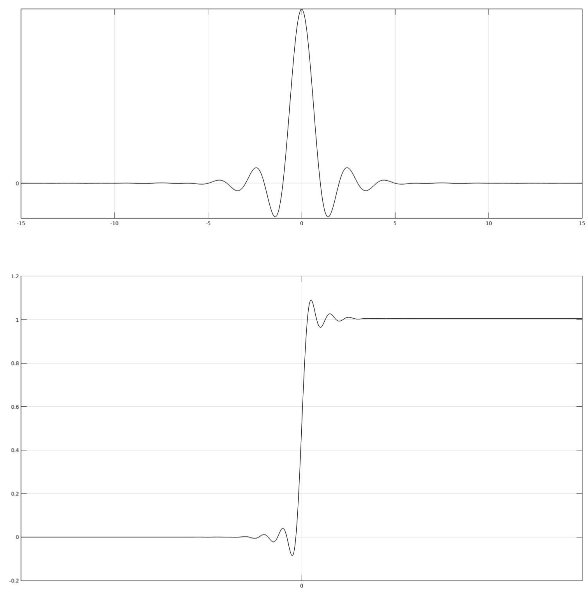


Figure 6.1.1: Representations of Dirac delta and Heaviside function

2. Analogously, we have $H(x) = 1$ if $x > r$ for some $r \in \mathbb{R}_{>0}$; $H(x) = 0$ if $x < -r$ for some $r \in \mathbb{R}_{>0}$, and finally $H(0) = \frac{1}{2}$ because of Lem. 64.1. By the intermediate value theorem, $H$ takes any value in the interval $[0, 1] \subseteq {}^\rho\widetilde{\mathbb{R}}$.

3. If $n = 1$, The composition $\delta \circ \delta \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$ is given by $(\delta \circ \delta)(x) = b\psi\left(b^2\psi(bx)\right)$ and is an even function. If $|x| > r$ for some $r \in \mathbb{R}_{>0}$, then $(\delta \circ \delta)(x) = b$. Since $(\delta \circ \delta)(0) = 0$, again using the intermediate value theorem, we have that $\delta \circ \delta$ takes any value in the interval $[0, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$. Suitably choosing the net $(\psi_\varepsilon)$ it is possible to have that if $0 \le x \le \frac{1}{kb}$ for some $k \in \mathbb{N}_{>1}$ (hence $x$ is infinitesimal), then $(\delta \circ \delta)(x) = 0$. If $x = \frac{k}{b}$ for some $k \in \mathbb{N}_{>0}$, then $x$ is still infinitesimal but $(\delta \circ \delta)(x) = b$. Analogously, one can deal with compositions such as $H \circ \delta$ and $\delta \circ H$.

See Fig. 6.1.1 for a graphical representations of $\delta$ and $H$. The infinitesimal oscillations shown in this figure can be proved to actually occur as a consequence of Lem. 64.5 which is a necessary property to prove Thm. 65.1, see [31, 32]. It is well-known that the latter property is one of the core ideas to bypass the Schwartz's impossibility theorem, see e.g. [37].

## 6.2 Functionally compact sets and multidimensional integration

### 6.2.1 Extreme value theorem and functionally compact sets

For GSF, suitable generalizations of many classical theorems of differential and integral calculus hold: intermediate value theorem, mean value theorems, suitable sheaf properties, local and global inverse function theorems, Banach fixed point theorem and a corresponding Picard-Lindelöf theorem both for ODE and PDE, see [29, 30, 31, 49, 32].

Even though the intervals $[a,b] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, $a$, $b \in \mathbb{R}$, are not compact in the sharp topology (see [29]), analogously to the case of smooth functions, a GSF satisfies an extreme value theorem on such sets. In fact, we have:

**Theorem 68.** *Let $f \in \mathcal{GC}^{\infty}(X, {}^{\rho}\widetilde{\mathbb{R}})$ be a GSF defined on the subset $X$ of ${}^{\rho}\widetilde{\mathbb{R}}^n$. Let $\emptyset \neq K = [K_\varepsilon] \subseteq X$ be an internal set generated by a sharply bounded net $(K_\varepsilon)$ of compact sets $K_\varepsilon \Subset \mathbb{R}^n$ , then*

$$\exists m, M \in K \, \forall x \in K : \; f(m) \leq f(x) \leq f(M). \tag{6.2.1}$$

We shall use the assumptions on $K$ and $(K_\varepsilon)$ given in this theorem to introduce a notion of "compact subset" which behaves better than the usual classical notion of compactness in the sharp topology.

**Definition 69.** A subset $K$ of ${}^{\rho}\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$, if there exists a net $(K_\varepsilon)$ such that

1. $K = [K_\varepsilon] \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$.
2. $\exists R \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} : \; K \subseteq B_R(0)$, i.e. $K$ is sharply bounded.
3. $\forall \varepsilon \in I : \; K_\varepsilon \Subset \mathbb{R}^n$.

If, in addition, $K \subseteq U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ then we write $K \Subset_{\mathrm{f}} U$. Finally, we write $[K_\varepsilon] \Subset_{\mathrm{f}} U$ if 2, 3 and $[K_\varepsilon] \subseteq U$ hold. Any net $(K_\varepsilon)$ such that $[K_\varepsilon] = K$ is called a *representative* of $K$.

We motivate the name *functionally compact subset* by noting that on this type of subsets, GSF have properties very similar to those that ordinary smooth functions have on standard compact sets.

*Remark* 70.

1. By Thm. 14.3, any internal set $K = [K_\varepsilon]$ is closed in the sharp topology and hence functionally compact sets are always closed. In particular, the open interval $(0,1) \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ is not functionally compact since it is not closed.
2. If $H \Subset \mathbb{R}^n$ is a non-empty ordinary compact set, then the internal set $[H]$ is functionally compact. In particular, $[0,1] = [[0,1]_{\mathbb{R}}]$ is functionally compact.

3. The empty set $\emptyset = \widetilde{\emptyset} \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}$.

4. ${}^\rho\widetilde{\mathbb{R}}^n$ is not functionally compact since it is not sharply bounded.

5. The set of compactly supported points $\mathrm{c}(\mathbb{R})$ is not functionally compact because the GSF $f(x) = x$ does not satisfy the conclusion (6.2.1) of Thm. 68.

In the present paper, we need the following properties of functionally compact sets.

**Theorem 71.**

1. *Let $K \subseteq X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $f \in \mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^d)$. Then $K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$ implies $f(K) \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^d$.*

2. *Let $K$, $H \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$. If $K \cup H$ is an internal set, then it is a functionally compact set. If $K \cap H$ is an internal set, then it is a functionally compact set.*

3. *Let $H \subseteq K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$, then if $H$ is an internal set, then $H \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$.*

As a corollary of this theorem and Rem. 70.2 we get

**Corollary 72.** *If $a$, $b \in {}^\rho\widetilde{\mathbb{R}}$ and $a \le b$, then $[a, b] \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}$.*

Let us note that $a, b \in {}^\rho\widetilde{\mathbb{R}}$ can also be infinite numbers, e.g. $a = \mathrm{d}\rho^{-N}$, $b = \mathrm{d}\rho^{-M}$ or $a = -\mathrm{d}\rho^{-N}$, $b = \mathrm{d}\rho^{-M}$ with $M > N$, so that e.g. $[-\mathrm{d}\rho^{-N}, \mathrm{d}\rho^{-M}] \supseteq \mathbb{R}$. Finally, in the following result we consider the product of functionally compact sets:

**Theorem 73.** *Let $K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$ and $H \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^d$, then $K \times H \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^{n+d}$. In particular, if $a_i \le b_i$ for $i = 1, \ldots, n$, then $\prod_{i=1}^n [a_i, b_i] \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$.*

Applying the extreme value theorem Thm. 68 to the first derivative, we also have the following

**Theorem 74.** *Let $a$, $b \in {}^\rho\widetilde{\mathbb{R}}^n$, $a < b$, $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ be a GSF. Then*

1. *$\exists c \in [a, b] :\ f(b) - f(a) = (b - a) \cdot f'(c)$.*

2. *Setting $M := \max_{c \in [a,b]} |f'(c)| \in {}^\rho\widetilde{\mathbb{R}}$, we hence have $\forall x, y \in [a, b] :\ |f(x) - f(y)| \le M \cdot |x - y|$.*

A theory of compactly supported GSF has been developed in [30], and it closely resembles the classical theory of LF-spaces of compactly supported smooth functions.

### 6.2.2 Multidimensional integration

Finally, to define FT of multivariable GSF we have to introduce multidimensional integration on suitable subsets of ${}^{\rho}\widetilde{\mathbb{R}}^n$ (see [31]).

**Definition 75.** Let $\mu$ be a measure on $\mathbb{R}^n$ and let $K$ be a functionally compact subset of ${}^{\rho}\widetilde{\mathbb{R}}^n$. Then, we call $K$ *$\mu$-measurable* if the limit

$$\mu(K) := \lim_{m\to\infty} [\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon))] \tag{6.2.2}$$

exists for some representative $(K_\varepsilon)$ of $K$. Here $m \in \mathbb{N}$, the limit is taken in the sharp topology on ${}^{\rho}\widetilde{\mathbb{R}}$, and $\overline{B^{\mathrm{E}}}_r(A) := \{x \in \mathbb{R}^n : d(x, A) \le r\}$.

Let $K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$. Let $(\Omega_\varepsilon)$ be a net of open subsets of $\mathbb{R}^n$, and $(f_\varepsilon)$ be a net of continuous maps $f_\varepsilon \colon \Omega_\varepsilon \longrightarrow \mathbb{R}$. Then we say that

$$(f_\varepsilon) \textit{ defines a generalized integrable map} : K \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$$

if

1. $K \subseteq \langle \Omega_\varepsilon \rangle$ and $[f_\varepsilon(x_\varepsilon)] \in {}^{\rho}\widetilde{\mathbb{R}}$ for all $[x_\varepsilon] \in K$.
2. $\forall (x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}^n_\rho : \ [x_\varepsilon] = [x'_\varepsilon] \in K \ \Rightarrow \ (f_\varepsilon(x_\varepsilon)) \sim_\rho (f_\varepsilon(x'_\varepsilon))$.

If $f \in \mathbf{Set}(K, {}^{\rho}\widetilde{\mathbb{R}})$ is such that

$$\forall [x_\varepsilon] \in K : \ f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \tag{6.2.3}$$

we say that $f : K \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ is a *generalized integrable function*.
We will again say that $f$ *is defined by the net* $(f_\varepsilon)$ or that the net $(f_\varepsilon)$ *represents* $f$. The set of all these generalized integrable functions will be denoted by ${}^{\rho}\mathcal{GI}(K, {}^{\rho}\widetilde{\mathbb{R}})$.

E.g., if $f = [f_\varepsilon(-)]|_K \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}})$, then both $f$ and $|f| = [|f_\varepsilon(-)|]|_K$ are integrable on $K$ (but note that, in general, $|f|$ is not a GSF).
In the following result, we show that this definition generates a correct notion of multidimensional integration for GSF.

**Theorem 76.** *Let $K \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be $\mu$-measurable.*

1. *The definition of $\mu(K)$ is independent of the representative $(K_\varepsilon)$.*
2. *There exists a representative $(K_\varepsilon)$ of $K$ such that $\mu(K) = [\mu(K_\varepsilon)]$.*
3. *Let $(K_\varepsilon)$ be any representative of $K$ and let $f = [f_\varepsilon(-)]|_K \in {}^{\rho}\mathcal{GI}(K, {}^{\rho}\widetilde{\mathbb{R}})$. Then*
$$\int_K f \,\mathrm{d}\mu := \lim_{m\to\infty} \left[ \int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} f_\varepsilon \,\mathrm{d}\mu \right] \in {}^{\rho}\widetilde{\mathbb{R}}$$
*exists and its value is independent of the representative $(K_\varepsilon)$.*

4. *There exists a representative* $(K_\varepsilon)$ *of* $K$ *such that*

$$\int_K f\,\mathrm{d}\mu = \left[\int_{K_\varepsilon} f_\varepsilon\,\mathrm{d}\mu\right] \in {}^\rho\widetilde{\mathbb{R}} \tag{6.2.4}$$

*for each* $f = [f_\varepsilon(-)]|_K \in {}^\rho\mathcal{GI}(K, {}^\rho\widetilde{\mathbb{R}})$*. From* (6.2.4)*, it also follows that* $\left|\int_K f\,\mathrm{d}\mu\right| \le \int_K |f|\,\mathrm{d}\mu$*.*

5. *If* $K = \prod_{i=1}^n [a_i, b_i]$*, then* $K$ *is* $\lambda$*-measurable (*$\lambda$ *being the Lebesgue measure on* $\mathbb{R}^n$*) and for all for each* $f = [f_\varepsilon(-)]|_K \in {}^\rho\mathcal{GI}(K, {}^\rho\widetilde{\mathbb{R}})$ *we have*

$$\int_K f\,\mathrm{d}\lambda = \left[\int_{a_{1,\varepsilon}}^{b_{1,\varepsilon}} dx_1 \ldots \int_{a_{n,\varepsilon}}^{b_{n,\varepsilon}} f_\varepsilon(x_1, \ldots, x_n)\,\mathrm{d}x_n\right] \in {}^\rho\widetilde{\mathbb{R}} \tag{6.2.5}$$

*for any representatives* $(a_{i,\varepsilon})$*,* $(b_{i,\varepsilon})$ *of* $a_i$ *and* $b_i$ *respectively. Therefore, if* $n = 1$*, this notion of integral coincides with that of Thm. 59 and Def. 60. Note that* (6.2.5) *also directly implies Fubini's theorem for this type of integrals.*

6. *Let* $K \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ *be* $\lambda$*-measurable, where* $\lambda$ *is the Lebesgue measure, and let* $\varphi \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ *be such that* $\varphi^{-1} \in {}^\rho\mathcal{GC}^\infty(\varphi(K), {}^\rho\widetilde{\mathbb{R}}^n)$*. Then* $\varphi(K)$ *is* $\lambda$*-measurable and*

$$\int_{\varphi(K)} f\,\mathrm{d}\lambda = \int_K (f \circ \varphi)\,|\det(\mathrm{d}\varphi)|\,\mathrm{d}\lambda$$

*for each* $f \in {}^\rho\mathcal{GI}(\varphi(K), {}^\rho\widetilde{\mathbb{R}})$*.*

In order to state a continuity property for this notion of integration, we have to introduce *hypernatural numbers and hyperlimits* as follows

## 6.3 Convolution on ${}^\rho\widetilde{\mathbb{R}}^n$

In this section, we define and study convolution $f * g$ of two GSF, where $f$ or $g$ is compactly supported. Compactly supported GSF were introduced in [28] for the gauge $\rho_\varepsilon = \varepsilon$. For an arbitrary gauge, we here define and study the notions needed for the HFT as well as for the study of convolution of GSF.

**Definition 77.** Assume that $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ and $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$, then

1. $\mathrm{supp}\,(f) := \overline{\{x \in X \mid |f(x)| > 0\}}$, where $\overline{(\cdot)}$ denotes the relative closure in $X$ with respect to the sharp topology, is called the *support* of $f$. We recall (see just after Def. 1 and Lem. 4) that $x > 0$ means that $x \in {}^\rho\widetilde{\mathbb{R}}_{\ge 0}$ is positive and invertible.

2. For $A \subseteq {}^\rho\widetilde{\mathbb{R}}$ we call the set $\mathrm{ext}\,(A) := \left\{x \in {}^\rho\widetilde{\mathbb{R}} \mid \forall a \in A:\ |x - a| > 0\right\}$ the *strong exterior of* $A$. Recalling Lem. 4, if $x \in \mathrm{ext}(A)$, then $|x - a| \ge \mathrm{d}\rho^q$ for all $a \in A$ and for some $q = q(a) \in \mathbb{N}$.

3. Let $H \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$, we say that $f \in {}^{\rho}\mathcal{GD}(H,Y)$ if $f \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^{n},Y)$ and $\mathrm{supp}(f) \subseteq H$. We say that $f \in {}^{\rho}\mathcal{GD}({}^{\rho}\widetilde{\mathbb{R}}^{n},Y)$ if $f \in {}^{\rho}\mathcal{GD}(H,Y)$ for some $H \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$. Such an $f$ is called *compactly supported*; for simplicity we set ${}^{\rho}\mathcal{GD}(H) := {}^{\rho}\mathcal{GD}(H, {}^{\rho}\widetilde{\mathbb{C}})$. Note that $\mathrm{supp}(f)$ is clearly always closed, and if $f \in {}^{\rho}\mathcal{GD}(H,Y)$ then it is also sharply bounded. However, in general it is not an internal set so it is not a functionally compact set. Accordingly, the theory of multidimensional integration of Sec. 6.2.2 does not allow us to consider $\int_{\mathrm{supp}(f)} f$ even if $f$ is compactly supported.

*Remark* 78.

1. Note that the notion of *standard support* $\mathrm{stsupp}(f)$ as defined in Thm. 65 and the present notion $\mathrm{supp}(f)$ of support, as defined above, are different. The main distinction is that $\mathrm{stsupp}(f) \subseteq \mathbb{R}^{n}$ while $\mathrm{supp}(f) \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$. Moreover if we consider a CGF $f \in {}^{\rho}\mathcal{GC}^{\infty}(\mathrm{c}(\Omega), {}^{\rho}\widetilde{\mathbb{R}}^{d})$, then $\mathrm{supp}(f) \cap \Omega \subseteq \mathrm{stsupp}(f)$.

2. Since $\delta(0) > 0$ then $\delta|_{B_r(0)} > 0$ for some $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ by the sharp continuity of $\delta$, i.e. Thm. 56.2, hence $B_r(0) \subseteq \mathrm{supp}(\delta)$, whereas $\mathrm{stsupp}(\delta) = \{0\}$. Example 67.1 also yields that $\mathrm{supp}(\delta) \subseteq [-r,r]^{n}$ for all $r \in \mathbb{R}_{>0}$.

3. Any rapidly decreasing function $f \in \mathcal{S}(\mathbb{R}^{n})$ satisfies the inequality $0 \le f(x) \le |x|^{-q}$, $\forall q \in \mathbb{N}$, for $|x|$ finite sufficiently large. Therefore, for all strongly infinite $x$, we have $f(x) = 0$ i.e., $f \in {}^{\rho}\mathcal{GD}\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$.

**Lemma 79.** *Let $\emptyset \neq H \Subset_{f} {}^{\rho}\widetilde{\mathbb{R}}^{n}$. Then $\mathrm{ext}(H)$ is sharply open.*

*Proof.* If $x = [x_\varepsilon] \in \mathrm{ext}(H)$, we set $d_\varepsilon := d(x_\varepsilon, H_\varepsilon)$ where $H = [H_\varepsilon]$ and $\emptyset \neq H_\varepsilon \Subset \mathbb{R}^{n}$ for all $\varepsilon$ (because $H \neq \emptyset$). Then $\exists h_\varepsilon \in H_\varepsilon :\ d := d(x_\varepsilon, h_\varepsilon)$, we set $h := [h_\varepsilon] \in H$ and $|x-h| = [d_\varepsilon] =: d > 0$ because $x \in \mathrm{ext}(H)$ and $h \in H$. Now, by taking $r := \frac{d}{2} > 0$, we prove that $B_r(x) \subseteq \mathrm{ext}(H)$. Pick $y \in B_r(x)$, then for all $a \in H$, we have $|y-a| \ge |x-a| - |y-x| \ge d - \frac{d}{2} > 0$. $\square$

**Theorem 80.** *Let $H \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$ and $f \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^{n}, {}^{\rho}\widetilde{\mathbb{C}})$, then the following properties hold:*

1. *$f \in {}^{\rho}\mathcal{GD}(H)$ if and only if $f|_{\mathrm{ext}(H)} = 0$.*

*If $f \in {}^{\rho}\mathcal{GD}(H)$, $x \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$ and $\alpha \in \mathbb{N}^{n}$, then:*

2. *$\partial^{\alpha} f(x) = 0$ for all $x \in \mathrm{ext}(H)$.*

3. *If $H \subseteq [-h,h]^{n}$ then $\partial^{\alpha} f(x) = 0$ whenever $x_p \ge h$ or $x_p \le -h$ for some $p = 1,\dots,n$.*

4. *If $H \subseteq [-h,h]^{n} \subseteq \prod_{p=1}^{n}[a_p,b_p]$, then*

$$\int_{a_1}^{b_1} \mathrm{d}x_1 \dots \int_{a_n}^{b_n} f(x)\, \mathrm{d}x_n = \int_{-h}^{h} \mathrm{d}x_1 \dots \int_{-h}^{h} f(x)\, \mathrm{d}x_n$$

*Proof.* 1: Assume that $\operatorname{supp}(f) \subseteq H$ and $x = [x_\varepsilon] \in \operatorname{ext}(H)$, but $f(x) \neq 0$. This implies that $|f(x)| \not\leq 0$ because always $|f(x)| \geq 0$. Thereby, Lem. 10 yields $|f(x)| >_L 0$ for some $L \subseteq_0 I$. Applying Lem. 4 for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$ we get $|f(x)| >_L \mathrm{d}\rho^q$ for some $q \in \mathbb{R}_{>0}$, i.e. $|f_\varepsilon(x_\varepsilon)| > \rho_\varepsilon^q$ for all $\varepsilon \in L_{\leq \varepsilon_0}$. Define $\bar{x}_\varepsilon := x_\varepsilon$ for all $\varepsilon \in L$ and $\bar{x}_\varepsilon := x_{\varepsilon_0}$ otherwise, so that $\bar{x} := [\bar{x}_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$ and $|f(\bar{x})| > \mathrm{d}\rho^q$. This yields $\bar{x} \in \operatorname{supp}(f) \subseteq H$, and hence $|x - \bar{x}| > 0$, which is impossible by construction because $\bar{x}|_L = x|_L$ and because of Lem. 4.

Vice versa, assume that $f|_{\operatorname{ext}(H)} = 0$ and take $x = [x_\varepsilon] \in \operatorname{supp}(f) \setminus H$. The property

$$\forall q \in \mathbb{R}_{>0}\, \forall^0 \varepsilon : \; d(x_\varepsilon, H_\varepsilon) \leq \rho_\varepsilon^q$$

cannot hold, because for $q \to +\infty$ Thm. 14.1 would imply $x \in H = [H_\varepsilon]$. Therefore, for some $q \in \mathbb{R}_{>0}$ and some $L \subseteq_0 I$, we have $d(x_\varepsilon, H_\varepsilon) \geq \rho_\varepsilon^q$ for all $\varepsilon \in L$. Thereby, if $a = [a_\varepsilon] \in H$ where $a_\varepsilon \in H_\varepsilon$ for all $\varepsilon$, we get $d(x_\varepsilon, a_\varepsilon) \geq d(x_\varepsilon, H_\varepsilon) \geq \rho_\varepsilon^q$ for all $\varepsilon \in L$, i.e. $x|_L \in \operatorname{ext}(H)|_L$. Applying Lem. 79 for the ring ${}^\rho\widetilde{\mathbb{R}}|_L$ we get

$$B_r(x)|_L \subseteq \operatorname{ext}(H)|_L \tag{6.3.1}$$

for some $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. From $x \in \operatorname{supp}(f)$, we get the existence of a sequence $(x_p)_{p\in\mathbb{N}}$ of points of $\left\{x \in {}^\rho\widetilde{\mathbb{R}}^n \mid |f(x)| > 0\right\}$ such that $x_p \to x$ as $p \to +\infty$ in the sharp topology. Therefore, $x_p \in B_r(x)$ for $p \in \mathbb{N}$ sufficiently large. Thereby, $x_p|_L \in \operatorname{ext}(H)|_L$ from (6.3.1) and hence $f(x_p)|_L = \left[\left(f_\varepsilon(x_{p\varepsilon})\right)_{\varepsilon\in L}\right] = 0$, which contradicts $|f(x_p)| > 0$.

Property 2 follows by induction on $|\alpha| \in \mathbb{N}$ using Thm. 57. We prove property 3 for the case $x_p \geq h$, the other case being similar. We consider

$$\bar{x}_q := (x_1, \overset{p-1}{\ldots}, x_{p-1}, x_p + \mathrm{d}\rho^q, x_{p+1}, \ldots, x_n) \quad \forall q \in \mathbb{N}.$$

Then $|\bar{x}_q - a| \geq |x_p + \mathrm{d}\rho^q - a_p| \geq \mathrm{d}\rho^q$ for all $a \in [-h,h]^n \supseteq H$ because $x_p \geq h \geq a_p$. Therefore, $\bar{x}_q \in \operatorname{ext}(H)$ and hence $\partial^\alpha f(\bar{x}_q) = 0$ from the previous 2. The conclusion now follows from the sharp continuity of the GSF $\partial^\alpha f$ (Thm. 56.2).

4: The inclusion $\pm(h,\ldots,h) \in [-h,h]^n \subseteq \prod_{p=1}^n [a_p, b_p]$ implies $a_p \leq -h$ and $b_p \geq h$ for all $p = 1,\ldots,n$. Using Thm. 76.5, we can write

$$\begin{aligned}
\int_{a_1}^{b_1} \mathrm{d}x_1 \ldots \int_{a_n}^{b_n} f(x)\, \mathrm{d}x_n = & \int_{a_1}^{b_1} \mathrm{d}x_1 \ldots \int_{a_{n-1}}^{b_{n-1}} \mathrm{d}x_{n-1} \int_{a_n}^{-h} f(x)\, \mathrm{d}x_n + \\
& \int_{a_1}^{b_1} \mathrm{d}x_1 \ldots \int_{a_{n-1}}^{b_{n-1}} \mathrm{d}x_{n-1} \int_{-h}^{+h} f(x)\, \mathrm{d}x_n + \\
& \int_{a_1}^{b_1} \mathrm{d}x_1 \ldots \int_{a_{n-1}}^{b_{n-1}} \mathrm{d}x_{n-1} \int_{h}^{b_n} f(x)\, \mathrm{d}x_n.
\end{aligned}$$

But if $x_n \in [a_n, -h]$ or $x_n \in [h, b_n]$, then property 3 yields $f(x) = 0$ and we obtain

$$\int_{a_1}^{b_1} \mathrm{d}x_1 \ldots \int_{a_n}^{b_n} f(x)\, \mathrm{d}x_n = \int_{a_1}^{b_1} \mathrm{d}x_1 \ldots \int_{a_{n-1}}^{b_{n-1}} \mathrm{d}x_{n-1} \int_{-h}^{h} f(x)\, \mathrm{d}x_n.$$

Proceeding in the same way with all the other integrals we get the claim. $\square$

In particular, if $T \in \mathcal{E}'(\Omega)$, then Thm. 80.1 implies that $\iota_\Omega^b(T) \in {}^\rho\mathcal{GD}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$. Also observe that $f(x) = e^{-x^2}$, $x \in \left\{x \in {}^\rho\widetilde{\mathbb{R}} \mid \exists N \in \mathbb{N} : \ x^2 \geq N \log \mathrm{d}\rho\right\}$, satisfies $f(x) \leq x^{-q}$ for all infinite $x$ and all $q \in \mathbb{N}$. Therefore

$$\forall Q \in \mathbb{N} : \ f \in {}^\rho\mathcal{GD}\left([-\mathrm{d}\rho^{-Q}, \mathrm{d}\rho^{-Q}]\right).$$

Based on these results, we can define

**Definition 81.** Let $f \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$, then

$$\int f := \int_{{}^\rho\widetilde{\mathbb{R}}^n} f := \int_{a_1}^{b_1} \mathrm{d}x_1 \ldots \int_{a_n}^{b_n} f(x)\, \mathrm{d}x_n \tag{6.3.2}$$

where $\mathrm{supp}(f) \subseteq \prod_{p=1}^n [a_p, b_p]$. This equality does not depend on $a_p$, $b_p$ because of Thm. 80.4.

Note that we can also write (6.3.2) as

$$\int f = \lim_{\substack{a_p \to -\infty \\ b_p \to +\infty \\ p=1,\ldots,n}} \int_{a_1}^{b_1} \mathrm{d}x_1 \ldots \int_{a_n}^{b_n} f(x)\, \mathrm{d}x_n = \lim_{h \to +\infty} \int_{-h}^{h} \mathrm{d}x_1 \ldots \int_{-h}^{h} f(x)\, \mathrm{d}x_n \tag{6.3.3}$$

even if we are actually considering limits of eventually constant functions. Using this notion of integral of a compactly supported GSF, we can also write the value of a distribution $\langle T, \varphi \rangle$ as an integral: let $b \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ be a strong infinite number, $\Omega \subseteq \mathbb{R}^n$ be an open set, $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, with $\mathrm{supp}(\varphi) \subseteq \prod_{i=1}^n [a_i, b_i]_{\mathbb{R}} =: J$. Then from Thm. 65.3 and Thm. 76.5 we get

$$\langle T, \varphi \rangle = \int_{[J]} \iota_\Omega^b(T)(x) \cdot \varphi(x)\, \mathrm{d}x = \int \iota_\Omega^b(T)(x) \cdot \varphi(x)\, \mathrm{d}x, \tag{6.3.4}$$

where the equalities are in ${}^\rho\widetilde{\mathbb{R}}$.

**Definition 82.** Let $f$, $g \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^n)$, with $f \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$ or $g \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$. In the former case, by Thm. 56.4 and Thm. 80.1, for all $x \in {}^\rho\widetilde{\mathbb{R}}^n$, $f \cdot g(x - \cdot) \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$ with $\mathrm{supp}\left(f \cdot g(x - \cdot)\right) \subseteq \mathrm{supp}(f) \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$. Moreover, $\mathrm{supp}\left(f(x - \cdot) \cdot g\right) \subseteq$

$x - \mathrm{supp}(f) \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$. Similarly, we can argue in the latter case, and we can hence define

$$(f * g)(x) := \int f(y)\, g(x-y)\ \mathrm{d}y = \int f(x-y)\, g(y)\ \mathrm{d}y \quad \forall x \in {}^{\rho}\widetilde{\mathbb{R}}^n. \tag{6.3.5}$$

Note that directly from Thm. 59 and Def. 81, it follows that $f * g \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^n)$. The next theorems provide the usual basic properties of convolution suitably formulated in our framework. We start by studying how the convolution is in relation to the supports of its factors:

**Theorem 83.** *Let $f$, $g$, $h \in {}^{\rho}\mathcal{GD}({}^{\rho}\widetilde{\mathbb{R}}^n)$. Then the following properties hold:*

1. *Let $\mathrm{supp}(f) \subseteq [-a,a]^n$, $\mathrm{supp}(g) \subseteq [-b,b]^n$, $a$, $b \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, and $x \in {}^{\rho}\widetilde{\mathbb{R}}^n$. Set $L_x := [-a,a]^n \cap (x - [-b,b]^n)$, then*

$$\mathrm{supp}\,(f \cdot g(x - \cdot)) \subseteq L_x = \prod_{p=1}^{n} [\max(-a, x_p - b), \min(a, x_p + b)] \tag{6.3.6}$$

$$(f * g)(x) = \int_{L_x} f(y)\, g(x-y)\ \mathrm{d}y. \tag{6.3.7}$$

2. $\mathrm{supp}(f * g) \subseteq \overline{\mathrm{supp}(f) + \mathrm{supp}(g)}$*, therefore* $f * g \in {}^{\rho}\mathcal{GD}({}^{\rho}\widetilde{\mathbb{R}}^n)$.

*Proof.* 1: If $|f(t)g(x-t)| > 0$, then $t \in \mathrm{supp}(f)$ and $x - t \in \mathrm{supp}(g)$. Therefore, $\mathrm{supp}\,(f \cdot g(x - \cdot)) \subseteq [-a,a]^n \cap (x - [-b,b]^n)$. As in the case of real numbers, we can say that if $t \in [-a,a]^n \cap (x - [-b,b]^n)$, then $-a \le t_p \le a$ and $-b \le x_p - t_p \le b$ for all $p = 1, \ldots, n$. Therefore, $t_p \in [\max(-a, x_p - b), \min(a, x_p + b)]$. Similarly, we can prove that also $L_x \subseteq [-a,a]^n \cap (x - [-b,b]^n)$. The conclusion (6.3.6) now follows from Def. 81. For completeness, recall that in general $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are not functionally compact sets and our integration theory allows to integrate only over the latter kind of sets. This justifies our formulation of the present property using intervals.

2: Since $f$ and $g$ are compactly supported, we have $\mathrm{supp}(f) \subseteq H$ and $\mathrm{supp}(g) \subseteq L$ for some $H$, $L \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$. Assume that $|(f * g)(x)| > 0$. Then, by Thm. 61.7, Thm. 76.5 and the extreme value Thm. 68, we get

$$0 < |(f * g)(x)| \le \lambda(H) \cdot \max_{y \in H} |f(y)g(x-y)|,$$

where $\lambda$ is the extension of the Lebesgue measure given by Def. 75. Therefore, there exists $y \in H$ such that $0 < \lambda(H) \cdot |f(y)g(x-y)|$. This implies that $y \in \mathrm{supp}(f)$ and $x - y \in \mathrm{supp}(g)$. Thereby, $x = y + (x-y) \in \overline{\mathrm{supp}(f) + \mathrm{supp}(g)}$. Taking the sharp closure we get the conclusion. Finally, $\overline{\mathrm{supp}(f) + \mathrm{supp}(g)} \subseteq \overline{H + L} = H + L$ and $H + L \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$ because it is the image under the sum $+$ of $H \times L$ (see Thm. 73 and Thm. 71). $\square$

Now, we consider algebraic properties of convolution and its relations with derivations and integration:

**Theorem 84.** *Let $f$, $g$, $h \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^{n})$ and assume that at least two of them are compactly supported. Then the following properties hold:*

1. $f * g = g * f$.
2. $(f * g) * h = f * (g * h)$.
3. $f * (h + g) = f * h + f * g$.
4. $\overline{f * g} = \overline{f} * \overline{g}$
5. $t \oplus (f * g) = (t \oplus f) * g = f * (t \oplus g)$ *where* $t \oplus f$ *is the translation of the function* $f$ *by* $t$ *defined by* $(t \oplus f)(x) = f(x - t)$ *(see Sec. 6.1.2).*
6. $\frac{\partial}{\partial x_p}(f * g) = \frac{\partial f}{\partial x_p} * g = f * \frac{\partial g}{\partial x_p}$ *for all* $p = 1, \ldots, n$.
7. $\int (f * g)(x)\, \mathrm{d}x = \left(\int f(x)\, \mathrm{d}x\right)\left(\int g(x)\, \mathrm{d}x\right)$

*Proof.* 1: We assume, e.g., that $f \in {}^{\rho}\mathcal{GD}({}^{\rho}\widetilde{\mathbb{R}}^{n})$. Take $h \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $\mathrm{supp}(f) \subseteq [-h, h]^{n}$. By (6.3.7) and Def. 81, we can write

$$(f * g)(x) = \int_{-h}^{h} \mathrm{d}y_1 \ldots \int_{-h}^{h} f(y)\, g(x - y)\, \mathrm{d}y_n.$$

We can now proceed as in the classical case, i.e. considering the change of variable $z = x - y$ (Thm. 62). We get

$$(f * g)(x) = \int_{x_1 - h}^{x_1 + h} \mathrm{d}z_1 \ldots \int_{x_n - h}^{x_n + h} f(x - z)\, g(z)\, \mathrm{d}z_n.$$

Taking the limit $h \to +\infty$ (see (6.3.3)), we obtain the desired equality. Similarly, we can also prove 2 and 3.

As usual, 4 is a straightforward consequence of the definition of complex conjugate.

5: The usual proof applies, in fact

$$\begin{aligned} t \oplus (f * g)(x) = (f * g)(x - t) &= \int f(y)\, g(x - t - y)\, \mathrm{d}y = \\ &= \int f(y)\, (t \oplus g)(x - y)\, \mathrm{d}y = (f * (t \oplus g))(x). \end{aligned} \tag{6.3.8}$$

Finally, the commutativity property 1 yields $(t \oplus f) * g = g * (t \oplus f)$ and applying (6.3.8) $g * (t \oplus f) = t \oplus (g * f) = t \oplus (f * g)$.

6: Set $h := f * g$ and take $x \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$. Using differentiation under the integral sign (Thm. 61.8) and Def. 81 we get

$$\frac{\partial}{\partial x_p} h(x) = \int_{{}^{\rho}\widetilde{\mathbb{R}}^{n}} f(y)\, \frac{\partial g}{\partial x_p}(x - y)\, \mathrm{d}y = \left(f * \frac{\partial g}{\partial x_p}\right)(x).$$

Using 1, we also have $\frac{\partial}{\partial x_p}h = \frac{\partial f}{\partial x_p} * g$.

To prove 7 we show the case $n = 1$, even if the general one is similar. Let $a, b \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ be such that $\operatorname{supp}(f * g) \subseteq [-a, a]$ (Thm. 83) and $\operatorname{supp}(f) \subseteq [-b, b]$. Then

$$\int (f * g)(x)\,\mathrm{d}x = \int_{-a}^{a} \mathrm{d}x \int_{-b}^{b} f(y)g(x-y)\,\mathrm{d}y.$$

Using Fubini's Thm. 76.5, we can write

$$\begin{aligned}\int (f * g)(x)\,\mathrm{d}x &= \int_{-b}^{b} f(y) \int_{-a}^{a} g(x-y)\,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_{-b}^{b} f(y) \int_{-a-y}^{a-y} g(z)\,\mathrm{d}z\,\mathrm{d}y = \\ &= \int_{-b}^{b} f(y)\,\mathrm{d}y \int_{-c}^{c} g(z)\,\mathrm{d}z,\end{aligned}$$

where we have taken $a \to +\infty$ or equivalently, considered any $c \geq a + b$. $\square$

Young's inequality for convolution is based on the generalized Hölder's inequality, on the inequality $\left|\int_K f\,\mathrm{d}\mu\right| \leq \int_K |f|\,\mathrm{d}\mu$ (see Thm. 76.4), monotonicity of integral (see Thm. 61.7) and Fubini's theorem (see Thm. 76.5). Therefore, the usual proofs can be repeated in our setting if we take sufficient care of terms such as $|f(x)|^p$ if $p \in {}^\rho\widetilde{\mathbb{R}}_{\geq 1}$:

**Definition 85.** Let $f \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$ and $p \in {}^\rho\widetilde{\mathbb{R}}_{\geq 1}$ be a finite number. Then, we set

$$\|f\|_p := \left(\int |f(x)|^p\,\mathrm{d}x\right)^{1/p} \in {}^\rho\widetilde{\mathbb{R}}_{\geq 0}.$$

Note that $|f|^p$ is a generalized integrable function (Def. 75) because $p$ is a finite number (in general the power $x^y$ is not well-defined, e.g. $\left(\frac{1}{\rho_\varepsilon}\right)^{1/\rho_\varepsilon} = \rho_\varepsilon^{-1/\rho_\varepsilon}$ is not $\rho$-moderate).

On the other hand, Hölder's inequality, if $\|f\|_p > 0$ and $\|g\|_q > 0$, is simply based on monotonicity of integral, Fubini's theorem and Young's inequality for products. The latter holds also in ${}^\rho\widetilde{\mathbb{R}}_{\geq 0}$ because it holds in the entire $\mathbb{R}_{\geq 0}$, see e.g. [63].

**Theorem 86** (Hölder)**.** *Let $f_k \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$ and $p_k \in {}^\rho\widetilde{\mathbb{R}}_{\geq 1}$ for all $k = 1, \dots, m$ be such that $\sum_{k=1}^m \frac{1}{p_k} = 1$ and $\|f_k\|_{p_k} > 0$. Then*

$$\left\|\prod_{k=1}^m f_k\right\|_1 \leq \prod_{k=1}^m \|f_k\|_{p_k}.$$

**Theorem 87** (Young)**.** *Let $f, g \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$ and $p, q, r \in {}^\rho\widetilde{\mathbb{R}}_{\geq 1}$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and $\|f\|_p$, $\|g\|_q > 0$, then $\|f * g\|_r \leq \|f\|_p \cdot \|g\|_q$.*

In the following theorem, we consider when the equality $(\delta * f)(x) = f(x)$ holds. As we will see later in Sec. 6.4.2, as a consequence of the Riemann-Lebesgue lemma we necessarily have a limitation concerning the validity of this equality.

**Theorem 88.** *Let $\delta$ be the $\iota^b_{\mathbb{R}^n}$-embedding of the $n$-dimensional Dirac delta (see Thm. 65). Assume that $f \in {}^\rho\mathcal{GC}^\infty\left({}^\rho\widetilde{\mathbb{R}}^n\right)$ satisfies, at the point $x \in {}^\rho\widetilde{\mathbb{R}}^n$, the condition*

$$\exists r \in \mathbb{R}_{>0}\, \exists M, c \in {}^\rho\widetilde{\mathbb{R}}\, \forall y \in \overline{B}_r(x)\, \forall j \in \mathbb{N} : \ \left|\mathrm{d}^j f(y)\right| \le M c^j, \tag{6.3.9}$$

$$\frac{b}{c} \text{ is a large infinite number}$$

*i.e. all its derivatives in a* finite *neighborhood of $x$ are bounded by a suitably small polynomial $Mc^j$ (such a function $f$ will be called* bounded by a tame polynomial at $x$*). Then $(\delta * f)(x) = f(x)$.*

*Proof.* Considering that $\delta(y) = b^n\psi(by)$, where $\psi$ is the considered $n$-dimensional Colombeau mollifier and $b$ is a strong infinite number. (see Example 67.1), we have:

$$\begin{aligned}
(\delta * f)(x) - f(x) &= \int f(x-y)\,\delta(y)\,\mathrm{d}y - f(x)\int \delta(y)\,\mathrm{d}y = \\
&= \int (f(x-y) - f(x))\,\delta(y)\,\mathrm{d}y = \\
&= \int_{\left[-\frac{r}{\sqrt{n}},\frac{r}{\sqrt{n}}\right]^n} (f(x-y) - f(x))\,\delta(y)\,\mathrm{d}y = \\
&= \int_{\left[-\frac{r}{\sqrt{n}},\frac{r}{\sqrt{n}}\right]^n} (f(x-y) - f(x))\,b^n\psi(by)\,\mathrm{d}y,
\end{aligned}$$

where $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ is the radius from (6.3.9), so that $\operatorname{supp}(\delta) \subseteq \left[-\frac{r}{\sqrt{n}}, \frac{r}{\sqrt{n}}\right]^n$ since $r \in \mathbb{R}_{>0}$. By changing the variable $by = t$, and setting $H := \left[-\frac{br}{\sqrt{n}}, \frac{br}{\sqrt{n}}\right]^n$ we have

$$(f * \delta)(x) - f(x) = \int_H \left(f\left(x - \frac{t}{b}\right) - f(x)\right)\psi(t)\,\mathrm{d}t.$$

Using Taylor's formula (Thm. 63.2) up to an arbitrary order $q \in \mathbb{N}$, we get

$$\begin{aligned}
\int_H \left(f\left(x - \frac{t}{b}\right) - f(x)\right)\psi(t)\,\mathrm{d}t &= \int_H \sum_{0<|\alpha|\le q} \frac{1}{\alpha!}\left(-\frac{t}{b}\right)^\alpha \partial^\alpha f(x)\,\psi(t)\,\mathrm{d}t + \\
&+ \int_H \frac{1}{(q+1)!}\int_0^1 (1-z)^q\,\mathrm{d}^{q+1} f\left(x - z\frac{t}{b}\right)\left(-\frac{t}{b}\right)^{q+1}\psi(t)\,\mathrm{d}z\,\mathrm{d}t. 
\end{aligned}\tag{6.3.10}$$

But 1 and 5 of Lem. 64 yield:

$$\int_H t^\alpha\psi(t)\,\mathrm{d}t = \left[\int_{\left[-\frac{b_\varepsilon r}{\sqrt{n}},\frac{b_\varepsilon r}{\sqrt{n}}\right]^n} t^\alpha\psi_\varepsilon(t)\,\mathrm{d}t\right] = \left[\int t^\alpha\psi_\varepsilon(t)\,\mathrm{d}t\right] = 0 \quad \forall|\alpha| \le q,$$

where we also used that $\frac{b_\varepsilon r}{\sqrt{n}} > 1$ for $\varepsilon$ sufficiently small because $b > 0$ is an infinite number and $r \in \mathbb{R}_{>0}$. Thereby, in (6.3.10) we only have to consider the remainder

$$\begin{aligned} R_q(x) &:= \int_H \frac{1}{(q+1)!} \int_0^1 (1-z)^q \, \mathrm{d}^{q+1} f\left(x - z\frac{t}{b}\right) \left(-\frac{t}{b}\right)^{q+1} \psi(t) \, \mathrm{d}z \, \mathrm{d}t = \\ &= \frac{(-1)^{q+1}}{b^{q+1}(q+1)!} \int_H \int_0^1 (1-z)^q \, \mathrm{d}^{q+1} f\left(x - z\frac{t}{b}\right) t^{q+1} \psi(t) \, \mathrm{d}z \, \mathrm{d}t. \end{aligned}$$

For all $z \in (0,1)$ and $t \in H = \left[-\frac{r}{\sqrt{n}}, \frac{r}{\sqrt{n}}\right]^n$, we have $\left|\frac{zt}{b}\right| \le \left|\frac{t}{b}\right| \le \frac{\sqrt{n}|t|_\infty}{b} \le \frac{rb}{b} = r$ and hence $x - z\frac{t}{b} \in \overline{B}_r(x)$. Thereby, assumption (6.3.9) yields $\mathrm{d}^{q+1} f\left(x - z\frac{t}{b}\right) \le Mc^{q+1}$, and hence

$$\begin{aligned} \left|R_q(x)\right| &\le b^{-q-1} \frac{Mc^{q+1}}{(q+1)!} \int_H \left|t^{q+1}\psi(t)\right| \, \mathrm{d}t = \\ &= \left(\frac{b}{c}\right)^{-q-1} \frac{M}{(q+1)!} \int_{[-1,1]^n} \left|t^{q+1}\psi(t)\right| \, \mathrm{d}t \le \\ &\le \left(\frac{b}{c}\right)^{-q-1} \frac{M}{(q+1)!} \int_{[-1,1]^n} |\psi(t)| \, \mathrm{d}t \le \\ &\le \left(\frac{b}{c}\right)^{-q-1} \frac{2M}{(q+1)!}, \end{aligned}$$

where we used 1 and 6 of Lem. 64 and $\frac{br}{\sqrt{n}} > 1$. We can now let $q \to +\infty$ considering that $\frac{b}{c} > \mathrm{d}\rho^{-s}$ for some $s \in \mathbb{R}_{>0}$, so that $\left|R_q(x)\right| \to 0$ and hence $(\delta * f)(x) = f(x)$. $\square$

**Example 89.**

1. If $f_\omega(x) = e^{-ix\omega}$, $b \ge \mathrm{d}\rho^{-r}$ and $\omega \in {}^\rho\widetilde{\mathbb{R}}$ satisfies $|\omega| \le \mathrm{d}\rho^{-s}$ with $s < r$ (e.g. if $\omega$ is a weak infinite number, see Def. 3), then $\frac{b}{|\omega|} \ge \mathrm{d}\rho^{-(r-s)}$ and $f_\omega$ is bounded by a tame polynomial at each point $x \in {}^\rho\widetilde{\mathbb{R}}$. On the contrary, e.g. if $b = \mathrm{d}\rho^{-r}$ and $|\omega| \ge \mathrm{d}\rho^{-r}$, then $\frac{b}{|\omega|} \le 1$ and $f_\omega$ is not bounded by a tame polynomial at any $x \in {}^\rho\widetilde{\mathbb{R}}$.

2. If $f \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^n)$ has always finite derivatives at a finite point $x \in {}^\rho\widetilde{\mathbb{R}}^n$ (e.g. it originates from the embedding of an ordinary smooth function), then it suffices to take $c = \mathrm{d}\rho^{-r-1}$ to prove that $f$ is bounded by a tame polynomial at $x$. Similarly, we can argue if $f$ is polynomially bounded for $x \to \infty$ and $x \in {}^\rho\widetilde{\mathbb{R}}^n$ is not finite.

3. The Dirac delta $\delta(x) = b^n\psi(bx)$ is not bounded by a tame polynomial at $x = 0$. This also shows that, generally speaking, the embedding of

a compactly supported distribution is not bounded by a tame polynomial. Below we will show that indeed $\delta * \delta \neq \delta$, even if we clearly have $(\delta * \delta)(x) = \delta(x) = 0$ for all $x \in {}^{\rho}\widetilde{\mathbb{R}}^n$ such that $|x| \geq r \in \mathbb{R}_{>0}$.

4. If $f \in {}^{\rho}\mathcal{GC}^{\infty}\left({}^{\rho}\widetilde{\mathbb{R}}^n\right)$ is bounded by a tame polynomial at 0, then since $\delta$ is an even function (see Example 67.1), we have:

$$\int \delta(x) \cdot f(x)\,\mathrm{d}x = \int \delta(0 - x) \cdot f(x)\,\mathrm{d}x = (f * \delta)(0) = f(0). \tag{6.3.11}$$

Finally, the following theorem considers the relations between convolution of distributions and their embedding as GSF:

**Theorem 90.** *Let $S \in \mathcal{E}'(\mathbb{R}^n)$, $T \in \mathcal{D}'(\mathbb{R}^n)$ and $b \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ be a strong positive infinite number, then for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$:*

1. $\langle S * T, \varphi\rangle = \int \iota^b_{\mathbb{R}^n}(S)(x) \cdot \iota^b_{\mathbb{R}^n}(T)(y) \cdot \varphi(x+y)\,\mathrm{d}x\,\mathrm{d}y = \int \left(\iota^b_{\mathbb{R}^n}(S) * \iota^b_{\mathbb{R}^n}(T)\right)(z) \cdot \varphi(z)\,\mathrm{d}z$.
2. $T * \varphi = \iota^b_{\mathbb{R}^n}(T) * \varphi$.

*Proof.* 1: Using (6.3.4), we have

$$\begin{aligned}\langle S * T, \varphi\rangle = \langle S(x), \langle T(y), \varphi(x+y)\rangle\rangle &= \langle S(x), \int \iota^b_{\mathbb{R}^n}(T)(y)\varphi(x+y)\,\mathrm{d}y\rangle = \\ &= \int \iota^b_{\mathbb{R}^n}(S)(x) \int \iota^b_{\mathbb{R}^n}(T)(y)\varphi(x+y)\,\mathrm{d}y\,\mathrm{d}x = \\ &= \int \left(\iota^b_{\mathbb{R}^n}(S) * \iota^b_{\mathbb{R}^n}(T)\right)(z)\varphi(z)\,\mathrm{d}z,\end{aligned}$$

where, in the last step, we used the change of variables $x = z - y$ and Fubini's theorem.

2: For all $x \in \mathrm{c}(\mathbb{R}^n)$, using again (6.3.4), we have $(T * \varphi)(x) = \langle T(y), \varphi(x - y)\rangle = \int \iota^b_{\mathbb{R}^n}(T)(y)\varphi(x-y)\,\mathrm{d}y = \left(\iota^b_{\mathbb{R}^n}(T) * \varphi\right)(x)$. ☐

We note that an equality of the type $\iota^b_{\mathbb{R}^n}(S * T) = \iota^b_{\mathbb{R}^n}(S) * \iota^b_{\mathbb{R}^n}(T)$ cannot hold because from Thm. 84.2 it would imply $1 * (\delta' * H) = (1 * \delta') * H$ as distributions. Considering their embeddings, we have $\iota^b_{\mathbb{R}^n}(1) * \left(\iota^b_{\mathbb{R}^n}(\delta') * \iota^b_{\mathbb{R}^n}(H)\right) = \iota^b_{\mathbb{R}^n}(1) * \left(\iota^b_{\mathbb{R}^n}(\delta) * \iota^b_{\mathbb{R}^n}(\delta)\right) = \left(\iota^b_{\mathbb{R}^n}(1) * \iota^b_{\mathbb{R}^n}(\delta')\right) * \iota^b_{\mathbb{R}^n}(H) = \left(\iota^b_{\mathbb{R}^n}(1') * \iota^b_{\mathbb{R}^n}(\delta)\right) * \iota^b_{\mathbb{R}^n}(H) = 0$. In particular, at the term $\iota^b_{\mathbb{R}^n}(\delta) * \iota^b_{\mathbb{R}^n}(\delta)$ we cannot apply Thm. 88 because $\delta^{(j)}(x) = b^{j+1}\psi^{(j)}(bx)$. This also implies that $\iota^b_{\mathbb{R}^n}(\delta) * \iota^b_{\mathbb{R}^n}(\delta) \neq \iota^b_{\mathbb{R}^n}(\delta)$ because otherwise we would have $0 = \iota^b_{\mathbb{R}^n}(1) * \left(\iota^b_{\mathbb{R}^n}(\delta) * \iota^b_{\mathbb{R}^n}(\delta)\right) = \iota^b_{\mathbb{R}^n}(1) * \iota^b_{\mathbb{R}^n}(\delta) = \int \delta = 1$.

## 6.4 Hyperfinite Fourier transform

**Definition 91.** Let $k \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ be a positive infinite number. Let $f \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{C}})$, we define the $n$-dimensional *hyperfinite Fourier transform (HFT) $\mathcal{F}_k(f)$ of $f$ on*

$K:=[-k,k]^{n}$ as follows:

$$\mathcal{F}_{k}\left(f\right)\left(\omega\right):=\int\limits_{K}f\left(x\right)e^{-ix\cdot\omega}\,\mathrm{d}x=\int\limits_{-k}^{k}\,\mathrm{d}x_{1}\ldots\int\limits_{-k}^{k}f\left(x_{1},\ldots,x_{n}\right)e^{-ix\cdot\omega}\,\mathrm{d}x_{n},\tag{6.4.1}$$

where $x=(x_{1}\ldots x_{n})\in K$ and $\omega=(\omega_{1}\ldots\omega_{n})\in{}^{\rho}\widetilde{\mathbb{R}}^{n}$. As usual, the product $x\cdot\omega$ on ${}^{\rho}\widetilde{\mathbb{R}}^{n}$ denotes the dot product $x\cdot\omega=\sum_{j=1}^{n}x_{j}\omega_{j}\in{}^{\rho}\widetilde{\mathbb{R}}$. For simplicity, in the following we will also use the notation ${}^{\rho}\mathcal{GC}^{\infty}(X):={}^{\rho}\mathcal{GC}^{\infty}(X,{}^{\rho}\widetilde{\mathbb{C}})$. If $f\in{}^{\rho}\mathcal{GD}(X)$ and $\mathrm{supp}(f)\subseteq K=[-k,k]^{n}$, based on Def. 81, we can use the simplified notation $\mathcal{F}(f):=\mathcal{F}_{k}(f)$.

In the following, $k=[k_{\varepsilon}]\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ will always denote a positive infinite number, and we set $K:=[-k,k]^{n}\Subset_{\mathrm{f}}{}^{\rho}\widetilde{\mathbb{R}}^{n}$.

The adjective *hyperfinite* can be motivated as follows: on the one hand, $k\in{}^{\rho}\widetilde{\mathbb{R}}$ is an infinite number, but on the other hand we already mentioned that GSF behave on a functionally compact set like $K$ as if it were a compact set. Similarly to the case of hyperfinite numbers ${}^{\rho}\widetilde{\mathbb{N}}$ (see Def. 16), the adjective *hyperfinite* is frequently used to denote mathematical objects which are in some sense infinite but behave, from several points of view, as bounded ones.

**Theorem 92.** *Let $f\in{}^{\rho}\mathcal{GC}^{\infty}\left(K\right)$, then the following properties hold:*

1. *Let $\omega=[\omega_{\varepsilon}]\in{}^{\rho}\widetilde{\mathbb{R}}^{n}$ and let $f$ be defined by the net $(f_{\varepsilon})$. Then we have:*

$$\mathcal{F}_{k}\left(f\right)\left(\omega\right)=\int\limits_{K}f\left(x\right)e^{-ix\cdot\omega}\,\mathrm{d}x=\left[\int\limits_{-k_{\varepsilon}}^{k_{\varepsilon}}\,\mathrm{d}x_{1}\ldots\int\limits_{-k_{\varepsilon}}^{k_{\varepsilon}}f_{\varepsilon}\left(x_{1},\ldots,x_{n}\right)e^{-ix\cdot\omega_{\varepsilon}}\,\mathrm{d}x_{n}\right]\in{}^{\rho}\widetilde{\mathbb{C}}.$$

2. *$\forall\omega\in{}^{\rho}\widetilde{\mathbb{R}}^{n}:\ |\mathcal{F}_{k}(f)(\omega)|\le\int_{K}|f(x)|\,\mathrm{d}x=\|f\|_{1}$, so that the HFT is always sharply bounded.*

3. *$\mathcal{F}_{k}:{}^{\rho}\mathcal{GC}^{\infty}\left(K\right)\longrightarrow{}^{\rho}\mathcal{GC}^{\infty}\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$.*

*Proof.* 1: For all $\omega\in{}^{\rho}\widetilde{\mathbb{R}}^{n}$ fixed, the map $x\in K\mapsto f\left(x\right)e^{-ix\cdot\omega}$ is a GSF by the closure with respect to composition, i.e. Thm. 56.4. Therefore, we can apply Thm. 76.5.

To prove 3, we have to show that $\mathcal{F}_{k}(f):{}^{\rho}\widetilde{\mathbb{R}}^{n}\longrightarrow{}^{\rho}\widetilde{\mathbb{C}}$ is defined by a net $\left(\mathcal{F}_{k}\right)_{\varepsilon}\in\mathcal{C}^{\infty}\left(\mathbb{R}^{n},\mathbb{C}\right)$ (see Def. 55). We can naturally define such a net as

$$\left(\mathcal{F}_{k}\right)_{\varepsilon}\left(y\right):=\int\limits_{-k_{\varepsilon}}^{k_{\varepsilon}}\,\mathrm{d}x_{1}\ldots\int\limits_{-k_{\varepsilon}}^{k_{\varepsilon}}f_{\varepsilon}\left(x_{1},\ldots,x_{n}\right)e^{-ix\cdot y}\,\mathrm{d}x_{n}\quad\forall y\in\mathbb{R}^{n},$$

and we claim it satisfies the following properties:

(a) $\left[\left(\mathcal{F}_{k}\right)_{\varepsilon}\left(\omega_{\varepsilon}\right)\right]\in{}^{\rho}\widetilde{\mathbb{C}}$, $\forall\left[\omega_{\varepsilon}\right]\in{}^{\rho}\widetilde{\mathbb{R}}^{n}$.

(b) $\forall\left[\omega_{\varepsilon}\right]\in{}^{\rho}\widetilde{\mathbb{R}}^{n}\,\forall\alpha\in\mathbb{N}^{n}:\ \left(\partial^{\alpha}\left(\mathcal{F}_{k}\right)_{\varepsilon}\left(\omega_{\varepsilon}\right)\right)\in\mathbb{C}_{\rho}$.

Claim (a) is justified by 1 above. From 1 it directly follows 2. In order to prove (b), we use the standard derivation under the integral sign to have

$$\partial^{\alpha}\left(\mathcal{F}_{k}\right)_{\varepsilon}\left(\omega_{\varepsilon}\right)=\int\limits_{-k_{\varepsilon}}^{k_{\varepsilon}}\mathrm{d}x_{1}\ldots\int\limits_{-k_{\varepsilon}}^{k_{\varepsilon}}f_{\varepsilon}\left(x_{1},\ldots,x_{n}\right)e^{-ix\cdot\omega_{\varepsilon}}\left(-ix^{\alpha}\right)\mathrm{d}x_{n}.$$

We can now proceed as above to prove (b) and hence the claim 3. $\square$

### 6.4.1 The heuristic motivation of the FT in a non-Archimedean setting

Frequently, the formula for the definition of the FT (e.g. for rapidly decreasing functions) is informally motivated using its relations with Fourier series. In order to replicate a similar argument for GSF, we need the notion of *hyperseries*. In fact, exactly as the ordinary limit $\lim_{n\in\mathbb{N}}a_{n}$ is not well suited for the sharp topology (because of its infinitesimal neighbourhoods) and we have to consider hyperlimits ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}a_{n}$ (see Def. 35), likewise to study series of $a_{n}\in{}^{\rho}\widetilde{\mathbb{C}}$, $n\in\mathbb{N}$, we have to consider

$$\begin{aligned}
{}^{\rho}\!\!\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}a_{n} & :={}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}}\sum_{n=0}^{N}a_{n}\in{}^{\rho}\widetilde{\mathbb{C}},\\
{}^{\rho}\!\!\sum_{n\in{}^{\sigma}\widetilde{\mathbb{Z}}}a_{n} & :={}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}}\sum_{n=-N}^{N}a_{n}\in{}^{\rho}\widetilde{\mathbb{C}},
\end{aligned}$$

where ${}^{\sigma}\widetilde{\mathbb{Z}}:={}^{\sigma}\widetilde{\mathbb{N}}\cup\left(-{}^{\sigma}\widetilde{\mathbb{N}}\right)\subseteq{}^{\sigma}\widetilde{\mathbb{R}}$. The main problem in this definition is how to define the *hyperfinite sums* $\sum_{n=M}^{N}a_{n}\in{}^{\rho}\widetilde{\mathbb{C}}$ for arbitrary hypernatural numbers $N$, $M\in{}^{\sigma}\widetilde{\mathbb{N}}$ and starting from suitable *ordinary* sequences $\left(a_{n}\right)_{n\in\mathbb{N}}$ of ${}^{\rho}\widetilde{\mathbb{C}}$. However, this can be done, and the resulting notion extends several classical theorems, see [69].

Only for this section, we hence assume that $f\in{}^{\rho}\mathcal{GD}([-T,T])$, $T\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$, can be written as a Fourier hyperseries

$$f(t)={}^{\rho}\!\!\sum_{n\in{}^{\sigma}\widetilde{\mathbb{Z}}}c_{n}e^{2\pi i\frac{n}{T}t}\quad\forall t\in(-T,T),$$

where $\sigma$ is another gauge such that $\sigma_{\varepsilon}\le\rho_{\varepsilon}^{q}$ for all $q\in\mathbb{N}$ and for $\varepsilon$ small (so that $\mathbb{R}_{\rho}\subseteq\mathbb{R}_{\sigma}$, see Def. 1). Using Thm. 41 to exchange hyperseries and integration, for each $h\in{}^{\sigma}\widetilde{\mathbb{Z}}$, we have

$$\int_{-T}^{T}f(t)e^{-2\pi i\frac{h}{T}t}\,\mathrm{d}t={}^{\rho}\!\!\sum_{n\in{}^{\sigma}\widetilde{\mathbb{Z}}}c_{n}\int_{-T}^{T}e^{2\pi i\frac{t}{T}(n-h)}\,\mathrm{d}t=2T\cdot c_{h}.$$

That is $c_h = \frac{1}{2T}\mathcal{F}(f)\left(2\pi\frac{h}{T}\right)$.

It is also well-known that, informally, if $T$ is "sufficiently large", then the Fourier coefficients $c_n$ "approximate" the FT scaled by $\frac{1}{2T}$ and dilated by $2\pi$. Using our non-Archimedean language, this can be formalized as follows: Let $\omega = [\omega_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, and assume that $T = [T_\varepsilon]$ is an infinite number, then setting $h_\omega := [\text{int}(\omega_\varepsilon \cdot T_\varepsilon)] \in {}^\rho\widetilde{\mathbb{Z}}$ (here we use $\mathbb{R}_\rho \subseteq \mathbb{R}_\sigma$), we have $\omega_\varepsilon \le \frac{h_{\omega\varepsilon}}{T_\varepsilon} \le \omega_\varepsilon + \frac{1}{T_\varepsilon}$, so that $\frac{h_\omega}{T} \approx \omega$ because $T$ is an infinite number. By Thm. 92, $\mathcal{F}(f)$ is a GSF. Let $a, b, c, d \in {}^\rho\widetilde{\mathbb{R}}$, with $a < c < d < b$, and set $M := \max_{\omega\in[2\pi a, 2\pi b]} \mathcal{F}(f)'(\omega)$. Using Lem. 4, we can find $q \in \mathbb{N}$ such that $c - a \ge \mathrm{d}\rho^q$ and $b - d \ge \mathrm{d}\rho^q$. Assume that $T$ is sufficiently large so that the following conditions hold

$$\frac{1}{T} \le \mathrm{d}\rho^q, \quad \frac{M}{T} \approx 0.$$

Then, for all $\omega \in [c, d]$, we have $\frac{h_\omega}{T} \le \omega + \frac{1}{T} \le d + \mathrm{d}\rho^q \le b$, and $\frac{h_\omega}{T} \ge \omega \ge c > a$, so that $\frac{h_\omega}{T}, \omega \in [a, b]$. From the mean value theorem Thm. 74, we hence have

$$\left|\mathcal{F}(f)\left(2\pi\frac{h_\omega}{T}\right) - \mathcal{F}(f)(2\pi\omega)\right| \le 2\pi M \left|\frac{h_\omega}{T} - \omega\right| \le 2\pi\frac{M}{T} \approx 0.$$

We hence proved that

$$\exists Q \in \mathbb{N}\, \forall T \ge \mathrm{d}\rho^{-Q}:\ c_{h_\omega} \approx \frac{1}{2T}\mathcal{F}(f)(2\pi\omega).$$

Finally, note that since $T$ is an infinite number, if $h_\omega \in \mathbb{Z}$, then necessarily $\omega$ must be infinitesimal; on the contrary, if $\omega \ge r \in \mathbb{R}_{\ne 0}$, then necessarily $h_\omega \in {}^\sigma\widetilde{\mathbb{Z}} \setminus \mathbb{Z}$ is an infinite integer number.

Therefore, with the precise meaning given above, the heuristic relations between Fourier coefficients and HFT holds also for GSF.

### 6.4.2 The Riemann-Lebesgue lemma in a non-linear setting

The following result represents the Riemann-Lebesgue lemma in our framework. It immediately highlights an important difference with respect to the classical approach since it states that the HFT of a very large class of compactly supported GSF is still compactly supported (see also Thm. 102 for a classical formulation of the uncertainty inequality for GSF).

**Lemma 93.** *Let $H \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$ and $f \in {}^\rho\mathcal{GD}(H)$ be a compactly supported GSF. Assume that*

$$\exists C, b \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall x \in H\, \forall j \in \mathbb{N}:\ \left|\mathrm{d}^j f(x)\right| \le C \cdot b^j. \tag{6.4.2}$$

*For all $N_1, \ldots, N_n \in \mathbb{N}$ and $\omega \in {}^\rho\widetilde{\mathbb{R}}^n$, if $\omega_1^{N_1} \cdot \ldots \cdot \omega_n^{N_n}$ is invertible, then*

$$|\mathcal{F}(f)(\omega)| \le \frac{1}{\left|\omega_1^{N_1} \cdot \ldots \cdot \omega_n^{N_n}\right|} \cdot \int_H \left|\partial_1^{N_1} \ldots \partial_n^{N_n} f(x)\right| \mathrm{d}x. \tag{6.4.3}$$

*Therefore*

$$\lim_{\omega\to\infty} |\mathcal{F}(f)(\omega)| = 0. \tag{6.4.4}$$

*Actually,* (6.4.3) *yields the stronger result:*

$$\exists Q \in \mathbb{N} : \ \mathcal{F}(f) \in {}^{\rho}\mathcal{GD}\left(\overline{B_{\mathrm{d}\rho^{-Q}}(0)}\right). \tag{6.4.5}$$

*Proof.* Let us apply integration by parts Thm. 61.6 at the $p$-th integral in (6.4.1) (assuming that $N_p > 0$):

$$\int_{-k}^{k} f(x)\, e^{-i\omega\cdot x}\, \mathrm{d}x_p = -\frac{f(x)}{i\omega_p} e^{-i\omega\cdot x}\Big|_{x_p=-k}^{x_p=k} + \frac{1}{i\omega_p}\int_{-k}^{k} \partial_p f(x)\, e^{-i\omega\cdot x}\, \mathrm{d}x_p =$$

$$= \frac{1}{i\omega_p}\int_{-k}^{k} \partial_p f(x)\, e^{-i\omega\cdot x}\, \mathrm{d}x_p.$$

because Thm. 80.3 yields $f(x) = 0$ if $x_p = \pm k$. Applying the same idea with $N_p \in \mathbb{N}$ repeated integrations by parts for each integral in (6.4.1), and using Thm. 80.3, we obtain

$$\mathcal{F}(f)(\omega) = \frac{1}{\omega_1^{N_1}\cdot\ldots\cdot\omega_n^{N_n} i^{N_1+\ldots+N_n}} \int_K \partial_1^{N_1}\ldots\partial_n^{N_n} f(x) e^{-ix\cdot\omega}\, \mathrm{d}\rho.$$

Claims (6.4.3) and (6.4.4) both follows from Thm. 76.4 and from the closure of GSF with respect to differentiation, i.e. Thm. 57.

To prove (6.4.5), we first recall (5.1.7), so that $\overline{B_{\mathrm{d}\rho^{-Q}}(0)} \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$. Let $C$, $b \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ from (6.4.2) and $\lambda(H) \in {}^{\rho}\widetilde{\mathbb{R}}$, where $\lambda$ is the Lebesgue measure. Therefore, $b \le \mathrm{d}\rho^{-R}$ for some $R \in \mathbb{N}$, and we can set $Q := R + 1$. We want to prove the claim using Thm. 80.1, so that we take $\omega = (\omega_1, \ldots, \omega_n) \in \mathrm{ext}\left(\overline{B_{\mathrm{d}\rho^{-Q}}(0)}\right)$. It cannot be $|\omega| <_{\mathrm{s}} \mathrm{d}\rho^{-Q}$ because this would yield $|\omega - a| =_{\mathrm{s}} 0$ for some $a \in \overline{B_{\mathrm{d}\rho^{-Q}}(0)}$; thereby, $|\omega| \ge \mathrm{d}\rho^{-Q}$ by Lem. 11. It always holds $\max_{l=1,\ldots,n} |\omega_l| \ge \frac{1}{n}|\omega|$, i.e. $[\max_{l=1,\ldots,n} |\omega_{l\varepsilon}|] \ge \frac{1}{n}[|\omega_\varepsilon|]$, where $\omega_l = [\omega_{l\varepsilon}]$ and $\omega_\varepsilon := |(\omega_{1\varepsilon}, \ldots, \omega_{n\varepsilon})|$. In general, we cannot say that $|\omega_p| = \max_{l=1,\ldots,n} |\omega_l|$ for some $p = 1, \ldots, n$ because at most this equality holds only for subpoints. In fact, set $L_p := \{\varepsilon \in I \mid \max_{l=1,\ldots,n} |\omega_{l\varepsilon}| = |\omega_{p\varepsilon}|\}$ and let $P \subseteq \{1, \ldots, n\}$ be the non empty set of all the indices $p = 1, \ldots, n$ such that $L_p \subseteq_0 I$. We hence have $|\omega_p| =_{L_p} \max_{l=1,\ldots,n} |\omega_l| \ge \frac{1}{n}|\omega| \ge \frac{1}{n}\mathrm{d}\rho^{-Q}$ for all $p \in P$, and

$$\forall^0 \varepsilon\, \exists p \in P : \ \varepsilon \in L_p. \tag{6.4.6}$$

We apply assumption (6.4.2) and inequality (6.4.3) with an arbitrary $N_p = N \in \mathbb{N}$, $p \in P$, and with $N_j = 0$ for all $j \ne p$ to get

$$|\mathcal{F}(f)(\omega)| \le \frac{1}{|\omega_p|^N} \cdot \int_H \left|\partial_p^N f(x)\right| \mathrm{d}x \le_{L_p} n^N \cdot \mathrm{d}\rho^{NQ} C b^N \lambda(H) \le$$

$$\le \mathrm{d}\rho^{-1} \cdot \mathrm{d}\rho^{N(Q-R)} C\lambda(H) = \mathrm{d}\rho^{N-1} C\lambda(H).$$

For $N \to +\infty$ (in the ring ${}^{\rho}\widetilde{\mathbb{R}}|_{L_p}$), we hence have that $\mathcal{F}(f)(\omega) =_{L_p} 0$. From (6.4.6) we hence finally get $\mathcal{F}(f)(\omega) = 0$. ☐

*Remark* 94.

1. Considering that $\delta(t) = b^n \psi(bt)$ and that $\psi$ is an even function (Lem. 64.1), we have

$$\mathcal{F}(\delta)(\omega) = \int \delta(t) e^{-it\omega} \, \mathrm{d}t = \int \delta(0-t) e^{-it\omega} \, \mathrm{d}t = \left(\delta * e^{-i(-)\omega}\right)(0). \tag{6.4.7}$$

We already know that if $b/|\omega|$ is a strong infinite number, then the function $f_\omega(t) = e^{-it\omega}$ is bounded by a tame polynomial. Thereby, using Thm. 88, we have $\mathcal{F}(\delta)(\omega) = f_\omega(0) = 1$; in particular, $\mathcal{F}(\delta)|_{\mathbb{R}} = 1$.

2. On the other hand, $\delta^{(j)}(t) = b^{j+n} \psi^{(j)}(bt)$ and hence Lem. 64.4 yields

$$\left|\delta^{(j)}(t)\right| \le b^{j+n} C b^{j+2} = C b^{n+2} \left(b^2\right)^j \quad \forall t \in {}^{\rho}\widetilde{\mathbb{R}}.$$

Thus, Dirac's delta satisfies condition (6.4.2) and hence

$$\exists Q \in \mathbb{N} : \ \mathcal{F}(\delta) \in {}^{\rho}\mathcal{GD}(\overline{B_{\mathrm{d}\rho^{-Q}}(0)}). \tag{6.4.8}$$

In the following, we will use the notation $\mathbb{1} := \mathcal{F}(\delta)$.

3. The previous result also yields that $f * \delta = f$ cannot hold in general since otherwise, we can argue as in (6.4.7) to prove that $\mathcal{F}(\delta)(\omega) = 1$ for all $\omega \in {}^{\rho}\widetilde{\mathbb{R}}$, in contradiction with (6.4.8).

Inequality (6.4.3) can also be stated as a general impossibility theorem (where we intuitively think $n = 1$).

**Theorem 95.** *Let $(R, \le)$ be an ordered ring and $G$ be an $R$-module. Assume that we have the following maps (for which we use notations aiming to draw the interpretation where $G$ is a space of GF)*

$$\begin{aligned} (-)' &: G \longrightarrow G \\ \int &: G \longrightarrow R \\ (-) \cdot \exp_\omega &: G \longrightarrow G \quad \forall \omega \in R \\ |-| &: R \longrightarrow R. \end{aligned}$$

*These maps satisfy the following integration by parts formula*

$$\int f \cdot \exp_\omega = \frac{1}{\omega} \int f' \cdot \exp_\omega \tag{6.4.9}$$

*for all invertible $\omega \in R^*$, $f \in G$, and*

$$|rs| = |r||s| \quad \forall r, s \in R \tag{6.4.10}$$

$$\forall f\in G\,\exists C\in R\,\forall\omega\in R^{*}:\ \left|\int f\cdot\exp_{\omega}\right|\le C. \tag{6.4.11}$$

*Then for all $f\in G$ and all $N\in\mathbb{N}_{>0}$ there exists $C=C(f,N)\in R$ such that*

$$\forall\omega\in R^{*}:\left|\int f\cdot\exp_{\omega}\right|\le\frac{C}{|\omega|^{N}}. \tag{6.4.12}$$

*Therefore, if $\delta\in G$ satisfies $\frac{C(\delta,N)}{|\omega|^{N}}<1$ for some $\omega\in R$ and some $N\in\mathbb{N}$, then*

$$\left|\int\delta\cdot\exp_{\omega}\right|<1.$$

*Proof.* For $f\in G$, in the usual way we recursively define $f^{(p)}\in G$ using the map $(-)':G\longrightarrow G$. Taking formula (6.4.9) for $N\in\mathbb{N}_{>0}$ times we get $\int f\cdot\exp_{\omega}=\frac{1}{\omega^{N}}\int f^{(N)}\cdot\exp_{\omega}$. Applying $|-|$ and using (6.4.10) and (6.4.11) we get the conclusion (6.4.12). □

Note that we can take $R=\left\{i\cdot r\mid r\in{}^{\rho}\widetilde{\mathbb{C}}\right\}$ to apply this abstract result to the case of Lem. 93. This result also underscore that in the case $G=\mathcal{D}'(\mathbb{R})$, $R=\mathbb{R}$ we cannot have an integration by parts formula such as (6.4.9). Once more, it also underscores that, since (6.4.9) holds in our setting, we cannot have $f*\delta=f$ without limitations because this would imply $\mathcal{F}(\delta)(\omega)=1$ for all $\omega\in{}^{\rho}\widetilde{\mathbb{R}}$.

**Example 96.** Let $f\left(x\right)=e^{x}$ for all $|x|\le k$, where $k:=-\log\left(\mathrm{d}\rho\right)$. The hyperfinite Fourier transform $\mathcal{F}_{k}$ of $f$ is

$$\begin{aligned}\mathcal{F}_{k}\left(f\right)\left(\omega\right)&=\frac{e^{k(1-i\omega)}-e^{-k(1-i\omega)}}{1-i\omega}=\frac{\mathrm{d}\rho^{(i\omega-1)}-\mathrm{d}\rho^{(1-i\omega)}}{1-i\omega}=\\&=\frac{1}{1-i\omega}\left(\frac{\mathrm{d}\rho^{i\omega}}{\mathrm{d}\rho}-\frac{\mathrm{d}\rho}{\mathrm{d}\rho^{i\omega}}\right)\quad\forall\omega\in{}^{\rho}\widetilde{\mathbb{R}}.\end{aligned}$$

Note that $1-i\omega$, $\omega\in{}^{\rho}\widetilde{\mathbb{R}}$, is always invertible with the usual inverse $\frac{1+i\omega}{1+\omega^{2}}$, moreover, $\mathrm{d}\rho^{i\omega}=e^{i\omega\log\mathrm{d}\rho}$ and hence $|\mathrm{d}\rho^{i\omega}|=1$. Therefore, $\mathcal{F}_{k}(f)(\omega)$ is always an infinite complex number for all finite numbers $\omega$. If $\omega\ge\mathrm{d}\rho^{-1-r}$, $r\in\mathbb{R}_{>0}$, then $\mathcal{F}_{k}(f)(\omega)$ is infinitesimal but not zero. Clearly, $f\notin{}^{\rho}\mathcal{GD}(K)$.

## 6.5 Elementary properties of the hyperfinite Fourier transform

In this section, we list and prove the elementary properties of the HFT.

**Theorem 97.** *(see Sec. 6.1.2 for the notations $\odot$ and $\oplus$) Let $f\in{}^{\rho}\mathcal{GC}^{\infty}\left(K\right)$ and $g:{}^{\rho}\widetilde{\mathbb{R}}^{n}\longrightarrow{}^{\rho}\widetilde{\mathbb{C}}$, then*

1. *$\mathcal{F}_{k}\left(f+g\right)=\mathcal{F}_{k}\left(f\right)+\mathcal{F}_{k}\left(g\right)$ if $g\in{}^{\rho}\mathcal{GC}^{\infty}(K)$.*

2. $\mathcal{F}_k(bf) = b\mathcal{F}_k(f)$ *for all* $b \in {}^{\rho}\widetilde{\mathbb{C}}$.

3. $\mathcal{F}_k\left(\overline{f}\right) = \overline{-1 \diamond \mathcal{F}_k(f)}$, *where* $-1 \diamond f$ *is the* reflection *of* $f$, *i.e.* $(-1 \diamond f)(x) := f(-x)$.

4. $\mathcal{F}_k(-1 \diamond f) = -1 \diamond \mathcal{F}_k(f)$

5. $\mathcal{F}_k(t \diamond g) = t \odot \mathcal{F}_{tk}(g)$ *for all* $t \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ *such that* $tk$ *is still infinite and* $g|_K \in {}^{\rho}\mathcal{GC}^{\infty}(K)$, $g|_{tK} \in {}^{\rho}\mathcal{GC}^{\infty}(tK)$. *Here,* $t \diamond g$ *is the* dilation *of* $f$, *i.e.* $(t \diamond g)(x) := g(tx)$.

6. *Let* $k > h > 0$ *be infinite numbers,* $s \in [-(k-h), k-h]^n$, $f \in {}^{\rho}\mathcal{GD}([-h,h]^n)$. *Then*

$$\mathcal{F}_k(s \oplus f) = e^{-is\cdot(-)}\mathcal{F}_k(f) = e^{-is\cdot(-)}\mathcal{F}_h(f) = e^{-is\cdot(-)}\mathcal{F}(f).$$

   *In particular, if* $h \geq \mathrm{d}\rho^{-p}$, $k \geq \mathrm{d}\rho^{-q}$, $p, q \in \mathbb{R}_{>0}$, $q > p$, *and* $s \in \mathrm{c}(\mathbb{R}^n)$, *then* $s \in [-(k-h), k-h]^n$. *In particular,* $\mathbb{R}^n \subseteq [-(k-h), k-h]^n$.

7. $\mathcal{F}_k\left(e^{is\cdot(-)}f\right) = s \oplus \mathcal{F}_k(f)$ *for all* $s \in {}^{\rho}\widetilde{\mathbb{R}}^n$.

8. *Let* $\omega \in {}^{\rho}\widetilde{\mathbb{R}}^n$ *and* $\alpha \in \mathbb{N}^n \setminus \{0\}$. *For* $p = 1, \ldots, |\alpha|$, *define* $\beta_p = (\beta_{p,q})_{q=1,\ldots,n} \in \mathbb{N}^n$ *with*

$$\beta_0 := \alpha$$
$$\beta_{p+1} := (0, \overset{j_p-1}{\ldots\ldots}, 0, \beta_{p,j_p} - 1, \beta_{p,j_p+1}, \ldots, \beta_{p,n}) \text{ if } j_p := \min\{q \mid \beta_{p,q} > 0\}.$$

   *Finally, for all* $\bar{f} \in {}^{\rho}\mathcal{GC}^{\infty}(K)$ *and* $j = 1, \ldots, n$, *set*

$$\Delta_{1k}\bar{f}(\omega) := \left[\bar{f}(x)e^{-ix\cdot\omega}\right]_{x_j=-k}^{x_j=k}$$
$$\Delta_{jk}\bar{f}(\omega) := \int_{-k}^{k} \mathrm{d}x_1 \ldots \int_{-k}^{k} \mathrm{d}x_{j-1} \int_{-k}^{k} \mathrm{d}x_{j+1} \ldots \int_{-k}^{k} \left[\bar{f}(x)e^{-ix\cdot\omega}\right]_{x_j=-k}^{x_j=k} \mathrm{d}x_n.$$

   *Then, we have*

$$\mathcal{F}_k(\partial_j f) = i\omega_j\mathcal{F}_k(f) + \Delta_{jk}f \quad \forall j = 1, \ldots, n \tag{6.5.1}$$
$$\mathcal{F}_k(\partial^{\alpha} f) = (i\omega)^{\alpha}\mathcal{F}_k(f) + \sum_{p=0}^{|\alpha|-1}(i\omega)^{\alpha-\beta_p}\Delta_{j_p k}(\partial^{\beta_{p+1}}f). \tag{6.5.2}$$

   *In particular, if*

$$f(x_1, \ldots, x_{j-1}, k, x_{j+1}) = f(x_1, \ldots, x_{j-1}, -k, x_{j+1}) = 0 \quad \forall x \in K, \tag{6.5.3}$$

   *then*

$$\mathcal{F}_k(\partial_j f) = i\omega_j\mathcal{F}_k(f).$$

9. $\frac{\partial}{\partial\omega_j}\mathcal{F}_k(f) = -i\mathcal{F}_k(x_j f)$ *for all* $j = 1, \ldots, n$.

10. *If* $f \in {}^\rho\mathcal{GD}(K)$ *or* $g \in {}^\rho\mathcal{GD}(K)$*, then* $\mathcal{F}_k(f * g) = \mathcal{F}_k(f)\,\mathcal{F}_k(g)$*. Therefore, if* $f \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$ *and* $g \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n)$*, then* $\mathcal{F}(f * g) = \mathcal{F}(f)\,\mathcal{F}(g)$.

11. $\mathcal{F}_k(s \odot g) = s \diamond \mathcal{F}_{\frac{k}{s}}(g)$ *for all invertible* $s \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ *such that* $\frac{k}{s}$ *is infinite,* $g|_K \in {}^\rho\mathcal{GC}^\infty(K)$ *and* $g|_{K/s} \in {}^\rho\mathcal{GC}^\infty(K/s)$.

*Proof.* Properties 1-5 can be proved like in the case of rapidly decreasing smooth functions. For 6, we have

$$\mathcal{F}_k(s \oplus f)(\omega) = \mathcal{F}_k(f(x-s))(\omega) = \int_K f(x-s)\,e^{-ix\cdot\omega}\,\mathrm{d}x =$$

$$= \int_{-k}^{k} \mathrm{d}x_1 \ldots \int_{-k}^{k} f(x-s)\,e^{-ix\cdot\omega}\,\mathrm{d}x_n.$$

Considering the change of variable $x - s = u$ we have

$$\mathcal{F}_k(s \oplus f)(\omega) = e^{-is\cdot\omega} \int_{-k-s_1}^{k-s_1} \mathrm{d}u_1 \ldots \int_{-k-s_n}^{k-s_n} f(u)\,e^{-iu\cdot\omega}\,\mathrm{d}u_n.$$

Finally, considering that $k > h$ and $s \in [-k+h, k-h]^n$ we have $k - s_i \geq h$, $-h \geq -k - s_i$ and $k + s_i \geq h$ for all $i = 1, \ldots, n$, so that

$$\int_{-k-s_1}^{k-s_1} \mathrm{d}u_1 \ldots \int_{-k-s_n}^{k-s_n} f(u)\,e^{-iu\cdot\omega}\,\mathrm{d}u_n = \int_{-h}^{h} \mathrm{d}u_1 \ldots \int_{-h}^{h} f(u)\,e^{-iu\cdot\omega}\,\mathrm{d}u_n =$$

$$= \int_{-k}^{k} \mathrm{d}u_1 \ldots \int_{-k}^{k} f(u)\,e^{-iu\cdot\omega}\,\mathrm{d}u_n$$

from Def. 81 since $f \in {}^\rho\mathcal{GD}([-h,h]^n)$.

7 is immediate from the Def. 91.

To prove 8, using integration by parts formula, we have

$$\mathcal{F}_k(\partial_j f)(\omega) = \int_K \partial_j f(x)\,e^{-ix\cdot\omega}\,\mathrm{d}x = \int_{-k}^{k} \mathrm{d}x_1 \ldots \int_{-k}^{k} \partial_j f(x)\,e^{-ix\cdot\omega}\,\mathrm{d}x_n =$$

$$= -\int_{-k}^{k} \mathrm{d}x_1 \ldots \int_{-k}^{k} f(x)\,(-i\omega_j)\,e^{-ix\cdot\omega}\,\mathrm{d}x_n +$$

$$+ \int_{-k}^{k} \mathrm{d}x_1 \ldots \int_{-k}^{k} \mathrm{d}x_{j-1} \int_{-k}^{k} \mathrm{d}x_{j+1} \ldots \int_{-k}^{k} \left[f(x)e^{-ix\cdot\omega}\right]_{x_j=-k}^{x_j=k} \mathrm{d}x_n =$$

$$= i\omega_j \mathcal{F}_k(f)(\omega) + \Delta_{jk} f(\omega).$$

Therefore, by applying this formula with $\partial_p f$ instead of $f$, we obtain

$$\mathcal{F}_k\left(\partial_j \partial_p f\right)(\omega)=-\omega_j \omega_p \mathcal{F}_k(f)(\omega)+i \omega_j \Delta_{pk}(f)(\omega)+\Delta_{jk}\left(\partial_p f\right)(\omega).$$

Proceeding similarly by induction on $|\alpha|$, we can prove the general claim.

To prove 9, we use Thm. 61.8, i.e. derivation under the integral sign:

$$\begin{aligned}
\frac{\partial}{\partial \omega_j} \mathcal{F}_k(f)(\omega) &= \frac{\partial}{\partial \omega_j}\left(\int_{-k}^{k} \mathrm{d} x_1 \ldots \int_{-k}^{k} f(x) e^{-i x \cdot \omega} \mathrm{d} x_n\right)= \\
&=\int_{-k}^{k} \mathrm{d} x_1 \ldots \int_{-k}^{k} \frac{\partial}{\partial \omega_j}\left(f(x) e^{-i x \cdot \omega}\right) \mathrm{d} x_n= \\
&=\int_{-k}^{k} \mathrm{d} x_1 \ldots \int_{-k}^{k}-i x_j f(x) e^{-i x \cdot \omega} \mathrm{d} x_n= \\
&=-i \mathcal{F}_k\left(x_j f\right)(\omega).
\end{aligned}$$

10:

$$\begin{aligned}
\mathcal{F}_k((f * g))(\omega) &= \int_K e^{-i x \omega}(f * g)(x) \mathrm{d} x= \\
&=\int_K e^{-i x \omega} \int_K f(y) g(x-y) \mathrm{d} y \mathrm{d} x.
\end{aligned}$$

Considering the change of variable $x-y=t$ and using Fubini's theorem, we have

$$\begin{aligned}
\int_K e^{-i(t+y) \omega} \int_K f(y) g(t) \mathrm{d} y \mathrm{d} t &= \int_K e^{-i y \omega} f(y) \mathrm{d} y \int_K e^{-i t \omega} g(t) \mathrm{d} t= \\
&=\mathcal{F}_k(f)(\omega) \mathcal{F}_k(g)(\omega).
\end{aligned}$$

Finally, we prove 11:

$$\mathcal{F}_k(s \odot g)(\omega)=\mathcal{F}_k\left(\frac{1}{s^n} g\left(\frac{x}{s}\right)\right)(\omega)=\int_K e^{-i x \cdot \omega} g\left(\frac{x}{s}\right) \frac{\mathrm{d} x}{s^n}.$$

Considering the change of variable $\frac{x}{s}=y$ we have

$$\begin{aligned}
\int_K e^{-i x \cdot \omega} g\left(\frac{x}{s}\right) \frac{\mathrm{d} x}{s^n} &= \int_{-k/s}^{k/s} \mathrm{d} y_1 \ldots \int_{-k/s}^{k/s} g(y) e^{-i s y \cdot \omega} \mathrm{d} y_n= \\
&=\int_{K/s} g(y) e^{-i y \cdot s \omega} \mathrm{d} y=\mathcal{F}_{k/s}(g)(s \omega)= \\
&=\left[s \diamond \mathcal{F}_{k/s}(g)\right](\omega).
\end{aligned}$$

$\square$

We will see in Sec. 6.8 that the additional term in (6.5.2) plays an important role in finding *non tempered* solutions of differential equations (like the exponentials of the trivial ODE $y' = y$). We also note that condition (6.5.3) is clearly weaker than asking $f$ compactly supported. For example, setting

$$l_j(x) := \frac{1}{2k}\left[f(x)|_{x_j=k} - f(x)|_{x_j=-k}\right]\cdot(x_j+k) + f(x)|_{x_j=-k},$$

then $\bar{f} := f - l_j$ satisfies (6.5.3).

## 6.6 The inverse hyperfinite Fourier transform, Parseval's relation, Plancherel's identity and the uncertainty principle

We naturally define the inverse HFT as follows:

**Definition 98.** Let $f \in {}^{\rho}\mathcal{GC}^{\infty}(K)$, we define the *inverse* HFT as

$$\mathcal{F}_k^{-1}(f)(x) := \frac{1}{(2\pi)^n}\int_K f(\omega)\, e^{ix\cdot\omega}\, \mathrm{d}\omega \tag{6.6.1}$$

for all $x \in {}^{\rho}\widetilde{\mathbb{R}}$. As we proved in Thm. 92, we have $\mathcal{F}_k^{-1} : {}^{\rho}\mathcal{GC}^{\infty}(K) \longrightarrow {}^{\rho}\mathcal{GC}^{\infty}\left({}^{\rho}\widetilde{\mathbb{R}}^n\right)$. We immediately note that the notation of the inverse function $\mathcal{F}_k^{-1}$ is an abuse of language because the codomain of $\mathcal{F}_k$ is larger than the domain of $\mathcal{F}_k^{-1}$ (and vice versa). When dealing with inversion properties, it is hence better to think at

$$\begin{aligned}
\mathcal{F}_k|_K &:= (-)|_K \circ \mathcal{F}_k : {}^{\rho}\mathcal{GC}^{\infty}(K) \longrightarrow {}^{\rho}\mathcal{GC}^{\infty}(K)\\
\mathcal{F}_k^{-1}|_K &:= (-)|_K \circ \mathcal{F}_k^{-1} : {}^{\rho}\mathcal{GC}^{\infty}(K) \longrightarrow {}^{\rho}\mathcal{GC}^{\infty}(K).
\end{aligned}$$

We will see in Sec. 6.8 that lacking this precision can easily lead to inconsistencies.

Note that

$$(2\pi)^n\, \mathcal{F}_k^{-1}(f) = \mathcal{F}_k(-1 \diamond f) = -1 \diamond \mathcal{F}_k(f), \tag{6.6.2}$$

where $-1\diamond$ denotes the reflection $(-1 \diamond g)(x) := g(-x)$.

Our main goal is clearly to investigate the relationship between HFT and its inverse HFT, i.e. to prove the Fourier inversion theorem for the HFT. Three important results used in the classical proof of the Fourier inversion theorem are: the application of approximate identities for convolution defined by Gaussian like functions, Lebesgue dominated converge theorem (we can replace it with Thm. 41), and the translation property of FT. In our setting, the last property corresponds to Thm. 97.6, which works only for compactly supported GSF. The idea is hence to avoid proving the inversion theorem firstly at the origin and then employing the translation property, but to prove it directly at an arbitrary interior point $y \in K$ using approximate identities obtained by mollification of a Gaussian function.

We hence start by the latter, explicitly noting that, contrary to the usual setting, considering Robinson-Colombeau generalized numbers, the Gaussian is compactly supported:

**Lemma 99.** *Let $f(x) = e^{-\frac{|x|^2}{2}}$ for all $x \in {}^\rho\widetilde{\mathbb{R}}^n$. Then $f \in {}^\rho\mathcal{GD}\left(\overline{B_h(0)}\right)$ for all strong infinite number $h \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. Moreover, $\mathcal{F}(f)(\omega) = (2\pi)^{\frac{n}{2}} e^{-\frac{|\omega|^2}{2}}$.*

*Proof.* The function $f$ satisfies the inequality $0 \le f(x) \le x^{-q}$, $\forall q \in \mathbb{N}$, for $|x|$ finite sufficiently large. Therefore, for all strongly infinite $x$, we have $f(x) = 0$ i.e., $f \in {}^\rho\mathcal{GD}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$. We first prove the second claim in dimension $n = 1$, by noting that $f(x) = e^{-\frac{|x|^2}{2}}$ satisfies the ODE

$$f'(x) + xf(x) = 0 \tag{6.6.3}$$

with the initial value $f(0) = 1$. By separation of variables, any solution of (6.6.3) is of the form $f(x) = c \cdot e^{-\frac{x^2}{2}}$, where $c = f(0)$. Applying the HFT to (6.6.3), and considering 8 and 9 of Thm. 97, we have

$$i\omega\mathcal{F}(f)(\omega) + i\mathcal{F}'(f)(\omega) = 0.$$

Thus $\mathcal{F}(f)$ also solves the ODE (6.6.3). Therefore we must have $\mathcal{F}(f)(\omega) = ce^{-\frac{\omega^2}{2}}$ and, taking as $h = [h_\varepsilon]$ any strong infinite number so that $\mathcal{F}_h(f) = \mathcal{F}(f)$, we have

$$\begin{aligned} c = \mathcal{F}(f)(0) &= \int_{-h}^{h} e^{-\frac{x^2}{2}}\,\mathrm{d}x = \left[\int_{-h_\varepsilon}^{h_\varepsilon} e^{-\frac{x^2}{2}}\,\mathrm{d}x\right] = \\ &= \left[\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x + \int_{-\infty}^{-h_\varepsilon} e^{-\frac{x^2}{2}}\,\mathrm{d}x + \int_{h_\varepsilon}^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x\right] = \sqrt{2\pi} \end{aligned}$$

since $\int_{-\infty}^{-h_\varepsilon} e^{-\frac{x^2}{2}}\,\mathrm{d}x$ and $\int_{h_\varepsilon}^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x$ are negligible: in fact, using L'Hôpital rule we can prove that $\lim_{y\to 0^+} \frac{\int_{\pm 1/y}^{\pm\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x}{y^q} = 0$ for all $q \in \mathbb{N}$. We note that naturally the equality above is in ${}^\rho\widetilde{\mathbb{R}}$. In dimension $n > 1$ we directly calculate using Fubini's theorem:

$$\begin{aligned} \mathcal{F}\left(e^{-\frac{|x|^2}{2}}\right)(\omega) &= \prod_{j=1}^{n} \int e^{-ix_j\cdot\omega_j} e^{-\frac{x_j^2}{2}}\,\mathrm{d}x_j \\ &= \prod_{j=1}^{n} \mathcal{F}(f)(\omega_j) = \prod_{j=1}^{n} (2\pi)^{\frac{1}{2}} e^{-\frac{\omega_j^2}{2}} = (2\pi)^{\frac{n}{2}} e^{-\frac{|\omega|^2}{2}}. \end{aligned}$$

□

In the following result, we use the notation $\forall^\infty p \in {}^\rho\widetilde{\mathbb{N}}$ to denote $\exists P \in {}^\rho\widetilde{\mathbb{N}}\, \forall p \in {}^\rho\widetilde{\mathbb{N}}_{\ge P}$, and we read it *for all* $p \in {}^\rho\widetilde{\mathbb{N}}$ *sufficiently large*.

**Theorem 100.** *Let $y$ be a sharply interior point of $K$, $f \in {}^\rho\mathcal{GC}^\infty(K)$, $G_p \in {}^\rho\mathcal{GD}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$, for $p \in {}^\rho\widetilde{\mathbb{N}}_{>0}$, satisfy*

*(a)* $\int G_p = 1$ *for* $p \in {}^\rho\widetilde{\mathbb{N}}_{>0}$ *sufficiently large.*

*(b) For $p$ sufficiently large, $(G_p)_{\in {}^\rho\widetilde{\mathbb{N}}_{>0}}$ is zero outside every ball $B_\delta(0)$, $\delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, i.e.,*

$$\forall \delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall^\infty p \in {}^\rho\widetilde{\mathbb{N}}\, \forall x : \ |x| \ge \delta \Rightarrow G_p(x) = 0. \tag{6.6.4}$$

*(c)* $\exists M \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall^\infty p \in {}^\rho\widetilde{\mathbb{N}} : \ \int |G_p(y)|\, \mathrm{d}y \le M.$

*Then*

1. $\int_{y+K} f(x)\, G_p(y-x)\, \mathrm{d}x = \int f(x)\, G_p(y-x)\, \mathrm{d}x = \int f(y-z)\, G_p(z)\, \mathrm{d}z$ *for all* $p \in {}^\rho\widetilde{\mathbb{N}}_{>0}$ *sufficiently large.*

2. ${}^\rho\lim_{p \in {}^\rho\widetilde{\mathbb{N}}} (f * G_p)(y) = f(y)$.

*Proof.* We only have to generalize the classical proof concerning limits of convolutions between continuous functions and approximate identities. Since $B_\delta(y) \subseteq K$ for some $\delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, we have that $\operatorname{supp}(G_p(y - \cdot)) \subseteq y + K$, for $p$ large, by condition (6.6.4). The remaining equality of property 1 is immediate by considering the change of variable $y - x = z$. For 2, we proceed as follows. Using (a), for $p$ large, let us say for $p \ge P \in {}^\rho\widetilde{\mathbb{N}}_{>0}$, we get

$$\begin{aligned}\left|\int f(x) G_p(y-x)\mathrm{d}x - f(y)\right| &= \left|\int \left[f(x) - f(y)\right] G_p(y-x)\, \mathrm{d}x\right| \\ &\le \int |f(x) - f(y)| \cdot |G_p(y-x)|\, \mathrm{d}x.\end{aligned}$$

Now, for each $l \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, sharp continuity of $f$ at $y$ yields $|f(x) - f(y)| < l$ for all $x$ such that $|x - y| < \delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. By (b), for $p$ large, we have

$$\left|\int f(x) G_p(y-x)\, \mathrm{d}x - f(y)\right| \le l \int_{B_\delta(0)} |G_p(y-x)|\, \mathrm{d}x \le l \cdot M, \tag{6.6.5}$$

where we have taken $p$ sufficiently large so that also (c) holds. The right hand side of (6.6.5) can be taken arbitrarily small in ${}^\rho\widetilde{\mathbb{R}}_{>0}$ because $l \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ is an arbitrary positive generalized number. $\square$

For example, we can set $t_p := \frac{1}{p}$ for $p \in {}^\rho\widetilde{\mathbb{N}}_{>0}$, $g(z) := e^{-|z|^2/2}$ and $G_p := t_p \odot g$, so that $G_p(z) = (2\pi)^{-n/2} \cdot e^{-\frac{|z|^2}{2t_p^2}} \cdot \frac{1}{t_p^n}$ for all $z \in {}^\rho\widetilde{\mathbb{R}}^n$. Let $\delta \ge \mathrm{d}\rho^Q$ and $|z| \ge \delta$;

for all $q \in \mathbb{N}$ and for $p \in {}^{\rho}\widetilde{\mathbb{N}}$ sufficiently large, we have

$$e^{-\frac{|z|^2}{2t_p^2}} \cdot \frac{1}{t_p^n} \le p^n e^{-\frac{1}{2}\delta^2 p^2} \le p^n \frac{1}{\delta^{2q} p^{2q}} \le \mathrm{d}\rho^q,$$

where the latter inequality holds e.g. for all $p \ge \mathrm{d}\rho^{-Q}$. This shows that property (6.6.4) holds in this case.

**Theorem 101.** *Let $h \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ be an infinite number and set $H := [-h, h]^n$. Let $f \in {}^{\rho}\mathcal{GC}^{\infty}(K)$ and $g \in {}^{\rho}\mathcal{GC}^{\infty}(H)$. Then*

1. $\int_H \mathcal{F}_k(f)(\omega)\, g(\omega)\, \mathrm{d}\omega = \int_K f(x)\, \mathcal{F}_h(g)(x)\, \mathrm{d}x$.
2. $\mathcal{F}_k^{-1}|_K (\mathcal{F}_k|_K(f)) = f = \mathcal{F}_k|_K \left(\mathcal{F}_k^{-1}|_K(f)\right)$.
3. $\mathcal{F}_k|_K(\mathcal{F}_k|_K(f)) = (2\pi)^n (-1 \diamond f)$, *where we recall that $-1 \diamond f$ is the reflection of $f$.*

*If $H = K$, then*

4. *(Parseval's relation)* $(2\pi)^n \int_K f\overline{g} = \int_K \mathcal{F}_k(f)\, \overline{\mathcal{F}_k(g)}$.
5. *(Plancherel's identity)* $(2\pi)^n \int_K |f|^2 = \int_K |\mathcal{F}_h(f)|^2$.
6. $\int_K fg = \int_K \mathcal{F}_k(f)\, \mathcal{F}_k^{-1}(g)$.

*Proof.* 1 follows from Def. 91 and Fubini's theorem.

2: We first prove the inversion theorem for sharply interior points $y \in K$. Hence, let $B_\delta(y) \subseteq K$ for some $\delta > 0$. Set $t_p$, $g$ and $G_p$ as above. Set $g_p(\omega, y) := \frac{e^{iy\cdot\omega}}{(2\pi)^{n/2}} \cdot e^{-\frac{|t_p\omega|^2}{2}}$ for all $\omega \in {}^{\rho}\widetilde{\mathbb{R}}^n$, and hence $g_p(-, y) = \frac{e^{iy\cdot(-)}}{(2\pi)^{n/2}} \cdot (t_p \diamond g)$. Thereby, from $g \in {}^{\rho}\mathcal{GD}({}^{\rho}\widetilde{\mathbb{R}}^n)$, we also get $g_p(-, y) \in {}^{\rho}\mathcal{GD}({}^{\rho}\widetilde{\mathbb{R}}^n)$. Since $k > 0$ and $\mathrm{supp}(G_p) \subseteq \overline{B_{t_p\cdot \mathrm{d}\rho^{-1}}(0)}$ (see Lem. 99), we get that

$$\mathrm{supp}(G_p) \subseteq K \tag{6.6.6}$$

for all $p$ sufficiently large. Let's compute the HFT $\mathcal{F}[g_p(-, y)]$ for an arbitrary $p \in {}^{\rho}\widetilde{\mathbb{N}}$:

$$\mathcal{F}[g_p(-, y)] = \frac{1}{(2\pi)^{n/2}} \mathcal{F}\left[e^{iy\cdot(-)} \cdot (t_p \diamond g)\right] = \frac{1}{(2\pi)^{n/2}} \cdot y \oplus \mathcal{F}(t_p \diamond g),$$

where we used Thm. 97.7. We have $\mathrm{supp}(t_p \diamond g) \subseteq \overline{B_{\mathrm{d}\rho^{-1}/t_p}(0)}$ because $\mathrm{supp}(g) \subseteq \overline{B_{\mathrm{d}\rho^{-1}}(0)}$. Set $h_p := \frac{\mathrm{d}\rho^{-1}}{t_p}$, and use Thm. 97.5 noting that $t_p h_p = \mathrm{d}\rho^{-1}$ is an infinite number:

$$\begin{aligned}
\mathcal{F}[g_p(-, y)](x) &= \left[\frac{1}{(2\pi)^{n/2}} \cdot y \oplus \mathcal{F}_{h_p}(t_p \diamond g)\right](x) = \\
&= \frac{1}{(2\pi)^{n/2}} \cdot [t_p \odot \mathcal{F}(g)](x - y) = \\
&= \frac{1}{(2\pi)^{n/2}} \cdot \left[t_p \odot (2\pi)^{n/2} g\right](x - y) = \\
&= [t_p \odot g](x - y) = G_p(x - y) = G_p(y - x).
\end{aligned}$$

Therefore, using 1, and for $p$ sufficiently large, we obtain

$$\int_K \mathcal{F}_k(f)(\omega)\cdot g_p(\omega,y)\,\mathrm{d}\omega = \int_K f(x)\cdot\mathcal{F}\left[g_p(-,y)\right](x)\,\mathrm{d}x = \\ = \int_K f(x)\cdot G_p(y-x)\,\mathrm{d}x = (f*G_p)(y), \tag{6.6.7}$$

where for $p$ large, we have $\operatorname{supp}\left(G_p(y-\cdot)\right)\subseteq B_\delta(y)\subseteq K$. Taking the hyperlimit for $p\in{}^\rho\widetilde{\mathbb{N}}$ of the right hand side of (6.6.7), Thm. 100 yields that it converges to $f(y)$. The same hyperlimit of the left hand side and Thm. 41 give

$$\begin{aligned}{}^\rho\lim_{p\in{}^\rho\widetilde{\mathbb{N}}}\int_K \mathcal{F}_k(f)(\omega)\cdot g_p(\omega,y)\,\mathrm{d}\omega &= \int_K \mathcal{F}_k(f)(\omega)\cdot{}^\rho\lim_{n\in{}^\rho\widetilde{\mathbb{N}}} g_p(\omega,y)\,\mathrm{d}\omega = \\ &= \int_K \mathcal{F}_k(f)|_K(\omega)\frac{e^{iy\cdot\omega}}{(2\pi)^n}\,\mathrm{d}\omega\end{aligned}$$

because of the definition of $g_p$. For boundary points of $K$, the identity follows using sharp continuity. This proves that $\mathcal{F}_k^{-1}\left(\mathcal{F}_k(f)|_K\right)=f$ on $K$, i.e. that $\mathcal{F}_k^{-1}|_K\left(\mathcal{F}_k|_K(f)\right)=f$. To prove the other identity, it suffices to apply this equality to $-1\diamond f$ and consider (6.6.2).

3 follows by (6.6.2) using 2 and the definition of reflection.

To prove 4, use 1 with $\overline{\mathcal{F}_k(g)}$ instead of $g$, then Thm. 97.3, and finally 3.

Plancherel's identity 5 is a trivial consequence of 4.

Finally, 6 follows from 2 and 1. □

We close this section with a proof of the uncertainty principle

**Theorem 102.** *If $\psi\in{}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}})$, then*

1. *If $\psi\in{}^\rho\mathcal{GD}(H)\cap{}^\rho\mathcal{GD}(K)$, then*

$$\int_H \omega^2\left|\mathcal{F}(\psi)(\omega)\right|^2\,\mathrm{d}\omega = \int_K \omega^2\left|\mathcal{F}(\psi)(\omega)\right|^2\,\mathrm{d}\omega =: \int \omega^2\left|\mathcal{F}(\psi)(\omega)\right|^2\,\mathrm{d}\omega$$

2. $\left(\int x^2\left|\psi(x)\right|^2\,\mathrm{d}x\right)\left(\int\omega^2\left|\mathcal{F}(\psi)(\omega)\right|^2\,\mathrm{d}\omega\right)\ge\frac{1}{4}\|\psi\|_2\|\mathcal{F}(\psi)\|_2$.

*Proof.* Properties 2 and 1 of Thm. 80 imply that also $\psi'\in{}^\rho\mathcal{GD}(H)$. Therefore, Plancherel's identity Thm. 101.5 yields

$$\int_H\left|\psi'\right|^2=\frac{1}{2\pi}\int_H\left|\mathcal{F}(\psi')\right|^2.$$

But $\left|\mathcal{F}(\psi')\right|^2=\omega^2\left|\mathcal{F}(\psi)\right|^2$ from Thm. 97.8 because $\psi$ is compactly supported and hence $\Delta_{1k}\psi=0$. Therefore

$$\int_H\left|\psi'\right|^2=\frac{1}{2\pi}\int_H\omega^2\left|\mathcal{F}(\psi)(\omega)\right|^2\,\mathrm{d}\omega. \tag{6.6.8}$$

At the same result we arrive considering $K$ instead of $H$. Finally, we apply Def. 81 of integral of a compactly supported GSF, which yields independence from the functionally compact integration domain.

To prove inequality 2, we assume that $\psi \in {}^{\rho}\mathcal{GD}(K)$; using integration by parts, we get:

$$\begin{aligned}\int x\overline{\psi(x)}\psi'(x)\,\mathrm{d}x &= \int_{-k}^{k} x\overline{\psi(x)}\psi'(x)\,\mathrm{d}x = \\ &= \left[x\overline{\psi(x)}\psi(x)\right]_{x=-k}^{x=k} - \int \psi(x)\left(\overline{\psi(x)} + x\overline{\psi'(x)}\right)\mathrm{d}x = \\ &= -\int \left[|\psi(x)|^2 + x\psi(x)\overline{\psi'(x)}\right]\mathrm{d}x.\end{aligned}$$

Thereby

$$\begin{aligned}\int |\psi(x)|^2\,\mathrm{d}x &= -2\mathrm{Re}\left(\int x\psi(x)\overline{\psi'(x)}\,\mathrm{d}x\right) \le \\ &\le 2\left|\mathrm{Re}\left(\int x\psi(x)\overline{\psi'(x)}\,\mathrm{d}x\right)\right| \le \\ &\le 2\int \left|x\psi(x)\overline{\psi'(x)}\right|\mathrm{d}x.\end{aligned}$$

Where we used the triangle inequality for integrals (see Thm. 76.4). Using Cauchy-Schwarz inequality (see Thm. 86), we hence obtain

$$\begin{aligned}\left(\int |\psi(x)|^2\,\mathrm{d}x\right)^2 &\le 4\left(\int \left|x\psi(x)\overline{\psi'(x)}\right|\mathrm{d}x\right)^2 \le \\ &\le 4\left(\int x^2|\psi(x)|^2\,\mathrm{d}x\right)\left(\int |\psi'(x)|^2\,\mathrm{d}x\right).\end{aligned}$$

From this, thanks to (6.6.8) and Plancherel's equality, the claim follows. □

Note that if $\|\psi\|_2 \in {}^{\rho}\widetilde{\mathbb{R}}$ is invertible, then also $\|\mathcal{F}(\psi)\|_2$ is invertible by Plancherel's equality, and we can hence write the uncertainty principle in the usual normalized form.

**Example 103.** On the contrary with respect the classical formulation in $L^2(\mathbb{R})$ of the uncertainty principle, in Thm. 102 we can e.g. consider $\psi = \delta \in {}^{\rho}\mathcal{GD}({}^{\rho}\widetilde{\mathbb{R}})$, and we have

$$\int x^2\delta(x)^2\,\mathrm{d}x = \left[\int_{-1}^{1} x^2 b_\varepsilon^2 \psi_\varepsilon(b_\varepsilon x)^2\,\mathrm{d}x\right]$$

where $\psi(x) = [\psi_\varepsilon(x_\varepsilon)]$ is a Colombeau mollifier and $b = [b_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ is a strong infinite number (see Example 67). Since normalizing the function $\varepsilon \mapsto b_\varepsilon^2\psi_\varepsilon(b_\varepsilon x)^2$ we get an approximate identity, we have $\lim_{\varepsilon\to 0^+}\int_{-1}^{1} x^2 b_\varepsilon^2\psi_\varepsilon(b_\varepsilon x)^2\,\mathrm{d}x = 0$, and hence $\int x^2\delta(x)^2\,\mathrm{d}x \approx 0$ is an infinitesimal. The uncertainty principle

Thm. 102 implies that it is an invertible infinitesimal. Considering the HFT $\mathbb{1} = \mathcal{F}(\delta) \in {}^\rho\mathcal{GD}(\widetilde{{}^\rho\mathbb{R}})$, we have

$$\int \omega^2 \mathbb{1}(\omega)^2 \, d\omega \geq \int_{-r}^{r} \omega^2 \, d\omega = 2\frac{r^3}{3} \quad \forall r \in \mathbb{R}_{>0}.$$

Thereby, $\int \omega^2 \mathbb{1}(\omega)^2 \, d\omega$ is an infinite number.

## 6.7 Preservation of classical Fourier transform

It is natural to inquire the relations between classical FT of tempered distributions and our HFT.

Let us start with a couple of exploring examples:

1. $\mathcal{F}_k(1)(\omega) = \int_{-k}^{k} 1 \cdot e^{-ix\omega} \, dx = \int_{-k}^{k} \cos(x\omega) \, dx$. If $L \subseteq_0 I$ and $\omega|_L$ is invertible (see Sec. 5.1.3 for the language of subpoints), then $\mathcal{F}_k(1)(\omega) =_L 2\frac{\sin(k\omega)}{\omega}$; if $\omega =_L 0$, then $\mathcal{F}_k(1)(\omega) = 2k$. Classically, we have $\hat{1} = 2\pi\delta$.
2. $\mathcal{F}_k(H)(\omega) = \int_{-k}^{k} H(x)e^{-ix\omega} \, dx$. Assuming that $\omega|_L$ is invertible on $L \subseteq_0 I$, we have $\mathcal{F}_k(H)(\omega) =_L \frac{i}{\omega}e^{-ik\omega} - \frac{i}{\omega}\mathbb{1}(\omega)$. Classically, we have $\hat{H} = \pi\delta - \frac{i}{\omega}$. Therefore, if $k$ is a strong infinite number and $\omega$ is far from the origin, $|\omega| \geq r \in \mathbb{R}_{>0}$, we have $\mathcal{F}_k(H)(\omega) = \iota^b_{\mathbb{R}}\left(\hat{H}\right)(\omega)$ (here $\iota^b_{\mathbb{R}}$ is an embedding of $\mathcal{D}'(\mathbb{R})$ into ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\mathbb{R}))$, see Sec. 6.7). However, the latter equality does not hold if $\omega \approx 0$.

Since classically we do not have infinite numbers, the former of these examples leads us to the following idea

$$\mathcal{F}(1 \cdot \mathbb{1}) = \mathcal{F}(\mathcal{F}(\delta)) = 2\pi\,(-1 \diamond \delta) = 2\pi\delta.$$

Note that if $f \in {}^\rho\mathcal{GC}^\infty(K)$, then $(f \cdot \mathbb{1})(\omega) = f(\omega)$ for all finite point $\omega \in K$. We therefore call $f \cdot \mathbb{1}$ the *finite part of* $f$. The same idea works for $e^{iax}$ and hence also for sin, cos. Let us now consider $\delta \cdot \mathbb{1}$:

$$\mathcal{F}(\delta \cdot \mathbb{1})(\omega) = \int \delta(x)\mathcal{F}(\delta)(x)e^{-ix\omega} \, dx.$$

We recall that integrating against $\delta$ yields the evaluation of the second factor at 0 only if the latter is bounded by a tame polynomial at 0 (see Example 89.4). But the function $x \mapsto \mathcal{F}(\delta)(x)e^{-ix\omega}$ is bounded by a tame polynomial at $x = 0$ for all $\omega$, and we get $\mathcal{F}(\delta \cdot \mathbb{1})(\omega) = 1$. Being bounded by a tame polynomial is, in general, a necessary assumption because

$$\begin{aligned}\mathcal{F}(H \cdot \mathbb{1})(\omega) &= \int H(x) \cdot \mathcal{F}(\delta)(x)e^{-ix\omega} \, dx = \\ &= \int \delta(x)\mathcal{F}_k(H \cdot e^{-i(-)\omega})(x) \, dx = \\ &= \int \delta(x)\mathcal{F}_k(H)(x+\omega) \, dx,\end{aligned}$$

but $\mathcal{F}_k(H)(x+\omega) = \frac{i}{x+\omega}e^{-ik(x+\omega)} - \frac{i}{x+\omega}\mathbb{1}(x+\omega)$ is not bounded by a tame polynomial at $x=0$ if $\omega \approx 0$ because of the terms $\frac{1}{\omega}$.

These exploratory examples lead us to the following

**Theorem 104.** *Let $f \in {}^\rho\mathcal{GC}^\infty(K)$, and assume that $\mathcal{F}_k(f)$ is bounded by a tame polynomial at $\omega \in {}^\rho\widetilde{\mathbb{R}}^n$. Then $\mathcal{F}(f\cdot\mathbb{1})(\omega) = \mathcal{F}_k(f)(\omega)$.*

*Proof.* It suffices to apply Thm. 101.1:

$$\begin{aligned}\mathcal{F}(f\cdot\mathbb{1})(\omega) &= \int f(x)\mathcal{F}(\delta)(x)e^{-ix\cdot\omega}\,\mathrm{d}x = \\ &= \int \delta(x)\mathcal{F}_k\left(f\cdot e^{-i(-)\cdot\omega}\right)(x)\,\mathrm{d}x = \\ &= \int \delta(x)\mathcal{F}_k\left(f\right)(x+\omega)\,\mathrm{d}x = \mathcal{F}_k\left(f\right)(\omega).\end{aligned}$$

□

Since $\frac{\partial}{\partial x_j}\left[\mathcal{F}_k(f)\right](\omega) = -i\mathcal{F}_k(x_j f)(\omega)$, we have the following sufficient condition for $\mathcal{F}_k(f)$ being bounded by a tame polynomial at $\omega \in {}^\rho\widetilde{\mathbb{R}}^n$:

**Theorem 105.** *Let $b \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ be a large infinite number, and let $f \in {}^\rho\mathcal{GC}^\infty(K)$ be uniformly bounded by a tame polynomial at $K$, i.e.*

$$\exists M, c \in {}^\rho\widetilde{\mathbb{R}}\,\forall y \in K\,\forall j \in \mathbb{N}: \ \left|\mathrm{d}^j f(y)\right| \le M\cdot c^j, \quad \frac{b}{c} \text{ is a large infinite number.} \tag{6.7.1}$$

*Then for all $\omega \in {}^\rho\widetilde{\mathbb{R}}^n$, the HFT $\mathcal{F}_k(f)$ is bounded by a tame polynomial at $\omega$. In particular, every $f \in \mathcal{S}(\mathbb{R}^n)$ satisfies condition (6.7.1), and hence*

$$\mathcal{F}(f) = \mathcal{F}(f\cdot\mathbb{1}) = \iota^b_{\mathbb{R}^n}(\hat{f}), \tag{6.7.2}$$

*where $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ is the classical FT of $f$.*

*Proof.* We have

$$\begin{aligned}\left|\mathrm{d}^j\mathcal{F}_k(f)(\omega)\right| &\le \max_{h\le j}\left|\mathcal{F}_k(x_h f)(\omega)\right| \le \max_{h\le j}\int_K \left|x_h f(x)\right|\,\mathrm{d}x \le \\ &\le Mc^j\max_{h\le j}\int_K |x_h|\,\mathrm{d}x =: \bar{M}c^j.\end{aligned}$$

If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\left|\mathrm{d}^j f(y)\right| \in \mathbb{R}$, so that if $b \ge \mathrm{d}\rho^{-r}$, $r \in \mathbb{R}_{>0}$, it suffices to take $c = \mathrm{d}\rho^{-r+s}$ where $0 < s < r$ to have that (6.7.1) holds. The last equality in (6.7.2) is equivalent to say that $\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\omega}\,\mathrm{d}x = \int_K f(x)e^{-ix\cdot\omega}\,\mathrm{d}x$, which can be proved as for the Gaussian, see Lem. 99. □

We can now consider the relations between $\iota^b_{\mathbb{R}^n}(\hat{T})$ and $\mathcal{F}_k(\iota^b_{\mathbb{R}^n}(T))$ when $T \in \mathcal{S}'(\mathbb{R}^n)$. A first trivial link is given by the so-called *equality in the sense of generalized tempered distributions*: For all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, from (6.3.4) we have

$$\int \iota^b_{\mathbb{R}^n}(\hat{T})\varphi = \langle\hat{T},\varphi\rangle = \langle T,\hat{\varphi}\rangle = \int \iota^b_{\mathbb{R}^n}(T)\hat{\varphi}.$$

Using the previous Thm. 105 we get $\hat{\varphi} = \mathcal{F}(\varphi)$ (identifying a smooth function with its embedding). Thereby

$$\int \iota^b_{\mathbb{R}^n}(\hat{T})\varphi = \int \iota^b_{\mathbb{R}^n}(T)\mathcal{F}(\varphi) = \int \mathcal{F}\left(\iota^b_{\mathbb{R}^n}(T)\right)\varphi \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{6.7.3}$$

In Colombeau's theory, this relation is usually written $\iota^b_{\mathbb{R}^n}(\hat{T}) =_{\text{g.t.d.}} \mathcal{F}\left(\iota^b_{\mathbb{R}^n}(T)\right)$.

In the following result, we give a sufficient condition to have a pointwise equality between the HFT of the finite part of $\iota^b_{\mathbb{R}^n}(T)$ and $\hat{T}$:

**Theorem 106.** *Let $b \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ be a large infinite number and $T \in \mathcal{S}'(\mathbb{R}^n)$. Assume that $\mathcal{F}(\iota^b_{\mathbb{R}^n}(T))$ is bounded by a tame polynomial at $\omega \in {}^\rho\widetilde{\mathbb{R}}^n$. Then*

$$\mathcal{F}_k(\iota^b_{\mathbb{R}^n}(T))(\omega) = \mathcal{F}(\iota^b_{\mathbb{R}^n}(T)\cdot \mathbb{1})(\omega) = \iota^b_{\mathbb{R}^n}(\hat{T})(\omega).$$

*Proof.* For simplicity of notation, we use $\iota := \iota^b_{\mathbb{R}^n}$. Using Thm. 104, we have

$$\mathcal{F}\left(\iota(T)\cdot \mathbb{1}\right)(\omega) = \mathcal{F}_k(\iota(T))(\omega).$$

Let $\psi(x) = [\psi_\varepsilon(x_\varepsilon)]$ *be an* $n$-dimensional Colombeau mollifier defined by $b$, and set $K_\varepsilon := [-k_\varepsilon, k_\varepsilon]^n$; we have

$$\begin{aligned}
\mathcal{F}_k\left(\iota(T)\right)(\omega) &= \left[\int_{K_\varepsilon} \langle T(y), \psi_\varepsilon(x-y)\rangle e^{-ix\cdot\omega_\varepsilon}\,\mathrm{d}x\right] = \\
&= \left[\langle T(y), \int \psi_\varepsilon(x-y) e^{-ix\cdot\omega_\varepsilon}\,\mathrm{d}x\rangle\right] = \\
&= \left[\langle T(y), \widehat{y\oplus\psi_\varepsilon}(\omega_\varepsilon)\rangle\right] = \\
&= \left[\langle \hat{T}(y), \left(y\oplus\psi_\varepsilon\right)(\omega_\varepsilon)\rangle\right] = \\
&= \left[\langle \hat{T}(y), \psi_\varepsilon(\omega_\varepsilon - y)\rangle\right] = \iota(\hat{T})(\omega).
\end{aligned}$$

□

### 6.7.1 Fourier transform in the Colombeau's setting

Only in this section we assume a very basic knowledge of Colombeau's theory.

Assume that $\Omega \subseteq \mathbb{R}^n$ be an open set. The algebra $\mathcal{G}^s_\tau(\Omega)$ of tempered generalized functions was introduced by J.F. Colombeau in [13], in order to develop a theory of Fourier transform. Since then, there has been a rapid development of the Fourier analysis, regularity theory and micro-local analysis in this setting.

**Definition 107.** The $\mathcal{G}^s_\tau(\Omega)$ algebra of *Colombeau tempered GF* (trivially generalized by using an arbitrary gauge $\rho$) is defined as follows:

$$\mathcal{E}^s_\tau(\Omega) := \Big\{ (u_\varepsilon) \in \left(\mathcal{C}^\infty(\Omega)\right)^I \mid \forall \alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} : \\ \sup_{x\in\Omega} (1+|x|)^{-N} \left|\partial^\alpha u_\varepsilon(x)\right| = O(\rho_\varepsilon^{-N}) \Big\},$$

$$\mathcal{N}_{\tau}^{s}\left(\Omega\right):=\Big\{(u_{\varepsilon})\in\left(\mathcal{C}^{\infty}(\Omega)\right)^{I}\mid\forall\alpha\in\mathbb{N}^{n}\,\exists p\in\mathbb{N}\,\forall m\in\mathbb{N}:$$

$$\sup_{x\in\Omega}\left(1+|x|\right)^{-p}|\partial^{\alpha}u_{\varepsilon}\left(x\right)|=O(\rho_{\varepsilon}^{m})\Big\},$$

$$\mathcal{G}_{\tau}^{s}(\Omega):=\mathcal{E}_{\tau}^{s}(\Omega)/\mathcal{N}_{\tau}^{s}(\Omega).$$

Colombeau tempered GF can be embedded as GSF, at least if the internal set $[\Omega]$ is sharply bounded:

**Theorem 108.** *Let $\Omega\subseteq\mathbb{R}^{n}$ be an open set such that $[\Omega]$ is sharply bounded. A Colombeau tempered GF $u=(u_{\varepsilon})+\mathcal{N}_{\tau}^{s}(\Omega)\in\mathcal{G}_{\tau}^{s}(\Omega)$ defines a GSF $u:[x_{\varepsilon}]\in[\Omega]\longrightarrow[u_{\varepsilon}(x_{\varepsilon})]\in{}^{\rho}\widetilde{\mathbb{C}}$. This assignment provides a bijection of $\mathcal{G}_{\tau}^{s}(\Omega)$ onto the space defined by $u\in{}^{\rho}\mathcal{GC}_{\tau}^{\infty}([\Omega])$ if and only if $u\in{}^{\rho}\mathcal{GC}^{\infty}([\Omega])$ and*

$$\forall\alpha\in\mathbb{N}^{n}\,\exists N\in\mathbb{N}\,\forall x\in[\Omega]:\ |\partial^{\alpha}u\left(x\right)|\le\frac{\left(1+|x|\right)^{N}}{\mathrm{d}\rho^{N}}.$$

Integration of a CGF $u=[u_{\varepsilon}]\in\mathcal{G}^{s}(\Omega)$ over a standard $K\Subset\Omega$ can be defined $\varepsilon$-wise as $\int_{K}u\left(x\right)\,\mathrm{d}x:=\left[\int_{K}u_{\varepsilon}\left(x\right)\,\mathrm{d}x\right]\in{}^{\rho}\widetilde{\mathbb{R}}$. Similarly we can proceed for $\int_{\Omega}u$ if $u$ is compactly supported and $\Omega\subseteq\mathbb{R}^{n}$ is an arbitrary open set. On the other hand, to define the Fourier transform, we have to integrate tempered CGF on the entire $\mathbb{R}^{n}$. Using this integration of CGF, this is accomplished by multiplying the generalized function by a so-called *damping measure* $\varphi$, see e.g. [39]:

**Definition 109.** Let $\varphi\in\mathcal{S}(\mathbb{R}^{n})$ with $\int_{\mathbb{R}^{n}}\varphi=1$, $\int_{\mathbb{R}^{n}}x^{\alpha}\varphi(x)\,\mathrm{d}x=0$ for all $\alpha\in\mathbb{N}^{n}\setminus\{0\}$, and set $\varphi_{\varepsilon}(x):=\rho_{\varepsilon}\odot\varphi(x)=\rho_{\varepsilon}^{-n}\varphi(\rho_{\varepsilon}^{-1}x)$. Let $u=[u_{\varepsilon}]\in\mathcal{G}_{\tau}(\mathbb{R}^{n})$, then $u_{\hat{\varphi}}:=[u_{\varepsilon}\widehat{\varphi_{\varepsilon}}]$, $\int_{\mathbb{R}^{n}}u(x)\,\mathrm{d}_{\hat{\varphi}}x:=\int_{\mathbb{R}^{n}}u_{\hat{\varphi}}\,\mathrm{d}x=\left[\int_{\mathbb{R}^{n}}u_{\varepsilon}(x)\widehat{\varphi_{\varepsilon}}(x)\,\mathrm{d}x\right]\in\widetilde{\mathbb{C}}$, where $\widehat{\varphi_{\varepsilon}}$ denotes the classical FT. In particular,

$$\mathcal{F}_{\hat{\varphi}}(u)(\omega):=\int_{\mathbb{R}^{n}}e^{-ix\omega}u(x)\,\mathrm{d}_{\hat{\varphi}}x=\left[\int_{\mathbb{R}^{n}}e^{-ix\omega}u_{\varepsilon}(x)\widehat{\varphi_{\varepsilon}}(x)\,\mathrm{d}x\right]$$

$$\mathcal{F}_{\hat{\varphi}}^{*}(u)(x):=(2\pi)^{-n}\int_{\mathbb{R}^{n}}e^{-ix\omega}u(\omega)\,\mathrm{d}_{\hat{\varphi}}\omega=\left[(2\pi)^{-n}\int_{\mathbb{R}^{n}}e^{-ix\omega}u_{\varepsilon}(\omega)\widehat{\varphi_{\varepsilon}}(\omega)\,\mathrm{d}\omega\right].$$

As a result, although this notion of Fourier transform in the Colombeau setting shares several properties with the classical one, it lacks e.g. the Fourier inversion theorem, which holds only at the level of equality in the sense of generalized tempered distributions [14, 16, 52], see also (6.7.3). See also [67] for a Paley-Wiener like theorem. In other words, we only have e.g. $\mathcal{F}_{\hat{\varphi}}(\partial^{\alpha}u)=_{\text{g.t.d.}}i^{|\alpha|}\omega^{\alpha}\mathcal{F}_{\hat{\varphi}}(u)$, $i^{|\alpha|}\mathcal{F}^{*}{}_{\hat{\varphi}}(\partial^{\alpha}u)=_{\text{g.t.d.}}x^{\alpha}\mathcal{F}^{*}{}_{\hat{\varphi}}(u)$, $\mathcal{F}_{\hat{\varphi}}\mathcal{F}^{*}{}_{\hat{\varphi}}u=_{\text{g.t.d.}}\mathcal{F}^{*}{}_{\hat{\varphi}}\mathcal{F}_{\hat{\varphi}}u$, where $\mathcal{F}_{\hat{\varphi}}(u)$ denotes the Fourier transform with respect to the damping measure. Moreover $\langle\iota_{\mathbb{R}}(\hat{T}),\psi\rangle\approx\langle\mathcal{F}_{\hat{\varphi}}\iota_{\mathbb{R}}(T),\psi\rangle$ for all $T\in\mathcal{S}'(\mathbb{R})$ and all $\psi\in\mathcal{S}(\mathbb{R})$, where $\iota_{\mathbb{R}}(T)$ is the embedding of Thm. 65. Intuitively, one could say that the use of the multiplicative damping measure introduces a perturbation of infinitesimal frequencies that inhibit several results that, on the contrary, hold for the HFT. On the other hand, HFT lies on a better integration theory that allows to integrate any GSF on the functionally compact set $K$. The only known possibility to obtain a strict Fourier inversion theorem in Colombeau's theory, is the approach used by [53], where smoothing kernels are used as index set (instead of the simpler $\varepsilon \in I$) and therefore the knowledge of infinite dimensional calculus in convenient vector spaces is needed. Unfortunately, the latter approach is not widely known, even in the community of CGF, and it can be considered as technically involved.

Finally, the following result links the HFT with the FT of tempered CGF as defined above.

**Theorem 110.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $[\Omega]$ is sharply bounded, and let $u \in {}^{\rho}\mathcal{GC}^{\infty}_{\tau}([\Omega])$ be a tempered CGF (identified with the corresponding GSF). Finally, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a dumping measure. Then*

$$\mathcal{F}_{\hat{\varphi}}(u) = \mathcal{F}\left[u \cdot \hat{\varphi}((-) \cdot \mathrm{d}\rho)\right] = \mathcal{F}\left[u \cdot \mathcal{F}(\varphi)((-) \cdot \mathrm{d}\rho)\right].$$

*Proof.* Def. (109) yields

$$\begin{aligned}
\mathcal{F}_{\hat{\varphi}}(u)(\omega) &= \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\omega}\, \mathrm{d}_{\hat{\varphi}} x = \\
&= \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\omega} \widehat{\mathrm{d}\rho \odot \varphi}(x)\, \mathrm{d}x = \\
&= \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\omega} \left(\mathrm{d}\rho \diamond \hat{\varphi}\right)(x)\, \mathrm{d}x = \\
&= \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\omega} \hat{\varphi}(\mathrm{d}\rho \cdot x)\, \mathrm{d}x = \\
&= \mathcal{F}\left[u \cdot \hat{\varphi}((-) \cdot \mathrm{d}\rho)\right](\omega) = \\
&= \mathcal{F}\left[u \cdot \mathcal{F}(\varphi)((-) \cdot \mathrm{d}\rho)\right](\omega),
\end{aligned}$$

where, in the last equality, we applied (6.7.2). ∎

## 6.8 Examples and applications

In this section we present an initial study of possible applications of HFT. Our aim is mainly to highlight the new potentialities of the theory.

### 6.8.1 Applications of HFT to ordinary differential equations

1) We first consider the following, apparently trivial but actually meaningful, example:

$$y' = y,\ \ y(0) = c,\ \ y \in {}^{\rho}\mathcal{GC}^{\infty}\left([-k, k]\right),\ \ c \in {}^{\rho}\widetilde{\mathbb{R}}, \tag{6.8.1}$$

where $k = -\log(\mathrm{d}\rho)$. Since we do not impose limitations on the initial value $c$, this simple example clearly shows the possibilities of the HFT to manage non tempered generalized functions. Applying the HFT to both sides of (6.8.1) and using the derivation formula (6.5.1), we get

$$\mathcal{F}_k(y) = \Delta_{1k} y + i\omega \mathcal{F}_k(y). \tag{6.8.2}$$

Set for simplicity $C(\omega) := \Delta_{1k} y(\omega) = y(k)e^{-ik\omega} - y(-k)e^{ik\omega}$ and note that the function $C$ does not depend on the whole function $y$ but only on the two values $y(\pm k)$. We get $\mathcal{F}_k(y)(\omega) = \frac{C(\omega)}{1-i\omega}$, and applying the inverse HFT Thm. 101.2, we obtain

$$y = \mathcal{F}_k^{-1}|_K \left( \left. \frac{C(\omega)}{1-i\omega} \right|_K \right). \tag{6.8.3}$$

Using the initial condition in (6.8.1), we have

$$y(0) = \mathcal{F}_k^{-1}\left(\frac{C(\omega)}{1-i\omega}\right)(0) = \int_{-k}^{k} \frac{C(\omega)}{1-i\omega}\,\mathrm{d}\omega = c. \tag{6.8.4}$$

Clearly, e.g. by separation of variables, (6.8.1) necessarily yields $y(x) = ce^x$ for all $x \in [-k,k]$. Therefore, $y(k) = ce^{-\log \mathrm{d}\rho} = \frac{c}{\mathrm{d}\rho}$, $y(-k) = ce^{\log \mathrm{d}\rho} = c\mathrm{d}\rho$ and $C(\omega) = c\mathrm{d}\rho^{i\omega-1} - c\mathrm{d}\rho^{-i\omega+1}$ because $\mathrm{d}\rho^{i\omega} = e^{-ik\omega}$. Vice versa, if (6.8.3) holds, using again Thm. 101.2, we obtain $\mathcal{F}_k|_K(y)(\omega) = \frac{C(\omega)}{1-i\omega}$ for all $\omega \in K$; from the differentiation formula (6.5.1) we hence get $\mathcal{F}_k(y)(\omega) - \mathcal{F}_k(y')(\omega) + C(\omega) = C(\omega)$. Another application of the inversion Thm. 101.2 yields $y' = y$ in $K$. We have hence proved that $y$ satisfies (6.8.1) if and only if

$$y(x) = ce^x = \mathcal{F}_k^{-1}|_K \left( c \left. \frac{\mathrm{d}\rho^{i\omega-1} - \mathrm{d}\rho^{-i\omega+1}}{1-i\omega} \right|_K \right)(x) \quad \forall x \in K. \tag{6.8.5}$$

We finally underscore that:

(a) In the classical theory, the lacking of the term $C(\omega)$ does not allow to obtain the non-tempered solution for $c \neq 0$: in other words, if $c \neq 0$, then (6.8.4) implies that $C \neq 0$.

(b) In the previous deduction, it is clearly important that the HFT can be applied to all the GF of the space ${}^\rho\mathcal{GC}^\infty(K)$.

(c) If we missed the restriction to $K$ in (6.8.3), we would wrongly obtained that $y = ce^x \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}})$, which necessarily implies $c = 0$ because the exponential function is not defined on the whole ${}^\rho\widetilde{\mathbb{R}}$.

(d) Compare (6.8.5) with Example 96 to note that if $c \geq r \in \mathbb{R}_{>0}$, then in (6.8.5) we are considering the inverse HFT of a GSF which always takes infinite values for all finite $\omega$. Clearly, this strongly motivates the use of a non-Archimedean framework for this type of problems.

(e) All our results, in particular the inversion Thm. 101.2, hold for an arbitrary infinite number $k$. In this particular case, we considered $k$ of logarithmic type to get moderateness of the exponential function.

2) Let us consider an arbitrary $n$-th order constant (generalized) coefficient ODE

$$a_n y^{(n)} + \ldots a_1 y^{(1)} + a_0 y = h,\ y, h \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k]),\ a_j \in {}^{\rho}\widetilde{\mathbb{R}},\ n \in \mathbb{N}_{\geq 1}. \tag{6.8.6}$$

Note that simply assuming to have a solution $y$ defined on the infinite interval $[-k,k]$ already yields an implicit limitation on the coefficients $a_j \in {}^{\rho}\widetilde{\mathbb{R}}$. In fact, the equation $y' - \frac{1}{\mathrm{d}\rho} y = 0$ has solution $y(x) = y(0)e^{x/\mathrm{d}\rho}$, which is defined only if $x \leq -N\mathrm{d}\rho \log \mathrm{d}\rho \approx 0$ for some $N \in \mathbb{N}$. By applying the HFT to both sides of equation (6.8.6), the differential equation is converted into the algebraic equation

$$P(\omega)\, \mathcal{F}_k(y) + C(\omega) = \mathcal{F}_k(h), \tag{6.8.7}$$

where

$$P(\omega) = \sum_{j=0}^{n} a_j (i\omega)^j,$$

and $C(\omega)$ is the sum of all the extra terms in Thm. 97.8, which in this case becomes

$$C(\omega) := \sum_{j=1}^{n} a_j \cdot \sum_{p=1}^{j} (i\omega)^{j-p} \Delta_{1k} y^{(p-1)}(\omega) \quad \forall \omega \in K.$$

Note that the function $C$ depends only on the points $y^{(p)}(\pm k)$ for $p = 0, \ldots, n-1$ and not on the whole function $y$. Assuming that $P(\omega)$ is invertible for all $\omega \in K$, from (6.8.7) and the inversion Thm. 101.2, we get

$$y = \mathcal{F}_k^{-1}|_K \left( \left. \frac{\mathcal{F}_k(h) - C}{P} \right|_K \right). \tag{6.8.8}$$

Proceeding as in the previous example, i.e. using again the inversion Thm. 101.2 and the differentiation formula (6.5.1), we can actually prove that (6.8.8) is equivalent to (6.8.6). For a generalization to GSF of the usual results about $n$-th order constant generalized coefficient ODE, see [48].

3) A simple example of non-constant coefficient linear ODE is given by the Airy equation

$$u''(x) - x \cdot u(x) = 0,\ u \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k], {}^{\rho}\widetilde{\mathbb{R}}). \tag{6.8.9}$$

By applying the derivative formulas Thm. 97.8 and Thm. 97.9, we get

$$-\omega^2 \mathcal{F}_k(u) + i\omega \Delta_{1k} u + \Delta_{1k} u' - i \mathcal{F}_k(u)' = 0$$

or dividing both sides by $i$

$$i\omega^2\mathcal{F}_k(u)+\omega\Delta_{1k}u-i\Delta_{1k}u'-\mathcal{F}_k'(u)=0$$

Let us now set $C(\omega):=\omega\Delta_{1k}u(\omega)-i\Delta_{1k}u'(\omega)$, $\forall\omega\in K$. Note, once again, that the function $C$ does not depend on the whole functions $u$ and $u'$ but only on the points $u(\pm k)$ and $u'(\pm k)$.

$$\mathcal{F}_k'(u)-i\omega^2\mathcal{F}_k(u)=C. \tag{6.8.10}$$

Equation (6.8.10) is a first order non-constant coefficient, non-homogeneous generalized ODE with respect to the variable $\omega$. We can solve it e.g. by considering the integrating factor $\mu(\omega):=e^{\int_0^\omega -iz^2\,\mathrm{d}z}=e^{-i\frac{\omega^3}{3}}$. Then, the solution of (6.8.10) is given by

$$\mathcal{F}_k(u)(\omega)=\frac{\int_0^\omega\mu(z)C(z)\,\mathrm{d}z+c}{\mu(\omega)}=\frac{\int_0^\omega e^{-\frac{iz^3}{3}}C(z)\,\mathrm{d}z+c}{e^{-\frac{i\omega^3}{3}}}\quad\forall\omega\in{}^\rho\widetilde{\mathbb{R}},$$

where $c:=\mathcal{F}_k(u)(0)\in{}^\rho\widetilde{\mathbb{R}}$. Finally, we apply the inversion Thm. 101.2 and substitute $C(\omega)$ to recover the original function

$$\begin{aligned}
u(x)&=\mathcal{F}_k^{-1}|_K\left(\left.\frac{\int_0^\omega e^{-\frac{iz^3}{3}}C(z)\,\mathrm{d}z+c}{e^{-\frac{i\omega^3}{3}}}\right|_K\right)(x)=\\
&=\mathcal{F}_k^{-1}|_K\left(\left.\frac{\int_0^\omega e^{-\frac{iz^3}{3}}C(z)\,\mathrm{d}z}{e^{-\frac{i\omega^3}{3}}}\right|_K\right)(x)+\frac{c}{\pi}\int_0^k\cos\left(\frac{\omega^3}{3}+\omega x\right)\mathrm{d}\omega\\
&=\frac{1}{\pi}\int_0^k\cos\left(\omega x+\frac{\omega^3}{3}\right)\int_0^\omega e^{-i\left(kz+\frac{z^3}{3}\right)}\left(zu(k)-iu'(k)\right)\mathrm{d}z\mathrm{d}\omega-\\
&-\frac{1}{\pi}\int_0^k\cos\left(\omega x+\frac{\omega^3}{3}\right)\int_0^\omega e^{-i\left(-kz+\frac{z^3}{3}\right)}\left(zu(-k)-iu'(-k)\right)\mathrm{d}z\mathrm{d}\omega\\
&+\frac{c}{\pi}\int_0^k\cos\left(\frac{\omega^3}{3}+\omega x\right)\mathrm{d}\omega\quad\forall x\in K.
\end{aligned}$$

If we assume that $u(\pm k)\approx 0$ and $u'(\pm k)\approx 0$, then $C(z)\approx 0$ for all $z\in{}^\rho\widetilde{\mathbb{R}}$ and we get the first Airy function up to infinitesimals $u(x)\approx c\cdot\mathrm{Ai}(x)$. Therefore, if $u(\pm k)\not\approx 0$ or $u'(\pm k)\not\approx 0$ and $c=0$, we must get, up to infinitesimals, a multiple of the second Airy function (see e.g. [1])

$$\exists d\in{}^\rho\widetilde{\mathbb{R}}:\ u(x)\approx\mathrm{Bi}(x)=\frac{d}{\pi}\int_0^{+\infty}\left\{\exp\left(-\frac{t^3}{3}+xt\right)+\sin\left(\frac{t^3}{3}+xt\right)\right\}\mathrm{d}t.$$

We explicitly recall that $\mathrm{Bi}(x)$ is of exponential order as $x\to+\infty$ and hence it is not a tempered distribution, so that classically we miss this solution.

### 6.8.2 Applications of HFT to partial differential equations

***The wave equation***

Let us consider the one dimensional (generalized) wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},\ \ c \in {}^{\rho}\widetilde{\mathbb{R}},\ \ u \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k] \times {}^{\rho}\widetilde{\mathbb{R}}_{\ge 0}), \tag{6.8.11}$$

where $c$ is invertible, and subject to the boundary conditions at $t = 0$ and $x = \pm k$

$$u(-,0) = f, \quad \partial_t u(-,0) = g, \tag{6.8.12}$$

$$u(\pm k,-) = F_{\pm}, \quad \partial_x u(\pm k,-) = G_{\pm}. \tag{6.8.13}$$

Explicitly note that all the GSF $f,\ g \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k])$, $F_+$, $F_-$, $G_+$, $G_- \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}_{\ge 0})$ are completely arbitrary. As usual, we directly apply the HFT with respect to the variable $x$ to both sides and then apply Thm. 97.8 to the right hand side

$$\mathcal{F}_k\left(\frac{\partial^2 u}{\partial t^2}\right) = c^2 \mathcal{F}_k\left(\frac{\partial^2 u}{\partial x^2}\right),$$

$$\frac{\partial^2 \mathcal{F}_k(u)}{\partial t^2} = -c^2\omega^2 \mathcal{F}_k(u) + i\omega \Delta_{1k} u + \Delta_{1k}(\partial_x u).$$

Note that also the $\Delta_{1k}$-terms refer to the variable $x$, but the result is a function of $t$. Set $C(\omega,t) := i\omega\Delta_{1k}(u(-,t)) + \Delta_{1k}(\partial_x u(-,t))$. The function $C$ does not depend on the whole functions $u$ and $\partial_x u$ but only on the functions of $t$: $u(\pm k,-) = F_{\pm}$ and $\partial_x u(\pm k,-) = G_{\pm}$:

$$C(\omega,t) = i\omega\left[F_+(t)e^{-ik\omega} - F_-(t)e^{ik\omega}\right] + G_+(t)e^{-ik\omega} - G_-(t)e^{ik\omega} \quad \forall \omega \in {}^{\rho}\widetilde{\mathbb{R}}. \tag{6.8.14}$$

Hence, we get

$$\frac{\partial^2 \mathcal{F}_k(u)}{\partial t^2}(\omega,-) + c^2\omega^2 \mathcal{F}_k(u)(\omega,-) = C(\omega,-) \quad \forall \omega \in {}^{\rho}\widetilde{\mathbb{R}}. \tag{6.8.15}$$

We obtain, for each fixed $\omega$, a constant (generalized) coefficient, non-homogeneous, second order ODE in the unknown $\mathcal{F}_k(u)(\omega,-)$. Clearly, (6.8.15) already highlights a difference with the classical FT, where $C = 0$. To solve equation (6.8.15), we can use the standard method of variation of parameters to get

$$\begin{aligned}\mathcal{F}_k(u)(\omega,t) = {} & d_2(\omega)tS(c\omega t) + d_1(\omega)\cos(c\omega t) + \\ & + tS(c\omega t)\int_1^t C(\omega,s)\cos(c\omega s)\,\mathrm{d}s - \qquad (6.8.16) \\ & - \cos(c\omega t)\int_1^t sC(\omega,s)S(c\omega s)\,\mathrm{d}s, \qquad (6.8.17)\end{aligned}$$

$$S(z) := \frac{1}{2}\int_{-1}^{1}\cos(zt)\,\mathrm{d}t. \tag{6.8.18}$$

More precisely, in the previous step we applied the general theory of linear constant generalized coefficient, non-homogeneous ODE developed in [48], which generalizes the classical theory proving that the space of all the solutions is a 2-dimensional ${}^{\rho}\widetilde{\mathbb{R}}$-module, generated in this case by $tS(c\omega t)$ and $\cos(c\omega t)$, and translated by a particular solution of (6.8.15). Explicitly note that every functions in (6.8.17) is a smooth function or a GSF; in particular, $S(z)$ is the smooth extension of $\frac{\sin(z)}{z}$. We also note that the functions $d_1$, $d_2$ satisfy

$$\begin{aligned}
d_1(\omega) &= \mathcal{F}_k(f)(\omega) - \int_0^1 sC(\omega,s)S(c\omega s)\,\mathrm{d}s \\
d_2(\omega) &= \mathcal{F}_k(g)(\omega) + \int_0^1 C(\omega,s)\cos(c\omega s)\,\mathrm{d}s.
\end{aligned}$$

They hence depend on all the functions of the boundary conditions. Finally, applying the inversion Thm. 101.2 we get

$$\begin{aligned}
u(x,t) = {}&\mathcal{F}_k^{-1}\big|_K \left( d_2(\omega)tS(c\omega t) + d_1(\omega)\cos(c\omega t)|_K \right)(x,t)+ \\
&+ \mathcal{F}_k^{-1}\big|_K \left( tS(c\omega t)\int_1^t C(\omega,s)\cos(c\omega s)\,\mathrm{d}s\bigg|_K \right)(x,t)- \\
&- \mathcal{F}_k^{-1}\big|_K \left( \cos(c\omega t)\int_1^t sC(\omega,s)S(c\omega s)\,\mathrm{d}s\bigg|_K \right)(x,t).
\end{aligned}$$

Following the usual calculations, the first summand yields the d'Alembert formula

$$\begin{aligned}
u(x,t) = {}&\frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')\,\mathrm{d}x'+ \\
&+ \mathcal{F}_k^{-1}\big|_K \left( tS(c\omega t)\int_1^t C(\omega,s)\cos(c\omega s)\,\mathrm{d}s\bigg|_K \right)(x,t)- \\
&- \mathcal{F}_k^{-1}\big|_K \left( \cos(c\omega t)\int_1^t sC(\omega,s)S(c\omega s)\,\mathrm{d}s\bigg|_K \right)(x,t). \qquad (6.8.19)
\end{aligned}$$

Given $f$, $g \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k])$ and $F_{\pm}$, $G_{\pm} \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}_{\geq 0})$, we can define $u(x,t)$ using (6.8.19) and reverse all the steps above to get a global solution of the wave equation subject to the given boundary conditions. This proves the following

**Theorem 111.** *Given $f$, $g \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k])$ and $F_{\pm}$, $G_{\pm} \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}_{\geq 0})$, there exists one and only one solution $u \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k]\times{}^{\rho}\widetilde{\mathbb{R}}_{\geq 0})$ of the wave equation subject to the boundary conditions (6.8.12). In particular, if $F_{\pm} = G_{\pm} = 0$, we get the usual solution, and if in addition we take $f = 0$, $g = \delta$, we get the wave kernel $u(x,t) = \frac{1}{2c}\left[H(x+ct) - H(x-ct)\right]$.*

### *The Heat equation*

Let us consider the one dimensional (generalized) heat equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad u \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k]\times{}^{\rho}\widetilde{\mathbb{R}}_{\geq 0}). \qquad (6.8.20)$$

where $a \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, $t \le \frac{N}{a^2k^2}\log(\mathrm{d}\rho)$, $N \in \mathbb{N}_{>0}$ and subject to the boundary conditions at $t = 0$ and $x = \pm k$

$$u(-,0) = f, \tag{6.8.21}$$
$$u(\pm k,-) = F_{\pm}, \quad \partial_x u(\pm k,-) = G_{\pm}. \tag{6.8.22}$$

Note, once again, that all the GSF $f \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k])$, $F_+$, $F_-$, $G_+$, $G_- \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}_{\ge 0})$ are completely arbitrary. Applying, as usual, the HFT with respect to the variable $x$ to both sides of (6.8.20) and Thm. 97.8 we get

$$\frac{\partial \mathcal{F}_k(u)}{\partial t} = -a^2\omega^2\mathcal{F}_k(u) + i\omega\Delta_{1k}u + \Delta_{1k}(\partial_x u).$$

For all $\omega \in {}^{\rho}\widetilde{\mathbb{R}}$, set

$$\begin{aligned} C(\omega,t) &:= i\omega\Delta_{1k}(u(-,t)) + \Delta_{1k}(\partial_x u(-,t)) = \\ &= i\omega\left[F_+(t)e^{-ik\omega} - F_-(t)e^{ik\omega}\right] + G_+(t)e^{-ik\omega} - G_-(t)e^{ik\omega}. \end{aligned}$$

Therefore, we get

$$\frac{\partial \mathcal{F}_k(u)}{\partial t}(\omega,-) + a^2\omega^2\mathcal{F}_k(u)(\omega,-) = C(\omega,-) \ \ \forall \omega \in {}^{\rho}\widetilde{\mathbb{R}}. \tag{6.8.23}$$

Solving (6.8.23) with the integrating factor $\mu(t) := e^{a^2\omega^2\int_0^t \mathrm{d}t} = e^{a^2\omega^2 t}$ (which is well-defined if $\omega \in K$ because we assumed that $t \le \frac{N}{a^2k^2}\log(\mathrm{d}\rho)$), we have

$$\mathcal{F}_k(u)(\omega,t) = \frac{\int_0^t e^{a^2\omega^2 t}C(\omega,t)\,\mathrm{d}t + c(\omega)}{e^{a^2\omega^2 t}},$$

where $c(\omega) := \mathcal{F}_k(u)(\omega,0) = \mathcal{F}_k(f)(\omega) \in {}^{\rho}\widetilde{\mathbb{R}}$, so that

$$\begin{aligned} \mathcal{F}_k(u)(\omega,t) &= e^{-a^2\omega^2 t}\int_0^t e^{a^2\omega^2 t}C(\omega,t)\,\mathrm{d}t + \mathcal{F}_k(f)(\omega)e^{-a^2\omega^2 t} = \\ &= e^{-a^2\omega^2 t}\int_0^t e^{a^2\omega^2 t}C(\omega,t)\,\mathrm{d}t + \mathcal{F}_k(f)(\omega)\mathcal{F}\left(\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\right)(\omega,t) =: \\ &=: e^{-a^2\omega^2 t}\int_0^t e^{a^2\omega^2 t}C(\omega,t)\,\mathrm{d}t + \mathcal{F}_k(f)(\omega)\mathcal{F}\left(H_t^a(x)\right)(\omega,t) = \\ &= e^{-a^2\omega^2 t}\int_0^t e^{a^2\omega^2 t}C(\omega,t)\,\mathrm{d}t + \mathcal{F}_k\left(f * H_t^a\right)(\omega,t), \end{aligned}$$

where $H_t^a(x) := \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$ is the heat kernel (which, in our setting, is a compactly supported GSF). Finally, applying the inversion Thm. 101.2 and the convolution formula Thm. 97.10 we get

$$u(x,t) = (f * H_t^a)(x,t) + \mathcal{F}_k^{-1}|_K \left( e^{-a^2\omega^2 t} \int_0^t e^{a^2\omega^2 t} C(\omega,t)\, \mathrm{d}t \right)(x,t).$$

As usual, if $C(\omega,t)$ equals zero, we obtain the classical solution. We can reverse all the steps above to get a global solution of the heat equation subject to the given boundary conditions. This proves the following

**Theorem 112.** *Given $f \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k])$ and $F_{\pm}$, $G_{\pm} \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}_{\geq 0})$, there exists one and only one solution $u \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k] \times {}^{\rho}\widetilde{\mathbb{R}}_{\geq 0})$ of the heat equation subject to the boundary conditions (6.8.21). In particular, if $F_{\pm} = G_{\pm} = 0$, we get the usual solution, and if in addition we take $f = \delta$, then we get the heat kernel $u(x,t) = H_t^a(x) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$.*

***Laplace's equation***

Let us consider the one dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ , u \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k] \times [\frac{N}{k}\log \mathrm{d}\rho, \frac{N}{k}\log \mathrm{d}\rho]), \tag{6.8.24}$$

where $N \in \mathbb{N}_{>0}$, and subject to the boundary conditions at $y = 0$ and $x = \pm k$

$$u(-,0) = f, \quad \partial_y u(-,0) = 0, \tag{6.8.25}$$
$$u(\pm k, -) = F_{\pm}, \quad \partial_x u(\pm k, -) = 0, \tag{6.8.26}$$

where $f \in {}^{\rho}\mathcal{GC}^{\infty}([-k,k])$, $F_+$, $F_- \in {}^{\rho}\mathcal{GC}^{\infty}(Y)$ and $Y := [\frac{N}{k}\log \mathrm{d}\rho, \frac{N}{k}\log \mathrm{d}\rho] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$. Actually, we show this example only for the sake of completeness, but we present here only a preliminary study. By applying the HFT with respect to $x$ and applying Thm. 97.8, the problem is converted into

$$\frac{\partial^2 \mathcal{F}_k(u)}{\partial y^2} = \omega^2 \mathcal{F}_k(u) + i\omega\Delta_{1k} u + \Delta_{1k}(\partial_x u)$$

Set

$$\begin{aligned} C(\omega,y) :&= i\omega\Delta_{1k}(u(-,y)) + \Delta_{1k}(\partial_x u(-,y)) = \\ &= i\omega\left[F_+(y)e^{-ik\omega} - F_-(y)e^{ik\omega}\right]. \end{aligned} \tag{6.8.27}$$

Note explicitly that $\frac{C(\omega,y)}{\omega}$ is a GSF exactly because of our boundary condition $\partial_x u(\pm k, -) = 0$ (compare (6.8.27) with (6.8.14)). Hence, we get

$$\frac{\partial^2 \mathcal{F}_k(u)}{\partial y^2}(\omega,y) - \omega^2 \mathcal{F}_k(u)(\omega,y) = C(\omega,y), \ \forall \omega \in {}^{\rho}\widetilde{\mathbb{R}}, \tag{6.8.28}$$

whose general solution is

$$
\begin{aligned}
\mathcal{F}_k(u)(\omega, y) = d_1(\omega)e^{\omega y} + d_2(\omega)e^{-\omega y} - \\
- e^{-\omega y}\int_1^y \frac{e^{z\omega}}{2} i\left[F_+(z)e^{-ik\omega} - F_-(z)e^{ik\omega}\right] \,\mathrm{d}z + \\
+ e^{\omega y}\int_1^y \frac{e^{-z\omega}}{2} i\left[F_+(z)e^{-ik\omega} - F_-(z)e^{ik\omega}\right] \,\mathrm{d}z =: \\
=: d_1(\omega)e^{\omega y} + d_2(\omega)e^{-\omega y} + D(\omega, y)
\end{aligned}
$$

where the functions $d_1$, $d_2$ satisfy

$$
\begin{aligned}
\mathcal{F}_k(f)(\omega) &= d_1(\omega) + d_2(\omega) + D(\omega, 0) \\
0 &= \omega d_1(\omega) - \omega d_2(\omega),
\end{aligned}
$$

because $\partial_y u(-, 0) = 0$ and $\partial_y D(\omega, 0) = 0$. Since the set of invertible numbers in ${}^\rho\widetilde{\mathbb{R}}$ is dense in the sharp topology, we hence have

$$
d_1(\omega) = d_2(\omega) = \frac{1}{2}\left[\mathcal{F}_k(f)(\omega) - D(\omega, 0)\right].
$$

Note that $e^{\pm\omega y}$ is well defined for all $\omega \in K$ and all $y \in Y = [\frac{N}{k}\log \mathrm{d}\rho, \frac{N}{k}\log \mathrm{d}\rho]$. Finally, applying the inversion Thm. 101.2 we get

$$
\begin{aligned}
u(x, y) = \mathcal{F}_k^{-1}\big|_K \left(d_1(\omega)e^{\omega y} + d_2(\omega)e^{-\omega y}\big|_K\right)(x, y) + \\
+ \mathcal{F}_k^{-1}\big|_K \left(D(\omega, y)|_K\right)(x, y).
\end{aligned}
$$

**Theorem 113.** *Given $f \in {}^\rho\mathcal{GC}^\infty([-k, k])$ and $F_\pm \in {}^\rho\mathcal{GC}^\infty([\frac{N}{k}\log \mathrm{d}\rho, \frac{N}{k}\log \mathrm{d}\rho])$, $N \in \mathbb{N}_{>0}$, there exists one and only one solution $u \in {}^\rho\mathcal{GC}^\infty([-k, k]\times[\frac{N}{k}\log \mathrm{d}\rho, \frac{N}{k}\log \mathrm{d}\rho])$ of the Laplace's equation subject to the boundary conditions (6.8.25). In particular, if $F_\pm = 0$, we get the usual solution, and if in addition we take $f = \delta$, then $u(x, y) = \mathcal{F}_k^{-1}\big|_K \left(\mathbb{1}(\omega)\cosh(\omega y)|_K\right)(x, y)$.*

### 6.8.3 Applications to convolution equations.

Consider the following convolution equation in $y$

$$
h = g + f * y, \tag{6.8.29}
$$

where we assume that $y$, $h$, $g \in {}^\rho\mathcal{GC}^\infty(K)$ and $f \in {}^\rho\mathcal{GD}({}^\rho\widetilde{\mathbb{R}})$. As in the classical theory, we apply the convolution Thm. 97.10 to get

$$
\mathcal{F}_k(h) = \mathcal{F}_k(g) + \mathcal{F}(f)\,\mathcal{F}_k(y).
$$

Assuming that $\mathcal{F}(f)(\omega)$ is invertible for all $\omega \in K$, the inversion Thm. 101.2 yields

$$
y = \mathcal{F}_k^{-1}\big|_K \left(\left.\frac{\mathcal{F}_k(h) - \mathcal{F}_k(g)}{\mathcal{F}(f)}\right|_K\right).
$$

For example, to highlight the differences with the classical theory, let us consider the convolution equation $(\delta'+\delta)*y=\delta$ with $y(-1)=0$. We have $h=\delta$, $g=0$ and $f=\delta'+\delta$ so that $\mathcal{F}(f)=i\omega\mathbb{1}+\mathbb{1}$, where as usual $\mathbb{1}=\mathcal{F}_k(\delta)$. Since $\mathbb{1}(\omega)\in{}^{\rho}\widetilde{\mathbb{R}}$ for all $\omega$, the quantity $i\omega\mathbb{1}(\omega)+\mathbb{1}(\omega)$ is always invertible, and we obtain

$$y=\mathcal{F}^{-1}\big|_K\left(\left.\frac{\mathbb{1}}{i\omega\mathbb{1}+\mathbb{1}}\right|_K\right).$$

It is easy to prove that $y(t)+y'(t)=\mathcal{F}^{-1}|_K\left(1|_K\right)(t)=\frac{1}{2\pi}\int_{-k}^{k}e^{i\omega t}\,\mathrm{d}t=\frac{k}{\pi}S(kt)$ (see (6.8.18)) and hence $y(t)=e^{-t}\frac{k}{\pi}\int_{-1}^{t}S(kx)e^{x}\,\mathrm{d}x$ e.g. for all $\log(\mathrm{d}\rho)\le t\le -\log(\mathrm{d}\rho)$. Therefore

$$y(t)=e^{-t}\int_{-1}^{t}\mathcal{F}^{-1}|_K\left(1|_K\right)(s)e^{s}\,\mathrm{d}s\approx e^{-t}\int_{-1}^{t}\delta(s)e^{s}\,\mathrm{d}s=e^{-t}H(t)$$

for all $t\in{}^{\rho}\widetilde{\mathbb{R}}$ which are far from the origin, i.e. such that $|t|\ge r\in\mathbb{R}_{>0}$ for some $r$.

## 6.9 Summary of the chapter 6

All in all, in this chapter, we have motivated the natural attempts of several authors to extend the domain of some kind of Fourier transform. The HFT presented in this chapter can be applied to the entire space of all the GSF defined in the infinite interval $[-k,k]^n$. These clearly include all tempered Schwartz distributions, all tempered Colombeau GF, but also a large class of non-tempered GF, such as the exponential functions, or non-linear examples like $\delta^a\circ\delta^b$, $\delta^a\circ H^b$, $a$, $b\in\mathbb{N}$, etc.

We want to summarize by listing some features of the theory that allow some of the main results that we have considered in this chapter:

1. The power of a non-Archimedean language permeates the whole theory since the beginning (e.g. by defining GF as set-theoretical maps with infinite values derivatives or in the use of sharp continuity). This power turned out to be important also for the HFT: see the heuristic motivation of the FT in Sec. 6.4.1, Example 103 about application of the uncertainty principle to a delta distribution, or the HFT of exponential functions in Example 96 and in Sec. 6.8.

2. The results presented here are deeply founded on a strong and flexible theory of multidimensional integration of GSF on functionally compact sets: the possibility to exchange hyperlimits and integration is an important step in the proof of the Fourier inversion theorem Thm. 101.2; the possibility to compute $\varepsilon$-wise integrals on intervals is another feature used in several theorems and a key step in defining integration of compactly supported GSF.

3. It can also be worth explicitly mentioning that the definition of HFT is based on the classical formulas used only for rapidly decreasing smooth functions and not on duality pairing. In our opinion, this is a strong simplification that even more underscores the strict analogies between ordinary smooth functions and GSF. All this in spite of the fact that the ring of scalars ${}^{\rho}\widetilde{\mathbb{R}}$ is not a field and is not totally ordered.

4. Important differences with respect to the classical theory result from the Riemann-Lebesgue Lem. 93 and the differentiation formula (6.5.1). In the former case, we explained these differences as a general consequence of integration by part formula, i.e. of the non-linear framework we are working in, see Thm. 95. The compact support of the HFT $\mathbb{1}$ of Dirac's delta reveals to be very important in stating and proving the preservation properties of HFT, see Sec. 6.7. Surprisingly (the classical formula dates back at least to 1822), in Sec. 6.8 we showed that the new differentiation formula is very important to get out of the constrained world of tempered solutions.

5. Finally, Example 103 of application of the uncertainty principle, further suggests that the space ${}^{\rho}\mathcal{GC}^{\infty}(K)$ may be a useful framework for quantum mechanics, so as to have both GF and smooth ones in a space sharing several properties with the classical $L^2(\mathbb{R}^n)$ (but which, on the other hand, is a *graded* Hilbert space).

# Chapter 7

# Fourier transform of rapidly decreasing generalized smooth functions.

## 7.1 Space of rapidly decreasing GSF

### 7.1.1 Definition, properties and examples

We introduce the space of rapidly decreasing generalized functions on ${}^{\rho}\widetilde{\mathbb{R}}^{n}$. The general idea of standard rapidly decreasing functions on $\mathbb{R}^{n}$ remains unchanged on ${}^{\rho}\widetilde{\mathbb{R}}^{n}$ i.e., they are smooth, and all of their derivatives decay faster than the reciprocal of any polynomial at infinity. More precisely, we give the following definition:

**Definition 114.** Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$, $Y \subseteq {}^{\rho}\widetilde{\mathbb{C}}^{n}$. We say that $f \in {}^{\rho}\mathcal{GS}(X, Y)$ if and only if

1. $f \in {}^{\rho}\mathcal{GC}^{\infty}(X, Y)$
2. $\forall \alpha \in {}^{\rho}\widetilde{\mathbb{N}}_{\mathrm{f}}^{n}\, \forall \beta \in \mathbb{N}^{n}\, \exists \rho_{k}(f) \in {}^{\rho}\widetilde{\mathbb{R}}$ such that $\rho_{k}(f) := \max_{x \in X} \left| x^{\alpha} \partial^{\beta} f(x) \right|$, where ${}^{\rho}\widetilde{\mathbb{N}}_{\mathrm{f}} := \left\{ n \in {}^{\rho}\widetilde{\mathbb{N}} \mid n \text{ is finite} \right\}$.

The space of all rapidly decreasing functions on ${}^{\rho}\widetilde{\mathbb{R}}^{n}$ denoted by ${}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$.

*Remark* 115. Note that $n \in {}^{\rho}\widetilde{\mathbb{N}}$ is finite if and only if

$$\exists N \in \mathbb{N} : \; |n| < N \text{ i.e., } \forall^{0}\varepsilon : \mathrm{ni}(n)_{\varepsilon} < N \tag{7.1.1}$$

We now define the following

**Definition 116.** Let $n \in {}^{\rho}\widetilde{\mathbb{N}}$, then the *image* of $n$ is $\mathrm{Im}(n) := \{k \in \mathbb{N} \mid \exists J \subseteq_{0} I\, \forall \varepsilon \in J : \mathrm{ni}(n)_{\varepsilon} = k\}$. Recall that $J \subseteq_{0} I$ means a cofinal set; see e.g. [31] and [50] for more details.

*Remark* 117. Note that we can have the following two cases:

1. $\mathrm{Im}\,(n)=\emptyset$, e.g. for $n_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$.
2. $\mathrm{Im}\,(n)$ is an infinite set if $\exists\,(J_p)_{p\in\mathbb{N}}\ \forall p\in\mathbb{N}:\ J_p\subseteq_0 I,\ \cup_{p\in\mathbb{N}}J_p=I$ such that $\forall p,q\in\mathbb{N}:\ p\neq q\Rightarrow J_p\cap J_q=\emptyset$. We set $n_\varepsilon:=p$ if $\varepsilon\in J_p$.

**Theorem 118.** *Let $n\in{}^\rho\widetilde{\mathbb{N}}$, then the following are equivalent:*

1. $n\in{}^\rho\widetilde{\mathbb{N}}_{\mathrm{f}}$.
2. $\exists M\in\mathbb{N}\,\forall^0\varepsilon:\ M=\max_{e\le\varepsilon}\mathrm{ni}\,(n)_e$.

Before proving this theorem we first prove the following result:

**Lemma 119.** *Let $x:=(x_\varepsilon):\ J\longrightarrow M\subseteq\mathbb{R}$, where $J\subseteq_0 I$, $M$ is finite. Then*

$$\exists m\in M:\ \{\varepsilon\in J|\,x_\varepsilon=m\}\subseteq_0 I.$$

*Proof.* Assume by contradiction that $\forall m\in K:\ \{\varepsilon\in J|\,x_\varepsilon=m\}\not\subseteq_0 I$. Therefore

$$\exists\varepsilon_m\in I:\ (0,\varepsilon_m)\cap\{\varepsilon\in J|\,x_\varepsilon=m\}=\emptyset. \tag{7.1.2}$$

We set that $\bar\varepsilon:=\min_{m\in M}\varepsilon_m>0$. Then (7.1.2) implies that $\forall\varepsilon\in(0,\bar\varepsilon)\ \forall m\in M:\ \varepsilon<\varepsilon_m\Rightarrow\varepsilon\notin\{\varepsilon\in J|\,x_\varepsilon=m\}$. But $J\subseteq_0 I\Rightarrow\exists e\in J\cap(0,\bar\varepsilon)$ and for $x_e=:m$ we have $e\in\{\varepsilon\in J|\,x_\varepsilon=m\}$ which is a contradiction. □

We can now prove Thm. 118.

*Proof.* (1⇒2): We first prove that $\mathrm{Im}\,(n)$ is finite and non-empty. (7.1.1) implies that,

$$\exists N\in\mathbb{N}:\ \forall^0\varepsilon:\ \mathrm{ni}\,(n)_\varepsilon\in\{0,\dots N-1\}=:K. \tag{7.1.3}$$

Using Lem. 119 with $x_\varepsilon:=\mathrm{ni}\,(n)_\varepsilon$, we get $\exists k\in K:\ \{\varepsilon\in I\,|\,\mathrm{ni}\,(n)_\varepsilon=k\}\subseteq_0 I$, hence $k\in\mathrm{Im}\,(n)$. Finally, $\forall k\in\mathrm{Im}\,(n):\ \exists J\subseteq_0 I\ \forall\varepsilon\in J:\ \mathrm{ni}\,(n)_\varepsilon=k$. From (7.1.1) we have that $\forall^0\varepsilon\in J:\mathrm{in}\,(n)_\varepsilon=k\in K$ i.e, $\mathrm{Im}\,(n)\in K$. We can hence set that $M:=\max\{\mathrm{Im}\,(n)\}\in\mathbb{N}$. We now prove that $\forall^0\varepsilon:\ \mathrm{ni}\,(n)_\varepsilon\in\mathrm{Im}\,(n)$. To prove this, we assume by contrdiction that $\exists L\subseteq_0 I\ \forall\varepsilon\in L:\ \mathrm{ni}\,(n)_\varepsilon\notin\mathrm{Im}\,(n)$. From (7.1.3) and Lem. 119 with $x_\varepsilon:=\mathrm{ni}\,(n)_\varepsilon$, $\forall\varepsilon\in L:\ \exists k\in K:\ \{\varepsilon\in L\mid\mathrm{ni}\,(n)_\varepsilon=k\}\subseteq_0 L$ yields that $k\in\mathrm{Im}\,(n)$. But this is a contradiction to $\exists\varepsilon\in L:\ \mathrm{ni}\,(n)_\varepsilon=k$. Therefore $\exists\varepsilon_0\,\forall\varepsilon\le\varepsilon_0$: $\mathrm{ni}\,(n)_\varepsilon\in\mathrm{Im}\,(n)$. In particular, $\forall e\le\varepsilon:\ \mathrm{ni}\,(n)_e\in\mathrm{Im}\,(n)$. On the other hand, since $\mathrm{ni}\,(n)_e\le M$ we have that $\sup_{e\le\varepsilon}\mathrm{ni}\,(n)_e\le M$. But $M\in\mathrm{Im}\,(n)$ implies that $\exists J\subseteq_0 I\ \forall\varepsilon\in J:\ \mathrm{ni}\,(n)_\varepsilon=M$. For any arbitrary $\bar\varepsilon\in J_{\le\varepsilon}$, $M=\mathrm{ni}\,(n)_{\bar\varepsilon}\le\sup_{e\le\varepsilon}\mathrm{ni}\,(n)_e$ which proves our claim above.

(2⇒1): From 2 we have that, $M\in\mathbb{N}$, $\exists\varepsilon_0\,\forall\varepsilon\le\varepsilon_0:\ M=\max\{\mathrm{Im}\,(n)\}$, $\mathrm{ni}\,(n)_\varepsilon\le\max\{\mathrm{Im}\,(n)\}=M$. Hence $n\in{}^\rho\widetilde{\mathbb{N}}_{\mathrm{f}}$. □

From the previous proof, we also obtain the following:

**Corollary 120.** *Let $n \in {}^{\rho}\widetilde{\mathbb{N}}_{f}$, then*

1. *$Im(n)$ is finite and non-empty.*
2. *$\forall^{0}\varepsilon:\ ni(n)_{\varepsilon} \in Im(n)$*
3. *Set $J_{k} := \{\varepsilon \in I \mid ni(n)_{\varepsilon} = k\}$, $\forall k \in Im(n)$ then $J_{k} \subseteq_{0} I$, $\forall k \forall^{0}\varepsilon:\ \varepsilon \in \cup_{k\in Im(n)} J_{k}$.*

Note that 3 holds because we have that $\forall^{0}\varepsilon:\ \mathrm{ni}(n)_{\varepsilon} = k \in \mathrm{Im}(n)$ yields $\varepsilon \in J_{k}$.

The results above show that we can have $\mathrm{Im}(n)$ finite and non-empty even if $n \in {}^{\rho}\widetilde{\mathbb{N}} \setminus {}^{\rho}\widetilde{\mathbb{N}}_{\mathrm{f}}$. It can be easily proved that the maximum in Thm. 118 is unique.

**Example 121.** Trivially, the function $f(x) = e^{-x^{2}}$ is in ${}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}\right)$ but $g(x) = e^{-|x|}$ is not even a GSF because the absolute value is not a GSF. The generalized smooth function $h(x) = \left(1 + |x|^{4}\right)^{-a}$, $a > 0$ is not in ${}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}\right)$ since it decays only like the reciprocal of a fixed polynomial at infinity.

Not every rapidly decreasing GSF has rapidly decreasing derivatives like the following example.

**Example 122.** Let $f : {}^{\rho}\widetilde{\mathbb{R}} \to {}^{\rho}\widetilde{\mathbb{R}}$. Then the function $f(x) := e^{-x^{2}} \sin\left(e^{x^{2}}\right)$ is rapidly decreasing but its derivative $f'(x) = -2xe^{-x^{2}} \sin\left(e^{x^{2}}\right) + 2xe^{-x^{2}} \cos\left(e^{x^{2}}\right)$ is asymptotically linearly increasing due to the second term.

*Remark* 123. If $f_{1} \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$ and $f_{2} \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^{m}\right)$, then the function of $n+m$ variables $f_{1}(x_{1}, \dots, x_{n}) f_{2}(x_{n+1}, x_{n+2} \dots x_{n+m}) \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^{n+m}\right)$. If $f(x) \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}\right)$ and $P(x)$ is a polynomial of $n$-variables, then $f(x) P(x) \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$. If $\alpha \in {}^{\rho}\widetilde{\mathbb{N}}_{\mathrm{f}}$, then $\partial^{\alpha} f \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$.

## 7.2 Fourier transform of the rapidly decreasing GSF

**Definition 124.** We define Fourier transform of $n$- dimensional rapidly decreasing GSF $f \in {}^{\rho}\mathcal{GS}({}^{\rho}\widetilde{\mathbb{R}}^{n}, \widetilde{\mathbb{C}}^{n})$ as follows:

$$\mathcal{F}(f)(\omega) := \int\limits_{{}^{\rho}\widetilde{\mathbb{R}}^{n}} f(x) e^{-ix\omega} \, \mathrm{d}x = \int \mathrm{d}x_{1} \dots \int f(x_{1} \dots x_{n}) e^{-ix\cdot\omega} \, \mathrm{d}x_{n}, \quad (7.2.1)$$

where $x = (x_{1} \dots x_{n}) \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$ and $\omega = (\omega_{1} \dots \omega_{n}) \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$. Note that the product $x \cdot \omega$ on ${}^{\rho}\widetilde{\mathbb{R}}^{n}$ as usual denotes the dot product:

$$x \cdot \omega = \sum_{j=1}^{n} (x_{j}\omega_{j}).$$

In the sequel, we denote ${}^\rho\mathcal{GS}({}^\rho\widetilde{\mathbb{R}}^n) := {}^\rho\mathcal{GS}({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{C}}^n)$.

**Theorem 125.** *Fourier transform of the rapidly decreasing GSF $\mathcal{F}$ maps ${}^\rho\mathcal{GS}\left({}^\rho\widetilde{\mathbb{R}}^n\right) \to {}^\rho\mathcal{GS}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$.*

*Proof.* Let $f \in {}^\rho\mathcal{GS}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$. In order to show that $\mathcal{F}(f) \in {}^\rho\mathcal{GS}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$ we need to establish an upper bound for the function $\omega^\alpha\partial^\beta\mathcal{F}(f)$ for each $\alpha \in {}^\rho\widetilde{\mathbb{N}}^n_f$, $\beta \in \mathbb{N}^n$. By Thm. 97.8 and 9 we have that

$$\omega^\alpha\partial^\beta_\omega\mathcal{F}(f)(\omega) = i^{|\alpha|+|\beta|}\int\limits_{{}^\rho\widetilde{\mathbb{R}}^n} x^\alpha\partial^\beta_x f(x)\, e^{-ix\omega}\,\mathrm{d}x. \tag{7.2.2}$$

To estimate, we set

$$M_{N,\alpha,\beta} = \max_{x\in{}^\rho\widetilde{\mathbb{R}}^n}\left|\left(1+|x|^2\right)^N x^\alpha\partial^\beta_x f\right|,$$

which is finite by Def. 114. Since the term $\left(1+|x|^2\right)^{-N}$ is integrable for $N$ sufficiently large, we can estimate (7.2.2) by

$$\omega^\alpha\partial^\beta_\omega\mathcal{F}(f)(\omega) \le M_{N,\alpha,\beta}\int\limits_{{}^\rho\widetilde{\mathbb{R}}^n}\frac{1}{\left(1+|x|^2\right)^N}\,\mathrm{d}x.$$

The right side is independent of the variable $\omega$, so this yields the required estimate. Now, assume that $f \in {}^\rho\mathcal{GS}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$ and set $k := \mathrm{d}\rho^{-1}$. Then

$$\begin{aligned}\mathcal{F}(f)(\omega) = \int f(x)\,e^{-ix\omega}\,\mathrm{d}x = \left[\int f(x)\,e^{-ix\omega_\varepsilon}\,\mathrm{d}x\right]\\ =: \left[\mathcal{F}^{\mathbb{R}^n}(f)(\omega_\varepsilon)\right] = \iota^b_{\mathbb{R}}\left(\mathcal{F}^{\mathbb{R}^n}(f)\right)(\omega).\end{aligned}$$

This implies that $\mathcal{F}(f) = \iota^b_{\mathbb{R}}\left(\mathcal{F}^{\mathbb{R}^n}(f)\right) = \mathcal{F}^{\mathbb{R}^n}(f)$. Note that the last equality comes from [31, Thm.31] . $\square$

**Theorem 126.** *Every compactly supported GSF is rapidly decreasing i.e., if $f \in {}^\rho\mathcal{GD}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$ then $f \in {}^\rho\mathcal{GS}\left({}^\rho\widetilde{\mathbb{R}}^n\right)$.*

*Proof.* We need to show that

1. $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$
2. $\forall\alpha \in {}^\rho\widetilde{\mathbb{N}}^n_f\,\forall\beta\,\exists\rho_k(f) \in {}^\rho\widetilde{\mathbb{R}}$ such that $\rho_k(f) = \max_{x\in{}^\rho\widetilde{\mathbb{R}}^n}\max_{|\alpha|+|\beta|\le k}\left|x^\alpha\partial^\beta f(x)\right|$.

The first condition is satisfied by definition of the compactly supported GSF. For the second condition, we assume that $f \in {}^\rho\mathcal{GD}\left({}^\rho\widetilde{\mathbb{R}}\right)$ and hence $\partial^\beta f \in {}^\rho\mathcal{GD}\left({}^\rho\widetilde{\mathbb{R}}\right)$, where $\beta$ is a multi-index. Then $\forall\beta\,\exists a\,\forall x \in {}^\rho\widetilde{\mathbb{R}}:\ |x| \le \mathrm{d}\rho^{-a} \Longrightarrow \partial^\beta f(x) = 0$. Then $\rho_k(f) \le \max_{x\in\left[-\mathrm{d}\rho^{-a},\mathrm{d}\rho^{-a}\right]^n}\max_{|\alpha|+|\beta|\le k}\left|x^\alpha\partial^\beta f(x)\right| = 0$. $\square$

## 7.3 Properties and the inverse of the Fourier transform of rapidly decreasing GSF and the inversion theorem

First, we state the main properties of the Fourier transform of rapidly decreasing GSF. Note that we do not prove these properties at this time, since the proofs are similar to those of HFT (see Thm. 97).

**Theorem 127.** *Let* $f, g \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$*, then*

1. $\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$.
2. *Let* $b \in \widetilde{\mathbb{C}}$*, then* $\mathcal{F}(bf) = b\mathcal{F}(f)$.
3. $\mathcal{F}\left(\overline{f}\right) = \overline{-1 \diamond \mathcal{F}_k(f)}$.
4. $\mathcal{F}_k(-1 \diamond f) = -1 \diamond \mathcal{F}_k(f)$.
5. $\mathcal{F}(t \diamond f) = t \odot \mathcal{F}(f)\,,\ t \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$.
6. $\mathcal{F}(s \oplus f) = e^{-is(-)}\mathcal{F}(f)\,,\ s \in {}^{\rho}\widetilde{\mathbb{R}}$.
7. $\mathcal{F}\left(e^{is(-)}f\right) = s \oplus \mathcal{F}(f)$ *for all* $s \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$.
8. $\mathcal{F}(\partial_j f)(\omega) = i\omega_j \mathcal{F}(f)(\omega)$.
9. $\frac{\partial}{\partial \omega_j}\mathcal{F}(f)(\omega) = -i\mathcal{F}(x_j f)(\omega)$.
10. $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.
11. $\mathcal{F}(s \odot g(x))(\omega) = \mathcal{F}\left(g\left(\frac{t}{s}\right)\right)(\omega)$*, for all invertible* $s \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$.

Next, we consider the inverse Fourier sransform of rapidly decreasing GSF. We prove that the inversion theorem, the Parseval's relation and the Plancherel's identity. We define the inverse hyperfinite Fourier transform as follows:

**Definition 128.** Let $f \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$, then

$$\mathcal{F}^{-1}(f)(x) = \left(\frac{1}{2\pi}\right)^{n} \int f(\omega)\, e^{ix\omega}\, \mathrm{d}\omega \tag{7.3.1}$$

for all $x, \omega \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$. The operation (7.3.1) is called the inverse of the FT of rapidly decreasing GSF $f$.

Note that

$$(2\pi)^{n}\,\mathcal{F}^{-1}(f) = \mathcal{F}_k(-1 \diamond f) = -1 \diamond \mathcal{F}(f), \tag{7.3.2}$$

where recall that $-1\diamond$ denotes the reflection $(-1 \diamond g)(x) := g(-x)$.

In fact, we can prove that the inverse FT of the rapidly decreasing GSF share the same properties as the hyperfinite Fourier transform does. In the next theorem, we state the main properties of FT of the rapidly decreasing GSF as follows but we do not prove them all since the proof is similar to those in HFT (see Thm.101).

**Theorem 129.** *Let $f,\ g,\ h \in {}^{\rho}\mathcal{GS}\left({}^{\rho}\widetilde{\mathbb{R}}^n\right)$ we have*

1. $\int f(x)\,\mathcal{F}(g)(x)\ \mathrm{d}x = \int \mathcal{F}(f)(\omega)\,g(\omega)\ \mathrm{d}\omega$.
2. $\mathcal{F}^{-1}[\mathcal{F}(f)] = f = \mathcal{F}\left[\mathcal{F}^{-1}(f)\right]$.
3. *(Parseval's relation)* $(2\pi)^n \int f(x)\,\overline{h}(x)\ \mathrm{d}x = \int \mathcal{F}(f)(\omega)\,\overline{\mathcal{F}(h)(\omega)}\,\mathrm{d}\omega$.
4. *(Plancherel's identity)* $(2\pi)^n \int |f(x)|^2\ \mathrm{d}x = \int |\mathcal{F}(f)(\omega)|^2\ \mathrm{d}\omega$.
5. $\int f(x)\,g(x)\ \mathrm{d}x = \int \mathcal{F}(f)(x)\,\mathcal{F}^{-1}(g)(x)$.

*Proof.* 1 follows immediately from Def. 124 and the Fubini's theorem. The inversion theorem 2 has already proved (see Thm.101.2). To prove 3, we use 1 with $g = \overline{\mathcal{F}(h)}$ and the fact that $\mathcal{F}(g) = \overline{h}$, which is a consequence of 127. 3 and 2. Plancherel's identity 4 is a trivial consequence of 3. Finally 5 easily follows from 1 and 2. ☐

## 7.4 Summary of the chapter 7

In this chapter, we have introduced a new space of rapidly decreasing GSF that preserves a general idea of the classical space of rapidly decreasing functions on $\mathbb{R}^n$ but extends it to ${}^{\rho}\widetilde{\mathbb{R}}^n$. We have defined a proper notion of Fourier transform in this space and have proved that it preserves the most of the basic and main properties of the standard FT of rapidly decreasing functions on $\mathbb{R}^n$. We have further proved that the FT of rapidly decreasing GSF is a continuous mapping from a space of rapidly decrasing GSF into itself. Moreover, we showed that the space of compactly supported GSF ${}^{\rho}\mathcal{GD}(-)$ is contained in the space of rapidly decreasing GSF ${}^{\rho}\mathcal{GS}(-)$ as a subspace and vice versa. We have also seen that the usual properties of the classical FT can be extended in the framework of rapidly decreasing GSF without any restrictions. This is one of the main advantages of the concept of FT of the rapidly decreasing GSF.

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