MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Wigner's friend and quantum computation "

verfasst von / submitted by

Clemens Lindner BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 876
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Physik
Betreut von / Supervisor:	Univ. Ass. Prof. Dr. Borivoje Dakic
Mitbetreut von / Co-Supervisor:	-

Zusammenfassung

Das Ziel dieser Arbeit ist es, eine neue Variante des Gedankenexperiments "Wigners Freund" einzuführen, in der wir es mit Quantencomputern kombinieren. In unserer Version des Experiments hat der Freund Zugang zu einem Computer, auf dem er eine Reihe von klassischen Operationen durchführen kann. Indem er auf das Labor des Freundes einwirkt, kann Wigner indirekt Quantenberechnungen auf dem Computer durchführen, was im Widerspruch zu der Überzeugung des Freundes steht, dass er nur klassische Operationen ausführt. Wir werden die Grundlagen der benötigten Konzepte erläutern und dann einen Quantenalgorithmus in unserer neuen Variante ausarbeiten. Des weiteren werden wir auch diskutieren, warum das vorgestellte Modell nur ein vorläufiger Versuch ist und keine tatsächliche physikalische Möglichkeit darstellt.

Acknowledgements

I would like to thank my advisor Borivoje Dakic for providing me with this interesting subject and Sebastian Horvath for the many discussions we had about the curiosities of quantum theory. I also want to thank my friend Alexander Preiss for his company throughout our studies and my parents for their continuous support.

Contents

1 Abstract

The aim of this thesis is to introduce a new variant of the Wigner's friend thought experiment in order to combine it with quantum computation. In our version of the experiment the friend has access to a computer on which he can do a set of classical operations. By acting on the friend's laboratory, Wigner can indirectly perform quantum computation on the computer, which appears to be in tension with the friend's belief that he is implementing classical operations. We will go over the basics of the needed concepts and then work out a quantum algorithm in our new framework. We are also going to discuss why the introduced framework is only a preliminary attempt and does not represent an actual physical possibility.

2 Introduction

Having started development in the beginning of the 20th century, quantum mechanics has since turned out to be one of the most successful physical theories ever produced. Generating explanations starting with the black-body radiation spectrum (which first necessitated the introduction of a quantized energy spectrum, see [1]) up to the celebrated standard model, quantum mechanics surely has a streak of successes in its backpack. However what would a great physical theory be if it didn't try to mess with our most basic understanding of reality? Like every other revolutionary physical theory, quantum mechanics had a lot of such changes in store.
Having not accepted those changes, Einstein, Podolsky and Rosen [2] in the year 1935 introduced one of the first formal criticisms of the new philosophical implications brought forward by quantum mechanics. They argued for the incompleteness of the theory and the potentially needed introduction of "hidden variables" to explain the non-local qualities of entangled states. Since the introduction of a set of inequalities by John Bell [3] and their experimental investigation we know that such hidden variables would come at a price and at least one of our prized intuitive notions of reality needs to be discarded. The question remains if we are dealing with a conflict between the theory and the universe or between the universe and its theorist's possible knowledge about it. Connected to the previous dilemma, another problem has repeatedly irritated physicists. In this case it was the nature of what a measurement truly is and, to an extent, who can measure the measurerer.

In the 60's Eugene Wigner first proposed [4] another form of the now famous Schrödinger´s cat experiment. Written as a short article he switched the cat out for another "conscious" observer capable of measurements. This of course immediately brought about a "who's right" paradox that hasn't entirely been solved to this very day. While Wigner argued for the incompleteness of quantum mechanics as soon as conscious beings enter the picture, others might argue by the general incompleteness of said theory. In the last years there has been an increasing interest in the Wigner's friend scenario as new publications show a range of No-go theorems tangled up in a plurality of Wigner's, Bob's, Alice's, Charlie's and friend's, see e.g. the recent work of Frauchinger and Renner [5]. This thesis tries to implement a new variant of the original Wigner's friend scenario which adds quantum computing in a way that seems paradoxical, more on that later.

Although the technological progress seems to speak for itself, there is still an enormous conceptual problem in the heart of quantum mechanics and quite a bit of time has been spent in the search for a satisfying solution regarding its implications. Whether the final straw will come from theoretical considerations or experimental evidence is anybody's guess.

In the following we will give a short overview of the necessary mathematics and introduce the concepts we are going to use throughout this thesis.

2.1 Quantum Mechanics

2.1.1 Mathematical framework

The main mathematical machinery of quantum mechanics stems from linear algebra. For our purposes [6] we will describe the state of a quantum mechanical system as a vector in a complex and finite dimensional Hilbert space, i.e. $\mathcal{H} \cong \mathbb{C}^d$. The elements of that space are so called "ket" vectors $|v\rangle \in \mathbb{C}^d$. We are also going to define a basis called the "computational basis" as

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad |1\rangle := \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \quad |d-1\rangle := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{1}$$

A general vector takes the form $|v\rangle = \sum_{i=0}^{d-1} v_i |i\rangle$, where v_i are the coefficients. The adjoint vector $(|v\rangle)^\dagger =: \langle v|$ is called a "bra", i.e. a row vector with complex conjugated coefficients. Given two vectors $|v\rangle$ and $|u\rangle = \sum_{j=0}^{d-1} u_j |j\rangle$, their dot product is called a "braket", i.e.

$$(|v\rangle)^\dagger \cdot |u\rangle = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \bar{v}_i u_j \langle i|j\rangle = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \bar{v}_i u_j \delta_{ij} = \sum_{i=0}^{d-1} \bar{v}_i u_i =: \langle v|u\rangle . \tag{2}$$

Using this dot product, we define the 2-norm $\| \, |v\rangle \, \|_2 := \sqrt{\langle v|v\rangle}$.

In order to act on our vectors we define a linear map $M : \mathcal{H} \to \mathcal{H}$. We can write any linear map in matrix form, i.e.

$$M = \sum_{i,j=0}^{d-1} M_{ij} |i\rangle\langle j| \quad \text{with} \quad M_{ij} = \langle i| M |j\rangle . \tag{3}$$

An example of such a map is the Identity $I = \sum_{i=0}^{d-1} |i\rangle\langle i|$ which acts as $I|v\rangle = |v\rangle$. A map $U : \mathcal{H} \to \mathcal{H}$ is called unitary if and only if $U^\dagger U = UU^\dagger = I$. The importance of unitary maps lies in the fact that they preserve angles, i.e.

$$(U|v\rangle)^\dagger \cdot U|u\rangle = \langle v| U^\dagger U |u\rangle = \langle v|u\rangle , \tag{4}$$

and therefore probabilities as we will see later.

Let's say we have two vectors $|v\rangle_A \in \mathcal{H}_A \cong \mathbb{C}^{d_A}$ and $|u\rangle_B \in \mathcal{H}_B \cong \mathbb{C}^{d_B}$ with their respective computational basis being $\{|i\rangle_A\}_{i=0}^{d_A-1}$ and $\{|j\rangle_B)\}_{j=0}^{d_B-1}$, i.e.

$$|v\rangle_A = \sum_{i=0}^{d_A-1} v_i |i\rangle_A , \quad |u\rangle_B = \sum_{j=0}^{d_B-1} u_j |j\rangle_B . \tag{5}$$

Their so called tensor product $|v\rangle_A \otimes |u\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B =: \mathcal{H}_{AB}$ is defined by choosing the basis for $\mathcal{H}_{AB}$ as the space with an orthonormal basis of tuples $(|i\rangle_A , |j\rangle_B)$ (denoted as $|i\rangle_A \otimes |j\rangle_B = |i\rangle_A |j\rangle_B = |ij\rangle_{AB}$, sometimes we will also omit the indices), where

$$(\langle i|_A \otimes \langle j|_B)(|k\rangle_A \otimes |l\rangle_B) = \langle i|k\rangle_A \langle j|l\rangle_B = \delta_{ik}\delta_{jl} \quad \text{and} \quad |v\rangle_A \otimes |u\rangle_B = \sum_{i,j=0}^{d_A-1,d_B-1} v_i u_j |ij\rangle . \tag{6}$$

This is not necessarily the correct form for a general vector which can be written as $|w\rangle_{AB} = \sum_{i,j=0}^{d_A-1,d_B-1} w_{ij} |ij\rangle$. A state where $w_{ij} = v_i u_j$ is called separable and a state where this doesn't hold entangled.

We can expand the concept to linear maps $M_A : \mathcal{H}_A \to \mathcal{H}_A$ and $N_B : \mathcal{H}_B \to \mathcal{H}_B$ by defining the map $M_A \otimes N_B : \mathcal{H}_{AB} \to \mathcal{H}_{AB}$ via

$$(M_A \otimes N_B)(|v\rangle_A \otimes |u\rangle_B) = (M_A |v\rangle_A) \otimes (N_B |u\rangle_B). \tag{7}$$

This can again be written in matrix form as

$$M_A \otimes N_B = \sum_{i,j,k,l} (M_A)_{ij} (N_B)_{kl} |ij\rangle \langle kl| . \tag{8}$$

2.1.2 Axioms of Quantum Mechanics

Having described the mathematical framework we are now going to use it and define the usual axioms of quantum mechanics. The state of a physical system is given as a vector $|\Psi\rangle \in \mathbb{C}^d$ with $\| |\Psi\rangle \|_2 = 1$. An example with $d = 2$ would be a state like

$$|\Psi\rangle = \sqrt{\frac{1}{3}} |0\rangle + \sqrt{\frac{2}{3}} |1\rangle , \tag{9}$$

in which case we would say that the state is in a superposition of the basis states $|0\rangle$ and $|1\rangle$. From the definition we also find that $|\Psi\rangle$ and $e^{i\phi} |\Psi\rangle$ (with $\phi \in \mathbb{R}$) represent the same physical state. The factor $e^{i\phi}$ is called a global phase and can be omitted. A local or relative phase would be the phase difference between two basis states, which becomes especially important in quantum computing because it's what allows us to amplify the probability of obtaining a certain result from a superposition of states.

Such a result would be obtained with a certain probability and requires a measurement. A measurement is defined by specifying an orthonormal basis $\{a_i\}$ in $\mathbb{C}^d$ and taking the outcome probabilities to be

$$p_i = |\langle a_i|\Psi\rangle|^2 \quad \text{with} \quad \sum_i p_i = 1. \tag{10}$$

We say that we "measure in the $\{a_i\}$-basis". As an example we can look at the state of equation 9 and measure in the basis $\{|0\rangle, |1\rangle\}$. With probability $p_0 = \frac{1}{3}$ we would obtain the post measurement state of $|\Psi\rangle = |0\rangle$. Any repeated measurement would result in the same state because $p_1 = |\langle 1|0\rangle|^2 = 0$. The superposition of the state has therefore "collapsed". Another way to characterize this is via the use of orthogonal projectors $E_i = |i\rangle\langle i|$. Defining $|\tilde{\Psi}_i\rangle = E_i |\Psi\rangle$ we get the probabilities and the post measurement state $|\Psi_i\rangle$ via

$$p_i = \| |\tilde{\Psi}_i\rangle \|_2^2 \quad \text{and} \quad |\Psi_i\rangle = \frac{|\tilde{\Psi}_i\rangle}{\| |\tilde{\Psi}_i\rangle \|_2} = \frac{|\tilde{\Psi}_i\rangle}{\sqrt{p_i}}. \tag{11}$$

A generalization of this would be a complete set of orthogonal projectors E_i with

$$E_i = E_i^\dagger, \quad E_i E_j = \delta_{ij} E_i \quad \text{and} \quad \sum_i E_i = I. \tag{12}$$

The evolution of a quantum state will be described as a linear process $|\Psi\rangle \mapsto U|\Psi\rangle$. Our only prerequisite is that probabilities stay conserved. This naturally happens when using unitary matrices, therefore any unitary matrix describes a valid evolution of a state.

If we consider two parties A and B with two different quantum states we can describe the composite system using the tensor product. If separable, their joint state is given by $|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \in \mathcal{H}_{AB}$, otherwise just $|\Psi_{AB}\rangle$. Let's say A and B want to do unitary evolutions X_A and Y_B on their corresponding states. This can be modelled by writing

$$(X_A \otimes I)(I \otimes Y_B)|\Psi_{AB}\rangle = (X_A \otimes Y_B)|\Psi_{AB}\rangle, \tag{13}$$

where I represents no action on the corresponding system.

This concludes the basic rules we are going to need for this thesis. It should be noted that this approach only characterizes so called pure states, i.e. states where we have full knowledge of the combined system. Another definition is that they are states on the surface on the Bloch sphere, which will be defined later. A more general version called mixed states exists in the form of density operators but we won't be needing it in the context of this thesis.

2.2 Wigner's friend

The idea behind the scenario of Wigner's friend [4] is to put the measurement process into a superposition itself. Wigner leads one of his disposable friends into an isolated box of sufficient size for a human and measurement apparatus. Any measurement apparatus needs something to measure of course, so he also inserts a qubit in some arbitrary superposition, e.g. $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. By the axiom of measurement, once the friend measures the qubit in the computational basis he will either obtain the state $|0\rangle$ or $|1\rangle$ with a common probability of $p = \frac{1}{2}$. The superposition therefore "collapses" by a projection onto one of the basis vectors. Wigner only knows that the friend did a measurement but not what the result turned out to be. For Wigner the whole system is now simply coupled to the friend. Mathematically we can describe the this system as a tensor product between Wigner, the friend, the qubit and the environment. In this case the friend acts as a measurement apparatus. We therefore have:

$$|\Psi\rangle = |W\rangle \otimes |F\rangle \otimes |\text{Qubit}\rangle \otimes |\text{Env.}\rangle \tag{14}$$

Now for the "paradoxical" part. We can take two views: That of Wigner and that of the friend. First we have a look at the friend who simply measures the qubit and gets a definitive outcome with probability $\frac{1}{2}$:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |F\rangle \rightarrow \begin{cases} |0\rangle \otimes |\text{F measured } 0\rangle \\ |1\rangle \otimes |\text{F measured } 1\rangle \end{cases} \tag{15}$$

This however stands in contrast with our second explanation of the whole process, in which we take Wigner's point of view. Instead of a probabilistic measurement, he will describe the whole process as a deterministic evolution of the whole system, i.e. as a coupling between the friend and the qubit:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |F\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle \otimes |\text{F measured } 0\rangle + |1\rangle \otimes |\text{F measured } 1\rangle) \tag{16}$$

As a sort of "superobserver" we know that this cannot be true because the measurement did take place and the qubit cannot therefore be in a superposition anymore. Both our parties however used a consistent quantum mechanical notion of the process so one of them has to be incorrect. To resolve the conundrum one can postulate two solutions:

1. Quantum mechanics doesn't truthfully describe both viewpoints (at least at the same time)
2. It is impossible to put an observer into a superposition and therefore the paradoxon never arises

Both solutions would be of great gravity to the whole of physics but it isn't easy to prove either. The first one might be addressed by introducing a concept of relative objectivity or all together creating a new theory. For the second one there has been an abundance of so called spontaneous collapse theories trying to give an explanation for why we never see macroscopic superpositions (for an introduction see [7]). None has yet been experimentally confirmed. On the other hand we are experiencing an increasing experimental effort to put larger and larger systems into superpositions (see e.g. [8]). For now it is unclear whether the superposition principle extends into the macroscopic realm but in this thesis we will still assume it does when taking Wigner's point of view.

In the next section we are going to introduce the concept of quantum computing and following that we will try to implement it into the Wigner's friend scenario.

2.3 Quantum Computing

The field of quantum computing is still in a state of rapid development. Some of the first theoretical considerations go back around 40 years, see e.g. Richard Feynman's ideas from 1982 [9], in which he discusses the problems that arise should one try so simulate the behavior of quantum systems using classical computers. The exponential increase in needed computational power made the invention of a new kind of computer that works with quantum particles to correctly simulate quantum particles almost a necessity for any such efforts.

Of course nowadays quantum computers are of global interest, not just for the simulation of quantum systems, but all the more for their ability to implement certain algorithms exceedingly faster than their classical counterparts. One of those algorithms is Shor's algorithm, which allows the factorization of large numbers in polynomial time compared to the practical inability of classical computers to do so. Such a feat becomes rather important in the field of classical cryptography, where the widely employed RSA encryption system uses this precise weakness of the classical world to encode messages and safely distribute them (see e.g. [12], p.640). It should however be noted that at the time of writing, there doesn't seem to be a precise proof for why classical computers aren't capable of finding the prime factors of large numbers efficiently. The faith in most cryptographic systems simply rests on the continuous inability to show that they can be efficiently broken.

Spending most of their existence in the theoretical world, the last years have shown an ever increasing success in actually building a quantum computer. One example would be the rapid increase in the number of qubits on a quantum processor. As of November 2022 the Osprey quantum processor from IBM implements 433 qubits with plans to increase that number in the coming years (see [10]). Although the first quantum computers have been build, the full range of their applications and especially their advantages compared to classical computers are not fully known. What is fully known is their basic theoretical implementation, which we are now going to give a quick overview of.

Compared to a classical computers bits, a quantum computer uses so called qubits which have the advantage that they can be in a superposition of the classical bits 0 or 1. First we use the computational basis where we define the following orthogonal row vectors:

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{17}$$

We can then write any qubit state as

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \quad \text{where} \quad \alpha, \beta \in \mathbb{C} \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = 1. \tag{18}$$

There is also a great general visualization of all the possible qubit states called the Bloch sphere. It is possible to write any pure quantum state as

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad \text{where} \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi. \tag{19}$$

This can then be visualized graphically by the Bloch sphere, as seen in figure 1. Pure states correspond to vectors going from the point of origin to the surface of the sphere.

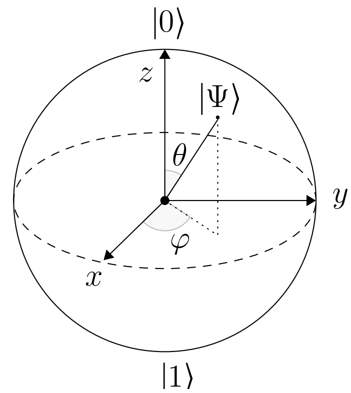


Figure 1: A Bloch sphere with exemplary state vector $|\Psi\rangle$ and orthogonal vectors $|0\rangle$ and $|1\rangle$ on the respective poles of the sphere

Two important quantum states are the

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{20}$$

states. Graphically they lay on the respective positive and negative surface points of the x-axis. Their importance comes from the fact that they are the result of acting with the so called Hadamard gate H on the computational basis states $|0\rangle$ and $|1\rangle$, e.g.

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}|0\rangle = |+\rangle , \tag{21}$$

and therefore creating in the process a superposition between the basis states.

Of great use in quantum mechanics are the well known Pauli matrices. As is common in quantum computing, we will call them X, Y and Z:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{22}$$

It can be easily checked that their squares are equal to the identity matrix, i.e. $X^2 = Y^2 = Z^2 = I$. We can also now see that $H = \frac{1}{\sqrt{2}}(X + Z)$.

Using the Pauli matrices it is possible to create a rotation operator that rotates one pure state on the Bloch sphere onto another. A rotation around the x-axis and by an angle θ can for example be accomplished by acting with the rotation operator

$$R_X(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) X = \begin{bmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \tag{23}$$

on the state. Using this operator, a rotation with $\theta = \pi$ will therefore turn the state vector $|0\rangle$ into $|1\rangle$. A more compact way to write the rotation operator is to remember that for any

INPUT		OUTPUT				
Bits		**AND($\wedge$)**	**NAND**	**OR($\vee$)**	**NOR**	**XOR($\oplus$)**
0	0	0	1	0	1	0
0	1	0	1	1	0	1
1	0	0	1	1	0	1
1	1	1	0	1	0	0

Table 1: Logic gates with their corresponding truth values for 2 bits

operator $A^2 = I$, the following identity holds:

$$e^{i\theta A} = \cos(\theta)I + i\sin(\theta)A. \tag{24}$$

We can therefore write $R_X(\theta) = \exp(-iX\theta/2)$. The goal of quantum computing is to execute any possible computational task using the qubits we have described in this chapter. For this we will need to show first how qubits can be turned into a computer and second how that computer guarantees their universal applicability to computational problems. In the following section both of these concepts will be elaborated upon in the form of quantum circuits and universal computation.

2.3.1 Universal Quantum Computation

There are a few models of computation available to us. One of the first was the concept of a Turing machine (named after it's creator Alan Turing) which is a simple set of rules that is in theory capable of simulating any algorithmic process. Another model is the circuit model, which is effectively equivalent to a Turing machine but allows a viewpoint that is maybe more intuitive, especially in the context of quantum computation.

Any computer can be modelled by a circuit model which includes a set of logic gates. These are elementary logical operations on a single bit or set of bits, e.g. the gate NOT which turns 0 to 1 and the other way around. A set of examples for classical gates and their truth tables (i.e. binary input and output correlations) can be seen in table 1.

In order to do an arbitrary kind of computation (i.e. express all possible truth tables) we need what is called a universal set of logic gates. One such set would be AND combined with NOT. It has also been shown that the gates NAND and NOR form a universal set by themselves.

In quantum computing the situation is the same but now we call it a quantum circuit model and we need logical gates that act on qubits. Because these gates are supposed to act under the laws of quantum mechanics, they need to be reversible and unitary. In order to depict them graphically we will use the most common approach. In this approach a circuit is read out from left to right. A single line depicts one qubit, while more lines depict many. As long as nothing disturbs a line, this can be thought of as the Identity-matrix continuously acting on the qubit. If we want a gate to act upon the qubit we introduce it by a square shaped object with the intended gate written inside the cube. For 2 or more qubits we can install gates that act on both of them. One of the most important ones is the controlled-U gate. This will act with any specified unitary operation U on the second qubit only if the first qubit is in the $|1\rangle$ state and otherwise just as the identity operation. In this case the first qubit is called a control-qubit and the second one a target-qubit.

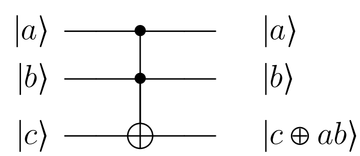
For convenience we will also denote $|a\rangle \otimes |b\rangle \otimes \cdots \otimes |c\rangle$ as $|a, b, \ldots, c\rangle = |ab \ldots c\rangle$. An example for a commonly used quantum gate is the CX or otherwise CNOT gate, where the first qubit acts as a control on the second one and swaps either $|0\rangle \to |1\rangle$ or $|1\rangle \to |0\rangle$. For $a, b, c \in \{0, 1\}$ it acts like the classical XOR gate:

$$\mathrm{CNOT}\,|a, b\rangle = |a, a \oplus b\rangle\,. \tag{25}$$

Using the CNOT gate we can construct one of the first known reversible gates, the so called Toffoli gate. This gate takes 3 input qubits and implements a controlled-CNOT operation on the third qubit, i.e. a NOT operation is applied on the third qubit if and only if the first and second are are set to $|1\rangle$. We therefore have:

$$\mathrm{Toffoli}\,|a, b, c\rangle = |a, b, c \oplus ab\rangle\,. \tag{26}$$

and graphically:



The reversibility of the Toffoli gate can be quickly seen by applying it two times in a row:

$$\mathrm{Toffoli}^2\,|a, b, c\rangle = \mathrm{Toffoli}\,|a, b, c \oplus ab\rangle = |a, b, c \oplus ab \oplus ab\rangle = |a, b, c\rangle\,. \tag{27}$$

It's importance stems from the fact that it acts as a universal gate for classical computation while being reversible. This quickly proves that any quantum computer is able to simulate a classical computer.

Another important family of gates are the phase shift gates which can be used to implement a relative phase shift in the quantum state. Generally they can be written as

$$P(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}, \tag{28}$$

with ϕ again encompassing a full longitudinal rotation on the bloch sphere. We are going to use a few elementary gates, so for convenience they are gathered in table 2.

It is still left to consider what it takes to perform universal quantum computation. There are a few universality constructions for quantum computing but the most used one and also

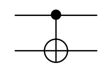
Gate	**I**	**X**	**Y**	**Z**	**H**
Matrix	$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Graph	—	—[X]—	—[Y]—	—[Z]—	—[H]—
Gate	**T**	$\mathbf{R_X}(\theta)$	$\mathbf{R_Y}(\theta)$	$\mathbf{R_Z}(\theta)$	**CNOT**
Matrix/ Exp. form	$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$	$\exp(-\frac{iX\theta}{2})$	$\exp(-\frac{iY\theta}{2})$	$\exp(-\frac{iZ\theta}{2})$	$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Graph	—[T]—	—[$R_X(\theta)$]—	—[$R_Y(\theta)$]—	—[$R_Z(\theta)$]—	

Table 2: Common quantum gates and their graphical depictions

the one we are going to use is the set given by H, T and CNOT (For a proof of this see [12], section 4.5 on universal quantum gates). Normally the phase gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{29}$$

is also included in this list but we can construct it using two T gates so unless explicitly needed, we can ignore it. Together these gates form a universal gate set for quantum computing and, should we be able to construct them, allow us to implement any quantum algorithm we want.

2.3.2 Universal quantum computation using classical operations

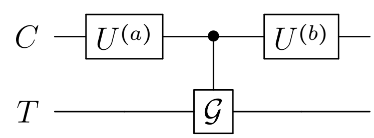
We now want to introduce a framework that implements universal quantum computation via quantum controlled classical operations. The end goal of this will be to use this concept in a new variant of the Wigner's friend scenario where we will let the friend act on a classical computer with operations that are specified by Wigner. Of importance is the fact that Wigner can send those specifications in a superposition to which the friend will then couple. More on that later. The main framework is already worked out and can in detail be looked at in [11]. We will however give an overview:

The model consists of a "quantum" control system that acts as a control on the "classical" target system (classical in the sense defined below). The 2^n-dimensional Hilbert space of the target system is spanned by the states $\{|\vec{x}\rangle \equiv |x_1, \ldots, x_n\rangle, \forall x_i = 0, 1\}$. Each of these states can be considered a qubit but we are going to limit the available set of operations from the full unitary group $U(2^n)$ to a finite subset $\mathcal{G} \subset U(2^n)$ where the operations $G_i \in \mathcal{G}$ are considered classical if $\mathcal{G} \subseteq S_{2^n}$, i.e. the symmetric group on 2^n elements. This means we are only considering reversible permutations of our basis states therefore making our n-qubits behave as n-bits.

Opposite to this is our control system's Hilbert space spanned by the states $\{|G_i\rangle, \forall G_i \in \mathcal{G}\}$. It controls the operations that act upon the target system with interactions between the two being modeled as

$$|G_i\rangle \otimes |\vec{x}\rangle \rightarrow |G_i\rangle \otimes G_i |\vec{x}\rangle. \tag{30}$$

The biggest difference to before is that we may act with any unitary operator upon the control system. Via an interaction it is now possible to apply operations on the target system that are not in the set $\mathcal{G}$. The basic process follows the following circuit graph:



Mathematically we initially start with some arbitrary state $|G_0\rangle \otimes |\vec{x}\rangle$, where $G_0 \in \mathcal{G}$. We apply the first unitary and let the systems interact. It will end up in

$$\sum_i u_{i0}^{(a)} |G_i\rangle \otimes G_i |\vec{x}\rangle, \quad \text{where} \quad u_{i0}^{(a)} = \langle G_i| U^{(a)} |G_0\rangle. \tag{31}$$

After the second unitary we gain the state

$$\sum_i |G_i\rangle \otimes O_j^{(ab)} |\vec{x}\rangle, \quad \text{where} \quad O_j^{(ab)} \equiv \sum_i u_{ji}^{(b)} u_{i0}^{(a)} G_i. \tag{32}$$

A projective measurement $\{|G_j\rangle \langle G_j|, \forall j\}$ on the control system then yields an outcome 'j' which leaves the target system in $O_j^{(ab)} |\vec{x}\rangle$ (up to normalization). This may may reach a state that was previously unobtainable by simply acting with any $G_i \in \mathcal{G}$. Repeating this entire process allows us to probabilistically implement products of $O_j^{(ab)}$. The probabilistic aspect of this scheme does dampen the fun a bit so it will be addressed in a later procedure.

As specified in [11] we need at least two target-qubits to implement any unitary transformation $U(2)$ on a two-dimensional subspace of the two qubit Hilbert space. This subspace will be called the *logical space* and consists of a computational basis $\{|0\rangle_L, |1\rangle_L\}$ defined via

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) \quad \text{and} \quad |1\rangle_L = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle) \tag{33}$$

It is therefore possible to create n logical qubits from $2n$ target ones. We will now also specify the set $\mathcal{G}$:

$$\mathcal{G} = \{G_0, G_1, G_2\} = \{\mathrm{I}, \mathrm{NOT}_1, \mathrm{CNOT}\}. \tag{34}$$

Here NOT_1 corresponds to the NOT-gate on the first qubit, i.e. $\mathrm{NOT}_1 |x_1 x_2\rangle = |(x_1 \oplus 1) x_2\rangle$.

For $y = 0, 1$ we can see that they act on the logical qubits as

$$G_1 |y\rangle_L = |y \oplus 1\rangle_L \quad \text{and} \quad G_2 |y\rangle_L = -(-1)^y |y\rangle_L, \tag{35}$$

which equal the actions of the Pauli gates X and $-Z$, i.e. $G_1 \cong X$ and $G_2 \cong -Z$. The same transformations are now available on each pair of target qubits and we label them via $G_i^{(l)}$ corresponding to a G_i action on the l-th pair $(2l-1, 2l)$ and the Identity on the others. For n pairs we therefore have the following available transformations, containing $(2n+1)$ elements:

$$\mathcal{G} = \bigcup_{l=1}^{n} \{G_1^{(l)}, G_2^{(l)}\} \cup \{I\}. \tag{36}$$

It follows from this that our control system needs to be $2n+1$-dimensional, i.e. consist of $\log(2n+1)$ qubits. Graphically the whole system is depicted in figure 2.

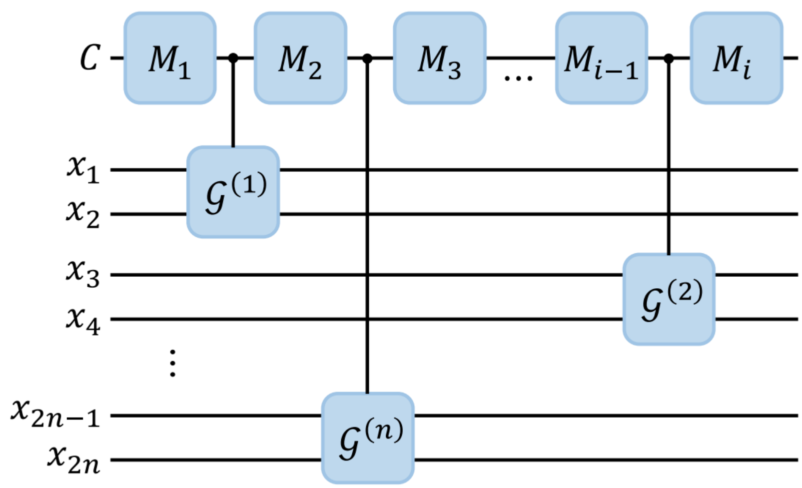


Figure 2: Full computational model where the operations M represent unitary transformation and/or projective measurements on the control system and $\mathcal{G}$ are the classical operations on each pair (figure taken from [11])

It can be shown that using this framework we can successfully implement the quantum gates H, T and CNOT which guarantees us the possibility of universal quantum computation. We will however only quickly show how to initialize the logical qubits and give an example of a single qubit gate.

Initialization

Our goal is to initialize the state $|0\rangle_L^{\otimes n}$. We begin with an arbitrary pairwise state of the target, e.g. $|10\rangle^{\otimes n}$ and the control in the state $|I\rangle$. In order to prepare one of the pairs in the $|0\rangle_L$ state we do the following procedure:

1. First a control Hadamard in the $\{|I\rangle, |G_2\rangle\}$ subspace leading to $\frac{1}{\sqrt{2}}(|I\rangle + |G_2\rangle)$. After an interaction this will lead to

$$\frac{1}{\sqrt{2}}(|I\rangle \otimes I\,|10\rangle + |G_2\rangle \otimes G_2\,|10\rangle) = \frac{1}{\sqrt{2}}(|I\rangle \otimes |10\rangle + |G_2\rangle \otimes |11\rangle) \tag{37}$$

2. Secondly, another equally acting Hadamard leading to

$$\frac{1}{2}[\,|I\rangle \otimes (|10\rangle + |11\rangle) + |G_2\rangle \otimes (|10\rangle - |11\rangle)] \tag{38}$$

3. A final projective measurement on the control system and its corresponding basis yields either $|I\rangle$ or $|G_2\rangle$. For an outcome G_2 the postselected state of the control system will be $\frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) = |0\rangle_L$, successfully implementing our initialization for one pair. The same can now be done for the others. Should the measurement yield I we simply perform target measurements that leave the state in either $|10\rangle$ or $|11\rangle$ and repeat the procedure.

Single qubit gate

As an example we will implement the Hadamard gate on some logical qubit $|\Psi\rangle_L$. We start with the control in $|G_1\rangle$ and execute a similar procedure:

1. First a Hadamard in the $\{|G_1\rangle, |G_2\rangle\}$ subspace. Following the interaction we get

$$\frac{1}{\sqrt{2}}(|G_1\rangle \otimes X|\Psi\rangle_L - |G_2\rangle \otimes Z|\Psi\rangle_L). \tag{39}$$

2. Another equally acting Hadamard then gets us

$$\frac{1}{2}(|G_1\rangle \otimes (X - Z)|\Psi\rangle_L + |G_2\rangle \otimes (X + Z)|\Psi\rangle_L) \tag{40}$$

3. Finally a measurement of $|G_2\rangle$ will leave the logical state in $\frac{1}{\sqrt{2}}(X + Z)|\Psi\rangle_L = H|\Psi\rangle_L$, successfully but probabilistically implementing H.

Of course every computational model needs a method of reading out the result. In our case this is made easy: In order to perform a measurement in the logical computational basis $\{|0\rangle_L, |1\rangle_L\}$ we simply perform one on the classical "bits". Should the outcome be $|10\rangle$ or $|11\rangle$ we have $|0\rangle_L$, otherwise for $|00\rangle$ or $|01\rangle$ we have $|1\rangle_L$.

Using the method shown in this section, we can implement any quantum gate and therefore any quantum algorithm with classically controlled operations. The algorithm we will be implementing is Grover's algorithm, which will be discussed in the following section.

2.3.3 Quantum algorithms

The two known main classes of quantum algorithms are first algorithms based on the quantum Fourier transform and second so called quantum search algorithms. The already mentioned Shor's algorithm for factoring large numbers is an example of the first class, we will however not go into any more detail. Our interest lies on the second class of algorithms, especially the so called Grover's algorithm (after Lov Grover, see [13]). The only reason for that however is the relative simplicity of creating a quantum circuit that successfully implements Grover's algorithm.

The main promise of any search algorithm is to find an element of an unsorted list that fulfills a certain condition. Classically this will be of the order $O(N)$ since we will need at most N tries of picking a random element and checking if it fulfills our condition. Quantum mechanically this can be achieved using just $O(\sqrt{N})$ tries as shown by Grover.

The way this is done is by introducing a black box, also called oracle, into the algorithm that "marks" the right answer by applying a relative phase shift to it. As an example (see [12], section 6.1) look at the function $f(x)$ which takes as input an integer x going from 0 to $N-1$ (Symbolizing the different index positions of the elements in the database). It's output can either be 0 or 1 and corresponds to:

$$f(x) = \begin{cases} 1 \Rightarrow x \text{ is a solution to the search problem} \\ 0 \Rightarrow x \text{ is not a solution} \end{cases} \tag{41}$$

Our oracle can now be described by it's action on an oracle qubit $|q\rangle$ (defined in the computational basis), i.e. as a unitary operator O:

$$|x\rangle\,|q\rangle \xrightarrow{O} |x\rangle\,|q \oplus f(x)\rangle \tag{42}$$

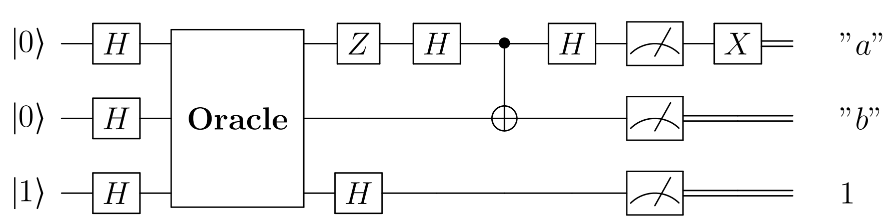
A solution is found if we initialize the state in $|x\rangle\,|0\rangle$ and check if the bit of the oracle qubit has been flipped after applying O. Even better, if we initialize it in $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, the oracle qubit won't change and can therefore be disregarded. The only thing that will happen is that the phase of the solution $|x\rangle$ state gets shifted. We have

$$|x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \xrightarrow{O} (-1)^{f(x)}\,|x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \tag{43}$$

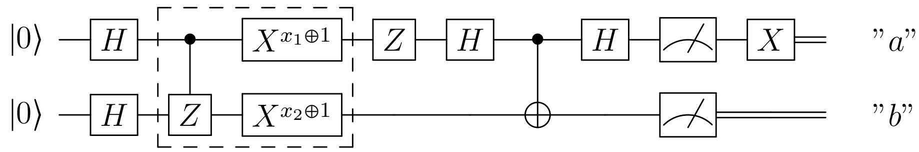
or just $O : |x\rangle \to (-1)^{f(x)}\,|x\rangle$. By evaluating $f(x)$ the oracle is therefore able to "recognize" and mark the correct answer to a given problem.

The rest of the algorithm now simply depends upon the fact that we can create a superposition in the elements of the database and via the introduced relative phase shift increase the amplitude of the correct solution. We won't go into specifics here but they can be found in [12]. What is important for our needs is that we find a quantum circuit that successfully implements an oracle and the rest of Grover's algorithm.

Normally we would implement the oracle using a Toffoli gate. Using quantum circuits such a gate is rather difficult to construct (we would need at least 6 CNOT and a few H and T gates for a single Toffoli gate) so as a proof of concept we will focus on a simplified version with $N = 4$ searchable entries $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$. Classically this would result in at most 4 queries to the oracle while quantum mechanically we can get the desired result with just one query (the typical order of $O(\sqrt{N})$ only becomes apparent at large N). An exemplary circuit we can use in the N=4 case looks like this:



Here the oracle is specified by an oracle qubit in the third row and the gates inside the box (e.g. a Toffoli gate). This however equals the following much more simplified circuit where the oracle is encompassed by the dashed line (see [14]):



The ending X-gate can be seen as a NOT operation on the first qubit and (x_1,x_2) correspond to the element of our given solution to the search problem. A correct result would therefore be $a = x_1$, $b = x_2$. We can explicitly see that this circuit works by letting the oracle choose $|11\rangle$ as the correct result, i.e. $x_1 = x_2 = 1$. In this case $X^2 = I$ and can be omitted. We quickly check that this gives indeed the correct result:

$$\begin{aligned}
|00\rangle &\xrightarrow{H^{\otimes 2}} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&\xrightarrow{C\text{-}Z} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \\
&\xrightarrow{Z\otimes I} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \\
&\xrightarrow{H\otimes I} \frac{1}{2\sqrt{2}}((|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)(|1\rangle - |0\rangle)) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
&\xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \\
&\xrightarrow{H\otimes I} \frac{1}{2}(|01\rangle + |11\rangle + |01\rangle - |11\rangle) = |01\rangle
\end{aligned} \tag{44}$$

A following measurement and application of X on the first qubit will lead to our desired result $ab = 11$.

The full gate order (omitting the measurement and X) we just calculated is

$$(H \otimes I)(\text{CNOT})(H \otimes I)(Z \otimes 1)(\text{CZ})(H^{\otimes 2})|00\rangle \tag{45}$$

In order to simplify the implementation in our framework presented in 2.3.2 we note that the CNOT-gate can otherwise be written as

$$\text{CNOT} = (I \otimes H)(\text{CZ})(I \otimes H) \Rightarrow (H \otimes I)(\text{CNOT})(H \otimes I) = (H^{\otimes 2})(\text{CZ})(H^{\otimes 2}) \tag{46}$$

Inserting this into equation 45 we get the following order:

$$(H^{\otimes 2})(\text{CZ})(H^{\otimes 2})(Z \otimes I)(\text{CZ})(H^{\otimes 2})|00\rangle \tag{47}$$

This can now be easily implemented in our framework as seen in the next section.

3 Grover's algorithm using quantum controlled classical operations

In section 2.3.3 we introduced a simplified N=4 Grover algorithm that can serve as a proof of concept in our framework and the coming connection to Wigner's Friend. We are now going to implement the gate order from equation 47 into the framework. First we define a new unitary T on the control system that switches from one pair of qubits to another, i.e. $T^{1\to 2} : |G_i^{(1)}\rangle \to |G_i^{(2)}\rangle$. Factually this will lead to a change from acting on the first logical qubit to acting on the second one.

We assume that the entire system has been initialized in $|G_1^1\rangle \otimes |00\rangle_L$. Next we define a few things to compress our equations:

1. $\xrightarrow{I}$ (I above an arrow) as representing an interaction between the control and the target systems (not to be confused with I:=Identity operation)

2. $H^{1,2}_{G_iG_j} = H^{1,2}_{ij}$:=Hadamard acting in the $\{G_i, G_j\}$ subspace on the first or second qubit (here $i, j \in 0, 1, 2$). We define the indices to act like $H_{ij}|G_i\rangle = \frac{1}{\sqrt{2}}(|G_i\rangle + |G_j\rangle)$ and $H_{ji}|G_i\rangle = \frac{1}{\sqrt{2}}(|G_i\rangle - |G_j\rangle)$.

3. $\mathbf{G}_i^{1,2}$:=Result after having measured the control system

4. $\xrightarrow{A,B}$ as representing first the application of B, then A

In each of the elements of the following list we will implement another gate, beginning with a Hadamard transform on both qubits where we need to measure in between both Hadamard applications to successfully implement the 2 qubit gate:

1. $(H^{\otimes 2})\left|00\right\rangle_L$

$$
\begin{aligned}
\left|G_1^1\right\rangle \otimes \left|00\right\rangle_L &\xrightarrow{H_{12}^1} \frac{1}{\sqrt{2}}(\left|G_1^1\right\rangle \left|00\right\rangle_L + \left|G_2^1\right\rangle \left|00\right\rangle_L) \qquad (48)\\
&\xrightarrow{I} \frac{1}{\sqrt{2}}(\left|G_1^1\right\rangle (X\otimes I)\left|00\right\rangle_L + \left|G_2^1\right\rangle (-Z\otimes I)\left|00\right\rangle_L)\\
&\xrightarrow{H_{12}^1} \frac{1}{2}((\left|G_1^1\right\rangle + \left|G_2^1\right\rangle)(X\otimes I)\left|00\right\rangle_L + (\left|G_1^1\right\rangle - \left|G_2^1\right\rangle)(-Z\otimes I)\left|00\right\rangle_L)\\
&= \frac{1}{2}(\left|G_1^1\right\rangle (\left|10\right\rangle_L - \left|00\right\rangle_L) + \left|G_2^1\right\rangle (\left|10\right\rangle_L + \left|00\right\rangle_L))\\
&\xrightarrow{\mathbf{G}_2^1} \frac{1}{\sqrt{2}}(\left|G_2^1\right\rangle (\left|10\right\rangle_L + \left|00\right\rangle_L)) = \left|G_2^1\right\rangle \otimes (H\otimes I)\left|00\right\rangle_L\\
&\xrightarrow{T^{1\to 2}} \frac{1}{\sqrt{2}}(\left|G_2^2\right\rangle (\left|10\right\rangle_L + \left|00\right\rangle_L))\\
&\xrightarrow{H_{21}^2} \frac{1}{2}(\left|G_1^2\right\rangle + \left|G_2^2\right\rangle)(\left|10\right\rangle_L + \left|00\right\rangle_L))\\
&\xrightarrow{I} \frac{1}{2}(\left|G_1^2\right\rangle (\left|11\right\rangle_L + \left|01\right\rangle_L) - \left|G_2^2\right\rangle (\left|10\right\rangle_L + \left|00\right\rangle_L)\\
&\xrightarrow{H_{12}^2} \frac{1}{2\sqrt{2}}((\left|G_1^2\right\rangle + \left|G_2^2\right\rangle)(\left|11\right\rangle_L + \left|01\right\rangle_L) - (\left|G_1^2\right\rangle - \left|G_2^2\right\rangle)(\left|10\right\rangle_L + \left|00\right\rangle_L))\\
&= \frac{1}{2\sqrt{2}}(\left|G_1^2\right\rangle (\left|11\right\rangle_L + \left|01\right\rangle_L - \left|10\right\rangle_L - \left|00\right\rangle_L) + \left|G_2^2\right\rangle (\left|11\right\rangle_L + \left|01\right\rangle_L + \left|10\right\rangle_L + \left|00\right\rangle_L))\\
&\xrightarrow{\mathbf{G}_2^2} \frac{1}{2}\left|G_2^2\right\rangle (\left|11\right\rangle_L + \left|01\right\rangle_L + \left|10\right\rangle_L + \left|00\right\rangle_L) = \left|G_2^2\right\rangle \otimes H^{\otimes 2}\left|00\right\rangle_L
\end{aligned}
$$

2. $(\mathrm{CZ})(H^{\otimes 2})\left|00\right\rangle_L$

$$
\begin{aligned}
&\xrightarrow{T^{2\to 1}} \frac{1}{2}\left|G_2^1\right\rangle (\left|11\right\rangle_L + \left|01\right\rangle_L + \left|10\right\rangle_L + \left|00\right\rangle_L) \qquad (49)\\
&\xrightarrow{I,H_{02}^1} \frac{1}{2\sqrt{2}}(\left|G_0^1\right\rangle (\left|11\right\rangle_L + \left|01\right\rangle_L + \left|10\right\rangle_L + \left|00\right\rangle_L) + \left|G_2^1\right\rangle (-\left|11\right\rangle_L + \left|01\right\rangle_L - \left|10\right\rangle_L + \left|00\right\rangle_L))\\
&\xrightarrow{H_{02}^1} \frac{1}{4}(\left|G_0^1\right\rangle + \left|G_2^1\right\rangle (\left|11\right\rangle_L + \left|01\right\rangle_L + \left|10\right\rangle_L + \left|00\right\rangle_L) + (\left|G_0^1\right\rangle - \left|G_2^1\right\rangle)(-\left|11\right\rangle_L + \left|01\right\rangle_L - \left|10\right\rangle_L + \left|00\right\rangle_L))\\
&= \frac{1}{2}(\left|G_0^1\right\rangle (\left|01\right\rangle_L + \left|00\right\rangle_L) + \left|G_2^1\right\rangle (\left|11\right\rangle_L + \left|10\right\rangle_L))\\
&\xrightarrow{I,T^{1\to 2}} \frac{1}{2}(\left|G_0^2\right\rangle (\left|01\right\rangle_L + \left|00\right\rangle_L) - \left|G_2^2\right\rangle (-\left|11\right\rangle_L + \left|10\right\rangle_L))\\
&\xrightarrow{H_{02}^2} \frac{1}{2\sqrt{2}}((\left|G_0^2\right\rangle + \left|G_2^2\right\rangle)(\left|01\right\rangle_L + \left|00\right\rangle_L) - (\left|G_0^2\right\rangle - \left|G_2^2\right\rangle)(-\left|11\right\rangle_L + \left|10\right\rangle_L))\\
&= \frac{1}{2\sqrt{2}}(\left|G_0^2\right\rangle (\left|01\right\rangle_L + \left|00\right\rangle_L + \left|11\right\rangle_L - \left|10\right\rangle_L) + \left|G_2^2\right\rangle (\left|01\right\rangle_L + \left|00\right\rangle_L - \left|11\right\rangle_L + \left|10\right\rangle_L))\\
&\xrightarrow{\mathbf{G}_2^2} \frac{1}{2}\left|G_2^2\right\rangle (\left|01\right\rangle_L + \left|00\right\rangle_L - \left|11\right\rangle_L + \left|10\right\rangle_L)
\end{aligned}
$$

3. $(Z \otimes I)(\mathrm{CZ})(H^{\otimes 2})\left|00\right\rangle_L$

$$\xrightarrow{I,T^{2\to 1}} -\frac{1}{2}\left|G_2^1\right\rangle(\left|01\right\rangle_L + \left|00\right\rangle_L + \left|11\right\rangle_L - \left|10\right\rangle_L) \tag{50}$$
$$\equiv \frac{1}{2}\left|G_2^1\right\rangle(\left|01\right\rangle_L + \left|00\right\rangle_L + \left|11\right\rangle_L - \left|10\right\rangle_L) \quad \text{(global phase)}$$

4. $(H^{\otimes 2})(Z \otimes I)(\mathrm{CZ})(H^{\otimes 2})\left|00\right\rangle_L$

$$\xrightarrow{I,H_{12}^1} \frac{1}{2\sqrt{2}}(\left|G_1^1\right\rangle(\left|11\right\rangle_L + \left|10\right\rangle_L + \left|01\right\rangle_L - \left|00\right\rangle_L) + \left|G_2^1\right\rangle(\left|01\right\rangle_L + \left|00\right\rangle_L - \left|11\right\rangle_L + \left|10\right\rangle_L)) \tag{51}$$
$$\xrightarrow{H_{12}^1} \frac{1}{4}((\left|G_1^1\right\rangle + \left|G_2^1\right\rangle)(\left|11\right\rangle_L + \left|10\right\rangle_L + \left|01\right\rangle_L - \left|00\right\rangle_L) + (\left|G_1^1\right\rangle - \left|G_2^1\right\rangle)(\left|01\right\rangle_L + \left|00\right\rangle_L - \left|11\right\rangle_L + \left|10\right\rangle_L))$$
$$= \frac{1}{2}(\left|G_1^1\right\rangle(\left|10\right\rangle_L + \left|01\right\rangle_L) + \left|G_2^1\right\rangle(\left|11\right\rangle_L - \left|00\right\rangle_L))$$
$$\xrightarrow{\mathbf{G}_1^1} \frac{1}{\sqrt{2}}(\left|G_1^1\right\rangle(\left|10\right\rangle_L + \left|01\right\rangle_L))$$
$$\xrightarrow{T^{1\to 2}} \frac{1}{\sqrt{2}}(\left|G_1^2\right\rangle(\left|10\right\rangle_L + \left|01\right\rangle_L))$$
$$\xrightarrow{I,H_{12}^2} \frac{1}{2}(\left|G_1^2\right\rangle(\left|11\right\rangle_L + \left|00\right\rangle_L) - \left|G_2^2\right\rangle(\left|10\right\rangle_L - \left|01\right\rangle_L))$$
$$\xrightarrow{H_{12}^2} \frac{1}{2\sqrt{2}}((\left|G_1^2\right\rangle + \left|G_2^2\right\rangle)(\left|11\right\rangle_L + \left|00\right\rangle_L) - (\left|G_1^2\right\rangle - \left|G_2^2\right\rangle)(\left|10\right\rangle_L - \left|01\right\rangle_L))$$
$$= \frac{1}{2\sqrt{2}}(\left|G_1^2\right\rangle(\left|11\right\rangle_L + \left|00\right\rangle_L - \left|10\right\rangle_L + \left|01\right\rangle_L) + \left|G_2^2\right\rangle(\left|11\right\rangle_L + \left|00\right\rangle_L + \left|10\right\rangle_L - \left|01\right\rangle_L))$$
$$\xrightarrow{\mathbf{G}_2^2} \frac{1}{2}(\left|G_2^2\right\rangle(\left|11\right\rangle_L + \left|00\right\rangle_L + \left|10\right\rangle_L - \left|01\right\rangle_L))$$

5. $(\text{CZ})(H^{\otimes 2})(Z \otimes I)(\text{CZ})(H^{\otimes 2})\left|00\right\rangle_L$

$$\begin{aligned}
&\xrightarrow{2\to 1} \frac{1}{2}(\left|G_2^1\right\rangle(\left|11\right\rangle_L + \left|00\right\rangle_L + \left|10\right\rangle_L - \left|01\right\rangle_L)) \qquad (52)\\
&\xrightarrow{I,H_{02}^1} \frac{1}{2\sqrt{2}}(\left|G_0^1\right\rangle(\left|11\right\rangle_L + \left|00\right\rangle_L + \left|10\right\rangle_L - \left|01\right\rangle_L) + \left|G_2^1\right\rangle(-\left|11\right\rangle_L + \left|00\right\rangle_L - \left|10\right\rangle_L - \left|01\right\rangle_L))\\
&\xrightarrow{H_{02}^1} \frac{1}{4}((\left|G_0^1\right\rangle + \left|G_2^1\right\rangle)(\left|11\right\rangle_L + \left|00\right\rangle_L + \left|10\right\rangle_L - \left|01\right\rangle_L) + (\left|G_0^1\right\rangle - \left|G_2^1\right\rangle)(-\left|11\right\rangle_L + \left|00\right\rangle_L - \left|10\right\rangle_L - \left|01\right\rangle_L))\\
&= \frac{1}{2}(\left|G_0^1\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L) + \left|G_2^1\right\rangle(\left|11\right\rangle_L + \left|10\right\rangle_L))\\
&\xrightarrow{I,T^{1\to 2}} \frac{1}{2}(\left|G_0^2\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L) - \left|G_2^2\right\rangle(-\left|11\right\rangle_L + \left|10\right\rangle_L))\\
&\xrightarrow{H_{02}^2} \frac{1}{2\sqrt{2}}((\left|G_0^2\right\rangle + \left|G_2^2\right\rangle)(\left|00\right\rangle_L - \left|01\right\rangle_L) - (\left|G_0^2\right\rangle - \left|G_2^2\right\rangle)(-\left|11\right\rangle_L + \left|10\right\rangle_L))\\
&= \frac{1}{2\sqrt{2}}(\left|G_0^2\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L + \left|11\right\rangle_L - \left|10\right\rangle_L) + \left|G_2^2\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L - \left|11\right\rangle_L + \left|10\right\rangle_L))\\
&\xrightarrow{\mathbf{G}_2^2} \frac{1}{2}(\left|G_2^2\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L - \left|11\right\rangle_L + \left|10\right\rangle_L))
\end{aligned}$$

6. $(H^{\otimes 2})(\text{CZ})(H^{\otimes 2})(Z \otimes I)(\text{CZ})(H^{\otimes 2})\left|00\right\rangle_L$

$$\begin{aligned}
&\xrightarrow{T^{2\to 1}} \frac{1}{2}(\left|G_2^1\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L - \left|11\right\rangle_L + \left|10\right\rangle_L)) \qquad (53)\\
&\xrightarrow{I,H_{12}^1} \frac{1}{2\sqrt{2}}(\left|G_1^1\right\rangle(\left|10\right\rangle_L - \left|11\right\rangle_L - \left|01\right\rangle_L + \left|00\right\rangle_L) + \left|G_2^1\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L + \left|11\right\rangle_L - \left|10\right\rangle_L))\\
&\xrightarrow{H_{12}^1} \frac{1}{4}((\left|G_1^1\right\rangle + \left|G_2^1\right\rangle)(\left|10\right\rangle_L - \left|11\right\rangle_L - \left|01\right\rangle_L + \left|00\right\rangle_L) + (\left|G_1^1\right\rangle - \left|G_2^1\right\rangle)(\left|00\right\rangle_L - \left|01\right\rangle_L + \left|11\right\rangle_L - \left|10\right\rangle_L))\\
&= \frac{1}{2}(\left|G_1^1\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L) + \left|G_2^1\right\rangle(\left|10\right\rangle_L - \left|11\right\rangle_L))\\
&\xrightarrow{\mathbf{G}_1^1} \frac{1}{\sqrt{2}}(\left|G_1^1\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L))\\
&\xrightarrow{T^{1\to 2}} \frac{1}{\sqrt{2}}(\left|G_1^2\right\rangle(\left|00\right\rangle_L - \left|01\right\rangle_L))\\
&\xrightarrow{I,H_{12}^2} \frac{1}{2}(\left|G_1^2\right\rangle(\left|01\right\rangle_L - \left|00\right\rangle_L) - \left|G_2^2\right\rangle(\left|00\right\rangle_L + \left|01\right\rangle_L))\\
&\xrightarrow{H_{12}^2} \frac{1}{2\sqrt{2}}((\left|G_1^2\right\rangle + \left|G_2^2\right\rangle)(\left|01\right\rangle_L - \left|00\right\rangle_L) - (\left|G_1^2\right\rangle - \left|G_2^2\right\rangle)(\left|00\right\rangle_L + \left|01\right\rangle_L))\\
&= \frac{1}{\sqrt{2}}(\left|G_1^2\right\rangle(-\left|00\right\rangle_L) + \left|G_2^2\right\rangle(\left|01\right\rangle_L))\\
&\xrightarrow{\mathbf{G}_2^2} \left|G_2^2\right\rangle \otimes \left|01\right\rangle_L
\end{aligned}$$

It should be noted that this is just one way of achieving it, there are others if we change the way some Hadamard gates act, which would then change the needed measurements on the control system. In the following we give the full necessary application of gates/measurements (47) to achieve the desired result:

$$
\begin{aligned}
\mathbf{Gates}(\left|G_1^1\right\rangle \otimes |00\rangle_L) = &(\mathbf{G}_2^2)(H_{12}^2)(\xrightarrow{I})(H_{12}^2)(T^{1\to 2})(\mathbf{G}_1^1)(H_{12}^1)(\xrightarrow{I})(H_{12}^1) \\
&(T^{2\to 2})(\mathbf{G}_2^2)(H_{02}^1)(\xrightarrow{I})(T^{1\to 2})(H_{02}^1)(\xrightarrow{I})(H_{02}^1)(T^{2\to 1}) \\
&(G_2^2)(H_{12}^2)(\xrightarrow{I})(H_{12}^2)(T^{1\to 2})(\mathbf{G}_1^1)(H_{12}^1)(\xrightarrow{I})(H_{12}^1) \\
&(\xrightarrow{I})(T^{2\to 1})(\mathbf{G}_\mathbf{2}^\mathbf{2})(H_{02}^2)(\xrightarrow{I})(T^{1\to 2})(H_{02}^1)(\xrightarrow{I})(H_{02}^1) \\
&(T^{2\to 1})(\mathbf{G}_2^2)(H_{12}^2)(\xrightarrow{I})(H_{21}^2)(T^{1\to 2})(\mathbf{G}_2^1)(H_{12}^1)(\xrightarrow{I}) \\
&(H_{12}^1)(\left|G_1^1\right\rangle \otimes |00\rangle_L) = \left|G_2^2\right\rangle \otimes |01\rangle_L
\end{aligned} \tag{54}
$$

A measurement in the target basis will therefore yield the logical result $|01\rangle_L$. After applying an X gate on the first qubit we will get the desired result $|11\rangle_L$. Because we got the result in just one query to the oracle our algorithm has successfully shown a quantum advantage using quantum controlled classical computation. Having already shown that universal quantum computation is possible in this framework, this comes as no surprise. Next however we are going to couple this mechanism with the Wigner's friend scenario.

4 Implementing quantum computation in Wigner's friend scenario

In the previous section we showed that universal quantum computation is possible using quantum controlled classical operations. With this in our toolbox, we will now try to figure out a way of combining this with the Wigner's friend scenario. Our desired result would be to implement a situation in which Wigner has control over the quantum mechanical control while our friend has access to the classical target system. The friend would then play a sort of middle-man between Wigner's commands and the input into the "classical part" of the computer, i.e. he would represent the interaction process. After introducing the basic idea we will then try to improve it by finding a similar framework for universal quantum computation that doesn't work probabilistically but in a deterministic way, meaning we could discard the final measurement process.

4.1 Computational framework

We are first looking at the basic scenario. Our three subsystems are Wigner's message M to the friend F inside an isolated laboratory with some environment E (combined as FE) and a classical computer C. The crucial part is that the entire system FEC can be described quantum mechanically from Wigner's point of view while the friend describes everything to be classical.

Wigner is able to write anything he wants on the message, it can even consist of statements in a superposition. Using the message, he will send instructions Π_i (only permutations) in a superposition to F. These correspond to the elements of our set $\mathcal{G}$. In incremental time steps, F will then read the message and find a specific instruction written on it, e.g. Π_1. He will push a button on the computer which corresponds to the execution of that specific permutation on the computer's bits. From Wigner's point of view the friend coupled to the superposition of all permutations that have been written on the message and therefore also executed every specified permutation on the computer. More specifically the entire laboratory with it's environment has coupled to the message. At the end of such an instruction process, F returns the message to W who then acts with a unitary U on the whole system and reverts F to his decoupled starting position. Wigner will then check the message (i.e. measuring G) and, if the measurement corresponds to the wanted action on the bits, send the message again with different instructions that continue e.g. an algorithm on the computer. This process is depicted in figure 3.

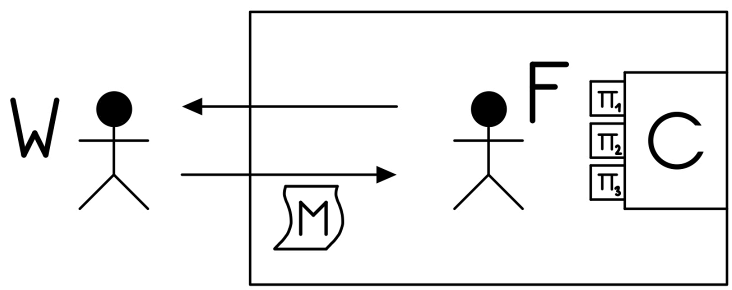


Figure 3: Wigner's friend scenario including a communication channel depicted as a message and a classical computer with an exemplary three different permutations Π_i inside some laboratory

The goal of Wigner is now to treat the computer in the laboratory as a quantum computer and execute a quantum algorithm on it. As an example he can execute our specific Grover's algorithm and find the correct solution in just one query to an oracle. This stands in direct contradiction to the friends point of view considering he only executed classical permutations on the computer.

We describe the initial state of FE with $|\alpha\rangle_{FE} := |\text{State of } FE \text{ at t=0}\rangle$ and that of the computer with $|x\rangle_C$ where x represents a string of bits. Wigner's message consists of a superposition of permutations Π_i, i.e. $\sum_i c_i |\Pi_i\rangle_M$ where the c_i are normalization constants.

As soon as the friend couples to the message, his state will undergo the change $|\alpha\rangle_{FE} \to |\phi_i\rangle_{FE}$ where ϕ_i describes F coupling to the permutation Π_i (from here on we treat FE verbally as just the friend). The friend then executes the permutation on the computer which induces a change in its state of $|x\rangle_C \to |\Pi_i(x)\rangle_C$ where $\Pi_i(x)$ describes the specific permutation Π_i acting on the string x. Ignoring normalization, our action of U therefore needs to be

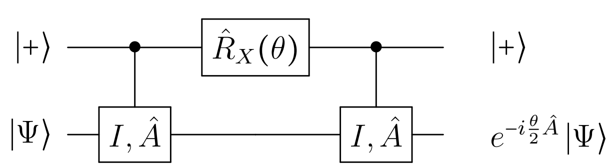
$$U : (\sum_i |\Pi_i\rangle_M) \otimes |\phi_i\rangle_{FE} \to (\sum_i |\Pi_i\rangle_M) \otimes |\alpha\rangle \, . \tag{55}$$

This unitary is of importance because it reverses the state of the friend to his initial one, decoupling him from the rest of the system. A measurement of the message by Wigner would then simply imply a setting of the state of the computer. The following describes the whole process beginning at some time $t = 0$:

$$
\begin{aligned}
&t = 0 : (\sum_i |\Pi_i\rangle_M) \otimes |\alpha\rangle_{FE} \otimes |x\rangle_C \\
&\qquad\qquad\downarrow \\
&t = 1 : \text{W sends M to F} \\
&\qquad\qquad\downarrow \\
&t = 2 : (\sum_i |\Pi_i\rangle_M) \otimes |\phi_i\rangle_{FE} \otimes |x\rangle_C \\
&\qquad\qquad\downarrow \\
&t = 3 : (\sum_i |\Pi_i\rangle_M) \otimes |\phi_i\rangle_{FE} \otimes |\Pi_i(x)\rangle_C \\
&\qquad\qquad\downarrow \\
&t = 4 : F \text{ sends } M \text{ to } W \\
&\qquad\qquad\downarrow \\
&t = 5 : \quad \xrightarrow{U} (\sum_i |\Pi_i\rangle_M) \otimes |\alpha\rangle_{FE} \otimes |\Pi_i(x)\rangle_C \\
&\qquad\qquad\downarrow \\
&t = 6 : \text{W does measurement on M}
\end{aligned}
\tag{56}
$$

The last step can be described by Wigner acting with an exemplary projector $|\Pi_j\rangle\langle\Pi_j|$ on the message M. Such a measurement will collapse the state of the message and leave the now decoupled computer in the state $|\Pi_j(x)\rangle_M$.

This is however just a general description of the process and if we want to implement an algorithm in this way, we will still have to measure the message after each implemented instruction/gate. In order to implement deterministic computation we will use a quantum circuit introduced in [15]. The quantum circuit we need consists of a rotation around the x-axis on the Bloch sphere on the control side and controlled $\hat{A}$ operations on the target. The circuit only works for $\hat{A}^2 = I$ and can be seen here:



If we describe the controlled gates as $\hat{C}_A$ we get the following mathematical representation:

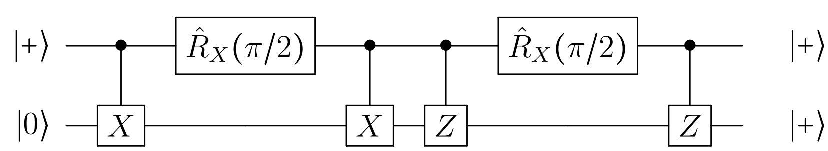
$$(\hat{C}_A \hat{R}_X(\theta) \hat{C}_A) |+\rangle \otimes |\Psi\rangle = (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \hat{A})(\hat{R}_X(\theta) \otimes I)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \hat{A}) |+\rangle \otimes |\Psi\rangle \tag{57}$$

Using equation 24 and the fact that $\hat{X}\left|+\right\rangle = \left|+\right\rangle$, this works out to

$$
\begin{aligned}
(\hat{C}_A \hat{R}_X(\theta) \hat{C}_A)\left|+\right\rangle \otimes \left|\Psi\right\rangle &= \begin{pmatrix} I & 0 \\ 0 & \hat{A} \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2})I & -i\sin(\frac{\theta}{2})I \\ -i\sin(\frac{\theta}{2})I & \cos(\frac{\theta}{2})I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \hat{A} \end{pmatrix} \left|+\right\rangle \otimes \left|\Psi\right\rangle \\
&= \begin{pmatrix} I & 0 \\ 0 & \hat{A} \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2})I & -i\sin(\frac{\theta}{2})\hat{A} \\ -i\sin(\frac{\theta}{2})I & \cos(\frac{\theta}{2})\hat{A} \end{pmatrix} \left|+\right\rangle \otimes \left|\Psi\right\rangle \\
&= \begin{pmatrix} \cos(\frac{\theta}{2})I & -i\sin(\frac{\theta}{2})\hat{A} \\ -i\sin(\frac{\theta}{2})\hat{A} & \cos(\frac{\theta}{2})I \end{pmatrix} \left|+\right\rangle \otimes \left|\Psi\right\rangle \quad (\hat{A}^2 = I) \\
&= (\cos\left(\frac{\theta}{2}\right) I \otimes I - i\sin\left(\frac{\theta}{2}\right) \hat{X} \otimes \hat{A}) \left|+\right\rangle \otimes \left|\Psi\right\rangle \\
&= \left|+\right\rangle \otimes (\cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) \hat{A})) \left|\Psi\right\rangle \\
&= \left|+\right\rangle \otimes e^{-i\frac{\theta}{2}\hat{A}} \left|\Psi\right\rangle .
\end{aligned}
\tag{58}
$$

We can now regard $\left|\Psi\right\rangle$ as a qubit and let Wigner have access to the ancilla state $\left|+\right\rangle$ which will take the role of the message in the further implementation. He also has the possibility of controlling the $\left|+\right\rangle$ state by acting with the rotation gates. Wigner is therefore able to implement any operation $e^{-i\frac{\theta}{2}\hat{A}}$ on the qubit. Because $X^2 = Z^2 = I$ he is especially able to implement any rotation on the Bloch sphere.

An exemplary circuit that applies a Hadamard gate ($H = \frac{1}{\sqrt{2}}(X + Z)$) on a qubit $\left|\Psi\right\rangle = \left|0\right\rangle$ can therefore be achieved by choosing $\theta = \frac{\pi}{2}$ and acting successively with the X and Z gate on the qubit:

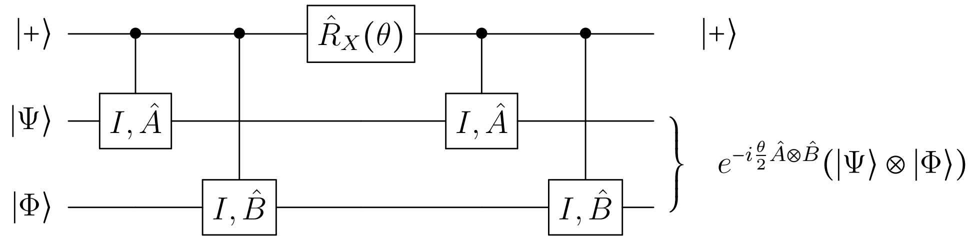


Mathematically this corresponds to

$$
\begin{aligned}
\left|+\right\rangle \otimes e^{-i\frac{\pi}{4}\hat{Z}} e^{-i\frac{\pi}{4}\hat{X}} \left|0\right\rangle &= \left|+\right\rangle \otimes \frac{1}{2}(I - i\hat{Z})(I - i\hat{X})\left|0\right\rangle \\
&= \left|+\right\rangle \otimes \frac{1}{2}(I - i\hat{Z})(\left|0\right\rangle - i\left|1\right\rangle) \\
&= \left|+\right\rangle \otimes \frac{1}{2}(\left|0\right\rangle - i\left|1\right\rangle - i\left|0\right\rangle + \left|1\right\rangle) \\
&= \left|+\right\rangle \otimes \frac{1-i}{2}(\left|0\right\rangle + \left|1\right\rangle)
\end{aligned}
\tag{59}
$$

We can see that for $\frac{1}{2}(1-i)$ there exists a global phase factor $e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$ such that $\frac{1}{2}(1-i)\frac{1}{\sqrt{2}}(1+i) = \frac{1}{2\sqrt{2}}(1-i)(1+i) = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$. Up to a global phase factor our result therefore corresponds to $\frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle) = H\left|0\right\rangle$.

This scheme can now rather easily be expanded, for our purposes we limit it to two qubits $\left|\Psi\right\rangle$ and $\left|\Phi\right\rangle$. For $\hat{A}^2 = \hat{B}^2 = I$, the following quantum circuit describes the two qubit process:



This is also confirmed by a quick calculation, with $I^{\otimes 2} = I \otimes I$ and

$$\begin{aligned}\hat{R}_X(\theta) &= (\hat{R}_X(\theta)\otimes I^{\otimes 2}),\\ \hat{C}_A &= (|0\rangle\langle 0|\otimes I^{\otimes 2} + |1\rangle\langle 1|\otimes \hat{A}\otimes I),\\ \hat{C}_B &= (|0\rangle\langle 0|\otimes I^{\otimes 2} + |1\rangle\langle 1|\otimes I\otimes \hat{B}),\end{aligned} \tag{60}$$

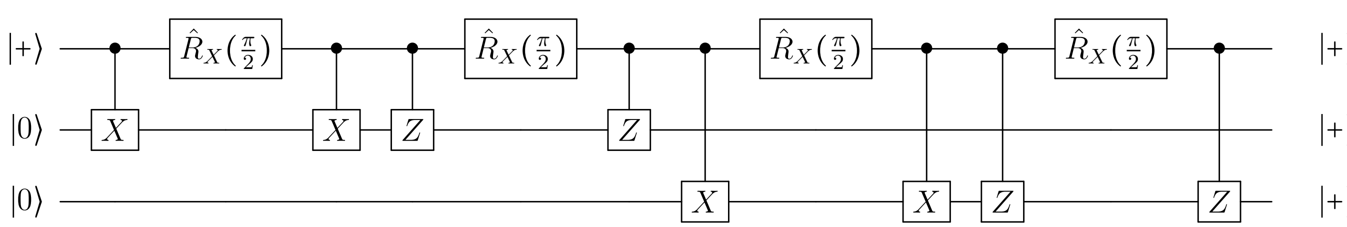
we have

$$(\hat{C}_B\hat{C}_A\hat{R}_X(\theta)\hat{C}_B\hat{C}_A)\,|+\rangle\otimes|\Psi\rangle\otimes|\Phi\rangle = \tag{61}$$

$$\begin{aligned}
&= \begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & \hat{A}\otimes I\end{pmatrix}\begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & I\otimes\hat{B}\end{pmatrix}\begin{pmatrix} \cos(\frac{\theta}{2})I^{\otimes 2} & -i\sin(\frac{\theta}{2})I^{\otimes 2} \\ -i\sin(\frac{\theta}{2})I^{\otimes 2} & \cos(\frac{\theta}{2})I^{\otimes 2}\end{pmatrix}\\
&\quad \begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & \hat{A}\otimes I\end{pmatrix}\begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & I\otimes\hat{B}\end{pmatrix}|+\rangle\otimes|\Psi\rangle\otimes|\Phi\rangle\\
&= \begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & \hat{A}\otimes I\end{pmatrix}\begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & I\otimes\hat{B}\end{pmatrix}\begin{pmatrix} \cos(\frac{\theta}{2})I^{\otimes 2} & -i\sin(\frac{\theta}{2})I^{\otimes 2} \\ -i\sin(\frac{\theta}{2})I^{\otimes 2} & \cos(\frac{\theta}{2})I^{\otimes 2}\end{pmatrix}\begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & \hat{A}\otimes\hat{B}\end{pmatrix}|+\rangle\otimes|\Psi\rangle\otimes|\Phi\rangle\\
&= \begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & \hat{A}\otimes I\end{pmatrix}\begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & I\otimes\hat{B}\end{pmatrix}\begin{pmatrix} \cos(\frac{\theta}{2})I^{\otimes 2} & -i\sin(\frac{\theta}{2})\hat{A}\otimes\hat{B} \\ -i\sin(\frac{\theta}{2})I^{\otimes 2} & \cos(\frac{\theta}{2})\hat{A}\otimes\hat{B}\end{pmatrix}|+\rangle\otimes|\Psi\rangle\otimes|\Phi\rangle\\
&\overset{(\hat{B}^2=I)}{=} \begin{pmatrix} I^{\otimes 2} & 0 \\ 0 & \hat{A}\otimes I\end{pmatrix}\begin{pmatrix} \cos(\frac{\theta}{2})I^{\otimes 2} & -i\sin(\frac{\theta}{2})\hat{A}\otimes\hat{B} \\ -i\sin(\frac{\theta}{2})I\otimes\hat{B} & \cos(\frac{\theta}{2})\hat{A}\otimes I\end{pmatrix}|+\rangle\otimes|\Psi\rangle\otimes|\Phi\rangle\\
&\overset{(\hat{A}^2=I)}{=} \begin{pmatrix} \cos(\frac{\theta}{2})I^{\otimes 2} & -i\sin(\frac{\theta}{2})\hat{A}\otimes\hat{B} \\ -i\sin(\frac{\theta}{2})\hat{A}\otimes\hat{B} & \cos(\frac{\theta}{2})I^{\otimes 2}\end{pmatrix}|+\rangle\otimes|\Psi\rangle\otimes|\Phi\rangle\\
&= (\cos\left(\frac{\theta}{2}\right)I\otimes I^{\otimes 2} - i\sin\left(\frac{\theta}{2}\right)\hat{X}\otimes\hat{A}\otimes\hat{B})\,|+\rangle\otimes|\Psi\rangle\otimes|\Phi\rangle\\
&= |+\rangle\otimes e^{-i\frac{\theta}{2}\hat{A}\otimes\hat{B}}(|\Psi\rangle\otimes|\Phi\rangle),
\end{aligned}$$

therefore confirming our 2 qubit circuit.

If we wish to implement a Hadamard transformation on both qubits we will have to do so one after the other, thus avoiding any possible coupling mechanism:

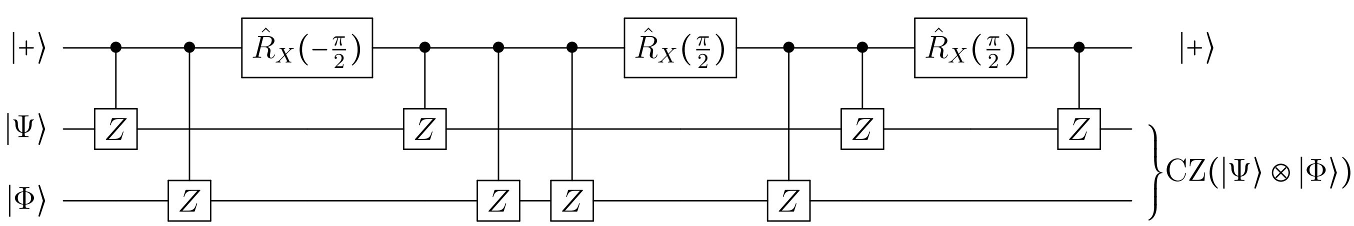


Again, mathematically this corresponds to

$$
\begin{aligned}
&|+\rangle \otimes e^{-i\frac{\pi}{4}(I\otimes\hat{Z})}e^{-i\frac{\pi}{4}(I\otimes\hat{X})}e^{-i\frac{\pi}{4}(\hat{Z}\otimes I)}e^{-i\frac{\pi}{4}(\hat{X}\otimes I)}|00\rangle = \qquad (62)\\
&= |+\rangle \otimes \frac{1}{4}(I^{\otimes 2} - i(I\otimes\hat{Z}))(I^{\otimes 2} - i(I\otimes\hat{X}))(I^{\otimes 2} - i(\hat{Z}\otimes I))(I^{\otimes 2} - i(\hat{X}\otimes I))|00\rangle\\
&= |+\rangle \otimes \frac{1}{4}(I^{\otimes 2} - i(I\otimes\hat{Z}))(I^{\otimes 2} - i(I\otimes\hat{X}))(I^{\otimes 2} - i(\hat{Z}\otimes I))(|00\rangle - i|10\rangle)\\
&= |+\rangle \otimes \frac{1}{4}(I^{\otimes 2} - i(I\otimes\hat{Z}))(I^{\otimes 2} - i(I\otimes\hat{X}))(|00\rangle - i|10\rangle - i|00\rangle + |10\rangle)\\
&= |+\rangle \otimes \frac{1}{4}(I^{\otimes 2} - i(I\otimes\hat{Z}))(I^{\otimes 2} - i(I\otimes\hat{X}))(1-i)(|00\rangle + |10\rangle)\\
&= |+\rangle \otimes \frac{1}{4}(I^{\otimes 2} - i(I\otimes\hat{Z}))(1-i)(|00\rangle + |10\rangle - i|01\rangle - i|11\rangle)\\
&= |+\rangle \otimes \frac{1}{4}(1-i)(|00\rangle + |10\rangle - i|01\rangle - i|11\rangle - i|00\rangle - i|10\rangle + |01\rangle + |11\rangle)\\
&= |+\rangle \otimes \frac{1}{4}(1-i)((1-i)|00\rangle + (1-i)|10\rangle + (1-i)|01\rangle + (1-i)|11\rangle)\\
&= |+\rangle \otimes -\frac{i}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle),
\end{aligned}
$$

which equals $(H\otimes H)|00\rangle$ up to a global phase factor $e^{i\frac{\pi}{2}} = i$.

Thus far, we have shown the implementation of rotations around any axis of the Bloch sphere which allows us for example a Hadamard transformation. In order to achieve universal quantum computation we still want to take a look at the CNOT and T-phase gates. First, a CNOT gate will be implemented via a number of rotations around the z-axis of the Bloch sphere that sums to a $CZ = \frac{1}{2}(I\otimes I + I\otimes Z + Z\otimes I - Z\otimes Z)$ gate, i.e. for some bipartite state $|\Psi\rangle\otimes|\Phi\rangle$:

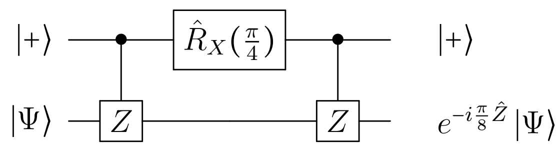


Mathematically we have

$$
\begin{aligned}
&|+\rangle \otimes e^{i\frac{\pi}{4}(\hat{Z}\otimes 1)}e^{i\frac{\pi}{4}(I\otimes\hat{Z})}e^{-i\frac{\pi}{4}(\hat{Z}\otimes\hat{Z})}|\Psi\rangle\otimes|\Phi\rangle = \qquad (63)\\
&= |+\rangle \otimes \frac{1}{2\sqrt{2}}(I^{\otimes 2} + i(\hat{Z}\otimes I))(I^{\otimes 2} + i(I\otimes\hat{Z}))(I^{\otimes 2} - i(\hat{Z}\otimes\hat{Z}))|\Psi\rangle\otimes|\Phi\rangle\\
&= |+\rangle \otimes \frac{1}{2\sqrt{2}}(I^{\otimes 2} + i(\hat{Z}\otimes I))(I^{\otimes 2} - i(\hat{Z}\otimes\hat{Z}) + i(I\otimes\hat{Z}) + (\hat{Z}\otimes I))|\Psi\rangle\otimes|\Phi\rangle\\
&= |+\rangle \otimes \frac{1}{2\sqrt{2}}(I^{\otimes 2} - i(\hat{Z}\otimes\hat{Z}) + i(I\otimes\hat{Z}) + (\hat{Z}\otimes I) + i(\hat{Z}\otimes I) + (I\otimes\hat{Z}) - (\hat{Z}\otimes\hat{Z}) + iI^{\otimes 2})|\Psi\rangle\otimes|\Phi\rangle\\
&= |+\rangle \otimes \frac{(1+i)}{2\sqrt{2}}(I^{\otimes 2} + I\otimes\hat{Z} + \hat{Z}\otimes I - \hat{Z}\otimes\hat{Z})|\Psi\rangle\otimes|\Phi\rangle,
\end{aligned}
$$

which equals the action of CZ up to a global phase factor $e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$. Now we only need to consider the fact that CNOT $= (I \otimes H)(CZ)(I \otimes H)$ and our already established possibility of implementing a Hadamard gate.

Second, a phase gate like T will be implemented via a rotation around the z-axis of the Bloch sphere by $\theta = \frac{\pi}{4}$, i.e.:



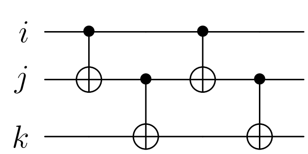
This works up to a global phase factor,

$$
\begin{aligned}
e^{-i\frac{\pi}{8}\hat{Z}} |\Psi\rangle &= \begin{pmatrix} \cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right) & 0 \\ 0 & \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right) \end{pmatrix} |\Psi\rangle \\
&= e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{8}+i\frac{\pi}{8}} \end{pmatrix} |\Psi\rangle \\
&= e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} |\Psi\rangle = e^{-i\frac{\pi}{8}} T |\Psi\rangle ,
\end{aligned} \tag{64}
$$

which can be discarded. We therefore have all the necessary gates available and can successfully implement universal quantum computation.

This framework gives us a deterministic way to implement quantum gates acting on the qubits, however in order to achieve our goal we will still need to find a mechanism that only allows classical gates to be used and explicitly find an implementation of the unitary U that "reverses" the friend. First we introduce the friend to our situation. In order to find a basic description we are going to simplify the friend into the finite system of a qubit. This will allow us to easily implement a version of U at the price of having to accept some physical problems addressed in section 4.2.

The qubit describing the friend couples first to Wigner's message (i.e. the ancilla $|+\rangle$) by a control-NOT gate, therefore in an abstract sense modeling the friends measurement process from Wigner's point of view (F couples to/measures 0 or 1). Depending on the state of the friend, another controlled gate dictates if something happens to the qubit or not. In the end we want the friend to be decoupled again. A clue of how to describe this comes from the concatenation of CNOT gates. Consider the following circuit:

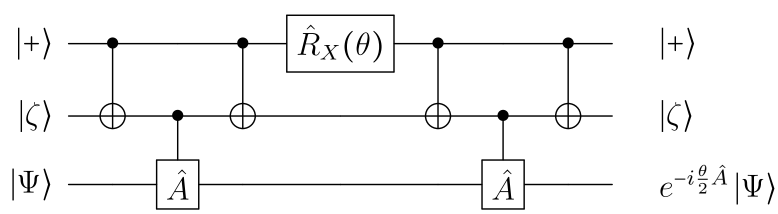


We can quickly create a truth table for all possible input values, see 3. If we define

ijk	C_{12}	C_{23}	C_{12}	C_{23}
000	000	000	000	000
001	001	001	001	001
010	010	011	011	010
100	110	111	101	101
110	100	100	110	111
101	111	110	100	100
011	011	010	010	011
111	101	101	111	110

Table 3: Truth table for inputs ijk, A column C_{ab} corresponds to a CNOT gate having acted between the a-th and b-th qubit

C_{ij} :=(CNOT gate between qubits i and j), we see that the middle qubit can be "skipped", i.e. $C_{23}C_{12}C_{23}C_{12} = C_{13}$. This would in theory model an interaction between 2 qubits with the middle one remaining separable in the end. If the goal is just to separate the second qubit after an interaction with the third one we can even omit the last CNOT gate. This leads us to a general framework which succeeds in doing computation on a qubit while also having a middle layer qubit that can represent the friends measurement/coupling process. The quantum circuit works again for any $\hat{A}^2 = I$ and is depicted here (for $\zeta \in \{0,1\}$, representing the two orthogonal states of $|\text{F measured } 0\rangle$ and $|\text{F measured } 1\rangle$):



Separating the two cases $\zeta \in \{0,1\}$ and defining the controlled-$\hat{A}$ operation as $C_{\hat{A}}$ we get

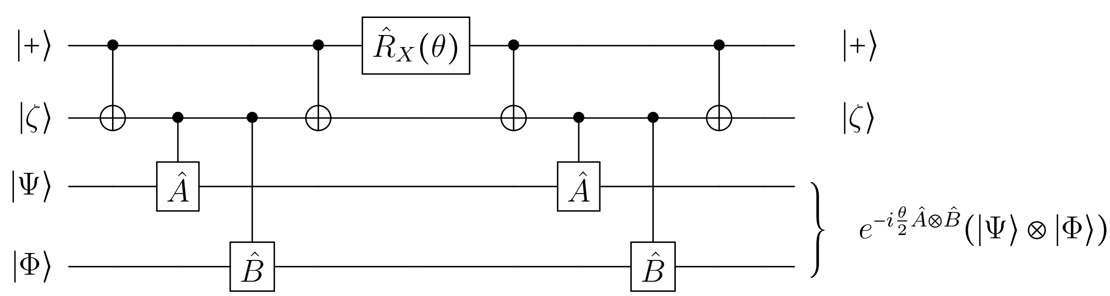
$$\begin{aligned}
|+\rangle \otimes |0\rangle \otimes |\Psi\rangle &\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|\Psi\rangle \xrightarrow{C_{\hat{A}}} \frac{1}{\sqrt{2}}(|00\rangle|\Psi\rangle + |11\rangle\hat{A}|\Psi\rangle) \qquad (65)\\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(|00\rangle|\Psi\rangle + |10\rangle\hat{A}|\Psi\rangle)\\
&\xrightarrow{\hat{R}_X(\theta)} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|00\rangle|\Psi\rangle + |10\rangle\hat{A}|\Psi\rangle) - i\sin\left(\frac{\theta}{2}\right)(|10\rangle|\Psi\rangle + |00\rangle\hat{A}|\Psi\rangle))\\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|00\rangle|\Psi\rangle + |11\rangle\hat{A}|\Psi\rangle) - i\sin\left(\frac{\theta}{2}\right)(|11\rangle|\Psi\rangle + |00\rangle\hat{A}|\Psi\rangle))\\
&\xrightarrow{C_{\hat{A}}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|00\rangle|\Psi\rangle + |11\rangle|\Psi\rangle) - i\sin\left(\frac{\theta}{2}\right)(|11\rangle\hat{A}|\Psi\rangle + |00\rangle\hat{A}|\Psi\rangle))\\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|00\rangle|\Psi\rangle + |10\rangle|\Psi\rangle) - i\sin\left(\frac{\theta}{2}\right)(|10\rangle\hat{A}|\Psi\rangle + |00\rangle\hat{A}|\Psi\rangle))\\
&= \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|0\rangle + |1\rangle)|0\rangle|\Psi\rangle - i\sin\left(\frac{\theta}{2}\right)(|0\rangle + |1\rangle)|0\rangle\hat{A}|\Psi\rangle)\\
&= |+\rangle \otimes |0\rangle \otimes e^{-i\theta\hat{A}}|\Psi\rangle
\end{aligned}$$

and

$$
\begin{aligned}
|+\rangle \otimes |1\rangle \otimes |\Psi\rangle &\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)|\Psi\rangle \xrightarrow{C_{\hat{A}}} \frac{1}{\sqrt{2}}(|01\rangle \hat{A}|\Psi\rangle + |10\rangle|\Psi\rangle) \\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(|01\rangle \hat{A}|\Psi\rangle + |11\rangle|\Psi\rangle) \\
&\xrightarrow{\hat{R}_X(\theta)} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|01\rangle \hat{A}|\Psi\rangle + |11\rangle|\Psi\rangle) - i\sin\left(\frac{\theta}{2}\right)(|11\rangle \hat{A}|\Psi\rangle + |01\rangle|\Psi\rangle)) \\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|01\rangle \hat{A}|\Psi\rangle + |10\rangle|\Psi\rangle) - i\sin\left(\frac{\theta}{2}\right)(|10\rangle \hat{A}|\Psi\rangle + |01\rangle|\Psi\rangle)) \\
&\xrightarrow{C_{\hat{A}}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|01\rangle|\Psi\rangle + |10\rangle|\Psi\rangle) - i\sin\left(\frac{\theta}{2}\right)(|10\rangle \hat{A}|\Psi\rangle + |01\rangle \hat{A}|\Psi\rangle)) \\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|01\rangle|\Psi\rangle + |11\rangle|\Psi\rangle) - i\sin\left(\frac{\theta}{2}\right)(|11\rangle \hat{A}|\Psi\rangle + |01\rangle \hat{A}|\Psi\rangle)) \\
&= \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|0\rangle + |1\rangle)|1\rangle|\Psi\rangle - i\sin\left(\frac{\theta}{2}\right)(|0\rangle + |1\rangle)|1\rangle \hat{A}|\Psi\rangle) \\
&= |+\rangle \otimes |1\rangle \otimes e^{-i\theta\hat{A}}|\Psi\rangle
\end{aligned}
\tag{66}
$$

We can therefore see that the first CNOT gate between the ancilla and F corresponds to the "measurement" where F gets coupled to Wigner's message whereas the second CNOT gate corresponds to the unitary U that decouples F from the message. This "measurement" process will further be elaborated upon in the next section.

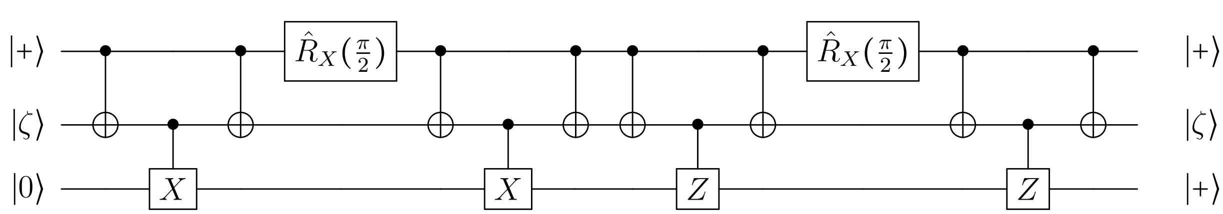
The following circuit corresponds to a 2-qubit case:



Defining $C_{\hat{A},\hat{B}}$ as the action of both controlled-operation gates we get:

$$
\begin{aligned}
|+\rangle \otimes |0\rangle \otimes |\Psi\rangle \otimes |\Phi\rangle &\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\,|\Psi\rangle\,|\Phi\rangle \xrightarrow{C_{\hat{A},\hat{B}}} \frac{1}{\sqrt{2}}(|00\rangle\,|\Psi\rangle + |11\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle) \qquad (67)\\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(|00\rangle\,|\Psi\rangle\,|\Phi\rangle + |10\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle)\\
&\xrightarrow{\hat{R}_X(\theta)} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|00\rangle\,|\Psi\rangle\,|\Phi\rangle + |10\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle)\\
&- i\sin\left(\frac{\theta}{2}\right)(|10\rangle\,|\Psi\rangle\,|\Phi\rangle + |00\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle))\\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|00\rangle\,|\Psi\rangle\,|\Phi\rangle + |11\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle)\\
&- i\sin\left(\frac{\theta}{2}\right)(|11\rangle\,|\Psi\rangle\,|\Phi\rangle + |00\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle))\\
&\xrightarrow{C_{\hat{A},\hat{B}}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|00\rangle\,|\Psi\rangle\,|\Phi\rangle + |11\rangle\,|\Psi\rangle\,|\Phi\rangle)\\
&- i\sin\left(\frac{\theta}{2}\right)(|11\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle + |00\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle))\\
&\xrightarrow{C_{12}} \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|00\rangle\,|\Psi\rangle\,|\Phi\rangle + |10\rangle\,|\Psi\rangle\,|\Phi\rangle)\\
&- i\sin\left(\frac{\theta}{2}\right)(|10\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle + |00\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle))\\
&= \frac{1}{\sqrt{2}}(\cos\left(\frac{\theta}{2}\right)(|0\rangle + |1\rangle)\,|0\rangle\,|\Psi\rangle\,|\Phi\rangle - i\sin\left(\frac{\theta}{2}\right)(|0\rangle + |1\rangle)\,|0\rangle\,(\hat{A}\otimes\hat{B})\,|\Psi\rangle\,|\Phi\rangle)\\
&= |+\rangle \otimes |0\rangle \otimes e^{-i\theta(\hat{A}\otimes\hat{B})}(|\Psi\rangle \otimes |\Phi\rangle).
\end{aligned}
$$

The case for $\zeta = 1$ follows the same pattern as equation 66. As a consequence it is possible to implement any $\hat{A}$ or $\hat{B}$ where $\hat{A}^2 = \hat{B}^2 = I$ on the qubits as long as we choose an angle of rotation θ s.t. $\cos(\frac{\theta}{2}) = 0$. The remaining factor can be seen as a global phase and therefore ignored. It also becomes possible to implement any rotation on the Bloch sphere which guarantees us e.g. the implementation of a Hadamard gate which looks identical to the case of equation 59 (up to the additional layer corresponding to F), e.g. for one qubit we have:



Should we want to implement a Hadamard on both qubits we have to concatenate the gates needed for both qubits which equally corresponds to 63.

In order to do any computation in our framework we still have to show how to implement classical operations. The basic idea goes as follows: We assume the target system to be inside the isolated laboratory with the friend. First Wigner initializes all qubits into their logical basis, i.e. $|\Psi\rangle$ and $|\Phi\rangle$ are already thought to be written in the logical basis $\{|0\rangle_L, |1\rangle_L\}$. Before the experiment even began, Wigner inserted extra mechanisms in the form of boxes connected to the classical computer which respond to the pressing of a button. These boxes correspond to e.g. the operations X and Z and act by implementing the required permutations on our classical qubits. The message entering the laboratory is now actually two messages which consist of the ancilla $|+\rangle$ and a description of which box to press e.g. $|+\rangle_{M_1}$ and $|\text{press Box Z}\rangle_{M_2}$. Again, M_2 is not allowed to be in a superposition. Before the experiment, Wigner told his friend that he should only do the instruction specified in message M_2 if the message M_1 is found to be in the $|1\rangle$ state. Our friend is therefore solely aware that he looked at a message and pressed depending on said message a button which implements (in the point of view of the friend) only classical permutations. Following this procedure we can implement the simplified Grover's algorithm from section 2.3.2 and in just one run of the algorithm get the desired result which our friend will then send to Wigner.

After the experiment has taken place (maybe even many times for good measure), Wigner and his friend reunite. Wigner tells the friend that he achieved a search result in just one run, surpassing the capabilities of a classical computer. This in turn seems impossible to our friend who assumed the entire procedure has taken place using only classical operations, hence the paradoxon.

Now obviously this situation is a vast simplification, especially because we characterize a macroscopic system like the friend as a single qubit. We could also argue that by continuously separating the friend we decouple his memory, therefore he wouldn't even remember any computation he made. In the next section we are going to elaborate further on this problem.

4.2 On the unitary U and it's physical interpretation

We can get a better understanding of the unitary U by treating it in a more physical example. Let's assume that Wigner's ancillary message is implemented using a photon in a superposition of horizontal and vertical polarization, i.e.

$$|\text{Photon}\rangle := \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle). \tag{68}$$

We will describe the friends state as a qubit with the orthogonal states being $|F_H\rangle :=$ (F measured H) and $|F_V\rangle :=$ (F measured V). Describing the process from Wigner's point of view, an interaction between the friend and the photon therefore leads to the state:

$$\frac{1}{\sqrt{2}}(|H\rangle|F_H\rangle + |V\rangle|F_V\rangle) \tag{69}$$

If we now repeat the interaction, then each "branch" of the superposition will stay the same because the photon itself didn't undergo any transformation and therefore a measurement should give the same result. That is not how our process works. In our process the measurement result will induce an action on the observer. Let us assume the friend can be sufficiently represented by a computer that stores it's memory of a measurement result in some register. Our interaction between the photon and the friend is then modelled by assuming a measurement result of $|V\rangle$ induces an action on all the necessary bits representing the memory in a way that switches that memory between $|F_H\rangle$ and $|F_V\rangle$, i.e. F retroactively believes to have measured a different result. Let us further assume that the friend previously obtained a horizontally polarized photon setting him into the initial state $|F_H\rangle$. Wigner then applies a gate that sets the photon into a superposition of polarizations and sends that same photon back to F, this interaction thus leads to:

$$|\text{Photon}\rangle \otimes |F_H\rangle \xrightarrow{I} \frac{1}{\sqrt{2}}(|H\rangle |F_H\rangle + |V\rangle |F_V\rangle) \tag{70}$$

This interaction is perfectly valid, F knows that there might have been a transformation of the photons state in the time he didn't interact with it. It can therefore be treated as a measurement process. The second interaction is where everything becomes tricky, we have:

$$\frac{1}{\sqrt{2}}(|H\rangle |F_H\rangle + |V\rangle |F_V\rangle) \xrightarrow{I} \frac{1}{\sqrt{2}}(|H\rangle |F_H\rangle + |V\rangle |F_H\rangle) = |\text{Photon}\rangle \otimes |F_H\rangle \tag{71}$$

Our friend measures the same photon state, he therefore knows there hasn't been any transformation on it. However, should we interpret the orthogonal states of the friend as measurement corresponding states, this will directly contradict the "after-interaction-state". These states can therefore not be seen as states corresponding to the results of measurements but to the memory of F regarding those same measurement results. This correlates to the fact that we cannot momentarily change the past, i.e. what factually happened, but we can change the memory that is supposed to store that information.

This second interaction (which corresponds to the second and fourth CNOT gate between Wigner's message and the friend in our quantum circuit framework) thus implements a unitary that reverses the memory of the friend to its initial position. It is an explicit form of the unitary U that was defined in equation 55. It's important to note that this is still a physical process because there has to be a physical implementation of said memory. What is unclear is the precise process of how U acts, in any real application this would require some extra mechanism.

A valid argument might now be: The goal was to make F think he only used classical permutations in his computation and then amaze him by revealing the result he got was only achievable in one run by him being in a superposition. If F forgets everything he did, how does the argument still hold? In order to settle that problem we need another way of storing information. More precisely we need F to be able to store the fact that he only used classical permutations on the computer. Whatever method we choose, F still cannot be sure that this information was written by him. One possibility might be to let F check the boxes he can use to input something into the computer. Once he realises that they only execute permutations, he might just be convinced. In this thesis we won't be elaborating further on this conundrum, it would however be of interest in some future considerations.

5 Grover's algorithm in Wigner's friend scenario

We now want to implement the entire Grover algorithm for the 2 qubit case. At each step we will specify the precise message of Wigner and what the friend is doing. As a quick reminder, the gate order we want to implement is:

$$(H^{\otimes 2})(\mathrm{CZ})(H^{\otimes 2})(Z \otimes I)(\mathrm{CZ})(H^{\otimes 2}) |00\rangle_L . \tag{72}$$

Should we however want to use just CNOT operations we can use $H^2 = I$ for equation 46 which gives us $\mathrm{CZ} = (I \otimes H)(\mathrm{CNOT})(I \otimes H)$ and therefore a gate order containing only CNOT gates:

$$(H \otimes I)(\mathrm{CNOT})(H \otimes I)(Z \otimes I)(I \otimes H)(\mathrm{CNOT})(H \otimes I) |00\rangle_L \tag{73}$$

To implement all operations classically, our friend has the following available four boxes connected to his computer:

- Box(G_1,$Q_{1/2}$): Implements the permutation $G_1 = \mathrm{NOT}_1$ on the 2 bits making up the first/second logical qubit
- Box(G_2,$Q_{1/2}$): Implements the permutation $G_2 = \mathrm{CNOT}$ on the 2 bits making up the first/second logical qubit

where our logical basis of 2 qubits was encoded in 4 classical bits, see section 2.3.2:

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) \quad \text{and} \quad |1\rangle_L = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle) \tag{74}$$

Mathematically we can model the box messages by introducing a new interaction between the friend and the messages. Instead of M_1 and M_2, for the entire state of the message including ancilla we will write:

$$|+\rangle_A \otimes |\text{Use Box}(G_i, Q_j)\rangle_M =: |+\rangle_A \otimes |G_i, Q_j\rangle_M \tag{75}$$

For $k \in \{0,1\}$, $i,j \in \{1,2\}$ and $r,s \in \{0,1,2\}$, the coupling between the ancilla state $|+\rangle$, the box message and the friend will be modelled by an interaction

$$I_{AMF} : |0\rangle_A \otimes |G_i, Q_j\rangle_M \otimes |k, (G_r, Q_s)\rangle_F \mapsto |0\rangle_A \otimes |G_i, Q_j\rangle_M \otimes |k, (G_i, Q_j)\rangle_F \tag{76}$$

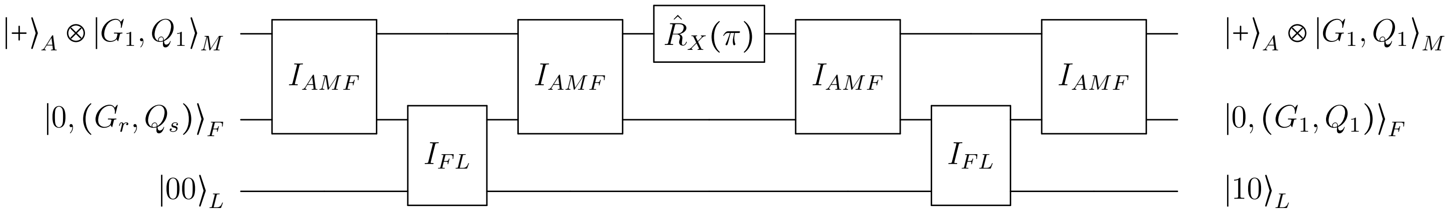
$$I_{AMF} : |1\rangle_A \otimes |G_i, Q_j\rangle_M \otimes |k, (G_r, Q_s)\rangle_F \mapsto |1\rangle_A \otimes |G_i, Q_j\rangle_M \otimes |k \oplus 1, (G_i, Q_j)\rangle_F \tag{77}$$

(The initial state $|k, (G_0, Q_0)\rangle$ corresponds to the friend having received no message yet), whereas for $a, b \in \{0,1\}$ the coupling between the friend and the logical qubits will be modelled as

$$I_{FL} : |0, (G_i, G_j)\rangle_F \otimes |ab\rangle_L \mapsto |0, (G_i, G_j)\rangle_F \otimes |ab\rangle_L \tag{78}$$

$$I_{FL} : |1, (G_i, G_j)\rangle_F \otimes |ab\rangle_L \mapsto |1, (G_i, G_j)\rangle_F \otimes \hat{A}(ij) |ab\rangle_L \tag{79}$$

where $\hat{A}(ij)$ is to the operator corresponding to the action of the (G_i, G_j) box. We also assume the identity operation on all other non-interacting parts of the whole system. As an example we can calculate the X-gate implementation using the interactions I_{AMF} and I_{FL}. Graphically they can simply be depicted by gates between the interacting parts:



While mathematically we have:

$$
\begin{aligned}
&|+\rangle_A \otimes |G_1,Q_1\rangle_M \otimes |0,(G_r,G_s)\rangle_F \otimes |00\rangle_L \qquad (80)\\
&\xrightarrow{I_{AMF}} \frac{1}{\sqrt{2}}(|0\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F + |1\rangle_A |G_1,Q_1\rangle_M |1,(G_1,Q_1)\rangle_F) \otimes |00\rangle_L\\
&\xrightarrow{I_{FL}} \frac{1}{\sqrt{2}}(|0\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |00\rangle_L + |1\rangle_A |G_1,Q_1\rangle_M |1,(G_1,Q_1)\rangle_F |10\rangle_L)\\
&\xrightarrow{I_{AMF}} \frac{1}{\sqrt{2}}(|0\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |00\rangle_L + |1\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |10\rangle_L)\\
&\xrightarrow{\hat{R}_X(\pi)} \frac{-i}{\sqrt{2}}(|1\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |00\rangle_L + |0\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |10\rangle_L)\\
&\xrightarrow{I_{AMF}} \frac{-i}{\sqrt{2}}(|1\rangle_A |G_1,Q_1\rangle_M |1,(G_1,Q_1)\rangle_F |00\rangle_L + |0\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |10\rangle_L)\\
&\xrightarrow{I_{FL}} \frac{-i}{\sqrt{2}}(|1\rangle_A |G_1,Q_1\rangle_M |1,(G_1,Q_1)\rangle_F |10\rangle_L + |0\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |10\rangle_L)\\
&\xrightarrow{I_{AMF}} \frac{-i}{\sqrt{2}}(|1\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |10\rangle_L + |0\rangle_A |G_1,Q_1\rangle_M |0,(G_1,Q_1)\rangle_F |10\rangle_L)\\
&\quad \equiv |+\rangle \otimes |G_1,Q_1\rangle_M \otimes |0,(G_1,Q_1)\rangle_F \otimes |10\rangle_L \quad \text{(Up to a global phase)}.
\end{aligned}
$$

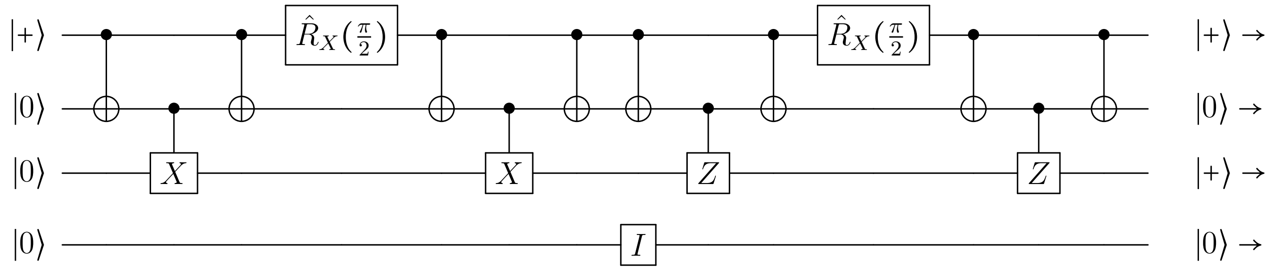
While tedious, this is a way of writing the entire state that encompasses all relevant systems. Of course we would also have an environment inside and outside the laboratory but these factors would unnecessarily complicate the equations. Also having implemented the interactions in this way, the friend's state still has information about the classical permutations he just applied. This only works for a one gate circuit though because the friend only actively stores information about the last gate he used. In a two or more gate circuit, Wigner would implement some unitary T that acts like (for $p, l \in \{1,2\}$)

$$T_{ij,pl} : |+\rangle \otimes |G_i, Q_j\rangle \mapsto |+\rangle \otimes |G_p, Q_l\rangle \qquad (81)$$

therefore changing e.g. the application of an X-gate on the logical qubit to the application of a Z-gate.

We now go through the full implementation of the Grover algorithm. For simplicity we represent the box-message only in wording and omit them graphically. Each step corresponds to another gate implementation (arrows at the end/beginning of the quantum circuit imply a continuation), we have:

1. $(H \otimes I)\,|00\rangle_L$



(a) W sends the following message to F: $|+\rangle \otimes |G_1, Q_1\rangle$

(b) F receives the message and couples ("measures") either to $|0\rangle$ and does nothing or to $|1\rangle$ and presses the button on the (G_1, Q_1) box

(c) The message gets returned to W who immediately sends $|+\rangle$ back to F

(d) F receives the message and couples either to $|0\rangle$ and does nothing or to $|1\rangle$ and "forgets" the previous "measurement"

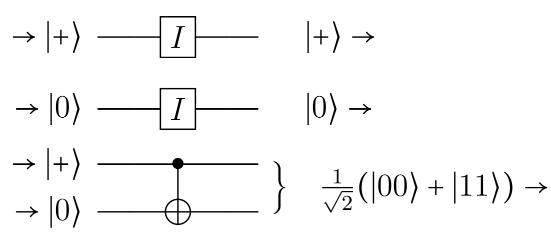
(e) The message gets returned to W, he applies $\hat{R}_X(\frac{\theta}{2})$ and sends it back to F with a specification of what box to press, i.e. $|G_1, Q_1\rangle$

(f) Repetition of step b

(g) Repetition of steps c and d and the message gets returned to W

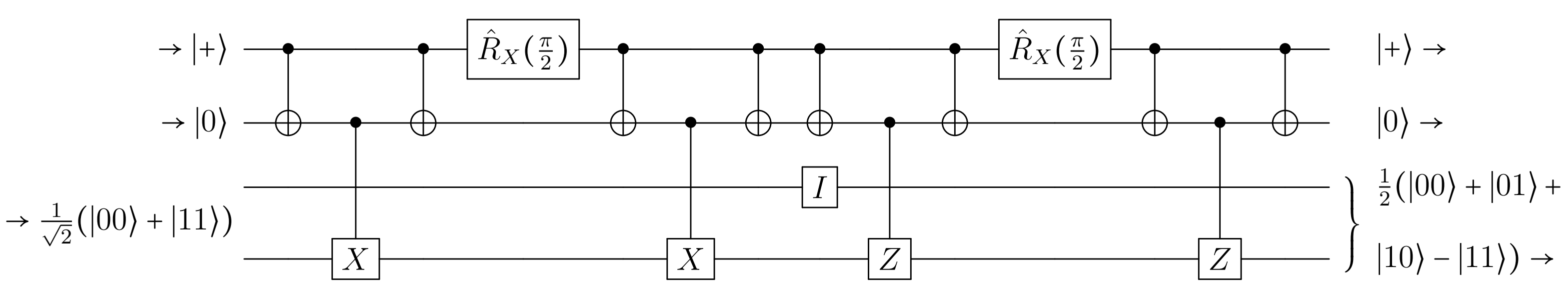
(h) Repetition of steps a-g using (G_2, Q_1) instead of (G_1, Q_1)

2. $(\text{CNOT})(H \otimes I)\,|00\rangle_L$



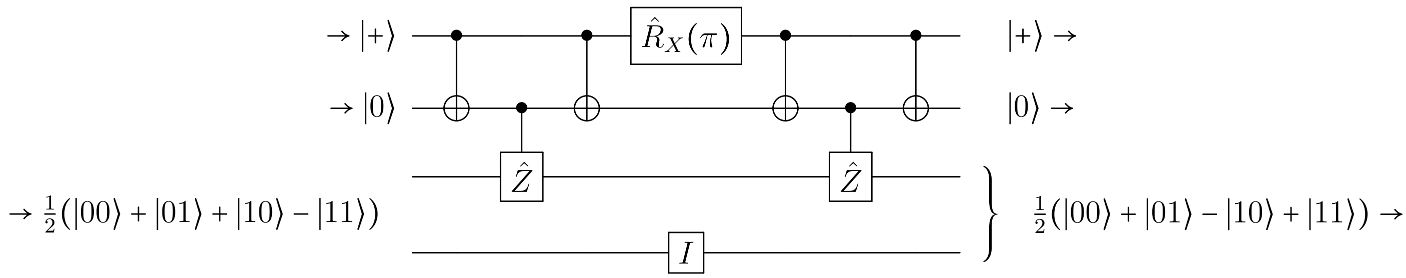
(a) For simplicity reasons we just apply a CNOT gate between the two qubit layers, the entire process would work similar to the others and involve two more Hadamard gates and a few rotations around the z-axis for the CZ gate

3. $(I \otimes H)(\text{CNOT})(H \otimes I)\,|00\rangle_L$



(a) Repetition of step (1.) using $|G_1, Q_2\rangle$ instead of $|G_1, Q_1\rangle$ and $|G_2, Q_2\rangle$ instead of $|G_2, Q_1\rangle$

4. $(Z \otimes I)(I \otimes H)(\text{CNOT})(H \otimes I)\,|00\rangle_L$



   (a) Repetition of (1.) steps a-g but using $|G_2, Q_1\rangle$ instead of $|G_1, Q_1\rangle$ and rotating around an angle $\theta = \pi$

5. $(H \otimes I)(Z \otimes I)(I \otimes H)(\text{CNOT})(H \otimes I)\,|00\rangle_L$

   (a) Full repetition of (1.), We have $\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \rightarrow \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$.

6. $(\text{CNOT})(H \otimes I)(Z \otimes I)(I \otimes H)(\text{CNOT})(H \otimes I)\,|00\rangle_L$

   (a) Full repetition of (2.). We have $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \rightarrow \frac{1}{\sqrt{2}}(|11\rangle + |01\rangle)$.

7. $(H \otimes I)(\text{CNOT})(H \otimes I)(Z \otimes I)(I \otimes H)(\text{CNOT})(H \otimes I)\,|00\rangle_L$

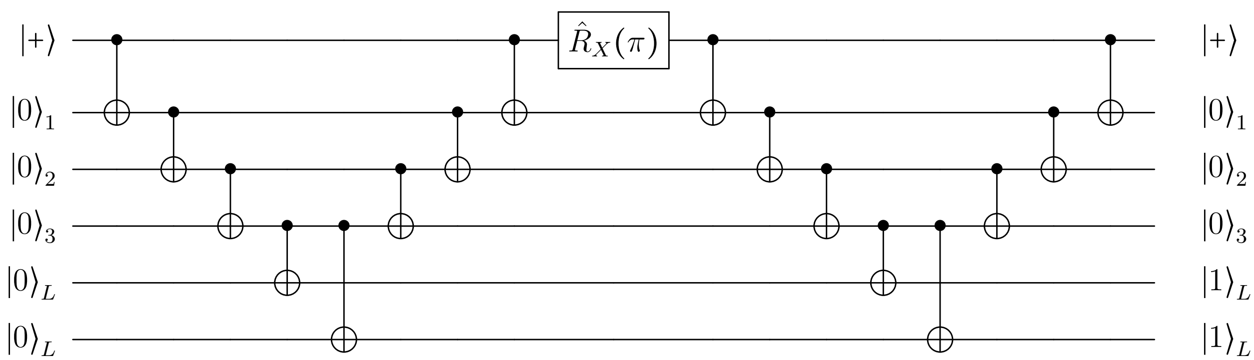
   (a) Full repetition of (1.). We have $\frac{1}{\sqrt{2}}(|11\rangle + |01\rangle) \rightarrow |01\rangle$.

8. F applies a NOT gate on first and second bit. A measurement will then imply that the logical qubits have been in the $|11\rangle$ state, successfully implementing the algorithm using only classical operations.

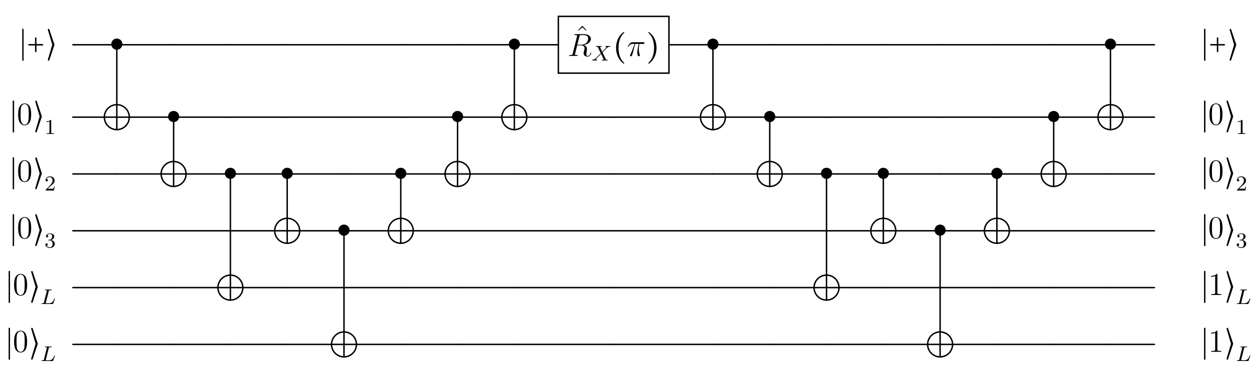
6 On the feasibility of experimental implementation

Up until now we derived a theoretical framework that allows the implementation of universal quantum computation in the Wigner's friend scenario. In this section we will talk about the experimental reality of our framework. First and foremost, the main problem is that of macroscopic superpositions. As in every other case of any Wigner's friend scenario it is entirely unclear if it is experimentally realizable to put an entire laboratory including a human being and computer in a sustained superposition. Even if it was a possibility we would still need to create a quantum channel for Wigner's message that somehow couples the entire laboratory to a single qubit. Therefore those questions will need to be set aside until the coming experimental effort has given a better understanding of macroscopic superpositions. What we can do however is to implement a more physical example of the entire framework.

First, we will exchange the friend with a machine whose only purpose is to realize if it is in a superposition or not (since that was the main purpose of our friend anyway, although it could be argued that a human may contemplate on the validity of it's used theory as well). We will describe the entirety of that machine using N qubits in the standard computational basis. From an information theoretic approach this should be an adequate description of any macroscopic object. We can think of the machine as including the classical computer and it's corresponding boxes of permutations. Any physical description of the interaction between the machine and Wigner's message therefore has to include a coupling between the message and every qubit making up the machine. The message cannot interact with the entire machine at once, so we have to assume that it couples with the "nearest" qubit at time $t = 1$ which then couples to the "nearest" machine qubit at time $t = 2$ and so on. This would describe a superposition cascade that will ultimately couple the entire macroscopic machine to the message in $t = N - 1$ time steps. For e.g. $N = 3$, this can be implemented in our framework in the following way:



Here we assumed that the "nearest" qubit to the classical computer will couple to the bits that make up the logical qubits. This might appear unsatisfying because it's not realistic that all bits are closest to a single machine qubit. Theoretically it doesn't matter however because we can model the interaction of the bits with the machines qubits on any layer, i.e. should the first bits that make up the first logical qubit be closer to the second qubit of the machine we have:



Which works just as well. The important part arises again when we want to reverse the machine. In the two previous circuit diagrams we modelled that reversion by a "reverse cascade" of interactions between the machines qubits (it could also be modelled by allowing the message to interact with every single one of the machines qubits again, although because of the different spatial distances that wouldn't be entirely physical). The question therefore becomes how the physical unitary U between two of the machine's qubits works.

One possibility of describing such a reverse cascade would be the following: In the first case we see that the reversal of the machine qubits acts in a way that is symmetric in time, we can therefore describe the reversal process with a unitary T that Wigner applies to the whole laboratory excluding the computer. This unitary T then acts on the machine qubits in a way that reverses their temporal interactions without needing to know their precise coupling mechanism. Something akin to this was recently published, see [16]. Unfortunately all this does is to exchange one problematic unitary with another. In the realm of this thesis it is therefore probably best to just accept that the message can via some mechanism interact with every single one of the machines qubits, oh well.

7 Conclusion

In this thesis we have created a framework that implements quantum computation in the Wigner's friend scenario. We have done this by first looking at quantum computation using classical operations and then, in a slightly changed and deterministic way, implementing this in a quantum circuit that represents the scenario. On the way we have discussed the physical aspects of the whole procedure and ended in some speculation about its experimental feasibility.

If one wishes to build upon this work one could in theory create a more elegant framework by finding some other way to implement the classical operations or improving different parts of the procedure. It would also be interesting to go into more detail on the memory aspect of the friend, i.e. maybe find an explicit way of working with a computer memory that models the friend. Here it should however be noted that one of the most important parts of the Wigner's friend scenario is to have somebody inside the laboratory who also does quantum theory. Fundamentally the paradoxon still only arises because it is unclear which description to apply to a certain situation and that easily gives rise to a conflict between two agents using the same theory. With that we conclude the conclusion and thank the reader for his or her attention.

References

[1] Max Planck, *Über das Gesetz der Energieverteiluzng im Normalspectrum*, Annalen der Physik, vol. 309, Issue 3, pp.553-563 (1901), https://doi.org/10.1002/andp.19013090310

[2] Einstein, A. and Podolsky, B. and Rosen, N. *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?* American Physical Society, Phys. Rev. Vol. 47, Iss. 10, pp. 777-780 (1935). https://doi.org/10.1103/PhysRev.47.777

[3] J. S. Bell, *On the Einstein Podolsky Rosen paradox*, Physics Physique Fizika 1 (3) ,pp.195 (1964), https://doi.org/10.1103/PhysicsPhysiqueFizika.1.195

[4] E.P.Wigner, *Remarks on the mind body question*, 1961 (The Scientist Speculates: An Anthology of Partly-Baked Ideas)

[5] Frauchiger, D., Renner, R. *Quantum theory cannot consistently describe the use of itself*, Nat Commun 9, 3711 (2018), https://doi.org/10.1038/s41467-018-05739-8

[6] This introduction to quantum physics follows the lecture notes by Norbert Schuch, *Quantum Information, Quantum Computing, and Quantum Algorithms*, winter of 2021

[7] Angelo Bassi, Kinjalk Lochan, Seema Satin, Tejinder P. Singh, Hendrik Ulbricht, *Models of Wave-function Collapse, Underlying Theories, and Experimental Tests* (2012), https://doi.org/10.48550/arXiv.1204.4325

[8] Markus Arndt, Klaus Hornberger, *Testing the limits of quantum mechanical superpositions* (2014), https://doi.org/10.48550/arXiv.1410.0270

[9] Feynman, R.P. *Simulating physics with computers*. Int J Theor Phys 21, 467–488 (1982), https://doi.org/10.1007/BF02650179

[10] IBM Osprey quantum processor, https://newsroom.ibm.com

[11] Sebastian Horvat, Xiaoqin Gao, Borivoje Dakic, *Universal quantum computation via quantum controlled classical operations*, 2022 (J. Phys. A: Math. Theor. 55 075301, 2022), https://doi.org/10.48550/arXiv.2104.06424

[12] Michael A. Nielsen, Isaac L. Chuang *Quantum Computation And Quantum Information 10th Anniversary Edition*, 2010 (Cambridge University Press)

[13] Lov K. Grover, *A Fast Quantum Mechanical Algorithm for Database Search*, 1996, (STOC '96: Proceedings of the twenty-eighth annual ACM symposium on Theory of Computing, p.212–219), https://doi.org/10.1145/237814.237866

[14] Eva Borbely, *Grover search algorithm*, https://doi.org/10.48550/arXiv.0705.4171

[15] Hoi-Kwan Lau and Martin B. Plenio *Universal Quantum Computing with Arbitrary Continous-Variable Encoding* (2016), https://doi.org/10.48550/arXiv.1605.09278

[16] P. Schiansky, T. Strömberg, D. Trillo, V. Saggio, B. Dive, M. Navascués, and P. Walther, *Demonstration of universal time-reversal for qubit processes* Optica 10, 200-205 (2023), https://doi.org/10.1364/OPTICA.469109