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„On T-dualities in Non-relativistic String Theory“

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Contents

Abstract

We verify the T-duality rules found in the worldsheet formulation of non-relativistic string theory in a target space formulation given by non-relativistic NS-NS gravity. This T-duality maps a non-relativistic string theory with a compact, spacial, longitudinal direction to a relativistic string theory with a compact, lightlike direction. As was shown recently, the non-relativistic limit of NS-NS gravity leads to a pseudo-action for non-relativistic NS-NS gravity, which encodes all but one equation of motion, the Poisson equation.
We employ this pseudo-action and verify the T-duality rules at the level of actions. We compare two different approaches: A Kaluza-Klein reduction along a spacelike longitudinal direction starting from the non-relativistic NS-NS pseudo-action versus a null reduction starting from the relativistic NS-NS action.
These two actions agree if we implement the T-duality rules as expected from the worldsheet formulation. We thus verified that the T-duality rules on target space are consistent with all, except one, equation of motion of non-relativistic NS-NS gravity.
Additionally, we have found the origin of the emergent dilatation symmetry which was the reason why the Poisson equation could not be derived from the non-relativistic limit of the action. This dilatation symmetry ultimately stems from the lightlike direction, which inherently has no notion of length.

Zusammenfassung

Wir verifizieren in einer Hintergrundformulierung, die durch nichtrelativistische NS-NS-Gravitation gegeben ist, die T-Dualitätsregeln, die in der Weltflächen-Wirkung der nicht-relativistischen Stringtheorie gefunden wurden. Diese T-Dualität bildet eine nicht- relativistische Stringtheorie mit einer kompakten, räumlichen, longitudinalen Richtung auf eine relativistische Stringtheorie mit einer kompakten, lichtartigen Richtung ab. Wie kürzlich gezeigt wurde, führt der nicht-relativistische Limes von NS-NS-Gravitation zu einer (Pseudo-)Wirkung für nicht-relativistische NS-NS-Gravitation, die alle Bewegungsgleichungen bis auf eine, die Poisson-Gleichung, kodiert.
Wir nutzen diese Pseudo-Wirkung und verifizieren die T-Dualitätsregeln auf der Ebene der Wirkungen. Wir vergleichen zwei verschiedene Ansätze: Eine Kaluza-Klein-Reduktion entlang einer raumartigen longitudinalen Richtung ausgehend von der nichtrelativistischen NS-NS-Pseudo-Wirkung gegenüber einer Null-Reduktion ausgehend von der relativistischen NS-NS-Wirkung.
Diese beiden Wirkungen stimmen überein, wenn wir die T-Dualitätsregeln anwenden, wie sie auch in der Weltfächen-Formulierung auftreten. Wir haben also überprüft, dass die T-Dualität im Hintergrund mit allen Bewegungsgleichungen der nicht-relativistischen NS-NS-Gravitation konsistent sind, bis auf Ausnahme der Poisson Gleichung.
Außerdem haben wir den Ursprung der unerwarteten Dilatationssymmetrie gefunden, die der Grund dafür war, dass die Poisson-Gleichung nicht aus dem nicht-relativistischen Grenzwert der Wirkung abgeleitet werden konnte. Diese Dilatationssymmetrie ergibt sich letztlich aus der lichtartigen Richtung, die von Natur aus keinen inherenten Begriff von Länge hat.

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1. Introduction

One of Einstein's genius thoughts in formulating general relativity was that physics should describe gravity not as a force in the Newtonian sense but rather geometrically caused by the curvature of spacetime [1, 2]. It was later discovered that a similar approach also allows us to geometrize other forces, such as electromagnetism, when we view them through the lens of symmetries and subsequently interpret the physical field strengths as a curvature in an abstract space related to the underlying symmetries of nature.
This geometrization of forces, known as gauge theories, is now widely believed to be the de facto way to describe physics, as it allows a formulation of physics in a way that is independent of the frame of reference for any physical observable. Such a procedure is formalized in the literature as a so-called gauging of the symmetries.

In such a framework of gauge theories, general relativity is the theory with the Lorentz group $SO(1, D-1)$ as underlying symmetry group, which is exactly the group that leaves the appropriate local notion of spacetime-distance, the metric g, invariant at each point of spacetime. Moreover, also Newtonian gravity can be formulated geometrically through the non-relativistic symmetries of the Galilei algebra, or more precisely, its central extension, the Bargmann algebra [3]. Instead of a single spacetime metric, these symmetries preserve the notion of time and space separately, which are encoded in a time-metric τ and a space-metric h. The corresponding geometry of Newtonian gravity is then called *Newton-Cartan (NC) geometry* and provides a formulation of Newtonian gravity that is also invariant under general coordinate transformations.
As expected from a non-relativistic theory, Newton-Cartan type geometries also arise as the large speed of light limits $c \to \infty$ [4] (or large c expansions [5]) of relativistic theories. Such limits then provide a guiding principle in how to obtain geometries, actions and equations of motion for non-relativistic theories.
While interesting in its own right from a gravitational and mathematical perspective, Newton-Cartan geometry was also applied to problems in condensed matter physics, such as the quantum hall effect in [6] and [7].

To this date, general relativity gives the most accurate description of gravitational dynamics, i.e. of large scale physics, while quantum field theory, guided by gauge theory, provides the most accurate description of microscopic physics in terms of fundamental particles. However, so far it is not clear how to combine the two into a combined theory of quantum gravity, working across all length scales. One candidate for such a theory of quantum gravity is given by string theory, where, instead of point-like particles, extended one-dimensional strings are considered to be fundamental. Formally, such string theories are formulated in a so-called worldsheet action, generalizing the notion of the worldline of a particle. As we will discuss in this work, the relativistic dynamics of the string moving through a spacetime, the so-called background or target space, gives rise to equations of motion for the background fields, one of which is a spacetime metric, thus giving rise to gravitational dynamics. In the relativistic case, these equations of motion can be accurately captured by an action principle.

In 2001, Gomis and Ooguri [8] pioneered a formulation of non-relativistic strings, referring to a certain non-relativistic limit of relativistic string theory. Such non-relativistic strings then also give rise to a corresponding target space formulation, which is geometrically encoded in the so-called *string Newton-Cartan (SNC) geometry*. This geometry is characterized by two distinguished so-called longitudinal directions along the worldsheet of the string and is thus distinct from the Newton-Cartan geometry in particle theories, where only the time direction was singled out.
Subsequent recent works [9, 10, 11] have expanded on the non-relativistic ansatz and its geometry and formulated closed bosonic non-relativistic string theory in general curved SNC backgrounds.

Similar to relativistic string theory, non-relativistic string theory is only consistent in $D = 26$ spacetime dimensions for the bosonic string and $D = 10$ spacetime dimensions for the supersymmetric string, leaving us with the task of connecting the 10- or 26-dimensional string to a theory in the familiar $D = 4$ spacetime dimensions. This is usually achieved through compactifications, an ansatz where some directions of spacetime are assumed to be small compact dimensions of size R. In relativistic string theory, surprisingly, the description of a string compactified over a single compact dimension of size R is related to the one of size $1/R$ if one interchanges the number of times the string winds around the compact direction with the (quantized) momentum number along the compact direction. The resulting relation between the seemingly distinct descriptions of the two models is called a *T-duality*. Such T-dualities give rules how to identify the fields of the two theories, often encoded in so-called Buscher-rules [12].
Similar T-dualities have been considered for the non-relativistic string in [10], where it was found that on the worldsheet the T-duality is more complex than in the relativistic case. It turns out that a non-relativistic string moving in a background with a compactified spacial direction is equivalently described by a relativistic string moving in a background with a null isometry, i.e. a distinguished direction in spacetime along a ray of light.

Analogously to how background dynamics followed in relativistic string theory, the non-relativistic string also gives rise to equations of motion for the non-relativistic background fields to which the string couples. Such equations of motion for the closed bosonic non-relativistic string were computed in [13] and one of the equations of motion was found to be the stringy analogue of the Poisson equation. In this sense, the non-relativistic string gives rise to (a version of) Newtonian gravity in the target space.
In the recent work [4], the non-relativistic limit of NS-NS gravity, the bosonic sector common to superstring theories, was constructed. As mentioned, the relativistic background dynamics could be accurately captured by the introduction of a corresponding NS-NS gravity action. The non-relativistic case is more subtle. It was shown that the non-relativistic limit of the equations of motion of NS-NS gravity agreed with the equations of motion deduced from the worldsheet formulation of the non-relativistic string. On the other hand, the limit of the NS-NS action led to a non-relativistic action that captured almost all corresponding equations of motion, yet, the string version of the Poisson equation could not be inferred from it. This is due to an unexpected dilatation

symmetry of the action that was not present in the relativistic action.

So far, T-dualities in non-relativistic string theory have been considered for closed [10] and open [14] strings in the literature. However, they have only been performed in a worldsheet formulation of string theory, not at the level of the target space. Expanding on this, in [15] corresponding target space T-duality rules have been assumed to hold to deduce non-relativistic string theory solutions from known appropriate relativistic ones, but it was not shown that these rules actually hold at the level of non-relativistic equations of motion.
It is thus the aim of this work to confirm the T-duality rules derived at the level of worldsheet in a target space formulation of non-relativistic string theory and subsequently show that these rules hold at the level of (pseudo-)actions between non-relativistic NS-NS gravity and relativistic NS-NS gravity with a null isometry. Verifying this relation then shows that under T-duality all but one pair of equations of motion (including the Poisson equation) correctly transform into each other.

This thesis is organized as follows. In section 2 we introduce the language of representation and gauge theory, which provide the foundation of all subsequent chapters. In section 3 we study general relativity in this language as the gauge theory of the Poincaré group. We introduce the notion of vielbeine, which is vital for the formulation of Newton-Cartan geometry and, furthermore, discuss several technicalities regarding torsion and diffeomorphisms that reappear similarly in the next part. In section 4 we motivate the formulation of Newtonian gravity through Newton-Cartan geometry and then follow the lines of the preceding sections to derive Newton-Cartan geometry as the gauge theory of the Bargmann algebra. Lastly, we explain how to take non-relativistic limits of relativistic theories and algebras. In section 5 we present different approaches to compactifications via dimensional reductions. First, we introduce the original Kaluza-Klein reductions, and then we explain null reductions. Both will be paramount in the results and calculations of this thesis. Next, in section 6 we discuss string theory. First, we consider the relativistic string to derive NS-NS gravity and discuss the general setup of T-dualities. Then, we specialize to non-relativistic theory and present its historical emergence in flat spacetime, as well as recent results in curved backgrounds. Finally, in section 7, we present the results of this work, where we applied the techniques of dimensional reductions of section 5 to non-relativistic NS-NS gravity.
In section 8, we summarize the results and give an outlook on how to further extend and apply the results of this thesis.

2. General Gauge Theory

Symmetries have always played a vital role in modern theoretical physics. It is not hard to discover that symmetries greatly simplify the description or solution of physical problems. One of the examples, taught as early as in undergraduate studies, is the solution of the two-body problem of Newtonian gravity through the use of the Lenz-Runge vector.
It was the genius work of Emmy Noether in 1918 that tied symmetries and physics closer together. In her work [16] she showed that so-called *global symmetries*, symmetries with a constant parameter, lead to conserved charges. However, in the same work, she also showed that *local symmetries*, i.e. with parameters being functions of spacetime, lead to relations between the equations of motion.
The theory of local symmetries is now called *gauge theory* and is one of the pillars of contemporary physics and research, with the most notable example being the standard model of particle physics.

In this chapter, we want to give a quick introduction to the language of representation theory, which encodes physical symmetries, and then develop the language and tools of gauge theory needed for the rest of this work. As this topic is very rich and broad, we can only present a small fragment here.

2.1. Groups and Representations

2.1.1. Groups

Symmetries in physics are encoded in the mathematical structure of groups. Many of these symmetries are *Lie groups*, groups whose elements are described by a simple set of (finite) parameters, that are related to each other smoothly. More technically speaking, Lie groups are groups that are also manifolds and whose group operations are smooth.
A nice and thorough mathematical treatment of such groups can be found in [17] and [18, chapter 16]. Here we will briefly outline some conventions and language used in physics, but made more clear in light of the mathematical background.

Usually, it suffices to only look at matrix Lie groups G, meaning Lie groups that are a closed subset of $GL(n, \mathbb{C})$, the group of invertible, complex $n \times n$-matrices.
Each Lie group comes associated with a Lie algebra $\mathfrak{g}$, a vector space with a bilinear, antisymmetric bracket $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying the Jacobi-identity.
This algebra can be derived from the tangent space of the group G at the identity $e \in G$, i.e. $\mathfrak{g} = T_eG$, which in the setting of matrix groups is always a subset of $Mat(n, \mathbb{C})$.
From this definition we can see that the dimension of $\mathfrak{g}$ as a vector space is the same as the dimension of G as a manifold.
We can think of the elements of $\mathfrak{g}$ as a linearization of the group, something that is often called an *infinitesimal symmetry*, while the elements of the group constitute *finite symmetries.*

In the practical calculations of this work, the algebra is often more important than the

group, but we should still describe the relationship between the two.
First note that, as is customary in physics, we can choose a basis on $\mathfrak{g}$ denoted by $\{T_A\}_{A\in I}$, where I is a (finite) index set. We then call the elements of this basis the *generators* of $\mathfrak{g}$. For an element $\xi \in \mathfrak{g}$ in the algebra, the expansion coefficients $\xi^A \in \mathbb{C}$ such that $\xi = \xi^A T_A$ are then called *parameters*. Since the bracket again gives an element of the algebra, we can express its action via the above basis as

$$[T_A, T_B] = f_{AB}{}^C T_C.$$

The numbers $f_{AB}{}^C$ are called *structure constants* and completely characterize the algebra (albeit in a basis dependent manner). It is for this reason that in physics usually only the generators and this relation are given as "the algebra". Algebras for which $f_{AB}{}^C \equiv 0$ are called *Abelian*.
We can relate the algebra to the group through the exponential map $\exp\colon \mathfrak{g} \to G$, which can be defined abstractly, but in the case of matrix groups and algebras is simply the matrix exponential

$$\exp(\xi) = \sum_{n\in\mathbb{N}} \frac{\xi^n}{n!}$$

for $\xi \in \mathfrak{g}$. It is then clear that $\exp(0) = e$ and that we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi)|_{t=0} = \xi \tag{1}$$

It is less clear that there is a neighborhood of $0 \in \mathfrak{g}$ and a neighborhood of $e \in G$, such that between these two the exp-map is a diffeomorphism. This further justifies, why the elements of the algebra are seen as "small" symmetries.

We can now tie this fact together with two further results. It can be shown [17, Prop. 2.5] that any connected Lie group is generated by any neighborhood of the identity. So, given a Lie group, we can consider its Lie algebra calculated as tangent space of the identity. If we then take a neighborhood around the zero vector in $\mathfrak{g}$ on which exp is a diffeomorphism, we find that for a connected Lie group we can always write any $g \in G$ through a finite product

$$g = \exp(\xi_1) \cdots \exp(\xi_p)$$

of exponents of p elements $\xi_i \in \mathfrak{g}$ of the algebra. Additionally, we have the *Baker-Campbell-Hausdorff-formula*,

$$\exp(\xi_1) \cdot \exp(\xi_2) = \exp(C(\xi_1, \xi_2)),$$

where $\xi_{1,2} \in \mathfrak{g}$ are near $0 \in \mathfrak{g}$ and $C\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a function that only involves $\xi_{1,2}$ and nested commutators of them. This means that near the identity, the product of elements in the group G is completely determined through the Lie algebra $\mathfrak{g}$.
So, for connected Lie groups we do not really need to distinguish between the group and

its algebra, and we will henceforth do so and just assume that for any symmetry $g \in G$ we have a corresponding $\xi \in \mathfrak{g}$, such that

$$g = \exp(\xi) = \exp\bigl(\xi^A T_A\bigr),$$

where we made the basis explicit.
We usually work over the connected component of a group, as we want to relate any symmetry smoothly to "doing nothing", i.e. to the identity.

Note that it is important that we start with a group and from there deduced the corresponding algebra. Given an abstract (finite dimensional) Lie algebra, it is not clear what group should be associated to its algebra. In general, different Lie groups might have the same Lie algebra, e.g. both $SU(2)$ and $SO(3)$ have the same abstract Lie algebra but are not isomorphic. However, if we require the group associated to the algebra to be simply connected, one can associate to each finite dimensional Lie algebra a unique (up to isomorphism) Lie group according to [17, Theorem 22.11]

2.1.2. Representations

A *representation of G on V*, where V is a vector space, is a Lie group homomorphism $\rho\colon G \to GL(V)$. We call dim V the dimension of the representation.
A representation is called *irreducible* if for any subspace $W \subset V$ that is an invariant subspace, meaning $gW \subset W$ for all $g \in G$, we find that either W is trivial $W = \{0\}$ or already the whole space $W = V$.
Irreducible representations are important, as they are the building blocks of other representations, due to the result [17, Cor. 23.11] that any representation of a compact Lie group is completely reducible, i.e. can be written as a direct sum of irreducible representations. Moreover, for semisimple Lie groups, i.e. Lie groups whose Lie algebra is semisimple, we have a similar, but more general, decomposition result. Every finite dimensional representation of a semisimple Lie algebra is completely reducible [19, Theorem 10.9]. For non-semisimple algebras this no longer holds true, and indeed, they are often reducible but indecomposable. This means that we can no longer find representation matrices that are block diagonal, but only upper triangle matrices.

From [17, Prop. 5.4] we know that any Lie group homomorphism ρ gives rise to a Lie algebra homomorphism $\rho^* = T_e\rho$, i.e. a linear map $\rho^*\colon \mathfrak{g} \to L(V)$ from the algebra to the linear maps on V, such that $\rho^*([\xi_1, \xi_2]) = [\rho^*(\xi_1), \rho^*(\xi_2)]$.
Thus, any group representation automatically gives an algebra representation, which for us is the main point of interest here. To expand on this, note that by [17, Theorem 8.8(i)], we have for any Lie group homomorphism ρ that

$$\rho \circ \exp = \exp \circ \rho^*,$$

or acting on elements of the algebra that

$$\rho(\exp(\xi)) = \exp(\rho^*(\xi)) = \exp\bigl(\xi^A \rho^*(T_A)\bigr).$$

This equation can be used to find the representation of the algebra. We can expand the exponential up to first order in the parameters, giving

$$\exp(\rho^*(\xi)) \simeq Id_V + \rho^*(\xi).$$

Thus, we can also define the action of $\rho^*(\xi)$ on an element $v \in V$ via

$$\delta(\xi)v \coloneqq v' - v \equiv \rho^*(\xi)v$$

where v' is the element that we get by applying the exponential and then truncate after the first order in the parameters. After a choice of basis on V, this can then naturally be written as the action on the components of v, denoted by v^i,

$$\delta(\xi)v^i = \xi^A \rho^*(T_A)^i{}_j v^j,$$

where $i = 1, ..., \dim(V)$ is the index enumerating the basis of the representation and $\rho^*(T_A)^i{}_j$ are the components of the matrix representing the action of the generator T_A. It is also common notation to drop the dependence on ξ altogether.
Furthermore, it is customary in physics to always choose a basis in the representation space. Given such bases, one can write down the transformation rules corresponding to the representation as acting on the indices of the basis. We then identify physical objects by their transformation rules (i.e. their representation), effectively making these rules their defining property[1]. We then also associate the indices to the representation, which might lead to statements like: the index i transforms under $SO(D)$ rotations.

For every Lie algebra there always exists at least one representation, the *adjoint representation*, that will be of great importance in gauge theory.
Via the Jacobi-identity, one can check that we always get a representation of $\mathfrak{g}$ onto itself via the map

$$\begin{aligned} \rho^* \colon \mathfrak{g} &\to L(\mathfrak{g}) \\ \xi &\mapsto [\ ,\xi], \end{aligned}$$

or applied to an element $\chi \in \mathfrak{g}$

$$\rho^*(\xi)[\chi] = [\chi, \xi].$$

After a choice of basis and using the delta symbol, this reads

$$\delta(\xi)\chi^A = f_{BC}{}^A \chi^B \xi^C.$$

So far, we have only talked about finite dimensional representations but in physics we usually consider infinite dimensional representations. As vector space, we take a vector space of functions over some spacetime M that takes values in a second vector space W that generally also carries a representation of some symmetry group. These functions

[1]Something often encountered in relativity, where a vector is introduced as transforming with the Lorentz matrix, while the co-vector transforms with the inverse.

we will call fields. After choosing a basis $\{w_i\}$ of W, we may write any field via its components $\phi^i(x)$.
With this we can distinguish between *internal symmetries*, acting only on the components of the fields $\phi(x) \mapsto \phi'(x)$ and *spacetime symmetries*, that act on the points of spacetime as well as the components, i.e. $\phi(x) \mapsto \phi'(x')$, where x' is the transformed coordinate of spacetime.
A simple example for an internal symmetry is given by fields with values in a vector space that also carry a representation of the rotation group $SO(D)$ for arbitrary dimension D. To illustrate this, we assume that this vector space is given by the defining representation, where an element $A \in SO(D)$ can act directly on the vector space by

$$\phi'(x) = A\phi(x).$$

Introducing a basis and hence the Latin $SO(D)$-indices $i, j = 1, ..., D$, we have the components $\phi^i(x)$ and the transformation rule for internal symmetries reads

$$\phi'^i(x) = A^i{}_j \phi^j(x).$$

A more complicated example for a spacetime symmetry comes in the form of the Dirac field $\psi^a(x)$, where the index a is often referred to as spinor index. This field transforms under an element Λ of the Lorentz group $SO(1, D-1)$ (or more precisely under the universal cover of it, for details see [20, Ch. 2&3]) as

$$\psi'^a(x') = \rho(\Lambda)^a{}_b \psi^b(\Lambda^{-1}x),$$

for $\rho(\lambda)$ a certain representation[2] matrix of an element of the Lorentz group.

We can now properly introduce the delta symbol acting on a field ϕ, which we will also call the *variation of* ϕ, given by

$$\delta\phi(x) = \phi'(x) - \phi(x).$$

While the action of an internal symmetry is immediately seen to be

$$\delta(\xi)\phi^i(x) = \rho^*(\xi)^i{}_j \phi^j(x),$$

the action of spacetime symmetries seems to be more complicated. However, we can also express them via actions which only act on the components of a field.
By linearizing the symmetry action on spacetime (i.e. truncating the exponential), we can always write

$$x' = x + \delta x,$$

where δx is assumed to be small, and we thus see that the symmetry acts as a translation. For simplicity, we assume that $\mathfrak{g}$ acts directly on the spacetime coordinates as a spacetime symmetry, i.e. that

$$\delta x^\mu = (\xi)^\mu{}_\nu x^\nu,$$

[2]Actually a projective representation, that is why one has to consider the universal cover.

and that it acts *only* on the coordinates and not on the components. We can also express that by saying that ϕ does not carry any indices and that $\phi'(x') = \phi(x)$. Again, we note that the indices displayed here indicate a (choice of) representation, in this case a defining representation on spacetime and a trivial representation on the components. The latter defines the field ϕ as a *scalar field*. We can then use a Taylor expansion to first order and thus get

$$\begin{aligned}\phi'(x) &= \phi'(x' - \delta x) \simeq \phi'(x') - \delta x^\mu \partial_\mu \phi'(x') \\ &= \phi(x) - \delta x^\mu \partial_\mu \phi(x).\end{aligned}$$

So, in total, we find that spacetime symmetries act again only on the component of the field ϕ as

$$\delta\phi(x) = -\delta x^\mu \partial_\mu \phi(x) = -(\xi)^\mu{}_\nu x^\nu \partial_\mu \phi(x).$$

One can proceed in the same way, including spacetime symmetries that also act on fields living in a representation with multiple components. In any case, in the linearized form, every symmetry will *only* act on the components. More precisely, every spacetime symmetry will contain a term as above and a term acting on the indices.
This is showcased considering the Dirac field again. By expansion, we have for an element λ of the Lie algebra of the Lorentz group

$$\delta\psi^a(x) = \rho_*(\lambda)^a{}_b \psi^b(x) + \lambda^\mu{}_\nu x^\nu \partial_\mu \psi^a(x).$$

Thus, how the first term acts distinguishes different field representations.
Further note, there seems to be a relation between spacetime symmetries and translations, something we will explore later when looking at general relativity as a gauge theory.

Further, note that the delta symbol has the properties that it is

1. linear in ϕ,
2. commutes with partial derivative, i.e.
$$\partial_\mu \delta\phi = \delta(\partial_\mu \phi),$$
3. a derivation
$$\delta(\phi_1(x)\phi_2(x)) = (\delta\phi_1(x))\phi_2(x) + \phi_1(x)(\delta\phi_2(x)),$$
4. a representation of the algebra
$$[\delta(\xi_1), \delta(\xi_2)]\phi(x) = \delta([\xi_1, \xi_2])\phi(x).$$

2.2. Basic Concepts of Gauge Theory

At every point of the last section, we always assumed that the symmetry parameters ξ^A were simply numbers. Gauge theory is the question of what happens if we instead allow the parameters to be arbitrary functions of spacetime, i.e. $\xi^A = \xi^A(x)$. Allowing this type of spacetime dependence and all the modifications it entails is called *gauging of the symmetry.*

From a mathematical perspective, however, this is a much more complicated process, as we switch from the language of pure representation theory to the language of principal fiber bundles. A thorough mathematical introduction can be found in [21] and [20]. We, however, will be more concise from this point on and follow a more heuristic approach (see [22, Chapter 11]).

Gauge theory is motivated by the idea that every physical quantity only makes sense in relation to some kind of abstract reference frame. The classic example is spin, where "spin up" and "spin down" are only a proper physical description of the system if one gives the direction relative to which "up" and "down" is defined. Gauge theory is the framework where we not only keep track of the physical quantity but also the frame relative to which it is defined. A priori these frames need not be constant (consider a rotating observer measuring "up" and "down"), hence it is fairly natural that frames are needed for each point of the spacetime over which we regard physics and, as we want to freely switch back and forth between different frames, that these have to be related by some group G at each point.

We thus see that a) symmetries take on a different role than before, as they are now redundancies *in our description* of the theory, and b) that we need to consider group actions that differ at each spacetime point.

Let us now see at which point difficulties arise. We will start from an action given by a Lagrangian

$$S[\phi] = \int d^D x \mathcal{L}(\phi(x), \partial\phi(x)).$$

The action depends on the collection of fields ϕ^i from before, where i might be a generic index to enumerate the fields or an index of a representation. An action is invariant if we have that

$$\delta S[\phi] = S[\phi + \delta\phi] - S[\phi] = 0. \tag{2}$$

When we naively plug in the now spacetime dependent transformation that acts only internally

$$\delta(\xi(x))\phi^i(x) = \rho^*(\xi(x))^i{}_j \phi^j(x) = \xi^A(x)\rho^*(T_A)^i{}_j \phi^j(x),$$

we see that while the fields transform as before, the same is not true for the partial derivative of a field

$$\delta(\xi(x))\partial_\mu\phi^i(x) = \xi^A(x)\rho^*(T_A)^i{}_j \partial_\mu\phi^j(x) + \partial_\mu[\xi^A(x)]\rho^*(T_A)^i{}_j \phi^j(x),$$

as it picks up an extra derivative term, spoiling invariance of the action. To remedy this, consider a 1-form field B_μ, taking values in the Lie algebra, i.e. $B_\mu = B^A T_A$, that transforms in the adjoint representation, but only up to a derivative of the symmetry parameter too

$$\delta(\xi) B_\mu^A = \partial_\mu \xi^A + f_{BC}{}^A B_\mu^B \xi^C$$

or in a basis and coordinate free notation

$$\delta(\xi) B = d\xi + [B, \xi].$$

A field with such variation is called a *gauge field* or *connection*. The combined expression

$$\mathcal{D}_\mu \phi^i(x) \coloneqq \partial_\mu \phi^i(x) - \delta(B_\mu)\phi^i(x) \equiv \partial_\mu \phi^i(x) - B_\mu^A \rho^*(T_A)^i{}_j \phi^j(x)$$

is called *covariant derivative* and has the property that

$$\delta(\xi)\mathcal{D}_\mu \phi^i(x) = \xi^A(x) \rho^*(T_A)^i{}_j \mathcal{D}_\mu \phi^j(x).$$

So, the covariant derivative of a field transforms as the field itself, and we say it "transforms covariantly".
If we then replace $\partial_\mu \mapsto \mathcal{D}_\mu$ in our Lagrangian, we indeed find that it is now invariant under spacetime dependent transformations. Note though that we had to pay a price for that, namely, we had to introduce an extra field in our theory.
This algorithm of replacing partial derivatives by covariant ones is called *minimal coupling*.

For more complicated examples such as torsional Newton-Cartan geometry, which we will deal with in section 4, we need to refine slightly what we mean by transforming covariantly, as "transforming the same as the field" will not suffice.
We rather define a *covariant quantity* as a quantity that under gauge symmetries does not transform with a derivative of the symmetry parameter. Clearly the connection is not such a quantity.
When we build Lagrangians, we want to make sure that we build it only from such covariant quantities, as physics should be invariant under gauge transformations.
The covariant derivative is such an object, thus, also subsequent applications of it should be one. There is one particular quantity of interest $F_{\mu\nu}(T^A)$, obtained by considering

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\phi^i =: F_{\mu\nu}(T^A) \rho^*(T_A)^i{}_j \phi^j.$$

This field $F_{\mu\nu}(T^A)$ is a Lie algebra valued 2-form called the *curvature of the generator(s)* T_A and can be explicitly calculated to be

$$F_{\mu\nu}(T^A) = 2\partial_{[\mu} B_{\nu]}^A + f_{BC}{}^A B_\mu^B B_\nu^C.$$

The name curvature stems from the similarity of this expression to the Riemann curvature tensor (which in fact, is an explicit example of this) and that it defines a notion of parallel transport, albeit not in spacetime but in an abstract space (for details see [21, Chapter 3.3&3.5]).

It can then be shown that F transforms in the adjoint representation, but more importantly, for any such curvature we get a so called *Bianchi-identity*, namely

$$\mathcal{D}_{[\mu}F_{\nu\rho]}(T^A) = 0.$$

The field content (ϕ^i, B_μ^A), their symmetries and gauging them specifies the *kinematics* of our theory, and a priori we get no dynamics purely by gauging an algebra. For dynamics, we need a Lagrangian and/or equations of motion.

The above version of a gauge theory is somewhat sanitized since in the transformation of B only other gauge fields appear. In reality, see supersymmetry and torsional Newton-Cartan theory, our field content may also contain non-gauge fields, which then also appear in the transformation of the gauge fields. This amounts to the more general transformation law

$$\delta(\xi)B_\mu^A = \partial_\mu\xi^A + f_{BC}{}^A B_\mu^B\xi^C + \xi^B\mathcal{M}_{\mu B}{}^A.$$

This also modifies the curvature, resulting in the corrected covariant curvature

$$\hat{F}_{\mu\nu}(T^A) = F_{\mu\nu}(T^A) - 2B^B_{[\mu}\mathcal{M}_{\nu]B}{}^A. \tag{3}$$

2.3. Noether Identities

As mentioned before, it is important to stress that gauge symmetries are not symmetries of the physical system but only symmetries, or rather redundancies, of our description of the system. It is customary to call degrees of freedom related to gauge symmetries *unphysical.*

One of the more important implications of this is encoded in Noether's second theorem, which leads to the concept of *Noether identities,* i.e. relations between the equations of motion due to gauge symmetries.
In modern language it is actually a direct consequence, as the invariance under a gauge symmetry in eq. (2) is exactly the same as the ansatz we use for computing equations of motion. It is thus clear that under a gauge symmetry we find

$$0 = \delta(\xi)S = \int d^Dx\frac{\delta\mathcal{L}}{\delta\phi^i}\delta(\xi)\phi^i.$$

If we then extract the parameters of ξ, and recall that they are arbitrary continuous functions, we know that the part not containing ξ has to vanish. The equations corresponding to the vanishing parts are then called *Noether identities*. More explicitly, for simplicity, assume that the ϕ do not contain any gauge fields among them. We then find

$$0 = \delta(\xi)S = \int d^Dx\frac{\delta\mathcal{L}}{\delta\phi^i}\rho^*(T_A)^i{}_j\phi^j\xi^A. \tag{4}$$

Since the above has to vanish for any ξ^A that is continuous, we find the Noether identities

$$\frac{\delta \mathcal{L}}{\delta \phi^i} \rho^*(T_A)^i{}_j \phi^j = 0,$$

relating the equations of motion $\frac{\delta \mathcal{L}}{\delta \phi^i}$ of different field components of ϕ. Note that the action could also include gauge fields, therefore the Noether identities can, after integration by parts, be differential instead of algebraic equations.
Consequently, we have started with say N field components, hence degrees of freedom, we see that we do not get an independent equation of motion for each component. For each gauge symmetry we lose an equation of motion, so we actually only have $N - r$ physical degrees of freedom, where r is the number of gauged symmetry parameters.
This may seem pathological at first but only represents that we have equivalence classes of field configurations that look different on paper but encode the same physical configuration. The representatives of the classes are related through gauge transformations. Picking one representative and therefore removing the unphysical degree of freedom from the mathematical description is called *gauge fixing*.
One can acquire another point of view in the Hamiltonian formulation, where gauge symmetries present themselves as constraints on the system. See [23, Section. 3] for more details.

We started from already knowing the symmetries and thus could find the independent degrees of freedom. In applications, we can go in the opposite direction as well. If we find (unexpectedly) a relation between equations of motion, then we know that a gauge symmetry has to be present. We refer to these unexpected gauge symmetries as *emergent symmetries*.

2.4. Example: The Relativistic Point Particle

Let us apply the abstract ideas of the previous section to the simple example of the relativistic point particle, as presented in [24]. If we start with Minkowski space $\mathbb{M}^{1,D-1}$ and fix a frame with coordinates $(t, \vec{x})$, we have the action

$$S[\vec{x}] = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}. \tag{5}$$

Computing the equations of motion and the energy, we find the known description of a relativistic point particle

$$\vec{p} = \frac{m\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}}, \quad E^2 = m^2 + \vec{p} \cdot \vec{p}.$$

So the action (5) describes relativistic dynamics, but it has some drawbacks. Time and space are not on equal footing in the action. While t is the curve parameter, the $D - 1$ positions $\vec{x}$ are the only dynamical variables of the theory. In this form, it is also not clear how Lorentz transformations are supposed to act on the system.

Let us instead start with the different action

$$\tilde{S}[X^\mu] = -m \int d\tau \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}. \tag{6}$$

Our dynamical degrees of freedom are now the D-many coordinates $X^\mu(\tau)$. Lorentz or rather Poincare symmetry is now immediately clear. But we now also have a gauge symmetry present, as the action is invariant under reparametrizations

$$\tau \mapsto \tau' = \tau'(\tau).$$

Infinitesimally, we can express reparametrizations as small time translations

$$\delta\tau = \varepsilon(\tau),$$

described by the single parameter ε. This immediately tells us, that not all of D components of X^μ are independent, and we should only have $D-1$ independent dynamical degrees of freedom.

We can now remove this artificial degree of freedom, by using the reparametrization invariance to set

$$\tau = X^0(\tau) \equiv t.$$

Doing this, we say that we fix the reparametrization invariance. In this gauge, the actions $\tilde{S}[X^\mu]$ and $S[\vec{x}]$ have the exact same form.

When not fixing the gauge, thus keeping the extra degree of freedom, we find a corresponding Noether identity. The variation of the coordinate reads

$$\delta X^\mu = -\dot{X}^\mu \varepsilon.$$

Hence, the variation of the action reads

$$\begin{aligned} 0 = \delta S[X] &= m \int d\tau \, \frac{d}{d\tau}\left(\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\right)\delta X^\mu \\ &= -m \int d\tau \, \frac{d}{d\tau}\left(\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\right)\dot{X}^\mu \varepsilon. \end{aligned}$$

This has to hold for any continuous function ε, thus we can read off the constraint

$$\frac{d}{d\tau}\left(\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\right)\dot{X}^\mu = 0,$$

which lets us remove exactly one equation of motion. Introducing the relativistic momenta

$$p_\mu = \frac{m\dot{X}_\mu}{\sqrt{-\dot{X}^\alpha \dot{X}_\alpha}},$$

we can also rephrase the Noether identity by writing the following relation for the momentum

$$\frac{d}{d\tau}\left(p_\mu p^\mu\right) = 0.$$

Ultimately, this amounts to the on-shell condition of special relativity

$$p_\mu p^\mu + m^2 = 0.$$

In a Hamiltonian formulation, we would have found the last equation as single constraint equation for the Hamiltonian system.

It is important to stress that the reparametrization symmetry does not map one solution of equations of motion to another but relates the same solution in different descriptions. Fixing a gauge may change the form of the action, but not the actual physics.

Also note that the action (6) can be easily coupled to an arbitrary background metric g via $\eta_{\mu\nu} \mapsto g_{\mu\nu}$.

3. General Relativity as a Gauge Theory

In this section, we want to introduce general relativity through the lens of gauge theories. This will give us a proper example of how symmetries give rise to the geometric structures underlying gravitational theories.
Here, we not only aim to introduce the basic notions of general relativity in a modern vielbein formalism but want to clarify in what sense the *Poincaré-symmetry* determines relativistic geometry.
This is important in its own right but also needed for our introduction to non-relativistic geometry in section 4, first to know what the proper starting point for any kind of non-relativistic limit should be and secondly, to reveal the changes resulting from the different choice of symmetries later.

3.1. General Relativity

3.1.1. The Vielbein Formalism

In this part, we may only cover the bare basics of general relativity in the vielbein formalism. For a thorough treatment see [22, Ch. 7]. Before we turn to the main objects of this part, let us first clarify some language.
It is often stated that general relativity is characterized by its diffeomorphism invariance. While we will see later that this needs to be refined, diffeomorphisms take a prime role in formulating the theory.
For a smooth manifold M of dimension D, a diffeomorphism $F\colon M \to M$ is a smooth bijective map, which also has a smooth inverse. It is best to think of diffeomorphisms as mere changes of coordinates, as we do not distinguish between diffeomorphic manifolds[3]. While they form a group, it is infinite-dimensional and thus quite complicated to work with.
We consequently only work with the linearized version, often called "infinitesimal diffeomorphisms", which are given by an arbitrary small coordinate change

$$\delta x^\mu = \xi^\mu(x).$$

These are naturally identified with vector fields on M and their action on any tensor T can be calculated to be the Lie derivative along ξ

$$\delta(\xi)T = \mathcal{L}_\xi T.$$

To fully step into the realm of general relativity, we need to introduce a metric on the manifold, a symmetric, non-degenerate 2-tensor $g\colon TM \times TM \to \mathcal{C}^\infty(M)$ of Lorentzian signature $(1, D-1)$, which we usually refer to by its components $g_{\mu\nu}$ with respect to some chart. It is only after the introduction of a metric that we have a notion of a (pseudo-)Riemannian geometry on a manifold. One of the key ideas of general relativity is to take the metric and hence the geometry to be a dynamical quantity.

[3] A fact which leads to some quite subtle questions, usually found under the word "hole problem".

As we try to keep track of symmetries, so far, we only have the action of diffeomorphisms on the metric via $\delta(\xi)g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$.

By Gram-Schmidt, we can always find an orthonormal basis for $g(x)$ for each tangent space T_xM. It turns out that we can extend this choice (locally) to a frame of the tangent bundle. This so-called *co-moving frame* is denoted by $\{E^\mu{}_A\}_{A=0}^{D-1}$ and its defining relation is

$$g_{\mu\nu}E^\mu{}_A E^\nu{}_B = \eta_{AB}.$$

This finally lets us introduce the *vielbeine* $E_\mu{}^A$ as the corresponding dual basis of the cotangent bundle: $E_\mu{}^A E^\mu{}_B = \delta^A{}_B$. The defining relation for the vielbeine is then

$$g_{\mu\nu} = \eta_{AB}E_\mu{}^A E_\nu{}^B. \tag{7}$$

This equation, however, seems to be problematic, as we have $D(D+1)/2$ independent components for g, while we apparently have D^2 independent components of the vielbein. As mentioned in the section on gauge theory, such redundancies should always catch our attention, as they are usually due to some symmetry in our description. Indeed, the vielbeine are not uniquely defined by eq. (7), as it is invariant under (gauged) Lorentz transformations of the form $E_\mu{}^A \mapsto A(x)^A{}_B E_\mu{}^B$. Since the dimension of this group is $D(D-1)/2$, it exactly cancels the undesired degrees of freedom in our description.
As the vielbeine are forms, we know how they transform under diffeomorphisms. We thus find their infinitesimal symmetries to be

$$\delta(\xi,\Lambda)E_\mu{}^A = \mathcal{L}_\xi E_\mu{}^A + \Lambda^A{}_B E_\mu{}^B,$$

for a vector field ξ and $\Lambda_{AB} = -\Lambda_{BA}$ a function of spacetime with values in $\mathfrak{so}(1, D-1)$, the Lie algebra of the Lorentz group $O(1, D-1)$.
Usually, we require the preservation of time orientation and space orientation of our spacetime, which leads us to the Lie group $SO^+(1, D-1)$. This, coincidentally, is exactly the connected component of $O(1, D-1)$. But as we are mainly interested in infinitesimal actions, we only need to consider the corresponding Lie algebras, which are naturally identical.
Hence, we usually drop the plus and, identifying algebra and group, we call this a $SO(1, D-1)$ symmetry.

While we introduced the index A simply to enumerate the frames, we now see that we can interpret it as a representation index transforming under (part of) the isometry-group of *flat* Minkowski space.
We thus call capital Latin indices $A, B, ...$ *flat* and Greek indices $\mu, \nu, ...$ are called *curved*.
By introducing co-moving frames and the vielbeine, we can express all tensors in terms of flat indices, i.e.

$$T_A := E^\mu{}_A T_\mu$$
$$T^A := E_\mu{}^A T^\mu,$$

and similarly for higher tensors. These tensors do not transform under diffeomorphisms anymore, but rather under (local) Lorentz transformations. We will call such a quantity a *spacetime scalar*. Switching to flat indices has the benefit that now all tensors are covariant quantities, unlike tensors with curved indices which transform under the Lie derivative, which always includes a derivative of a parameter.
Clearly, we can then express the metric acting on two vector fields V, W as

$$g_{\mu\nu} V^\mu W^\nu = \eta_{AB} V^A W^B.$$

Thus, we have found an isomorphism between each tangent space and flat Minkowski space.

As we are now firmly in the realm of gauge theory, we want to identify covariant quantities. Clearly, the exterior derivative of the vielbein does not transform covariantly, as it picks up a derivative of the parameter of Lorentz transformations $\delta dE^A = d(\Lambda^A{}_B E^B)$. As discussed in sec. 2, this may be resolved by introducing a $\mathfrak{so}(1, D-1)$-gauge field ${\Omega_\mu}^{AB}$ transforming as

$$\delta {\Omega_\mu}^{AB} = \partial_\mu \Lambda^{AB} - 2\Lambda^{[A}{}_C {\Omega_\mu}^{B]C}.$$

This connection is also called *spin connection*, a name that becomes clear when studying spinors (for details see [20]). We can then define the following covariant quantity in terms of the covariant derivative $\mathcal{D}$ with respect to Ω as

$$2\mathcal{D}_{[\mu} {E_{\nu]}}^A = 2\partial_{[\mu} {E_{\nu]}}^A - 2{\Omega_{[\mu}}^{AC} E_{\nu]C} =: {\mathcal{T}_{\mu\nu}}^A \tag{8}$$

or coordinate free as

$$\mathcal{D}E^A = dE^A - \Omega^A{}_C \wedge E^C = \mathcal{T}^A.$$

This equation is known as *Cartan's first structure equation.* The 2-form $\mathcal{T}^A$ is called *torsion* and transforms under diffeomorphisms, as well as Lorentz transformations

$$\delta(\Lambda)\mathcal{T}^A = \Lambda^A{}_B \mathcal{T}^B.$$

We will see how this tensor relates to the torsion known from general relativity in metric formulation and how it can be interpreted as the curvature of a connection later. It is also interesting that torsion appears more generally when studying so called G-structures, for a brief introduction see [25, Sec. 2.1].
Cartan's first structure equation is also referred to as a *conventional constraint.* Since it contains the spin connection only algebraically, we can solve for it in terms of the vielbein E and the torsion $\mathcal{T}$, consequently rendering the constraint an identity.

By requiring that $\mathcal{D}$ acts as a derivative on functions $f\colon M \to \mathbb{R}$ and that it is compatible with the tensor product, we can extend it such that it maps tensors to tensors. However, this sentence is imprecise, as we have introduced two different kinds of indices, flat and curved ones.

The action on tensors with all flat indices is completely clear by requiring the resulting tensor to be a covariant quantity, resulting in

$$\begin{aligned}\mathcal{D}_\mu T^{A_1\cdots A_p}{}_{B_1\cdots B_q} =&\partial_\mu T^{A_1\cdots A_p}{}_{B_1\cdots B_q} - \Omega_\mu{}^{A_1}{}_C T^{C\cdots A_p}{}_{B_1\cdots B_q} - \cdots - \Omega_\mu{}^{A_p}{}_C T^{A_1\cdots C}{}_{B_1\cdots B_q}\\ &- \Omega_{\mu B_1}{}^C T^{A_1\cdots A_p}{}_{C\cdots B_q} - \cdots - \Omega_{\mu B_q}{}^C T^{A_1\cdots A_p}{}_{B_1\cdots C}\,.\end{aligned}$$

But on the other hand, we already know that in the metric formulation, we have a covariant derivative ∇, with affine connection form $\Gamma^\alpha{}_{\mu\nu}$, that precisely covariantizes curved indices. As we can freely switch from curved to flat indices via the vielbein, we require that these two notions of covariant derivatives agree, i.e. that

$$\nabla_\mu V^\alpha = E^\alpha{}_A \mathcal{D}_\mu V^A.$$

This condition can be formulated in terms of the connection forms and the vielbein as the so called *vielbein postulate*

$$\partial_\mu E_\nu{}^A - \Omega_\mu{}^A{}_C E_\nu{}^C - E_\alpha{}^A \Gamma^\alpha{}_{\mu\nu} = 0.$$

If we anti-symmetrize this in the curved indices, we get the relation

$$2E_\alpha{}^A \Gamma^\alpha{}_{[\mu\nu]} = \mathcal{T}_{\mu\nu}{}^A. \tag{9}$$

So indeed, the torsion $\mathcal{T}$ is precisely the torsion of the connection that covariantizes curved indices.

From the vielbein postulate eq. (9) we can solve Γ in terms of the vielbein and the spin connection as

$$\Gamma^\alpha{}_{\mu\nu} = E^\alpha{}_A (\partial_\mu E_\nu{}^A - \Omega_\mu{}^A{}_C E_\nu{}^C).$$

Since ∇ naturally acts on an affine vector bundle, i.e. the tangent bundle, we call such a connection Γ an *affine connection*.

As mentioned before, we can also solve Cartan's first structure equation (8) to express the spin connection in terms of the vielbein and the torsion. If we introduce the notation

$$E_{\mu\nu}{}^A := \partial_{[\mu} E_{\nu]}{}^A$$

we can express the, now dependent on E and $\mathcal{T}$, spin connection as

$$\Omega_\mu{}^{AB} = E_{\mu C} E^{ABC} - 2E_\mu{}^{[AB]} + \mathcal{T}_\mu{}^{[AB]} - \frac{1}{2} E_{\mu C} \mathcal{T}^{ABC}.$$

Consequently, we can also express the corresponding affine connection via the vielbein and torsion or, equivalently, via the metric and the torsion $\Gamma = \Gamma(g, \mathcal{T})$.

If we require vanishing torsion of the connection, the unique torsion free solution, called *Levi-Civita spin connection*, reads

$$\Omega_\mu{}^{AB} = E_{\mu C} E^{ABC} - 2E_\mu{}^{[AB]}. \tag{10}$$

The corresponding affine connection $\Gamma(g)$ then corresponds to the infamous, unique, torsion free Levi-Civita connection. As we will see in section 3.1.2, this is the connection governing general relativity.

Independent of whether we use the torsionless or torsional connection, we can assemble another covariant quantity, the curvature

$$R_{\mu\nu}{}^{AB} = 2\partial_{[\mu}\Omega_{\nu]}{}^{AB} + 2\Omega_{[\mu}{}^{AC}\Omega_{\nu]}{}^{B}{}_{C}; \tag{11}$$

Or in a basis-free notation, we get the following equation

$$R^{AB} = d\Omega^{AB} + \Omega^{AC} \wedge \Omega^{B}{}_{C},$$

called *Cartan's second structure equation.* When we express this abstract Lorentz curvature in terms of the dependent spin connection, we recover precisely the Riemann curvature tensor known from general relativity. It is this notion of curvature that we utilize when formulating dynamics, as of now, we only have quantified the kinematics of general relativity.

3.1.2. Dynamics for General Relativity and the Palatini Formalism

So far we have introduced 4 fields: The vielbein, the spin connection, torsion and the curvature. We have seen that only the vielbein and the torsion are independent. A priori we do not know if we should assume zero torsion and furthermore, we want and need to prescribe dynamics, most conveniently through the use of an action principle. To this end, we introduce the *Ricci scalar* or scalar curvature as

$$R \coloneqq -E^{\mu}{}_{A}E^{\nu}{}_{B}R_{\mu\nu}{}^{AB}.$$

The governing action of general relativity, thought of by Hilbert and called the *Einstein-Hilbert action*, is given as the one extremizing the scalar curvature i.e.

$$S_{\text{EH}} = \frac{1}{2\kappa^2}\int d^D x E R. \tag{12}$$

Here we have introduced $E \coloneqq \det\big[(E_{\mu}{}^{A})\big]$ and the gravitational constant κ, to render the action dimensionless.
Note that we do not prescribe the variables of this action yet, as there are two ways to think about this.

The first way is the so-called *second order formalism.* Here we assume a torsion free connection, which is entirely determined by the vielbein via eq. (10) as $\Omega = \Omega(E)$. Thus, also our action has only the vielbein as dynamical variable $S_{\text{EH}}[E] = S_{\text{EH}}[E, \Omega(E)]$. Varying the action then gives

$$\delta_E S_{\text{EH}}[E] = \frac{1}{\kappa^2}\int d^D x E \left\{ R_{\mu\nu}{}^{AB}E^{\nu}{}_{B} - \frac{1}{2}E_{\mu}{}^{A}R \right\} \delta E^{\mu}{}_{A}. \tag{13}$$

We see in the bracket the well-known *vacuum Einstein equations*

$$G_\mu{}^A := R_{\mu\nu}{}^{AB} E^\nu{}_B - \frac{1}{2} E_\mu{}^A R = 0,$$

with G the corresponding *Einstein Tensor*. Note that in the derivation of this equation, we use that the variation of $R_{\mu\nu}{}^{AB}$ is proportional to a divergence, hence we can discard it by integration by parts.

The Einstein-Hilbert Lagrangian also provides another example of a Noether identity, as introduced in section 2.3. We have the SO-gauge symmetry for the co-moving frame, i.e. $\delta(\Lambda) E^\mu{}_A = -\Lambda^B{}_A E^\mu{}_B$, and plugging this into the variation of the action in eq. (13) leads to

$$0 = \frac{1}{\kappa^2} \int d^D x E \left(G_\mu{}^A \right) \delta E^\mu{}_A = -\frac{1}{\kappa^2} \int d^D x E \left(G_\mu{}^A \right) \Lambda^B{}_A E^\mu{}_B = -\frac{1}{\kappa^2} \int d^D x E \left(G_{AB} \right) \Lambda^{AB}.$$

This has to hold for any (continuous) $\Lambda \in \mathfrak{so}$, i.e. $\Lambda_{AB} = \Lambda_{[AB]}$, hence we can conclude

$$G_{[AB]} \equiv 0,$$

so that the Einstein Tensor G is symmetric.

We can couple gravity to bosonic matter, adding a term $S_{\text{boson}} = \int d^D x E L$ to the action, making sure we use minimal coupling, and varying the total expression accordingly. But this only gives a contribution in terms of an *energy-momentum tensor*

$$T_\mu{}^A := -2 \frac{1}{E} \frac{\delta(EL)}{\delta E^\mu{}_a}.$$

Then we arrive at the full Einstein equations

$$G_\mu{}^A = R_{\mu\nu}{}^{AB} E^\nu{}_B - \frac{1}{2} E_\mu{}^A R = \kappa^2 T_\mu{}^A.$$

The second point of view, known as *first order formalism*, assumes the spin connection Ω is an independent field too. We thus arrive at the *Palatini action*

$$S_{\text{P}}[E, \Omega] := S_{\text{EH}}[E, \Omega].$$

We now get additional equations of motion when varying with respect to the spin connection namely

$$\delta_\Omega S_{\text{P}}[E, \Omega] = \frac{1}{2\kappa^2} \int d^D x E \; 3 \left\{ \mathcal{D}_\mu E_\nu{}^A E^\mu{}_{[A} E^\nu{}_B E^\alpha{}_{C]} \right\} \delta \Omega_\alpha{}^{BC}.$$

Which can be seen to result in the following set of equations

$$G_\mu{}^A := R_{\mu\nu}{}^{AB} E^\nu{}_B - \frac{1}{2} E_\mu{}^A R = 0, \tag{14}$$

$$\mathcal{T}_{\mu\nu}{}^A \equiv 2\partial_{[\mu} E_{\nu]}{}^A - 2\Omega_{[\mu}{}^{AC} E_{\mu]C} = 0. \tag{15}$$

Hence, we see that we derive the zero torsion condition we imposed earlier in the second order formalism. Solving the torsion equation for the connection and substituting this in the Palatini action recovers the second order formalism.

3.2. Gauging the Poincaré Algebra

In the last section, we have seen how Lorentz symmetries arise naturally when describing general relativity via Lorentzian geometry. We will now turn this logic around and find a way to recover Lorentzian geometry if we start from a set of symmetries and consider a gauge theory of them.

3.2.1. The Symmetries

As starting point we take the isometries of standard flat Minkowski space $\mathbb{M}^{1,D-1}$, meaning the most general set of transformations $F\colon \mathbb{R}^D \to \mathbb{R}^D$ that leave the standard constant Minkowski metric η invariant:

$$F^*\eta = \eta.$$

Here, we denoted the pullback under F as F^*. These symmetries are then given by maps

$$F_{A,a}(x) = Ax + a,$$

where $A \in O(1, D-1)$ is a Lorentz transformation and $a \in \mathbb{R}^{\mathbb{D}}$ a spacetime translation. They form the *Poincaré group*

$$\mathcal{P} = \{(A, a) | A \in O(1, D-1),\ a \in \mathbb{R}^{\mathbb{D}}\},$$

with the composition law

$$(A, a) \cdot (B, b) = (AB, Ab + a).$$

Thus, we see that the Poincaré group has the form of a semi-direct product

$$\mathcal{P} = O(1, D-1) \ltimes \mathbb{R}^D.$$

We turn our attention to the infinitesimal action on Minkowski space, which is given by

$$\delta(\Lambda, \xi)x^A = \Lambda^A{}_B x^B + \xi^A, \tag{16}$$

for $\Lambda \in \mathfrak{so}(1, D-1)$ and $\xi \in \mathbb{R}^D$. This is precisely the case of spacetime symmetries we discussed in section 2, where the parameters could act directly on spacetime.
Of course, we still have the abstract algebra, and we associate, or rather have already associated implicitly, a basis to the parameters. This basis is usually labeled as

$$\{J_{[AB]}, P_C\}$$

and is of dimension $D(D-1)/2 + D$. The P's are the generators of translations, while the J's are the generators of Lorentz transformations. We conveniently grouped them together in an antisymmetric form. This gives us the combined index $[AB]$, which corresponds to a single Lie algebra index in section 2 on representation theory.

In any case, we can now read off the commutation relations of the generators by using equation (16), its linearity, that it is a representation[4] of the algebra and find the commutation relations to be

$$
\begin{aligned}
[P_A, P_B] &= 0 \\
[P_A, J_{[BC]}] &= 2\eta_{A[B}P_{C]} \\
[J_{[AB]}, J_{[CD]}] &= 4\eta_{[A[C}J_{D]B]}.
\end{aligned}
$$

From this we can immediately read off the structure constants as

$$
\begin{aligned}
f_{AB}{}^{C} &= 0 \\
f_{A[BC]}{}^{D} &= 2\eta_{A[B}\delta^{D}{}_{C]} \\
f_{[AB][CD]}{}^{[EF]} &= 8\eta_{[A[C}\delta^{[E}{}_{D]}\delta^{F]}{}_{B]}.
\end{aligned} \tag{17}
$$

From the commutation relation we can infer two more things. First, recall that the commutator defines the adjoint representation of the algebra onto itself. When we now consider the action of a Lorentz transformation on the generator of translations, we find

$$\rho^*(\Lambda)[P_A] = \frac{1}{2}\Lambda^{BC}[P_A, J_{[BC]}] = \Lambda_A{}^{C}P_C.$$

Thus, we see that P^A in the adjoint representation transforms exactly as x^A in the defining representation. As we identify objects with their representation, we say that P^A *is* a vector due to its commutation relation.
Note, we introduced a factor of 2! when summing over the grouped index $[BC]$ to avoid over-counting.

Secondly, we know that structure constants are basis dependent. If we choose a different basis or simply relabel it, we will find different structure constants. One such relabeling is given by

$$
\begin{aligned}
H &\coloneqq P_0 \\
P_a &\equiv P_a \\
B_a &\coloneqq J_{[a0]} \\
J_{[ab]} &\equiv J_{[ab]}
\end{aligned}
$$

where $a = 1, ...D-1$. This relabeling makes clear the difference between the zero coordinate, i.e. the time, and space. H is the generator of time translations, which can be identified with the total energy or, in other words, with the Hamiltonian.
The B_a are called the *boosts*. Their action can best be understood via their defining representation on Minkowski space, when splitting space and time. It is easy to see, that they correspond to the Lorentz transformation with the parameters $\Lambda^a{}_b = 0$ and $\Lambda^a{}_0 \coloneqq \Lambda^a$. Their action is then given by

$$
\begin{aligned}
\delta_B x^0 &= -\Lambda_a x^a \\
\delta_B x^a &= \Lambda^a x^0.
\end{aligned}
$$

[4] In fact, it is the defining representation.

Note that space and time appear on equal footing as it should be for a relativistic theory. When considering non-relativistic symmetries later, this will no longer be the case. We will understand the implications of that in section 4.
In terms of the relabeled generators we have the new non-zero commutation relations

$$\begin{aligned}
[J_{[ab]}, J_{[cd]}] &= \delta_{[a[c} J_{d]b]}, \\
[P_a, J_{[bc]}] &= 2\delta_{a[b} P_{c]} \\
[B_a, J_{[bc]}] &= 2\delta_{a[b} B_{c]} \\
[B_a, B_b] &= -J_{ab} \\
[P_a, B_b] &= \delta_{ab} H \\
[H, B_a] &= P_a.
\end{aligned} \tag{18}$$

We see that the boosts transform as vectors under rotations, and they equally mix space and time translations into each other.
Commutation relations of similar form will appear in non-relativistic gravity again, and it is noteworthy that they can be interpreted in the theory of *kinematical Lie algebras*. This theory allows for a nice treatment and classification of possible spacetime symmetries (for details see [26]).

3.2.2. Gauging of the Poincaré Algebra

After having specified our desired symmetries, let us now gauge the algebra and consider its kinematics.
Again, formally we just replace the symmetry parameters with spacetime functions, but from section 2 we know the machinery to do this consistently. We start off with associating a gauge field to each generator of the symmetry algebra:

Generator	Parameter	Gauge field
P_A	ξ^A	$E_\mu{}^A$
$J_{[AB]}$	Λ^{AB}	$\Omega_\mu{}^{AB}$

In principle, we could view the two gauge fields as a single Poincaré-algebra-valued gauge field via

$$B_\mu = E_\mu{}^A P_A + \frac{1}{2}\Omega_\mu{}^{AB} J_{[AB]}.$$

This makes the relation to the algebra much clearer, and we can employ the language of Lie algebras directly, which has its benefits as well. For a short example see [27, Section 2]. In this thesis, however, we choose to work in all index notation. The variation of the gauge fields, as given by the structure constants in eq. (17), reads

$$\delta(\xi,\Lambda) E_\mu{}^A = \partial_\mu \xi^A - \Omega_\mu{}^A{}_B \xi^B + \Lambda^A{}_B E_\mu{}^B \equiv \mathcal{D}_\mu \xi^A + \Lambda^A{}_B E_\mu{}^B, \tag{19}$$

$$\delta(\xi,\Lambda) \Omega_\mu{}^{AB} = \partial_\mu \Lambda^{AB} - 2\Omega_\mu{}^{C[A} \Lambda^{B]}{}_C \equiv \mathcal{D}_\mu \Lambda^{AB}. \tag{20}$$

If we now compute the associated curvatures of the connections, we find

$$R_{\mu\nu}(P^A) = 2\partial_{[\mu}E_{\nu]}{}^A - 2\Omega_{[\mu}{}^{AC}E_{\mu]C},$$
$$R_{\mu\nu}(J^{AB}) = 2\partial_{[\mu}\Omega_{\nu]}{}^{AB} + 2\Omega_{[\mu}{}^{AC}\Omega_{\nu]}{}^{B}{}_{C}.$$

These are exactly the objects appearing in Cartan's first and second structure equations (8) and (11), which explains our suggestive notation for the connection of translations with the same symbol as the vielbein.
However, it is important to stress that the field $E_\mu{}^A$ of the current section is *not* the vielbein at this point. This can be inferred from eq. (19), as $E_\mu{}^A$ transforms under translations, a symmetry only approximately present in general relativity, but not under diffeomorphisms, one of the key symmetries of GR.
We will deal with this discrepancy soon, but if we ignore it for a moment, we can see that torsion is nothing else but the curvature of translations as can be inferred from section 3.1.1.

While discussing how spacetime symmetries act in field theories in section 2, we noticed that there seems to be a close relation to infinitesimal translations. This becomes more apparent by the following calculation. Recall that translations act on $E_\mu{}^A$ via the covariant derivative of the parameter. Assuming diffeomorphism with parameter ζ^μ act on the translation connection, we find

$$\begin{aligned}
\delta(\zeta)E_\mu{}^A &= \mathcal{L}_\zeta E_\mu{}^A = \zeta^\alpha\partial_\alpha E_\mu{}^A + \partial_\mu\zeta^\alpha E_\alpha{}^A \\
&= \partial_\mu(\zeta^\alpha E_\alpha{}^A) - \zeta^\alpha\partial_\mu E_\alpha{}^A + \zeta^\alpha\partial_\alpha E_\mu{}^A \\
&= \partial_\mu(\zeta^\alpha E_\alpha{}^A) - \Omega_\mu{}^A{}_C\zeta^\alpha E_\alpha{}^C + \Omega_\mu{}^A{}_C\zeta^\alpha E_\alpha{}^C - 2\zeta^\alpha\partial_{[\mu}E_{\alpha]}{}^A \\
&= \mathcal{D}(\zeta^\alpha E_\alpha{}^A) + \zeta^\alpha\Omega_\alpha{}^A{}_C E_\mu{}^C - \zeta^\alpha\left(2\partial_{[\mu}E_{\alpha]}{}^A - \Omega_\mu{}^A{}_C E_\alpha{}^C + \Omega_\alpha{}^A{}_C E_\mu{}^C\right) \\
&= \mathcal{D}(\zeta^\alpha E_\alpha{}^A) + \zeta^\alpha\Omega_\alpha{}^A{}_C E_\mu{}^C - \zeta^\alpha\left(2\partial_{[\mu}E_{\alpha]}{}^A - 2\Omega_{[\mu}{}^A{}_C E_{\alpha]}{}^C\right) \\
&= \mathcal{D}(\zeta^\alpha E_\alpha{}^A) + \zeta^\alpha\Omega_\alpha{}^A{}_C E_\mu{}^C - \zeta^\alpha R_{\mu\alpha}(P^A).
\end{aligned}$$

Henceforth, we can write the action of diffeomorphism as a translation by $\xi^A := \zeta^\alpha E_\alpha{}^A$, a rotation by $\Lambda^{AB} := \zeta^\alpha\Omega_\alpha{}^{AB}$ and an extra shift by $\zeta^\alpha R_{\mu\alpha}(P^A)$.
It is in this sense that we can interpret the translation connection as the vielbein, since it transforms precisely as required for a vielbein. Furthermore, it turns curved spacetime indices into flat algebra indices. This is apparent when we denote the vector field and its corresponding translation with the same letter

$$\xi^A := \xi^\alpha E_\alpha{}^A.$$

The extra term involving the torsion seems to spoil the exact correspondence. However, we can deal with it in two ways.

The first would be to assume vanishing torsion $R_{\mu\alpha}(P^A) = 0$. While this seems restrictive, we have already seen that in general relativity this is a quite natural choice. It further allows us to express the connection of Lorentz transformations via $E_\mu{}^A$, leaving only the connection of translations as the only independent field. Additionally, we can then interpret the curvature of Lorentz transformations as the Riemann curvature, as discussed in section 3.1.1 for $\mathcal{T} = 0$.

If we nonetheless want to avoid this restriction, we can define a modified variation under diffeomorphisms schematically as

$$\bar{\delta}(\xi)E = \mathcal{L}_\xi E + i_\xi R(P), \tag{21}$$

where i is the insertion operator on forms. This procedure is found sometimes in the literature (see e.g. [27]), it is however not completely clear how such a transformation should be interpreted or if it indeed defines a representation of the algebra. It should thus be interpreted as merely a formal procedure, which should correspond to a formulation of the problem in terms of $SO(1, D-1)$-structures. In such a formulation, diffeomorphisms are naturally included, and torsion, as well as curvature, remain completely arbitrary a priori.

Whichever variant we choose, we have now introduced diffeomorphisms into our gauge theory. We can interpret them via a kind of change of basis in the symmetry algebra, as they arise as a combination of translations and rotations. In conclusion, the full variation of the connections is given by[5]

$$\begin{aligned}
\delta(\xi, \Lambda){E_\mu}^A &= \mathcal{L}_\xi {E_\mu}^A + {\Lambda^A}_B {E_\mu}^B \\
\delta(\xi, \Lambda){\Omega_\mu}^{AB} &= \mathcal{L}_\xi {\Omega_\mu}^{AB} + \mathcal{D}_\mu \Lambda^{AB}.
\end{aligned}$$

As now ${E_\mu}^A$ transforms as required for a vielbein, we will no longer make the distinction and rightfully call it like that.
There is, however, still a discrepancy, as we do not know if ${E_\mu}^A$ can be inverted to give rise to a co-moving frame. We can just declare it to be invertible, then ${E_\mu}^A$ satisfies the definition of a *soldering form* and hence can be viewed as an isomorphism of the tangent bundle onto itself (for further details see [28]).

In conclusion, we have seen how Lorentzian geometry arises through the gauge theory of the Poincaré algebra. It is this connection of symmetry to the theory of gravity that will guide us in the non-relativistic case. One can continue from here, imposing dynamics via an appropriate gauge-invariant action or via equations of motion.

[5] Omitting bars if we have chosen the modified variation.

4. Non-Relativistic Geometry

In the last section, we have seen the close relationship between geometry and gravity in the relativistic case. In this section, we want to first motivate the correct symmetries and preserved structures for the non-relativistic geometry and discuss the corresponding algebra. We then turn the logic around, starting from the symmetries and gauging them, leaving us with a genuine understanding and description of non-relativistic gravity that is completely analogous to relativistic gravity.
Furthermore, we want to discuss how to derive the non-relativistic theory and symmetries by an appropriate infinite speed of light limit.

4.1. Motivation

The general motion of a free-falling observer $x^\mu(s)$, $s \in I \subseteq \mathbb{R}$, $\mu = 0, ...D-1$, i.e. in the case of pure gravity in the absence of any external forces, is governed by the geodesic (or autoparallel) equation

$$\nabla_{\dot{x}}\dot{x} = 0,$$

which in coordinates reads

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0.$$

We compare this to the equations of motion for a particle in Newtonian gravity

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}t^2} + \delta^{ab}\partial_b\phi = 0,$$

where the Latin index is considered to be a spacial one, taking values in $a = 1, ..., D-1$ and ϕ is the Newtonian potential. Originally, this was interpreted as the deviation from a straight line in space due to the presence of a gravitational force. However, following Einsteins train of thought, we could imagine that the deviation is due to a curvature in some kind of Newtonian spacetime. If we assume the Newtonian equation of motion is just formulated in the special spacetime coordinates $(x^\mu(t)) = (t, x^a(t))$, $\mu = 0, ..., D-1$ but otherwise completely analogous to GR, we can read off the only non-zero Christoffel symbols to be

$$\Gamma^a{}_{00} = \delta^{ab}\partial_b\phi. \tag{22}$$

The corresponding non-zero components of the curvature tensor are

$$R^a{}_{0b0} = \delta^{ac}\partial_c\partial_b\phi.$$

Imposing the equation of motion

$$R^a{}_{0a0} = 4\pi G_N\rho,$$

for G_N the Newton constant and ρ the mass density, recovers the Poisson equation of Newtonian gravity.

As of now, we have only rewritten Newtonian gravity as to mimic GR. It is not clear how to interpret this result geometrically, as we do not know any further geometric structure. Additionally, the curvature tensor differs in its index-structure from what we expect coming from GR. Nonetheless, one could follow this different approach, as was done in [29, Ch. 12].

4.1.1. From Non-Relativistic Symmetries to Geometry

Here however, we choose a more modern approach that, again, is strongly connected to the underlying symmetries. In what follows, we will mainly refer to [30].
First, it is well known that Newtonian gravity should contain the Galilei symmetries as structure group. We consider the Galilei transformations over a $D-$dimensional spacetime, and if we split space and time, it is given by the transformations

$$\begin{aligned} t' &= t + \zeta^0 \\ x'^a &= A^a{}_b x^b + v^a t + \zeta^a, \end{aligned}$$

where ζ^0 is a time translation, ζ^a a space translation, v^a is the boost parameter and A a spacial rotation $SO(D-1)$. This of course gives rise to an infinitesimal action of the Galilei algebra, which we will introduce later. At this point it is only important that boosts and translations commute with each other.
While splitting space and time is completely natural in Newtonian geometry, as we will see later, it is still convenient to consider these transformations on spacetime, where they take the form

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + \zeta^\mu,$$

with

$$\Lambda = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}.$$

If we examine which symmetric 2-tensors are preserved by the Galilei group, we do not find a proper metric and a proper inverse metric but rather a *temporal metric*

$$(t_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 0_{D-1} \end{pmatrix},$$

with 0_{D-1} the $D-1$-dimensional square zero matrix, and a *spacial co-metric*

$$(h^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{D-1} \end{pmatrix}.$$

So the Galilei group keeps two degenerate "metrics" invariant that are mutually orthogonal

$$t_{\mu\nu}h^{\nu\rho} = 0.$$

It is this "metric" structure that is the starting point for describing Newtonian gravity analogously to general relativity.
Similarly, we can also introduce two kinds of vielbeine associated to each metric, by defining the *temporal vielbein* or *clock form* τ_μ and the *spacial vielbein* ${e_\mu}^a$ via

$$\begin{aligned} t_{\mu\nu} &= \tau_\mu \tau_\nu, \\ \delta^{ab} &= h^{\mu\nu} {e_\mu}^a {e_\nu}^b. \end{aligned}$$

Now, we carefully observe what the most general transformations rules are that leave this relation invariant. While the temporal vielbein does not seem to have any viable transformations, the index structure of the spacial vielbein hints at several symmetries. First, since δ^{ab} can be seen as the defining structure of spacial rotational symmetry $SO(D-1)$, we can also assume that e transforms accordingly. Additionally, the second relation also remains invariant, if e transforms under boosts to something proportional to τ_μ, due to the orthogonality with h. Since we assumed that both vielbeine are forms, we find in total the transformation rules

$$\begin{aligned} \delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu, \\ \delta {e_\mu}^a &= \mathcal{L}_\xi {e_\mu}^a + {\lambda^a}_b {e_\mu}^b + \lambda^a \tau_\mu, \end{aligned} \tag{23}$$

for an infinitesimal spacial rotation ${\lambda^a}_b$, an infinitesimal boost λ^a and an infinitesimal diffeomorphisms ξ. Thus, we have formalized Newtonian geometry in a coordinate invariant fashion.

The spacial vielbein can then be interpreted as transforming the spacelike directions in tangent space to flat $\mathbb{R}^{D-1}$ with the standard inner product. Since both metrics are degenerate, they do not have proper inverses, but we can at least define *projective inverses* $\tau^\mu, {e^\mu}_a$ characterized by

$${e_\mu}^a \tau^\mu = 0, \quad {e^\mu}_a \tau_\mu = 0, \quad {e_\mu}^a {e^\mu}_b = {\delta^a}_b, \quad \tau_\mu \tau^\mu = 1, \quad {\delta^\mu}_\nu = \tau^\mu \tau_\nu + {e^\mu}_a {e_\nu}^a. \tag{24}$$

This now allows us to split tensors with curved Greek indices $\mu, \nu, ...$ into a temporal and a spacelike part due to the last equation, since

$$T_\mu = {\delta^\nu}_\mu T_\nu = (\tau^\nu T_\nu)\tau_\mu + ({e^\nu}_a T_\nu){e_\mu}^a =: T_0 \tau_\mu + T_a {e_\mu}^a$$

for an arbitrary form T, and analogously for arbitrary tensors. We will see later that the temporal and the spacelike part of a tensor then transform exactly like the corresponding (inverse) vielbein and that they will be hugely beneficial in calculations.

4.1.2. Completing the Symmetries

Ultimately, the question remains whether the Galilei transformations are the entire symmetry group of Newtonian physics. To answer this question, we analyze one of the simplest systems available, the free Newtonian particle. This matter model can latter be coupled to Newtonian gravity and its action is given in spacetime coordinates $(x^\mu) = (x^0, x^a)$ by

$$S[x^\mu] = \frac{m}{2} \int d\tau \frac{\delta_{ab}\dot{x}^a\dot{x}^b}{\dot{x}^0}. \tag{25}$$

The inclusion of $\dot{x}^0$ appears strange at first but ensures invariance of the action under change of parametrization $\tau \to \tau'(\tau)$, similar to the relativistic particle in section 2. We recover the more familiar expression of the kinetic energy upon imposing the gauge $x^0 = \tau$.

The Galilei transformations take the infinitesimal form

$$\begin{aligned} \delta x^0 &= \zeta^0 \\ \delta x^a &= \lambda^a{}_b x^b + \lambda^a x^0 + \zeta^a, \end{aligned}$$

where $\zeta^\mu \in \mathbb{R}^D$ is a spacetime translation, $\lambda^a{}_b \in \mathfrak{so}(D-1)$ an infinitesimal rotation and $\lambda^a \in \mathbb{R}^{D-1}$ an infinitesimal boost. Again, we stress that in this case boosts and translations commute.
From a quick calculation we can infer that the above action (25) remains inert under translations and rotations but is only *quasi invariant* under boosts, i.e. only invariant up to the total derivative

$$\delta_B S = \int d\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\left(m\lambda_i x^i\right) \equiv 0.$$

It is well known (see [23]) that for quasi invariance the corresponding conserved Noether charge for the generator T is given by

$$Q_T = K_T - p_\mu \delta_T x^\mu,$$

where $K_T\colon \mathbb{R} \to \mathbb{R}$ is such that $\delta_T \mathcal{L} = \dot{K}_T$ and $p_\mu := \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$ is the momentum. Furthermore, the Noether charges generate the corresponding symmetries when turned into a vector field via the Poisson bracket, i.e.

$$\delta_T f = \{f, Q_T\},$$

for an arbitrary function $f\colon M \to \mathbb{R}$, in particular for $f = x^\mu$. Notably, we get a representation of the symmetries via the charges and the Poisson bracket (for details see [31, Ch. 11]) as

$$\{Q_{T_A}, Q_{T_B}\} = f_{AB}{}^C Q_{T_C}.$$

Naively, we expect this to be just the Galilean symmetries, where we noticed that boost and translations were commuting. However, if we consider the Noether charges of the spacial translations and of the boosts we find from the action, i.e.

$$Q_P = -p_a \zeta^a$$
$$Q_B = m\lambda_a x^a - p_a \lambda^a x^0$$

and calculate their Poisson bracket, we find that

$$\{Q_B, Q_P\} = -m\lambda_a \zeta^a.$$

Contrasting this with the Galilei algebra, where the boosts and translations were commuting, we see the emergence of an additional charge.
This novel charge commutes with all other Noether charges and is thus represented by something proportional to the identity. Such a charge is called a *central charge*. If we relate this charge to a new generator M, commuting with all other generators, we find an extension of the Galilei algebra with central charge M, known as a *central extension*. This central extension of the Galilei algebra is called *Bargmann algebra*, and by our analysis above, it is the correct choice for the full symmetries of Newtonian physics, i.e. Newtonian gravity. The extra generator is related to the fact that in Newtonian physics mass is a conserved quantity as well. In fact, we will see this relation when gauging the Bargmann algebra in section 4.2, where the corresponding gauge field will play a prime role in encoding the gravitational dynamics.

For now, note that we get an invariant action if we introduce an extra coordinate s to our spacetime and include it in the action as

$$S[x, s] = \frac{m}{2} \int d\tau \left(\frac{\delta_{ab} \dot{x}^a \dot{x}^b}{\dot{x}^0} + 2\dot{s} \right), \tag{26}$$

provided that s is inert under rotations, transforms into a constant under translations and under boosts transforms as

$$\delta s = -\lambda_a x^a.$$

Importantly, the momentum conjugate to s is the mass m, i.e.

$$p_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} \equiv m,$$

and since s is a cyclic coordinate, the mass m is a conserved quantity. This shows how the extra symmetry given by the central charge implements mass conservation in Newtonian physics.

4.2. Gauging of the Bargmann Algebra

In the last section we have motivated that the correct set of symmetries in Newtonian gravity is the Bargmann algebra. In 4.3.2 we will show how to properly derive the algebra, but for now, we only want to present it and describe the geometry that follows from gauging it, which we will call *Newton-Cartan geometry*.

4.2.1. The Bargmann Algebra

The Bargmann algebra $\mathfrak{barg}$ is given by a choice of basis J_{ab} for rotations, P_a for spacial translations, H for time translations, B_a for boosts and M for the central extension generator, where indices $a = 1, ...D-1$ denote spacial indices. To make it an algebra, we define its following non-zero commutation relations

$$\begin{aligned}
[J_{ab}, J_{cd}] &= 4\delta_{[a[c}J_{d]b]}, \\
[P_a, J_{bc}] &= 2\delta_{a[b}P_{c]}, \\
[B_a, J_{bc}] &= 2\delta_{a[b}B_{c]}, \\
[H, B_a] &= P_a, \\
[P_a, B_b] &= \delta_{ab}M.
\end{aligned} \tag{27}$$

Again, we can see that the boosts and translations transform as vectors under rotations, and they commute into central charge transformations. Comparing the above commutation relations of the Bargmann algebra with the ones of the Poincaré algebra in eq. (18) we can already infer that now space and time are not on equal footing, as time translations transform into space translations under boosts, but space translations under boosts transform into central charge transformations and not into time translations.

4.2.2. Gauging Procedure

We now apply the same gauging procedure we have seen when gauging the Poincaré symmetries, and start by associating gauge fields to each generator:

Generator	Parameter	Gauge field
H	ξ^0	τ_μ
P_a	ξ^a	$e_\mu{}^a$
$J_{[ab]}$	λ^{ab}	$\omega_\mu{}^{ab}$
B_a	λ^a	$\omega_\mu{}^a$
M	σ	m_μ

It is again no coincidence that the gauge field of the time and space translations are denoted by the same symbols as the non-relativistic vielbeine, and we will see the relation once we introduce diffeomorphisms into the theory.
From their definition as gauge fields and the commutation relations of $\mathfrak{barg}$ we can immediately write down their variations

$$\begin{aligned}
\delta\tau_\mu &= \partial_\mu\xi^0, \\
\delta e_\mu{}^a &= \partial_\mu\xi^a - \omega_\mu{}^a{}_b\xi^b + \lambda^a{}_b e_\mu{}^b + \lambda^a\tau_\mu - \omega_\mu{}^a\xi^0, \\
\delta\omega_\mu{}^a &= \partial_\mu\lambda^a - \omega_\mu{}^a{}_b\lambda^b + \omega_\mu{}^b\lambda^a{}_b, \\
\delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab} - 2\lambda^{[a}{}_c\omega_\mu{}^{b]c}, \\
\delta m_\mu &= \partial_\mu\sigma - \omega_\mu{}^a\xi_a + \lambda_a e_\mu{}^a.
\end{aligned} \tag{28}$$

We again see that space and time are not on equal footing, as the spacial vielbein transforms into the temporal vielbein under boosts but not vice versa. It is rather the central charge vielbein which under boosts transforms into the spacial vielbein.

Given the variations of the gauge fields, we can also immediately compute their curvatures

$$\begin{aligned} R_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]}, \\ R_{\mu\nu}(P^a) &= 2\partial_{[\mu}e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}e_{\nu]b} - 2\omega_{[\mu}{}^{a}\tau_{\nu]}, \\ R_{\mu\nu}(J^{ab}) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]}{}^{b}{}_{c}, \\ R_{\mu\nu}(B^a) &= 2\partial_{[\mu}\omega_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{ab}\omega_{\nu]b}, \\ R_{\mu\nu}(M) &= 2\partial_{[\mu}m_{\nu]} - 2\omega_{[\mu}{}^{a}e_{\nu]a}. \end{aligned} \tag{29}$$

As in general relativity, we want to have diffeomorphisms as symmetries and not translations. Again, in complete analogy, we can compute how we can express diffeomorphisms acting as a Lie derivative via other symmetries. If we compute the variation of (τ, e, m) under diffeomorphisms (ξ^μ), we find that

$$\begin{aligned} \delta_\xi\tau_\mu &\equiv \mathcal{L}_\xi\tau_\mu = \partial_\mu\left(\xi^\alpha\tau_\alpha\right) - \xi^\alpha R_{\mu\alpha}(H), \\ \delta_\xi e_\mu{}^a &= \partial_\mu\left(\xi^\alpha e_\alpha{}^a\right) - \omega_\mu{}^a{}_b\xi^\alpha e_\alpha{}^b + \xi^\alpha\omega_\alpha{}^a{}_b e_\mu{}^b + \xi^\alpha\omega_\alpha{}^a\tau_\mu - \omega_\mu{}^a\xi^\alpha\tau_\alpha - \xi^\alpha R_{\mu\alpha}(P^a), \\ \delta_\xi m_\mu &= \partial_\mu\left(\xi^\alpha m_\alpha\right) - \omega_\mu{}^a\xi^\alpha e_{\mu a} + \xi^\alpha\omega_{\alpha a}e_{\mu^a} - \xi^\alpha R_{\mu\alpha}(M). \end{aligned}$$

By comparison to eq. (28), we again see that up to curvature terms, the action of the diffeomorphisms can be written as the action of a time translations with parameter $\xi^0 := \xi^\alpha\tau_\alpha$, a space translation with parameter $\xi^a := \xi^\alpha e_\alpha{}^a$, a central charge transformation $\sigma := \xi^\alpha m_\alpha$, a rotation $\lambda^{ab} := \xi^\alpha\omega_\alpha{}^{ab}$ and a boost $\lambda^a := \xi^\alpha\omega_\alpha{}^a$. As the transformation rules of τ and e take exactly the form (23) of the defining fields of Newton-Cartan geometry and as they transform curved indices μ into flat indices $(0, a)$, we now again rightfully call them temporal and spacial vielbeine, with the corresponding "inverses" as in eq. (24).
Again, we see that we have encoded the geometric structure of non-relativistic gravity through various gauge fields, and up to the curvature terms, diffeomorphisms are now represented by merely a change of basis in the Bargmann algebra.
Completely analogously to the gauging procedure of the Poincaré algebra in section 3.2.2, we have two choices of dealing with the curvature terms. Either setting them to zero or redefine the variation of the gauge fields to include the curvature term, as was done in [27] for Carollian geometry. As discussed in section 3.2.2, the latter approach leaves some open questions, thus we will focus on the former approach, trying to find appropriate curvature constraints.
We see that in the curvatures appearing in this derivation, i.e. $R(H)$, $R(P)$ and $R(M)$, the latter two contain the connections of boosts and rotations only algebraically (the same is technically also true for $R(H)$). Therefore, we can impose curvature constraints, similar to Cartan's structure equations in general relativity in section 3.1.1. This allows us to introduce diffeomorphisms into our gauge theory and to solve for the connections in

terms of the fields (τ, e, m). Such constraints which only include gauge fields algebraically are called *conventional*, and there is usually very little reason to not impose.

After imposing the conventional constraints, the upshot is that we only have (τ, e, m) as independent fields of our theory, and they now transform as

$$\begin{aligned}
\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu, \\
\delta e_\mu{}^a &= \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu, \\
\delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e_\mu{}^a.
\end{aligned}$$

We can use the equations for spacial translations $R(P) = 0$ and charge transformations $R(M) = 0$ to express the dependent connection forms, which we will call *spin connections*, as

$$\omega_\mu{}^a = 2\tau_\mu m_0{}^a + e_{\mu c}\left(-2e_0{}^{(ca)} + m^{ca}\right), \tag{30}$$

$$\omega_\mu{}^{ab} = -2e_\mu{}^{[ab]} + e_{\mu c}e^{abc} - \tau_\mu m^{ab}, \tag{31}$$

where $e_{\mu\nu}{}^a := \partial_{[\mu} e_{\nu]}{}^a$ and $m_{\mu\nu} = \partial_{[\mu} m_{\nu]}$ and for compact notation we have turned curved indices into flat ones.

4.2.3. The Intrinsic Torsion of Newtonian Spacetime

We want to interpret the conventional constraints in the context of an affine connection and its corresponding torsion. If we introduce the affine connection Γ via a two-fold vielbein postulate for the temporal and spacelike vielbein

$$\begin{aligned}
\nabla_\mu \tau_\nu &= \partial_\mu \tau_\nu - \Gamma^\rho{}_{\mu\nu}\tau_\rho = 0, \\
\nabla_\mu e_\nu{}^a &= \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\mu{}^a \tau_\nu - \Gamma^\rho{}_{\mu\nu} e_\rho{}^a = 0,
\end{aligned}$$

we find that the temporal and spacial part of the torsion of the affine connection reads

$$\begin{aligned}
\mathcal{T}^0_{\ \mu\nu} &:= 2\Gamma^\rho{}_{[\mu\nu]}\tau_\rho \equiv R_{\mu\nu}(H), \\
\mathcal{T}^a_{\ \mu\nu} &:= 2\Gamma^\rho{}_{[\mu\nu]} e_\rho{}^a \equiv R_{\mu\nu}(P^a).
\end{aligned}$$

While not fully obvious in this formulation, these two torsion terms should be complemented by an additional torsion

$$\mathcal{T}_{\mu\nu} := R_{\mu\nu}(M),$$

that is given by the curvature of the central charge. This stems from the fact (see [32]) that the central charge gauge field m_μ can be viewed as another vielbein, if we consider the corresponding Bargmannian G-structure. Thus, also $R(M)$ is part of the torsion of this structure.

As it turns out (see [25, Thm. 6]), $R(H) = d\tau$ is the only relevant equation for describing the so-called *intrinsic torsion* of a Galilean spacetime. Intrinsic torsion is characterized as the part of the torsion (in the sense of G-structures), which cannot be absorbed in a redefinition of the connections ω. This can be seen directly, since $R(H)$ does not contain the connections at all, thus prescribing different values for $\mathcal{T}^0$ leads to different theories.

In full generality, it is not necessary to assume the conventional constraints. Instead, on can prescribe general torsion for time and space translations and curvature for the central charge transformations, i.e.

$$\begin{aligned}\mathcal{T}^0_{\ \mu\nu} &\stackrel{!}{=} R_{\mu\nu}(H),\\ \mathcal{T}^a_{\ \mu\nu} &\stackrel{!}{=} R_{\mu\nu}(P^a),\\ \mathcal{T}_{\mu\nu} &\stackrel{!}{=} R_{\mu\nu}(M).\end{aligned}$$

However, the choice of torsion is constrained by boost invariance. A quick calculation (or see [32]) shows that under boosts the curvatures transform into each other, thus also the prescribed torsion has to do so. More precisely

$$\begin{aligned}\delta_{\mathrm{B}}R_{\mu\nu}(M) = \lambda_a R_{\mu\nu}(P^a) &\implies \delta_{\mathrm{B}}\mathcal{T}_{\mu\nu} = \lambda_a \mathcal{T}^a_{\ \mu\nu}\ ,\\ \delta_{\mathrm{B}}R_{\mu\nu}(P^a) = \lambda^a R_{\mu\nu}(H) &\implies \delta_{\mathrm{B}}\mathcal{T}^a_{\ \mu\nu} = \lambda_a \mathcal{T}^0_{\ \mu\nu}\ ,\\ \delta_{\mathrm{B}}R_{\mu\nu}(H) = 0 &\implies \delta_{\mathrm{B}}\mathcal{T}^0_{\ \mu\nu} = 0.\end{aligned} \tag{32}$$

One can then solve for the rotation and boost connections in terms of the geometric data e, τ, m and the torsion $\mathcal{T}$. Interestingly, by the above transformation rules for the torsion and curvature, it is not consistent to set the curvature of M and the spacial torsion to zero, while giving the intrinsic torsion a non-zero value. Consequently, the resulting affine connection Γ can be not invariant under boosts, if torsion is set to zero inconsistently, see [32] for further details.

We now give the variations of the spin connections in this general setting[6]. These variations can be inferred from eq. (32), expanding the curvatures in terms of the connections and vielbeine and by using the transformation rules of the latter. The variations contain terms proportional to the intrinsic torsion $\tau_{\mu\nu} := (d\tau)_{\mu\nu}$ and read

$$\begin{aligned}\delta\omega_\mu{}^a &= \partial_\mu\lambda^a - \omega_\mu{}^{ac}\lambda_c + e_{\mu c}2\lambda^{(c}\tau^{a)}{}_0 + \lambda^a{}_b\omega_\mu{}^b,\\ \delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab} - 2\omega_\mu{}^{c[a}\lambda^{b]}{}_c + 2\lambda^{[a}\tau_\mu{}^{b]} + e_{\mu c}\lambda^c\tau^{ab}.\end{aligned} \tag{33}$$

We note that these are no longer the variations inferred from the commutation relations of the Bargmann algebra as in eq. (29). This is due to the fact that we made the connections dependent on the other fields. Consequently, also the curvatures of these connections get modified, and we find the generalized curvature (in the sense of eq. (3)) to be

$$\begin{aligned}\hat{R}_{\mu\nu}(J^{ab}) &:= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]}{}^b{}_c - 4\omega_{[\mu}{}^{[a}\tau_{\nu]}{}^{b]} - 2\omega_{[\mu}{}^c e_{\nu]c}\tau^{ab},\\ \hat{R}_{\mu\nu}(B^a) &:= 2\partial_{[\mu}\omega_{\nu]}{}^a - 2\omega_{[\mu}{}^{ac}\omega_{\nu]c} - 4\omega_{[\mu}{}^{(c}\tau^{a)}{}_{|0|}\, e_{\nu]c}.\end{aligned}$$

[6]We assume that we absorbed any non-intrinsic torsion terms into a redefinition of the boost and rotation connections.

Note that this reduces to the curvatures inferred from the Bargmann algebra in eq. (29) upon imposing, $\tau_{\mu\nu} = 0$. Hence, this underlies a more general theory of gravity than standard Newton-Cartan geometry.

As was shown in [25] the intrinsic torsion $R(H) \equiv d\tau$ can only take thee different, inequivalent values, each defining a different type of geometry depending on this torsion.

1. Torsionless Newton-Cartan (NC) Geometry This geometry is characterized by setting the intrinsic torsion to zero, i.e.

$$R(H) = d\tau = 0.$$

By the Lemma of Poincaré (assuming our manifold is simply connected), zero torsion implies that there is a global function $t\colon M \to \mathbb{R}$, such that

$$\tau = dt.$$

The function t then defines a notion of *absolute time*, as the time difference

$$T := \int_\gamma \tau \equiv t(x) - t(y),$$

between two points $x, y \in M$ of the underlying Newtonian spacetime is independent of the chosen path γ connecting the points.

2. Twistless Torsional Newton-Cartan (TTNC) Geometry This geometry has non-vanishing torsion, but it is *twistless*, i.e.

$$\begin{aligned} d\tau &\neq 0, \\ d\tau \wedge \tau &= 0. \end{aligned}$$

This gives us the following geometric consequences. First, we see that we lose the notion of absolute time, as now time differences T will depend on the path. The upshot is that the twistless condition is exactly the Frobenius condition from the Frobenius theorem for forms (see [33, Appendix]), stating that a one-form τ defines a co-dimension one foliation via its kernel, if it holds that

$$d\tau \wedge \tau = 0.$$

We thus obtain a notion of absolute simultaneity; two events are simultaneous if they lie in the same leaf of the foliation. Furthermore, the spacial vielbeine give the leaves the structure of Riemannian manifolds.
Clearly, zero intrinsic torsion is already sufficient to fulfill this condition, thus we also get absolute simultaneity in the torsionless NC geometry. Overall, the torsionless setting is the one most closely resembling what we naively expect from a Newtonian theory.

TTNC geometry, on the other hand, seems best to capture the post-Newtonian i.e.

the large speed of light expansion of general relativity (something we will explore in section 4.3) in the presence of strong gravitational fields (see [34] for details). It was encountered in applications to the quantum Hall effect (see [7]) and it is also the correct type of geometry in the presence of dilatations, for example in the Schrödinger algebra [35].

3. Torsional Newton-Cartan (TNC) geometry This geometry has neither absolute time nor simultaneity, i.e. it is characterized by

$$\begin{aligned} d\tau &\neq 0, \\ d\tau \wedge \tau &\neq 0. \end{aligned}$$

It is the most general description of Newton-Cartan geometry, as the first and second case can be recovered on imposing the correct torsion constraints. It is generally most useful in applications in condensed matter physics.

4.2.4. Galilean Gravity

Finally, let us connect back to the beginning of this section, and show that Newton-Cartan geometry in fact encodes Newtonian gravity. In the following we will only present a summary, for the full details see [36]. To do so, we first impose the constraints of case 1, i.e. torsionless Newton-Cartan geometry, where we have set the conventional constraints

$$R(P) = 0, \quad \text{and} \quad R(M) = 0,$$

together with imposing zero torsion

$$R(H) = 0$$

to recover the notion of absolute time. Furthermore, we want to describe a flat space, which is implemented upon imposing

$$R(J) = 0.$$

So, only $R(B)$, the curvature of boosts, remains non-zero and thus has to encode the gravitational dynamics.

Upon imposing vanishing of the curvature, i.e. assuming that our manifold is flat (in the sense of G-structures), we can, analogously to flat Riemannian manifolds, choose a flat frame

$$\tau_\mu = \delta^0{}_\mu, \quad e_\mu{}^a = \delta^a{}_\mu, \quad m_a = 0, \quad \omega_\mu{}^{ab} = 0,$$

and corresponding so-called *Galilean coordinates* $(x^\mu) = (t, x^a)$, satisfying all the above constraints. These choices break part of the gauge symmetries, such that only the following ones are preserved

$$\xi^0(x) = \xi^0, \quad \lambda^{ab}(x) = \lambda^{ab}, \quad \xi^a(x) = \xi^a(t) - \lambda^a{}_b \delta^b{}_\mu x^\mu, \quad \lambda^a(x) = -\dot{\xi}^a(t), \quad \sigma = 0.$$

Note that time translations and spacial rotations are now constant, while boosts are dependent on the (time derivative) of $\xi^a(t)$, which is the only explicitly spacetime- or rather time-dependent parameter.
Consequently, m_0 is the only independent field left, thus has to equal a function

$$m_0(x) \equiv \phi(x)$$

that we will identify with the Newtonian potential of gravity. To do so, note that the only non-zero component of the boost spin connection is given by

$$\omega_0{}^a(x) = -\partial^a \phi(x),$$

which should immediately remind us of eq. (22). Consequently, if we impose the equation of motion

$$R_{0a}(B^a) = 0,$$

and plug in the connection in terms of ϕ, we precisely recover the Poisson equation

$$R_{0a}(B^a) \equiv \partial_a \partial^a \phi = 0.$$

We see that we had to take the Bargmann algebra, i.e. the central extension of the Galilei algebra, to be able to encode gravitational dynamics, where the gauge field of the central extension generator carried the Newtonian potential. Furthermore, we can then interpret the gravitational dynamics as a curvature constraint on the curvature of boosts.

4.3. Non-Relativistic Limits

By gauging the Bargmann algebra, we have found the kinematics of Newtonian gravity. Similar to the relativistic case, for dynamics we need equations of motion or an action and some kind of guiding principle on how to obtain them, as non-relativistic boost invariance is often highly non-trivial, thus difficult to guess.
Additionally, it is not clear how to properly motivate the constraints and equations of motion we imposed at the end of the last section to derive the Poisson equation from the curvatures.

Intuitively, we expect that actions, equations of motion and constraints can be inferred from general relativity. To gain insight into this, observe that, after introducing explicit factors of c, the speed of light, we can write the Minkowski metric and its inverse as

$$\frac{1}{c^2}\eta = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{c^2}\mathbb{1}_{D-1} \end{pmatrix}, \quad \eta^{-1} = \begin{pmatrix} -\frac{1}{c^2} & 0 \\ 0 & \mathbb{1}_{D-1} \end{pmatrix}. \tag{34}$$

Consequently, in the limit $c \to \infty$ we recover the defining geometric objects of Newton-Cartan geometry from section 4.1.1, i.e.

$$\lim_{c\to\infty} \frac{1}{c^2}\eta = t,$$
$$\lim_{c\to\infty} \eta^{-1} = h.$$

This now allows us to generate non-relativistic theories from relativistic ones. The formal process and extensions to the geometric structures, as presented in [5, Sec. 2], is quite involved and leads to more general or different theories than what is considered in this work. It is important to note that taking non-relativistic limits is usually non-trivial, as often divergences appear that have to be cancelled correctly by choosing the appropriate ansatz.

4.3.1. Limits of Lagrangians

For now, we want to present a simple example taken from [37] showcasing the process and some of its difficulties, but more interesting examples can be found in [38, 39, 40] and references therein.
Note that there are two different approaches to extracting non-relativistic theories from relativistic ones.
The first approach is via expansions and is reflected in [5]. In non-relativistic expansions, the speed of light c is taken as an expansion parameter, thus has a finite value. Such expansions usually require the introduction of extra fields at each order. These extra fields are needed to capture the corresponding dynamics, but they also render the relations to the relativistic fields non-invertible. At full order, such expansions are equivalent to GR and can also capture relativistic features such as time dilation, etc. (see [5]).
The second approach, which is the one taken in this thesis, is non-relativistic limits or $c \to \infty$ limits. This approach is analogously to the Inönü-Wigner contractions we will present in 4.3.2. Here a redefinition of fields and transformation rules is chosen that is *invertible* for finite speed of light. Then, leading order contributions of Lagrangians, equations of motion, etc. are taken, where some subtleties and cancellations need to be taken into account, which we will see in the following example. In the $c \to \infty$ limit, the resulting theory is fully non-relativistic and thus is distinctly different from general relativity.

We start with the action of a relativistic point particle moving in a gravitational field

$$S_{\text{Rel}}[x] = -mc \int ds \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}.$$

Here, $s \mapsto x^\mu(s)$ is the world line of the particle. Inspired by eq. (34), we introduce an invertible (for finite c) ansatz for the relativistic vielbeine

$$E_\mu{}^0 = c\tau_\mu, \quad E_\mu{}^a = e_\mu{}^a.$$

Note that the fields τ, e and their inverses satisfy the relations in eq. (24). Notably, so far we have only found a reformulation that is fully relativistic but includes factors of the speed of light.
If we now expand this action to first order in $\frac{1}{c^2}$ we find

$$S_{\text{Rel}}[x] \simeq_2 -mc^2 \int ds \tau_\mu \dot{x}^\mu + \frac{m}{2} \int ds \frac{\delta_{ab} e_\mu{}^a e_\nu{}^b \dot{x}^\mu \dot{x}^\nu}{\tau_\alpha \dot{x}^\alpha}.$$

Where the symbol $\simeq_2$ denotes that we excluded terms $\mathcal{O}(c^{-2})$. As stated above, the non-relativistic expansion amounts to taking the leading order contribution of the expansion, and in this case the result seems disappointing. The leading order is given by the Lagrangian

$$\overset{(-2)}{\mathcal{L}} = -m\tau_\mu \dot{x}^\mu,$$

and its dynamics are rather boring. In flat space, they amount to the single equation $\dot{x}^0 \equiv 0$. This of course is not an interesting theory.
Furthermore, if we want to take the limit $c \to \infty$, the leading order contribution is divergent, and additionally, we see that the next to leading order term looks much more promising as a theory. Thus, we need to find a procedure how to deal with the leading order term. This phenomenon often appears in relativistic expansions and is caused by the rest mass, or rather, rest energy of the particle $E = mc^2$. There are usually two approaches to resolve the divergence.

First, we can try to cancel the divergence by introducing extra constraints. The leading order contribution certainly only gives a total derivative, consequently not contributing to the equations of motion, if the clock form would be exact, i.e.

$$\tau_\mu = \partial_\mu t,$$

for a function t. On a simply connected spacetime, this is equivalent to $d\tau = 0$. Then the next to leading order term is equivalent to the Newtonian particle action in eq. (25), but with the torsion $d\tau$ set to zero.

Secondly, we could modify our theory and add additional fields to try and cancel the divergence. If the particle is charged, we would add a so-called Wess-Zumino term to the relativistic action including a $U(1)$ gauge field A_μ

$$S_{\text{Rel}}[x] = -mc \int ds \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - q \int ds A_\mu \dot{x}^\mu. \tag{35}$$

Expanding this, while keeping A untouched for now, we find the leading order contributions

$$S_{\text{Rel}}[x] \simeq_2 -mc^2 \int ds \left(\tau_\mu + \frac{q}{mc^2} A_\mu\right) \dot{x}^\mu + \frac{m}{2} \int ds \frac{\delta_{ab} e_\mu{}^a e_\nu{}^b \dot{x}^\mu \dot{x}^\nu}{\tau_\alpha \dot{x}^\alpha}.$$

Notably, we can cancel the divergence if we assume a *critical value* for the electromagnetic part of the theory, i.e. assume that the charge is equal to the mass $q = m$ and that we have the ansatz for the electromagnetic field

$$A_\mu = -c^2 \tau_\mu + m_\mu.$$

Then the divergent term cancels, and we find the non-relativistic action

$$S_{\text{rel}} \simeq_2 S_{\text{NR}}[x] = \frac{m}{2} \int ds \frac{\delta_{ab} \dot{x}^a \dot{x}^a}{\dot{x}^0} - m \int ds \; m_\mu \dot{x}^\mu,$$

where we have introduced a notation as in Newton-Cartan geometry, i.e. $\dot{x}^0 := \tau_\mu \dot{x}^\mu$ and $\dot{x}^a := e_\mu{}^a \dot{x}^\mu$. Note that this looks like the invariant action of eq. (26) for $\dot{s} = -m_\mu \dot{x}^\mu$, i.e.

$$S_{\text{NR}}[x] = \frac{m}{2} \int d\tau \left(\frac{\delta_{ab} \dot{x}^a \dot{x}^b}{\dot{x}^0} - 2 m_\mu \dot{x}^\mu \right).$$

Indeed, this form of the action ensures boost invariance, and in the flat case reduces precisely to eq. (26).

While we could now infer from the action that the non-relativistic fields (τ, e, m) are exactly the gauge fields from the Bargmann algebra, and thus transform correspondingly (to leave the action invariant), we can also derive their variations directly from the relativistic ansatz, since we chose it invertible. We know how the relativistic fields transform under Lorentz transformations and correspondingly find

$$\delta e_\mu{}^a \equiv \delta E_\mu{}^a = \Lambda^a{}_B E_\mu{}^B = \Lambda^a{}_b E_\mu{}^b + \Lambda^a{}_0 E_\mu{}^0 = \Lambda^a{}_b e_\mu{}^b + \Lambda^a{}_0 c \tau_\mu,$$

where the "relativistic" index $A = 0, ..., D-1$ splits into 0 and $a = 1, ..., D-1$. We see that the second term is proportional to c and thus would diverge in the $c \to \infty$ limit. To cancel this, we introduce the boost parameter

$$\Lambda^a{}_0 =: \frac{1}{c} \lambda^a,$$

while the spacial rotations remain inert, i.e.

$$\Lambda^a{}_b \equiv \lambda^a{}_b.$$

Overall, after taking the limit, we find that

$$\delta e_\mu{}^a = \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu.$$

Therefore, the spacial vielbein derived from the expansion transforms exactly as the spacial vielbein derived from gauging the Bargmann algebra in section 4.2, hence the two fields are the same. Repeating the same process for $E_\mu{}^0$ and A_μ lets us derive the variations (without diffeomorphisms)

$$\begin{aligned} \delta \tau_\mu &= 0, \\ \delta e_\mu{}^a &= \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu, \\ \delta m_\mu &= \partial_\mu \sigma + \lambda_a e_\mu{}^a. \end{aligned}$$

So, the non-relativistic limit coincides with the gauging procedure of the Bargmann algebra. Additionally, we also obtain a non-relativistic Lagrangian and therefore also dynamics consistent with a limit of the relativistic theory.

Note however, that if we first take the equations of motion and then apply the non-relativistic limit to them, we might end up with a bigger set of equations then first taking the limit and then computing the equations of motion gives. Put differently, computing equations of motion and taking non-relativistic limits is non-commuting. This has to do with the degenerate nature of Newton-Cartan theory and is something we will explore further in section 5.4.

Lastly, there is one more interesting detail in the non-relativistic action, namely an *emergent symmetry*, i.e. an unexpected symmetry. The action S_{NR} is inert under gauged *dilatations* with parameter λ, provided that

$$\begin{aligned}\delta_D e_\mu{}^a &= \lambda e_\mu{}^a, \\ \delta_D \tau_\mu &= 2\lambda \tau_\mu.\end{aligned}$$

This dilatation symmetry was not present in the relativistic case and thus emerged in taking the non-relativistic limit. We will find a partial explanation in section 5.4 and again encounter this phenomenon in 7.1 when we examine NS-NS gravity in string theory.

4.3.2. Deriving the Bargmann Algebra

So far, we have not properly derived the Bargmann algebra but have only given motivations as to why it is the correct choice for non-relativistic geometry. Now, equipped with more motivation and intuition from the non-relativistic large speed of light expansion from the last section, we will fill in the blanks.

We want to consider so-called *Lie algebra contractions*, for a proper introduction see [41, Ch. 10]. Consider the following setting of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ and generators $\{T_A\}_{A=1}^{\dim \mathfrak{g}}$, together with a family of invertible maps $U(\varepsilon)$, $\varepsilon \in (0,1]$, such that $U(1) = Id_{\mathfrak{g}}$ and the limit $\lim_{\varepsilon\to 0} U(\varepsilon)$ is degenerate, i.e. not invertible.
We can then construct the Lie algebra $\mathfrak{g}_\varepsilon$, with the same vector space underlying as $\mathfrak{g}$, but with bracket

$$[T_A, T_B]_\varepsilon \coloneqq f_{AB}{}^C(\varepsilon) T_C,$$

with the structure constants

$$f_{AB}{}^C(\varepsilon) \coloneqq U(\varepsilon)_A{}^D U(\varepsilon)_B{}^E f_{DE}{}^F \left(U(\varepsilon)^{-1}\right)_F{}^C,$$

where $f_{DE}{}^F$ are the structure constants of $\mathfrak{g}$. For $\varepsilon > 0$, $U(\varepsilon)$ defines a Lie algebra isomorphism between $\mathfrak{g}_\varepsilon$ and $\mathfrak{g}$, which is not necessarily true for $\varepsilon \to 0$. We then say that the Lie algebra $(\mathfrak{g}' = \text{Span}\{T_A\}, [\cdot,\cdot]')$ is the *contraction of* $\mathfrak{g}$ if the limit

$$[T_A, T_B]' \coloneqq \lim_{\varepsilon\to 0}[T_A, T_B]_\varepsilon = \lim_{\varepsilon\to 0} f_{AB}{}^C(\varepsilon) T_C,$$

exists and defines a Lie algebra. Note, the contraction has the same dimension as the initial algebra, but it is by construction generally not isomorphic to it.
Depending on the proposed form of $U(\varepsilon)$, one can define different forms of contractions, and usually the initial algebra has to fulfill extra conditions to make sure the contraction

exists.

One such contraction is given by the *Inönü-Wigner contraction* [42], which is defined on a Lie algebra $\mathfrak{g}$ that as a vector space splits into the direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

such that contraction map is given by

$$U(\varepsilon)|_{\mathfrak{g}_1} = Id_{\mathfrak{g}_1}, \quad U(\varepsilon)|_{\mathfrak{g}_2} = \varepsilon Id_{\mathfrak{g}_2}.$$

It can be shown (see [41, Ch. 10]) that this only gives a well-defined contraction if $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}$. If we define the generators of $\mathfrak{g}_1$ as $\{T_I\}$ and the generators of $\mathfrak{g}_2$ as $\{T_\alpha\}$, the contraction has the form of a semi-direct sum of the subalgebra $\mathfrak{g}_1$, with the Abelian ideal $\mathfrak{g}_2$, given by the commutation relations

$$\begin{aligned}
[T_I, T_J] &= f_{IJ}{}^K T_K \\
[T_I, T_\alpha] &= f_{I\alpha}{}^J T_J \\
[T_\alpha, T_\beta] &= 0.
\end{aligned}$$

We now want to relate this to the non-relativistic limit. For deriving the Galilean algebra from the Poincaré algebra, this exactly holds, but for the derivation of the Bargmann algebra, one has to expand the definition.

As motivation, note that we can expand the zero component of the momentum four-vector due to the on-shell condition

$$p_0 = \sqrt{c^2 p_a p^a + c^4 m^2} \simeq_2 mc^2 + \frac{1}{2m} p_a p^a, \tag{36}$$

Where we have a sum of the rest energy and the kinetic energy. Furthermore, as Minkowski space is a homogeneous spacetime, we can identify the 4-momentum with the translation generators, i.e. $p_\mu \leftrightarrow P_\mu$. We then trivially extend the Poincaré algebra given by the generators $\{H, P_a, J_{ab}, B_a\}$ as in eq. (18) by a $U(1)$ generator M that commutes with all other generators. Additionally, we perform a mere change of basis in the relativistic algebra motivated by eq. (36) and given by

$$\begin{aligned}
H &\mapsto \frac{1}{c} H + cM, \\
M &\mapsto \frac{1}{c} H - cM, \\
B_a &\mapsto cB_a,
\end{aligned}$$

and all others remaining inert. So we see, that we are not quite in the setting of an Inönü-Wigner contraction but in a generalization of it. Here, we allow for different subspaces being scaled by different powers of the contraction parameter. This generalization is then known as a Weimar-Woods contraction (see [43]). Computing the commutation relations and taking the limit $c \to \infty$, we recover the Bargmann algebra with the commutation relations in eq. (27).

5. Dimensional Reduction

From what we experience in everyday life and also on the scale of current experiments, the physical world presents itself in a 4-dimensional spacetime.
However, as modern theoretical physics searches for new ways to grapple with unsolved problems such as quantum gravity, this experience has been questioned and more general dimensions are now under consideration. The most notable cases of this being string theories, which we will introduce in section 6, that are only consistent in $D > 4$ dimensions. The question thus arises, how a lower-dimensional theory may be recovered from a higher dimensional one, ensuring consistency of new and old physics.

This can be done via a procedure called *dimensional reduction* and was first considered by Kaluza [44] in 1921 and expanded upon by Klein in 1926 [45]. Klein applied it to 5-dimensional general relativity and could reproduce (a form of) 4-dimensional gravity and electromagnetism, hence unifying them.

In the context of string theory, dimensional reduction is a necessity to recover 4-dimensional physics through a kind of "projection" from higher dimensions. However, it can also be used as a tool to embed a lower-dimensional theory into a higher-dimensional one, giving a new perspective and sometimes an easier description of the former theory.

This part is based upon [46, Chapter 11.2], [22, Chapter 5.3] and [47, Part 5].

5.1. Kaluza-Klein Reduction on Scalar Fields

To motivate the definitions and approaches we will consider later, let us first take a look at what happens to a scalar field in Kaluza-Klein (KK) theory.

The idea of KK theory is that we have one extra compact dimension of size L, usually assumed to be very small. This assumption explains why we cannot observe the extra dimensions, as by the uncertainty principle we would need energies of $E \approx 1/L$ to probe structures of spacial size L and for small L, this may get unreasonably large.

Formally, we assume our $D+1$-dimensional flat spacetime M^{D+1} to be the Cartesian product of a Minkowski space $\mathbb{M}^{1,D-1}$ and a circle S^1, i.e. $M^{D+1} = \mathbb{M}^{1,D-1} \times S^1$.
This gives natural coordinates on M via (x^μ, z), where the former are the coordinates of Minkowski space and the latter is the coordinate of the circle that is periodically identified as

$$z \sim z + 2\pi L.$$

The natural choice of metric on M^{D+1} is then

$$g = \eta_{\mu\nu} dx^\mu dx^\nu + dz^2,$$

introducing an extra spacelike dimension. Extra timelike dimensions have been considered, but as was shown in [48] they usually lead to problems with causality and/or tachyons.

Consider now a massless Klein-Gordon scalar field $\phi(x, z)$ and observe the splitting of the Klein-Gordon equation in these coordinates

$$0 = \Box_g \phi \equiv (\Box_\eta + \partial_z^2)\phi, \tag{37}$$

where $\Box_g$ is the Laplacian with respect to the metric g. By the periodicity of the extra dimension, we naturally have a Fourier decomposition of ϕ as

$$\phi(x, z) = \sum_{k \in \mathbb{Z}} e^{i\frac{kz}{L}} \phi_k(x).$$

Substituting the ansatz back into eq. (37), we gather an infinite number[7] of equations for the Fourier coefficients of ϕ, namely

$$\left[\Box_\eta - \left(\frac{k}{L}\right)^2\right] \phi_k = 0.$$

So we see that starting from a *massless* scalar field in $D+1$ dimensions, we end up with an infinite number of *massive* scalar fields in D dimensions, which acquired masses $m_k := \frac{|k|}{L}$ through the extra compact dimension. Note that these infinitely many fields $\phi_k(x)$ in D dimensions are *equivalently* described by a single one $\phi(x, z)$ in $D+1$ dimensions, giving a much simpler description in higher dimensions.

Interestingly, the masses are scaled by a factor $1/L$. So for a small extra dimension, i.e. $L \to 0$, they become incredibly massive, which makes it reasonable to truncate the expansion to only the zero mode $k = 0 : \ \phi_0(x)$. Note that this truncation is characterized by the condition

$$\partial_z \phi(x, z) = 0. \tag{38}$$

5.2. Dimensional Reduction of the Geometry

Let us now turn to the more complicated setting of an arbitrary $D + 1$-dimensional spacetime M^{D+1}, with a general metric $\hat{g}$ and non-scalar quantities, as well as one compact dimension. Even imposing the zero mode condition, the interpretation of the results is not straight forward, as the resulting fields now also carry a representation of spacetime symmetries. We first have to clarify the role of these symmetries and can then interpret the resulting fields as an appropriate representation over a D-dimensional spacetime.

Our starting point is the vielbein of the metric $\hat{g}$, transforming under diffeomorphisms ξ and $SO(1, D+1)$ transformations Λ as

$$\begin{aligned} \delta E_{\hat{\mu}}{}^{A} &= \mathcal{L}_\xi E_{\hat{\mu}}{}^{A} + \Lambda^{A}{}_{B} E_{\hat{\mu}}{}^{B} \\ &= \xi^{\hat{\alpha}} \partial_{\hat{\alpha}} E_{\hat{\mu}}{}^{A} + \partial_{\hat{\mu}} \xi^{\hat{\alpha}} E_{\hat{\alpha}}{}^{A} + \Lambda^{A}{}_{B} E_{\hat{\mu}}{}^{B}, \end{aligned}$$

[7] Often called "Kaluza-Klein tower".

note that a priori the indices $\hat{\mu}$ and A range from 0 to $D+1$.

Recall the zero-mode-condition eq. (38) that amounts to finding a coordinate z such that the vielbein is independent of it. We want to formulate this geometrically and thus assume the existence of an isometry along the compact dimension. This is equivalent to assuming that there exists a spacelike Killing vector field χ tangent to the compact dimension, i.e. that

$$\mathcal{L}_\chi E_{\hat{\mu}}{}^A = 0.$$

By invoking the Frobenius-theorem for a single vector field (for details look at [17, Corollary 17.4]), there exists a foliation with codimension 1 and (locally) adapted coordinates $(x^{\hat{\mu}}) = (x^\mu, z)$, where $\mu = 0, ...D$, s.t.

$$\chi = \partial_z.$$

So far, it is not clear that the compact dimension, i.e. the integral curves of χ, are closed and that we can interpret the compact dimension as an S^1. From here one, we will always assume that this holds and that the extra dimension is periodic[8].

Assuming the correct boundary conditions, we find that the leaves of the foliation are circles S^1, while the directions orthogonal to χ are the desired D-dimensional spacetime M^D. This lower-dimensional spacetime carries the induced metric

$$\Pi_{\hat{\mu}\hat{\nu}} = \hat{g}_{\hat{\mu}\hat{\nu}} - \frac{\chi_{\hat{\mu}}\chi_{\hat{\nu}}}{\hat{g}(\chi, \chi)}.$$

Note that something interesting is happening for the case of χ being a null vector. We will come back to this case in section 5.4, as it will give another way to derive Newton-Cartan geometry, and it will play a central role in the results of this thesis. The degeneracy of the induced metric then naturally connects to the degenerate metrics of Newton-Cartan geometry.

Let us now analyze what happens to the spacetime symmetries under KK reduction. To preserve the foliation and hence the adapted coordinates, we have to break part of the diffeomorphism invariance by requiring that for a diffeomorphism ξ we have

$$0 \stackrel{!}{=} \mathcal{L}_\xi \chi^{\hat{\mu}} = -\partial_z \xi^{\hat{\mu}}.$$

Thus, diffeomorphisms do not depend on the compact direction z. Now as the left-hand side of $\delta E_{\hat{\mu}}{}^A$ does not depend on z, the same is true for the right-hand side, but the only parameter left with an a priori z-dependency is $\Lambda^A{}_B$. Hence, it cannot depend on the compact direction either, so nothing is z-dependent.

[8]Indeed, it is a periodicity condition we have to impose by hand, which is essential to the KK dimensional reduction, as different boundary conditions will change the mass values and most boundary conditions don't even allow for massless modes (for details see [49]).

Though there is no geometric reason for it, we also assume that the zero-mode-condition holds for any tensor T, which means that

$$\mathcal{L}_\chi T \stackrel{!}{=} 0.$$

Effectively, this restricts all lower-dimensional fields from gaining mass through the compact dimension as in section 5.1.

To investigate further symmetry breaking, first note that our foliation allows us to split vector fields and hence diffeomorphisms as $(\xi^{\hat{\mu}}(x)) = (\xi^\mu(x), \xi^z(x))$. Taking the z-independence into account in the variation of the vielbein, we see that

$$\delta_\xi E_\mu{}^A \equiv \mathcal{L}_\xi E_\mu{}^A = \xi^\alpha \partial_\alpha E_\mu{}^A + \partial_\mu \xi^\alpha E_\alpha{}^A + (\partial_\mu \xi^z) E_z{}^A, \tag{39}$$
$$\delta_\xi E_z{}^A \equiv \mathcal{L}_\xi E_z{}^A = \xi^\alpha \partial_\alpha E_z{}^A. \tag{40}$$

Due to its action on the vielbeine, it is natural to interpret ξ^μ as D-dimensional diffeomorphisms of the hypersurface M^D orthogonal to the compact dimension S^1, while ξ^z appears similar to a $U(1)$ parameter.
Further, note that $E_z{}^A$ transforms as a scalar field over M^D under these diffeomorphisms, while $E_\mu{}^A$ correctly transforms as a one-form on M^D.

We now also split the algebra index $A = (a, \bar{z})$, $a = 0, ..., D-1$ and see how our choices affect the $SO(1, D+1)$ symmetry. Consider the transformations of the z-component of the vielbein

$$\begin{aligned} \delta E_z{}^{\bar{z}} &= \Lambda^{\bar{z}}{}_b E_z{}^b, \\ \delta E_z{}^a &= \Lambda^a{}_b E_z{}^b + \Lambda^a{}_{\bar{z}} E_z{}^{\bar{z}}. \end{aligned}$$

As we already know, $E_z{}^{\bar{z}}$ is a scalar under diffeomorphisms, hence the expression $C^a := \Lambda^a{}_{\bar{z}} E_z{}^{\bar{z}}$ can be seen as an arbitrary collection of scalar functions $\{C^a\}$. These functions act on $E_z{}^a$ by an arbitrary shift of the field components. Such arbitrary shifts are also called *Stückelberg symmetry*[9], and allow us to arbitrarily change the field value at any given point. They indicate that the theory in question does not depend on the field after all, and we can consistently set the corresponding field components to zero. Thus, we may fix this symmetry in Kaluza-Klein reduction by setting

$$E_z{}^a \stackrel{!}{=} 0.$$

To preserve this choice of gauge, we break part of the $SO(1, D+1)$ symmetry, as we have to require

$$\Lambda^a{}_{\bar{z}} = 0,$$

[9]They were first introduced 1938 in [50] by Stückelberg, and allowed for gauge invariant mass terms for gauge fields. This was later rediscovered in 1964 with the now infamous Higgs-mechanism. For more mathematical details see [51] and for a review and history of the matter see [52].

effectively reducing to $SO(1, D)$ symmetries as $\Lambda^a{}_b$ stays unaffected. This gives the variation for the remaining components of the vielbein

$$\begin{aligned}
\delta_{\text{SO}} E_\mu{}^a &= \Lambda^a{}_b E_\mu{}^b \\
\delta_{\text{SO}} E_\mu{}^{\bar{z}} &= 0 \\
\delta_{\text{SO}} E_z{}^{\bar{z}} &= 0.
\end{aligned}$$

So we see, that $E_z{}^{\bar{z}} =: k$ is a scalar under Lorentz transformations, as well as diffeomorphisms, making it a genuine scalar field that we will call the *KK scalar*. To explain its role, note that the induced metric on the compact z-direction, i.e. on S^1 at a fixed point $x \in M^D$ of the D-dimensional spacetime, is

$$g_{S^1} = k^2(x) dz^2.$$

This now lets us integrate out the appropriate volume of the compact direction as

$$\int_{S^1} d\,\text{vol}(g_{S^1}) = \int_0^{2\pi L} k(x) dz = 2\pi L k(x). \tag{41}$$

Thus, we see that we get an effective size of the circle $\tilde{L}(x) := Lk(x)$. Thus, the KK scalar "dilates" the size of the compact dimension. If k is a dynamical field, then the size of the compact dimension is dynamical as well.

As the KK scalar k multiplies the $U(1)$ parameter ξ^z in the transformation of $E_\mu{}^{\bar{z}}$ in eq. (39), it is reasonable to define $E_\mu{}^{\bar{z}} =: kA_\mu$, with A_μ a standard $U(1)$ gauge field over M^D.

In conclusion, this allows us to express the $D+1$-dimensional vielbein as

$$(E_{\hat{\mu}}{}^A) = \begin{array}{c c} & \begin{array}{cc} a & \bar{z} \end{array} \\ \begin{array}{c} \mu \\ z \end{array} & \begin{pmatrix} E_\mu{}^a & kA_\mu \\ 0 & k \end{pmatrix} \end{array}, \tag{42}$$

transforming under diffeomorphisms ξ^μ of the D-dimensional hypersurface M^D, $U(1)$-transformations $\theta := \xi^z$ and D-dimensional Lorentz transformations $\lambda^a{}_b$ as

$$\begin{aligned}
\delta E_\mu{}^a &= \xi^\alpha \partial_\alpha E_\mu{}^a + \partial_\mu \xi^\alpha E_\alpha{}^a + \Lambda^a{}_b E_\mu{}^b, \\
\delta A_\mu &= \xi^\alpha \partial_\alpha A_\mu + \partial_\mu \xi^\alpha A_\alpha + \partial_\mu \theta, \\
\delta k &= \xi^\alpha \partial_\alpha k,
\end{aligned}$$

Note that $E_\mu{}^a$ exactly transforms as a vielbein over the D-dimensional spacetime M^D and A_μ and k also transform correctly as a one-form and a scalar field over M^D. As none of the fields and transformation parameters depend on the extra compact dimension, we indeed gain a gauge theory over the spacetime M^D.

The appearance of a $U(1)$ field seems mysterious at first, but recall that $U(1)$ is diffeomorphic to S^1. Therefore, our $D+1$-dimensional spacetime looks locally like $M^D \times U(1)$, i.e. a $U(1)$ principle fiber bundle, where the emergence of a $U(1)$ connection is completely natural.

Let us note the inverses of this ansatz as well

$$(E^{\hat{\mu}}{}_{A}) = \begin{array}{c c} & \begin{array}{cc} \mu & z \end{array} \\ \begin{array}{c} a \\ \bar{z} \end{array} & \begin{pmatrix} E^{\mu}{}_{a} & -E^{\alpha}{}_{a} A_{\alpha} \\ 0 & \frac{1}{k} \end{pmatrix} \end{array},$$

with $E^{\mu}{}_{a} E_{\mu}{}^{b} = \delta^{b}{}_{a}$, the correct D-dimensional inverse vielbein.

For any other p-form field T, we then define

$$\begin{aligned} T_{\mu_1 \cdots \mu_{p-1}} &:= T_{\mu_1 \cdots \mu_{p-1} z}, \\ \hat{T}_{\mu_1 \cdots \mu_p} &:= T_{\mu_1 \cdots \mu_p}, \end{aligned}$$

where the hat indicates, that we have a quantity remaining from higher dimensions, but usually the unhatted quantity will be more relevant. It is important to stress, that after the initial ansatz, we will never mix higher-dimensional indices $\hat{\mu}$,... with lower-dimensional indices μ,..., etc. The same holds true in coordinate free notation, which will only be considered over M^D. Thus, the hatted quantities allow us to distinguish the quantities when not using coordinates.
A quick calculation then shows, that under the remainder of the diffeomorphisms in the z-direction, i.e. $U(1)$-transformations with parameter ξ^z, we have

$$\delta(\xi^z)\hat{T} = T \wedge d\xi^z. \tag{43}$$

This means that the $U(1)$-transformations mix components of the hatted and unhatted components and there usually is no trivial way to get rid of this.

Finally, we reconsider the scalar field ϕ and its Fourier decomposition, if we do not truncate to massless modes

$$\phi(x, z) = \sum_{n \in \mathbb{Z}} e^{i\frac{nz}{R}} \phi_n(x).$$

Notably, the Fourier modes are now also charged under A_μ. Recall that the $U(1)$ transformations of A_μ came from the diffeomorphisms along the compact direction, i.e. for $\delta z = \theta(x)$, we have $\delta A = d\theta$. Consequently, since ϕ as a scalar field is inert under such transformations, the Fourier decomposition lets us infer that

$$\delta_\theta \phi_n = -i\frac{n}{R}\theta\phi_n, \tag{44}$$

i.e. ϕ_n carries charge $\frac{n}{R}$ under the $U(1)$ field A_μ.

5.3. Truncations and Subtleties

Let us now explore some subtleties of KK compactification. Through the use of our ansatz in eq. (42), we can relate the lower-dimensional fields back to the higher-dimensional ones.
We can also expand the equations of motion of the $D+1$-dimensional theory over M^{D+1} in terms of the fields over M^D. This lets us infer under which conditions the equations of motion over M^D are consistent with the higher-dimensional ones, once expressed through the ansatz (42).
Consequently, we find that truncation of equations by setting some fields to a particular value will usually yield constraints (to retain compatibility with the original EOMs) that cannot be reproduced from a truncated action.

We can see this, when looking at the original example of pure Einstein gravity in dimension $D+1$. Starting from the Einstein-Hilbert action over M^{D+1}

$$S = \frac{1}{2\kappa^2}\int d^{D+1}x\sqrt{|\hat{g}|}\hat{R},$$

where the hatted quantities are the metric and the Ricci scalar of the $D+1$-dimensional theory.
Proceeding as shown above, but with a metric ansatz, it is not too hard but tedious (for details see [46, P. 300]) to show that this reduces to

$$S = \frac{2\pi L}{2\kappa^2}\int d^D x\sqrt{|g|}k\left[R - \frac{1}{4}k^2 F_{\mu\nu}F^{\mu\nu}\right],$$

where the unhatted quantities now are the appropriate metric and Ricci scalar of the D-dimensional theory, k is the KK scalar as above and $F_{\mu\nu} := 2\partial_{[\mu}A_{\nu]}$.

Now, the KK scalar appears in a rather unusual way. It looks as if it does not have a kinetic term. However, in deriving the Einstein equations, one has to integrate by parts, picking up derivatives of k. In total, the equations of motion read

$$\begin{aligned} 0 &= G_{\mu\nu} + (\partial_\mu \ln k \partial_\nu \ln k - g_{\mu\nu}\partial_\alpha \ln k \partial^\alpha \ln k) + (\nabla_\mu \partial_\nu \ln k - g_{\mu\nu}\nabla_\alpha \ln k \nabla^\alpha \ln k) && (45)\\ &\quad - \frac{1}{2}k^2\left(F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), && (46)\\ 0 &= R - \frac{3}{4}k^2 F_{\alpha\beta}F^{\alpha\beta}, && (47)\\ 0 &= \nabla_\mu(k^3 F^{\mu\alpha}). && (48)\end{aligned}$$

Taking the trace of the Einstein equation and combining it with eq. (47), we get a proper equation of motion for k, namely

$$\nabla_\mu \nabla^\mu k = -\frac{D-1}{4}k^3 F_{\alpha\beta}F^{\alpha\beta}.$$

We thus see, that setting $k = \text{const}$ is only consistent if one also sets the field strength $F = 0$, i.e. assuming A to be flat, which is generally not true. Thus, the KK scalar

cannot be simply ignored.
Setting on the other hand $A = 0$ is completely consistent with the other equations of motion, so truncating it in the action will still yield equations of motion that are consistent with the ones in higher dimensions.

The issue of truncations becomes even more subtle if other tensor fields are present. This issue stems largely from the mixing of components seen in equation (43).
Take the example of a Kalb-Ramond field, i.e. a two-form field B that transforms such that its symmetry parameter is a one-form $\theta_{\hat{\mu}}$

$$\delta B_{\hat{\mu}\hat{\nu}} = \mathcal{L}_\xi B_{\hat{\mu}\hat{\nu}} + (d\theta)_{\hat{\mu}\hat{\nu}}.$$

It is straight forward to show that the one-form symmetry reduces to

$$\begin{aligned}\delta(\theta)B_\mu &= \partial_\mu \theta_z \\ \delta(\hat{\theta})\hat{B}_{\mu\nu} &= 2\partial_{[\mu}\hat{\theta}_{\nu]}.\end{aligned}$$

So B is a standard $U(1)$ gauge field, while $\hat{B}$ still carries the remainder of the one-form symmetry. If we, however, add the diffeomorphisms in z-direction, we are again mixing with B, i.e. in the notation of section 5.2 and eq. (43)

$$\delta\hat{B} = d\hat{\theta} + B \wedge d\xi^z, \tag{49}$$

where all quantities are defined over M^D. Thus, if we want to truncate $\hat{B}$, we find that due to the mixing in eq. (49) our symmetries are constrained to satisfy

$$d\hat{\theta} = -B \wedge d\xi^z.$$

This however immediately implies via a second differentiation that

$$0 = dB \wedge d\xi^z = F(B) \wedge d\xi^z. \tag{50}$$

Which in particular is fulfilled if either B is flat or we need to gauge fix $d\xi^z = 0$. So both options restrict the $U(1)$ fields B and/or A.

More generally, as stated in [33, Appendix, Lemma 2], the truncation condition (50) implies (provided that $d\xi^z \neq 0$) that there exists a one-form α such that

$$F(B) \equiv \alpha \wedge d\xi^z.$$

Note that this relation is invariant under a shift in α under $d\xi^z$, i.e. we have the $U(1)$ symmetry

$$\delta\alpha = d\xi^z.$$

Thus, we can readily identify it with the $U(1)$ field A gathered from the reduction ansatz of the metric in eq. (42), i.e.

$$\alpha \equiv A \implies F(B) \equiv A \wedge d\xi^z.$$

This in particular implies that $A \wedge F(B) \equiv 0$. While this identity seems obscure at first, it ensures consistency of truncation. For example, the dimensional reduction of curvatures of the higher-dimensional Kalb-Ramond field $B_{\hat{\mu}\hat{\nu}}$ often contains terms involving the lower-dimensional fields B and $\hat{B}$, as well as A in the form of a covariant derivative

$$\mathcal{D}\hat{B} \coloneqq d\hat{B} - A \wedge F(B),$$

to ensure invariance under ξ^z diffeomorphism. Thus, if we truncate the first term, the second then also vanishes automatically, thus in total ensuring vanishing of the whole covariant derivative.

Finally, observe what happens to generic connection forms $B_{\hat{\mu}}{}^A$, where A now is a generic algebra index. As $\partial_z T = 0$ for all objects T, we can easily deduce that

$$\delta B_z^A = \partial_z \lambda^A + f_{BC}{}^A B_z^B \lambda^C = f_{Bc}{}^A B_z^B \lambda^c,$$

if we assume that B just transforms in the adjoint representation, and we denote summation over the non-zero symmetry parameters with a small c. We see thus, that gauge fields of the z-direction become ordinary fields.

5.4. Scherk-Schwarz and Null Reductions

As we imposed the zero-mode-condition for any tensor, it is clear that we cannot generate massive fields via KK dimensional reduction. Through the use of extra symmetries, we can still introduce a mass through a process called Scherk-Schwarz dimensional reduction, which we will outline here briefly.
More importantly, we will also analyze how a dimensional reduction changes, if, instead of a spacial direction, we consider a null direction along which we reduce. These reductions will play a central role in the results of this thesis in section 7.

The former was originally introduced in [53] under the name "generalized dimensional reduction" to introduce mass parameters in supersymmetry. The basic idea is that if the theory has a global symmetry group G, we can consistently allow for a z-dependence during dimensional reduction, as long as it is realized by a group action depending on z, i.e. that

$$\phi(x, z) = g(z)(\psi(x))$$

where $g(z) \in G$ and $\psi(x)$ is the resulting field over M^D. This ansatz together with invariance under G of the higher-dimensional theory will guarantee that the resulting theory is independent of z and hence is a theory over D dimensions.

We will illustrate this procedure together with ideas of the second type of dimensional reduction, the so-called *null reduction*. Following [54], this gives us another way to derive (torsional) Newton-Cartan geometry from Lorentzian geometry. Here we assume that the compact direction is null instead of spacelike, which means that the killing vector field χ tangent to the compact direction is lightlike, i.e. $\hat{g}(\chi, \chi) = 0$.

Again, we first look at a massless, but now complex scalar field ϕ in a flat background, to get some intuition and motivation.
Let us first change to lightcone coordinates, where we have the two null directions (x^+, x^-) given by $x^\pm := x^0 \pm x^{D-1}$, together with the spacial directions x^a, where $a = 1, ..., D-2$. This means, that the Minkowski metric now reads $\eta_{ab} = \delta_{ab}$, $\eta_{+-} = -1$ and all others zero (in particular $\eta_{--} = \eta_{++} = 0$). Assuming that x^+ is the compact direction and invoking the $U(1)$ symmetry of the complex scalar field, we may take the Scherk-Schwarz ansatz

$$\phi(x, x^+, x^-) = g(x^+)(\psi(x, x^-)) = e^{-imx^+}\psi(x, x^-),$$

for $m \in \mathbb{R}$ a real parameter. Then the Klein-Gordon equation for the massless scalar ϕ reads:

$$\Box_{D+1}\phi = \Delta_x\phi - 2\partial_+\partial_-\phi = 0$$

and plugging in the ansatz for ϕ, we get, renaming $x^- =: t$, that

$$i\partial_t\psi(x, t) = -\frac{1}{2m}\Delta_x\psi(x, t),$$

i.e. the free Schrödinger equation with $\hbar = 1$.

So, we see that one of the null directions becomes a time direction and that we can embed the free Schrödinger equation, an equation invariant under the non-relativistic Bargmann algebra, into the massless relativistic Klein-Gordon equation.

Let us now focus solely on the null reduction and apply it to a higher-dimensional vielbein. We assume the existence of null Killing vector field χ of the metric $\hat{g}$ in $D+1$-dimensions, i.e.

$$\mathcal{L}_\chi\hat{g} = 0, \quad \hat{g}(\chi, \chi) = 0.$$

Again, by the Frobenius theorem we find adapted coordinates $(x^{\hat{\mu}}) = (x^\mu, v)$, s.t. $\chi = \partial_v$, which implies

$$\partial_v\hat{g}_{\hat{\mu}\hat{\nu}} = 0, \quad \hat{g}_{vv} = 0. \tag{51}$$

Null directions in tangent space always come in pairs, so we may write the vielbein index as $A = (a, +, -)$, where $a = 1, ...D-1$ and $\pm$ denote the null directions.
Then, analogously to the KK reduction, we can infer that the fields ${E_v}^\pm$ transform as scalar fields under diffeomorphisms. We can then combine the null condition on g, i.e.

$$\hat{g}_{vv} = 0 \iff \eta_{ab}{E_v}^a{E_v}^b - 2{E_v}^+{E_v}^- = 0,$$

with the transformation property under $SO(1, D)$-transformations

$$\delta{E_v}^a = {\lambda^a}_b{E_v}^b + {\lambda^a}_+{E_v}^+ + {\lambda^a}_-{E_v}^-.$$

Again as in section 5.2, since ${E_v}^+$ is a scalar field, we observe a kind of Stückelberg symmetry for ${E_v}^a$ with parameter $C^a := {\lambda^a}_+ {E_v}^+$. Consequently, we can use this shift symmetry to consistently set

$${E_v}^a = 0,$$

and the null condition then implies

$${E_v}^+ {E_v}^- = 0,$$

which will certainly hold if

$${E_v}^- \equiv 0.$$

To preserve the gauge ${E_v}^a = 0$, we have to set ${\lambda^a}_+ \equiv 0$, leading to the variation

$$\delta {E_v}^+ = {\lambda^+}_+ {E_v}^+,$$

i.e. the scalar field ${E_v}^+$ transforms under dilatations with weight 1 and parameter ${\lambda^+}_+$. The condition ${E_v}^- \equiv 0$ does not require any further restrictions on parameters, and the remaining variations take the form

$$\begin{aligned}
\delta {E_\mu}^a &= \mathcal{L}_\xi {E_\mu}^a + {\lambda^a}_- {E_\mu}^-, \\
\delta {E_\mu}^+ &= \mathcal{L}_\xi {E_\mu}^+ + \partial_\mu \xi^v {E_v}^+ + \lambda_{a-} {E_\mu}^a + {\lambda^+}_+ {E_\mu}^+, \\
\delta {E_\mu}^- &= \mathcal{L}_\xi {E_\mu}^- - {\lambda^+}_+ {E_\mu}^-.
\end{aligned}$$

We summarize the above in the below ansatz (see [55] for a thorough treatment) of a higher-dimensional vielbein under null reduction as

$$({E_{\hat\mu}}^A) = \begin{array}{c c} & \begin{array}{ccc} a & - & + \end{array} \\ \begin{array}{c} \mu \\ v \end{array} & \begin{pmatrix} {e_\mu}^a & s^{-1}\tau_\mu & s m_\mu \\ 0 & 0 & s \end{pmatrix} \end{array}, \quad ({E^{\hat\mu}}_A) = \begin{array}{c c} & \begin{array}{cc} \mu & v \end{array} \\ \begin{array}{c} a \\ - \\ + \end{array} & \begin{pmatrix} {e^\mu}_a & -{e^\mu}_a m_\mu \\ s\tau^\mu & -s\tau^\mu m_\mu \\ 0 & s^{-1} \end{pmatrix} \end{array}. \tag{52}$$

Note that the fields e and τ lead to the projective inverse relations we saw in eq. (24). The ansatz breaks the full $SO(1,D)$ symmetry, and we are left with spacial rotations ${\lambda^a}_b$, boosts $\lambda^a := {\lambda^a}_-$ and dilatations of the *null dilaton* or *null scalar* s with parameter $\lambda := {\lambda^+}_+$. Note that, again, the component of diffeomorphisms in the v-direction appears as a kind of $U(1)$ parameter for m_μ, so defining $\sigma := \xi^v$, we find the transformation rules for the D-dimensional theory

$$\begin{aligned}
\delta \tau_\mu &= \mathcal{L}_\xi \tau_\mu \\
\delta {e_\mu}^a &= \mathcal{L}_\xi {e_\mu}^a + {\lambda^a}_b {e_\mu}^b + s^{-1} \lambda^a \tau_\mu \\
\delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + s^{-1} \lambda_a {e_\mu}^a \\
\delta s &= \mathcal{L}_\xi s + \lambda s.
\end{aligned} \tag{53}$$

We can gauge fix the dilatation symmetry with parameter λ by setting $s = 1$ to recover precisely Newton-Cartan geometry in D-dimensions, as discussed in Section 4.2. However, for the calculations of this thesis it is important to *not* fix this degree of freedom and instead carry it through the calculations. The dilatation symmetry arises due to the degeneracy of the metric in the lightlike direction. Thus, there is no inherent notion of length on the lightcone, and, consequently, also the radius of the compact direction has no inherent meaning.
Note that at no point any constraint on the torsion $\partial_{[\mu}\tau_{\nu]}$ had to be imposed, resulting in a torsional Newton-Cartan geometry.

More importantly though, deriving an action through a null reduction has an important downside. Since we set one component of the metric equivalently to zero, i.e. $g_{vv} \equiv 0$, it will not appear in the action. It is in general not advisable to put on-shell information back into the action. In this case, we lose the corresponding equation of motion, formally speaking the part involving R^{vv}. Thus, one should rather take the null reduction of the equations of motion and not of the action itself, as to not lose information. This exact derivation and consequent applications of non-relativistic geometry through a null reduction was already considered in [56], where it was also shown how to deal with the degeneracy at the level of actions, which then leads to the natural inclusion of the mass density in the equations of motion.

6. Non-relativistic String Theory

String theory is built upon the idea that the fundamental elements of our physical world are not particles, i.e. 0-dimensional objects, but rather extended 1-dimensional objects, i.e. strings. Their different vibrational states can then be interpreted as different types of particles, of which some can be interpreted as particles mediating gravity. Hence, string theory is a theory of quantum-gravity and thus promises to unify the standard model of particle physics with Einstein's theory of general relativity. While string theory so far did not make any measurable predictions, due to its high energy nature[10], it is still an inspiration for numerous advances in both theoretical physics and mathematics, such as *Calabi-Yau manifolds*, *Mirror symmetry* and *AdS/CFT-correspondence.*
The term "string theory" is more of an umbrella term for multiple related theories, but, usually, it refers to relativistic theories. In recent years, however, interest emerged in the study of string theories that exhibit non-Lorentzian spacetime symmetries and in particular Galilean symmetries [8, 57].

As string theory is under active research, there are plenty of references and resources to study it. The "standard" Lorentzian formulation of string and superstring theory can be found in the books [58, 59, 60], as well as in the lecture notes [24] which this section will mostly follow. A concise overview and the history of string theory can be found in the reviews [61, 62]. Material on non-relativistic string theory is not as expansive, but [8, 63, 4, 37] and the references therein provide an introduction to the matter.

In this section, we will first introduce relativistic string theory to develop the underlying concepts and language. For that, we will study string theory from the string perspective and motivate how to infer the corresponding gravitational field theory relevant to the results of this thesis. Then we will explain and examine the concept of T-dualities in string theory, to motivate the relevance of this thesis. We will then discuss non-relativistic string theory and how corresponding gravitational theories and T-dualities can be inferred in this setting.

6.1. A Hint of Relativistic String Theory

6.1.1. Non-Linear Sigma Models

String theories are formulated as so-called *non-linear sigma models* (see [22, Chapter 7.11] for details). The setup of non-linear sigma models is given by two (pseudo-)Riemannian Manifolds (M_W, g) and (M_T, G), where M_W is frequently called either *worldline* ($\dim M_W = 1$), *worldsheet* ($\dim M_W = 2$), or *worldvolume* ($\dim M_W > 2$), depending on the dimension of M_W, while M_T is usually called the *target space*, sometimes also referred to as *background.* We then view a set of coordinates of the target space M_T as maps from the worldvolume M_W to the target space, i.e. the coordinates $X^\mu, \mu = 0, ..., \dim M_T - 1$ of M_T are fields over M_W with coordinates σ^i,

[10] Arguably it makes such a prediction in the sense that it predicts a dimension of spacetime other than four.

$i = 0, ..., \dim M_W - 1$:

$$X^\mu \equiv X^\mu(\sigma).$$

Overall, the action governing such a system is

$$S[X] = \kappa \int d^D\sigma \sqrt{-\det g} g^{ij} \partial_i X^\mu \partial_j X^\nu G_{\mu\nu}, \tag{54}$$

where κ is an appropriate constant to render the action dimensionless, $G \equiv G(\sigma) \equiv G(X(\sigma))$ a symmetric two-tensor on target space, which can be interpreted as a metric on target space, while $g(\sigma)$ is the metric on the worldsheet, both of which are a priori independent. We see that the Lagrangian is just the trace with respect to g of the pullback of G to the worldvolume. In case g is defined as the pullback of G, the action equals the volume of the target space worldvolume.
The simplest example is given if the worldvolume is one-dimensional, i.e. a subset of $\mathbb{R}$, and the target space is flat Minkowski space. Provided we equip the worldvolume with the pullback of the Minkowski metric, the resulting sigma model action is

$$S[X] = \kappa \int d\tau \sqrt{-\det(X^*\eta)} \equiv -m \int d\tau \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu},$$

which is just the action of the relativistic point particle, eq. (6), that we studied as an example in section 2.4.

6.1.2. The Nambu-Goto and the Polyakov Action

String theory is another example of such a model. A string is a one-dimensional object moving in some spacetime of dimension D, which for now we assume to be flat Minkowski space. The string sweeps out a two-dimensional worldsheet, parametrized by the timelike coordinate $\tau \in \mathbb{R}$ and the spacelike coordinate $\sigma \in [0, 2\pi)$. Consequently, a point on the worldsheet is given by the embedding of a string into target space via the embedding functions

$$X^\mu(\tau, \sigma).$$

It is common notation to define $(\sigma^\alpha) \coloneqq (\tau, \sigma),\ \alpha = 0, 1$.

Either guided by our studies of sigma models or the relativistic point particle, we assume that the dynamics of the string are such that it extremizes the volume[11] of the worldsheet. If we define the pullback of the Minkowski metric under X, i.e.

$$\gamma \coloneqq X^*\eta \iff \gamma_{\alpha\beta} \coloneqq \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu},$$

[11]In the sense of its intrinsic volume, i.e. the area in this case.

The action is given in the *Nambu-Goto* form as proportional to the area of the worldsheet,

$$S_{NG}[X] \coloneqq -T\int d^2\sigma\sqrt{-\det\gamma} \equiv -T\int d^2\sigma\sqrt{-(\dot{X})^2(X')^2+(\eta_{\mu\nu}\dot{X}^\mu X'^\nu))^2},$$

where $\dot{X} \coloneqq \partial_\tau X$ and $X' \coloneqq \partial_\sigma X$. The proportionality constant T is the *tension* of the string. It is common to express the tension through the *Regge slope*

$$T =: \frac{1}{2\pi\alpha'}.$$

This notation rose out of the history of string theory, as it was invented to explain effects of the strong interaction (see [24, Section 3.1.3] for details). α' defines a length scale, the so-called *string scale* l_s defined by

$$\alpha' =: l_s^2,$$

which is the natural length scale of string theory.

The Nambu-Goto action enjoys a global Poincaré symmetry $X^\mu \mapsto A^\mu{}_\nu X^\nu + \xi^\mu$, as well as a gauge reparametrization symmetry $\sigma \mapsto \tilde{\sigma}(\sigma)$, but as it is non-polynomial in derivatives of fields, it is hard to quantize. Therefore, we introduce the equivalent *Polyakov action*

$$S_P[X] \coloneqq -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu},$$

which we immediately recognize as the two-dimensional sigma model. Thus, we have introduced the auxiliary field g onto the worldsheet, and the Polyakov action can be interpreted as a collection of scalar fields X coupled to a two-dimensional gravity background.

We have discussed how the Polyakov action coincides with the Nambu-Goto action, provided that g is the pullback of η. However, when solving the (algebraic) equations of motion for g, we find the result

$$g_{\alpha\beta} = 2f\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu},$$

where

$$\frac{1}{f} \coloneqq g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}.$$

Thus, g is not necessarily the exact pullback metric but only a scaled version of it. Notably, the scaling f cancels when plugging the resulting g in the Polyakov action and gives exactly the Nambu-Goto action. This cancelation relies on the dimension of the worldsheet being two.
It also hints at an important symmetry of the action. Firstly, the Polyakov action enjoys the same symmetries as the Nambu-Goto action, i.e. a global Lorentz symmetry,

as well as the reparametrization invariance, which we from now on recognize as two-dimensional diffeomorphisms. The worldsheet metric g transforms correctly as 2-tensor under these diffeomorphisms. More importantly, due to the fact that the worldsheet is two-dimensional, we have a gauge symmetry involving a rescaling of the metric, i.e. *Weyl symmetry*, acting as

$$\begin{aligned} X(\sigma) &\mapsto X(\sigma), \\ g(\sigma) &\mapsto \Omega^2(\sigma) g(\sigma). \end{aligned}$$

The latter takes the infinitesimal form

$$\delta g = 2\omega g,$$

for $\Omega^2 = e^{2\omega}$. We can now fix a gauge for the symmetries. The metric has three independent components. Using our two diffeomorphism degrees of freedom, we impose the *conformal gauge*, i.e. we fix the metric to be locally conformally flat

$$g = e^{2\omega}\eta,$$

for ω a function on the worldsheet. Invoking Weyl invariance, we can also gauge fix $\omega \equiv 0$ and thus end up with the flat Minkowski metric on the worldsheet.

Nonetheless, we still have some gauge freedom left. Due to the Weyl invariance, we can take any conformal diffeomorphism, i.e. a diffeomorphism s.t. $g \mapsto e^{2\omega} g$, and undo its action by rescaling the metric.

As the metric defines the geometry of the manifold, the Polyakov string does not distinguish between geometries that are deformed locally, provided the deformation preserves angles. Such an invariance poses a strong restriction on the theory and is generally known as a *conformal symmetry*. A field theory with such a symmetry is called *conformal field theory (CFT)* and is now recognized to be one of the de facto ways to describe string theory. The study of CFTs is quite expansive, and hence, cannot be presented here. A standard introduction can be found in [64].

6.1.3. The Classical Equations of Motion of the Closed String

Given the flat metric on the worldsheet, the Polyakov action simplifies tremendously, as it takes the form of a theory of D free scalar fields. The equations of motion for the string reduce to the free wave equation

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{55}$$

However, by introducing the worldsheet metric g, we also introduced an auxiliary field with its own equations of motion. They are given by requiring that the energy momentum tensor vanishes, i.e.

$$T_{\alpha\beta} \coloneqq -\frac{2}{T}\frac{1}{\sqrt{-\det g}}\frac{\partial S}{\partial g^{\alpha\beta}} = 0.$$

This results in the two constraints

$$\begin{aligned} T_{01} &= \dot{X} \cdot X' = 0, \\ T_{00} = T_{11} &= \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0, \end{aligned} \tag{56}$$

where we denoted the inner product on target space with the usual dot. The first constraint imposes that the string moves perpendicular to itself, and the second constraint results in a well-defined notion of a string length (for details see [24, Section 1.3.2]).

The two-dimensional wave equation eq. (55) can be easily solved by introducing light-cone coordinates

$$\sigma^{\pm} := \tau \pm \sigma,$$

where the general solution splits into a left and a right moving part

$$X^\mu(\tau, \sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-).$$

For now, we are mainly interested in the case of closed strings, i.e. the case in which we impose periodic boundary conditions

$$X(\tau, \sigma) = X(\tau, \sigma + 2\pi),$$

which allows us to write the general solution in terms of Fourier modes as

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{n}\tilde{a}^\mu_n e^{-in\sigma^+}, \\ X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{n}a^\mu_n e^{-in\sigma^-}. \end{aligned} \tag{57}$$

Here, x^μ and p^μ correspond to the position and momentum of the center of mass of the string. The normalization is chosen for later convenience. Furthermore, it is also often convenient to define the zero modes

$$\begin{aligned} a^\mu_0 &:= \sqrt{\frac{\alpha'}{2}}p^\mu, \\ \tilde{a}^\mu_0 &:= \sqrt{\frac{\alpha'}{2}}p^\mu, \end{aligned}$$

for left and right moving modes. Finally, by requiring the solution to be real, we find the relation for the Fourier modes

$$\begin{aligned} a^\mu_n &= \left(a^\mu_{-n}\right)^*, \\ \tilde{a}^\mu_n &= \left(\tilde{a}^\mu_{-n}\right)^*. \end{aligned}$$

Defining the sum of oscillator modes as

$$L_n \coloneqq \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{n-m} \cdot a_m,$$

and equivalently for the tilded oscillators, the constraints (56) take the form

$$L_n = \tilde{L}_n = 0, \ \forall n \in \mathbb{Z}.$$

The constraints for $n = 0$ are special, since they involve the a_0 mode, and hence, by definition, the momentum p^μ. Thus, the constraints can be rewritten as the on-shell condition $p \cdot p = -M^2$, where the effective mass M^2 can be expressed in terms of the oscillator modes as

$$M^2 = \frac{4}{\alpha'} \sum_{n=1}^{\infty} a_n \cdot a_{-n} = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \tilde{a}_n \cdot \tilde{a}_{-n}. \tag{58}$$

We see that the oscillator modes determine the mass, and even though we have not quantized the theory yet, we may interpret the different excitations of the string as particles of different masses.
We see that we can express the mass either via the left- or the right-moving modes, but they have to match. If we define the *level operators*

$$N \coloneqq \sum_{n=1}^{\infty} a_n \cdot a_{-n},$$
$$\tilde{N} \coloneqq \sum_{n=1}^{\infty} \tilde{a}_n \cdot \tilde{a}_{-n},$$

the condition

$$N = \tilde{N}$$

is called *level matching*, and it has interesting consequences when quantizing the closed string.

6.1.4. Quantization of the Closed String

As this thesis is not directly concerned with quantum-aspects of string theory, we will only provide an idea of the quantization procedure here. The full details can, however, be found in [24, Chaper 2].

Taking the usual route of canonical quantization leads to the commutation relations

$$[x^\mu, p_\nu] = i\delta^\mu{}_\nu$$
$$[a_n^\mu, a_m^\nu] = [\tilde{a}_n^\mu, \tilde{a}_m^\nu] = n\eta^{\mu\nu}\delta_{m+n,0}.$$

The commutation relations for x and p give exactly what we expect from operators realizing momentum and position of a point-like object like the center of mass, while the Fourier modes can be interpreted as creation and annihilation operators upon defining

$$c_n := \frac{a_n}{\sqrt{n}},\ c_n^\dagger := \frac{a_{-n}}{\sqrt{n}},\ n > 0,$$

for each target space direction. Then we can build the Fock space as usual, but filled with two different towers of states, by introducing a vacuum that obeys

$$a_n^\mu \left|0\right\rangle = \tilde{a}_n^\mu \left|0\right\rangle,\ \forall n > 0.$$

Importantly, this is *not* the vacuum of spacetime but the vacuum of a string, i.e. describes an unexcited string. This can be seen from the center of mass position and momentum operators, as they give this vacuum extra structure in the form of momentum eigenvalues p, i.e. $|0;p\rangle$. A generic state then takes the form of an excited state of a string

$$\left(a_{-1}^{\mu_1}\right)^{n_{\mu_1}} \left(a_{-2}^{\mu_2}\right)^{n_{\mu_2}} ... \left(\tilde{a}_{-1}^{\nu_1}\right)^{n_{\nu_1}} \left(\tilde{a}_{-2}^{\nu_2}\right)^{n_{\nu_2}} ... \left|0;p\right\rangle,$$

where each of the *infinitely-many* excitations can be interpreted as a different particle in spacetime.

A problem arises due to the Lorentzian signature in target space. It results in negative norm states, so-called *ghosts*, from the zero-direction

$$\left\langle p';0\right| a_1^0 a_{-1}^0 \left|0;p\right\rangle \sim -\delta^D(p-p').$$

There are different approaches to remedy this, and we choose the so-called *lightcone quantization*. The procedure starts with the introduction of lightcone coordinates on target space

$$X^\pm := \sqrt{\frac{1}{2}}\left(X^0 \pm X^{D-1}\right).$$

It turns out that we can use the remnant reparametrization freedom from section 6.1.2 to set the Fourier modes in the plus-direction to zero, resulting in the solution in the *lightcone gauge*

$$X^+ = x^+ + \alpha' p^+ \tau,$$

or put differently, we identify the plus-direction of target space with the time-direction of the worldsheet. Furthermore, due to the constraints in eq. (56), the Fourier modes and momentum in the minus-direction can be expressed in terms of the *transverse* quantities p^i and a_n^i, where $i = 1, ..., D-2$.
Furthermore, the *classical* spectrum and level-matching condition then read

$$M^2 = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n=1}^{\infty} a_{-n}^i a_n^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n=1}^{\infty} \tilde{a}_{-n}^i \tilde{a}_n^i.$$

In total, the general classical solution consists of transverse oscillator modes a_n^i, $\tilde{a}_n^i$ and the zero modes x^i, p^i, p^+ and x^- that characterize the motion of the center of mass. x^+ can be thought of as time and p^- as the lightcone Hamiltonian, since it generates translations in x^+, i.e. in time.

Correspondingly, we impose the following commutation relations for quantization

$$\begin{aligned}[x^i, p^j] = i\delta^{ij},\ [x^+, p^-] = [x^-, p^+] = -i,\\ [a_n^i, a_m^j] = [\tilde{a}_n^i, \tilde{a}_m^j] = n\delta^{ij}\delta_{m+n,0}.\end{aligned}$$

String states are again built from the vacuum $|0;p\rangle$ with momentum eigenvalues of p^μ but only by acting with the transverse creation operators a_{-n}^i, $n > 0$. Thus, by construction, no ghosts appear in the Fock space.

Still, we have to impose the level-matching condition. But quantization amounts to introducing non-commuting objects, hence, we also have to consider the correct ordering to have a well-defined quantization. We will use normal ordering, where creation operators are sorted to the left. Defining the quantum level operator

$$N \coloneqq \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} : a_{-n}^i a_n^i :$$

and similarly for the tilded modes, the spectrum and level-matching condition then suffers from an ordering-ambiguity, which we can encode in the ordering constant $a \in \mathbb{R}$. Consequently, we postulate the ansatz for the mass formula to be

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\tilde{N} - a).$$

Through a heuristic calculation (see [24, Equation (2.26)]) one can show that

$$a = \frac{D-2}{24}.$$

This finally lets us analyze the spectrum of the closed string. The ground state $|0;p\rangle$ corresponds to a *tachyon*, as it has negative mass squared

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6}.$$

In quantum field theories, tachyons indicate an instability, as they arise from expansions around a maximum of the potential. In String theory, the tachyon is poorly understood and leads to some undesired effects, but it is not present in *superstring theory*, i.e. string theory including fermions, thus it can be ignored for now.

Overall, the first excited states are more interesting anyway. Because of the level-matching condition, we have to act with both a and $\tilde{a}$, resulting in $(D-2)^2$ particle states of the form

$$\tilde{a}_{-1}^i a_{-1}^j |0;p\rangle .$$

They are of mass

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right). \tag{59}$$

This presents a problem, however, as by Wigner's method of induced representations (see [65]), massive states should transform under the group $SO(D-1) \subset SO(1, D-1)$ of spatial rotations. However, our states are built from the oscillators a^i, $i = 1, ..., D-2$ which transform under $SO(D-2)$ and the $(D-2)^2$-many states cannot fit in such a massive representation.
If, on the other hand, the first excited states were massless, by Wigner's Method they would fall under the massless representation of the group $SO(D-2)$ we already have present.
Thus, to implement Lorentz symmetry, we need the mass in eq. (59) to vanish, which in turn requires that the dimension of spacetime is

$$D = 26.$$

This so-called *critical dimension* is an infamous result of string theory and also one notable prediction of string theory. It is also one of the reasons why the techniques of dimensional reduction we studied in section 5 became interesting again for theoretical physicists.
It turns out that the critical dimension can also be inferred from requiring that the Noether charges of the Lorentz transformations form a representation of the algebra (see [24, Section 2.4]).

Altogether, the massless states in string theory split into a direct sum of irreducible representations of $SO(D-2)$ given by the *traceless symmetric*, the *antisymmetric* and the *singlet/trace* representation. To each mode there is a corresponding target space field given by

$$G_{\mu\nu}(X),\ B_{\mu\nu}(X),\ \Phi(X).$$

The first field can be identified as the spacetime metric of the target space, as it is a massless spin 2 particle, the second field is called *Kalb-Ramond* field and the third, the scalar field, is called the *dilaton*.

Thus, we see that string theory naturally includes a graviton in its spectrum, giving a theory of quantum gravity.

6.1.5. The Open String and Branes

So far, we have only considered the closed string, i.e. strings with periodic boundary conditions. If we now consider a general string, the variation of the action is only stationary if the following boundary conditions are fulfilled

$$\partial_\sigma X^\mu \delta X_\mu = 0, \quad \text{at } \sigma = 0, \pi.$$

Thus, we can have two different boundary conditions for the endpoints of the string

1. Neumann boundary conditions

$$\partial_\sigma X^\mu = 0, \quad \text{at } \sigma = 0, \pi.$$

   Here, the endpoints are free to move as δX is unconstrained.

2. Dirichlet boundary conditions

$$\delta X^\mu = 0, \quad \text{at } \sigma = 0, \pi. \tag{60}$$

   Here, the endpoints are fixed at a constant position $X = c$.

The Dirichlet boundary conditions appear rather strange, but have ultimately led to an important ingredient of string theory. Assume, that we impose Neumann boundary conditions on the time and the first $p \leq D-1$ coordinates and Dirichlet on the remaining directions, i.e.

$$\begin{aligned} \partial_\sigma X^a = 0, &\quad a = 0, ..., p, \\ X^I = C^I, &\quad I = p+1, ..., D-1. \end{aligned}$$

This effectively constrains the endpoints to move on a $p+1$-dimensional hypersurface of spacetime, breaking the Lorentz symmetry $SO(1, D-1)$ into $SO(1,p) \times SO(D-p-1)$. We call this hypersurface a *Dp-brane*, where the p stands for its spacial dimension. Notably, *Dp*-branes also become dynamical quantities. This revelation led to the second superstring revolution and allowed for the construction of realistic cosmological models.

Solving the equations of motion leads to the same mode expansion as in eq. (57), but the different boundary conditions lead to

$$\begin{aligned} &a_n^a = \tilde{a}_n^a, \\ &x^I = c^I, \ p^I = 0, \ a_n^I = -\tilde{a}_n^I, \end{aligned}$$

thus we have only one set of oscillators remaining.

Quantization then proceeds analogously to the closed string via lightcone quantization, but as the momentum operator p^I is constrained to be zero, the wave functions of such string modes depend only on the first x^a coordinates, thus effectively live on the *Dp*-brane.
The mass formula is also modified slightly

$$M^2 = \frac{1}{\alpha'} \left(\sum_{i=1}^{p-1} \sum_{n=1}^{\infty} a_{-n}^i a_n^i + \sum_{I=p+1}^{D-1} \sum_{n=1}^{\infty} a_{-n}^I a_n^I - 1 \right).$$

It turns out that quantization again leads to the critical dimension $D = 26$. This indicates that open and closed strings are not disjoint but form a common theory.

We introduce a ground state that is annihilated by all annihilation operators, i.e.

$$a_n^\mu \left|0;p\right\rangle = 0, \quad n > 0.$$

Again, this ground state is tachyonic and appears to be unstable.

The first excited states are more interesting as they are again massless. They split into two distinct classes

1. Longitudinal:

$$a_{-1}^i \left|0;p\right\rangle, \quad i = 1, ..., p-1.$$

   The index i transforms under the rotation subgroup $SO(p-2)$ of the Lorentz group $SO(1,p)$ on the brane, thus they fit into a massless representation. The corresponding gauge field A_a, $a = 0, ..., p$ living on the brane can then be interpreted as a photon.

2. Transverse

$$a_{-1}^I \left|0;p\right\rangle, \quad I = p+1, ..., D-1.$$

   These states are scalars under the Lorentz group of the brane and transform under the remaining $SO(D-p-1)$ rotations. They can be interpreted naturally as the fluctuations of the brane in the orthogonal direction.

6.1.6. Superstring Theory

In this section we follow [62, Section 3.1], but only present the very minimum to introduce language.

Superstring theory relies on introducing fermionic degrees of freedom ψ onto the worldsheet, resulting in the action (in the conformal gauge)

$$S = -\frac{T}{2}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right).$$

We consider this theory over flat Minkowski space, with α a two-dimensional index on the worldsheet, while μ is a D-dimensional index on target space. ρ^α are two 2×2-dimensional matrices, which satisfy the Clifford relation $\left\{\rho^\alpha, \rho^\beta\right\} = 2\eta^{\alpha\beta}$ and the D-many massless spinors $\psi^\mu = (\psi_-^\mu, \psi_+^\mu)$ are Majorana. Again, introducing lightcone coordinates on the worldsheet, we have the spinor equations of motion

$$\partial_+ \psi_-^\mu = \partial_- \psi_+^\mu = 0.$$

Thus, ψ_- corresponds to right-moving modes and ψ_+ to left-moving. The action is invariant under (worldsheet) supersymmetry (see [62, Equation (74)]), a symmetry exchanging fermionic and bosonic degrees of freedom, hence the name *super*string.

Due to the extra fermionic degrees of freedom present, the critical dimension is now reduced to

$$D = 10.$$

The quantization and the spectrum of superstring theory are more complicated than the bosonic string (for details see [62, 59]). However, through the so-called GSO projection introduced in [66] one can consistently truncate different modes from the spectrum, in particular the tachyonic mode. Furthermore, it also implies that supersymmetry is implemented on the target space as well.

One can then find different supersymmetric string theories, for example Type IIA, Type IIB, Type I, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic. At the level of massless modes, the Type II string theories have a common sector, the so-called NS-NS-sector (where NS stands for Neveu-Schwarz). It looks analogous to the massless modes of bosonic string theory, i.e. we have the graviton, Kalb-Ramond field and dilaton

$$G_{\mu\nu}, B_{\mu\nu}, \Phi.$$

Additionally, superstring theories typically contain extra massless modes, including the higher gauge fields A for Type II given by

Name	Extra massless modes
Type IIA	A_μ, $A_{\mu\nu\rho}$
Type IIB	A, $A_{\mu\nu}$, $A_{\mu\nu\rho\sigma}$

6.1.7. Low Energy Actions and Scaling Symmetry

We now return to the case of the bosonic string in $D = 26$ dimensions, following [24, 67, Chapter 6&7]. So far, we have only considered a string moving in a flat background, which ultimately led to a theory of D free scalar fields. In the following we want to give a glimpse of how to do perturbative expansions in string theory, how this reproduces a theory of gravity at low energies and finally, how we can couple a string to such a curved background.

We can compute scattering amplitudes in string theory through the path integral formalism, in which we integrate $e^{S_\mathrm{P}}V$ over all possible metrics, where V is a so-called *vertex operator*. We will not need the precise details of such operators in this thesis (see [24, Section 5.4]), just note that they represent asymptotic in- and outgoing states on a worldsheet with fixed topology. Additionally, to take interactions into account, we need to also consider all different possible topologies of the worldsheet, which we then summarize in a perturbative series. A visual example of this is given by the scattering of two closed strings, visualized as in fig. 1 below:

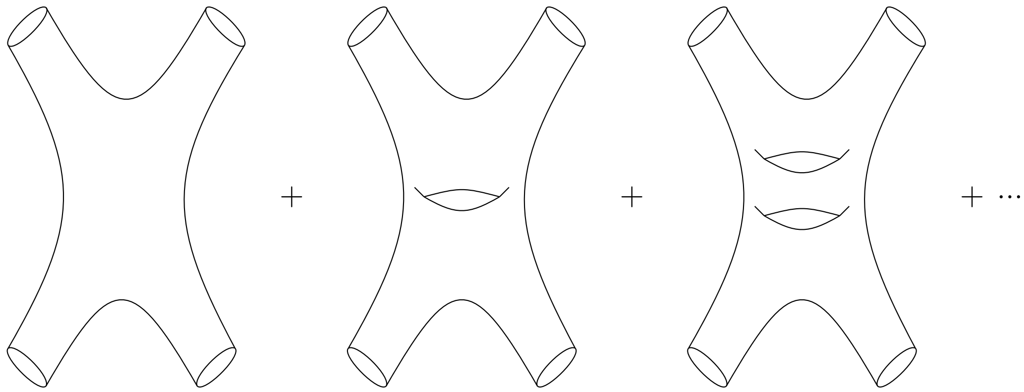

Figure 1: Visualization of string scattering given in terms of a perturbation series over different topologies.

Contributions are weighted differently with a *string coupling* constant

$$g_s \coloneqq e^{\lambda},$$

where λ is a real number. It is useful, because it allows for an asymptotic series for $g_s \ll 1$. In practice, the string coupling is introduced by an ad-hoc modification of the Polyakov action given by the string action

$$S_{\text{string}} = S_{\text{P}} + \lambda\chi, \tag{61}$$

where χ is the Euler characteristic, which in two dimensions can be expressed via the Ricci scalar as

$$\chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\det h}R^{(2)}(h). \tag{62}$$

This term is invariant under diffeomorphisms as well as Weyl transformations. As it is also independent of the particular worldsheet metric h, it only amounts to a topological term[12], which turns out to be the number of holes, the genus g of the worldsheet, i.e.

$$\chi = 2(1-g).$$

Thus, the summation over topologies translates into a summation over the genera, i.e. we have the perturbative expansion

$$\sum_g e^{-2\lambda(1-g)}\int \mathcal{D}X\mathcal{D}he^{-iS_{\text{P}}(X,h)}V.$$

This allows us to compute the scattering amplitudes order by order, and as it turns out (see [24, Section 6.2.3]), the amplitude for tachyonic in- and out-goin states has poles exactly at the physical masses of the string modes, which we identified as different

[12]And thus does not make gravity dynamical.

particles.
Recall, the mass was proportional to $1/\alpha'$, thus, this leads to the idea of a *double expansion*, where we expand the amplitudes in powers of α'.
It is then only natural to consider *low energy effective actions*, i.e. theories with an action that effectively reproduces the string scattering amplitudes in the lowest order of α', while also respecting the gauge symmetries of the amplitudes.
Remarkably, as it turns out (see [68, Section 16.3]), at lowest order in α' the effective action of string scattering is given exactly by a modified version of Einstein gravity with the Lagrangian

$$S_{\text{NS-NS}} = \frac{1}{2\kappa^2}\int d^D X\sqrt{-\det G}e^{-2\Phi}\left(R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\right), \tag{63}$$

where G is a Lorentzian metric, $H \coloneqq dB$ is the field strength of the Kalb-Ramond field B and Φ is the dilaton. This theory of gravity is the common sector of all supergravity effective actions.

So far, we viewed everything in the setting of the massless sector of the bosonic string. However, the same formula holds in the case of the superstring and gives the low energy effective action of the massless NS-NS sector we briefly saw in section 6.1.6, thus, for $D = 10$ we will also call this theory *NS-NS gravity*.

The formalism to derive such effective actions is quite involved, however, there is an alternative approach to derive similar results. For this approach, we consider a string moving in a *curved background*, i.e. a sigma model with a general background metric G, given by the action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G(X)_{\mu\nu}.$$

We interpret the target space metric $G(X)$ as a self-coupling of the fields X and as it turns out (see [24, Section 7]) the inclusion of G can be interpreted as a coherent graviton state.

The non-linear sigma model of the string has a well-known perturbation theory, but the quantum theory has divergences that have to be regularized. Thus, we have to introduce cut-offs during regularization and consequently also a scale μ for a given process. However, we have seen that classical string theory is conformally invariant, i.e. scale independent, and we want to keep this gauge symmetry in the quantum theory.

Now, given a coupling g and an energy scale μ, the dependence of the coupling on the scale is encoded in the *beta function* by

$$\beta(g) \coloneqq \mu\frac{\partial g}{\partial\mu}.$$

If a process or theory is scale invariant, the beta function of its coupling should be zero, $\beta(g) \equiv 0$. In our case, the coupling is the background metric G, and its beta function can be computed order by order in perturbation theory (see [24, Section 7.1.1]). Remarkably, doing so to first order for string theory in a curved background reveals that the theory is scale invariant at the quantum level, i.e. conformally invariant, if we have

$$\beta_{\mu\nu}(G) \equiv \alpha' R_{\mu\nu} + \mathcal{O}(\alpha'^2) = 0.$$

But for $D \neq 2$ these are *exactly* the vacuum Einstein-equations (14). Thus, requiring conformal invariance on the string worldsheet is *equivalent* to requiring that the target space follows Einstein's theory of general relativity. Thus, we see the emergence of D-dimensional gravity from a two-dimensional string. Consequently, at low energies, the string coupled to a curved background can be effectively described by the modified version of the Einstein-Hilbert action (12).

We can continue and couple the string to the other massless background fields B, Φ we have found in the quantum string theory. The fully coupled string is given by the action

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det g} \left(g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G(X)_{\mu\nu} - \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B(X)_{\mu\nu} + \alpha' \Phi(X) R^{(2)} \right), \tag{64}$$

where $R^{(2)}$ is the Ricci scalar of the two-dimensional worldsheet, i.e. of the metric g. The coupling of B to the string is realized through the introduction of a so-called *Wess-Zumino term*[13] and amounts to considering a charged string (see [59, Chapter 16]). The Wess-Zumino term remains invariant under transformations of the two-form field B under the one-form symmetry with parameter θ given by

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\theta_{\nu]}. \tag{65}$$

Ultimately, the beta functions corresponding to the action (64) are

$$\begin{aligned}
\beta(G)_{\mu\nu} &= \alpha' \left(R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\lambda\rho} H_\mu{}^{\lambda\rho} \right) \\
\beta(B)_{\mu\nu} &= \alpha' \left(-\frac{1}{2} \nabla_\rho H^\rho{}_{\mu\nu} + \nabla_\rho \Phi H^\rho{}_{\mu\nu} \right) \\
\beta(\Phi) &= \alpha' \left(\frac{1}{2} \nabla^2 \Phi + \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} \right),
\end{aligned}$$

where $H \coloneqq dB$ is the gauge invariant field strength of B. A background to string theory is only consistent with Weyl symmetry on the quantum level if the above beta functions vanish. We can interpret the vanishing of the beta functions as equations of motion for the background, and they follow precisely from the low energy effective action eq. (63).

The dilaton coupling in the full string action (64) takes on an interesting role. Comparing to eq. (62) and (61), we see that the dilaton Φ appears in the place of λ, thus making

[13] For more details see 6.2.4.

the string coupling also a *dynamical* quantity. More precisely, the expectation value of the dilaton determines the string coupling, hence the coupling is not and independent parameter of the theory.

Remarkably, we have derived how spacetime and other fields behave in $D = 26$ or $D = 10$ just from the dynamics of a string moving in a flat spacetime.

6.2. T-dualities

In section 5 we studied the impact that compact dimensions of spacetime had on theories formulated on them. There, we mostly used it as a tool to derive theories or embed theories in higher dimensions. As mentioned, in string theory compactifications take on a different role. We have seen that string theories are only consistent in dimension $D = 26$ or $D = 10$, and as our world appears four-dimensional, we need a way to derive lower-dimensional theories from string theory. Here, we want to investigate the interplay between the compact dimensions and string theory. This will lead to a surprising *duality*, i.e. a relation between distinct physical systems that are formulated in very different ways but described by the same physics. This section follows mostly [24, Chapter 8] and [59, Chapter 17].

6.2.1. Compactifications on Target Space

We thoroughly described the implications of compactifications on background fields in section 5, but to connect to our current notation, we review some notions here.

The setup, again, is that we assume that one of the spacelike directions of target space is compact and periodic, i.e. a circle of radius R. Without loss of generality, we assume that direction to be the last one, i.e. in the setting of bosonic string theory, we assume that

$$x^{25} = x^{25} + 2\pi R.$$

Such a compactification meant that the 26-dimensional metric $G_{\hat{\mu}\hat{\nu}}$ gave rise to a 25-dimensional metric $G_{\mu\nu}$, a $U(1)$ gauge field A_μ and a scalar k. Furthermore, as we have seen in section 6.1.4, we also have the Kalb-Ramond field $B_{\hat{\mu}\hat{\nu}}$ present which splits into a two-form field $B_{\mu\nu}$ and another $U(1)$ gauge field $\tilde{A}_\mu =: B_{\mu 25}$ (see the end of section 5.3 for details). All of these fields live on the 24-dimensional spacetime M^D orthogonal to the compact direction.

6.2.2. Compactifications on the Worldsheet

We consider the setting of closed strings, where we analyze the string embedding in the compact direction, i.e. X^{25} for fixed $\tau \in \mathbb{R}$. We thus have the (differentiable) map

$$\begin{aligned} X^{25}(\tau, \cdot) \colon S^1 &\to S^1, \\ \sigma &\mapsto X^{25}(\tau, \sigma). \end{aligned}$$

But continuous maps between S^1 are characterized by an integer $m \in \mathbb{Z}$, called the *winding number*, and in our setting this amounts to the inclusion of more general periodicity conditions than in section 6.1.3, namely

$$X^{25}(\tau, 2\pi) = X^{25}(\tau, 0) + 2\pi R m.$$

This number describes how many times the string winds around the compact dimension, hence the name. Furthermore, having a compact direction present quantizes the momentum along the compact direction, as to leave the corresponding plane-wave eigenfunctions well-defined. Thus, the momentum takes the form

$$p^{25} = \frac{n}{R}, \quad n \in \mathbb{Z}.$$

Note, that this is exactly the charge for massive Fourier modes we found in eq. (44). Ultimately, while all directions orthogonal to the compact direction have the same expansion as before, the embedding function of the compact direction now reads

$$X^{25}(\tau, \sigma) = x^{25} + \frac{\alpha' n}{R}\tau + mR\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{e^{-ik\tau}}{k} \left(\tilde{a}_k^{25} e^{-ik\sigma} + a_k^{25} e^{ik\sigma}\right). \tag{66}$$

We see that the expansion includes the momentum proportional to n multiplying τ, but also a term involving the winding number, multiplying σ.

It is useful to split this expression into a left- and right-moving part, introducing the quantities

$$\begin{aligned} p_L &:= \frac{n}{R} + \frac{mR}{\alpha'}, \quad p_R := \frac{n}{R} - \frac{mR}{\alpha'}, \\ x_L &:= x + q, \quad x_R := x - q. \end{aligned}$$

The position q seems to be meaningless at this point, as it will cancel right away, but it will be useful later on. Using this notation, we have the results

$$\begin{aligned} X_L^{25}(\sigma^+) &= \frac{1}{2}x_L + \frac{1}{2}\alpha' p_L \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{k}\tilde{a}_k^{25} e^{-ik\sigma^+}, \\ X_R^{25}(\sigma^-) &= \frac{1}{2}x_R + \frac{1}{2}\alpha' p_R \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{k} a_k^{25} e^{-ik\sigma^-}. \end{aligned}$$

Quantization proceeds as before, but we now have to view formulas from the viewpoint of an observer living in the spacetime x^μ, $\mu = 0, ..., 24$. The corresponding mass is

$$M^2 = p_L^2 + \frac{4}{\alpha'}(\tilde{N} - 1) = p_R^2 + \frac{4}{\alpha'}(N - 1),$$

where the level operators are defined as before including all directions. It turns out that level matching now is implemented by the requirement

$$N - \tilde{N} = nm.$$

Plugging in the momenta into the mass formula, we find that

$$M^2 = \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(\tilde{N} + N - 2). \tag{67}$$

So, we see the effect of the quantized momentum, but also a new contribution from the winding number, due to the tension of the string stretched around the compact direction.

We can now investigate the massless modes for zero momentum $n = 0$ and zero winding $m = 0$, i.e. at level $N = \tilde{N} = 1$. They arrange in the three groups

1. $a^\mu_{-1}\tilde{a}^\nu_{-1}\left|0;p\right\rangle$ - These fit in a massless representation of $SO(1,24)$, resulting again in a metric $G_{\mu\nu}$, a 2-form $B_{\mu\nu}$ and a scalar Φ.
2. $\left(a^\mu_{-1}\tilde{a}^{25}_{-1}\left|0;p\right\rangle, a^{25}_{-1}\tilde{a}^\nu_{-1}\left|0;p\right\rangle\right)$ - These give two massless vector fields, but remembering the decomposition of the metric and the Kalb-Ramond field under dimensional reduction, we identify the sum

$$\left(a^\mu_{-1}\tilde{a}^{25}_{-1} + a^{25}_{-1}\tilde{a}^\nu_{-1}\right)\left|0;p\right\rangle$$

with the vector A_μ coming from the metric. As it turns out (see [24, Section 8.2.2]) fields with momentum number n are charged under A_μ with charge $p_L + p_R \sim \frac{n}{R}$. Furthermore, we identify the difference

$$\left(a^\mu_{-1}\tilde{a}^{25}_{-1} - a^{25}_{-1}\tilde{a}^\nu_{-1}\right)\left|0;p\right\rangle$$

with the vector field $\tilde{A}_\mu$ coming from the Kalb-Ramond field. Fields with winding number m are now charged under $\tilde{A}_\mu$ with charge $p_L - p_R \sim \frac{mR}{\alpha'}$. The latter seems unexpected, but as mentioned, the string is charged under the Kalb-Ramond field and this is merely a lower dimensional artifact of this charge.
3. $a^{25}_{-1}\tilde{a}^{25}_{-1}\left|0;p\right\rangle$ - This is simply a scalar field and can be identified with the Kaluza-Klein scalar k.

6.2.3. T-Duality for Closed Strings

The spectrum of the closed string (67) has an interesting property. It remains inert if we change the radius to

$$R \to \tilde{R} \coloneqq \frac{\alpha'}{R},$$

provided we also exchange winding and momentum number $m \leftrightarrow n$. Consequently, string theory does not distinguish between large and small circles for compactifications, provided it also exchanges the notion of momentum and winding. This phenomenon is known as *T-duality*.
Furthermore, through T-duality there is also a minimal length scale in string theory,

since one can only shrink the circle until the self dual radius $R = \tilde{R} = \sqrt{\alpha'}$, after which shrinking ultimately amounts to increasing the size of the circle, but for momentum and winding exchanged.

In bosonic string theory, T-duality does not give a mapping between different theories, but can be interpreted as merely an interpretative ambiguity of a single theory. This ambiguity is related to the coordinate q introduced earlier, which canceled in the expression for the string coordinate and thus at that point seemed pointless. For superstrings T-duality is more complicated, as it maps different superstring theories into each other, as we will explain briefly in section 6.2.6.
Exchanging winding and momentum number amounts to mapping the left- and right-moving momenta to

$$p_L \mapsto p_L, \quad p_R \mapsto -p_R.$$

Correspondingly, we then define the dual string coordinate

$$\begin{aligned} Y^{25} &:= X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-) \\ &\equiv q + mR\tau + \frac{\alpha' n}{R}\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{-ik\tau}}{k} \left(\tilde{a}_k^{25} e^{-ik\sigma} - a_k^{25} e^{ik\sigma}\right). \end{aligned}$$

Thus, compared to the initial expansion of the compact coordinate (66), we see that x^{25} is replaced by q and that the momentum and winding are exchanged. Furthermore, the sign of the right-moving oscillators is changed. Interestingly, this is still a fully generic solution to the classical wave equation, and as a quantum theory, it leads to the exact same CFT as X^{25}, provided we assume the compactification has the dual radius $\tilde{R} = \frac{\alpha'}{R}$. Hence, we see that by two equivalent choices, we could extract two seemingly different theories from the same starting action, expressing that the two compactifications are dual.

In the quantum theory, the exchange of winding and momentum implies the exchange of the two $U(1)$ gauge fields related to these charges, i.e. T-duality means $A_\mu \leftrightarrow \tilde{A}_\mu$. This duality will also be present in the effective action, and we will rediscover this in the context of non-relativistic strings in section 7.

Finally, we also need to consider the dilaton. As mentioned above, the dilaton is related to the coupling strength of string theory. Now, by considering the low energy effective actions derived from the beta-functions, one finds that the coupling under dimensional reduction reads $\frac{R}{g_s^2}$. Thus, if the coupling g_s were to remain inert under T-duality, we could experimentally distinguish between large and small radii of compactifications, which would contradict T-duality on the worldsheet. Thus, also the coupling strength, and hence the dilaton, has to shift under T-duality, leading to

$$g_s \mapsto \tilde{g}_s := \frac{\sqrt{\alpha'}}{R} g_s.$$

6.2.4. T-Dualities of Open Strings

We now consider open strings. First, assume we have a string whose ends move freely throughout spacetime, i.e. we have a *space-filling D25-brane* present. Again, we assume that the 25th coordinate is compactified on a circle of radius R. The momentum of the open string in this direction is quantized to $p^{25} = \frac{n}{R}$, but contrary to before, as the endpoints are free to move, the string can always just shrink to unwind itself, hence there is no winding contribution, and thus no winding number.

Now, assume we have an open string moving on a D25-brane and spacetime compactified over the dual radius $\tilde{R} = \frac{\alpha'}{R}$, which for the closed string was physically equivalent to R. But we now see, that the resulting quantized momentum is $p^{25} = \frac{\alpha'}{R}n$. Thus, the open string seems to be able to distinguish between the dual radii, defying our notion of T-duality.

However, we made the assumption that D-branes remain inert under T-dualities, and this, in fact, does not hold. To see this, recall that we had the left- and right-moving decomposition

$$X^{25}(\tau,\sigma) = X_L^{25}(\tau+\sigma) + X_R^{25}(\tau-\sigma).$$

The Neumann boundary condition for this direction then leads us to

$$\partial_\sigma X^{25}(\tau,\sigma) = X_L^{25}(\tau+\sigma)' - X_R^{25}(\tau-\sigma)' \equiv \partial_\tau Y^{25}(\tau,\sigma).$$

We can also write this more compactly with the epsilon tensor as

$$\partial_\alpha X^{25} = \varepsilon_{\alpha\beta}\partial^\beta Y^{25}. \tag{68}$$

As Y^{25} represents the direction that led to the dual description over $\tilde{R}$, we see that under T-duality, the boundary condition in the compact direction gets flipped from Neumann to Dirichlet and vice versa, effectively mapping the $D25$-brane to a $D24$-brane. With Dirichlet boundary conditions in place, the open string is now constrained to have zero momentum in the compact direction, i.e. its endpoints stay at a fixed position, and thus the open string can now wind around the compact direction, closely resembling the closed string, only that the endpoints do not have to coincide.

We can see this more directly by writing out the mode expansions. We take a similar ansatz as in the closed case and find for the standard expansion

$$X^{25} = x^{25} + \alpha'\frac{n}{R}\tau + \text{oscillators}.$$

The dual expansion then reads

$$Y^{25} = q + \alpha'\frac{n}{R}\sigma + \text{oscillators}.$$

Again, we see that in the dual world, the momentum is mapped to a winding, provided the radius R is mapped to the dual radius $\tilde{R}$. Thus, we fully recover T-duality also for the open string, provided we switch the boundary conditions of branes in the compact direction, i.e. we map a Dp-brane to a $D(p\pm 1)$-brane depending on the boundary conditions.

6.2.5. T-Duality on the Worldsheet

Let us now briefly state how T-duality might be formulated directly through the world-sheet action. For this, we consider a simplified string model, where we have a single string coordinate X, which winds once around the only compact direction, i.e.

$$X(2\pi) = X(0) + 2\pi R.$$

To make the radius dependence explicit, we introduce a single scalar field φ as

$$X := R\varphi, \implies \varphi(2\pi) = \varphi(0) + 2\pi,$$

and consequently have the flat Polyakov action

$$S = \frac{R^2}{4\pi\alpha'} \int d^2\sigma \partial_\alpha \varphi \partial^\alpha \varphi. \tag{69}$$

We introduce the ad-hoc parent action with the new fields b_α and $\tilde{\varphi}$ as

$$S_{\text{Parent}} := \frac{R^2}{4\pi\alpha'} \int d^2 b_\alpha b^\alpha - \frac{1}{2\pi} \int d^2\sigma \varepsilon^{\alpha\beta} b_\alpha \partial_\beta \tilde{\varphi}$$

If we compute the equations of motion for the field $\tilde{\varphi}$, we see that it amounts to

$$db = 0.$$

Now, if we would consider our theory with a non-compact direction, this would amount to b being exact due to the vanishing first de Rham cohomology. However, due to the compactification, we have the topology of a circle. While far from obvious, as it turns out (see [69, From eq. (11.86) onwards]), this amounts to the existence of a coordinate function φ of the unit circle that is periodic up to a shift of 2π, such that

$$b_\alpha = \partial_\alpha \varphi. \tag{70}$$

Plugging this solution into the parent action, we effectively recover the Polyakov action for the single coordinate φ from eq. (69), as the second term in S_{Parent} is only a total derivative.
Conversely, if we vary the action with respect to the field b_α, we find the relation

$$Rb^\alpha = \frac{\alpha'}{R} \varepsilon^{\alpha\beta} \partial_\beta \tilde{\varphi}. \tag{71}$$

It is again not obvious, but it turns out, that $\tilde{\varphi}$ is again a coordinate function of the unit circle, periodic up to a shift of 2π (again, see [69]). If we plug the above relation into the parent action (using $\varepsilon^{\alpha\gamma}\varepsilon_\gamma{}^\beta = \eta^{\alpha\beta}$), we find the dual action

$$S_{\text{dual}} = \frac{1}{4\pi\alpha'} \left(\frac{\alpha'}{R}\right)^2 \int d^2\sigma \partial_\alpha \tilde{\varphi} \partial^\alpha \tilde{\varphi}.$$

Comparing to eq. (69), we see that the dual action describes a single coordinate function

$$Y := \frac{\alpha'}{R}\tilde{\varphi}, \tag{72}$$

i.e. a dual coordinate with periodicity condition

$$Y(2\pi) = Y(0) + 2\pi\frac{\alpha'}{R},$$

that now includes the dual radius. Thus, we have shown that both the compactification over radius R and the one over the dual radius α'/R are contained in the parent action. In this sense, T-duality is implemented in a worldsheet formulation. Furthermore, recalling equations (70), (72) and (71), we recover the relation between the derivatives of coordinate and its dual from equation (68), i.e.

$$\partial_\alpha X = \varepsilon_{\alpha\beta}\partial^\beta Y.$$

6.2.6. T-Duality in Type II Superstrings

In superstring theory T-duality becomes more complicated, as it now does not map the same theory to itself, but instead it maps Type IIA theory compactified over radius R to Type IIB theory compactified over radius $\tilde{R}$. Again, this duality shows, that we are overall not dealing with distinct theories and that they appear to be different sides of the same coin.

To swiftly motivate this, we have to expand our discussion of branes and their charge. This will be brief again, so for a full introduction see [59, Chapter 16]. Dp-branes can be described by one time coordinate τ and p space coordinates σ^i, $i = 1, ..., p$, and their corresponding embedding into target space

$$(\tau, \sigma^i) \mapsto X^\mu(\tau, \sigma^i).$$

In this sense, a particle traces a $D0$-brane, while a string traces a $D1$-brane, etc. We recall that the string naturally coupled to the two-form field $B_{\mu\nu}$ and from electrodynamics that particles naturally couple to a one-form gauge field A_μ. This coupling can be generalized to Dp-branes coupling to a $p+1$-form $A_{\mu\mu_1,...,\mu_p}$ and is realized through the action

$$S_p = \int d\tau d\sigma^p A_{\mu\mu_1,...,\mu_p}\partial_\tau X^\mu \partial_{\sigma^1} X^{\mu_1} ... \partial_{\sigma^p} X^{\mu_p}.$$

Therefore, Dp-branes can only be charged in the presence of massless forms. Recall, that in Type IIA theory, we had the extra massless forms A_μ and $A_{\mu\nu\rho}$ from the R-R sector, and correspondingly we have charged $D0$- and $D2$-branes in Type IIA. Furthermore, through the magnetic dual of the A's, one can also show that in Type IIA all Dp-branes with p even are electrically or magnetically charged, and thus, only such branes are

present in Type IIA. The situation then reverses for Type IIB theory, where only Dp-branes with p odd are present.
Since T-duality reduces or increases the degree of a Dp-brane by one, it exactly maps the even to the odd branes and vice versa. While this does not give the exact mapping, this motivates that under T-duality we map Type IIA string theory to Type IIB and vice versa.

6.3. Non-relativistic String Theory

Non-relativistic string theory provides a simplified version of relativistic string theory, and thus presents a model that can be used to gather hints and ideas on aspects of the full theory, such as non-perturbative formulations or the AdS/CFT-correspondence. Non-relativistic string theory is characterized by a two-dimensional *relativistic* CFT on the worldsheet together with global *string-Galilean* symmetry on target space. Similar to the relativistic case, where the global Poincaré invariance on the target space ultimately led to the emergence of Lorentzian geometry and general relativity, the symmetries of non-relativistic string theory give rise to target space fields that now encode non-relativistic string-Galilean geometry. The non-relativistic string then naturally couples to the geometry and extra fields, such as the (non-relativistic) Kalb-Ramond and dilaton field. Furthermore, to ensure Weyl invariance of the quantized theory, the corresponding beta-functions need to vanish, giving rise to equations of motion for the non-relativistic background theory.
Non-relativistic string theory was originally introduced and studied as a conformal field theory in [8], since then much work was done on the quantum aspects, as well as on the geometry and backgrounds, see [4, 63, 10, 9, 11, 70] and references therein.

6.3.1. Non-relativistic Limit of Flat Spacetime

We try to gain intuition about the non-relativistic[14] limit of string theory by first studying the simplest example, a string moving in flat spacetime as presented in [11]. We start with the Nambu-Goto action

$$S_{\text{NG}}[X] \coloneqq -T \int d^2\sigma \sqrt{-\det \gamma}$$

with $\gamma \coloneqq X^*\eta$. As was shown in [71], consistent limits of objects with a $p+1$-dimensional worldvolume (such as p-branes) require singling out and rescaling one timelike and p spacial directions along the worldvolume of the brane, called *longitudinal* directions, while the $D-p-1$ remaining so-called *transverse* directions are left unchanged. As the string has a two-dimensional worldvolume, we scale (adopting flat space index notation) the longitudinal coordinates $X^A \mapsto cX^A$, $A = 0, 1$ and leave the transverse directions $X^{A'}$, $A' = 2, ..., D-1$ inert[15]. Plugging this redefinition into the Nambu-Goto action

[14] Note that this is a serious abuse of language, as the ansatz is notably different to what is considered a non-relativistic limit. It is rather a low energy limit.
[15] Again, we emphasize that the contraction parameter c is *not* the usual speed of light.

gives

$$S_{\mathrm{NG}}[X] \simeq_2 -Tc^2 \int d^2\sigma \sqrt{-\det \bar{\gamma}} \left(1 + \frac{1}{2c^2} \bar{\gamma}^{\alpha\beta} \partial_\alpha X^{A'} \partial_\beta X^{B'} \delta_{A'B'}\right),$$

where now $\bar{\gamma}$ is the pullback of the longitudinal metric η_{AB} under the longitudinal coordinates X^A, i.e. $\bar{\gamma}_{\alpha\beta} = \partial_\alpha X^A \partial_\beta X^B \eta_{AB}$, with inverse $\bar{\gamma}^{\alpha\beta}$. It turns out that due to the dimension of the worldsheet being two (see [11, Equation (3.7)]), the divergent term is a total derivative, leaving us with the flat non-relativistic Nambu-Goto action

$$S_{\text{NG-NR-flat}} = -\frac{T}{2} \int d^2\sigma \sqrt{-\det \bar{\gamma}} \bar{\gamma}^{\alpha\beta} \partial_\alpha X^{A'} \partial_\beta X^{B'} \delta_{A'B'}. \tag{73}$$

This action is invariant under reparametrizations of the worldsheet, as well as under an analog of boosts and rotations, the global *string Galilei* transformations

$$\begin{aligned} \delta X^A &= \lambda^A{}_B X^B + \zeta^A, \\ \delta X^{A'} &= \lambda^{A'}{}_{B'} X^{B'} + \lambda^{A'}{}_A X^A + \zeta^{A'}, \end{aligned}$$

where $\lambda^A{}_B$ are $SO(1,1)$ transformations on the worldsheet, $\lambda^{A'}{}_{B'}$ are $SO(D-2)$ transverse rotations, the $\lambda^{A'}{}_A$ are "string" Galilean boosts and $\zeta^A, \zeta^{A'}$ are longitudinal and transverse translations. Making the spacial rotations and string boosts local provides a guiding principle for constructing arbitrary string Newton-Cartan backgrounds[16].

The geometric structure preserved by these transformations again consists of a degenerate *longitudinal metric* $t_{\mu\nu}$ and a degenerate spacial co-metric $h^{\mu\nu}$ that are orthogonal. Now, however, the kernel of h is two-dimensional, i.e. the rank of t is two. We can again introduce two types of vielbeine. First, the *longitudinal vielbein* by the relation

$$t_{\mu\nu} = \eta_{AB} \tau_\mu{}^A \tau_\nu{}^B$$

and secondly, the spacial vielbein as before

$$\delta^{A'B'} = h^{\mu\nu} e_\mu{}^{A'} e_\nu{}^{B'}.$$

The above relations are certainly invariant under the following variations of the vielbeine

$$\begin{aligned} \delta\tau_\mu{}^A &= \mathcal{L}_\xi \tau_\mu{}^A + \lambda^A{}_B \tau_\mu{}^B, \\ \delta e_\mu{}^{A'} &= \mathcal{L}_\xi e_\mu{}^{A'} + \lambda^{A'}{}_{B'} e_\mu{}^{B'} + \lambda^{A'}{}_A \tau_\mu{}^A, \end{aligned}$$

Where ξ is an infinitesimal diffeomorphism, $\lambda^A{}_B$ an infinitesimal longitudinal Lorentz transformation, $\lambda^{A'}{}_{B'}$ an infinitesimal transverse rotation $SO(D-2)$ and $\lambda^{A'}{}_A$ an infinitesimal string boost. We will henceforth call this geometry a *string Newton-Cartan (SNC)*

[16] In doing so, an extension of the above symmetries has to be considered (see [11]), similar to the central extension of the Galilei algebra to the Bargmann algebra we saw in section 4.1.2. The extension is necessary, since, again, the action is only quasi-invariant under boosts. However, in the string case it is not entirely clear what the correct extension should be, see [11, 9, 70, 37, 4].

geometry.

We can then define projective inverses given by the relations

$$e_\mu{}^{A'}\tau^\mu{}_A = 0, \quad e^\mu{}_{A'}\tau_\mu{}^A = 0, \quad e_\mu{}^{A'}e^\mu{}_{B'} = \delta^{A'}{}_{B'}, \quad \tau_\mu{}^A\tau^\mu{}_B = \delta^A{}_B, \quad \delta^\mu{}_\nu = \tau^\mu{}_A\tau_\nu{}^B + e^\mu{}_a e_\nu{}^a. \tag{74}$$

Furthermore, we can split arbitrary tensor indices into longitudinal and transverse parts

$$T_\mu = T_A\tau_\mu{}^A + T_{A'}e_\mu{}^{A'},$$

where

$$T_A \coloneqq T_\mu \tau^\mu{}_A \tag{75}$$
$$T_{A'} \coloneqq T_\mu e^\mu{}_{A'}. \tag{76}$$

6.3.2. The Gomis-Ooguri String and the Non-Relativistic Spectrum

We now want to derive a Polyakov-type action following the lines of the original paper [8], where an $\alpha' \to 0$ limit was considered. This starting point is useful to see the connection to the relativistic theory and to derive the appropriate spectrum.

We start from the closed relativistic string in conformal gauge coupled to a curved background and a Kalb-Ramond field

$$S_{\text{Rel}} = \frac{1}{4\pi\hat{\alpha}'}\int d^2\sigma\left(\partial_\alpha X^\mu\partial^\alpha X^\nu G_{\mu\nu} - i\varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial^\alpha X^\nu B_{\mu\nu}\right)$$

with the background fields

$$G = \begin{pmatrix}\eta_{AB} & 0\\ 0 & \frac{\hat{\alpha}'}{\alpha'}\delta_{A'B'}\end{pmatrix}, \quad B = \begin{pmatrix}-\varepsilon_{AB} & 0\\ 0 & 0\end{pmatrix}. \tag{77}$$

Here, $\hat{\alpha}'$ is the relativistic string tension which we want to send to zero, while α' is the effective tension of the non-relativistic string[17]. This results in the action

$$S_{\text{Rel}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_\alpha X^{A'}\partial^\alpha X_{A'} - \frac{\alpha'}{\hat{\alpha}'}\bar{\partial}X\partial\bar{X}\right),$$

where we introduced the lightcone coordinates $X \coloneqq X^0 + X^1$, $\bar{X} \coloneqq X^0 - X^1$ and the complex derivatives $\partial \coloneqq \partial_\tau + i\partial_\sigma$, $\bar{\partial} \coloneqq \partial_\tau - i\partial_\sigma$.
Sending $\hat{\alpha}' \to 0$ would lead to divergences in the action, however we can introduce Lagrange multipliers $\lambda, \bar{\lambda}$ to make this limit well-defined. We thus write

$$S_{\text{Rel}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_\alpha X^{A'}\partial^\alpha X_{A'} + \lambda\bar{\partial}X + \bar{\lambda}\partial\bar{X} + \frac{\hat{\alpha}'}{\alpha'}\lambda\bar{\lambda}\right).$$

[17] To do this appropriately one has to simultaneously also send the string coupling $\hat{g}_s$ to infinity.

It is important to stress that this action is still fully relativistic, and that the term $\lambda\bar{\lambda}$ governs how the theory is deformed from the non-relativistic to the relativistic regime. Taking the limit $\hat{\alpha}' \to 0$ leads to the *Gomis-Ooguri* action for non-relativistic strings

$$S_{\text{GO}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_\alpha X^{A'} \partial^\alpha X_{A'} + \lambda \bar{\partial} X + \bar{\lambda} \partial \bar{X} \right) \tag{78}$$

The action S_{GO} is the Polyakov form of the action (73). We still have the Galilei symmetries present in this action, provided that the Lagrange multipliers also transform under boosts as

$$\delta_{\text{B}} \lambda = (\lambda_{0A'} + \lambda_{1A'})\, \partial X^{A'}, \quad \delta_{\text{B}} \bar{\lambda} = (\lambda_{0A'} - \lambda_{1A'})\, \bar{\partial} X^{A'}.$$

They also transform as one-forms under worldsheet diffeomorphisms, thus are overall part of the geometric data. The relation will become more clear in the context of null reductions.

To derive the spectrum of the theory, first note that the ansatz for the KR field is pure gauge, i.e. gauge equivalent to the zero form under the one-form transformations with parameter θ we introduced in eq. (65)

$$\theta_A = \frac{1}{2} \varepsilon_{AB} X^B, \quad \theta_{A'} = 0$$

if we assume that all directions are non-compact. However, the non-zero contribution by the KR field was important to take the correct limit. Thus, to have a non-trivial form field, we need to assume that the X^1 direction is compactified with radius R, which renders the above one-form parameter discontinuous due to X^1 being discontinuous. One can see this more concretely by computing how the presence of the KR field changes the energy of the relativistic string. Following [72], in the presence of the constant KR field from (77), the relativistic energy of the string changes to

$$\tilde{P}_0 = E + \frac{1}{2\pi\hat{\alpha}'} \int d\sigma \partial_\sigma X^1 \equiv E + \frac{1}{2\pi\hat{\alpha}'} \left[X^1(2\pi) - X^1(0) \right] \equiv E + \frac{Rw}{\hat{\alpha}'}.$$

We denote by E the kinetic energy of the string and by w the winding number of the string. The latter is of course only present in the case of a compact X^1-direction. Put differently, the KR field we introduced is only physically relevant if we also have non-trivial fundamental group, and we can implement this by introduction of a compact direction. It turns out that the relativistic on-shell condition analogously to eq. (67) is given by

$$-\tilde{P}_\mu \tilde{P}^\mu \equiv \left(E + \frac{Rw}{\hat{\alpha}'} \right)^2 - \frac{\alpha'}{\hat{\alpha}'} P^{A'} P_{A'} = \frac{n^2}{R^2} + \frac{w^2 R^2}{(\hat{\alpha}')^2} + \frac{2}{\hat{\alpha}'} (\tilde{N} + N - 2),$$

Together with the level matching $N - \tilde{N} = nw$. Taking the limit $\hat{\alpha}' \to 0$ gives the non-relativistic dispersion relation for a 9-dimensional particle with mass wR/α'

$$E = \frac{\alpha'}{2wR} \left[P^{A'} P_{A'} + \frac{2}{\alpha'} \left(N + \tilde{N} - 2 \right) \right].$$

It is important to note that this relation is only well-defined for strings with winding around the compact direction, i.e. $w \neq 0$, and taking the limit consistently even requires the condition $w > 0$ (see the discussion around eq. (25) in [57])[18]. This can be interpreted as anti-particles decoupling from the theory, similarly to how anti-particles decouple in the non-relativistic limits of QFTs. Thus, strings in non-relativistic string theory that are on-shell necessarily have winding, effectively only allowing states with a mass. In particular this does not allow a graviton in the non-relativistic spectrum.
However, as was shown in [8, Section 4], unwound states do appear *off-shell* as intermediate states in scattering of wound strings that do not exchange winding number. The long-range, leading-order behavior of their propagators is proportional to $(P_{A'}P^{A'})^{-1}$, corresponding to the Green's function of the Poisson equation. As a consequence (see also [73, Section 4] for extensive details), these unwound states correspond to a Newton-like potential, introducing an instantaneous gravitational force between wound strings.

6.3.3. T-Dualities for the Gomis-Ooguri String

We can give an interpretation to the Lagrange multipliers $\lambda, \bar{\lambda}$ through T-dualities. Note that we can have T-dualities along a spacial[19] longitudinal or transverse directions. Here, we are following an approach as in [10]. We implement the T-duality along the compact longitudinal spacelike direction X^1, similarly to section 6.2.5, via a parent action

$$S_{\text{Parent}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_\alpha X^{A'}\partial^\alpha X_{A'} + \lambda(\bar{\partial}X^0 + \bar{v}) + \bar{\lambda}(\partial X^0 - v) + 2Y_1(\bar{\partial}v - \partial\bar{v})\right).$$

Integrating out Y_1 gives the equation $\bar{\partial}v = \partial\bar{v}$, which is solved by $v = \partial X^1$ and $\bar{v} = \bar{\partial}X^1$, recovering the original S_{GO} from eq. (78). Conversely, we can integrate out $v, \bar{v}$, which gives $\lambda = -2\partial Y_1$ and $\bar{\lambda} = -2\bar{\partial}Y_1$, which gives the action

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_\alpha X^{A'}\partial^\alpha X_{A'} - 2\partial Y_1\bar{\partial}X^0 - 2\bar{\partial}Y_1\partial X^0\right).$$

We recognize this as the action of a *relativistic* string theory in a flat background with spatial directions $X^{A'}$ and lightlike directions Y_1 and X^0, where Y_1 is compactified, as it is dual to X^1.[20] We will encounter this relation again in section 7, when we consider the effect of this T-duality on the target space in NS-NS gravity. Consequently, we can interpret the Lagrange multipliers as conjugate to the longitudinal string winding, characterized by a lightlike direction.
Quantizing such a theory is known as discrete lightcone quantization (DLCQ), which is related to matrix models of string and M-theory, usually defined via a limiting procedure of spacelike circles (see [74] and [8, Section 6]). Thus, non-relativistic closed string theory provides an approach to define DLCQ of relativistic strings in a rigorous way.

[18]Note that what we call non-relativistic string theory is called NRCOS there.
[19]There also seem to be lightlike longitudinal T-dualities (see [10]), their interpretation is not completely clear, however.
[20]In the literature often the radius of compactification is given as well, but as noted in section 5.4, null directions have no inherent notion of length, hence also no radius.

If instead of a longitudinal, a transverse direction is chosen to be compactified, we recover a similar notion of T-duality as in the relativistic case, mapping one non-relativistic string theory to another with dual radius (see [10]).

6.3.4. Curved Non-relativistic Limit

There are several approaches to introduce curved target spaces into non-relativistic string theory. Here, we want to follow along the lines of [4] and consider a non-relativistic limit.

We start from the curved Nambu-Goto action coupled to a Kalb-Ramond field for the moment omitting the dilaton coupling

$$S_{\text{NG}} = -T \int d^2\sigma \sqrt{-\det\left(\eta_{\hat{A}\hat{B}} E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} \partial_\alpha X^\mu \partial_\beta X^\nu\right)} - \frac{T}{2} \int d^2\sigma \varepsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{79}$$

The hatted indices take values $\hat{A} = 0, ..., D-1$ in the full tangent space. Note that the target space fields E and B are functions of the string coordinates X.
Furthermore, it is important to recall from section 6.3.2 that the non-relativistic limit required a compact (longitudinal) direction and our physical states all carried winding number. On the contrary, the massless metric, i.e. vielbein, KR field and dilaton were all part of the zero-winding sector (see discussion after (67)). Thus, taking this limit does not yield physical non-relativistic states. However, as remarked at the end of section 6.3.2, these unwound intermediate states, corresponding to the limit of the above fields, were mediating an instantaneous force between asymptotic states that carried winding number. Thus, we expect that the limits of these fields are the correct ones to describe the non-relativistic geometry of the target space.

We have already learned from the non-relativistic flat case in section 6.3.1 that we have to introduce a splitting in longitudinal and transverse directions, i.e. $\hat{A} \mapsto A = 0, 1$, and $A' = 2, ..., D-1$, and we correspondingly split the vielbeine and make the ansatz for the vielbeine and KR field

$$\begin{aligned} E_\mu{}^A &= c\tau_\mu{}^A + \frac{1}{c} m_\mu{}^A, \\ E_\mu{}^{A'} &= e_\mu{}^{A'}, \\ B_{\mu\nu} &= -c^2 \tau_\mu{}^A \tau_\mu{}^B \varepsilon_{AB} + b_{\mu\nu}. \end{aligned} \tag{80}$$

This closely resembles the ansatz for the particle in section 4.3.1, however, we included the field m, that renders the above relations non-invertible. This introduction is not strictly necessary, but, as we will comment on later, it will provide useful. The ansatz for the KR field is chosen to, again, cancel a divergence coming from the rest tension of the string. Furthermore, the fields $(\tau_\mu{}^A, e_\mu{}^{A'})$ admit (projective) inverses as in eq. (74).

Defining the pullback metric

$$\gamma_{\alpha\beta} \coloneqq \eta_{AB}\tau_\mu{}^A\tau_\nu{}^B\partial_\alpha X^\mu\partial_\beta X^\nu,$$

and the boost-invariant spacial metric

$$H_{\mu\nu} \coloneqq \delta_{A'B'}e_\mu{}^{A'}e_\nu{}^{B'} + 2\tau_{(\mu}{}^A m_{\mu)}{}^B\eta_{AB},$$

we find that the non-relativistic Nambu-Goto string action from the $c \to \infty$ limit is given by

$$S_{\text{NG-NR}} = -\frac{T}{2}\int d^2\sigma\left\{\sqrt{-\det\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu H_{\mu\nu} + \varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu b_{\mu\nu}\right\}.$$

The action is invariant under an Abelian one-form symmetry that is inherited from the relativistic KR field

$$\delta b = d\theta,$$

for an arbitrary one-form θ. Furthermore, the geometric fields transform under longitudinal $SO(1,1)$ transformations with parameter $\lambda^A{}_B$, transverse $SO(D-2)$ transformations with parameter $\lambda^{A'}{}_{B'}$, as well as string boosts with parameter $\lambda^{AA'}$ as

$$\begin{aligned}\delta\tau_\mu{}^A &= \lambda^A{}_B\tau_\mu{}^B,\\ \delta e_\mu{}^{A'} &= \lambda^{A'}{}_{B'}e_\mu{}^{B'} + \lambda^{A'}{}_A\tau_\mu{}^A,\\ \delta m_\mu{}^A &= \lambda^A{}_B m_\mu{}^B + \lambda^A{}_{A'}e_\mu{}^{A'}.\end{aligned} \tag{81}$$

Due to the form of H this action is manifestly invariant under all these symmetries. Additionally, we observe an extra emergent symmetry in the form of a Stückelberg shift

$$\begin{aligned}\delta b_{\mu\nu} &= 2c_{[\mu}{}^A\tau_{\nu]}{}^B\varepsilon_{AB},\\ \delta m_\mu{}^A &= -c_\mu{}^A.\end{aligned} \tag{82}$$

This symmetry is due to the fact that with the introduction of m we introduced more non-relativistic than relativistic fields, hence we over-parametrized our theory. It can be used to relate the different approaches to non-relativistic string theory (see [70]). Overall, this shift symmetry lets us gauge fix the extra field m to zero, i.e.

$$m_\mu{}^A \equiv 0.$$

As can be seen from eq. (82) and (81), keeping this gauge requires a compensating transformation, which makes the non-relativistic KR field part of the geometric data, as it now transforms under boosts as

$$\delta_{\text{B}} b_{\mu\nu} = -2\varepsilon_{AB}\lambda^A{}_{A'}\tau_{[\mu}{}^B e_{\nu]}{}^{A'}.$$

After gauge fixing, the Newtonian potential is now also part of the KR field, more precisely it resides in the longitudinal components b_{AB} (see [4] for details).

We now also include the dilaton into the theory, where its non-relativistic ansatz is given by a shift in c by

$$\Phi = \phi + \ln c,$$

resulting in the non-relativistic Nambu-Goto action of a string moving in a curved non-relativistic string Newton-Cartan background

$$S_{\text{NG-NR}} = -\frac{T}{2}\int d^2\sigma \left\{ \sqrt{-\det\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \delta_{A'B'} e_\mu{}^{A'} e_\nu{}^{B'} + \varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu b_{\mu\nu} \right\}$$
$$+\frac{1}{4\pi}\int d^2\sigma \sqrt{-\det\gamma} R^{(2)}(\gamma)\phi.$$

Similar to the particle action in section 4.3.1, this string action also has an emergent dilatation symmetry, which has some interesting consequences in the target space formulation, as we will see in section 7.1. The dilatation symmetry manifests as

$$\delta\tau_\mu{}^A = \lambda_D \tau_\mu{}^A,$$
$$\delta\phi = \lambda_D.$$

Note that we can equivalently also introduce a non-relativistic Polyakov action (see [10])

$$S_{\text{P-NR}} = -\frac{T}{2}\int d^2\sigma \left\{ \sqrt{-\det g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu H_{\mu\nu} + \varepsilon^{\alpha\beta}\left(\lambda e_\alpha \tau_\mu + \bar{\lambda}\bar{e}_\alpha \bar{\tau}_\mu\right)\partial_\beta X^\mu \right\} \tag{83}$$

$$-\frac{T}{2}\int d^2\sigma \varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu b_{\mu\nu} + \frac{1}{4\pi}\int d^2\sigma \sqrt{-\det\gamma} R^{(2)}(\gamma)\phi. \tag{84}$$

Here, g is the worldsheet metric, for which we introduced a zweibein $e_\alpha{}^A$, that we then expressed in lightcone coordinates as $e, \bar{e}$. Similarly, we introduced lightcone coordinates for the longitudinal vielbein[21] $\tau_\mu{}^A$ as $\tau, \bar{\tau}$. Again, we have the Lagrange multipliers $\lambda, \bar{\lambda}$ as in the flat non-relativistic string action in eq. (78).

As can be shown by a gauging of the SNC algebra (see [11]) the intrinsic torsion of such a geometry is given by

$$R_{\mu\nu}(H^A) \coloneqq 2\mathcal{D}_{[\mu}\tau_{\nu]}{}^A.$$

Here, the covariant derivative is covariant with respect to the worldsheet Lorentz transformations, i.e. $\mathcal{D}_\mu \tau_\nu = \partial_\mu \tau_\nu - \omega_\mu{}^A{}_B \tau_\mu{}^B$. This provides an analog of the intrinsic torsion we have found for standard Newton-Cartan geometry in section 4.2.3.
Originally, earlier works on SNC imposed a "zero torsion constraint", given by

$$\mathcal{D}_{[\mu}\tau_{\nu]}{}^A = 0,$$

leading to a torsionless SNC geometry. This can also be reformulated as

$$d\tau^A = \omega^A{}_B \wedge \tau^B,$$

[21] Recall that this lives on target space, contrary to the zweibein of g, which lives on the worldsheet.

which in turn by the Frobenius theorem for forms implies that our target space admits a codimension 2 foliation into two longitudinal directions and $D-2$ transverse directions. Contracting with all possible combinations of longitudinal and transverse vielbeine, we see that part of the zero torsion constraint is conventional, while others pose a genuine constraint on the geometry.

Recent works ([39, 4, 15, 37, 70]) have shown that it is relevant to relax the zero torsion condition to allow for more general torsion, leading to different types of *torsional string Newton-Cartan (TSNC) geometries*.

Furthermore, torsion constraints also appear when taking the non-relativistic string limit in the target space without including the KR field, as then the divergent leading order terms of the relativistic spin connections include the torsion.
Additionally, this constraint does not emerge as an equation of motion, hence has to be imposed by hand (see [4] and references therein for details).
Introducing the KR field with the ansatz as above in eq. (80) results in the cancelation of the divergences and thus allows to take the non-relativistic limit including torsion.

Thus, we see that the non-relativistic limit of a string moving in curved relativistic background naturally leads to non-relativistic strings coupled to arbitrary curved TSNC backgrounds.

6.3.5. T-Duality on the worldsheet

We have seen in section 6.2.5 how T-dualities in relativistic string theory can be formulated purely in the worldsheet action. Following allowing the lines of [10], we want to take the same approach for introducing T-dualities in the non-relativistic setting, now, however, we consider a curved SNC background.

First, recall that for the non-relativistic limit to be well-defined, we necessarily needed one direction of target space compactified. In string Newton-Cartan theory, we have three different choices for the causal structure of the compact direction. This is encoded in the direction of the Killing field. Here, we consider spacial longitudinal directions[22].

We thus assume the existence of a longitudinal spacelike Killing vector χ, and as in section 5 this amounts to adapted coordinates $(X^{\hat{\mu}}) = (X^{\mu}, z)$, $\hat{\mu} = 0, ..., D$ and $\mu = 0, ..., D-1$, such that $\chi = \partial_z$. The longitudinal vielbein in these adapted coordinates fulfills (for details see 7.3)

$$\tau_z{}^0 = 0, \quad \tau_z{}^1 \neq 0, \quad e_z{}^{A'} = 0.$$

We assume all fields and parameters to be independent of the coordinate z. Defining

$$u_\alpha \coloneqq \partial_\alpha z,$$

[22] Other types were considered in [10].

we can write a parent action that is equivalent to the non-relativistic Polyakov action (83), given by

$$
\begin{aligned}
S_{\text{Parent}} =& -\frac{T}{2}\int d^2\sigma\sqrt{-\det g}g^{\alpha\beta}\left[u_\alpha u_\beta H_{zz} + 2u_\alpha\partial_\beta X^\mu H_{\mu z} + \partial_\alpha X^\mu\partial_\beta X^\nu H_{\mu\nu}\right] \\
& -\frac{T}{2}\int d^2\sigma\varepsilon^{\alpha\beta}\left[\lambda e_\alpha\left(u_\beta\tau_z + \partial_\beta X^\mu\tau_\mu\right) + \bar\lambda\bar e_\alpha\left(u_\beta\bar\tau_z + \partial_\beta X^\mu\bar\tau_\mu\right)\right] \\
& -\frac{T}{2}\int d^2\sigma\varepsilon^{\alpha\beta}\left[2u_\alpha\partial_\beta X^\mu b_{z\mu} + \partial_\alpha X^\mu\partial_\beta X^\nu b_{\mu\nu} + 2v\partial_\alpha u_\beta\right] \\
& +\frac{1}{4\pi}\int d^2\sigma\sqrt{g}R^{(2)}\phi
\end{aligned}
$$

In this action, we consider u_α to be an independent field and introduced v as a Lagrange multiplier. Its equations of motion impose $du = 0$, which locally is solved by $u_\alpha = \partial_\alpha z$, giving our original Polyakov action.

Integrating out first u_α and then the original Lagrange multipliers $\lambda, \bar\lambda$, results in the T-dual action

$$
\begin{aligned}
S_{\text{dual}} =& -\frac{T}{2}\int d^2\sigma\left(\sqrt{g}g^{\alpha\beta}\partial_\alpha Y^{\hat\mu}\partial^\alpha Y^{\hat\nu}\tilde G_{\hat\mu\hat\nu} + \varepsilon^{\alpha\beta}\partial_\alpha Y^{\hat\mu}\partial_\beta Y^{\hat\nu}\tilde B_{\hat\mu\hat\nu}\right) \\
& + \frac{1}{4\pi}\int d^2\sigma R^{(2)}\Phi,
\end{aligned}
\tag{85}
$$

where we introduced the dual coordinates $(Y^{\hat\mu}) \coloneqq (X^\mu, v)$ and the non-relativistic *Buscher rules* [12]

$$
\begin{aligned}
\tilde G_{vv} &= 0, & \Phi &= \phi - \frac{1}{2}\ln t_{zz}, \\
\tilde G_{v\mu} &= \frac{\tau_\mu{}^A\tau_z{}^B\varepsilon_{AB}}{t_{zz}}, & \tilde B_{v\mu} &= \frac{t_{z\mu}}{t_{zz}}, \\
\tilde G_{\mu\nu} &= H_{\mu\nu} + \frac{2b_{z(\mu}\tau_{\nu)}{}^A\tau_z{}^B\varepsilon_{AB} + H_{zz}t_{\mu\nu} - 2H_{z(\mu}t_{\nu)z}}{t_{zz}}, \\
\tilde B_{\mu\nu} &= b_{\mu\nu} + \frac{2b_{z[\mu}t_{\nu]z} - (H_{zz}\tau_\mu{}^A\tau_\nu{}^B + 2H_{z[\mu}\tau_{\nu]}{}^A\tau_z{}^B)\varepsilon_{AB}}{t_{zz}}.
\end{aligned}
$$

We recognize that the action (85) represents the action of a string moving in a *relativistic* background with null isometry along the Killing vector ∂_v. We will discuss this relation and the Buscher rules in depth in section 7 in a target space formulation.

6.3.6. Weyl Invariance and Beta functions

We have seen that in the case of the relativistic string, the target space equations of motion, i.e. the Einstein equations were given by requiring that the sigma model coupled to curved spacetime preserves the Weyl invariance at the quantum level.

The same logic applies to the non-relativistic string coupled to a curved non-relativistic (but torsionless) background. The vanishing of the beta functions leads to equations of motion that were computed in [13]. In [4] it was shown that the same equations of motion could be generated from a non-relativistic limit of NS-NS gravity, showing that taking the limit and computing the beta functions commutes upon assuming zero intrinsic torsion.
The resulting non-relativistic version of NS-NS gravity describes the dynamics of the zero-winding sector in non-relativistic string theory, which encodes the instantaneous Newtonian gravitational force. The corresponding non-relativistic action will be the starting point for the calculations in this thesis, therefore, we will start the next section by briefly reviewing the result.

7. T-Dualities in Non-relativistic Target Space

We have seen in the previous section that the non-relativistic limit of string theory necessarily needs a compactified direction. Thus, it is fairly natural to consider T-dualities in this setting. We have already seen that a longitudinal T-duality maps the non-relativistic string worldsheet action to the worldsheet action of the relativistic string with a null direction. While this worldsheet point of view is well understood and was shown to hold in [10, 70], the corresponding reductions in the target space formulation of effective actions was only considered recently in [15] for some solutions of non-relativistic (super)string theories and without verifying that the target space equations of motion indeed are mapped correctly under T-duality.
As was shown in [4], most of the bosonic equations of motion could be inferred from the non-relativistic limit of the NS-NS gravity action. However, compared to the direct limit of the equations of motion, one equation of motion does not follow from the action, and it turns out to be the covariant version of the Poisson equation. This entails that the action does not describe a dynamical gravitational background, and the source of the discrepancy could ultimately be traced back to an emergent dilatation symmetry.
The main goal of this work is to provide a check that the longitudinal, spacial Kaluza-Klein reduction of the non-relativistic NS-NS action of [4] is equal to a null reduction of the action of the relativistic NS-NS action (86). As mentioned in the previous paragraph, this shows that all but one equation of motion are mapped to each other under T-duality[23]. This verifies that the T-duality on the target space is compatible with T-duality on the worldsheet. Along this way, we will see how and why the emergent dilatation symmetry comes into the non-relativistic limit, ultimately explaining why the limit cannot generate the Poisson equation.

7.1. Non-relativistic NS-NS Gravity

We want to briefly review [4], as it is the starting point for the calculations of this thesis. The action of NS-NS gravity is given by

$$S_{\text{NS-NS}} = \frac{1}{2\kappa^2}\int d^{10}xEe^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}H^2\right), \tag{86}$$

where κ is the gravitational coupling constant, $H = dB$ the KR field strength, with kinetic term $H^2 := \frac{1}{3!}H_{\mu\nu\rho}H^{\mu\nu\rho}$, $E = \det\left(E_\mu{}^{\hat{A}}\right)$ and R is the Ricci scalar of the background metric (or equivalently of the vielbein E). Greek curved indices and hatted capital flat Latin indices run from $0, ..., D-1$.

The non-relativistic limit ansatz corresponds to the one from the worldsheet action in section 6.3.4 with the gauge $m = 0$ in eq. (80). It is noteworthy, as was shown in [4], that this ansatz had to be chosen exactly as in the worldsheet action. In the current setting,

[23] Recall from the discussion at the end of section 5.4 that also the null reduction of an action could not give rise to all equations of motion, due to the degenerate nature of the metric.

it was needed to ensure a fine-tuned cancellation between a divergent term stemming from the Ricci tensor with a divergence coming from the kinetic term H^2 of the KR field. Thus, the non-relativistic string action leads exactly to the same gravitational theory as the non-relativistic limit of the target space equations of motion. The corresponding non-relativistic transformations follow as

$$\begin{aligned}
\delta\tau_\mu{}^A &= \lambda_M \varepsilon^A{}_B \tau_\mu{}^A, \\
\delta e_{\mu A'} &= -\lambda_A{}^{A'}\tau_\mu{}^A + \lambda^{A'}{}_{B'} e_\mu{}^{B'}, \\
\delta b_{\mu\nu} &= 2\partial_{[\mu}\theta_{\nu]} - 2\varepsilon_{AB}\lambda^A{}_{A'}\tau_{[\mu}{}^B e_{\nu]}{}^{A'}, \\
\delta\phi &= 0.
\end{aligned} \tag{87}$$

The resulting non-relativistic action is

$$\overset{(0)}{S} = \frac{1}{2\kappa^2}\int d^{10}x \det(e,\tau) e^{-2\phi}\Big\{ R(J) + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}h_{A'B'C'}h^{A'B'C'} \tag{88}$$

$$- 4\mathcal{D}_{A'}d^{A'} - 4d_{A'}d^{A'} - 4\tau_{A'\{AB\}}\tau^{A'\{AB\}}\Big\}. \tag{89}$$

We have turned curved indices into flat ones using τ and e according to eq. (75), introduced the notation $\tau_{\mu\nu}{}^A = \partial_{[\mu}\tau_{\nu]}{}^A$ and the traceless symmetric part of a tensor by $T_{\{AB\}}$. Furthermore, we encoded the geometric data in a dependent dilatation connection d_μ, rotation and boost connections $\omega_\mu{}^{A'B'}$, $\omega_\mu{}^{A'}$, as well as a worldsheet spin connection ω_μ as detailed in [4, Equation (39)][24]. Additionally, we introduced a scalar curvature for the rotations, as well as a covariant derivative[25] for the dilatation connection that is covariant with respect to rotations and boosts

$$\begin{aligned}
R(J) &:= -2e^\mu{}_{A'}e^\nu{}_{B'}\left(\partial_{[\mu}\omega_{\nu]}{}^{A'B'} + \omega_{[\mu}{}^{A'C'}\omega_{\nu]}{}^{B'}{}_{C'}\right) - 4\omega^{A'BB'}\tau_{A'B'B}, \\
\mathcal{D}_\mu d^{A'} &:= \partial_\mu d^{A'} - \omega_\mu{}^{A'B'}d_{B'} - \omega_\mu{}^{AB'}\tau_{A'B'A}.
\end{aligned}$$

These fields correspond to a string Galilei symmetry, extended by dilatations in τ and ϕ. Together with the intrinsic torsion constraint

$$\tau_{[\mu^-}\partial_\mu\tau_{\rho]}{}^- = 0,$$

the resulting geometry is called Dilatation invariant SNC (DSNC$^-$) geometry (see Appendix B and [39]).
The torsion constraint is needed for consistency under supersymmetry transformations when considering the supergravity version of non-relativistic NS-NS gravity. Furthermore, this constraint corresponds to a twistless torsional Newton-Cartan geometry, defining a co-dimension one foliation of spacetime. This constraint is also inert under supersymmetry transformations and dilatations, thus it does not give rise to further

[24] Note that they denote the dilatation connection d by b, which we do not, to avoid confusion with the KR field b.
[25] Note that this is not a proper covariant derivative, but more of a notational device.

constraints. While this constraint plays an important role in the supergravity setting, it will play no further role in this thesis.
The action (88) is invariant under the expected string Newton-Cartan symmetries (87) but also under the emergent dilatation symmetry

$$\delta\tau_\mu{}^A = \lambda_D \tau_\mu{}^A, \\ \delta\phi = \lambda_D.$$

We can bring the non-relativistic action into a form where the dilatation symmetry is manifest

$$S_{\text{NR}} = \frac{1}{2\kappa^2}\int d^{10}x \det(e,\tau) e^{-2\phi}\Big\{ R(J) + 4\nabla_{A'}\phi\nabla^{A'}\phi - \frac{1}{12}h_{A'B'C'}h^{A'B'C'} \\ - 4\tau_{A'\{AB\}}\tau^{A'\{AB\}} + 4\omega^{A'BB'}\tau_{A'B'B}\Big\}, \tag{90}$$

where we introduced the dilation covariant derivative $\nabla_\mu\phi := \partial_\mu\phi - d_\mu$.

This emergent dilatation symmetry has profound consequences, as due to its Noether identity[26], we lose one equation of motion. In [4] the equations of motion from the non-relativistic action were compared to the non-relativistic limit of the equations of motion of NS-NS gravity, and it was found that the lost equation of motion is precisely the covariant Poisson equation, deleting the Newtonian gravitational dynamics from the action.
Thus, the above functional gives only rise to a *pseudo-action*. It is not a proper action for non-relativistic NS-NS gravity, as it does not give rise to the (arguably) most important equation of motion. However, it still gives almost all equations of motion and thus provides a solid base to study the role of T-dualities.

7.2. Particle Limit and Geometry

This section is a technical, preliminary introduction into the geometry we will encounter during the spacelike and the null reduction. It will prove useful to organize and identify fields and connections we encounter during the reductions and give their corresponding geometry.

7.2.1. The Particle Limit of String Theory

We start with the relativistic closed string in the Nambu-Goto action coupled to a target space metric G and a KR field B, i.e. we start from eq. (79). We assume that the string appears as a particle in the reduced theory, i.e. that our coordinates take the special form

$$(X^{\hat\mu}(\tau,\sigma)) = (X^\mu(\tau), R\sigma),$$

[26] See 2.3 for an in depth explanation of such identities.

where $\hat{\mu} = 0, ..., D-1, z$ and $\mu = 0, ..., D-1$. This ansatz means that the string winds exactly once around the compact direction with radius R and only moves along the non-compact directions. Using these coordinates and the Kaluza-Klein ansatz (42), we find that the pullback under X of the metric to the two-dimensional worldsheet reads

$$X^*G = \begin{pmatrix} \dot{X}^A \dot{X}_A + k^2(A_\mu \dot{X}^\mu) & Rk^2 A_\mu \dot{X}^\mu \\ Rk^2 A_\mu \dot{X}^\mu & R^2 k^2 \end{pmatrix}.$$

Here, we already used $T^A := E_\mu{}^A T^\mu$, where $A = 0, ..., D-1$ takes values in the tangent space transverse to the compact direction. A quick calculation reveals thus

$$\det X^*G = R^2 k^2 \dot{X}^A \dot{X}_A.$$

A similar calculation for the KR field B gives

$$X^*B = \begin{pmatrix} 0 & RB_\mu \dot{X}^\mu \\ -RB_\mu \dot{X}^\mu & 0 \end{pmatrix},$$

where we defined $B_\mu := B_{\mu z}$. Putting this back into the Nambu-Goto action and defining $m := TR$, we find the particle action

$$S_{\text{Rel}} = -mc \int ds \left(k\sqrt{-\eta_{AB} \dot{X}^A \dot{X}^B} - B_\mu \dot{X}^\mu \right).$$

Up to a sign and a factor of k, this is exactly the critical action (35) of a relativistic particle moving in a curved background and electromagnetic field that was tuned such that the electric charge was equal to the mass, i.e. $q = m$. As such, we can immediately write down its non-relativistic limit

$$S_{\text{NR}}[x] = \frac{m}{2} \int d\tau \left(k \frac{\delta_{ab} \dot{X}^a \dot{X}^b}{\dot{X}^0} - 2b_\mu \dot{X}^\mu \right).$$

Here we have split already $T^a := e_\mu{}^a T^\mu$ and $T^0 := \tau_\mu T^\mu$. Note that we have kept the name b_μ instead of introducing the field m_μ. This is to remind us of the origin of the $U(1)$ potential from the KR field and its corresponding relation to the winding. Furthermore, we have kept the KK scalar k explicit which we could have also absorbed in a redefinition of e and τ.

Keeping it explicit has the benefit that the emergent dilatation symmetry is now encoded in the variations

$$\delta_{\text{D}} k = \lambda_D k,$$
$$\delta_{\text{D}} \tau_\mu = \lambda_D \tau_\mu.$$

Recall that in the non-relativistic string case, the KR field became part of the geometric data. This is matched in the particle case, as we have the variations under central charge transformations, boosts and spacial rotations (omitting diffeomorphisms and dilatations)

$$\begin{aligned}\delta\tau_\mu &= 0,\\ \delta e^a_\mu &= \lambda^a{}_b e^b_\mu + \lambda^a \tau_\mu,\\ \delta b_\mu &= \partial_\mu\theta + k\lambda_a e^a_\mu.\end{aligned}$$

Note, the presence of k in the variations of b will slightly complicate the computations of curvatures and connections in the next section.

7.2.2. The Reduced Geometry (DTNC)

With the above variations under boosts and rotations, we may introduce the curvatures (or rather torsion) for the geometric data

$$\begin{aligned}R_{\mu\nu}(H) &= 2\tau_{\mu\nu}\\ R_{\mu\nu}(Q) &= F(b)_{\mu\nu} - 2k\omega_{[\mu}{}^c e_{\nu]c}\\ R_{\mu\nu}(P^a) &= 2e_{\mu\nu}{}^a - 2\omega_{[\mu}{}^{ac} e_{\nu]c} - 2\omega_{[\mu}{}^a\tau_{\nu]},\end{aligned} \tag{91}$$

where we defined $\tau_{\mu\nu} := \partial_{[\mu}\tau_{\nu]}$, $e_{\mu\nu}{}^a := \partial_{[\mu}e_{\nu]}{}^a$ and $F(b)_{\mu\nu} := 2\partial_{[\mu}b_{\nu]}$. We want to emphasize that we do not introduce a connection for dilatations, as it is not necessary or enlightening for this work.
Note, we want to leave the intrinsic torsion unconstrained, but imposing the conventional constraints

$$R_{\mu\nu}(Q) \stackrel{!}{=} 0, \quad R_{\mu\nu}(P^a) \stackrel{!}{=} 0, \tag{92}$$

we can solve for the boost and spin connection. This is analogously to eq. (30) but is modified by terms including the KK scalar, reading

$$\omega_\mu{}^a = \tau_\mu \frac{1}{k} F(b)_0{}^a + e_{\mu c}\left(-2e_0{}^{(ca)} + \frac{1}{k}\frac{F(b)^{ca}}{2}\right), \tag{93}$$

$$\omega_\mu{}^{ab} = -2e_\mu{}^{[ab]} + e_{\mu c}e^{abc} - \tau_\mu \frac{1}{k}\frac{F(b)^{ab}}{2}. \tag{94}$$

Consequently, also the variations of the dependent connections are modified and now take the more complicated form

$$\begin{aligned}\delta\omega_\mu{}^a &= \partial_\mu\lambda^a - \omega_\mu{}^{ac}\lambda_c + \tau_\mu\lambda^a\partial_0\ln k + e_{\mu c}\left(2\lambda^{(c}\tau^{a)}{}_0 + \partial^{[c}\ln k\lambda^{a]}\right)\\ &\quad + \lambda^a{}_b\omega_\mu{}^b - \lambda_D\omega_\mu{}^a,\\ \delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab} - 2\omega_\mu{}^{c[a}\lambda^{b]}{}_c + 2\lambda^{[a}\tau_\mu{}^{b]} + e_{\mu c}\lambda^c\tau^{ab} - \tau_\mu\partial^{[a}\ln k\lambda^{a]}.\end{aligned} \tag{95}$$

This in turn leads to the modified[27] $SO(D-1)$ rotation curvature

$$\begin{aligned}\hat{R}_{\mu\nu}(J^{ab}) &:= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]}{}^b{}_c\\ &\quad - 4\omega_{[\nu}{}^{[a}\tau_{\nu]}{}^{b]} - 2\omega_{[\mu}{}^c e_{\nu]c}\tau^{ab} + 2\partial^{[a}\ln k\,\omega_{[\mu}{}^{b]}\tau_{\nu]},\\ \hat{R}(J) &:= -e^\mu_a e^\nu_b R_{\mu\nu}(J^{ab}) = -2e^\mu_a e^\nu_b(\partial_{[\mu}\omega_{\nu]}{}^{ab} + \omega_{[\mu}{}^{ac}\omega_{\nu]}{}^b{}_c) + 4\omega_{ab}\tau^{ab}.\end{aligned} \tag{96}$$

[27]As in the sense of eq. (3).

We see, that upon imposing $k = 1$, we recover the standard version of TNC we have seen in eq. (33). Furthermore, note that part of the intrinsic torsion transforms as the spacial part of a connection for dilatations, i.e.

$$\delta_{\mathrm{D}} 2\tau_{a0} = \partial_a \lambda_D .$$

It will prove useful later to also introduce a corresponding covariant derivative for this field

$$\mathcal{D}_\mu \tau_0{}^a = \overset{\mathrm{SO}}{\mathcal{D}}_\mu \tau_0{}^a - \omega_{\mu b} \tau^{ab}, \tag{97}$$

where $\overset{\mathrm{SO}}{\mathcal{D}}$ is covariant with respect to spacial rotations $SO(D-2)$. We could thus define a dilatation connection $d_a \coloneqq 2\tau_{a0}$. However, for the applications of this thesis it is not necessary.

We will call the geometry derived from the Kaluza-Klein reduction *dilatation invariant torsional Newton-Cartan (DTNC) geometry.*

7.3. Kaluza-Klein Reduction of Non-Relativistic NS-NS Gravity

We now want to perform a KK[28] reduction along a spacelike longitudinal direction as in section 5.2. Even though we are in the setting of a Newton-Cartan type geometry, we can still use most of the results of section 5, as the longitudinal vielbein $\tau_{\hat{\mu}}{}^A$ remains Lorentzian. Our starting point is the geometric data of [4], i.e. section 7.1, namely the non-relativistic vielbeine $\tau_{\hat{\mu}}{}^A$, $e_{\hat{\mu}}{}^{A'}$ and the KR field $b_{\hat{\mu}\hat{\nu}}$ that transform under longitudinal Lorentz $SO(1,1)$ λ_M, transverse rotations $SO(D-2)$ $\lambda^{A'}{}_{B'}$, string boosts $\lambda_A{}^{A'}$, dilatations λ_D and one-form symmetries $\theta_{\hat{\mu}}$ as

$$\begin{aligned}
\delta\tau_{\hat{\mu}}{}^A &= \lambda_D \tau_{\hat{\mu}}{}^A + \lambda_M \varepsilon^A{}_B \tau_{\hat{\mu}}{}^A, \\
\delta e_{\hat{\mu}A'} &= -\lambda_A{}^{A'} \tau_{\hat{\mu}}{}^A + \lambda^{A'}{}_{B'} e_{\hat{\mu}}{}^{B'}, \\
\delta b_{\hat{\mu}\hat{\nu}} &= 2\partial_{[\hat{\mu}} \theta_{\hat{\nu}]} - 2\varepsilon_{AB} \lambda^A{}_{A'} \tau_{[\hat{\mu}}{}^B e_{\hat{\mu}]}{}^{A'}, \\
\delta\phi &= \lambda_D \phi.
\end{aligned} \tag{98}$$

Now, we again assume the existence of a spacelike Killing vector field χ for the longitudinal vielbein, i.e. $\mathcal{L}_\chi \tau_{\hat{\mu}}{}^A = 0$ and $\eta_{AB}\chi^A\chi^B = 1$. By the Frobenius theorem there exist coordinates $(x^{\hat{\mu}}) = (x^\mu, z)$, such that $\chi \equiv \partial_z$. Correspondingly, the Killing equation becomes

$$\partial_z \tau_{\hat{\mu}}{}^A = 0.$$

Furthermore, we impose the zero-mode condition, i.e. all fields and parameters should be independent of z.

[28]Note that while the null reduction is technically also a KK reduction, we will henceforth always refer to a spacelike KK reduction.

Following the lines of section 5, we split the target space diffeomorphisms as $(\xi^{\hat{\mu}}(x,z)) = (\xi^{\mu}(x), \xi^{z}(x))$ and require that they leave the adapted coordinates inert.
Furthermore, we can use part of the string Galilei symmetries to gauge fix ${\tau_z}^1 = 0$, leading to restricting $\lambda_M = 0$ and gauge fix ${e_z}^{A'} = 0$, leading to ${\lambda_1}^{A'} = 0$. This amounts to the ansatz

$$(\tau_{\hat{\mu}}{}^A) = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} \mu \\ z \end{array} & \begin{pmatrix} \tau_\mu & k m_\mu \\ 0 & k \end{pmatrix} \end{array}, \quad (e_{\hat{\mu}^{A'}}) = \begin{array}{c c} & A' \\ \begin{array}{c} \mu \\ z \end{array} & \begin{pmatrix} {e_\mu}^{A'} \\ 0 \end{pmatrix} \end{array}.$$

The corresponding string projective inverses (74) imply the Newton-Cartan projective inverses (24) if we take the following ansatz for the inverses

$$(\tau^{\hat{\mu}}{}_A) = \begin{array}{c c} & \begin{array}{cc} \mu & z \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \tau^\mu & -\tau^\alpha m_\alpha \\ 0 & \frac{1}{k} \end{pmatrix} \end{array}, \quad (e^{\hat{\mu}}{}_{A'}) = \begin{array}{c c} & \begin{array}{cc} \mu & z \end{array} \\ A' & \begin{pmatrix} e^{\mu}{}_{A'} & -e^{\alpha}{}_{A'} m_\alpha \end{pmatrix} \end{array}.$$

Here, we have defined the KK scalar $k := {\tau_z}^0$ and the $U(1)$ field $km_\mu := {\tau_\mu}^1$, which transforms under diffeomorphisms in the compact z-direction as

$$\delta_{\xi^z} m_\mu = \partial_\mu \xi^z.$$

Recall that this implies that m_μ is related to the momentum in the compact direction, i.e. fields with such momentum are charged under m_μ.

Introducing the reduced boost parameter

$$\lambda^{A'} = -{\lambda_0}^{A'}$$

and the reduced KR field

$$\begin{aligned} b_\mu &:= b_{\mu z}, \\ \delta b_\mu &= \partial_\mu \theta_z + k \lambda_a {e^a}_\mu, \end{aligned} \tag{99}$$

we recover exactly the transformation rules of DTNC we found in section 7.2.1 from reducing the variations in eq. (98) in this gauge (upon identifying $\theta_z \equiv \theta$).

We want to emphasize that in all subsequent discussions, we will never discuss higher dimensional fields over the $D+1$-dimensional spacetime, but only fields *after* we have performed the KK reduction. Consequently, the symbol b will *always* refer to the reduced KR field (99). Index free notation will also *always* refer to $D-$dimensional fields carrying spacetime indices $\mu, \nu, ... = 0, ..., D-1$.

Contrary to the particle case, we also keep the remainder of the KR field[29] which transforms non-trivially under both boosts and ξ^z transformations

$$\begin{aligned} \hat{b}_{\mu\nu} &:= b_{\mu\nu}, \\ \delta \hat{b}_{\mu\nu} &= 2\partial_{[\mu}\theta_{\nu]} - 2k\lambda_{A'} m_{[\mu} {e_{\nu]}}^{A'} - 2\partial_{[\mu}\xi^z b_{\nu]}. \end{aligned}$$

[29]We introduce this redundant notation for clarity in index free notation.

Thus, we see that boosts mix winding modes with compact momentum, while diffeomorphisms in z-direction mix $\hat{b}$ and b, as we have already discussed at the end of section 5.3.

Note that we can introduce a composite field containing the remainder of the KR field, such that it only transforms under $U(1)$ and one-form symmetries

$$\hat{b}'_{\mu\nu} := \hat{b}_{\mu\nu} + 2m_{[\mu}b_{\nu]}, \tag{100}$$

$$\delta\hat{b}'_{\mu\nu} = 2\partial_{[\mu}\theta_{\nu]} + 2m_{[\mu}\partial_{\nu]}\theta_z. \tag{101}$$

It is also convenient to give the results in coordinate free notation, where its transformation rule is given by

$$\delta\hat{b}' = d\theta - d\theta_z \wedge m. \tag{102}$$

This will be useful later, when comparing to the result of the null reduction. Again, in coordinate free notation, we can write

$$\begin{aligned} \hat{b}' &= \hat{b} + m \wedge b, \\ d\hat{b}' &= d\hat{b} + F(m) \wedge b - m \wedge F(b), \end{aligned}$$

with the field strengths $F(m) := dm$ and $F(b) := db$. For compact notation we also introduce the corresponding covariant derivative

$$\mathcal{D}\hat{b}' := d\hat{b}' - F(m) \wedge b = d\hat{b} - m \wedge F(b). \tag{103}$$

Due to the structure of the KK reduction, we are always guaranteed to reduce higher-dimensional curvatures to lower-dimensional covariant derivatives. It has to be the case as we need to preserve invariance under the $U(1)$ symmetry remnant of the z-diffeomorphisms.
Note that this definition is indeed covariant, however, it transforms non trivially under boosts

$$\delta_{\mathrm{B}}\mathcal{D}\hat{b}' = -F(m) \wedge \delta_{\mathrm{B}} b = -k\lambda_{A'} F(m) \wedge e^{A'}$$

In total, we have thus shown that we recovered DTNC geometry from section 7.2.2 substituted with a $U(1)$ field m and the remainder $\hat{b}$ of the KR field.

For notational convenience, we also introduce a notation that lets us distinguish between contractions with the higher-dimensional vielbeine, i.e. with index $\hat{\mu}$ and the reduced vielbeine, i.e. with index μ

$$\begin{aligned} &T_a := e^{\mu}{}_{A'} T_\mu \qquad && T_{A'} := e^{\hat{\mu}}{}_{A'} T_{\hat{\mu}} \equiv T_a - T_z m_a, \\ &T_0 := \tau^\mu T_\mu && T_{\hat{0}} := \tau^{\hat{\mu}}{}_0 T_{\hat{\mu}} = T_0 - T_z m_0, \\ &T_{\hat{1}} := \tau^{\hat{\mu}}{}_1 T_{\hat{\mu}} = \frac{1}{k} T_z \end{aligned}$$

for an arbitrary tensor T. The spacial index takes values $a = 2, .., D-1$. Thus, in places where there is no danger of confusion, we will freely identify $A' = a$, etc.

7.3.1. KK Reduction of Curvatures and Connections

When we consider the spacial reduction of the conventional constraints in DSNC (122), we find that they reduce to a torsional version of DTNC, as can be seen in Appendix C.2. However, these torsion terms are largely non-intrinsic and can thus be seen as a simple redefinition of the connections (something similar was observed in [35] for the null reduction of conformal gravity).
There are terms involving the intrinsic torsion, which we could also absorb into a reduced dilatation connection, but again, such a connection will not be relevant for this thesis.
Continuing with the Kaluza-Klein reduction of the DSNC connections (123) (for full details see Appendix C), we can express the torsional connections $\hat{\omega}$ inferred from the DSNC as

$$
\begin{aligned}
\hat{\omega}_\mu{}^a &= -2e_{\mu b}e_0{}^{(ab)} + \frac{1}{k}\frac{F(b)_\mu{}^a}{2} + W_\mu{}^{0A'} \\
&= \omega_\mu{}^a - \tau_\mu \frac{1}{2k} F(b)_0{}^a + W_\mu{}^{0A'}, \\
\hat{\omega}_\mu{}^{ab} &= -2e_\mu{}^{[ab]} + e_{\mu c}e^{abc} - \tau_\mu \frac{1}{k}\frac{F(b)^{ab}}{2} + m_\mu \hat{\omega}_z{}^{ab} \\
&= \omega_\mu{}^{ab} + m_\mu \hat{\omega}_z{}^{ab}, \\
\hat{\omega}_z{}^{ab} &= \frac{1}{2} k \mathcal{D}\hat{b}'^{0ab}.
\end{aligned}
$$

Here, the connections ω are now precisely the dependent connections of boosts and rotations of DTNC transforming as in eq. (95) (see Appendix C.4). All other connection coefficients (see Appendix C.3) not indicated here, as well as the undetermined components W, will ultimately drop out in the final expression for the action, thus are not important for this thesis.

7.3.2. KK Reduction of the Non-relativistic NS-NS Action

Applying the Kaluza-Klein reduction to the non-relativistic NS-NS gravity action (90), we find the reduced action

$$
\begin{aligned}
S_{\mathrm{KK}} &:= \overset{(0)}{S}|_{KK} \\
&= \frac{L}{2\kappa^2}\int d^9x \det(e,\tau) k e^{-2\phi}\Big\{ \hat{R}(J,\omega) - \frac{1}{12}\mathcal{D}\hat{b}'_{abc}\mathcal{D}\hat{b}'^{abc} - \frac{1}{2}kF(m)_{ab}\mathcal{D}\hat{b}'^{0ab} + \frac{k^2}{2}F(m)_{a0}F(m)^a{}_0 \\
&\qquad + 4\partial_a\phi\left(\partial^a\phi - \partial \ln k\right) + \frac{1}{2}\partial_a \ln k \partial^a \ln k + 2\partial_a \ln k \tau^a{}_0 - 6\tau_{a0}\tau^a{}_0 \\
&\qquad - 4\mathcal{D}_a\tau^a{}_0 \Big\}.
\end{aligned}
\tag{104}
$$

Here, all fields and indices appearing correspond to DTNC from section 7.2.2, $\hat{R}(J,\omega)$ corresponds to the DTNC curvature (96), while the covariant derivatives act as indicated in eq. (97) and (103). Note that there is no explicit Yang-Mills-like term present for the reduced KR field b.

This action is invariant under all transformations of DTNC, i.e. rotations, boosts, central charge transformations and also dilatations, the latter only up to a boundary term. We can bring the action also to an equivalent, manifestly dilatation invariant form by integration by parts

$$\begin{aligned} S_{\text{KK-dil}} = \frac{L}{2\kappa^2} \int d^9x \det(e,\tau) k e^{-2\phi} \Big\{ & \hat{R}(J,\omega) - 4\omega_{ab}\tau^{ab} \\ & - \frac{1}{12}\mathcal{D}\hat{b}'_{abc}\mathcal{D}\hat{b}'^{abc} - \frac{1}{2}kF(m)_{ab}\mathcal{D}\hat{b}'^{0ab} + \frac{k^2}{2}F(m)_{a0}F(m)^a{}_0 \\ & + 4\nabla_a\phi(\nabla^a\phi - \nabla^a \ln k) + \frac{1}{2}\nabla_a \ln k \nabla^a \ln k \Big\}, \end{aligned} \tag{105}$$

where we introduced the dilatation invariant covariant derivative $\nabla_a f = \partial_a f - 2\tau_{a0}$, for $f \in \{\phi, \ln k\}$. Note that this action now is no longer manifestly boost invariant as can be seen by the bare boost connection.

7.4. Null Reduction of NS-NS Gravity

In accordance with the worldsheet T-duality 6.3.5, we want to perform a null reduction of the relativistic NS-NS gravity action (86). We choose our ansatz exactly as in 5.4, i.e. we choose the form of the metric (52) with variations (53). However, we will adopt a slightly different notation that aims at clarifying the T-duality rules we expect to find. To see the emergence of the dilatation symmetry, it is important to keep the null reduction scalar s, which transforms under said dilatations.

The full details of the calculation can be found in Appendix D. For now, we assume adapted coordinates $(X^{\hat{\mu}}) = (X^\mu, X^v)$, where X^v is the lightlike isometry direction and $\mu = 0, ...D-1$. Additionally, we introduce lightcone coordinates in tangent space and split its index $\hat{A} = (a, +, -)$, in which the Minkowski metric reads $\eta_{ab} = \delta_{ab}$ and $\eta_{+-} = -1$.

We repeat the ansatz for the metric in our adapted notation

$$(E_{\hat{\mu}}{}^A) = \begin{array}{c} \\ \mu \\ v \end{array} \!\!\begin{array}{c} \begin{array}{ccc} a & - & + \end{array} \\ \begin{pmatrix} \tilde{e}_\mu{}^a & s^{-1}\tilde{\tau}'_\mu & s\tilde{m}_\mu \\ 0 & 0 & s \end{pmatrix} \end{array}, \quad (E^{\hat{\mu}}{}_A) = \begin{array}{c} \\ a \\ - \\ + \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mu & v \end{array} \\ \begin{pmatrix} \tilde{e}^\mu{}_a & -\tilde{e}^\mu{}_a\tilde{m}_\mu \\ s\tilde{\tau}'^\mu & -s\tilde{\tau}'^\mu\tilde{m}_\mu \\ 0 & s^{-1} \end{pmatrix} \end{array}$$

and its corresponding variations under spacial rotations $\tilde{\lambda}^a{}_b$, boosts $\tilde{\lambda}^a$, dilatations $\tilde{\lambda}_D$ and ξ^v the remainder of diffeomorphisms in v-direction as

$$
\begin{aligned}
\delta \tilde{\tau}'_\mu &= 0 \\
\delta \tilde{e}_\mu{}^a &= \tilde{\lambda}^a{}_b \tilde{e}_\mu{}^b + s^{-1}\tilde{\lambda}^a \tilde{\tau}'_\mu, \\
\delta \tilde{m}_\mu &= \partial_\mu \xi^v + s^{-1}\tilde{\lambda}_a \tilde{e}_\mu{}^a, \\
\delta s &= \tilde{\lambda}_D s.
\end{aligned}
$$

Recall, the $U(1)$ field $\tilde{m}$ is related to the momentum along the null direction along which we reduce. Furthermore, the prime on the temporal vielbein is foreshadowing an upcoming redefinition that is necessary to compare to the results of the KK reduction.

We again split the KR field in a remainder $\tilde{\tilde{b}}'$ and a reduced part $\tilde{b}$ defined as

$$
\begin{aligned}
\tilde{\tilde{b}}'_{\mu\nu} &:= B_{\mu\nu}, \\
\tilde{b}_\mu &:= B_{\mu v}.
\end{aligned}
$$

These fields transform under the reduced diffeomorphisms ξ^μ, $U(1)$ transformations θ_v and ξ^v, as well as one form symmetries θ as

$$
\begin{aligned}
\delta \tilde{b} &= \mathcal{L}_\xi \tilde{b} + d\theta_v, \\
\delta \tilde{\tilde{b}}' &= \mathcal{L}_\xi \tilde{\tilde{b}}' + d\theta - d\xi^v \wedge \tilde{b}.
\end{aligned}
$$

Thus, we see that $\tilde{b}$, which is related to the winding along the null direction, transforms as a $U(1)$ connection under the reduction of the one-form symmetry θ_v. The remainder of the KR field $\tilde{\tilde{b}}'$ transforms under the remainder of the one-form symmetry θ and also experiences mixing with $\tilde{b}$ under diffeomorphisms in the compact direction for ξ^v.
Note that the transformation of $\tilde{\tilde{b}}'$ here corresponds to the variation of the remainder of the KR field under KK reduction (100), if we identify the parameters ξ^v from null reductions and θ_z from KK reduction. This is already the first hint of the upcoming T-duality.
This does not hold, however, for the reduced KR fields b from the KK reduction and $\tilde{b}$ from the null reduction. While the former also transforms under boosts, the latter does not.

7.4.1. Null Reduction of the Connections

We can express the reduction of the relativistic spin connections Ω from eq. (129) in the following form

$$
\begin{aligned}
\Omega_\mu{}^{a+} &= -s\tilde{\omega}'_\mu{}^a - s\tilde{m}_\mu \tilde{\tau}'_0{}^a, \\
\Omega_\mu{}^{ab} &= \tilde{\omega}'_\mu{}^{ab} - \tilde{m}_\mu \tilde{\tau}'^{ab}.
\end{aligned}
$$

This matches a similar result in [35] upon gauge fixing $s = 1$. Here, we identified the spin and boost connections $\tilde{\omega}'$ that represent TNC connections built from $\tilde{e}$, $\tilde{\tau}'$ and $\tilde{m}$ as in eq. (30). Note, however, that they covariantize boosts with parameter $\tilde{\lambda}^a/s$ and thus transform as

$$\delta\tilde{\omega}'_\mu{}^a = \overset{\text{SO}}{\mathcal{D}}_\mu\left(\frac{\tilde{\lambda}^a}{s}\right) + \tilde{e}_{\mu c}2\frac{\tilde{\lambda}^{(c}\tilde{\tau}'^{a)}{}_0}{s} + \tilde{\lambda}^a{}_b\tilde{\omega}'_\mu{}^b$$

$$\delta\tilde{\omega}'_\mu{}^{ab} = \overset{\text{SO}}{\mathcal{D}}_\mu\tilde{\lambda}^{ab} + 2\frac{\tilde{\lambda}^{[a}\tilde{\tau}'_\mu{}^{b]}}{s} + \tilde{e}_{\mu c}\frac{\tilde{\lambda}^c}{s}\tilde{\tau}'^{ab}.$$

We introduced the notation $\overset{\text{SO}}{\mathcal{D}}$ for covariantizing spacial rotations $SO(D-2)$ only. Given these connections and variations, we can immediately write down the corresponding TNC geometry if we just absorb s into $\tilde{\lambda}^a$.

7.4.2. Null Reduction of the Action

Using the results from Appendix D for the expression of the curvatures under null reduction, we derive the null-reduced action

$$\begin{aligned}
S_{\text{Null}} &:= S_{\text{NS-NS}}|_{G_{vv}=0} \\
&= \frac{L}{2\kappa^2}\int d^9x\ \det\left(\tilde{e}, \frac{1}{s}\tilde{\tau}'\right)e^{-2\Phi}s\Big\{\hat{R}(J,\tilde{\omega}') - \frac{1}{12}(d\tilde{\hat{b}}' - \tilde{m}\wedge F(\tilde{b}))_{abc}(d\tilde{\hat{b}}' - \tilde{m}\wedge F(\tilde{b}))^{abc} \\
&\qquad -\frac{1}{2}F(\tilde{b})_{ab}\left(d\tilde{\hat{b}}' - \tilde{m}\wedge F(\tilde{b})\right)^{ab0} + \frac{1}{2}F(\tilde{b})_{a0}F(\tilde{b})^a{}_0 \\
&\qquad +4\partial_a\Phi\partial^a\Phi - 6\tilde{\tau}'^a{}_0\tilde{\tau}'_{a0} - 4\tilde{\mathcal{D}}_a\tilde{\tau}'^a{}_0\Big\}.
\end{aligned}$$

Here, all fields and connections correspond to a version of TNC from section 4.2.3 and $F(\tilde{b}) = d\tilde{b}$. This action is fully invariant under the Bargmann symmetries as well as dilatations of s.

As can be inferred from the determinant, the above action is not in the typical form we expect from a non-relativistic limit. We can write it in a known form if we redefine the temporal vielbein to include the null reduction scalar s

$$\tilde{\tau}_\mu := \frac{1}{s}\tilde{\tau}'_\mu \implies \tilde{\tau}'^\mu = s\tilde{\tau}^\mu. \tag{106}$$

It is important that the temporal vielbein $\tilde{\tau}$ is completely equivalent to $\tilde{\tau}'$ and fulfills the same completeness and degeneracy relations (24), if we introduce an according inverse temporal vielbein.
Importantly though, it changes its behavior under dilatations and is no longer inert

$$\delta_{\text{D}}\tilde{\tau}_\mu = -\tilde{\lambda}_D\tilde{\tau}_\mu.$$

Furthermore, the spacial vielbein now transforms under boost as expected in a DTNC geometry

$$\delta_{\mathrm{B}}\tilde{e}_\mu{}^a = \tilde{\lambda}^a\tilde{\tau}_\mu.$$

Combined, these changes have subtle implications. The connections can now be written as

$$\begin{aligned}\tilde{\omega}'_\mu{}^{ab} &= -2\tilde{e}_\mu{}^{[ab]} + \tilde{e}_{\mu c}\tilde{e}^{abc} - s\tilde{\tau}_\mu\tilde{m}^{ab} =: \tilde{\omega}_\mu{}^{ab} \\ \tilde{\omega}'_\mu{}^a &= \frac{1}{s}\left[\tilde{\tau}_\mu 2\ s\ \tilde{m}_{\tilde{0}}{}^a + \tilde{e}_{\mu c}\left(-2\tilde{e}_{\tilde{0}}{}^{(ac)} + s\tilde{m}^{ac}\right)\right] \\ &=: \frac{1}{s}\tilde{\omega}_\mu{}^a.\end{aligned}$$

The tilde over the 0 signals that we used $\tilde{\tau}$ as projector, and we recall that $\tilde{m}_{\mu\nu} = \partial_{[\mu}\tilde{m}_{\nu]}$. By comparison to eq. (93), we see that the connections without tilde correspond exactly to the connections of DTNC in terms of $\tilde{\tau}_\mu$ and $\tilde{e}_\mu{}^a$, given we also identify $k \equiv s^{-1}$ and $\tilde{m} \equiv b$, which is the next hint at the presence of a T-Duality.

The above expression lets us read off the altered transformations

$$\begin{aligned}\delta\tilde{\omega}_\mu{}^a &= \partial_\mu\tilde{\lambda}^a - \tilde{\omega}_\mu{}^{ac}\tilde{\lambda}_c + \tilde{\tau}_\mu\tilde{\lambda}^a\partial_{\tilde{0}}\ln\frac{1}{s} + \tilde{e}_{\mu c}\left(2\tilde{\lambda}^{(c}\tilde{\tau}^{a)}{}_{\tilde{0}} + \partial^{[c}\ln\frac{1}{s}\tilde{\lambda}^{a]}\right) \\ &\quad + \tilde{\lambda}^a{}_b\tilde{\omega}_\mu{}^b + \tilde{\lambda}_D\tilde{\omega}_\mu{}^a \end{aligned} \tag{107}$$

$$\delta\tilde{\omega}_\mu{}^{ab} = \partial_\mu\tilde{\lambda}^{ab} - 2\tilde{\omega}_\mu{}^{c[a}\tilde{\lambda}^{b]}{}_c + 2\tilde{\lambda}^{[a}\tilde{\tau}_\mu{}^{b]} + \tilde{e}_{\mu c}\tilde{\lambda}^c\tilde{\tau}^{ab} - \tilde{\tau}_\mu\partial^{[a}\ln\frac{1}{s}\tilde{\lambda}^{b]}. \tag{108}$$

Clearly, these are exactly the variations of the DTNC connections in eq. (95) under the identification of k and s^{-1}, so we can apply all other results for curvatures, variations, etc. from section 7.2.2.

Now, observe how the intrinsic torsion changes under the redefinition of the temporal vielbein

$$\tilde{\tau}'_{\mu\nu} = \partial_{[\mu}s\tilde{\tau}_{\nu]} + s\tilde{\tau}_{\mu\nu}.$$

Therefore, also the covariant derivative changes to

$$\mathcal{D}_a\tilde{\tau}'^a{}_0 = \mathcal{D}_a\tilde{\tau}^a_{\tilde{0}} + \frac{1}{2}\overset{\text{so}}{\mathcal{D}}_a\partial^a\ln s,$$

where on the right-hand side we covariantized with $\tilde{\omega}$ instead of $\tilde{\omega}'$. From now on, we drop the tilde on indices and remember that such indices are no longer inert under dilatations.

Furthermore, we also introduce a shift of the relativistic dilaton Φ by s (similarly to the Buscher rules in [39]), given as

$$\Phi := \tilde{\phi} + \ln s.$$

Consequently, the non-relativistic dilaton $\tilde{\phi}$ is no longer inert under dilatations and transforms as

$$\delta_{\mathrm{D}}\tilde{\phi} = -\tilde{\lambda}_D.$$

Finally, we can rewrite the null-reduced action in a form that resembles a non-relativistic limit, given by

$$\begin{aligned}
S_{\mathrm{Null}} = \frac{L}{2\kappa^2}\int d^9x \det(\tilde{e},\tilde{\tau})e^{-2\tilde{\phi}}\frac{1}{s}\Big\{&\hat{R}(J,\tilde{\omega}) - \frac{1}{12}(d\tilde{\tilde{b}}' - \tilde{m}\wedge F(\tilde{b}))_{abc}(d\tilde{\tilde{b}}' - \tilde{m}\wedge F(\tilde{b}))^{abc} \\
&- \frac{1}{2}\frac{1}{s}F(\tilde{b})_{ab}\left(d\tilde{\tilde{b}}' - \tilde{m}\wedge F(\tilde{b})\right)^{ab0} + \frac{1}{2}\frac{1}{s^2}F(\tilde{b})_{a0}F(\tilde{b})^a{}_0 \\
&+ 4\partial_a\tilde{\phi}\left(\partial^a\tilde{\phi} - \partial_a\ln\frac{1}{s}\right) + \frac{1}{2}\partial_a\ln\frac{1}{s}\partial^a\ln\frac{1}{s} + 2\partial_a\ln\frac{1}{s}\tilde{\tau}^a{}_0 - 6\tilde{\tau}^a{}_0\tilde{\tau}_{a0} - 4\mathcal{D}_a\tilde{\tau}^a{}_0\Big\}.
\end{aligned} \tag{109}$$

Here, $\hat{R}(J,\tilde{\omega})$ corresponds to the DTNC curvature (96). This action is by construction invariant under all Bargmann symmetries, as well as dilatations, diffeomorphisms on the lower-dimensional spacetime and $U(1)$ transformations along the null direction.

7.5. T-Dualities and Emergent Dilatations

Let us now compare the two types of geometries we have found. Under the spacial Kaluza-Klein reduction of non-relativistic NS-NS gravity, we found the geometric data $\tau_\mu, e_\mu{}^a, b_\mu, k$ and the matter fields $m_\mu, \hat{b}'_{\mu\nu}$ and ϕ, with the following variations under dilatations λ_D, spacial rotations $\lambda^a{}_b$, boosts λ^a, $U(1)$ transformations θ_z and ξ^z, as well as one-form symmetries θ_μ

$$\begin{aligned}
\delta\tau_\mu &= \lambda_D\tau_\mu, & \delta m_\mu &= \partial_\mu\xi^z, \\
\delta e_\mu{}^a &= \lambda^a{}_b e_\mu{}^b + \lambda^a\tau_\mu, & \delta\hat{b}_{\mu\nu} &= 2\partial_{[\mu}\theta_{\nu]} + 2m_{[\mu}\partial_{\nu]}\theta_z, \\
\delta b_\mu &= \partial_\mu\theta_z + k\lambda_a e_\mu{}^a, & \delta\phi &= \lambda_D, \\
\delta k &= \lambda_D k.
\end{aligned}$$

This geometric data defines a DTNC geometry (see section 7.2.2) with connections ω depending on the geometric data $\tau_\mu, e_\mu{}^a, b_\mu, k$, transforming as

$$\begin{aligned}
\delta\omega_\mu{}^a &= \partial_\mu\lambda^a - \omega_\mu{}^{ac}\lambda_c + \tau_\mu\lambda^a\partial_0\ln k + e_{\mu c}\left(2\lambda^{(c}\tau^{a)}{}_0 + \partial^{[c}\ln k\lambda^{a]}\right) + \lambda^a{}_b\omega_\mu{}^b - \lambda_D\omega_\mu{}^a, \\
\delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab} - 2\omega_\mu{}^{c[a}\lambda^{b]}{}_c + 2\lambda^{[a}\tau_\mu{}^{b]} + e_{\mu c}\lambda^c\tau^{ab} - \tau_\mu\partial^{[a}\ln k\lambda^{b]}.
\end{aligned} \tag{110}$$

We can express the spacial reduction of the non-relativistic NS-NS gravity action through this data by the action (104), where we expand the covariant derivative on $\hat{b}'$ and find

$$
\begin{aligned}
S_{\text{KK}} = \frac{L}{2\kappa^2}\int d^9x \det(e,\tau)e^{-2\phi}k\Big\{ & \hat{R}(J,\omega) - \frac{1}{12}(d\hat{b}' - b\wedge F(m))_{abc}(d\hat{b} - b\wedge F(m))^{abc} \\
& - \frac{1}{2}kF(m)_{ab}\left(d\hat{b}' - b\wedge F(m)\right)^{ab0} + \frac{1}{2}kF(m)_{a0}F(m)^{a}{}_{0} \\
& + 4\partial_a\phi\left(\partial^a\phi - \partial_a \ln k\right) + \frac{1}{2}\partial_a \ln k\partial^a \ln k + 2\partial_a \ln k\tau^{a}{}_{0} - 6\tau^{a}{}_{0}\tau_{a0} - 4\mathcal{D}_a\tau^{a}{}_{0}\Big\}.
\end{aligned}
$$

In comparison, the null reduction of relativistic NS-NS gravity leads us to the geometric data $\tilde{\tau}_\mu, \tilde{e}_\mu{}^a, \tilde{m}_\mu, s$ and the matter fields $\tilde{b}_\mu, \tilde{\hat{b}}'_{\mu\nu}$ and $\tilde{\phi}$, transforming under dilatations $\tilde{\lambda}_D$, spacial rotations $\tilde{\lambda}^a{}_b$, boosts $\tilde{\lambda}^a$, $U(1)$ transformations θ_v and ξ^v, as well as one-form symmetries θ_μ as

$$
\begin{aligned}
&\delta\tilde{\tau}_\mu = -\tilde{\lambda}_D\tilde{\tau}_\mu, && \delta\tilde{b}_\mu = \partial_\mu\theta_v, \\
&\delta\tilde{e}_\mu{}^a = \tilde{\lambda}^a{}_b\tilde{e}_\mu{}^b + \tilde{\lambda}^a\tilde{\tau}_\mu, && \delta\tilde{\hat{b}}'_{\mu\nu} = 2\partial_{[\mu}\theta_{\nu]} + 2\tilde{b}_{[\mu}\partial_{\nu]}\xi^v, \\
&\delta\tilde{m}_\mu = \partial_\mu\xi^v + s^{-1}\tilde{\lambda}_a\tilde{e}_\mu{}^a, && \delta\tilde{\phi} = -\tilde{\lambda}_D, \\
&\delta s = \tilde{\lambda}_D s.
\end{aligned}
$$

The null reduction defines another DTNC geometry in terms of the geometric data $\tilde{\tau}_\mu, \tilde{e}_\mu{}^a, \tilde{m}_\mu, s$. The connections $\tilde{\omega}$, which are dependent on the geometric data of this geometry, transform as

$$
\begin{aligned}
\delta\tilde{\omega}_\mu{}^a &= \partial_\mu\tilde{\lambda}^a - \tilde{\omega}_\mu{}^{ac}\tilde{\lambda}_c + \tilde{\tau}_\mu\tilde{\lambda}^a\partial_0 \ln\frac{1}{s} + \tilde{e}_{\mu c}\left(2\tilde{\lambda}^{(c}\tilde{\tau}^{a)}{}_0 + \partial^{[c}\ln\frac{1}{s}\tilde{\lambda}^{a]}\right) + \tilde{\lambda}^a{}_b\tilde{\omega}_\mu{}^b + \tilde{\lambda}_D\tilde{\omega}_\mu{}^a \\
\delta\tilde{\omega}_\mu{}^{ab} &= \partial_\mu\tilde{\lambda}^{ab} - 2\tilde{\omega}_\mu{}^{c[a}\tilde{\lambda}^{b]}{}_c + 2\tilde{\lambda}^{[a}\tilde{\tau}_\mu{}^{b]} + \tilde{e}_{\mu c}\tilde{\lambda}^c\tilde{\tau}^{ab} - \tilde{\tau}_\mu\partial^{[a}\ln\frac{1}{s}\tilde{\lambda}^{b]}.
\end{aligned}
$$

We then have the action of NS-NS gravity under the null reduction (109), i.e.

$$
\begin{aligned}
S_{\text{Null}} = \frac{L}{2\kappa^2}\int d^9x \det(\tilde{e},\tilde{\tau})e^{-2\tilde{\phi}}\frac{1}{s}\Big\{ & \hat{R}(J,\tilde{\omega}) - \frac{1}{12}(d\tilde{\hat{b}}' - \tilde{m}\wedge F(\tilde{b}))_{abc}(d\tilde{\hat{b}}' - \tilde{m}\wedge F(\tilde{b}))^{abc} \\
& - \frac{1}{2}\frac{1}{s}F(\tilde{b})_{ab}\left(d\tilde{\hat{b}}' - \tilde{m}\wedge F(\tilde{b})\right)^{ab0} + \frac{1}{2}\frac{1}{s^2}F(\tilde{b})_{a0}F(\tilde{b})^{a}{}_{0} \\
& + 4\partial_a\tilde{\phi}\left(\partial^a\tilde{\phi} - \partial_a \ln\frac{1}{s}\right) + \frac{1}{2}\partial_a \ln\frac{1}{s}\partial^a \ln\frac{1}{s} + 2\partial_a \ln\frac{1}{s}\tilde{\tau}^{a}{}_{0} - 6\tilde{\tau}^{a}{}_{0}\tilde{\tau}_{a0} - 4\mathcal{D}_a\tilde{\tau}^{a}{}_{0}\Big\}.
\end{aligned}
$$

From this comparison, we can now immediately infer how we can identify the two geometries, fields and actions under the T-duality in target space. The duality is given in two parts. The more obvious one is implemented, if we naturally identify the fields

$$\tau \leftrightarrow \tilde{\tau}, \tag{111}$$
$$e \leftrightarrow \tilde{e}, \tag{112}$$
$$\hat{b}' \leftrightarrow \tilde{\hat{b}}', \tag{113}$$
$$\phi \leftrightarrow \tilde{\phi}, \tag{114}$$

and symmetry parameters

$$\lambda^a{}_b \leftrightarrow \tilde{\lambda}^a{}_b,$$
$$\lambda^a \leftrightarrow \tilde{\lambda}^a.$$

On the left-hand side we always have the fields and parameters from the KK reduction, while on the right-hand side we have the fields from the null reduction.

The second and more interesting part is given if we identify the fields

$$k \leftrightarrow \frac{1}{s}, \tag{115}$$
$$b \leftrightarrow \tilde{m}, \tag{116}$$
$$m \leftrightarrow \tilde{b}. \tag{117}$$

Furthermore, we identify the symmetry parameters of KK reduction on the left with the symmetry parameters of the null reduction on the right as

$$\theta_z \leftrightarrow \xi^v, \tag{118}$$
$$\xi^z \leftrightarrow \theta_v, \tag{119}$$
$$\lambda_D \leftrightarrow -\tilde{\lambda}_D. \tag{120}$$

Given these identifications, the actions, fields and variations agree, thus also the corresponding geometry, i.e. the DTNC connections ω and $\tilde{\omega}$. Note, for this identification to hold, we had to use the redefined null reduction vielbein $\tilde{\tau}$ that included the null reduction scalar s and thus transformed under dilatations.

In conclusion, we see how the T-duality on the worldsheet, as discussed in section 6.2.3 and 6.3.5, is implemented on the target space.
Recall that the fields b and $\tilde{b}$ correspond to a winding of the string along the compact spacial and null directions, while the fields m and $\tilde{m}$ correspond to the momentum along said directions. Under the T-duality, b and $\tilde{m}$, as well as m and $\tilde{b}$ where interchanged. Thus, we observe the interchange of momentum and winding under T-duality.

Furthermore, recall from eq. (41) that under KK reduction, we could interpret the KK scalar k as the effective size of the compact direction. Thus, under the T-duality map, it seems that k also corresponds to the inverse dual size $1/s$, mimicking the dual radius. However, this interpretation is flawed, as a lightlike direction simply does not have a well-defined length. Due to the degeneracy of the metric along this direction, the corresponding size (41) would be equivalently zero.
This is the source of the emergent dilatation symmetry. Under T-duality, the spacelike longitudinal direction corresponds to a lightlike isometry direction. As the latter does not have a notion of length or scale, thus having an inherent dilatation symmetry, also the former cannot have a notion of scale either. That the dilatations involve the temporal vielbein τ is then just a happy accident, due to the redefinition (106).
As the dilatation acts as a Stückelberg shift on the non-relativistic dilaton ϕ, we can

simply gauge it to zero. This signals that the corresponding (pseudo-)action was independent of ϕ after all.

More importantly, we have now shown explicitly that under T-duality the Kaluza-Klein reduction along a longitudinal, spacelike reduction of the pseudo-action of non-relativistic NS-NS gravity maps to the null reduction of the relativistic NS-NS gravity action.

By showing that these two ansätze lead to the same lower-dimensional theory, we have found an effective way to implement the T-duality in the higher-dimensional theories. This could be further encoded in Buscher rules as in relativistic string theory. However, as non-relativistic string theory always necessitates compactified directions to be present, we think that the T-duality rules given in the lower-dimensional formulation are sufficient.

As mentioned before and shown in [4], the pseudo-action (90) gives rise to almost all equations of motion of non-relativistic NS-NS gravity, except for the equation of motion that corresponds to a string version of the Poisson equation. Thus, we have verified and shown, at least up to one equation, the T-duality between non-relativistic NS-NS gravity with a compactified longitudinal spacial direction and relativistic NS-NS gravity with a compactified null direction also holds true on the level of equations of motion of the target space formulation.

8. Outlook

In this thesis, we have investigated how T-dualities of non-relativistic string theory are realized in a target space formulation. We could verify in the target space formulation the results of [10], where it was found that the worldsheet formulation of non-relativistic string theory in a curved SNC background with a compactified longitudinal spacelike direction is T-dual to a relativistic string theory in a curved background with a null isometry.
We have shown that the Kaluza-Klein reduction of non-relativistic NS-NS gravity along a longitudinal spacial direction leads to the same 9-dimensional action as the null reduction of relativistic NS-NS gravity. Since the actions agree, the equivalence between the two models also holds at the level of equations of motion, except for the string version of the Poisson equation. Recall that the information about this equation was lost in the action due to the emergent dilatation symmetry, which we now understand to be a consequence of the vanishing metric component in the null direction. The almost equivalence in the lower-dimensional equations of motion then strongly suggests that we can lift the equivalence in form of a T-duality to the full 10-dimensional gravitational theories.

Therefore, we verified the approach of [15], where the above T-duality[30] was assumed and used as a tool to map relativistic string solutions with null directions to non-relativistic string solutions. There it was shown that the so-called *pp-wave*, corresponding to a relativistic non-winding string, was mapped to a non-relativistic solution that is purely winding and sources a Newtonian potential corresponding to a massive Newtonian particle. Furthermore, the *fundamental NS string* solution with pure winding was shown to map to a non-winding non-relativistic mode. As mentioned in section 6.3.2, unwound non-relativistic string modes mediate the instantaneous Newtonian gravity. Thus, this mode was interpreted as a massless Galilei particle sourcing a torsional Newton-Cartan geometry.

Even though the worldsheet T-duality tells us that the corresponding target space duality should also hold for the missing Poisson equation, further work should verify the T-duality for the full set of equations of motion of non-relativistic NS-NS gravity, including the Poisson equation.
Furthermore, in [15] the T-duality was conjectured to also hold for the supersymmetric extension of non-relativistic NS-NS gravity, corresponding to a supersymmetric DSNC$^-$ geometry. Since we have verified the duality already for the bosonic part of such a model, the natural next step is to apply the above reduction techniques also to the fermionic part, thus deriving T-duality rules for the fermionic degrees of freedom.
Such T-duality rules can then be used to further chart non-relativistic string theory in terms of the discrete lightcone quantization (DLCQ) of the relativistic string, i.e. with a null isometry, and vice versa.

In subsequent works it could also prove valuable to try and refine the null reduction

[30] Complemented with torsion constraints.

(and consequently also spacial reduction). As was shown in [32], the Bargmann algebra, which was part of the geometry we found after both reductions, acts only naturally on spacetimes including a compact $U(1)$ direction. Since, we assumed the zero mode condition in our reduction ansätze, we lost the dependence on this extra coordinate in our theory. Importantly, this extra coordinate can be interpreted as a "mass direction" and as such could be related to the gravitational dynamics encoded in the Poisson equation. Choosing an ansatz that incorporates the extra dimension could then result in an action that encodes all equations of motion.

A. Conventions

Throughout this work we always assume the Einstein summation convention.

For the Minkowski metric we choose the mostly plus convention, i.e.

$$\eta = \mathrm{diag}(-1, 1, ..., 1).$$

Correspondingly, we define the two-dimensional Levi-Civita symbols as

$$\varepsilon_{01} = 1, \quad \varepsilon^{01} = -1.$$

Symmetrizations and antisymmetrizations are defined with weight 1, i.e.

$$A_{[\mu_1 \dots \mu_p]} := \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sign}(\sigma) A_{\mu_{\sigma(1)} \dots \mu_{\sigma(p)}},$$
$$A_{(\mu_1 \dots \mu_p)} := \frac{1}{p!} \sum_{\sigma \in S_p} A_{\mu_{\sigma(1)} \dots \mu_{\sigma(p)}},$$

where S_p is the permutation group of p elements. We also introduce the traceless symmetric part of the tensor by defining

$$A_{\{\mu\nu\}} := A_{(\mu\nu)} - \frac{1}{D} g_{\mu\nu} A^{\alpha}{}_{\alpha}.$$

Components of forms are defined without the weight of the permutation of the basis, i.e. for a p-form

$$\omega^{(p)} = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}.$$

We define the exterior derivative without a combinatorial factor, i.e.

$$d\omega^{(p)} := \frac{1}{p!} \partial_{\mu_1} \omega_{\mu_2 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}},$$

or in components

$$(d\omega^{(p)})_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]}.$$

Similarly, we have for the wedge product

$$(\omega^{(p)} \wedge \alpha^{(q)})_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} \omega^{(p)}_{[\mu_1 \dots \mu_p} \alpha^{(q)}_{\mu_{p+1} \dots \mu_{p+q}]}.$$

Our conventions for Lorentzian geometry can be found in the corresponding section 3.

B. DSNC

The DSNC geometry with the string critical dimension $D = 10$ introduced in [4] is a string Newton-Cartan geometry characterized by eq. (87) with an added dilatation symmetry

$$\begin{aligned}\delta\tau_\mu{}^A &= \lambda_D \tau_\mu{}^A,\\ \delta\phi &= \lambda_D.\end{aligned}$$

The corresponding curvatures are given as

$$\begin{aligned}R_{\mu\nu}(H^A) &= 2\tau_{\mu\nu}{}^A - 2\varepsilon^A{}_B\omega_{[\mu}\tau_{\nu]}{}^B - 2d_{[\nu}\tau_{\nu]}{}^A,\\ R_{\mu\nu}(P^{A'}) &= 2e_{\mu\nu}{}^{A'} - 2\omega_{[\mu}{}^{A'B'}e_{\nu]B'} + 2\omega_{[\mu}{}^{AA'}\tau_{\nu]A},\\ H_{\mu\nu\rho} &= h_{\mu\nu\rho} + 6\varepsilon_{AB}\omega_{[\mu}{}^{AB'}\tau_\nu{}^B e_{\rho]B'},\end{aligned} \tag{121}$$

where

$$\tau_{\mu\nu}{}^A := \partial_{[\mu}\tau_{\nu]}{}^A, \quad e_{\mu\nu}{}^{A'} := \partial_{[\mu}e_{\nu]}{}^{A'}, \quad h_{\mu\nu\rho} := 3\partial_{[\mu}b_{\mu\nu]}.$$

We introduced the SNC connections ω_μ for $SO(1,1)$ transformations, $\omega_\mu{}^{A'}$ for string boosts and $\omega_\mu{}^{A'B'}$ for $SO(8)$ rotations. This is supplemented by the dilatation connection d_μ and the dilatation covariant derivative

$$\nabla_\mu\phi = \partial_\mu\phi - d_\mu.$$

We have turned curved indices into flat ones by virtue of the vielbeine as in eq. (75). To express the connection forms in terms of the geometric data τ, e, ϕ, b we want to impose conventional constraints that at the same time do not restrict the torsion of the geometry. Such constraints are given by

$$\begin{aligned}\varepsilon^{AB}R_{A'A}(H_B) &= 0, \quad & R_{\mu\nu}(P^{A'}) &= 0,\\ \eta^{AB}R_{A'A}(H_B) &= 0, & H_{AA'B'} &= 0,\\ \varepsilon^{AB}R_{AB}(H_C) &= 0, & \nabla_A\phi &= 0.\end{aligned} \tag{122}$$

These only constitute 444 algebraic equations for the 460 components of the connection forms. Thus, there are 16 undetermined components of $\omega_{\{AB\}}{}^A$ that we will denote by $W_{\mu A}{}^{A'} = \tau_\mu{}^B\omega_{\{AB\}}{}^A$. Note that these drop out in most of the results and are thus not important for most applications.

The most general solution of the above constraints is given by

$$\begin{aligned}d_\mu &= e_\mu{}^{A'}\tau_{A'A}{}^A + \tau_\mu{}^A\partial_A\phi,\\ \omega_\mu &= \left(\tau_\mu{}^{AB} - \frac{1}{2}\tau_\mu{}^C\tau^{AB}{}_C\right)\varepsilon_{AB} - \tau_\mu{}^A\varepsilon_{AB}\partial^B\phi,\\ \omega_\mu{}^{AA'} &= -e_\mu{}^{AA'} + e_{\mu B'}e^{AA'B'} + \frac{1}{2}\varepsilon^A{}_B h_\mu{}^{BA'} + W_\mu{}^{AA'},\\ \omega_\mu{}^{A'B'} &= -2e_\mu{}^{[A'B']} + e_{\mu C'}e^{A'B'C'} - \frac{1}{2}\tau_\mu{}^A\varepsilon_{AB}h^{BA'B'}.\end{aligned} \tag{123}$$

The corresponding transformations can be calculated from requiring that the constraints remain invariant under variations. Thus, under dilatations, transverse rotations $SO(8)$ and longitudinal $SO(1,1)$ the connections transform as

$$
\begin{aligned}
\delta_{\mathrm{SO,R,D}} d_\mu &= \partial_\mu \lambda_D, \\
\delta_{\mathrm{SO,R,D}} \omega_\mu &= \partial_\mu \lambda_M, \\
\delta_{\mathrm{SO,R,D}} {\omega_\mu}^{A'B'} &= \partial_\mu \lambda^{A'B'} - 2{\omega_\mu}^{C'[A'} {\lambda^{B']}}_{C'}, \\
\delta_{\mathrm{SO,R,D}} {\omega_\mu}^{AA'} &= \lambda_M {\varepsilon^A}_B {\omega_\mu}^{BA'} + {\lambda^{A'}}_{B'} {\omega_\mu}^{AB'} - \lambda_D {\omega_\mu}^{AA'}.
\end{aligned}
\tag{124}
$$

Under string boosts we find the boost transformations

$$
\begin{aligned}
\delta_{\mathrm{B}} d_\mu &= {e_\mu}^{A'} \tau_{A'B'B} \lambda^{BB'} + {\tau_\mu}^A \lambda_{AA'} \nabla^{A'} \phi, \\
\delta_{\mathrm{B}} \omega_\mu &= -{e_\mu}^{A'} \varepsilon^{AB} \tau_{A'B'A} {\lambda_B}^{B'} - 2{\tau_\mu}^A \left(\varepsilon^{BC} \lambda_{BB'} \tau_{B'\{AC\}} + \frac{1}{2} \varepsilon_{AB} \lambda^{BB'} \nabla_{B'} \phi \right), \\
\delta_{\mathrm{B}} {\omega_\mu}^{AA'} &= \nabla_\mu \lambda^{AA'} + 2{e_\mu}^{B'} \left(\lambda_{BB'} \tau^{A'\{AB\}} + \frac{1}{4} \varepsilon^{AB} \lambda_{BC'} h^{A'B'C'} \right), \\
\delta_{\mathrm{B}} {\omega_\mu}^{A'B'} &= 4{\tau_\mu}^A \left(\lambda^{B[A'} {\tau^{B']}}_{\{AB\}} - \frac{1}{8} \varepsilon^{AB} \lambda_{BC'} h^{A'B'C'} \right) - {e_\mu}^{C'} \left(\lambda_{CC'} \tau^{A'B'C} - 2{\lambda_C}^{[A'} \tau^{B']C'C} \right),
\end{aligned}
\tag{125}
$$

where a “covariant derivative” for the boost parameter was introduced as

$$
\nabla_\mu \lambda^{AA'} \coloneqq \partial_\mu \lambda^{AA'} - \omega_\mu {\varepsilon^A}_B \lambda^{BA'} - {\omega_\mu}^{A'B'} {\lambda^A}_{B'} + d_\mu \lambda^{AA'}.
$$

C. Kaluza-Klein Reduction Results

With the ansatz and conventions as in section 7.3, we can express the components of the intrinsic torsion as

$$
\begin{aligned}
{\tau_{\mu\nu}}^0 &= \partial_{[\mu} {\tau^0}_{\nu]} = \tau_{\mu\nu}, \\
{\tau_{\mu\nu}}^1 &= \partial_{[\mu} k m_{\nu]} + \frac{k}{2} F(m)_{\mu\nu}, \\
{\tau_{\mu z}}^0 &= 0, \\
{\tau_{\mu z}}^1 &= \frac{1}{2} \partial_\mu k,
\end{aligned}
$$

where $F(m) = dm$. The spacial vielbein then reads

$$
\begin{aligned}
{e_{\mu\nu}}^{A'} &= \partial_{[\mu} {e_{\nu]}}^{A'} \eqqcolon {e_{\mu\nu}}^a \\
{e_{\mu z}}^{A'} &= 0
\end{aligned}
$$

C.1. Curvatures

Here, we present the KK reduction of the curvatures of DSNC in eq. (121). We start with the curvature of H:

$$R_{\hat{\mu}\hat{\nu}}(H^A) = 2\tau_{\hat{\mu}\hat{\nu}}{}^A - 2\varepsilon^A{}_B\hat{\omega}_{[\hat{\mu}}\tau_{\hat{\nu}]}{}^B - 2\hat{d}_{[\hat{\nu}}\tau_{\hat{\nu}]}{}^A.$$

We distinguish the following 4 cases:

$$\begin{aligned}
R_{\mu\nu}(H^0) &= 2\tau_{\mu\nu} - 2\hat{d}_{[\mu}\tau_{\nu]} + 2k\hat{\omega}_{[\mu}m_{\nu]} \\
&= R_{\mu\nu}(H) - 2\hat{d}_{[\mu}\tau_{\nu]} + 2k\hat{\omega}_{[\mu}m_{\nu]} \\
R_{\mu\nu}(H^1) &= k\left(F(m)_{\mu\nu} - 2\hat{d}_{[\mu}m_{\nu]}\right) + 2\partial_{[\mu}km_{\nu]} + 2\hat{\omega}_{[\mu}\tau_{\nu]} \\
R_{\mu z}(H^0) &= k\left(\hat{\omega}_\mu - \hat{\omega}_z m_\mu\right) + \hat{d}_z\tau_0 \\
R_{\mu z}(H^1) &= \partial_\mu k + k(\hat{d}_z m_\mu - \hat{d}_\mu) - \hat{\omega}_z\tau_\mu,
\end{aligned}$$

Where we have indicated with hats on the connections that they may be different compared to the DTNC connections in eq. (93).

We turn to the curvature of $P^{A'}$

$$R_{\hat{\mu}\hat{\nu}}(P^{A'}) = 2e_{\hat{\mu}\hat{\nu}}{}^{A'} - 2\hat{\omega}_{[\hat{\mu}}{}^{A'B'}e_{\hat{\nu}]B'} + 2\hat{\omega}_{[\hat{\mu}}{}^{AA'}\tau_{\hat{\nu}]A}.$$

Here we only need to distinguish two cases (with $\hat{\omega}_\mu{}^a := \hat{\omega}_\mu{}^a{}_0$)

$$\begin{aligned}
R_{\mu\nu}(P^{A'}) &= 2e_{\mu\nu}{}^a - 2\hat{\omega}_{[\mu}{}^{ab}e_{\nu]b} - 2\hat{\omega}_{[\mu}{}^a\tau_{\nu]} + 2k\hat{\omega}_{[\mu}{}^{1A'}m_{\nu]} \\
&= R_{\mu\nu}(P^a, \hat{\omega}) + 2k\hat{\omega}_{[\mu}{}^{1A'}m_{\nu]} \\
R_{\mu z}(P^{A'}) &= \hat{\omega}_z{}^a{}_b e_\mu{}^b + k\left(\hat{\omega}_\mu{}^{1a} - \hat{\omega}_z{}^{1a}m_\mu\right).
\end{aligned}$$

While the torsions take the functional form (91) of DTNC, we explicitly denoted the dependence on the hatted connections to remind us that there might be torsion present for the connections stemming from the reduction of DSNC, which would give a different result than DTNC.
At last, we look at the curvature of the KR field

$$H_{\hat{\mu}\hat{\nu}\hat{\rho}} = h_{\hat{\mu}\hat{\nu}\hat{\rho}} + 6\varepsilon_{AB}\hat{\omega}_{[\hat{\mu}}{}^{AB'}\tau_{\hat{\nu}}{}^B e_{\hat{\rho}]B'},$$

where

$$h_{\hat{\mu}\hat{\nu}\hat{\rho}} := 3\partial_{[\hat{\mu}}b_{\hat{\mu}\hat{\nu}]}.$$

This results in the two cases:

$$\begin{aligned}
H_{\mu\nu\rho} &= h_{\mu\nu\rho} + 6k\hat{\omega}_{[\mu}{}^b m_\nu e_{\rho]b} - 6\hat{\omega}_{[\mu}{}^{1b}\tau_\nu e_{\rho]b} \\
H_{\mu\nu z} &= F(b)_{\mu\nu} + 2k\left(\hat{\omega}_{zb}m_{[\mu}e_{\nu]}{}^b - \hat{\omega}_{[\mu}{}^b e_{\nu]b}\right) + 2\hat{\omega}_z{}^{1b}\tau_{[\mu}e_{\nu]b} \\
&= R_{\mu\nu}(Q, \hat{\omega}) + 2k\hat{\omega}_{zb}m_{[\mu}e_{\nu]}{}^b + 2\hat{\omega}_z{}^{1b}\tau_{[\mu}e_{\nu]b}
\end{aligned}$$

C.2. Constraints

Starting from the full conventional constraints (122) of DSNC, we may write them in terms of the reduced curvatures (91) of DTNC.
We start with the dilaton field

$$\begin{aligned}
&0 = \nabla_A \phi \\
&A = \hat{0}: \quad \nabla_0 \phi - m_0 \hat{d}_z = \partial_0 \phi - \hat{d}_0 - m_0 \hat{d}_z \overset{!}{=} 0 \\
&A = \hat{1}: \quad -k\hat{d}_z \overset{!}{=} 0
\end{aligned}$$

We continue with the constraints on the curvature of longitudinal translations H^A:

$$\begin{aligned}
R_{A'A}(H^A) &= R_{a0}(H) + \partial_a \ln k \equiv 2\tau_{a0} - 2\hat{d}_a + \partial_a \ln k \overset{!}{=} 0 \\
\varepsilon^{AB} R_{A'A}(H_B) &= -kF(m)_{a0} - 2\hat{\omega}_a + 2\hat{\omega}_z m_a \overset{!}{=} 0 \\
\varepsilon^{AB} R_{AB}(H^C) &\overset{!}{=} 0 \\
&C = \hat{0}: \quad k(\hat{\omega}_0 - \hat{\omega}_z m_0) + \hat{d}_z = 0 \overset{!}{=} 0 \\
&C = \hat{1}: \quad \partial_0 \ln k + k(\hat{d}_z m_0 - \hat{d}_0) - \hat{\omega}_z \overset{!}{=} 0.
\end{aligned}$$

For the constraints on longitudinal translations $P^{A'}$ we find:

$$\begin{aligned}
R_{\mu\nu}(P^{A'}) &= R_{\mu\nu}(P^a, \hat{\omega}) + 2k\hat{\omega}_{[\mu}{}^{1A'} m_{\nu]} \overset{!}{=} 0 \\
R_{\mu z}(P^{A'}) &= \hat{\omega}_z{}^a{}_b e_\mu{}^b + \hat{\omega}_z{}^a \tau_\mu + k\left(\hat{\omega}_\mu{}^{1a} - \hat{\omega}_z{}^{1a} m_\mu\right) \overset{!}{=} 0
\end{aligned}$$

Finally we turn to the constraints on the curvature of b:

$$\begin{aligned}
&H_{AA'B'} \overset{!}{=} 0 \\
&\quad A = \hat{0}: \quad 0 \overset{!}{=} h_{0ab} + 2\hat{\omega}_{[ab]}{}^1 - 2m_{[a}F(b)_{b]0} - 2\hat{\omega}_z{}^1{}_{[a} m_{b]} - m_0 F(b)_{ab} \\
&\qquad\qquad\quad = \frac{1}{2}\mathcal{D}\hat{b}'{}_{0ab} + 2\hat{\omega}_{[ab]}{}^1 - 2\hat{\omega}_z{}^1{}_{[a} m_{b]} \\
&\quad m = \hat{1}: \quad \frac{1}{k}\left[R_{ab}(Q,\hat{\omega}) - 2k\hat{\omega}_{z[a} m_{b]}\right] \overset{!}{=} 0 \\
&H_{ABA'} \overset{!}{=} 0 \\
\iff &\varepsilon^{AB} H_{ABA'} \sim \frac{1}{k} H_{a0z} = \frac{1}{k} R_{a0}(Q,\hat{\omega}) - m_0 \hat{\omega}_{za} - \frac{1}{k}\hat{\omega}_z{}^1{}_a \\
&\qquad\qquad = \frac{1}{k} F(b)_{a0} + \hat{\omega}_{0a} - \hat{\omega}_{za} m_0 - \frac{1}{k}\hat{\omega}_z{}^1{}_a \overset{!}{=} 0
\end{aligned}$$

So we see that the KK reduction of DSNC corresponds to a torsional version of DTNC.

C.3. The connections

Under the KK reduction the DSNC connections (123) reduce to the following:
For the connection of dilatations we find

$$
\begin{aligned}
\hat{d}_z &= 0 \\
\hat{d}_\mu &= e_\mu{}^a \left[\tau_{a0} + \frac{1}{2}\partial_a \ln k\right] + \tau_\mu \partial_0 \phi.
\end{aligned}
$$

For the connection of longitudinal $SO(1,1)$ we find

$$
\begin{aligned}
\hat{\omega}_z &= k(\partial_0 \ln k - \partial_0 \phi) \\
\hat{\omega}_0 &= k m_0(\partial_0 \ln k - \partial_0 \phi) \\
\hat{\omega}_a &= k\left[-\frac{1}{2}F(m)_{a0} + m_a(\partial_0 \ln k - \partial_0 \phi)\right].
\end{aligned}
$$

For the connection of boosts we find

$$
\begin{aligned}
\hat{\omega}_z{}^a &= W_z{}^{0A'} = -k\hat{\omega}_{\{00\}}{}^{A'} \\
\hat{\omega}_z{}^{1A'} &= \frac{1}{2}F(b)_0{}^a + W_z{}^{1A'} = \frac{1}{2}F(b)_0{}^a + k\hat{\omega}_{\{10\}}{}^{A'} \\
\hat{\omega}_\mu{}^{1A'} &= \frac{1}{2}\left[F(b)_\mu{}^a - F(b)_{\mu 0}m^a + F(b)_\mu{}^a m_0\right] + W_\mu{}^{1A'} \\
\hat{\omega}_\mu{}^a &= -2e_{\mu b}e_0{}^{(ab)} + \frac{1}{k}\frac{F(b)_\mu{}^a}{2} + W_\mu{}^{0A'} \\
&= \omega_\mu{}^a - \tau_\mu \frac{1}{2k}F(b)_0{}^a + W_\mu{}^{0A'}
\end{aligned}
$$

Finally for the connection of transverse rotations we find that

$$
\begin{aligned}
\hat{\omega}_z{}^{ab} &= -\frac{1}{2}k\left[h_0{}^{ab} - 2m^{[a}F(b)^{b]}{}_0 - F(b)^{ab}m_0\right] = \frac{1}{2}k\left[h^{0ab} - 3m^{[0}F(b)^{ab]}\right] = \frac{1}{2}k\mathcal{D}\hat{b}'^{0ab}, \\
\hat{\omega}_\mu{}^{ab} &= -2e_\mu{}^{[ab]} + e_{\mu c}e^{abc} - \tau_\mu \frac{1}{k}\frac{F(b)^{ab}}{2} - m_\mu \frac{1}{2}k\left[h_0{}^{ab} - 2m^{[a}F(b)^{b]}{}_0 - F(b)^{ab}m_0\right] \\
&= -2e_\mu{}^{[ab]} + e_{\mu c}e^{abc} - \tau_\mu \frac{1}{k}\frac{F(b)^{ab}}{2} + m_\mu \hat{\omega}_z{}^{ab} \\
&= \omega_\mu{}^{ab} + m_\mu \hat{\omega}_z{}^{ab}.
\end{aligned}
$$

Thus, we recover the reduced DTNC geometry from 7.2.2 encoded in the connections ω, up to extra torsion terms and corrections from the reduction of the DSNC connections. However, these corrections will drop out in the final result and thus do not play a role.

C.4. Transformations

To compute the reduction of the variations of the DSNC connections (124) and (125) we need

$$
\tau_{A'\{AB\}} = \tau_{a\{AB\}} - m_a \tau_{z\{AB\}}
$$

and find

$$
\begin{aligned}
\tau_{A'\{00\}} &= \frac{1}{2}\left[\frac{1}{2}\partial_a \ln k - \tau_{a0}\right] \\
\tau_{A'\{01\}} &= \frac{k}{4}F(m)_{a0} \\
\tau_{A'\{11\}} &= \frac{1}{2}\left[\frac{1}{2}\partial_a \ln k - \tau_{a0}\right].
\end{aligned}
$$

We start with the transformation of the reduced connections under dilations and rotations

$$
\begin{aligned}
\delta_{\mathrm{D,R}}\hat{d}_z &= 0 \\
\delta_{\mathrm{D,R}}\hat{d}_\mu &= \partial_\mu \lambda_D \\
\delta_{\mathrm{D,R}}\hat{\omega}_{\hat{\mu}} &= 0 \\
\delta_{\mathrm{D,R}}\hat{\omega}_z{}^{ab} &= \lambda^a{}_c\hat{\omega}_z{}^{cb} + \lambda^b{}_c\hat{\omega}_z{}^{ac} \\
\delta_{\mathrm{D,R}}\hat{\omega}_\mu{}^{ab} &= \partial_\mu\lambda^{ab} - 2\hat{\omega}_\mu{}^{c[a}\lambda^{b]}{}_c \\
\delta_{\mathrm{D,R}}\hat{\omega}_z{}^{AA'} &= \lambda^a{}_b\hat{\omega}_z{}^{ab} - \lambda_D\hat{\omega}_z{}^{AA'} \\
\delta_{\mathrm{D,R}}\hat{\omega}_\mu{}^{AA'} &= \lambda^a{}_b\hat{\omega}_\mu{}^{ab} - \lambda_D\hat{\omega}_\mu{}^{AA'}.
\end{aligned}
$$

The boosts are much more intricate:

$$
\begin{aligned}
\delta_{\mathrm{B}}\hat{d}_z &= 0 \\
\delta_{\mathrm{B}}\hat{d}_\mu &= -e_\mu{}^a\tau_{ab}\lambda^b - \tau_\mu\lambda^a\nabla_a\phi \\
\delta_{\mathrm{B}}\hat{\omega}_{\hat{\mu}}{}^{AA'} &= \nabla_{\hat{\mu}}\lambda^{AA'} - 2e_{\hat{\mu}}{}^b\left\{\lambda_b\tau^{A'\{A0\}} + \frac{1}{4}\varepsilon^{A0}\lambda_c\left[h^{abc} - 3m^{[a}F(a)^{bc]}\right]\right\} \\
\delta_{\mathrm{B}}\hat{\omega}_z{}^{AA'} &= \nabla_z\lambda^{AA'} = -\hat{\omega}_z\varepsilon^A{}_0\lambda^a - \delta^A{}_0\hat{\omega}_z{}^{A'B'}\lambda_{B'} \\
\delta_{\mathrm{B}}\hat{\omega}_\mu{}^a &= \overset{\mathrm{SO}}{\mathcal{D}}_\mu\lambda^a - e_\mu{}^b\lambda_b\left(\frac{1}{2}\partial^a\ln k - \tau^a{}_0\right) \\
\delta_{\mathrm{B}}\hat{\omega}_\mu{}^{1A'} &= \hat{\omega}_\mu\lambda^a + e_\mu{}^b\lambda_b k\frac{F(m)^a{}_0}{2} - \frac{1}{4}\lambda_c\mathcal{D}\hat{b}'^{abc} \\
\delta_{\mathrm{B}}\hat{\omega}_z{}^{A'B'} &= k^2\lambda^{[a}F(m)_0{}^{b]} + \frac{1}{4}k\lambda_c\mathcal{D}\hat{b}'^{abc} \\
\delta_{\mathrm{B}}\hat{\omega}_\mu{}^{A'B'} &= 2\tau_\mu\lambda^{[a}\left(\frac{1}{2}\partial^{b]}\ln k - \tau^{b]}{}_0\right) + e_\mu{}^c\lambda_c\tau^{ab} - 2e_{\mu c}\lambda^{[a}\tau^{b]c} + m_\mu\delta_{\mathrm{B}}\hat{\omega}_z{}^{A'B'}.
\end{aligned}
$$

Note that this gives us in particular that the connections ω transform exactly as connections of DTNC.

C.5. The Action

We expand each ingredient of the NS-NS action (86) separately:

$$
\det\left(e_{\hat{\mu}}{}^{A'}, \tau_{\hat{\mu}A}\right) = k\det\left(e_\mu{}^a, \tau_\mu\right) \equiv k\det(e,\tau).
$$

The curvature of rotation becomes

$$\begin{aligned}R(J,\hat{\omega}) &= R(J,\omega) - F(m)_{ab}\omega_z{}^{ab} + 4W^{a0b}\tau_{ab}\\ &= R(J,\omega) - \frac{1}{2}kF(m)_{ab}\mathcal{D}\hat{b}'^{0ab} + 4W^{a0b}\tau_{ab},\end{aligned}$$

Where $R(J,\hat{\omega})$ is the DSNC curvature of rotations, while $R(J,\omega)$ now is the proper curvature of DTNC rotations (96). The kinetic term for the dilaton reads

$$\partial_{A'}\phi\partial^{A'}\phi = \partial_a\phi\partial^a\phi,$$

while the curvature of the KR-field reads

$$h_{A'B'C'} = h_{abc} - 3m_{[a}F(B)_{bc]} = \mathcal{D}\hat{b}'_{abc},$$

with the covariant derivative $\mathcal{D}\hat{b}$ as in eq. (103). The mass term of the dilatation connection reduces to

$$\hat{d}_{A'}\hat{d}^{A'} = \hat{d}_a\hat{d}^a = \tau_{a0}\tau^a{}_0 + \tau_{a0}\partial^a\ln k + \frac{1}{4}\partial_a\ln k\partial^a\ln k.$$

And the covariant derivative of the dilatation connection reads

$$\mathcal{D}_{A'}\hat{d}^{A'} = \overset{\text{SO}}{\mathcal{D}}_a\tau^a{}_0 + \omega_{ab}\tau^{ab} + \frac{1}{2}\overset{\text{SO}}{\mathcal{D}}_a\partial^a\ln k + W^{a0b}\tau_{ab} = \mathcal{D}_a\tau^a{}_0 + \frac{1}{2}\overset{\text{SO}}{\mathcal{D}}_a\partial^a\ln k + W^{a0b}\tau_{ab},$$

where $\overset{\text{SO}}{\mathcal{D}}_\mu$ is covariant with respect to transverse rotations and $\mathcal{D}\tau$ is the covariant derivative (97).
Finally, the traceless symmetric part in the action reduces to

$$-4\tau_{A'\{AB\}}\tau^{A'\{AB\}} = -\frac{1}{2}\partial_a\ln k\partial^a\ln k + 2\partial_a\ln k\tau^a{}_0 - 2\tau_{a0}\tau^a{}_0 + \frac{k^2}{2}F(m)_{a0}F(m)^a{}_0.$$

Altogether, these reductions amount to the total reduced action

$$\begin{aligned}\overset{(0)}{S} = \frac{L}{2\kappa^2}\int d^9x\det(e,\tau)ke^{-2\phi}\Big\{&R(J,\omega) - \frac{1}{12}\mathcal{D}\hat{b}'_{abc}\mathcal{D}\hat{b}'^{abc} - \frac{1}{2}kF(m)_{ab}\mathcal{D}\hat{b}'^{0ab} + \frac{k^2}{2}F(m)_{a0}F(m)^a{}_0\\ &+ 4\partial_a\phi\partial^a\phi - \frac{3}{2}\partial_a\ln k\partial^a\ln k - 2\partial_a\ln k\tau^a{}_0 - 6\tau_{a0}\tau^a{}_0\\ &- 4\mathcal{D}_a\tau^a{}_0 - 2\overset{\text{SO}}{\mathcal{D}}_a\partial^a\ln k\Big\}\end{aligned}$$

C.6. Modified integration by parts

In the above action we integrate by parts for an arbitrary Tensor T as

$$\begin{aligned}\overset{\text{SO}}{\mathcal{D}}_aT^a &\overset{\text{P.I.}}{=} -2\tau_{a0}T^a + (2\partial_a\phi - \partial_a\ln k)T^a\\ &\equiv (2\nabla_a\phi - \nabla_a\ln k)T^a,\end{aligned}$$

where $\overset{\text{P.I.}}{=}$ indicates that this only holds under the integral with measure $d^9x\det(e,\tau)ke^{-2\phi}$, and we discarded boundary terms.

C.7. Final form of action

Integrating by parts, we find

$$\overset{(0)}{S} = \frac{L}{2\kappa^2}\int d^9x\,\det(e,\tau)ke^{-2\phi}\Big\{R(J,\omega) - \frac{1}{12}\mathcal{D}\hat{b}'_{abc}\mathcal{D}\hat{b}'^{abc} - \frac{1}{2}kF(m)_{ab}\mathcal{D}\hat{b}'^{0ab} + \frac{k^2}{2}F(m)_{a0}F(m)^a{}_0$$
$$+ 4\partial_a\phi\,(\partial^a\phi - \partial\ln k) + \frac{1}{2}\partial_a\ln k\partial^a\ln k + 2\partial_a\ln k\tau^a{}_0 - 6\tau_{a0}\tau^a{}_0$$
$$- 4\mathcal{D}_a\tau^a{}_0\Big\}. \tag{126}$$

This is not obvious, but it is boost invariant and also dilatation invariant.
Furthermore, we can bring the action into a manifest dilatation invariant form by integrating by parts, giving

$$S_{\mathrm{KK}} = \frac{L}{2\kappa^2}\int d^9x\,\det(e,\tau)ke^{-2\phi}\Big\{R(J,\omega) - 4\omega_{ab}\tau^{ab}$$
$$- \frac{1}{12}\mathcal{D}\hat{b}'_{abc}\mathcal{D}\hat{b}'^{abc} - \frac{1}{2}kF(m)_{ab}\mathcal{D}\hat{b}'^{0ab} + \frac{k^2}{2}F(m)_{a0}F(m)^a{}_0$$
$$+ 4\nabla_a\phi(\nabla^a\phi - \nabla^a\ln k) + \frac{1}{2}\nabla_a\ln k\nabla^a\ln k\Big\}. \tag{127}$$

Due to the presence of a bare boost connection term, this is no longer manifestly boost invariant.

D. Null Reduction of NS-NS gravity

D.1. Ansatz

We start from the Lagrangian of NS-NS gravity

$$S_{\text{NS-NS}} = \frac{L}{2\kappa^2}\int d^{10}xEe^{-2\Phi}\left\{R + 4\partial_{\hat{A}}\Phi\partial^{\hat{A}}\Phi - \frac{1}{12}H_{\hat{A}\hat{B}\hat{C}}H^{\hat{A}\hat{B}\hat{C}}\right\}.$$

The hatted Latin indices run from $0,...,D$ and take values in the full tangent space. We single out the lightlike isometry direction X^v and find adapted coordinates $(X^{\hat{\mu}}) = (X^\mu, X^v)$. Furthermore, we work in lightcone coordinates in tangent space, where we split $\hat{A} = (a,+,-)$, such that $\eta_{ab} = \delta_{ab}$ and $\eta_{+-} = -1$. This gives the ansatz for the metric as

$$(E^{\hat{\mu}}{}_{A}) = \begin{array}{c c} & \begin{array}{ccc} a & - & + \end{array} \\ \begin{array}{c} \mu \\ v \end{array} & \begin{pmatrix} \tilde{e}_\mu{}^a & s^{-1}\tilde{\tau}'_\mu & s\tilde{m}_\mu \\ 0 & 0 & s \end{pmatrix} \end{array}, \quad (E^{\hat{\mu}}{}_{A}) = \begin{array}{c c} & \begin{array}{cc} \mu & v \end{array} \\ \begin{array}{c} a \\ - \\ + \end{array} & \begin{pmatrix} \tilde{e}^\mu{}_a & -\tilde{e}^\mu{}_a\tilde{m}_\mu \\ s\tilde{\tau}'^\mu & -s\tilde{\tau}'^\mu\tilde{m}_\mu \\ 0 & s^{-1} \end{pmatrix} \end{array},$$

and corresponding variations as in section 5.4

$$\begin{aligned}
\delta\tilde{\tau}'_{\mu} &= 0\\
\delta\tilde{e}_{\mu}{}^{a} &= \tilde{\lambda}^{a}{}_{b}\tilde{e}_{\mu}{}^{b} + s^{-1}\tilde{\lambda}^{a}\tilde{\tau}'_{\mu},\\
\delta\tilde{m}_{\mu} &= \partial_{\mu}\xi^{v} + s^{-1}\tilde{\lambda}_{a}\tilde{e}_{\mu}{}^{a},\\
\delta s &= \tilde{\lambda}_{D} s,
\end{aligned}$$

D.2. Contractions

For two vectors V, W:

$$\eta_{\hat{A}\hat{B}} V^{\hat{A}} W^{\hat{B}} = \eta_{ab} V^{a} W^{b} - (V^{+}W^{-} - V^{-}W^{+}).$$

We introduce a notation that lets us distinguish between contractions with the full vielbein and the reduced vielbein

$$\begin{aligned}
&T_{A'} := E^{\hat{\mu}}{}_{a} T_{\hat{\mu}} = T_{a} - \tilde{m}_{a} T_{v},\\
&T_{-} = s(T_{0} - \tilde{m}_{0} T_{v}),\\
&T_{+} = \frac{1}{s} T_{v},
\end{aligned}$$

where

$$\begin{aligned}
T_{a} &:= \tilde{e}^{\mu}{}_{a} T_{\mu},\\
T_{0} &:= \tilde{\tau}'^{\mu} T_{\mu}.
\end{aligned}$$

D.3. The dilaton

The kinetic term for the dilation reduces to

$$\partial_{A}\Phi\partial^{A}\Phi = \partial_{a}\Phi\partial^{a}\Phi,$$

and we will also have to introduce a shift in the dilaton to make contact to the KK reduction

$$\Phi := \tilde{\phi} + \ln s \tag{128}$$

D.4. The Kalb Ramond field

First we split the KR field

$$\begin{aligned}
\tilde{\tilde{b}}'_{\mu\nu} &:= B_{\mu\nu},\\
\tilde{b}_{\mu} &:= B_{\mu v}.
\end{aligned}$$

Note that for $\tilde{b}$ the one-form symmetry reduces to a standard $U(1)$ symmetry, while for $\tilde{\tilde{b}}'$ the remainder of diffeomorphisms mixes it with $\tilde{b}$

$$\begin{aligned}\delta\tilde{b} &= \mathcal{L}_\xi \tilde{b} + d\theta_v \\ \delta\tilde{\tilde{b}}' &= \mathcal{L}_\xi \tilde{\tilde{b}}' + d\hat{\theta} - d\xi^v \wedge \tilde{b}\end{aligned}$$

where $\hat{\theta}$ is the remainder of the one-form symmetry. Note that $\tilde{\tilde{b}}'$ transforms exactly like the boost invariant two-form (100) we introduced during the KK reduction. Furthermore, we denoted the reduction of the KR field with $\tilde{b}$, as we cannot identify it with the reduced KR field from the KK reduction, due to the different variation under boosts. The kinetic term for the KR field then reduces to

$$\begin{aligned}H_{\hat{A}\hat{B}\hat{C}}H^{\hat{A}\hat{B}\hat{C}} =&(d\tilde{\tilde{b}}' - \tilde{m} \wedge F(\tilde{b}))_{abc}(d\tilde{\tilde{b}}' - \tilde{m} \wedge F(\tilde{b}))^{abc} \\ &+ 6F(\tilde{b})_{ab}\left(d\tilde{\tilde{b}}' - \tilde{m} \wedge F(\tilde{b})\right)^{ab0} - 6F(\tilde{b})_{a0}F(\tilde{b})^{a}{}_{0},\end{aligned}$$

with $F(\tilde{b}) = d\tilde{b}$.

D.5. The geometry

D.5.1. The connections

We solve Cartan's first structure equation

$$R_{\hat{\mu}\hat{\nu}}(P^{\hat{A}}) = 2E_{\hat{\mu}\hat{\nu}}{}^{\hat{A}} - 2{}_{[\hat{\mu}}{}^{\hat{A}c}E_{\nu]c} + 2\left(\Omega_{[\hat{\mu}}{}^{\hat{A}+}E_{\hat{\nu}]}{}^{-} + \Omega_{[\hat{\mu}}{}^{\hat{A}-}E_{\hat{\nu}]}{}^{+}\right),$$

where $E_{\hat{\mu}\hat{\nu}}{}^{\hat{A}} = \partial_{[\mu}E_{\nu]}{}^{\hat{A}}$ and Ω the full relativistic spin connections. We infer from this that

$$\begin{aligned}\Omega_v{}^{+-} &= 0, \\ \Omega_\mu{}^{+-} &= \tilde{\tau}'{}_{\mu 0} - \frac{1}{s}\partial_\mu s, \\ \Omega_v{}^{a-} &= 0, \\ \Omega_\mu{}^{a-} &= -\frac{1}{s}\tilde{\tau}'{}_\mu{}^a, \\ \Omega_v{}^{a+} &= -s\tilde{\tau}'{}_0{}^a, \\ \Omega_\mu{}^{a+} &= -s\tilde{\omega}'{}_\mu{}^a - s\tilde{m}_\mu\tilde{\tau}'{}_0{}^a, \\ \Omega_v{}^{ab} &= -\tilde{\tau}'^{ab}, \\ \Omega_\mu{}^{ab} &= \tilde{\omega}'{}_\mu{}^{ab} - \tilde{m}_\mu\tilde{\tau}'^{ab}.\end{aligned} \tag{129}$$

Note that this is always of the form $T_\mu \mapsto t_\mu + \tilde{m}_\mu T_v$, guaranteeing that it is covariant under $U(1)$ transformations along the lightlike direction.
We introduced the spin and boost connections $\tilde{\omega}'$ which are now exactly the TNC connections built of $\tilde{e}, \tilde{\tau}'$, and $\tilde{m}$ as in eq. (30).

It is important to Note that $\tilde{\omega}'{}_\mu{}^a$ covariantizes boosts with $\tilde{\lambda}^a/s$. Consequently, the variations of the connections are given by

$$\begin{aligned}
\delta\tilde{\omega}'{}_\mu{}^a &= \overset{\text{SO}}{\mathcal{D}}_\mu\left(\frac{\tilde{\lambda}^a}{s}\right) + \tilde{e}_{\mu c}2\frac{\tilde{\lambda}^{(c}\tilde{\tau}'^{a)}{}_0}{s} + \tilde{\lambda}^a{}_b\omega_\mu{}^b \\
\delta\tilde{\omega}'{}_\mu{}^{ab} &= \overset{\text{SO}}{\mathcal{D}}_\mu\tilde{\lambda}^{ab} + 2\frac{\tilde{\lambda}^{[a}\tilde{\tau}'{}_\mu{}^{b]}}{s} + \tilde{e}_{\mu c}\frac{\tilde{\lambda}^c}{s}\tilde{\tau}'^{ab}.
\end{aligned}$$

We have exactly the same curvatures as in normal TNC, as we could always just absorb s into $\tilde{\lambda}^a$.

D.5.2. The curvature

We expand the Riemann tensor as

$$R_{\hat{\mu}\hat{\nu}}{}^{\hat{A}\hat{B}} = 2\partial_{[\hat{\mu}}\Omega_{\hat{\nu}]}{}^{\hat{A}\hat{B}} + 2\Omega_{[\hat{\mu}}{}^{\hat{A}c}\Omega_{\hat{\nu}]}{}^{\hat{B}}{}_c - 2\Omega_{[\hat{\mu}}{}^{\hat{A}+}\Omega_{\hat{\nu}]}{}^{\hat{B}-} - 2\Omega_{[\hat{\mu}}{}^{\hat{A}-}\Omega_{\hat{\nu}]}{}^{\hat{B}+}.$$

The Ricci Scalar reads

$$\begin{aligned}
R &= -R_{\hat{A}\hat{B}}{}^{\hat{A}\hat{B}} \\
&= -\left(R_{ab}{}^{ab} - 2\tilde{m}_b R_{av}{}^{ab}\right) - 2s\left(R_{a0}{}^{a-} + \tilde{m}_a R_{0v}{}^{a-} - \tilde{m}_0 R_{av}{}^{a-}\right) \\
&\quad - 2\frac{1}{s}R_{av}{}^{a+} + 2R_{0v}{}^{+-}
\end{aligned}$$

The terms needed to express the Ricci scalar are

$$\begin{aligned}
R_{\mu v}{}^{+-} &= -\tilde{\tau}'{}_{\mu a}\tilde{\tau}'{}_0{}^a \\
R_{\mu v}{}^{a+} &= s\left(\mathcal{D}_\mu\tau^a_{\ 0} + \tau_{\mu 0}\tau^a_{\ 0}\right) \\
R_{av}{}^{a-} &= \frac{1}{s}\tilde{\tau}'{}_{\mu b}\tilde{\tau}'^{ab} \\
R_{\mu\nu}{}^{a-} &= -\frac{1}{s}\left[2\partial_{[\mu}\tilde{\tau}'{}_{\nu]}{}^a - 2\tilde{\omega}'{}_{[\mu}{}^{ac}\tilde{\tau}'{}_{\nu]c} + 2\tilde{m}_{[\mu}\tilde{\tau}'{}_{\nu]b}\tilde{\tau}'^{ab} + 2\tilde{\tau}'{}_{[\mu}{}^a\tilde{\tau}'{}_{\nu]0}\right] \\
R_{\mu z}{}^{ab} &= -\overset{\text{SO}}{\mathcal{D}}_\mu\tilde{\tau}'^{ab} - \tilde{m}_\mu 2\tilde{\tau}'^{[a}{}_c\tilde{\tau}'^{b]c} + 2\tilde{\tau}'{}_0{}^{[a}\tilde{\tau}'{}_\mu{}^{b]} \\
\hat{R}_{\mu\nu}{}^{ab} &= R_{\mu\nu}(J^{ab},\tilde{\omega}') + 2\tilde{\omega}'{}_{[\mu}{}^c\tilde{e}_{\nu]c}\tilde{\tau}'^{ab} - 2\tilde{m}_{\mu\nu}\tilde{\tau}'^{ab} - 2\overset{\text{SO}}{\mathcal{D}}_{[\mu}\tilde{\tau}'^{ab}\tilde{m}_{\nu]} - 4\tilde{m}_{[\mu}\tilde{\tau}'{}_0{}^{[a}\tilde{\tau}'{}_{\nu]}{}^{a]},
\end{aligned}$$

where $\hat{R}(J,\tilde{\omega}')$ corresponds to the TNC curvature (33), and we have the covariant derivative

$$\mathcal{D}_\mu\tilde{\tau}'^a{}_0 := \overset{\text{SO}}{\mathcal{D}}_\mu\tilde{\tau}'^a{}_0 + \tilde{\omega}'{}_{\mu b}\tilde{\tau}'^{ab}.$$

In total, the Ricci scalar reads

$$R = \hat{R}(J,\tilde{\omega}') - 4\mathcal{D}_a\tilde{\tau}'^a{}_0 - 6\tilde{\tau}'^a{}_0\tilde{\tau}'{}_{a0}.$$

D.5.3. Null Reduction of the Action of NS-NS gravity

So we arrive at the action

$$
\begin{aligned}
S_{\text{Null}} = \frac{L}{2\kappa^2}\int d^9x \det\left(\tilde{e}, \frac{1}{s}\tilde{\tau}'\right) e^{-2\Phi} s \Big\{ & \hat{R}(J, \tilde{\omega}') - \frac{1}{12}(d\tilde{\tilde{b}}' - \tilde{m}\wedge F(\tilde{b}))_{abc}(d\tilde{\tilde{b}}' - \tilde{m}\wedge F(\tilde{b}))^{abc} \\
& - \frac{1}{2}F(\tilde{b})_{ab}\left(d\tilde{\tilde{b}}' - \tilde{m}\wedge F(\tilde{b})\right)^{ab0} + \frac{1}{2}F(\tilde{b})_{a0}F(\tilde{b})^{a}{}_{0} \\
& + 4\partial_a\Phi\partial^a\Phi - 6\tilde{\tau}'^{a}{}_{0}\tilde{\tau}'_{a0} - 4\mathcal{D}_a\tilde{\tau}'^{a}{}_{0}\Big\}.
\end{aligned}
$$

It is hard to see, but the above is in fact invariant under boost (and rotations), whereas it is immediately clear that it is invariant under dilatations of s.

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