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„Spherically symmetric geometries in the framework of matrix models"

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Abstract

This work gives an introduction into a special class of solutions of the IKKT matrix model with mass-term, called covariant quantum space-times. The focus lies on the semi-classical limit, which consists of a 6-dimensional symplectic manifold, which can be seen to be an equivariant fibre bundle over a cosmological space-time with $k = -1$ FLRW metric, where the fibres are interpreted as internal degrees of freedom. The symplectic form on this fibre bundle is studied in more detail and an explicit coordinate expression is derived. Furthermore, we explicitly check a consistency condition on the holonomy of the symplectic form coming from the quantization condition. On the base, the internal degrees of freedom give rise to a higher-spin gauge theory, whose geometrical sector can be described by a dynamical frame generated by the semi-classical limit of the matrix configuration and additional constraints coming from the matrix model. Considering local perturbations of the cosmological background, it is shown that the present framework can describe any static spherically symmetric metric. The problem of reconstructing semi-classical generators for the frame describing such geometries is discussed and it is suggested that higher-spin contributions to the frame are unavoidable.

Zusammenfassung

Diese Arbeit gibt eine Einführung in eine spezielle Klasse von Lösungen des IKKT Matrix Modells mit Massenterm, welche als kovariante Quanten-Raumzeiten bezeichnet werden. Hierbei liegt der Fokus auf dem semiklassischen Limes, der gegeben ist durch eine 6-dimensionale symplektische Mannigfaltigkeit, welche ein äquivariantes Faserbündel über eine kosmologischen Raumzeit mit $k = -1$ FLRW Metrik darstellt, wobei die Fasern als interne Freiheitsgrade interpretiert werden. Die symplektische Form auf dem Faserbündel wird genauer analysiert und eine explizite Koordinatenform wird hergeleitet. Weiters wird eine Konsistenzbedingung an die Holonomie der symplektischen Form, welche von einer Quantisierungsbedingung stammt, explizit überprüft. Die internen Freiheitsgrade auf dem Basisraum führen zu einer „Higher-spin"-Eichtheorie, deren geometrischer Sektor durch einen dynamischen Rahmen, welcher durch das semiklassiche Limit der Matrixkonfiguration generiert wird, sowie zusätzliche Identitäten, kommend vom Matrixmodell, beschrieben wird. Es werden lokale Störungen des kosmologischen Hintergrunds betrachtet und es wird gezeigt, dass jede statische, sphärisch symmetrische Metrik innerhalb dieses Modells beschrieben werden kann. Das Problem der Rekonstruktion semiklassischer Generatoren für Rahmen, welche solche Geometrien beschreiben, wird diskutiert und es wird gefolgert, dass „Higher-spin"-Beiträge zum Rahmen unvermeidbar sind.

Contents

1. Introduction

Spherically symmetric geometries form a class of extremely well studied models of space-time and are among the first exact solutions of Einstein's theory of general relativity to have been found. They have been used to explain effects like the perihelion precession of Mercury, which have been experimentally studied to high levels of precision, always finding agreement with the theoretical predictions [1]. Therefore, any theory of gravity that is more fundamental, as would be the case for a theory of quantum gravity, should, in some regime, be able to give a description of spherically symmetric space-times, that reproduces these thoroughly tested results.

One candidate for this, are so called matrix models. Having been introduced as non-perturbative approaches to superstring theory, they provide a unique angle from which one can study the extensive body of structures that is string theory [2]. Although they can be stated in terms of extremely simple objects, namely a finite collection of matrices, they can yield configurations that give rise to interesting physics.

One particular matrix model is the so called IKKT or IIB matrix model for which an interesting type of solution was found and described in [3, 4, 5]. They are 3+1 dimensional, exhibit a global $SO(3,1)$ symmetry and have consequently been dubbed *covariant quantum space-times*. As solutions to the matrix model equations of motion, they are dynamical and give rise to a higher-spin gauge theory for fluctuations on this background. Their geometrical content is governed by a dynamical frame, which is generated by the dynamical matrices of the matrix model, at the semi-classical level. In a second step, this description was cast into a covariant form based on a Weitzenböck connection and its torsion [6], which allows one to look for frames that describe geometries that are known from other theories of gravity, like general relativity.

The aim of this thesis is to study spherically symmetric static geometries inside this matrix model framework and investigate, how they can be realized as local perturbations of the background provided by the semi-classical limit of the covariant quantum space-time. Concretely, we will look for spherically symmetric frames that satisfy necessary constraints originating from the matrix model, without considering the actual equations of motion following from the action of the IKKT model. This will allow to get more insight into the classes of spherically symmetric geometries lending themselves to a description in the matrix model framework and help guide the development of the matrix models, so that they reproduce physically relevant geometries.

To achieve this and give more exposition to the relevant theory, this thesis is organised in the following fashion:

Section 2 gives an introduction to the broader topic of non-commutative geometry, which matrix models and related concepts are an example of. Section 3 explains the idea of quantum geometry as an physics-based approach to non-commutative geometry and gives the definitions of the basis concepts of matrix geometry, a framework that allows to extract geometrical content from certain sets of matrices. This is based on the concept of *quasi-coherent states*, which will be discussed in this section as well. Having such a procedure , as should be no surprise, is essential in obtaining a semi-classical description of the covariant quantum space-times.

The following, section 4, discusses the relationship between matrix geometries and matrix models, where the later provide a mechanism that gives preferred sets of matrices for which the former give the procedure of constructing a well-behaved semi-classical limit. This is achieved via minimization of a certain energy functional. The second part of this section describes the aforementioned covariant cosmological quantum space-times in more detail. The basis for this is given by the fuzzy hyperboloid H_n^4, whose corresponding semi-classical geometry is $\mathbb{C}P^{1,2}$, an indefinite complex projective space. This space has the structure of a symplectic manifold and a S^2-fibre bundle over the semi-classical cosmological background $\mathcal{M}^{3,1}$, where the fibre coordinates describe internal degrees of freedom.

Section 5 concerns a more detailed study of this space. We determine an exact expression for the symplectic form, using appropriately chosen local coordinates and check a consistency condition that is implied from the fact that this space is the semi-classical limit of a matrix configuration.

Having these basics at hand, section 6 introduces the emergent higher-spin gauge theory on the cosmological background $\mathcal{M}^{3,1}$. It is shown how the dynamical frame arises as a collection of Hamiltonian vector fields and how the effective metric arises as the metric governing the propagation of all fields on the background. The final part of this section gives the derivation of the dilaton identity and the divergence constraint, conditions coming from the matrix model, but independent of the actual equations of motion.

Section 7 sets the stage for the investigation of static spherically symmetric geometries in this higher-spin theory, by giving an Ansatz for the dynamical frame and calculating the dilaton identity in spherical coordinates.

The final two sections 8 and 9 present the main findings of this work. We construct several spherically symmetric metrics by using the Ansatz for the frame and show that the most general spherically symmetric metric can indeed be realized in the present framework, also when one accounts for the dilaton identity. We then turn to the problem of actually constructing the semi-classical generators for such frames and discuss the problem of higher-spin terms arising in the expression for the frame.

We conclude with section 10, where we recap the content of the thesis and give directions for further investigations. Appendix A gives the convention for the gamma matrices used in section 4, while appendix B gives the explicit constructions of the representations used in defining the fuzzy hyperboloid H_n^4. Appendix C gives a more detailed explanation of the fibre bundle structure on the semi-classical cosmological background $\mathcal{M}^{3,1}$.

2. Non-commutative spaces

The following chapter will give an introduction to the concept of non-commutative spaces and matrix geometries as a simple, but very rich and useful example of the former. A short introduction to the underlying ideas will be given, as well as a more thorough discussion of how these ideas can be combined with matrix models to obtain matrix geometries with desirable properties and additional structure.

2.1. First ideas behind non-commutative spaces

As discussed previously, there is good reason to suspect, that a fundamental model of space-time and gravity has to incorporate quantum features at some scale. One of these features is fuzziness. Space-time should stop being a smooth manifold at a fundamental level and exhibit uncertainty relations between its degrees of freedom, analogously as in other quantum theories. This can be captured by the concept of *non-commutative spaces*.

The rough idea of non-commutative spaces can be explained by first considering some well-known results about ordinary (i.e. commuting) spaces. First we will recall some definitions regarding certain types of associative algebras:

Definition 2.1. *An unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ is a $\mathbb{C}$-vector-space $\mathcal{A}$, together with a bilinear map*

$$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A},\ (a,b) \mapsto a \cdot b \tag{1}$$

such that

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for all $a, b, c \in \mathcal{A}$ and a neutral element e with the property that

$$a \cdot e = a = e \cdot a$$

for all $a \in \mathcal{A}$. A associative algebra is furthermore called commutative if $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{A}$.

It is common notation to suppress the explicit use of the algebra multiplication symbol and simply write $a \cdot b = ab$. Next one can introduce some extra structure on these algebras:

Definition 2.2. *A complex Banach algebra $\mathcal{A}$, is an associative algebra $\mathcal{A}$ over $\mathbb{C}$, together with a norm $||\cdot||$ on $\mathcal{A}$ such that the underlying vector space is complete in the metric induced by the norm and*

$$||ab|| \leq ||a||\ ||b||$$

for all $a, b \in \mathcal{A}$. A complex Banach algebra is further called unital, if it is an unital associative algebra and $||e|| = 1$.

Finally, we want to consider a special class of Banach algebras, namely C*-algebras. The origin of this area of mathematics and mathematical physics in particular, goes back to the work of von Neumann, who tried to construct a rigorous mathematical framework of quantum mechanics [7]. There they play an important role, as they give a model for the algebra of observables of a quantum system. Their first abstract characterization was given by Gelfand and Naimark in 1943 [8]:

Definition 2.3. *A C^*-algebra $\mathcal{A}$, is a complex Banach algebra, together with a map*

$$^* : \mathcal{A} \to \mathcal{A}, \ a \mapsto a^* \tag{2}$$

such that

1. $(a^*)^* = a$ *for all* $a \in \mathcal{A}$.
2. $(a+b)^* = a^* + b^*, \ (ab)^* = b^* a^*$ *for all* $a, b \in \mathcal{A}$.
3. $(\lambda a)^* = \overline{\lambda} a^*$ *for all* $\lambda \in \mathbb{C}$ *and* $a \in \mathcal{A}$.
4. $||a^* a|| = ||a|| \, ||a^*||$ *for all* $a \in \mathcal{A}$.

Note that the fourth identity is called the C^* identity and can equally be stated as $||aa^*|| = ||a||^2$. Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, a bounded linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called a *-homomorphism if

1. $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$.
2. $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathcal{A}$.

C^*-algebras have rich structure and are useful in a variety of contexts, for more information, see the introductory book [9]. The example of C^*-agebras first considered is also the one most relevant to physics, namely the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators acting on some complex Hilbert space $\mathcal{H}$. Here, the algebra multiplication is given by composition of linear maps, the norm is the operator-norm and the $*$-map is given by taking the adjoint of operators. Note that this is an example of a non-commutative C^*-algebra. Another example is given by the set of continuous functions $\mathcal{C}^0(X)$ from a compact Hausdorff space to the complex numbers. This is a unital commutative C^*-algebra, where multiplication and addition are given by the corresponding point-wise operations for functions, the unit element is the identity function on X, the norm is taken to be the supremum norm and the *-map is given by complex conjugation.

In the context of non-commutative spaces, the following result gives the motivation for the approach that will be studied in this thesis. It goes under the name of commutative Gelfand-Naimark theorem. Let $\mathcal{A}$ be a C^*-algebra and consider the set

$$X_{\mathcal{A}} = \{\varphi : \mathcal{A} \to \mathbb{C} \ : \ \varphi \neq 0, \ \varphi \text{ is a } ^*\text{-homomorphism}\} \tag{3}$$

called the spectrum of $\mathcal{A}$. When $\mathcal{A}$ is unital and commutative, it can be shown that $X_{\mathcal{A}}$ is a compact Hausdorff space. The commutative Gelfand-Naimark theorem then states:

Theorem 2.1. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $X_\mathcal{A}$ its spectrum. Then there exists an isometric *-isomorphism γ from $\mathcal{A}$ onto $\mathcal{C}^0(X_\mathcal{A})$.*

One can use this result to set up a correspondence between compact Hausdorff spaces and unital commutative C^*-algebras.

On the one hand, it says that any unital commutative C^*-algebra can equally be thought of as a algebra of functions on some compact Hausdorff space.

On the other hand consider a compact Hausdorff space Y and let $\mathcal{C}^0(Y)$ be the space of continuous functions from Y into the complex numbers. Since it its an unital commutative C^*-algebra, it makes sense to define its spectrum $X_{\mathcal{C}^0(Y)}$. So one is then in the setting of the above theorem, which tells one that $\mathcal{C}^0(Y)$ is isometric *-isomorphic to $\mathcal{C}^0(X_{\mathcal{C}^0(Y)})$. Further consider an arbitrary $y \in Y$ and let φ_y be defined by $\varphi_y(f) = f(y)$ for all $f \in \mathcal{C}^0(Y)$. Then $\varphi_y \in X_{\mathcal{C}^0(Y)}$ and it turns out that the map $y \mapsto \varphi_y$ is actually a homeomorphism between Y and $X_{\mathcal{C}^0(Y)}$.

What this tells us is that starting from a topological space Y, we can consider its algebra of continuous functions $\mathcal{C}^0(Y)$ and then pass to an equivalent description given by $(X_{\mathcal{C}^0(Y)}, \mathcal{C}^0(X_{\mathcal{C}^0(Y)}))$. We thus have a topological space, together with a algebra of functions on it, that is completely determined by the original's space of continuous functions $\mathcal{C}^0(Y)$, but carries the same information as the original space Y. The assertion that compact topological spaces are characterized by their algebra of continuous functions is further backed by the following: Two compact compact Hausdorff spaces X and Y are homeomorphic if and only if, $\mathcal{C}^0(X)$ and $\mathcal{C}^0(Y)$ are $*$-isomorphic.

The technical statement of the above is the fact that these considerations can be used to construct a contravariant functor from the category Cpct of compact Hausdorff spaces with their continuous functions, to the category CommC*1 of unital commutative C^*-algebras and *-homomorphisms between them and this functor is actually an equivalence of categories [10].

However, these technicalities are not important for the considerations of this thesis. What is important, is the fact that compact Hausdorff spaces can equally well be thought of as unital commutative C^*-algebras. This gives the starting point to wonder what one could maybe consider as a non-commutative space.

While it is not at all obvious, how one could come up with a meaningful definition of a non-commutative version of a space whose underlying set is, for example, a point set, there is a well-defined meaning of what a non-commutative algebra is. Since we have just established that topological spaces correspond to unital commutative C^*-algebras, the natural choice would be to pass to unital non-commutative C^*-algebras and, by remembering the above correspondence, view them as non-commutative, or *quantized* algebras of functions on some non-commutative space. This point of view is also the main one taken in the framework of quantum, an in particular, matrix geometries.

While certainly interesting, topological spaces do not carry enough structure to be used as realistic models for space-time and gravity. For this, one has to consider smooth Lorentzian manifolds. Luckily, a similar characterization as for compact topological spaces by functions on them exists for smooth manifolds. Consider a smooth n-dimensional manifold $\mathcal{M}$ together with the set $\mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ of all smooth functions from $\mathcal{M}$ into $\mathbb{R}$. Then $\mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a unital, commutative and associative algebra over $\mathbb{R}$ and it carries all information about $\mathcal{M}$, meaning it uniquely determines the underlying topological space, as well as the smooth structure [11]. Thus we can again think about non-commutative analogs of smooth manifolds by considering non-commutative algebras of functions. Note however, that information about the metric is not encoded in the algebra of smooth functions and thus has to be considered separately. Dealing with the problem of introducing geometrical structure onto these non-commutative spaces is the topic of *non-commutative geometry*. In the following, a short overview of some approaches is given, before the main topic of this section, quantum geometry, is discussed.

2.2. Overview of approaches to non-commutative geometry

Before we discuss the topic of quantum geometries, we want to give a brief overview of other approaches to the topic of non-commutative spaces. The most famous of which is the spectral triple approach envisioned by Alain Connes. It is a fully rigorous mathematical framework that concerns itself with constructing non-commutative versions of Riemannian manifolds, via a so called spectral triple. The basic ideas mirror our previous considerations. Consider a smooth Riemannian manifold M and a smooth vector bundle E over M. Then one considers the Hilbert space $L^2(M, E)$ of square integrable sections of E together with an unbounded operator D that satisfies certain properties. There is then a reconstruction theorem, stating that M as a Riemannian manifold can be fully recovered from the triple $(E, L^2(M, E), D)$ [12]. Thus, one again has a duality between a space and an algebra of functions on it. In the same spirit as our discussion above, it is then proposed to define a non-commutative Riemannian manifold as a collection $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, called a *spectral triple*, where $\mathcal{A}$ is a representation (rep) of a C^*-algebra on a Hilbert space $\mathcal{H}$ and $\mathcal{D}$ is an unbounded operator on $\mathcal{H}$, again satisfying certain properties. *Spectral geometry* then concerns itself with extracting spaces with a suitable sense of metric from these spectral triples. In physics, the spectral triple approach has found its application mostly in studying the internal structure of the standard model of particle physics, see for example [13],[14].

Another approach to non-commutative spaces is *deformation quantization*. There, one starts with a Poisson manifold $(M, \{\cdot,\cdot\})$[1], but instead of considering its algebra of smooth functions, one instead defines the algebra of formal power series $\mathcal{A}[[\theta]] \coloneqq \oplus_n f_n \theta^n$ where $f_n : M \to \mathbb{C}$ is smooth for each $n \in \mathbb{N}$. One notes that the $n = 0$ sector of $\mathcal{A}[[\theta]]$ is exactly given by $\mathcal{C}^\infty(M)$. On $\mathcal{C}^\infty(M) \subset \mathcal{A}[[\theta]]$ one then defines a unital associative, but not commutative, product $\star$, via

$$f \star g = fg + \sum_{n=1}^{\infty} B_n(f, g)\theta^n \tag{4}$$

[1]For a definition of Poisson manifolds, see section 3.

where for each $n \in \mathbb{N}$, $f_n \in \mathcal{C}^\infty(M)$ and $B_n : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a bilinear differential operator such that

1. $\star$ is associative
2. $B_1(f,g) - B_1(g,f) = \{f,g\}$
3. $1 \star f = f \star 1 = f$.

This definition is then extended onto all of $\mathcal{A}[[\theta]]$ by linearity over $\mathbb{R}[[\theta]]$ and θ-adic continuity [15],[16]. The $\star$-product can be used to define a bracket

$$[f,g] = f \star g - g \star f = \theta\{f,g\} + \mathcal{O}(\theta^2) \tag{5}$$

where the second equality holds due to the second condition above. Eq.(4) and eq.(5) then tell you that

$$f \star g \xrightarrow{\theta \to 0} fg$$
$$\frac{1}{\theta}[f,g] \xrightarrow{\theta \to 0} \{f,g\}$$

which means that in this limit, the undeformed algebra of functions $\mathcal{C}^\infty(M)$ with its commutative product, as well as the Poisson bracket on it is recovered. Again this procedure is in the spirit of the discussion in section2.1. One has a commutative space, here the Poisson manifold M and a commutative algebra of functions on it, here $\mathcal{C}^\infty(M)$, one passes to a non-commutative algebra, here the deformed algebra $\mathcal{A}[[\theta]]$ together with the $\star$-product, and views this as a quantized algebra of functions on some non-commutative space.

Deformation quantization has the merit of not having to choose a Hilbert space rep and work with an operator algebra. All the quantum features are build into the $\star$-product, which can be viewed as a simple non-commutative product of classical functions. This makes it more accessible for mathematical investigation, so that there are proves of theorems concerning the existence and classification of $\star$-products. In mathematical physics, deformation quantization and $\star$-products are related to statistical mechanics and path integrals, as well as field theory, where they can be used to construct non-commutative field theories by replacing point-wise multiplication of fields by an appropriate $\star$-product. This approach is not free of divergences as they typically appear in interacting theories, but the cohomological features of deformation theory allow one to deal with them in a novel way, termed cohomological renormalization. For an overview on the topic of deformation quantization, see i.e. the review [17].

Finally, we want to mention the area of quantum groups. It was thought that (semi)simple Lie groups and their Lie algebras are rigid objects, e.g. they do not amid themselves to the process of deformation well. However, consider a slightly "bigger" object, namely the algebra of functions on some Lie group. This is then an example of a more general algebraic object, called a *Hopf algebra*. Such Hopf algebras can be continuously deformed and the resulting objects are, in a sense, very closely connected to the original groups, hence their name quantum groups. Again, one can then interpret them as a non-commutative algebra of functions of some non-commutative space. For more information on them, see i.e. [18].

3. Quantum geometry

While some of the possible ways to approach non-commutative geometry have been discussed in the previous section, we have yet to explore the approach that much of the later results of this thesis are based on. Many of the approaches mentioned above, try to establish firm mathematical axioms for their framework of non-commutative geometry first. This certainly makes sense from a mathematicians point of view, however it is somewhat contrary to the way physical theories have developed in the past, where there have been observations followed by models being proposed to describe these specific observations. Generalization and axiomatization of physical theories always only happens after the fact, when a certain model has been proven to accurately describe a certain set of observations. In this spirit, the following approach to non-commutative geometry is based on models and examples related to fundamental physics and is constructed to incorporate all necessary features to do physics in the context of gauge theory and gravity.

Due to its inspiration from other quantum theories, this framework is called *quantum geometry*. The mathematical structure on which it is build is the concept of a quantized symplectic space, akin to the phase-space formulation of quantum mechanics. However, one would ultimately like to do gauge theory and in particular gravity on these quantum geometries, so that necessarily, more general symplectic spaces have to be considered, which will also be equipped with additional geometrical structures [19].

In the following, quantized symplectic spaces will be discussed in more detail.

Mathematical interlude: Symplectic and Poisson manifolds The prototypical example for a quantum geometry is the quantized phase space of ordinary quantum mechanics. To properly understand how this can be seen as a non-commutative version of Hamiltonian mechanics, one first needs some mathematical background.

Consider a *symplectic Manifold* $(\mathcal{M}, \omega)$, which consists of a smooth manifold $\mathcal{M}$, together with a 2-form ω on $\mathcal{M}$, that satisfies the following:

1. $d\omega = 0$, e.g. ω is closed.
2. For every $x \in \mathcal{M}$, $\omega(x) : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ is non-degenerate.

In local coordinates (x^a), the symplectic form is given as

$$\omega = \frac{1}{2}\omega_{ab} dx^a \wedge dx^b, \tag{6}$$

while being closed is equivalent to

$$\partial_a\omega_{bc} + \partial_b\omega_{ca} + \partial_c\omega_{ab} = 0,\ \omega_{ab} = -\omega_{ba} \tag{7}$$

and non-degeneracy is equivalent to $\omega_{ab}(x)$ being a non-degenerate matrix for every point $x \in \mathcal{M}$. Due to these properties, every symplectic manifold has a natural volume form Ω, e.g. a nowhere vanishing top-degree form, which can be constructed from the symplectic form ω as

$$\Omega \coloneqq \frac{1}{n!}\omega^{\wedge n} \tag{8}$$

for dim $\mathcal{M} = 2n$. Furthermore, consider the (commutative) algebra of smooth functions from $\mathcal{M}$ into $\mathbb{C}$ and let it be denoted by $\mathcal{C}^\infty(\mathcal{M})$. One can use the symplectic form to define a pairing between these functions, called a *Poisson bracket.* In general, a Poisson bracket on a smooth manifold $\mathcal{M}$ is a bilinear and antisymmetric map

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M}) \tag{9}$$

such that

1. $\{fg, h\} = f\{g, h\} + \{f, h\}g$ (Leibnitz identity)
2. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity)

hold for all $f, g, h \in \mathcal{C}^\infty(\mathcal{M})$. Such a pair $(\mathcal{M}, \{\cdot,\cdot\})$ is then called a *Poisson manifold.* Again consider local coordinates (x^a), then any Poisson bracket can be written as

$$\{f, g\} = \theta^{ab}\partial_a f \partial_b g, \tag{10}$$

where $\theta^{ab} = \{x^a, x^b\}$. This in turn defines a bi-vector field Π, called *Poisson tensor*, which reads

$$\Pi = \theta^{ab}\partial_a \otimes \partial_b. \tag{11}$$

The Jacobi identity can then be expressed in local coordinates as

$$\theta^{ad}\partial_d\theta^{bc} + \theta^{bd}\partial_d\theta^{ca} + \theta^{cd}\partial_d\theta^{ab} = 0 \tag{12}$$

and we call a Poisson bracket non-degenerate, if the coefficient matrix $\theta^{ab}(x)$ is non-degenerate in every point $x \in \mathcal{M}$.

On a symplectic manifold, having the symplectic form at hand, a canonical Poisson bracket is induced by $\{f, g\} \coloneqq -\omega(V_f, V_g)$, where V_f is the vector field associated to f by the isomorphism $\Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ implicitly defined by $\omega(V_f, W) = df(W)$ for all $W \in \mathfrak{X}(\mathcal{M})$. Conversely, every Poisson manifold can be decomposed into regularly immersed symplectic manifolds, called *symplectic leaves.* On these, the Poisson bracket can be "inverted" to give the symplectic form. However, Poisson manifolds are the more general concept, while symplectic manifolds are examples of the former. This connection becomes more explicit in local coordinates (x^a), where given a symplectic form $\omega_{ab}(x)$, one can define its inverse $\theta^{ab}(x)$ at every point $x \in \mathcal{M}$:

$$\theta^{ac}(x)\omega_{cb}(x) = \delta^a{}_b. \tag{13}$$

This then defines a Poisson tensor as in eq.(11). In turn, if one has a Poisson bracket such that the corresponding Poisson tensor Π is non-degenerate, then eq.(13) can similarly be used to define a symplectic form ω.

Note that in the following, we will freely switch between an index notation and a coordinate- free notation and use the two interchangeably, whenever there is no ambiguity. Having these mathematical notions at hand, we can now discuss some of the parallels between Hamiltonian and quantum mechanics.

3.1. "The wrong direction": Quantum mechanics and quantization maps

Dirac was the first to comment on the similarities between the mathematical structures appearing in analytical mechanics and quantum mechanics [20]. In Hamiltonian mechanics, all possible configurations of a system are represented as points of a symplectic manifold $\mathcal{M}$, called *phase space*. Physical observables are then modeled by real functions, i.e. $f \in \mathcal{C}^\infty(\mathcal{M})$, $f^* = f$, whose dynamics can be calculated via taking the Poisson bracket with a *Hamiltonian function* H: $\dot{f} = \{f, H\}$. (Mixed) states of the physical system are, in general, described by real functions $\rho \geq 0$ on $\mathcal{M}$ such that $\int_\mathcal{M} \Omega\rho = 1$, where Ω is the symplectic volume form on $\mathcal{M}$.

All these structures are mirrored in the Heisenberg picture of quantum mechanics. Instead of phase space, one has a Hilbert space $\mathcal{H}$ together with an algebra of endomorphisms $\mathrm{End}(\mathcal{H})$. Physical observables are the self-adjoint elements of this algebra, e.g. $F = F^\dagger$ and their dynamics can be calculated from the commutator with the Hamiltonian $\hat{H}$: $\dot{F} = \frac{-i}{\hbar}[F, \hat{H}]$, while (mixed) states of the quantum system are given by positive semi-definite hermitian operators $\hat{\rho}$ with $\mathrm{Tr}(\hat{\rho}) = 1$.

The main idea of quantum mechanics can thus be roughly stated as follows: Find an appropriate Hilbert space $\mathcal{H}$ and replace the commutative algebra $\mathcal{C}^\infty(\mathcal{M})$ by the non-commutative algebra $\mathrm{End}(\mathcal{H})$, the Poisson bracket $\{\cdot, \cdot\}$ by the commutator $\frac{-i}{\hbar}[\cdot, \cdot]$ and try to preserve as many of the above features as possible.

This idea of relating symplectic spaces and their commutative algebra of smooth functions to a Hilbert space and its non-commutative algebra of operators is at the heart of quantum geometry. Being inspired by the quantization procedure of going from Hamiltonian mechanics to quantum mechanics, one now tries to formulate a general way of establishing such a procedure. Taking a Hilbert space $\mathcal{H}$ and its endomorphism algebra $\mathrm{End}(\mathcal{H})$ as our prime candidates for a quantum geometry, table 1 shows a comparison between structures on symplectic manifolds and on our quantum geometries.

structure	symplectic manifold	quantum geometry
algebra	$\mathcal{C}^\infty(\mathcal{M})$	$\mathrm{End}(\mathcal{H})$
alg. multiplication	pointwise product of functions (comm.)	matrix product (non-comm.)
bracket	$\{f, g\}$	$[F, G]$
conjugation	$f \mapsto f^*$	$F \mapsto F^\dagger$
inner prod.	$\langle f \vert g \rangle_{L^2} = \frac{1}{(2\pi\alpha)^n} \int_\mathcal{M} \Omega f^* g$	$\langle F \vert G \rangle_{HS} = \mathrm{Tr}(F^\dagger G)$

Table 1: Comparison between similar structures on classical spaces and quantum geometries [21].

In the spirit of non-commutative geometry as discussed in section2.1, we say that quantizing a symplectic manifold $\mathcal{M}$ consists of finding an appropriate Hilbert space $\mathcal{H}$ and mapping elements of the commutative algebra $\mathcal{C}^\infty(\mathcal{M})$ to elements of the non-commutative algebra $\mathrm{End}(\mathcal{H})$ in a way such that the correspondences in table 1, as well as any additional structures are preserved, at least in some useful fashion. This plan is formalized by the idea of a *quantization map* $\mathcal{Q}$. A good working definition of such a quantization map is given by the following [22]:

Definition 3.1. *Let* $(\mathcal{M}, \{\cdot,\cdot\})$ *be a Poisson manifold and let* $\mathcal{H}$ *be a separable Hilbert space. A quantization map* $\mathcal{Q}$ *is then a linear map*

$$\mathcal{Q} : \mathcal{C}^\infty(\mathcal{M}) \to End(\mathcal{H})$$

depending on a parameter θ*, such that the following hold:*

1. $\mathcal{Q}(id_{\mathcal{M}}) = \mathbb{1}_{\mathcal{H}}$.
2. $\mathcal{Q}(f^*) = \mathcal{Q}(f)^\dagger$*, for all* $f \in \mathcal{C}^\infty(\mathcal{M})$.
3. $\mathcal{Q}(fg) - \mathcal{Q}(f)\mathcal{Q}(g) \xrightarrow{\theta\to 0} 0$,
   $\frac{1}{\theta}\big(\mathcal{Q}(i\{f,g\}) - [\mathcal{Q}(f),\mathcal{Q}(g)]\big) \xrightarrow{\theta\to 0} 0$ *for all* $f, g \in \mathcal{C}^\infty(\mathcal{M})$.
4. *If* $[\mathcal{Q}(f), F] = 0$ *for all* $f \in \mathcal{C}^\infty(\mathcal{M})$*, then* $F = c \cdot \mathbb{1}_{\mathcal{H}}$ *for some* $c \in \mathbb{C}$.

Linearity and condition 2 make sure that the vector space structure, as well as the *-map of the two algebras is respected, while condition 4 means that the Hilbert space is as small as possible, while being as big as necessary, as there are no sectors of End($\mathcal{H}$) that are invariant under the action of the quantization of $\mathcal{C}^\infty(\mathcal{M})$. The really interesting condition is the third, as it makes clear how the Poisson structure and commutative multiplication on the classical side should be connected to the commutator and the non-commutative multiplication on the quantum side. Namely, that as the parameter θ approaches zero, the commutative multiplication and the Poisson bracket on $\mathcal{C}^\infty(\mathcal{M})$ are recovered.

While definition 3.1 gives all the necessary conditions for a meaningful quantization map, one might hope for even more properties, two of which are particularly interesting, as they happen to be fulfilled for the quantization of coadjoint orbits, an important example of quantum geometries that will be discussed in section 3.2.2. Having the symplectic volume form Ω on $\mathcal{M}$ and the trace on End($\mathcal{H}$), one can define inner products on both spaces, as in the last row of table 1. One can then demand that $\mathcal{Q}$ is an isometry, e.g.

$$\langle \mathcal{Q}(f)|\mathcal{Q}(g)\rangle_{HS} = \langle f|g\rangle_{L^2} \tag{14}$$

for all $f, g \in \mathcal{C}^\infty(\mathcal{M})$. Furthermore, if both $\mathcal{C}^\infty(\mathcal{M})$ and End($\mathcal{H}$) carry a group action of some Lie group G, then one could demand that the quantization map respects that group action, e.g. is an *intertwiner*:

$$\mathcal{Q}(g \cdot f) = g \cdot \mathcal{Q}(f) \tag{15}$$

for all $g \in G$ and $f \in \mathcal{C}^\infty(\mathcal{M})$, where $\cdot$ is used to denote the group action on both spaces. We will see examples of quantum geometries where these two properties hold. While there is in general no clear choice of quantization map for a given symplectic manifold, in the presence of sufficient symmetries[2], demanding eq.(14) and eq.(15) lets one, in some cases,

[2]For example, the action of a Lie group.

choose a unique quantization map.

Finally we note a corollary of demanding that a quantization map satisfies eq.(14). Together with condition 1 in definition 3.1, one has that

$$\dim\mathcal{H} = \mathrm{Tr}(\mathbb{1}) = \frac{1}{(2\pi\alpha)^n}\int_{\mathcal{M}}\Omega = \frac{1}{(2\pi\alpha)^n}\mathrm{Vol}_\omega(\mathcal{M}). \tag{16}$$

Eq.(16) says that, for finite dimensional $\mathcal{H}$ and compact $\mathcal{M}$, the symplectic volume of $\mathcal{M}$ must always be an integer multiple of $(2\pi\alpha)^n$. This is related to the Bohr-Sommerfeld quantization condition used in the early stages of the development of quantum mechanics [23]. There is an underlying geometrical reason for this equation, as the symplectic form ω on a symplectic manifold that can be quantized, can be seen to arise as the curvature of a $U(1)$ bundle. In this picture, the quantization condition eq.(16) is simply a consistency condition on the holonomy along a closed path γ on a 2-cycle S^2 [19].

For a more thorough discussion of what requirements one might have on a quantization map, from a physical as well as a mathematical point of view, see for example [24].

3.1.1. Quantization vs. de-quantization

Why is the train of thought of the previous section going the wrong way? While the concept of an quantization map is built upon the idea of quantizing a known classical system (or in our case geometry) and finding a corresponding quantum system (or quantum geometry), it is a physical fact that quantum mechanics is actually more fundamental than classical physics. This means one should actually be concerned with the process of *de-quantization*, e.g. starting from a quantum system and finding an appropriate classical system related to it, and it should be clear that any correspondence found, can only hold approximately and at a certain scale, as classical physics can only be an effective description of quantum physics. In the same spirit, one should consider quantum geometry as more fundamental and, starting from it, try to extract a classical geometry as an effective description of it, that holds in some regime. Such a classical geometry will be called a *semi-classical limit* of a quantum geometry.

We have seen that a non-commutative space corresponding to a symplectic manifold can be constructed via elements of a non-commutative algebra $\mathrm{End}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$, e.g. matrices. It is thus natural to consider such configurations as working definitions of quantum geometries and try to extract a semi-classical limit for a given non-commutative algebra. This framework goes under the name of *matrix geometry* and will be further discussed in the following.

3.2. Matrix geometries

Let $\mathcal{H}$ be a finite dimensional, separable Hilbert space and consider a set of D hermitian matrices $\{X^a\}_{a=1}^D \subset \mathrm{End}(\mathcal{H})$, which will be called a *matrix configuration*. Such a matrix configuration will be the non-commutative space we consider. While one can not hope to find a sensible semi-classical limit for any randomly given matrix configuration, we will see in section 4.1, how matrix models give a natural selection criterion for useful matrix configurations, that induce well-behaved semi-classical manifolds.

How can one now extract a semi-classical limit from a given matrix configuration? As seen in the previous section, quantization maps give the correspondence between quantum geometries and commutative functions. Thus, a natural way to relate some classical geometry to a matrix configuration is by viewing the matrices X^a as *quantized embedding functions* for some submanifold of target space $\mathbb{R}^D$. More precisely, let $\sim$ denote the semi-classical limit. Then given a matrix configuration $\{X^a\}_{a=1}^D$ one defines a quantization map $\mathcal{Q}$ such that

$$X^a = \mathcal{Q}(x^a) \sim x^a : \ \mathcal{M} \hookrightarrow \mathbb{R}^D \tag{17}$$

where the x^a are the Cartesian coordinate functions on target space $\mathbb{R}^D$ pulled back on the variety $\mathcal{M}$. In general, the term embedding in this context is to be understood in a lose sense, as sometimes, the maps x^a might be degenerate in various ways, however, in the explicit cases that are of interest here, the embedding is an actual one. On $\mathcal{M}$, one can immediately define a Poisson tensor by recalling condition 3 in definition 3.1:

$$\theta^{ab} = \{x^a, x^b\} \sim -i[X^a, X^b]. \tag{18}$$

This construction is very close to the concept of D-branes embedded in target space as known from string theory, however in this thesis, we will not focus on this connection and only consider it in its own right [23].

Having these ideas at hand, the program of matrix geometries can thus be stated as follows:

Given a matrix configuration $\{X^a\}_{a=1}^D$, can one find a symplectic manifold $\mathcal{M}$ embedded into $\mathbb{R}^D$ via maps $x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D$ in such a way that $X^a \sim x^a$, e.g. the X^a can be viewed as quantization of the embedding functions, in a suitable sense.

Surprisingly, the answer is yes. There indeed is a general procedure to associate geometric structure to arbitrary matrix configurations. This can be achieved via *quasi-coherent states*, which give a general definition of quantum geometry and are based on concepts from geometrical phases in quantum mechanics [19]. While being general in their applicability, for arbitrary matrix configuration, the extracted geometries will be far from well-behaved. Possible criteria for the quality of the semi-classical limit of a matrix configuration will be discussed in section 4.1. However, if we consider a matrix configuration for which $\mathrm{End}(\mathcal{H})$ is a rep of a Lie group G, then one can always find a semi-classical symplectic manifold $\mathcal{M}$ that also carries an action of G such that the quantization map is an intertwiner of the group action. This approach is based on the rep theory of the underlying symmetry group G and is known as *quantized coadjoint orbits.* These two approaches are, in general, different, however in the presence of sufficient symmetries, the two quantization methods agree. In the following, both approaches will be reviewed, with a focus on the cases where they both are applicable, adopting a best of both worlds point of view.

3.2.1. Quasi-coherent states

The following is a short and rough overview over the framework of quasi-coherent states as an approach to matrix geometries based on [19]. For a far more detailed discussion, see the source and references therein.

Consider a matrix configuration $\{X^a\}_{a=1}^D \subset \mathrm{End}(\mathcal{H})$. A matrix configuration is called irreducible, if for every $F \in \mathrm{End}(\mathcal{H})$, $[F, X^a] = 0$ for $a = 1, ..., D$ implies that $F = c \cdot \mathbb{1}$ for some $c \in \mathbb{C}$. Given an irreducible matrix configuration and a point $x = (x^1, ..., x^D) \in \mathbb{R}^D$, one defines the *displacement Hamiltonian*

$$H_x := \frac{1}{2}\sum_{a=1}^{D}\left(X^a - x^a\mathbb{1}\right)^2. \tag{19}$$

H_x then is a positive definite hermitian operator on $\mathcal{H}$ and one wants to study its eigenvectors. For this, let $\lambda(x) > 0$ be the lowest eigenvalue of H_x. One then defines the quasi-coherent state $|x\rangle$ at x as a normalized eigenvector

$$H_x |x\rangle = \lambda(x) |x\rangle \tag{20}$$

to the displacement Hamiltonian. For simplicity, one assumes that the corresponding eigenspaces E_x are one dimensional, except for some singular subset $\mathcal{K} \subset \mathbb{R}^D$. To keep things well-defined and unambiguous, one thus considers

$$\tilde{\mathbb{R}}^D := \mathbb{R}^D \setminus \mathcal{K}. \tag{21}$$

Why are we interested in the smallest eigenvalue of H_x? This is due to the following considerations. For a given matrix configuration, look at the maps

$$\begin{aligned} \mathbf{x}^a : \tilde{\mathbb{R}}^D &\to \mathbb{R} \\ x &\mapsto \langle x| X^a |x\rangle . \end{aligned} \tag{22}$$

For arbitrary matrix configurations, $\mathbf{x} = (\mathbf{x}^1, ..., \mathbf{x}^a) \neq x$ and one can measure the deviation by the *displacement*:

$$d^2(x) := \sum_{a=1}^{D}(\mathbf{x}^a(x) - x^a)^2. \tag{23}$$

The quality of the matrix configuration itself, which ultimately relates to the quality of the semi-classical limit, is measured by the *dispersion*:

$$\begin{aligned} \delta^2(x) &:= \sum_{a=1}^{D}(\Delta X^a)^2(x) \\ (\Delta X^a)^2(x) &:= \langle x| \left(X^a - \mathbf{x}^a(x)\right)^2 |x\rangle . \end{aligned} \tag{24}$$

If now both displacement $d^2(x)$ and dispersion $\delta^2(x)$ are small, one has that $X^a \approx \mathbf{x}^a \approx x^a$, so that one can most likely find a suitable semi-classical limit such that $X^a \sim x^a$. Using the defining equation eq.(20) together with eq.(23) and eq.(24), one can show that

$$\lambda(x) = \delta^2(x) + d^2(x), \tag{25}$$

hence, both, displacement and dispersion are bounded by $\lambda(x)$. So we see that the distinguishing feature of quasi-coherent states is, that they are the unique states with minimal dispersion and displacement for a given matrix configuration.

Next, in order to construct a semi-classical limit, one considers the union of the normalized quasi-coherent states

$$\mathcal{B} := \cup_{x\in\tilde{\mathbb{R}}^D} |x\rangle \subset \mathcal{H} \cong \mathbb{C}^N \tag{26}$$

and views this as a $U(1)$ bundle over a space $\mathcal{M}'$, where $\mathcal{M}'$ is

$$\mathcal{M}' := \mathcal{B}/U(1) \hookrightarrow \mathbb{C}P^{N-1} \tag{27}$$

and points in $\mathcal{M}'$ are therefore unit rays $|x\rangle_{U(1)}$. This however, is in general not a smooth manifold. In order to construct a smooth manifold from $\mathcal{M}'$, one can use the following map

$$q : \tilde{\mathbb{R}}^D \to \mathcal{M}' \tag{28}$$

$$x \mapsto |x\rangle_{U(1)} \tag{29}$$

which is a globally defined, smooth and surjective map [21]. Whenever q has constant rank, we can use it to construct local coordinates, so that one defines

$$\hat{\mathbb{R}}^D := \{x \in \tilde{\mathbb{R}}^D : \mathrm{rk}(T_x q) = k\} \subset \tilde{\mathbb{R}}^D \tag{30}$$

which is open and k is the maximal rank of q. Then the space

$$\mathcal{M} := q|_{\hat{\mathbb{R}}^D}(\hat{\mathbb{R}}^D) \tag{31}$$

can be given the structure of a smooth manifold, using q and is refereed to as *abstract quantum space*. More concretely, it is a smooth immersed submanifold of $\mathbb{C}P^{N-1}$ with (real) dimension k [21]. Note that if q is actually a bijection, e.g. has constant maximal rank on all of $\tilde{\mathbb{R}}^D$, then $\mathcal{M}' = \mathcal{M}$ is a D dimensional, immersed submanifold of $\mathbb{C}P^{N-1}$.

The maps eq.(22) are naturally extended to $\mathcal{M}'$:

$$\begin{aligned} \mathbf{x}^a : \mathcal{M}' &\to \mathbb{R}^D \\ |x\rangle_{U(1)} &\mapsto \mathbf{x}^a := \langle x| X^a |x\rangle \end{aligned} \tag{32}$$

as any phase ambiguity drops out. These then give some variety $\tilde{\mathcal{M}} = \mathbf{x}(\mathcal{M}') \subset \mathbb{R}^D$ in target space $\mathbb{R}^D$, which is denoted as *embedded quantum space*. This is exactly the embedding we wish eq.(17) to hold for, however, recall that this embedding might be degenerate.

The abstract quantum space $\mathcal{M}$ then has a symplectic form $\omega_\mathcal{M}$ and a metric $g_\mathcal{M}$ defined on it, both of which can be extracted from $\mathbb{C}P^{N-1}$ in the following way:

Recall that

$$T_{|\psi\rangle_{U(1)}}\mathbb{C}P^{N-1} \cong \{|v\rangle \in \mathbb{C}^N : \langle\psi|v\rangle = 0\} \tag{33}$$

e.g. the tangent space at a point of $\mathbb{C}P^{N-1}$, can be identified with vectors perpendicular to a representative[3]. There exists a hermitian form h on $T\mathbb{C}P^{N-1}$, given by

$$h(|\psi\rangle_{U(1)})(|v\rangle\,,|w\rangle) = \langle v|w\rangle\,, \tag{34}$$

that can be split into real and imaginary parts $h = g + i\omega$, where g and ω are constant multiples of the Fubini-Study metric and the Kirillov-Kostant-Souriau (KKS) symplectic form on $\mathbb{C}P^{N-1}$ respectively [25]. Since $\mathcal{M}$ is an immersed submanifold, one can pull-back h onto $\mathcal{M}$, giving $h_{\mathcal{M}} = g_{\mathcal{M}} + i\omega_{\mathcal{M}}$, equipping $\mathcal{M}$ with a metric $g_{\mathcal{M}}$ and a closed 2-form $\omega_{\mathcal{M}}$. In general, no statement about the degeneracy of $\omega_{\mathcal{M}}$ can be made, however for the following considerations, we will assume that $\omega_{\mathcal{M}}$ is non-degenerate, giving us a symplectic form on $\mathcal{M}$. We will see that this assumption is satisfied for the case of quantized coadjoint orbits, which will be the class of matrix geometries that are studied in the later sections of this thesis. Furthermore, one notes that it can be shown that for $\omega_{\mathcal{M}}$ non-degenerate, the induced Poisson tensor (θ^{ab}) agrees with the Poisson tensor obtained from the matrix configuration in the semi-classical limit via

$$\theta^{ab}(|x\rangle_{U(1)}) \approx -i\,\langle x|\,[X^a, X^b]\,|x\rangle \tag{35}$$

as it should for a unique Poisson structure. Note that eq.(35) holds only approximately in general, with the quality of the approximation increasing for larger sized matrices [19].

Having this, one can define the following quantization map:

$$\begin{aligned} \mathcal{Q}:\ \mathcal{C}^\infty(\mathcal{M}) &\to \mathrm{End}(\mathcal{H}) \\ \phi \mapsto \Phi &:= \int_{\mathcal{M}} \phi(x)\,|x\rangle\,\langle x| \end{aligned} \tag{36}$$

where the integral is defined as

$$\int_{\mathcal{M}} \phi(x) := \frac{1}{(2\pi\alpha)^n}\int_{\mathcal{M}} \Omega_{\mathcal{M}}\phi(x) \tag{37}$$

where $\Omega_{\mathcal{M}}$ is the symplectic volume form, α is a normalization constant to be determined on a case-to-case basis for every matrix configuration and $\dim \mathcal{M} = 2n$.

One can further define a de-quantization map, also called the *symbol map*

$$\begin{aligned} \mathrm{Sym} : \mathrm{End}(\mathcal{H}) &\to \mathcal{C}^\infty(\mathcal{M}) \\ \Phi &\mapsto \mathrm{Sym}(\Phi) \end{aligned} \tag{38}$$

where

$$\mathrm{Sym}(\Phi)(|x\rangle_{U(1)}) = \langle x|\,\Phi\,|x\rangle =: \phi(x), \tag{39}$$

which can only be an approximate inverse to the quantization map, as $\mathrm{End}(\mathcal{H})$ is finite dimensional and $\mathcal{Q}$ can thus not be injective. One would now have to check, whether

[3] Again, note that there is no phase ambiguity.

the conditions in definition 3.1 are satisfied. While that is an interesting and non-trivial question, it will not be answered in full generality here, rather, we will investigate it in section 3.2.2 for the case important for this thesis, namely quantized coadjoint orbits.

Using the quasi-coherent state approach has many advantages. One can straightforwardly extend the formalism to spinors and Dirac-type operators and thus consider fermionic quantities. Furthermore, the formalism lends itself excellently to numerical implementation. Multiple investigations have shown that using quasi-coherent states, it is possible to extract high-quality numerical simulations of the classical geometries associated to a given matrix configuration, not only in the case of well-studied examples of quantum geometries, but also for deformations thereof [21, 22].

3.2.2. Quantized coadjoint orbits

The following section consist of a short summary of the properties of coadjoint orbits, with a focus on the features that make them suitable for quantization, followed by a rough explanation of the quantization procedure, which is, in simple terms, based on a matching of irreducible representations (irreps) of the underlying Lie group G.

Mathematical interlude: Coadjoint orbits Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. G acts on itself via conjugation, e.g. let $g \in G$, then

$$\begin{aligned} \mathrm{conj}_g : G &\to G \\ h &\mapsto ghg^{-1}. \end{aligned} \tag{40}$$

The identity element e of G is a fixed point of this map for every $g \in G$ and thus gives a preferred point for differentiation, which gives the adjoint action:

$$\begin{aligned} \mathrm{Ad}(g) : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto \mathrm{Ad}(g)(X) \coloneqq T_e \mathrm{conj}_g(X). \end{aligned} \tag{41}$$

Letting the element one conjugates by vary, one obtains the adjoint rep of the Lie group G on its Lie algebra $\mathfrak{g}$:

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}(g). \end{aligned} \tag{42}$$

One can dualize this construction, yielding the coadjoint rep of a Lie group G on the dual of its Lie algebra $\mathfrak{g}^*$,

$$\begin{aligned} \mathrm{Ad}^* : G &\to \mathrm{Aut}(\mathfrak{g}^*) \\ g &\mapsto \mathrm{Ad}^*(g) \end{aligned} \tag{43}$$

where for every $\mu \in \mathfrak{g}^*$

$$\begin{aligned} \mathrm{Ad}^*(g)(\mu) : \mathfrak{g} &\to \mathbb{R} \\ X &\mapsto \mathrm{Ad}^*(g)(\mu)(X) \coloneqq \mu(\mathrm{Ad}(g^{-1})(X)) \end{aligned} \tag{44}$$

as by the usual definition of a dual rep. The same considerations lead to the coadjoint rep ad^* of the Lie algebra $\mathfrak{g}$ on its dual $\mathfrak{g}^*$, which is defined as

$$\begin{aligned} \mathrm{ad}^*(X)(\mu) : \mathfrak{g} &\to \mathbb{R} \\ Y &\mapsto \mathrm{ad}^*(X)(\mu)(Y) := -\mu([X,Y]) \end{aligned}$$

for $X, Y \in \mathfrak{g}, \mu \in \mathfrak{g}^*$. Using the coadjoint rep Ad^*, one can act on elements of the dual space $\mathfrak{g}^*$ and consider the following orbit trough such elements,

$$\mathcal{O}_\mu := \{\mathrm{Ad}^*(g)\mu : \ \forall g \in G\} \subset \mathfrak{g}^* \tag{45}$$

which is then called *coadjoint orbit through* μ. These orbits are obviously invariant under the action of the coadjoint rep

$$\mathrm{Ad}^*(g)(\mathcal{O}_\mu) \subset \mathcal{O}_\mu \tag{46}$$

and as for every orbit of the action of a Lie group, one has that the coadjoint orbits are homogeneous spaces, e.g.

$$\mathcal{O}_\mu \cong G/G_\mu \tag{47}$$

where

$$G_\mu := \{g \in G : \ \mathrm{Ad}^*(g)(\mu) = \mu\} \tag{48}$$

is called the *stabilizer* of μ. From eq.(47), standard theorems show that $\mathcal{O}_\mu$ is naturally a submersed submanifold of $\mathfrak{g}^*$.

The importance of this construction in the context of quantum geometries stems from the following fact: On each coadjoint orbit $\mathcal{O}_\mu$, there is a G-invariant symplectic form ω, defined by

$$\omega(\nu)(\mathrm{ad}^*_X(\nu), \mathrm{ad}^*_Y(\nu)) := \nu([X,Y]) \tag{49}$$

for all $\nu \in \mathcal{O}_\mu, X, Y \in \mathfrak{g}$. That this map is well-defined can be seen from the following characterization of the tangent spaces,

$$T_\nu\mathcal{O}_\mu = \{-\mathrm{ad}^*(X)(\nu) : \ X \in \mathfrak{g}\} \tag{50}$$

which can be identified with $\mathfrak{g}/\mathfrak{g}_\nu$, where $\mathfrak{g}_\nu$ is the Lie algebra of the stabilizer G_ν, as the coadjoint action is transitive on each $\mathcal{O}_\mu$. This construction is canonical and the resulting symplectic form ω is referred to as Kirillov-Kostant-Souriau (KKS) symplectic form[4] [26].

Semi-classical limit via irreducible representations We will now turn to the problem of constructing a matrix geometry, whose semi-classical limit is given by a coadjoint orbit. This construction makes heavy use of the representation theory of the associated Lie group G and its Lie algebra $\mathfrak{g}$ and while it is rather technical, we will see in the following paragraph how using quasi-coherent states in the context of coadjoint orbits leads to a

[4] Compare with section 3.2.1.

more clear and accessible approach.

First, we choose a maximal Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a positive root system. Let then $\lambda \in \mathfrak{g}^*$ be a dominant integral element. The theorem of highest weight tells us that there is a unique finite dimensional irrep $\mathcal{H}_\lambda$ of $\mathfrak{g}$, together with an inner product that is invariant under the action of $\mathfrak{g}$, which has highest weight λ. Furthermore, the non-commutative algebra $\mathrm{End}(\mathcal{H}_\lambda)$ inherits an action of $\mathfrak{g}$, as well as an invariant inner product. As $\mathcal{H}_\lambda$ is finite dimensional, $\mathrm{End}(\mathcal{H}_\lambda)$ can be identified with square-matrices acting on $\mathcal{H}_\lambda$ and this space will thus serve as our matrix geometry. Choosing a basis of $\mathrm{End}(\mathcal{H}_\lambda)$ gives the corresponding matrix configuration. The crucial point is now the following observation:

$$\mathrm{End}(\mathcal{H}_\lambda) \cong \mathcal{H}_\lambda \otimes \mathcal{H}_\lambda^* \cong \bigoplus_{\mu} N_{\lambda\lambda^+}^{\mu} \mathcal{H}_\mu \tag{51}$$

where $\mathrm{End}(\mathcal{H}_\lambda)$ is decomposed into harmonics via the fusion rules of highest weight reps of $\mathfrak{g}$ and $\mathcal{H}_\mu$ are the corresponding highest weight G-modules. The idea is now to find a semi-classical limit carrying a suitable rep of $\mathfrak{g}$ and construct the quantization map by mapping the corresponding irreps. Such a space is given precisely by a coadjoint orbit of G. To see this, one considers the coadjoint orbit $\mathcal{O}_\lambda$ through λ:

$$\mathcal{M} \coloneqq \mathcal{O}_\lambda \cong G/G_\lambda. \tag{52}$$

Then the space of smooth functions $\mathcal{C}^\infty(\mathcal{O}_\lambda)$ on $\mathcal{O}_\lambda$ inherits an action of the group G and thus decomposes into irreps in the following way:

$$\mathcal{C}^\infty(\mathcal{O}_\lambda) = \mathcal{C}^\infty(\mathcal{M}) \cong \bigoplus_{\lambda \in P^+} \mathrm{mult}_{\lambda^+}^{(G_\lambda)} \mathcal{H}_\lambda \tag{53}$$

Here P^+ is the set of all dominant integral weights and $\mathrm{mult}_{\lambda^+}^{(G_\lambda)}$ is the dimension of the subspace of $\mathcal{H}_\lambda$ invariant under the stabilizer G_λ. λ^+ denotes the conjugate weight to λ, so that $\mathcal{H}_{\lambda^+} \cong \mathcal{H}_\lambda^*$. In order to view $\mathcal{M}$ as the semi-classical limit of the matrix geometry $\mathcal{H}_\lambda$, one needs to construct a suitable quantization map $\mathcal{Q}$. Having G-invariance as the guiding principle, this is achieved by matching harmonics in $\mathrm{End}(\mathcal{H}_\lambda)$ with harmonics in $\mathcal{C}^\infty(\mathcal{M})$. Since $\dim(\mathcal{H}_\lambda) < \infty$, this can only be done up to cut off. To identify the cut off, one notes that when μ is smaller then all nonzero Dynkin labels of λ, then

$$N_{\lambda\lambda^+}^{\mu} = \mathrm{mult}_{\mu^+}^{(G_\lambda)} \tag{54}$$

so that the cut off is precisely at the highest weight $\bar{\mu}$ such that it is still smaller then all non-zero Dynkin labels of λ. One then defines $\mathcal{Q}$ as the isomorphism between the irreps of same highest weight μ, which has the additional property of being norm-preserving [21, 26, 27, 28].

While structurally beautiful, this construction is not very explicit and needs quite some structure theory of semi-simple Lie algebras. In the following, we will see how quasi-coherent states can be used to give an explicit and more transparent construction.

Semi-classical limit via quasi-coherent states As in section 3.2.2, one starts with a semi-simple Lie group G, and its Lie algebra $\mathfrak{g}$, where dim $G = D$. Furthermore, one again chooses a maximal Cartan subalgebra $\mathfrak{h}$, a set of positive roots and a dominant integral element $\mu \in \mathfrak{h}^*$. By the theorem of highest weight, we get a unique finite dimensional irrep $\mathcal{H}_\mu$, that we again take as our Hilbert space. Choosing orthogonal generators T^a, $a = 1, ..., D$ in $\mathfrak{g}$ and an ONB $v^1, ..., v^N$ of $\mathcal{H}_\mu$, the generators then act as Hermitian matrices T^a_μ on $\mathcal{H}_\mu$.

As G is semi-simple, we have that for any irrep with highest weight μ

$$\sum_{a=1}^{D} T^a T^a = C^2_\mu \mathbb{1} \tag{55}$$

e.g. the quadratic Casimir acts as a scalar multiple of the identity operator and we denote this constant by C^2_μ, which only depends on the weight. Observing this, we define the matrix configuration $\{X^a\}_{a=1}^D$ that we want to consider by

$$X^a := \frac{1}{C_\mu} T^a_\mu. \tag{56}$$

Due to this normalization, the displacement Hamiltonian eq.(19) takes a simple form:

$$H_x = \frac{1}{2}(1 + |x|^2)\mathbb{1} - \sum_{a=1}^{D} x^a X^a. \tag{57}$$

Next, one has to find the quasi-coherent states of the matrix configuration, which can be achieved by using the available symmetries. First, one notes that the eigenstate of H_x with lowest eigenvalue, will be the eigenstate of $\sum_a x^a X^a$ with highest eigenvalue. Secondly, note that using the dual T^{a*} of T^a w.r.t. the Killing form, one can view each $x \in \tilde{\mathbb{R}}^D$ as an element $x \in \mathfrak{g}^*$ by

$$x := \sum_{a=1}^{D} x^a T^{a*}. \tag{58}$$

Then one has that for any $g \in G$,

$$\begin{aligned}\sum_{a=1}^{D} x^a X^a &= \sum_{a=1}^{D} \mathrm{Ad}(g)^{-1}\big(\mathrm{Ad}(g)(x^a X^a)\big) = \sum_{a,b=1}^{D} \mathrm{Ad}(g)^{-1}\big(x^a \mathrm{Ad}(g)^{ab} X^b\big) \\ &= \sum_{a=1}^{D} \mathrm{Ad}(g)^{-1}\big(\mathrm{Ad}^*(g)(x)^a X^a\big)\end{aligned} \tag{59}$$

which implies that for every $x \in \tilde{\mathbb{R}}^D$, also $(\mathrm{Ad}^*(g)(x)^a) \in \tilde{\mathbb{R}}^D$[5], as well as

$$|\mathrm{Ad}^*(g)(x)\rangle_{U(1)} = g \cdot |x\rangle_{U(1)}. \tag{60}$$

[5] Here $\mathrm{Ad}^*(g)(x) = \sum_a \mathrm{Ad}^*(g)(x)^a T^{a*}$, as before.

Here $\cdot$ denotes the action of G corresponding to the representation on the highest weight space $\mathcal{H}_\mu$ [6]. With this, it is natural to define the following orbits

$$\begin{aligned}\mathcal{O}_x &:= \{\mathrm{Ad}^*(g)(x) :\ g \in G\} =: \mathrm{Ad}^*(G)(x) \subset \mathfrak{g}^* \\ \mathcal{O}_{|\psi\rangle} &:= \{g \cdot |\psi\rangle_{U(1)} :\ g \in G\} =: G \cdot |\psi\rangle_{U(1)} \subset \mathbb{C}P^{N-1},\end{aligned} \tag{61}$$

for every $x \in \tilde{\mathbb{R}}^D$ and $|\psi\rangle_{U(1)} \in \mathcal{H}_\mu$. From eq.(60) it can be seen, that if one considers the orbits through quasi-coherent states, then for every $x \in \tilde{\mathbb{R}}^D$, the stabilizers of the orbits $\mathcal{O}_x$ and $\mathcal{O}_{|x\rangle}$ agree.

Next, consider the weight basis $\{|\mu^a\rangle\}$ of $\mathcal{H}_\mu$, ordered such that $|\mu^0\rangle = |\mu\rangle$. Lie algebra theory tells us that every orbit $\mathcal{O}_x$ contains at least one x_0 that lies in the closure of the fundamental Weyl chamber in $\mathfrak{h}^*$ [29], which implies that then $\sum_a x_0^a X^a \in \mathfrak{h}$ and thus, by the property of the weight vectors,

$$\sum_{a=1}^D x_0^a X^a \left|\mu^b\right\rangle = \mu^b\left(\sum_{a=1}^D x_0^a X^a\right) = \frac{\left(\mu^b, x_0\right)}{C_\mu}\left|\mu^b\right\rangle, \tag{62}$$

where $\left(\cdot,\cdot\right)$ is the dual to the Killing form on $\mathfrak{h}^*$. Due to μ being the highest weight, we have that

$$\left(\mu^b, x_0\right) \leq \left(\mu, x_0\right) \tag{63}$$

and thus the smallest eigenvalue of H_{x_0} is given by

$$\lambda(x_0) = \frac{1}{2}(1 + |x_0|^2) - \frac{\left(\mu, x_0\right)}{C_\mu} \tag{64}$$

corresponding to the lowest eigenstate

$$|x_0\rangle := |\mu\rangle\,. \tag{65}$$

If x_0 lies in the interior of the fundamental Weyl chamber, this is the unique (up to a phase) quasi-coherent state at x_0, however if x_0 lies on the border, then the eigenspace E_{x_0} can be degenerate [21]. This however concludes the search for the quasi-coherent states of this matrix configuration, as we note that for any $x \in \mathbb{R}^D$, one can always write $x = \mathrm{Ad}^*(g)(x_0)$ for some $g \in G$ and x_0 in the closure of the fundamental Weyl chamber. If x_0 lies on the border, it is possible that $x \in \mathcal{K}$, otherwise $x \in \tilde{\mathbb{R}}^D$ and

$$|x\rangle = g \cdot |\mu\rangle \ \text{ with } \lambda(x) = \frac{1}{2}(1 + |x|^2) - \frac{\left(\mu, x_0\right)}{C_\mu} \tag{66}$$

is the quasi-coherent state at x. Eq.(66) shows that

$$\mathcal{M} = \mathcal{O}_{|\mu\rangle} \cong \mathcal{O}_\mu \tag{67}$$

[6] Note that here one assumes the action of G on $\mathcal{H}_\mu$ is well-defined, e.g. G is simply-connected. Otherwise one has to restrict oneself to the connected component of the identity of G.

e.g. the semi-classical limit is given by the coadjoint orbit through the highest weight μ. Thus, $\omega_{\mathcal{M}}$ as obtained in section 3.2.1, agrees with the canonical KKS symplectic form on $\mathcal{O}_\mu$ as described in section 3.2.2, up to a constant factor. In particular, this means that $\omega_{\mathcal{M}}$ is G-invariant and therefore $\Omega_{\mathcal{M}}$ is too. The quasi-coherent state quantization map $\mathcal{Q}$ as defined in eq.(36), is thus an intertwiner of the action of G on $\mathcal{C}^\infty(\mathcal{M})$ and $\mathrm{End}(\mathcal{H}_\mu)$, which implies that it maps irreps to irreps, e.g. it respects the decomposition of the two spaces as discussed in section 3.2.2. This immediately implies the following:

$\mathcal{Q}$ maps $\mathrm{id}_{\mathcal{M}}$ to $\mathbb{1}_{\mathcal{H}_\mu}$, as they both lie in the trivial rep. Similarly, one has that $\mathcal{Q}(\mathbf{x}^a) = c \cdot X^a$, for a scalar c, as both lie in the adjoint rep. Indeed, all conditions in definition 3.1 are satisfied, but $\mathcal{Q}$ is not an isometry. This is a manifestation of the fact, that although the semi-classical limit $(\mathcal{M}, \omega_{\mathcal{M}})$ and the non-commutative algebra $\mathrm{End}(\mathcal{H}_\mu)$ agree for the two constructions discussed in section 3.2.2 and section 3.2.2, the quantization maps are different. This means that as non-commutative geometries, the two constructions do not fully coincide, but are rather two different objects [19, 21, 22].

Before discussing the covariant quantum space-times, it is instructive to discuss this formalism for a well-known example of matrix geometries, namely the *fuzzy sphere.*

3.2.3. Example: The fuzzy sphere S^2_N

The fuzzy sphere S^2_N [30, 31] is the prototypical example of a matrix geometry and the non-commutative analogue to the regular sphere S^2. Here the corresponding symmetry group is $SU(2)$ and the matrix configuration is defined by

$$X^a := \frac{1}{C_N} J^a_N, \ a = 1, 2, 3 \tag{68}$$

where the J^a_N are the generators of the N-dimensional irrep of $\mathfrak{su}(2)$ on $\mathcal{H}_N := \mathbb{C}^N$ and the quadratic Casimir takes the value $C_N = \frac{1}{4}(N^2-1)$. The Lie algebra relation and the normalized quadratic Casimir then read:

$$[X^a, X^b] = \frac{i}{C_N} \sum_{c=1}^{3} \varepsilon^{abc} X^c, \quad \sum_{a=1}^{3} X^a X^a = \mathbb{1}. \tag{69}$$

The second equation in eqs.(69) is the non-commutative version of the definition of a sphere $\sum_a x^a x^a = 1$. This is the first obvious hint as to how this matrix configuration relates to the classical commutative sphere. As discussed in section 3.2.2, the displacement Hamiltonian $H_x, \ x \in \mathbb{R}^3$ takes the simple form as given in eq.(19) and the quasi-coherent state at x is determined by the highest weight vector $|N\rangle := \left|\frac{N-1}{2}, \frac{N-1}{2}\right\rangle$ of $\mathcal{H}_N$ via the following group action:[7]

Consider the adjoint rep. of $SU(2)$, e.g. $SO(3)$. For arbitrary $x \in \mathbb{R}^3$, let $R_x \in SO(3)$ be the unique rotation such that $x = |x| R_x^{-1} n_3$ where $n_3 = (0, 0, 1)$. As $SU(2)$ is the double

[7]See section 3.2.2 for the general discussion of this and note that since $SU(2)$ is compact, we have identified $\mathfrak{su}(2)$ and $\mathfrak{su}(2)^*$ via the Killing form, and therefore also the adjoint and coadjoint reps.

cover of $SO(3)$, we find some (up to a sign) unique $U_x \in SU(2)$, such that $R_x = \mathrm{Ad}(U_x)$ and the quasi-coherent state together with the corresponding eigenvalue at x is therefore given by

$$|x\rangle = U_x |N\rangle\,,\ \lambda(x) = \frac{1}{2}(1+|x|^2) - |x|\frac{N-1}{2C_N} = \frac{1}{2}(1+|x|^2) - |x|\sqrt{\frac{N-1}{N+1}}. \tag{70}$$

Note that U_x is independent of $|x|$ and thus $|x\rangle = |\alpha x\rangle$ for any $\alpha > 0$.

It is easy to see that the stabilizer of $|N\rangle_{U(1)}$ is given by the one-parameter subgroup generated by J^3_N, which can be identified with $U(1)$, so that one has

$$\mathcal{M} = SU(2)\cdot |N\rangle_{U(1)} \cong SU(2)/U(1) \cong S^2 \tag{71}$$

e.g. the semi-classical limit truly recovers the commutative geometry of the sphere, as it is diffeomorphic to it. The embedding into target space is then given by

$$\begin{aligned} \langle x| X^a |x\rangle &= \langle N| U_x^\dagger X^a U_x |N\rangle = \langle N| \mathrm{Ad}(U_x)(X^a) |N\rangle \\ &= \sum_{b=1}^{3} (R_x^{-1})^{ab} \langle N| X^b |N\rangle = \sqrt{\frac{N-1}{N+1}}(R_x^{-1} n_3)^a = \sqrt{\frac{N-1}{N+1}}\frac{x^a}{|x|} \end{aligned} \tag{72}$$

so that $\tilde{\mathcal{M}}$ also forms a sphere, but instead of unit radius, has radius $\sqrt{\frac{N-1}{N+1}}$.

The closed 2-form $\omega_{\mathcal{M}}$ can be explicitly calculated (see Appendix C in [21]) and is given by

$$(\omega_{\mathcal{M}})_{ab} = \frac{N-1}{4|x|^2}\sum_{c=1}^{3}\varepsilon^{abc}\frac{x^c}{|x|}, \tag{73}$$

while the Poisson tensor as obtained by the semi-classical limit of the matrix commutator[8] reads

$$\theta^{ab} = \frac{2}{N+1}\sum_{c=1}^{3}\varepsilon^{abc}\frac{x^c}{|x|}. \tag{74}$$

Note that $\omega_{\mathcal{M}}$ is actually degenerate, so that these two structures are not inverse to each other. However, for the embedding $\tilde{\mathcal{M}}$ into target space, one has that for the cartesian embedding functions $\mathbf{x}^a$, as given by eq.(32),

$$\{\mathbf{x}^a, \mathbf{x}^b\} = -\frac{4}{N+1}\sum_{c=1}^{3}\varepsilon^{abc}\frac{x^c}{|x|} = -2\theta^{ab}, \tag{75}$$

so that the two Poisson structures agree up to a constant factor, that could be absorbed in a redefinition of $\omega_{\mathcal{M}}$ [19, 21].

Apart from the round case, also a deformed version of this matrix geometry, coined *squashed fuzzy sphere*, has been studied. In order to investigate these deformations, different numerical procedures based on the idea of quasi-coherent states haven been implemented, see for example [21, 22].

[8] See eq.(35).

4. Matrix models and special solutions

In the previous chapter, we discussed how the formalism of matrix geometry relates non-commutative spaces given by matrix configurations, to classical geometries. Recall that while the formalism works for arbitrary matrix configurations[9], random ones will not give rise to a well-behaved semi-classical limit. Thus one has the problem of finding "useful" matrix configurations. One way to address this, is by matrix models, which provide a mechanism that gives well-behaved matrix configurations by making the matrices dynamical objects for which an action functional is stated and a variational principle can be invoked. More precisely, a matrix model is defined in terms of an action functional

$$S:\ \mathrm{End}(\mathcal{H})^{\times D} \to \mathbb{R} \tag{76}$$

applied to an (irreducible) matrix configuration $\{Y^a\}_{a=1}^D \subset \mathrm{End}(\mathcal{H})$. Variation of this action functional then leads to matrix equations of motion (e.o.m.), whose solutions are regarded as defining an emergent geometry through the semi-classical limit and and as giving the background for fields on this emergent space. Similarly, one can fix a matrix configuration, i.e. fix a background and consider a field theory on this non-commutative space. The fields of such a theory are then naturally also given by matrices, so that such theories are also referred to as matrix field theories. For more details on general matrix models and matrix field theory, see for example [32] and sources therein.

4.1. Preferred configurations and quasi-coherent states

First, we will shortly study a simple matrix model. Let $\{Y^a\}_{a=1}^D \subset \mathrm{End}(\mathcal{H})$ be an arbitrary irreducible matrix configuration. We consider the following action

$$S_0[Y] \coloneqq \mathrm{Tr}\Bigg(\delta_{ac}\delta_{bd}[Y^a, Y^b][Y^c, Y^d]\Bigg) \tag{77}$$

which defines a matrix model with euclidean signature, that only consists of a kinematic term. As S_0 is basically the trace of the square of the commutator of the matrices, this action will be minimized by matrix configurations that are almost-commuting, e.g. for which their commutator is small, meaning that they can almost be diagonalized simultaneously. This is where the connection to quasi-coherent states is drawn, as these are precisely the approximate common eigenstates to the Y^a that are optimally localized[10] [19].

In the case of Minkowski signature[11], this argument has to be augmented, as potentially large space-like commutators $[Y^i, Y^j]$ could be compensated by potentially equally large space-time commutators $[Y^j, Y^0]$. To see how, note that such matrix configurations would be solutions with high energy, as measured by the time-like component $E = T^{00}$ of the matrix energy-momentum tensor T^{ab} [33], which for the action S_0 with Minkowski signature, reads

$$E = [Y^0, Y^i][Y^0, Y_i] + \frac{1}{2}[Y^i, Y^j][Y_i, Y_j]. \tag{78}$$

[9] For the case of quantized coadjoint orbits, sufficient symmetries are available.

[10] See the discussion in section 3.2.1.

[11] This is obtained by exchanging δ_{ab} with $\eta_{ab} = \mathrm{diag}(-1, 1, ..., 1)$.

Based on physical grounds, one would argue that the most fundamental and significant solutions are those with minimal energy, which are then exactly those that are almost commuting and the quasi-coherent states formalism yields a semi-classical Poisson manifold [34].

4.2. The IKKT matrix model

The main concern of the present thesis is the emergent geometry of a certain class of solutions of the e.o.m. of the so-called *IKKT matrix model*. This model, which is also called the *IIB matrix model*, was first put forward in 1996, as a constructive definition of superstring theory [2]. It was originally conceived by considering a certain regularization of the worldsheet action of the superstring; however, it can alternatively be viewed as the dimensional reduction of ten-dimensional super-Yang-Mills theory all the way to the point [35]. Even tough it is related to highly non-trivial constructions in superstring theory, as a matrix model, it is extremely easy to define. In this work, we will consider the IKKT matrix model plus an additional mass term, for which the action functional reads:

$$S[Z,\Psi] = \mathrm{Tr}\Big([Z^{\dot\alpha}, Z^{\dot\beta}][Z_{\dot\alpha}, Z_{\dot\beta}] + 2m^2 Z^{\dot\alpha} Z_{\dot\alpha} + \overline{\Psi}\Gamma_{\dot\alpha}[Z^{\dot\alpha}, \Psi]\Big). \tag{79}$$

Here $Z^{\dot\alpha}$, $\dot\alpha = 0, ..., 9$ is a set of 10 hermitian matrices, acting on some separable Hilbert space $\mathcal{H}$, comprising the matrix configuration with respect to which one varies the action. The model exhibits a local $SO(9,1)$ symmetry and indices are contracted with $\eta_{\dot\alpha\dot\beta}$ accordingly. Furthermore, the model is invariant under the following gauge transformations

$$Z^{\dot\alpha} \mapsto U^{-1} Z^{\dot\alpha} U \tag{80}$$

where U is an arbitrary unitary matrix in $\mathrm{End}(\mathcal{H})$.

The mass parameter m^2 sets the scale of the theory and will play an important role in later considerations. The model further includes a fermionic term, expressed trough Majorana-Weyl spinors Ψ of $SO(9,1)$, whose entries are Grassmann-valued matrices, and the gamma matrices $\Gamma_{\dot\alpha}$, which generate the Clifford algebra of $SO(9,1)$. For vanishing mass-parameter $m^2 = 0$, the model is maximally supersymmetric [2]. While general matrix models have loops involving certain string-like modes, which lead to non-locality known as UV/IR mixing [36], the maximally supersymmetric version of the IKKT model avoids these problems [34]. An important feature of matrix models is that they are rather straightforward to quantize, simply by an path integral approach, which for the case of finite dimensional matrices, is only an integration over the space of matrices. In particular consider the IKKT matrix model eq.(79) with vanishing fermionic term. For euclidean signature, the partition function is defined as

$$\mathcal{Z} = \int dZ e^{-S_E[Z]} \tag{81}$$

where dZ is invariant under the gauge transformations in eq.(80). Eq.(81) is well-defined and finite for traceless matrices $Z^{\dot\alpha}$ [37, 38]. In the case of Minkowsi signature, the integral in eq.(81) is oscillating and thus ill-behaved, but can be regularized, by giving the mass-parameter an imaginary part:

$$S_\epsilon[Z] = \mathrm{Tr}\Big([Z^{\dot\alpha}, Z^{\dot\beta}][Z_{\dot\alpha}, Z_{\dot\beta}] + 2m^2(-e^{i\epsilon}(Z^0)^2 + e^{-i\epsilon}Z^i Z_i)\Big)$$
$$\xrightarrow{\epsilon\mapsto 0} \mathrm{Tr}\Big([Z^{\dot\alpha}, Z^{\dot\beta}][Z_{\dot\alpha}, Z_{\dot\beta}] + 2m^2 Z^{\dot\alpha} Z_{\dot\alpha}\Big). \tag{82}$$

Then the partition function

$$\mathcal{Z}_\epsilon = \int dZ e^{iS_\epsilon[Z]} \tag{83}$$

is absolutely convergent for $\epsilon \in (0, \frac{\pi}{2})$[12] and it turns out that this regularization actually exactly realizes Feynman's $i\epsilon$-prescription [34]. While the quantum theory is interesting to study, the remainder of this thesis will focus on the classical level of the IKKT model and we will further restrict ourselves to the bosonic sector as given by the second line in eq.(82).

Varying the bosonic action yields the following e.o.m.,

$$[Z^{\dot\alpha}, [Z_{\dot\alpha}, Z^{\dot\beta}]] + m^2 Z^{\dot\beta} = \Box_Z Z^{\dot\beta} + m^2 Z^{\dot\beta} = 0 \tag{84}$$

where we have defined the *matrix laplacian* $\Box_Z := [Z^{\dot\alpha}, [Z_{\dot\alpha}, \cdot]]$. In the following, we will not address the issue of finding the most general solution of eq.(84). Instead we will focus on a particular kind of solution that yields interesting physics, namely covariant cosmological quantum space-times. Note that in general, if one fixes a background $Z^{\dot\alpha}$ of the matrix model[13] then the fluctuations

$$Y^{\dot\alpha} := Z^{\dot\alpha} + A^{\dot\alpha},\ \ A^{\dot\alpha} \in \mathrm{End}(\mathcal{H}) \tag{85}$$

transform under gauge transformations as

$$A^{\dot\alpha} \mapsto U^{-1} A^{\dot\alpha} U + U^{-1}[Z^{\dot\alpha}, U] \tag{86}$$

which is clearly the matrix analogue to the transformation law for gauge fields in a Yang-Mills type gauge theory. Thus, in matrix models, fluctuations on the underlying space are automatically gauge fields and both are dynamically generated and treated on equal footing. [39].

In order to define the covariant quantum-space times, one first has to study the following solution of the IKKT model.

4.3. Fuzzy hyperboloid H_n^4

The fuzzy hyperboloid H_n^4 was first introduced in [40] while first being studied as a solution of the IKKT model in [4]. In this thesis, we will discuss it as the basis for covariant cosmological quantum space-times, which give the setting for the results presented in sections

[12] This still holds when including the fermionic terms.
[13] E.g. solutions to eq.(84).

8 and 9.

In order to define H_n^4, consider first the $SO(4,2)$ invariant metric $(\eta_{ab}) = \text{diag}(-1,1,1,1,1,-1)$ and let $\mathcal{M}^{ab}$ be the Hermitian generators of the Lie algebra $\mathfrak{so}(4,2)$, which thus satisfy

$$[\mathcal{M}_{ab}, \mathcal{M}_{cd}] = i(\eta_{ac}\mathcal{M}_{bd} - \eta_{ad}\mathcal{M}_{bc} - \eta_{bc}\mathcal{M}_{ad} + \eta_{bd}\mathcal{M}_{ac}), \tag{87}$$

for $a,b = 0,1,2,3,4,5$. In order to define the matrix configuration constituting H_n^4, one chooses a particular type of (series of) unitary irreps $\mathcal{H}_n,\ n \in \mathbb{N}$ called *minireps* [41, 42], which have the property that they stay irreducible under the restriction to the subgroup $SO(4,1) \subset SO(4,2)$ [41, 43, 44]. Furthermore, on the minireps $\mathcal{H}_n$[14], one has that

$$\text{spec}(\mathcal{M}^{05}) = \left\{\frac{n}{2}+1, \frac{n}{2}+2, ...\right\} \tag{88}$$

e.g. the generator $\mathcal{M}^{05}$ has positive discrete spectrum. Next, one defines the following two sets of Hermitian generators

$$\begin{aligned} X^a &:= \tilde{R}\mathcal{M}^{a5}, \quad a = 0,1,2,3,4 \\ T^a &:= \frac{1}{R}\mathcal{M}^{a4}, \quad a = 0,1,2,3,5 \end{aligned} \tag{89}$$

where $\tilde{R}$ and R both have dimension of length and which, as a consequence of eq.(87) satisfy

$$[X^a, X^b] = -i\tilde{R}^2\mathcal{M}^{ab} =: i\Theta^{ab} \tag{90a}$$

$$[T^\mu, X^\nu] = \frac{\tilde{R}}{R}[\mathcal{M}^{\mu 4}, \mathcal{M}^{\nu 5}] = \frac{i}{R}\eta^{\mu\nu}X^4 \tag{90b}$$

$$[T^\mu, X^4] = \frac{\tilde{R}}{R}[\mathcal{M}^{\mu 4}, \mathcal{M}^{45}] = \frac{-i}{R}X^\mu \tag{90c}$$

$$[T^5, X^\mu] = \frac{\tilde{R}}{R}[\mathcal{M}^{54}, \mathcal{M}^{\mu 5}] = -i\tilde{R}T^\nu \tag{90d}$$

$$[T^\mu, T^\nu] = \frac{i}{R^2}\mathcal{M}^{\mu\nu} = \frac{-i}{\tilde{R}^2\tilde{R}^2}\Theta^{\mu\nu} \tag{90e}$$

$$\sum_{b=0}^{4}\left(\mathcal{M}^{ab}X_b + X_b\mathcal{M}^{ab}\right) = 0, \quad a = 0,1,2,3,4 \tag{90f}$$

$$\sum_{b=0,1,2,3,5}\left(\mathcal{M}^{ab}T_b + T_b\mathcal{M}^{ab}\right) = 0, \quad a = 0,1,2,3,5 \tag{90g}$$

where Greek indices run from 0 to 3. Furthermore, if one now restricts to $SO(4,1)$, the X^a transform as vectors under the adjoint action:

$$[\mathcal{M}_{ab}, X_c] = i(\eta_{ac}X_b - \eta_{bc}X_a) \tag{91}$$

[14]Note, that from this point forward, we will use the same symbol for the represented generators on the minireps $\mathcal{H}_n$, as for the abstract ones.

for $a,b,c = 0,...,4$ and since, as mentioned above, the restriction remains irreducible, it follows that the X^a form a hyperboloid

$$\sum_{a,b=0}^{4} \eta_{ab} X^a X^b = \sum_{a=1}^{4} (X^a)^2 - (X^0)^2 = -R^2 \mathbb{1} \tag{92}$$

where the radius is given by [45]

$$R^2 = \frac{\tilde{R}^2}{4}(n^2 - 4). \tag{93}$$

Note that since X^0 has positive spectrum, eq.(92) describes a non-commutative version of a one-sided hyperboloid in 5 dimensions, which is exactly H_n^4. Similarly, the T^a transform as vectors under the restriction to $SO(3,2) \subset SO(4,2)$, which also remains irreducible, so that one has

$$\sum_{a,b=0,1,2,3,5} \eta_{ab} T^a T^b = \sum_{i=1}^{3} (T^i)^2 - (T^5)^2 - (T^0)^2 = \frac{1}{\tilde{R}^2} \mathbb{1} \tag{94}$$

and thus the T^a form the non-commutative version $H^{2,2}$ of a hyperboloid with signature $(--++)$ [45]. For these two irreducible matrix configurations, one has, as a consequence of eq.(90a)-(90e) that

$$\begin{aligned} \Box_X X^a &= -4\tilde{R}^2 X^a \\ \Box_T T^a &= \frac{4}{R^2} T^a \end{aligned} \tag{95}$$

so that both matrix configurations are solutions to the e.o.m. eq.(84). Note that neither the T^a nor the X^a form an algebra that is closed under the commutator[15]. Thus, both sets of generators are needed to fully describe the non-commutative space.

As we have seen in section 3.2.2, the non-commutative algebra $\mathrm{End}(\mathcal{H}_n)$ can be decomposed into unitary irreps, in the present case this can even be done for the subgroups $SO(3,1) \subset SO(4,1) \subset SO(4,2)$, which are organized and interpreted as higher-spin modes. For this, one defines the following spin Casimir [45]:

$$\begin{aligned} \mathcal{S}^2 &:= \frac{1}{2} \sum_{a,b=0}^{4} [\mathcal{M}_{ab}, [\mathcal{M}^{ab}, \cdot]] + \tilde{R}^{-2} [X_a, [X^a, \cdot]] \\ &= 2C^2[\mathfrak{so}(4,1)] - C^2[\mathfrak{so}(4,2)] \end{aligned} \tag{96}$$

where $C^2[\mathfrak{g}]$ denotes the Casimir operator on the corresponding $\mathfrak{g}$-irrep. This satisfies

$$[\mathcal{S}^2, \Box_X] = 0 \tag{97}$$

which means that the two can be simultaneously diagonalized, giving the spin s modes $\mathcal{C}_n^s$

$$\mathrm{End}(\mathcal{H}_n) = \bigoplus_{s=0}^{n} \mathcal{C}_n^s, \quad \mathcal{S}^2|_{\mathcal{C}_n^s} = 2s(s+1) \tag{98}$$

[15] This can be seen from eq.(90a) and eq.(90e).

as has been shown in [45]. In addition, we also have that

$$[\mathcal{S}^2, \Box_T] = 0 \tag{99}$$

so that $\Box_T$ also respects the decomposition in eqs.(98). This decomposition is also given in the semi-classical limit, as will be discussed in section 4.5, where it builds the basis for the higher-spin theory.

4.3.1. Semi-classical geometry $\mathbb{C}P^{1,2}$

Next we want to extract the semi-classical limit associated to H^4_n. For this, we first have to consider the lowest weight state $|\Omega\rangle$ [16] of the minirep $\mathcal{H}_n$, from which all other quasi-coherent states are obtained via the group action: $|m\rangle = g \cdot |\Omega\rangle$, for suitable $g \in SO(4,2)$. As discussed in section 3.2.2, the Poisson manifold $\mathcal{M}$ is then given by the group orbit of the corresponding Lie group G on the lowest (highest) weight state, which in the present case would be

$$\mathcal{M} = SO(4,2) \cdot |\Omega\rangle_{U(1)} \,. \tag{100}$$

Note that it can be shown that the stabilizer K of $|\Omega\rangle$ under the $SO(4,2)$ action is given by $K \cong U(1,2)$ [40], so that

$$\mathcal{M} \cong SO(4,2)/U(1,2) \cong \mathbb{C}P^{1,2} \tag{101}$$

where $\mathbb{C}P^{1,2}$ is the indefinite complex projective space of signature $(p,q) = (1,2)$. We will study this space in more detail in the following.

$\mathbb{C}P^{1,2}$ as a $SO(4,1)$-equivariant bundle over the hyperboloid H^4 Let $H^{4,3}$ denote the 7-hyperboloid

$$H^{4,3} = \{\psi \in \mathbb{C}^4 \ : \ \psi^\dagger \gamma^0 \psi =: \bar{\psi}\psi = 1\} \tag{102}$$

where

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \tag{103}$$

is one of the $SO(4,1)$ gamma matrices $\gamma^a,\ a = 0,1,2,3,4$ [17] and $\mathbb{1}_N$ is the $N \times N$ unit matrix. Using them, one can define the following Hopf-type maps

$$\begin{aligned} (x^a):\ & H^{4,3} \to H^4 \subset \mathbb{R}^{1,4} \\ & \psi \mapsto (x^a) := \frac{\tilde{R}}{2}(\bar{\psi}\gamma^a\psi) \end{aligned} \tag{104}$$

where $a = 0,1,2,3,4$. One can explicitly check that $x^a \in \mathbb{R}$ and

$$\sum_{a,b=0}^{4} \eta_{ab} x^a x^b = -\frac{\tilde{R}}{4} \tag{105}$$

[16]See appendix B for a definition of $|\Omega\rangle$ and a more detailed construction of the lowest weight rep. $\mathcal{H}_n$.
[17]See appendix A for a definition of the whole collection of gamma matrices.

so that (x^a) actually lies in H^4. The map (104) is a non-compact version of the Hopf map $S^7 \to S^4$ and respects $SO(4,1)$, due to the transformation property of the gamma matrices [45]. As the global phase of ψ drops out, the map (104) factorizes and we have that

$$x^a : \; \mathbb{C}P^{1,2} \to H^4 \subset \mathbb{R}^{1,4} \tag{106}$$

where $\mathbb{C}P^{1,2} := H^{4,3}/U(1)$. Since the x^a are intertwiner of $SO(4,1)$, this means that one can see $\mathbb{C}P^{1,2}$ as an $SO(4,1)$-equivariant bundle[18] over H^4. This however, can be used to explicitly exhibit the fibres[19] and one concludes that $\mathbb{C}P^{1,2}$ is a fibre bundle over H^4 with S^2-fibres. Using the quasi-coherent state quantization map $\mathcal{Q}$ as defined in eq.(36) one obtains a $SO(4,2)$-equivariant quantization map

$$\begin{aligned} \mathcal{Q} : \mathcal{C}^\infty(\mathbb{C}P^{1,2}) &\to \mathrm{End}(\mathcal{H}_n) \\ f &\mapsto \int_{\mathbb{C}P^{1,2}} d\mu f(m) \left|m\right\rangle \left\langle m\right| \end{aligned}$$

which in particular satisfies $\mathcal{Q}(x^a) = X^a$, for the bundle projection x^a defined in (104) and furthermore $\mathcal{Q}(T^a) = t^a$ where t^a can be expressed as

$$t^a = \frac{1}{R} \bar{\psi} \Sigma^{a4} \psi \tag{107}$$

where

$$\Sigma^{\mu 4} = -\frac{i}{2} \gamma^\mu \gamma^4, \quad \Sigma^{54} = \frac{1}{2} \gamma^4 \tag{108}$$

are part of the generators of the spinorial rep. of $\mathfrak{so}(4,1)$. Thus H^4_n gives a quantization of the S^2-bundle $\mathbb{C}P^{1,2}$ over H^4.

4.4. Covariant cosmological quantum space-time $\mathcal{M}^{3,1}_n$

In the next step to obtain a physically relevant background, one uses H^4_n to construct a cosmological quantum space-time $\mathcal{M}^{3,1}_n$. This is done in the following way:

From the $SO(4,1)$-covariant objects discussed above, we want to obtain $SO(3,1)$-covariant ones. This is achieved by simply "projecting". e.g. only considering the generators $X^\mu, \ \mu = 0,1,2,3$, which then satisfy

$$\eta_{\mu\nu} X^\mu X^\nu = -R^2 - (X^4)^2 \tag{109a}$$

$$\eta_{\mu\nu} T^\mu T^\nu = \frac{1}{\tilde{R}^2 R^2} \left(R^2 + (X^4)^2\right) \tag{109b}$$

$$X_\mu T^\mu + T^\mu X_\mu = 0 \tag{109c}$$

[18] An G-equivariant bundle $\pi : \; E \to B$ is a fibre bundle where both, the total space E and the base space B are equipped with a group action $\rho_E : \; G \times E \to E$, $\rho_B : \; G \times B \to B$ such that the bundle projection π is an intertwiner, e.g. $\pi \circ \rho_E = \rho_B \circ \pi$.

[19] See appendix C for a more detailed discussion of the fibres.

as a consequence of eq.(92), eq.(94) and eq.(90g) respectively. On the semi-classical level, the resulting space-time $\mathcal{M}^{3,1}$ is given by

$$\begin{aligned} \Pi : \mathbb{C}P^{1,2} &\to \mathcal{M}^{3,1} \subset \mathbb{R}^{3,1} \\ \psi_{U(1)} \mapsto (x^\mu) &= \frac{\tilde{R}}{2}(\bar{\psi}\gamma^\mu\psi) \end{aligned} \tag{110}$$

which can be viewed as a projection of H^4 onto the 0123-plane. As discussed in [5], $\mathcal{M}^{3,1}$ is a cosmological FLRW space-time with $k = -1$. The restriction of eq.(107) to indices running from 0 to 3 gives additional $SO(3,1)$-covariant functions t^μ on $\mathbb{C}P^{1,2}$, which will be discussed in more detail in the following. Finally note that the projection (110) respects the bundle structure, so that $\mathbb{C}P^{1,2}$ is also a S^2-bundle over $\mathcal{M}^{3,1}$.

4.5. Semi-classical cosmological background $\mathcal{M}^{3,1}$

As seen in the previous section, we have obtained an $SO(3,1)$ covariant matrix geometry that is the non-commutative analogue of a cosmological FLRW space-time. The following section is dedicated to the study of the corresponding semi-classical limit $\mathcal{M}^{3,1}$.

First one should note the objects on $\mathcal{M}^{3,1}$, stemming from the associated bundle structure. Recall that the space[20] $\mathcal{C}$ of functions on $\mathbb{C}P^{1,2}$ encode the structure of the equivariant bundle over the space-time $\mathcal{M}^{3,1}$. The functions on the bundle space $\mathbb{C}P^{1,2}$ as obtained in the semi-classical limit then split as follows [3]:

The x^μ are viewed as cartesian embedding functions on $\mathcal{M}^{3,1}$ and generate the algebra $\mathcal{C}^0 \subset \mathcal{C}$ of functions on $\mathcal{M}^{3,1}$, while the t^μ generate the rest of the algebra $\mathcal{C}$, viewed as a module over $\mathcal{C}^0$. In the semi-classical limit, the relations satisfied by the Hermitian generators eq.(90a)-(90g), as well as eq.(92) and eq.(94), amount to the following,

$$x_\mu x^\mu = -R^2 - x_4^2 =: -R^2 \cosh^2 \eta, \tag{111a}$$

$$t_\mu t^\mu = \frac{1}{\tilde{R}^2} \cosh^2 \eta, \tag{111b}$$

$$t_\mu x^\mu = 0, \tag{111c}$$

$$t_\mu \theta^{\mu\alpha} = -\sinh \eta \; x^\alpha \tag{111d}$$

$$x_\mu \theta^{\mu\alpha} = -\tilde{R}^2 R^2 \sinh \eta \; t^\alpha \tag{111e}$$

$$\eta_{\mu\nu}\theta^{\mu\alpha}\theta^{\nu\beta} = R^2\tilde{R}^2\eta^{\alpha\beta} - R^2\tilde{R}^4 t^\alpha t^\beta + \tilde{R}^2 x^\alpha x^\beta \tag{111f}$$

where, using $SO(3,1)$ covariance and the above relations, one can explicitly express the projected Poisson structure $\theta^{\mu\nu}$ in terms of the generators x^μ, t^ν [3]:

$$\theta^{\mu\nu} = \frac{\tilde{R}^2}{\cosh^2 \eta}\bigg(\sinh \eta (x^\mu t^\nu - x^\nu t^\mu) + \varepsilon^{\mu\nu\alpha\beta} x_\alpha t_\beta \bigg). \tag{112}$$

As can be seen from eq.(111a), the cartesian embedding functions

$$X^\mu \sim x^\mu : \; \mathcal{M}^{3,1} \hookrightarrow \mathbb{R}^{3,1} \tag{113}$$

[20] In order to declutter notation, we set $\mathcal{C} := \mathcal{C}^\infty(\mathbb{C}P^{1,2})$.

map onto the region $-x_\mu x^\mu \geq R^2$, while η is a global time coordinate defined by

$$R\sinh\eta = \pm\sqrt{-x_\mu x^\mu - R^2} = x^4 \tag{114}$$

which describes the scale parameter of the universe via [46]

$$a(t)^2 = R^2\cosh^2\eta\sinh\eta. \tag{115}$$

As it turns out, $\mathcal{M}^{3,1}$ is a two-sheeted space-time and the two possible signs of η correspond to the two distinct sheets, while $\eta = 0$ describes a big bounce [5]. As mentioned above, the x^μ are viewed to generate the algebra $\mathcal{C}^0$ of functions on the space-time $\mathcal{M}^{3,1}$, while the t^μ are viewed as extra generators, describing internal degrees of freedom. More precisely, they describe the internal S^2 fibre over every point of $\mathcal{M}^{3,1}$, which is space-like, due to eq.(111c) and has radius $\tilde{R}^{-2}\cosh^2\eta$. This setup is schematically shown in figure 1.

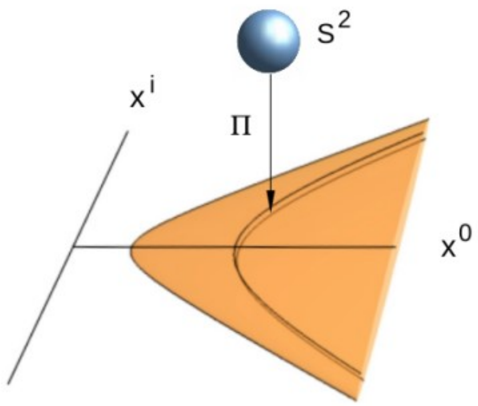

Figure 1: Cosmological space-time $\mathcal{M}^{3,1}$, the internal S^2 over every point and the bundle projection Π [6].

As can be done in the non-commutative case, the algebra of functions $\mathcal{C}$ can be decomposed into spin s sectors. This is done by expanding $\mathcal{C}$ into polynomials of minimal degree s in t^μ, giving

$$\mathcal{C} = \bigoplus_{s=0}^{\infty}\mathcal{C}^s \tag{116}$$

where $\mathcal{C}^0$ is given by functions of x alone and $\mathcal{C}^s \ni \phi_\alpha(x)t^\alpha$ where $\alpha = \alpha_1...\alpha_s$ [6].

The Poisson bracket, as obtained from the commutators eqs.(90a),(90b) and (90e) in the semi-classical limit satisfies

$$\begin{aligned}\{x^\mu, x^\nu\} &= \theta^{\mu\nu} = -\tilde{R}^2R^2\{t^\mu, t^\nu\}\\ \{t^\mu, x^\nu\} &= \sinh\eta\;\eta^{\mu\nu}.\end{aligned} \tag{117}$$

As can be seen from the explicit expression for $\theta^{\mu\nu}$ in eq.(112), the Poisson structure mixes different spin sectors of $\mathcal{C}$,

$$\begin{aligned}\{t, \mathcal{C}^s\} &\in \mathcal{C}^s\\ \{x, \mathcal{C}^s\} &\in \mathcal{C}^{s-1}\oplus\mathcal{C}^{s+1}\end{aligned} \tag{118}$$

and for general elements $\phi_\alpha(x)t^\alpha \in \mathcal{C}^s$, this gives

$$\{\phi_\alpha(x)t^\alpha, \mathcal{C}^s\} = \phi(x)_\alpha\{t^\alpha, \mathcal{C}^s\} + t^\alpha\{\phi_\alpha(x), \mathcal{C}^s\} \in \mathcal{C}^{s-2} \oplus \mathcal{C}^s \oplus \mathcal{C}^{s+2} \tag{119}$$

which can be seen by using the derivation property of the Poisson bracket.

Bundle structure and higher-spin fields Since the bundle projection (110) is not injective, its push-forward does not map vector fields on $\mathbb{C}P^{1,2}$ to vector fields on $\mathcal{M}^{3,1}$. However, one can view the push-forward of Π as mapping vector fields on $\mathbb{C}P^{1,2}$ to higher-spin valued vector fields on $\mathcal{M}^{3,1}$,

$$\Pi_* : \ \mathfrak{X}(\mathbb{C}P^{1,2}) \to \mathfrak{X}(\mathcal{M}^{3,1}) \otimes \mathfrak{hs} \tag{120}$$

where $\mathfrak{hs}$ denotes the space of functions on the S^2-fibres and is spanned by polynomials in t^μ. Then $\mathcal{C}$ is an algebra of $\mathfrak{hs}$-valued functions on $\mathcal{M}^{3,1}$. This is the point of view needed to translate the gauge theory on $\mathbb{C}P^{1,2}$ into a higher-spin theory on $\mathcal{M}^{3,1}$, that in turn can be viewed as a generalized gravity theory [6]. This higher-spin gravity theory will be discussed more in section 6. Note that we can use the push-forward on the Poisson structure on $\mathbb{C}P^{1,2}$ in order to obtain a $\mathfrak{hs}$-valued bracket on $\mathcal{M}^{3,1}$ given by[21]

$$\{f, g\} = \theta^{\mu\nu}\partial_\mu f \partial_\nu g \tag{121}$$

where $\theta^{\mu\nu} = \{x^\mu, x^\nu\} \in \mathcal{C}^1$ and one could use any local coordinates on $\mathcal{M}^{3,1}$. This bracket respects the Jacobi identity on $\mathcal{C}^0$ but in general, it only satisfies an approximate Jacobi identity for functions f in the asymptotic regime, which is defined by the condition $R\cosh\eta\,\partial f \gg f$ [6]. In the following we will analyze the underlying symplectic structure of the Poisson bracket on $\mathbb{C}P^{1,2}$ in more detail, before turning to the emergent higher-spin gravity theory.

[21]Note that we use the same symbol for the Poisson structure on $\mathbb{C}P^{1,2}$ and the bracket on $\mathcal{M}^{3,1}$.

5. Symplectic structure on the total space

In order to understand the cosmological space-time $\mathcal{M}^{3,1}$ better, one would like to study the symplectic structure on the total space $\mathbb{C}P^{1,2}$ in more detail[22]. In order to make the S^2 fibres more explicit, it is useful to take a local approach where $\mathbb{C}P^{1,2}$ is diffeomorphic to a subset U of the Cartesian product

$$\mathbb{C}P^{1,2} \cong U \subset \mathbb{R}^3_x \times \mathbb{R}^3_p \tag{122}$$

described by the space-like Cartesian embedding functions x^i, p^j, while the time-like ones x^0, p^0 are determined by the constraints [47]. The p^μ are re-scaled generators

$$p^\mu = \tilde{R}Rt^\mu \tag{123}$$

which now have the same dimension as the x^μ. The Poisson brackets is then given by

$$\begin{aligned}\{p^\mu, x^\nu\} &= \tilde{R}x^4\eta^{\mu\nu} = \tilde{R}R\sinh\eta\ \eta^{\mu\nu}\\ \{x^\mu, x^\nu\} &= \theta^{\mu\nu} = -\{p^\mu, p^\nu\}\end{aligned} \tag{124}$$

and writing[23] $r^2 = x^i x^i$, the constraints eq.(111a), eq.(111b) and eq.(111c) can be expressed as

$$\begin{aligned}p_0 x^0 &= p^k x^k\\ -(p^0)^2 + p^i p^i &= (x^0)^2 - r^2.\end{aligned} \tag{125}$$

Using these constraints, one has that[24]

$$\begin{aligned}(p^0 + x^0)^2 &= |\vec{x} + \vec{p}|^2\\ (p^0 - x^0)^2 &= |\vec{x} - \vec{p}|^2\end{aligned} \tag{126}$$

which under certain conditions can be solved for x^0 and p^0. For example, if one restricts to the region where $x^0 \geq p^0$, one has that

$$\begin{aligned}x^0 &= \frac{1}{2}\big(|\vec{x} + \vec{p}| + |\vec{x} - \vec{p}|\big)\\ p^0 &= \frac{1}{2}\big(|\vec{x} + \vec{p}| - |\vec{x} - \vec{p}|\big).\end{aligned} \tag{127}$$

From eq.(127), one concludes that $x^0 > 0$, while p^0 can have any sign, and, by using the triangle inequality, that $|p^0| \leq |\vec{x}|$.

Having x^0, p^0 expressed trough $\vec{x}, \vec{p}$, we are effectively considering an open subset of $\mathbb{R}^3_x \times \mathbb{R}^3_p$ equipped with the Poisson structure

$$\begin{aligned}\{x^i, x^j\} &= \theta^{ij} = -\{p^i, p^j\}\\ \{x^i, p^j\} &= -\tilde{R}R\sinh\eta\ \delta^{ij}\end{aligned} \tag{128}$$

[22]Recall that $\mathbb{C}P^{1,2}$ is a coadjoint orbit of $SO(4,2)$ and as such, is canonically equipped with the KKS symplectic form.

[23]Sum over repeated indices is understood from now on, Latin indices run from 1 to 3 and indices are raised and lowered with η =diag(-1,1,1,1).

[24]Here $\vec{x} = (x^1, x^2, x^3)$ and similarly for $\vec{p}$.

where

$$\theta^{ij} = \frac{\tilde{R}R}{R^2\cosh^2\eta}\big[\sinh\eta\ (x^ip^j - x^jp^i) + \varepsilon^{ijk}(x_0p_k - x_kp_0)\big], \tag{129}$$

and $x^0(\vec{x},\vec{p}), p^0(\vec{x},\vec{p})$ as given in eq.(127). Similarly, applying eq.(127) to eq.(111a), one can express the hyperbolic functions of the cosmic time parameter η as

$$R^2\cosh^2\eta = \frac{1}{2}\big(|\vec{x}+\vec{p}||\vec{x}-\vec{p}| - (\vec{x}+\vec{p})\cdot(\vec{x}-\vec{p})\big) = R^2 + R^2\sinh^2\eta. \tag{130}$$

In order to explicitly calculate the corresponding symplectic form, it is useful to perform the following change of coordinates:

$$\begin{aligned}\bar{x}^i &= \frac{1}{\sqrt{2}}(x^i + p^i)\\ \bar{p}^i &= \frac{1}{\sqrt{2}}(x^i - p^i),\end{aligned} \tag{131}$$

where $i = 1,2,3$. Note that at the matrix level, these linear combinations correspond to defining generators of the Poincaré algebra inside of $\mathfrak{so}(4,2)$, e.g. as a subalgebra [45]. Eq.(130) then reads

$$R^2\cosh^2\eta = |\vec{\bar{x}}||\vec{\bar{p}}| - \vec{\bar{x}}\cdot\vec{\bar{p}} \geq R^2 \tag{132}$$

which gives a condition on the angle between $\vec{\bar{x}}$ and $\vec{\bar{p}}$.

In these new coordinates, the Poisson structure reads

$$\begin{aligned}\{\bar{x}^i,\bar{x}^j\} &= 0 = \{\bar{p}^i,\bar{p}^j\}\\ \{\bar{x}^i,\bar{p}^j\} &= \sinh\eta\ \delta^{ij} + \theta^{ij}\end{aligned} \tag{133}$$

where

$$\theta^{ij}(\bar{x},\bar{p}) = \frac{\tilde{R}R}{R^2\cosh^2\eta}\big[\sinh\eta(\bar{p}^i\bar{x}^j - \bar{p}^j\bar{x}^i) + \varepsilon^{ijk}(\bar{x}_k\bar{p} - \bar{p}_k\bar{x})\big] \tag{134}$$

and we have defined $|\vec{\bar{x}}| =: \bar{x}$ and similarly for $\vec{\bar{p}}$. We have now obtained a Poisson structure in block form, as can be explicitly seen by defining

$$Y^i := \begin{cases}\bar{x}^i, & \text{if } i = 1,2,3\\ \bar{p}^{i-3}, & \text{if } i = 4,5,6\end{cases} \tag{135}$$

with which one can write

$$\{Y^i, Y^j\} = J^{ij} \tag{136}$$

where the skew-symmetric 6×6 block-matrix J is given by

$$J = \begin{pmatrix} 0 & \theta + \tilde{R}R\sinh\eta\ \mathbb{1}_3\\ \theta - \tilde{R}R\sinh\eta\ \mathbb{1}_3 & 0\end{pmatrix} =: \begin{pmatrix} 0 & \Theta_+\\ \Theta_- & 0\end{pmatrix}, \tag{137}$$

and $\theta := (\theta^{ij})$. The corresponding symplectic form, in the local coordinates (Y^i), is then obtained by inverting J, which amounts to inverting $\Theta_\pm$. This can be done in closed form and one obtains

$$\Theta_\pm^{-1} = \frac{1}{\mathcal{N}}\Big(\pm\tilde{R}R\sinh\eta\ \mathbb{1}_3 \pm \frac{1}{\tilde{R}R\sinh\eta}S - \theta\Big) \tag{138}$$

where the normalization factor $\mathcal{N}$ and the symmetric matrix S are given by

$$\begin{aligned}\mathcal{N} &= (\theta^{12})^2 + (\theta^{13})^2 + (\theta^{23})^2 + \tilde{R}R\sinh^2\eta = 2\tilde{R}^2\bar{x}\bar{p}\\ S_{ij} &= \frac{1}{4}\varepsilon_{imn}\theta^{mn}\varepsilon_{jkl}\theta^{kl} =: v_i v_j.\end{aligned} \tag{139}$$

Thus, we can explicitly calculate the symplectic form and using the block-form of J^{-1}, one has that

$$\begin{aligned}\omega &= \frac{1}{2}(J^{-1})_{ij}dY^i \wedge dY^j = \frac{1}{2}\left[(\Theta_-^{-1})_{ij}d\bar{x}^i \wedge d\bar{p}^j + (\Theta_+^{-1})_{ij}d\bar{p}^i \wedge d\bar{x}^j\right]\\ &= (\Theta_+^{-1})_{ij}d\bar{p}^i \wedge d\bar{x}^j\end{aligned}$$

which written out fully reads

$$\begin{aligned}\omega = \frac{1}{2\tilde{R}^2\bar{x}\bar{p}}\Bigg\{&\tilde{R}R\sinh\eta\; d\bar{p}^i \wedge d\bar{x}_i - \frac{\tilde{R}R}{R^2\cosh^2\eta}(\bar{x}^k\bar{p} - \bar{p}^k\bar{x})\varepsilon_{ijk}d\bar{p}^i \wedge d\bar{x}^j\\ &+ \frac{\tilde{R}R}{R^4\cosh^4\eta}\left[\sinh\eta(\vec{\bar{x}}\times\vec{\bar{p}})_i(\vec{\bar{x}}\times\vec{\bar{p}})_j + (\vec{\bar{x}}\times\vec{\bar{p}})_i(\bar{p}_j\bar{x} - \bar{x}_j\bar{p}) + (\bar{p}_i\bar{x} - \bar{x}_i\bar{p})(\vec{\bar{x}}\times\vec{\bar{p}})_j\right]d\bar{p}^i \wedge d\bar{x}^j\\ &+ \left[\frac{\tilde{R}R\sinh\eta}{R^2\cosh^2\eta}(\bar{x}_i\bar{p}_j - \bar{x}_j\bar{p}_i) + \frac{\tilde{R}R}{R^4\cosh^4\eta\sinh\eta}(\bar{p}_i\bar{x} - \bar{x}_i\bar{p})(\bar{p}_j\bar{x} - \bar{x}_j\bar{p})\right]d\bar{p}^i \wedge d\bar{x}^j\Bigg\}.\end{aligned} \tag{140}$$

One notes a few things about ω. For starters, ω is invariant under the $SO(3)$-action

$$(\bar{x}^i, \bar{p}^j) \mapsto (R^i{}_k\bar{x}^k, R^j{}_l\bar{p}^l) \tag{141}$$

for R in the vector rep. of $SO(3)$. In 6 dimensions, there are exactly 12 possible mixed differentials[25] that are invariant under this group action. As ω contains all 12 such terms, it is in this sense the most general $SO(3)$-invariant symplectic form, up to the prefactors of the individual terms. Closedness of ω was explicitly checked, using Mathematica. However, as will become clear from the next considerations, ω is not exact. This can already be seen from eq.(140), as, for example, the third term, containing components of the cross-product $(\vec{\bar{x}} \times \vec{\bar{p}})$, can not be exact.

5.1. Integration of the symplectic form over spheres

In order to further investigate the symplectic form, we calculate its integral over a 2-cycle S^2 embedded into $\mathbb{R}^3_x \times \mathbb{R}^3_p$. Explicitly, we consider a sphere in the p^i, e.g.

$$(p^1)^2 + (p^2)^2 + (p^3)^2 = 1,\; x^1 = x^2 = x^3 = 0 \tag{142}$$

which in the new coordinates $\bar{x}^i, \bar{p}^j$ is realized by the embedding

$$\begin{aligned}\iota_p :\; &S^2 \hookrightarrow \mathbb{R}^3_x \times \mathbb{R}^3_p\\ (q^1, q^2, q^3) &\mapsto \frac{1}{\sqrt{2}}(q^1, q^2, q^3, -q^1, -q^2, -q^3).\end{aligned} \tag{143}$$

[25] Mixed differentials means differentials of the form $dx^i \wedge dy^j$ for an arbitrary division of coordinates $(z^k) = (x^i, y^j)$.

Due to the symmetric form of the embedding (143), most terms of the symplectic form ω are mapped to zero under the pull-back along ι_p, so that the pulled back form $\iota_p^*(\omega)$ is much simpler:

$$\iota_p^*(\omega) = \frac{R}{2\tilde{R}} \varepsilon_{ijk} q^k dq^i \wedge dq^j. \tag{144}$$

Up to the constant in front, this is exactly the unique $SO(3)$-invariant symplectic form on S^2, given in Cartesian coordinates, which also coincides with the volume form on S^2, so that one straightforwardly calculates

$$\int_{S^2} \iota_p^*(\omega) = \frac{R}{2\tilde{R}} \int_{S^2} \varepsilon_{ijk} q^k dq^i \wedge dq^j = \frac{R}{2\tilde{R}} \int_{S^2} \Omega = \frac{R}{2\tilde{R}} 8\pi = 4\pi \frac{R}{\tilde{R}}. \tag{145}$$

Recalling the connection between R and $\tilde{R}$, eq.(93), we see that in the semi-classical limit, which corresponds to $n \to \infty$, we have that

$$\int_{S^2} \iota_p^*(\omega) = 2\pi\sqrt{n^2 - 4} \sim 2\pi n \tag{146}$$

in agreement with the quantization condition described at the end of section 3.1. Eq.(146) shows that at least for this particular choice of 2-cycle S^2, the integral of ω does not vanish, which implies, via Stokes theorem, that ω can not be exact, in agreement with our previous observations.

While it would seem natural to also consider a sphere in the x^i, the condition given in eq.(132) shows that such a sphere would not be contained in the subset U of the Cartesian product space, so that one would have to consider a different solution to the constraints in eq.(126).

6. Emergent higher-spin gravity

In the following we will elaborate on the higher-spin gravity theory on the FLRW background $\mathcal{M}^{3,1}$ obtained in the semi-classical limit of the covariant quantum space-time $\mathcal{M}^{3,1}_n$. After giving the underlying geometric description, which is based on the idea of a frame generated by the semi-classical limit of the generators of the background, section 7 will discuss an Ansatz for spherically symmetric local perturbations of the background. We will see that this Ansatz, together with a divergence constraint coming from the underlying matrix model, is, in principle, able to reproduce arbitrary static spherically symmetric metrics, at least in the regime of large cosmic time $\eta \to \infty$. In addition, we will construct and analyze further, more specialized metrics compatible with the present framework. Finally, section 9 will deal with the problem of reconstructing semi-classical generators of the frame for a given geometry.

6.1. Notes on the matrix model background

The background of the theory is defined by the following solution of the bosonic IKKT matrix model (82), e.g. solutions of the e.o.m. (84):

$$Z^{\dot\alpha} = \begin{cases} T^{\dot\alpha}, \ \dot\alpha = 0,1,2,3 \\ 0, \ \dot\alpha = 4,...,9 \end{cases} \quad . \tag{147}$$

One notes that the $T^{\dot\alpha}$ do not close under the commutator $[\cdot,\cdot]$, but rather require the additional generators $X^{\dot\alpha}$ to close the algebra and give a well-defined semi-classical limit with Poisson bracket $i\{\cdot,\cdot\} \sim [\cdot,\cdot]$. Thus one considers the $X^{\dot\alpha}$ to describe the covariant quantum space-time $\mathcal{M}^{3,1}_n$ from section 4.4, while the $T^{\dot\alpha}$ define the matrix Laplacian $\Box = [T^{\dot\alpha},[T_{\dot\alpha},\cdot]]$, which governs the propagation of the fluctuations in the matrix model gauge theory [6]. In the semi-classical limit, this exactly gives the cosmological space-time $\mathcal{M}^{3,1}$ with the additional S^2-bundle structure as described in section 4.5.

6.2. Frame and effective metric on $\mathcal{M}^{3,1}$

Up until now, we have not discussed how one can obtain a metric on the semi-classical space-time $\mathcal{M}^{3,1}$. This can be done by considering a solution $Z^{\dot\alpha} \in \mathrm{End}(\mathcal{H}_n), \ \dot\alpha = 0,1,2,3$ to the matrix e.o.m living in the non-commutative algebra. Take now a scalar matrix $\Phi = \mathrm{End}(\mathcal{H}_n)$, coupled to the matrix background via a kinetic term, for which one has that[3]

$$S_{K.E.}[\phi] = \mathrm{Tr}\Big([Z^{\dot\alpha},\phi][Z_{\dot\alpha},\phi]\Big) = -\mathrm{Tr}\Big(\phi[Z^{\dot\alpha},[Z_{\dot\alpha},\phi]]\Big). \tag{148}$$

In the semi-classical limit, we write $Z^{\dot\alpha} \sim z^{\dot\alpha} \in \mathcal{C}$, $\Phi \sim \phi$ and the above translates to [3]

$$\begin{aligned}\mathrm{Tr}\Big([Z^{\dot\alpha},\phi][Z_{\dot\alpha},\phi]\Big) &\sim -\int_{\mathbb{C}P^{1,2}} \Omega\{z^{\dot\alpha},\phi\}\{z_{\dot\alpha},\phi\} \\ &= -\int_{\mathcal{M}^{3,1}} dx^0...dx^3 \rho_M(x)\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\int_{\mathcal{M}^{3,1}} dx^0...dx^3\sqrt{|G|}G^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\end{aligned} \tag{149}$$

where the last expression is manifest covariant and some dimensionful constants where absorbed into φ. Here

$$\gamma^{\mu\nu} = \eta_{\dot\alpha\dot\beta}\{z^{\dot\alpha}, x^\mu\}\{z^{\dot\beta}, x^\nu\}, \ G^{\mu\nu} := \frac{1}{\rho^2}\gamma^{\mu\nu} \tag{150}$$

are the auxiliary and the effective metric on $\mathcal{M}^{3,1}$ and we used that one can write the symplectic volume form Ω on $\mathbb{C}P^{1,2}$ as

$$\Omega = \rho_M(x)dx^0...dx^3\Omega_{S^2}, \ \rho_M(x) = \frac{1}{\sinh\eta} \tag{151}$$

where Ω_{S^2} is the volume form of the unit 2-sphere which corresponds to the local fibre and $\rho_M(x)$ is the 4-density corresponding to the base [6]. Auxiliary and effective metric are conformally equivalent, with the conformal factor ρ^2 given by

$$\rho^2 = \rho_M\sqrt{|\gamma|} \tag{152}$$

where one notes that eq.(152) shows that ρ^2 is actually a scalar field, rather than a density. One therefore refers to the scalar field ρ as *dilaton* [47]. Eq.(149) shows that G is the metric that governs the propagation of scalar fields in the semi-classical limit. One can even show that it governs all, i.e. scalar, gauge and fermion, fields, so that necessarily, G has to be interpreted as the effective gravitational metric [48].

Let now be $\phi \in \mathcal{C}$ be an arbitrary function on $\mathbb{C}P^{1,2}$. One can use the background solution $Z^{\dot\alpha} \sim z^{\dot\alpha}$ to define a vector field

$$E_{\dot\alpha}(\phi) := \{z_{\dot\alpha}, \phi\} \tag{153}$$

which transforms under the global $SO(3,1)$. Using the push-forward to $\mathcal{M}^{3,1}$, we obtain a $\mathfrak{hs}$-valued frame on $\mathcal{M}^{3,1}$

$$E_{\dot\alpha}{}^\mu\partial_\mu, \quad E_{\dot\alpha}{}^\mu := \{z_{\dot\alpha}, x^\mu\} \tag{154}$$

which is an $\mathcal{C}$-valued 4×4 matrix. Since we are using the push-forward, the definition of the frame is independent of the x^μ and we could use any coordinates on $\mathcal{M}^{3,1}$. One immediately notes that

$$\gamma^{\mu\nu} = \eta^{\dot\alpha\dot\beta}E_{\dot\alpha}{}^\mu E_{\dot\beta}{}^\nu = \rho^2 G^{\mu\nu} \tag{155}$$

which means that both metrics are $\mathfrak{hs}$-valued as well and correspond to the frame $E_{\dot\alpha}{}^\mu$. This is the chief insight of the present formalism and allows us to describe the non-linear regime of the present higher-spin gravity theory.

For the background solution $\bar Z_{\dot\alpha} = T_{\dot\alpha}$ the above quantities are given by

$$\begin{aligned} \bar E_{\dot\alpha}{}^\mu &= \sinh\eta\ \delta_{\dot\alpha}{}^\mu, \ \bar\gamma^{\mu\nu} = \sinh^2\eta\ \eta^{\mu\nu} \\ \bar\rho^2 &= \sinh^3\eta, \ \bar G^{\mu\nu} = \frac{1}{\sinh\eta}\eta^{\mu\nu}. \end{aligned} \tag{156}$$

From this, one can now explicitly see how $\mathcal{M}^{3,1}$ is a cosmological FLRW space-time. The metric $\bar G$ is a hyperbolic $k=-1$ FLRW metric and can be written as

$$\bar G = \bar G_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 d\Sigma^2 \tag{157}$$

with $a(t)$ being the cosmic scale factor as a function of comoving time t and $d\Sigma^2$ being the $SO(3,1)$-invariant metric on H^3. The cosmic scale factor and comoving time are determined by

$$a(t)^2 = R^2 \sinh\eta \cosh^2\eta, \;\; dt = R \sinh^{3/2}\eta \, d\eta. \tag{158}$$

For the geometric description of the non-linear sector, one focuses on the asymptotic regime $a(t) \to \infty$, i.e., $\eta \to \infty$ and considers only perturbations of the geometry on scales much smaller than the cosmic scale. This is also the regime for which the dominating term of the Poisson structure on $\mathbb{C}P^{1,2}$ agrees exactly with the $\mathfrak{hs}$-valued bracket on $\mathcal{M}^{3,1}$ [47].

6.3. Gauge transformations in the higher-spin theory

Next, we want to discuss how the gauge transformations of the matrix model are reflected in the higher-spin theory on $\mathcal{M}^{3,1}$. Recall from section 4.2 that the action of the IKKT matrix model is invariant under gauge transformations $\Phi \to U\Phi U^\dagger$ for $\Phi, U \in \mathrm{End}(\mathcal{H}_n)$ where $U^\dagger = U^{-1}$. For unitary matrices U such that $U = e^{i\Lambda}$ where $\Lambda = \Lambda^\dagger \in \mathrm{End}(\mathcal{H}_n)$, the infinitesimal gauge transformations are given as

$$\Phi \to -i[\Lambda, \Phi]. \tag{159}$$

Letting now $\Phi \sim \phi, \Lambda \sim \lambda$ denote the corresponding functions in the semi-classical limit, i.e., $\phi, \lambda \in \mathcal{C}$ and $\lambda = \lambda^*$, the semi-classical analogue of the matrix model infinitesimal gauge transformations is given by

$$\delta_\lambda \phi = \{\lambda, \phi\} = \mathcal{L}_\xi \phi \tag{160}$$

which is exactly the Lie derivative of ϕ along the Hamiltonian vector field $\xi := \{\lambda, \cdot\}$. Restricting this to functions on $\mathcal{M}^{3,1}$, i.e., $\phi \in \mathcal{C}^0$, the infinitesimal gauge transformations are given by

$$\delta_\lambda \phi = \xi^\mu \partial_\mu \phi = \mathcal{L}_\xi \phi \tag{161}$$

with the natural interpretation as a Lie derivative along the $\mathfrak{hs}$-valued vector field on $\mathcal{M}^{3,1}$, given by

$$\xi = \xi^\mu \partial_\mu, \quad \xi^\mu = \{\lambda, x^\mu\}. \tag{162}$$

For the more general case of arbitrary functions $\phi \in \mathcal{C}$, interpreted as $\mathfrak{hs}$-valued functions on $\mathcal{M}^{3,1}$, we have that

$$\delta_\lambda \phi = \{\lambda, \phi\} = \{\lambda, x^\mu\}\partial_\mu \phi = \xi^\mu \partial_\mu \phi, \tag{163}$$

which holds when ϕ and the ξ^μ are in the asymptotic regime. In the same spirit, this is then interpreted as Lie derivative of the $\mathfrak{hs}$-valued function ϕ along the $\mathfrak{hs}$-valued vector field ξ.

We have seen how infinitesimal gauge transformations in the matrix model give rise to Hamiltonian vector fields on $\mathbb{C}P^{1,2}$ and in turn, to $\mathfrak{hs}$-valued vector fields on $\mathcal{M}^{3,1}$. As the diffeomorphisms given by the flow of the Hamiltonian vector fields preserve the symplectic volume on $\mathbb{C}P^{1,2}$, making them symplectomorphisms, one expects the $\mathfrak{hs}$-diffeomorphisms obtained from the flow of the $\mathfrak{hs}$-valued vector fields on $\mathcal{M}^{3,1}$ to also be volume-preserving in a suitable sense. This can indeed be shown to hold [6]. Thus the above analysis shows how the gauge invariance of the matrix model corresponds to generalized diffeomorphisms under which the fundamental geometric objects, namely the generators of the frame, transform, rather than under local Lorentz transformations. This means that the present higher-spin theory is not just some reformulation of general relativity [6]. This last point will be discussed further in this thesis.

6.4. Weitzenböck connection and torsion

As we have seen in the previous section, the fundamental degrees of freedom of the matrix model, the matrix configurations $Z_{\dot\alpha}$ give a notion of geometry in the semi-classical limit via the frame eq.(154). We will now discuss this geometric picture further.

One starts by defining the inverse frame $E^{\dot\alpha}{}_{\mu}$ in the usual way, via demanding that

$$E^{\dot\alpha}{}_{\mu}E_{\dot\beta}{}^{\mu} = \delta^{\dot\alpha}{}_{\dot\beta}, \; E^{\dot\alpha}{}_{\nu}E_{\dot\alpha}{}^{\mu} = \delta^{\mu}{}_{\nu} \tag{164}$$

holds. As the frame is the fundamental geometric object of the present theory, one follows the spirit of teleparallel gravity and uses it to define a Weitzenböck connection ∇. On a metric manifold, the Weitzenböck connection is the unique metric compatible connection with vanishing curvature, but non-vanishing torsion[26] [49]. The Weitzenböck connection is defined by demanding that it respects the frame:

$$0 = \nabla_\nu E_{\dot\alpha}{}^{\mu} = \partial_\nu E_{\dot\alpha}{}^{\mu} + \Gamma_{\nu\rho}{}^{\mu}E_{\dot\alpha}{}^{\rho}, \tag{165}$$

where $\Gamma_{\nu\rho}{}^{\mu}$ denote the connection coefficients. This then immediately implies that the auxiliary metric $\gamma^{\mu\nu}$ is also respected: $\nabla\gamma^{\mu\nu} = 0$. As mentioned, and like eq.(165) shows, the frame is parallel and thus the connection is flat. However, it has non-vanishing torsion T, which can be computed to be [6]:

$$\begin{aligned} T_{\mu\nu}{}^{\rho} &= \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho} \\ T_{\mu\nu}{}^{\dot\alpha} &= T_{\mu\nu}{}^{\rho}E^{\dot\alpha}{}_{\rho} = \partial_\mu E^{\dot\alpha}{}_{\nu} - \partial_\nu E^{\dot\alpha}{}_{\mu}. \end{aligned} \tag{166}$$

Similarly, for the connection coefficients, one has that

$$\Gamma_{\mu\nu}{}^{\rho} = -E^{\dot\alpha}{}_{\nu}\partial_\mu E_{\dot\alpha}{}^{\rho} \tag{167}$$

by eq.(165). The frame-valued[27] torsion $T_{\mu\nu}{}^{\dot\alpha}$ satisfies a Bianchi identity

$$\partial_\mu T_{\nu\rho}{}^{\dot\alpha} + \partial_\nu T_{\rho\mu}{}^{\dot\alpha} + \partial_\rho T_{\mu\nu}{}^{\dot\alpha} = 0 \tag{168}$$

which can be seen as a direct consequence of eq.(166). This can also be seen by viewing the $T_{\mu\nu}{}^{\dot\alpha}$ as components of a two-form $T^{\dot\alpha}$, referred to as *torsion two-from*, which, by eq.(166)

[26] In comparison to the Levi-Civita connection, which is the unique metric compatible connection with vanishing torsion, but non-vanishing curvature.

[27] Note, that all quantities derived from the $\mathfrak{hs}$-valued frame, are themselves $\mathfrak{hs}$-valued.

is exactly the exterior derivative of the frame and the Bianchi identity eq.(168) is simply the statement that $T^{\dot\alpha}$ is closed,

$$T^{\dot\alpha} = dE^{\dot\alpha} = \frac{1}{2} T_{\mu\nu}{}^{\dot\alpha} dx^\mu \wedge dx^\nu, \quad dT^{\dot\alpha} = 0. \tag{169}$$

One can also consider the Levi-Civita connection $\nabla^{(G)}$ associated to the effective metric $G^{\mu\nu}$ which, as usual, can be defined via the Christoffel symbols

$$\begin{aligned}\Gamma^{(G)}{}_{\mu\nu}{}^{\rho} &= \frac{1}{2} G^{\rho\sigma}\Big(\partial_\mu G_{\sigma\nu} + \partial_\nu G_{\sigma\mu} - \partial_\sigma G_{\mu\nu}\Big)\\ &= \frac{1}{2}\rho^{-2}\Big(\delta^\rho_\nu \partial_\mu \rho^2 + \delta^\rho_\mu \partial_\nu \rho^2 - \gamma_{\mu\nu}\gamma^{\sigma\rho}\partial_\sigma \rho^2\Big) + \frac{1}{2}\gamma^{\rho\sigma}\Big(\partial_\mu \gamma_{\sigma\nu} + \partial_\nu \gamma_{\sigma\mu} - \partial_\sigma \gamma_{\mu\nu}\Big),\end{aligned} \tag{170}$$

so that one has that

$$\Gamma^{(G)}{}_{\mu\nu}{}^{\rho} = \Gamma^{(\gamma)}{}_{\mu\nu}{}^{\rho} + \frac{1}{2}\rho^{-2}\Big(\delta^\rho_\nu \partial_\mu \rho^2 + \delta^\rho_\mu \partial_\nu \rho^2 - \gamma_{\mu\nu}\gamma^{\sigma\rho}\partial_\sigma \rho^2\Big) \tag{171}$$

where $\Gamma^{(\gamma)}{}_{\mu\nu}{}^{\rho}$ are the Christoffel symbols associated to the auxiliary metric γ. The relation between the Weitzenböck ∇ and Levi-Civita $\nabla^{(G)}$ connection is given by

$$\Gamma^{(G)}{}_{\mu\nu}{}^{\sigma} = \Gamma_{\mu\nu}{}^{\sigma} + K_{\mu\nu}{}^{\sigma} - \frac{1}{2}\rho^{-2}\Big(\delta^\rho_\nu \partial_\mu \rho^2 + \delta^\rho_\mu \partial_\nu \rho^2 - \gamma_{\mu\nu}\gamma^{\sigma\rho}\partial_\sigma \rho^2\Big) \tag{172}$$

where

$$K_{\mu\nu}{}^{\sigma} = \frac{1}{2}\Big(T_{\mu\nu}{}^{\sigma} + T^{\sigma}{}_{\mu\nu} - T_{\nu}{}^{\sigma}{}_{\mu}\Big) \tag{173}$$

is the contorsion of the Weitzenböck connection [6].

Using the Hodge star, one can define a one-form $\tilde T$,

$$\tilde T_\mu := \frac{1}{2\sqrt{|\gamma|}}\gamma_{\mu\rho}\varepsilon^{\nu\sigma\rho\kappa}T_{\nu\sigma\rho} = \frac{1}{2\sqrt{|G|}}\rho^2 G_{\mu\rho}\varepsilon^{\nu\sigma\rho\kappa}T_{\nu\sigma\rho} \tag{174}$$

which is denoted as axion one-form [47]. Note that this is determined by the fully antisymmetric part of the torsion.

In [6] the e.o.m. of the matrix model where translated into the semi-classical limit and formulated in terms of the torsion and consequently in terms of the frame. They amount to a set of non-linear PDE's and were subsequently solved in [47, 50], making an Ansatz for a static spherically symmetric frame. While in the first of the two works, only a special solution, corresponding to the linearized Schwarzschild geometry, was found, the later work describes the most general static spherically symmetric solution. However, we will not consider these e.o.m. in the following. Instead, we will simply investigate what kind of geometries admit themselves to a description in the emergent higher-spin theory, independent of the concrete details of the matrix model, with a focus on spherically symmetric, static metrics.

6.5. Dilaton identity and divergence constraint

The torsion of the Weitzenböck connection satisfies two properties which lead to important results. Firstly, that the dilaton and the torsion are intimately connected and secondly, that the frame satisfies a divergence constraint. These two results, together with the necessary identities will be derived in this section, based on results in [6].

We claim that the following two identities hold:

$$\Gamma_{\rho\mu}{}^{\rho} = \frac{\partial_\mu \rho_M}{\rho_M} \tag{175a}$$
$$\Gamma_{\mu\nu}{}^{\nu} = -\partial_\mu \ln\sqrt{|\gamma|}. \tag{175b}$$

To show eq.(175b), we simply write out the definitions and use the derivation property of the Poisson bracket:

$$\begin{aligned}\Gamma_{\dot\gamma\mu}{}^{\mu} &= \Gamma_{\dot\gamma\dot\beta}{}^{\mu}E^{\dot\beta}{}_{\mu} = -\{z_{\dot\gamma}, E_{\dot\beta}{}^{\mu}\}E_{\dot\beta}{}^{\nu}\gamma_{\nu\mu}\\ &= -\{z_{\dot\gamma}, E_{\dot\beta}{}^{\mu}E_{\dot\beta}{}^{\nu}\}\gamma_{\nu\mu} + \{z_{\dot\gamma}, E_{\dot\beta}{}^{\nu}\}E_{\dot\beta}{}^{\mu}\gamma_{\nu\mu}\\ &= \{z_{\dot\gamma}, \gamma^{\mu\nu}\}\gamma_{\nu\mu} + \{z_{\dot\gamma}, E_{\dot\beta}{}^{\nu}\}E^{\dot\beta}{}_{\nu}\\ &= \{z_{\dot\gamma}, \gamma^{\mu\nu}\}\gamma_{\nu\mu} - \Gamma_{\dot\gamma\mu}{}^{\mu}.\end{aligned} \tag{176}$$

Thus, we have that

$$2\Gamma_{\dot\gamma\mu}{}^{\mu} = \{z_{\dot\gamma}, \gamma^{\mu\nu}\}\gamma_{\nu\mu} = -E_{\dot\gamma}{}^{\rho}\gamma_{\nu\mu}\partial_\rho\gamma^{\mu\nu} = -E_{\dot\gamma}{}^{\rho}\partial_\rho \ln|\gamma| \tag{177}$$

which upon contraction with the frame and applying logarithm rules gives exactly the desired identity. In order to show eq.(175a), we need the following fact about Poisson tensors:

Let θ^{ab} be any Poisson tensor on a n-dimensional Poisson manifold and let $\rho := \det(\theta^{ab})^{-n}$. Then

$$\partial_a(\rho\theta^{ab}) = 0 \tag{178}$$

for all $b = 1, ..., n$ [23].

Evidently, this does also apply to the Poisson tensor on $\mathbb{C}P^{1,2}$ and upon restricting to the asymptotic regime $\eta \to \infty$, one has a similar identity for the reduced Poisson structure on $\mathcal{M}^{3,1}$:

$$\partial_\mu(\rho_M\theta^{\mu\nu}) \sim_\eta 0. \tag{179}$$

Here ρ_M is the 4-density corresponding to the base, as in eq.(151) and $\sim_\eta$ denotes the behaviour for large η. With this at hand, one observes that one can write

$$E_{\dot\beta}{}^{\mu} = \{z_{\dot\beta}, x^\mu\} = -\theta^{\mu\nu}\partial_\nu z_{\dot\beta} \tag{180}$$

with which one has that

$$\partial_\mu(\rho_M E_{\dot\beta}{}^{\mu}) = -\partial_\mu(\rho_M\theta^{\mu\sigma}\partial_\sigma z_{\dot\beta}) = -\partial_\mu(\rho_M\theta^{\mu\sigma})\partial_\sigma z_{\dot\beta} = 0, \tag{181}$$

by eq.(179). From eq.(167) it follows that one can write $\Gamma_{\mu\dot{\beta}}{}^{\mu} = -\partial_\mu E_{\dot{\beta}}{}^{\mu}$ which together with the above gives

$$\rho_M \Gamma_{\mu\dot{\beta}}{}^{\mu} = \rho_M \partial_\mu E_{\dot{\beta}}{}^{\mu} = E_{\dot{\beta}}{}^{\mu} \partial_\mu \rho_M \tag{182}$$

which is equivalent to eq.(175a). Using eq.(166), and the two identities, one can evaluate the contraction of the torsion:

$$\begin{aligned} T_{\mu\nu}{}^{\mu} &= \Gamma_{\mu\nu}{}^{\mu} - \Gamma_{\nu\mu}{}^{\mu} = \partial_\nu \Big(\ln \sqrt{|\gamma|} + \ln \rho_M \Big) \\ &= \partial_\nu \ln \rho_M \sqrt{|\gamma|} = \partial_\nu \ln \rho^2 \end{aligned}$$

where eq.(152) was used in the last equality. This gives an important identity, dubbed the *dilaton constraint*

$$-\frac{2}{\rho} \partial_\mu \rho = T_{\mu\nu}{}^{\nu} \tag{183}$$

which will play an important role in the analysis of spherically symmetric geometries in the next section. Note that eq.(183) is independent of the explicit form of the matrix model, but is purely a consequence of having a background solution of the matrix model which defines a frame and the Poisson structure obtained in the semi-classical limit.

By considering eq.(152), one sees that eq.(181) also implies the divergence constraint

$$\partial_\mu (\sqrt{|G|} \rho^{-2} E_{\dot{\alpha}}{}^{\mu}) = \nabla_\mu^{(G)} (\rho^{-2} E_{\dot{\alpha}}{}^{\mu}) = 0 \tag{184}$$

which means that $\rho^{-2} E_{\dot{\alpha}}$ are volume-preserving vector fields. Furthermore, this has the following important consequence: In the traditional Cartan formalism of general relativity, one has that local Lorentz transformations of the frame leave the metric invariant. However, in the present framework, not all frames are permissible, but only those satisfying the divergence constraint eq.(184). Thus, the local Lorentz invariance is broken, leading to extra degrees of freedom, like the dilaton ρ. This marks a fundamental difference between general relativity and the present theory, where the frame is the fundamental object. Nonetheless, this does not lead to the exclusion of metrics arising as solutions to the Einstein equations in general relativity. Indeed, there is strong evidence [47, 51] that starting from any metric $\tilde{G}$, one can find a frame $E_{\dot{\alpha}}{}^{\mu}$ and a scalar function ρ such that

$$\tilde{G}^{\mu\nu} = \frac{1}{\rho^2} \eta^{\dot{\alpha}\dot{\beta}} E_{\dot{\alpha}}{}^{\mu} E_{\dot{\beta}}{}^{\nu} \text{ and } \nabla_\mu^{(\tilde{G})} (\rho^{-2} E_{\dot{\alpha}}{}^{\mu}) = 0, \tag{185}$$

so that any metric can be realized in the matrix model framework. However, explicit realisations of specific frames via generators from the matrix model as in eq.(154) are still difficult to obtain, as will be discussed in section 9. Finally one should note that, as is necessary for consistency reasons, the $\mathfrak{hs}$-valued diffeomorphisms discussed in section 6.3, preserve the divergence constraint, i.e. they map divergence free frames to frames that are still divergence free [47].

After having discussed the emergent higher-spin theory of gravity, the next section will be focused on investigating spherically symmetric static geometries in the present framework.

7. Spherically symmetric static geometries in the matrix model framework

The setup described in this section follows the Ansatz presented in [47] and [50].

We wish to find all spherically symmetric, static geometries that are admissible to a description via a frame induced by generators of a matrix model. In order to implement the first condition, we demand that the frame one-forms $E^{\dot{\alpha}} = E^{\dot{\alpha}}{}_{\mu} dx^{\mu}$ transform in the vector representation of $SO(3)$. The easiest way to achieve this, is to go to Cartesian coordinates (t, x^1, x^2, x^3) which we centre around $r = \sqrt{x^i x^i} = 0$, and make the following Ansatz for the frame one-forms:

$$\begin{aligned}
E^0{}_0 &= A(r) \\
E^i{}_0 &= E(r)x^i \\
E^0{}_i &= D(r)x^i \\
E^i{}_j &= F(r)x^i x^j + B(r)\delta^i{}_j + S(r)\varepsilon_{ijk}x^k.
\end{aligned} \tag{186}$$

This Ansatz is then also static, as none of the coefficients of the frame one-forms depend on the coordinate t. While this is the most general spherically symmetric, static Ansatz for the frame one-forms, one can quickly see that by performing a change of coordinates $t \to t + f(r)$ and $x^i \to g(r)x^i$ with suitable functions $f(r), g(r)$, one can eliminate D and F. We adopt this change of coordinates from here on out. Then the frame one-forms are given by

$$\begin{aligned}
E^0 &= A(r)dt \\
E^i &= B(r)dx^i + E(r)x^i dt + S(r)\varepsilon_{ijk}x^k dx^j
\end{aligned} \tag{187}$$

and the corresponding torsion two-forms $T^{\dot{\alpha}}$ read

$$\begin{aligned}
T^0 &= A' dr \wedge dt \\
T^i &= B' dr \wedge dx^i + E dx^i \wedge dt + x^i E' dr \wedge dt + S\varepsilon_{ijk} dx^k \wedge dx^j + S' \varepsilon_{ijk} x^k dr \wedge dx^j,
\end{aligned} \tag{188}$$

where $'$ denotes the derivative with respect to r. Since we want to study spherically symmetric geometries, spherical coordinates $(t, r, r, \theta, \varphi)$ are best suited to analyse the effective metric, which is determined using eq.(155) and reads

$$(G_{\mu\nu}) = \rho^2 \begin{pmatrix} -(A^2 - r^2E^2) & rBE & 0 & 0 \\ rBE & B^2 & 0 & 0 \\ 0 & 0 & r^2(B^2 + r^2S^2) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta(B^2 + r^2S^2) \end{pmatrix} \tag{189}$$

with

$$\sqrt{|\gamma|} = |AB|(B^2 + r^2S^2)r^2 \sin\theta. \tag{190}$$

Using eq.(152), the dilaton can be calculated and is given by

$$\rho^2 = \frac{\rho_M}{|AB|(B^2 + r^2S^2)}. \tag{191}$$

Using this Ansatz, we can calculate the dilaton constraint eq.(183) which will amount to two coupled differential equations for the dilaton ρ and the coefficient functions A, B, E, S. Due to the imposed spherical symmetry and staticity, it suffices to consider the time and radial equations, which amount to

$$\frac{E}{B}\left(\left(\ln\left|\frac{A}{rE}\right|\right)' - \frac{2B^2}{r(B^2+r^2S^2)}\right) = 0$$
$$\left(\ln|A\rho^2(B^2+r^2S^2)|\right)' + \frac{2rS^2}{B^2+r^2S^2} = 0. \tag{192}$$

The eq.(192) can be seen to restrict the frame component functions and dilaton, leaving two independent degrees of freedom. Similarly, one can calculate the components of the axion one-form $\tilde{T}_\mu$, which are given by

$$\tilde{T}_t = -\frac{2rBS}{|AB|(B^2+r^2S^2)}\left(r^2E^2\left(\ln\left|\frac{B}{rS}\right|\right)' - A^2\left(\ln\left|\frac{B}{r^3S}\right|\right)'\right)$$
$$\tilde{T}_r = -\frac{2r^2EB^2S}{|AB|(B^2+r^2S^2)}\left(\ln\left|\frac{rS}{B}\right|\right)' \tag{193}$$

while all other components vanish in spherical coordinates. In the discussion of spherically symmetric static geometries admissible to a description in the present framework, we will make use of isotropic coordinates, which will shortly be introduced in the following.

Mathematical interlude: Isotropic coordinates Spherically symmetric static space-times admit coordinates $(t, \bar{r}, \theta, \phi)$ such that the metric takes on the following form

$$g = -a(\bar{r})^2 dt^2 + b(\bar{r})^2[d\bar{r}^2 + \bar{r}^2 d\Omega^2], \tag{194}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the standard surface element on the sphere. We will henceforth refer to eq.(194) as isotropic form of the metric. Writing the metric in this form, one can use the parametrized post-Newtonian formalism to compare the theory to other metric theories of gravity, such as general relativity. For more details, see for example [52]. One notes that eq.(194) has only two degrees of freedom, namely the two functions $a(\bar{r})$ and $b(\bar{r})$. If one considers the family of nested coordinate spheres, defined by $t = t_0, \bar{r} = \bar{r}_0$ for some constants $t_0, \bar{r}_0$, the metric eq.(194) becomes

$$g|_{t=t_0,\bar{r}=\bar{r}_0} = b(\bar{r}_0)^2\bar{r}_0^2 d\Omega^2 \tag{195}$$

which shows that these surfaces are indeed geometrical spheres. However, the appearance of $b(\bar{r}_0)^2\bar{r}_0^2$ shows that $\bar{r}$ does not have the natural interpretation of the radius of these spheres. This is in contrast to spherically symmetric static metrics in Schwarzschild coordinates, where r does indeed have the interpretation of the radius of these nested coordinate spheres.

Furthermore, the Lie algebra of Killing vector fields of a spherically symmetric static space-time in isotropic coordinates is generated by one timelike and three spacelike vector fields,

$$\left.\partial_t\right\}\text{timelike}$$

$$\left.\begin{array}{l}\partial_\phi \\ \sin(\phi)\partial_\theta + \cot(\theta)\cos(\phi)\partial_\phi \\ \cos(\phi)\partial_\theta - \cot(\theta)\sin(\phi)\partial_\phi\end{array}\right\}\text{spacelike} \tag{196}$$

which are noted to take the same form as in Schwarzschild coordinates.

As an example for a well known metric in isotropic form, consider the Schwarzschild metric

$$g_S = -\left(\frac{1-\frac{M}{2\bar{r}}}{1+\frac{M}{2\bar{r}}}\right)^2 dt^2 + \left(1+\frac{M}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2) \tag{197}$$

which is obtained from its usual form in Schwarzschild coordinates

$$g_S = -\left(1-\frac{2M}{r}\right) dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2 \tag{198}$$

by the following coordinate transformation:

$$r = \bar{r}\left(1+\frac{M}{2\bar{r}}\right)^2. \tag{199}$$

The event horizon $r = 2M$, in isotropic coordinates, is given by $\bar{r} = \frac{M}{2}$, in accordance to the fact that the norm of the timelike Killing vector field ∂_t vanishes on this hypersurface. One can solve eq.(199) for $\bar{r}$, which yields

$$\bar{r} = \frac{1}{2}\left(r - M + \sqrt{r(r-2M)}\right) \tag{200}$$

from which we can see that the new coordinates are only well-defined for $r \geq 2M$.

8. Finding and classifying possible geometries

The following section is committed to finding and investigating different geometries that can be described by the matrix model framework using the static, spherically symmetric Ansatz for the frame discussed in the previous section The main result is that the frame actually consists of unnecessary many non-trivial components and it is shown that an even simpler Ansatz suffices to describe the most general static, spherically symmetric metric. Still, we investigate how the different components of the frame contribute to the metric by studying various examples relating to the vanishing of the axion one-form.

8.1. $E = 0 = S$ system

In the case of $E = 0 = S$, the axion one-form, given in eq.(193), vanishes and the first dilaton constraint in eq.(192) is trivially satisfied. The radial dilaton constraint then simplifies significantly, and one obtains

$$(\ln |A|B^2\rho^2)' = \frac{1}{|A|B^2\rho^2}(|A|B^2\rho^2)' = 0, \tag{201}$$

which is solved by

$$\rho^2 = \frac{c_\rho}{|A|B^2}, \tag{202}$$

for a positive constant of integration c_ρ. The effective metric, using eq.(202), can be written as

$$G = -\frac{c_\rho|A|}{B^2}dt^2 + \frac{c_\rho}{|A|}\left(dr^2 + r^2 d\Omega^2\right). \tag{203}$$

By identifying $a(r)^2 = \frac{c_\rho|A|}{B^2}$ and $b(r)^2 = \frac{c_\rho}{|A|}$ and noting that A and B are independent degrees of freedom, we see that we have obtained the most general spherically symmetric static metric in isotropic coordinates. In the following, we want to study some well known examples of metrics on spherically symmetric static space-times and determine the exact expressions for the coefficient functions A and B, as well as for the dilaton ρ.

8.1.1. Schwarzschild metric

For simplicity, we will choose $c_\rho = 1$ in the following and denote the radial variable by $\bar{r}$, as to avoid confusion with the radial variable of the Schwarzschild coordinates. Recall the Schwarzschild metric eq.(197) in isotropic form. Comparing with the form of the effective metric eq.(203), we see that the coefficient functions A and B must satisfy

$$\frac{|A|}{B^2} = \left(\frac{1 - \frac{M}{2\bar{r}}}{1 + \frac{M}{2\bar{r}}}\right)^2, \quad \frac{1}{|A|} = \left(1 + \frac{M}{2\bar{r}}\right)^4 \tag{204}$$

which can be solved in a straightforward manner, yielding

$$
\begin{aligned}
A &= \frac{1}{\left(1+\frac{M}{2\bar{r}}\right)^4} \\
B &= \frac{1}{\left(1-\frac{M^2}{4\bar{r}^2}\right)}
\end{aligned}
\tag{205}
$$

where we imposed that $A, B > 0$ as $\bar{r} \to \infty$ since we are considering perturbations of the background solution eq.(156) and have restricted ourselves to the region outside the horizon, given by $\bar{r} > \frac{M}{2}$. Using eq.(202), one determines the dilaton to be

$$
\rho = \left(1+\frac{M}{2\bar{r}}\right)^2 \left(1-\frac{M^2}{4\bar{r}^2}\right)^2 . \tag{206}
$$

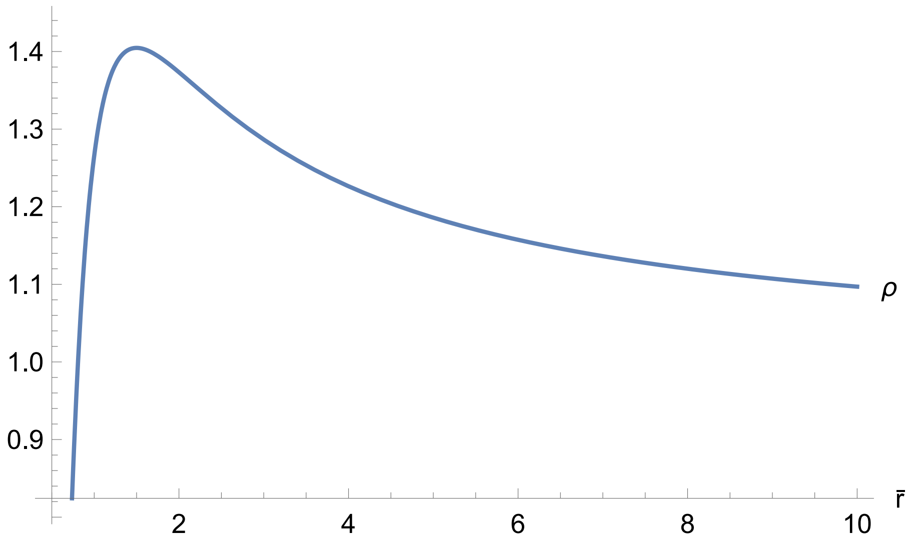

Figure 2: The dilaton in the region outside the Schwarzschild horizon $\bar{r} = M/2$ for $M = 1$.

As can be seen in figure 2, the dilaton ρ vanishes at the horizon, while reaching its maximum for $\bar{r} = M$. Its asymptotic behaviour is given by $\rho \to 1$ as $\bar{r} \to \infty$ in accordance with the fact that the frame considered in our Ansatz should approach the background frame eq.(156) for large $\bar{r}$ [50].

8.1.2. Reissner-Nordström metric

As another example, we want to consider the Reissner-Nordström metric, which in natural units is given by

$$
g = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{207}
$$

in a Schwarzschild chart. Here M is the mass of the system and Q its charge. The metric becomes singular at two horizons, corresponding to

$$
r_{\pm} = M \pm \sqrt{M^2 - Q^2} \tag{208}
$$

where r_+ is the event horizon, while r_- defines an inner horizon, called Cauchy horizon. In order to describe this metric in our framework, we will change coordinates to an isotropic chart. This is achieved by the following redefinition of the radial variable

$$r = \tilde{r}\left[\left(1+\frac{M}{2\tilde{r}}\right)^2 - \frac{Q^2}{4\tilde{r}^2}\right] \tag{209}$$

with which the metric can be written as

$$g = -\frac{\left[\left(1-\frac{M}{2\tilde{r}}\right)\left(1+\frac{M}{2\tilde{r}}\right)+\frac{Q^2}{4\tilde{r}^2}\right]^2}{\left[\left(1+\frac{M}{2\tilde{r}}\right)^2-\frac{Q^2}{4\tilde{r}^2}\right]^2}dt^2 + \left[\left(1+\frac{M}{2\tilde{r}}\right)^2 - \frac{Q^2}{4\tilde{r}^2}\right]^2(d\tilde{r}^2+\tilde{r}^2 d\Omega^2). \tag{210}$$

One can invert the relation eq.(209), obtaining

$$\tilde{r} = \frac{1}{2}\left(r - M + \sqrt{Q^2+r^2-2Mr}\right) \tag{211}$$

from which one can see that this coordinate change is valid for $Q^2 + r(r-2M) \geq 0$, which corresponds to the region outside the event horizon. In isotropic coordinates, this corresponds to the region described by $2\tilde{r} \geq \sqrt{M^2-Q^2}$. For simplicity, we again assume that $c_\rho = 1$, then comparison with eq.(203) shows that

$$\frac{|A|}{B^2} = -\frac{\left[\left(1-\frac{M}{2\tilde{r}}\right)\left(1+\frac{M}{2\tilde{r}}\right)+\frac{Q^2}{4\tilde{r}^2}\right]^2}{\left[\left(1+\frac{M}{2\tilde{r}}\right)^2-\frac{Q^2}{4\tilde{r}^2}\right]^2}, \quad \frac{1}{|A|} = \left[\left(1+\frac{M}{2\tilde{r}}\right)^2 - \frac{Q^2}{4\tilde{r}^2}\right]^2 \tag{212}$$

which again by demanding that $A, B > 0$ for $\tilde{r} \to \infty$, gives

$$A = \left[\left(1+\frac{M}{2\tilde{r}}\right)^2 - \frac{Q^2}{4\tilde{r}^2}\right]^{-2}, \quad B = \left[\left(1-\frac{M}{2\tilde{r}}\right)\left(1+\frac{M}{2\tilde{r}}\right)+\frac{Q^2}{4\tilde{r}^2}\right]^{-1} \tag{213}$$

from which, with eq.(202) the dilaton is calculated to be

$$\rho = \left[1+\frac{M}{\tilde{r}}+\frac{M^2-Q^2}{4\tilde{r}^2}\right]\left[1-\frac{M^2-Q^2}{4\tilde{r}^2}\right], \tag{214}$$

and is shown in figure 3.

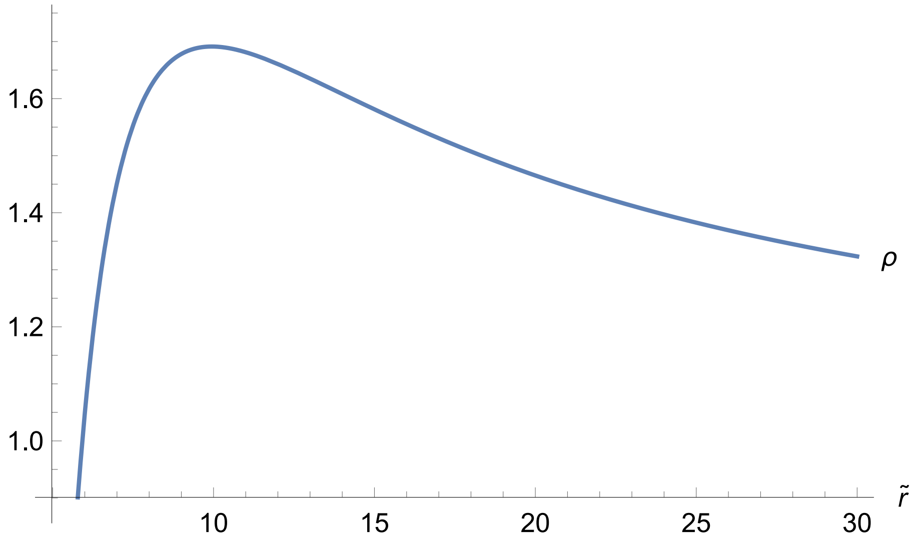

Figure 3: The dilaton in the region outside the event horizon $\tilde{r} = \sqrt{M^2 - Q^2}/2$, for $M = 10$, $Q = 1$.

As can bee seen in figure 3, the dilaton vanishes at the horizon, quickly obtains it maximum and goes to 1, as $\tilde{r} \to \infty$. Similarly, A and B approach 1 for $\tilde{r} \to \infty$, as we want for pertubations of the background frame eq.(156).

Extremal case $|Q| = M$ In the extremal case, the Reissner-Nordström metric takes a much simpler form,

$$g_{ext} = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \tag{215}$$

and the two horizon collaps to a single one, located at

$$r_{ext} = M. \tag{216}$$

The coordinate change to an isotropic chart also simplifies,

$$r = \tilde{r} + M \tag{217}$$

and the metric in isotropic coordinates then reads

$$g_{ext} = -\left(1 + \frac{M}{\tilde{r}}\right)^{-2} dt^2 + \left(1 + \frac{M}{\tilde{r}}\right)^2 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2). \tag{218}$$

The horizon is described by $\tilde{r}_{ext} = 0$ and the region outside the horizon corresponds to $\tilde{r} > 0$. The frame component functions can be directly determined from eq.(213)

$$A_{ext} = \left(1 + \frac{M}{\tilde{r}}\right)^{-2}, \quad B_{ext} = 1 \tag{219}$$

while the dilaton reads

$$\rho_{ext} = 1 + \frac{M}{\tilde{r}}. \tag{220}$$

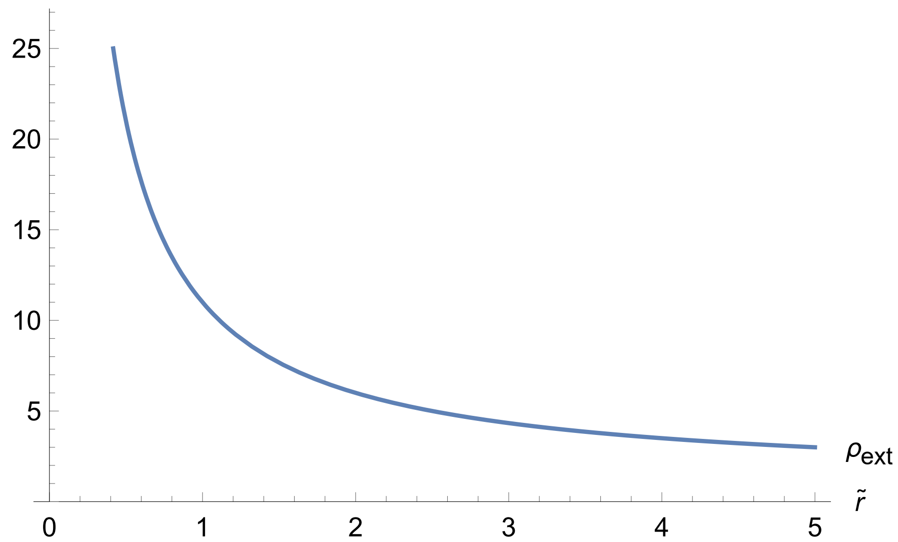

Figure 4: The dilaton in the extremal case $M = Q = 10$ in the region outside the event horizon $\tilde{r}_{ext} = 0$.

As can be seen in figure 4, the dilaton behaves qualitatively different in the extremal case. It does not stay bounded as one approaches the horizon, but rather diverges. However, it still exhibits the right behaviour for large $\tilde{r}$.

8.2. Axion free system

From eq.(193), we see that there are two possibilities for a vanishing axion one-form, either the trivial $S = 0$ case, or a condition that relates the two coefficient functions B and S. We will analyze these two cases in the following, in order to investigate how the coefficient functions E and S contribute to the metric in the current setting.

8.2.1. Case of $S = 0$

In this case, one first notices that the dilaton constraints eqs.(192) simplify and can both be directly integrated, yielding

$$|E| = \frac{c_E |A|}{r^3},\ \rho^2 = \frac{c_\rho}{|A| B^2} \tag{221}$$

where both c_E and c_ρ are positive integration constants. The effective metric is then given by

$$G = \rho^2 \left[-\left(A^2 - r^2 E^2\right) dt^2 + 2rEBdtdr + B^2 \left(dr^2 + r^2 d\Omega^2\right)\right] \tag{222}$$

which we wish to diagonalize by making the Ansatz $t \to t + f(r)$. One sees that for

$$f'(r) = \frac{rEB}{A^2 - r^2E^2} = \frac{B}{|A|} \frac{c_E}{r^2 - \frac{c_E^2}{r^2}} \tag{223}$$

this is achieved, where the dilaton constraint eqs.(221) was used. One notes that this coordinate transformation could become ill-behaved as $r \to \sqrt{c_E}$, so that we will constraint ourselves to $r > \sqrt{c_E}$. Again, using eqs.(221), the effective metric can be written as

$$G = -\frac{c_\rho |A|}{B^2}\left(1 - \frac{c_E^2}{r^4}\right) dt^2 + \frac{c_\rho}{|A|}\left(\frac{1}{1 - \frac{c_E^2}{r^4}}\right) dr^2 + \frac{c_\rho}{|A|} r^2 d\Omega^2. \tag{224}$$

In order to investigate the influence of the E frame coefficient function, we will choose $A, B, \rho = 1$, such that the effective metric becomes:

$$G = -\left(1 - \frac{c_E^2}{r^4}\right) dt^2 + \left(\frac{1}{1 - \frac{c_E^2}{r^4}}\right) dr^2 + r^2 d\Omega^2. \tag{225}$$

Upon inspection, one sees that for $r \to \sqrt{c_E}$, the metric becomes singular, indicating some sort of horizon-like structure. To further investigate this metric, we will perform a second coordinate transformation in order to obtain the effective metric in isotropic form. Comparing with eq.(194), we see that the conditions for isotropic form of the metric under a coordinate change $r \to r(\bar{r})$ are

$$\begin{aligned} b(\bar{r})^2 d\bar{r}^2 &= \frac{dr^2}{1 - \frac{c_E^2}{r^4}} \\ b(\bar{r})^2 \bar{r}^2 &= r^2. \end{aligned} \tag{226}$$

Taking the square root of their quotient, these can be integrated yielding

$$\ln\left(\bar{r}^4 / c_E^2\right) = \ln\left(\frac{r^2 + \sqrt{r^4 - c_E^2}}{r^2 - \sqrt{r^4 - c_E^2}}\right) \tag{227}$$

where we chose the integration constant such that the argument of the lhs becomes dimensionless, noting that c_E sets the only comparable length scale of our problem. Solving this for r gives

$$r = \frac{\sqrt{\bar{r}^4 + c_E^2}}{\sqrt{2}\bar{r}} \tag{228}$$

and the effective metric reads

$$G = -\left(\frac{\bar{r}^4 - c_E^2}{\bar{r}^4 + c_E^2}\right)^2 dt^2 + \left(\frac{\bar{r}^4 + c_E^2}{2\bar{r}^4}\right)(d\bar{r}^2 + \bar{r}^2 d\Omega^2). \tag{229}$$

A few notes on the preceding calculations: From eq.(228) we see that $\bar{r} \to \sqrt{c_E}$ as $r \to \sqrt{c_E}$, however the effective metric eq.(229) experiences no singular behaviour for $\bar{r} \to \sqrt{c_E}$. Of course, however, the horizon is still encoded in the fact that the norm of the timelike Killing vector ∂_t vanishes at $\bar{r} = \sqrt{c_E}$, making this a Killing horizon.

8.2.2. Axion condition

Now we want to turn our attention to the case $E = 0$ but $S \neq 0$. As seen by inspecting eq.(193), $\tilde{T}_r = 0$ and $\tilde{T}_t$ will also vanish, if

$$\left(\ln\left|\frac{B}{rS}\right|\right)' = \frac{2}{r} \tag{230}$$

which can be straightforwardly integrated to yield $S = \frac{c_S B}{r^3}$, where c_S is a positive integration constant. For the following analysis, we will adapt eq.(230), so that the axion one-form vanishes, but $S \neq 0$. Plugging this into the second of eqs.(192), one can integrate the resulting equation and solve for the dilaton, giving

$$\rho^2 = \frac{c_\rho}{|A|B^2\sqrt{1+\frac{c_S^2}{r^4}}} \tag{231}$$

where $c_S > 0$ is an integration constant. With this, the effective metric becomes

$$G = c_\rho\left[-\frac{|A|}{B^2\sqrt{1+\frac{c_S^2}{r^4}}}dt^2 + \frac{dr^2}{|A|\sqrt{1+\frac{c_S^2}{r^4}}} + \frac{\sqrt{1+\frac{c_S^2}{r^4}}}{|A|}r^2 d\Omega^2\right]. \tag{232}$$

Again, for our analysis of how imposing this condition affects the effective metric, we will assume $A, B, c_\rho = 1$. In order to transform the effective metric into isotropic form, recall eq.(226) and note that the same logic can be used to to arrive at the following equation of differentials

$$\frac{d\bar{r}}{\bar{r}} = \frac{dr}{r^2(1+\frac{c_S^2}{r^4})} \tag{233}$$

which can be integrated to give

$$\bar{r}^4/c_S^2 = \frac{\sqrt{r^4+c_S^2}+r^2}{\sqrt{r^4+c_S^2}-r^2} \tag{234}$$

which, under the condition that $\bar{r} > \sqrt{c_S}$, can be inverted, yielding the following coordinate transformation:

$$r = \frac{1}{\sqrt{2}}\frac{\sqrt{\bar{r}^4-c_S^2}}{\bar{r}}. \tag{235}$$

Using this, the effective metric can be written as

$$G = -\frac{\bar{r}^4-c_S^2}{\bar{r}^4+c_S^2}dt^2 + \frac{\bar{r}^4+c_S^2}{2\bar{r}^4}(d\bar{r}^2+\bar{r}^2 d\Omega^2). \tag{236}$$

We note that, as can be seen from eq.(235), $\bar{r} \to \sqrt{c_S}$ corresponds to $r \to 0$, corresponding to the origin, where the metric becomes singular.

8.3. General coordinate transformation into isotropic form

For completion, we want to look at how the metric eq.(189) in its full extent looks like when transformed into isotropic coordinates. Consider the full form of the metric in spherical coordinates:

$$\gamma = -(A^2 - r^2E^2)dt^2 + 2rBEdtdr + B^2dr^2 + r^2(B^2 + r^2S^2)d\Omega^2. \tag{237}$$

In order to make comparisons with other metrics easier, we will now consider a general Ansatz for a coordinate transformation, such that the metric takes the isotropic form eq.(194). To achieve this, we consider two coordinate transformations, the first of which diagonalizes the metric, while the second one transforms the metric into isotropic form. First, we make the following transformation of the t coordinate.

$$\bar{t} = t + f(r) \tag{238}$$

where f is an undetermined function of r at first. Calculating the differential gives

$$dt = d\bar{t} - f'(r)dr \tag{239}$$

and the off-diagonal term vanishes if

$$f'(r) = \frac{rEB}{r^2E^2 - A^2} \tag{240}$$

which we will choose now. The metric then becomes diagonal:

$$\gamma = -(A^2 - r^2E^2)d\bar{t}^2 + \left(\frac{A^2B^2}{A^2 - r^2E^2}\right)dr^2 + (B^2 + r^2S^2)r^2d\Omega^2. \tag{241}$$

While this first coordinate transformation is relatively straightforward, the second one into isotropic form is more complicated. We make the Ansatz $r = g(\bar{r})$ for some undetermined function g, so that the relevant part of the metric becomes

$$\left(\frac{\bar{A}^2\bar{B}^2}{\bar{A}^2 - g^2\bar{B}^2}\right)g'^2d\bar{r}^2 + (\bar{B}^2 + g^2\bar{S}^2)g^2d\Omega^2 \tag{242}$$

where the bar denotes the coefficient functions after the coordinate transformation, i.e. $\bar{A}(\bar{r}) = A(g(\bar{r}))$. Then the condition on g can be read off of eq.(242):

$$\frac{\bar{A}^2\bar{B}^2}{\bar{A}^2 - g^2\bar{B}^2}g'^2 = \frac{(\bar{B}^2 + g^2\bar{S}^2)g^2}{\bar{r}^2}. \tag{243}$$

While this differential equation is quite involved, in general solutions should exist, if we assume that the coefficient functions A, B, E and S are such that it has smooth coefficients. This will formally be assumed for now. The isotropic form of the metric then reads:

$$\gamma = -\left(\bar{A}^2 - g^2\bar{E}^2\right)d\bar{t}^2 + \left(\frac{\bar{A}^2\bar{B}^2g'^2}{\bar{A}^2 - g^2\bar{E}^2}\right)(d\bar{r}^2 + \bar{r}^2d\Omega^2). \tag{244}$$

One should note however, that eq.(244) is still constraint, as the dilaton identity eq.(192) has not been used in its derivation yet. As discussed in [50], if all four component functions A, B, E, S are non vanishing, one can integrate the difference of the two eq.(192) to obtain

$$E = \frac{c_E}{r^3 \rho^2 (B^2 + r^2 S^2)} \tag{245}$$

where again $c_E > 0$ is an integration constant. Inserting this into the first of the equations gives

$$\left(\ln \left| \frac{A}{rE} \right| \right)' = \frac{2}{c_E} r^2 \rho^2 B^2 E. \tag{246}$$

Eq.(245) and eq.(246) can alternatively be written as

$$B^2 = \frac{c_E}{2r^2 \rho^2} \frac{\left(\ln \left| \frac{A}{rE} \right| \right)'}{E}$$

$$S^2 = -\frac{c_E}{2r^4 \rho^2} \frac{\left(\ln \left| \frac{A}{r^3 E} \right| \right)'}{E}$$

which shows that B and S can be fully determined from the two arbitrary functions A and E. This is in accordance with what was discussed in section 8.1, namely that the matrix model framework leaves exactly two degrees of freedom undetermined, which is necessary to produce any static spherically symmetric space-time.

9. The problem of constructing semi-classical generators for the frame

While section 8 was devoted to finding geometries that suit a description in the present framework, one question that was not yet considered is, whether given a fixed frame $E_{\dot\alpha}{}^{\mu}$, there exists a matrix configuration $Z_{\dot\alpha}$, such that in the semi-classical regime,

$$\{z_{\dot\alpha}, x^{\mu}\} = E_{\dot\alpha}{}^{\mu} \tag{247}$$

where $Z_{\dot\alpha} \sim z_{\dot\alpha}$ and x^{μ} are (general) coordinates on $\mathcal{M}^{3,1}$. This question was already addressed in [47], where for diagonal frames of the form

$$\begin{aligned} E_0{}^0 &= \sinh\eta\ A^{-1}(r) \\ E_i{}^j &= \sinh\eta\ b_0 \delta_i{}^j \end{aligned} \tag{248}$$

with $r^2 = x^i x^i$ and Cartesian coordinates x^i, it was shown that it was always possible to find such generators $z_{\dot\alpha}$, at least in the asymptotic regime $\eta \to \infty$. One notes that these generators have an infinite tower of higher-spin corrections, showing that even in this simple case, one needs to include terms with arbitrary spin components.

9.1. Generators for static spherically symmetric frames

In this section, we will discuss the problem of constructing semi-classical generators $z_{\dot\alpha}$ for the frame corresponding to the most general static spherically symmetric metric as discussed in section 8.1. This frame is diagonal with two radial coefficient functions:

$$E_0{}^0 = A^{-1}(r^2),\ E_i{}^j = B^{-1}(r^2)\delta_i{}^j \tag{249}$$

with all other components of the frame being zero. As the relations discussed in section 4.5 for the semi-classical space-time are given for x^{μ} seen as Cartesian coordinates, we will adopt this choice for these considerations. Therefore, the problem can be stated as follows: One must find functions $\tilde{z}_{\dot\alpha} \in \mathcal{C}$ such that

$$\begin{aligned} \left\{\tilde{z}_0, x^0\right\} &= -\sinh\eta\ A^{-1}(r^2),\ \left\{\tilde{z}_0, x^i\right\} = 0 \\ \left\{\tilde{z}_i, x^0\right\} &= 0,\ \left\{\tilde{z}_i, x^j\right\} = \sinh\eta\ B^{-1}(r^2)\delta_i{}^j \end{aligned} \tag{250}$$

where we inserted a factor of $\sinh\eta$ in order to obtain the background frame eq.(156) in the large r limit. In general, it is not possible to satisfy all four eqs.(250) simultaneously in a strict sense. However since the diagonal brackets grow exponentially in the late-time regime $\eta \to \infty$, one can hope to at least satisfy eqs.(250) up to terms that stay bounded as η grows large. In order to analyze the late-time behaviour of the different contributions, one will need the following relations

$$\begin{aligned} |\vec{t}| \sim_{\eta} \tilde{R}^{-1}\cosh\eta,\ |x_0| \sim_{\eta} R\cosh\eta \\ |t_0| \sim_{\eta} 1,\ |\vec{x}| = r \sim_{\eta} 1 \end{aligned} \tag{251}$$

where $\sim_{\eta}$ denotes the behaviour for large η. These relations are simple consequences of the relations eq.(111a) and eq.(111b).

9.2. Naive Ansatz

To find the three spatial generators $\tilde{z}_i$, we make the following Ansatz:

$$\tilde{z}_i = h(r^2)t_i + k(r^2)x_i. \tag{252}$$

In order to compute the bracket with x^j, we require the following two relations:

$$\begin{aligned} \left\{r^2, x^j\right\} &= 2x_k\theta^{kj} = \frac{2\tilde{R}\sinh\eta}{\cosh^2\eta}\left(r^2t^j - x_kt^kx^j\right) \\ \theta^{ij} &= \frac{\tilde{R}}{\cosh^2\eta}\left[\sinh\eta\left(x^it^j - t^ix^j\right) + \varepsilon^{ijk}\left(x_0t_k - x_kt_0\right)\right] \end{aligned} \tag{253}$$

which are easily seen to be consequences of the explicit form of the Poisson tensor given by eq.(112). Using these, and the Poisson brackets eqs.(117) one can calculate the brackets obtaining

$$\begin{aligned} &\left\{z_i, x^j\right\} = \sinh\eta\; h(r^2)\delta_i^{\,j} + \left\{r^2, x^j\right\}\left(\partial_{r^2}h(r^2)t_i + \partial_{r^2}k(r^2)x_i\right) + k(r^2)\theta^{ij} \\ &= \sinh\eta\; h(r^2)\delta_i^{\,j} + \frac{2\tilde{R}^2}{\cosh^2\eta}\Bigg[\sinh\eta\Bigg(\left(r^2k'(r^2) + \frac{1}{2}k(r^2)\right)x_it^j - \left(x_kt^kh'(r^2) + \frac{1}{2}k(r^2)\right)t_ix^j \\ &+ r^2h'(r^2)t_it^j - x_kt^kk'(r^2)x_ix^j\Bigg) + \frac{1}{2}k(r^2)\varepsilon^{ijk}\left(x_0t_k - x_kt_0\right)\Bigg] \end{aligned} \tag{254}$$

from which one would now try to obtain relations determining the functions h and k. In order to do this systematically in the late-time regime, one considers the behaviour of the different contributions as η grows. One sees that the first term on the lhs of eq.(254) looks exactly like the one needed to satisfy the bottom-right eq. of eqs.(250). Thus, one focuses on the terms following. Using eqs.(251) one sees that

$$\begin{aligned} &\left|\left\{z_i, x^j\right\} - \sinh\eta\; h(r^2)\delta_i^{\,j}\right| \le \frac{2\tilde{R}}{\cosh^2\eta}\Bigg[|\sinh\eta|\Bigg(\left|r^2k'(r^2) + \frac{1}{2}k(r^2)\right|r|\vec{t}| \\ &+ \left|x_kt^kh'(r^2) + \frac{1}{2}k(r^2)\right|r|\vec{t}| + r^2|h'(r^2)||\vec{t}|^2 + r^3k'(r^2)|\vec{t}|\Bigg) + \frac{1}{2}|k(r^2)|\Big(x_0||\vec{t}| + r|t_0|\Big)\Bigg] \\ &\sim_\eta 2\tilde{R}\Bigg[\tanh\eta\left|r^2k'(r^2) + \frac{1}{2}k(r^2) + r^3k'(r^2)\right| + \tanh\eta\left|x_kt^kh'(r^2) + \frac{1}{2}k(r^2)\right| \\ &+ \tilde{R}^{-1}\sinh\eta\; r^2|h'(r^2)| + \frac{1}{2}k(r^2) + \frac{k(r^2)}{2\cosh^2\eta}\Bigg], \end{aligned} \tag{255}$$

where the only two terms that do not stay bounded as η grows are

$$\tanh\eta\left|x_kt^kh'(r^2)\right|,\; \tilde{R}^{-1}\sinh\eta\; r^2|h'(r^2)| \tag{256}$$

which both grow exponentially as η grows large and can therefore not be neglected in the late-time regime. Of course setting $h'(r^2) = 0$ would make both these terms vanish, but then the leading contribution to the frame would become trivial. This suggest that one has to make a more general Ansatz for the generators $\tilde{z}_{\dot{\alpha}}$ and additionally for the coordinates x^μ so that the unwanted higher-spin contributions to the frame cancel.

9.3. General Ansatz

In order to compensate for the exponentially large, higher spin contributions, we will consider the following more general Ansatz,

$$\begin{aligned} \bar{x}^0 &= \bar{x}^0(r^2, \chi, x^0), \ \bar{x}^i = a(r^2, \chi)t^i + b(r^2, \chi)x^i \\ z^0 &= g(r^2, \chi)t^0, \ z^i = c(r^2, \chi)t^i + d(r^2, \chi)x^i \end{aligned} \tag{257}$$

where, since we consider perturbations of the background frame eq.(156), we impose the following asymptotic conditions,

$$\begin{aligned} a &\to 0, \ b \to 1 \\ c &\to 1, \ d \to 0 \end{aligned} \tag{258}$$

for $r \to \infty$. Here χ is the central element of the $SO(3)$-invariant subalgebra of functions generated by r^2, x^0, p^0 and is defined as $\chi := r^2 - (p^0)^2$ [47]. This and the tensor structure of the generators are chosen so as to transform in the vector rep. of $SO(3)$, in hopes of recovering the diagonal part of the static spherically symmetric frame of eq.(186). Moreover, as can be seen from eq.(251), $a(r^2, \chi)t^i$ would be the dominant contribution to the new coordinate $\bar{x}^i$ in the large-η limit. As we do not want the internal d.o.f. to dominate, we will further impose that $a \sim_\eta \cosh\eta^{-1}$. The redefinition of the coordinates x^μ of the semi-classical cosmological space-time amounts to a different projection $\tilde{\Pi}$ from the six-dimensional bundle space $\mathbb{C}P^{1,2}$. Thus, the newly defined $\tilde{x}^\mu$ are to be seen as embedding functions $\mathcal{M}^{3,1} \hookrightarrow \mathbb{R}^{3,1}$ and accordingly, the decomposition of $\mathcal{C}$ into functions on the base and higher-spin sectors will change. Therefore, we have to express the frame obtained by this Ansatz as functions of only the new embedding coordinates $\tilde{x}^\mu$, in order for it to be well-defined on the base.

In order to calculate the spacial bracket $\{z^i, \bar{x}^j\}$, we will, in addition to the ones in eq.(253), need the following basic brackets

$$\begin{aligned} \{x^i, \chi\} &= \{x^i, r^2\}, \ \{t^i, r^2\} = 2\sinh\eta x^i \\ \{t^i, \chi\} &= \frac{2\tilde{R}^2}{\cosh^2\eta}\left(\sinh\eta(t^2x^i - x_k t^k t^i) - \varepsilon^{imn}x_m t_n t^0\right) \end{aligned} \tag{259}$$

which are derived from the basic brackets and constraints of the coordinate functions given in section 4.5. Using these, one obtains[28]

$$\begin{aligned}\{z^i,\bar{x}^j\} &= -\sinh\eta\ (ad-bc)\delta^{ij} + \left(bd - \frac{ac}{\tilde{R}R}\right)\theta^{ij} + F_{11}t^it^j + F_{22}x^ix^j - F_{12}t^ix^j + F_{21}x^it^j\\ &- \frac{\tilde{R}t^0}{\cosh^2\eta}\left(\varepsilon^{imn}x_mt_n\Big((\partial_\chi a)ct^j + (\partial_\chi b)cx^j\Big) - \varepsilon^{jmn}x_mt_n\Big((\partial_\chi c)at^i + (\partial_\chi d)ax^i\Big)\right) \qquad (260)\end{aligned}$$

where, by introducing the directional derivative $D := \partial_{r^2} + \partial_\chi$, the four scalar functions F_{kl} can be written as

$$\begin{aligned}F_{11} &= \frac{2\tilde{R}^2\sinh\eta}{\cosh^2\eta}\left(\Big((Dc)b-(Da)d\Big)r^2 + \Big((\partial_\chi c)a - (\partial_\chi a)c\Big)x_kt^k\right)\\ F_{22} &= 2\sinh\eta\left(\frac{\tilde{R}^2}{\cosh^2\eta}\Big(\big((Db)d-(Dd)b\big)x_kt^k + \big((\partial_\chi b)c - (\partial_\chi d)a\big)t^2\Big) + (\partial_{r^2}b)c - (\partial_{r^2}d)a\right)\\ F_{12} &= 2\sinh\eta\left(\frac{\tilde{R}^2}{\cosh^2\eta}\Big(\big((Dc)b-(\partial_\chi b)c\big)x_kt^k + (Db)dr^2 + (\partial_\chi c)at^2\Big) + (\partial_{r^2}c)a\right)\\ F_{21} &= 2\sinh\eta\left(\frac{\tilde{R}^2}{\cosh^2\eta}\Big(\big((Da)d-(\partial_\chi d)a\big)x_kt^k + (Dd)br^2 + (\partial_\chi a)ct^2\Big) + (\partial_{r^2}a)c\right). \qquad (261)\end{aligned}$$

While the expression for the bracket is rather involved, things get a little less complicated once one goes to the large-η limit. Considering the asymptotic behaviour depicted in eq.(251) and recalling that the coefficient function a is asymptotically suppressed, one arrives at the following expression

$$\{z^i,\bar{x}^j\} \sim_\eta \sinh\eta\ bc\delta^{ij} + \tilde{F}_{11}t^it^j + \tilde{F}_{22}x^ix^j - \tilde{F}_{12}t^ix^j + \tilde{F}_{21}x^it^j \qquad (262)$$

with the four slightly different scalar functions

$$\begin{aligned}\tilde{F}_{11} &= \frac{2\tilde{R}^2\sinh\eta}{\cosh^2\eta}\left((Dc)br^2 + \Big((\partial_\chi c)a - (\partial_\chi a)c\Big)x_kt^k\right)\\ \tilde{F}_{22} &= 2\sinh\eta(Db)c\\ \tilde{F}_{12} &= 2\sinh\eta\left(\frac{\tilde{R}^2}{\cosh^2\eta}\big((Dc)b - (\partial_\chi b)c\big)x_kt^k + (\partial_{r^2}c)a\right)\\ \tilde{F}_{21} &= 2\sinh\eta(Da)c. \qquad (263)\end{aligned}$$

Since we are interested in obtaining a frame that is diagonal, we demand that the $\tilde{F}$'s vanish, which is equivalent to the following set of coupled PDE's in the coefficient functions

$$\begin{aligned}&(Dc)br^2 + \Big((\partial_\chi c)a - (\partial_\chi a)c\Big)x_kt^k = 0,\ (Db)c = 0\\ &\frac{\tilde{R}^2x_kt^k}{\cosh^2\eta}\big((Dc)b - (\partial_\chi b)c\big) + (Dc)a = 0,\ (Da)c = 0. \qquad (264)\end{aligned}$$

The two right-most equations in eq.(264) are rather trivial to solve and give that

$$b(r^2,\chi) = b_0(r^2-\chi),\ a(r^2,\chi) = \frac{a_0(r^2-\chi)}{\cosh\eta} \qquad (265)$$

[28]Note that again we repressed the explicit arguments of the coefficient functions of the Ansatz.

where $a_0, b_0 : [0, r] \to \mathbb{R}$ are smooth and we made the large-η behaviour of a explicit. Before we proceed with our analysis, let us briefly recall our goal, which is to realize a diagonal frame that is only dependent on a radial coordinate, i.e. $\{z^i, \bar{x}^j\} = \sinh\eta\ \bar{B}(\bar{r}^2)\delta^{ij}$, where $\bar{r}^2 = \bar{x}_k \bar{x}^k$ and hence $\bar{B}(\bar{r}^2) = B(r^2, \chi)$. Comparison with eq.(262) shows that thus,

$$c(r^2, \chi) = \frac{B(r^2, \chi)}{b_0(r^2 - \chi)} \tag{266}$$

which by plugging into the remaining two PDE's of eqs.(264) gives

$$\begin{aligned} (DB)r^2 + \frac{B}{b_0 \cosh\eta}\left(\left(\frac{\partial_\chi B}{B} + \frac{b_0'}{b_0}\right)a_0 + a_0'\right)x_k t^k &= 0 \\ \frac{\tilde{R}^2 x_k t^k b_0}{\cosh\eta}\left(\frac{DB}{B} + \frac{b_0'}{b_0}\right) + \frac{DB}{B}a_0 &= 0. \end{aligned} \tag{267}$$

We have thus obtained a problem that is equivalent to finding a diagonal frame only dependent on a radial variable, namely the following: Given an arbitrary function $\bar{B}(\bar{r}^2(r^2, \chi))$, can we find functions $a_0(r^2 - \chi), b_0(r^2 - \chi)$ such that eqs.(267) are satisfied?

Recalling that $\chi \coloneqq r^2 - (p^0)^2$, one sees that a necessary condition for the existence of a solution to this system of ODE's would be that both eqs.(267) have to be writable as ODE's with the only independent variable given by p^0. This however, seems to not be possible, so that in general, no solutions to this system exist.

9.4. Simpler Ansatz

To see what kind of frames one can build in this framework in general, we'll study a simplified Ansatz for the projection to a four-dimensional space-time. To keep the spatial components of the frame simpler, we will not change the projection to the spatial coordinates of the background, instead we will make a general Ansatz for the temporal coordinate and generator:

$$\begin{aligned} &\tilde{x}^0 = \tilde{x}^0(r^2, \chi, x^0),\ \tilde{x}^i = x^i \\ &\tilde{z}^i = c(r^2, \chi)t^i + d(r^2, \chi)x^i,\ \tilde{z}^0 = a(r^2, \chi)t^0. \end{aligned} \tag{268}$$

With this Ansatz, the spatial components of the frame in the asymptotic regime are

$$\begin{aligned} E_i^{\ j} = \{\tilde{z}^i, \tilde{x}^j\} \sim_\eta \sinh\eta\ c(r^2, \chi)\delta^{ij} + \frac{2\tilde{R}^2 \sinh\eta}{\cosh^2\eta}(Dc)(r^2, \chi)r^2 t^i t^j \\ - \frac{2\tilde{R}^2 R^2 \sinh\eta}{R\cosh\eta}(Dc)(r^2, \chi)t^0 t^i x^j - \frac{2\tilde{R}^2 R^2 x_0}{R^2 \cosh^2\eta}\varepsilon^{imn} t_m x_n t^j (Dc)(r^2, \chi). \end{aligned} \tag{269}$$

From eq.(269) it immediately becomes obvious, that the only way to obtain a frame that is only a function of the new space-time coordinates $\tilde{x}^\mu$, is by demanding that $Dc = 0$, e.g., c is only a function of p^0. Unfortunately, this does not fully resolve the issue, as we now have to define the temporal coordinate $\tilde{x}^0$ in such a way, that we can express p^0 as a function of the new space-time coordinates $\tilde{x}^\mu$.

With this in mind, we calculate the corresponding temporal component of the frame:

$$E_0{}^0 = -\{\tilde{z}^0, \tilde{x}^0\} = \sinh\eta\big[a(r^2,\chi) + 2(r^2-\chi)\partial_{r^2}a(r^2,\chi)\big]\partial_{x^0}\tilde{x}^0(r^2,\chi,x^0). \tag{270}$$

This expression is similar to the one obtained in [47], so that one can employ the solution found there to the problem at hand. Let g be an arbitrary function of r^2. Then

$$a(r^2,\chi) = \frac{1}{2p^0}\int_\chi^{r^2}\frac{g(u)}{\sqrt{u-\chi}}du, \tag{271}$$

solves the following differential equation:

$$a(r^2,\chi) + 2(r^2-\chi)\partial_{r^2}a(r^2,\chi) = g(r^2). \tag{272}$$

This is a promising first step to obtaining a space-time with spherically symmetric metric on it, as, apart from how one defines the temporal coordinate $\tilde{x}^0$, this gives a diagonal component of the frame that is purely a function of the radial variable r^2. One further notes, that this holds generally and not only in the late-time regime.

Thus, defining a in that way, one obtains that

$$E_0{}^0 = \sinh\eta\; g(r^2)\partial_{x^0}\tilde{x}^0(r^2,\chi,x^0), \tag{273}$$

as is desired. The analysis of the temporal off-diagonal components of the frame gives the following two expressions in the late-time regime:

$$\begin{aligned}
E_i{}^0 &= \{\tilde{z}^i, \tilde{x}^0\} \sim_\eta 2\sinh\eta\Bigg(\Big[(D\tilde{x}^0)(r^2,\chi,x^0)c((p^0)^2) - \tilde{R}R\partial_{r^2}d(r^2,\chi)\partial_{x^0}\tilde{x}^0(r^2,p^0,x^0)\Big]\tilde{x}^i \\
&\quad - \Big[\frac{\partial_\chi\tilde{x}^0(r^2,\chi,x^0)c((p^0)^2)}{R\cosh\eta} + \partial_{x^0}\tilde{x}^0(r^2,\chi,x^0)c'((p^0)^2) + \frac{\tilde{R}Rd(r^2,\chi)\partial_{x^0}\tilde{x}^0(r^2,\chi,x^0)}{2R\cosh\eta}\Big]p^i\Bigg) \\
E_0{}^i &= -\{\tilde{z}^0, \tilde{x}^i\} \sim_\eta 1.
\end{aligned} \tag{274}$$

While $E_0{}^i$ stays bounded in the late-time regime and can thus be neglected, the analysis of $E_i{}^0$ is somewhat more complicated. Recall that in order to have a well-defined frame on our four-dimensional space-time $\mathcal{M}^{3,1}$, it needs to be expressed purely as a function of the space-time coordinates $\tilde{x}^\mu$. Considering our Ansatz eq.(268), one sees that in order to achieve this, all terms proportional to p^i in eq.(274) have to vanish. To make this possible, we make the choice of demanding that $\partial_{x^0}\tilde{x}^0 = 1$, or in other words $\tilde{x}^0 = x^0 + f(r^2,\chi)$, which is quite natural, considering we are looking at perturbations of the background.

Since the coefficient function d does not contribute to any of the other frame components in the late-time regime, we have freedom to choose it such that the following expression vanishes:

$$\frac{c((p^0)^2)}{R\cosh\eta} + c'((p^0)^2) + \frac{\tilde{R}Rd(r^2,\chi)}{2R\cosh\eta}. \tag{275}$$

In order to further simplify the frame and the relation between new and old space-time coordinates, one can make the additional demand that $D\tilde{x}^0 = 0$, so that $\tilde{x}^0 = x^0 + f((p^0)^2)$. To summarize, the chosen Ansatz becomes

$$\begin{aligned}&\tilde{x}^0 = x^0 + f((p^0)^2),\ \ \tilde{x}^i = x^i \\ &\tilde{z}^0 = a(r^2,\chi)t^0,\ \ \tilde{z}^i = c((p^0)^2)t^i + d(r^2,\chi)x^i\end{aligned} \tag{276}$$

where

$$\begin{aligned}a(r^2\chi) &= \frac{1}{2p^0}\int_\chi^{r^2} \frac{g(u)}{\sqrt{u-\chi}}du \\ d(r^2,\chi) &= -\frac{2}{\tilde{R}R}\Big(c((p^0)^2) + R\cosh\eta c'((p^0)^2)\Big)\end{aligned} \tag{277}$$

for arbitrary functions of one variable g and c. In the late-time regime, this then gives the following frame:

$$\begin{aligned}&E_0{}^0 = \sinh\eta\ g(r^2),\ \ E_i{}^j \sim_\eta -\sinh\eta\ c((p^0)^2)\delta_i^j \\ &E_0{}^i \sim_\eta 0,\ \ E_i{}^0 \sim_\eta -2\tilde{R}R\sinh\eta\ \partial_{r^2}d(r^2,\chi)\tilde{x}^i.\end{aligned} \tag{278}$$

However, there is still the issue of expressing p^0 as a function of the new space-time coordinates $\tilde{x}^\mu$, in order to make the frame a well-defined object on the base $\mathcal{M}^{3,1}$, independent of the fibre, e.g. the d.o.f. on the internal S^2. From eqs.(276) it is immediately obvious, that this is not possible for this choice of coordinates $\tilde{x}^\mu$. However, in the above section 9.4, all additional constraints or assumptions where made in order to obtain a diagonal frame, that is only a function of the space-time coordinates, so that the conclusion must be that it is not possible with this Ansatz.

The above considerations lead to the following take-home message regarding the issue of reconstructing semi-classical generators $z_{\dot\alpha}$ for $\mathfrak{hs}$-valued frame fields on $\mathcal{M}^{3,1}$:

It does not seem to be possible to choose appropriate generators $z_{\dot\alpha}$ and space-time coordinates $\tilde{x}^\mu$ such that the corresponding frame has no higher-spin contributions. The higher-spin theory seems to give a novel deviation from general relativity and one has to find an appropriate physical interpretation of the $\mathfrak{hs}$-valued objects arising.

Note that this does not mean that one has to disregard the emergent higher-spin theory, as at every point of the base, one can always get rid of the higher-spin components of the frame and the metric, reproducing the known laws of gravity, at least locally [6]. Similarly, there are results, showing that for any divergence-free frame $E_{\dot\alpha}{}^\mu \in \mathcal{C}^0$, one can construct a suitable generator $z^{\dot\alpha} \in \mathcal{C}^1$ that satisfies $[\{z^{\dot\alpha}, x^\mu\}]_0 = E_{\dot\alpha}{}^\mu$, where $[\cdot]_0$ denotes the projection onto $\mathcal{C}^0$ [50].

9.5. Gauge transformations of the generators

In all the calculations above, we have not used the gauge transformations of the emergent higher-spin theory, which are a result of the symmetries of the underlying matrix model.

So, for the final considerations of this section, lets look at how the gauge symmetries can be used to simplify a general Ansatz for the generators.

Recall from section 6.3 that under an infinitesimal gauge transformation with gauge potential $\lambda \in \mathcal{C}$, any scalar function $\phi \in \mathcal{C}$ transform as given in eq.(160). We want to investigate how this infinitesimal gauge transformation acts on general $SO(3)$-covariant[29] functions on the total space. For this, consider the following Ansatz for generators z^μ:

$$\begin{aligned} z^0 &= A(r^2, \chi, x^0) \\ z^i &= B(r^2, \chi, x^0)t^i + C(r^2, \chi, x^0)x^i. \end{aligned} \tag{279}$$

Since we want the transformed generators to also be $SO(3)$-covariant in this way, we require the gauge potential to transform as a scalar under $SO(3)$, e.g.

$$\lambda = \lambda(r^2, \chi, x^0). \tag{280}$$

With this gauge potential, the timelike generator z^0 transforms as

$$\delta_\lambda z^0 = \{\lambda, z^0\} = 2\tilde{R}R \sinh\eta\, \sqrt{r^2 - \chi}(\partial_{x^0}\lambda \partial_{r^2} A - \partial_{x^0} A \partial_{r^2}\lambda), \tag{281}$$

while the spacelike generators z^i have a more complicated transformation, which in the asymptotic regime is given by

$$\delta_\lambda z^i = \{\lambda, z^i\} \sim_\eta 2\tilde{R}R \sinh\eta \left(G t^i + H x^i \right) \tag{282}$$

where G and H are the following two scalar functions:

$$\begin{aligned} G &= \sqrt{r^2 - \chi}(\partial_{x^0}\lambda \partial_{r^2} B - \partial_{x^0} B \partial_{r^2}\lambda) + \frac{\tilde{R}R}{R\cosh\eta}\partial_\chi \lambda B \\ H &= \sqrt{r^2 - \chi}(\partial_{x^0}\lambda \partial_{r^2} C - \partial_{x^0} C \partial_{r^2}\lambda) - \frac{B}{\tilde{R}R} D\lambda. \end{aligned} \tag{283}$$

Now one can ask if it is possible to gauge away the contribution from H, in order to see if a fully general Ansatz as in eq.(257) requires a term proportional to x^i. Since we consider infinitesimal gauge transformations, this is equivalent to asking if for any given B and C, one can find a gauge potential λ as in eq.(280), such that

$$2\tilde{R}R \sinh\eta\, \sqrt{r^2 - \chi}(\partial_{x^0}\lambda \partial_{r^2} C - \partial_{x^0} C \partial_{r^2}\lambda) - \frac{B}{\tilde{R}R} D\lambda = F(r^2, \chi, x^0) \tag{284}$$

for an arbitrary smooth scalar function F, that stays bounded in the asymptotic regime. As eq.(284) is a quasi-linear PDE in the gauge potential λ, this is in general possible. Thus one can argue that, with infinitesimal gauge transformations, one can make the contribution proportional to x^i in the spacelike generators z^i arbitrarily small, so that when one passes to full gauge transformations, it is also possible to completely gauge away any term proportional to x^i. Thus one can consider the case $C = 0$ in eq.(279), without a loss of generality.

[29] Meaning they transform in the vector rep. of $SO(3)$, as explained in section 7.

10. Conclusion

This thesis expanded on the work done in [6, 47, 50], by further investigating the emergent higher-spin theory on the cosmological background solution $\mathcal{M}^{3,1}$ and local perturbations thereof. It was shown that a description of arbitrary static spherically symmetric space-times in 3+1 dimensions via a frame field $E^{\dot{\alpha}}$ consisting of Hamiltonian vector fields is compatible with the constraints coming from the matrix model in the semi-classical limit. However, the explicit reconstruction of generators $z^{\dot{\alpha}}$ of the frame vector fields $E^{\dot{\alpha}} = \{z^{\dot{\alpha}}, \cdot\}$ as arising in the semi-classical limit of the matrix model has proven difficult, with the main obstacle being the elimination of the higher-spin contributions to the frame. The work presented here suggests that, in general, some higher-spin contributions are unavoidable for static spherically symmetric perturbations of the cosmological background, even for late cosmic times $\eta \to \infty$.

Sections 2 and 3 introduced general aspects of non-commutative geometry and their relevance to physical theories of the fundamental interactions, with an emphasis on aspects important to quantum gravity. In particular, section 3 discussed the ideas of matrix geometries as a rich and surprisingly straightforward approach to non-commutative geometry. In addition, we introduced the framework of quasi-coherent states, as systematized in [19] and their relation to the well known construction of quantized coadjoint orbits of Lie groups. It was shown how they provide a mechanism to obtain a well-behaved semi-classical limit from matrix configurations, which is the mechanism employed for constructing the semi-classical cosmological background $\mathcal{M}^{3,1}$, introduced in section 4. We saw how the non-commutativity at the matrix level is encoded in a Poisson structure at the semi-classical level.

Furthermore, section 4 discussed how the framework of matrix geometries can be combined with the idea of matrix models as fundamental theories of (quantum) space-time and interactions, in order to extract a semi-classical model of space-time. The focus of this thesis was on a particular solution to the e.o.m. of the bosonic IKKT matrix model with additional mass-term, the fuzzy hyperboloid H_n^4, which exhibits a high degree of symmetry, making it extremely well-behaved when passing to its corresponding semi-classical limit $\mathcal{M}^{3,1}$. It was shown how this manifold exhibits a global $SO(3,1)$ symmetry, as well as admitting a $k = -1$ FLRW metric, qualifying it as a cosmological background solution.

Section 5 was dedicated to studying the Poisson and corresponding symplectic structure of the total space $\mathbb{C}P^{1,2}$ in more detail. An explicit calculation of the KKS form ω on $\mathbb{C}P^{1,2}$ on was given, by using local coordinates that block-diagonalize the Poisson tensor θ. In addition, we explicitly checked the consistency condition on the holonomy of ω. It was shown that in the semi-classical limit $n \to \infty$, the integral of ω over a sphere S^2 indeed is an integer multiple of 2π, as expected from the quasi-coherent state framework, where the symplectic form ω can be seen to arise as the curvature of a $U(1)$ connection [19, 21].

The following, section 6, described the emergent higher-spin theory and introduced the geometric description with which one can access the non-linear regime of the gravity theory. The main object of interest in this construction is the frame induced by the matrix model generators, which, due to the divergence constraint eq.(184), breaks the local Lorentz invariance present in the usual vielbein formalism of general relativity. This gives the frame more physical content and leads to the presence of additional degrees of freedom, like the dilaton. In order to study some explicit examples, section 7 introduced a static spherically symmetric Ansatz for a frame that describes local perturbations much smaller then the cosmic scale.

While section 8 discussed how different static spherically symmetric geometries can be described via the Ansatz presented in the previous section, the following, section 9, showed the aforementioned difficulties in actually constructing the semi-classical generators of the frame. One should note however, that this is not the end of the story. The cosmological background $\mathcal{M}^{3,1}$ investigated here is only a solution of the semi-classical e.o.m. of the classical IKKT matrix model, we have not yet considered the quantized theory. There it is expected that corrections to the action at one-loop induce an Einstein-Hilbert term [53], which will lead to a different class of solutions, for which it may be possible to find explicit generators of the frame. The construction and more detailed study of such solutions poses an interesting opportunity for further research.

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A. Gamma matrices

For this thesis, we adopted the conventions from [45]. With the following sign convention for the Minkowski metric $(\eta_{ab}) = \mathrm{diag}(-1,1,1,1,1)$, the Gamma matrices of $\mathfrak{so}(4,1)$ are defined to satisfy

$$\{\gamma_a, \gamma_b\} = -2\eta_{ab}, \quad a,b = 0,...,4 \tag{A.1}$$

where $\{A,B\} = AB + BA$ is the anti-commutator of matrices. This can be achieved by choosing the following convention:

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \ \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \ \gamma_4 = i\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{A.2}$$

Here $\sigma_i, \ i = 1,2,3$ are the usual Pauli matrices.

Spinorial reps of $SO(4,2)$ can be obtained by using the following matrices:

$$\Sigma_{\mu\nu} := \frac{1}{4i}[\gamma_\mu, \gamma_\nu], \ \Sigma_{\mu 4} := -\frac{i}{2}\gamma_\mu\gamma_4, \ \Sigma_{\mu 5} := -\frac{1}{2}\gamma_\mu, \ \Sigma_{45} := -\frac{1}{2}\gamma_4. \tag{A.3}$$

with the convention for the gamma matrices as given in eq.(A.2). One can check that they satisfy the $\mathfrak{so}(4,2)$ algebra

$$[\Sigma_{ab}, \Sigma_{cd}] = i(\eta_{ac}\Sigma_{bd} - \eta_{ad}\Sigma_{bc} + \eta_{bd}\Sigma_{ac} - \eta_{bc}\Sigma_{ad}). \tag{A.4}$$

B. Oscillator construction of minireps $\mathcal{H}_n$

This appendix will give the explicit construction of the minireps $\mathcal{H}_n$, as discussed in section 4, based on the presentation in [45].

$\mathcal{H}_n$ is a highest-weight unitary rep of $SU(2,2)$ and can be constructed by quantizing the definition of $\mathbb{C}P^{1,2}$ in terms of a quotient space of $H^{4,3}$. For this, let $\psi_\alpha, \ \alpha = 1,2,3,4$ be four operators satisfying the bosonic commutation relation

$$[\psi_\alpha, \bar{\psi}^\beta] = \delta^\beta_\alpha. \tag{B.1}$$

Then one can define the associated bilinears

$$\mathcal{M}^{ab} := \bar{\psi}\Sigma^{ab}\psi \tag{B.2}$$

which can be seen to realize the commutation relations of $\mathfrak{so}(4,2)$, as given in eq.(87), due to the following:

$$[\mathcal{M}^{ab}, \mathcal{M}^{cd}] = [\bar{\psi}\Sigma^{ab}\psi, \bar{\psi}\Sigma^{cd}\psi] = \bar{\psi}[\Sigma^{ab}, \Sigma^{cd}]\psi. \tag{B.3}$$

Furthermore, as

$$\Sigma^{ab\dagger} = \gamma^0\Sigma^{ab}(\gamma^0)^{-1} \tag{B.4}$$

one sees that the $\mathcal{M}^{ab}$ are self-adjoint operators, which means that they implement unitary reps of $SU(2,2)$. These can be constructed explicitly as follows:

The algebra eq.(B.1) can be realized by considering bosonic creation and annihilation operators a_i, b_j satisfying

$$[a_i, a_j^\dagger] = \delta_i^j,\ [b_i, b_j^\dagger] = \delta_i^j,\ i, j = 1, 2 \tag{B.5}$$

from which one can build the spinorial operators

$$\psi := \left(a_1^\dagger, a_2^\dagger, b_1, b_2\right)^T \tag{B.6}$$

and their Dirac conjugates

$$\bar{\psi} = \psi^\dagger\gamma^0 = \left(-a_1, -a_2, b_1^\dagger, b_2^\dagger.\right) \tag{B.7}$$

These then satisfy eq.(B.1) and one defines the generators of $\mathfrak{so}(4, 2)$ as in eq.(B.2). One furthermore has the bosonic number operators

$$N_a := a_1^\dagger a_1 + a_2^\dagger a_2,\ N_b := b_1^\dagger b_1 + b_2^\dagger b_2 \tag{B.8}$$

and the invariant operator

$$N := \bar{\psi}\psi = N_b - N_a - 2. \tag{B.9}$$

The maximal compact subgroup of $SO(4, 2)$ is $SU(2)_L \times SU(2)_R \times U(1)_E$ and the generators of the various groups in the product can be defined as

$$L_i^k := a_k^\dagger a_i - \frac{1}{2}\delta_i^k N_a,\ R_j^i := b_i^\dagger b_j - \frac{1}{2}\delta_j^i N_b, \tag{B.10}$$

for $SU(2)_L$ and $SU(2)_R$ respectively, while the $U(1)_E$ generator E is given by

$$E = \mathcal{M}^{05} = \frac{1}{2}\psi^\dagger\psi = \frac{1}{2}\left(N_a + N_b + 2\right). \tag{B.11}$$

The non-compact generators are then linear combinations of products of creation and annihilation operators, i.e. terms of the form $a_i^\dagger b_j^\dagger$ and $a_i b_j$.

The minireps can then be constructed as follows: The lowest weight space is given by the Fock vacuum $a_i\,|0\rangle = 0 = b_i\,|0\rangle$ from which one builds the minireps by acting with creation operators on the lowest weight vectors:

$$\begin{aligned} |\Omega_n^a\rangle &:= \left|E, \frac{n}{2}, 0\right\rangle := a_{i_1}^\dagger ... a_{i_n}^\dagger\,|0\rangle\,, \quad E = 1 + \frac{n}{2}, j_L = \frac{n}{2}, j_R = 0 \\ \left|\Omega_n^b\right\rangle &:= \left|E, 0, \frac{n}{2}\right\rangle := b_{i_1}^\dagger ... b_{i_n}^\dagger\,|0\rangle\,, \quad E = 1 + \frac{n}{2}, j_L = 0, j_R = \frac{n}{2}. \end{aligned} \tag{B.12}$$

Here j_L and j_R are the quantum numbers of the $SU(2)_L$ and $SU(2)_R$ rep respectively. The lowest weight vectors $|\Omega_n^a\rangle\,, \left|\Omega_n^b\right\rangle$ are annihilated by all operators of the form $a_i b_j$ and one has that

$$n^2 := \left(N^2 + 2\right)^2 = \left(N_a - N_b\right)^2, \quad n = 0, 1, 2, ... \,. \tag{B.13}$$

Acting with all operators of the form $a_i^\dagger b_j^\dagger$ on $|\Omega_n^a\rangle$ or $\left|\Omega_n^b\right\rangle$, one obtains positive energy discrete series unitary reps $\mathcal{H}_{\mu_{a,b}}$ of $SU(2, 2)$, with lowest weight given by $\mu_a = (E, \frac{n}{2}, 0)$ and $\mu_b = (E, 0, \frac{n}{2})$. Since the distinction is not important in this thesis, we will ignore it and simply write $\mathcal{H}_n$.

C. S^2-fibres of $\mathbb{C}P^{1,2}$ over H^4 and $\mathcal{M}^{3,1}$

In the following we want do explicitly exhibit the S^2-fibre of $(x^a) : \mathbb{C}P^{1,2} \to H^4$, following the presentation of this given in [45]. For this, recall that $SO(4,1)$ acts transitively on H^4, so that it suffices to consider a base point. Let $\psi \in H^{4,3}$ be arbitrary. Then

$$x^0 = \frac{\tilde{R}}{2}\psi^\dagger\psi > 0 \tag{C.1}$$

and thus, there exists a $SO(4,1)$ transformation such that

$$(x^a) = R\big(1,0,0,0,0\big) \tag{C.2}$$

which gives the aforementioned base point in H^4. The stabilizer under $SO(4,1)$ of this point is given by

$$H = \{h \in SO(4,1) \ : \ [h,\gamma_0] = 0\} \cong SU(2) \times SU(2) \subset SO(4,1). \tag{C.3}$$

In order to further analyze this, one can write

$$\psi = \big(a_1^*, a_2^*, b_1, b_2\big)^T \tag{C.4}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{C}$. Then since we are considering ψ that give eq.(C.2), one has that

$$\begin{aligned} \bar{\psi}\psi = 1 &= -|a_1|^2 - |a_2|^2 + |b_1|^2 + |b_2|^2 \\ \psi^\dagger\psi = 1 &= |a_1|^2 + |a_2|^2 + |b_1|^2 + |b_2|^2 \end{aligned} \tag{C.5}$$

which implies that $a_1 = 0 = a_2$ and $|b_1|^2 + |b_2|^2 = 1$. Thus by acting with $SU(2)$ on ψ, one can achieve that

$$\psi = \big(0,0,0,1\big)^T. \tag{C.6}$$

The fibre is then obtained by acting on ψ with the second copy of $SU(2)$, i.e.

$$(x^a)^{-1}\big(\{R(1,0,0,0,0)\}\big) \cong SU(2)\cdot\psi \cong S^2. \tag{C.7}$$

For the cosmological background $\mathcal{M}^{3,1}$, the fibres can be exhibited in the following way: Again, it suffices to consider a reference point, which due to $SO(3,1)$-invariance, can be chosen to be

$$\xi = (x^0,0,0,0,x^4) \xrightarrow{\Pi} (x^0,0,0,0), \ \ x^0 = R\cosh\eta, \ \ x^4 = R\sinh\eta. \tag{C.8}$$

Then t^μ reads

$$t^0 = 0, \ \ t_i t^i = \tilde{R}^{-2}\cosh^2\eta \tag{C.9}$$

which describes the S^2-fibre over $\mathcal{M}^{3,1}$.