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Contents

Abstract

The study of noetherian rings plays a central role in commutative algebra. One of the most important invariants is the Krull dimension, assigning to every ring a (possibly infinite) value. This concept was defined by the German mathematician Wolfgang Krull, whose study of the dimension theory for noetherian rings was one of his greatest achievements. The aim of this thesis is to give a detailed overview on this classical area of research. There are many beautiful theorems which are worth knowing about. **Chapter 1** gives an introduction to the notion of dimension, both in the general sense and specifically for noetherian rings. We also explain an axiomatic approach to the Krull dimension. In **Chapter 2** we are dealing with the fundamental theorem of dimension theory, stating three equivalent characterizations of the Krull dimension for local rings. This has a number of important consequences. Afterwards, we describe in **Chapter 3** different classes of noetherian rings in terms of properties involving the dimension, height and chain conditions of prime ideals. In **Chapter 4**, we study the behaviour of the dimension under ring morphisms. Finally, we take a closer look at the Hilbert function in **Chapter 5**.

Zusammenfassung

Das Studium von noetherschen Ringen spielt eine zentrale Rolle in der kommutativen Algebra. Eine der wichtigsten Invarianten eine Rings ist die Krulldimension, welche zu jedem Ring einen Wert zuordnet. Dieses Konzept wurde definiert von dem deutschen Mathematiker Wolfgang Krull, dessen Forschung über die Dimensionstheorie von noetherschen Ríngen als eines seiner größten Leistungen gepriesen wird. Das Ziel dieser Thesis ist es, einen detaillierten Überblick über dieses klassische Forschungsgebiet zu geben. Es gibt viele wissenswerte Theoreme aus diesem Fachgebiet. In **Kapitel 1** geben wir einen generellen Überblick über den Dimensionsbegriff in der Mathematik, und spezialiseren uns dann auf noethersche Ringe. Mittels eines axiomatischen Zuganges werden wir die Krulldimension definieren. In **Kapitel 2** beschäftigen wir uns mit dem Fundamentalsatz der Dimensionstheorie, welches eine Aussage über drei äquivalente Charakterisierungen des Dimensionsbegriffes für lokale Ringe beschreibt. Dies wird einige wichtige Konsequenzen mit sich ziehen. Danach beschreiben wir in **Kapitel 3** verschiedene Klassen von noetherschen Ringen mithilfe von Eigenschaften, welche die Dimension, Höhe und Kettenbedingungen von Idealen verwenden. Im **Kapitel 4** studieren wir das Verhalten der Krulldimension unter Ringmorphismen. Zum Schluss betrachten wir die Hilbertfunktion im **Kapitel 5** im Detail.

Chapter 1

Introduction

Let me start this thesis by explaining how I chose the topic. In a course about algebraic geometry I first learned about Krull's principal ideal theorem, which is a statement about the relation between a geometric and an algebraic notion for ideals of a noetherian ring. Subsequently, I read about the biography of Wolfgang Krull, in which I learned that his principal ideal theorem is seen as an essential theorem for the dimension theory of noetherian rings as well as that he made fundamental contributions to this area of research. While getting acquainted with rings and modules in a commutative algebra course by Ass.-Prof. Anton Mellit, my advisor Herwig Hauser encouraged me to take a closer look at this area, ultimately helping me to make the decision of writing about dimension theory of noetherian rings as my master's thesis. Indeed, there are a number of beautiful theorems which would be suitable for presenting. In the end, I settled on five related topics which should give the reader a vast overview.

1.1 Notation

We will try to use a consistent notation over the whole thesis. The following gives a brief explanation: All rings are considered to be commutative and unital, and ring morphisms are also assumed to be unital. We will use the letters R, S, or T for rings. Most often they are assumed to be noetherian, however we will always indicate this property. The symbol $\cong$ indicates isomorphisms, and ideals are denoted by I, J, p, q, P or Q. The arrows $\hookrightarrow$ and $\twoheadrightarrow$ stand for injective respectively surjective maps. When speaking about a field, we will use the letters k or K. $\mathbb{N}$ stands for the set of natural numbers (including 0). Finally, we will assume the axiom of choice.

1.2 The notion of dimension in mathematics

One of the most fundamental notions in mathematics is that of dimension. Depending on the area, this concept has different definitions. For vector spaces, the dimension is equal to the cardinality of a basis. Notice that in this case the dimension uniquely determines the vector space. For metric spaces, the Hausdorff dimension is defined as the minmal real number d such that the corresponding d-dimensional Hausdorff measure of the metric space is equal to zero. For topologcal spaces, the Lebesgue covering dimension is the minimal value n such that every open cover of the space has a refinement of order $n+1$ or smaller. A natural question to ask is how one should algebraically define the dimension of a ring. As is common in mathematics in general, it took quite some time until the modern definition of dimension for commutative rings was established. Without revealing this definition of the Krull dimension yet, we want to look at the past, to see for which class of rings the dimension was originally used, and then describe the shift to the present time.

1.3 Transcendence degree and similarites/differences to Krull dimension

Let us look at the topological dimension of a space at first. Geometric ideas were the beginning of the concept of dimension. In former times, this notion was used intuitively. For example, the ancient greeks did take for granted that the dimension of euclidean n-space was n. In general, an object was considered to be n-dimensional if the minimal number of parameters describing it points was n. Ideas of Canter, Peano and Hilbert showed that this intuitive use lead to problems, and was a stepping stone into the direction of giving the dimension a precise definition.

For commutative algebra, the definition of the dimension of a ring is based on geometric ideas of algebraic geometry. From 1800 on, algebraic geometry was the study of real and complex algebraic curves. After the introduction of coordinates, these curves where defined by one equation, and seen as 1-dimensional. Together with the introduction of the complex numbers and the complex projective plane in the late nineteenth century, different objects were considered. After some time the idea emerged that the dimension of a curve should have a meaning independent of the field used for the coordinates of the points as well as the chosen embedding into some space. Finally, the point of focus was based on the study of the zero-loci of equations (although just for complex 3-space in the beginning). The notion of transcendence degree took a central role. Together with the consensus that affine d-space should have dimension d, the dimension of an irreducible variety in n-space over a field k was defined as the transcendence degree of the field of rational functions on the variety over k. But this is just the transcendence degree over k of the ring $k[x_1, ..., x_n]/I]$, where I is the ideal of all vanishing functions on the variety. This was the reason why one defined the dimension of a ring as the transcendende degree at that time. This works for rings over a ground field k, but is insufficient in general. After different definitions were made for rings where the transcendence degree did not make sense, Wolfgang Krull proposed in 1937 the following definition:

The Krull dimension of a ring R, written as $\dim(R)$, is the supremum of the length of all

strictly ascending chains of prime ideals in R. It should be noted that not every maximal ascending chain of prime ideals must have the same length. This definition provides the following advantages: First of all it can be defined for all rings, secondly it mimics the dimension of a vector space, which can be seen as the length of the longest chain of proper subspaces. And thirdly: it can be seen as an extension of the transcendence degree, because in many cases their values coincide.
We recall the definition of the transendence degree: Let R be an algebra over a field K. A subset $\{r_1, ..., r_n\}$ of R of size n is called algebraically independent over K if for all non-zero polynomials $f \in K[x_1, ..., x_n]$, $f(r_1, ..., r_n) \neq 0$. The transcendence degree of R is defined as the supremum of the cardinalities of all finite sets contained in R whose elements are algebraically independent over K. If R is the zero ring, then one defines it as -1. The following relation between the transcendence degree and the Krull dimension is known:

Lemma 1.3.1. *(Upper bound of the dimension for K-algebras) Let R be an algebra over a field K. Then* $\dim(R) \leq \operatorname{trdeg}(R)$.

Proof. We will show the following stronger result: Given any generating set S of R, the Krull dimension of R is smaller than or equal to the supremum of the cardinalities of all finite subsets T whose elements are algebraically independent and contained in S. If $R = \{0\}$ or if the right hand side of the inequality is ∞, then the claim is trivially true. We therefore assume that the supremum of the cardinalities of all algebraically independent sets is finite equal to n. Clearly it is enough to show that the dimension of the quotient ring R/P is bounded from above by n for all prime ideals P of R. By a similar argument n cannot increase if we replace the set S by $\{r + P \mid r \in S\}$. Therefore we may assume that R is an integral domain.
We will prove our claim by induction on n: If $n = 0$, then all elements of S are algebraic over K. It follows that $\operatorname{Quot}(R)/K$ is an algebraic field extension, and therefore R is algebraic over K too. But every integral domain that is algebraic over a field must be a field as well. We conclude that $\dim(R)= 0$.
Now we let $n > 0$ and assume that the claim is true for all smaller n. Let $P_0 \subsetneq P_1 \subsetneq ... \subsetneq P_m$ be a maximal chain of prime ideals in R. Factoring by P_1 yields a chain of size $m - 1$ in R/P_1. If we could show that all algebraically independent subsets $T \subseteq \{r + P_1 \mid r \in S\}$ have size smaller or equal to n, then by induction we could conclude that the dimension of R/P_1 is smaller than n and therefore also the dimension of R is bounded from above by n.
Suppose this is wrong: Let $\{r_1, ..., r_n\}$ be a subset of S such that the corresponding set $\{r_1 + P_1, ..., r_n + P_1\}$ in R/P_1 is an algebraically independent set of size n. Then $\{r_1, ..., r_n\}$ is also algebraically independent over K. Now let L be the quotient field of $K[r_1, ..., r_n]$. Per construction we know that all elements of S are algebraic over L, and therefore the same must hold for R and $\operatorname{Quot}(R)$.
Since P_1 is not the zero ideal, there exists a non-zero element r in P_1 and a polynomial $G(x)$ in $L[x]$ of the form $G(x) = \sum_{i=0}^{k} g_i x^i$ with $G(r) = 0$. We know that the coefficient g_0 is non-zero because R is an integral domain, and we can assume without loss of generality that all the coefficients lie in $K[r_1, ..., r_n]$. Then we get the equation $g_0 = -\sum_{i=1}^{k} g_i r^i \in P_1$. We now view g_0 as a polynomial in n indeterminates over K, to obtain that $g_0(r_1 + P_1, ..., r_n + P_1) = 0 + P_1$.

However, this is a contradiction to the algebraic independence of the set $\{r_1 + P_1, ..., r_n + P_1\}$, so we are done. $\square$

For affine algebras this result can even be strengthened:

Proposition 1.3.2. *Let R be a finitely generated algebra over a field K. Then* $\dim(R) = \operatorname{trdeg}(R)$.

Proof. Again we will show the following stronger result: Given any generating subset S of R, the Krull dimension of R is equal to the supremum of the cardinalities of all finite subsets T whose elements are algebraically independent and contained in S. By the proposition above it is enough to show that the transcendence degree of R is smaller than or equal to the Krull dimension of R. We will show per induction on n that if $\operatorname{trdeg}(R) \geq n$, than $\dim(R) \geq n$. Without loss of generality assume that n is positive.
Let $\{r_1, r_2, ..., r_n\}$ in R be algebraically independent. By the Hilbert basis theorem R is noetherian, and has therefore only finitely many minimal prime ideals $m_1, ..., m_t$. Claim: there exists an i such that the elements $r_1 + m_i, r_2 + m_i, ..., r_n + m_i$ in R/m_i are algebraically independent. Assume this is wrong. Then there exist non-zero polynomials f_1,...,f_n in $K[x_1, x_2, ..., x_n]$ such that $f_i(r_1, r_2, ..., r_n)$ is in m_i for all i. But then the product of these polynomials is contained in the nilradical of R, which is the intersection of all minimal prime ideals. However then we would have $\{\prod_{i=1}^{t} f_i(r_1, r_2, ..., r_n)\}^k$=0 for large enough k and the elements $r_1, r_2, ..., r_n$ would not be algebraically independent. Now it suffices to show that the dimension of R/m_i s bigger or equal to n. Replacing R by R/m_i we can assume that our ring is an integral domain.
Now we consider the quotient field $L = \operatorname{Quot}(K[r_1])$, which is a subfield of $\operatorname{Quot}(R)$, and the subalgebra $R' := LR \in \operatorname{Quot}(R)$. We know that R' is a finitely generated algebra over L, and that the elements $r_2, ..., r_n$ are algebraically independent over L.
By induction the dimension of $\dim(R')$ is bigger than or equal to $n-1$, so there exists a chain of prime ideals $p'_0 \subsetneq ... \subsetneq p'_{n-1}$ of length $n-1$ in R'. Let $p_i = p'_i \cap R$ be the contraction of p'_i in R. Since the product of L and p_i equals p'_i, we know that the inclusion of p_{i-1} in p_i is strict. Furthermore the intersection of L with p_{n-1} has to be 0, otherwise p'_{n-1} would contain an invertible element from L, which would mean $p'_{n-i} = R'$. It follows that $r_1 + p_{n-1}$ in R/p_{n-1} is not algebraic over K. But then R/p_{n-1} is not a field, so p_{n-1} is not a maximal ideal. We denote by p_n a maximal ideal of R containing p_{n-1}. We have constructed a strictly increasing chain of prime ideals $p_0 \subsetneq ... \subsetneq p_n$ of length n, therefore the dimension of R is bigger than or equal than n. $\square$

1.4 Properties of the Krull dimension for noetherian rings

Before giving a reason why the proposed notion of Krull should is the correct definition, we want to collect basic definitions and properties regarding the dimension and ideals of a ring R. Given any prime ideal p of R, the height of p is defined as the supremum of all lengths of ascending chains of prime ideals in R of the form $p_0 \subseteq p_1 \subseteq \cdots \subseteq p_n = p$. For the dual notion of coheight one takes the supremum of all lengths of chains of prime ideals of the form $p = p_0 \subseteq p_1 \subseteq \cdots \subseteq p_n$. This definition can be extende to arbitrary ideals by substituting for any ideal I a minimal prime ideal p laying above I. It is easy to see that these definition correspond to the Krull dimension of R_p respectively R/p. Most often we will work with noetherian rings. These rings satisfy the condition that all ideals are finitely generated or equivalently that the asscending chain condition on ideals is fulfilled. This notion says that every ascending chain must eventually become stationary. At first glance one could believe that noetherian rings have only finite Krull dimension. However, this is not the case. The first counterexample was given by Nagata in 1962. Nevertheless, pretty much every noetherian ring which one encounters in algebraic geometry and commutative algebra has fnite dimension.

1.5 Axiomatic characterization of the Krull dimension for noetherian rings

Now comes the justification for the definition of the Krull dimension for rings. We give two axiomatic characterizations: one for local rings, which we will prove rigorously, and one for general rings, where we only sketch the proof.
Let us consider a function f from the set of noetherian local rings to the positive integers. We claim that any function which satisfies the following three properties is in fact equal to the Krull dimension:
i) $f(R)$ is equal to the supremum over all values $f(R/p)$ for minimal prime ideals p of R.
ii) For all fields R we have $f(R) = 0$.
iii) For all domains R and $x \in R \setminus 0$ we have $f(R) - f(R/\langle x \rangle) = 1$.

Proof. By induction on the value for R: Because of the first condition we can assume without loss of generality that R is a domain. If R is a field, we have nothing to prove. Assume therefore that $f(R) > 0$. Now we can assume the third condition to reduce to a case for which we already know the uniqueness off. We still have to show that the Krull dimension fulfills these properties. By the correspondence between chains of ideals in a ring and chains of ideals in a quotient ring the first property is easily seen to be true. Equally trivial is the claim that the dimension of a field is zero. The last property will be proved in a more general setting in chapter 2. $\square$

In greater detail, one can identify sufficient and necessary axioms based on geometric considerations for general noetherian rings that make the Krull dimension unique. The first thing one could require is that the dimension should be a local property. In particular, this means that the dimension of a ring should be the supremum of the dimensions of all its localizations. Additionally, the dimension should not change when passing to the completion, since this corresponds geometrically to looking at a smaller neighborhood. By a more involved geometric argument, it seems reasonable to require that the dimension of a ring should stay the same when modding out nilpotent elements. Comparable to manifolds, the dimension of a noetherian ring should not be affected by morphisms with finite fibers. Finally, for specific rings one wants the dimension to be a specific number, for example the dimension of a field should be zero and the ring of algebraic integers of a number field should have dimension one. This leads to the following list of axioms:

Axiom 1: (Dimension is a local property) $\dim(R) = \sup_{p \ prime} R_p$, $\dim(R) = \dim(\widehat{R})$.
Axiom 2: (Nilpotents do not affect the dimension) If I is a nilpotent ideal of R, than $\dim(R)$=$\dim(R/I)$.
Axiom 3: (Dimension is preserved by morphisms with finite fibres) If $R \subset S$ are rings such that S is a finitely generated R-module, than $\dim(R) = \dim(S)$.
Axiom 4: (Calibration) If R is a complete discrete valuation ring, then $\dim(R[x_1, \ldots, x_{n-1}]) = n$.

We want to give a rough argument why these axioms uniquely define the Krull dimension for noetherian rings:
In chapter 2 we will show that the completion of a noetherian ring is still noetherian, and that the Krull dimension does not change when passing to it. Using axiom 1, it is enough to consider complete local noetherian rings. By axiom 2 we can assume that there are no nilpotent elements. It can be shown that a complete noetherian ring containing a field is a finitely generated module over some subring isomorphic to a power series ring in finitely many variables over a field. Similarly, it can be shown that a complete local ring without nilpotents that does not contain a field is a finitely generated module over a subring isomorphic to a power series ring over a complete discrete valuation ring. In both cases axioms 3 and 4 completely determine its dimension. (This uses the Cohen Structure theorem ([C]), which we want to take for granted.)

Chapter 2

The dimension of local rings

This chapter is devoted to arguably the most beautiful theorem in dimension theory of noetherian rings. We remark that many problems about rings can be reduced to the study of the local case. This is the reason why the fundamental theorem of dimension theory, which shows the equivalence of the Krull dimension with two other notions, is so important. Moreover, there are a number of applications of great significance. In the following everything will be stated for modules, and then specialized for noetherian rings. For one thing because it is more general, for another thing that the proofs are equally difficult. Before giving the precise statement of the fundamental theorem, we collect a few lemmas.
We start by reviewing relevant notions for modules which we gradually need:
Let M be a finitely generated module over a noetherian local ring R, and I an ideal of R. The length of M is defined as the number n of elements in the longest strictly ascending chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ of submodules of M. A possibly infinite descending chain of submodules of the form $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$ is called a filtration of M. This filtration is called a filtration with respect to I or I-filtration if additionally $IM_n \subset M_{n+1}$, and stable with respect to I or stable I-filtration if $IM_n = M_{n+1}$ for large enough n.
A graded ring is a ring R together with a family of $(R_n)_{n\geq 0}$ submodules of the additive group of R such that $R = \bigoplus\limits_{n\geq 0} R_n$ and $R_m R_n \subset R_{m+n}$ for all $m, n \geq 0$. Similarly, a graded module M over a graded ring R is graded if $M = \bigoplus\limits_{n\geq 0} M_n$ and $R_m M_n \subset M_{m+n}$ for all $m, n \geq 0$. An element $x \in M$ is homogeneous of degree n if $x \in M_n$.
For noetherian graded rings we have:

Lemma 2.0.1. *let $R = \bigoplus\limits_{n\geq 0} R_n$ be a graded ring. Then the following are equivalent:*
i) R is noetherian.
ii) R_0 is noetherian and R is a finitely generated R_0-module.

Proof. i) $\Rightarrow$ ii): Assume that $n > 0$. Let $R_+ = \bigoplus\limits_{n\geq 0} R_n$, then R_0 is equal to $R \setminus R_+$, and therefore also noetherian. Since R_+ is an ideal of R, it is generated by finitely many elements $x_1, \ldots, x_s$, which we can assume to be homogeneous of degree $k_1, \ldots, k_s$. We want to show that these elements actually generate R as an R_0-module. We will show by induction on n

that R_n is contained in $\langle x_1, \ldots, x_s \rangle$ for all n. For $n = 0$ this is trivially true. Assume that n is bigger than 0 and that the element y is contained in R_n. We write $y = \sum r_i x_i$ with $r_i \in R_{n-k_i}$ homogeneous. By the hypothesis every element r_i is a polynomial in the elements $x_1, ..., x_s$ with coefficients in R_0. Therefore the same has to be the case for y, and therefore $R_n \subset \langle x_1, ..., x_s \rangle$. ii) $\Rightarrow$ i): This is just an application of Hilbert's basis theorem. $\square$

Theorem 2.0.2. *Let (R, m) be a noetherian local ring with maximal ideal m, q an m-primary ideal, M a finitely generated R-module, and $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$ a stable q-filtration of M. Then the following holds:*
a) M/M_n is of finite length for all $n \geq 0$.
b) For all sufficiently large n the length of M/M_n is equal to a polynomial $g(n)$ of degree $\leq \mu(q)$, where $\mu(q)$ is the minimal number of generators of q.
c) The degree and leading coefficient of g depend only on M and q.

Proof. a) Let $G_q(R) = \bigoplus\limits_{n=0}^{\infty} q^n/q^{n+1}$ be the associated graded ring and $G(M) = \bigoplus\limits_{n=0}^{\infty} M_n/M_{n+1}$ the $G_q(R)$-module corresponding to the stable q-filtration chosen above. Since R/q is a noetherian local ring, $G_q(R)$ is noetherian as well. Because the filtration is stable with respect to q, we know that $G(M)$ is a finitely generated $G_q(R)$-module. Since each homogeneous part M_n/M_{n+1} is a noetherian R-module annihilated by q, it is also a noetherian R/q-module. Therefore it is of finite length. This implies that $l_n := l(M/M_n) = \sum\limits_{i=1}^{n} l(M_{i-1}/M_i)$, is also of finite length.
b) Let q be generated by the elements $x_1, \ldots, x_s$. Then the images of these generators in the quotient ring q/q^2 generate $G_q(R)$ as an R/q-algebra, additionally all of them have degree 1. By a theorem of Hilbert and Serre (see chapter 5) there is a polynomial f of degree smaller than or equal to s such that $l(M_n/M_{n+1}) = f(n)$ for all large n. Since $l_{n+1} - l_n$ is equal to $f(n)$, it follows that there exists a polynomial g of degree smaller than or equal to s which is identical to l_n for large n.
c) Let $M = M'_0 \supseteq M'_1 \supseteq \ldots$ be another stable q-filtration of M, and denote by $g\prime$ the corresponding polynomial such that $g\prime(n) = l(M/M'_n)$ for large n. We claim that these two stable q-filtrations have a bounded difference, meanong that there exists an integer n_0 such that $M_{n+n_0} \subseteq M'_n$ and $M'_{n+n_0} \subseteq M_n$ for all $n \geq 0$. To show that we assume that the second filtration is maximal. Than we have $M'_n = q^n M$. But we know that $q^n M \subset M_n$ and $qM_n = M_{n+1}$ for all n larger than some value n_0, therefore $M_{n+n_0} = q^n M_{n_0} \subset q^n M$. Therefore there exists an integer n_0 such that $M_{n+n_0} \subseteq M'_n$ and $M'_{n+n_0} \subseteq M_n$. We conclude that $g(n + n_0) \geq g'(n)$ and $g'(n + n_0) \geq g(n)$ for all large n. For polynomials this implies that $\lim_{n\to\infty} \frac{g(n)}{g'(n)}$ is equal to 1. Therefore the degree and leading coefficient coincide. $\square$

It is usual to denote the polynomial $g(n)$ corresponding to the stable q-filtration $M \supseteq qM \supseteq q^2M \supseteq \ldots$ by $\chi_q^M(n)$. In the special case $M = R$, we omit the module and write $\chi_q(n)$. This polynomial is then called the characteristic polynomial of the m-primary ideal q.
We have just shown that for large n there exists a polynomial $\chi_q(n)$ of degree smaller than or equal to the minimal number of generators of q such that the length of R/q^n is equal to $\chi_q(n)$.

We can say even more:
The choice of the m-primary ideal q does not matter for the degree of $\chi_q(n)$:
For every m-primary ideal q there exists a natural number t such that $m^t \subset q \subset m$. Therefore we get the inclusions $m^{nt} \subset q^n \subset m^n$ and estimates $\chi_m(n) \leq \chi_q(n) \leq \chi_m(nt)$. This can only be fulfilled if the polynomials have the same degree.

2.1 The fundamental theorem

We are now in the position to state and prove the fundamental theorem of dimension theory:

Theorem 2.1.1. *Let (R, m) be a noetherian local ring with maximal ideal m, q an m-primary ideal and $\mu(q)$ the minimal number of generators of q. The following equalities hold:*

$$\dim(R) = \mu(q) = \deg(\chi_q(n))$$

In order to prove this theorem, we need one more proposition:

Proposition 2.1.2. *Let (R, m) be a noetherian local ring with maximal ideal m, q an m-primary ideal, M a finitely generated R-module and $x \in R$ a not a zero-divisor. Then $\deg(\chi_q^{M/xM}(n)) \leq \deg(\chi_q^M(n)) - 1$.*

Proof. Since x is a not a zero-divisor of m, we have an isomorphism between M and xM as R-modules. We denote xM by N and $xM \cap q^n M$ by N_n. Then we get the following short exact sequence:

$$0 \to N/N_n \to M/q^n M \to (M/xM)/q^n(M/xM) \to 0$$

This gives us the equality $l(N/N_n) - \chi_q^M(n) + \chi_q^{M/xM}(n) = 0$. The strong version of the Artin-Rees lemma ([R]) tells us that (N_n) is a stable q-filtration of $N = xM$. The isomorphism between M and N implies that the corresponding polynomial of the length function $l(N/N_n)$ and the characteristic polynomial $\chi_q^M(n)$ have the same leading term. $\square$

We can finally give the proof of the fundamental theorem:

Proof. We have already seen that the degree of the characteristic polynomial is bounded from above by the minimal number of generators of q. We want to show the inequality $\deg \leq \deg(\chi_q(n))$: We do this by induction on the degree of the characteristic polynomial $\chi_q(n)$: If the degree of $\chi_q(n)$ is zero, then this means that $l(R/q^n)$ is constant for large n and we get $m^n = m^{n+1}$. We apply the Nakayama lemma ([N]) to get $m^n = 0$. In chapter 3 we will show that this means that R has dimension 0.
Now assume that the degree of $\chi_q(n)$ is positive. Let $p_0 \subset p_1 \subset \cdots \subset p_r$ be an arbitrary chain of prime ideals in R. Let x be an element of the set $p_1 \setminus p_0$, and denote by x' the corresponding element $x + p_0$ in p_1/p_0. Since x' is non-zero in the integral domain $R' = R/p_0$, we have by the proposition above that the degree of $\chi_q^{R'/x'R'}(n)$ is bounded from above by the degree of $\chi_q^{R'}(n) - 1$. Let m' be the maximal ideal of R'. Since $R'/(m')^n$ is a homomorphic image of R/m^n, we get the inequality $l(R'/(m')^n) \leq l(R/m^n)$ and hence $\deg(\chi_q^{R'}(n)) \leq \deg(\chi_q^R(n))$.

We put everything together to get $\deg(\chi_q^{R'/x'R'}(n)) \leq \deg(\chi_q^R(n)) - 1$.
By the hypothesis, the length of any chain of prime ideals in $R'/x'R'$ is smaller than or equal to $\deg(\chi_q(n)) - 1$. But the images of the chain of prime ideals $p_1 \subset \cdots \subset p_r$ in R' give rise to a chain of length $r-1$ in $R'/x'R'$. Hence $r - 1 \leq \deg(\chi_q(n)) - 1$, and therefore $\deg(R) \leq \deg(\chi_q(n)) - 1$. $\square$

Proposition 2.1.3. *Let (R, m) be a noetherian local ring with maximal ideal m of dimension d. There exists an m-primary ideal q generated by elements $x_1, ..., x_d$. In particular $\mu(q) \leq$ dim(R).*

Proof. We construct the elements $x_1, ..., x_d$ such that the following property is fulfiled:
For $i \in \{1, ..., d\}$ we want that every prime ideal containing $x_0, x_1, ..., x_i$ to have height $\geq i$.
Assume $i \geq 1$ and that x_1 up to x_{i-1} have been constructed. Let T be the finite set of prime ideals p which lie minimally above $\langle x_1, x_2, ..., x_{i-1}\rangle$ and have exactly the height $i-1$. Since the height of the maximal ideal m is equal to the dimension d of R and bigger then or equal to $i-1$, we know that $m \notin T$. The prime avoidance lemma ([N]) gives us that m is not contained in the union of all elements of T.
Now we choose an element x_i in the set-theoretic difference $m \setminus \bigcup_{p \in T} p$ and let q be a prime ideal containing $x_1, ..., x_i$. Such a prime ideal q must strictly contain a prime ideal p lying minimally over $\langle x_1, x_2, ..., x_{i-1}\rangle$.
If p is contained in T, then the height of q is bigger than the height of $p = i - 1$.
If p is not contained in T, then the height of p is at least i and we can conclude again that $\mathrm{ht}(q) \geq i$. But the constructed ideal $\langle x_1, ..., x_d\rangle$ is clearly m-primary, since every prime ideal containing it must have height at least d. But this can only be fulfilled by the maximal ideal m. This completes the proof, because we have just shown that the minimal number of generators of q is smaller than or equal than the Krull dimension of R. Therefore all three notions coincide. $\square$

We want to stress that the noetherian property is absolutely crucial. Things behave much worse if the ascending chain condition is not satisfied. This suggests that the noetherian property is reasonable and extremely useful, but also quite restrictive when studying rings.

2.2 System of parameters of a local ring

Before giving applications of the fundamental theorem of dimension theory, we want to consider the notion of system of parameters. Given a noetherian local ring (R, m) of dimension d, a sequence of elements $x_1, \ldots, x_d$ generating an m-primary ideal is called a system of parameters. The fundamental theorem asserts that a system of parameters always exists. Many questions about ideals in commutative algebra can be restricted to parameter ideals. One interesting result that will be proven in chapter 3 is the following: A sequence $x_1, \ldots, x_d$ is a parameter system if and only if the quotient ring $R/x_1, ..., x_d$ satisfies the ascending chain condition on its ideals. There exist also much more subtle statements about these systems: In the 1970's Mel Hochster asked what properties system of parameters must fulfill. Motivated by connections to homological algebra, the so-called monomial conjecture of Hochster did arise. It can be stated as follows:
Let (R, m) be a noetherian local ring of Krull dimension d and $x_1, ..., x_d$ a system of parameters. Then the monomial conjecture of Hochster asserts that for all positive integers t the following holds:

$$(x_1 \cdots x_d)^t \notin \langle x_1, \ldots, x_d \rangle^{t+1}$$

This innocent looking question took more than 50 years to be answered. For a long time only partial results by Heitmann for rings of dimension 3 were known. In 2016, Yves André was successful in applying algebraic-geometric tools by Peter Scholze to give a proof of an equivalent statement.
The converse is also natural to think about: Since systems of parameters are rare, one can pursue the idea to find relations that can only be satisfied by them. The most fundamental theorem about this is due to Dutta and Roberts: Suppose that (R, m) is a noetherian Cohen-Macaulay local ring (see Chapter 3) of dimension n. Given elements $y_1, \ldots, y_n$ which are contained in the ideal $\langle x_1, \ldots, x_n \rangle$ generated by a parameter system one can write $y_j = \sum\limits_{i=1}^{n} s_i x_i$ for all $j = 1, \ldots, n$. If we denote the corresponding matrix of coefficients by S, the multiplication by the determinant of S gives a well defined map between $R/\langle x_1, \ldots, x_n \rangle$ and $R/\langle y_1, \ldots, y_n \rangle$. The beautiful theorem of Dutta and Robert states that this map is injective if and only if the elements $y_1, \ldots, y_n$ are also a system of parameters. One should keep in mind that both of these theorems were not just proved using tools from commutative algebra, but rather a combination involving algebraic geometry and homological algebra.
We will see later in chapter 3 how system of parameters can be used to describe an important class of noetherian rings.
Using the characterization of the dimension of local rings, we can describe a certain independence property that systems of parameters always fulfill. We need a few results of chapter 5: For now we want to take for granted that the generating function of the length function $P(R, t)$ of an associated graded ring has a special form. It is enough for us to know that the dimension of the ring R is equal to the order of $P(R, t)$ at $t = 1$, denoted by $d(G_q(R))$.

Theorem 2.2.1. *Let (R, m) be a noetherian local ring of dimension d, $q = \langle x_1, \ldots, x_d \rangle$ an*

m-primary ideal generated by a system of parameters, and f a homogeneous polynomial of degree n with d variables and coefficients in R. Assume that $f(x_1, \ldots, x_d) \in q^{n+1}$. Then all coefficients of f lie in m.

Proof. We consider the ring map $g : (R/q)[t_1, \ldots, t_d] \to G_q(R)$ that maps each variable t_i to the element x_i/q. This function is actually an epimorphism of graded rings. The definition of f tells us that $f(x_1, \ldots, x_d)$ modulo q is contained in the kernel of g. Assume that one of the coefficients of f is a unit. Then the reduction of f modulo q, denoted by $\bar{f}$, cannot be a zero-divisor in this case. We get the following inequality:

$$d(G_q(R)) \leq d((R/q)[t_1, \ldots, t_d]/\langle \bar{f} \rangle).$$

But the right hand term is equal to $d((R/q)[t_1, \ldots, t_d]) - 1$, because $\bar{f}$ is not a zero-divisor. We see in chapter 5 that the integer $d((R/q)[t_1, \ldots, t_d])$ of a polynomial ring in d variables over an artinian ring is d. But since $d(G_q(R))$ is equal to the dimension of R, we get a contradiction, therefore f only contains coefficients of m. □

This statement looks more appealing under the following assumption:

Corollary 2.2.2. *Let (R, m) be a noetherian local ring, and assume that there exists a field $k \subset R$ which is isomorphic to the residue field R/m. Let $x_1, \ldots, x_d$ be a system of parameters. Then $x_1, \ldots, x_d$ are algebraically independent over k.*

Proof. Assume that the parameter system is algebraically dependent, and let f be a polynomial with coefficients in k such that $f(x_1, \ldots, x_d) = 0$. Let us consider only the terms of f of smallest degree s. Denote the sum of these by f_s. By the theorem before, we know that all the coefficients of f_s are in m. But this implies that the coefficient of f_s and therefore f itself is equal to 0. □

One could say much more about system of parameters, they frequently come up when studying properties of rings which can be derived from properties of parameter ideals. This is often the case when a definition only depends on parameter ideals. We will get to know such an example when talking about Cohen-Macaulay rings later.

2.3 Consequences of the fundamental theorem

We now want to list several statements that rely heavily on the fundamental theorem. The most immediate consequence is that the Krull dimension of a noetherian local domain is finite. This is clear because the maximal ideal can be generated by finitely many elements. It is also easy to see that the prime ideals of a noetherian ring satisfy the descending chain condition. This is because of the correspondence between chains of prime ideals in a ring and chains of prime ideals in a localized ring and the fact that this statement is clear for local rings.
A milestone in the dimension theory of noetherian rings was the following theorem:

Theorem 2.3.1. *(Krull's principal ideal theorem, general case) Let R be a noetherian ring, and $I = \langle x_1, ..., x_r\rangle$ an ideal generated by r elements. If p is a prime ideal that lies minimal above I, then the height of p is smaller or equal to r.*

Proof. Since the height of p is the same as the dimension of the localized ring R_p, and the ideal IR_p is a primary ideal belonging to the unique maximal ideal pR_p, we can use the fact that this ideal can be generated by maximal r elements. Therefore the height of p must be bounded from above by r. □

The first proof of this theorem was given by Krull in 1928 using symbolic powers of ideals and the theory of artinian rings. It gives a relation between of the number of generators of an ideal and its height, in particular the height of every ideal must be finite. As before, the noetherian property is essential. To see a counterexample for non-noetherian rings. consider the algebra $K[x, xy, xy^2, \dots]$ over a field with variables x and y. An easy calculation shows that the minimal prime ideal containing x is the maximal ideal $\langle x, xy, xy^2, \dots\rangle$, which has height 2.
There is also a converse of the principal ideal theorem:

Corollary 2.3.2. *Let R be a noetherian ring, and p a prime ideal of R with height r. Then:*
i) There exist elements $x_1, ..., x_r$ such that p is minimal above the ideal generated by these elements.
ii) Let $y_1, \dots, y_s$ be arbitrary elements of R, then the height of the ideal $p/\langle y_1, \dots, y_s\rangle$ is at least $r - s$.
iii) Let $x_1, \dots, x_r$ be chosen as in i). Then the height of $p/\langle x_1, \dots, x_i\rangle$ is equal to $r - i$ for all i.

Proof. i) Since R_p is an r-dimensional local ring, we can find an ideal generated by r elements that is primary to the unique maximal ideal. But we can assume without loss of generality that all the generators lie already in R, which means that we found the desired elements.
ii) Let $\bar{R}$ be the quotient ring $R/\langle y_1, \dots, y_s\rangle$, and $\bar{p}$ the ideal $p/\langle y_1, \dots, y_s\rangle$ of $\bar{R}$. Assume that the height of $\bar{R}$ is equal to t. Then we know there exist elements $z_1, \dots, z_t$ of $\bar{R}$ such that $\bar{p}$ lies minimally above the ideal generated by these elements. This implies that P lies minimally above the ideal $\langle y_1, \dots, y_s, z_1, \dots, z_t\rangle$. Therefore r is smaller than or equal to $s + t$.
iii) We use the fact that the prime ideal $p/\langle x_1, \dots, x_i\rangle$ lies minimally above the ideal $\langle x_{i+1}, \dots, x_r\rangle$ (as an ideal in $R/\langle x_1, \dots, x_i\rangle$). We conclude that $\mathrm{ht}(p/\langle x_1, \dots, x_i\rangle) \leq r - i$. □

We see that the fundamental theorem gives a way to bound the number of generators of the radical of ideals. A much more subtle question is to bound the number of generators of ideals in noetherian local rings itself. For rings of dimension ≤ 1, there exists a fixed bound for all ideals. For dimension 2 or higher, no such bound can exist. To give some insight on what results are published in this area, we state the following interesting result which is due to McLean from 1990:

Theorem 2.3.3. *Let (R, m) be a noetherian local ring with maximal ideal m. Then every ideal of R can be generated by maximal n elements if and only if the ideal m^n can be generated by n elements.*

We can also prove a claim made in chapter 1 about the expected behaviour of the dimension of a ring in relation to non-zero-divisors:

Corollary 2.3.4. *Let (R, m) be a noetherian local ring, and $x \in R$ non-zero-divisor. Then* $\dim(R/\langle x \rangle) = \dim(R) - 1$.

Proof. Let d be the Krull dimension of $R/\langle x \rangle$. Clearly $d \leq \dim(R) - 1$. We choose a sequence of elements $x_1, \ldots, x_d$ from R such that their image in $R/\langle x \rangle$ generate an $m/\langle x \rangle$-primary ideal. Then the ideal generated by x and the elements $x_1, \ldots, x_d$ is m-primary, so $d + 1 \geq \dim(R)$. This completes the proof. □

Another application of the fundamental theorem gives a relation between the Krull dimension of a local ring and the embedding dimension, defined as the dimension of the R/m-vector space m/m^2.

Corollary 2.3.5. *Let (R, m) be a noetherian local ring. Then* $\dim(R) \leq \operatorname{embdim}(R)$.

Proof. Let $x_1, ..., x_s$ be elements of m such that their image form a basis of the R/m-vector space m/m^2. As the composition $\langle x_1, \ldots, x_s \rangle \to m \to m/m^2$ maps $\langle x_1, \ldots, x_s \rangle$ onto m/m^2, we conclude that $\langle x_1, \ldots, x_s \rangle + m/m^2 = m$, and a variant of Nakayama's lemma tells us that $\langle x_1, \ldots, x_s \rangle$ is equal to m. By the fundamental lemma we know that the Krull dimension of R is smaller than or equal to s. □

One of the most important properties of the Krull dimension is that it stays invariant under completion. This was one of the presumed properties which lead to the modern definition of the notion of dimension. Since this analytic concept of completion is more involved, we give a reminder on how the construction works and prove basic properties:
Consider a topological abelian group $(G, +)$, so that the two functions $G \times G \to G, (x, y) \to x + y$ and $G \to G, x \to -x$ are continuous. If 0 is closed in G, then G is hausdorff. Since the translation $T_r(x) = r + x$ is a homeomorphism from G to G, the topology of G is determined by the neighborhoods of 0 in G. It is easy to see that G is hausdorff if and only if 0 is closed. Suppose this is the case from now on. If we assume that 0 has a countable system of fundamental neighborhoods, then the completion is usually defined in terms of Cauchy sequences. A Cauchy sequence in G is defined to be a sequence (x_k) of elements such that

for all neighborhoods U of 0 there exists an integer s depending on U such that the difference between two elements x_i, x_j of the sequence is contained in U for $i, j \geq s$. As usual two Cauchy sequences are considered equivalent if their difference goes to 0 in G. One denotes the set of all equivalence classes of Cauchy sequences by $\widehat{G}$. The group law of G naturally gives a group law on he comppletion $\widehat{G}$. Identifying each element x of G with the constant sequence $(x, x, x, ...)$ gives a homomorphism from G to $\widehat{G}$. Since the kernel of this map is exactly the intersection of all neighborhoods of 0, this is injective if and only if G is hausdorff.
We want to consider the restricted case that 0 has a fundamental neighborhood consisting only of subgroups G_n. Then we get a descending chain $G \supseteq G_1 \supseteq ... \supseteq G_n \supseteq ...$ and U is a neighborhood of 0 if and only if it contains some G_n. For topologies given by these kind of subgroups one defines the completion purely algebraically in terms of inverse limits. Given a Cauchy sequence (x_k) in G, the image of x_k in G/G_n eventually becomes constant, which we denote by ζ_n. One can consider the projection $G/G_{n+1} \to G/G_n$, mapping ζ_{n+1} to ζ_n. A sequence (ζ_k) with this properties is called a coherent sequence. Conversely, every coherent sequence can be used to construct a Cauchy sequence. One must only define (x_k) by requiring that x_n is contained in a neighborhood of ζ_n. This just means that the sequence (x_k) must fulfill $x_{n+1} - x_n \in G_n$. A sequence of subgroups G_n of G and morphisms between them is called an inverse system, and the group of all coherent sequences of this system is then called the inverse limit. With this notation, one defines the completion $\widehat{G}$ to be this inverse limit: $\widehat{G} \cong \varprojlim G/G_n$.
Now one can study completions effectively via inverse limits. Given three inverse systems $\{A_n\}, \{B_n\}$ and $\{C_n\}$ and an exact sequence $0 \to A_n \to B_n \to C_n \to 0$, then the corresponding sequence $0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$ is also exact. If the three systems are surjective, this is also the case. This has an important consequence. If G, G' and G'' are topological abelian groups and $0 \to G' \to G \to G'' \to 0$ an exact sequence with $\theta G \to G''$ such that the topology of G is induced by a sequence of subgroups G_n, and the topologies of G' and G'' are induced by G, then the corresponding sequence $0 \to \widehat{G'} \to \widehat{G} \to \widehat{G''} \to 0$ is also exact. This can be specifically used for the case $G' = G_n$ and $G'' = G/G_n$. Then G'' has the discrete topology such that $\widehat{G''} = G''$. This implies that $\widehat{G_n}$ is a subgroup of $\widehat{G}$ and that $\widehat{G}/\widehat{G_n}$ is isomorphic to G/G_n. It is also clear then that the completion is an idempotent operation.
The most important example of such a topological group is that of a ring R together with the topology induced by the power of an ideal I. This topology is called the I-adic topology. Similarly, given a module M over a ring R one can take $G = M$ and $G_n = I^n M$. This defines an I-topology on M, and the resulting module $\hat{M}$ is a topological $\hat{R}$-module. Using tensor products, it is not hard to see that for finitely generated modules M over a noetherian ring R we have an isomorphism between $\hat{M}$ and $\hat{R} \otimes_I M$. It can also be shown that for a noetherian ring R and ideal I with the I-adic topology, the ring $\hat{R}$ is a flat R-algebra.
This gives the following properties for a noetherian ring R and its I-adic completion $\bar{R}$:
i) $\widehat{I} = \widehat{R}I \cong \widehat{R} \otimes_R I$, ii) $\widehat{I^n} = (\widehat{I})^n$
iii) $I^n/I^{n+1} \cong \widehat{I^n}/\widehat{I^{n+1}}$, iv) $\widehat{I}$ is contained in the Jacobson ideal of $\bar{R}$.
Now the following is immediate:

Corollary 2.3.6. *Let (R, m) be a noetherian local rin, and $\widehat{R}$ be the m-adic completion of R. Then the dimensions of R and $\widehat{R}$ are the same.*

Chapter 3

Important classes of noetherian rings

Many interesting classes of noetherian rings can be characterized by properties concerning notions such as dimension, height and chain conditions. The following chapter surveys artinian rings, catenary rings, Cohen-Macaulay rings and regular rings.

3.1 Artinian rings

One can consider artinian rings to be one of the simplest rings. We remind the reader that these are rings that satisfy the descending chain condition for its ideals. The definition seems really similar to the noetherian property, but is in fact much more restrictive. Before stating the main theorem we collect some important properties of these rings:

Proposition 3.1.1. *Let R be an artinian ring. Then:*
i) Every prime ideal is maximal.
ii) R has only finitely many maximal ideals.
iii) The product of all maximal ideals is nilpotent.

Proof. i): Let p be a prime ideal of R and consider the integral domain R/p. By the descending chain condition there exists for all non-zero x in this quotient ring an $n \in \mathbb{N}$ such that $\langle x^n \rangle = \langle x^{n+1} \rangle$. This implies that can write 0 as $x^n(1 - xy)$ for some $y \in R/p$. But then y is the inverse of x, and R/p is a field. Therefore p is maximal.
ii): Assume there are infinitely many maximal ideals $m_1, m_2, \ldots$ and define J_n to be the intersection of the maximal ideals $m_1, m_2, \ldots, m_n$. By assumption we have $J_n = J_{n+1}$ for n big enough. This implies that the intersection of these ideals $\cap_{i=1}^{n} m_i$ is contained in m_{n+1}. But then there exists an index j such that m_j is contained in $m_n + 1$. This however is a contradiction to the prime avoidance lemma (Matsumura, 1986, Exercise 16.8.).
iii): Let I be the product of all maximal ideals of R. By the descending chain condition there exists $n \in \mathbb{N}$ such that $I^n = I^{n+i} \forall i \geq 0$. We claim that I^n is equal to the zero ideal.
We assume that this is wrong, and consider the non-empty set of ideals $M = \{J \subset R \mid I^n J \neq 0\}$.
Since M is non-empty, it contains a minimal element $\widehat{J}$ with respect to the inclusion. It is clear that this ideal must be a principal: $\widehat{J} = \langle x \rangle$. By the minimality of $\widehat{J}$ and the definition

of M we have $= I^n\langle x\rangle = I^{2n}\langle x\rangle \neq 0$, but this implies $I^n\langle x\rangle = \langle x\rangle$. Therefore there exists an element y in I^n such that $(y-1)x = 0$. Since y is contained in all maximal ideals we know that $1-y$ is invertible in R, so x has to be 0, a contradiction. □

One more lemma will be needed:

Lemma 3.1.2. *Let R be a ring such that 0 is the product of finitely many (not necessarily different) maximal ideals $m_1, ...m_n$. Then R is noetherian if and only if R is artinian.*

Proof. We use the following fact about R-modules: Let M be a R-module and $N \subset M$ be a submodule. Then the following statements are equivalent: M is noetherian if and only if N and the quotient module M/N are noetherian. The same statement also holds for the property Artinian instead of Noetherian.
Now denote by I_i the product of the first i maximal ideals and consider the chain $0 \subset I_i \subset I_{i-1} \subset \cdots \subset I_1 \subset R$. Repeatedly applying the lemma above we have that R is noetherian (resp. artinian) if and only if every quotient module I_{i-1}/I_i is noetherian (resp. artinian). Since $m_i I_{i-1}/I_i = 0$, I_{i-1}/I_i is an R/m_i vector space. We know that a subset of I_{i-1}/I_i is an R-module if and only if it is an R/m_i-subspace. Therefore I_{i-1}/I_i is noetherian/artinian if and only if $\dim_{R/m_i} < \infty$. □

We can know state the essential theorem:

Theorem 3.1.3. *For a non-zero ring R the following are equivalent:*
i) R is artinian.
ii) R is noetherian and the Krull dimension of R is zero.

Proof. i) $\Rightarrow$ ii): We have last seen that $\dim(R) = 0$ for artinian rings. Since $\{0\}$ is the product of finitely many maximal ideals, R is noetherian by the lemma.
ii) $\Rightarrow$ i): The nilradical of R is the intersection of all prime ideals. Of course, it is enough to take this intersection only over all minimal prime ideals. We use the fact that a noetherian ring has only finitely many minimal prime ideals. This follows from considerations about the Zariski-topology on $\mathrm{Spec}(R)$: Namely, the minimal prime ideals of R are in bijection with the irreducible components of $\mathrm{Spec}(R)$. But for noetherian R the topological space $\mathrm{Spec}(R)$ is noetherian too. However, such a space consists of only finitely many irreducible components. Therefore the set of all minimal prime ideals is finite. Thus we get $\prod m = Nil(R) = \sqrt{0}$, where the product runs over all maximal ideals m. We now use the following equivalence which gives a relation between ideal powers and radical ideals: For an arbitrary ring R and ideals I, J such that I is finitely generated, we have: I is contained in the radical of J if and only if a power of I is contained in J. Using this we see that the product of all maximal ideals of R is nilpotent and therefore R is artinian. □

3.2 Catenary rings

Catenary rings are described in terms of chains of prime ideals. These chains fulfill a specific regularity condition. Many important noetherian rings belong to this class:

Definition. Let R be a notherian ring. R is called catenary if for any two prime ideals $p \subset q$ all maximal chains of prime ideals between p and q have the same length. Furthermore R is called universally catenary if all finitely generated R-algebras are catenary.

By the correspondence between ideals of a ring and ideals in a quotient ring or localized ring it is clear that catenary rings behave well under taking quotients and localizing:

Proposition 3.2.1. *Let R be a noetherian catenary ring, $I \subset R$ and ideal and $S \in R$ a multiplicatively closed set. Then the following holds:*
i) R/I is catenary.
ii)$S^{-1}R$ is catenary.

Lemma 3.2.2. *(Local-global principle for catenary rings) Let R be a noetherian ring. The following are equivalent:*
i) R is catenary.
ii) R_p is catenary for all prime ideals p.
iii) R_m is catenary for all maximal ideals m.

Proof. $i) \Rightarrow ii)$: by the proposition, $ii) \Rightarrow iii)$: immediate, $iii) \Rightarrow i)$: Assume R_m is catenary for all maximal ideals m of R. For all prime ideals $p \subset q$ choose a maximal ideal m with $q \subset m$. Since chains of prime ideals between p and q are in one-to-one correspondence with chains of prime ideals between pR_m and qR_m R is catenary. □

A noetherian domain is catenary if and only if a specific formula for the height and coheight of prime ideals is fulfilled.

Theorem 3.2.3. *(Ratkiff, 1972) Let (R, m) be a noetherian local domain. The following are equivalent:*
i) R is catenary.
ii) For all prime ideals p of R we have: $\mathrm{ht}(R) + \mathrm{coht}(R) = \mathrm{ht}(m) = \dim(R)$.

The proof requires the following two theorems which we just state:

Theorem 3.2.4. *(Ratkiff's strong existence theorem, [M, p. 247-250])*
Let R be a noetherian integral domain, $p \subset P$ prime ideals, $\mathrm{ht}(p) = h > 0$ *and* $\mathrm{ht}(P/p) = d$. *Then for all $0 \le i < d$ the set*

$$\{p' \in \mathrm{Spec}(R) \mid p' \subset P, \mathrm{ht}(P/p') = d - i \textit{ and } \mathrm{ht}(p') = h + i\}$$

is infinite.

Theorem 3.2.5. *([A]) Let R be a noetherian ring and P a prime ideal. Then there are at most finitely many prime ideals P' of R satisfying $P \subset P'$,* $\mathrm{ht}(P'/P) = 1$ *and* $\mathrm{ht}(P') > \mathrm{ht}(P) + 1$.

Proof. $i) \Rightarrow ii)$: Let R be catenary. Per definition all maximal chains between 0 and m have the same length.
$ii) \Rightarrow i)$: Let $\dim(R) = n$ and assume that R is not catenary. Then there exist prime ideals $p \subset q$ such that $\mathrm{ht}(P/p) = 1$ and $\mathrm{ht}(P) > \mathrm{ht}(p) + 1$. Set $\mathrm{ht}(m/P) = d$. We apply Ratkiff's strong existence theorem to the ring R/p to get that there exist infinitely many prime ideals p' such that $p \subset p'$, $\mathrm{ht}(p'/p) = 1$ and $\mathrm{ht}(m/p') = d$. However per assumption we also have $\mathrm{ht}(m/p') + \mathrm{ht}(p') = n$, therefore $\mathrm{ht}(p') = n - d = \mathrm{ht}(P) > \mathrm{ht}(p) + 1$. However the second theorem tells us that there are only finitely many such p', a contradiction. $\square$

Universally catenary rings can be described in terms of an dimension inequality: For a noetherian domain A and a domain B containing A that is finitely generated over A and prime ideals P of B and $p = P \cap A$ of A the dimension formula is the following inequality:
$\mathrm{ht}(P) \leq \mathrm{ht}(p) + \mathrm{trdeg}_A(B) - \mathrm{trdeg}_{\kappa(p)}(\kappa(P))$

Theorem 3.2.6. *(Ratkiff) Let R be a noetherian ring. Then R is universally catenary if and only if the dimension formula holds between R/p and S, for every prime ideal p of R and every finitely generated domain S that contains R.*

Proof. $\Rightarrow$: If R is universally catenary, then this also applies to R/p and we can assume without loss of generality that R is an integral domain. Write $S = R[X_1, \ldots, X_m]/Q$, then if $p = p'/Q$, we have $\mathrm{ht}(p) = \mathrm{ht}(p\prime) - \mathrm{ht}(Q)$ because $R[X_1, \ldots, X_m]$ is catenary as well. But then we have equality in the dimension formula.
$\Leftarrow$: Assume that R is not universally catenary. Then there exists a finitely generated R-algebra S which is not catenary. We denote by p the corresponding kernel of the morphism from R to S. Since S is not catenary, there exists prime ideals $q_1 \subset q_2$ of S such that $\mathrm{ht}(q_2/q_1) = d$ and $\mathrm{ht}(q_2) > d + \mathrm{ht}(q_1)$. Let $\mathrm{ht}(q_1) = h$. We choose elements $s_1, \ldots, s_h$ from q_1 such that $\mathrm{ht}(\langle s_1, \ldots, s_h \rangle) = h$ and q_1 is a minimal prime divisor of this ideal. Let $\langle s_1, \ldots, s_h \rangle = Q_1 \cap \cdots \cap Q_k$ be a shortest primary decomposition of I such that $\sqrt{Q_1} = q_1$. For $s \in q_2 Q_2 \cdots Q_k / q_1$ we have $(\langle s_1, \ldots, s_h \rangle : \langle s \rangle^i) = Q_1$. We set $\frac{s_i}{s} = t_1$, $T = S[t_1, \ldots, t_h]$, $J = \langle t_1, \ldots, t_h \rangle_T$ and $M = J + \langle q_2 \rangle_T$. Every element of T can be written as $\frac{u}{t^k}$, where $u \in (\langle s_1, \ldots, s_h \rangle + tS)^k$, so that $z \in J \cap S$ implies $zt^k \in \langle s_1, \ldots, s_h \rangle$ for k big enough and therefore $z \in Q_1$. But $Q_1 \subset J \cap S$ so that $Q_1 = J \cap S$. We have $M \cap S = J + \langle q_2 \rangle_T \cap S = J \cap S + \langle q_2 \rangle$, $T/J \cong S/Q_1$ and $T/M \cong S/q_2$. Therefore $T_M/JT_M \cong S_{q_2}/Q_1 S_{q_2}$ is a d-dimensional noetherian local ring, and J is generated by h elements, so that $\mathrm{ht}(M) = \dim(T_M) \leq h + d < \mathrm{ht}(q_2)$. But S and T have the same field of fractions, and furthermore $\kappa(M) = \kappa(q_2)$. This implies that the dimension formula does not hold between S and T, but by our assumption the formula holds between R/p and B and R/p and C. $\square$

3.3 Cohen-Macaulay rings

The class of Cohen Macaulay rings and the subclass of regular rings are of great importance not only in commutative algebra, but also homological algebra, algebraic geometry and algebraic combinatorics. To cite Mel Hochster: "Life is really worth living in a Cohen Macaulay ring." Their definition is closely related to system of parameters, but there are connections to other concepts of dimension theory as well. Before stating the definition, we consider the notion of depth.

Let R be a noetherian ring. A sequence of elements $x_1, \dots, x_n$ of R is called a regular sequence if the following two conditions are satisfied:

i) The elements $x_1, \dots, x_{i-1}$ satisfy only trivial relations.

ii) $R/\langle x_1, \dots, x_n\rangle \neq 0$

The depth of R, denoted by $\mathrm{depth}(R)$, is defined as the maximal number of elements of a regular sequence in R. Similarly, the depth of an ideal I of R is the maximal number of elements of a regular sequence that is contained in I.

It is useful to notice that if R is local and $\langle x_1, \dots, x_n\rangle$ is contained in the maximal ideal, then the second condition is fulfilled by default because of Nakayama's lemma.

Let us consider the local case first: If (R, m) is a noetherian local ring, then R is called Cohen Macaulay if there exists a system of parameters that is a regular sequence. More generally, a noetherian ring is called Cohen Macaulay if the localizations by all prime ideals gives a Cohen Macaulay local ring.

Cohen Macaulay local rings satisfy interesting properties:

Theorem 3.3.1. *Let (R, m) be a Cohen Macaulay local ring.*

1) The following are equivalent for a sequence of elements $x_1, \dots, x_n$ of m:

i) $x_1, \dots, x_n$ is a regular sequence in R;

ii) $\mathrm{ht}(\langle x_1, \dots, x_i\rangle) = i$ for all i;

iii) $\mathrm{ht}(\langle x_1, \dots, x_n\rangle) = n$;

iv) $x_1, \dots, x_n$ is part of a system of parameters of R;

2) For all ideals I of R we have: $\mathrm{ht}(I) = \mathrm{depth}(I), \mathrm{ht}(I) + \dim(R/I) = \dim(R)$

3) R is catenary.

Proof. 1) i) $\Rightarrow$ ii): Assume that $x_1, \dots, x_n$ is a regular sequence. By definition we know that $0 < \mathrm{ht}(x_1) < \mathrm{ht}(x_1, x_2) < \dots$. On the contrary, Krull's principal ideal theorem tells us that the height of the ideal $\langle x_1, \dots, x_i\rangle$ is at most i.

ii) $\Rightarrow$ iii): Obvious.

iii) $\Rightarrow$ iv): If the dimension of R is n, then this is automatically true. Assume that $\dim(R)$ is strictly bigger than n. Then we know that the maximal ideal is not a minimal prime divisor of $\langle x_1, \dots, x_n\rangle$, so we can find an element $x_{n+1} \in m$ such that x_{n+1} is not contained in any minimal prime divisor of $\langle x_1, \dots, x_n\rangle$. Then we have $\mathrm{ht}(\langle x_1, \dots, x_{n+1}\rangle) = n + 1$ We can do this finitely many times until we have a parameter system of R which contains the elements x_1 up to x_n.

iv) $\Rightarrow$ i): We show the stronger result that any system of parameters is a regular sequence. Assume that $x_1, \dots, x_n$ is a parameter system of R. We know that the set of all zero divisors of a noetherian local ring is the union of all associated prime ideals of R. But R is Cohen

Macaulay, so we also have the equality $\dim(R) = \dim(R/p)$ for all associated prime ideals p. Therefore x_1 cannot be contained in any associated prime, otherwise Krull's principal ideal theorem would yield a contradiction. So x_1 is a regular element. We can consider the quotient ring $R/\langle x_1\rangle$. We use the definition of a regular sequence and Corollary 2.3.3. about the difference between the dimension of a noetherian ring and the quotient ring by a non zero-divisor. This tells us that the image of the elements $x_2, \ldots, x_n$ in $R/\langle x_1\rangle$ form a regular sequence again. But this means precisely that $R/\langle x_1\rangle$ is Cohen Macaulay as well. Now we use induction on the dimension of R to prove the claim.
2) Let $\mathrm{ht}(I) = r$. By the converse of Krull's principal ideal theorem we can find elements $x_1, \ldots, x_r$ such that $\mathrm{ht}(\langle x_1, \ldots, x_i\rangle) = i$ for all i. Therefore $x_1, \ldots, x_r$ is a regular sequence in I and $\mathrm{depth}(I) \geq \mathrm{ht}(I)$. Now assume that $y_1, \ldots, y_s$ is a regular sequence in I. We have seen that this implies $s \leq \mathrm{ht}(I)$, which proves the assertion.
For the second claim it is enough to show that $\mathrm{ht}(p) + \dim(R/p) = \dim(R)$ for all minimal prime divisors p of I. Let $\mathrm{ht}(p) = r = \dim(R_p)$ and $\dim(R) = n$. It is easy to see that the image of a regular sequence of elements in p forms again a regular sequence of elements in R_p. Therefore R_p is Cohen Macaulay as well. Let now $x_1, \ldots, x_r$ be a maximal p-regular sequence, then p has to be minimal above $\langle x_1, \ldots, x_r\rangle$ because of 1), and we compute that $\dim(R/p) = \dim(R/\langle x_1, \ldots, x_r\rangle) = n - r$.
3) Let $q \subset p$ be two prime ideals of R. Since R_p is a Cohen Macaulay ring, we have $\dim(R_p) = \mathrm{ht}(qR_p) + \dim(R_p/qR_p)$. This is equivalent to $\mathrm{ht}(p) - \mathrm{ht}(q) = \mathrm{ht}(p/q)$. □

The proof of this theorem tells us that in a Cohen Macaulay local ring every system of parameters is a regular sequence, and that all maximal regular sequences have the same maximal length, namely the dimension of R. So we could alter the definition slightly.
While the definition of a Cohen Macaulay ring is easy to grasp, it can be notoriously difficult to determine whether a given ring is actually of this type. One of the classical problems involving this question concerns the so-called homological conjectures by Hochster (Homological conjectures, old and new, Hochster, Illinois Journal of Mathematics Volume 51, Number 1 (2007), 151-169.).
There exists a remarkable statement by Cohen from 1916 which gives an equivalent characterization for a noetherian ring to be Cohen-Macaulay. This needs the concept of unmixedness: An ideal I of a noetherian ring R is called unmixed if the height of all its prime divisors are equal. This means that there exists no embedded prime divisor of I. One says that the unmixedness theorem holds for R if for all $r \geq 0$ the ideals of height r, which can be generated by r elements, are unmixed.

Theorem 3.3.2. *The following are equivalent for a noetherian ring R:*
i) R is Cohen-Macaulay.
ii) The ring R satisfies the unmixedness theorem.

Proof. i) $\Rightarrow$ ii): Let R be Cohen-Macaulay and assume that I is an ideal of height r generated by r elements $x_1, \ldots, x_r$. Assume that p is an embedded prime divisor of I. We can localize R at the prime ideal p and assume without loss of generality that R is local. By the theorem before the quotient ring $R/\langle x_1, \ldots, x_r\rangle$ is Cohen-Macaulay as well. But this means that I does

not possess any embedded prime divisors, a contradiction.
ii) $\Rightarrow$ i): Suppose the unmixedness theorem holds for R. By the converse of Krull's principal ideal theorem we can find for the prime ideals p of height r from before elements $x_1, \ldots, x_r$ that generate p and satisfy $\mathrm{ht}(\langle x_1, \ldots, x_i \rangle) = i$ for all i. By the unmixedness theorem, all prime divisors of $\langle x_1, \ldots, x_i \rangle$ have height i, so that this ideal cannot contain the element x_{i+1}. Therefore x_{i+1} is a regular element of the quotient ring $R/\langle x_1, \ldots, x_i \rangle$, which means that $x_1, \ldots, x_r$ is a regular sequence. Since the depth of the ring R_p coincides with its dimension, R_p is a Cohen-Macaulay local ring. Due to the fact that the choice of p was arbitrary, we conclude that R is Cohen-Macaulay. □

Similarly to the noetherian property in Hilbert's basis theorem a polynomial ring over a Cohen-Macaulay ring is Cohen-Macaulay as well:

Theorem 3.3.3. *Let R be a noetherian Cohen-Macaulay ring. Then the same holds for $R[x_1, \ldots, x_n]$.*

Proof. By induction it is enough to consider the case $n = 1$. Let $S = R[x]$ and P a maximal ideal of S. Let m be the contraction of P in R. Then S_P is a localization of the ring $R_m[x]$. Therefore we can replace R by R_m and assume without loss of generality that R is Cohen-Macaulay with maximal ideal m. In this case we just need to show that S_P is Cohen-Macaulay. If we denote by k the residue field of R, than we have $S/mS = k[x]$, P/mS is a principal ideal of $k[x]$ generated by an irreducible monic polynomial $f(x)$. Let $g(x) \in R[x]$ be the polynomial that is equivalent to f modulo mB, then P is generated by m and g. We now choose a system of parameters $r_1, \ldots, r_n$ of R such that $r_1, \ldots, r_n, f$ is a system of parameters for R_P. Since a polynomial ring is flat over its coefficient ring, the regular sequence of elements $r_1, \ldots, r_n$ of R is also a regular sequence of S. Since the image of f in the quotient ring $R/\langle r_1, \ldots, r_n \rangle[x]$ is not a zero-divisor, $r_1, \ldots, r_n, f$ is a regular sequence in S too. We get the inequality $\mathrm{depth}(P) \geq n + 1 = \dim(S_P)$. Therefore S_P is Cohen-Macaulay. □

Using all the theorems we have shown, it is easy to see the following:

Corollary 3.3.4. *Let R be a Cohen-Macaulay ring. Then any quotient of R is universally catenary.*

Another possible way to characterize Cohen-Macaulay local rings is via the connection between the multiplicity of the ring and the coefficient of the Hilbert polynomial. We will show this connection in chapter 5.

3.4 Regular rings

Regular rings play an important role in algebraic geometry. They correspond to nonsingular points of an affine variety. There structure is fascinating, in a way they behave like polynomial rings.
As usual we cover the local case first: A noetherian local ring (R, m) is called regular if the maximal ideal can be generated by a system of parameters. This distinguishes them by far from general local rings, where one can only ensure that an m-primary ideal can be generated by a system of parameters. This means that regular local rings are precisely the rings for which the Krull dimension and the embedding dimension coincide.
As usual, a noetherian ring is called regular if the localization at every prime ideals is regular. They form a main subclass of Cohen-Macauly rings: Indeed, a system of parameters that generates m is also a regular sequence.
Regular local rings can be characterized by their associated graded ring:

Theorem 3.4.1. *Let (R, m) be a noetherian regular local ring of dimension d. Then the associated graded ring $G_m(R)$ is equal to the polynomial ring $(R/m)[x_1, \ldots, x_d]$ in d variables.*

Proof. We can use Theorem 2.2.1. from chapter 2: Let $r_1, \ldots, r_d$ be a system of parameters of R. We already have seen that there exists a surjection from $(R/m)[x_1, \ldots, x_d]$ to $G_m(R)$, where every variable x_i is mapped to the image of r_i in m/m^2. If the parameter system is chosen such that it generates the maximal ideal, then of course this morphism is injective as well. $\square$

This result can be used to prove that a regular local ring is in fact a domain:

Corollary 3.4.2. *Let (R, m) be a noetherian regular local ring. Then R is a domain.*

Proof. We use Krull's intersection theorem, telling us that the intersection of all powers of the maximal ideal is equal to 0. Now assume there exist two non-zero elements x and y of R such that $xy = 0$. Let us assume that n and o are the maximal integers such that $x \in m^n$ and $y \in m^o$. By choice of n and o the images of the elements x and y in the quotient rings m^n/m^{n+1} and m^o/m^{o+1} are non-zero. Since the polynomial ring is an integral domain, the product of these two images in the associated graded ring is non-zero as well. Other than that, the product of the elements xy is contained in m^{n+o+1}. But then xy cannot be zero, a contradiction. $\square$

In a regular local ring it is easy to see if a sequence of elements form part of a system of parameters or not:

Proposition 3.4.3. *Let (R, m) be a noetherian regular local ring, and $x_1, \ldots, x_m$ a sequence of elements in m. Then the following are equivalent:*
i $x_1, \ldots, x_m$ is a subset of a system of parameters of R.
ii) The images of $x_1, \ldots, x_m$ in m/m^2 are linearly independent over R/m.
iii) The quotient ring $R/\langle x_1, \ldots, x_m \rangle$ is regular of dimension $dim(R) - n$.

Proof. i) $\Leftrightarrow$ ii): We already know that there is a bijection between regular system of parameters and R/m-bases of m/m^2. i) $\Rightarrow$ iii): Since $x_1, ..., x_m$ is a regular sequence, we know that $R/\langle x_1, \ldots, x_m \rangle$ is noetherian local of dimension $\dim(R) - m$. By definition of i) there exist $\dim(R) - m$ elements that generate the maximal ideal of this ring, so it has to be regular.
iii) $\Rightarrow$ ii): Since $R/\langle x_1, \ldots, x_m \rangle$ is regular, we can lift these elements to a larger sequence of elements containing them which generate the maximal ideal m and form a regular sequence. $\square$

Theorem 3.4.4. *Let (R, m) be a noetherian local ring. Then R is regular if and only if $\widehat{R}$ is regular.*

Proof. This can be proved easily by passing to the associated graded rings. In chapter 2 we already learned that the graded rings $G_m(R)$ and $G_{\widehat{m}}(\widehat{R})$ are isomorphic. But Theorem 3.4.1 tells us that this associated graded ring is isomorphic to a polynomial ring if and only if R is regular, the assertion follows. $\square$

Maybe the most important result related to regular rings is Cohen's structure theorem. We have already used it to suggest why certain geometric properties of the Krull dimension are fulfilled. This theorem is of such importance because it essentially asserts that complete noetherian regular local rings are effectively completions of polynomial rings. One can take advantage of this surprising feature. Often, a problem about rings does not change when passing to the completion, so it is possible that one can reduce a problem to this case.
Hereafter, we want to state this theorem in its most used form. We have to relinquish of a proof because of its complexity.

Theorem 3.4.5. *Let R be a complete noetherian regular local ring of dimension d. If R contains a coefficient field, then R is isomorphic to the ring of power series in d variables over this field. ([C])*

Another aspect of regular rings is their connection to homological algebra. One can use the homological invariant of global dimension to characterize this class of rings. For example, all proofs showing that any localization of a regular ring is again regular uses this technical notion. Since proofs involving this notion are quite different to standard commutative algebra techniques, we decided not to use it.
We want to end this chapter by stating an essential property which is always fulfilled for regular local rings. The proof of this theorem uses global dimension:

Theorem 3.4.6. *(Auslander, Maurice; Buchsbaum, D. A. (1959), "Unique factorization in regular local rings", Proceedings of the National Academy of Sciences of the United States of America, 45: 733–734)*
Let R be a noetherian regular local ring. Then R is factorial.

In chapter 5, we will characterize regular local rings via the Hilbert polynomial.

Chapter 4

Maps between rings

Given the notion of dimension for rings, a natural question to ask is how the dimension behaves under ring morphisms. Given no restriction on the ring homomorphism no apparent relations between the dimension of the rings can be made: One just has to consider the inclusion of an integral domain of arbitrary Krull dimension into its field of fraction. By forcing specific assumptions on the ring maps subtle assertions can be made. At first we consider integral ring morphisms, for which many statements about the dimension and height of prime ideals can be made. Afterwards, we look at arbitrary ring morphisms, and summarize what can be said about them. In the end, we make a statement about the behaviour of the dimension of fibres. We recall basic definitions needed for this chapter: A ring homomorphism $\phi : R \to S$ is called finite if S is finitely generated as an R-module. It is called of finite type if there exists a natural number n and a surjection of R-algebras $R[x_1, \ldots, x_n] \to S$. An element $s \in S$ is called integral over R if there exists a monic polynomial $P(x) \in R[x]$ such that $P^{\phi}(s) = 0$, where $P^{\phi}(x) \in S[x]$ is the associated polynomial from the image of P under the induced mapping $\phi : R[x] \Rightarrow S[x]$. The morphism ϕ is called integral if all elements of S are integral over R. Any ring homomorphism $\phi : R \to S$ induces a function on the spectra of these rings in the opposite direction. Namely, a prime ideal q of S is mapped to its contraction $P^c = f^{-1}(q)$. We assign to each point $p \in R$ the inverse image $\{q \in \mathrm{Spec}(S) \mid q^c = p\}$ If we denote by $\kappa(p) = R_p/pR_p$ the residue field of p, then the spectrum of the ring $S \otimes_R \kappa(p)$ is called the fibre of ϕ over p. This notation is justified because there is a homeomorphism between $\mathrm{Spec}(S \otimes_R \kappa(p))$ and the set of prime ideals of S that contract to p.

4.1 Integral ring homomorphism

Lemma 4.1.1. *Let $\phi : R \to S$ be a ring morphism, and $y \in S$. Assume that there exists a finitely generated R-submodule M of S containing* 1 *such that $yM \subset M$. Then y is integral over R.*

Proof. Let $x_1 = 1$ and assume that M is generated as an R-module by the elements $x_2, ..., x_n$. Then consider the sequence $x_1, \ldots, x_n$. Every product yx_i can be written as $\sum \phi(r_{ij})x_j$ with coefficients r_{ij} from R. Let $P(x) \in R[x]$ be the characteristic polynomial of the matrix

$\widetilde{R} = (r_{ij})_{i,j}$. By a variant of the Cayley-Hamilton theorem we know that $P(\tilde{R}) = 0$. By construction the natural map $\pi : R^n \to M$ sending a tupel $(r_1, \ldots, r_n)$ to $\sum_{i]\phi(r_i} x_i$ commutes with $\widetilde{R} : R^n \to R^n$ and the multiplication by y: $\pi(\tilde{R}x) = y\pi(x)$. Therefore $P(y) = P(y)x_1 = P(y)\pi((1, ...0)) = \pi(P(A)(1, ..., 0)) = 0$, and we see that y is integral over R. □

Lemma 4.1.2. *Let $\phi : R \to S$ be a morphism of rings. Let $s_1, \ldots, s_n$ be elements of S. Then they are integral over R if and only if there exists a subalgebra S' of S that contains $s_1, \ldots, s_n$ and is finite over R.*

Proof. Assume that all S_i are integral. Then we can consider the subalgebra S' of S generated by $\phi(R)$ and s_1 up to s_n. This algebra is finite: If we denote by k_i the degree of the equation of integral dependence satisfied by s_i, then S' is generated as an R-module by elements of the form $s_1^{l_1}...s_n^{l_n}$, where l_i ranges from 0 to $k_i - 1$. The converse is true by the lemma above. □

The following characterizes integral ring morphisms, which are of finite type:

Lemma 4.1.3. *Let $\phi : R \to S$ be a ring morphism, then the following are equivalent:*
i) ϕ is finite.
ii) ϕ is integral and of finite type.
iii) S is generated as an R-algebra by integral elements $x_1, \ldots, x_n$.

Proof. i) $\Rightarrow$ ii): Let $\phi : R \to S$ be finite. Then S itself is a finitely generated R-module that satisfies $yS \in S$ for all $y \in S$. By the previous lemma ϕ is integral. It is clear that every finite ring map is of finite type.
ii) $\Rightarrow$ iii): Since ϕ is integral and of finite type, we can find finitely many integral elements $x_1, \ldots, x_n$ of S that generate S as an R-algebra.
iii) $\Rightarrow$ i): Obvious. □

Being integral is a transitive property:

Lemma 4.1.4. *Let $R \to S$ and $S \to T$ be two integral ring maps. Then the composition $R \to T$ is also integral.*

Proof. Let t be an element of T. Since T is integral over S, the exists a monic polynomial $P(x) \in S[x]$ that annihilates t. We look at the finite set of coefficients of this polynomial, and find a finite R-subalgebra S' of S that contains all these elements. Per construction t is integral over S' as well. Now we find a finite S-subalgebra T^* of T which contains t. Since the maps $R \to S' \to T'$ are finite, the composition map $R \to T'$ is finite as well. Applying the last lemma 4.1.3. once more gives that t is integral over R. □

It is immediate now, that for a given morphism $\phi : R \to S$ the set of all integral elements of S over R is an R-subalgebra of S. This algebra is called the integral closure of R in S. One says that R is integrally closed in S if and only if R equals its integral closure.
Integral dependence behaves well under taking quotients and localization:

Lemma 4.1.5. *Let $\phi : R \to T$ be a morphism of rings, and T' the integral closure of R in T. If S is a multiplicatively closed set of R, then the integral closure of $S^{-1}R$ in $S^{-1}T$ is given by $S^{-1}T'$. Additionally, if the morphism ϕ is injective and q denotes an ideal of T that is contained in T', then T'/q is integral over R/q^c, where q^c denotes the contraction of q.*

Proof. We use the fact that localization is an exact functor. Therefore $S^{-1}T'$ is contained in $S^{-1}T$. Let x be an integral element of T^* and let $f \in S$. We get an equation of the form $x^n + \sum_{i=1}^{n-1} r_i x^{n-i} = 0$ in T, with coefficients r_i from R. Dividing both sides of this equation by f^n gives $(\frac{x}{f})^n + \sum_{i=1}^{n} \frac{r_i}{f^i}(\frac{x}{f})^{d-1} = 0$ in $S^{-1}T$. From this it is evident that $S^{-1}T'$ is contained in the integral closure of $S^{-1}R$ in $S^{-1}T$. On the other hand, suppose that $\frac{x}{f} \in S^{-1}T$ is integral over $S^{-1}R$. Then we have an equation of the form $(\frac{x}{f})^n + \sum_{i=1} d\frac{r_i}{f_i}(\frac{x}{f})^{d-i} = 0$ with coefficients $\frac{r_i}{f_i} \in S^{-1}R$. If we take the product of all involved elements f_i from S, then there exists $f' \in S$ such that we get an equation of integral dependence in T for the element $f' f_1 \cdots f_n x$ is integral over T. This allows us to conclude that $\frac{x}{f}$ is contained in $S^{-1}T'$.
Assume that ϕ is even injective. Without loss of generality we assume that R is a subring of T. Then if $x \in q$ is an integral element, we can reduce the equation of integral dependence $x^n + \sum_{i=1}^{n-1} r_i x^{n-i} = 0$ in T by q to get the desired result. □

An application of the previous lemma tells us that integral dependence is a local property:

Corollary 4.1.6. *Let $\phi : R \to S$ be a morphism of rings, and x an element of S. Then the following are equivalent:*
i) x is integral over R.
ii) $\frac{x}{1} \in S_p$ is integral over R_p for all prime ideals p of R.

Proof. i) ⇒ ii): This is clear. ii) ⇒ i): We consider the subalgebra S' of S generated by $\phi(R)$ and x. Let p be a prime ideal of R. Then we know that $x^n + \sum\limits_{i=1}^{n-1} \phi(r_i)x^{n-i} = 0$ in S_p for some elements r_i of R_p. Because there are only finitely many denominators, we can find an element f of R/p such that the same equation $x^n + \sum\limits_{i=1}^{n-1} \phi(r_i)x^{n-i} = 0$ also holds in S_f. Hence the coefficients r_i can be assumed to be from R_f.
By the lemmas before, we know that S'_f is finite over R_f. We use the fact that the spectrum of a noetherian ring is quasi-compact, and deduce that we can find elements $f_1, ..., f_d$ of R which generate the unit ideal of R and such that S'_{f_i} is finite over R'_{f_i}. In particular, S'_{f_i} are also of finite type. We want to show that this implies that S' is also finite over R. Since S' is integral over R, we know that it is enough to show that S' is of finite type over R.
We do the follwing: Denote by X_i a finite set that generates S_{f_i} as an R'_{f_i}-algebra. Without loss of generality we can assume that the elements of this set are in the image of the natural map $S \to S_{f_i}$. We then denote by Y_i the corresponding preimages. If Y is the union of all these sets, we get an R-algebra homomorphism $R[x_y]_{y \in Y} \to S$ (which depends on Y). The image of this map gives an R-module T which is an R-submodule of S. Therefore it makes sense to look at the quotient module S/T. Since the f_i generate the unit ideal, this module becomes 0 if we localize by these elements. This tells us that $S/T = 0$, which means that S is an R-algebra of finite type. Therefore S' is finite over R and x is integral over R. □

Theorem 4.1.7. *Let $R \in S$ be an extension of rings and assume that $\phi : R \to S$ is integral. Then the map* $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ *is surjective.*

Proof. We already know that a prime ideal p of R is in the image of this map if and only if $\operatorname{Spec}(S \otimes_R \kappa(p))$ is nonzero. Equivalently, $S_p/pS_p = (S/pS)_p \neq 0$ or $p = \phi^{-1}(pS)$. We want to use the last characterization. Since the localized ring R_p embeds into S_p and is also integral, we can replace R and S by R_p and S_p and assume that $\phi : R \to S$ is an integral morphism of local rings (R, m) and (S, n). Now it is enough to show that $m \neq mS$. Suppose that this is wrong, and write $1 = \sum f_i s_i$ as a linear combination of elements of m and S. Since all the involved elements s_i are integral over R, there exists a subalgebra S' of S that contains these elements and is finite over R. But the equation $1 = \sum f_i s_i$ implies that $S' = S'm$. Then we can apply Nakayama's lemma to get $S' = 0$, a contradiction. □

Not only is the map of spectra surjective for integral ring morphisms, there are also restrictions to different prime ideals having the same image:

Lemma 4.1.8. *Let $\phi : R \to S$ be integral. Then two distincet prime ideals q, q' of S which map to the same prime ideal by the corresponding map of spectra cannot be contained in each other.*

Proof. If q and q' map both to the prime ideal p of R, then by the discussion at the beginning these two correspond to ideals of the ring $S \otimes_R \kappa(p)$. Thus, it suffices to prove the following assertion: Let k be a field, and S an k-algebra over k. If S is integral over k and a domain, then it is already a field. To see this, let s be an element of S. Since s is integral over k we can find a finite dimensional subalgebra S' containing S. But a finite dimensional domain over a field is a field because the multiplication by every nonzero element induces an isomorphism. This completes the proof. □

Corollary 4.1.9. *Let $\phi : R \to S$ be integral, and let $p \subset p'$ be prime ideals of R. Assume that there exists a prime ideal q of S that gets mapped to p. Then there also exists a prime ideal q' of S that contains q and gets mapped to p'.*

Proof. By similar considerations as before we can replace R and S by R/p and S/q. Therefore we can assume that both rings are domains and that p' is a prime ideal of R. We need to show that there exists a prime ideal q' that is in the inverse image of p'. But this is the case since our morphism was integral. □

We can say even more about these maps:

Lemma 4.1.10. *Let $\phi : R \to S$ be a finite morphism. Then the fibers of* $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ *are finite. In particular, every integral morphism of rings has only finite fibers.*

Proof. For every prime ideal p of R we have to investigate the spectrum of the ring $S \otimes_R \kappa(p)$. Since the morphism from R to S is finite, the ring $S \otimes_R \kappa(p)$ is finite over R as well. Notably, it is finitely generated as well, which tells us that it must be noetherian since R was assumed to be noetherian. By the lemma above every prime ideal of $S \otimes_R \kappa(p)$ is minimal. But it is well known that noetherian rings only have finitely many such prime ideals, therefore the fibers must be finite. □

We need a few notions. Let $\phi : R \to S$ be a ring map. Going up: We say that ϕ satisfies the going up-property if for all prime ideals $p \subset p'$ of R and any prime ideal q of S lying above p there exists a prime ideal q' of S that contains q and lies above p'. Going down: Similarly we say that ϕ satisfies the going down-property if for all prime ideals $p \subset p'$ of R and any prime ideal q' lying above p' there exists a prime ideal q of S that is contained in q' and lies above p. With this terminology, we have proven that an integral ring map satisfies the going up property. This implies:

Corollary 4.1.11. *Let $\phi : R \to S$ be integral. Then* $\dim(R) \leq \dim(S)$.

Proof. Since the map ϕ is integral, the going up property tells us that we can find for every chain of prime ideals of length n also an associated chain of prime ideals in S of the same length. The claim is then obvious. □

An interesting question arises: When is the converse true? Which morphisms satisfy the going down property? The most important case are flat morphisms. Before stating the theorem and proof we need a few lemmas:

Lemma 4.1.12. *(Flatness is a local property) Let $\phi : R \to S$ be a ring map, and M an S-module. Then M is a flat R-module if and only if the localization M_m is a flat R_p-module for all (prime) maximal ideals m of S ($p = m \cap S$).*

Proof. We use the fact that localization is exact and commutes with tensor products: Assume that M_m is a flat R_p-module for all (prime) maximal ideals m of S. Let I be an ideal of R. We need to show that the map $I \otimes_R M \to M$ is injective. Let us consider this as a map of S-modules. By assumption the localization $(I \otimes_R M)_m \to M_m$ is injective because $(I \otimes_R M)_m = I_p \otimes_{R_p} M_m$. Therefore the kernel of $I \otimes_R M \to M$ is 0, so M is a flat R-module. Conversely, let M be a flat R-module. Let p be a prime ideal of R and let q be a prime ideal of S that lies above p. Let I be an ideal of R_p. Similarly as before we need to show that $I \otimes_{R_p} M_q \to M_q$ is injective. If we write I as J_p for some ideal J of R, then we see that localizing the map $J \otimes_R M \to M$ at p gives exactly the map we had before. Since localization is exact, M_q is a flat R_p-module. □

Theorem 4.1.13. *Let $\phi : R \to S$ be a flat morphism. Then the going down property holds:*

Proof. We assume that we have prime ideals $p \subset p'$ of R and a prime ideal q' of S that lies above p'. First we want to reduce to the local case. Since ϕ is a flat morphism, we apply the theorem from before to get that the local ring morphism $R_{p'} \to S_{q'}$ is also flat. We know that the module $S \otimes_R \kappa(m)$ is nonzero. Since the R-module M is faithfully flat if and only if the module $mM \neq M$ for all maximal ideals m of R, then it is clear that ϕ is even faithfully flat. In particular, this means that for the prime ideal pR'_p of $R_{p'}$ the ring $S_{q'} \otimes_R \kappa(pR'_p)$ is nonzero. But the spectrum of this ring consists of exactly the prime ideals that lie above $pR_{p'}$. Per construction, the inverse image of this ideal in S is then mapped to p. □

Corollary 4.1.14. *Let $\phi : R \to S$ be a flat morphism. Then* $\dim(R) \geq \dim(S)$.

The next logical question to ask would be whch assumptions on a ring map $R \to S$ ensure that the Krul ldimension of R and S are equal. Clearly a necessary and sufficient condition is that both the going up and the going down property are satisfied. For integral ring morphisms we were able to ensure the going up property. In addition, if we assume that the integral morphism $\phi : R \to S$ embeds into S and that R is normal, then the going down property is also satisfied:

Theorem 4.1.15. *Let $\phi : R \in S$ be an integral ring extension, assume that R is normal and S an integral domain. Then the dimensions of R and S coincide.*

Proof. Clearly it is enough to show that the going down property is satisfied between R and S. Let us assume therefore that $p \subset p'$ are two prime ideals in R, and that q' is a prime ideal of S that contracts to p'. We need to show there exists a prime ideal $q \subset q'$ that also contracts to p.

The map $S \to S_{q'}$ is injective, so we can look at the composition map $R \to S_{q'}$. If we can find a prime ideal that lies over p under this morphism, the contraction of this ideal in S would give us the desired ideal q. By similar considerations as before p has a non-empty preimage in $S_{q'}$ if and only if $\phi^{-1}(pS_q) = pS_q \cap R = p$. It is clear that p is contained in $pS_q \cap R$. Let x be an element of $pS_q \cap R$. Then x can be written as $= \frac{s}{t}$ with $s \in pS$ and $t \in S\prime$. Equivalently, we can write $s = tx$.

Because ϕ is integral, every element of pS is integral over p, so that we get a monic polynomial P with coefficients from p that annihilates s. We now use the fact that the minimal polynomial of a ring extension of a normal ring has coefficients which belong all to the normal ring. We denote this polynomial by $P(z) = z^m + r_1 z^{m-1} + ... + r_m \in R[z]$. Similarly, the minimal polynomial $\widetilde{P}$ of the element t in the integral extension only has coefficients in R. If we write $\widetilde{P}$ as $z^n + \widetilde{r_1} z^{n-1} + ... + \widetilde{r_n}$, then we can observe that not all the coefficients $\widetilde{r_i}$ can lie in p. Otherwise $t^n = \sum_{i<n} r_j t^j \in pS \subset p'S \subset q'$, a contradiction.

Since $s = tx$, the polynomial $Q(z) = z^n + x\tilde{r_1} z^{n-1} + ... + x^n \tilde{r_n}$ is the minimal polynomial of s. But then Q divides P, and it can be shown that this implies the following for the coefficients: $x^i \tilde{r_i}$ is contained in the radical ideal generated by $r_1, \ldots, r_m$. Because not all the coefficients of $\widetilde{P}$ belong to p, this implies $x \in p$. □

4.2 Morphisms between rings

Theorem 4.2.1. *Let $\Phi : R \to S$ be a morphism between noetherian rings, q a prime ideal of S and $p = q^c$ the corresponding prime ideal in R. Then* $\mathrm{ht}(q) \leq \mathrm{ht}(p) + \dim(S_q/pS_q)$.

Proof. First we reduce the inequality to the local case and then use the varying characterizations of the Krull dimension for local rings. Replacing R and S by R_p and S_q we can assume that (R, m) and (S, n) are noetherian local rings, and that $\phi(m) \subset n$. Thus we have to prove the inequality $\dim(S) \leq \dim(R) + \dim(S/mS)$. Let $\dim(R) = s$ and $r_1, \ldots, r_s$ a system of parameter. Then there exists an integer $k \in \mathbb{N}$ such that $m^k \subset \langle r_1, \ldots, r_s \rangle_R$. But then $(mS)^k \subset \langle \phi(r_1), ..., \phi(r_s) \rangle_S$. Let $\dim(S/mS) = t$ and $s_1, \ldots, s_t \in S$ such that $s_1 + mS, \ldots, s_t + mS$ is a parameter system of S/mS. Claim: $n = \sqrt{\phi(r_1), ..., \phi(r_s), s_1, \ldots, s_t}$.

It is equivalent to show that these s+t elements $\phi(r_1), \ldots, \phi(r_s), s_1, ..., s_t$ are a parameter system of S. Let therefore $x \in n$. Then there exists an integer $l \in \mathbb{N}$ such that $(x + mS)^l \in < s_1 + mS, ..., s_t + Ms >_{S/mS}$. Therefore we have $x^l \in \langle s_1, ..., s_t \rangle_S + mS$ and thus $x^{kl} \in \langle s_1, ..., s_t \rangle_S + (mS)^k \subset \langle \phi(r_1), ..., \phi(r_s), s_1, ..., s_t \rangle$. Therefore $\dim(S) \leq s + t$. □

The inequality above can be made into an equality under specific conditions. Such a sufficient condition is the going-down theorem between R and S:

Theorem 4.2.2. *Let $\Phi : R \to S$ be a morphism between noetherian rings, q a prime ideal of S and $p = q^c$ the corresponding prime ideal in R. Assume that the going-down theorem holds between R and S. Then* $\mathrm{ht}(q) = \mathrm{ht}(p) + \dim(S_q/pS_q)$.

Proof. By the theorem above it is enough to show that $\dim(S) \geq \dim(R) + \dim(S/mS)$.
Let $\dim(S/mS) = t$ and $q_t \subset \cdots \subset q_0 = n$ be a strictly increasing chain of prime ideals in S between mS and n. We know that $q_i \cap R = m$ for all i. Let $\dim(R) = s$ and $p_r \subset ... \subset p_0 = m$ a maximal chain of prime ideals in R. By the going-down theorem we can construct a strictly increasing chain of prime ideals $q_{s+t} \subset ... \subset q_{t+1} \subset q_t$ in S such that $q^c_{t+i} = p_i$. Thus $\dim(S) \geq s + t$. □

4.3 Dimension inequality for noetherian integral domains

The following theorem by Cohen describes a dimension inequality relating the height of a prime ideals in an extension of a domain to the height of the corresponding contracted prime ideal.

Theorem 4.3.1. *Let $R \subset S$ be two noetherian domains, q a prime ideal of S and $p = q^c = q \cap R$ the corresponding contracted prime ideal. Then the following inequality holds:*

$$\mathrm{ht}(q) \leq \mathrm{ht}(p) - \mathrm{trdeg}_{\kappa(p)}\kappa(q) + \mathrm{trdeg}_R S.$$

Proof. Without loss of generality we assume that S is finitely generated over R. Assume that the right hand side is finite. If $m \in \mathbb{N}$ such that $m \leq \mathrm{ht}(q)$, then there exists a chain of prime ideals $q_0 \subsetneq \cdots \subsetneq q_m = q$ in T, and we can take $a_i \in q_i/q_{i-1}$ for all $i = 1, \ldots, m$. If $n \in \mathbb{N}$ such that $n \leq \mathrm{trdeg}_{\kappa(p)}\kappa(q)$ then let $b_i, ..., b_n \in B$ such that their images modulo q are algebraically independent over A/p. Set $S' = A[a_1, \ldots, a_m, b_1, \ldots, b_n]$. If the theorem holds for S', then we have $m + n \leq \mathrm{ht}(q) + \mathrm{trdeg}_R S\prime \leq \mathrm{ht}(p) + \mathrm{trdeg}_R S$. Letting m and n vary we get the desired inequality.
By induction we can assume that S is generated over R by one element: $R \subset S = R[x]$. Replacing R by R_p and S by $S_p = R_p[x]$ e we can assume that (R, p) is a local ring with maximal ideal p. Now let $k = R/p$ and write $S = R[X]/Q$. If $Q = 0$, then $S = R[X]$ and by the theorem above we have $\mathrm{ht}(q) = \mathrm{ht}(p) + \dim(q/pS)$. But we know that $S/pS = k[x]$, therefore we have either $q = pS$ or $\dim(q/pS) = 1$. In both cases the inequality holds. Suppose that $Q \neq \{0\}$. Then $trdeg_R S = 0$. Since $R \subset S$ we have $Q \cap R = 0$. Write $K = Quot(R)$, then $\mathrm{ht}(Q) = \mathrm{ht}(QK[X]) = 1$. Let $q\prime$ be the inverse image of q in $R[x]$. Then $q = q\prime/p$, and $\kappa(q) = \kappa(q\prime)$. We conclude that $\mathrm{ht}(q) \leq \mathrm{ht}(q\prime) - \mathrm{ht}(Q) = \mathrm{ht}(q\prime) - 1 = \mathrm{ht}(p) + 1 - \mathrm{trdeg}_{\kappa(p)}\kappa(p\prime) - 1 = \mathrm{ht}(p) - \mathrm{trdeg}_{\kappa(p)}\kappa(p)$. □

With the same conditions as above, one says that this dimension formula holds between R and S if one has equality for all prime ideals q in S. We have used it in chapter 3 to characterize universally catenary rings.

4.4 The dimension of fibers

The behaviour of the dimension of fibers of ring morphisms is a well-studied topic. We will use a lemma of Grothendieck to ultimately proof that under mild assumptions the fiber dimension is upper-semicontinuous.
The following two lemmas will be needed:

Lemma 4.4.1. *(Generic Freeness lemma) Let R be a noetherian domain and S a finitely generated R-algebra. If M is a finitely generated S-module, then there exists a nonzero element $a \in R$ such that $M[a^{-1}]$ is a free $R[a^{-1}]-$module. If in addition $S = S_0 \oplus S_1 \oplus \dots$ is positively graded with R acting in degree 0, and M is a graded S-module, then one can take a such that each graded component of $M[a^{-1}]$ is free over R.*

Proof. Let $\kappa(R)$ be the quotient field of R. We apply induction on the dimension of $S \otimes_R \kappa(R)$. We start with the case $S \otimes_R \kappa(R) = 0$, so the dimension can be seen as 1. Then $1 \in S$ is annihilated by some nonzero element a of R, and therefore M is annihilated by a too. Therefore $M[a^{-1}] = 0$, and the claim is fulfilled.
Let us apply the Noether normalization theorem to $S \otimes_R \kappa(R)$, which is finitely generated over the field $\kappa(R)$. Then there exists elements algebraically independent elements $x_1, \dots, x_n$ of $S \otimes_R \kappa(R)$ such that $S \otimes_R \kappa(R)$ is a finitely generated module over $K[x_1, \dots, x_n]$. If S is graded as written in the statement, we can choose without loss of generality the elements x_i to be homogeneous. But then we can further assume that they all belong to S after multiplying them by suitable elements from R. Let S be generated as an R-algebra by $r_1, ..., r_t$, then each b_i satisfies an integral equation over $K[x_1, ..., x_n]$. We can write this as a polynomial equation over R by clearing the denominator. Let c_i be the leading coefficient, and let a be the product of all of them. Then we deduce that $S[a^{-1}]$ is integral and therefore a finitely generated module over $S' = R[a^{-1}][x_1, ..., x_n]$. The module $M' = M[a^{-1}]$ is then finitely generated over S' as well.
It is possible to find a finite filtration of M' by S'-submodules $M' = M'_1 \supset \dots \supset M'_{s+1}$ such that $M'_i/M'_{i+1} = S'/q'_i$ for prime ideals q_i of S'. If q_i is nonzero, then we have the strict inequality $\dim(S'/q_i \otimes_R \kappa(R)) < n$, so by induction there is an element a_i of R such that $S'/q_i[a_i^{-1}]$ is free over $R[a_i^{-1}]$. If $q_i = 0$, then $S'/q_i = S'$ is a free $R[a^{-1}]$-module, and we set $a_i = a$ in this case. Over the ring $R[(a_1 a_2 \cdots a_s)^{-1}]$, the module $M[(a_1 a_2 \cdots a_s)^{-1}]$ has a finite filtration by free R-modules and is therefore free. If S and M are graded, then the q_i can be taken to be homogeneous. In this case the homogeneous components of $S'/q_i[a^{-1}]$ are free over R, and the claim is true. □

Lemma 4.4.2. *Let R be a noetherian domain, and T a domain that contains R which is also finitely generated as an R-algebra. If q is a prime ideal of T and p the contraction of q in R,* *then we have the following inequality:*

$$\dim(T_q) \leq \dim(R_p) + \dim(\kappa(R) \otimes_R T).$$

With these lemmas, we can give a first corollary that strengthens a theorem which we had earlier:

Corollary 4.4.3. *Let R be a noetherian domain and S a finitely generated R-algebra containing R. Then there exists a nonzero element $a \in R$ such that for every prime ideal p of R that does not contain a and every prime ideal q of S that contracts to p we have the equality*

$$\dim(S_q) = \dim(R_p) + \dim(S_q/pS_q).$$

Proof. By the generic freeness lemma we can choose $a \in R$ such that $S[a^{-1}]$ is free over $R[a^{-1}]$. Let p be a prime ideal of R that does not contain a. The local ring R_p is a further localization of $R[a^{-1}]$, so that we have equality between $R_p \otimes_R S$ and $R_p \otimes_{R[a^{-1}]} S[a^{-1}]$. This is free as an R_p-module. It follows that $\kappa(R_p \otimes_R S) \neq (R_p \otimes_R S)$. Therefore it is contained in a prime ideal q' whose intersection with S is the desired q. The further localization$(R_p \otimes_R S)_{q'} = S_q$ gives a flat $(R_p \otimes_R S)-$module, and is therefore also flat over R_p. Since a flat module satisfies the going down property, we get the desired equality. □

This can be even more specialized:

Corollary 4.4.4. *Let $R \in S$ be an inclusion of affine domains over a field K. Let $d =$ $\mathrm{trdeg}_R(S.)$ Then there is an element a of R such that for each maximal ideal p of R that does not contain a there exists prime ideals q of S that contain p such that if q is minimal above pS, then we have dim$(S/q) = d$.*

Proof. We choose a as in the corollary above. Let p be a maximal ideal of R not containing a. Let $q, q_1, \ldots, q_n$ be the minimal prime ideals containing pS. By the prime avoidance lemma we can choose for each i an element that is contained in q_i but not in q. Now we use Hilbert's Nullstellensatz, so q is the intersection of maximal ideals. Let q' be a maximal ideal that contains q but not the product of the elements $f_1, \ldots, f_n$. It follows that $\dim(S'_q/pS'_q) = \dim(S'_q) - \dim(R_p) = d$. But by construction q' does not contain the prime ideals $q_1, \ldots, q_n$, so we get $\dim(S'_q/pS'_q) = \dim(S'_q/q'_q)$. But the dimension of S'_q/q'_q is the same as the dimension of S/q. This means that $\dim(S/q) = d$. □

The general statement about the dimension of fibers is as follows:

Theorem 4.4.5. *Let R be a noetherian universally catenary ring and S a finitely generated R-algebra. Then for each integer e we have:*
i)There is an ideal I_e of S such that if q is a maximal ideal of S and p the contraction of q in R, then $\dim(S_q/pS_q) \geq e \Leftrightarrow q \supset I_e$.
ii) If $S = S_0 \oplus S_1 \oplus \ldots$ is a positively graded algebra which is finitely generated over $R = S_0$, then there is an ideal J of R such that for all prime ideals p of R we have: $\dim(S \otimes_R \kappa(R/p)) \geq e \Leftrightarrow p \supset J_e$.

We remark that the assumption of R being universally catenary is not necessary, but simplifies the proof. One could alternatively reduce the general case to the one we cover.

Proof. It is clear that the statement is true if R is a field. Now we argue by induction as follows: If the theorem is false for some R and R-algebra S, then there exists an ideal I of R that is maximal among all those ideals such that the result is false for R/I and some R/I-algebra S'. We then replace R by R/I and assume that for all nonzero ideals J the result is true for R/J and every R/J-algebra S'. We notice that we can assume that R is contained in S, otherwise we can just factor out the kernel of the map $R \to S$.
i) For any prime ideal q of S we have that the dimension of S_q/pS_q is equal to the maximum of the dimension of $S_q/(pS_q + q')$, where the index runs over all minimal prime ideals q' of S. Therefore the set of all prime ideals q such that $\dim(S_q/pS_q)$ is bigger than or equal to e is the same as the union over the set of all prime ideals q that contain a minimal prime ideal q' of S for which $\dim(S_q/(pS_q + q'))$ is bigger than or equal to e. This is the reason why we can factor out one minimal prime ideal of S and the corresponding preimage in R to be able to assume that R and S are both domains.
Let now $d = \dim(\kappa(R) \otimes_R S)$. For every maximal ideal q of S we have $\dim(S_q/pS_q) \geq \dim(S_q) - \dim(R_p)$ by the lemmas and corollaries before. We can therefore take I_d to be the zero ideal. We choose a nonzero element a of R such that if a is not contained in q, then $\dim(S_q/pS_q) = \dim(S[a^{-1}]) - \dim(R[a^{-1}])$. We see that the strict inequality $\dim(S_q/pS_q) > \dim(S) - \dim(R)$ is only satisfied if a is contained in q.
By induction, this is true for the R/aR-algebra S/aS. If I'_e is the ideal corresponding to the induced map, then for all integers $e > d$ we can take I_e to be the preimage of I'_e in S.
ii) We notice that R can be taken to be a domain. Indeed, if R was not a domain, then by induction there would be for every minimal prime ideal p of R and integer e an ideal $I_{e,p}$ containing p such that the dimension of $S \otimes_R \kappa(R/p')$ would be bigger than or equal to e if and only if p' contains $I_{e,p}$ for prime ideals p' of S that contain pS. In this case we could take I_e to be the intersection of all ideals of the form $I_{e,p}$ for minimal prime ideals p of R.
We have already seen that the dimension of $S \otimes_R \kappa(R/p)$ can be computed by the degree of the Hilbert polynomial that agrees with the numerical function $\dim_{\kappa(R/p)}(S_n \otimes_R \kappa(R/p))$ for large enough n. Using Nakayama's lemma we see that this value $\dim_{\kappa(R/p)}(S_n \otimes_R \kappa(R/p))$ is the number of elements which are needed to generate the R_p-module $(S_n)_p$. This gives the estimate $\dim_{\kappa(R/p)}(S_n \otimes_R \kappa(R/p)) \geq \dim_{\kappa(p)}(S_n \otimes_R \kappa(R))$. We conclude that for all prime ideals p of R we have $\dim(S \otimes_R \kappa(R/p)) \geq \dim(\kappa(p) \otimes_R S)$. If $d = \dim(\kappa(p) \otimes_R S)$, then we can assume $I_d = 0$.
We now choose a nonzero element a of R such that if Q does not contain a than each $S_n[a^{-1}]$ is free over R. This implies that for each prime ideal p of R that does not contain a the R_p-module $(S_n)_p$ is free of rank $\dim_{\kappa(R)}(S_n \otimes_R \kappa(R))$. Therefore the Hilbert polynomials of $S \otimes_R \kappa(R/p)$ and of $S \otimes_R \kappa(R)$ are the same. Evidently, the dimension of $S \otimes_R \kappa(R/p)$ and $S \otimes_R \kappa(R)$ are also the same. This implies that any prime ideal p of R that satisfies $S \otimes_R \kappa(R/p) > S \otimes_R \kappa(R)$ must contain a.
By induction, the result is true for the R/aR-algebra S/aS. As before, if I'_e is the ideal of S/aS corresponding to the induced map, then for $e > \dim(S \otimes_R \kappa(R))$ we can take I_e to be the preimage of I'_e in S. □

Chapter 5

Hilbert function, Hilbert polynomial and multiplicity

We devote the last chapter to the theory of Hilbert functions. By adding additional structure to rings and modules, namely a grading, one can use tools from homological algebra as well as the theory of Gröbner bases to study the dimension of aa ring in grater detail. There are a number of remarkable results that benefited from the assistance of computer algebra systems. One should be aware that nevertheless there are a number of unsolved problems that resist a solution for quite some time now.

In our case, we want to look at noetherian graded rings $R = \sum_{i=0}^{\infty} R_i$. As seen in chapter 2, these rings R are then finitely generated as an algebra over R_0 by elements $x_1, ..., x_s$ of degree $k_1, \ldots, k_s$. In addition, we will assume R_0 to be artinian and local. For our purposes, this restriction will always be fullfilled, and is generally an accepted standard. Graded R-Modules $M = \sum_{i=0}^{\infty} M_i$ will be assumed to be finitely generated. In that case the research is particularly fruitful.

5.1 Hilbert functions and polynomials

Let $R = \sum_{i=0}^{\infty} R_i$ be a noetherian graded ring, and $M = \sum_{i=0}^{\infty} M_i$ a finitely generated graded module over R. Let λ be an additive function on the class of finitely generated R_0-modules (with values in $\mathbb{Z}$). Then one can form the generating function of $\lambda(M_n)$:

$$P(M,t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n \in \mathbb{Z}.$$

The resulting formal power series is called the Poincaré series of M with respect to λ. By a theorem of Hilbert and Serre, this power series can be written in a special form:

Theorem 5.1.1. *$P(M,t)$ is a rational function in t of the form $\frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}$, where $f(t) \in \mathbb{Z}[t]$, s denotes the minimal number of generators $x_1, \ldots, x_s$ of R as an R_0-algebra with degrees $k_1, \ldots, k_s$.*

Proof. By induction on s. The result is clear for $s = 0$.
Let now s be bigger than 0, and assume that the claim is true for $s-1$. The multiplication by the homogeneous element x_s of degree k_s gives a homomorphism of R-modules $M_n \to M_{n+k_s}$. We get the short exact sequence $0 \to L_n \to M_n \to M_{n+k_s} \to N_{n+k_s} \to 0$, where K_n is the submodule of elements of M_n that are annihilated by x_s and N_{n+k_s} is the quotient module obtained by modding out x_{n+k_s} from M_{n+k_s}.
The modules $L = \sum_{n=0}^{\infty} L_n$ respectively $N = \sum_{n=0}^{\infty} N_n$ are by construction $R_0[x_1, ..., x_{s-1}]$-modules. We can apply the additive function λ to the sequence and obtain $\lambda(L_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(N_{n+k_s}) = 0$. This gives $(1-t^{k_s})P(M,t) = P(N,t) - t^{k_s}P(L,t) + g(t)$, where $g(t)$ is a polynomial. Applying the hypothesis give the desired result. □

One denotes the order of the pole of $P(M,t)$ at $t = 1$ by $\mathrm{d}(M)$. The following case is of great importance:

Corollary 5.1.2. *If we assume that the degree k_i of each generator x_i of R is equal to 1, then $\lambda(M_n)$ is a polynomial in n with rational coefficients of degree $d(M) - 1$ for sufficiently large n.*

Proof. By the theorem above we know that $\lambda(M_n)$ is the coefficient of t^n of $f(t)(1-t)^{-s}$. If we write the polynomial $f(t)$ as $\sum_{i=0}^{s} a_i t^i$, then we can assume without loss of generality that $s = d(M)$ and $f(1) \neq 0$. From the identity

$$(1-t)^{-d(M)} = \sum_{k=0}^{\infty} \binom{d(M)+k-1}{d-1} t^k$$

we compute

$$\lambda(M_n) = \sum_{k=0} N a_k \binom{d(M)+n-k-1}{d-1}$$

for all $n \geq N$.
The right-hand side is a polynomial in n of degree $d(M)-1$ with leading coefficient $\frac{(\sum a_k)n^{d-1}}{(d-1)!}$. □

The polynomial which obtains the values of $\lambda(M_n)$ is called the Hilbert polynomial with respect to λ. Although it is a function with values in $\mathbb{Z}$, the coefficients are almost always from $\mathbb{Q}$. Integer-valued polynomials have a very rstricted form:

Lemma 5.1.3. *Let $f(x)$ be a polynomial in $\mathbb{Q}[x]$. The following are equivalent:*
i) f is a $\mathbb{Z}$-linear combination of binomial polynomials of the form $\binom{x}{k} = \frac{x(x-1)...(x-k+1)}{k!}$
ii) f has values in $\mathbb{Z}$ for all integers $n \in \mathbb{Z}$.
iii) f has values in $\mathbb{Z}$ for all large enough integers $n \in \mathbb{Z}$.
iv) The standard difference operator $\triangle$ applied to f is a function that satisfies i) and for which at least one value of $\triangle(f)$ lies in $\mathbb{Z}$.

Proof. i) $\Rightarrow$ ii): It is clear that the binomial polynomial Q_k has only value in $\mathbb{Z}$.
ii) $\Rightarrow$ iii): clear.
i) $\Rightarrow$ iv): clear.
iv) $\Rightarrow$ i): If $\triangle\ (f)$ is a $\mathbb{Z}$-linear combination in the $\binom{x}{k}$, then we can write $\triangle\ f(x) = \sum e_k\binom{x}{k}$ with $e_k \in \mathbb{Z}$. Since the binomial polynomials satisfy $\triangle\ (\binom{x}{k}) = \binom{x}{k-1}$ for $k > 0$, we can write $f(x) = \sum e_k\binom{x}{k+1} + e_0$, where we now have to assume that $e_0 \in \mathbb{Q}$. But we can use the fact that f obtains the value of an integer at least once, which implies that e_0 lies actually in $\mathbb{Z}$.
iii) $\Rightarrow$ iv): We argue by induction on the degree of f. If we apply the induction assumption to $\triangle\ (f)$, then we see that $\triangle\ (f)$ satisfies property i), and therefore also property iv). This completes the proof. □

We have just seen that a polynomial $f \in \mathbb{Q}[x]$ satisfying one of the equivalent statements of the lemma can be written as $\sum e_k\binom{x}{k}$. The coefficients are usually written as $e_k(f)$ to stress for which polynomial f they are used in the decomposition.
The coefficients $e_k(f) \in \mathbb{Z}[x]$ can be written as $\triangle^k\ (f)$ and therefore we have $f(x) = e_k(x)\frac{x^k}{k!} + g(x)$ with $\deg(g) < \deg(f)= k$. This implies that $f(n)$ is asympthotically equal to $e_k(f)\frac{n^k}{n!}$ as n. Therefore the leading coefficient of f is positive if and only if the evaluation of the polynomial is bigger than 0 for all large enough n.
A polynomial with values in $\mathbb{Z}$ that agrees with an integer-valued polynomial for large enough n is called a polynomial-like function. We have seen that any additive function with values in $\mathbb{Z}$ on the class of finitely-generated modules over a noetherian graded ring is an example of such a polynomial-like function.
There is a close connection between functions of this special form and the difference operator $\triangle$:

Lemma 5.1.4. *The following are equivalent: i)f is polynomial-like.*
ii) $\triangle\ (f)$ is polynomial-like.
iii) There exists an integer $r \geq 0$ such that $\triangle^r\ (f(n)) = 0$ for sufficiently large n.

Proof. i) $\Rightarrow$ ii): clear.
ii) $\Rightarrow$ iii): clear.
ii) $\Rightarrow$ i): Assume that $\triangle\ (f)$ is equal to an integer-valued polynomial $p_{\triangle(f)}$ for large enough n. By lemma 5.1.3. there exists an integer-valued function $q(x)$ that is equal to $\triangle\ (p_{\triangle(f)})$. The function $g : n \rightarrow f(n) - q(n)$ satisfies $\triangle\ (g) = 0$ for high enough n. Therefore it is a constant equal to e_0 for large enough values. We can therefore write $f(n) = g(n) + e_0$ for large n enough, which implies that f is a polynomial-like function.
iii) $\Rightarrow$ i): This follows from applying the argument of ii) $\rightarrow$ i r times. □

By abuse of notation one usually writes $\deg(f) = \deg(p_f)$ for a polynomial-like function f that agrees with the integer-valued polynomial p_f from a point on. Similarly, one denotes by $e_k(f)$ the coefficient $e_k(p_f)$. Consider the following twoi cases:
Let M be a finitely generated module over a noetherian graded ring R, let l be the length function which assigns to each graded part M_n of M the length $l(M_n)$. The function $n \rightarrow l(M_n)$ is denoted by $\chi(M, n)$. Its properties were used in chapter 2 to prove the fundamental theorem of dimension theory. if R is assumed to be a quotient of a polynomial ring, then additional

statements can be proven. We will do this later. Now let M be a finitely generated module over the noetherian ring R, and q an ideal of R. We assume that the length of M/qM is finite. If $M \supset M_1 \supset ... \supset M_n \supset ...$ denotes a q-filtration of M, then the R-modules M/qM_n all have finite length, and therefore the function $n \to l(M/M_n)$ is well-defined. We have already seen that this is a polynomial-like function by passing to the associated graded rings.
In the special case that (R, m) is a noetherian local ring, q an m-primary ideal, and the filtration of M is given by setting $M_n = q^n M$, then one calls the function $n \to l(M/q^n M)$ the Hilbert-Samuel function of M. The corresponding integer-valued polynomial is denoted by χ_q^M, it is called the Hilbert-Samuel polynomial or the characteristic polynomial of the m-primary ideal q.
We have already seen the relation of these notions to the concept of dimension. However, the relation between the Hilbert function and the Hilbert polynomial itself is a subject of interest. Although the definition is so simple, many problems immediately arise. For example, a sharp bound for the minimal number $n \in \mathbb{N}$ such that the Hilbert function and polynomial coincide fro mthis point onwards is not known. The coefficients of these functions are also of great interest. They can be used to characterize certain classes of local rings. The coefficient of the highest term even has its own name; it is called the multiplicity of the module, and will be discussed separately later. Two well-known questions that arise from the study of Hilbert functions are the following:
Given a function $f : \mathbb{Z} \to \mathbb{Z}$, does there exist a module whose Hilbert function equals f? Which modules possess the same Hilbert polynomial?
Both are open problems, nevertheless partial results are known. We will prove a significant statement related to the second question later.

5.2 The multiplicity of a module

The multiplicity of a finitely generated module over a noetherian local ring is an important notion with a number of applications. It can be defined via the corresponding Hilbert polynomial of the Hilbert function. We will treat the graded and the nongraded case separately.
Graded case: Let R be a noetherian graded ring, and let M be a finitely generated graded R-module. Assume that the dimension of M is equal to d. Then the Hilbert polynomial $p_M(x) \in \mathbb{Q}$, for which the length of M_n is equal to $p_M(n)$ for large n enough, can be written as

$$p_M(x) = \sum_{i=0}^{d-1} (-1)^{d-i-1} e_{d-i-1} \binom{x+i}{i}.$$

The multiplicity of M is defined as $e(M) = e_0$ if d is larger than 0, and otherwise as the length of M. The coefficients of the Hilbert polynomial, in particular also the multiplicity, can be recovered from the Hilbert series, which is another name for the Poincaré series, used when the additive function λ is the length function of a module.

Lemma 5.2.1. *Let R be a noetherian graded ring such that R_0 is artinian and R is finitely generated R_0-algebra. Let M be a finitely generated module over R of dimension d. Let $\frac{f_M(t)}{(1-t)^d}$*

be the Hilbert series of M with polynomial $f_M(t) \in \mathbb{Q}[t]$. Then $e_i = \frac{f_M^{(i)}(1)}{i!}$ In particular, the multiplicity of M is given by $f_M(1)$.

Proof. Let us write

$$\frac{f_M(t)}{(1-t)^d} - \sum_{i=0}^{d-1} \frac{(-1)^i}{i!} \frac{f_M^{(i)}(1)}{(1-t)^{d-1}} = \frac{D(t)}{(1-t)^d},$$

where $D(t) = f_M(t) - \sum_{i=0}^{d-1} \frac{(-1)^i}{i!} \frac{f_M^{(i)}(1)}{(1-t)^{d-1}}$ denotes the remainder term of the Taylor series of $f_M(t)$ up to degree d. Since $D^{(j)}(1) = 0$ for $j = 0, ..., d-1$, the element $D(t) \in \mathbb{Z}[t, t^{-1}]$ is divisible by $(1-t)^d$. We conclude that the coefficients of the Hilbert series of M and of the expression $\sum_{i=0}^{d-1} \frac{(-1)^i}{i!} \frac{f_M^{(i)}(1)}{(1-t)^{d-1}}$ coincide for large enough n. Therefore

$$\sum_{i=0}^{d-1} \frac{(-1)^i}{i!} \frac{f_M^{(i)}(1)}{(1-t)^{d-1}} = \sum_{n \geq 0} p_M(n) t^n,$$

since the coefficients on both sides are polynomials in n which are equal for sufficiently large n, and therefore must be equal in general. Rewriting the left-hand side as a power series in t and comparing coefficients yields $e_i = \frac{f_M^{(i)}(1)}{i!}$. This shows that the multiplicity of M is indeed equal to $f_M(1)$. If d is equal to 0, then the Hilbert function of M is equal to the polynomial $f_M(t)$. In this case the identity $f_M(1) = l(M) = e(M)$ still holds. □

Ungraded case: Let (R, m) be a noetherian local ring of dimension d, q an m-primary ideal, and M a finitely generated R-module. We have seen that the Hilbert-Samuel function $n \to l(M/q^n M)$ is equal to the Hilbert-Samuel polynomial χ_M^q for large enough n. Let us write $\chi_M^q(n) = \frac{e_0}{d!} n^d +$ (terms of lower order). The multiplicity of M is defined as the coefficient e_0. We will denote it by $e(q, M)$. From the definition we get the following formula: $e(q, M) = \lim_{n \to \infty} \frac{d!}{n^d} l(M/q^n M)$. It is easy to see that the multiplicity is an additive function:
Indeed, if $0 \to M \to N \to O \to 0$ is a short exact sequences of finitely generated R-modules, than $e(q, N) = e(q, M) + e(q, O)$. Furthermore, there is the following decomposition into products of multiplicities and length of related modules:

Theorem 5.2.2. *Let (R, m) be a noetherian local ring with m-primary ideal q and minimal prime ideals $p_1, ..., p_r$ such that $dim(R/p_i) = d$. If we denote by $\bar{q}_i$ the image of q in R/p_i and by $l(M_{p_i})$ the length of M_{p_i} as an R_{p_i}-module, then*

$$e(q, M) = \sum_{i=1}^{r} e(\bar{q}_i, R/p_i) l(M_{p_i}).$$

Proof. By induction on the value of $\sum_i l(M_{p_i})$. If this sum is 0, then the dimension of M is smaller than d, and both sides are 0. Suppose the sum is equal to $\sigma > 0$. Then there exists an element p of the list $p_1, ..., p_r$ such that the localized module M_p is nonzero. It follows that p is a minimal element of the support of M, and therefore also an associated prime.

Therefore M contains a submodule N that is isomorphic to R/p. We apply the additivity of the multiplicity to get $e(q, M) = e(q, N) + e(q, M/N)$. If we localize N at p, we get the module $N_p = R_p/pR_p$. However localizing N at any other minimal prime ideal of the list gives the zero module. Therefore the length of N_p is equal to 1.
Since the value of the sum has decreased by one, we can apply the induction hypothesis and assume that the theorem holds for the module M/N. Then we have the equality $e(q, N) = e(q, R/q) = e(\bar{q}, R/)$. Applying once more the additivity of the multiplicity gives the formula for M. □

There are many theorems relating the multiplicity of a module with the length of specific quotient modules. Next we will show one such important formula, and state a similar but much more stronger result afterwards:

Theorem 5.2.3. *Let (R, m) be a noetherian local ring of dimension d, q an m-primary ideal, $x_1, ..., x_d$ a system of parameters that is contained in q, M a finitely generated R-module and $v_1, \dots, v_d \in \mathbb{N}$. Suppose that $x_i \in q^{v_i}$ for all i. Then for $s = 1, \dots, d$ we have*

$$e(q/\langle x_1, \dots, x_s\rangle, M/\langle x_1, \dots, x_s\rangle M) \geq v_1 \cdot \dots \cdot v_s e(q, M).$$

In particular, for $s = d$, we have $l(M/\langle x_1, \dots, x_s\rangle M) \geq v_1 \cdot \dots \cdot v_d e(q, M)$.

Proof. By induction it is enough to prove this for $s = 1$. Let $R^{'} = R/\langle x_1\rangle$, $q^{'} = q/\langle x_1\rangle$, $M^{'} = M/\langle x_1\rangle M$ and $v = v_1$. Then the dimension of $R^{'}$ is equal to $d - 1$. The length of the module $M^{'}/q^{'}M^{'}$ is equal to the length of $M/(x_1M + q^nM)$. In turn this is the same as the length of $(x_1M + q^nM)/q^nM$. Now we use the fact that $(x_1M + q^nM)/q^nM$ is isomorphic to $x_1M/x_1M \cap q^nM$ and $M/(q^nM : x_1)$ and that $q^{n-v}M \subset (q^nM : x_1)$ to get the inequality $-l(x_1M + q^nM)/q^nM) \geq -l(M/q^{n-v}M)$.
The length of $M^{'}/q^{'}M^{'}$ is therefore bounded from below by the difference between the length of M/q^nM and of $M/q^{n-v}M$. The right-hand side is of the following form:

$$\frac{e(q, M)}{d!}(n^d - (n - v)^d) + \text{(polynomial in n of degree d-2)} =$$
$$\frac{e(q, M)}{d!}vn^{d-1} + \text{(polynomial in n of degree d-2)}$$

Therefore the claim is true. □

This theorem has an appealing form in the following special case:

Corollary 5.2.4. *Let (R, m) be a noetherian local ring of dimension d, let $x_1, ..., x_d$ be a system of parameters, let q be the ideal generated by these elements and $v \in \mathbb{N}$. Then $l(R/q) \geq e(q, R)$. If in addition $x_i \in m^v$ for all i, then $l(R/q) \geq v^d e(q, R)$.*

One can describe many other formulas using multiplicities of a module, we want to end this section by introducing Lech's lemma. It describes the multiplicity as a limit involving the length of quotient modules:

Lemma 5.2.5. *([L]) Let (R, m) be a noetherian local ring of dimension d, and let $x_1, \dots, x_d$ be a system of parameters. Let q be the ideal generated by these elements and assume that M is a finitely generated R-module. Then*

$$e(q, M) = \lim_{min\ v_i \to \infty} \frac{l(M/(x_1^{v_1}, ..., x_d^{v_d})M)}{v_1 ... v_d}$$

5.3 Characterizing classes of rings via multiplicities

Now we can show that Cohen-Macaulay local rings can be characterized by the multiplicity:

Theorem 5.3.1. *The following are equivalent:*
i) (R, m) is a noetherian Cohen-Macaulay local ring.
ii) For all parameter ideals q of R the equality $l(R/q) = e(q)$ holds.
iii) There exists a parameter ideal q of R such that $l(R/q) = e(q)$.

Proof. *i)* $\to$ *ii)*: Let $x_1, \dots, x_d$ be a system of parameters and q the parameter ideal generated by these elements. Because R is Cohen-Macaulay, we know that the associated graded ring $G_q(R)$ is equal to $(R/q)[X_1, \dots, X_d]$. We can calculate that $\chi_M^q(n) = l(R/q)\binom{n+d}{d}$, so that $l(R/q) = e(q)$ follows.
ii) $\Rightarrow$ *iii)*: Obvious.
i) $\Rightarrow$ *ii)*: Let $q = \langle x_1, \dots, x_d \rangle$ be a parameter ideal satisfying $l(R/q) = e(q)$. We denote by $B := (R/q)[X_1, ..., X_d]$. Then there exists a homogeneous ideal b of B such that $G_q(R) \cong B/b$. Denote by ϕ_B and ϕ_b the Hilbert polynomials of B and B/b. Computation shows that $\phi_B(n) = l(R/q)\binom{n+d-1}{d-1}$ and that for large enough n one has $l(q^n/q^{n+1}) = \phi_B(n) - \phi_b(n)$. The left-hand side is a polynomial in n of degree $d-1$, and the coefficient of m^{d-1} is equal to $\frac{e(q)}{(d-1)!}$. Since $e(q) = l(R/q)$ by the assumption, it follows that the degree of the polynomial ϕ_b can be at most $d-2$. We assume that b is nonzero, and take an element $f(x)$ from b. Without loss of generality we assume that $f \neq 0$ but $m/qf = 0$. This is possible for the following reason: Let $q^r \subset m$. If we denote by $\widehat{m} = m/q$, then in B we have $\widehat{m}^r = \{0\}$. Therefore we can replace f by the product of f with a nonzero element of $\widehat{m}$ to get the desired f. We have $b \supset fB \cong (R/m)[X_1, ..., X_d]$. If the degree of f is p, then we get the inequality $\phi_b(n) \geq \binom{n-p+d-1}{d-1}$. But the right-hand side is the length of the homogeneous components of degree $n-p$ in $(R/m)[X_1, \dots, X_d]$. This is impossible if the degree of ϕ_b is smaller than $d-1$. Therefore b must be zero, and the associated graded ring $G_q(R)$ is isomorphic to $B = (R/q)[X_1, \dots, X_d]$. But then $x_1, \dots, x_d$ is a regular sequence in R, and we are done. $\square$

5.4 Special case: Polynomial rings

Let us consider in the following only finitely generated modules M over polynomial rings $R = k[x_1, ..., x_n]$. This is an interesting case per se, furthermore there are more methods available to compute the Hilbert function.
Remark : In this section we denote the Hilbert function with respect to λ = length function by Hilb(t). This way it is distinguised to the other additive functions.
We will use free resolutions to calculate them. We remember that given a short exact sequence of finitely generated R-modules $0 \to M \to N \to O \to 0$, the corresponding Hilbert functions satisfy $Hilb_N(t)(t) = Hilb_M(t)(t) + Hilb_O(t)(t)$. This is clear because of the additivity of the length function. Given a finitely generated graded module M over the noetherian graded ring $R = k[x_1, \ldots, x_n]$ (with k a field), we consider a free resolution of M: This is an exact sequence of free R-modules of the form

$$\mathbb{F} : ... \to F_i \to F_{i-1} \to ... \to F_1 \to F_0 \to M \to 0.$$

We will restrict to the case that the complex is graded, all of the finite morphisms are of degree 0 and the modules F_i are free. This means that every F_i is of the form $\sum\limits_{j\in\mathbb{Z}} R^{c_{i,j}}(-j)$.
Here $R(-j)$ denotes the module R shifted by j degrees: the graded component $(R(-j))_k$ is equal to R_{k-j}. As a consequence we get the identity $Hilb_{R(-j)}(t) = t^j Hilb_R(t)$. Hilbert used resolutions to calculate the Hilbert function in the following way:

Theorem 5.4.1. *Let M be a finitely generated module over the ring R/I, where R is a polynomial ring $k[x_1, ..., x_n]$ in finitely many variable over a field and I a graded ideal. Let $\mathbb{F}$ be a graded free resolution of M, write $F_i = \sum\limits_{j\in\mathbb{Z}} R^{c_{i,j}}(-j)$. Suppose that for all j the coefficients $c_{i,j}$ become zero when i tends to infinity. Then*

$$Hilb_M(t) = Hilb_R(t) \sum_{i\geq 0} \sum_{j\in\mathbb{Z}} (-1)^j c_{i,j} t^j.$$

In particular, if $I = 0$, then $Hilb_M(t) = \frac{\sum\limits_{i\geq 0}\sum\limits_{j\in\mathbb{Z}} (-1)^j c_{i,j} t^j}{(1-t)^n}$.

Proof. Since each F_i is graded, we can write it in the form $\sum\limits_{j\geq 0} F_{i,j}$. By assumption $F_{i,j} = 0$ for large enough i. If we take the j'th graded component of every element F_i of the complex $\mathbb{F}$, then we get a fintie exact sequence of k-vector spaces:

$$0 \to ... \to F_{i,j} \to F_{i-1,j} \to ... \to F_{1,j} \to F_{0,j} \to M_j \to 0$$

Therefore the dimension of the k-vector space M_j is equal to $\sum_{i\geq 0} (-1)^i \dim_k(F_{i,j})$.
By assumtion this is always a finite sum. Thus,

$$\dim_k(M_j)t^j = \sum_{i\geq 0} (-1)^i \dim_k(F_{i,j}) t^j.$$

We sum over all j to get

$$Hilb_M(t) = \sum_j \dim_k(M_j)t^j = \sum_j (\sum_{i\geq 0} (-1)^i \dim_k(F_{i,j})t^j)$$

$$= \sum_{i\geq 0} (-1)^i (\sum_j \dim_k(F_{i,j})t^j) = \sum_{i\geq 0} (-1)^i Hilb_{F_i}(t).$$

We use the fact that $Hilb_R(-j)(t) = t^j Hilb_R(t)$ and obtain

$$Hilb_{F_i}(t) = \sum_j c_{i,j} Hilb_R R(-j)(t) = \sum_j c_{i,j} t^j Hilb_R(t) = Hilb_R(t) \sum_j c_{i,j} t^j.$$

Therefore

$$Hilb_M(t) = Hilb_R(t) \sum_{i\geq 0} \sum_{j\in\mathbb{Z}} (-1)^j c_{i,j} t^j$$

. □

This theorem can be applied whenever the resolution $\mathbb{F}$ is finite or minimal free. This is always satisfied whenever M is equal to R/I. It shifts the problem of finding the Hilbert function to finding the coefficients $c_{i,j}$ of a free resolution. These coefficients are called Betti numbers, and a whole subfield of homological algebra deals with its properties. We have seen that the Betti numbers can be used to compute the Hilbert series of M and therefore also the dimension, multiplicity and the Hilbert polynomial.
To understand why the study of Hilbert functions of ideals in polynomial rings is useful, we need some theory about Gröbner bases. This is an important part of computational commutative algebra. It will allow us to find families of ideals with the same Hilbert function.
Let R be the polynomial ring $k[x_1, ..., x_n]$. An element of R is called a monomial if it is of the form $x_1^{t_1} \cdot x_n^{t_n}$ and a term if it has the form $cx_1^{t_1} \cdot x_n^{t_n}$ with $c \in k\{0\}$. In the latter case the coefficient c is called the leading coefficient of the term. Let $\leq$ be a monomial ordering: This is a total order on the set of monomials respecting the multiplication which is also a well-order. For an element $f(x) \in R$ let $\mathrm{T}(f)$ be the set of all terms of f and $\mathrm{Mon}(f)$ be the set of all monomials of f. We denote by $LM(f)$ the leading monomial of f, by $LC(f)$ the leading coefficient of f and by $LT(f)$ the leading term of f. With this notation $LT(f) = LC(f)LM(f)$.
Given a set S of polynomials in R, the leading ideal $L(S)$ of S is the ideal generated by elements of the form $LM(f)$ with $f \in S$. If I is an ideal of R, a finite subset G of I is called a Gröbner basis of I with respect to a monomial ordering if the leading ideal of I is equal to the leading ideal of G. It is clear that for a given ideal there is no unique Gröbner basis. However, under additional assumptions uniqueness can be guaranteed. This requires the notion of a normal form:
Let $S = \{g_1, ..., g_r\} \in R$ be a finite set of elements, and let $f \in R$. Then f is called normal with respect to S if no monomial of $\mathrm{Mon}(f)$ is divisible by the leading monomial of an element of S.
A polynomial f^* is said to be the normal form of f with respect to S if f^* is in normal form and there exist elements $h_1, ..., h_r$ of R such that $f - f^* = \sum h_i g_i$ and $LM(h_i g_i) \leq LM(f)$ for

all i. It is enough for us to know that given S and f there is an algorithm that computes f^*. Of course, at this point the element f^* is not unique. However, for Gröbner bases we have the following remarkable result:

Theorem 5.4.2. *Let I be an ideal of R and let G be a Gröbner bases of I. Then*
i) Every element f of R has a unique normal form f^* with respect to G.
ii) The normal form map $NF_G : R \to R, f \to f^*$ is k-linear, with kernel I.
iii) If $\widetilde{G}$ is another Gröbner bases of I, then $NF_G = NF_{\widetilde{G}}$.

Proof. i) We can give an algorithm that finds a normal form: The following does not depend on G being a Gröbner basis. Assume that G consists of the elements $g_1, ..., g_r$:
Step 1: Set $f^* := f$ and $h_i := 0$ for all i.
Step 2: Set $T = \{(t, i) \mid t \text{ is a monomial contained in } f^* \text{ such that } LM(t) \text{ divides } t\}$.
Step 3: If $T = \emptyset$, then terminate and return f^*. Otherwise choose (t, m) of T with t maximal and let c be the coefficient of t in f^*.
Step 4: Set $f^* := f^* - \frac{ct}{LT(g_i)} g_i$ and $h_i := h_i + \frac{ct}{LT(g_i)}$.
Step 5: Repeat Step $2 - 4$.

iii)Let f^* and $\hat{f}$ be two normal forms of f. Then $f^* - \hat{f}$ is an element of I, so that $LM(f^* - \hat{f}) \in L(I) = L(G) = L(\tilde{G})$. If $f^* \neq \hat{f}$, then there exist elements $g \in G$ and $\tilde{g} \in \widetilde{G}$ such that $LM(g)$ and $LM(\tilde{g})$ both divide $LM(f^* - \hat{f})$. But the leading monomial of $f^* - \hat{f}$ is a monomial that is either contained in $\mathrm{Mon}(f^*)$ or in $\mathrm{Mon}(\hat{f})$. However, this contradicts the definition of a normal form, so that f^* has to be equal to $\hat{f}$.
ii) Let f, g be from R, let c be an element from k. Consider $h := NF_G(f + cg) - NF_G(f) - cNF_G(g)$. This is congruent to 0 modulo the ideal generated by G, so $h \in I$. If h is non-zero, than the leading monomial of h would be divisible by the leading monomial of some element of G. But this is a contrdiction to the fact that h is in normal form with respect to G. Therefore $h = 0$, and NF_G is k-linear. It remains to show that the kernel of this map is exactly I. Indeed, if $f \in \ker(NF_G)$, then $f = f - NF_G(f) \in I$. Otherwise, if $f \in I$, then also the normal form of F is contained in I. □

This theorem already gives us a test which shows if a given element f of R is contained in I or not. It is also used in the proof of the next corollary:

Theorem 5.4.3. *Let I be an ideal of R, and let G be a Gröbner bases of I with respect to a chosen monomial order. Then the ideal generated by the elements of G is exactly I.*

Proof. Since G is contained in I, one direction is trivial. On the other hand, let f be an element of I. Then $NF_G(f) = 0$, so that the normal form of f is 0. Since a monomial order is a also well-order, f is contained in the ideal generated by G. □

We have just seen the dominant behaviour of the leading ideal. The next theorem, which shows an even stronger connection between an ideal and a corresponding Gröbner bases, is of similar type. The only caveat is that the given monomial ordering must even be a total degree ordering, which means that whenever $g \leq g'$ also $\deg(g) \leq \deg(g')$ follows.

Theorem 5.4.4. *Let I be an ideal of R. Assume that $\leq$ is a total degree ordering on the monomials. Then the Hilbert series of I and of $L(I)$ coincide.*

Proof. Let G be a Gröbner bases of I. The normal form map induces a non-trivial linear embedding of R/I into R. We denote by $\phi_d : (R/I)_{\leq d} \to R$ the restriction of NF_G to $R/I \cap k[x_1, \dots, x_d]$. Let V_d be the subspace of R generated by all monomials of degree smaller than or equal to d which are not contained in the leading ideal of I. Since all monomials of V_d are in normal form with respect to G, we get the equality $f = NF_G(f) = \phi_d(f+I)$. Therefore V_d is contained in the image of ϕ_d.
On the other hand, by the definition of a normal form and the properties of $\leq$ we also get the inclusion $\mathrm{Im}(\phi_d) \subset V_d$. This implies that the Hilbert series of I is the same as the dimension of V_d. However, V_d depends only on the leading monomials of I, so every ideal that generates the same leading ideal as I must have the same Hilbert series. In particular, this holds for $L(I)$ and every Gröbner basis G. $\square$

This justifies why the study of ideals or more generally finitely generated modules of polynomial rings is sensible. The computational aspect can nowadays be done by a computer program, which is a huge advantage. Via Buchberger's algorithm one can quickly construct Gröbner bases corresponding to different monomial orders.

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