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Abstract

An iterative process for constructing formal solutions of irregular singular differential equations at 0 is presented. This is achieved by conjugating the differential operator with an exponential function. In doing so we generalise the concept of local exponents and show that they are the coefficients of the term containing the essential singularity of the solution. By this method one obtains a basis of solutions over the complex numbers. Moreover the convergence of solutions of irregular singular differential equations is discussed.

Zusammenfassung

Ein iterativer Prozess zur Konstruktion formaler Lösungen irregulär singulärer Differentialgleichungen in 0 wird präsentiert. Dies wird durch die Konjugation des Differentialoperators mit einer Exponentialfunktion erreicht. Dabei wird das Konzept von lokalen Exponenten verallgemeinert und gezeigt, dass sie die Koeffizienten des Terms sind, welcher die wesentliche Singularität der Lösung beinhaltet. Mit dieser Methode erhält man eine Basis von Lösungen über den komplexen Zahlen. Außerdem wird die Konvergenz von Lösungen irregulär singulärer Differentialgleichungen besprochen.

Contents

1 Introduction

Lazarus Fuchs introduced the concept of *regular singular* differential equations [Fuc66]. A differential equation with holomorphic coefficients is said to be regular singular at 0 if it admits a full basis of *regular* solutions. Solutions are said to be regular at 0 if their growth is at most polynomial as they approach 0 in sectors. Fuchs characterised regular singular differential equations by an algebraic bound on the order of the coefficients of the differential equation, now known as *Fuchs' criterion*. In subsequent works Fuchs, Thomé, and Frobenius developed methods for constructing solutions of regular singular differential equations [Fuc68, Tho72, Fro73]. Frobenius in particular developed a prominent method using *local exponents*, known as the *Frobenius method*. In later works Thomé and Fabry constructed solutions for *irregular singular* differential equations, equations where the Fuchs criterion is violated [Tho73, Fab85]. The way these solutions were constructed is by first calculating the *determining factor*, the exponential of a rational function, and then deriving an equation with a regular solution. Thomé denoted solutions of this type as *normal* solutions [Tho73]. Fabry generalises this approach and constructs a new type of solutions he called *subnormal* solutions, normal solutions in the variable $x^{1/t}$ [Fab85]. By this method one obtains at most as many $\mathbb{C}$-linearly independent solutions as the order of the differential equation indicates. It turns out that the existence of normal and subnormal solutions is rather exceptional for irregular singular differential equations, as it requires convergence of the occurring power series [Inc56]. When also considering formal solutions one again obtains a basis of solutions for irregular singular differential equations. Though not necessarily convergent, Maillet showed that power series solutions of irregular singular differential equations are *Gevrey* [Mai03].

As an alternative formulation to differential equations one can focus on the study of the differential operator itself. Following this approach, Hauser obtained a normal form theorem for differential operators, reformulating the Frobenius method in a modern fashion. When applying the normal form theorem to a regular singular differential operator at 0 one obtains a $\mathbb{C}$-basis of local solutions of the associated differential equation. This is not the case for irregular singular differential operators at 0. Though not providing a basis, one may still get solutions by applying the normal form theorem. In fact, one obtains a solution for every local exponent of the differential operator [Hau22].

In this work we generalise the concept of local exponents and show that every differential operator has as many *generalised local exponents* as its order indicates, counting with multiplicity. Furthermore we develop an algorithm using the generalised local exponents to stepwise reduce a given differential operator to one where solutions of the associated differential equation can be obtained by applying the normal form theorem of Hauser. Then, backtracking our algorithm, we reconstruct solutions of the original differential equation. We show that for every generalised local exponent of the differential operator we obtain a number of $\mathbb{C}$-linearly independent solutions of the associated equation equal to its multiplicity. The theorem of Fabry is thereby proven in a more conceptual manner [Fab85]:

Theorem 1.1. *Let $L \in \mathbb{C}((x))[\partial]$ be a linear differential operator with formal power series coefficients. A $\mathbb{C}$-basis of solutions of the associated linear differential equation $Ly = 0$ is given by a set of functions of the form*

$$y = e^{P(x^{-1/t})} \cdot x^{\rho} \cdot [f_0(x^{1/t}) + f_1(x^{1/t}) \log(x) + \cdots + f_m(x^{1/t}) \log^m(x)],$$

where $P \in \mathbb{C}[X]$ is a polynomial, $f_0, \ldots, f_m \in \mathbb{C}[[X]]$ are formal power series, $t \in \mathbb{Z}_+$ a positive integer and $\rho \in \mathbb{C}$ a complex number.
A more comprehensive version of this theorem is presented by Theorem 3.13.

For the algorithm to construct solutions we first introduce generalised local exponents. This is done by varying the underlying basis of the vector space of differential operators. To this end we introduce the derivations $\delta_r = x^{r+1}\partial$ for rational numbers $r \in \mathbb{Q}$. We may then rewrite the differential operator $L = f_n\partial^n + f_{n-1}\partial^{n-1} + \cdots + f_0$ in terms of δ_r, i.e., $L = g_n\delta_r^n + g_{n-1}\delta_r^{n-1} + \cdots + g_0$. Now one can compute local exponents from the coefficients $g_n, g_{n-1}, \ldots, g_0$ in a similar fashion as one did classically from the coefficients $f_n, f_{n-1}, \ldots, f_0$. We denote them as *generalised local exponents*. We show that, disregarding generalised local exponents equal to 0, every differential operator has a number of generalised local exponents equal to the order of the operator, counting with multiplicity.

We then go on to construct solutions of the differential equation $Ly = 0$ corresponding to the generalised local exponents of the associated differential operator L. This is achieved by conjugating the differential operator L with the multiplication operator $\varphi : f \mapsto \exp(\frac{\sigma}{-r}x^{-r}) \cdot f$, where σ is a generalised local exponent of L in the derivation δ_r. This conjugation $\gamma : L \mapsto \varphi^{-1} \circ L \circ \varphi$ is an automorphism of the space of differential operators and induces an isomorphism on the kernels, again given by the multiplication operator $\varphi : \ker \gamma(L) \to \ker L$. We show that an iterated process of conjugating the differential operator L with multiplication operators corresponding to the correct generalised local exponents terminates, yielding a differential operator $\widetilde{L}$ that has a *classical* local exponent ρ. One can then apply the normal form theorem to the operator $\widetilde{L}$ to construct a solution $\widetilde{y}$ of the associated equation $\widetilde{L}\widetilde{y} = 0$, cf. [Hau22]. Backtracking the iterated conjugations, a solution y of the equation $Ly = 0$ is given by applying a corresponding series of induced isomorphisms to $\widetilde{y}$, i.e.,

$$y = \prod_{i=0}^{N} \exp(\frac{\sigma_i}{-r_i}x^{-r_i}) \cdot \widetilde{y} = \exp(\sum_{i=0}^{N} \frac{\sigma_i}{-r_i}x^{-r_i}) \cdot \widetilde{y} = e^{P(x^{-1/t})} \cdot \widetilde{y}.$$

We show that for every generalised local exponent of L we obtain a number of $\mathbb{C}$-linearly independent solutions of $Ly = 0$ equal to its multiplicity. Thus, ranging over all generalised local exponents, a $\mathbb{C}$-basis of solutions is constructed.

2 Differential Operators

The objects of interest are formal differential operators with Puiseux series coefficients. The field of formal Puiseux series $\mathbb{C}\langle x\rangle$ is defined as the union of the fields of formal fractional Laurent series $\mathbb{C}((x^{\frac{1}{t}}))$,

$$\mathbb{C}\langle x\rangle = \bigcup_{t\in\mathbb{Z}_+} \mathbb{C}((x^{\frac{1}{t}})).$$

The motivation for considering Puiseux series coefficients is that, as we will later show, we can in general not expect the differential equation to have power series solutions, even for differential operators with power series coefficients.
We equip the field of formal Puiseux series with the $\mathbb{C}$-derivations

$$\begin{aligned} \delta_r : \mathbb{C}\langle x\rangle &\to \mathbb{C}\langle x\rangle, \\ \sum_{s\in\mathbb{Q}} a_s x^s &\mapsto \sum_{s\in\mathbb{Q}} s a_s x^{s+r}, \end{aligned}$$

for all rationals $r \in \mathbb{Q}$. This turns the field of formal Puiseux series into a differential field $(\mathbb{C}\langle x\rangle, \delta_r)$ for every $r \in \mathbb{Q}$.
For $r = -1$ we have $\delta_{-1} = \partial$ the usual derivative and for $r = 0$ the derivation $\delta = \delta_0$ is the *logarithmic derivative*. In the following we are going to omit the index of the derivation $\delta_0 = \delta$.
Note that as operators on $\mathbb{C}\langle x\rangle$ we can write $\delta_r = x^r\delta$ and thus we will refer to δ_r as the *derivation with shift* $r \in \mathbb{Q}$.
Set $r = \frac{p}{q} \in \mathbb{Q}$ for integers $p, q \in \mathbb{Z}$. The field of formal fractional Laurent series $\mathbb{C}((x^{\frac{1}{q}}))$ can be seen as a differential subfield of $\mathbb{C}\langle x\rangle$,

$$(\mathbb{C}((x^{\frac{1}{q}})), \delta_r) \subseteq (\mathbb{C}\langle x\rangle, \delta_r).$$

For $r \in \mathbb{Q}$ we will define $\delta_r^j = \delta_r \circ \cdots \circ \delta_r$ as the j-fold composition of δ_r on $\mathbb{C}\langle x\rangle$ and further define a *differential operator of index* r as an element

$$L = f_n\delta_r^n + f_{n-1}\delta_r^{n-1} + \cdots + f_0 \in \mathbb{C}\langle x\rangle[\delta_r].$$

The differential operator $L \in \mathbb{C}\langle x\rangle[\delta_r]$ can be seen as an operator on the function space $\mathbb{C}\langle x\rangle$, hence we will associate to $L \in \mathbb{C}\langle x\rangle[\delta_r]$ the $\mathbb{C}$-linear map

$$\begin{aligned} L : \mathbb{C}\langle x\rangle &\to \mathbb{C}\langle x\rangle \\ h &\mapsto \sum_{j=0}^{n} f_j\delta_r^j h. \end{aligned}$$

Together with the term-wise addition, scalar multiplication with elements in $\mathbb{C}$, and the multiplication defined via the composition of the associated $\mathbb{C}$-linear maps on $\mathbb{C}\langle x\rangle$, the differential operators of index r form a $\mathbb{C}$-algebra $(\mathbb{C}\langle x\rangle[\delta_r], +, \cdot, \circ)$. Alternatively, one can view $\mathbb{C}\langle x\rangle[\delta_r]$ as a $\mathbb{C}\langle x\rangle$-vector space, where δ_r^j form a basis.

2.1 Change of Basis for Differential Operators

Our first goal is to establish a relation between the $\mathbb{C}$-algebras of differential operator $\mathbb{C}\langle x\rangle[\delta_r]$ for all $r \in \mathbb{Q}$. To this end first recall the Stirling numbers:

Definition 2.1. The *Stirling numbers of the first kind* and the *Stirling numbers of the second kind* are recursively defined numbers. They can be constructed in the following way:
First define $\begin{bmatrix}0\\0\end{bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix} = 1$ and for all integers $j \in \mathbb{Z}\setminus\{0\}$ and for all naturals $n \in \mathbb{N}\setminus\{0\}$

$$\begin{bmatrix}0\\j\end{bmatrix} = \begin{bmatrix}n\\0\end{bmatrix} = \begin{Bmatrix}0\\j\end{Bmatrix} = \begin{Bmatrix}n\\0\end{Bmatrix} = 0.$$

For the Stirling number of the first kind we define

$$\begin{bmatrix}n+1\\j\end{bmatrix} = n\begin{bmatrix}n\\j\end{bmatrix} + \begin{bmatrix}n\\j-1\end{bmatrix}$$

and analogously for the Stirling numbers of the second kind we define

$$\begin{Bmatrix}n+1\\j\end{Bmatrix} = j\begin{Bmatrix}n\\j\end{Bmatrix} + \begin{Bmatrix}n\\j-1\end{Bmatrix}.$$

One can consider the Stirling numbers of the first and second kind as inverse to each other, the following two identities hold:

$$\sum_j (-1)^{n-j}\begin{bmatrix}n\\j\end{bmatrix}\begin{Bmatrix}j\\i\end{Bmatrix} = \begin{cases}1, & \text{for } n = i,\\ 0, & \text{for } n \neq i,\end{cases}$$

and

$$\sum_j (-1)^{j-i}\begin{Bmatrix}n\\j\end{Bmatrix}\begin{bmatrix}j\\i\end{bmatrix} = \begin{cases}1, & \text{for } n = i,\\ 0, & \text{for } n \neq i.\end{cases}$$

Lemma 2.1. *Let δ_r be the derivation with shift $r \in \mathbb{Q}$. Seen as an operator on $\mathbb{C}\langle x\rangle$ the n-fold composition of δ_r equals*

$$\delta_r^n = x^{rn}\sum_{j=0}^{n} r^{n-j}\begin{bmatrix}n\\j\end{bmatrix}\delta^j.$$

Conversely, as operators on $\mathbb{C}\langle x\rangle$ the n-fold composition of δ equals

$$\delta^n = \sum_{j=0}^{n} (-r)^{n-j} \left\{ {n \atop j} \right\} x^{-rj} \delta_r^j.$$

Proof. We are going to prove the statement by induction on $n \in \mathbb{N}$.
For $n = 0$ the statement clearly holds by the definitions of $\left[{0 \atop j} \right]$ and $\left\{ {0 \atop j} \right\}$.
Now for $n \geq 0$ we get

$$\begin{aligned} \delta_r^{n+1} &= \delta_r \delta_r^n = \delta_r x^{rn} \sum_{j\in\mathbb{Z}} r^{n-j} \left[{n \atop j} \right] \delta^j \\ &= x^{r(n+1)} \sum_{j\in\mathbb{Z}} r^{n-j} \left[{n \atop j} \right] \delta^{j+1} + rn\; x^{r(n+1)} \sum_{j\in\mathbb{Z}} r^{n-j} \left[{n \atop j} \right] \delta^j \\ &= x^{r(n+1)} \sum_{j\in\mathbb{Z}} r^{n+1-j} (\left[{n \atop j-1} \right] + n \left[{n \atop j} \right]) \delta^j \\ &= x^{r(n+1)} \sum_{j\in\mathbb{Z}} r^{n+1-j} \left[{n+1 \atop j} \right] \delta^j, \end{aligned}$$

where we used the induction hypothesis in the second equality and Leibniz' rule in the third. Similarly we also get

$$\begin{aligned} \delta^{n+1} &= \delta\delta^n = \delta \sum_{j\in\mathbb{Z}} (-r)^{n-j} \left\{ {n \atop j} \right\} x^{-rj} \delta_r^j \\ &= \sum_{j\in\mathbb{Z}} (-r)^{n-j} \left\{ {n \atop j} \right\} x^{-r(j+1)} \delta_r^{j+1} - rj \sum_{j\in\mathbb{Z}} (-r)^{n-j} \left\{ {n \atop j} \right\} x^{-rj} \delta_r^j \\ &= \sum_{j\in\mathbb{Z}} (-r)^{n+1-j} (\left\{ {n \atop j-1} \right\} + j \left\{ {n \atop j} \right\}) x^{-rj} \delta_r^j \\ &= \sum_{j\in\mathbb{Z}} (-r)^{n+1-j} \left\{ {n+1 \atop j} \right\} x^{-rj} \delta_r^j, \end{aligned}$$

which concludes the proof. QED

Proposition 2.2. *Let $r \in \mathbb{Z}$ be a rational. There is an* action preserving $\mathbb{C}$-*algebra isomorphism*

$$\alpha_r : \mathbb{C}\langle x\rangle[\delta] \to \mathbb{C}\langle x\rangle[\delta_r]$$

between the algebras of differential operators of index 0 and r. Let $L = f_{0,n}\delta^n + f_{0,n-1}\delta^{n-1} + \cdots + f_{0,0} \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator of index 0, then $\alpha_r(L)$ is given by

$$\alpha_r(L) = f_{r,n}\delta_r^n + f_{r,n-1}\delta_r^{n-1} + \cdots + f_{r,0},$$

where the coefficients equal

$$f_{r,j}(x) = x^{-rj} \sum_{i=j}^{n} (-r)^{i-j} \left\{ {i \atop j} \right\} f_{0,i}(x) \in \mathbb{C}\langle x \rangle$$

for $j = 0, \ldots, n$.
The inverse

$$\alpha_r^{-1} : \mathbb{C}\langle x \rangle[\delta_r] \to \mathbb{C}\langle x \rangle[\delta]$$

of a differential operator $M = f_{r,n}\delta_r^n + f_{r,n-1}\delta_r^{n-1} + \cdots + f_{r,0} \in \mathbb{C}\langle x \rangle[\delta_r]$ *of index* r *is given by*

$$\alpha_r^{-1}(M) = f_{0,n}\delta^n + f_{0,n-1}\delta^{n-1} + \cdots + f_{0,0},$$

where the coefficients equal

$$f_{0,i}(x) = \sum_{j=i}^{n} r^{j-i} \begin{bmatrix} j \\ i \end{bmatrix} x^{rj} f_{r,j}(x) \in \mathbb{C}\langle x \rangle$$

for $i = 0, \ldots, n$.

Remark. This isomorphism α_r is *action preserving* when viewing the associated $\mathbb{C}$-linear map on the field of Puiseux series $\mathbb{C}\langle x \rangle$ as the action of the multiplicative group $(\mathbb{C}\langle x \rangle[\delta_r], \circ)$ on $\mathbb{C}\langle x \rangle$. Let $L \in \mathbb{C}\langle x \rangle[\delta]$ and $\alpha_r(L) \in \mathbb{C}\langle x \rangle[\delta_r]$ be differential operators acting on $\mathbb{C}\langle x \rangle$, then

$$\alpha_r(L)h = Lh$$

for all $h \in \mathbb{C}\langle x \rangle$.

Proof. It is easy to see that the map $\alpha_r : \mathbb{C}\langle x \rangle[\delta] \to \mathbb{C}\langle x \rangle[\delta_r]$ is $\mathbb{C}$-linear, in fact even $\mathbb{C}\langle x \rangle$-linear. We are first going to show that α_r and α_r^{-1} are actually inverse to each other. To this end we look at the transformation of the coefficients $f_{0,i}(x)$, for $i = 0, \ldots, n$, when applying the composition $\alpha_r^{-1} \circ \alpha_r$ to a differential operator $L \in \mathbb{C}\langle x \rangle[\delta]$.

$$f_{0,i}(x) \mapsto \sum_{j=i}^{n} r^{j-i} \begin{bmatrix} j \\ i \end{bmatrix} x^{rj} x^{-rj} \sum_{l=j}^{n} (-r)^{l-j} \left\{ {l \atop j} \right\} f_{0,l}(x).$$

Rearranging yields

$$\sum_{l=i}^{n} \sum_{j=i}^{l} (-1)^{l-j} \left\{ {l \atop j} \right\} \begin{bmatrix} j \\ i \end{bmatrix} r^{l-i} f_{0,l}(x) = f_{0,i}.$$

Hence $\alpha_r^{-1} \circ \alpha_r = \mathrm{Id}$ on $\mathbb{C}\langle x \rangle[\delta]$. Similarly for the transformation of coefficients $f_{r,j}(x)$ under the composition $\alpha_r \circ \alpha_r^{-1}$ of $L \in \mathbb{C}\langle x \rangle[\delta_r]$,

$$f_{r,j}(x) \mapsto x^{-rj} \sum_{i=j}^{n} (-r)^{i-j} \left\{ {i \atop j} \right\} \sum_{l=i}^{n} r^{l-i} \begin{bmatrix} l \\ i \end{bmatrix} x^{rl} f_{r,l}(x).$$

We can again rearrange, yielding

$$x^{rj}\sum_{l=i}^{n}\sum_{i=j}^{l}(-1)^{i-j}\begin{bmatrix} l \\ i \end{bmatrix}\begin{Bmatrix} i \\ j \end{Bmatrix}r^{l-j}x^{rl}f_{r,l}(x) = f_{r,j}.$$

Therefore α_r is a $\mathbb{C}$-vector space isomorphism and its inverse is given by α_r^{-1}. In fact, we have even shown that α_r is a $\mathbb{C}\langle x\rangle$-vector space isomorphism.
To see that α_r preserves the action of $\mathbb{C}\langle x\rangle[\delta]$ on $\mathbb{C}\langle x\rangle$ we use Lemma 2.1.
Let

$$L = f_{0,n}\delta^n + f_{0,n-1}\delta^{n-1} + \cdots + f_{0,0} \in \mathbb{C}\langle x\rangle[\delta]$$

be a differential operator of index 0 and $h \in \mathbb{C}\langle x\rangle$ a Puiseux series, then

$$\begin{aligned} Lh &= \sum_{i=0}^{n} f_{0,i}\delta^i h \\ &= \sum_{i=0}^{n} f_{0,i}\sum_{j=0}^{i}(-r)^{i-j}\begin{Bmatrix} i \\ j \end{Bmatrix}x^{-rj}\delta_r^j h \\ &= \sum_{j=0}^{n} \underbrace{x^{-rj}\sum_{i=j}^{n}(-r)^{i-j}\begin{Bmatrix} i \\ j \end{Bmatrix}f_{0,i}}_{=f_{r,j}}\ \delta_r^j h \\ &= \alpha_r(L)h. \end{aligned}$$

Hence α_r is action preserving. Since the multiplication on $\mathbb{C}\langle x\rangle[\delta_r]$ is defined via the composition of operators and

$$\alpha_r(L\circ M)h = (L\circ M)h = (\alpha_r(L)\circ\alpha_r(M))h$$

we conclude that α_r is a $\mathbb{C}$-algebra isomorphism, thus proving the claim. QED

Note that we can replace $\mathbb{C}\langle x\rangle[\delta_r]$ with $\mathbb{C}((x^{\frac{1}{q}}))[\delta_r]$ in the statement of Proposition 2.2 for any $q \in \mathbb{Z}$ such that there exists a $p \in \mathbb{Z}$, $r = \frac{p}{q} \in \mathbb{Q}$. Additionally, when $\mathbb{C}\langle x\rangle[\delta_r]$ is seen as a $\mathbb{C}\langle x\rangle$-vector space, the isomorphism $\alpha_r : \mathbb{C}\langle x\rangle[\delta_r] \to \mathbb{C}\langle x\rangle[\delta]$ is a change-of-basis transformation of the space of differential operators. We therefore regard them as different representations of the same object.
In the following we often view differential operators $L \in \mathbb{C}\langle x\rangle[\delta]$ of index 0 as differential operators of index $r \in \mathbb{Q}$, i.e., we regard L as an element in $\mathbb{C}\langle x\rangle[\delta_r]$ for all $r \in \mathbb{Q}$ and refrain from using $\alpha_r : \mathbb{C}\langle x\rangle[\delta] \to \mathbb{C}\langle x\rangle[\delta_r]$. We therefore omit the index when there is no need for specification.
When talking about the coefficients of $L \in \mathbb{C}\langle x\rangle[\delta]$ seen as a differential operator of index $r \in \mathbb{Q}$ we will call the coefficients $(f_{r,j})_{j=0,\dots,n}$ of $L \in \mathbb{C}\langle x\rangle[\delta_r]$ the *coefficients of index r* of L.

Recall that the field $\mathbb{C}\langle x \rangle$ of formal Puiseux series is a *valued field*. The *valuation* ν on $\mathbb{C}\langle x \rangle$ is given by

$$\begin{aligned} \nu : \mathbb{C}\langle x \rangle &\to \mathbb{Q} \cup \{\infty\}, \\ \sum_j a_j x^{\frac{j}{t}} &\mapsto \frac{\min\{j \in \mathbb{Z} : a_j \neq 0\}}{t}, \end{aligned}$$

and coincides with the *order* on formal Laurent series $\mathbb{C}((x)) \subseteq \mathbb{C}\langle x \rangle$. The valuation of Puiseux series is independent of the choice of the *ramification index* $t \in \mathbb{Z}_+$. Throughout this work we will need to study the valuation of the coefficients of our differential operator $L \in \mathbb{C}\langle x \rangle[\delta]$. Thus we will make a few initial remarks on the order of the coefficients of index $r \in \mathbb{Q}$:

Remarks. Let $L \in \mathbb{C}\langle x \rangle[\delta]$ be a differential operator and $(f_{r,j})_{j=0,\ldots,n}$ be the coefficients of index r for all $r \in \mathbb{Q}$.

1. The first coefficient, $f_{r,0}$, is unaltered by the change-of-basis transformation α_r. Since $\left[\begin{smallmatrix} j \\ 0 \end{smallmatrix}\right] = 1$ if and only if $j = 0$, the coefficient $f_{r,0} = f_{0,0}$ is independent of the choice or $r \in \mathbb{Q}$.

2. For a fixed $j = 1, \ldots, n$ the valuation ν of the coefficient $f_{r,j}$ of index $r \in \mathbb{Q}$ has the tendency to decrease when increasing r. To be precise, let $L = f_{0,n}\delta^n + f_{0,n-1}\delta^{n-1} + \cdots + f_{0,0} \in \mathbb{C}\langle x \rangle[\delta]$ be a differential operator of index 0. The transformation of the coefficients under α_r is given by

$$\alpha_r(L) = \underbrace{x^{-rn} f_{0,n}}_{=f_{r,n}} \delta_r^n + \underbrace{x^{-r(n-1)}(f_{0,n-1} - r\begin{Bmatrix} n \\ n-1 \end{Bmatrix} f_{0,n})}_{=f_{r,n-1}} \delta_r^{n-1} + \cdots + \underbrace{f_{0,0}}_{=f_{r,0}}.$$

   Thus for $j = 1, \ldots, n$ the coefficient $f_{r,j}$ of index r of L is a linear combination of the coefficients $\{f_{0,i} \text{ for } i \geq j\}$ of index 0 with the prefactor x^{-rj}. Thus we get a lower bound for the valuation by

$$\nu(f_{r,j}) \geq \min_{i \geq j} \nu(f_{0,i}) - rj.$$

   Without any further knowledge of the coefficients $f_{0,i}$ of index 0 we cannot get an exact formula for the valuation of coefficients of index r in terms of the valuation of coefficients of index 0. Some cancellation in the sum of coefficients of index 0 can occur, thus increasing the valuation.

Example 1. Consider the differential operator

$$L = \delta^2 + \delta + 1 \in \mathbb{C}[\delta]$$

with constant coefficients $f_{0,j}(x) = 1$ of index 0, for $j = 0, 1, 2$. Seen as a differential operator of index 1

$$\begin{aligned} L &= x^{-2}\delta_1^2 + x^{-1}(1-1)\delta_1 + 1 \\ &= x^{-2}\delta_1^2 + 1. \end{aligned}$$

The valuation of the coefficient $f_{1,1}(x) = 0$ of L is $\nu(f_{1,1}) = \infty$.

3. For the last coefficient, $f_{r,n}$, one can be precise about the valuation,

$$\nu(f_{r,n}) = \nu(f_{0,n}) - rn.$$

With this in mind it is easy to see that for every differential operator $L \in \mathbb{C}\langle x\rangle[\delta]$ there is a rational $r \in \mathbb{Q}$ such that the order of $f_{r,n}$ is minimal in the set of coefficients of index r. This leads to the following definition:

Definition 2.2. Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and let for all rationals $r \in \mathbb{Z}$ the coefficients of index r of L be $(f_{r,j})_{j=0,\dots,n}$. The *rank* of L is defined as

$$\operatorname{rank}(L) = \min\{r \in \mathbb{Z} : \nu(f_{r,n}) \le \nu(f_{r,j}) \text{ for all } j \in \{0,\dots,n\}\}.$$

Remark. Since the valuation of the first coefficient $f_{r,n}$ decreases faster than the valuation of all other coefficients of index r when r increases, the condition

$$\nu(f_{r,n}) \le \nu(f_{r,j}) \text{ for all } j \in \{0,\dots,n\}$$

is also met by all $r \in \mathbb{Q}$ larger than the rank of L.
This notion of rank is defined slightly differently, but equivalent to the *Poincaré rank* of differential operators [Poi00].

Definition 2.3. Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and let for $r \in \mathbb{Q}$ and $j = 0,\dots,n$

$$f_{r,j} = \sum_i a_{r,j,i}\, x^i \in \mathbb{C}\langle x\rangle$$

be the j^{th} coefficient of index r of L. We define the *shift of index* r of L as the minimal valuation of the coefficients,

$$\tau_r(L) = \min_{j\in\{0,\dots,n\}} \nu(f_{r,j}).$$

Furthermore we define the *initial operator of index* r as the projection

$$\begin{aligned} I_r &: \mathbb{C}\langle x\rangle[\delta_r] \to \mathbb{C}[\delta_r], \\ I_r(L) &= \sum_{j=0}^{n} a_{j,\tau_r(L)}\delta_r^j \end{aligned}$$

onto the *differential operators of index* r *with constant coefficients.* We will call $I_r(L)$ the *initial form of index* r of L.

Remarks. Let $r \in \mathbb{Q}$ be a rational.

1. We extend the definition of shift and initial operator of index $r \in \mathbb{Q}$ to differential operators $L \in \mathbb{C}\langle x\rangle[\delta]$ of arbitrary index via the coefficients of index r.
2. The initial form operator $I_r : \mathbb{C}\langle x\rangle[\delta_r] \to \mathbb{C}[\delta_r]$ is idempotent. Clearly when restricting to operators with constant coefficients the initial operator equals the identity
$$I_r\big|_{\mathbb{C}[\delta_r]} = \mathrm{Id}_{\mathbb{C}[\delta_r]}\,.$$
3. A differential operator $L \in \mathbb{C}((x^{\frac{1}{t}}))[\delta_r]$ of index r can be decomposed into the sum
$$L = x^{\tau_r(L)}[L_0 + x^{\frac{1}{t}}L_1 + x^{\frac{2}{t}}L_2 + \dots],$$
where the terms $L_k \in \mathbb{C}[\delta_r]$ for $k \in \mathbb{N}$ are differential operators of index r with constant coefficients and $L_0 = I_r(L)$ is the initial form of L.
4. Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be an n^{th} order differential operator. The initial form $I_r(L) \in \mathbb{C}[\delta_r]$ of index r is again a differential operator of order n if and only if $\mathrm{rank}(L) \leq r$.

Example 2. Consider the differential operator
$$L = x\delta^2 - \delta + 1.$$
Seen as an operator of index $r = -1$
$$L = x^3\partial^2 + (x^2 - x)\partial + 1,$$
in terms of $\partial = \delta_{-1}$, the *actual* derivative, and as an operator of index $r = 1$
$$L = x^{-1}\delta_1^2 - (x^{-1} + 1)\delta_1 + 1.$$
Observe that the rank of the differential operator L equals $\mathrm{rank}(L) = 1$. Furthermore the initial forms of indices $-1, 0,$ and 1 are
$$I_{-1}(L) = 1 \in \mathbb{C}[\partial], \qquad I_0(L) = -\delta + 1 \in \mathbb{C}[\delta], \qquad I_1(L) = \delta_1^2 - \delta_1 \in \mathbb{C}[\delta_1].$$

2.2 Extension of the Function Space $\mathcal{F}$

In this section we are going to enlarge the function space $\mathcal{F}$ we let our differential operators $L \in \mathbb{C}\langle x\rangle[\delta]$ act on. Towards this goal we are first going to define exponential functions with respect to the derivations δ_r.
Exponential function with respect to δ_r. Let $r \in \mathbb{Q} \setminus \{0\}$ be a rational other than 0 and $\rho \in \mathbb{C}$ a complex number. Denote by $z_r(x)$ the weighted monomials
$$z_r(x) = -\frac{1}{r}x^{-r},$$

and define the functions

$$e_r^{\rho x} = e^{\rho z_r(x)}.$$

Observe that the monomials $z_r(x)$ are defined such that $\delta_r z_r = 1$. By the chain rule for the usual derivative $\delta_{-1} = \partial$ for holomorphic functions f, g we get

$$\begin{aligned}\delta_r(f \circ g) &= x^{r+1}\partial(f \circ g) = x^{r+1}\partial g \cdot (\partial f \circ g)\\ &= \delta_r g \cdot (\partial f \circ g).\end{aligned}$$

Now since $z_r(x)$ is holomorphic on any simply connected open subset of $\mathbb{C} \setminus \{0\}$ we get

$$\delta_r e_r^{\rho x} = \rho \delta_r z_r \cdot e_r^{\rho x} = \rho e_r^{\rho x}.$$

The function e_r^x can therefore be seen as an *exponential function* with respect to the derivation δ_r.

Extension of the function space $\mathcal{F}$. For every integer $r \in \mathbb{Q} \setminus \{0\}$ and $\rho \in \mathbb{C}$ we can enlarge the function space $\mathcal{F}$ on which the differential operator $L \in \mathbb{C}\langle x \rangle[\delta_r]$ acts. By formally defining

$$\delta_r e_r^{\rho x} = \rho e_r^{\rho x},$$

we can again associate to $L \in \mathbb{C}\langle x \rangle[\delta_r]$ the linear map

$$\begin{aligned}L : e_r^{\rho x}\mathbb{C}\langle x \rangle &\to e_r^{\rho x}\mathbb{C}\langle x \rangle,\\ e_r^{\rho x} h(x) &\mapsto L(e_r^{\rho x} h(x)),\end{aligned}$$

on the extended function space $\mathcal{F} = e_r^{\rho x}\mathbb{C}\langle x \rangle$.

Here, we can regard $e_r^{\rho x}$ either as new symbols subject to the differentiation rules we defined or as holomorphic functions on any simply connected open subset of $\mathbb{C} \setminus \{0\}$. There our definition coincides with the analytic interpretation of the derivative.

The logarithm. For $r = 0$ we can first extend our differential ring $(\mathbb{C}\langle x \rangle, \delta)$ to account for the logarithm. To this end, we may introduce a new variable $z = z_0 = \log(x)$ and identify it with the logarithm. Note that $\log(x)$ again satisfies

$$\delta \log(x) = 1$$

on any simply connected open subset of $\mathbb{C} \setminus \{0\}$. We can equip the polynomial ring $\mathbb{C}\langle x \rangle[z]$ over the field of Puiseux series with the $\mathbb{C}$-derivation

$$\delta : \mathbb{C}\langle x \rangle[z] \to \mathbb{C}\langle x \rangle[z],$$

where we define

$$\delta(x^i z^j) = (i + jz^{-1})x^i z^j,$$

for $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ and extend this linearly to the whole space. This again turns $(\mathbb{C}\langle x \rangle[z], \delta)$ into a differential ring. By our definition $(\mathbb{C}\langle x \rangle[z], \delta)$ is an extension of $(\mathbb{C}\langle x \rangle, \delta)$ as differential rings.

Furthermore we may enlarge the function space to $\mathcal{F} = x^\rho \mathbb{C}\langle x\rangle[z]$, where we write x^ρ for $e_0^{\rho z} = e^{\rho \log(x)} = x^\rho$ and define $\delta x^\rho = \rho x^\rho$. To $L \in \mathbb{C}\langle x\rangle[\delta]$ we associate the $\mathbb{C}$-linear map

$$\begin{aligned} L : x^\rho \mathbb{C}\langle x\rangle[z] &\to x^\rho \mathbb{C}\langle x\rangle[z], \\ x^\rho h(x) z^j &\mapsto L(x^\rho h(x) z^j). \end{aligned}$$

We again treat x^ρ and z as symbols subject to their differentiation rules, motivated by the analytic interpretation.

Differential subrings of $(\mathbb{C}\langle x\rangle, \delta)$**.** When treating differential operators with coefficients in the ring of formal power series $\mathbb{C}[[x]]$, the field of formal Laurent series $\mathbb{C}((x))$, ring of germs of holomorphic functions $\mathbb{C}\{x\}$, or the ring of germs of meromorphic functions $\mathcal{M}(x) = \mathrm{Quot}(\mathbb{C}\{x\})$ we may also replace the field of Puiseux series $\mathbb{C}\langle x\rangle$ with $\mathbb{C}[[x]]$, $\mathbb{C}((x))$, $\mathbb{C}\{x\}$, or $\mathcal{M}(x)$, respectively, in the function spaces $\mathcal{F}$ the differential operator acts on. Note that the rings above are all closed under δ and therefore differential subrings of $(\mathbb{C}\langle x\rangle, \delta)$.

2.3 The Conjugation Operator

The goal of this section is to introduce the conjugation operator on the $\mathbb{C}$-algebra of differential operators. We then go on to show some useful properties of the conjugation.

Definition 2.4. Let $r \in \mathbb{Q}$ be a rational and $\rho \in \mathbb{C}$ be a complex number. The *multiplication operator* $\varphi_{\rho,r}$ on $\mathbb{C}\langle x\rangle$ is the canonical isomorphism

$$\begin{aligned} \varphi_{\rho,r} : \mathbb{C}\langle x\rangle &\to e_r^{\rho x} \mathbb{C}\langle x\rangle, \\ h(x) &\mapsto e_r^{\rho x} h(x). \end{aligned}$$

Definition 2.5. Let $r \in \mathbb{Q}$ be a rational and $\rho \in \mathbb{C}$ be a complex number. The *conjugation* $\gamma_{\rho,r}$ of a differential operator $L \in \mathbb{C}\langle x\rangle[\delta]$ with $\varphi_{\rho,r}$ is defined as the map

$$\gamma_{\rho,r} : L \mapsto \varphi_{\rho,r}^{-1} \circ L \circ \varphi_{\rho,r}.$$

When the differential operator L is seen as an operator on the function space $e_r^{\rho x}\mathbb{C}\langle x\rangle$, the conjugated operator $\gamma_{\rho,r}(L)$ is defined as the operator on $\mathbb{C}\langle x\rangle$ such that the diagram

$$\begin{tikzcd} \mathbb{C}\langle x\rangle \arrow[r, "\gamma_{\rho,r}(L)"] \arrow[d, "\varphi_{\rho,r}"'] & \mathbb{C}\langle x\rangle \arrow[d, "\varphi_{\rho,r}"] \\ e_r^{\rho x}\mathbb{C}\langle x\rangle \arrow[r, "L"] & e_r^{\rho x}\mathbb{C}\langle x\rangle \end{tikzcd}$$

commutes. Given the differentiation rule $\delta_r e_r^{\rho x} = \rho e_r^{\rho x}$ and the Leibniz rule we get

$$\delta_r^j(e_r^{\rho x} h) = e_r^{\rho x} \sum_{i=0}^{j} \binom{j}{i} \rho^{j-i} \delta_r^i h,$$

for all $h \in \mathbb{C}\langle x\rangle$. Hence the differential operator $L = \sum_{j=0}^{n} f_{r_j} \delta_r^j$ applied to $e_r^{\rho x} h$ equals

$$\begin{aligned}\sum_{j=0}^{n} f_{r,j}\delta_r^j(e_r^{\rho x}h) &= e_r^{\rho x}\sum_{j=0}^{n} f_{r,j}\sum_{i=0}^{j}\binom{j}{i}\rho^{j-i}\delta_r^i h\\ &= e_r^{\rho x}\sum_{j=0}^{n}\Big(\sum_{i=j}^{n}\binom{i}{j}\rho^{i-j}f_{r,i}\Big)\delta_r^j.\end{aligned}$$

The conjugation operator $\gamma_{\rho,r}$ on $\mathbb{C}\langle x\rangle[\delta_r]$ is therefore given by the $\mathbb{C}$-algebra automorphism

$$\begin{aligned}\gamma_{\rho,r} : \mathbb{C}\langle x\rangle[\delta_r] &\to \mathbb{C}\langle x\rangle[\delta_r],\\ \sum_{j=0}^{n} f_{r,j}\delta_r^j &\mapsto \sum_{j=0}^{n}\Big(\sum_{i=j}^{n}\binom{i}{j}\rho^{i-j}f_{r,i}\Big)\delta_r^j.\end{aligned}$$

For differential operators $L \in \mathbb{C}\langle x\rangle[\delta]$ written as operators of index 0 the conjugation $\gamma_{\rho,r}$ is given as the composition of automorphisms

$$\alpha_r^{-1}\circ\gamma_{\rho,r}\circ\alpha_r : \mathbb{C}\langle x\rangle[\delta] \to \mathbb{C}\langle x\rangle[\delta].$$

Remarks. Let $r \in \mathbb{Q}$ be a rational and $\rho \in \mathbb{C}$ be a complex number.

1. The conjugation $\gamma_{\rho,r} : \mathbb{C}\langle x\rangle[\delta] \to \mathbb{C}\langle x\rangle[\delta]$ is indeed a $\mathbb{C}$-algebra automorphism of $\mathbb{C}\langle x\rangle[\delta]$. It is easy to see that $\gamma_{\rho,r}$ is linear. For the associated $\mathbb{C}$-linear maps to the differential operators $L, M \in \mathbb{C}\langle x\rangle[\delta]$ it holds that

   $$\gamma_{\rho,r}(LM) = \varphi_{\rho,r}^{-1}LM\varphi_{\rho,r} = \varphi_{\rho,r}^{-1}L\varphi_{\rho,r}^{-1}\varphi_{\rho,r}M\varphi_{\rho,r} = \gamma_{\rho,r}(L)\gamma_{\rho,r}(M).$$

   Lastly the conjugation operator $\gamma_{\rho,r}$ is invertible and the inverse is given by

   $$\gamma_{\rho,r}^{-1} = \gamma_{-\rho,r},$$

   since the inverse of the multiplication operator $\varphi_{\rho,r}$ is given by $\varphi_{\rho,r}^{-1} = \varphi_{-\rho,r}$.

2. When the space of differential operators $\mathbb{C}\langle x\rangle[\delta]$ is seen as a $\mathbb{C}\langle x\rangle$-vector space the conjugation $\gamma_{\rho,r}$ is a $\mathbb{C}\langle x\rangle$-vector space automorphism. The key aspect of this remark is that the conjugation also respects the left-multiplication with Puiseux series.

3. The conjugation operator $\gamma_{\rho,r}$ restricted to the sub-algebra $\mathbb{C}[\delta_r] \subseteq \mathbb{C}\langle x\rangle[\delta]$ of differential operators with constant coefficients,

   $$\gamma_{\rho,r}\big|_{\mathbb{C}[\delta_r]} : \mathbb{C}[\delta_r] \to \mathbb{C}[\delta_r],$$

   is a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[\delta_r]$.

4. For $r = 0$ the same holds when $\gamma_{\rho,0}(L)$ and L act on the extended function spaces $\mathbb{C}\langle x\rangle[z]$ and $x^\rho\mathbb{C}\langle x\rangle[z]$, respectively.

The motivation for considering the conjugation of differential operators is the following:

Lemma 2.3. *Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator. The conjugation $\gamma_{\rho,r}$ induces an isomorphism*

$$\begin{aligned} \varphi_{\rho,r} : \ker \gamma_{\rho,r}(L) &\to \ker L, \\ y &\mapsto e_r^{\rho x} y, \end{aligned}$$

on the subspaces given by the respective kernels of the operators $\gamma_{\rho,r}(L)$ on $\mathbb{C}\langle x\rangle$ and L on $\mathbb{C}\langle x\rangle$.

Proof. This follows directly from the commutative diagram

$$\begin{tikzcd} \mathbb{C}\langle x\rangle \arrow[r, "\gamma_{\rho,r}(L)"] \arrow[d, "\varphi_{\rho,r}"'] & \mathbb{C}\langle x\rangle \arrow[d, "\varphi_{\rho,r}"] \\ e_r^{\rho x}\mathbb{C}\langle x\rangle \arrow[r, "L"] & e_r^{\rho x}\mathbb{C}\langle x\rangle. \end{tikzcd}$$

QED

This lemma will turn out to be crucial for our construction of solutions of differential equations. It lets us recover solutions of the differential equation $Ly = 0$ from solutions of the differential equation $\gamma_{\rho,r}(L)y = 0$. We will revisit this in later chapters, for now we show the following:

Lemma 2.4. *Let $r \in \mathbb{Q}$ be a rational and $\rho \in \mathbb{C}$ a complex number. The conjugation $\gamma_{\rho,r}$ and the initial form I_r commute as operators on $\mathbb{C}\langle x\rangle[\delta_r]$. Write $\gamma_{\rho,r}$ for the restricted conjugation operator $\gamma_{\rho,r}\big|_{\mathbb{C}[\delta_r]}$, then the diagram*

$$\begin{tikzcd} \mathbb{C}\langle x\rangle[\delta_r] \arrow[r, "\gamma_{\rho,r}"] \arrow[d, "I_r"'] & \mathbb{C}\langle x\rangle[\delta_r] \arrow[d, "I_r"] \\ \mathbb{C}[\delta_r] \arrow[r, "\gamma_{\rho,r}"] & \mathbb{C}[\delta_r] \end{tikzcd}$$

commutes.

Proof. Let $L \in \mathbb{C}((x^{\frac{1}{t}}))$ be a differential operator of index r. We can decompose the operator L into the sum

$$L = x^{\tau_r(L)}[L_0 + x^{\frac{1}{t}}L_1 + x^{\frac{2}{t}}L_2 + \dots],$$

where $L_0 = I_r(L)$ and $L_i \in \mathbb{C}[\delta_r]$ are differential operators with constant coefficients for $i \in \mathbb{N}$. Now by linearity, the conjugation of L equals

$$\begin{aligned}\gamma_{\rho,r}(L) &= x^{\tau_r(L)}\gamma_{\rho,r}([L_0 + x^{\frac{1}{t}}L_1 + x^{\frac{2}{t}}L_2 + \dots]) \\ &= x^{\tau_r(L)}[\gamma_{\rho,r}(L_0) + \underbrace{x^{\frac{1}{t}}\gamma_{\rho,r}(L_1) + x^{\frac{2}{t}}\gamma_{\rho,r}(L_2) + \dots}_{\text{higher shifts}}],\end{aligned}$$

thus

$$I_r(\gamma_{\rho,r}(L)) = \gamma_{\rho,r}(L_0) = \gamma_{\rho,r}(I_r(L)).$$

QED

2.4 Differential Operators with Constant Coefficients

In the following we want to show how the conjugation can be used to construct the kernel of differential operators with constant coefficients. This can be seen as an easy application of isomorphism $\varphi_{\rho,r}$ induced by the conjugation $\gamma_{\rho,r}$, for ρ a *local exponent* of the differential operator. We start by formally defining local exponents.

Definition 2.6. Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and

$$I_r(L) = \sum_{j=0}^{n} a_j \delta_r^j \in \mathbb{C}[\delta_r]$$

the initial form of index $r \in \mathbb{Q}$. The *characteristic polynomial of index r of L* is defined as the polynomial

$$\chi_r(L)(\lambda) = \sum_{j=0}^{n} a_j \lambda^j \in \mathbb{C}[\lambda].$$

Roots $\rho \in \mathbb{C}$ of the characteristic polynomial $\chi_r(L)$ are called a *local exponents of index r of L*. Furthermore we denote the set of local exponents of index r by

$$\Omega_r(L) = \{\rho \in \mathbb{C} : \chi_r(L)(\rho) = 0\}.$$

We consider sets of local exponents as *multisets*, i.e. $\Omega_r(L)$ contains the local exponent ρ of index r a number of times equal to the multiplicity of ρ as a local exponent.

Remark. For differential operators $L \in \mathbb{C}[\delta_r]$ with constant coefficients the characteristic polynomial $\chi_r(L)$ is given as the image of mapping $(\delta_r \mapsto \lambda)$, or equivalently, the operator viewed as a polynomial.

Lemma 2.5. *Let $L \in \mathbb{C}[\delta_r]$ be a differential operator of index $r \in \mathbb{Q}$ with constant coefficients and $\rho \in \mathbb{C}$ a complex number. The conjugation $\gamma_{\rho,r}$ of L is given by the explicit formula*

$$\gamma_{\rho,r}(L) = \sum_{j=0}^{n} \frac{\chi_r(L)^{(j)}(\rho)}{j!}\delta_r^j \in \mathbb{C}[\delta_r],$$

where $\chi_r(L)^{(j)}$ denotes the j^{th} derivative of $\chi_r(L)$.

Proof. Let $L = \sum_{i=0}^{n} a_i \delta_r^i \in \mathbb{C}[\delta_r]$ be our differential operator with constant coefficients. The j^{th} derivative of the characteristic polynomial $\chi_r(L)$ of L equals

$$\chi_r(L)^{(j)}(\lambda) = \sum_{i=j}^{n} a_i (i)_j \lambda^{i-j},$$

where $(i)_j = i(i-1)\cdots(i-j+1)$ denotes the falling factorial. Now the conjugated operator $\gamma_{\rho,r}(L)$ is given by

$$\begin{aligned}\gamma_{\rho,r}(L) &= \sum_{j=0}^{n} \Big(\sum_{i=j}^{n} a_i \binom{i}{j} \rho^{i-j} \Big) \delta_r^j \\ &= \sum_{j=0}^{n} \frac{\chi_r(L)^{(j)}(\rho)}{j!} \delta_r^j\end{aligned}$$

hence the claim. QED

Corollary 2.6. *Let $L \in \mathbb{C}[\delta_r]$ be a differential operator of index $r \in \mathbb{Q}$ with constant coefficients and $\Omega_r(L)$ be the set of local exponents of index r. Depending on r, set $\mathcal{F}_r$ to be*

$$\mathcal{F}_r = \begin{cases} \mathcal{M}(x), & \text{for } r \in \mathbb{Q}_-, \\ \sum\limits_{\rho \in \Omega_0(L)} x^\rho \mathbb{C}\{x\}[z], & \text{for } r = 0, \\ \bigoplus\limits_{\rho \in \Omega_r(L)} e_r^{\rho x} \mathcal{M}(x), & \text{for } r \in \mathbb{Q}_+. \end{cases}$$

The kernel of L as an operator on $\mathcal{F}_r$ equals

$$\ker L = \bigoplus_{\rho \in \Omega_r(L)} \bigoplus_{m=0}^{\mu_\rho - 1} \mathbb{C} e_r^{\rho x} z_r^m,$$

where μ_ρ is the multiplicity of $\rho \in \Omega_r(L)$ as a local exponent.

Proof. Consider the conjugated operator

$$\gamma_{\rho,r}(L) = \sum_{j=0}^{n} \frac{\chi_r(L)^{(j)}(\rho)}{j!} \delta_r^j \in \mathbb{C}[\delta_r]$$

for $\rho \in \Omega_r(L)$ a local exponent of index r. Recall that we defined $z_r(x) = -\frac{1}{r} x^{-r}$ and thus $\delta_r z_r^m = m\, z_r^{m-1}$. Now let $m < \mu_\rho$, then

$$\gamma_{\rho,r}(L)(z_r^m) = \sum_{j=0}^{n} \frac{\chi_r(L)^{(j)}(\rho)}{j!} \delta_r^j z_r^m = \sum_{j=0}^{n} \frac{\chi_r(L)^{(j)}(\rho)}{j!} m_{(j)} z_r^{m-j}.$$

Since ρ is a local exponent of multiplicity μ_ρ, ρ is a root of $\chi_r(L)^{(j)}$ for $j < \mu_\rho$. Furthermore since $m < \mu_\rho$ we also have $m_{(j)} = 0$ for $j \geq \mu_\rho > m$. Therefore $m_{(j)}\chi_r(L)^{(j)}(\rho) = 0$ for all $j = 0, \ldots, n$ and thus $z_r^m \in \ker \gamma_{\rho,r}(L)$. By Lemma 2.3 we know that

$$\varphi_{\rho,r}(z_r^m) = e_r^{\rho x} z_r^m \in \ker L.$$

Ranging over all local exponents $\rho \in \Omega_r(L)$ and $m < \mu_\rho$, the kernel of L is given by

$$\ker L = \bigoplus_{\rho \in \Omega_r(L)} \bigoplus_{m=0}^{\mu_\rho - 1} \mathbb{C} e_r^{\rho x} z_r^m,$$

which concludes the proof. QED

Example 3. Consider again the differential operator of Example 2,

$$L = x\delta^2 - \delta + 1.$$

We can apply Corollary 2.6 to compute the kernels of the initial forms of L. Recall that the initial forms of indices $r = -1, 0, 1$ are

$$I_{-1}(L) = 1, \qquad I_0(L) = -\delta + 1, \text{ and } \qquad I_1(L) = \delta_1^2 - \delta_1.$$

The corresponding sets of local exponents are given by

$$\Omega_{-1}(L) = \emptyset, \qquad \Omega_0(L) = \{1\}, \qquad \Omega_1(L) = \{0, 1\}.$$

Corollary 2.6 now yields the kernels, or equivalently, solutions of the differential equations $I_r(L)y = 0$.
For $r = -1$ the solution space is trivial, i.e., $\{0\}$. For $r = 0$ the solution space of the differential equation $-\delta y + y = 0$ is given by $\mathbb{C}x$. Finally for $r = 1$ the solution of the differential equation $\delta_1^2 y - \delta_1 y = 0$ equals $\mathbb{C} \oplus \mathbb{C}e^{x^{-1}}$.

3 Solutions of Differential Equations

Differential operators $L \in \mathbb{C}\{x\}[\partial]$ of $\operatorname{rank}(L) < 0$ are commonly referred to as *non-singular* differential operators. As a special case of Corollary 2.6 for $r = -1$ we recover a basis of solutions for the differential equation $Ly = 0$, where

$$L = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in \mathbb{C}[\partial]$$

is an ordinary differential operator with constant coefficients. A basis of solutions is given by functions of the form

$$y_{\rho,m} = e^{\rho x}x^m,$$

for $\rho \in \mathbb{C}$ a root of the *characteristic polynomial*

$$\chi_{-1}(L)(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 \in \mathbb{C}[\lambda]$$

and $m < \mu_\rho$, the multiplicity of ρ as a local exponent of L.
The general case for local solutions of non-singular differential equations was solved by Cauchy and Kowalevski. For a differential operator

$$L = f_n\partial^n + f_{n-1}\partial^{n-1} + \cdots + f_0 \in \mathcal{M}(x)[\partial]$$

the condition $\operatorname{rank}(L) < 0$ implies that $\nu(f_n) \leq \nu(f_j)$ for the coefficients f_j of L. The kernel of L as an operator on $\mathcal{M}(x)$ is unaltered by the left multiplication of meromorphic functions. Therefore we can study the differential operator L in the normal form

$$\widetilde{L} = \frac{1}{f_n}L = \partial^n + \frac{f_{n-1}}{f_n}\partial^{n-1} + \cdots + \frac{f_0}{f_n} \in \mathbb{C}\{x\}[\partial],$$

where the coefficients do not have a pole at 0 by the assumption on the rank. The theorem of Cauchy-Kowalevski states:

Theorem 3.1 (Cauchy-Kowalevski, [Kow75])**.** *Let $g_0(x), \ldots, g_{n-1}(x) \in \mathbb{C}\{x\}$ be holomorphic functions at* 0*. The differential equation*

$$\partial^n y + g_{n-1}(x)\partial^{n-1}y + \cdots + g_0(x)y = 0$$

has a local basis of n holomorphic solutions.

3.1 A Normal Form for Differential Operators

Differential operators with meromorphic coefficients in 0 of $\operatorname{rank}(L) = 0$ are commonly referred to as *regular singular* differential operators at 0. For *Euler operators*, differential operators $L \in \mathbb{C}[\delta]$ of rank 0 with constant coefficients, Corollary 2.6 is a well established fact about their local solutions. A $\mathbb{C}$-basis of local solutions of the differential equation $Ly = 0$ is given by the functions

$$y_{\rho,i} = x^{\rho} \log(x)^{i},$$

for $\rho \in \Omega_0(L)$ a local exponent of L and $i < \mu_\rho$, the multiplicity of ρ as a local exponent. The general case, finding a $\mathbb{C}$-basis of local solutions for regular singular differential equations was solved by Fuchs [Fuc66, Fuc68] and later revised by Thomé and Frobenius [Tho72, Fro73]. As an elegant reformulation of the *Frobenius method* H. Hauser established a normal form of differential operators $L \in \mathbb{C}[[x]][\delta]$, up to an isomorphism v of the function space $\mathcal{F}$ the operator acts,

$$L \circ v = I_0(L),$$

where $I_0(L) \in \mathbb{C}[\delta]$ is the initial form of L [Hau22]. Solutions of the differential equation $Ly = 0$ can then be computed from solutions of the differential equation $I_0(L)y = 0$, when accounting for the automorphism v.
For this we partition the set of local exponents $\Omega_0(L)$ into sets $\omega = \{\rho_0, \dots, \rho_k\}$ with integer difference, i.e., such that $\rho_j - \rho_i \in \mathbb{Z}$ for all $i \leq j \leq k$. This was already done classically by Fuchs and Frobenius [Fuc66, Fuc68, Fro73].

Theorem 3.2 (Normal Form Theorem, [Hau22])**.** *Let $L \in \mathbb{C}[[x]][\delta]$ be a differential operator and $\omega = \{\rho_1, \dots, \rho_m\} \subseteq \Omega_0(L)$ a set of increasingly ordered local exponents ρ_k with integer differences. Denote by μ_k the multiplicity of ρ_k and set $n_k = \mu_1 + \cdots + \mu_k$. Let L and the initial form $I_0(L)$ act on the function space*

$$\mathcal{F} = \sum_{k=1}^{m} x^{\rho_j} \mathbb{C}[[x]][z]_{<n_k} = \bigoplus_{k=1}^{m} \bigoplus_{i=n_{k-1}}^{n_k-1} x^{\rho_j} \mathbb{C}[[x]] z^i.$$

1. *The map L sends $\mathcal{F}$ into*

$$x\mathcal{F} = \sum_{k=1}^{m} x^{\rho_k+1} \mathbb{C}[[x]][z]_{<n_j}.$$

2. *The map $I_0(L)$ has image $\operatorname{Im}(I_0(L)) = x\mathcal{F}$ and kernel $\ker(I_0(L)) = \bigoplus_{k=1}^{m} \mathbb{C}x^{\rho_k}[z]_{<\mu_k}$. The kernel has direct complement*

$$\mathcal{H} = \bigoplus_{k=2}^{m} \bigoplus_{i=\mu_k}^{n_k-1} \mathbb{C}x^{\rho_k} z^i \oplus \bigoplus_{k=1}^{m-1} \bigoplus_{l=1}^{\rho_{k+1}-\rho_k-1} \bigoplus_{i=0}^{n_k-1} \mathbb{C}x^{\rho_k+l} z^i \oplus \bigoplus_{i=0}^{n_m-1} x^{\rho_r} \mathbb{C}[[x]] z^i$$

   in $\mathcal{F}$. Thus the restriction $I_0(L)\big|_{\mathcal{H}}$ defines a linear isomorphism between $\mathcal{H}$ and $x\mathcal{F}$.

3. *The composition of the inverse* $(I_0(L)\big|_{\mathcal{H}})^{-1} : x\mathcal{F} \to \mathcal{H}$ *of* $I_0(L)\big|_{\mathcal{H}}$ *with the canonical embedding* $\mathcal{H} \hookrightarrow \mathcal{F}$ *defines a right inverse* $S : x\mathcal{F} \to \mathcal{F}$ *of* $I_0(L)$*. Let* T *be the map* $T = (I_0(L) - L) : \mathcal{F} \to x\mathcal{F}$*. The map*

$$u = \mathrm{Id}_{\mathcal{F}} - S \circ T : \mathcal{F} \to \mathcal{F}$$

is a linear automorphism of $\mathcal{F}$*, with inverse*

$$v = u^{-1} = \sum_{j=0}^{\infty} (S \circ T)^j : \mathcal{F} \to \mathcal{F}.$$

4. *The automorphism* v *of* $\mathcal{F}$ *transforms* L *into* $I_0(L)$*,*

$$L \circ v = I_0(L).$$

5. *If the differential operator* L *has holomorphic coefficients and* $\mathrm{rank}(L) \leq 0$*, one can replace the ring of formal power series* $\mathbb{C}[[x]]$ *with the ring of germs of holomorphic functions* $\mathbb{C}\{x\}$ *in the construction of* $\mathcal{F}$ *and* $\mathcal{H}$*.*

As a consequence of the normal form theorem we recover the theorem of local solutions due to Fuchs, Frobenius and Thomé [Fuc68, Tho72, Fro73].

Theorem 3.3 (local solutions)**.** *Let* $L \in \mathbb{C}\{x\}[\delta]$ *be a differential operator and assume* $\mathrm{rank}(L) \leq 0$*. For each set* $\omega = \{\rho_1, \dots, \rho_m\} \subseteq \Omega_0(L)$ *of increasingly ordered local exponents* ρ_k *of* L *with integer differences we denote by* μ_k *the multiplicity of* ρ_k *and set* $n_k = \mu_1 + \cdots + \mu_k$*. Let* $v_\omega : \mathcal{F}_\omega \to \mathcal{F}_\omega$ *be the automorphism of assertion 4 of the normal form theorem, where* $\mathcal{F}_\omega$ *is the function space*

$$\mathcal{F}_\omega = \sum_{k=1}^{m} x^{\rho_k} \mathbb{C}\{x\}[z]_{<n_k}.$$

Let $\mathcal{F} = \bigoplus_\omega \mathcal{F}_\omega$ *be the direct sum of the function spaces* $\mathcal{F}_\omega$ *and* $v : \mathcal{F} \to \mathcal{F}$ *be the automorphism of* $\mathcal{F}$*, where the restriction* $v\big|_{\mathcal{F}_\omega} = v_\omega$*.*

1. *A* $\mathbb{C}$*-basis of local solutions of* $Ly = 0$ *at* 0 *is given by*

$$y_{\rho,i}(x) = v(x^\rho z^i)\big|_{z=\log(x)},$$

for $\rho \in \Omega_0(L)$ *a local exponent of* L *of multiplicity* μ_ρ*, and* $i < \mu_\rho$*.*

2. *Let* $\omega = \{\rho_1, \dots, \rho_m\} \subseteq \Omega_0(L)$ *be an increasingly ordered set of the partition of* $\Omega_0(L)$*. Each solution related to* ω *is of the form, for* $1 \leq k \leq m$ *and* $0 \leq i < \mu_k$*,*

$$y_{\rho_k,i}(x) = x^{\rho_k}[f_{k,i}(x) + \cdots + f_{k,0}(x)\log(x)^i] + \sum_{l=k+1}^{m} x^{\rho_l} \sum_{j=n_{l-1}}^{n_l - 1} h_{k,i,j}(x) log(x)^j,$$

with holomorphic $f_{k,i}$ *and* $h_{k,i,j}$ *in* 0 *with non-zero constant term.*

A proof can be found in [Hau22], as a consequence of the normal form theorem.
In the case of differential operators $L \in \mathbb{C}\{x\}[\delta]$ of $\operatorname{rank}(L) > 0$, the differential equation $Ly = 0$ is classically referred to as *irregular singular* at the point 0. To find solutions of irregular singular differential equations we can also look to the normal form theorem, though in general we will not get a complete set of n linearly independent solutions over the complex numbers $\mathbb{C}$.

Corollary 3.4. *Let $L \in \mathbb{C}[[x]][\delta]$ be a differential operator and $\omega = \{\rho_1, \dots, \rho_m\} \subseteq \Omega_0(L)$ a set of increasingly ordered local exponents ρ_k with integer differences. Denote by μ_k the multiplicity of $\rho_k \in \omega$ and set $n_k = \mu_1 + \cdots + \mu_k$. Formal solutions of $Ly = 0$ related to ω are given by functions of the form, for $1 \leq k \leq m$ and $0 \leq i < \mu_k$,*

$$y_{\rho_k,i}(x) = x^{\rho_k}[f_{k,i}(x) + \cdots + f_{k,0}(x)\log(x)^i] + \sum_{l=k+1}^{m} x^{\rho_l} \sum_{j=n_{l-1}}^{n_l-1} h_{k,i,j}(x) log(x)^j,$$

with formal power series $f_{k,i}, h_{k,i,j} \in \mathbb{C}[[x]]$ with non-zero constant term.
The total number of such $\mathbb{C}$-linearly independent solutions is $|\Omega_0(L)|$, one for each local exponent of index 0 of L, counted with multiplicity.

Unlike in the case for differential operators of $\operatorname{rank}(L) \leq 0$, the convergence of the coefficients of L does in general not imply the convergence of the power series occurring in the solutions constructed.
By a theorem of Maillet the power series solutions of $Ly = 0$ for a differential operator $L \in \mathbb{C}\{x\}[\delta]$ are *Gevrey-series* [Mai03]. A power series $f(x) \in \mathbb{C}[[x]]$ is said to be *Gevrey* if there exists an $m \in \mathbb{N}$ such that the m^{th}-Borel transform of $f(x)$ converges, i.e., $\mathcal{B}_m f(x)$ converges where

$$\begin{aligned} \mathcal{B}_m : \mathbb{C}[[x]] &\to \mathbb{C}[[x]] \\ \sum_{k=0}^{\infty} a_k x^k &\mapsto \sum_{k=0}^{\infty} \frac{a_k}{(k!)^m} x^k. \end{aligned}$$

This can also be seen as a consequence of the normal form theorem. Set n and n' to be the orders of L and the initial form $I_0(L)$ of L, respectively, and $m = n - n'$. The m^{th}-Borel transform of the power series f_i and $h_{i,j} \in \mathbb{C}[[x]]$ of Corollary 3.4 can be shown to converge [Hau22].
Now, since the number of local exponents $|\Omega_0(L)| = n$ if and only if $\operatorname{rank}(L) < 0$, we will not get a full basis of solutions and in general we can not guarantee even a single solution in the function space we defined. The following should exemplify how we can try to remedy this situation:

Example 4. Consider again the differential operator of Example 2 and 3

$$L = x\delta^2 - \delta + 1 \in \mathbb{C}[[x]][\delta].$$

Recall that L is an operator of $\operatorname{rank}(L) = 1$ with the sets of local exponents $\Omega_0(L) = \{1\}$ and $\Omega_1(L) = \{0, 1\}$. The normal form theorem therefore yields one solution of $Ly = 0$

corresponding to the local exponent $\rho = 1$ of index 0. One may verify easily that the series

$$y(x) = x\sum_{k=0}^{\infty} k!x^k \in \mathbb{C}[[x]]$$

satisfies the equation. For a second linearly independent solution we can look at the conjugated operator

$$\gamma_{1,1}(L) = x\delta^2 + \delta \in \mathbb{C}[[x]][\delta]$$

corresponding to the local exponent $\rho = 1$ of index 1 of L. Since the conjugated operator $\gamma_{1,1}(L)$ has no constant term we know $\mathbb{C} \subseteq \ker \gamma_{1,1}(L)$ and hence by the induced isomorphism $\varphi_{1,1} : \ker \gamma_{1,1}(L) \to \ker L$ a second solution of $Ly = 0$ is given by

$$\varphi_{1,1}(1) = e^{-x^{-1}}.$$

Conversely, since we know $y(x) = x\sum_{k=0}^{\infty} k!x^k$ is a solution of $Ly = 0$, the solution space of the differential equation $\gamma_{1,1}(L)y = x\delta^2 y + \delta y = 0$ is given by

$$\mathbb{C} \oplus \mathbb{C}e^{x^{-1}} x\sum_{k=0}^{\infty} k!x^k.$$

3.2 Local Exponents

In the previous example we obtained a solution of the differential equation $Ly = 0$ from the conjugated operator $\gamma_{\rho,r}(L)$ for an local exponent of L. In the following we are going to show that there are always *enough* local exponents. First we need some technical lemmas.

Lemma 3.5. *Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and $(f_{r,j})_{j=0,\dots n}$ the coefficients of index $r \in \mathbb{Q}$ of L. If for a fixed $j = 0, \dots, n$ the valuation $\nu(f_{0,j}) < \nu(f_{0,i})$ for all $i > j$, then the valuation of the j^{th} coefficient of index r equals*

$$\nu(f_{r,j}) = \nu(f_{0,j}) - rj,$$

for all $r \in \mathbb{Q}$.
Furthermore, if for a $j = 0, \dots, n$ there exists an $r \in \mathbb{Q}_+$ such that the shift $\tau_r(L)$ is attained by $f_{r,j}$ then

$$\nu(f_{0,j}) < \nu(f_{0,i})$$

holds for all $i > j$.

Proof. By Proposition 2.2 we know that

$$f_{r,j}(x) = x^{-rj}\sum_{i=j}^{n}(-r)^{i-j}\left\{ {i \atop j} \right\} f_{0,i}(x).$$

and thus in general we know that

$$\nu(f_{r,j}) \geq \min_{i \geq j} \nu(f_{0,i}) - rj$$

is a lower bound on the valuation of $f_{r,j}$. Furthermore if $\nu(f_{0,j}) < \nu(f_{0,i})$ holds for all $i > j$ then

$$\nu(f_{r,j}) = \nu(f_{0,j}) - rj$$

must follow since no cancellation of the valuation coefficient of $f_{0,j}$ can occur.
Now let $r \in \mathbb{Q}_+$ be a positive index and $j = 0, \ldots, n$ such that $\nu(f_{r,j}) = \tau_r(L)$, i.e., the shift of index r is attained by $f_{r,j}$. Assume towards a contradiction that there exists an $i > j$ violating the inequality of the lemma. Set $m = \min_{i \geq j} \nu(f_{0,i})$ and consider i_j, the largest $i > j$ such that $\nu(f_{0,i_j}) = m$ holds. Since we chose i_j to be maximal it follows that $\nu(f_{0,i_j}) < \nu(f_{0,i})$ for all $i > i_j$. Hence we can estimate

$$\nu(f_{r,i_j}) = \nu(f_{0,i_j}) - ri_j < \min_{i > j} \nu(f_{0,i}) - rj \leq \nu(f_{r,j}),$$

which is a contradiction to the assumption that $\tau_r(L)$ is attained at $f_{r,j}$. QED

Note that for the second part of this lemma we require the index r to be positive. In general this will not hold for negative indices. We can use Lemma 3.5 to give a definition of the shift $\tau_r(L)$ of index r in terms of the coefficients of index 0 of the differential operator L.

Lemma 3.6. *Let $L \in \mathbb{C}\langle x \rangle[\delta]$ be a differential operator and $r \in \mathbb{Q}_+$ a positive rational. Let $(f_{0,j})_{j=0,\ldots,n}$ be the coefficients of index 0 of L. The shift of index r of L equals*

$$\tau_r(L) = \min_{j \in \{0,\ldots,n\}} (\nu(f_{0,j}) - rj).$$

Proof. Recall the definition of the shift of index r,

$$\tau_r(L) = \min_{j \in \{0,\ldots,n\}} \nu(f_{r,j}).$$

If the shift is attained by $f_{r,j}$ we know by Lemma 3.5 that $\nu(f_{0,i}) > \nu(f_{0,j})$ for all $i > j$ and thus by Lemma 3.5

$$\tau_r(L) = \nu(f_{r,j}) = \nu(f_{0,j}) - rj.$$

Now if the shift of index r is not attained at $f_{r,j}$ then one of two things must hold. Either

1. the valuation of $f_{r,j}$ is greater than $\nu(f_{r,j}) \geq \nu(f_{0,j}) - rj > \tau_r(L)$, or
2. $\nu(f_{0,j}) - rj < \nu(f_{r,j})$. Then there exists an $i > j$ such that $\nu(f_{0,i}) \leq \nu(f_{0,j})$. Set $m = \min_{i \geq j} \nu(f_{0,i})$ then for $i_j := \max(\{i > j : \nu(f_{0,i}) = m\})$ the inequality

   $$\nu(f_{0,j}) - rj > \nu(f_{0,i_j}) - ri_j = \nu(f_{r,i_j}) \geq \tau_r(L)$$

   holds.

In conclusion,for all $j = 0, \dots, n$, we have showed

$$\nu(f_{0,j}) - rj \geq \tau_r(L),$$

and there exists a $j \in \{0, \dots, n\}$ such that $\tau_r(L) = \nu(f_{0,j}) - rj$, hence proving the claim. QED

In the following we will denote by $\operatorname{ord}(\chi)$ the *order* of the polynomial $\chi \in \mathbb{C}[\lambda]$ considered as a power series. As exemplified in Example 4, we are not that interested in local exponents $\rho = 0$ of positive rank $r \in \mathbb{Q}_+$ since the conjugation $\gamma_{0,r}$ equals the identity on $\mathbb{C}\langle x\rangle[\delta]$. This also gives reason to look at the order of characteristic polynomials. The multiplicity of $\rho = 0$ as a local exponent or index r is exactly $\operatorname{ord}(\chi_r(L))$. Equivalently, the number of local exponents ρ *not equal* to 0 is $\deg(\chi_r(L)) - \operatorname{ord}(\chi_r(L))$, when counting with multiplicity.
The next lemma sheds light on how the degree and order of characteristic polynomials of different indices relate.

Lemma 3.7. *Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and $r < s \in \mathbb{Q}_{\geq 0}$ two non-negative rationals. For the characteristic polynomials of indices r and s of L it holds that*

$$\deg(\chi_r(L)) \leq \operatorname{ord}(\chi_s(L)).$$

Proof. By definition of the characteristic polynomial the shifts $\tau_r(L)$ and $\tau_s(L)$ are certainly attained by the coefficients $f_{r,\deg(\chi_r(L))}$ and $f_{s,\operatorname{ord}(\chi_s(L))}$, respectively. Thus by Lemma 3.5 the shifts of index r and s equal

$$\tau_r(L) = \nu(f_{r,\deg(\chi_r(L))}) = \nu(f_{0,\deg(\chi_r(L))}) - r\deg(\chi_r(L))$$

and

$$\tau_s(L) = \nu(f_{s,\operatorname{ord}(\chi_s(L))}) = \nu(f_{0,\operatorname{ord}(\chi_s(L))}) - s\operatorname{ord}(\chi_s(L)).$$

Furthermore by Lemma 3.5 we also get $\nu(f_{s,\deg(\chi_r(L))}) = \nu(f_{0,\deg(\chi_r(L))}) - s\deg(\chi_r(L))$ and $\nu(f_{r,\operatorname{ord}(\chi_s(L))}) = \nu(f_{0,\operatorname{ord}(\chi_s(L))}) - r\operatorname{ord}(\chi_s(L))$. Since the minimal shift of index r is attained by $f_{r,\deg(\chi_r(L))}$ we deduce

$$\begin{aligned}
\nu(f_{s,\deg(\chi_r(L))}) + (s-r)\deg(\chi_r(L)) &= \nu(f_{r,\deg(\chi_r(L))}) \\
&\leq \nu(f_{r,\operatorname{ord}(\chi_s(L))}) \\
&= \nu(f_{s,\operatorname{ord}(\chi_s(L))}) + (s-r)\operatorname{ord}(\chi_s(L)).
\end{aligned}$$

Now since the minimal shift of index s is attained by $\nu(f_{s,\operatorname{ord}(\chi_s(L))})$ we conclude

$$\deg(\chi_r(L)) \leq \operatorname{ord}(\chi_s(L))$$

and therefore the proof. QED

Now using this lemma we can show an even stronger connection between the degree and order of characteristic polynomials of different indices.

Proposition 3.8. *Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and $0 \leq s < t \in \mathbb{Q}$ two non-negative rationals. There is a finite strictly increasing sequence*

$$s = r_0 < r_1 < \cdots < r_N = t$$

of rational numbers $r_i \in \mathbb{Q}$ such that

$$\deg(\chi_{r_i}(L)) = \operatorname{ord}(\chi_{r_{i+1}}(L))$$

for all $i = 0\ldots, N-1$.

Proof. We prove the assertion by an induction on $m = \operatorname{ord}(\chi_t(L)) - \deg(\chi_s(L)) \in \mathbb{N}$. First let $m = 0$, then $\deg(\chi_s(L)) = \operatorname{ord}(\chi_t(L))$ and hence the sequence $s < t$ suffices. For $m > 0$ assume the statement holds for all naturals smaller than m. Consider the rational

$$r = \frac{\nu(f_{0,\operatorname{ord}(\chi_t(L))}) - \nu(f_{0,\deg(\chi_s(L))})}{\operatorname{ord}(\chi_t(L)) - \deg(\chi_s(L))} \in \mathbb{Q}.$$

We are going to first show the inequality $s < r < t$.
Recall that by definition of the characteristic polynomial $\chi_s(L)$ the shift $\tau_s(L)$ is definitely attained at $f_{s,\deg(\chi_s(L))}$, the $\deg(\chi_s(L))^{\text{th}}$ coefficient of index s of L. Furthermore by the assumption $m > 0$ the shift $\tau_s(L)$ is certainly not attained at the coefficient $f_{s,\operatorname{ord}(\chi_t(L))}$. Combining the two facts we know

$$\nu(f_{s,\operatorname{ord}(\chi_t(L))}) - \nu(f_{s,\deg(\chi_s(L))}) > 0.$$

Now rearranging under the use of Lemma 3.5 yields

$$s < \frac{\nu(f_{0,\operatorname{ord}(\chi_t(L))}) - \nu(f_{0,\deg(\chi_s(L))})}{\operatorname{ord}(\chi_t(L)) - \deg(\chi_s(L))} = r.$$

A similar line of reasoning yields the inequality

$$\nu(f_{t,\operatorname{ord}(\chi_t(L))}) - \nu(f_{t,\deg(\chi_s(L))}) < 0$$

and hence by Lemma 3.5 $r < t$.
Next we are going to argue that the shift $\tau_r(L)$ will be attained by the coefficient $f_{r,\operatorname{ord}(\chi_t(L))}$ if and only if it is attained by the coefficient $f_{r,\deg(\chi_s(L))}$.
If $\tau_r(L)$ is attained at $f_{r,\deg(\chi_s(L))}$ Lemma 3.5 shows

$$\nu(f_{r,\deg(\chi_s(L))}) = \nu(f_{0,\deg(\chi_s(L))}) - r\deg(\chi_s(L)).$$

Thus by our construction of the index r one may verify

$$\begin{aligned} &\nu(f_{r,\deg(\chi_s(L))})[\operatorname{ord}(\chi_t(L)) - \deg(\chi_s(L))] \\ &= \nu(f_{0,\deg(\chi_s(L))})\operatorname{ord}(\chi_t(L)) - \nu(f_{0,\operatorname{ord}(\chi_t(L))})\deg(\chi_s(L)) \\ &= \nu(f_{r,\operatorname{ord}(\chi_t(L))})[\operatorname{ord}(\chi_t(L)) - \deg(\chi_s(L))], \end{aligned}$$

where the last equality is again due to Lemma 3.5. Hence we showed that

$$\tau_r(L) = \nu(f_{r,\deg(\chi_s(L))}) = \nu(f_{r,\operatorname{ord}(\chi_t(L))}).$$

The other direction follows by symmetry.
Now there are two cases to consider:

1. The shift $\tau_r(L)$ of index r is simultaneously attained by $f_{r,\deg(\chi_s(L))}$ and $f_{r,\operatorname{ord}(\chi_t(L))}$. We can thus estimate the order and the degree of $\chi_r(L)$ by

$$\operatorname{ord}\chi_r(L) \leq \deg\chi_s(L) < \operatorname{ord}\chi_t(L) \leq \deg\chi_r(L)$$

and hence by Lemma 3.7

$$\deg\chi_s(L) = \operatorname{ord}\chi_r(L), \quad \deg\chi_r(L) = \operatorname{ord}\chi_t(L).$$

We have found our desired sequence $s < r < t$.

2. The shift $\tau_r(L)$ of index r is attained at neither of the two. Then by Lemma 3.7

$$\deg\chi_s(L) < \operatorname{ord}\chi_r(L) \leq \deg\chi_r(L) < \operatorname{ord}\chi_t(L).$$

Now by the induction hypothesis we can find sequences

$$s = r_{0,0} < r_{0,1} < \cdots < r_{0,N_0} = r$$

and

$$r = r_{1,0} < r_{1,1} < \cdots < r_{1,N_1} = t$$

for $m_0 = \operatorname{ord}\chi_r(L) - \deg\chi_s(L) < m$ and $m_1 = \operatorname{ord}\chi_t(L) - \deg\chi_r(L) < m$, respectively. Combining the sequences yields the result.

QED

As we have noted before, for our purposes, we are only interested in *non-zero* local exponents of positive index. We therefore propose the following notions:

Definition 3.1. For a differential operator $L \in \mathbb{C}\langle x\rangle[\delta]$ and a rational $r \in \mathbb{Q}_+$ we denote the set of *non-zero* local exponents of index r of L by

$$\Omega'_r(L) = \{\rho \in \mathbb{C} \setminus \{0\} : \chi_r(L)(\rho) = 0\}.$$

Here, we again regard $\Omega'_r(L)$ as a *multiset*. Additionally we define the following sets of local exponents:

$$\Omega_{<r}(L) = \bigsqcup_{s\in\mathbb{Q}_+,\ s<r} \Omega'_s(L) \sqcup \Omega_0(L), \qquad \Omega_{>r}(L) = \bigsqcup_{r\in\mathbb{Q}_+,\ s>r} \Omega'_s(L),$$

$$\Omega_{\leq r}(L) = \bigsqcup_{s\in\mathbb{Q}_+,\ s\leq r} \Omega'_s(L) \sqcup \Omega_0(L), \qquad \Omega_{\geq r}(L) = \bigsqcup_{s\in\mathbb{Q}_+,\ s\geq r} \Omega'_s(L),$$

$$\Omega(L) = \bigsqcup_{r\in\mathbb{Q}_+} \Omega'_r(L) \sqcup \Omega_0(L).$$

We use the symbol $\bigsqcup$ for the disjoint union of multisets.

Corollary 3.9. *Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator. The cardinalities of the sets of local exponents, when counting with multiplicities, are:*

$$
\begin{aligned}
|\Omega_{<r}(L)| &= \operatorname{ord}(\chi_r(L)), & |\Omega_{>r}(L)| &= \deg(\chi_r(L)),\\
|\Omega_{\leq r}(L)| &= \deg(\chi_r(L)), & |\Omega_{\geq r}(L)| &= \operatorname{ord}(\chi_r(L)),\\
|\Omega(L)| &= n.
\end{aligned}
$$

Proof. We are going to show the last item of the list, the rest can be shown in a similar fashion. First note that the cardinality of $\Omega'_r(L)$ equals

$$|\Omega'_r(L)| = \deg(\chi_r(L)) - \operatorname{ord}(\chi_r(L)).$$

Apply Proposition 3.8 to $s = 0$ and $t = \operatorname{rank}(L)$. There is a finite sequence

$$0 = r_0 < r_1 < \cdots < r_N = \operatorname{rank}(L)$$

such that

$$\deg(\chi_{r_i}(L)) = \operatorname{ord}(\chi_{r_{i+1}}(L))$$

for $i = 0, \ldots, N-1$.
For every positive rational $r < \operatorname{rank}(L)$, where $r \neq r_i$ for all $i = 0, \ldots, N$, there has to be an i such that $r_i \leq r \leq r_{i+1}$, thus by Lemma 3.7

$$\deg(\chi_{r_i}(L)) \leq \operatorname{ord}(\chi_r(L)) \leq \deg(\chi_r(L)) \leq \operatorname{ord}(\chi_{r_{i+1}}(L)) = \deg(\chi_{r_i}(L)).$$

For rationals $r > \operatorname{rank}(L)$ Lemma 3.7 yields

$$n = \operatorname{ord}(\chi_r(L)) = \deg(\chi_r(L)).$$

Thus $\Omega'_r(L) = \emptyset$ for all $r \neq r_i$.
We therefore conclude

$$
\begin{aligned}
|\Omega(L)| &= |\Omega_0(L)| + \sum_{i=1}^{N} |\Omega'_{r_i}(L)|\\
&= \deg(\chi_0(L)) + \sum_{i=1}^{N} \deg(\chi_{r_i}(L)) - \operatorname{ord}(\chi_{r_i}(L))\\
&= \deg(\chi_{\operatorname{rank}(L)}(L)) = n,
\end{aligned}
$$

which proves the claim. QED

We are therefore always given a number of local exponents consistent with the order of the differential operator L. The importance of this corollary is that we are going to construct a solution of the differential equation $Ly = 0$ for every local exponent $\rho \in \Omega(L)$. Given that these solutions are linearly independent over the complex numbers, we find a basis of solutions.

3.3 Theorem of Formal Solutions

In this section we are going to show how one can obtain solutions of the differential equation $Ly = 0$ for local exponents $\rho \in \Omega(L)$ of the differential operator $L \in \mathbb{C}\langle x\rangle[\delta]$. The way this is done is through the conjugated operator $\gamma_{\rho,r}(L)$. The next proposition deals with how the local exponents of an operator L change under the conjugation $\gamma_{\rho,r}$.

Proposition 3.10. *Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and $\rho \in \Omega_r(L)$ be a local exponent of index $r \in \mathbb{Q}_+$ of L with multiplicity μ_ρ. The following hold for the local exponents of the conjugated operator $\gamma_{\rho,r}(L) \in \mathbb{C}\langle x\rangle[\delta]$*

1. *The local exponents of index $> r$ are unchanged,*

$$\Omega_{>r}(\gamma_{\rho,r}(L)) = \Omega_{>r}(L).$$

2. *The local exponents of index r get shifted by ρ,*

$$\Omega_r(\gamma_{\rho,r}(L)) = \Omega_r(L) - \rho.$$

3. *There are μ_ρ local exponents of index $< r$.*

$$|\Omega_{<r}(\gamma_{\rho,r}(L))| = \mu_\rho$$

Proof. For the first item decompose the differential operator $L \in \mathbb{C}((x^{\frac{1}{t}}))[\delta] \subseteq \mathbb{C}\langle x\rangle[\delta_r]$ into

$$L = x^{\tau_r(L)}[L_0 + x^{\frac{1}{t}}L_1 + x^{\frac{2}{t}}L_2 + \dots],$$

where the operators $L_i \in \mathbb{C}[\delta_r]$ have constant coefficients for all $i \in \mathbb{N}$. Let $s \in \mathbb{Q}_+$ be an index such that $s > r$. To transform $L \in \mathbb{C}\langle x\rangle[\delta_r]$ into an operator of index s we may apply the change-of-basis transformation $\alpha_{r,s} := \alpha_s \circ \alpha_r^{-1}$ to the operators of constant coefficients separately, i.e.,

$$\alpha_{r,s}(L) = x^{\tau_r(L)}[\alpha_{r,s}(L_0) + x^{\frac{1}{t}}\alpha_{r,s}(L_1) + x^{\frac{2}{t}}\alpha_{r,s}(L_2) + \dots] \in \mathbb{C}\langle x\rangle[\delta_s].$$

Set for $i \in \mathbb{N}$ the operator $L_i \neq 0$ equal

$$L_i = \sum_{j=0}^{m_i} a_{i,j}\delta_r^j,$$

where $m_i \leq n$ is the largest natural such that $a_{i,m_i} \neq 0$. The change-of-basis $\alpha_{r,s}$ applied to L_i equals

$$\alpha_{r,s}(L_i) = \sum_{j=0}^{m_i}\Big(\sum_{k=j}^{m_i}(-s)^{k-j}\begin{Bmatrix}k\\ j\end{Bmatrix}\sum_{l=k}^{m_i} r^{l-k}\begin{bmatrix}l\\ k\end{bmatrix}a_{i,l}x^{rl-sj}\Big)\delta_s^j.$$

The coefficients can only contribute to the initial form $I_s(L)$ of index $s > r$ if the exponent $rl - sj$ of x is minimal. One can easily observe that for fixed j the exponent is minimal

for $l = j$. Hence for varying $l \leq j \leq m_i$ it is minimal for $l = j = m_i$, since we assumed $s > r$. The coefficient of $x^{(r-s)m_i}\delta_s^{m_i}$ trivialises to be $a_{i,m_i} \neq 0$.
The key observation now is that the coefficients a_{i,m_i} for $i \in \mathbb{N}$ stay unaltered by $\gamma_{\rho,r}$. Since $\gamma_{\rho,r}$ is a $\mathbb{C}\langle x\rangle$-isomorphism we can apply $\gamma_{\rho,r}$ to the operators of constant coefficients $L_i \in \mathbb{C}[\delta_r]$ separately

$$\gamma_{\rho,r}(L_i) = \sum_{j=0}^{m_i}\Big(\sum_{k=j}^{m_i}\binom{k}{j}\rho^{k-j}a_{i,k}\Big)\delta_r^j.$$

The coefficient of $\delta_r^{m_i}$ again equals a_{i,m_i}. Therefore the initial forms of L and $\gamma_{\rho,r}(L)$ must be equal,

$$I_p(L) = I_p(\gamma_{\rho,r}(L)).$$

For the second item we can combine Lemma 2.4 and Lemma 2.5 to show

$$I_r(\gamma_{\rho,r}(L)) = \gamma_{\rho,r}(I_r(L)) = \sum_{j=0}^{n}\frac{\chi_r(L)^{(j)}(\rho)}{j!}\delta_r^j,$$

where the second equality holds since $I_r(L) \in \mathbb{C}[\delta_r]$ is a differential operator with constant coefficients and $\chi_r(I_r(L)) = \chi_r(L)$.
Therefore the characteristic polynomial of the conjugated operator is given by

$$\chi_r(\gamma_{\rho,r}(L)) = \sum_{j=0}^{n}\frac{\chi_r(L)^{(j)}(\rho)}{j!}\lambda^j = \chi_r(L)(\lambda+\rho),$$

by the power series expansion of the polynomial $\chi_r(L)(\lambda+\rho)$. Hence $\chi_r(\gamma_{\rho,r}(L))$ vanishes for $\sigma \in \Omega_r(L) - \rho$.
For the last item observe that the multiplicity of 0 as a root of $\chi_r(\gamma_{\rho,r}(L))$ has to be equal to μ_ρ, the multiplicity of ρ as a root of $\chi_r(L)$. Hence $\operatorname{ord}(\chi_r(\gamma_{\rho,r}(L)) = \mu_\rho$ and by Corollary 3.9 we know

$$|\Omega_{<r}(\gamma_{\rho,r}(L))| = \operatorname{ord}(\chi_r(\gamma_{\rho,r}(L)) = \mu_\rho,$$

which concludes the proof. QED

The idea for constructing solutions of $Ly = 0$ is to apply the the normal form theorem on the conjugated operator $\gamma_{\rho,r}(L)$, thereby yielding a solution $\widetilde{y}$ of the equation $\gamma_{\rho,r}(L)\widetilde{y} = 0$ for local exponents of index 0. Then, according to Lemma 2.3, a solution of $Ly = 0$ is given by the multiplication operator $\varphi_{\rho,r}$ applied to $\widetilde{y}$.
Now the existence of a local exponent of the conjugated operator is by no means guaranteed. Proposition 3.10 only shows the existence of local exponents of index $< r$ of $\gamma_{\rho,r}(L)$. In general we will therefore need to iterate this process and apply multiple conjugations to arrive at an operator admitting a local exponent of index 0. To show that this nevertheless is a finite process we can look to the next lemma, showing the ramification of local exponents.

Lemma 3.11. *Let $L \in \mathbb{C}((x^{\frac{1}{t}}))[\delta] \subseteq \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and $\rho \in \Omega_r(L)$ be a local exponents of index $r \in \mathbb{Q}_+$ of L. Set $r = \frac{p}{q}$ for coprime integers $p, q \in \mathbb{Z}_+$ and $i = \frac{\operatorname{lcm}(t,q)}{t} \in \mathbb{Z}_+$. Then $\zeta_i \rho \in \Omega_r(L)$ is a local exponent of index r for all i^{th} roots of unity $\zeta_i \in \{\zeta \in \mathbb{C} : \zeta^i = 1\}$.*

Proof. Let $(f_{0,j})_{j=0,\dots,n}$ and $(f_{r,j})_{j=0,\dots,n}$ be the coefficients of index 0 and r, respectively. We are going to show the following:
For any two $j_0, j_1 \in \{0, \dots, n\}$, such that the shift $\tau_r(L)$ is attained by f_{r,j_0} and f_{r,j_1}, the naturals j_0 and j_1 must be congruent modulo i.
Then the characteristic polynomial $\chi_r(L)(\lambda)$ can be decomposed into

$$\chi_r(L)(\lambda) = \lambda^{\operatorname{ord}(\chi_r(L))} P(\lambda^i).$$

for some polynomial $P \in \mathbb{C}[\lambda]$. A complex $\rho \in \mathbb{C} \setminus \{0\}$ is a local exponent of index r of L if and only if $P(\rho^i) = 0$, hence $\rho \in \Omega_r(L)$ if and only if $\zeta_i \rho \in \Omega_r(L)$ for ζ_i an i^{th} root of unity.
To this end let $j_0, j_1 \in \{0, \dots, n\}$ be such that the shift $\tau_r(L)$ is attained by both coefficients f_{r,j_0} and f_{r,j_1}. By Lemma 3.5 the shift $\tau_r(L)$ of index r equals

$$\tau_r(L) = \nu(f_{0,j_0}) - rj_0 = \nu(f_{0,j_1}) - rj_1.$$

Since we assumed $f_{0,j_0}, f_{0,j_1} \in \mathbb{C}((x^{\frac{1}{t}}))$, there exist integers $k_0, k_1 \in \mathbb{Z}$ such that $\nu(f_{0,j_0}) = \frac{k_0}{t}$ and $\nu(f_{0,j_1}) = \frac{k_1}{t}$. Now multiplying the equation $\nu(f_{0,j_0}) - r(j_0) = \nu(f_{0,j_1}) - rj_1$ with $\operatorname{lcm}(t, q)$ yields the Diophantine equation

$$ik_0 - \frac{\operatorname{lcm}(t,q)}{q} pj_0 = ik_1 - \frac{\operatorname{lcm}(t,q)}{q} pj_1.$$

In particular this equation must also hold modulo $i = \frac{\operatorname{lcm}(t,q)}{t} \in \mathbb{Z}$, thus

$$\frac{\operatorname{lcm}(t,q)}{q} pj_0 \equiv \frac{\operatorname{lcm}(t,q)}{q} pj_1 \mod i.$$

Since we assumed p and q to be coprime and i is a divisor of q, the integers p and i must be coprime as well. Furthermore the integers $\frac{\operatorname{lcm}(t,q)}{q}$ and $\frac{\operatorname{lcm}(t,q)}{t}$ are coprime by construction. Hence the congruence

$$j_0 \equiv j_1 \mod i,$$

which concludes the proof. QED

Using this lemma we can show that one arrives at a differential operator admitting a local exponent of index 0 after a finite process of iterated conjugations.

Lemma 3.12. *Let $L \in \mathbb{C}\langle x\rangle[\delta]$ be a differential operator and $\rho \in \Omega(L)$ be a local exponent of index $r \in \mathbb{Q}_{\geq 0}$. There exists a finite sequence of local exponents $(\rho = \rho_0, \rho_1, \dots, \rho_N)$, a* *corresponding strictly descending sequence of indices* $r = r_0 > r_1 > \cdots > r_N = 0$*, and a sequence of differential operators* $(L = L_0, L_1, \ldots, L_N)$*, such that, for* $j = 0, \ldots, N-1$*, the differential operator* L_{j+1} *is the conjugated operator*

$$L_{j+1} = \gamma_{\sigma_j, r_j}(L_j),$$

$\sigma_j \in \Omega'_{r_j}(L_j)$ *is a non-zero local exponent of* L_j *of index* r_j*, and* $\sigma_N \in \Omega_0(L_N)$ *is a local exponent of* L_N *of index* $r_N = 0$*.*
Furthermore, for every local exponent $\rho \in \Omega(L)$ *the number of* distinct *sequences of this form equals* μ_ρ*, the multiplicity of* ρ *as a local exponent, when counting sequences with multiplicity of* ρ_N*.*

Proof. By item 3 of Proposition 3.10 the number of local exponents of index $< r$ of the conjugated operator $\gamma_{\rho,r}(L)$ equals μ_ρ. Hence, in particular, there exists a local exponent of $\gamma_{\rho,r}(L)$ with an index $< r$. We can use this to construct a chain of local exponents $(\rho_j)_j$ and differential operators $(L_j)_j$ such that $L_{j+1} = \gamma_{\rho_j, r_j}(L)$, where $(r_j)_j$ is the strictly decreasing chain of indices of ρ_j. As long as ρ_j is indeed a local exponent of L_j (with a positive index r_j), the conjugated operator $\gamma_{\rho_j, r_j}(L)$ must have a local exponent of index $< r_j$. If the sequence terminates after a finite number of steps, it has to terminate at an local exponent of index 0. We are going to show that any such sequence must eventually terminate.
Set μ_j to be the multiplicity of ρ_j as a local exponent of L_j. Let $t_0 \in \mathbb{Z}_+$ be the ramification index of $L_0 \in \mathbb{C}((x^{\frac{1}{t_0}}))[\delta]$, i.e. the coefficients of L_0 are formal Laurent series of $x^{\frac{1}{t_0}}$. Additionally set $r_0 = \frac{p_0}{q_0}$ for relative prime $p_0, q_0 \in \mathbb{Z}_+$, $t_1 = \operatorname{lcm}(t_0, q_0)$ and $i_0 = \frac{t_1}{t_0}$. Recall that by Lemma 3.11 $\zeta_{i_0}\rho_0 \in \Omega_{r_0}(L)$ is also a local exponent of L_0 of index r_0 for all i_0^{th} roots of unity. Furthermore recall that by Corollary 3.9 the number of local exponents of L_0 equals $|\Omega(L_0)| = n$. Since $i_0 \cdot \mu_0 \le |\Omega_{r_0}(L)| \le n$ the denominator q_0 of the index r_0 is bounded by

$$q_0 \le t_1 \le \frac{n}{\mu_0} t_0 \le n t_0.$$

We are going to inductively show that the denominators q_j of index r_j are uniformly bounded by

$$q_j \le t_{j+1} \le \frac{n}{\mu_j} t_0 \le n t_0$$

for all j.
To this end set for $j > 0$ the indices $r_j = \frac{p_j}{q_j}$ for relative prime $p_j, q_j \in \mathbb{Z}_+$, $t_{j+1} = \operatorname{lcm}(t_j, q_j)$, and $i_j = \frac{t_{j+1}}{t_j}$.

Note that by construction the differential operator $L_{j+1} = \gamma_{\rho_j, r_j}(L_j) \in \mathbb{C}((x^{\frac{1}{t_{j+1}}}))[\delta]$ has coefficients in the field of formal Laurent series of $x^{\frac{1}{t_{j+1}}}$. This is due to the fact that γ_{ρ_j, r_j} is an automorphism of $\mathbb{C}((x^{\frac{1}{\operatorname{lcm}(t_j, q_j)}}))[\delta]$. Lemma 3.11 and Proposition 3.10 now yield $i_{j+1} \cdot \mu_{j+1} \le \mu_j$ and thus

$$q_{j+1} \le t_{j+2} \le \frac{\mu_j}{\mu_{j+1}} t_{j+1}.$$

By the induction hypothesis $t_{j+1} \leq \frac{n}{\mu_j} t_0$ and hence

$$\frac{\mu_j}{\mu_{j+1}} t_{j+1} \leq \frac{n}{\mu_{j+1}} t_0 \leq n t_0.$$

We conclude that the denominators q_j of the indices r_i are uniformly bounded by nt_0. We are therefor given a strictly decreasing sequence of indices $(r_j)_j$ of non-negative rationals $r_j \geq 0$ with uniformly bounded denominators, hence the sequence must terminate. Furthermore the terminating index $r_N = 0$, as previously noted.
Finally we are going to prove by induction that there are exactly μ_ρ distinct sequences, when counting with multiplicity.
First let $\mu_\rho = 1$, then there can only exist one such sequence of local exponents since $|\Omega_{<r_j}(L_{j+1})| = 1$ for all $j = 0, \ldots, N-1$. Consequently also the multiplicity of the local exponent ρ_N of index 0 equals $\mu_N = 1$.
Next we assume the multiplicity of ρ equals $\mu_\rho = m > 1$. We consider the following two cases:

1. The set $\Omega_{<r_j}(L_{j+1})$ consists of a single local exponent ρ_{j+1} for all $j = 0, \ldots, N-1$. Then the multiplicity of the local exponent ρ_N of index 0 equals $\mu_N = \mu_\rho$.

2. There exists an j such that $\Omega_{<r_j}(L_{j+1})$ is not a singleton. Assume j to be minimal, i.e., for all $k < j$ the set of local exponents $\Omega_{<r_k}(L_{k+1})$ consist of exactly one element. Then, as a consequence of Proposition 3.10, for the multiplicities μ_σ of local exponents $\sigma \in \Omega_{<r_j}(L_{j+1})$ holds

$$\sum_{\sigma \in \Omega_{<r_j}(L_{j+1})} \mu_\sigma = \mu_\rho.$$

Now for every $\sigma \in \Omega_{<r_j}(L_{j+1})$, by the induction hypothesis for $\mu_\sigma < m$, there are exactly μ_σ distinct sequences of local exponents $(\sigma = \sigma_0, \sigma_1, \ldots, \sigma_N)$, counted with multiplicity μ_N. Therefore there are also μ_σ distinct sequences of the form

$$(\rho = \rho_0, \rho_1, \ldots, \rho_{j+1} = \sigma = \sigma_0, \sigma_1, \ldots, \sigma_N)$$

for every $\sigma \in \Omega_{<r_j}(L_{j+1})$ and consequently μ_ρ distinct sequences of local exponents, counted with multiplicity.

QED

Theorem 3.13 (Theorem of formal solutions)**.** *Let $L \in \mathbb{C}\langle x \rangle[\delta]$ be a differential operator. Consider a sequence of local exponents $\pi = (\sigma_0, \ldots, \sigma_N)$, a sequence of descending indices $r_0 > \cdots > r_N = 0$ and a sequence of differential operators $(L = L_0, \ldots, L_N)$, such that, for $j = 0, \ldots, N-1$, the differential operator L_{j+1} is the conjugated operator*

$$L_{j+1} = \gamma_{\sigma_j, r_j}(L_j),$$

σ_j is a non-zero local exponent of L_j of index r_j, and σ_N a local exponent of L_N of index $r_N = 0$. Let $\omega_\pi = \{\rho_0, \ldots, \rho_m\}$ be a set of increasingly ordered local exponents ρ_k

of L_N of index 0 with integer differences. Denote by μ_k the multiplicity of ρ_k and set $n_k = \mu_1 + \cdots + \mu_k$.
Formal solutions of the differential equation $Ly = 0$ related to ω_π are given by functions of the form, for $0 \le k \le m$ and $0 \le i < \mu_k$,

$$\widetilde{y_{k,i}} = \exp(-\sum_{j=0}^{N-1} \frac{\sigma_j}{r_j} x^{-r_j}) \cdot y_{k,i},$$

for the solutions

$$y_{k,i}(x) = x^{\rho_k}[f_{k,i}(x^{\frac{1}{t}}) + \cdots + f_{k,0}(x^{\frac{1}{t}})\log(x)^i] + \sum_{l=k+1}^{m} x^{\rho_l} \sum_{j=n_{l-1}}^{n_l-1} h_{k,i,j}(x^{\frac{1}{t}}) log(x)^j$$

of the differential equation of $L_N y = 0$, where $f_{k,i}, h_{k,i,j} \in \mathbb{C}[[x]]$ are formal power series with non-zero constant term and $t \in \mathbb{Z}_+$.
Ranging over all sequences of local exponents π and sets of local exponents ω_π with integer differences a $\mathbb{C}$-basis of formal solutions is given for the differential equation $Ly = 0$.

Proof. Let $\pi = (\sigma_0, \ldots, \sigma_N)$ be a sequence of local exponents corresponding to L as constructed in Lemma 3.12. Set

$$\gamma_\pi = \gamma_{\sigma_{N-1}, r_{N-1}} \circ \cdots \circ \gamma_{\sigma_0, r_0}$$

to be the composition of conjugations corresponding to the local exponents $\sigma_i \in \pi$ then $\sigma_N \in \Omega_0(\gamma_\pi(L))$ is a local exponent of index 0 of $\gamma_\pi(L)$. In particular, the set $\Omega_0(\gamma_\pi(L))$ of local exponents of index 0 of $\gamma_\pi(L)$ is not empty. We can partition $\Omega_0(\gamma_\pi(L))$ into sets of local exponents with integer differences and apply Corollary 3.4. For a set $\omega_\pi = \{\rho_0, \ldots, \rho_m\} \subseteq \Omega_0(\gamma_\pi(L))$ of increasingly ordered local exponents ρ_k with integer differences Corollary 3.4 yields a set of $\mathbb{C}$-linearly independent formal solutions of $\gamma_\pi(L)y = 0$,

$$y_{k,i}(x) = x^{\rho_k}[f_{k,i}(x^{\frac{1}{t}}) + \cdots + f_{k,0}(x^{\frac{1}{t}})\log(x)^i] + \sum_{l=k+1}^{m} x^{\rho_l} \sum_{j=n_{l-1}}^{n_l-1} h_{k,i,j}(x^{\frac{1}{t}}) log(x)^j,$$

for $0 \le k \le m$ and $0 \le i \le \mu_k$, the multiplicity of ρ_k. Here the functions $f_{k,i}, h_{k,i,j} \in \mathbb{C}[[x^{\frac{1}{t}}]]$ are formal power series in $x^{\frac{1}{t}}$, where $t \in \mathbb{Z}_+$ is the ramification index of the differential operator $\gamma_\pi(L) \in \mathbb{C}((x^{\frac{1}{t}}))[\delta]$.
Recall that by Lemma 2.3 the automorphism $\gamma_{\rho,r}$ on $L \in \mathbb{C}\langle x \rangle[\delta]$ induces an isomorphism on the kernels,

$$\varphi_{\rho,r} : \ker \gamma_{\rho,r}(L) \to \ker L,$$

where $\varphi_{\rho,r}$ is the multiplication operator multiplying with the exponential $e_r^{\rho x}$. Therefore for the composition of multiplication operators

$$\varphi_\pi = \varphi_{\sigma_{N-1}, r_{N-1}} \circ \cdots \circ \varphi_{\sigma_0, r_0}$$

it holds that $\varphi_\pi(y_{k,i})$ is a solution of the differential equation $Ly = 0$. Hence the solutions are of the form

$$\varphi_\pi(y_{k,i}) = \prod_{i=0}^{N-1} e_{r_i}^{\sigma_i x} \cdot y_{k,i} = \exp(\sum_{i=0}^{N-1} \sigma_i z_i) \cdot y_{k,i}$$
$$= \exp(-\sum_{i=0}^{N-1} \frac{\sigma_i}{r_i} x^{-r_i}) \cdot y_{k,i}.$$

Now by Lemma 3.12 there are, counted with multiplicity, exactly n such distinct sequences π of local exponents. Additionally for every distinct sequence we get exactly as many linearly independent solutions as the multiplicity of the last local exponent σ_N indicates. Hence we have found a full basis of solutions. QED

Similar as the case in Corollary 3.4, we can in general not expect the power series $f_{k,i}, h_{k,i,j}$ occurring in the formal solutions of Theorem 3.13 to be convergent at 0. This holds true even for differential operators $L \in \mathcal{M}(x)[\delta]$ with meromorphic coefficients.
Even though the conjugated differential operator L_N of Theorem 3.13 has a solution free of the exponential term, it can only be regular singular at 0 if every solution of $Ly = 0$ has the same exponential term. This is the case if, for all $j < N$, the differential operators L_j admit only a single local exponent of multiplicity n, the order of the differential operator.
Thought not necessarily convergent, when the differential operator L has meromorphic coefficients, the power series $f_{k,i}, h_{k,i,j}$ of the formal solutions are *Gevrey*, cf. Corollary 3.4.
For differential operators with meromorphic coefficients we recover Fuchs' theorem as a consequence of the theorem of formal solutions:

Theorem 3.14 (Theorem of Fuchs, [Fuc66, Fuc68])**.** *Let $L \in \mathcal{M}(x)[\delta]$ be a differential operator with meromorphic coefficients in* 0*. The following are equivalent:*

1. *The differential operator L is* regular singular *at* 0*.*
2. *The differential equation $Ly = 0$ has a basis of* regular *solutions at* 0*, i.e., for any holomorphic solution y near* 0 *there exists a natural number $N \in \mathbb{N}$ such that for any $-\pi \leq \phi_0 < \phi_1 \leq \pi$*
$$|x^N||y(x)| \to 0 \text{ for } |x| \to 0,$$
when one bounds the argument $\phi_0 < \arg(x) < \phi_1$.

Proof. By Theorem 3.13 the differential equation $Ly = 0$ has a solution with an essential singularity at 0 if and only if the differential operator L has a local exponent of positive index. For the differential operator L this is equivalent to having a positive rank which implies L is not regular singular at 0.
The regularity of the solutions of regular singular differential equations at 0 follows by the convergence of the power series $f_{k,i}, h_{k,i,j}$ of Theorem 3.13, cf. Corollary 3.3. QED

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