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Last but not least, it may sound stupid but I want to thank mathematics itself for enriching my life, by sparking my passion and bringing lots of joy as well as new friends along this miraculous journey up to this very moment.

Abstract

Historically the interest in rook- and file-numbers arose, when researchers tried to get a grip on the hit- and fit-numbers. Instead of following this approach, we will motivate the definitions to come, in a rather simplistic and playful fashion in chapter 1.2, using the game of chess and some basic graph theory. This will allow is to introduce the objects of interest one after another in part 2, instead of having to define them all at once, in order to understand their connections. Nevertheless those connections between them, discovered by big names from the past, will be very important and some of our main theorems in part 2. This part will start by examining the rook-numbers closer in section 2.1.1 and observing some important properties. Led by an example, we will encounter the hit-numbers and their connection to the rook-numbers in section 2.1.2. Afterwards, we will imitate our approach to the rook-numbers in order to investigate the file-numbers in section 2.2.1. Again led by an example, we will peal out their connection to the fit-numbers in section 2.2.2. We close the section on the classical theory with 2.3, a chapter on further recursion-formulae and generating functions for rook-, file-, hit- and fit-numbers.
Still, the rook- and file-numbers are very interesting objects on their own without having to consider the hit- and fit-numbers. One reason for that is, that they generalise some of the special combinatorial numbers, like for example the Stirling- or Lah-numbers as seen in the Examples 2.5 & 2.26. This is also one aspect the thesis will focus on in the later sections of part 3. I also chose this structure for the thesis, since there do not exist weighted analogues of the Hit- and fit-numbers in the most general setting. Since the idea of this thesis is generalising the basic theory, I thought that this path is the most promising one. In part 3 we will first discuss the concept of weights and the inherent weight hierarchy in chapter 3.1, before we introduce the weighted rook- and file-numbers in chapter 3.2. We manage to generalise nearly all of the results from the sections 2.1.1, 2.2.1, 2.1.3, 2.3.2 & 2.3.5 to the weighted setting in the chapters 3.3 & 3.4, thereby creating a new weighted rook- and file-theory, generalising earlier results of A. M. Garsia and J. B. Remmel from [8] as well as M. J. Schlosser and M. Yoo from [9]. This is similar in style to results of J. Küstner, M. J. Schlosser and M. Yoo on lattice paths as in [15], [16] and [10]. Last but not least, we explain some problems arising with weighted hit- and fit-numbers and give an outlook on further interesting questions in chapter 3.5.

Zusammenfassung

In der Geschichte der Mathematik entwickelte sich das Interesse an den rook- und file-Zahlen im Zuge der Untersuchungen der hit- und fit- Zahlen. Ich wähle allerdings einen anderen Ansatz. Im Kapitel 1.2 werden wir einen eher spielerischen Zugang, über Schach und einfache Graphentheorie, zu den rook- und file-Zahlen schaffen, der gleichzeitig auch Motivation für die folgenden Untersuchungen sein soll. Diese alternative Herangehensweise erlaubt es uns, die Objekte von Interesse im zweiten Teil 2 eines nach dem anderen einzuführen, anstatt sie alle geblockt zu definieren, wie es oftmals in der Literatur gemacht wird. Anschließend werden wir die Verbindungen dieser Objekte untersuchen und dabei auf einige Resultate von großen Mathematikern der Vergangenheit stoßen. Wir beginnen diesen Abschnitt mit Untersuchungen der rook-Zahlen und einiger ihrer Eigenschaften im Abschnitt 2.1.1. Geleitet von einem Beispiel, entdecken wir die hit-Zahlen und ihre Verbindung zu den rook-Zahlen im Abschnitt 2.1.2. Im Abschnitt 2.2.1 imitieren wir unsere Herangehensweise an die rook-Zahlen, um die file-Zahlen näher zu untersuchen. Im Abschnitt 2.2.2 wird uns erneut ein Beispiel zur Entdeckung der fit-Zahlen und ihrer Verbindung zu den file-Zahlen führen. Wir schließen den zweiten Teil dieser Arbeit mit 2.3, einem Kapitel über weitere Rekursionsformeln und erzeugende Funktionen der rook, file-, hit- und fit- Zahlen.
Trotz all dieser Verbindungen, sind die rook- und file-Zahlen auch interessante mathematische Objekte, ohne die hit- und fit-Zahlen zu betrachten. Ein Grund dafür ist, dass sie Verallgemeinerungen einiger kombinatorischer Zahlen, wie etwa der Stirling- oder Lah-Zahlen, darstellen. Wir werden dies in den Beispielen 2.5 & 2.26 genauer untersuchen.
Im dritten Teil der Arbeit 3, finden wir einen weiteren Grund dafür, dass ich diesen Ansatz für die ersten Teil gewählt habe. Es gibt nämlich noch keine gewichtete Verallgemeinerung der hit- und fit-Zahlen im allgemeinsten Sinn. Da das eigentliche Ziel dieses zweiten Abschnitts jedoch die Verallgemeinerung der Resultate des ersten Teils auf das gewichtete Level ist, dachte ich, dass dies die sinnvollste Herangehensweise sei. Im Kapitel 3.1 besprechen wir nun zunächst das Konzept der Gewichte und die Hierarchie dieser, bevor wir die gewichteten rook- und file-Zahlen im Abschnitt 3.2 einführen. In den beiden Abschnitten 3.3 & 3.4 gelingt es uns, nahezu alle Resultate aus den vorherigen Abschnitten 2.1.1, 2.2.1, 2.1.3, 2.3.2 & 2.3.5 auf den gewichteten Fall zu verallgemeinern. Wir erschaffen somit eine neue, gewichtete rook- und file-Theorie, im Stil ähnlich zu Resultaten von J. Küstner, M. J. Schlosser und M. Yoo zu Gitterpunktwegen, wie etwa in [15], [16] and [10]. Wir verallgemeinern damit frühere Resultate von A. M. Garsia und J. B. Remmel aus [8], sowie von M. J. Schlosser und M. Yoo aus [9]. Zu guter Letzt, versuchen wir uns an einer Erklärung der Schwierigkeiten bei der Verallgemeinerung von hit- und fit-Zahlen und geben einen Ausblick auf weitere interessante, offenen Probleme in Kapitel 3.5.

Contents

1 Introduction

1.1 Notation

Throughout this thesis I will always use *us* and *we* instead of *my* and *I*, as I think this makes the reading experience much better as one feels somewhat involved in the development and more addressed by the text. Furthermore we make use of the following notation:

- By RHS, LHS we abbreviate *right hand side* and *left hand side* respectively.
- By WLOG we abbreviate *without loss of generality.*
- $\mathbb{N} \overset{\text{def}}{=} \{1, 2, 3, ...\}$
- $\mathbb{N}_0 \overset{\text{def}}{=} \{0, 1, 2, 3, ...\}$
- $[n] \overset{\text{def}}{=} \{1, 2, ..., n\}$
- $[n]_0 \overset{\text{def}}{=} \{0, 1, 2, ..., n\}$
- By $\mathbb{Z}$ we denote the set of integers.
- By $\mathbb{C}$ we denote the set of complex numbers.
- $\mathbb{C}^* \overset{\text{def}}{=} \mathbb{C} \setminus \{0\}$ denotes the punctured complex plane.
- For a set S we denote 2^S its power set.
- We use the convention that $|\varnothing| \overset{\text{def}}{=} 0$.
- For $a_1, ..., a_n \in \mathbb{Z}$ not necessarily different, denote $\{\{a_1, ..., a_n\}\}$ the multiset containing $a_1, ..., a_n$.
- For $n, m \in \mathbb{N}$ we denote $[m]^{[n]}$ the set of functions from $[n]$ to $[m]$.
- For all $n \in \mathbb{N}$ we denote S_n the set of permutations of n elements.
- Let $n \in \mathbb{N}$. We denote by $!n$ the number of derangements in S_n, i.e. the number of elements $\sigma \in S_n$ without fixed points.
- Denote $\Gamma(z)$ the analytic continuation of $\int_0^\infty x^{z-1}e^{-x}dx$ to $\mathbb{C}\setminus\mathbb{Z}_{<0}$, the so called Euler-Gamma-Function.
- For an element $x \in \mathbb{C} \setminus \mathbb{Z}_{<-1}$ and $n \in \mathbb{C}$ such that $x - n + 1 \notin \mathbb{Z}_{<0}$ denote $(x)_n \overset{\text{def}}{=} \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$ the falling factorial. If $n \in \mathbb{N}$ this yields $(x)_n = \prod_{k=0}^{n-1}(x - k)$.
- For an element $x \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ and $n \in \mathbb{C}$ such that $x + n \notin \mathbb{Z}_{<0}$ denote $x^{(n)} \overset{\text{def}}{=} \frac{\Gamma(x+n)}{\Gamma(x)}$ the rising factorial. If $n \in \mathbb{N}$ this yields $x^{(n)} = \prod_{k=0}^{n-1}(x + k)$.
- For $n, k \in \mathbb{N}_0$ denote $S_{n,k}$ the Stirling-numbers of second kind, for example determined by $\sum_{k=0}^{n} S_{n,k}(x)_k = x^n$, counting the number of unordered set partitions of $[n]$ into k nonempty subsets.

- For $n, k, r \in \mathbb{N}_0$, with $n \geq r \geq 0$, denote $S_{n,k}^{(r)}$ the r-restricted Stirling-numbers of second kind, for example determined by $\sum_{k=0}^{n} S_{n,k}^{(r)}(x)_k = x^{n-r}(x)_r$ counting the number of unordered set partitions of $[n]$ into k nonempty subsets, where the numbers $1, ..., r$ lie in different subsets.

- For $n, k \in \mathbb{N}_0$ denote $s_{n,k}$ the signed Stirling-numbers of first kind, for example determined by $\sum_{k=0}^{n} s_{n,k}x^k = (x)_n$. We call $c_{n,k} = |s_{n,k}|$ the unsigned Stirling-numbers of first kind, counting for example the number of permutations $\sigma \in S_n$ with k disjoint cycles. These satisfy the recurrence relation $c_{n,k} = c_{n-1,k-1} + (n-1)c_{n-1,k}$ with initial conditions $c_{0,0} = 1$ and $c_{n,0} = c_{0,n} = 0$ for $n > 0$ or defined by $\sum_{k=0}^{n} c_{n,k}x^k = x^{(n)}$.

- For $n, k, r \in \mathbb{N}_0$, with $n \geq r \geq 0$, denote $s_{n,k}^{(r)}$ the signed, r-restricted Stirling-numbers of first kind, for example determined by $\sum_{k=0}^{n} s_{n,k}^{(r)}x^k = x^r(x+r)_{n-r}$. We also define the unsigned, r-restricted Stirling numbers of first kind $c_{n,k}^{(r)}$ by $\sum_{k=0}^{n} c_{n,k}^{(r)}x^k = x^r(x+r)^{(n-r)}$, counting the number of permutations $\sigma \in S_n$ with exactly k-cycles and where the elements $1, ..., r$ are in different cycles.

- For $n, k \in \mathbb{N}_0$ denote $L_{n,k}$ the unsigned Lah-numbers, for example determined by $\sum_{k=0}^{n} L_{n,k}(x)_k = x^{(n)}$, counting the number of partitions of of $[n]$ into k nonempty, linearly ordered subsets.

- For $n, k, r \in \mathbb{N}_0$, with $n \geq r \geq 0$, denote $L_{n,k}^{(r)}$ the unsigned, r-restricted Lah-numbers, for example determined by $\sum_{k=0}^{n+r-1} L_{n,k}^{(r)}(x)_k = (x+n-r)_{n+r-1}$, counting the number of partitions of of $[n]$ into k nonempty, linearly ordered subsets, where the numbers $1, ..., r$ are in distinct subsets.

- Liouville's theorem: Every bounded, entire function is constant.

- For an integral domain R we denote by $F(R)$ its field of fractions. For an integral polynomial ring $R[x]$ we denote its field of fractions by $R(x)$.

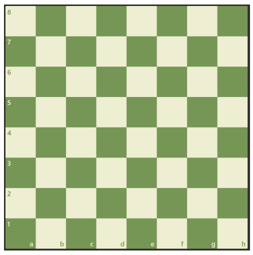
- The polynomial argument: Let $(R, +, \cdot)$ be a commutative, unitary, factorial integral domain, $p(x) \in R[x]$ a polynomial of degree $n \in \mathbb{N}$. If $p(x)$ has at least different $n+1$ zeroes then $p(x) \equiv 0$.

- Let $k \in \mathbb{N}$, $x_1, ..., x_k$ independent variables over $\mathbb{C}$ and $n \in [k]$ then we define the n-th complete homogeneous symmetric polynomial in k variables $h_n(k) \stackrel{\text{def}}{=} h_n(x_1, ..., x_k)$ by the recursion $h_n(k) = h_n(k-1) + x_k h_{n-1}(k)$ with initial conditions $h_0(k) = 1$ and $h_n(k) = 0$ for $n < 0$ or $n > k$.

- Let $k \in \mathbb{N}$, $x_1, ..., x_k$ independent variables over $\mathbb{C}$ and $n \in [k]$ then we define the n-th power sum symmetric polynomial in k-variables simply by $p_n(x_1, ..., x_k) \stackrel{\text{def}}{=} \sum_{j=1}^{k} x_j^n$.

1.2 Motivation

So let us start with a question motivated by the game of chess. How many ways are there, to place a number of rooks on a chess board, such that no two of them are attacking each other?



For the reader not familiar with the game of chess, we give a quick explanation. A chess board consists of two times 32 squares of different colors, classically black and white, forming the shape of an 8×8 square. Here on the right we are using two different shades of green for the squares as usual for school boards though. The game is played by two players, one of them controlling 16 white figures and the other one controlling 16 black figures. There are 6 different figures in the game of chess but for our purposes we are only interested in a figure called "rook".

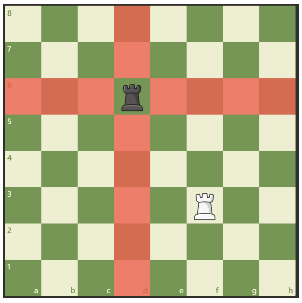


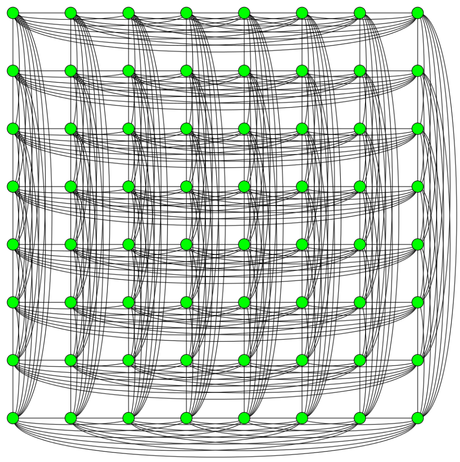
Figure 1: The squares attacked by the black rook.

The rook is a figure, that can move any number of squares vertically or horizontally. Since in chess figures of the opponent are captured, if one of our figures can move to a square with an opponents figure on it, we say that a figure attacks all squares to which it can move, also you cannot capture your own figures, hence you are only attacking figures of the other colour. As it usually is a disadvantage if your figures get captured, we would like to place our rook on a square not attacked by the enemy rook, as for example in the picture to the left. Then clearly our rook can not be attacking the enemy rook either. Now this simple question for a white rook and a black rook is easy to answer. We first place the black rook, for this there are 64 possibilities since we have no restriction for placing it on the board. This black rook now attacks 14 squares and occupies one square himself, hence there are 49 squares remaining, on which we may place the white rook. Now one can ask, how we could possible make this question a bit more interesting. There are two easy generalisations:

- What if we do not restrict ourselves to two colors for the rooks? Or what if we only consider one colour but we allow capturing our own rooks. This is obvious nonsense when thinking of chess but leads to some very interesting mathematics, as we will see shortly.
- What if we consider not only 8×8 boards? This gives rise to the question what a mathematically meaningful definition of a "board" could be.

For the reader with some background in graph theory, we can give another motivating question. Consider the (8×8) rooks-graph G_R with vertex set $V(G_R) = \{(i,j) \mid i,j \in [8]\}$ and an edge connecting (i_1, j_1) and (i_2, j_2) if either $i_1 = i_2$ or $j_1 = j_2$. So the vertices represent the squares of the 8×8 chess board and two squares are connected by an edge, if and only if they are either in the same row or in the same column. See the picture below. Now the question of how many ways there are, to place k non-attacking rooks on the 8×8 chess board, is equivalent to the question of how

Figure 2: The 8×8 rook-graph G_R from [2]



many sets of k-non adjacent vertices of this graph, i.e. independent sets also called matchings, there are. The generalizations hinted at above, are also applicable here, i.e. what if we consider the $[n] \times [m]$ rook-graph etc.

1.3 Boards

We will first answer the question regarding boards. For this consider the $\mathbb{N} \times \mathbb{Z}$-grid, where we label the columns from left to right and the rows from bottom to top. We identify the points (i, j) of the grid with the "boxes"/"squares" to the left and below each point, i.e. we obtain the following grid:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$\dots$
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$\dots$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$\dots$
$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$\dots$
$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$\dots$
$(1,-1)$	$(2,-1)$	$(3,-1)$	$(4,-1)$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Definition 1.1. We call the line, separating the boxes with positive second coordinate from the ones with non-positive coordinate, the ground.

Now given the above setup, we can make a meaningful definition of a "board":

Definition 1.2. Let $n, l \in \mathbb{N}_0$, with $l \leq n^2$. A set of cells $B = \{(i_1, j_1), ..., (i_l, j_l)\} \subset [n] \times [n]$ is called a board of size l.

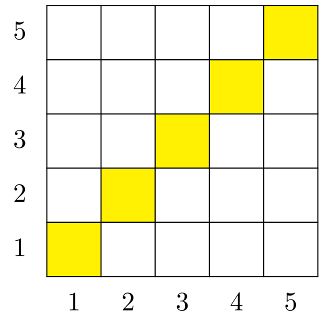
We will be particularly interested in boards of a "nice" form:

Definition 1.3. We call $B(b_1, ..., b_n) = \{(i, j) \mid 1 \leq i \leq n \;\&\; 1 \leq j \leq b_i\}$, i.e. a finite subset of the $\mathbb{N} \times \mathbb{Z}$ grid consisting of n columns, where the i-th column has height b_i, a skyline-board. Furthermore we call it Ferrers-board, if $b_i \leq b_{i+1}$ for all $i \in [n-1]$ and denote it by $F(b_1, ..., b_n)$.

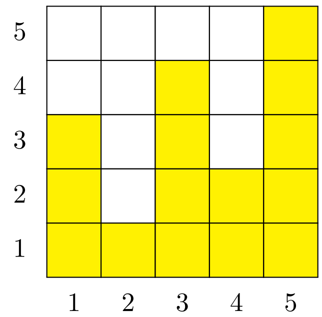
Remark 1.4. When we depict boards, then the boxes coloured in yellow will give the board, whereas the numbers to the right of the board and below it will always indicate the indices of the rows and columns respectively. Furthermore, we will abbreviate rectangular boards $B(m, ..., m)$ consisting of n rows of height m by $[n] \times [m]$.

Let us look at a few examples to convince ourselves that, this definition actually does what we want it to do.

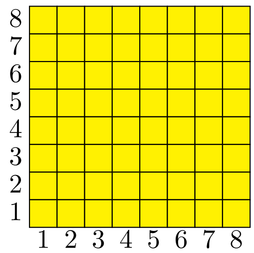
Example 1.5. First consider the board $B = \{(1,1),(2,2),(3,3),(4,4),(5,5)\} \subset [5] \times [5]$:



Also consider the sky-line board $B(3,1,4,2,5)$:



This hints at why it is called a skyline board. It looks like the New-Yorker skyline, doesn't it?
Now consider the board $[8] \times [8]$:

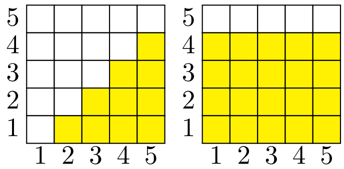


This clearly is our chess board from the motivational section, although colored fully in yellow. But for our question the colour of the squares did not matter anyways, so we can happily ignore that fact.

Now, the next two boards will appear many times throughout this thesis and have names attached to them, hence we introduce them as a definition.

Definition 1.6. For an integer $n \in \mathbb{N}$, we define the n-th Staircase-board by $St_n = F(0,1,...,n-1)$ and the n-th Laguerre-board by $L_n = [n] \times [n-1]$.

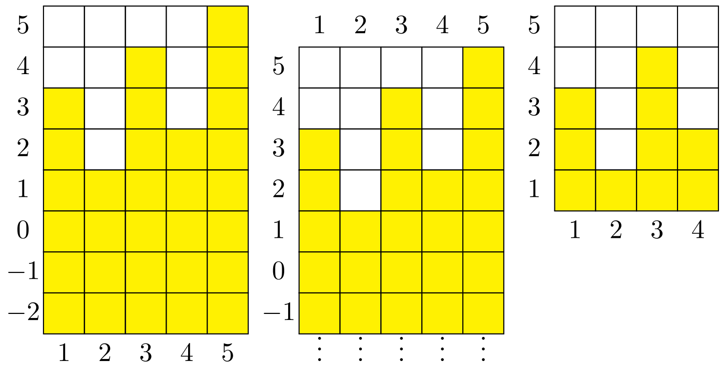
Example 1.7. Here the 5-th Staircase and Laguerre board:



Notation 1.8. Later on, it will be convenient to consider three types of boards constructed out of a skyline board $B = B(b_1, ..., b_n)$. If we attach to it a $[n] \times [k]$ board below the ground, we will denote this board by B_k and if we attach the infinite $[n] \times \mathbb{Z}_{\leq 0}$ board to it below the ground, we will denote this by B_∞. Furthermore we denote by B^- the board obtained from B, by deleting the last column. So we would end up with the following boards:

- $B_k \stackrel{\text{def}}{=} \{(i,j) \mid 1 \leq i \leq n \ \& \ -(k-1) \leq j \leq b_i\}$
- $B_\infty \stackrel{\text{def}}{=} \{(i,j) \mid 1 \leq i \leq n \ \& \ j \leq b_i\}$
- $B^- \stackrel{\text{def}}{=} \{(i,j) \mid 1 \leq i \leq n-1 \ \& \ 1 \leq j \leq b_i\}$

For our board $B = B(3,1,4,2,5)$ from above and $k = 3$ we would obtain the following three boards B_3, B_∞ & B^-:

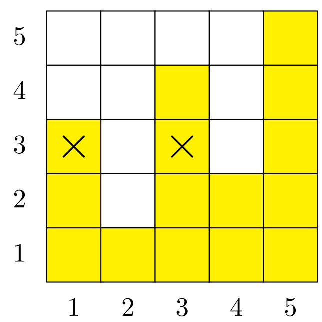


1.4 Rook placements

Now, that we have a mathematical object inheriting the spirit of the chess board, we have to come up with a generalisation of the rook-figure. Well, we can simply choose a square which we want to put the figure on and identify it with the rook. This way, all our rooks have the same colour and we allow ourselves to attack our own rooks. But we know how to get from an unlabelled model to a labelled, in this case coloured, one, so this is no restriction at all. Then, such a chosen square, attacks all squares in its row and in its column. This would lead to the following definition:

Given a board B, a non-attacking rook placement of k-rooks on B is a subset of k cells $P \subset B$, such that no two cells in P have the same first or the same second coordinate.

This hides a certain problem. We have not specified the type of board that B could be. If it were a skyline board and we would place rooks as below, marked by an $\times$, then in the above sense they would end up attacking each other...but can rooks really teleport over abysses?



This rather pseudo-philosophical question shall be our justification as to why one may consider it necessary, or at least interesting enough, to come up with the two following definitions:

Definition 1.9. Given a board B, a non-attacking rook placement of k-rooks on B is a subset of k cells $P \subset B$, such that no two cells in P have the same first or the same second coordinate.

This is exactly the definition from above, so in this case we allow our rooks to "teleport". In the second definition we restrict our rook movement to up and down in order of avoiding the teleportation problem.

Definition 1.10. Given a board B, a file-placement of k rooks on B is a subset of k cells $P \subset B$, such that no two cells in P have the same second coordinate.

Since one of our interests lies in counting such placements, it will come in handy to have the following notation available.

Notation 1.11. Let B be a board, then we will write

1. $\mathcal{N}_k(B)$ for the set of non-attacking rook placements of k rooks on B,

2. $r_k(B) \overset{\text{def}}{=} |\mathcal{N}_k(B)|$ and call this the k-th rook-number of the board B,

3. $\mathcal{F}_k(B)$ for the set of file placements of k rooks on B,

4. $f_k(B) \stackrel{\text{def}}{=} |\mathcal{F}_k(B)|$ and call this the <u>k-th file-number</u> of the board B.

2 Classical Rook Theory

In this chapter, we will discuss some of the results of the classical rook theory covered in [1]. Although, we will not just follow along the lines of [1], but rather follow our curiosity motivated by chapter 1.2 and discover the most important results along the way. The idea is, to develop some insight on the theory, before we get a grip on the weighted version, since, the proof-ideas in the weighted case are oftentimes closely related to the classical proofs. So the goal is, to develop some intuition how to tackle the proofs in the technically more involved, weighted case. Surely, there are many more interesting aspects that one could cover, but then the length of this thesis would get out of hand. We really only cover the parts, that we want to generalise to the weighted setting and some closely related topics to them. For a more thorough coverage of the topic take a look at [1] and the series of papers by J. R. Goldman, J. T. Joichi and D. E. White, starting with [20].

2.1 The rook- & hit-numbers

This section will treat the classical results on rook-numbers and hence we will naturally come across hit-numbers which we will define later on.

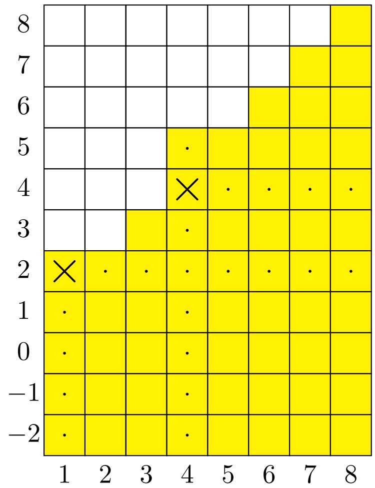
2.1.1 A recursive approach to rook-numbers

There lie some subtle problems in the structure of the board when it comes to certain ideas on how to mathematically handle rook placements. For Ferrers-boards this is a bit easier, so we will first focus on that case. As our interest lies in counting the number of placements of a given number of rooks on a Ferrers-board $F = F(b_1, ..., b_n)$, we should think of a way to construct such placements. The first basic observation is, that there has to be a rightmost rook on F, at least if we place more than 0 rooks. If this rook is in the, say, i-th column, we can remove it and obtain a placement of rooks one less than before on F. But how many ways where there to actually place this one rook in the i-th column? If we can answer this question, we can recursively construct a rook placement by adding rooks to the board from left to right. The next lemma actually gives us this desired answer.

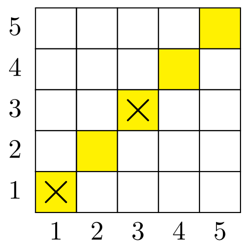
Lemma 2.1. Let $F = F(b_1, ..., b_n)$ be a Ferrers-board, $z \in \mathbb{N}_0$, $k \in [n]_0$, $P \in \mathcal{N}_k(F_z)$ a placement of k-rooks in the first $k \leq i - 1 < n$ columns of F_z, $\mathcal{P}(P)$ the set of placements obtained from P, by adding a rook in the i-th column of F_z, then we have that

$$|\mathcal{P}(P)| = b_i + z - k.$$

Proof. Note that F_z is a Ferrers-board, hence $b_i \geq b_{i-1} \geq ... \geq b_1$, thus every one of the k rooks placed in the first $i - 1$ columns attacks exactly one cell in the i-th column, precisely the one in its row. Since there are k rooks placed, there remain $b_i + z - k$ squares, non-attacked, to place the new rook as the i-th column of F_z consists of $b_i + z$ cells. For example, take a look at the placement of the two rooks below and say you want to add a third rook in one of the columns from 5 to 8, where the two rooks attack all of the cells marked by a dot. □



See that, in general, for an arbitrary board, the above lemma would simply be wrong, as the cells cancelled by the rooks already placed, need not lie on the board. Take for example the diagonal board given by $D = \{(i,i) \mid i \in [n]\} \subset [n] \times [n]$, then no matter where we placed the k rooks in the first $i-1$ columns, we always have exactly one choice for placing the $k+1$-th rook in the i-th column. Take for example $n = 5, k = 2, i = 4$ and the placement below:



The square in the 4-th column will never be attacked by the rooks placed to the left of it. But since this lemma will be the key to proving our next few statements, we get an idea on how problematic this obstacle, mentioned at the beginning, is.
The following theorem now formalizes our intuitive idea of constructing a placement by putting down the rooks one after another from left to right.

Theorem 2.2. Let $F = F(b_1, ..., b_n)$ be a Ferrers-board, $z \in \mathbb{N}_0$, $k \in [n]_0$ then

$$r_k(F_z) = r_k(F_z^-) + (b_n + z - (k-1))r_{k-1}(F_z^-),$$

with $r_0(\varnothing) = 1$, $r_k(F(b_1, ..., b_j)_z) = 0$ if $b_j + z < k$ and $r_k(F_z) = 0$ if $n < k$.

Proof. First it is clear, that there is no way to place k rooks on a board with n columns, in a way, such that there is at most one rook in each column if $n < k$. It is also clear, that there is no way to place k rooks on a board with b_j rows, in such a way, that there is at most one rook in each row if $b_j < k$. The other initial condition is just convention.
Now we come to the recursion. For this, we consider the partition of $\mathcal{N}_k(F_z)$ into the two sets $\{P \in \mathcal{N}_k(F_z) \mid \text{there is no rook in the } n\text{-th column of } F_z\}$ and $\{P \in \mathcal{N}_k(F_z) \mid \text{there is a rook in the } n\text{-th column of } F_z\}$. Then the first set is clearly in

bijection with $\mathcal{N}_k(F_z^-)$, simply by erasing the n-th column of F_z, giving the first term in the recurrence. The second term comes from the second set by using lemma 2.1. □

Now that we have constructed a recursion, we should go on a generating-function-hunt. As it turns out, the smartest way, is to not consider an ordinary or exponential generating function, but rather one in the basis of the falling factorials. The following theorem is due to Goldman, Joichi and White from 1975, see [20].

Theorem 2.3. (Goldman, Joichi and White)
Let $F = F(b_1, ..., b_n)$ be a Ferrers-board, then

$$\sum_{k=0}^{n} r_{n-k}(F)(x)_k = \prod_{i=1}^{n}(x + b_i - (i-1)), \tag{2.1}$$

as polynomial in x.

Proof. First of all note that both sides are actually polynomials of degree n in x, hence it suffices to check the identity for all $x \in \mathbb{N}$. We will do this via double counting. So let $x \in \mathbb{N}$ and consider the board F_x and the number of ways to place n rooks on it, then

1. the RHS is obtained by placing the rooks column after column. This follows directly by lemma 2.1 when applying it iteratively from left to right.
2. the LHS is obtained by first placing $n-k$ rooks on F and then placing the remaining k rooks below the ground, for every choice of $k \in [n]_0$. More formally, consider the partition $\dot{\bigcup}_{k=0}^{n} N_k = \mathcal{N}_n(F_x)$ where $N_k = \{P \in \mathcal{N}_n(F_x) \mid$ exactly $n-k$ rooks are on $F\}$. Then for each $k \in [n]$ we have that N_k is in bijection with $\mathcal{N}_{n-k}(F) \times \mathcal{N}_k([k] \times [x])$. Hence the statement follows by considering the cardinality of both sides and applying lemma 2.1 iteratively again. □

Example 2.4. Recall our question from the beginning: how many ways are there to place a number of rooks on a chess board, such that no two of them are attacking each other? By now we can answer this quite easily. With the help of our good friend *the computer algebra*, we can calculate the product on the RHS in theorem 2.3 and write it in the base of falling factorials. That is, for the Ferrrers-board $[8] \times [8]$ we have that:

$$\sum_{k=0}^{8} r_{8-k}([8] \times [8])(x)_k \overset{2.3}{=} \prod_{i=1}^{8}(x + 8 - (i-1)) = \prod_{i=1}^{8}(x + i) =$$
$$1(x)_8 + 64(x)_7 + 1568(x)_6 + 18816(x)_5 + 117600(x)_4 +$$
$$376320(x)_3 + 564480(x)_2 + 322560(x)_1 + 40320$$

So by comparing coefficients, we see that:

$$\begin{array}{ccc} r_0([8] \times [8]) = 1 & r_1([8] \times [8]) = 64 & r_2([8] \times [8]) = 1568 \\ r_3([8] \times [8]) = 18816 & r_4([8] \times [8]) = 117600 & r_5([8] \times [8]) = 376320 \\ r_6([8] \times [8]) = 564480 & r_7([8] \times [8]) = 322560 & r_8([8] \times [8]) = 40320 \end{array}$$

Example 2.5. To the reader with a more combinatorial background, the formula (2.1) may look suspiciously familiar. It is indeed a generalisation of two very well known formulae, as we will discuss now:

- Let $n \in \mathbb{N}_0$ and consider the Staircase-board St_n, as defined in 1.6, then we see that formula (2.1) becomes

$$\sum_{k=0}^{n} r_{n-k}(St_n)(x)_k = \prod_{i=1}^{n}(x + (i-1) - (i-1)) = x^n, \tag{2.2}$$

one of the defining relations for the Stirling-numbers of second kind $S_{n,k}$ and so $r_{n-k}(St_n) = S_{n,k}$ by the linear independence of $\{(x)_k \mid k \in \mathbb{N}_0\}$. We could also deduce this from theorem 2.2, since for St_n we would obtain, by setting $k = n - k$, that

$$r_{n-k}(St_n) = r_{n-k}(St_{n-1}) + (n - 1 - (n - k - 1))r_{n-k-1}(St_{n-1}),$$

with initial conditions $r_0(St_n) = 1$ and $r_{n-k}(St_n) = 0$ if $n - k < 0$. So setting $r_{n-k}(St_n) = R(n, k)$ we get the equivalent statement:

$$R(n, k) = R(n - 1, k - 1) + kR(n - 1, k)$$

with initial conditions $R(n, n) = 1$ and $R(n, k) = 0$ if $n < k$, which is exactly the defining recurrence of the Stirling-numbers of second kind.

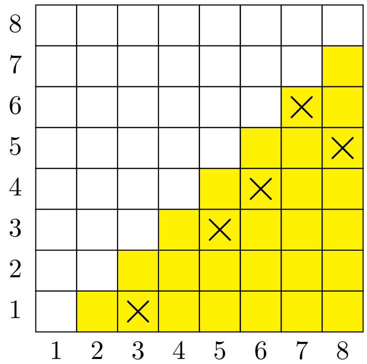
- Let then again $n \in \mathbb{N}_0$ and now consider the Laguerre-board L_n, as defined in 1.6, then formula (2.1) is just

$$\sum_{k=0}^{n} r_{n-k}(L_n)(x)_k = \prod_{i=1}^{n}(x + (n-1) - (i-1)) = \prod_{i=0}^{n-1}(x + i) = x^{(n)}, \tag{2.3}$$

the defining relation for the unsigned Lah-numbers $L_{n,k}$, hence again, by the linear independence of $\{(x)_k \mid k \in \mathbb{N}_0\}$, we obtain $L_{n,k} = r_{n-k}(L_n)$.

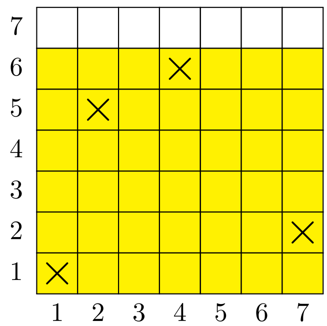
Note that for both of these equations there also exist bijective proofs. We give a sketch of these.

- (Sketch of the proof of equation (2.2))
For this denote $\mathcal{SP}_{n,k}$ the set of set partitions of $[n]$ into k parts. Then $S_{n,k} = |\mathcal{SP}_{n,k}|$ and we claim that there exists a bijection from $\mathcal{SP}_{n,k}$ to $\mathcal{N}_{n-k}(St_n)$.
This map takes an element $C = (C_1, ..., C_k) \in \mathcal{SP}_{n,k}$, where $C_1, ..., C_k$ are the blocks of C which WLOG, we ordered by their minimal element, i.e. $\min(C_1) < \min(C_2) < ... < \min(C_k)$, and maps it to a placement $P(C)$ defined as follows: for each $i \in [k]$ consider the block C_i then, if $|C_i| = 1$ we do nothing whereas if $|C_i = \{c_1^i < ... < c_{s_i}^i\}| = s_i > 1$ we place rooks on the cells $(c_1^i, c_2^i), (c_2^i, c_3^i), ..., (c_{s_i-1}^i, c_{s_i}^i)$. For example considering the following partition $(\{1, 3, 5, 8\}, \{2\}, \{4, 6, 7\},)$ we obtain the placement:



The inverse map is then given by taking a rook placement $P \in \mathcal{N}_{n-k}(St_n)$ and placing two numbers i & j in the same block of the partition if there is a rook on the cell (i,j). Numbers where there is neither a rook in the column nor the row indexed by that number will be placed in their own, single element blocks.

- (Sketch of the proof of equation (2.3))
For this denote $\mathcal{T}_{n,k}$ the set of placements of n labelled balls into k unlabelled, nonempty tubes, i.e. the order in which the balls are placed into the tubes matters. Then $L_{n,k} = |\mathcal{T}_{n,k}|$ and we claim, that there exists a bijection from $\mathcal{T}_{n,k}$ to $\mathcal{N}_{n-k}(L_n)$.
For this procedure consider such a placement Q of n labelled balls into k unlabelled, nonempty tubes and WLOG, order the tubes by increasing bottom elements, i.e. writing the tubes as tuples $T^1, ..., T^k$, we can assume that $T^1_1 < T^2_2 < ... < T^k_k$. Now we place rooks onto L_n in the following way: we start with $i = 1$ then if $|T^1| = 1$ we do nothing and leave the first row empty, whereas if $|T^1| > 1$ then we place a rook on the cell (t, T^1_{t+1}) for each $t \in [|T^1| - 1]$ and then we leave the $|T^1|$-th row empty. In general the tuples $T^1, ..., T^j$ will have determined the placements or non placements of rooks in the first $|T^1| + ... + |T^j|$- rows. Now if $|T^{j+1}| = 1$, we do nothing and leave the $|T^1| + ... + |T^j| + 1$-th row empty, whereas if $|T^{j+1}| > 1$ then we place a rook on the cell $(|T^1| + ... + |T^j| + t, T^{j+1}_{t+1})$ for all $t \in [|T^{j+1}| - 1]$ and then we leave the $|T^1| + ... + |T^j| + |T^{j+1}|$-th row empty. For example given the tubes $((3,1,6),(5),(7,2,4))$ we obtain the placement:



Also see, that for a suitable choice of boards, we can even obtain the r-restricted cases of the two special numbers above:

- Let now $n, r \in \mathbb{N}_0$, with $n \geq r \geq 0$, and consider a Staircase-board cut off at place r, i.e. $St_n^{(r)} = B(0, ..., 0, r, r+1, ..., n-1) \subset [n] \times [n]$ then we obtain from formula (2.1)

$$\sum_{k=0}^{n} r_{n-k}(St_n^{(r)})(x)_k = \prod_{i=1}^{r}(x-(i-1)) \prod_{i=r+1}^{n}(x+(i-1)-(i-1)) = (x)_r x^{n-r},$$

which is exactly one of the defining properties of the r- restricted Stirling-numbers of second kind and so again, by the linear independence of $\{(x)_k \mid k \in \mathbb{N}_0\}$ we obtain that $r_{n-k}(St_n^{(r)}) = S_{n,k}^{(r)}$.

- Let again $n, r \in \mathbb{N}_0$, with $n \geq r \geq 0$, and consider the board $L_n^{(r)} = [n+r-1] \times [n-r]$ then we obtain from formula (2.1)

$$\sum_{k=0}^{n+r-1} r_{n+r-1-k}(L_n^{(r)})(x)_k = \prod_{i=1}^{n+r-1}(x+n-r-(i-1)) = (x+n-r)_{n+r-1},$$

which is exactly one of the defining properties of the unsigned, r- restricted Lah-numbers and so again, by the linear independence of $\{(x)_k \mid k \in \mathbb{N}_0\}$ we obtain that $r_{n+r-1-k}(L_n^{(r)}) = L_{n,k}^{(r)}$.

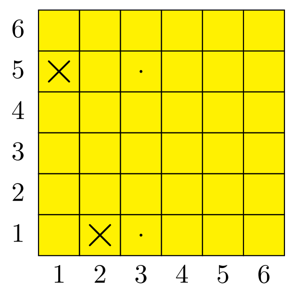
As in the classical case also these identities for the r-restricted versions can be proven bijectively. For those proofs we refer to [9].

2.1.2 Introduction of the hit-numbers

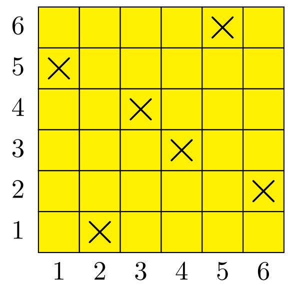
Let $n \in \mathbb{N}$ and consider the $[n] \times [n]$ board, then it is easy to see that

$$r_n([n] \times [n]) = n!.$$

Since we have to place exactly one rook in each column, we can just place the rooks column after column. But then the result is clear because for the first column we have n possible squares to place the first rook, for the second column we only have $n-1$ options since the rook in the first column cancels a cell from the second column and so on. See for example below, the two already placed rooks cancel two of the cells in the third column:



This gives rise to the natural question whether there is a bijection between $\mathcal{N}_n([n] \times [n])$ and the set of permutations S_n. Sometimes questions of this type are hard to answer but in this case it is really easy, since such a placement $P \in \mathcal{N}_n([n] \times [n])$ can directly be interpreted as the graph of an element $\sigma \in S_n$. See for example the placement below, which corresponds to $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 4 & 3 & 6 & 2 \end{pmatrix} \in S_6$.



We denote such a placement corresponding to a permutation $\sigma \in S_n$ by P_σ.
We quickly recall the following terminology:

Definition 2.6. Let $n \in \mathbb{N}$. A derangement in S_n is an element $\sigma \in S_n$ without fixed points.

Now in terms of our interpretation of permutations above, we obtain the following equivalence:

Lemma 2.7. Let $n \in \mathbb{N}$, then $\sigma \in S_n$ is a derangement in S_n if and only if $P_\sigma \cap \{(i,i) \mid i \in [n]\} = \varnothing$.

So to obtain a derangement, we want to avoid a certain board, given by $\{(i,i) \mid i \in [n]\}$, inside $[n] \times [n]$.
Note that this is actually the first time we are talking about non-Ferrers-boards. Up till now one might have thought about the general notion of a board as rather useless, but we will shortly see that is not only of great use but also really necessary for our next results!
But let us continue with our thought from above. We just saw, that we can encode the number of fixed points of a permutation $\sigma \in S_n$ by the number of rooks on the board $D = \{(i,i) \mid i \in [n]\}$ in P_σ. Now observe, that we can clearly partition S_n into the sets $(O_k^n)_{k \in [n]_0}$ where $O_k^n \overset{\text{def}}{=} \{\sigma \in S_n \mid \sigma \text{ has } k \text{ fixed points}\}$. So in particular O_0^n is the number of derangements of S_n. But this induces a partition of $\mathcal{N}_n([n] \times [n])$ into the sets $(Q_k^n)_{k \in [n]_0}$ where $Q_k^n \overset{\text{def}}{=} \{P \in \mathcal{N}_n([n] \times [n]) \mid |P_\sigma \cap D| = k\}$, the images of the $(O_k^n)_{k \in [n]_0}$ under the bijection from above.
This gives us a structural decomposition of our placements $P \in Q_k^n$, since such an element decomposes into the following two parts:

- a placement $P_1 \in \mathcal{N}_k(D)$ and,
- a placement P_2 of $n-k$ remaining rooks on $[n] \times [n] \setminus D$, such that there is exactly one rook in each row and column when looking at $P_1 \cup P_2$ on $[n] \times [n]$.

But we know how to calculate the number of these placements:

- we just have $r_k(D) = \binom{n}{k}$ as we choose k out of the n squares to place rooks on,
- in the second case we want to calculate $|\{P \in \mathcal{N}_{n-k}([n-k] \times [n-k]) \mid P_\sigma \cap \{(j,j) \mid j \in [n-k]\} = \varnothing\}|$. Give it a little thought. We placed k-rooks on the diagonal in the first step, each of those cancels its column and its cell, so we remain with a board of size $[n-k] \times [n-k]$ where the diagonal corresponds to the $n-k$ empty diagonal squares from the initial board. But this is just the number of

derangements in S_{n-k} by the lemma from before. This is well known to be equal to

$$!(n-k) = (n-k)! \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}.$$

Hence we see that

$$|Q_k^n| = \binom{n}{k}(n-k)! \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} = \sum_{j=k}^{n} \frac{(-1)^{n-j} n!}{(n-j)!k!} = \sum_{j=k}^{n} (-1)^{n-j} \binom{j}{k}\binom{n}{j}(j-i)!.$$

And more importantly we obtain the following equalities for generating functions:

Proposition 2.8. For $n \in \mathbb{N}$ we have that

$$\sum_{k=0}^{n} |O_k^n| x^k = \sum_{k=0}^{n} |Q_k^n| x^k = \sum_{k=0}^{n} r_k(D)(!(n-k))x^k,$$

which in particular reduces to the following nice and well known summation formula, upon setting $x = 1$:

Corollary 2.9. For $n \in \mathbb{N}$ we have that

$$n! = \sum_{k=0}^{n} \binom{n}{k}(!(n-k)).$$

So we see that we can connect the rook-numbers of the board D, to the numbers of permutations with a certain number of fixed points. What if we are no-more interested in the number of fixed points but some other properties? Say we want to know the number of permutations with a given number of j-excedances in S_n. Where we quickly recall the following definition:

Definition 2.10. Let $n \in \mathbb{N}$, $j \in \mathbb{N}_0$ with $j < n$, $\sigma = \begin{pmatrix} 1 & \dots & n \\ \sigma_1 & \dots & \sigma_n \end{pmatrix} \in S_n$. A pair (i, σ_i) for $i \in [n]$ is called j-excedance pair and i is called j-excedance if $\sigma_i - i = j$. We denote $Exc_j(\sigma)$ the set of j-excedances and $Exc(\sigma) = \{i \in [n] \mid \sigma_i > i\}$ the set of all excedances of σ which are not fixed points.

In particular, we obtain for $j = 0$ the number of fixed points of permutations. Note that the spelling in this case is actually *excedance* and not *exceedance* as one would guess. This is due to the originator of that term, Richard Stanley. He said that the double "e" just did not feel right and wasn't to his liking, as he explains in [3]. But coming back to mathematics, it is appealing to rewrite this condition again in terms of rook placements on the $[n] \times [n]$ board, as this was leading to some interesting results earlier.

Lemma 2.11. Let $n \in \mathbb{N}$, $j \in [n-1]_0$, $m \in [n]_0$, then $\sigma \in S_n$ has exactly m j-excedances if and only if $|P_\sigma \cap \{(i, i+j) \mid i \in [n-j]\}| = m$.

This again easily follows when looking at the graph of σ, given by P_σ. Now we could follow the same procedure as in the fixed point problem above, but before we do that, let me ask you a question. Why should we restrain ourselves to such nice looking boards? What if we ask for the number of placements $P \in \mathcal{N}_n([n] \times [n])$, such that $|P \cap \{(3,5),(3,7),(6,8)\}| = m$ or rather $|P \cap B| = m$, for some arbitrary board $B \subset [n] \times [n]$, $n \geq 8$, $0 \leq m \leq n$. Which in the first case would correspond to the permutations $\sigma \in S_n$ with the following properties:

- for $m = 0$ we would have that 3 would neither get mapped to 5 nor to 7 and 6 does not get mapped to 8,
- for $m = 1$ we would have that either 3 gets mapped to 5 or to 7 or 6 gets mapped to 8 but not both,
- for for $m = 2$ we would have that 3 gets mapped to 5 or to 7 and 6 gets mapped to 8,
- and for $m > 2$ there would be no such permutations at all due to the shape of the board.

So we see, that this concept makes sense for arbitrary boards $B \subset [n] \times [n]$. Interested in generalizing as far as possible, we make the following definition.

Definition 2.12. Let $n, k \in \mathbb{N}$, with $k \leq n$, $B \subset [n] \times [n]$ a board, then

$$H_{k,n}(B) \stackrel{\text{def}}{=} |\{\sigma \in S_n \mid |P_\sigma \cap B| = k\}|$$

is called the k-hit-number of the board B. For $\sigma \in S_n$ an element of $P_\sigma \cap B$ is called a hit of the board B.

So in our calculations before, we should have written $|O_k^n| = H_{k,n}(D)$, using our new notation. Our most powerful result deduced until now was by far proposition 2.8, so if we can generalise this, then all the other results will follow along, since those were just corollaries of proposition 2.8. Let us state this again, using our new notation:

Proposition 3.9*. For $n \in \mathbb{N}$ we have that

$$\sum_{k=0}^{n} H_{k,n}(D)x^k = \sum_{k=0}^{n} r_k(D)(!(n-k))x^k.$$

As in general it might not be so easy to calculate the number of placements of the $n-k$ remaining rooks on $[n] \times [n] \setminus B$ as in the case of derangements, we have to come up with a clever trick. The following is due to Irving Kaplansky and John Riordan from 1946, see [19].

Theorem 2.13. (Kaplansky, Riordan)
Let $n \in \mathbb{N}$, $B \subset [n] \times [n]$ a board, then

$$\sum_{k=0}^{n} H_{k,n}(B)x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k, \tag{2.4}$$

or equivalently upon multiplying out the monomial $(x-1)^k$

$$\sum_{k=0}^{n} H_{k,n}(B)x^k = \sum_{k=0}^{n} \left(\sum_{j=0}^{n} (-1)^{j-k} r_j(B)(n-j)! \binom{j}{k} \right) x^k \tag{2.5}$$

as generating functions in x. Or equivalently upon setting $y = x - 1$:

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{n}\binom{j}{k}H_{j,n}(B)\right)y^k = \sum_{k=0}^{n} r_k(B)(n-k)!y^k \tag{2.6}$$

as generating functions in y.

Proof. We will prove formula (2.6). It thus suffices to check, that for all $k \in [n]_0$ we have that $\sum_{j=0}^{n}\binom{j}{k}H_{j,n}(B) = r_k(B)(n-k)!$.
We will first give a combinatorial proof in words and then a "formalisation", or rather more formulae oriented version, of that exact same proof.
Both sides can be interpreted as the number of ways to choose a placement of n rooks on $[n] \times [n]$ with j elements on B, at least k, and choosing k of those j elements and color them red. The LHS does this by first fixing a $k \leq j \leq n$ and then choosing the placement in $H_{j,n}(B)$ ways, after which we choose k of the j rooks to be colored red in $\binom{j}{k}$ ways. Whereas the RHS first places the k red rooks on B in $r_k(B)$ ways and then places the other $n-k$ rooks onto some of the $[n-k] \times [n-k]$ legal spots remaining of $[n] \times [n]$ in $(n-k)!$ ways.
Now for the more symbolical proof recall, that our claim is somewhat similar to the statement in proposition 2.8. This time instead of considering the partition $S_n = \dot{\bigcup}_{i=0}^{n}\{\sigma \in S_n \mid |P_\sigma \cap B| = i\}$, we now consider for a fixed $k \in [n]_0$ the partition

$$\begin{gathered}\{(\sigma, T) \in S_n \times 2^{[n]} \mid T \subset [|P_\sigma \cap B|] \text{ with } |T| = k\} = \\ \dot{\bigcup}_{j=k}^{n}\{(\sigma, T) \mid |P_\sigma \cap B| = j \;\&\; T \subset [|P_\sigma \cap B|] \text{ with } |T| = k\} \cong \\ \dot{\bigcup}_{j=k}^{n}\{(\sigma, T) \mid |P_\sigma \cap B| = j \;\&\; T \subset [j] \text{ with } |T| = k\},\end{gathered}$$

where the bijection follows by linear projection of $[|P_\sigma \cap B|]$ onto $[j]$. Now evaluating $|\{(\sigma, T) \in S_n \times 2^{[n]} \mid T \subset [|P_\sigma \cap B|] \text{ with } |T| = k\}|$ in two different ways will yield our result.

1. Consider $\sum_{j=k}^{n} |\{(\sigma, T) \mid |P_\sigma \cap B| = j \;\&\; T \subset [j] \text{ with } |T| = k\}|$ then for each $j \in [n]$ with $j \geq k$ we have that

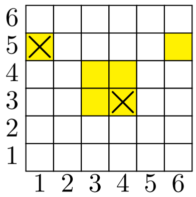
$$\begin{gathered}\{(\sigma, T) \mid |P_\sigma \cap B| = j \;\&\; T \subset [j] \text{ with } |T| = k\} \cong \\ \{\sigma \in S_n \mid |P_\sigma \cap B| = j\} \times \{T \subset [j] \mid |T| = k\}\end{gathered}$$

bijectively as sets. But by definition of the hit-numbers and the binomial coefficient we now know that then

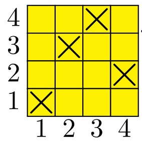
$$\begin{gathered}|\{(\sigma, T) \in S_n \times 2^{[n]} \mid T \subset [|P_\sigma \cap B|] \text{ with } |T| = k\}| = \\ \sum_{j=k}^{n} |\{(\sigma, T) \mid |P_\sigma \cap B| = j \;\&\; T \subset [j] \text{ with } |T| = k\}| = \\ \sum_{j=k}^{n} |\{\sigma \in S_n \mid |P_\sigma \cap B| = j\} \times \{T \subset [j] \mid |T| = k\}| = \sum_{j=k}^{n} H_{j,n}(B)\binom{j}{k},\end{gathered}$$

which is our desired LHS.

2. We can also consider the bijection from $\{(\sigma, T) \in S_n \times 2^{[n]} \mid T \subset [|P_\sigma \cap B|] \text{ with } |T| = k\}$ to $\mathcal{N}_k(B) \times S_{n-k}$, taking an element (σ, T) and mapping it to the element given by the placement $P \in \mathcal{N}_k(B)$, obtained when erasing all the rooks from P_σ not on B and not being indexed by an element of T, when counting the rooks on B from left to right, times the permutation $\pi \in S_{n-k}$, obtained from P_σ, when erasing all the rooks kept in the first step and the cells they are cancelling too. For example consider the board $B = \{(1,5), (3,3), (3,4), (4,3), (4,4), (6,5)\} \subset [6] \times [6]$ then the following pair $(\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 4 & 3 & 6 & 2 \end{pmatrix}, \{1,3\})$ gets mapped to the pair, given by:

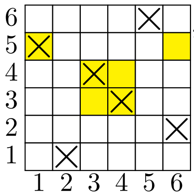


and



.

Recall that the rook-placement corresponding to σ is given by



,

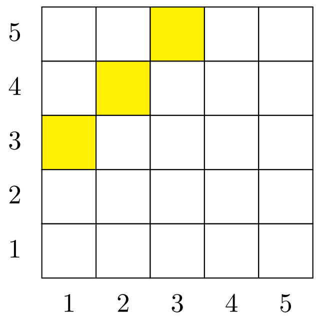
where we already highlighted the board B in yellow.
The inverse map is given by taking a pair $(P, \pi) \in \mathcal{N}_k(B) \times S_{n-k}$ and mapping it to the permutation $\sigma \in S_n$ corresponding to the rook placement constructed by inserting the rooks of P_π into P, when only considering the board obtained after cancelling all the cells attacked by the rooks of P, times the set of column indices of the rooks of P, when only counting the columns with non-empty intersection with B.
But the we know by definition of the rook-numbers that

$$|\{(\sigma, T) \in S_n \times 2^{[n]} \mid T \subset [|P_\sigma \cap B|] \text{ with } |T| = k\}| = \\ |\mathcal{N}_k(B) \times S_{n-k}| = r_k(B)(n-k)!,$$

as desired. □

So theorem 2.13 tells us, that as soon as we can calculate either the hit-numbers of a board or the rook-numbers of that exact same board, we also get the others. This also allows us to answer our last question from our considerations in the beginning.

Example 2.14. We wanted to find the number of permutations with a certain number of j-excedances. Recall that in lemma 2.11 we connected this to a condition in terms of hits of a certain board given by $EX_{n,j} = \{(i, i+j) \mid i \in [n-j]\}$. Take for example $n = 5, j = 2$ to obtain:



But it is just as easy as for the diagonal board to give the rook-number of this, since clearly $r_k(EX_{n,j}) = \binom{n-j}{k}$. Therefore by theorem 2.13, more concretely formula (2.5), we obtain:

$$|\{\sigma \in S_n \mid \sigma \text{ has } k\ j\text{-excedances}\}| = H_{k,n}(EX_{n,j}) = \sum_{s=0}^{n} (-1)^{s-k} \binom{n-j}{s} (n-s)! \binom{s}{k}$$

Example 2.15. Now what if we want to specify not only the number of k-excedances, for some $k \in \mathbb{N}$, but rather the number of all excedances? Well the following lemma again follows easily by looking at the rook placement corresponding to the permutation.

Lemma 2.16. *Let $n \in \mathbb{N}$ then $\sigma \in S_n$ has exactly m excedances, this time also counting fixed points, if and only if $|P_\sigma \cap St_n| = n - m$.*

Therefore we see that the number of permutations in S_n, for some $n \in \mathbb{N}$, with exactly m excedances, also counting fixed points, is given by $H_{n-m,n}(St_n)$. Fortunately, we can even calculate these hit-numbers! For this recall that in example 2.5 we had seen that $r_{n-k}(St_n) = S_{n,k}$ so using formula (2.5) we get that

$$H_{n-m,n}(St_n) = \sum_{j=0}^{n} (-1)^{j+m-n} r_j(St_n)(n-j)! \binom{j}{n-m},$$

hence reversing the order of summation yields

$$|\{\sigma \in S_n \mid \sigma \text{ has m excedances}\}| = H_{n-m,n}(St_n) = \sum_{j=0}^{n} (-1)^{m-j} S_{n,j} j! \binom{n-j}{n-m},$$

a well known formula.

2.1.3 An interlude on an equivalence relation for boards

In this section we will try to classify the set of boards contained in $[n] \times [n]$, for some $n \in \mathbb{N}$, in a meaningful way.
Given a certain board $B \subset [n] \times [n]$, for some $n \in \mathbb{N}$, the main objects of our studies so far have been the rook-numbers $(r_k(B))_{k=0}^n$. Those clearly are uniquely determined by the board B, but does the converse also hold? I.e. given a sequence $(r_k)_{k=0}^n$ is there a unique board B such that $r_k = r_k(B)$ for all $k \in [n]_0$? This is a natural

question to ask so lets us investigate a bit closer.
First of all, we should probably ask ourselves, whether the existence of board B with $r_k = r_k(B)$ for all $k \in [n]_0$ is even a given fact, before we can dive into the realms of uniqueness. Sadly enough this does not hold in general. Take for example any sequence of the form $(1, a, 0, b, ...)$ with $b > a > 0$. If there would exist such a board then it would consist of exactly a cells and all of them would have to be in the same row or column, as otherwise there would at least be 1 way to place two rooks on it, but then we clearly cannot place three rooks in a non-attacking way on it either. So we see, that in the general case there are some problems. Maybe it is a good idea to focus on Ferrers-boards as in the beginning, where things got easier.
But if there does not even exist an arbitrary board in $[n] \times [n]$ for certain sequences there surely does not exist a Ferrers-board either. In this case we can, at least, identify the sequences for which we can construct a Ferrers-board quite easily. For this recall theorem 2.2. Now the following lemma won't be a surprise anymore:

Lemma 2.17. Let $n \in \mathbb{N}$, $(r_k)_{k=0}^n \subset \mathbb{N}_0$ then the following two are equivalent:

1. There exists a Ferrers-board $F = F(b_1, ..., b_n) \subset [n] \times [n]$ such that $r_k = r_k(F)$ for all $k \in [n]_0$.

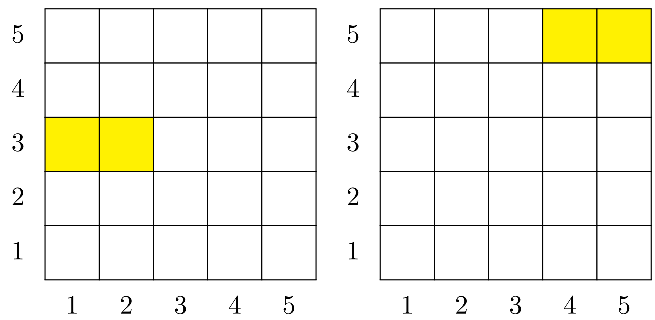
2. There exist numbers $c_1 \leq c_2 \leq ... \leq c_n \in [n]$ such that $r_k = r_{k,n}$ for all $k \in [n]_0$. Where $(r_{k,j})_{k,j=0}^n$ is a sequence defined recursively by

$$r_{k,j} = r_{k,j-1} + (c_j - (k-1))r_{k-1,j-1},$$

   with initial conditions $r_{0,j} = 1$ for all $j \in [n]_0$, $r_{k,j} = 0$ if $k > c_j$ and $r_{k,j} = 0$ if $k > j$.

Proof. First note that the recursion given in 2. is actually well defined and determines a sequence of non-negative integers $(r_{k,j})_{k,j=0}^n$ since by the initial conditions we are only adding non-negative terms to each other.
<u>1. $\Rightarrow$ 2.</u>: This is the exact statement of theorem 2.2 when setting $c_j = b_j$ for all $j \in [n]_0$ and denoting $r_k(F(b_1, ..., b_j)) = r_{k,j}$.
<u>2. $\Rightarrow$ 1.</u>: Consider the board $F = F(c_1, ..., c_n)$, then this clearly is a Ferrers-board and again by theorem 2.2 we obtain that $r_k = r_{k,n} = r_k(F)$. □

So now that we have answered the question of existence, at least in the case of Ferrers-boards, we can try to handle the uniqueness. The problem here is that uniqueness is quite obviously not given. Just consider the following two boards:



They both have the rook-numbers $(1, 2, 0, 0, 0, 0)$ which is quite clear if you give it a short thought. But maybe we can classify the set of boards that share the same sequence of rook-numbers? Let us formulate that concept more thoroughly.

Definition 2.18. Let $B, B' \subset [n] \times [n]$ be two boards then we call B and B' rook-equivalent, denoted by $B \overset{R}{\sim} B'$, if $r_k(B) = r_k(B')$ for all $k \in [n]$. Note that this is an equivalence relation on the set of boards contained in $[n] \times [n]$.

Now we would like to give an actual description of the equivalence classes of this relation restricted to the set of Ferrers-boards, as that would answer our question from before. The next theorem does exactly that.

Theorem 2.19. Let $F = F(b_1, ..., b_n), F' = F(b'_1, ..., b'_n) \subset [n] \times [n]$ be two Ferrers-boards then the following are equivalent:

1. $F \overset{R}{\sim} F'$
2. $\{\{b_1, b_2 - 1, ..., b_n - (n-1)\}\} = \{\{b'_1, b'_2 - 1, ..., b'_n - (n-1)\}\}$ as multisets.

Proof. By definition $F \overset{R}{\sim} F'$ if and only if $r_k(F) = r_k(F')$ for all $k \in [n]$, but by theorem 2.3 we know that this is the case if and only if $\prod_{i=1}^{n}(x + b_i - (i-1)) = \prod_{i=1}^{n}(x + b'_i - (i-1))$. But this in turn holds if and only if $\{\{b_1, b_2 - 1, ..., b_n - (n-1)\}\} = \{\{b'_1, b'_2 - 1, ..., b'_n - (n-1)\}\}$ as multisets. Hence we are done. □

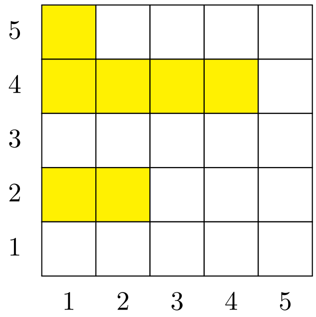
So at least for Ferrers-boards we were able to answer our initial question completely.

Remark 2.20. Note that for lemma 2.17, definition 2.18 and theorem 2.19 the condition of the board being contained in $[n] \times [n]$ is not necessary as it was nowhere used in the proofs of theorem 2.2 and theorem 2.3. We still kept it around to not deviate too much from the motivating question. Nevertheless this is still worth mentioning explicitly!

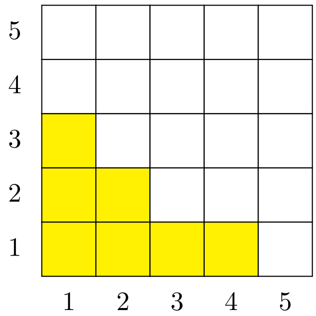
Now recall that in our approach to handling the rook-numbers of general boards contained in $[n] \times [n]$, we came across the hit-numbers, which established a connection between the rather well known set S_n, some of its properties and the rook-numbers. So even though we might not be able to give an existence theorem in the general case, we could still try to use this connection, in the form of theorem 2.13, to try and give a classification of the existing boards in terms of their rook-numbers. For this the next lemma will be quite useful:

Lemma 2.21. Let $n \in \mathbb{N}$, $B \subset [n] \times [n]$ a board, $B' \subset [n] \times [n]$ obtained from B by permuting the rows and columns of $[n] \times [n]$. Then $H_{k,n}(B) = H_{k,n}(B')$ for all $k \in [n]_0$.

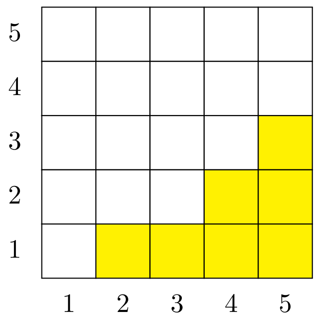
Proof. First note, that we can WLOG assume, that we first permute the rows and then permute the columns, just like in the case of matrices when showing independence of the determinant up to sign under permutations of rows and columns. Now let $\sigma_c, \sigma_r \in S_n$ be permutations of the columns and rows of B to obtain B' respectively and $\sigma \in S_n$ with k hits on B. Then we claim that $\sigma_r \circ \sigma \circ \sigma_c^{-1}$ is an element of S_n with k hits on B'. If this holds we are done, since concatenation from the left and right are isomorphisms of S_n, in particular bijections, hence also $\sigma_r \circ - \circ \sigma_c^{-1}$ is and by the claim it preserves the hit statistic.
It remains to prove the claim. For this let us first consider an example. Take the board:



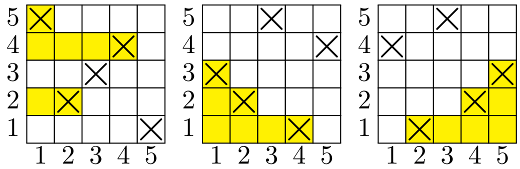
Then by first permuting the rows via $\sigma_r = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$ we obtain the board:



And by permuting the columns via $\sigma_c = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ we even obtain a Ferrers-board:



Now consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix}$, then we get that $\sigma_r \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix}$ and $\sigma_r \circ \sigma \circ \sigma_c^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$, drawing them on the boards, we get the hits of σ on the initial board, $\sigma_r \circ \sigma$ on the board after permuting the rows, $\sigma_r \circ \sigma \circ \sigma_c^{-1}$ on the final Ferrers-board from left to right:



Now it is intuitively clear how that works, the conjugation of σ by σ_r, σ_c^{-1} ensures that our rooks determined by σ get permuted together with the rows and columns of the board. It is as if they were glued onto the board!
So let us transform our intuition into a precise proof. For this let $\sigma \in S_n$ with k hits on B. Let $(i,j) \in B$ be one of them, i.e. $j = \sigma(i)$.
Now what does permutation of the rows and columns of $[n] \times [n]$ by σ_r, σ_c respectively, mean? Well it tells us that an element $(s,t) \in [n] \times [n]$ gets mapped to the element $(\sigma_c(s), \sigma_r(t)) \in [n] \times [n]$, so in particular our board $B = \{(s_l, t_l) \mid l = 1, ..., m\}$, for a suitable $m \in [n^2]$, gets mapped to the board $B' = \{(\sigma_c(s_l), \sigma_r(t_l)) \mid l = 1, ..., m\}$ bijectively.
So we would like to check that our hit $(i, \sigma(i)) \in B$ gets mapped to a hit $(i', \sigma_r \circ \sigma \circ \sigma_c^{-1}(i')) \in B'$ under the above map. Then we would be done, as the map is a bijection, i.e. $(i, \sigma(i)) \in B$ is a hit if and only if its image under the map above is a hit $(i', \sigma_r \circ \sigma \circ \sigma_c^{-1}(i')) \in B'$ on B'.
But our hit $(i, \sigma(i)) \in B$ gets mapped to the element $(\sigma_c(i), \sigma_r \circ \sigma(i)) \in B'$. Since $(\sigma_c(i), \sigma_r \circ \sigma(i)) = (\sigma_c(i), \sigma_r \circ \sigma \circ \sigma_c^{-1}(\sigma_c(i)))$, we see that the image is a hit $(i', \sigma_r \circ \sigma \circ \sigma_c^{-1}(i')) \in B'$ with $i' = \sigma_c(i)$ for our permutation $\sigma_r \circ \sigma \circ \sigma_c^{-1}$ on B' and we are done. □

Now we can use this lemma and theorem 2.13, to give at least a sufficient condition for rook equivalence in the general case.

Theorem 2.22. Let $n \in \mathbb{N}$, $B \subset [n] \times [n]$ a board, $B' \subset [n] \times [n]$ obtained from B by permuting the rows and columns of $[n] \times [n]$, then $B \overset{R}{\sim} B'$.

Proof. By definition we know that $B \overset{R}{\sim} B'$ if and only if $r_k(B) = r_k(B')$ for all $k \in [n]_0$, but by theorem 2.13 we know that this is the case if and only if $H_{k,n}(B) = H_{k,n}(B')$ for all $k \in [n]_0$. Now lemma 2.21 tells us that this holds if our assumption is given, so we are done. □

2.2 The file- & fit-numbers

In this section we will follow along the ideas of the first chapter 2.1 and use them for our file-placement model. Along this road we will establish the classical results for file-numbers and introduce the fit-numbers.

2.2.1 A recursive approach to file-numbers

Contrary to the case considered in section 2.1.1, there lie no problems in considering arbitrary boards when it comes to the results of file-numbers. We will still restrict ourselves to the case of skyline-boards first, since the classical theorems are all stated in terms of skyline-boards. Later on we will explain how we can easily generalise all of the proofs given.
Recall that in section 2.1.1 we started our journey with the question, how we can

construct a non-attacking rook placement in the best possible way. We came up with the solution of placing rooks from left to right, column after column. We can do the same thing for file-placements. So again we have to ask, how many ways there are to place the right-most rook on a skyline-board $B = B(b_1, ..., b_n)$, say in the i-th column, if we have placed rooks in the columns $1, ..., i-1$ already. So we come up with a lemma analogous to lemma 2.1.

Lemma 2.23. Let $B = B(b_1, ..., b_n)$ be a skyline-board, $z \in \mathbb{N}_0$, $k \in [n]_0$, $P \in \mathcal{F}_k(B_z)$ a file-placement of k-rooks in the first $k \leq i-1 < n$ columns of B_z, $\mathcal{P}(P)$ the set of file-placements obtained from P, by adding a rook in the i-th column of B_z, then we have that

$$|\mathcal{P}(P)| = b_i + z.$$

Proof. In the file placement model we have no restrictions where to place the new rook in the i-th column. Since the i-th column of B_z consists of $b_i + z$ cells we have that many options to place the new rook and hence the result follows. □

Again we formalize our intuitive idea of constructing a file-placement by putting down the rooks column by column, from left to right in a theorem analogous to theorem 2.2.

Theorem 2.24. Let $B = B(b_1, ..., b_n)$ a skyline-board, $z \in \mathbb{N}$,$k \in [n]_0$ then

$$f_k(B_z) = f_k(B_z^-) + (b_n + z)f_{k-1}(B_z^-),$$

with initial conditions $f_0(\varnothing) = 1$, $f_k(B_z) = 0$ if $n < k$ and $f_k(B(b_1, ..., b_j)) = 0$ if $|\{b_i + z > 0 \mid i \in [j]\}| < k$.

Proof. Let us begin with the initial conditions. First of all there is clearly no way to place k rooks on a board with n columns with at most one rook in each column if $n < k$. Furthermore if there are less than k columns of the board with at least one cell in them, we also clearly cannot place k rooks on the board. The last initial condition is just convention.
For the recursion we again, as in theorem 2.2, distinguish between the two cases of having a rook in the last column or not. So we consider the partition of $\mathcal{F}_k(B_z)$ into $\{P \in \mathcal{F}_k(B_z) \mid \text{there is no rook in the } n\text{-th column of } B_z\}$ and $\{P \in \mathcal{F}_k(B_z) \mid \text{there is a rook in the } n\text{-th column of } B_z\}$. Then by erasing the last column/ appending a column of height b_n we obtain a bijection of the first set and $\mathcal{F}_k(B_z^-)$, which gives the first term in the recurrence. For the second term we use lemma 2.23 on the second set. □

And yet again we strive for a generating function of the file-numbers. This time we do not need a clever choice of basis as in the rook-numbers case, but rather are able to give a nice formula for an ordinary generating function. This is due to Miceli and Remmel from 1986.

Theorem 2.25. (Miceli and Remmel)
Let $B = B(b_1, .., b_n)$ be a skyline board, then

$$\sum_{k=0}^{n} f_{n-k}(B)x^k = \prod_{i=1}^{n}(x + b_i), \tag{2.7}$$

as polynomials in x.

Proof. The proof strategy is the same as in the proof of theorem 2.3. Note that again both sides are polynomials in x, hence it suffices to check the equality for all $x \in \mathbb{N}$. We will also use double counting of the file placements of n rooks on B_x.

1. For the RHS we place the rooks column by column. Then the formula follows by iteratively using lemma 2.23.

2. For the LHS we first place $n-k$ rooks on the board B and then k rooks on the part of B_z below the ground. So we partition $\mathcal{F}_n(B_x) = \dot{\bigcup}_{k=0}^{n} F_k$, where $F_k = \{P \in \mathcal{F}_n(B_x) \mid \text{exactly } n-k \text{ rooks are on } B\}$. Then for each $k \in [n]_0$ we have that F_k is in bijection with $\mathcal{F}_{n-k}(B) \times \mathcal{F}_k([k] \times [x])$, so the claim follows.

□

As in the case of the rook-numbers, this theorem, similar to theorem 2.13, leads to a connection with some well known combinatorial numbers.

Example 2.26. We let $n \in \mathbb{N}_0$ and consider again the Staircase-board St_n, as defined in 1.6. For this board, formula (2.7) becomes

$$\sum_{k=0}^{n} f_{n-k}(St_n)x^k = \prod_{i=1}^{n}(x + (i-1)) = x^{(n)}.$$

Now we can multiply both sides by $(-1)^n$ and replace $-x$ by x to obtain

$$\sum_{k=0}^{n}(-1)^{n-k} f_{n-k}(St_n)x^k = (x)_n,$$

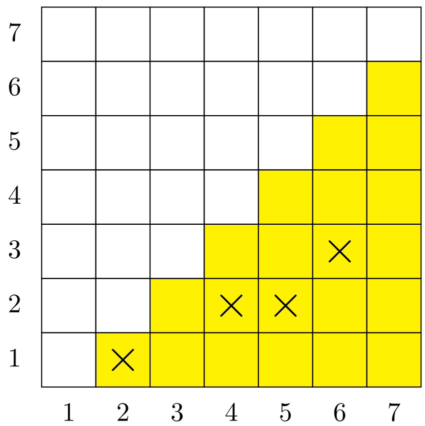
which is just the defining equation for the signed Stirling-numbers of first kind. Hence we see that $(-1)^{n-k} f_{n-k} = s_{n,k}$ by the linear independence of $\{x^k \mid k \in \mathbb{N}_0\}$. In particular we obtain that $f_{n-k}(St_n) = c_{n,k}$ equal the unsigned Stirling-numbers of first kind, by comparing coefficients when using the first equality and the defining relation of the unsigned Stirling-numbers of first kind.
We could also make this same observation by using theorem 2.24. The recursion for the file-numbers of the Staircase board is precisely

$$f_{n-k}(St_n) = f_{n-1-(k-1)}(St_{n-1}) + (n-1)f_{n-1-k}(St_{n-1}),$$

with initial conditions $f_0(St_n) = 1$ and $f_{n-k}(St_n) = 0$ if $n-k \leq 0$. Which is equivalent to the defining recurrence of the unsigned Stirling-numbers of first kind.
This can also be proven bijectively. For this denote $Perm_{n,k}$ the set of permutations $\sigma \in S_n$ with exactly k-disjoint cycles. We want a bijection from $Perm_{n,k}$ to $\mathcal{F}_{n-k}(St_n)$. So let $\sigma = (C_1, ..., C_k) \in S_n$, where $C_1, ..., C_k$ denote the disjoint cycles of σ, written WLOG with their minimal element in first place and ordered by their minimal elements. For example consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 6 & 1 & 4 & 3 & 7 \end{pmatrix}$, then we would write it as $\sigma = ((1,2,5,4),(3,6),(7))$. Now we consider the permutations $\sigma^{(1)}, ..., \sigma^{(n)}$, where $\sigma^{(i)}$ is obtained from σ by erasing all the elements $i+1, ..., n$ in the cycle structure of σ. In our example we would obtain the following elements:

$$\sigma^{(1)} = (1)$$
$$\sigma^{(2)} = (1,2)$$
$$\sigma^{(3)} = (1,2)(3)$$
$$\sigma^{(4)} = (1,2,4)(3)$$
$$\sigma^{(5)} = (1,2,5,4)(3)$$
$$\sigma^{(6)} = (1,2,5,4)(3,6)$$
$$\sigma^{(7)} = (1,2,5,4)(3,6)(7)$$

Now out of this list of permutations we construct a file placement in the following way:
We place no rook in column i if the element i is in a 1-cycle in the permutation $\sigma^{(i)}$ and we place a rook in the i-th column in row j if i immediately follows j in the cycle structure of $\sigma^{(i)}$. This gives a placement of $n-k$ rooks on St_n, since each cycle gives one empty column in the construction. So for our permutation we would obtain the file-placement:



For the inverse map we start with a file-placement of $n-k$ rooks on St_n and map it to an element $\sigma \in S_n$. For this we inductively construct a sequence $\sigma^{(1)}, ..., \sigma^{(n)}$ of permutations with $\sigma^{(i)} \in S_i$, in the following way:
We set $\sigma^{(1)} = (1)$ and assume that $\sigma^{(1)}, ..., \sigma^{(i)}$ have been constructed already for some $i \in [n-1]$. We write again, WLOG, the elements $\sigma^{(1)}, ..., \sigma^{(i)}$ in their cycle decompositions, where we start each cycle with its minimal element and order the cycles by their minimal elements. Then we obtain $\sigma^{(i+1)}$ from $\sigma^{(i)}$, by inserting a new cycle with the element $i+1$ if there is no rook in column $i+1$ or by inserting the element $i+1$ after the element j in the corresponding cycle, if there is a rook in the $i+1$-th column and j-th row. By construction this map is clearly inverse to the one given before.
Note that, as in example 2.5, we can also obtain the r-restricted Stirling-numbers of the first kind as file-numbers. For this let $n, r \in \mathbb{N}_0$, with $n \geq r \geq 0$, and consider a Staircase-board cut off at place r, i.e. $St_n^{(r)} = B(0, ..., 0, r, r+1, ..., n-1) \subset [n] \times [n]$, then formula (2.7) tells us that:

$$\sum_{k=0}^{n} f_{n-k}(St_n^{(r)}) x^k = x^r \prod_{i=r+1}^{n} (x + (i-1)) = x^r (x+r)^{(n-r)},$$

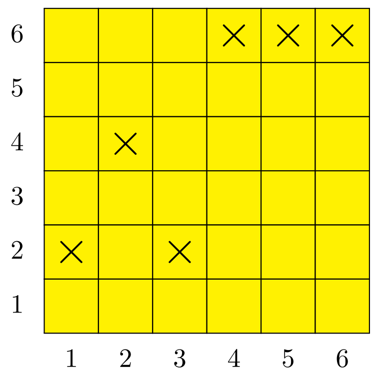
which is exactly the determining equation of the unsigned, r-restricted Stirling-numbers of first kind. By the linear independence of $\{x^k \mid k \in \mathbb{N}_0\}$ we obtain $f_{n-k}(St_n^{(r)}) = c_{n,k}^{(r)}$.

2.2.2 Introduction of the fit-numbers

Let $n \in \mathbb{N}$ and consider the board $[n] \times [n]$, then it is easy to see that

$$f_n([n] \times [n]) = n^n.$$

Since we have to place exactly one rook in each column and in each of those columns there are n squares available to place the rook. This gives rise to the question whether there is a bijection between $\mathcal{F}_n([n] \times [n])$ and the set $[n]^{[n]}$. As in the case of rook-numbers, this question has a similarly easy answer, since a file placement $P \in \mathcal{F}_n([n] \times [n])$ can directly be interpreted as the graph of an element $f \in [n]^{[n]}$. We denote a file-placement corresponding to a function $f \in [n]^{[n]}$ by $P_f \in \mathcal{F}_n([n] \times [n])$. See for example the placement below which corresponds to the function $f : 1 \to 2, 2 \to 4, 3 \to 2, 4 \to 6, 5 \to 6, 6 \to 6 \in [6]^{[6]}$:



Similarly to the rook-number case, we might ask questions along the lines of:

How many fix-point free functions $f \in [n]^{[n]}$ are there?

Or more general:

How many functions $f \in [n]^{[n]}$ with P_f avoiding or even hitting a certain board are there?

Which leads us to the following definition:

Definition 2.27. Let $n, k \in \mathbb{N}$, with $k \leq n$ $B \subset [n] \times [n]$ a board, then

$$F_{k,n}(B) \overset{\text{def}}{=} |\{f \in [n]^{[n]} \mid |P_f \cap B| = k\}|$$

is called the <u>k-th fit-number</u> of the board B. For $f \in [n]^{[n]}$ an element of $P_f \cap B$ is called a <u>fit</u> of the board B.

Since in the case of rook- and hit-numbers it worked out so well, giving us the strong theorem 2.13, we have high hopes to find an as meaningful connection in the file- and fit-number case. Using the naive approach of more or less copying the proof-idea from theorem 2.13 works perfectly fine and gives us the following theorem, due to Garsia and Remmel.

Theorem 2.28. Let $n \in \mathbb{N}$, $B \subset [n] \times [n]$ a board, then

$$\sum_{k=0}^{n} F_{k,n}(B)x^k = \sum_{k=0}^{n} f_k(B)n^{n-k}(x-1)^k, \tag{2.8}$$

or equivalently upon multiplying out the monomial $(x-1)^k$,

$$\sum_{k=0}^{n} F_{k,n}(B)x^k = \sum_{k=0}^{n} \left(\sum_{j=0}^{n} f_j(B)n^{n-j}(-1)^{j-k}\binom{j}{k} \right) x^k \tag{2.9}$$

as generating functions in x. Or equivalently upon setting $y = x - 1$:

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{n} \binom{j}{k} F_{j,n}(B) \right) y^k = \sum_{k=0}^{n} f_k(B)n^{n-k}y^k \tag{2.10}$$

as generating functions in y.

Proof. As hinted before, we will follow along the proof of theorem 2.13, so we will prove equation (2.10). It again suffices to check that for all $k \in [n]_0$ we have that

$$\sum_{j=0}^{n} \binom{j}{k} F_{j,n}(B) = f_k(B)n^{n-k}$$

We will restrict ourselves to a combinatorial proof in words as the formulae oriented version works completely analogous as in the proof of theorem 2.13.
We claim that both sides can be interpreted as the number of ways to choose a file-placement of n rooks on $[n] \times [n]$, with j elements on B, at least k, and choosing k of those j elements on B to color them red.
The LHS is obtained by first choosing an element $j \in \{k, ..., n\}$, then a placement of the n rooks with j of them on B in $F_{j,n}(B)$ ways and finally choosing the k rooks to be colored in $\binom{j}{k}$ ways. Whereas the RHS is obtained by first placing k rooks on B and coloring all of them red in $f_k(B)$ ways and then placing $n-k$ rooks in the remaining $n-k$ columns of height n on B in n^{n-k} ways. □

Remark 2.29. As promised in the introduction, we will now explain how to transform the results obtained for file-numbers of skyline boards, into results for file-numbers of arbitrary boards. For this we let $B \subset [n] \times [n]$ be a board and denote by $b_i^* \overset{\text{def}}{=} |\{(i,j) \in B\}|$ for all $i \in [n]$. Then giving it a little thought, all of the results obtained above, hold for arbitrary boards $B \subset [n] \times [n]$ as well, simply replacing b_i by b_i^* in all formulae.

2.3 Further recursion-formulae

In this section we will establish further recursion-formulae for the rook-, file-, hit- and fit-numbers.

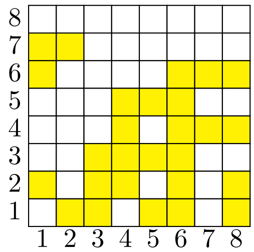
2.3.1 Deletion/Contraction

We start by introducing two new concepts related to boards.

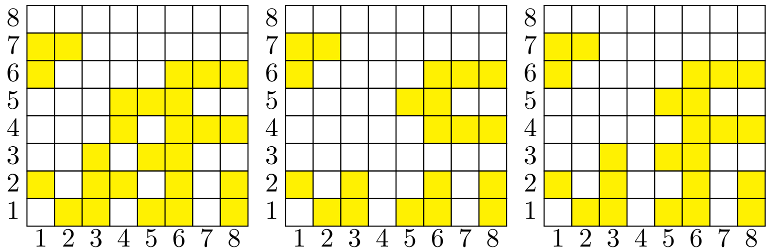
Definition 2.30. Let $B \subset [n] \times [n]$ be a board, $(i, j) \in B$ a cell, then we denote by

- B/c the board obtained from B by deleting c. We call B/c the <u>deletion</u> of c from B.
- $B \setminus_r c$ the board obtained from B by deleting all cells contained in B that are in the same row or column as c. We call $B \setminus_r c$ the <u>rook-contraction</u> of B by c.
- $B \setminus_f c$ the board obtained from B by deleting all cells contained in B that are in the same column as c. We call $B \setminus_f c$ the <u>file-contraction</u> of B by c.

Example 2.31. For example from the board



we obtain the board B/c to the left and $B \setminus_r c$ in the middle and $B \setminus_f c$ to the right:



for the cell $c = (4, 3)$.

2.3.2 Deletion/Contraction recurrences

We state another recurrence for rook- and file-numbers each.

Proposition 2.32. Let $n \in \mathbb{N}$, $k \in [n]_0$, $B \subset [n] \times [n]$ a board, $c \in B$ a cell, then we have that

$$r_k(B) = r_k(B/c) + r_{k-1}(B \setminus_r c),$$

with initial conditions $r_k(\varnothing) = 0$ if $k > 0$ and $r_0(\varnothing) = 1$.

Proof. The initial conditions are just convention. Consider the partition of the set $\mathcal{N}_k(B)$ into the two parts, given by $\{P \in \mathcal{N}_k(B) \mid \text{there is no rook on } c\}$ and $\{P \in \mathcal{N}_k(B) \mid \text{there is a rook on } c\}$. Then by definition, the first set gives the first term in the recursion and the second set gives the second term in the recurrence. Simply speaking, we either put no rook on c and thus k rooks on B/c or we put a rook on c, which cancels all cells in the column and row of c and so we have to put the $k-1$ remaining rooks on the board $B \setminus_r c$. □

The recursion for the file-numbers is given by:

Proposition 2.33. Let $n \in \mathbb{N}$, $k \in [n]_0$, $B \subset [n] \times [n]$ a board, $c \in B$ a cell, then we have that

$$f_k(B) = f_k(B/c) + f_{k-1}(B \setminus_f c),$$

with initial conditions $f_k(\varnothing) = 0$ if $k > 0$ and $f_0(\varnothing) = 1$

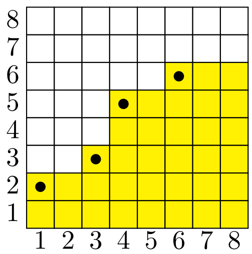
Proof. The initial conditions are just convention. Consider the partition of the set $\mathcal{F}_k(B)$ into the two parts, given by $\{P \in \mathcal{F}_k(B) \mid \text{there is no rook on } c\}$ and $\{P \in \mathcal{F}_k(B) \mid \text{there is a rook on } c\}$. Then by definition, the first set gives the first term in the recursion and the second set fives the second term in the recurrence. Simply speaking, we either put no rook on c and thus k rooks on B/c or we put a rook on c, which cancels all cells in the column of c and so we have to put the $k-1$ remaining rooks the board $B \setminus_f c$. □

2.3.3 Corner- and top-squares

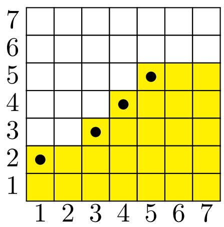
We start by introducing two new notions connected to Ferrers- and skyline-boards.

Definition 2.34. Let $F = F(b_1, ..., b_n)$ be a Ferrers-board. A cell (i, b_i) of F, for some $i \in [n]_0$ is called corner-square if either $i > 1$ and $b_{i-1} < b_i$ or if $i = 1$ and $0 < b_1$.

Example 2.35. In the board below all cells marked by a dot are corner-squares.



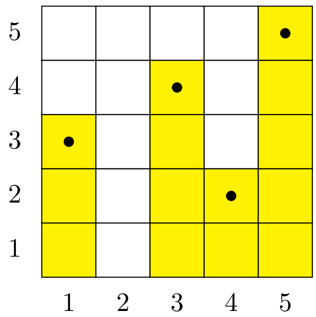
Notation 2.36. Using this new notion of a corner-square, we want to construct some new Ferrers-boards out of given a Ferrers-board $F = F(b_1, ..., b_n)$. For this let $c = (i, b_i)$ be a corner square of F, then we denote $F/\bar{c}$ the Ferrers-board obtained from F, by removing the column and the row containing the cell c and treating $F/\bar{c}$ as a board of length $n-1$, i.e. shifting all cells of F above c one row down and all cells of F to the right of c one column to the left.
Considering the board from example 2.35 and the corner-square $c = (4, 5)$ we obtain the board $F/\bar{c}$, with corner squares again indicated by a dot, as below:



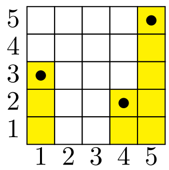
Considering the picture above, one could arrive at the conclusion, that $F/\bar{c}$ deserves the name *contraction* more than $F \setminus c$, since the board $F/\bar{c}$ is actually smaller, i.e. contracted. Nevertheless we stick to our naming from before as this is often used in the literature.

Definition 2.37. Let $B = B(b_1, ..., b_n)$ be a skyline-board. A cell (i, b_i) is called top-square if $b_i > 0$.

Example 2.38. All cells marked by a dot on the board below are top-squares.



Notation 2.39. We also want to use this new concept of a top-cell to construct some new skyline-boards out of a given skyline-board $B = B(b_1, ..., b_n)$. This time let $c = (i, b_i)$ be a top cell of B. We denote $B/\overline{\overline{c}}$ the skyline-board obtained from B, by removing all the cells in the column containing c, but not the column itself! So this is just the file-contraction of B by c. Nevertheless we will use this notation from the literature in the case of skyline-boards for the classical section.
Below we can see the board $B/\overline{\overline{c}}$, obtained from the board in example 2.38 when $c = (3, 4)$. Again the top-squares are marked by a dot.



2.3.4 Corner- and top-square recursions

We begin by stating a recursion for the rook- and file-numbers each and then we will make use of theorem 2.13 and theorem 2.28 to deduce recursions for the hit- and fit-numbers. We will also give combinatorial proofs of the deduced identities.

Corollary 2.40. Let $n \in \mathbb{N}$, $k \in [n]_0$, $F = F(b_1, ..., b_n)$ a Ferrers-board, c a corner-square of F then we have that

$$r_k(F) = r_k(F/c) + r_{k-1}(F/\overline{c}),$$

with initial conditions $r_k(\varnothing) = 0$ if $k > 0$ and $r_0(\varnothing) = 1$.

Proof. Again the initial conditions are just convention. Note that this follows directly from proposition 2.32, since $\mathcal{N}_{k-1}(F \setminus_r c)$ is in bijection with $\mathcal{N}_{k-1}(F/\overline{c})$, giving the second term in the recurrence. □

Note that for a Ferrers board $F \subset [n] \times [n]$ and a corner-square c of F the Ferrers-board $F/\overline{c}$ is contained in $[n-1] \times [n-1]$. Using this and theorem 2.13 we obtain the following chain of equalities:

$$\sum_{k=0}^{n} H_{k,n}(F)x^k \overset{(2.4)}{=} \sum_{k=0}^{n} r_k(F)(n-k)!(x-1)^k \overset{2.40}{=}$$

$$\sum_{k=0}^{n}\Big(r_k(F/c)+r_{k-1}(F/\overline{c})\Big)(n-k)!(x-1)^k \overset{r_{-1}(F/\overline{c})=0}{=}$$

$$\sum_{k=0}^{n} r_k(F/c)(n-k)!(x-1)^k + \sum_{k=1}^{n} r_{k-1}(F/\overline{c})(n-k)!(x-1)^k \overset{(2.4)}{=}$$

$$\sum_{k=0}^{n} H_{k,n}(F/c)x^k + (x-1)\sum_{k=1}^{n} r_{k-1}(F/\overline{c})(n-1-(k-1))!(x-1)^{k-1} \overset{k\to k+1}{=}$$

$$\sum_{k=0}^{n} H_{k,n}(F/c)x^k + (x-1)\sum_{k=0}^{n-1} r_k(F/\overline{c})(n-1-k)!(x-1)^k \overset{(2.4)}{=}$$

$$\sum_{k=0}^{n} H_{k,n}(F/c)x^k + (x-1)\sum_{k=0}^{n-1} H_{k,n-1}(F/\overline{c})x^k$$

Now comparing coefficients of x^k on both sides we obtain the following recursion for the hit-numbers of a Ferrers board.

Theorem 2.41. Let $n \in \mathbb{N}$, $F = F(b_1, ..., b_n) \subset [n] \times [n]$ a Ferrers-board, $c \in F$ a corner-square, then for all $k \in \mathbb{N}_0$ we have that

$$H_{k,n}(F) = H_{k,n}(F/c) + H_{k-1,n-1}(F/\overline{c}) - H_{k,n-1}(F/\overline{c}),$$

with initial conditions $H_{k,n}(\varnothing) = 0$ if $k > 1$, $H_{0,n}(\varnothing) = 1$ and $H_{k,n}(F) = 0$ if $k > n$.

We can also give a combinatorial proof of this recursion.

Proof. The first two initial conditions are rather conventions than something to prove. The third one is clear, since there cannot be a permutation in S_n with k hits on a board contained in $[n] \times [n]$ if $k > n$. The idea of the proof is to partition two sets of suitable cardinality into two subsets each. One of which has a cardinality appearing in the recursion again and the second one beeing equal in both cases. This allows us to compare the cardinalities of the partitioned sets minus the cardinalities of one part of the partition with each other.
Now let $n \in \mathbb{N}$, $k \in [n]_0$, $F = F(b_1, ..., b_n) \subset [n] \times [n]$ a Ferrers-board and $c \in F$ a corner-square, then we consider the partition of $T \overset{\text{def}}{=} \{\sigma \in S_n \mid |P_\sigma \cap F| = k\}$ into the two sets $T_1 \overset{\text{def}}{=} \{\sigma \in T \mid \text{there is a rook on } c\}$ and $T_2 \overset{\text{def}}{=} \{\sigma \in T \mid \text{there is no rook on } c\}$. Clearly

$$|T| = H_{k,n}(F)$$

and the set T_1 is in bijection with the set $\{\sigma \in S_{n-1} \mid |P_\sigma \cap F/\overline{c}| = k-1\}$ simply by taking $\sigma \in T_1$ and removing the rook on c, as well as the column and row containing it from P_σ. Therefore by definition of the hit-numbers

$$|T_1| = H_{k-1,n-1}(F/\overline{c}).$$

So we see that

$$H_{k,n}(F) - H_{k-1,n-1}(F/\overline{c}) = |T_2|.$$

Similarly we can partition the set $R \overset{\text{def}}{=} \{\sigma \in S_n \mid |P_\sigma \cap F/c| = k\}$ into $R_1 \overset{\text{def}}{=} \{\sigma \in R \mid \text{there is a rook on } c\}$ and $R_2 \overset{\text{def}}{=} \{\sigma \in R \mid \text{there is no rook on } c\}$. Then

$$|R| = H_{k,n}(F/c)$$

and the set R_1 is in bijection with the set $\{\sigma \in S_{n-1} \mid |P_\sigma \cap F/\overline{c}| = k\}$, simply by removing the rook on c, as well as the row and column containing it in P_σ. Therefore

$$|R_1| = H_{k,n-1}(F/\overline{c}).$$

So we see that

$$H_{k,n}(F/c) - H_{k,n-1}(F/\overline{c}) = |R_2|.$$

We now claim that $T_2 = R_2$. Then we are done, since by the above we obtain

$$H_{k,n}(F/c) - H_{k,n-1}(F/\overline{c}) = |R_2| = |T_2| = H_{k,n}(F) - H_{k-1,n-1}(F/\overline{c}),$$

which is exactly what we wanted to show. So it remains to check the claim. But this is easy since

$$\begin{gathered}
T_2 = \{\sigma \in T \mid \text{there is no rook on } c\} = \\
\{\sigma \in S_n \mid |P_\sigma \cap F| = k \text{ \& there is no rook on } c\} = \\
\{\sigma \in S_n \mid |P_\sigma \cap F/c| = k \text{ \& there is no rook on } c\} = \\
\{\sigma \in R \mid \text{there is no rook on } c\} = R_2.
\end{gathered}$$

□

Note that for a skyline-board $B \subset [n] \times [n]$ and a top-square c of B, the board $B/\overline{c}$ is again a skyline board contained in $[n] \times [n]$. Using this, theorem 2.33 and theorem 2.28 we obtain the following chain of equalities:

$$\begin{gathered}
\sum_{k=0}^{n} F_{k,n}(B)x^k \overset{(2.8)}{=} \sum_{k=0}^{n} f_k(B)n^{n-k}(x-1)^k \overset{2.33}{=} \\
\sum_{k=0}^{n} \Big(f_k(B/c) + f_{k-1}(B/\overline{c})\Big)n^{n-k}(x-1)^k \overset{f_{-1}(B/\overline{c})=0}{=} \\
\sum_{k=0}^{n} f_k(B/c)n^{n-k}(x-1)^k + \sum_{k=1}^{n} f_{k-1}(B/\overline{c})n^{n-k}(x-1)^k \overset{(2.8)}{=} \\
\sum_{k=0}^{n} F_{k,n}(B/c)x^k + \frac{(x-1)}{n}\sum_{k=1}^{n} f_{k-1}(B/\overline{c})n^{n-(k-1)}(x-1)^{k-1} \overset{k\to k+1}{=} \\
\sum_{k=0}^{n} F_{k,n}(B/c)x^k + \frac{(x-1)}{n}\sum_{k=0}^{n-1} f_k(B/\overline{c})n^{n-k}(x-1)^k \overset{f_n(B/\overline{c})=0}{=} \\
\sum_{k=0}^{n} F_{k,n}(B/c)x^k + \frac{(x-1)}{n}\sum_{k=0}^{n} f_k(B/\overline{c})n^{n-k}(x-1)^k \overset{(2.8)}{=} \\
\sum_{k=0}^{n} F_{k,n}(B/c)x^k + \frac{(x-1)}{n}\sum_{k=0}^{n} F_{k,n}(B/\overline{c})x^k
\end{gathered}$$

So comparing coefficients of x^k on both sides, we obtain the following recursion for the fit-numbers of a skyline board.

Theorem 2.42. Let $n \in \mathbb{N}$, $B = B(b_1, ..., b_n) \subset [n] \times [n]$ a skyline-board, $c \in B$ a top-square, then for all $k \in \mathbb{N}_0$ we have that

$$F_{k,n}(B) = F_{k,n}(B/c) + \frac{1}{n}F_{k-1,n}(B/\overline{\overline{c}}) + \frac{1}{n}F_{k,n}(B/\overline{\overline{c}}),$$

with initial conditions $F_{k,n}(\varnothing) = 0$ if $k > 1$, $F_{0,n}(\varnothing) = 1$ and $F_{k,n}(B) = 0$ if $k > n$.

Again we will give a combinatorial proof of this as well.

Proof. The first two initial conditions are again rather conventions than something to prove. The third one follows, since there exists no function $[n]^{[n]}$ with k fits on a board B contained in $[n] \times [n]$ if $k > n$. The rest of the proof is very similar to the idea of the proof of theorem 2.41.
Now fix $n \in \mathbb{N}$, $k \in [n]_0$, $B = B(b_1, ..., b_n) \subset [n] \times [n]$ as skyline board and $c \in B$ a top square. We consider the partition of $T \overset{\text{def}}{=} \{f \in [n]^{[n]} \mid |P_f \cap B| = k\}$ into the sets $T_1 \overset{\text{def}}{=} \{f \in T \mid \text{there is a rook on } c\}$ and $T_2 \overset{\text{def}}{=} \{f \in T \mid \text{there is no rook on } c\}$. Then clearly

$$|T| = F_{k,n}(B)$$

and the set $T_1 \times [n]$ is in bijection with $\{f \in [n]^{[n]} \mid |P_f \cap B/\overline{\overline{c}}| = k - 1\}$, simply by removing the rook on the cell c, erasing all the cells in the column containing c of the board and choosing one of the n cells in this column to place a new rook onto. Therefore we see that

$$|T_1| = \frac{1}{n}F_{k-1,n}(B/\overline{\overline{c}}).$$

And so in particular

$$F_{k,n}(B) - \frac{1}{n}F_{k-1,n}(B/\overline{\overline{c}}) = |T_2|.$$

But we can also partition the set $R \overset{\text{def}}{=} \{f \in [n]^{[n]} \mid |P_f \cap B/c| = k\}$ into the two sets $R_1 \overset{\text{def}}{=} \{f \in R \mid \text{there is a rook on } c\}$ and $R_2 \overset{\text{def}}{=} \{f \in R \mid \text{there is no rook on } c\}$. Then

$$|R| = F_{k,n}(B/c)$$

and the set $R_1 \times [n]$ is in bijection with the set $\{f \in [n]^{[n]} \mid |P_f \cap B/\overline{\overline{c}}| = k\}$, simply by removing the rook on the cell c, erasing all the cells in the column containing c of the board and choosing one of $[n]$ new cells to place a rook on. Hence we obtain

$$|R_1| = \frac{1}{n}F_{k,n}(B/\overline{\overline{c}}).$$

We claim that $T_2 = R_2$ and then we are done, since by the above we obtain

$$F_{k,n}(B) - \frac{1}{n}F_{k-1,n}(B/\overline{\overline{c}}) = |T_2| = |R_2| = F_{k,n}(B/c) - \frac{1}{n}F_{k,n}(B/\overline{\overline{c}}),$$

which is exactly what we wanted to show. So it remains to check the claim. But this is easy since

$$\begin{gathered} T_2 = \{f \in T \mid \text{there is no rook on } c\} = \\ \{f \in [n]^{[n]} \mid |P_f \cap B| = k \;\&\; \text{there is no rook on } c\} = \\ \{f \in [n]^{[n]} \mid |P_f \cap B/c| = k \;\&\; \text{there is no rook on } c\} = \\ \{f \in R \mid \text{there is no rook on } c\} = R_2 \end{gathered}$$

□

Remark 2.43. It is oddly interesting, that in order to deduce theorem 2.41, it was not sufficient to consider the rook-contraction $F \setminus_r c$ of F by a corner square c but we actually needed to contract the board to $F/\bar{c}$, to make use of theorem 2.13 a second time, whereas in the file-number case we simply could use the file-contraction $B \setminus_f c$ of B by a top square without contracting the board. Note that in both cases the recurrences were only needed for the special corner-/top-squares. This observation will shed a light on some of the problems discussed in chapter 3.5 later on.

2.3.5 Another generating function and a formula of Frobenius

In this section we will first introduce a new statistic on rook placements and a new generating function using this said statistic, before we combine this with some results from earlier sections to deduce a formula of Frobenius.
So to start we state a definition:

Definition 2.44. Let $F = F(b_1, ..., b_n)$ be a skyline-board, $k \in \mathbb{N}$, $P \in \mathcal{F}_k(F_\infty)$ then we define the maximum of the placement P by

$$\max(P) \stackrel{\text{def}}{=} |\min\{0, -1 + \min\{j \in \mathbb{Z} \mid \text{there is a rook in row } j \text{ of } F \text{ in } P\}\}|.$$

So it equals the number of rows below the ground up to the "lowest" placed rook in the placement, still counting its row, or zero if there is no rook below the ground.

We first restrict ourselves to Ferrers-boards again and only consider rook-numbers, since this will lead to the formula of Frobenius. Now we want to consider a generating function with respect to that new statistic max. We could just consider the ordinary generating function $\sum_{P \in \mathcal{N}_n(F)} x^{\max(P)}$. But what do the coefficients actually tell us? Well, they give us the number of placements for which $\max(P) = k$ for some $k \in \mathbb{N}_0$. But this is just

$$|\{P \in \mathcal{N}_n(F_\infty) \mid \max(P) = k\}| = |\{P \in \mathcal{N}_n(F_k) \mid \max(P) = k\}|.$$

Now the equality $\max(P) = k$ is quite a strong restriction, what if we just leave it away? Then we are left with $|\{P \in \mathcal{N}_n(F_k)\}| = r_n(F_k)$. So we already know these objects then, but how are they related to the coefficients we wanted to consider? By definition

$$P \in \{P \in \mathcal{N}_n(F_\infty) \mid \max(P) = k\} \Leftrightarrow P \in \mathcal{N}_n(F_k) \setminus \mathcal{N}_n(F_{k-1}),$$

so that our initial coefficients are just given by $r_n(F_k) - r_n(F_{k-1})$. So we see, that it is equivalent to consider the ordinary generating function $\sum_{P \in \mathcal{N}_n(F)} x^{\max(P)}$ and the rather familiar object $\sum_{k \geq 0} r_n(F_k) x^k$. This gives us hope to actually find a nice closed form for the coefficients or even the whole generating function. Making this intuition more precise, we obtain the following theorem, giving not only one but three(!) expressions for the desired generating function.

Theorem 2.45. Let $F = F(b_1, ..., b_n)$ be a Ferrers board then

$$\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(F_\infty)} x^{\max(P)} = \sum_{k \geq 0} r_n(F_k) x^k = \sum_{k \geq 0} \left(\prod_{i=1}^{n} (k + b_i - (i-1)) \right) x^k =$$
$$\sum_{k=0}^{n} \frac{r_{n-k}(F) k! x^k}{(1-x)^{k+1}} \stackrel{(*)}{=} \frac{1}{(1-x)^{n+1}} \sum_{k=0}^{n} H_{k,n}(F) x^{n-k},$$

where $(*)$ holds if and only if $b_n \leq n$.

Proof. Let us start with the first new equality. This is just the formal statement of the outline we gave before. Why? Well if we expand $\frac{1}{1-x}$ as a geometric series and multiply out the LHS then this just equals

$$\sum_{k\geq 0} |\{P \in \mathcal{N}_n(F_\infty) \mid \max(P) \leq k\}| x^k.$$

But now recall our key lemma 2.1 which tells us how to calculate $|\{P \in \mathcal{N}_n(F_\infty) \mid \max(P) \leq k\}|$! Note that

$$P \in |\{P \in \mathcal{N}_n(F_\infty) \mid \max(P) \leq k\}| \Leftrightarrow P \in \mathcal{N}_n(F_k),$$

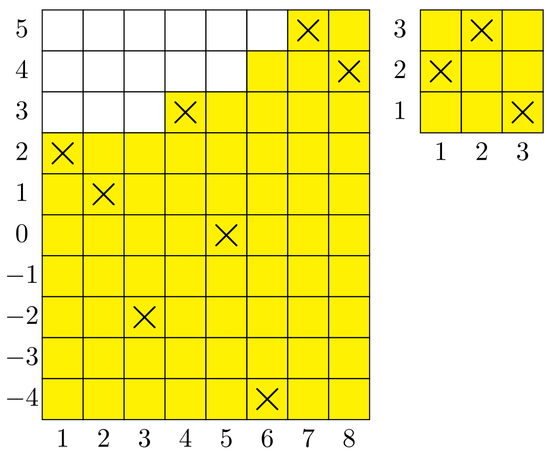
hence using lemma 2.1 iteratively we obtain we obtain that

$$|\{P \in \mathcal{N}_n(F_\infty) \mid \max(P) \leq k\}| = |\mathcal{N}_n(F_k)| = r_n(F_k) = \prod_{i=1}^{n}(k + b_i - (i-1)).$$

Giving the second equality.
To check the third one we classify our placements by how the rooks are arranged above the ground, i.e. we observe that:

$$\frac{1}{1-x}\sum_{P\in\mathcal{N}_n(F_\infty)} x^{\max(P)} = \sum_{k=0}^{n}\sum_{Q\in\mathcal{N}_{n-k}(F)} \frac{1}{1-x}\sum_{P\in\mathcal{N}_n(F_\infty) \text{ s.t. } P\cap F=Q} x^{\max(P)}$$

Let us fix $Q \in \mathcal{N}_{n-k}(F)$ and denote $S_Q(x) = \sum_{P\in\mathcal{N}_n(F_\infty) \text{ s.t. } P\cap F=Q} x^{\max(P)}$. We can code such a placement $P \in \mathcal{N}_n(F_\infty)$ with $P \cap F = Q$ by a tuple $(p_1, ..., p_k) \in \mathbb{N}^k$ and a permutation $\sigma \in S_k$, where p_1 denotes the number of rows between the ground and the row containing the first rook below the ground, p_2 is the number of rows between the first row below the ground containing a rook and the second row below the ground containing a rook, etc. The permutation $\sigma \in S_k$ is given by the rook placement, obtained when only considering the rows below the ground containing a rook and deleting all the columns containing a rook on the board. So for example the following placement below on the left, would be coded by the sequence $(0, 1, 1)$ and the permutation below on the right, i.e. $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.



This gives a bijection between $\{P \in \mathcal{N}_n(F_\infty) \mid P \cap F = Q\}$ and $\mathbb{N}^k \times S_k$. Note that under this map, we have that $\max(P) = p_1 + ... + p_k + k$. So we can rewrite the sum in the following way:

$$S_Q(x) = \sum_{P \in \mathcal{N}_n(F_\infty) \text{ s.t. } P \cap F = Q} x^{\max(P)} = \sum_{((p_1,...,p_k),\sigma) \in \mathbb{N}^k \times S_k} x^{p_1+...+p_k+k} =$$

$$k!x^k \sum_{p_1 \geq 0} ... \sum_{p_k \geq 0} x^{p_1+...+p_k} = k!x^k \prod_{j=1}^{k} \left(\sum_{p_j \geq 0} x^{p_j} \right) = \frac{k!x^k}{(1-x)^k}$$

Inserting this expression for $S_Q(x)$ into our initial equality, we obtain

$$\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(F_\infty)} x^{\max(P)} = \sum_{k=0}^{n} \sum_{Q \in \mathcal{N}_{n-k}(F)} \frac{1}{1-x} S_Q(x) =$$

$$\sum_{k=0}^{n} \sum_{Q \in \mathcal{N}_{n-k}(F)} \frac{k!x^k}{(1-x)^{k+1}} = \sum_{k=0}^{n} \frac{r_{n-k}(F)k!x^k}{(1-x)^{k+1}},$$

which is just the desired third equality.
Note that the hit numbers of a board $F = F(b_1, ..., b_n)$ are defined if and only if $b_n \leq n$, which justifies the equivalence part, so we only have to check $(*)$. Now to do this recall theorem 2.4, stating that:

$$\sum_{k=0}^{n} H_{k,n}(B)x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k$$

Substituting x to $\frac{1}{y}$ and then multiplying this equation by y^n we obtain that:

$$\sum_{k=0}^{n} H_{k,n}(B)y^{n-k} = \sum_{k=0}^{n} r_k(B)(n-k)!y^n \Big(\frac{1}{y} - 1\Big)^k =$$

$$\sum_{k=0}^{n} r_k(B)(n-k)!y^n \Big(\frac{1-y}{y}\Big)^k = \sum_{k=0}^{n} r_k(B)(n-k)!(1-y)^k y^{n-k}$$

Now replacing k by $n-k$ on the RHS and dividing the equation by $(1-y)^{n+1}$ before substituting y to x again we get

$$\frac{1}{(1-x)^{n+1}} \sum_{k=0}^{n} H_{k,n}(B)x^{n-k} = \sum_{k=0}^{n} \frac{r_{n-k}(B)k!x^k}{(1-x)^{k+1}},$$

which is exactly $(*)$, hence we are done. □

In order to state the formula of Frobenius, to which we referred in the first paragraph, we need to introduce another statistic on permutations and its connection to the excedances, defined in 2.10, first.

Definition 2.46. Let $n \in \mathbb{N}$, $\sigma = \begin{pmatrix} 1 & ... & n \\ \sigma_1 & ... & \sigma_n \end{pmatrix} \in S_n$, $k \in [n-1]$ then a pair (σ_i, σ_{i+1}) for $i \in [n-1]$ is called k-descent pair and i is called k-descent if $\sigma_i - \sigma_{i+1} = k$. We denote $Des_k(\sigma)$ the set of k descents and $Des(\sigma) = \{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}$ the set of all descents of σ.

Recall the statistic of excedances defined in 2.10. For any $n \in \mathbb{N}$ there is a bijection $\Phi : S_n \to S_n$ of fundamental importance, showing that the distributions of k-descents and k-excedances for a given $k \in [n-1]$ are equal. This very useful bijection is called Foata's first fundamental transform. We shall state this more formally and prove it in the following lemma.

Lemma 2.47. (Foata's first fundamental transform)
Let $n \in \mathbb{N}$, $k \in [n-1]$ then

$$\sum_{\sigma \in S_n} x^{|Exc_k(\sigma)|} = \sum_{\sigma \in S_n} x^{|Des_k(\sigma)|}.$$

Proof. As announced we will show this, by constructing a self-bijection that maps k-excedances onto k-descents and vice versa. For $\sigma = \begin{pmatrix} 1 & ... & n \\ \sigma_1 & ... & \sigma_n \end{pmatrix} \in S_n$ we will call σ_j a left-to-right maximum/minimum if for all $i < j$ we have that $\sigma_j > \sigma_i$/ $\sigma_j < \sigma_i$ respectively.
Now let us start with constructing the bijection. For this take some $\sigma \in S_n$ and write it in cycle form $\sigma = C_1...C_l$, where WLOG, we write the cycles with their largest element last and order them by increasing largest elements. Now we reverse each cycle and delete the parentheses, to obtain a new permutation, $\Phi(\sigma)$. Now assume that (i, σ_i) is a k-excedance pair of σ, then when writing σ in cycle form i will always be to the left of σ_i as $i + k = \sigma_i$, so i will never be the largest element in its cycle. Hence, after reversing the cycles, we will have i to the right of σ_i, thus after deleting the parenthesis, the pair $(\sigma_i, i) = (\Phi(\sigma)_j, \Phi(\sigma)_{j+1})$, for some $j \in [n]$, is clearly a k-descent.
To construct an inverse mapping, take some $\sigma = \sigma_1...\sigma_n \in S_n$ and write a vertical line before each left-to-right maximum, $\sigma = \sigma_1...\sigma_{m_1}|....\sigma_{m_l}|\sigma_{m_l+1}...\sigma_n$. Now put brackets around each block and reverse the order of its elements, to obtain the cycles of $\Phi^{-1}(\sigma)$. That this is an inverse to the map from before, follows by construction. We still have to check, that this maps k-descents to k-excedances. So consider a k-descent pair (σ_i, σ_{i+1}), then clearly σ_i and σ_{i+1} will be in the same block after cutting before each left-to-right maximum. Because the only way they would not end up in the same block, would be to cut before σ_{i+1}, but that would imply that σ_{i+1} is a left-to-right maximum, contradicting the fact that $\sigma_i = \sigma_{i+1} + k$. Hence after reversing the elements, we see that σ_{i+1} is on the left of σ_i in the cycle decomposition of $\Phi^{-1}(\sigma)$, i.e. $(\sigma_i, \sigma_{i+1}) = (\sigma_i, \Phi^{-1}(\sigma)(\sigma_i))$ is a k-excedance pair of $\Phi^{-1}(\sigma)$.
To illustrate the above process consider the following example. We start with $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 3 & 7 & 2 & 5 & 8 \end{pmatrix} \in S_8$, with its excedance pairs $(1, 6), (3, 4)\&\ (5, 7)$. Let us apply Φ to σ. So we write σ in its cycle form $(34)(216)(57)(8)$, where we ordered the cycles by increasing largest elements and wrote them such that the largest element is last. Now we reverse each cycle and delete the parentheses so obtain $\Phi(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 6 & 1 & 2 & 7 & 5 & 8 \end{pmatrix}$ with its descent pairs $(4, 3), (6, 1), (7, 5)$. Now we will apply Φ^{-1} to $\Phi(\sigma) = 43612758$. The left-to-right maxima are given by $6, 7, 8$, so we insert vertical lines to obtain $43|612|75|8$. Reversing the blocks and interpreting them as cycles we obtain $(34)(216)(57)(8)$ which is exactly $\sigma \in S_8$ again. □

We recall that for $n \in \mathbb{N}$ and $\sigma \in S_n$, we have that $P_{\sigma^{-1}}$ is just the placement obtained from P_σ by reflection along the main diagonal. From this the next lemma is easily deduced.

Lemma 2.48. Let $n \in \mathbb{N}$, $m \in [n]_0$ then $\sigma \in S_n$ has exactly m, non-fix point excedances if and only if $|P_{\sigma^{-1}} \cap St_n| = m$.

Since the non-fix point excedances are just the hits of the board $\{(i, i+j) \mid 1 \leq i \leq n \ \& \ 1 \leq j \leq n-i\}$. So we see that:

$$|\{\sigma \in S_n \mid \sigma \text{ has } m \text{ non-fix point excedances}\}| = |\{\sigma^{-1} \in S_n \mid |P_{\sigma^{-1}} \cap St_n| = m\}| = \\ |\{\sigma \in S_n \mid |P_\sigma \cap St_n| = m\}| = H_{m,n}(St_n)$$

Combining this with lemma 2.47 we get that $\{\sigma \in S_n \mid \sigma \text{ has } m \text{ descents}\} = \{\sigma \in S_n \mid \sigma \text{ has } m \text{ non-fix point excedances}\}$ and so

$$|\{\sigma \in S_n \mid \sigma \text{ has } m \text{ descents}\}| = H_{m,n}(St_n).$$

Now let us introduce the complimentary permutation of an element of the symmetric group.

Definition 2.49. Let $n \in \mathbb{N}$, $\sigma = \begin{pmatrix} 1 & \dots & n \\ \sigma_1 & \dots & \sigma_n \end{pmatrix} \in S_n$, then the permutation $\sigma^c = \begin{pmatrix} 1 & \dots & n \\ n+1-\sigma_1 & \dots & n+1-\sigma_n \end{pmatrix} \in S_n$ is called the complementary permutation of σ.

Note that $S_n \to S_n, \sigma \mapsto \sigma^c$ is a bijection. The following lemma follows directly from the definition.

Lemma 2.50. Let $n \in \mathbb{N}$, $m \in [n-1]_0$ then $\sigma \in S_n$ has exactly m descents if and only if σ^c has exactly $n-1-m$ descents.

Combining this with the results from before we obtain that:

$$H_{m,n}(St_n) = \\ |\{\sigma \in S_n \mid \sigma \text{ has } m \text{ descents}\}| = |\{\sigma^c \in S_n \mid \sigma^c \text{ has } n-1-m \text{ descents}\}| = \\ |\{\sigma \in S_n \mid \sigma \text{ has } n-1-m \text{ descents}\}|$$

So we see that:

$$\sum_{k=0}^{n} H_{k,n}(St_n) x^{n-k} = \sum_{\sigma \in S_n} x^{|\{Des(\sigma)\}|+1}$$

Now we are finally ready to state Frobenius' formula and prove it.

Corollary 2.51. (A formula of Frobenius)
Let $n \in \mathbb{N}$ then

$$\frac{1}{(1-x)^{n+1}} \sum_{\sigma \in S_n} x^{|\{Des(\sigma)\}|+1} = \sum_{k=0}^{n} \frac{S_{n,k} k! x^k}{(1-x)^{k+1}}.$$

Proof. This follows directly from $(*)$ in theorem 2.45 and the discussion above when considering the Ferrers board St_n. □

We see, that this formula was obtained by specialising the Ferrers board in theorem 2.45, to the Staircase board. Now one might ask, if we can obtain other interesting results by specialising the board in a different way and to the mindful reader it might not come as a surprise, that choosing our other board of special interest, the Laguerre board, yields another useful formula.

Corollary 2.52. Let $n \in \mathbb{N}$ then

$$\frac{n!x}{(1-x)^{n+1}} = \sum_{k=1}^{n} \frac{L_{n,k}k!x^k}{(1-x)^{k+1}}.$$

Proof. For $k \in [n]_0 \setminus \{n-1\}$ we have that $H_{k,n}(L_n) = 0$, since $|P_\sigma \cap L_n| = n-1$ for all $\sigma \in S_n$. Therefore $H_{n-1,n}(L_n) = n!$ and so using $(*)$ in theorem 2.45 we obtain the result. □

Until now, we only considered Ferrers boards and talked about rook-numbers in this chapter. But we have another similar object, the file-numbers. So it is quite natural to ask, whether there is an analogous result to theorem 2.45 in terms of skyline-boards and file-numbers. The following theorem shall answer this question.

Theorem 2.53. Let $B = B(b_1, ..., b_n)$ be a skyline board, then

$$\frac{1}{1-x} \sum_{P \in \mathcal{F}_n(B_\infty)} x^{\max(P)} = \sum_{k \geq 0} f_n(B_k)x^k = \sum_{k \geq 0} \left(\prod_{i=1}^{n} (k + b_i) \right) x^k =$$
$$\sum_{k=0}^{n} \frac{1}{1-x} \left(\sum_{j=1}^{k} \frac{j!S_{j,k}}{(1-x)^j} \right) f_{n-k}(B).$$

Proof. The proof is very similar to the one of theorem 2.45. The first equality follows completely analogous. For the second equality we will again make use of the key lemma 2.23 of the introductory section. Now again, we have

$$\frac{1}{1-x} \sum_{P \in \mathcal{F}_n(B_\infty)} x^{\max(P)} = \sum_{k \geq 0} |\{P \in \mathcal{F}_n(B_\infty) \mid \max(P) \leq k\}| x^k =$$
$$\sum_{k \geq 0} |\{P \in \mathcal{F}_n(B_k)\}| x^k = \sum_{k \geq 0} f_n(B_k)x^k = \sum_{k \geq 0} \left(\prod_{i=1}^{n} (k + b_i) \right) x^k,$$

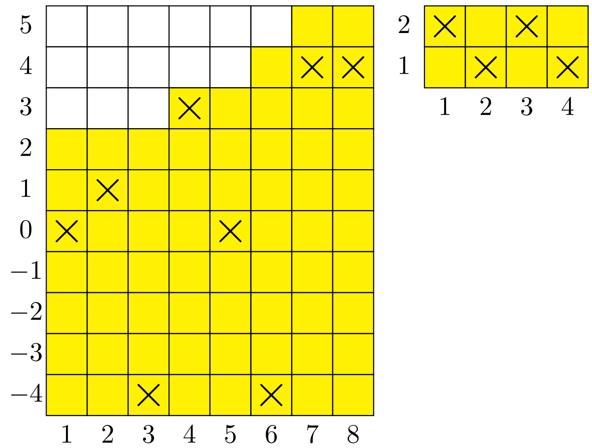
where the last equality is due to iterative use of lemma 2.23.
For the third equality, we will distinguish the placements by how the rooks are placed above the ground, to obtain the result in a similar fashion as in the second part of the proof of theorem 2.45. We again make the observation, that:

$$\frac{1}{1-x} \sum_{P \in \mathcal{F}_n(B_\infty)} x^{\max(P)} = \sum_{k=0}^{n} \sum_{Q \in \mathcal{F}_{n-k}(B)} \frac{1}{1-x} \sum_{P \in \mathcal{F}_n(B_\infty) \text{ s.t. } P \cap B = Q} x^{\max(P)}$$

So let us fix some $Q \in \mathcal{F}_{n-k}(B)$ for some $k \in [n]$ and denote

$$T_Q(x) = \sum_{P \in \mathcal{F}_n(B_\infty) \text{ s.t. } P \cap B = Q} x^{\max(P)}.$$

Now we can code such a placement $P \in \mathcal{F}_n(B_\infty)$, with $P \cap B = Q$, by a tuple $(p_1, ..., p_j)$ and a surjective function $f \in [j]^{[k]}$, where p_1 denotes the number of rows between the ground and the first row containing a rook or rooks. Note that we are looking at file placements, hence there can be multiple rooks in a row. p_2 is the number of rows between the first row below the ground containing rooks and the second row below the ground containing rooks etc. The surjective function is given by the file placement, obtained when only considering the rows below the ground containing rooks and deleting all the columns containing a rook above the ground. So for example the following placement below on the left, would be coded by the tuple $(0,3)$ and the surjective function below on the right, i.e. $f:[4]\to[2], 1\mapsto 2, 2\mapsto 1, 3\mapsto 2, 4\mapsto 1$.



This gives a bijection between $\{P\in\mathcal{F}_n(B_\infty)\mid P\cap B=Q\}$ and $\prod_{j=1}^{k}(\mathbb{N}^j\times[j]^{[k]})$ under which we have that $\max(P)=p_1+\ldots+p_j+j$. Now quickly recall that $|[j]^{[k]}|=j!S_{j,k}$ as we can identify each such surjective function $f\in[j]^{[k]}$ by the ordered set partition $(f^{-1}(1),\ldots,f^{-1}(j))$ of k. Therefore we get that

$$T_Q(x)=\sum_{P\in\mathcal{F}_n(B_\infty)\text{ s.t. }P\cap B=Q}x^{\max(P)}=$$

$$\sum_{j=1}^{k}\sum_{((p_1,\ldots,p_j),f)\in\mathbb{N}^j\times[j]^{[k]}}x^{p_1+\ldots+p_j+j}=\sum_{j=1}^{k}S_{j,k}x^j\sum_{p_1\geq 0}\ldots\sum_{p_j\geq 0}x^{p_1+\ldots+p_j}=$$

$$\sum_{j=1}^{k}S_{j,k}x^j\prod_{i=1}^{j}\left(\sum_{p_i\geq 0}x^{p_i}\right)=\sum_{j=1}^{k}\frac{S_{j,k}x^j}{(1-x)^j},$$

hence we see that

$$\frac{1}{1-x}\sum_{P\in\mathcal{F}_n(B_\infty)}x^{\max(P)}=\sum_{k=0}^{n}\sum_{Q\in\mathcal{F}_{n-k}(B)}\frac{1}{1-x}T_Q(x)=$$

$$\sum_{k=0}^{n}\sum_{Q\in\mathcal{F}_{n-k}(B)}\frac{1}{1-x}\sum_{j=1}^{k}\frac{S_{j,k}x^j}{(1-x)^j}=\sum_{k=0}^{n}\frac{1}{1-x}\left(\sum_{j=1}^{k}\frac{S_{j,k}x^j}{(1-x)^j}\right)f_{n-k}(B),$$

as desired. □

Remark 2.54. Note that the generating function $\sum_{k\geq 0}f_n(B_k)x^k$ is independent under the permutation of columns since the file-numbers are, hence we can WLOG always consider Ferrers-boards instead of skyline-boards when dealing with this object.

Remark 2.55. Although we just have proven theorem 2.53, we might still feel a bit empty inside, as if we broke down at the last 100 meter mark of a marathon. This is probably due to us missing a version of $(*)$ from theorem 2.45 for the file-number case. But why did we not just go for a similar approach as in the rook-number case? We might ask ourselves. The answer is simple, the essential trick for proving $(*)$ was to use theorem 2.13. This was possible because our generating function was of the form $\frac{Q(x)}{(1-x)^{n+1}}$, where $Q(x)$ was a polynomial in $\mathbb{N}[x]$ of a form, which we could use theorem 2.4 on. Now we could say, that our generating function in theorem 2.53 is of the form $\frac{Q'(x)}{(1-x)^{n+1}}$ as well. But, you see, $Q'(x)$ need not even be an element of $\mathbb{N}[x]$, so it cannot be of the form $\frac{1}{(1-x)^{n+1}}\sum_{k=0}^{n} F_{k,n}(B)x^{n-k}$ as we would desire. An easy example where one can see this is the board $B = B(0,2)$. Then $Q'(x) = \sum_{k=1}^{2} f_{2-k}(B)\sum_{j=1}^{k} j!S_{j,k}x^j(1-x)^{2-j} = 3x - x^2 \notin \mathbb{N}[x]$. So we loose all hope on finding a general analogue of $(*)$ in the file-number case and give up, right? Well...no, we should at least try to see, if we can fix this under certain restrictions. Luckily enough this is possible in some way or another. One of them we will present below.

So as noted before, we will consider WLOG a Ferrers-board $F = F(b_1, ..., b_n)$. We restrict to the case where $b_1 = 0$ and $b_{i+1} \leq b_i + 1$, so we have at most increases by 1 in the columns. Then we can consider the board $F' = B(a_1, ..., a_n)$, where $a_i = b_{n+1-i} + i - 1$ for all $i \in [n]$, i.e. $F' = F$"+"$B(n-1, n-2, ..., 1, 0)$ if "+" means stacking the columns of the second board onto the first one. Now by our assumptions on F, we have that $a_i = b_{n+1-i} + i - 1 \leq n - i + i - 1 \leq n - 1$, i.e. $F' \subset [n] \times [n-1]$ and $a_i = b_{n+1-i} + i - 1 \leq b_{n+1-(i+1)} + 1 + i - 1 = a_{i+1}$, so that F' is a Ferrers-board as well. So this process might at first seem a bit ad-hoc, but the following equality will hopefully enlighten our minds and make clear why this is actually a quite natural, yet still clever trick. So see that for the above constructed sequences and any $k \in \mathbb{N}_0$, we have that

$$\prod_{i=1}^{n}(k + a_i - (i-1)) = \prod_{i=1}^{n}(k + b_i).$$

So this equality relates the rook-numbers $r_n(F'_k)$ and the file-numbers $f_n(F_k)$, which where the coefficients of our generating functions of interest. It is true that we could have gotten this equality easier by simply defining $a_i = b_i + (i-1)$, but then we would not have ended up with F' being contained in $[n] \times [n]$, which is essential for $(*)$ to hold and which we shall make use of right now.

The following chain of equalities shall prove that in the above setting, the term $Q'(x) = (1-x)^{n+1}\sum_{k=0}^{n}\frac{1}{1-x}(\sum_{j=1}^{k}\frac{j!S_{j,k}x^j}{(1-x)^j})f_{n-k}(F)$ is a polynomial in $\mathbb{N}[x]$ and give a closed form for it:

$$\frac{\sum_{k=0}^{n} H_{k,n}(F')x^{n-k}}{(1-x)^{n+1}} \overset{2.45(*)}{=}$$

$$\sum_{k\geq 0}\left(\prod_{i=1}^{n}(k + a_i - (i-1))\right)x^k \overset{\text{above}}{=} \sum_{k\geq 0}\left(\prod_{i=1}^{n}(k + b_i)\right)x^k \overset{2.53}{=}$$

$$\frac{\sum_{k=0}^{n}\left(\sum_{j=1}^{k} j!S_{j,k}x^j(1-x)^{n-j}\right)f_{n-k}(F)}{(1-x)^{n+1}}$$

We still could not relate the RHS with the fit-numbers as desired though, this is a problem that will come up in chapter 3.5 again. Summing up all our results we obtain the following proposition.

Proposition 2.56. Let $F = F(b_1, ..., b_n)$ be a Ferrers board with $b_1 = 0$ and $b_{i+1} \leq b_i + 1$, then $F' = B(a_1, ..., a_n)$, where $a_i = b_{n+1-i} + i - 1$ for all $i \in [n]$, is a Ferrers board contained in $[n-1] \times [n]$ and

$$\sum_{k=0}^{n} \left(\sum_{j=1}^{k} j! S_{j,k} x^j (1-x)^{n-j} \right) f_{n-k}(F) = \sum_{k=0}^{n} H_{k,n}(F') x^{n-k} \in \mathbb{N}[x].$$

Using the fact that for $F = St_n$ we obtain for F' the Laguerre board L_n we derive the following interesting formula, which shall conclude our section.

Corollary 2.57. Let $n \in \mathbb{N}$ then

$$\frac{n!x}{(1-x)^{n+1}} = \sum_{k \geq 0} f_n\big((St_n)_k\big) x^k.$$

Proof. By corollary 2.52 we have that

$$\frac{n!x}{(1-x)^{n+1}} = \frac{\sum_{k=1}^{n} H_{k,n}(L_n) x^{n-k}}{(1-x)^{n+1}}.$$

So by proposition 2.56 and the correlation between St_n and L_n, we get that

$$\frac{\sum_{k=1}^{n} H_{k,n}(L_n) x^{n-k}}{(1-x)^{n+1}} = \frac{\sum_{k=0}^{n} \big(\sum_{j=1}^{k} j! S_{j,k} x^j (1-x)^{n-j} \big) f_{n-k}(St_n)}{(1-x)^{n+1}},$$

hence we are done by theorem 2.53. □

3 Weighted Rook Theory

In this section we will first introduce the concept of weights and study two important special cases, prominent in all of combinatorics, and their properties. Afterwards we will analyze, to which extent the results from the section on classical rook theory can be generalized to the weighted case. We shall always state the most general version of the result. The hierarchy which we are going to follow, will be made clear in the section on weights 3.1.5. This will also explain how we can obtain certain weights from the others. In the last portion we will discuss problems arising in this process and why certain theorems are not fully generalized yet.

3.1 On weights

3.1.1 q-Weights

In this section we will give a recap on the well known theory of q-weights and introduce some notation. We will only state results useful for our investigations and will not motivate any of the theory. If you are completely new to the topic, then, in my opinion, a nice introductory read would be chapter 10 of [4]. So let us start with an indeterminate q over the field $\mathbb{C}$. The idea is, to replace the number k by the polynomial $1 + q + ... + q^{k-1} = \frac{1-q^k}{1-q}$. This leads to the following definitions.

Definition 3.1. Let q be an indeterminate over $\mathbb{C}$, $k \in \mathbb{Z}$, then we define the small q-weight by

$$w_q(k) \stackrel{\text{def}}{=} q$$

and the big q-weight by

$$W_q(k) \stackrel{\text{def}}{=} q^k$$

Note that for any $k \in \mathbb{N}_0$ we have that

$$W_q(k) = \prod_{j=1}^{k} w_q(j), \tag{3.1}$$

which might seem like a trivial statement in this setting. But it is an reoccurring theme throughout this section so worth keeping in mind.

Definition 3.2. Let q be an indeterminate over $\mathbb{C}$, $k \in \mathbb{C}$, then we define the q-number of k by $[k]_q \stackrel{\text{def}}{=} \frac{1-q^k}{1-q}$.

Note that for $k \in \mathbb{N}$, we have that $[k]_q = 1 + q + ... + q^{k-1}$. Here we can make again a trivial observation, which will motivate certain definitions in a more general setting, that is for $k \in \mathbb{N}$ we have

$$[k]_q = \sum_{j=0}^{k-1} W_q(j). \tag{3.2}$$

We can also define a generalization of the factorial of a number.

Definition 3.3. Let q be an indeterminate over $\mathbb{C}$, $n \in \mathbb{N}$, then we define the q-factorial of n by $[n]_q! \stackrel{\text{def}}{=} [n]_q[n-1]_q...[2]_q[1]_q$.

3.1.2 On Theta-functions

Before we can introduce our next layer in the weight hierarchy, we have to get used to an object from complex analysis: Theta functions. Before we can define them, we will need to deal with some technicalities. This chapter will mainly follow the first four chapters of [6]. We assume some familiarity with basic complex analysis and will not be self contained in this section.

Definition 3.4. A complex function $f : \mathbb{C} \to \mathbb{C}$ is called elliptic function if it is meromorphic on the whole complex plane $\mathbb{C}$ and has two periods η and τ, i.e for all $z \in \mathbb{C}$ one has:

$$f(z+\eta) = f(z) = f(z+\tau)$$

Remark 3.5. We always assume $\eta \neq 0 \neq \tau$ as these cases are trivial. First we observe that upon interchanging $\pm\eta$ and $\pm\tau$, we can always assume $\Im(\frac{\eta}{\tau}) \geq 0$ and if $\lambda = \frac{\eta}{\tau} \in \mathbb{R}$, we get that for all $z \in \mathbb{C}$:

$$f(z+\tau) = f(z) = f(z+\lambda\tau)$$

so our f is either just a periodic function or constant. To see this, denote the set of non trivial periods of f by $P(f) = \{\omega \in \mathbb{C}^* \mid f(z+\omega) = f(z)\ \forall z \in \mathbb{C}\}$. Now observe that if this contains elements of arbitrarily small absolute values, then it contains in particular a sequence $(\omega_n)_{n\in\mathbb{N}} \subset P(f)$ with $\lim_{n\to\infty} |\omega_n| = 0$ but then, we have for all $z \in \mathbb{C}$ that

$$f'(z) = \lim_{n\to\infty} \frac{f(z+\omega_n) - f(z)}{\omega_n} = 0,$$

so f is constant. Similarly if $P(f)$ contains a limit point in the finite portion of the complex plane, say ω, then it contains a sequence $(\omega_n)_{n\in\mathbb{N}} \subset P(f)$ with $\lim_{n\to\infty} \omega_n = \omega$ so the sequence $(\omega_n - \omega)_{n\in\mathbb{N}} \subset P(f)$ satisfies $\lim_{n\to\infty} |\omega_n - \omega| = 0$, so in this case f is constant too.

So we deduce that if f is not constant, then, for any $R > 0$, the set $P(f) \cap \overline{B(0,R)}$ is discrete, where $B(0,R)$ denotes the open disk of radius R with center 0, as any infinite subset of a compact set has a limit point. Therefore, if f has a nontrivial period the set $\{|\omega| \mid \omega \in P(f)\}$ has a minimal element, call it τ. Now clearly for every integer $n \in \mathbb{Z}$ also $n\tau \in P(f)$. If this exhausts the set $P(f)$, i.e. if $P(f) = \{n\tau \mid n \in \mathbb{Z} \setminus \{0\}\}$ then f has only one period. Otherwise we can find a minimal element in $\{|\omega| \mid \omega \in P(f) \setminus \{n\tau \mid n \in \mathbb{Z} \setminus \{0\}\}\}$, say η. If we would have $\lambda = \frac{\eta}{\tau} \in \mathbb{R}$, then we would have $\eta - \{\lambda\}\tau = \lambda\tau - \{\lambda\}\tau = (\lambda - \{\lambda\})\tau \in P(f)$ which contradicts the minimality of τ, if $\lambda \notin \mathbb{Z}$.

So if $\lambda = \frac{\eta}{\tau} \in \mathbb{R}$ we either have that f is constant or that $\lambda \in \mathbb{Z}$ and hence f has only one period.

So to avoid more trivialities we can even assume $\frac{\eta}{\tau} \in \mathbb{C} \setminus \mathbb{R}$, i.e. $\Im(\frac{\eta}{\tau}) > 0$.

We can simplify our definition even more by the change of variables $z \mapsto z' = \tau z$ which yields, denoting $\alpha = \frac{\eta}{\tau}$:

$$f(z'+1) = f(z+\tau) = f(z) = f(z+\eta) = f(z'+\alpha)$$

Which leads to our next definition.

Definition 3.6. A complex function $f : \mathbb{C} \to \mathbb{C}$ is called additively elliptic function if it is non constant and meromorphic on the whole complex plane $\mathbb{C}$ and there exists $\alpha \in \mathbb{C}$ with $\Im(\alpha) > 0$, such that for all $z \in \mathbb{C}$ one has:

$$f(z+1) = f(z) = f(z+\alpha)$$

Remark 3.7. Given an additive elliptic function f, we can introduce a new function $g(e^{2\pi iz}) \overset{\text{def}}{=} f(z)$ or equivalently $g(z) \overset{\text{def}}{=} f\left(\frac{\log(z)}{2\pi i}\right)$. Note that the function g is well defined in this case due to the condition $f(z+1) = f(z)$, which tells us, that for any logarithm $\ln(r) + i(\theta + 2\pi k)$ of $z = re^{i\theta}$ we obtain

$$f\left(\frac{\ln(r) + i(\theta + 2\pi k)}{2\pi i}\right) = f\left(\frac{\ln(r) + i\theta}{2\pi i} + 1\right) = f\left(\frac{\ln(r) + i\theta}{2\pi i}\right),$$

so our function is independent of the choice of the branch for the logarithm. Now our function g is meromorphic on the punctured complex plane $\mathbb{C}^*$, as the logarithm is holomorphic on this domain. Our second additive period α of f now becomes a multiplicative period $\rho \overset{\text{def}}{=} e^{2\pi i\alpha}$ of g, since for all $z = re^{i\theta} \in \mathbb{C}$:

$$g(\rho z) = f\left(\frac{\log(re^{2\pi i(\alpha+\theta)})}{2\pi i}\right) = f\left(\frac{\ln(r) + i2\pi(\alpha + \theta)}{2\pi i}\right) =$$
$$f\left(\frac{\ln(r) + i2\pi\theta}{2\pi i} + \alpha\right) = f\left(\frac{\ln(r) + i2\pi\theta}{2\pi i}\right) = f\left(\frac{\log(re^{2\pi i\theta})}{2\pi i}\right) = g(z)$$

Since we assumed $\Im(\alpha) > 0$, we have $0 < |\rho| = e^{-2\pi\Im(\alpha)} < 1$. This leads to our third definition.

Definition 3.8. A complex function $f : \mathbb{C} \to \mathbb{C}$ is called multiplicatively elliptic function, if it is non constant, meromorphic on the punctured complex plane $\mathbb{C}^*$ and there exists a $\rho \in \mathbb{C}$ with $0 < |\rho| < 1$, such that for all $z \in \mathbb{C}$ we have:

$$f(\rho z) = f(z)$$

Definition 3.9. For complex numbers $a, q \in \mathbb{C}$, we define the q-shifted factorial recursively by

$$(a; q)_0 \overset{\text{def}}{=} 1,$$
$$(a; q)_k \overset{\text{def}}{=} (a; q)_{k-1}(1 - aq^{k-1}),$$

so $(a; q)_k = \prod_{n=0}^{k-1}(1 - aq^n)$ and if $|q| < 1$ we define

$$(a; q)_\infty \overset{\text{def}}{=} \prod_{n=0}^{\infty}(1 - aq^n),$$

which converges absolutely, since $\sum_{n=0}^{\infty} |aq^n|$ is just a geometric series and $|q| < 1$. We also introduce the following shorthand notation for all $a_1, .., a_n \in \mathbb{C}$ and $k = 1, 2, ..., \infty$:

$$(a_1, ..., a_n; q)_k \overset{\text{def}}{=} (a_1; q)_k ...(a_n; q)_k$$

Lemma 3.10. For complex numbers $a, q \in \mathbb{C}$, we have for $k \in \mathbb{N}$ that

$$(1 - a)(qa; q)_k = (a; q)_{k+1}.$$

Proof. We calculate $(qa; q)_k = \prod_{n=0}^{k-1}(1 - (qa)q^n) = \prod_{n=1}^{k}(1 - aq^n)$, so upon multiplying both sides by $(1 - a)$ the claim follows. ☐

Definition 3.11. We define the modified Jacobi-Theta function with argument $x \in \mathbb{C}^*$ and nome $p \in \mathbb{C}$ with $0 < |p| < 1$, by

$$\theta(x;p) \stackrel{\text{def}}{=} (x;p)_\infty (\frac{p}{x};p)_\infty$$

and observe that this is well defined by definition 3.9. We again introduce the following shorthand notation for $x_1, .., x_n \in \mathbb{C}^*$ and $p \in \mathbb{C}$ with $0 < |p| < 1$:

$$\theta(x_1, ..., x_n; p) \stackrel{\text{def}}{=} \theta(x_1;p)...\theta(x_n;p)$$

Lemma 3.12. For $p \in \mathbb{C}$ with $0 < |p| < 1$, $(x;p)_\infty$ is an entire function in x, with zeroes precisely $\{p^k \mid 0 \geq k \in \mathbb{Z}\}$.

Proof. Recall the Taylor-expansion of the logarithm for complex numbers:

$$\log\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

which converges uniformly for $|x| < 1$. Luckily this area of convergence is a possible branch of $\log(z)$, so we can exponentiate both sides to obtain:

$$1 - x = \exp\left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right)$$

Now for any $x \in \mathbb{C}$, we can find a $N \in \mathbb{N}$, such that $|xp^{N+1}| < 1$, so that we can then write:

$$\begin{aligned}
(x;p)_\infty = \prod_{j=0}^{N}(1 - xp^j) \prod_{j=N+1}^{\infty}(1 - xp^j) &\stackrel{(1)}{=} \\
\prod_{j=0}^{N}(1 - xp^j) \prod_{j=N+1}^{\infty} \exp\left(-\sum_{n=1}^{\infty} \frac{(xp^j)^n}{n}\right) &\stackrel{(2)}{=} \\
\left(\prod_{j=0}^{N}(1 - xp^j)\right) \exp\left(-\sum_{j=N+1}^{\infty} \sum_{n=1}^{\infty} \frac{(xp^j)^n}{n}\right) &\stackrel{(3)}{=} \\
\left(\prod_{j=0}^{N}(1 - xp^j)\right) \exp\left(-\sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{j=N+1}^{\infty} (p^n)^j\right) &= \\
\left(\prod_{j=0}^{N}(1 - xp^j)\right) \exp\left(-\sum_{n=1}^{\infty} \frac{x^n p^{(N+1)n}}{n(1-p^n)}\right)&,
\end{aligned}$$

where we justify the steps done, as follows:

1. We can use the above identity, since by assumption $|xp^j| < 1$ for all $j > N$.

2. As our product $\prod_{j=N+1}^{\infty} \exp\left(-\sum_{n=1}^{\infty} \frac{(xp^j)^n}{n}\right) = \prod_{j=N+1}^{\infty}(1 - xp^j)$ converges absolutely, since $(x;p)_\infty$ does, we can use the continuity of the exponential to pull the product into the exponential and write it as a sum.

3. We claim that the double series $\sum_{j=N+1}^{\infty}\sum_{n=1}^{\infty}\frac{(xp^j)^n}{n}$ converges absolutely, which allows us to interchange the two sums as we did. To prove the claim we bound a partial sum for $M > N+1$ and $m > 0$

$$\Big|\sum_{j=N+1}^{M}\sum_{n=1}^{m}\frac{(xp^j)^n}{n}\Big| \le \sum_{j=N+1}^{M}\sum_{n=1}^{m}\Big|\frac{(xp^j)^n}{n}\Big| = \sum_{n=1}^{m}\Big|\frac{x^n}{n}\Big|\sum_{j=N+1}^{M}|p^n|^j =$$
$$\sum_{n=1}^{m}\Big|\frac{(xp^{N+1})^n}{n}\Big|\sum_{j=0}^{M-N-1}|p^n|^j = \sum_{n=1}^{m}\Big|\frac{(xp^{N+1})^n}{n}\Big|\cdot\frac{1-|p^n|^{M-N-1}}{1-|p^n|} \le$$
$$\frac{1}{1-|p|}\sum_{n=1}^{m}\Big|\frac{(xp^{N+1})^n}{n}\Big| \le \frac{1}{1-|p|}\sum_{n=1}^{\infty}\frac{(|xp^{N+1}|)^n}{n} =$$
$$\frac{1}{1-|p|}\log\Big(\frac{1}{1-|xp^{N+1}|}\Big) < \infty.$$

As this bound is independent of M and m, we can take the double limit to deduce that
$$\sum_{j=N+1}^{\infty}\sum_{n=1}^{\infty}\Big|\frac{(xp^j)^n}{n}\Big| < \infty,$$
hence the double series converges absolutely, as it is bounded and all summands are non-negative.

The fact that $(x;p)_\infty$ is holomorphic at every point $x \in \mathbb{C}$, is now obvious, as locally we can express it as the above product of holomorphic functions. The fact that $\{p^k \mid 0 \ge k \in \mathbb{Z}\}$ is part of the zero set is also clear from the expansion. We still have to check that no other zeroes can appear:
So assume $z \in \mathbb{C} \setminus \{p^k \mid 0 > k \in \mathbb{Z}\}$ then we can again expand the shifted factorial locally into the above product and observe that the finite part is clearly nonzero as $z \notin \{p^k \mid 0 > k \in \mathbb{Z}\}$ and the exponential part is nonzero too, since the sum in the exponent is bounded. □

Corollary 3.13. For $p \in \mathbb{C}$ with $0 < |p| < 1$, the modified theta function $\theta(x;p)$ is holomorphic on $\mathbb{C}^*$ and has zeroes precisely $\{p^k \mid k \in \mathbb{Z}\}$.

Proof. By definition $\theta(x;p) = (x;p)_\infty(\frac{p}{x};p)_\infty$, so the function is holomorphic on $\mathbb{C}^*$, since by lemma 3.12 the two factors are holomorphic on the punctured complex plane. Furthermore we know that the zeroes of $(x;p)_\infty$ are given by $\{x = p^k \mid 0 \ge k \in \mathbb{Z}\}$ and the zeroes of $(\frac{p}{x};p)_\infty$ are given by $\{\frac{p}{x} = p^k \mid 0 \ge k \in \mathbb{Z}\}$, i.e. by $\{x = p^k \mid 0 < k \in \mathbb{Z}\}$ which implies the second statement. □

Remark 3.14. The first important observation about the theta function we can make, is, that it is <u>not</u> multiplicatively elliptic. If it were, the only possible period is p, as clearly any other period would contradict the result on our set of zeroes. But now we can simply calculate

$$\theta(px;p) = (px;p)_\infty(\frac{p}{px};p)_\infty = \prod_{n=0}^{\infty}(1-xp^{n+1})\prod_{n=0}^{\infty}\left(1-\frac{p}{px}p^n\right) =$$

$$\prod_{n=1}^{\infty}(1-xp^n)\prod_{n=-1}^{\infty}\left(1-\frac{p}{x}p^n\right) = \left(\frac{1}{1-x}\right)\prod_{n=0}^{\infty}(1-xp^n)\prod_{n=0}^{\infty}\left(1-\frac{p}{x}p^n\right)\left(1-\frac{1}{x}\right) =$$

$$\left(\frac{1}{1-x}\right)(x;p)_\infty(\frac{p}{x};p)_\infty\left(\frac{1-x}{-x}\right) = -\frac{1}{x}\theta(x;p)$$

and see that this is not a period either. Nevertheless

$$\theta(px;p) = -\frac{1}{x}\theta(x;p) \tag{3.3}$$

is an important property and called quasi-periodicity of the theta function.
We oftentimes encounter expressions of the form $\theta(\frac{1}{x};p)$ and so it would be desirable to relate this somehow to $\theta(x;P)$. Indeed a little calculation shows that

$$\theta(\frac{1}{x};p) = (\frac{1}{x};p)_\infty(px;p)_\infty = \prod_{n=0}^{\infty}\left(1-\frac{p^n}{x}\right)\prod_{n=0}^{\infty}(1-xp^{n+1}) =$$

$$\prod_{n=0}^{\infty}\left(1-\frac{pp^{n-1}}{x}\right)\prod_{n=1}^{\infty}(1-xp^n) = \left(1-\frac{1}{x}\right)\prod_{n=0}^{\infty}\left(1-\frac{p}{x}p^n\right)\prod_{n=0}^{\infty}(1-xp^n)\left(\frac{1}{1-x}\right) =$$

$$\left(\frac{1}{1-x}\right)(\frac{p}{x};p)_\infty(x;p)_\infty\left(\frac{1-x}{-x}\right) = -\frac{1}{x}\theta(x;p).$$

So we obtain another important formula for later on:

$$\theta(\frac{1}{x};p) = -\frac{1}{x}\theta(x;p) \tag{3.4}$$

The most fundamental result for theta-functions will follow now.

Theorem 3.15. (Three-term-identity): Let $p \in \mathbb{C}$ with $0 < |p| < 1$, then for all $u, x, y, z \in \mathbb{C}^*$, we have that

$$\theta(xy, \frac{x}{y}, uz, \frac{u}{z};p) - \theta(xz, \frac{x}{z}, uy, \frac{u}{y};p) = \frac{u}{y}\theta(yz, \frac{y}{z}, xu, \frac{x}{u};p). \tag{3.5}$$

Proof. We will use Liouville's theorem. Let us define the auxiliary function:

$$f(y) \stackrel{\text{def}}{=} \theta(xy, \frac{x}{y}, uz, \frac{u}{z};p) - \theta(xz, \frac{x}{z}, uy, \frac{u}{y};p) - \frac{u}{y}\theta(yz, \frac{y}{z}, xu, \frac{x}{u};p)$$

Then we can observe the following:

- $f(z) = \theta(xz, \frac{x}{z}, uz, \frac{u}{z};p) - \theta(xz, \frac{x}{z}, uz, \frac{u}{z};p) - \frac{u}{z}\theta(z^2, 1, xu, \frac{x}{u};p) = 0$, since $\theta(1) = 0$.
- $f(z^{-1}) = \theta(\frac{x}{z}, xz, uz, \frac{u}{z};p) - \theta(\frac{x}{z}, xz, uz, \frac{u}{z};p) - uz\theta(1, z^2, xu, \frac{x}{u};p) = 0$

- By formulae (3.3) & (3.4) we see that

$$f(py) =$$
$$\theta(xpy, \frac{x}{py}, uz, \frac{u}{z}; p) - \theta(xz, \frac{x}{z}, upy, \frac{u}{py}; p) - \frac{u}{py}\theta(pyz, \frac{py}{z}, xu, \frac{x}{u}; p) =$$
$$\frac{1}{py^2}\theta(xy, \frac{x}{y}, uz, \frac{u}{z}; p) - \frac{1}{py^2}\theta(xz, \frac{x}{z}, uy, \frac{u}{y}; p) - \frac{1}{p}y^2\frac{u}{y}\theta(yz, \frac{y}{z}, xu, \frac{x}{u}; p) =$$
$$\frac{1}{py^2}f(y)$$

Therefore, our function f vanishes at all of $zp^{\mathbb{Z}} \cup z^{-1}p^{\mathbb{Z}}$. By lemma 3.12, the following auxiliary function is holomorphic on $\mathbb{C}^*$:

$$g(y) \stackrel{\text{def}}{=} \frac{f(y)}{\theta(zy, zy^{-1}; p)}$$

Now again by formulae (3.3) & (3.4), we obtain that $g(py) = g(y)$, so that our function is multiplicatively elliptic. Hence it is bounded and so by Liouville's theorem it is constant. Since in particular for $u \neq z^{\pm 1}$ we have $f(u) = 0$, also $g(u) = 0$ and so $f(y) \equiv 0$ on $\mathbb{C}^*$, hence we are done. □

3.1.3 Elliptic Weights

In this section we will introduce the elliptic weights defined by M. J. Schlosser in [5]. We will make use of various results from the previous section on Theta-functions to deduce certain properties of these weights. So let us begin with a definition.

Definition 3.16. Let $a, b, q, p \in \mathbb{C}^*$ with $0 < |p| < 1$, $k \in \mathbb{C}$, then we define the small elliptic weight by

$$w_{a,b;q,p}(k) \stackrel{\text{def}}{=} \frac{\theta(aq^{2k+1}, bq^k, \frac{aq^{k-2}}{b}; p)}{\theta(aq^{2k-1}, bq^{k+2}, \frac{aq^k}{b}; p)}q$$

and we define the big elliptic weight by

$$W_{a,b;q,p}(k) = \frac{\theta(aq^{2k+1}, bq, bq^2, \frac{aq^{-1}}{b}, \frac{a}{b}; p)}{\theta(aq, bq^{k+1}, bq^{k+2}, \frac{aq^{k-1}}{b}, \frac{aq^k}{b}; p)}q^k.$$

These are the weights as defined by M. J. Schlosser. Note that for $k \in \mathbb{N}_0$ we have, similar to formula (3.1), that

$$W_{a,b;q,p}(k) = \prod_{j=1}^{k} w_{a,b;q,p}(j) \tag{3.6}$$

since the RHS is a telescoping product by definition of the small elliptic weights. And where the empty product equals 1 and indeed $W_{a,b;q,p}(0) = 1$ also by definition of the elliptic big weight. Furthermore observe, that for two arbitrary $k, y \in \mathbb{C}$, we have that

$$w_{a,b;q,p}(k + j) = w_{aq^{2k},bq^k;q,p}(j), \tag{3.7}$$

so that we can "hide" one summand in the argument of the elliptic small weight. This is an important property constantly made use of in [9] to simplify all sorts of formulae. We will also make use of it a few times, so keep it in the back of your head.

Remark 3.17. Let us give a quick remark on the history of these weights. The goal in [5] was, to generalize the q-binomial theorem to the elliptic setting. In [7] appeared binomial coefficients generalised to the elliptic setting, but these were not elliptic functions in the sense of the previous section. M. J. Schlosser managed to resolve this problem with his, actually elliptic, weights, hence, why they deserve their name. One way to see this, is to write the parameters in polar coordinates, i.e. $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, $a = q^{\alpha}$ and $b = q^{\beta}$, where $\sigma, \tau, \alpha, \beta \in \mathbb{C}$. Then the small weight $w_{a,b;q,p}(k)$ is seen to be periodic in α, β and k with periods σ^{-1} and $\tau\sigma^{-1}$, upon making use of formulae (3.3) and (3.4). For more details check [5].

Definition 3.18. Let $z \in \mathbb{C}$, $a, b, q, p \in \mathbb{C}^*$ with $0 < |p| < 1$, then we define the <u>elliptic number</u> of z, by

$$[z]_{a,b;q,p} \overset{\text{def}}{=} \frac{\theta(q^z, aq^z, bq^2, \frac{a}{b}; p)}{\theta(q, aq, bq^{z+1}, \frac{aq^{z-1}}{b}; p)}.$$

We also obtain a similar property to formula (3.2) in this elliptic setting.

Lemma 3.19. Let $z \in \mathbb{C}$, $a, b, q, p \in \mathbb{C}^*$ with $0 < |p| < 1$, then

$$[z]_{a,b;q,p} = [z-1]_{a,b;q,p} + W_{a,b;q,p}(z-1) \tag{3.8}$$

Proof. Rewriting the equality in terms of theta-functions, we have to show that:

$$\frac{\theta(q^z, aq^z, bq^2, \frac{a}{b}; p)}{\theta(q, aq, bq^{z+1}, \frac{aq^{z-1}}{b}; p)} =$$

$$\frac{\theta(q^{z-1}, aq^{z-1}, bq^2, \frac{a}{b}; p)}{\theta(q, aq, bq^z, \frac{aq^{z-2}}{b}; p)} + \frac{\theta(aq^{2(z-1)+1}, bq, bq^2, \frac{aq^{-1}}{b}, \frac{a}{b}; p)}{\theta(aq, bq^z, bq^{z+1}, \frac{aq^{z-2}}{b}, \frac{aq^{z-1}}{b}; p)} q^{z-1}$$

Therefore, upon multiplying both sides of the equation by their common denominator $\theta(q, aq, bq^z, bq^{z+1}, \frac{aq^{z-1}}{b}, \frac{aq^{z-2}}{b}; p)$ and dividing out the common factor $\theta(bq^2, \frac{a}{b}; p)$, we have:

$$\theta(q^z, aq^z, bq^z, \frac{aq^{z-2}}{b}; p) =$$

$$\theta(q^{z-1}, aq^{z-1}, bq^{z+1}, \frac{aq^{z-1}}{b}; p) + q^{z-1}\theta(aq^{2(z-1)+1}, bq, \frac{aq^{-1}}{b}, q; p)$$

Now using formula (3.4) on the last summand twice, we obtain:

$$\theta(q^z, aq^z, bq^z, \frac{aq^{z-2}}{b}; p) =$$

$$\theta(q^{z-1}, aq^{z-1}, bq^{z+1}, \frac{aq^{z-1}}{b}; p) + \frac{aq^{z-1}}{b}\theta(aq^{2(z-1)+1}, bq, \frac{b}{aq^{-1}}, \frac{1}{q}; p)$$

Recall formula (3.5). Then the above follows directly by setting $x = a^{\frac{1}{2}}q^{z-1}$, $y = ba^{-\frac{1}{2}}q$, $z = a^{\frac{1}{2}}$, $u = a^{\frac{1}{2}}q^z$ in formula (3.5). □

In particular, for $z \in \mathbb{N}$, using formula (3.8) iteratively, yields

$$[z]_{a,b;q,p} = \sum_{k=0}^{z-1} W_{a,b;q,p}(k),$$

which is similar to formula (3.2).

Remark 3.20. Note that elliptic functions form a field and satisfy various explicit addition theorems such as formula (3.5). More on this can be found in [6]. Thus it is natural to use elliptic weights, since by those addition theorems we can handle all sorts of expressions involving those weights, meaning we can obtain closed forms when counting those weights w.r.t. a certain statistic. Moreover, the theories of hypergeometric and basic hypergeometric series are known to extend to elliptic hypergeometric series. This makes it natural to consider elliptic weights in settings (such as rook theory) where hypergeometric and basic hypergeometric series are involved, as one can hope that many results that hold in the ordinary and basic cases admit extensions to the elliptic case. Now it is again natural to ask if one can generalize (parts of the theory) even further to weights involving arbitrary sequences of variables, not restricted to the elliptic case. The next section introduces the language and machinery needed for this project.

3.1.4 General weights

In this section we introduce the general concept of weights and some notation which we will constantly make use of. We start with our definition of weights.

Definition 3.21. Let $(R, +, \cdot)$ be a commutative, unitary, integral domain, then a function $\mathrm{w} : \mathbb{Z} \to F(R)$ is called <u>small weight</u>.

The idea now is, to reproduce the properties (3.1), (3.2), (3.6) and (3.8) of the previous sections on weights. The following definitions are exact mirror images of this mindset.

Definition 3.22. Let $(R, +, \cdot)$ be a commutative, unitary, integral domain. Given a small weight $\mathrm{w} : \mathbb{Z} \mapsto F(R)$, we define $\mathrm{W} : \mathbb{Z} \times \mathbb{Z} \to F(R)$ by $\mathrm{W}(k,l) \stackrel{\text{def}}{=} \prod_{j=l}^{k} \mathrm{w}(j)$, where for $k > l$ the empty product equals 1 by definition. We call this W <u>big weight</u>.

Definition 3.23. Let $(R, +, \cdot)$ be a commutative, unitary, integral domain. Given a small weight $\mathrm{w} : \mathbb{Z} \mapsto F(R)$, we define the <u>weighted difference</u> $[-,-] : \mathbb{Z} \times \mathbb{Z} \to R$ by $[n,m] \stackrel{\text{def}}{=} \sum_{k=m-1}^{n-1} \mathrm{W}(k,m)$, where the empty sum equals 0 by definition.

Definition 3.24. Let $(R, +, \cdot)$ be a commutative, unitary, integral domain. Given a small weight $\mathrm{w} : \mathbb{Z} \mapsto F(R)$ and $n \in \mathbb{N}$ we define the <u>weighted natural number</u> by $[n] \stackrel{\text{def}}{=} [n, 1]$.

We close this section with a lemma summing up a few other/new properties of these just defined objects. The proofs are all straightforward manipulations of the terms. Nevertheless these basic properties will be made use of quite heavily later on, so they are worth remembering.

Lemma 3.25. Let $(R, +, \cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, then:

1. For three integers $n \geq m \geq l$ we have that

$$\mathrm{W}(n,l) = \mathrm{W}(n,m)\mathrm{W}(m-1,l). \tag{3.9}$$

2. For two integers $n \geq m \in \mathbb{N}$ we have that

$$[n,m] = \frac{1}{\mathrm{W}(m-1,1)}([n]-[m-1]). \tag{3.10}$$

3. For two integers $n \in \mathbb{N}$ and $0 \geq m \in \mathbb{Z}$ we get that

$$[n,m] = \mathrm{W}(0,m)[n,1] + [0,m]. \tag{3.11}$$

Proof. 1. Let $n \geq m \geq l \in \mathbb{Z}$, then by definition, we see that:

$$\mathrm{W}(n,l) = \prod_{k=l}^{n} \mathrm{w}(k) = \prod_{k=m}^{n} \mathrm{w}(k) \prod_{k=l}^{m-1} \mathrm{w}(k) = \mathrm{W}(n,m)\mathrm{W}(m-1,l)$$

2. Let $n \geq m \in \mathbb{N}$, then by definition, we have that:

$$[n,m] = \sum_{k=m-1}^{n-1} W(k,m) = \frac{1}{W(m-1,1)} \sum_{k=m-1}^{n-1} W(k,m)W(m-1,1) \overset{(3.9)}{=}$$

$$\frac{1}{W(m-1,1)} \sum_{k=m-1}^{n-1} W(k,1) =$$

$$\frac{1}{W(m-1,1)}\left(\sum_{k=0}^{n-1} W(k,1) - \sum_{k=0}^{m-2} W(k,1)\right) = \frac{1}{W(m-1,1)}([n,1]-[m,1])$$

3. Let $n \in \mathbb{N}$ and $0 \leq m \in \mathbb{Z}$, then by definition, we obtain:

$$[n,m] = \sum_{k=m-1}^{n-1} \mathrm{W}(k,m) = \sum_{k=0}^{n-1} \mathrm{W}(k,m) + \sum_{k=m-1}^{-1} \mathrm{W}(k,m) \overset{(3.9)}{=}$$

$$\sum_{k=0}^{n-1} \mathrm{W}(k,1)\mathrm{W}(0,m) + [0,m] = \mathrm{W}(0,m)[n,1] + [0,m]$$

□

3.1.5 The weight hierarchy

As promised at the beginning of the chapter, we are now going to relate all of the previously defined weights with each other. The following should not be very surprising, given the fact that we built our general weights upon the properties of the previously defined ones.

Proposition 3.26. Let q be an indeterminate over $\mathbb{C}$. Setting $\mathrm{w} : \mathbb{Z} \to \mathbb{C}(q), j \mapsto q = w_q(j)$, we obtain back the small q-weight. Furthermore:

- $\mathrm{W} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}(q), (j,k) \mapsto \prod_{l=k}^{j} \mathrm{w}(l) = q^{j-(k-1)} = W_q(j-k+1)$ is a shifted big q-weight.

- $[-,-] : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}(q), (n,m) \mapsto \sum_{k=m-1}^{n-1} \mathrm{W}(k,m) = \sum_{k=m-1}^{n-1} q^{k-(m-1)} = \sum_{k=0}^{n-1-(m-1)} q^k = [n-m+1]_q$ is just the q-number of the difference $n-m+1$.
- For $n \in \mathbb{N}$, the weighted natural number $[n]$ equals $1+q+...+q^{n-1} = [n]_q$ the q-number of n.

Proof. This follows directly from the definition of the general weights and the q-weights. ☐

Proposition 3.27. Let $a,b,q,p \in \mathbb{C}^*$ with $0 < |p| < 1$. Setting our small weight to $\mathrm{w} : \mathbb{Z} \to \mathbb{C}(\{w_{a,b;q,p}(k) \mid k \in \mathbb{Z}\}), j \mapsto w_{a,b;q,p}(j)$, we obtain back the elliptic small weight. Furthermore:

- $\mathrm{W} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}(\{w_{a,b;q,p}(k) \mid k \in \mathbb{Z}\})$ given by $(j,k) \mapsto \prod_{l=k}^{j} w_{a,b;q,p}(l) = W_{aq^{2(k-1)},bq^{k-1};q,p}(j-(k-1))$ is a shifted version of the elliptic big weight. In particular for $k=1$ we obtain back the unshifted elliptic big weight.
- $[-,-] : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}(\{w_{a,b;q,p}(k) \mid k \in \mathbb{Z}\}), (n,m) \mapsto \sum_{k=m-1}^{n-1} \mathrm{W}(k,m) = \sum_{k=m-1}^{n-1} W_{aq^{2(m-1)},bq^{m-1};q,p}(k-(m-1)) = [n-m+1]_{aq^{2(m-1)},bq^{m-1};q,p}$ is a shifted version of the elliptic number.
- For $n \in \mathbb{N}$, the weighted natural number $[n]$ equals the elliptic number $[n]_{a,b;q,p}$.

Proof. For the first item let $j \geq k \in \mathbb{Z}$, then by definition we have that:

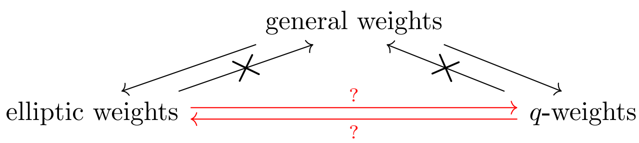
$$\mathrm{W}(j,k) = \prod_{l=k}^{j} \mathrm{w}(l) = \prod_{l=k}^{j} w_{a,b;q,p}(l)$$

Now making use of equation (3.7), we see that this equals

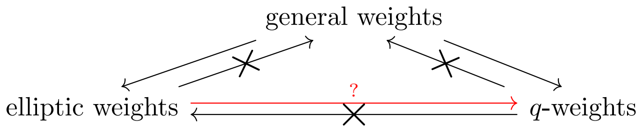
$$\mathrm{W}(j,k) = \prod_{l=k}^{j} w_{a,b;q,p}(j) = \prod_{l=1}^{j-(k-1)} w_{a,b;q,p}(l+(k-1)) \overset{(3.7)}{=}$$
$$\prod_{l=1}^{j-(k-1)} w_{aq^{2(k-1)},bq^{(}k-1);q,p}(l) = W_{aq^{2(k-1)},bq^{k-1};q,p}(j-(k-1)).$$

Now the second item follows directly from equation (3.8) and the definition of the weighted difference. The third item follows directly from the second one. ☐

So we see, that we can get back the full q- and elliptic-setting from the general weighted case, by specifying the small weights accordingly. Since we clearly cannot obtain the general weights from neither the q-weights nor the elliptic weights, this can be seen by simply comparing the domains of the small weights, we can picture this in the following diagram:



As the red arrows indicate, we are still missing a connection between the elliptic and q-weights. Again, by looking at the domain of the small weights, it is clear, that the elliptic weights are not a special case of the q- weights. Hence we are only left with one question-mark remaining:



The following proposition and corollary shall give an answer to this.

Proposition 3.28. Let $a, b, p, q \in \mathbb{C}^*$ with $0 < |p| < 1$, $k \in \mathbb{Z}$, then

$$\lim_{b\to 0}\lim_{a\to 0}\lim_{p\to 0} w_{a,b;q,p}(k) = q = \lim_{a\to\infty}\lim_{b\to 0}\lim_{p\to 0} w_{a,b;q,p}(k).$$

Proof. We just make the straightforward calculations. First note that, for any $x \in \mathbb{C}$, we have that $\lim_{p\to 0}\theta(x;p) = 1 - x$ by definition of the theta function. Thus

$$\lim_{p\to 0} w_{a,b;q,p}(k) = \frac{(1-aq^{2k+1})(1-bq^k)(1-\frac{aq^{k-2}}{b})}{(1-aq^{2k-1})(1-bq^{k+1})(1-\frac{aq^k}{b})}q.$$

Now we only have to take the two remaining limits in the two different ways:

$$\lim_{b\to 0}\lim_{a\to 0}\frac{(1-aq^{2k+1})(1-bq^k)(1-\frac{aq^{k-2}}{b})}{(1-aq^{2k-1})(1-bq^{k+1})(1-\frac{aq^k}{b})}q =$$
$$\lim_{b\to 0}\frac{1-bq^k}{1-bq^{k+1}}q = q$$

and

$$\lim_{a\to\infty}\lim_{b\to 0}\frac{(1-aq^{2k+1})(1-bq^k)(1-\frac{aq^{k-2}}{b})}{(1-aq^{2k-1})(1-bq^{k+1})(1-\frac{aq^k}{b})}q =$$
$$\lim_{a\to\infty}\left(\frac{(1-aq^{2k+1})}{(1-aq^{2k-1})}q\lim_{b\to 0}\frac{b-aq^{k-2}}{b-aq^k}\right) = \lim_{a\to\infty}\frac{(1-aq^{2k+1})}{(1-aq^{2k-1})q^2}q =$$
$$\lim_{a\to\infty}\frac{(\frac{1}{a}-q^{2k+1})}{(\frac{q^2}{a}-q^{2k+1})}q = q.$$ ∎

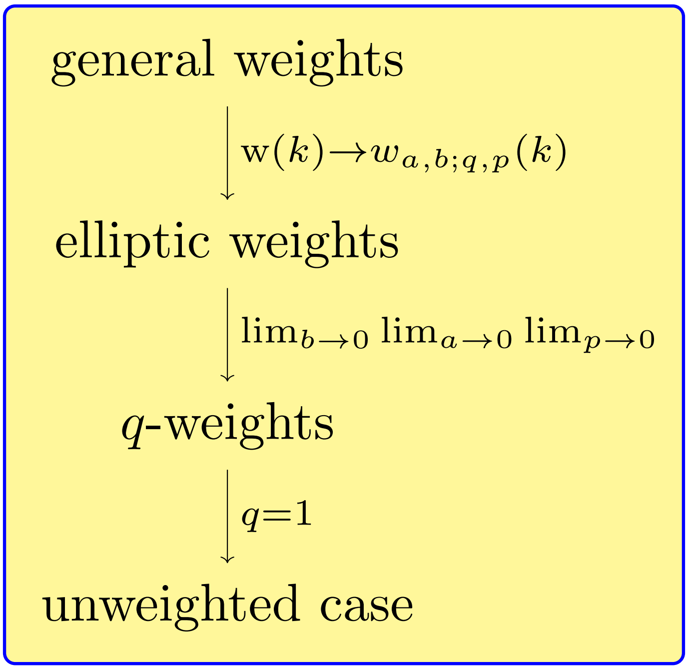
So we see, that taking limits in the elliptic case, yields the q case.

Corollary 3.29. Let $a, b, p, q \in \mathbb{C}^*$ with $0 < |p| < 1$, $k \in \mathbb{Z}$, then upon taking the limits $p \to 0$, $a \to 0$ and $b \to 0$ in this exact order, or equivalently $p \to 0$, $b \to 0$ and $a \to \infty$ and denoting this for simplicity by lim, we obtain for $k \in \mathbb{Z}$ and $c \in \mathbb{C}$ that

- $\lim w_{a,b;q,p}(k) = q$,
- $\lim W_{a,b;q,p}(k) = q^k$,
- $\lim [c]_{a,b;q,p} = [c]_q$.

Proof. This follows directly from proposition 3.28, when using equations (3.6), (3.8) and (3.1), (3.2). The last point for general $c \in \mathbb{C}$ follows, by directly taking the limits as in proposition 3.28. $\square$

Note that upon setting $q = 1$ in the q-case, we always obtain back the unweighted, classical case. All-together, we obtain the following weight hierarchy:



Remark 3.30. Note that that there are many more interesting weights that are worth considering. One might for example consider the symmetric q-weights used in mathematical physics, more precisely in quantum groups, where one defines the symmetric q-number of $n \in \mathbb{N}$ via

$$[n]_q^s \overset{\text{def}}{=} \frac{q^n - q^{-n}}{q - q^{-1}}.$$

This is symmetric w.r.t. $q \Leftrightarrow q^{-1}$, which is a property that comes in handy quite often. Also note that this is closely related to the q-numbers we introduced, since:

$$[n]_q^s = \frac{q^{-n}}{q^{-1}} \frac{q^{2n} - 1}{q^2 - 1} = q^{1-n} [n]_{q^2}$$

This definition of a symmetric q-number now leads to the big symmetric q-weights and the small symmetric q-weights via the definitions 3.22 and 3.23. One can also obtain these weights in a simpler fashion as a specialization of the elliptic weights. For this one considers the limit $p \to 0$ and afterwards lets $b \to 0$, or equivalently $b \to \infty$, in the small elliptic weights $w_{a,b;q,p}(k)$, to obtain the small $a;q$-weights $w_{a;q}(k) = \frac{1-aq^{2k+1}}{1-aq^{2k-1}} q^{-1}$. Now one sets a to -1 to obtain the small symmetric q-weight

$$w_q^s(k) \overset{\text{def}}{=} \frac{1 + q^{2k+1}}{1 + q^{2k-1}} q^{-1}.$$

Observe that these are also symmetric w.r.t. $q \Leftrightarrow q^{-1}$. See that these $a;q$-weights somehow interpolate between the symmetric and ordinary q-weights as both can be obtained from them when specializing a. These $a;q$-weights are of interest on their own too, oftentimes giving the first insight into an approach to the elliptic setting when only the q-case is known.
Another weight of interest is the one encountered in example 3.52 and similar ones connected to all sorts of symmetric functions. Due to space limitations we will not be able to cover all of them here in detail. This remark is just to show that there are many more interesting applications of the general weighted setting and that the weight hierarchy depicted above is just a small branch of the big tree spanned by the interconnections of all possible weights.

3.2 On statistics & weighted rook- and file-numbers

Now that we have introduced certain weights, we will also need a statistic in order to do some weighted counting. In this section we will first introduce two statistics defined by A. M. Garsia and J. B. Remmel in [8] and then combine them with the different weights from the previous chapter, to define different weighted versions of the rook- and the file-numbers from definition 1.11.
As usual, we begin with the case related to the rook-numbers.

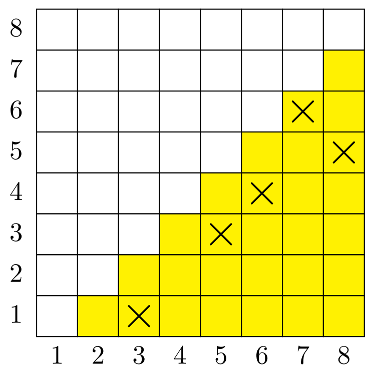
Definition 3.31. Let $B \subset \mathbb{Z} \times [n]$ be a board, $k \in [n]_0$, $P \in \mathcal{N}_k(B)$. Then, for every rook in P place dots in all the cells below the rook in its column and all the cells to the right of the rook in its row. Now denote by

- $\mathrm{Inv}_B(P)$ the set of cells of B not containing a dot or a rook in P,
- $\mathrm{inv}_B(P) \stackrel{\text{def}}{=} |\mathrm{Inv}_B(P)|$,
- $\mathrm{TL}_{(i,j)}(P)$ the number of rooks to the top left of the cell (i,j) of B in P or equivalently the number of dots in the i-column above the cell (i,j).

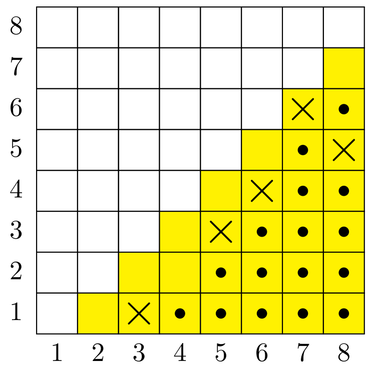
Then $\mathrm{inv}(P)$ is the inversion type statistic introduced by Garsia and Remmel in [8], whereas the definition of $\mathrm{TL}_{(i,j)}(P)$ is due to Schlosser and Yoo in [9].

Before examining these new objects any further, let us look at some examples.

Example 3.32. Let us consider the following placement of 5 rooks on St_8:



Now inserting the dots as described above, we obtain the following picture:

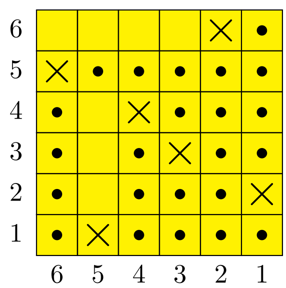


Hence we obtain $\mathrm{Inv}(P) = \{(2,1),(3,2),(4,2),(4,3),(5,4),(6,5),(8,7)\}$ and $\mathrm{inv}(P) = 7$ for this placement. For example choosing the cells $(4,1),(8,2)$, we obtain that $\mathrm{TL}_{(4,1)}(P) = 0$ and $\mathrm{TL}_{(8,2)}(P) = 3$ respectively.

Remark 3.33. Considering a placement corresponding to a permutation explains why our statistic is denoted by Inv. Take for example $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 3 & 4 & 1 & 5 \end{pmatrix} \in S_6$ and quickly recall the following definition:

Definition 3.34. Let $n \in \mathbb{N}$, $\sigma \in S_n$. A pair $(i,j) \in [n] \times [n]$ with $i < j$ is called inversion of σ if $\sigma(i) > \sigma(j)$.

If we now index our columns of the board from right to left and insert dots into the corresponding placement as described, we obtain the following:



In this case, we get that $\mathrm{Inv}(P_\sigma) = \{(3,6),(4,6),(5,2),(5,3),(5,4),(5,6),(6,6)\}$. But this is exactly the set of inversions of σ! A little thought about how inversions are displayed in the graph of the permutation tells us, that this is no coincidence and indeed always holds. So we see, that this statistic of Garsia and Remmel is some kind of generalisation of the inversion statistic for permutations.

Now we will consider the case related to file-numbers. The following definition is also due to Garsia and Remmel, but did not make it into the final version of [8] as mentioned on page 30 of [9].

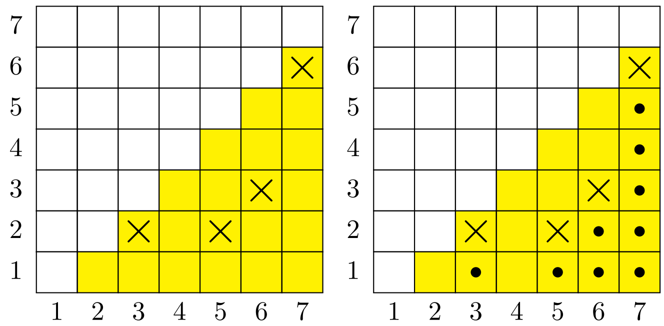
Definition 3.35. Let $B \subset \mathbb{Z} \times [n]$ be a skyline-board, $k \in [n]_0$, $P \in \mathcal{F}_k(B)$. Then, for every rook in P place dots in all cells below the rook in its column. Now denote by

- $U_B(P)$ the set of cells of B not containing a dot or a rook in P,

- $u_B(P) \stackrel{\text{def}}{=} |U_B(B)|$.

Again we consider an example.

Example 3.36. The file placement below on the left leads to the diagram below on the right, when inserting all the dots as described.



So that $U_{St_7}(P) = \{(2,1),(4,1),(4,2),(4,3),(5,3),(5,4),(6,4),(6,5)\}$ and hence in this case $u_{St_7}(P) = 8$.

Now that we found our statistics, we can define the weighted versions of the rook- and file- numbers. By our weight hierarchy, it suffices to define them in the general weighted setting, as this will give us the others for free. For this we will make use of a clever trick from [9].

Definition 3.37. Let $(R,+,\cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $B \subset \mathbb{Z} \times [n]$ a board, $k \in [n]_0$. Then we define the <u>k-th weighted rook-number</u> of the board B, by

$$r_k^{\mathrm{w}}(B) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{N}_k(B)} \prod_{(i,j) \in \mathrm{Inv}_B(P)} \mathrm{w}\big(i - j - \mathrm{TL}_{(i,j)}(P)\big). \tag{3.12}$$

We often denote by $\mathrm{Wt}_r^{\mathrm{w}}(P \mid B)$ the product $\prod_{(i,j)\in \mathrm{Inv}_B(P)} \mathrm{w}\big(i - j - \mathrm{TL}_{(i,j)}(P)\big)$ for a non-attacking rook placement $P \in \mathcal{N}_k(B)$.

This subtraction of the term $\mathrm{TL}_{(i,j)}(P)$ in the argument of the weights, is similar to that one used by Schlosser and Yoo in [9]. It allowed them to factor their elliptic rook-numbers in a nice way and will play an analogous role in our investigations.

Definition 3.38. Let $(R,+,\cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $B \subset \mathbb{Z} \times [n]$ a board, $k \in [n]_0$. Then we define the <u>k-th weighted file-number</u> of the board B, by

$$f_k^{\mathrm{w}}(B) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{F}_k(B)} \prod_{(i,j) \in U_B(P)} \mathrm{w}(1 - j). \tag{3.13}$$

We often denote by $\mathrm{Wt}_f^{\mathrm{w}}(P \mid B)$ the product $\prod_{(i,j)\in U_B(P)} \mathrm{w}(1-j)$ for a file- placement $P \in \mathcal{F}_k(B)$.

Now using our weight hierarchy we obtain the following elliptic rook- and file-numbers. These agree with the ones introduced by Schlosser and Yoo in [9].

Definition 3.39. Let $a, b, p, q \in \mathbb{C}^*$ with $0 < |p| < 1$, $B \subset \mathbb{Z} \times [n]$ a board, $k \in [n]_0$. Then we define the <u>k-th elliptic rook-number</u> of the board B, by

$$r_k(a, b; q, p \mid B) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{N}_k(B)} \prod_{(i,j) \in \operatorname{Inv}_B(P)} w_{a,b;q,p}\big(i - j - \operatorname{TL}_{(i,j)}(P)\big). \tag{3.14}$$

Definition 3.40. Let $a, b, p, q \in \mathbb{C}^*$ with $0 < |p| < 1$, $B \subset \mathbb{Z} \times [n]$ a board, $k \in [n]_0$. Then we define the <u>k-th elliptic file-number</u> of the board B, by

$$f_k(a, b; q, p \mid B) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{F}_k(B)} \prod_{(i,j) \in U_B(P)} w_{a,b;q,p}(1 - j). \tag{3.15}$$

Making use of the weight hierarchy once again, we obtain the q-rook- and q-file-numbers introduced by Garsia and Remmel in [8].

Definition 3.41. Let q be an indeterminate over $\mathbb{C}$, $B \subset \mathbb{Z} \times [n]$ a board, $k \in [n]_0$. Then we define the <u>k-th q-rook-number</u> of the board B, by

$$r_k(q \mid B) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{N}_k(B)} \prod_{(i,j) \in \operatorname{Inv}_B(P)} q = \sum_{P \in \mathcal{N}_k(B)} q^{\operatorname{inv}(P)}. \tag{3.16}$$

Definition 3.42. Let q be an indeterminate over $\mathbb{C}$, $B \subset \mathbb{Z} \times [n]$ a board, $k \in [n]_0$. Then we define the <u>k-th q-file-number</u> of the board B, by

$$f_k(q \mid B) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{F}_k(B)} \prod_{(i,j) \in U_B(P)} q = \sum_{P \in \mathcal{F}_k(B)} q^{u_B(P)}. \tag{3.17}$$

3.3 The weighted rook-numbers

In this section we will try to generalise all of the results of section 2.1 on rook-numbers from the classical case, to the weighted one. We will state the highest version w.r.t. weight hierarchy available and indicate which result of the section on the classical rook-numbers we are generalising, by referring to it in brackets next to the number of our new result. Furthermore we will indicate by a q, e, w in the bracket whether this is now the q-, elliptic- or general weighted case. All versions considering weights of lower order in the hierarchy can then be deduced. We will not state them here, as otherwise this thesis would have gotten even longer than it already is. We will although remark most of the time, who discovered the lower order result first.

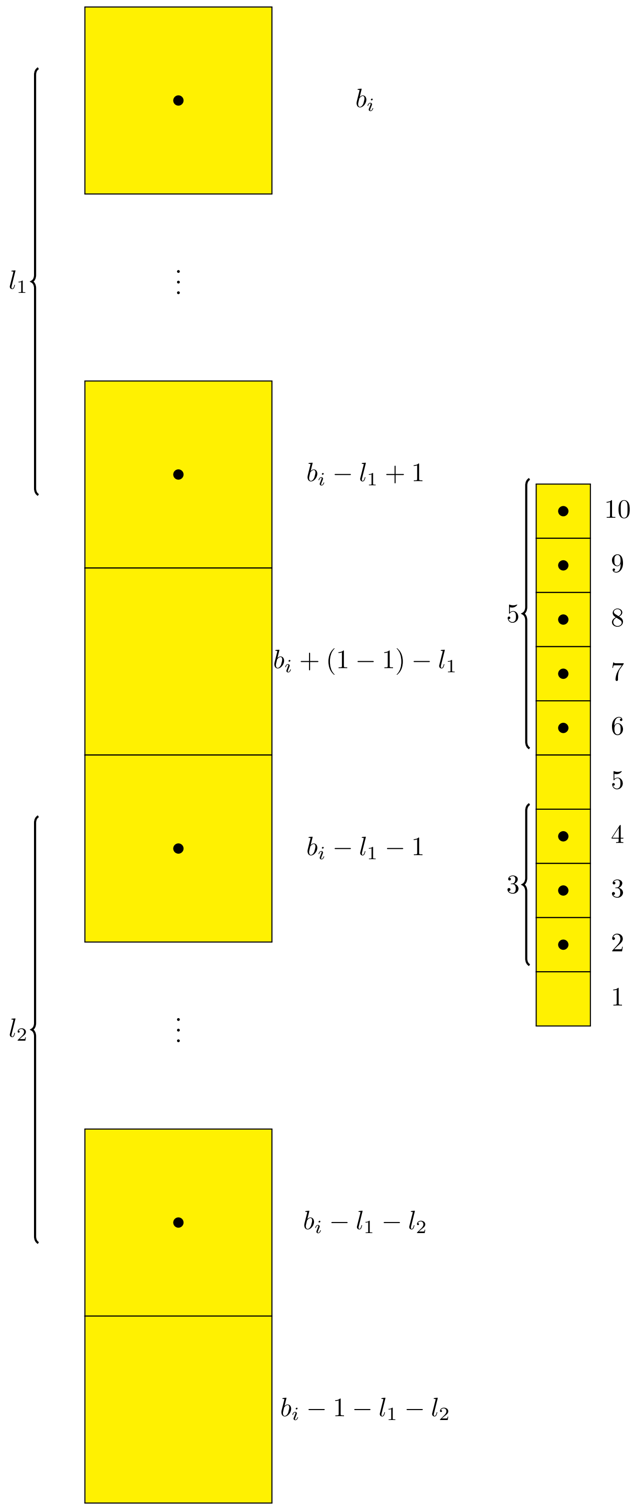
3.3.1 A recursive approach to weighted rook-numbers

Lemma 3.43. (Lemma 2.1, w)
Let $(R, +, \cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $F = F(b_1, ..., b_n)$ a Ferrers-board, $z \in \mathbb{N}_0$, $k \in [n]_0$, $P \in \mathcal{N}_k(F_z)$ a placement of k-rooks in the first $k \leq i-1 < n$ columns of F_z, $\mathcal{P}(P) \subset \mathcal{N}_{k+1}(F(b_1, ..., b_i)_z)$ the set of placements obtained from P by adding a rook in the i-th column of F_z then we have that

$$\sum_{Q \in \mathcal{P}(P)} \mathrm{Wt}_r^{\mathrm{w}}\big(Q \mid F(b_1, ..., b_i)_z\big) = \mathrm{Wt}_r^{\mathrm{w}}\big(P \mid F(b_1, ..., b_{i-1})_z\big)[i + z - k - 1, i - b_i].$$

The elliptic version of the above lemma is due to Schlosser and Yoo and can be found as lemma 10 in [9]. The q-version has never been stated explicitly but was made use of first by Garsia and Remmel in [8]. Before we prove this, let us introduce some notation that will simplify a lot of the proofs to come.

Notation 3.44. Let $F = F(b_1, ..., b_n)$ a Ferrers-board, $z \in \mathbb{N}_0$, $k \in [n]_0$, $i \in [n]$, $P \in \mathcal{N}_k(F_z)$ such that the i-th column does not contain a rook. Now mark all the cells below and to the right of some rook with a dot as described in definition 3.31. Denote the j-th empty cell, i.e. without a rook, counted from top to bottom, in the i-th column by $\big(i, b_i - (j-1) - (l_1 + ... + l_j)\big)$. Where l_1 is the number of dots in the i-th column before the first empty cell, l_2 is the number of dots in the i-th column between the first empty cell and the second empty cell, etc. Zooming in on the i-th column, this might look like the following, really large picture on the next page. Also consider the explicit column on the right of height 10 with the first empty cell being the cell $(1, 5)$ and the second empty cell being the cell $(1, 1)$, hence $l_1 = 10 - (1-1) - 5 = 5$ and $l_2 = 10 - (2-1) - 5 - 1 = 3$. Note that then, by definition $\mathrm{TL}_{(i, b_i-(j-1)-(l_1+...+l_j))} = l_1 + ... + l_j$, since this is just the number of dots above the cell $\big(i, b_i - (j-1) - (l_1 + ... + l_j)\big)$.


b_i
l_1
$b_i - l_1 + 1$
$b_i + (1-1) - l_1$
$b_i - l_1 - 1$
l_2
$b_i - l_1 - l_2$
$b_i - 1 - l_1 - l_2$
10
9
8
7
6
5
4
3
2
1
5
3

Now let us prove the lemma.

Proof. For $i \in [n]$ let us abbreviate $F(b_1,...,b_i)_z$, the board F_z cut off at the i-th column, by F_i. Recall that by definition, for any $Q \in \mathcal{P}(P)$:

$$\mathrm{Wt}_r^{\mathrm{w}}(Q \mid F_i) = \prod_{(l,j)\in \mathrm{Inv}_{F_i}(Q)} \mathrm{w}\big(l-j-\mathrm{TL}_{(l,j)}(Q)\big)$$

Now we can split the product into two parts. One ranging over all cells of $\mathrm{Inv}_{F_i}(Q)$, contained in F_{i-1} and the second product ranging over all cells of $\mathrm{Inv}_{F_i}(Q)$ in the i-th column of F_z. But since by assumption $Q \cap F_{i-1} = P$ we know that $\mathrm{Inv}_{F_i}(Q) \cap F_{i-1} = \mathrm{Inv}_{F_{i-1}}(P)$, since the one additional rook that Q has, compared to P, sits in the rightmost column of F_i, the i-th column, i.e. does not add any dots to F_{i-1}. So we obtain that

$$\mathrm{Wt}_r^{\mathrm{w}}(Q \mid F_i) = \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_{i-1}) \prod_{(i,j)\in \mathrm{Inv}_{F_i}(Q)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(Q)\big),$$

therefore

$$\sum_{Q\in\mathcal{P}(P)} \mathrm{Wt}_r^{\mathrm{w}}(Q \mid F_i) =$$
$$\mathrm{Wt}_r^{\mathrm{w}}(P \mid F_{i-1}) \sum_{Q\in\mathcal{P}(P)} \prod_{(i,j)\in \mathrm{Inv}_{F_i}(Q)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(Q)\big),$$

hence it remains to check that

$$\sum_{Q\in\mathcal{P}(P)} \prod_{(i,j)\in \mathrm{Inv}_{F_i}(Q)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(Q)\big) = [i+z-k-1, i-b_i].$$

Now let us think for a moment ... what do the placements $Q \in \mathcal{P}(P)$ look like? We are adding one new rook to the i-th column, so we are choosing one empty cell of the i-th column, i.e. without a dot, on which we place a rook. Since there are k-rooks to the left of the i-th column, there are $b_i + z - k$ empty cells available. Here is where our notation from 3.44 comes in handy. Let us denote the j-th empty cell, counted from top to bottom, of the i-th row by $\big(i, b_i-(j-1)-(l_1+...+l_j)\big)$. Hence

$$\mathcal{P}(P) = \Big\{P \cup \big\{\big(i, b_i-(j-1)-(l_1+...+l_j)\big)\big\} \mid j \in [b_i+z-k]\Big\},$$

as these are just all the placements obtained from P by placing a rook on one of the empty cells of the i-th column. If we place a rook on the j-th empty cell of the i-th column, then all cells below it will receive a dot and only the $j-1$ empty cells, remaining empty, above it will add to the Inv statistic, i.e. for any $j \in [b_i+z-k]$:

$$\mathrm{Inv}\Big(P \cup \big\{\big(i, b_i-(j-1)-(l_1+...+l_j)\big)\big\}\Big) =$$
$$\big\{\big(i, b_i-(m-1)-(l_1+...+l_m)\big) \mid m \in [j-1]\big\}$$

So we have that

$$\sum_{Q\in\mathcal{P}(P)} \prod_{(i,j)\in \mathrm{Inv}_{F_i}(Q)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(Q)\big) =$$
$$\sum_{j\in[b_i+z-k]} \prod_{m=1}^{j-1} \mathrm{w}\big(i-b_i+(m-1)\big),$$

since for the m-th empty cell, counted from top to bottom, we have

$$\mathrm{w}\big(i-(b_i-(m-1)-(l_1+...+l_m))-\mathrm{TL}_{(i,b_i-(m-1)-(l_1+...+l_m))}(Q)\big) = \\ \mathrm{w}\big(i-b_i+(m-1)\big)$$

by the clever choice of our notation in 3.44. But now this is an easy calculation by our section on the general weights:

$$\sum_{j\in[b_i+z-k]}\prod_{m=1}^{j-1}\mathrm{w}\big(i-b_i+(m-1)\big) \overset{3.22}{=} \sum_{j\in[b_i+z-k]}\mathrm{W}\big(i-b_i+(j-2),i-b_i\big) = \\ \sum_{j=i-b_i-1}^{i+z-k-2}\mathrm{W}\big(j,i-b_i\big) \overset{3.23}{=} [i+z-k-1,i-b_i]$$

So we are done. □

Example 3.45. Note that, by the previous lemma, we can already calculate $r_n^{\mathrm{w}}(F_z)$ for any Ferrers-board $F=F(b_1,...,b_n)$ and $z\in\mathbb{N}_0$. That is we apply the lemma 3.43 iteratively as follows. We keep the notation from lemma 3.43 to write:

$$r_n^{\mathrm{w}}(F_z) = \\ \sum_{Q^{(0)}\in\mathcal{N}_1(F_1)}\sum_{Q^{(1)}\in\mathcal{P}(Q^{(0)})}\sum_{Q^{(2)}\in\mathcal{P}(Q^{(1)})}\cdots\sum_{Q^{(n-1)}\in\mathcal{P}(Q^{(n-2)})}\mathrm{Wt}_r^{\mathrm{w}}\big(Q^{(n-1)}\mid F_z\big) \overset{3.43}{=} \\ \sum_{Q^{(0)}\in\mathcal{N}_1(F_1)}\cdots\sum_{Q^{(n-2)}\in\mathcal{P}(Q^{(n-3)})}\mathrm{Wt}_r^{\mathrm{w}}\big(Q^{(n-2)}\mid F_{z-1}\big)[n+z-(n-1)-1,n-b_n] \\ = \sum_{Q^{(0)}\in\mathcal{N}_1(F_1)}\cdots\sum_{Q^{(n-2)}\in\mathcal{P}(Q^{(n-3)})}\mathrm{Wt}_r^{\mathrm{w}}\big(Q^{(n-2)}\mid F_{z-1}\big)[z,n-b_n] \overset{3.43}{=} ... = \\ \prod_{i=1}^{n}[z,i-b_i]$$

In particular, using 3.26, we obtain for the q-case, that

$$r_n(q\mid F)=\prod_{i=1}^{n}[z+b_i-(i-1)]_q,$$

which we will use later on.

Theorem 3.46. (Theorem 2.2, w)
Let $(R,+,\cdot)$ be a commutative, unitary, integral domain, $\mathrm{w}:\mathbb{Z}\to R$ a small weight, $F=F(b_1,...,b_n)$ a Ferrers board, $z\in\mathbb{N}_0$, $k\in[n]_0$, then

$$r_k^{\mathrm{w}}(F_z)=\mathrm{W}(n+z-k-1,n-b_n)r_k^{\mathrm{w}}(F_z^-)+[n+z-k,n-b_n]r_{k-1}^{\mathrm{w}}(F_z^-),$$

with $r_0^{\mathrm{w}}(\varnothing)=1$, $r_k^{\mathrm{w}}(F(b_1,...,b_j)_z)=0$ if $b_j+z<k$ and $r_k^{\mathrm{w}}(F_z)=0$ if $n<k$.

Again, the elliptic version of the above theorem is due to Schlosser and Yoo and can be found as theorem 14 in [9]. The q-version is due to Garsia and Remmel. It is their theorem 1.1 in [8].

Proof. The initial conditions follow by the same reasoning as in theorem 2.2. For the actual proof, we partition $\mathcal{N}_k(F_z)$ into the two sets

$$\mathcal{P}_1 \overset{\text{def}}{=} \{P \in \mathcal{N}_k(F_z) \mid \text{there is no rook in the } n\text{-th column of } F_z\}$$

and

$$\mathcal{P}_2 \overset{\text{def}}{=} \{P \in \mathcal{N}_k(F_z) \mid \text{there is a rook in the } n\text{-th column of } F_z\}.$$

Then by definition

$$r_k^{\mathrm{w}}(F_z) = \sum_{P \in \mathcal{N}_k(F_z)} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_z) = \sum_{P \in \mathcal{P}_1} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_z) + \sum_{P \in \mathcal{P}_2} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_z).$$

Now for $i \in [n]$ let us again abbreviate $F(b_1, ..., b_i)_z$, the board F_z cut off at the i-th column, by F_i and denote, for fixed $P \in \mathcal{P}_1$, the j-th empty cell, counted from top to bottom, of the n-th row by $(n, b_n - (j-1) - (l_1 + ... + l_j))$. Then note that, for any $P \in \mathcal{P}_1$, we have

$$\mathrm{Inv}_{F_z}(P) = \mathrm{Inv}_{F_{n-1}}(P) \cup \big\{ \big(n, b_n - (j-1) - (l_1 + ... + l_j)\big) \mid j \in [b_n + z - k] \big\},$$

since $P \in \mathcal{P}_1$ is just a placement of k rooks on F_{n-1}. So we see that

$$\sum_{P \in \mathcal{P}_1} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_z) = \sum_{P \in \mathcal{P}_1} \prod_{(i,j) \in \mathrm{Inv}_{F_z}(P)} \mathrm{w}\big(i - j - \mathrm{TL}_{(i,j)}(P)\big) = \sum_{P \in \mathcal{P}_1} \prod_{(i,j) \in \mathrm{Inv}_{F_{n-1}}(P)} \mathrm{w}\big(i - j - \mathrm{TL}_{(i,j)}(P)\big) \prod_{j \in [b_n + z - k]} \mathrm{w}\big(n - b_n + (j-1)\big),$$

since again by our clever choice of notation

$$\mathrm{w}\big(n - (b_n - (j-1) - (l_1 + ... + l_j)) - \mathrm{TL}_{(n, b_n - (j-1) - (l_1 + ... + l_j))}(P)\big) = \mathrm{w}\big(n - b_n + (j-1)\big).$$

But now, this easily simplifies:

$$\sum_{P \in \mathcal{P}_1} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_z) = \sum_{P \in \mathcal{P}_1} \prod_{(i,j) \in \mathrm{Inv}_{F_{n-1}}(P)} \mathrm{w}\big(i - j - \mathrm{TL}_{(i,j)}(P)\big) \prod_{j \in [b_n + z - k]} \mathrm{w}\big(n - b_n + (j-1)\big) \overset{3.22}{=} \sum_{P \in \mathcal{P}_1} \mathrm{W}(n + z - k - 1, n - b_n) \prod_{(i,j) \in \mathrm{Inv}_{F_{n-1}}(P)} \mathrm{w}\big(i - j - \mathrm{TL}_{(i,j)}(P)\big) \overset{3.12}{=} \mathrm{W}(n + z - k - 1, n - b_n) r_k^{\mathrm{w}}(F_{n-1}) = \mathrm{W}(n + z - k - 1, n - b_n) r_k^{\mathrm{w}}(F_z^-),$$

where the last equality is just a change of notation. Now it remains to calculate the second sum. We recall the notation from lemma 3.43. Again, denote for $P \in \mathcal{N}_{k-1}(F_z^-)$, by $\mathcal{P}(P) \subset \mathcal{N}_k(F_z)$ the set of non-attacking rook placements obtained from P by adding a rook in the n-th column. Then clearly

$$\mathcal{P}_2 \cong \coprod_{P \in \mathcal{N}_{k-1}(F_z^-)} \mathcal{P}(P),$$

so using lemma 3.43, we can rewrite the second sum in the following way:

$$\sum_{P\in\mathcal{P}_2} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_z) = \sum_{P\in\mathcal{N}_{k-1}(F_z^-)} \sum_{Q\in\mathcal{P}(P)} \mathrm{Wt}_r^{\mathrm{w}}(Q \mid F_z) \overset{3.43}{=}$$

$$\sum_{P\in\mathcal{N}_{k-1}(F_z^-)} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_z^-)[n+z-(k-1)-1, n-b_n] \overset{3.12}{=}$$

$$[n+z-k, n-b_n] r_{k-1}^{\mathrm{w}}(F_z^-),$$

as desired. □

Before we state the next theorem, we need to generalise a definition from before.

Definition 3.47. Let $(R,+,\cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$, $m \in \mathbb{Z}$, then we define the generalised weighted difference of z and m by

- $[z,m]_g \overset{\mathrm{def}}{=} \frac{1}{\mathrm{W}(m-1,1)}(z-[m-1])$ if $m \in \mathbb{N}$,
- $[z,m]_g \overset{\mathrm{def}}{=} \mathrm{W}(0,m)z + [0,m]$ if $0 \geq m \in \mathbb{Z}$.

Remark 3.48. Note that for $n \in \mathbb{N}$, $m \in \mathbb{Z}$ with $n \geq m$ we obtain by lemma 3.25, that

$$[[n],m]_g = [n,m],$$

so this is a proper generalisation of the concept of weighted differences that we were using until now. The point of this is to make use of the fact that, since $\mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$ is a field, we can use the polynomial argument. This will play an important role in the proof of the next theorem.

Theorem 3.49. (Theorem 2.3, w)
Let $(R,+,\cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $F = F(b_1,...,b_n)$ a Ferrers board, then for all $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$ we have that

$$\sum_{k=0}^{n} r_{n-k}^{\mathrm{w}}(F) \prod_{j=1}^{k} [z,j]_g = \prod_{i=1}^{n} [z, i-b_i]_g.$$

And yet again, the elliptic version of the above theorem is due to Schlosser and Yoo and can be found as theorem 12 in [9]. The q-version is due to Garsia and Remmel. It is their formula (1.3) in [8].

Proof. First of all note that both sides are polynomials in z of degree at most n, by definition of the generalised weighted difference. By our remark 3.48 before, we can use the polynomial argument. So if we can show the equality for at least $n+1$ values in $\mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$, we are done. We will in fact show it for infinitely many values. For this let $n < x \in \mathbb{N}$. We check the equality for $[x] \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$. So by our remark it remains to prove that:

$$\sum_{k=0}^{n} r_{n-k}^{\mathrm{w}}(F) \prod_{j=1}^{k} [x,j] = \prod_{i=1}^{n} [x, i-b_i]$$

for all $n < x \in \mathbb{N}$. We will do this as in the classical case by double counting the placements of n rooks on F_x.

- RHS: We place the rooks column by column. Using the notation from lemma 3.43 and the lemma itself iteratively yields, analogously to example 3.45:

$$r_n^{\mathrm{w}}(F_x) =$$
$$\sum_{Q^{(0)}\in\mathcal{N}_1(F_1)} \sum_{Q^{(1)}\in\mathcal{P}(Q^{(0)})} \sum_{Q^{(2)}\in\mathcal{P}(Q^{(1)})} \dots \sum_{Q^{(n-1)}\in\mathcal{P}(Q^{(n-2)})} \mathrm{Wt}_r^{\mathrm{w}}(Q^{(n-1)} \mid F_x) \overset{3.43}{=}$$
$$\sum_{Q^{(0)}\in\mathcal{N}_1(F_1)} \dots \sum_{Q^{(n-2)}\in\mathcal{P}(Q^{(n-3)})} \mathrm{Wt}_r^{\mathrm{w}}(Q^{(n-2)} \mid F_{x-1})[n+x-(n-1)-1, n-b_n]$$
$$= \sum_{Q^{(0)}\in\mathcal{N}_1(F_1)} \dots \sum_{Q^{(n-2)}\in\mathcal{P}(Q^{(n-3)})} \mathrm{Wt}_r^{\mathrm{w}}(Q^{(n-2)} \mid F_{x-1})[x, n-b_n] \overset{3.43}{=} \dots =$$
$$\prod_{i=1}^{n}[x, i-b_i]$$

- LHS: We first place $n-k$ rooks onto the board F and then the remaining k rooks below the ground. So we consider, as in the classical case, the partition $\dot\bigcup_{k=0}^{n} N_k = \mathcal{N}_n(F_x)$ where $N_k = \{P \in \mathcal{N}_n(F_x) \mid \text{exactly } n-k \text{ rooks are on } F\}$. Then for each $k \in [n]$ we have that N_k is in bijection with $\mathcal{N}_{n-k}(F) \times \mathcal{N}_k([k] \times [x])$. Let us call this bijection ϕ. So we can rewrite the rook-number in the following way:

$$r_n^{\mathrm{w}}(F_x) = \sum_{k=0}^{n} \sum_{(P_1,P_2)\in\mathcal{N}_{n-k}(F)\times\mathcal{N}_k([k]\times[x])} \mathrm{Wt}_r^{\mathrm{w}}\Big(\phi^{-1}\big((P_1,P_2)\big) \mid F_x\Big)$$

Let us fix $(P_1, P_2) \in \mathcal{N}_{n-k}(F) \times \mathcal{N}_k([k] \times [x])$ and now focus on the term $\mathrm{Wt}_r^{\mathrm{w}}\Big(\phi^{-1}\big((P_1,P_2)\big) \mid F_x\Big)$. Let us denote $P = \phi^{-1}\big((P_1,P_2)\big)$. Clearly the rooks placed below the ground do not cause any dots on the board itself. So the inversions on the Ferrers board itself all come from P_1. So we see that

$$\mathrm{Inv}_{F_x}(P) = \big(\mathrm{Inv}_{F_x}(P) \cap F\big) \cup \big(\mathrm{Inv}_{F_x}(P) \cap (F_x \setminus F)\big) =$$
$$\mathrm{Inv}_F(P_1) \cup \big(\mathrm{Inv}_{F_x}(P_1) \cap (F_x \setminus F)\big),$$

hence by definition of the weight, we get that

$$\mathrm{Wt}_r^{\mathrm{w}}(P \mid F_x) = \prod_{(i,j)\in\mathrm{Inv}_{F_x}(P)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(P)\big) =$$
$$\prod_{(i,j)\in\mathrm{Inv}_F(P_1)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(P)\big) \prod_{(i,j)\in\mathrm{Inv}_{F_x}(P)\cap(F_x\setminus F)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(P)\big).$$

We can use this to rewrite the rook-number once again:

$$r_n^{\mathrm{w}}(F_x) = \sum_{k=0}^{n} \sum_{(P_1,P_2)\in\mathcal{N}_{n-k}(F)\times\mathcal{N}_k([k]\times[x])} \mathrm{Wt}_r^{\mathrm{w}}\Big(\phi^{-1}\big((P_1,P_2)\big) \mid F_x\Big) =$$

$$\sum_{k=0}^{n} \sum_{P_1\in\mathcal{N}_{n-k}(F)} \sum_{P_2\in\mathcal{N}_k([k]\times[x])} \mathrm{Wt}_r^{\mathrm{w}}\Big(\phi^{-1}\big((P_1,P_2)\big) \mid F_x\Big) =$$

$$\sum_{k=0}^{n} \sum_{P_1\in\mathcal{N}_{n-k}(F)} \left(\prod_{(i,j)\in\mathrm{Inv}_F(P_1)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(P_1)\big) \right) \cdot$$

$$\sum_{P_2\in\mathcal{N}_k([k]\times[x])} \prod_{(i,j)\in\mathrm{Inv}_{F_x}(\phi^{-1}((P_1,P_2)))\cap(F_x\setminus F)} \mathrm{w}\Big(i-j-\mathrm{TL}_{(i,j)}\Big(\phi^{-1}\big((P_1,P_2)\big)\Big)\Big)$$

Let us ignore the first two sums and first product for a second. It is clear that this will be the part causing the rook-numbers to enter the game. The most inner sum seems more mysterious. But we claim, that we can actually calculate this one. For this fix some $P_1 \in \mathcal{N}_{n-k}(F)$ and note, that there are k columns containing a rook below the ground on F_x in $\phi^{-1}((P_1, P_2))$. Let us denote them by $i_1, ..., i_k$. Then it is clear, that only cells from these k columns contribute to $\mathrm{Inv}_{F_x}(\phi^{-1}((P_1, P_2))) \cap (F_x \setminus F)$, since a rook on the board cancels all the cells in its column below the ground. Furthermore, only the cells below the ground of these columns contribute, since the cells above the ground are already contained in $\mathrm{Inv}_F(P_1)$. So P_2 fully determines the set of cells $\mathrm{Inv}_{F_x}(\phi^{-1}((P_1, P_2)))\cap(F_x\setminus F)$. We can obtain all placements $P_2 \in \mathcal{F}([k]\times[x])$, by placing the rooks column by column and hence calculate the sum in this way. For this denote, similar to 3.44, the j-th empty cell below the ground of the i_t-th column, counted from ground to bottom, by $(i_t, -(j-1)-(l_1+...+l_j))$. With a difference though. Here l_1 denotes the number of dots between the ground and the first empty cell below the ground, l_2 denotes the number of dots between the first and the second empty cell below the ground, etc. After a second of thought it becomes clear, that we can write $i_t = (t-1)+1+r_1+...r_t$, where r_1 is the number of rooks placed on F before column i_1, r_2 is the number of rooks placed between column i_1 and i_2 on F, etc. So in particular for the j-th empty cell of the i_t-th column, counted from ground to bottom, we see that

$$\mathrm{TL}_{(i_t,-(j-1)-(l_1+...+l_j))} = r_1 + ... + r_t + l_1 + ... + l_t,$$

since $r_1 + ... + r_t$ counts exactly the rooks to the top-left of $(i_t, -(j-1) - (l_1 + ... + l_j))$ on the board and $l_1 + ... + l_j$ counts exactly the rooks to the top left of $(i_t, -(j-1) - (l_1 + ... + l_j))$ below the ground. Hence each of these cells $(i_t, -(j-1) - (l_1 + ... + l_j))$, for some j, has the weight

$$\mathrm{w}\big(i_t - (-(j-1) - (l_1 + ... + l_j)) - \mathrm{TL}_{(i_t,-(j-1)-(l_1+...+l_j))}\big) = \mathrm{w}\big(j + (t-1)\big),$$

hence if we place a rook on the j-th empty cell below the ground of the i_t-th column, counted from ground to bottom, then this column contributes the weight

$$\prod_{m\in[j-1]} \mathrm{w}\big(m + (t-1)\big) = \mathrm{W}(j + t - 2, t).$$

Placing the rooks from left to right on the k available columns below the ground, we see that for the i_t-th row we have $x-(t-1)$ cells available, since we placed $t-1$ rooks already to the left of it. So summing over these $x-(t-1)$ cells we obtain the weight

$$\sum_{j=1}^{x-(t-1)} \mathrm{W}(j+t-2,t) = [x,t].$$

So, since there are k columns available and upon summing over all possible placements of rooks below the ground $P_2 \in \mathcal{N}_k([k]\times[x])$, we get that:

$$\sum_{P_2\in\mathcal{N}_k([k]\times[x])} \prod_{(i,j)\in \mathrm{Inv}_{F_x}(\phi^{-1}((P_1,P_2)))\cap(F_x\setminus F)} \mathrm{w}\Big(i-j-\mathrm{TL}_{(i,j)}\Big(\phi^{-1}\big((P_1,P_2)\big)\Big)\Big)$$
$$=\prod_{j=1}^{k}[x,j]$$

Plugging this into the above, we see that:

$$r_n^{\mathrm{w}}(F_x) =$$
$$\sum_{k=0}^{n}\sum_{P_1\in\mathcal{N}_{n-k}(F)}\prod_{(i,j)\in\mathrm{Inv}_F(P_1)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(P_1)\big)\prod_{j=1}^{k}[x,j] =$$
$$\sum_{k=0}^{n}\sum_{P_1\in\mathcal{N}_{n-k}(F)}\prod_{(i,j)\in\mathrm{Inv}_F(P_1)} \mathrm{w}\big(i-j-\mathrm{TL}_{(i,j)}(P_1)\big)\prod_{j=1}^{k}[x,j] \overset{3.12}{=}$$
$$\sum_{k=0}^{n} r_{n-k}^{\mathrm{w}}(F)\prod_{j=1}^{k}[x,j]$$

So we are done. □

Example 3.50. We should, at least once, show, how to deduce the lower order versions of a theorem using the weight hierarchy. We will use theorem 3.49 as an example. Recall that we had just shown, that

$$\sum_{k=0}^{n} r_{n-k}^{\mathrm{w}}(F)\prod_{j=1}^{k}[z,j]_g = \prod_{i=1}^{n}[z,i-b_i]_g,$$

for all $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$. Now by proposition 3.27, we can replace the expressions in the above formula accordingly. That means we set $\mathrm{w}(j)$ to $w_{a,b;q,p}(j)$ for all $j \in \mathbb{Z}$. Let us denote by $[z,m]_{g,e}$

- $[z,m]_{g,e} \overset{\mathrm{def}}{=} \frac{1}{W_{a,b;q,p}(m-1)}(z-[m-1]_{a,b;q,p})$ if $m\in\mathbb{N}$,
- $[z,m]_{g,e} \overset{\mathrm{def}}{=} W_{aq^{2(m-1)},bq^{(m-1)};q,p}\big(-(m-1)\big)z + [-m+1]_{aq^{2(m-1)},bq^{m-1};q,p}$ if $0 \geq m \in \mathbb{Z}$.

then, in particular for $z = [c]_{a,b;q,p}$, for some complex $c \in \mathbb{C}$, we see that

$$[[c]_{a,b;q,p}, m]_{g,e} = [c - m + 1]_{aq^{2(m-1)},bq^{m-1};q,p},$$

for all $m \in \mathbb{Z}$ by formula (3.5). Using all of this

- $r^{\mathrm{w}}_{n-k}(F)$ becomes $r_{n-k}(a, b; q, p \mid F)$ for all $k \in [n]_0$,
- $\prod_{j=1}^{k}[z, j]_g$ becomes $\prod_{j=1}^{k}[z, j]_{g,e}$
- $\prod_{i=1}^{n}[z, i - b_i]_g$ becomes $\prod_{i=1}^{n}[z, i - b_i]_{g,e}$

So inserting into the formula we obtain

$$\sum_{k=0}^{n} r_{n-k}(a, b; q, p \mid F) \prod_{j=1}^{k}[z, j]_{g,e} = \prod_{i=1}^{n}[z, i - b_i]_{g,e},$$

for all $z \in \mathbb{C}((w_{a,b;q,p}(j))_{j\in\mathbb{Z}})$. In particular for any natural number $n \in \mathbb{N}$, we get that $[n]_{a,b;q,p} \in \mathbb{C}((w_{a,b;q,p}(j))_{j\in\mathbb{Z}})$ and so

$$\sum_{k=0}^{n} r_{n-k}(a, b; q, p \mid F) \prod_{j=1}^{k}[z - j + 1]_{aq^{2(j-1)},bq^{j-1};q,p} = \prod_{i=1}^{n}[z - i + b_i + 1]_{aq^{2(i-b_i-1)},bq^{i-b_i-1};q,p},$$

for all $z \in \mathbb{N}$. But since both sides are analytic in z, we can use analytic continuation to deduce the formula for all $z \in \mathbb{C}$. This is exactly theorem 12 of [9] as claimed.
If we want to conclude the q-version instead, we make use of corollary 3.29 instead. Hence, upon taking the appropriate limits, the above formula simplifies to

$$\sum_{k=0}^{n} r_{n-k}(q \mid F) \prod_{j=1}^{k}[z - j + 1]_q = \prod_{i=1}^{n}[z - i + b_i + 1]_q$$

for all $z \in \mathbb{C}$, which is exactly (1.3) in [8].

Example 3.51. (Example 2.5, w)
Recall that in 2.5 we encountered the Stirling and Laguerre Numbers as special cases of rook-numbers. Specialising the boards in the same way as back then, we obtain weighted analogues of these special numbers.

- Weighted Stirling-numbers of second kind: Let $n \in \mathbb{N}_0$ and consider the Staircase-board St_n. Then by theorem 3.46, the weighted rook-numbers of this board satisfy the recursion given by:

  $$r^{\mathrm{w}}_k(St_n) = \mathrm{W}(n - k - 1, 1)r^{\mathrm{w}}_k(St_{n-1}) + [n - k]r^{\mathrm{w}}_{k-1}(St_{n-1})$$

  with initial conditions $r^{\mathrm{w}}_0(St_0) = 1$ and $r^{\mathrm{w}}_k(St_n) = 0$ if $n < k$. So upon replacing k by $n - k$, we obtain the recursion:

  $$r^{\mathrm{w}}_{n-k}(St_n) = \mathrm{W}(k - 1, 1)r^{\mathrm{w}}_{n-k}(St_{n-1}) + [k]r^{\mathrm{w}}_{n-k-1}(St_{n-1})$$

with initial conditions $r_0^{\mathrm{w}}(St_0) = 1$ and $r_{n-k}^{\mathrm{w}}(St_n) = 0$ if $n < k$. So setting $S_{n,k}^{\mathrm{w}} \stackrel{\text{def}}{=} r_{n-k}^{\mathrm{w}}(St_n)$, this sequence $(S_{n,k}^{\mathrm{w}})_{n,k\in\mathbb{N}_0}$ is a weighted analogue to the q-Stirling numbers introduced by Garsia and Remmel in [8]. As they satisfy the recursion

$$S_{n,k}^{\mathrm{w}} = \mathrm{W}(k-1,1)S_{n-1,k-1}^{\mathrm{w}} + [k]S_{n-1,k}^{\mathrm{w}}, \tag{3.18}$$

with initial conditions $S_{0,0}^{\mathrm{w}} = 1$ and $S_{n,k}^{\mathrm{w}} = 0$ if $n < k$. This simplifies to the defining recurrence of the Stirling numbers by Garsia and Remmel in the q-case, when using the weight hierarchy. They also generalise the elliptic Stirling numbers introduced by Schlosser and Yoo in [9] and the classical Stirling-numbers of second kind introduced in example 2.5. Furthermore, they are closely related to the weighted-Stirling numbers of second kind $(S_{\mathrm{w}}(n,k))_{n,k\in\mathbb{N}_0}$ introduced by Küstner, Schlosser and Yoo in [10]. In this paper, they obtain the following recursion for their version of the Stirling-numbers:

$$S_{\mathrm{w}}(n,k) = S_{\mathrm{w}}(n-1,k-1) + [k]S_{\mathrm{w}}(n-1,k)$$

with initial conditions $S_{\mathrm{w}}(0,0) = 1$ and $S_{\mathrm{w}}(n,k) = 0$ if $n < k$. So upon multiplying their recurrence by $\prod_{j=1}^{k-1}\mathrm{W}(j,1)$ and writing

$$S_{\mathrm{w}}^*(n,k) \stackrel{\text{def}}{=} \left(\prod_{j=1}^{k-1}\mathrm{W}(j,1)\right)S_{\mathrm{w}}(n,k),$$

we obtain that

$$S_{\mathrm{w}}^*(n,k) = \mathrm{W}(k-1,1)S_{\mathrm{w}}^*(n-1,k-1) + [k]S_{\mathrm{w}}^*(n-1,k),$$

with initial conditions $S_{\mathrm{w}}^*(0,0) = 1$ and $S_{\mathrm{w}}(n,k) = 0$ if $n < k$, which is exactly the recurrence of our weighted Stirling-numbers of the second kind. So we see that

$$S_{n,k}^{\mathrm{w}} = S_{\mathrm{w}}^*(n,k) = \left(\prod_{j=1}^{k-1}\mathrm{W}(j,1)\right)S_{\mathrm{w}}(n,k). \tag{3.19}$$

Reducing this to the q-case via the weight hierarchy, we see that the Stirling-numbers of Küstner, Schlosser and Yoo are weighted analogues to the q-Stirling-numbers defined by H. W. Gould in [11]. These satisfy a similar formula connecting them to the q-Stirling-numbers of Garsia and Remmel like the connection between the two weighted versions above. This is described in more detail in the last section of [8]. So we see that also in the weighted setting, there are these two closely connected, yet different, versions of the Stirling-numbers of second kind appearing.

Also theorem 3.49 tells us, that:

$$\sum_{k=0}^{n} S_{n,k}^{\mathrm{w}} \prod_{j=1}^{k}[z,j]_g = z^n,$$

for all $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$, by definition 3.47 and the fact, that $\mathrm{W}(0,1) = 1$ and $[0,1] = 0$. Rephrasing this for the Stirling-numbers of second kind by Küstner, Schlosser and Yoo, we obtain:

$$\sum_{k=0}^{n}\left(\prod_{j=1}^{k-1}\mathrm{W}(j,1)\right)S_{\mathrm{w}}(n,k)\prod_{j=1}^{k}[z,j]_g = \prod_{i=1}^{n}[z,1]_g = z^n$$

Now note that by the definition of the generalised weighted difference, we have for $k \in [n]_0$, that:

$$\left(\prod_{j=1}^{k-1} \mathrm{W}(j,1)\right)\left(\prod_{j=1}^{k}[z,j]_g\right) = [z,1]_g \prod_{j=2}^{k} \mathrm{W}(j-1)[z,j]_g \overset{3.47}{=}$$

$$z\prod_{j=2}^{k} \mathrm{W}(j-1)\frac{1}{\mathrm{W}(j-1)}(z-[j-1]) = \prod_{j=1}^{k}(z-[j-1]),$$

hence our equation from theorem 3.49 transforms into:

$$\sum_{k=0}^{n} S_{\mathrm{w}}(n,k)\prod_{j=1}^{k}(z-[j-1]) = z^n,$$

for all $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$. So we obtain a combinatorial proof for equation (3.19) from [10]. There they used this property as defining equation for their Stirling-numbers of second kind. So we just have shown, that the definition via the recurrence relation, the definition via this factorization property and the definition as weighted rook-numbers are all equivalent.

- Weighted Lah-numbers: If we let $n \in \mathbb{N}_0$ again and consider the Laguerre-board L_n, then theorem 3.49 tells us, that

$$\sum_{k=0}^{n} r^{\mathrm{w}}_{n-k}(L_n)\prod_{j=1}^{k}[z,j]_g = \prod_{i=1}^{n}[z,i-(n-1)]_g,$$

for all $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$. Here the weighted Lah-numbers $L^{\mathrm{w}}_{n,k} \overset{\mathrm{def}}{=} r^{\mathrm{w}}_{n-k}(L_n)$ reduce to the elliptic Lah-numbers introduced by Schlosser and Yoo in [9], the q-Lah-numbers introduced by Garsia and Remmel in [12], which can easily be seen when looking at their rook-theoretic interpretation of these same numbers given in [8], as well as the Lah-numbers introduced in example 2.5, all upon using the weight hierarchy.

Example 3.52. Another interesting application of the results established so far comes from the last example. We will work with the weighted Stirling numbers defined by Küstner, Schlosser and Yoo since they are more handy for this example. We could also work with the ones that we obtained, since they are related by formula (3.19), but this would be a bit more cumbersome in this case. Recall that the complete homogeneous symmetric polynomials satisfy the recurrence relation $h_n(k) = h_n(k-1) + x_k h_{n-1}(k)$ with initial conditions $h_0(k) = 1$ and $h_n(k) = 0$ for $n < 0$ or $n > k$. Upon interchanging n and $n-k$ this becomes

$$h_{n-k}(k) = h_{n-1-(k-1)}(k-1) + x_k h_{n-1-k}(k),$$

with initial conditions $h_0(k) = 1$ and $h_{n-k}(k) = 0$ for $n < k$. As mentioned in section 3.3. of [10], we obtain

$$h_{n-k}([1],...,[k]) = S_{\mathrm{w}}(n,k)$$

simply by comparing the defining recurrence relations. So by formula (3.19) we also obtain that

$$S^{\mathrm{w}}_{n,k} = \left(\prod_{j=1}^{k-1}\mathrm{W}(j,1)\right)h_{n-k}([1],...,[k]).$$

Now we claim that we cannot only obtain the weighted Stirling-numbers of second kind as specialization of the complete homogeneous symmetric polynomials, but can do this the other way around as well. That is, we obtain the complete homogeneous symmetric polynomials as specialization of the weighted Stirling-numbers. To see this, let $..., x_{-1}, x_0, x_1, ...$ be indeterminates over $\mathbb{C}$. Consider the small weight:

$$\mathrm{w} : \mathbb{Z} \to \mathbb{C}((x_i)_{i\in\mathbb{Z}}) \ ; \ j \mapsto \frac{x_{j+1} - x_j}{x_j - x_{j-1}}$$

Then we obtain that

- For $k \geq l \in \mathbb{Z}$
$$\mathrm{W}(k,l) = \frac{x_{k+1} - x_k}{x_l - x_{l-1}}.$$
In particular for $k \in \mathbb{N}$
$$\mathrm{W}(k,1) = \frac{x_{k+1} - x_k}{x_1 - x_0}.$$
- For $n \geq m \in \mathbb{Z}$
$$[n,m] = \sum_{k=m-1}^{n-1} \frac{x_{k+1} - x_k}{x_m - x_{m-1}} = \frac{1}{x_m - x_{m-1}} \sum_{k=m-1}^{n-1} x_{k+1} - x_k = \frac{1}{x_m - x_{m-1}}(x_n - x_{m-1}).$$
In particular for $n \in \mathbb{N}$
$$[n] = \frac{1}{x_1 - x_0}(x_n - x_0),$$

hence the recursion for $S_{\mathrm{w}}(n,k)$ becomes

$$S_{\mathrm{w}}(n,k) = S_{\mathrm{w}}(n-1,k-1) + \frac{1}{x_1 - x_0}(x_k - x_0) S_{\mathrm{w}}(n-1,k),$$

with initial conditions $S_{\mathrm{w}}(0,0) = 1$ and $S_{\mathrm{w}}(n,k) = 0$ if $n < k$. Now specializing $x_0 = 0$, this transforms into

$$S_{\mathrm{w}}(n,k) = S_{\mathrm{w}}(n-1,k-1) + \frac{x_k}{x_1} S_{\mathrm{w}}(n-1,k),$$

with initial conditions $S_{\mathrm{w}}(0,0) = 1$ and $S_{\mathrm{w}}(n,k) = 0$ if $n < k$. So now doing the change of variables $x_k = x_k \cdot x_1$, we obtain the recursion

$$S_{\mathrm{w}}(n,k) = S_{\mathrm{w}}(n-1,k-1) + x_k S_{\mathrm{w}}(n-1,k),$$

with initial conditions $S_{\mathrm{w}}(0,0) = 1$ and $S_{\mathrm{w}}(n,k) = 0$ if $n < k$. Which is exactly the defining recurrence of the complete homogeneous symmetric functions. Hence in this setting $S_{\mathrm{w}}(n,k) = h_{n-k}(k)$. So we see that, in fact the weighted Stirling numbers and the complete homogeneous symmetric functions are in some sense equivalent.
It might also be interesting to further examine the factorization property from theorem 3.49 in this setting. In the last example we had seen, that:

$$\sum_{k=0}^{n} S_{\mathrm{w}}(n,k) \prod_{j=1}^{k} (z - [j-1]) = z^n,$$

for all $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$. Now note that after all our substitutions above we have $[k] = x_k$ for all $k \in \mathbb{N}$ and $[0] = x_0 = 0$. Now note that $h_n(0) = 0$, so our formula from above becomes

$$\sum_{k=1}^{n} h_{n-k}(k) \prod_{j=1}^{k-1} (z - x_j) = z^{n-1},$$

so upon interchanging $n-k$ and k we see that:

$$\sum_{k=0}^{n-1} h_k(n-k) \prod_{j=1}^{n-k-1} (z - x_j) = z^{n-1}$$

Now setting n to $n+1$ yields:

$$\sum_{k=0}^{n} h_k(n-k+1) \prod_{j=1}^{n-k} (z - x_j) = z^n = p_n(z),$$

for all $z \in \mathbb{C}(x_j)_{j\in\mathbb{Z}_{\neq 0}}$, which is exactly the formula relating the complete homogeneous symmetric functions with the power sum symmetric functions, stated on page 208 of [13].

Example 3.53. (Example 2.5, w)(continued)
Recall that in example 2.5 we also introduced the r-restricted Stirling-numbers of second kind and r-restricted Lah-numbers. We can also obtain weighted analogues of these numbers.

- Weighted r-restricted Stirling-numbers of second kind: Let $n, r \in \mathbb{N}_0$, with $n \geq r \geq 0$, and consider a Staircase-board cut off at place r, i.e.

  $$St_n^{(r)} = B(0, ..., 0, r, r+1, ..., n-1) \subset [n] \times [n]$$

  then, upon denoting the weighted r-restricted Stirling-numbers of second kind by $S_{n,k}^{(r),\mathrm{w}} \stackrel{\text{def}}{=} r_{n-k}^{\mathrm{w}}(St_n^{(r)})$, theorem 3.49 tells us that

  $$\sum_{k=0}^{n} S_{n,k}^{(r),\mathrm{w}} \prod_{j=1}^{k} [z, j]_g =$$
  $$\prod_{i=1}^{r} [z, i]_g \prod_{i=r+1}^{n} [z, i-(i-1)]_g = z^{n-r} \prod_{i=1}^{r} [z, i]_g,$$

  for all $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$. Furthermore we obtain the following recursion from theorem 3.46:

  $$S_{n,k}^{(r),\mathrm{w}} = \mathrm{W}(k-1, 1) S_{n-1,k-1}^{(r),\mathrm{w}} + [k] S_{n-1,k}^{(r),\mathrm{w}}, \tag{3.20}$$

  with initial conditions $S_{r-1,r-1}^{\mathrm{w}} = 1$ and $S_{n,k}^{\mathrm{w}} = 0$ if $n < k$ or $k < r-1$. Using the weight hierarchy, these reduce to the elliptic r-restricted Stirling-numbers of second kind, introduced by Schlosser and Yoo in [9], the q-r-restricted Stirling-numbers of second kind and the r-restricted Stirling-numbers of second kind described in example 2.5.

- <u>Weighted r-restricted Lah-numbers</u>: Let again $n, r \in \mathbb{N}_0$, with $n \geq r \geq 0$, and consider the board $L_n^{(r)} = [n+r-1] \times [n-r]$, then upon denoting the weighted r-restricted Lah-number by $L_{n,k}^{(r),\mathrm{w}} \stackrel{\text{def}}{=} r_{n-k}^{\mathrm{w}}(L_n^{(r)})$ we obtain from theorem 3.49, that

$$\sum_{k=0}^{n} L_{n,k}^{(r),\mathrm{w}} \prod_{j=1}^{k} [z, j]_g = \prod_{i=1}^{n+r-1} [z, i-n+r]_g,$$

for all $z \in \mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$. Again by the weight hierarchy, these reduce in the elliptic setting to the elliptic r-restricted Lah-numbers introduced by Schlosser and Yoo in [9], the q-r-restricted Lah-numbers and the r-restricted Lah-numbers described in example 2.5.

3.3.2 On another generating function

Before we can talk about weighted hit-numbers, we will need access to some weighted analogues of the results from section 2.3.5. In particular we are interested in theorem 2.45.

Theorem 3.54. ((partially) Theorem 2.45, w)
Let $(R, +, \cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $F = F(b_1, ..., b_n)$ a Ferrers board, then:

$$\frac{1}{1-x} \sum_{P\in\mathcal{N}_n(F_\infty)} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_\infty) x^{\max(P)} = \sum_{k\geq 0} r_n^{\mathrm{w}}(F_k) x^k = \sum_{k\geq 0} \left(\prod_{i=1}^{n} [k, i-b_i] \right) x^k$$

as generating functions in x.

The elliptic version of this theorem is due to Schlosser and Yoo and can be found as proposition 11 in [9]. The q-version was stated first by Garsia and Remmel in [8] and will be our next theorem 3.55.

Proof. The first equality follows by expanding $\frac{1}{1-x}$ into a geometric series, then multiplying out and by the fact, that for $P \in \mathcal{N}_n(F_\infty)$, we have $\max(P) \leq k \Leftrightarrow P \in \mathcal{N}_n(F_k)$ as seen in the proof of theorem 2.45.

$$\frac{1}{1-x} \sum_{P\in\mathcal{N}_n(F_\infty)} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_\infty) x^{\max(P)} =$$
$$\sum_{k\geq 0} \left(\sum_{P\in\mathcal{N}_n(F_\infty) \text{ s.t. } \max(P)\leq k} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_\infty) \right) x^k =$$
$$\sum_{k\geq 0} \left(\sum_{P\in\mathcal{N}_n(F_k)} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_k) \right) x^k \stackrel{3.12}{=} \sum_{k\geq 0} r_n^{\mathrm{w}}(F_k) x^k$$

The second equality follows by using lemma 3.43, iteratively, to calculate $r_n^{\mathrm{w}}(F_k)$ for any $k \in \mathbb{N}$. Note that, since we are placing n rooks on a board with n columns, we can place the rooks column by column, as in the proof of theorem 3.49 and example

3.45, to obtain:

$$r_n^{\mathrm{w}}(F_k) =$$

$$\sum_{Q^{(0)}\in\mathcal{N}_1(F_1)} \sum_{Q^{(1)}\in\mathcal{P}(Q^{(0)})} \sum_{Q^{(2)}\in\mathcal{P}(Q)} \dots \sum_{Q^{(n-1)}\in\mathcal{P}(Q^{(n-2)})} \mathrm{Wt}_r^{\mathrm{w}}(Q^{(n-1)} \mid F_k) \overset{3.43}{=}$$

$$\sum_{Q^{(0)}\in\mathcal{N}_1(F_1)} \dots \sum_{Q^{(n-2)}\in\mathcal{P}(Q^{(n-3)})} \mathrm{Wt}_r^{\mathrm{w}}(Q^{(n-2)} \mid F_{k-1})[n+k-(n-1)-1, n-b_n]$$

$$= \sum_{Q^{(0)}\in\mathcal{N}_1(F_1)} \dots \sum_{Q^{(n-2)}\in\mathcal{P}(Q^{(n-3)})} \mathrm{Wt}_r^{\mathrm{w}}(Q^{(n-2)} \mid F_{k-1})[k, n-b_n] \overset{3.43}{=} \dots =$$

$$\prod_{i=1}^{n}[k, i-b_i] \qquad \square$$

Theorem 3.55. (Theorem 2.45, q)
Let q be an indeterminate over $\mathbb{C}$, $F = F(b_1, ..., b_n)$ a Ferrers board, then:

$$\frac{1}{1-x} \sum_{P\in\mathcal{N}_n(F_\infty)} q^{\mathrm{inv}(P)} x^{\max(P)} = \sum_{k\geq 0} r_n(q \mid F_k)x^k =$$

$$\sum_{k\geq 0}\left(\prod_{i=1}^{n}[k+b_i-(i-1)]_q\right)x^k = \sum_{k=0}^{n} \frac{r_{n-k}(q \mid F)[k]_q!x^k}{(1-x)(1-qx)...(1-q^kx)} \overset{(*)}{=}$$

$$\frac{1}{(1-x)(1-qx)...(1-q^nx)} \sum_{k=0}^{n} H_k(q \mid F)x^{n-k}$$

as generating functions in x. Where $H_k(q \mid F)$ are the q-hit-numbers, which we will discuss in the last chapter.

We will ignore $(*)$ for now and refer to the last chapter 3.5 for a more thorough discussion on this equality.

Proof. By theorem 3.54, it remains to check the third equality, as the first two follow by using the weight hierarchy. To prove this, we first rearrange the rook-placements $P \in \mathcal{N}_n(F_\infty)$ that we are summing over, by how many rooks are above the ground and how they are arranged, that is:

$$\frac{1}{1-x} \sum_{P\in\mathcal{N}_n(F_\infty)} q^{\mathrm{inv}(P)} x^{\max(P)} =$$

$$\sum_{k=0}^{n} \sum_{Q\in\mathcal{N}_{n-k}(F)} \frac{1}{1-x} \sum_{P\in\mathcal{N}_n(F_\infty) \text{ s.t. } P\cap F=Q} q^{\mathrm{inv}(P)} x^{\max(P)}$$

Now let us fix some $Q \in \mathcal{N}_{n-k}(F)$ and denote

$$S_Q(x, q) \overset{\mathrm{def}}{=} \sum_{P\in\mathcal{N}_n(F_\infty) \text{ s.t. } P\cap F=Q} q^{\mathrm{inv}(P)} x^{\max(P)}.$$

Recall that in the proof of theorem 2.45, we had a bijection between the two sets $\{P \in \mathcal{N}_n(F_\infty) \mid P \cap F = Q\}$ and $\mathbb{N}^k \times S_k$. We denote this bijection by ψ. Also recall, that then

$$\max\left(\psi^{-1}\Big(\big((p_1, ..., p_k), \sigma\big)\Big)\right) = p_1 + ... + p_k + k.$$

The question now becomes: what is $\text{inv}\Big(\psi^{-1}\Big(((p_1,...,p_k),\sigma)\Big)\Big)$? To answer this, let us fix some $((p_1,...,p_k),\sigma) \in \mathbb{N}^k \times S_k$ and denote $P = \psi^{-1}\Big(((p_1,...,p_k),\sigma)\Big)$. Then we can partition $\text{Inv}_{F_\infty}(P)$ into three parts:

$$\begin{gathered}\text{Inv}_{F_\infty}(P) = \{(i,j) \in \text{Inv}_{F_\infty}(P) \mid (i,j) \in F\} \cup \\ \{(i,j) \in \text{Inv}_{F_\infty}(P) \setminus F \mid \text{there is a rook in row } j\} \cup \\ \{(i,j) \in \text{Inv}_{F_\infty}(P) \setminus F \mid \text{there is no rook in row } j\}.\end{gathered}$$

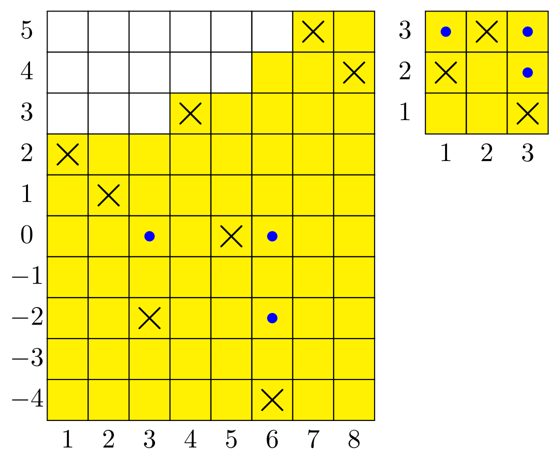
Taking a closer look we clearly identify

$$\text{Inv}_F(Q) = \{(i,j) \in \text{Inv}_{F_\infty}(P) \mid (i,j) \in F\}$$

Since $\{(i,j) \in \text{Inv}_{F_\infty}(P) \mid (i,j) \in F\} = \text{Inv}_{F_\infty}(P) \cap F = \text{Inv}_F(P \cap F) = \text{Inv}_F(Q)$. The second set is just the elements corresponding to the elements of $\text{Inv}_{[k]\times[k]}(P_\sigma)$ under ψ. Recall, when applying ψ we just erase all the columns not containing a rook below the ground and then all the rows not containing a rook below the ground from F, which leaves us with P_σ. So we see that:

$$\text{Inv}_{[k]\times[k]}(P_\sigma) \cong \{(i,j) \in \text{Inv}_{F_\infty}(P) \setminus F \mid \text{there is a rook in row } j\}$$

See for example below, where we marked the elements of the second set with a blue bullet • in the original placement and see how they correspond to the Inv's of the corresponding permutation:



Now for the third set, we see that these are just the empty cells below the ground in the rows not containing a rook. But we know how many of these rows there are: $p_1+...+p_k$. Before placing the k rooks below the ground these rows have $kp_1+...+kp_k$ cells without a dot. Since we already placed $n-k$ rooks above the ground, putting dots in all of the cells below the ground in these $n-k$ columns. We can never put a rook above the $-p_1$-th row, so the kp_1 cells coming from the first p_1 rows below the ground, will remain empty. We place the rooks now row by row. After placing a rook into the $-p_1$-th row, which is the first available row, we have to place dots into one more column all the way till the end of the board. Hence we are left with $(k-1)p_2$ many empty cells, of the rows $-(p_1+1),...,-(p_1+p_2+1)$, staying empty after placing the first rook below the ground. Going on with this argument, we see that, no matter how we place the rooks below the ground, the third set will always have cardinalty $\sum_{j=0}^{k-1}(k-j)p_{j+1}$. Therefore we get that:

$$\begin{gathered}
\mathrm{inv}(P)_{F_\infty} = |\{(i,j) \in \mathrm{Inv}_{F_\infty}(P) \mid (i,j) \in F\} \cup \\
\{(i,j) \in \mathrm{Inv}_{F_\infty}(P) \setminus F \mid \text{there is a rook in row } j\} \cup \\
\{(i,j) \in \mathrm{Inv}_{F_\infty}(P) \setminus F \mid \text{there is no rook in row } j\}| = \\
|\mathrm{Inv}_F(Q)| + |\mathrm{Inv}_{[k]\times[k]}(P_\sigma)| + \sum_{j=0}^{k-1}(k-j)p_{j+1} = \\
\mathrm{inv}_F(Q) + \mathrm{inv}_{[k]\times[k]}(P_\sigma) + \sum_{j=0}^{k-1}(k-j)p_{j+1}
\end{gathered}$$

Plugging this into the above we see that:

$$\begin{gathered}
S_Q(x,q) = \sum_{(p_1,\dots,p_k)\in\mathbb{N}^k} \sum_{\sigma\in S_k} q^{\mathrm{inv}_F(Q)+\mathrm{inv}_{[k]\times[k]}(P_\sigma)+\sum_{j=0}^{k-1}(k-j)p_{j+1}} x^{p_1+\dots+p_k+k} = \\
x^k q^{\mathrm{inv}_F(Q)} \left(\sum_{\sigma\in S_k} q^{\mathrm{inv}_{[k]\times[k]}(P_\sigma)} \right) \left(\sum_{(p_1,\dots,p_k)\in\mathbb{N}^k} q^{\sum_{j=0}^{k-1}(k-j)p_{j+1}} x^{p_1+\dots+p_k} \right) = \\
x^k q^{\mathrm{inv}_F(Q)} \left(\sum_{P\in\mathcal{N}_k([k]\times[k])} q^{\mathrm{inv}_{[k]\times[k]}(P)} \right) \left(\sum_{(p_1,\dots,p_k)\in\mathbb{N}^k} q^{\sum_{j=0}^{k-1}(k-j)p_{j+1}} x^{p_1+\dots+p_k} \right) \overset{3.41}{=} \\
x^k q^{\mathrm{inv}_F(Q)} r_k(q \mid [k]\times[k]) \left(\sum_{(p_1,\dots,p_k)\in\mathbb{N}^k} q^{\sum_{j=0}^{k-1}(k-j)p_{j+1}} x^{p_1+\dots+p_k} \right) \overset{3.45}{=} \\
x^k q^{\mathrm{inv}_F(Q)} [k]_q \prod_{j=1}^{k} \left(\sum_{p_j\geq 0} (q^{k+1-j}x)^{p_j} \right) = \frac{x^k q^{\mathrm{inv}_F(Q)}[k]_q}{(1-xq)\dots(1-xq^k)}
\end{gathered}$$

And now plugging this into our initial equation, we see that

$$\begin{gathered}
\frac{1}{1-x} \sum_{P\in\mathcal{N}_n(F_\infty)} q^{\mathrm{inv}(P)} x^{\max(P)} = \\
\sum_{k=0}^{n} \sum_{Q\in\mathcal{N}_{n-k}(F)} \frac{1}{1-x} S_Q(x,q) = \\
\sum_{k=0}^{n} \sum_{Q\in\mathcal{N}_{n-k}(F)} \frac{1}{1-x} \frac{x^k q^{\mathrm{inv}_F(Q)}[k]_q}{(1-xq)\dots(1-xq^k)} = \\
\sum_{k=0}^{n} \frac{x^k [k]_q}{(1-x)(1-xq)\dots(1-xq^k)} \left(\sum_{Q\in\mathcal{N}_{n-k}(F)} q^{\mathrm{inv}_F(Q)} \right) \overset{3.41}{=} \\
\sum_{k=0}^{n} \frac{r_{n-k}(q \mid F)[k]_q! x^k}{(1-x)(1-qx)\dots(1-q^k x)},
\end{gathered}$$

as desired. □

3.3.3 Another short remark on rook-equivalence

Recall that in section 2.1.3 we talked about certain conditions for the notion of rook-equivalence of Ferrers-boards, defined in 2.18. The main result of that section was theorem 2.19. Recall that the proof was basically just us realising, that the rook-numbers are fully encoded in the multiset $\{\{b_1-0,...,b_n-(n-1)\}\}$ for a Ferrers-board $F(b_1,...,b_n)$ and vice versa. This was just a reformulation of Goldman's, Joichi's and White's result theorem 2.3. We may have a weighted analogue in theorem 3.49, but we can only obtain one direction of the above result. This is due the fact, that our generalised weighted differences $[z,j]_g$ are uniquely determined by the integer j but, they may not determine the integer j uniquely in general as our weights are arbitrary. Nevertheless, we still obtain the following theorem:

Theorem 3.56. ((partially) Theorem 2.19, w)
Let $(R,+,\cdot)$ be a commutative, unitary, integral domain, $\mathrm{w}:\mathbb{Z}\to R$ a small weight, $F=F(b_1,...,b_n)$, $F'=F(b'_1,...,b'_n)\subset[n]\times[n]$ two Ferrers-boards. If $F\overset{R}{\sim}F'$, then $r_k^{\mathrm{w}}(F)=r_k^{\mathrm{w}}(F')$ for all $k\in[n]_0$.

Proof. Note that by theorem 2.19, we have that $F\overset{R}{\sim}F'$ if and only if $\{\{b_1-0,...,b_n-(n-1)\}\}=\{\{b'_1-0,...,b'_n-(n-1)\}\}$ as multisets. But if this holds, then by theorem 3.49:

$$\sum_{k=0}^{n} r_{n-k}^{\mathrm{w}}(F)\prod_{j=1}^{k}[z,j]_g=\prod_{i=1}^{n}[z,i-b_i]_g=\prod_{i=1}^{n}[z,i-b'_i]_g=\sum_{k=0}^{n} r_{n-k}^{\mathrm{w}}(F')\prod_{j=1}^{k}[z,j]_g$$

Hence upon comparing coefficients of z^{n-k} on both sides for all $k\in[n]_0$ and multiplying with $\prod_{j=1}^{n-k}\frac{1}{\mathrm{W}(j-1,1)}$ we obtain the result. □

3.4 The weighted file-numbers

In this section we will try to generalise all of the results of section 2.2 on file-numbers from the classical case to the weighted one. We will state the highest version w.r.t. weight hierarchy available and indicate which result of the section on classical file-numbers we are generalising, by referring to it in brackets next to the number of our new result. Furthermore we will indicate by a q, e, w in the bracket whether this is now the q-, elliptic- or general weighted case. All versions considering weights of lower order in the hierarchy can then be deduced. We will not state them here as otherwise this thesis would have gotten even longer than it already is. We will although remark most of the time who discovered the lower order result first.

3.4.1 A recursive approach to weighted file-numbers

Lemma 3.57. (Lemma 2.23, w)
Let $(R,+,\cdot)$ be a commutative, unitary, integral domain, $\mathrm{w}:\mathbb{Z}\to R$ a small weight, $B=B(b_1,...,b_n)$ be a skyline-board, $z\in\mathbb{N}_0$, $k\in[n]_0$, $P\in\mathcal{F}_k(B_z)$ a file-placement of k-rooks in the first $k\le i-1<n$ columns of B_z, $\mathcal{P}(P)$ the set of file-placements obtained from P, by adding a rook in the i-th column of B_z then we have that

$$\sum_{Q\in\mathcal{P}(P)}\mathrm{Wt}_f^{\mathrm{w}}\big(Q\mid B(b_1,...,b_i)_z\big)=\mathrm{Wt}_f^{\mathrm{w}}\big(P\mid B(b_1,...,b_{i-1})_z\big)[z,1-b_i].$$

Neither the elliptic, nor the q-version of this lemma have been stated explicitly yet, but the first one is made use of extensively in [9] and the second one is made use of in [8], although none of these parts about the q-file-numbers made it to the final version of the paper.

Proof. For $i \in [n]$ let us abbreviate $B(b_1, ..., b_i)_z$, the board B_z cut off at the i-th column, by B_i. Recall that by definition

$$\mathrm{Wt}_f^{\mathrm{w}}(Q \mid B_i) = \prod_{(i,j) \in U_{B_i}(Q)} \mathrm{w}(1-j).$$

Now we can split the product into two parts. One ranging over all cells of $U_{B_i}(Q)$ contained in B_{i-1} and the second product ranging over all cells of $U_{B_i}(Q)$ in the i-th column of B_z. But since by assumption $Q \cap B_{i-1} = P$, we know that $U_{B_i}(Q) \cap B_{i-1} = U_{B_{i-1}}(P)$, since the one additional rook that Q has, compared to P, sits in the rightmost column of B_i, the i-th column, i.e. does not add any dots to B_{i-1}. So we obtain that

$$\mathrm{Wt}_f^{\mathrm{w}}(Q \mid B_i) = \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_{i-1}) \prod_{(i,j) \in U_{B_i}(Q)} \mathrm{w}(1-j),$$

therefore we see

$$\sum_{Q \in \mathcal{P}(P)} \mathrm{Wt}_f^{\mathrm{w}}(Q \mid B_i) = \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_{i-1}) \sum_{Q \in \mathcal{P}(P)} \prod_{(i,j) \in U_{B_i}(Q)} \mathrm{w}(1-j),$$

hence it remains to check that

$$\sum_{Q \in \mathcal{P}(P)} \prod_{(i,j) \in U_{B_i}(Q)} \mathrm{w}(1-j) = [z, 1-b_i].$$

Now observe that

$$\mathcal{P}(P) = \big\{ P \cup \{(i, b_i - j)\} \mid j \in [b_i + z - 1]_0 \big\},$$

since any $Q \in \mathcal{P}(P)$ is obtained from P by adding a rook to the last column. Since we are in the file placement case, any cell of the last column is eligible for this. As there are $b_i + z$ cells available, the above follows. Now by construction of our statistic U, only the cells above the cell of the last column which we placed a rook onto, will contribute to the weight. Hence we see that, for all $j \in [b_i + z - 1]_0$

$$U_{B_i}\big(P \cup \{(i, b_i - j)\}\big) = \{(i, b_i - k) \mid k \in [j-1]_0\},$$

therefore

$$\sum_{Q \in \mathcal{P}(P)} \prod_{(i,j) \in U_{B_i}(Q)} \mathrm{w}(1-j) = \sum_{j=0}^{b_i+z-1} \prod_{k=0}^{j-1} \mathrm{w}(1+k-b_i) \overset{3.22}{=} \sum_{j=0}^{b_i+z-1} \mathrm{W}(j-b_i, 1-b_i) \overset{3.23}{=} [z, 1-b_i].$$

So we are done. ☐

Theorem 3.58. (Theorem 2.24, w)
Let $(R, +, \cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $B = B(b_1, ..., b_n)$ a skyline-board, $z \in \mathbb{N}$, $k \in [n]_0$ then

$$f_k^{\mathrm{w}}(B_z) = \mathrm{W}(z, 1-b_n) f_k^{\mathrm{w}}(B_z^-) + [z, 1-b_n] f_{k-1}^{\mathrm{w}}(B_z^-),$$

with initial conditions $f_0^{\mathrm{w}}(\varnothing) = 1$, $f_k^{\mathrm{w}}(B_z) = 0$ if $n < k$ and $f_k^{\mathrm{w}}(B(b_1, ..., b_j)) = 0$ if $|\{b_i + z > 0 \mid i \in [j]\}| < k$.

The elliptic version of this theorem is due to Schlosser and Yoo. It is their theorem 27 in [9]. The q-version is due to Garsia and Remmel, again it did not make it into the final version of [8] though.

Proof. The initial conditions follow by the same reasoning as in theorem 2.24.
For the actual proof, we partition $\mathcal{F}_k(B_z)$ into the two sets

$$\mathcal{P}_1 \stackrel{\text{def}}{=} \{P \in \mathcal{F}_k(B_z) \mid \text{there is no rook in the } n\text{-th column of } B_z\}$$

and

$$\mathcal{P}_2 \stackrel{\text{def}}{=} \{P \in \mathcal{F}_k(B_z) \mid \text{there is a rook in the } n\text{-th column of } B_z\}.$$

Then by definition

$$f_k^{\mathrm{w}}(B_z) = \sum_{P \in \mathcal{F}_k(B_z)} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_z) = \sum_{P \in \mathcal{P}_1} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_z) + \sum_{P \in \mathcal{P}_2} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_z).$$

Since, by definition of our U statistic, dots only appear in columns with a rook in them, we have for any $P \in \mathcal{P}_1$

$$U_{B_z}(P) = U_{B_z^-}(P) \cup \{(n, b_n - j) \mid j \in [b_n + z - 1]_0\},$$

simply by splitting the cells without a dot on them into the set of cells contained in B_z^- and the cells contained in the last column of B_z and because P is just a file-placement of k-rooks on B_z^-. Therefore we calculate:

$$\sum_{P \in \mathcal{P}_1} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_z) = \sum_{P \in \mathcal{P}_1} \prod_{(i,j) \in U_{B_z}(P)} \mathrm{w}(1-j) = \sum_{P \in \mathcal{P}_1} \prod_{(i,j) \in U_{B_z^-}(P)} \mathrm{w}(1-j) \prod_{j=0}^{b_n+z-1} \mathrm{w}(1+j-b_n) \stackrel{3.22}{=} \sum_{P \in \mathcal{P}_1} \Big(\prod_{(i,j) \in U_{B_z^-}(P)} \mathrm{w}(1-j) \Big) \mathrm{W}(z, 1-b_n) \stackrel{3.38}{=} \mathrm{W}(z, 1-b_n) f_k^{\mathrm{w}}(B_z^-)$$

Now for the second sum we make use of lemma 3.57. We keep the notation from that lemma as well, that is, for $P \in \mathcal{F}_{k-1}(B_z^-)$, we denote by $\mathcal{P}(P) \subset \mathcal{F}_k(B_z)$, the set of file-placements obtained from P by adding a rook in the n-th column. Clearly

$$\mathcal{P}_2 \cong \coprod_{P \in \mathcal{F}_{k-1}(B_z^-)} \mathcal{P}(P),$$

hence we can rewrite the second sum in the following way:

$$\sum_{P\in\mathcal{P}_2}\mathrm{Wt}_f^{\mathrm{w}}(P\mid B_z)=\sum_{P\in\mathcal{F}_{k-1}(B_z^-)}\sum_{Q\in\mathcal{P}(P)}\mathrm{Wt}_f^{\mathrm{w}}(Q\mid B_z)\overset{3.57}{=}$$
$$\sum_{P\in\mathcal{F}_{k-1}(B_z^-)}\mathrm{Wt}_f^{\mathrm{w}}(P\mid B_z^-)[z,1-b_n]=[z,1-b_n]f_{k-1}^{\mathrm{w}}(B_z^-),$$

which gives the second term as desired. □

Theorem 3.59. (Theorem 2.25, w)
Let $(R,+,\cdot)$ be a commutative, unitary, integral domain, $\mathrm{w}:\mathbb{Z}\to R$ a small weight, $B=B(b_1,...,b_n)$ a skyline board, then for all $z\in\mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$:

$$\sum_{k=0}^{n}f_{n-k}^{\mathrm{w}}(B)z^k=\prod_{i=1}^{n}[z,1-b_i]_g.$$

The elliptic version of this theorem is Schlosser's and Yoo's theorem 26 in [9]. The q-version is again due to Garsia and Remmel, but does not appear in the final version of [8].

Proof. First of all note that both sides are polynomial in z of degree at most n, by definition of the generalised weighted difference. By our remark 3.48 before we can use the polynomial argument. So if we can show the equility for at least $n+1$ values in $\mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$, we are done. We will in fact show it for infinitely many values. For this let $n<x\in\mathbb{N}$. We check the equality for $[x]\in\mathbb{C}((\mathrm{w}(j))_{j\in\mathbb{Z}})$. So by our remark it remains to prove that:

$$\sum_{k=0}^{n}f_{n-k}^{\mathrm{w}}(B)[x]^k=\prod_{i=1}^{n}[x,1-b_i]$$

for all $n<x\in\mathbb{N}$. We will do this as in the classical case by double counting the placements of n rooks on B_x.

- <u>RHS</u>: We place the rooks column by column. Using the notation from lemma 3.57 and the lemma itself iteratively yields:

$$f_n^{\mathrm{w}}(B_x)=$$
$$\sum_{Q^{(0)}\in\mathcal{F}_1(B_1)}\sum_{Q^{(1)}\in\mathcal{P}(Q^{(0)})}\sum_{Q^{(2)}\in\mathcal{P}(Q^{(1)})}\cdots\sum_{Q^{(n-1)}\in\mathcal{P}(Q^{(n-2)})}\mathrm{Wt}_f^{\mathrm{w}}(Q^{(n-1)}\mid F_x)\overset{3.57}{=}$$
$$\sum_{Q^{(0)}\in\mathcal{N}_1(B_1)}\cdots\sum_{Q^{(n-2)}\in\mathcal{P}(Q^{(n-3)})}\mathrm{Wt}_r^{\mathrm{w}}(Q^{(n-2)}\mid F_{x-1})[x,1-b_n]\overset{3.57}{=}\dots=$$
$$\prod_{i=1}^{n}[x,1-b_i]$$

- <u>LHS</u>: We first place $n-k$ rooks onto the board B and then the remaining k rooks below the ground. So as in the classical case, we consider the partition $\mathcal{F}_n(B_x)=\dot{\bigcup}_{k=0}^{n}F_k$, where $F_k=\{P\in\mathcal{F}_n(B_x)\mid \text{exactly } n-k \text{ rooks are on } B\}$. Recall that then for each $k\in[n]_0$, we have that F_k is in bijection with $\mathcal{F}_{n-k}(B)\times$

$\mathcal{F}_k([k] \times [x])$. Let us call this bijection ϕ. So we can rewrite the file-number in the following way:

$$f_n^{\mathrm{w}}(B_x) = \sum_{k=0}^{n} \sum_{(P_1,P_2)\in\mathcal{F}_{n-k}(B)\times\mathcal{F}_k([k]\times[x])} \mathrm{Wt}_f^{\mathrm{w}}\Big(\phi^{-1}\big((P_1,P_2)\big) \mid B_x\Big)$$

Let us fix $(P_1, P_2) \in \mathcal{F}_{n-k}(B) \times \mathcal{F}_k([k] \times [x])$ and now focus on the term $\mathrm{Wt}_f^{\mathrm{w}}\Big(\phi^{-1}\big((P_1,P_2)\big) \mid B_x\Big)$. Let us denote $P = \phi^{-1}\big((P_1,P_2)\big)$. Clearly the rooks placed below the ground do not cause any dots on the board itself. So the cells in $U_{B_x}(P)$ on the board itself all come from P_1. So we see that

$$U_{B_x}(P) = \big(U_{B_x}(P) \cap B\big) \cup \big(U_{B_x}(P) \cap (B_x \setminus B)\big) = \\ U_B(P_1) \cup \big(U_{B_x}(P_1) \cap (B_x \setminus B)\big),$$

hence by definition of the weight, we get that

$$\mathrm{Wt}_f^{\mathrm{w}}(P \mid B_x) = \prod_{(i,j)\in U_{B_x}(P)} \mathrm{w}(1-j) = \\ \prod_{(i,j)\in U_B(P_1)} \mathrm{w}(1-j) \prod_{(i,j)\in U_{B_x}(P)\cap(B_x\setminus B)} \mathrm{w}(1-j).$$

We can use this to rewrite the file-number once again:

$$f_n^{\mathrm{w}}(B_x) = \sum_{k=0}^{n} \sum_{(P_1,P_2)\in\mathcal{F}_{n-k}(B)\times\mathcal{F}_k([k]\times[x])} \mathrm{Wt}_f^{\mathrm{w}}\Big(\phi^{-1}\big((P_1,P_2)\big) \mid B_x\Big) = \\ \sum_{k=0}^{n} \sum_{P_1\in\mathcal{F}_{n-k}(B)} \sum_{P_2\in\mathcal{F}_k([k]\times[x])} \mathrm{Wt}_f^{\mathrm{w}}\Big(\phi^{-1}\big((P_1,P_2)\big) \mid B_x\Big) = \\ \sum_{k=0}^{n} \sum_{P_1\in\mathcal{F}_{n-k}(B)} \Bigg(\prod_{(i,j)\in U_B(P_1)} \mathrm{w}(1-j)\Bigg) \\ \sum_{P_2\in\mathcal{F}_k([k]\times[x])} \prod_{(i,j)\in U_{B_x}(\phi^{-1}((P_1,P_2)))\cap(B_x\setminus B)} \mathrm{w}(1-j)$$

As in the rook-number case and proof of theorem 3.49, we will first focus on the inner sum. Again, for a fixed $P_1 \in \mathcal{F}_{n-k}(B)$ the set $\phi^{-1}((P_1,P_2))) \cap (B_x \setminus B)$ if fully determined by $P_2 \in \mathcal{F}([k] \times [x])$, since the cells in this set, are just the cells in the columns, that contain a rook below the ground, between the ground and that rook. So now fix some $P_1 \in \mathcal{F}_{n-k}(B)$. We denote the k columns that contain a rook below the ground by $i_1, ..., i_k$. We can obtain all placements $P_2 \in \mathcal{F}([k] \times [x])$, by placing the rooks column by column. Note that the j-th cell below the ground $(i_t, 1-j)$ of the i_t-th column, counted from ground to bottom, has the weight $\mathrm{w}(1-(1-j)) = \mathrm{w}(j)$. So if we place a rook in the j-th cell below the ground $(i_t, 1-j)$ of the i_t-th column, counted from ground to bottom, then only the $j-1$ cells above it will contribute to the weight with a factor of:

$$\prod_{k\in[j-1]} \mathrm{w}(j) = \mathrm{W}(j-1, 1)$$

Since we have all x cells below the ground available for all of the k columns (recall we are dealing with file-placements!) summing over these x choices for a fixed column, we obtain the weight of the column:

$$\sum_{j=1}^{x} \mathrm{W}(j-1,1) = [x,1] = [x]$$

Since there are k columns available, we see that

$$\sum_{P_2 \in \mathcal{F}_k([k]\times[x])} \prod_{(i,j)\in U_{B_x}(\phi^{-1}((P_1,P_2)))\cap(B_x\setminus B)} \mathrm{w}(1-j) = [x]^k,$$

by summing over all possible placements. Plugging this into the above and using the definition of our weighted file-numbers we obtain

$$f_n^{\mathrm{w}}(B_x) = \sum_{k=0}^{n} \sum_{P_1\in\mathcal{F}_{n-k}(B)} \left(\prod_{(i,j)\in U_B(P_1)} \mathrm{w}(1-j) \right) [x]^k \overset{3.38}{=} \sum_{k=0}^{n} f_{n-k}^{\mathrm{w}}(B)[x]^k,$$

as desired. □

Example 3.60. (Example 2.26, w)
In example 2.26 we had obtained the Stirling-numbers of first kind as a special case of file-numbers. In example 3.51 we obtained a weighted analogue of the Stirling-numbers of second kind by specialising the board in our weighted rook-numbers, so it should be no surprise, that something analogous works in the weighted file-number case. So let $n \in \mathbb{N}_0$ and consider the Staircase-board St_n. Now by theorem 3.58, the weighted file-numbers of this board satisfy the recurrence given by

$$f_k^{\mathrm{w}}(St_n) = \mathrm{W}(0, 1-(n-1)) f_k^{\mathrm{w}}(St_{n-1}) + [0, 1-(n-1)] f_{k-1}^{\mathrm{w}}(St_{n-1}),$$

with initial conditions $f_0^{\mathrm{w}}(St_0) = 1$ and $f_k^{\mathrm{w}}(St_n) = 0$ if $n < k$. So replacing k by $n-k$ we obtain

$$f_{n-k}^{\mathrm{w}}(St_n) = \mathrm{W}(0, 2-n) f_{n-1-(k-1)}^{\mathrm{w}}(St_{n-1}) + [0, 2-n] f_{n-1-k}^{\mathrm{w}}(St_{n-1}),$$

with initial conditions $f_0^{\mathrm{w}}(St_0) = 1$ and $f_{n-k}^{\mathrm{w}}(St_n) = 0$ if $n-k < 0$. Now denoting by $s_{n,k}^{\mathrm{w}} \overset{\mathrm{def}}{=} f_{n-k}^{\mathrm{w}}(St_n)$, this sequence $(s_{n,k}^{\mathrm{w}})_{n,k\in\mathbb{N}_0}$ is a weighted analogue to the unsigned q-Stirling-numbers of first kind defined in [8], the elliptic unsigned Stirling-numbers of first kind defined in [9] and the unsigned Stirling-numbers of first kind introduced in example 2.26. This can be seen by reducing their defining recurrence

$$s_{n,k}^{\mathrm{w}} = \mathrm{W}(0, 2-n) s_{n-1,k-1}^{\mathrm{w}} + [0, 2-n] s_{n-1,k}^{\mathrm{w}},$$

with initial conditions $s_{0,0} = 1$ and $s_{n,k} = 0$ if $n < k$, via the weight hierarchy. By theorem 3.59 they satisfy

$$\sum_{k=0}^{n} s_{n,k}^{\mathrm{w}} z^k = \prod_{i=1}^{n} [z, 2-i]_g,$$

which is a weighted analogon to formula (5.8) in [9]. Note that Küstner, Schlosser and Yoo also defined weighted Stirling-numbers of first kind $s_{\mathrm{w}}(n,k)$ in [10]. They defined them via the relation:

$$\sum_{k=0}^{n} s_{\mathrm{w}}(n,k) z^k = \prod_{i=1}^{n} (z - [i-1])$$

Then this looks very different from our Stirling-numbers of first kind. This is due to the fact, that their Stirling numbers reduce to the signed q-Stirling- and signed Stirling-numbers of first kind upon using the weight hierarchy, whereas ours reduce to the unsigned Stirling-numbers of first case. In the q-case they not only differ by a sign, but theirs also reduce again, to the q-version introduced by Gould in [11], whereas ours reduce to the ones introduced by Garsia and Remmel in [8]. In the q-setting they are related similarly as in the case of the Stirling-numbers of second kind. This is again described more thoroughly in the last section of [8]. Recall that in the basic case, the signed $s^*_{n,k}$ and unsigned $s_{n,k}$ Stirling-numbers of first kind are related via:

$$s^*_{n,k} = (-1)^{n-k} s_{n,k}$$

Now one might ask if a similar relation also holds in the weighted case. It seems that this is only possible in the q-case without some further restrictions on the weights. Although it surely might be an interesting question what condition would be sufficient or necessary for a relation like that, we will not answer this here due to space limitations.

3.4.2 Deletion/Contraction recurrence

In this section we will analyze the recurrence for file-numbers from section 2.3.2 in order to try and generalize it to the weighted setting.

Proposition 3.61. (almost Proposition 2.33, w)
Let $(R, +, \cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $n \in \mathbb{N}$, $k \in [n]_0$, $B = B(b_1, ..., b_n)$ a skyline-board, $c = (c_i, c_j) \in B$ a top-square, then we have that

$$f^{\mathrm{w}}_k(B) = \mathrm{w}(1 - c_j) f^{\mathrm{w}}_k(B/c) + f^{\mathrm{w}}_{k-1}(F/\overline{\overline{c}}),$$

with initial conditions $f^{\mathrm{w}}_k(\varnothing) = 0$ if $k > 0$ and $f^{\mathrm{w}}_0(\varnothing) = 1$.

There has not been stated an elliptic or q-version of this theorem yet, but using our weight hierarchy we can obtain both versions.

Proof. The initial conditions are just convention. Now consider the partition of the set $\mathcal{F}_k(B)$ into the two parts, given by $S_1 \overset{\text{def}}{=} \{P \in \mathcal{F}_k(B) \mid \text{there is no rook on } c\}$ and $S_2 \overset{\text{def}}{=} \{P \in \mathcal{F}_k(B) \mid \text{there is a rook on } c\}$ as in the classical case. Then by definition:

$$f^{\mathrm{w}}_k(B) = \sum_{P \in \mathcal{F}_k(B)} \mathrm{Wt}^{\mathrm{w}}_f(P \mid B) =$$

$$\sum_{P \in S_1} \mathrm{Wt}^{\mathrm{w}}_f(P \mid B) + \sum_{P \in S_2} \mathrm{Wt}^{\mathrm{w}}_f(P \mid B)$$

But now for $P \in S_1$ we always have $c \in U_F(P)$, since there is no rook on c and c is a top square, i.e. there is no rook above c in P on F. Also we have that

$$U_B(P) \setminus c = U_{B/c}(P)$$

and $S_1 = \mathcal{F}_k(B/c)$. Hence we see that

$$\sum_{P \in S_1} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B) = \sum_{P \in S_1} \prod_{(i,j) \in U_B(P)} \mathrm{w}(1-j) =$$
$$\sum_{P \in \mathcal{F}_k(B/c)} \mathrm{w}(1-c_j) \prod_{(i,j) \in U_{B/c}(P)} \mathrm{w}(1-j) \overset{3.38}{=}$$
$$\mathrm{w}(1-c_j) f_k^{\mathrm{w}}(B/c),$$

which is the desired first summand. Now for $P \in S_2$, the rook on c puts dots in the full column of F containing c, hence

$$U_B(P) = U_{B/\overline{\overline{c}}}(P \setminus c).$$

Since $\{P \setminus c \mid P \in S_2\} = \mathcal{F}_{k-1}(F/\overline{\overline{c}})$ we have that

$$\sum_{P \in S_2} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B) = \sum_{P \in S_2} \prod_{(i,j) \in U_B(P)} \mathrm{w}(1-j) =$$
$$\sum_{P \in \mathcal{F}_{k-1}(F/\overline{\overline{c}})} \prod_{(i,j) \in U_{F/\overline{\overline{c}}}(P)} \mathrm{w}(1-j) \overset{3.12}{=} f_{k-1}^{\mathrm{w}}(F/\overline{\overline{c}}),$$

so we are done. □

Remark 3.62. Note that proposition 3.61 only holds if c is a top-square. This is the big difference to the classical version proposition 2.33. This is due to the fact that, if c is not a top square, then one cannot control the first summand in the proof, as then one need not have $c \in U_F(P)$ for $P \in S_1$ as there might be a rook above c in P_1.

3.4.3 On another generating function

In this section we are interested in some weighted analogues of the results from section 2.3.5. In particular, since this chapter is dealing with weighted file-numbers, we would like to generalise theorem 2.53.

Theorem 3.63. ((partially) Theorem 2.53, w)
Let $(R, +, \cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $B = B(b_1, ..., b_n)$ a skyline board, then:

$$\frac{1}{1-x} \sum_{P \in \mathcal{F}_n(B_\infty)} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_\infty) x^{\max(P)} = \sum_{k \geq 0} f_n^{\mathrm{w}}(B_k) x^k = \sum_{k \geq 0} \left(\prod_{i=1}^{n} [k, 1-b_i] \right) x^k$$

as generating functions in x.

An elliptic version of this theorem has not been stated yet, the q-version is due to Garsia and Remmel but did not make it into the final version of [8].

Proof. The first equality follows by expanding $\frac{1}{1-x}$ into a geometric series, then multiplying out and by the fact, that for $P \in \mathcal{F}_n(B_\infty)$, we have $\max(P) \leq k \Leftrightarrow P \in \mathcal{F}_n(B_k)$ as seen in the proof of theorem 2.45. So

$$\frac{1}{1-x} \sum_{P \in \mathcal{F}_n(B_\infty)} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_\infty) x^{\max(P)} =$$
$$\sum_{k \geq 0} \Bigg(\sum_{P \in \mathcal{F}_n(B_\infty) \text{ s.t. } \max(P) \leq k} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_\infty) \Bigg) x^k =$$
$$\sum_{k \geq 0} \Bigg(\sum_{P \in \mathcal{F}_n(B_k)} \mathrm{Wt}_f^{\mathrm{w}}(P \mid B_k) \Bigg) x^k \overset{3.38}{=} \sum_{k \geq 0} f_n^{\mathrm{w}}(B_k) x^k.$$

The second equality follows by using lemma 3.57, iteratively, to calculate $f_n^{\mathrm{w}}(B_k)$ for any $k \in \mathbb{N}$. Note that, since we are placing n rooks on a board with n columns, we can place the rooks column by column, as in the proof of theorem 3.59, to obtain:

$$f_n^{\mathrm{w}}(B_k) =$$
$$\sum_{Q^{(0)} \in \mathcal{F}_1(B_1)} \sum_{Q^{(1)} \in \mathcal{P}(Q^{(0)})} \sum_{Q^{(2)} \in \mathcal{P}(Q)} \cdots \sum_{Q^{(n-1)} \in \mathcal{P}(Q^{(n-2)})} \mathrm{Wt}_f^{\mathrm{w}}(Q^{(n-1)} \mid F_k) \overset{3.57}{=}$$
$$\sum_{Q^{(0)} \in \mathcal{N}_1(B_1)} \cdots \sum_{Q^{(n-2)} \in \mathcal{P}(Q^{(n-3)})} \mathrm{Wt}_r^{\mathrm{w}}(Q^{(n-2)} \mid F_{k-1})[k, 1-b_n] \overset{3.57}{=} \ldots =$$
$$\prod_{i=1}^{n} [k, 1-b_i]$$

□

3.5 The problems with weighted hit- and fit-numbers

3.5.1 A short overview on A. M. Garsia's and J. B. Remmel's approach to the q-hit-numbers

In this section we will do a quick recap on the results established on q-hit-numbers in [8] and M. Dworkin's results from [14]. The paper [8] roughly splits into two parts. The first one is establishing results similar to ours in chapter 3.3 but of lower order in the hierarchy and the second one is on q-hit-numbers. The heart of the second part is our theorem 3.55, which they proved in their first part. Let us state the formula here once again for completeness: Let q be an indeterminate over $\mathbb{C}$, $F = F(b_1, ..., b_n)$ a Ferrers board, then:

$$\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(F_\infty)} q^{\mathrm{inv}(P)} x^{\max(P)} = \sum_{k \geq 0} r_n(q \mid F_k) x^k =$$
$$\sum_{k \geq 0} \left(\prod_{i=1}^{n} [k + b_i - (i-1)]_q \right) x^k = \sum_{k=0}^{n} \frac{r_{n-k}(q \mid F)[k]_q! x^k}{(1-x)(1-qx)...(1-q^k x)}$$

as generating functions in x.
Note that we left out $(*)$ here since this is part of what will be proven. Now they

define a polynomial $Q_B(x,q)$, for any board B, by the following equation:

$$\sum_{k=0}^{n} \frac{r_{n-k}(q \mid B)[k]_q! x^k}{(1-x)(1-qx)...(1-q^k x)} = \frac{Q_B(x,q)}{(1-x)(1-qx)...(1-q^n x)} \tag{3.21}$$

In other words

$$Q_B(x,q) = \sum_{k=0}^{n} r_{n-k}(q \mid B)[k]_q! x^k (1-q^{k+1}x)...(1-q^n x) \tag{3.22}$$

Now they move on to prove the following theorem:

Theorem 3.64. Let q be an indeterminate over $\mathbb{C}$, $B \subset [n] \times [n]$ a board, then $Q_B(x,q) = \sum_{k=0}^{n} r_{n-k}(q \mid B)[k]_q! x^k (1-q^{k+1}x)...(1-q^n x)$ has non-negative integer coefficients.

They actually show even more. They recursively construct a pair of statistics $d_B(\sigma)$ and $m_B(\sigma)$ depending on the board B, such that

$$Q_B(x,q) = \sum_{\sigma \in S_n} x^{d_B(\sigma)} q^{m_B(\sigma)},$$

by defining the three operations FLIP, RAISE and ADD for boards and interpreting the impact of these operations on $Q_B(x,q)$, as operators on $Q_B(x,q)$ treated as polynomial in x. By specializing $q=1$ in equation (3.22), we obtain back theorem 2.13. So we see that $Q_B(x,q)$ is a q-analogue of the hit-polynomial.
Now M. Dworkin takes this one step further in [14]. Given a board $B \subset [n] \times [n]$ and some q-rook-numbers $(r_k(q \mid B))_{k \in [n]_0}$, he defines two different versions of q-hit-numbers $\tilde{H}_i(q \mid B), \hat{H}_i(q \mid B)$ for $i \in [n]_0$, via the relations:

$$r_k(q \mid F)[n-k]_q! = \sum_{i=k}^{n} \frac{[i]_q!}{[k]_q![i-k]_q!} \tilde{H}_i(q \mid B) q^{(n-k)(i-k)} \text{ for all } k \in [n]_0$$

$$r_k(q \mid F)[n-k]_q! = \sum_{i=k}^{n} \frac{[i]_q!}{[k]_q![i-k]_q!} \hat{H}_i(q \mid B) \text{ for all } k \in [n]_0$$

Now he uses the two different q-Vandermonde identities to deduce a q-analogue of theorem 2.13 for each of his q-analogues. He proceeds with a few calculations and comes to the conclusion, that

$$Q_B(x,q) = \sum_{i=0}^{n} \tilde{H}_i(q \mid B) x^{n-i}.$$

So one of his q-hit-numbers coincides with the one (implicitly) defined by Garsia and Remmel. Which also proves $(*)$ in theorem 3.55. He proceeds by proving a recursion for the q-rook-numbers similar in style to corollary 2.40:

$$r_k(q \mid B) = q r_k(q \mid B/c) + r_{k-1}(B/\bar{c}),$$

with initial condition $r_k(q \mid \varnothing) = \delta_{k,0}$, for a board $B \subset [n] \times [n]$, c a corner square of B and $k \in [n]_0$. Similarly to what we did in the classical case, he obtains a recursion for his q-hit-numbers from this:

$$\tilde{H}_k(q \mid B) = q\tilde{H}_k(q \mid B/c) + \tilde{H}_{k-1}(B/\bar{c}) - q^n \tilde{H}_k(B/\bar{c}),$$

with initial condition $\tilde{H}_k(\varnothing) = [n]_q!\delta_{k,0}$. Now he defines a new statistic $\zeta(\sigma, B)$ on S_n, also including some placements of dots, depending on where the rooks are relative to a given board. For details on this please take a look at his paper [14]. The important thing here is, that this statistic is explicit and only depends on the board B, not like the recursively defined statistic that Garsia and Remmel gave. One very useful fact is, that this statistic is independent under column permutations. He makes use of this independence and the recurrence relation established above to show that:

$$\tilde{H}_k(q \mid B) = \sum_{\sigma \in S_n \text{ s.t. } |P_\sigma \cap B| = k} q^{\zeta(\sigma, B)}$$

So he obtains a combinatorial interpretation for these q-hit-numbers defined by Garsia and Remmel.

3.5.2 The failure of this approach in the weighted case

The first approach to a generalisation to the weighted case would be, as in the q-case, to imitate the behaviour of lower order analogues. In the q-case this worked out perfectly for Dworkin as described in the last chapter. Trying to follow a similar plan for the weighted setting leads to some severe obstacles. We will at least try to explain some of them in this section.
If we take the same steps as Garsia and Remmel, we soon encounter our first problem: we were only able to obtain a partial weighted analogue in theorem 3.54 of theorem 2.45. The important third equality of theorem 3.55 was not obtained in the general weighted setting. This is not due to some technical difficulties, but rather because we loose the nice properties of the geometric series that we made use of in the classical and q-setting as soon as our weights start to depend on the cells. It is best understood if one sees it. So let us just follow along the approach of the proof of theorem 3.55.
So let $(R, +, \cdot)$ be a commutative, unitary, integral domain, $\mathrm{w} : \mathbb{Z} \to R$ a small weight, $F = F(b_1, ..., b_n)$ a Ferrers board, then we want to calculate

$$\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(F_\infty)} \mathrm{Wt}_r^{\mathrm{w}}(P) x^{\max(P)}$$

in similar style as in the proof of theorem 3.55. The first few steps work out completely analogous:

$$\frac{1}{1-x} \sum_{P \in \mathcal{N}_n(F_\infty)} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_\infty) x^{\max(P)} =$$
$$\sum_{k=0}^{n} \sum_{Q \in \mathcal{N}_{n-k}(F)} \frac{1}{1-x} \sum_{P \in \mathcal{N}_n(F_\infty) \text{ s.t. } P \cap F = Q} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_\infty) x^{\max(P)}$$

We again define for a fixed $Q \in \mathcal{N}_{n-k}(F)$

$$S_Q^{\mathrm{w}}(x) \overset{\text{def}}{=} \sum_{P \in \mathcal{N}_n(F_\infty) \text{ s.t. } P \cap F = Q} \mathrm{Wt}_r^{\mathrm{w}}(P \mid F_\infty) x^{\max(P)}$$

and make use of the same bijection ψ. Fixing some $((p_1, ..., p_k), \sigma) \in \mathbb{N}^k \times S_k$ and denoting $P = \psi^{-1}(((p_1, ..., p_k), \sigma))$, we can partition $\mathrm{Inv}_{F_\infty}(P)$ into the same three parts:

$$\mathrm{Inv}_{F_\infty}(P) = \{(i,j) \in \mathrm{Inv}_{F_\infty}(P) \mid (i,j) \in F\} \cup$$
$$\{(i,j) \in \mathrm{Inv}_{F_\infty}(P) \setminus F \mid \text{there is a rook in row } j\} \cup$$
$$\{(i,j) \in \mathrm{Inv}_{F_\infty}(P) \setminus F \mid \text{there is no rook in row } j\}$$

By the same reasoning $\mathrm{Inv}_F(Q) = \{(i,j) \in \mathrm{Inv}_{F_\infty}(P) \mid (i,j) \in F\}$ and so we see that

$$\mathrm{Wt}_r^{\mathrm{w}}(P \mid F_\infty) = \mathrm{Wt}_r^{\mathrm{w}}(Q \mid F) \prod_{(i,j)\in \mathrm{Inv}_{F_\infty}(P)\setminus F} w\big(i-j-\mathrm{TL}_{(i,j)}(P)\big).$$

So we can control the first part as easily as in the q-setting. The nice thing in the q-case was, that one part of this product was independent of σ and when summing over all $\sigma \in S_k$ the second one reduced to a rook-number. In the general setting this does not work out. It rather needs some more notation and a lot of indices. A picture guiding you through the notation is present at the end of the discussion. So fix some $((p_1,...,p_k),\sigma) \in \mathbb{N}^k \times S_k$ and denote by $i_1,...,i_k$ the columns below the ground that contain rooks in $P = \psi^{-1}\Big(((p_1,...,p_k),\sigma)\Big)$. Denote

$$C = \Big\{\Big(i_t, -\big(p_1+...+p_j+(j-1)\big)\Big) \mid t \in [k], j \in [k]\Big\}$$

the set of cells below the ground, whose row contains a rook and whose column contains a rook below the ground. For $t \in [k]$ denote $l^\sigma_{t,1}$ the number of dots in the i_t-th column between the first empty cell contained in C and the ground, $l^\sigma_{t,2}$ the number of dots in the i_t-th column between the first empty cell contained in C and the second empty cell contained in C, etc. Note that when we say j-th empty cell, we always count from top/ground to bottom as usual.
Now say we placed $t-1$ rooks already below the ground, from left to right, to construct our permutation σ. Then the j-th empty cell of C in column i_t is given by

$$\Big(i, -\big(p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)} + (l^\sigma_{t,1}+...+l^\sigma_{t,j}) + (j-1)\big)\Big),$$

since we have to skip the first $j-1+(l^\sigma_{t,1}+...+l^\sigma_{t,j})$ cells of C, so it is the $j+(l^\sigma_{t,1}+...+l^\sigma_{t,j})$-th cell of C. Using our description of C we obtain the above. So we see that the j-th empty cell of C is the $p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)} + (l^\sigma_{t,1}+...+l^\sigma_{t,j}) + j$-th empty cell of the column i_t. Now recall our notation from the proof of theorem 3.49, where we denoted by l_1 the number of dots between the ground and the first empty cell of the i_t-th column, etc. Then clearly for any $j \in [k-t+1]$

$$l_1+...+l_{p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)}+(l^\sigma_{t,1}+...+l^\sigma_{t,j})+j} = l^\sigma_{t,1}+...+l^\sigma_{t,j},$$

as both sides of the equality just give the number of dots between the ground and the $p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)} + (l^\sigma_{t,1}+...+l^\sigma_{t,j}) + j$-th cell of the i_t-th column. As in the proof of theorem 3.49 we write $i_t = (t-1)+1+r_1+...+r_t$. Recall that we had seen, that then:

$$\mathrm{TL}_{(i,-(p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)}+(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)))}(P) =$$
$$r_1+...r_t+l_1+...+l_{p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)}+(l^\sigma_{t,1}+...+l^\sigma_{t,j})+j}$$
$$= r_1+...r_t+l^\sigma_{t,1}+...+l^\sigma_{t,j}$$

So placing a rook on the j-th empty cell of C all the $p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)}+(l^\sigma_{t,1}+...+l^\sigma_{t,j})+j-1$ empty cells of the i_t-th column contribute to the weight with the factor

$$\prod_{m=1}^{p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)}+(l^\sigma_{t,1}+...+l^\sigma_{t,j})+j-1} w\big(m+(t-1)\big) =$$
$$\mathrm{W}\Big(p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,j})+(j-1)}+(l^\sigma_{t,1}+...+l^\sigma_{t,j})+j+t-2,t\Big),$$

just like in the proof of theorem 3.49.
Since a permutation $\sigma=\sigma_1...\sigma_k\in S_k$ places a rook on the $k+1-\sigma_1$-th empty cell of C in the i_1-th column, a rook on the $k+1-\sigma_2$-th empty cell of C in the i_2-th column, etc, we see that

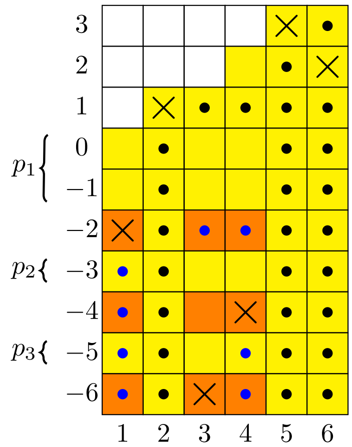
$$\prod_{(i,j)\in \mathrm{Inv}_{F_\infty}(P)\setminus F} w\big(i-j-\mathrm{TL}_{(i,j)}(P)\big) =$$
$$\prod_{j=1}^{k}\mathrm{W}\Big(p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,k+1-\sigma_j})+(k+1-\sigma_j-1)}+$$
$$(l^\sigma_{t,1}+...+l^\sigma_{t,k+1-\sigma_j})+k+1-\sigma_j+t-2,t\Big)=$$
$$\prod_{j=1}^{k}\mathrm{W}\Big(p_1+...+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,k+1-\sigma_j})+(k-\sigma_j)}+$$
$$(l^\sigma_{t,1}+...+l^\sigma_{t,k+1-\sigma_j})+k-\sigma_j+t-1,t\Big).$$

Now inserting this we obtain the following expression for $S^{\mathrm{w}}_Q(x)$:

$$S^{\mathrm{w}}_Q(x)=\mathrm{Wt}^{\mathrm{w}}_r(Q\mid F)x^k\sum_{\sigma\in S_n}\sum_{p_1\geq 0}...\sum_{p_k\geq 0}\prod_{j=1}^{k}\Bigg(x^{p_j}\mathrm{W}\Big(p_1+...$$
$$+p_{(l^\sigma_{t,1}+...+l^\sigma_{t,k+1-\sigma_j})+(k-\sigma_j)}+(l^\sigma_{t,1}+...+l^\sigma_{t,k+1-\sigma_j})+k-\sigma_j+t-1,t\Big)\Bigg)$$

So you see that there is no really nice way to simplify these sums any further. Even if one could manage to handle the sum over the $\sigma\in S_k$, we would still have to deal with a k-fold product of infinite series, which are not geometric anymore, since the sum over the symmetric group is quite likely to depend on the p_j's.
So we see that we cannot obtain the weighted analogue of the LHS of formula (3.21) via a weighted analogue of theorem 3.55, but rather have make an ansatz for that. Hence one has to walk the path of Dworkin. Sadly we do not have a weighted version of Vandermonde-identities at our disposal yet, so it is hard to make the right choice of where to insert the weights, as Dworkin calls it in his paper. Then one would have to come up with a clever way to obtain an analogue of theorem 2.13 as well, since in Dworkins paper this follows from the Vandermonde-identities. Say one managed to do all of that, then one still cannot obtain a weighted analogue of corollary 2.40, yet not even of proposition 2.32, so the weighted analogue of theorem 2.13 would somehow have to resolve that problems as well, or one would need a different approach to obtaining a combinatorial interpretation then. Maybe if some of these problems are

resolved, then one could manage to obtain a weighted analogue of the hit-numbers. Or one needs a completely different idea, which would be very interesting as well.
Here is the picture, trying to give an insight into the notation, as promised. The placement below corresponds via ψ to $\left((2,1,1), \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right)$, when fixing $Q = \{(2,1),(5,3),(6,2)\}$. The set of cells C is colored orange. The dots coming from the rooks above the ground are colored black. The dots coming only from the rooks below the ground are colored blue. Below we have that $l^{\sigma}_{2,1} = 1$.



3.5.3 The struggle with the fit-numbers

First of all we should mention, that there is not even a q-analogue for the fit-numbers yet. A few pages ago it was already foreshadowed, that the fit-numbers might be a bit different or might need a completely different approach. This was in remark 2.55. As you saw in the last section, the obstacle denying us even entering the game was the missing weighted analogue of $(*)$. In the fit-number case, we do not even have a formula similar to $(*)$ at our disposal in the classical case. This makes it impossible to use a similar approach as for the hit-numbers, which already needed a lot of ingenuity and clever tricks. So again one would have to make an ansatz for an analogue of theorem 2.28. So you see that the path to a solution here is quite foggy, but there is some shimmer of hope glancing at the horizon. Contrary to the hit-number case we did obtain a partial weighted analogue of proposition 2.33 in proposition 3.61, which generalizes all that was needed in order to deduce theorem 2.42 in the classical case, as remarked in 2.43, to the weighted setting. So it seems that the problems with the fit-numbers are fully focused in the weighted analogue of theorem 2.28. This definitely seems like an interesting problem for further study.

3.5.4 Outlook

We will close this thesis with an overview of a few open problems and questions that could be of interest. Surely there are many more available but these are the ones that seem doable for me (except for 1. most likely).

1. We have talked two whole chapters about weighted hit- and fit- numbers already, but nevertheless we still mention them once again since they are probably the

hardest part of our list.

2. Is there a similar connection between our weighted Stirling-numbers of first kind from example 3.60 and the weighted Stirling-numbers of first kind defined by Küstner, Schlosser and Yoo in [10] as in the case of the Stirling-numbers of second kind, discussed in example 3.51?

3. Is there a similar connection between our weighted Stirling-numbers of first kind from example 3.60 and the elementary symmetric functions as between the Stirling-numbers of second kind and the complete homogeneous symmetric functions, discussed in example 3.52? Intuitively there should be, at least if the second question can be answered with yes. This suspicion is due to the connection between the weighted Stirling-numbers of first kind and the elementary symmetric functions as discussed in [10].

4. Develop a weighted theory for the $\hat{\jmath}$-attacking rook-model of Remmel and Wachs from [21], similar to the elliptic version of Schlosser and Yoo discussed in [9].

5. Develop a weighted theory for the q-rook theory for matchings of graphs of Haglund and Remmel from [22] and the elliptic extension of Schlosser and Yoo from [23].

6. Develop a weighted theory for rook placements on augmented boards of Miceli and Remmel from [24] also generalising the formula of Schlosser and Yoo from [25].

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