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Matrix iterations and filter combinatorics

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Zusammenfassung

Diese Arbeit untersucht Matrix-Iterationstechniken des Forcings und diskutiert einige Anwendungen, die es ermöglichen, kardinale Invarianten des Kontinuums und Filterkombinatorik zu analysieren. Insbesondere nutzen wir diese Techniken, um drei Konstruktionen von Modellen der Mengenlehre zu betrachten, in denen bestimmte Beziehungen zwischen kardinalen Invarianten entschieden werden können. Die erste Konstruktion, von Blass-Shelah (1989), zeigte die Konsistenz von $\mathfrak{u} < \mathfrak{d}$, also zwischen der Ultrafilter-erzeugenden und der dominierenden Zahl. Die zweite, von Fischer-Friedman-Mejía-Montoya (2018), zeigte die Konsistenz von $\mathfrak{b} = \mathfrak{a} < \mathfrak{d}$, also zwischen der beschränkenden, fast disjunkten und der dominierenden Zahl. Die dritte, von Brendle-Fischer (2011), berücksichtigt zusätzlich die Spaltungszahl in der Relation und zeigt die Konsistenz von $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$. Wir schließen mit einer Diskussion über einige Implikationen dieser Konstruktionen in anderen Bereichen der Mengenlehre, insbesondere in der Filterkombinatorik.

Abstract

The thesis reviews matrix iteration techniques of forcing and discusses some applications that allow to analyze cardinal invariants of the continuum and filter combinatorics. In particular, we will use these techniques to review three constructions of models of set theory in which some relations between cardinal invariants can be decided. The first one, from Blass-Shelah (1989) showed the consistency of $\mathfrak{u} < \mathfrak{d}$, between the ultrafilter generating and dominating numbers. The second one, from Fischer-Friedman-Mejía-Montoya (2018), showed the consistency of $\mathfrak{b} = \mathfrak{a} < \mathfrak{d}$, between the bounding, almost disjoint and dominating numbers. The third one, from Brendle-Fischer (2011) additionally considers the splitting number in the relation, showing the consistency of $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$. We conclude with a discussion about some implications of these constructions in other areas of set theory, in particular, in filter combinatorics.

Contents

Chapter 1

Introduction

"Infinity." This, like "God," "spirit," and some other expressions of which the equivalents exist in all languages, is by no means the expression of an idea—but of an effort at one. It stands for the possible attempt at an impossible conception. Man needed a term by which to point out the direction of this effort—the cloud behind which lay, forever invisible, the object of this attempt. A word, in fine, was demanded, by means of which one human being might put himself in relation at once with another human being and with a certain tendency of the human intellect. Out of this demand arose the word, "Infinity"; which is thus the representative but of the thought of a thought.

–Edgar Allan Poe, *Eureka: A Prose Poem* (1848)

The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds.

– Georg Cantor, as quoted in *Infinity and the Mind* (1995) by Rudy Rucker

The Infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite.

– David Hilbert, *Über das Unendliche* (1925)

The nature of infinity has fascinated and haunted humankind since ancient times, capturing the imagination of philosophers, mathematicians, poets, and mystics alike. A rich history of thought has emerged surrounding this concept, which continues to influence the development of mathematics and philosophy. For a brief history of infinity with special pauses for its role in set theory and mathematical practice we refer to [1, 2, 3]. This prelude briefly traces some milestones to provide some context and motivation for the problems addressed in this work.

The idea of infinity already makes an appereance in some paradoxes concerning the impossibility of motion, which can be traced back to **Zeno of Elea** (495–435 BC). **Aristotle of Stagira** (384–322 BC) later proposed a resolution treating the infinite as a *potentiality*, a concept that prevailed for centuries. Early Greek mathematicians such as **Eudoxus of Cnidus** (390-340 BC) and **Archimedes of Syracuse** (287-212 BC) were already aware that this potentiality could nonetheless be leveraged for computations, such as volumes of geometrical shapes. The modern understanding of infinity began to take shape with **Galileo Galilei** (1564-1642), who anticipated the notion of infinity later introduced in set theory by **Richard Dedekind** (1831-1916) as the featuring property of collections that can be put into a one-to-one correspondence with a proper subcollection. Similarly, **Bernard Bolzano** (1781-1848) in his work *Paradoxien des Unendlichen* (1830) laid important groundwork in understanding the paradoxes surrounding infinity, he introduced the notion of sets and argued for the acceptance of *actual* infinity.

Set theory, as we understand it today, was largely developed by **Georg Cantor** (1845-1918), with significant contributions from Dedekind. One of Cantor's key discoveries was a procedure to generate a larger infinite set from any given infinite set, simply by taking its powerset (i.e., the set of all subsets). In particular, he showed that the cardinality of the continuum is greater than that of the natural numbers, which is captured in the inequality $\aleph_0 < 2^{\aleph_0} = \mathfrak{c}$. Cantor also posed the famous question of whether there exists an infinity between the natural numbers and the continuum, $\aleph_0 < \aleph_1 < 2^{\aleph_0}$. This question, known as the *continuum hypothesis* (CH) was later shown to be independent of the axioms of Zermelo-Fraenkel set theory (ZFC), thanks to the combined efforts of **Kurt Gödel** (1906-1978) and **Paul Cohen** (1934-2007).

Notably, Cohen's method of *forcing*, developed to prove the consistency of the negation of the continuum hypothesis, has had far-reaching implications in set theory and other mathematical fields, enabling the derivation of a plethora of consistency results. Forcing shows that there can be, indeed, many different sizes of infinity between the natural numbers and the continuum. Remarkably, many of these intermediate infinities can be characterized via combinatorial

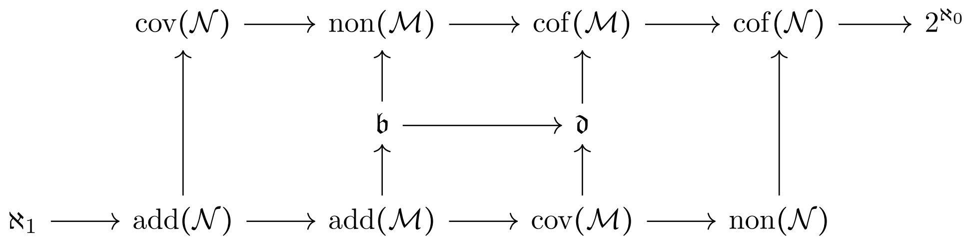

Figure 1.1: **Cichoń's diagram** captures relations between ten cardinal invariants, an arrow $\kappa \longrightarrow \lambda$ indicates that ZFC implies $\kappa \leq \lambda$. All definitions and relations can be found in [5, §1] and [6, Theorem 3.11]. Here, $\mathcal{N}$ and $\mathcal{M}$ are the ideals of *null* and *meager* subsets of the real line. For an ideal I: $\text{add}(I)$ is the smallest number of sets in I whose union is not in I; $\text{cov}(I)$ is the smallest number of sets in I whose union is $2^{\aleph_0}$; $\text{non}(I)$ is the smallest size of a set not in I; and $\text{cof}(I)$ is the cofinality of $(I, \subset)$. Last, the *bounding number* $\mathfrak{b}$ is the smallest size of an unbounded family of reals and the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family of reals.

properties, which are known as *cardinal characteristics* or *cardinal invariants* of the continuum (although they vary from one model of set theory to another).

One field of current research in set-theory revolves around better understanding the mutual interdependencies of cardinal invariants. For instance, Cichoń diagram [4, 5] consists of a selection of ten such invariants, as shown in Figure 1.1. A particular *constellation* assigns cardinals to each of them consistently with some model of set theory, typically obtained with forcing arguments. The main theme of this work is the analysis of the interdependencies of selected collections of cardinal invariants and some other combinatorial properties of ultrafilters.

1.1 Outline and contributions

We review the technique of matrix iterations of forcing introduced in [7] and further developed in [8, 9]. Chapter 1 reviews preliminaries about preorders and filters, and provides an introduction to a selection of cardinal invariants and their interconnections in ZFC. Chapter 2 reviews some standard forcing notions and presents the notion of coherent systems of finite support iterations from [9], which generalize finite support iterations and include matrix iterations as a particular case. Finally, Chapter 3 illustrates how to use these techniques to establish three independence results about cardinal invariants.

The first result, Theorem 66, involves the ultrafilter number $\mathfrak{u}$ (Definition 9) and the dominating number $\mathfrak{d}$ (Definition 14), from [7, Main theorem].

Theorem. *Let ν and δ be uncountable regular cardinals in a model of ZFC + GCH. There is a c.c.c. forcing extension such that $\mathfrak{u} = \nu$ and $\mathfrak{d} = \delta$.*

The second one, Theorem 73, involves the bounding $\mathfrak{b}$ (Definition 12) and the almost disjoint $\mathfrak{a}$ (Definition 18) numbers, from [9, Theorem 5.1].

Theorem. *Let $\kappa < \lambda$ be regular uncountable cardinals in a model of ZFC + GCH. There is a c.c.c. forcing extension such that $\mathfrak{b} = \mathfrak{a} = \kappa$ and $\mathfrak{d} = \mathfrak{c} = \lambda$.*

The third one, Theorem 83, is a strengthening of the previous one that involves the bounding number $\mathfrak{b}$ and the splitting number $\mathfrak{s}$ (Definition 21), from [8, Theorem 17].

Theorem. *Let $\kappa < \lambda$ be regular uncountable cardinals in a model of ZFC + GCH. There is a c.c.c. forcing extension such that $\mathfrak{b} = \mathfrak{a} = \kappa$ and $\mathfrak{s} = \mathfrak{c} = \lambda$.*

We conclude with a discussion about some implications of these constructions in filter combinatorics. In particular, we will obtain a generalization of the following result [10, Theorem 10] about Mathias forcing $\mathbb{M}_{\mathcal{U}}$ (Section 2.2).

Theorem. *Assume $\mathfrak{d} = \mathfrak{c}$. Then there exists an ultrafilter $\mathcal{U}$ with the property that $\mathbb{M}_{\mathcal{U}}$ does not introduce a dominating real.*

1.2 Preliminaries

We refer to [11, §III.3] and [12, §1] for the basic definitions introduced in this section. In this thesis, we will consider forcing poset as preorders [11, Definition III.3.1] and [12, Definition 1.2], although these give no more generality than partial orders [12, Claim 5.6]. Key concepts for forcing purposes include those of: antichains [11, Definition 3.3] and [12, Definition 1.15]; dense subsets [11, Definition III.3.8] and [12, Definition 1.3]; and predense subsets [12, Definition 1.15].

Definition 1 (Preorder)**.** *A* preorder $(\mathbb{P}; \leq)$ *is a set equipped with a reflexive and transitive relation, typically called* partial order *when the relation is additionally antisymmetric.* Forcing notions *are preorders with maximal element*

$(\mathbb{P}; \leq, \mathbf{1})$. *For* $p, q \in \mathbb{P}$, *we say* p extends q *if* $p \leq q$*; we say* p *and* q *are compatible if there exists a common extension* $r \leq p$ *and* $r \leq q$*; we write* $p \perp q$ *if* p *and* q *are* incompatible. *A subset* $D \subset \mathbb{P}$ *is said* $\mathbb{P}$-dense *if every* $p \in \mathbb{P}$ *admits an extension* $q \in D$. *A subset* $P \subset \mathbb{P}$ *is* $\mathbb{P}$-predense *if every* $p \in \mathbb{P}$ *is compatible with some* $q \in P$. *A* $\mathbb{P}$-antichain *is a collection* $A \subset \mathbb{P}$ *of pairwise incompatible elements. We say* $\mathbb{P}$ *has* countable chain condition (c.c.c.) *if there are no uncountable* $\mathbb{P}$*-antichains.*

The fundamental objects to obtain forcing extensions from forcing posets are filters [11, Definition III.3.10]; in particular, generic filters [11, Lemma III.3.14] and [12, Definition 1.4A].

Definition 2 (Filter)**.** *A non-trivial collection* $G \subset \mathbb{P}$ *is a* filter *if (i) downward directed, any* $p, q \in G$ *admit a common extension* $r \in G$*, and (ii) upward closed,* $q \in G$ *when* $p \leq q$ *for some* $p \in G$. *A filter is* $\mathbb{P}$-generic *when it* meets *every dense subset of* $\mathbb{P}$*, i.e.* $G \cap D \neq \varnothing$ *for each* $D \subset \mathbb{P}$ *dense.*

We will often use the following observation [12, Claim 1.20].

Lemma 3. *Let* $G \subset \mathbb{P}$ *be a filter. The following are equivalent:*
(i) G *meets every* $D \subset \mathbb{P}$ *dense.*
(ii) G *meets every* $P \subset \mathbb{P}$ *predense.*
(iii) G *meets every* $A \subset \mathbb{P}$ *maximal antichain.*

Proof. Dense sets and maximal antichain are predense, thus (ii) implies (i) and (iii). Now let P be $\mathbb{P}$-predense and imagine G meets every $\mathbb{P}$-dense set. The set $D_P = \{p \in \mathbb{P} : \exists q \in P(p \leq q)\}$ is $\mathbb{P}$-dense, since each $r \in \mathbb{P}$ is compatible with some $q \in P$. That is, there exists an extension t of r and q, which means $t \in D_P$. This proves that (i) implies (ii). Last, let D be $\mathbb{P}$-dense and imagine G meets every maximal antichain. By Zorn's lemma, there is an antichain $A_D \subset D$ maximal in D, which is also maximal in $\mathbb{P}$. Otherwise, there is some $r \in \mathbb{P}$ incompatible with A_D, which has an extension $t \leq r$ in D incompatible with A_D, contradicting maximality of A_D in D. Thus, (iii) implies (i). □

1.3 Cardinal invariants of the continuum

We review some cardinal invariants of the continuum. These are cardinals κ satisfying $\aleph_1 \leq \kappa \leq \mathfrak{c} = 2^{\aleph_0}$ and characterized by some combinatorial property. For a more detailed exposition we refer to [13]. In particular, we focus on the ultrafilter, dominating and bounded numbers considered in [7]; and the almost disjoint and splitting numbers additionally considered in [8, 9]. Most of their properties can be found in [13], but we adapt some proofs. The next definition particularizes Definition 2 for subsets of ω and introduces the notion of ultrafilters.

Definition 4 (Filters). *A non-trivial collection $\mathcal{F}$ of subsets of ω is a* filter over ω *if it is a filter in the partial order $(\mathcal{P}(\omega), \subset)$. That is, (i) downward directed, for $A, B \in \mathcal{F}$, there is $C \in \mathcal{F}$ such that $C \subset A \cap B$, and (ii) upward closed, for $A \in \mathcal{F}$ and $B \supset A$ also $B \in \mathcal{F}$. A filter $\mathcal{F}$ is* free *or* non-principal *if $\bigcap \mathcal{F} = \varnothing$. Filters maximal under inclusion are called* ultrafilters*. We say $\mathcal{G} \subset \mathcal{F}$* generates *$\mathcal{F}$ if every set in $\mathcal{F}$ containes one in $\mathcal{G}$.*

We will make use later of the following elementary observations.

Claim 5. *Any filter $\mathcal{F}$ is contained in an ultrafilter.*

Proof. Let F be the collection of filters containing $\mathcal{F}$ ordered under inclusion. Clearly, $\mathcal{F} \in F$ and $\bigcup C \in F$ for any chain $C \subset F$. By Zorn's lemma, F has a maximal element $\mathcal{U}$, which is a maximal filter containing $\mathcal{F}$. That is, $\mathcal{U}$ is an ultrafilter containing $\mathcal{F}$. □

Ultrafilters are often characterized with the following property.

Claim 6. *$\mathcal{U}$ is an ultrafilter if and only if $A \in \mathcal{U}$ or $\omega - A \in \mathcal{U}$ for each $A \subset \omega$.*

Proof. It is enough to show that if $A \notin \mathcal{U}$ and $\omega - A \notin \mathcal{U}$, we can extend $\mathcal{U}$ to a filter containing A. Indeed, $\mathcal{V} = \{B \subset \omega : \exists U \in \mathcal{U}(A \cap U \subset B)\}$ is clearly downward directed and upward closed. Moreover, $\varnothing \notin \mathcal{V}$, because $A \cap U \neq \varnothing$ for each $U \in \mathcal{U}$. Otherwise, $U \subset \omega - A \in \mathcal{U}$, contradicting the hypothesis. □

Lemma 7. *Let κ be an uncountable cardinal and $\langle s_\alpha \rangle_{\alpha<\kappa}$ an almost decreasing sequence of subsets of ω. The sets s_α together with their intersection with cofinite subsets of ω generate a filter.*

Here, the sequence $\langle s_\alpha \rangle_{\alpha<\kappa}$ being *almost decreasing* means that s_α is almost contained in s_β when $\alpha < \beta$, denoted $s_\alpha \subset^* s_\beta$. That is, $s_\beta - s_\alpha$ is finite.

Proof. Let $\mathcal{G}$ be the collection of s_α and their intersection with cofinite sets. The collection $\mathcal{F} = \{A \subset \omega : \exists s \in \mathcal{G}(s \subset A)\}$ is a filter generated by $\mathcal{G}$. Indeed, for $A, B \in \mathcal{F}$ there are $s_A, s_B \in \mathcal{G}$ such that $s_A \subset A$, $s_B \subset B$ and $s_A \subset^* s_B$. The last inclusion means $s_A \cap c_A \subset s_B$ for some cofinite set c_A. Thus, $s_A \cap c_A$ is an element of $\mathcal{G}$ contained in $A \cap B$, so $A \cap B \in \mathcal{F}$. Last, for $A \in \mathcal{F}$ there is $s_A \in \mathcal{G}$ contained in A. Thus, $A \subset B$ readily implies $B \in \mathcal{F}$. Hence, $\mathcal{F}$ is a filter. □

Given models $M \subseteq N$ of ZFC, we will need to extend an ultrafilter $\mathcal{U}$ in M to an ultrafilter $\mathcal{V}$ in N while *avoiding* some undesirable sets. For that purpose, we introduce the following lemma.

Lemma 8. *Let $\mathcal{A}$ and $\mathcal{Z}$ be collections of subsets of ω such that $\mathcal{A}$ is closed under intersections and no element of $\mathcal{A}$ is covered with finitely many elements of $\mathcal{Z}$. Then, there exists an ultrafilter containing $\mathcal{A}$ and avoiding $\mathcal{Z}$.*

Proof. Let $\mathcal{F}$ be the collection of finite unions of elements of $\mathcal{Z}$ and take $\mathcal{U} = \{U \subset \omega : \exists A \in \mathcal{A}, Z \in \mathcal{F}\,(A - Z \subset U)\}$. Clearly $\mathcal{A} \subset \mathcal{U}$ and $\mathcal{Z} \cap \mathcal{U} = \varnothing$. Additionally, $\mathcal{U}$ is not trivial because no $A \in \mathcal{A}$ is contained in some $Z \in \mathcal{F}$. For $A - Z_A, B - Z_B \in \mathcal{U}$ we have $(A - Z_A) \cap (B - Z_B) = A \cap B - Z_A \cup Z_B \in \mathcal{U}$, thus $\mathcal{U}$ is closed under intersections. Last, $\mathcal{U}$ is upward closed by construction. Thus, $\mathcal{U}$ is a filter containing $\mathcal{A}$ and avoiding $\mathcal{Z}$. By Zorn's lemma, $\mathcal{U}$ extends to a maximal filter $\mathcal{V}$ with these properties. We show that $\mathcal{V}$ is an ultrafilter. Otherwise, by Claim 6, there is some $B \subset \omega$ such that neither B nor $\omega - B$ are in $\mathcal{V}$. Either B or $\omega - B$ is not covered with finitely many elements of $\mathcal{Z}$, call it B'. Otherwise, every subset $A \subset \omega$ can be covered with finitely many elements of $\mathcal{Z}$, contradicting the hypothesis. Repeating the the first part of the proof, we obtain a filter containing $\mathcal{V} \cup \{B'\}$ and avoiding $\mathcal{Z}$, which contradicts maximality of $\mathcal{V}$. □

Definition 9 ($\mathfrak{u}$)**.** *The* ultrafilter *number $\mathfrak{u}$ is the smallest cardinality of a family $\mathcal{G}$ generating a non-principal ultrafilter of subsets of ω.*

Proposition 10. $\aleph_1 \leq \mathfrak{u} \leq \mathfrak{c}$*.*

Proof. Let $\mathcal{G}$ be a family of subsets of ω generating a non-principal ultrafilter $\mathcal{F}$. First, notice that $\mathcal{G} \subset \mathcal{P}(\omega)$ implies $|\mathcal{G}| \leq |\mathcal{P}(\omega)| = 2^{\aleph_0}$. For the other inequality, assume without loss of generality that $\mathcal{G} = \{G_i : i < \omega\}$ is a decreasing generating family, i.e. $G_j \subsetneq G_i$ when $i < j$. Then take $a_i \in G_i - G_{i+1}$ for each $i < \omega$. The set $A = \{a_{2k} : k < \omega\}$ does not contain any G_i since $a_{2i+1} \in G_i - A$ for each $i < \omega$. Similarly, the set $\omega - A$ does not contain any G_i, since $a_{2i} \in G_i \cap A$ for each $i < \omega$. Thus, $\mathcal{G}$ does not generate an ultrafilter. □

Given a set A and a cardinal κ, we will denote through this thesis by A^κ the set of functions from κ to A; and by $[A]^\kappa$ the set of subsets of A of size κ. Similarly, $A^{<\kappa}$ denotes the set of sequences of A with length smaller than κ; and $[A]^{<\kappa}$ the set of of subsets of A with size smaller than κ.

Definition 11 (Dominating functions)**.** *We say $f \in \omega^\omega$* dominates *$g \in \omega^\omega$ if $g(n) \leq f(n)$ for all but finitely many $n < \omega$, denoted $g \leq^* f$. A family $\mathcal{B} \subset \omega^\omega$*

is said unbounded *if no function* $f \in \omega^\omega$ *dominates all functions in* $\mathcal{B}$*. A family* $\mathcal{D} \subset \omega^\omega$ *is said* dominating *if every function* $f \in \omega^\omega$ *is dominated by some function in* $\mathcal{D}$*.*

Definition 12 ($\mathfrak{b}$)**.** *The* bounding *number* $\mathfrak{b}$ *is the smallest cardinality of an unbounded family.*

Proposition 13. $\mathfrak{b} \geq \aleph_1$ *is regular (no* κ*-sequence in* $\mathfrak{b}$ *is cofinal when* $\kappa < \mathfrak{b}$*).*

Proof. If $\mathcal{B} = \{g_i\}_{i<\omega} \subset \omega^\omega$, the function $f(n) = \max\{g_i(k) : i \leq n\}$ dominates every $g_i \in \mathcal{B}$. Thus, no countable family of functions is unbounded. Now let $\delta = \mathrm{cf}(\mathfrak{b})$ and imagine $\mathcal{B} = \bigcup_{\alpha<\delta} B_\alpha$ is an unbounded family, where $|B_\alpha| < \mathfrak{b}$. By minimality of $\mathfrak{b}$, each B_α is bounded by some function h_α. The family $\mathcal{C} = \{h_\alpha : \alpha < \delta\}$ is unbounded, since a function dominating $\mathcal{C}$ would dominate $\mathcal{B}$, which is unbounded. Thus, by minimality, $\mathfrak{b} = \delta = \mathrm{cf}(\mathfrak{b})$. □

Definition 14 ($\mathfrak{d}$)**.** *The* dominating *number* $\mathfrak{d}$ *is the smallest cardinality of a dominating family.*

Proposition 15. $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$.

Proof. Let $\delta = \mathrm{cf}(\mathfrak{d})$ and $\mathcal{D} = \bigcup_{\alpha<\delta} D_\alpha$ a dominating family, where $|D_\alpha| < \mathfrak{d}$. By minimality of $\mathfrak{d}$, there is a function h_α not dominated by D_α for each $\alpha < \delta$. Imagine $\delta < \mathfrak{b}$, then $\mathcal{C} = \{h_\alpha : \alpha < \delta\}$ is bounded by minimality of $\mathfrak{b}$ for some function f. For each $\alpha < \delta$, D_α does not dominate h_α, thus neither f. This contradicts $\mathcal{D}$ being dominating, since it is covered with D_α. □

Combining the previous observations we obtain the following relations.

Proposition 16. $\aleph_1 \leq \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$.

Next, we introduce almost disjoint families [13, Definitions 6.16] and the almost disjoint number [13, Definitions 8.3].

Definition 17 (Almost disjoint family)**.** *We say* $\mathcal{A} \subset [\omega]^\omega$ *is an* almost disjoint family *when any two elements of* $\mathcal{A}$ *have a finite intersection. A* maximal almost disjoint *(mad) family is an almost disjoint family maximal under inclusion.*

Definition 18 ($\mathfrak{a}$)**.** *The* almost disjoint number $\mathfrak{a}$ *is the smallest cardinality of an infinite maximal almost disjoint family.*

The following relation can be found in [13, Proposition 8.4].

Proposition 19. $\mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$.

Proof. Let $\mathcal{A}$ be a maximal almost disjoint family of size $\mathfrak{a}$. By modifying finitely many elements, we can assume that there exists a disjoint subfamily $\mathcal{A}_0 = \{A_n\}_{n<\omega} \subset \mathcal{A}$ such that $\omega = \bigcup_{n<\omega} A_n$, and take a bijection $h{:}\,\omega \to \omega \times \omega$ such that $h(A_n) = \{n\} \times \omega$. For each $A \in \mathcal{A} - \mathcal{A}_0$, take the function $f_A \in \omega^\omega$ defined as $f_A(n) = \sup\{m \in \omega : (n,m) \in h(A_n \cap A)\} < \omega$ (the intersection is finite). Imagine that the family $\mathcal{F} = \{f_A : A \in \mathcal{A} - \mathcal{A}_0\}$ is bounded. There exists $g \in \omega^\omega$ such that $f_A < g$ for each such A. Then, $B = \{h^{-1}(n, g(n))\}_{n<\omega} \in [\omega]^\omega$ is almost disjoint with each $A_n \in \mathcal{A}_0$, since $B \cap A_n = \{h^{-1}(n, g(n))\}$. Moreover, $A \cap B = \varnothing$ for each $A \in \mathcal{A} - \mathcal{A}_0$, since $h(A \cap A_n \cap B) = \varnothing$ for each $n < \omega$ and $\mathcal{A}_0$ partition ω. This contradicts the maximality of $\mathcal{A}$. Therefore, the family $\mathcal{F}$ is unbounded and hence $\mathfrak{b} \leq \mathfrak{a}$. □

The last invariant we introduce is the splitting number [13, Definition 3.1].

Definition 20 (Splitting family)**.** *We say $\mathcal{S} \subset [\omega]^\omega$ is a* splitting family *if for each $A \in [\omega]^\omega$ there is some $B \in \mathcal{S}$ such that $|A \cap B| = \omega = |A \cap (\omega - B)|$. Such B is said to* split *A.*

Definition 21 ($\mathfrak{s}$)**.** *The* splitting *number $\mathfrak{s}$ is the smallest cardinality of a splitting family.*

The next theorem can be found in [13, Theorem 3.3].

Theorem 22. $\aleph_1 \leq \mathfrak{s} \leq \mathfrak{d}$.

Proof. First notice that for each $X, S \in [\omega]^\omega$, we have $|X \cap S| + |X \cap (\omega - S)| = \omega$. Thus, for a countable family $\mathcal{S} \subset [\omega]^\omega$, we can iteratively intersect X with either S or $\omega - S$ for each $S \in \mathcal{S}$, so that X is not split by any $S \in \mathcal{S}$.

Let $\mathcal{D}$ be a dominating family. For $X = \{x_0 < x_1 < \dots\} \in [\omega]^\omega$ consider the increasing function $f_X(n) = x_n$. Take some $d \in \mathcal{D}$ dominating f_X, that is, $f_X(n) \leq d(n)$ for each $n \geq n_0$. In particular, $x_{n+1} - x_n \geq d(n+1) - n$ for $n \geq n_0$. Recursively define $s \in \omega^\omega$ via $s(0) = 0$ and $s(n+1) = s(n) + d(n+1)$. The intervals $I_n = [s(n), s(n+1))$ partition ω, thus $x_{n_0} \in I_{k_0}$ for some $k_0 < \omega$ which we can assume to be greater than n_0. Then, by the dominating condition and our choice of end points for the intervals, the sequence of $\langle x_n : n \geq n_0 \rangle$ cannot skip any interval I_k with $k_0 \leq k$. In particular, the union of even-indexed intervals $S_d = \bigcup_n I_{2n}$ splits X. Any dominating family $\mathcal{D}$ gives rise to a splitting family $\mathcal{S} = \{S_d : d \in \mathcal{D}\}$ with its same size, so $\mathfrak{s} \leq \mathfrak{d}$. □

The relations established in this section are summarized in Figure 1.2.

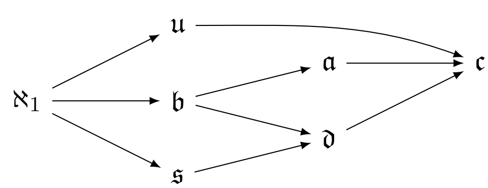

Figure 1.2: **Cardinal invariants** of the continuum that we will consider in Chapter 3: the ultrafilter $\mathfrak{u}$, bounding $\mathfrak{b}$, dominating $\mathfrak{d}$, almost disjoint $\mathfrak{a}$ and splitting $\mathfrak{s}$ numbers. An arrow $\kappa \longrightarrow \lambda$ indicates that ZFC implies $\kappa \leq \lambda$.

Chapter 2

Forcing

2.1 Introduction to forcing

We review the basics of the theory of forcing [14, 11, 15] and finite support iterations [12]. Then we present coherent systems of finite support iterations [9], generalizing matrix iterations of finite support that appeared for the first time in [7]. The following definition just particularizes Definition 1.

Definition 23 (Forcing notion)**.** *A* forcing notion *is a bounded preorder* $(\mathbb{P}; \leq, \mathbf{1})$*. For* $p, q \in \mathbb{P}$*, we say* p extends q *if* $q \leq p$*; we say* p *and* q *are* compatible *if there exists a common extension* $r \leq p$ *and* $r \leq q$*; we write* $p \perp q$ *if* p *and* q *are* incompatible*. A* $\mathbb{P}$-antichain *is a collection* $A \subset \mathbb{P}$ *of pairwise incompatible elements. We say* $\mathbb{P}$ *has* countable chain condition (c.c.c.) *if there are no uncountable* $\mathbb{P}$*-antichains.*

Forcing notions with c.c.c. have special properties. In particular, they preserve cardinals and cofinalities [11, Definition IV.3.2, Theorem IV.3.4] (also [15, Theorem 13.3]). The following combinatorial lemma [11, Lemma III.2.6] will be used to show that some of the forcing notions discussed later are c.c.c.

Lemma 24 (Δ-system)**.** *Let* X *be a family of finite sets with uncountable regular size. Then, there exists an uncountable subfamily* $Y \subset X$ *with the same cardinality such that* $A \cap B = R$ *for each pair of distinct elements* $A, B \in Y$*. We say* Y *is a* Δ*-system for* X *with root* R*.*

Proof. Let $X = \{x_\alpha\}_{\alpha<\kappa}$ and $\kappa_n = \{\alpha < \kappa : |x_\alpha| = n\}$ for each $n < \omega$. Since $\kappa = \bigcup_n \kappa_n$ is a regular cardinal, $|\kappa_n| = \kappa$ for some $n < \omega$. Thus we may assume $|x_\alpha| = n$ for each $\alpha < \kappa$ and argue by induction on n. For $n = 1$, X is already a Δ-system with root $R = \varnothing$. Now assume the statement holds for all $m < n$.

If there exists some a contained in κ-many x_α, take Y that collection. By induction, there is a root R such that $(x_\alpha - \{a\}) \cap (x_\beta - \{a\}) = R$ for each $\alpha < \beta < \kappa$. Thus, Y gives a Δ-system with root $R \cup \{a\}$. Otherwise, start with $Y_0 \in X$ and iteratively pick $Y_\alpha \in X$ disjoint with Y_β for $\beta < \alpha$ to get $Y = \{Y_\alpha\}_{\alpha<\kappa}$. We can always pick an element since κ is regular and each Y_α is finite. Thus, Y is a Δ-system with root $R = \varnothing$. □

Forcing notions allow to *label* sets in such a way that their content is decided by generic filters. These labels are called *names* [11, Definition IV.2.5] and [12, Definition 1.8].

Definition 25 (Name)**.** *A* $\mathbb{P}$-name τ *is a collection of pairs* $\langle\sigma, p\rangle$*, where* σ *is a* $\mathbb{P}$*-name and* $p \in \mathbb{P}$*. These can be defined with transfinite recursion over the step* α *at which the name is built, which is called the* name rank*. We denote* $V^{\mathbb{P}}$ *the universe of* $\mathbb{P}$*-names over the ground model* V*, and* $V^{\mathbb{P}}_\alpha$ *those with rank no greater that* α*. The* evaluation *of a* $\mathbb{P}$*-name* τ *on* $G \subset \mathbb{P}$ *is recursively defined as* $\tau^G = \{\sigma^G : \exists p \in G(\langle\sigma, p\rangle \in \tau)\}$*. Each* $x \in V$ *has a* canonical name $\check{x} = \{\langle\check{y}, p\rangle : y \in x, p \in \mathbb{P}\}$*, that evaluates* $\check{x}^G = x$ *for every* $G \subset \mathbb{P}$*.*

The following observation finds several applications: names for subsets can be taken with moderate size [15, Lemma 20.6].

Lemma 26. *Let* G *be* $\mathbb{P}$*-generic,* $A \in V$ *and* $B \subset A$ *in* $V[G]$*. If* $|A| \le \kappa$ *and each antichain* $W \subset \mathbb{P}$ *has size* $|W| \le \kappa$*, there exists a* $\mathbb{P}$*-name* τ *such that* $\tau^G = B$ *and* $|\tau| \le \kappa$*.*

Proof. For each $x \in A$, let $W_x \subset P$ be an antichain of elements $p \in \mathbb{P}$ forcing that $x \in B$, and which is maximal with this property (Zorn's lemma). Then consider the $\mathbb{P}$-name $\tau = \{\langle\check{x}, p\rangle : x \in A, p \in W_x\}$. First, notice that $|\tau| = |\bigcup_{x\in A} W_x| \le |A|\kappa = \kappa$, since $|W_x| \le \kappa$ and $|A| \le \kappa$. Given $\langle\check{x}, p\rangle \in \tau$ with $p \in G$, we have that $\check{x}^G = x$ which is forced by $p \in W_x$ to belong to B, thus $\tau^G \subset B$. Conversely, for $x \in B$ there exists some $p \in G$ forcing it, by Lemma 3. Then, $p \in W_x$ and $x \in \tau^G$, which shows that $B \subset \tau^G$. □

When $A = \omega$, we can understand such $B \in [\omega]^\omega$ as a real in the generic extension $V[G]$. Lemma 26 tells that for each $n < \omega$ there is an antichain W_n that decides if $n \in B$. Alternatively, understanding reals as functions $f \in \omega^\omega$, we can assign to f a *normal name* that consists of a sequence of pairs $(W_n, f_n)_{n<\omega}$ [12, Lemma 1.9], where W_n is a maximal antichain and $f_n\colon W_n \to \omega$ decides the value $f_n(p)$ that p forces $f(n)$ to have. Given a generic filter G there is a unique $p_n \in W_n \cap G$ for each n (Lemma 3), thus the value of the name is the function sending n to $f_n(p_n)$. Any name for a real is equivalent to one with this form.

Lemma 27 (GCH). *Let $\mathbb{P}$ be a c.c.c. forcing notion with uncountable size, $|\mathbb{P}| = \kappa$, and G a $\mathbb{P}$-generic filter. Then $V[G] \vDash \mathfrak{c} \leq \kappa$.*

Proof. Each real $B \subset \omega$ has a normal name $\tau_B = \{\langle \check{n}, p\rangle : n \in \omega, p \in W_n\}$, where W_n is a maximal antichain in $\mathbb{P}$. Thus, $|\tau_B| = \aleph_0$. There are $|[\kappa]^{\leq\omega}| = \kappa$ countable subsets of $\mathbb{P}$, thus there are at most κ normal names for reals in $V[G]$. Hence, $\mathfrak{c} \leq \kappa$. □

We can use names to talk about the generic extension within the ground model. This leads to the notion of forcing language [11, Definition IV.2.20] and [12, Definition 1.13].

Definition 28 (Forcing language). *Let $\mathbb{P}$ be a forcing notion and $V^{\mathbb{P}}$ the collection of $\mathbb{P}$-names over the ground model V. The* forcing language *consists of the membership relation and one constant symbol $\mathring{\tau}$ for each $\mathbb{P}$-name τ, $\mathcal{L}^{\mathbb{P}} = \{\in, \mathring{\tau}\}_{\tau\in V^{\mathbb{P}}}$. A* forcing formula *is a formula φ in the forcing language. Given a $\mathbb{P}$-generic filter G, we write $V[G] \vDash \varphi^G$ when the formula φ^G obtained from φ replacing each $\mathring{\tau}$ by τ^G is true in $V[G]$. We say that p* forces *φ, and write $p \Vdash \varphi$, if $V[G] \vDash \varphi^G$ for every generic filter G containing p. We write $\Vdash \varphi$ when $V[G] \vDash \varphi$ for every generic filter G.*

We repeatedly use the following result, whose proof we omit, sometimes called truth lemma [11, IV.2.24] or fundamental theorem of forcing [12, Theorems 1.5, 1.12 & 1.14] and [15, Theorem 11.1]. It says that everything that holds in a generic extension $V[G]$ is already forced by some condition $p \in G$.

Theorem 29. *Let G be a $\mathbb{P}$-generic filter and φ a forcing formula. Then, $V[G] \vDash \varphi^G$ if and only if $p \Vdash \varphi$ for some $p \in G$.*

2.2 Review of some forcing notions

We recall some basic properties of Cohen, random and Mathias forcing notions. These are used in Section 3.1 to present the independence result for the dominating $\mathfrak{d}$ and ultrafilter $\mathfrak{u}$ numbers [7]. We also recall two Hechler forcing notions, one introducing dominating reals and the other almost disjoint families. These are used in Sections 3.2 and 3.3 to obtain independence results about the almost disjoint $\mathfrak{a}$ and splitting $\mathfrak{s}$ numbers [8, 9].

Cohen reals

The poset is $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ [11, p.272], the set of finite partial functions $\kappa \times \omega \to 2$ ordered under reversed inclusion. That is, $f \le g$ whenever $f \supset g$. We will show that $\mathbb{P}$ is c.c.c. and a generic filter introduces κ unbounded and not-dominating reals. So $\aleph_1 \le \mathfrak{b} < \mathfrak{d} = \kappa = \mathfrak{c}$ holds in the extension.

Claim 30. *Cohen forcing is c.c.c.*

Proof. Take $F \subset \mathrm{Fn}(\kappa \times \omega, 2)$ uncountable (regular) and $X = \{\mathrm{dom}\, f : f \in F\}$. Notice that X is uncountable, since there are only finitely many functions with the same domain. By Lemma 24, there exists $Y \subset X$ with $|Y| = |X|$ such that $A \cap B = R$ for each pair of distinct $A, B \in Y$. For each $A \in Y$, pick $f_A \in F$ with $\mathrm{dom}\, f_A = A$. Then, $\mathrm{dom}\, f_A \cap \mathrm{dom}\, f_B = R$. There are only finitely many functions with domain R, thus $f_A|_R = f_B|_R$ for some $A, B \in Y$. That is, F contains compatible elements. Thus, no uncountable set can be an antichain. □

Claim 31. *Cohen forcing introduces κ reals $\langle r_\alpha \rangle_{\alpha<\kappa}$.*

Proof. Let G be a $\mathrm{Fn}(\kappa \times \omega, 2)$-generic filter over V. Then $r_G = \bigcup G$ defines a total function $r_G\colon \kappa \times \omega \to 2$, since G intersects each dense subset $D(\alpha, n) = \{f \in \mathrm{Fn}(\kappa \times w, 2) : (\alpha, n) \in \mathrm{dom}\, f\}$ and any two finite approximations agree on their intersection. Now, $r_\alpha(n) = r_G(\alpha, n)$ define κ reals $r_\alpha\colon \omega \to 2$ that belong to the generic extension $V[G]$.

Conversely, the generic filter G can be recovered from r_G. Indeed, take the filter $H = \{f \in \mathrm{Fn}(\kappa \times \omega, 2) : f \subset r_G\}$ in $V[r_G]$. Clearly, $G \subset H$. Conversely, take $f \subset r_G$ with finite domain $\{(\alpha_i, n_i)\}_{i \in I}$. Since G is generic, for each $i \in I$ there is some $g_i \in G$ such that $g_i(\alpha_i, n_i) = f(\alpha_i, n_i) = r_G(\alpha_i, n_i)$. Thus, $f \subset \bigcup\{g_i : i \in I\} \in G$, which implies $f \in G$ since G is a filter. □

For this reason we sometimes write $V[G] = V[r_G] = V[r_\alpha]_{\alpha<\kappa}$.

Claim 32. *Cohen reals $\langle r_\alpha \rangle_{\alpha<\kappa}$ are unbounded.*

Proof. For each $h\colon \omega \to \omega$, $\alpha < \kappa$ and $m < \omega$ in the ground model take the dense set $D(h, \alpha, m) = \{f \in \mathrm{Fn}(\kappa \times \omega, \omega) : \exists n > m[h(n) < f(\alpha, n)]\}$. Since $G \cap D(h, \alpha, m) \neq \varnothing$ for each generic filter G, for each $m < \omega$ and $\alpha < \kappa$ there is some $n > m$ such that $h(n) < r_\alpha(n)$. Thus, h does not dominate r_α. □

Claim 33. *Cohen reals $\langle r_\alpha \rangle_{\alpha<\kappa}$ are not dominating.*

Proof. For each $h{:}\omega \to \omega$ in the ground model, $\alpha < \kappa$ and $m < \omega$, consider the dense set $D(h, \alpha, m) = \{f \in \mathrm{Fn}(\kappa \times \omega, \omega) : \exists n > m[f(\alpha, n) < h(n)]\}$. Since $G \cap D(h, \alpha, m) \neq \varnothing$ for each generic filter G, for each $m < \omega$ and $\alpha < \kappa$ there is some $n > m$ such that $r_\alpha(n) < h(n)$. Thus, r_α does not dominate h. □

Even more, Cohen forcing preserves unbounded families [16, Lemma 3.5.3]. Consider Cohen forcing $\mathbb{P} = \mathrm{Fn}(\omega, 2)$ adding one real.

Lemma 34. *Let H be an unbounded family of reals in the ground model V. Then H remains unbounded in $V[G]$ for any $\mathbb{P}$-generic filter G.*

Proof. Let $\mathring{f}$ be a $\mathbb{P}$-name for a real in ω^ω. For each $p \in \mathbb{P}$, consider the minimum value $f_p(i)$ some extension $q \leq p$ forces for $\mathring{f}$ at $i < \omega$. Namely, $f_p(i) = \min\{j : \exists q \leq p\,(q \Vdash \text{“}\mathring{f}(i) = j\text{”})\}$. Since $\{f_p : p \in \mathbb{P}\}$ is countable, take some $g \in \omega^\omega$ in V such that $f_p \leq^* g$ for all $p \in \mathbb{P}$. On the other hand, since H is unbounded there is some $h \in H$ such that $A = \{i : g(i) < h(i)\}$ is infinite. We show that $\mathbb{P}$ forces that “$\mathring{f}(i) \leq \check{g}(i)$ for infinitely many $i \in \check{A}$”, thus h is not bounded by the real introduced with $\mathring{f}$. If not, there is some $p \in \mathbb{P}$ such that $p \Vdash$ “$\check{g}(i) < \mathring{f}(i)$ for all $i \in \check{A}$ big enough”. At the same time, there exists an extension $q \leq p$, such that $q \Vdash$ “$\mathring{f}(i) = \check{f}_p(i)$”, which gives a contradiction since $f_p(i) \leq g(i)$ for all i big enough. □

We can use Lemma 26 to obtain normal names for Cohen reals.

Example 35 (Cohen real)**.** $\mathbb{P} = \mathrm{Fn}(\omega, 2)$ *consists of finite partial functions* $p{:}\omega \to \{0, 1\}$. *For each $n < \omega$, the set $W_n = \{(n, 0), (n, 1)\}$ is a maximal antichain that decides the value of the Cohen real at n. Namely, condition (n, a) forces the Cohen real f to take value $f(n) = a$, thus the assignments $f_n((n, a)) = a$ give the normal name $(W_n, f_n)_{n<\omega}$.*

Random reals

The preorder is $\mathbb{P} = \mathrm{MB}(2^{\kappa\times\omega}, \mu)$ [11, Definition IV.7.33], the *measure algebra* of equivalence classes modulo null sets in $2^{\kappa\times\omega}$ with respect to the product measure μ of the uniform measure over 2, ordered under inclusion up to null sets. Namely, $[S] \leq [T]$ if $\mu(T - S) = 0$. We will show that random forcing is c.c.c. and introduces κ dominated reals. Additionally, neither a random real nor its compliment include any infinite set from the ground model.

Claim 36. *Random forcing is c.c.c.*

Proof. Let $A \subset \mathrm{MB}(2^{\kappa\times\omega},\mu)$ be an antichain. The intersection of any two elements $[S],[T] \in A$ must have zero measure, otherwise $[S\cap T]$ gives a common extension. For any $n<\omega$ take $A_n = \{[S]\in A : \mu(S)\geq 1/n\}$. Then, $|A_n|\leq n$, since $1\geq \mu(\bigcup A_n)\geq |A_n|/n$. Thus, $A=\bigcup_n A_n$ is the countable union of finite sets and hence countable.

□

Claim 37. *Random forcing introduces κ random reals $\langle a_\alpha\rangle_{\alpha<\kappa}$.*

Proof. Let G be a $\mathrm{MB}(2^{\kappa\times\omega},\mu)$ generic filter over the ground model V. Then $a_G = \bigcap\{S : [S]\in G\}$ can be seen as a characteristic function $a_G\colon \kappa\times\omega\to 2$. Hence, $a_\alpha(n) = a_G(\alpha,n)$ define κ reals $a_\alpha\colon\omega\to 2$ in $V[G]$. Conversely, G can be recovered from a_G. Indeed, $H=\{[S]\in \mathrm{MB}(2^{\kappa\times\omega},\mu) : a_G\subset S\}$ is clearly a filter containing G. Take $T\supset a_G = \bigcap\{S\cap T : [S]\in G\}$, thus $[S\cap T]\in G$ is a non-null extension of $[T]$. Since G is a filter, $[T]\in G$. □

Thus we sometimes write $V[G]=V[a_G]=V[a_\alpha]_{\alpha<\kappa}$.

Claim 38. *Random forcing does not introduce unbounded reals.*

Proof. Let $[P]\Vdash \mathring{f}\colon\omega\to\omega$, where $\mathring{f}$ is a forcing name. For $n,m<\omega$ let $[Q]$ be the largest element in $\mathrm{MB}(2^{\kappa\times\omega},\mu)$ forcing $\mathring{f}(n)=m$ and take $R(n,m) = [P]\wedge[Q]$. For each $n<\omega$ pick some $h(n)<\omega$ such that $\mu(\bigvee\{R(n,m) : m\leq h(n)\})\geq \mu(P)(1-1/2^{n+2})$. Then, $[S]=\bigwedge_{n<\omega}\bigwedge\{R(n,m): m\leq h(n)\}$ extends $[P]$ and forces that $\mathring{f}$ is dominated by $\mathring{h}$, which is a forcing name for $h\colon\omega\to\omega$. Moreover, $[S]\neq 0$ since $\mu(S)\geq\mu(P)(1-\sum_n(1/2^{n+2})) = \mu(P)/2>0$. □

Claim 39. *Neither a random real nor its complement contain an infinite set from the ground model.*

Proof. Let $a_G=\bigcap\{S : [S]\in G\}$ the random real introduced via a filter G generic for $\mathbb{P}=\mathrm{MB}(2^\omega,\mu)$. We can see $a\subset\omega$ as a characteristic function $a\in 2^\omega$ and take $C(a)=\{x\in 2^\omega : a\leq x\}$, the set of characteristic functions of reals containing a. For $a\in[\omega]^\omega$, let $C_n(a) = C(a|_n)$, where $a|_n$ denote the first n-elements of a. We have $\mu(C_n)=1/2^n$, since μ is the product of the uniform measure over 2. Moreover, $\mu(C(a))=\mu(\bigcap_n C_n)=\lim_n 1/2^n = 0$. This means that a is not contained in a_G. Otherwise, $a\subset a_G$ which means $C(a)\subset S$ for each $[S]\in G$. However, $\omega - C(a)$ is dense when $\mu(C(a))=0$, thus there exists $[S]\in G$ such that $S\subset\omega - C(a)$, which leads to the contradiction $C(a)\subset S\subset \omega - C(a)$. The same argument shows that $D(a)=\{x\in 2^\omega : a\leq \mathbf{1}-x\}$ has measure zero, where $\mathbf{1}\in 2^\omega$ denotes the constant function taking value 1, thus a is not contained in $\omega - a_G$. □

We can use Lemma 26 to obtain normal names for random reals.

Example 40 (Random real). $\mathbb{P} = \mathrm{MB}(2^\omega, \mu)$ *consists of equivalence classes of subsets of* 2^ω *modulo* μ*-null sets, partially ordered under inclusion modulo* μ*-null sets. Maximal antichains* $A \subset \mathbb{P}$ *consist of countable collections* $([S_i])_{i \in I}$ *of equivalence classes of not null disjoint subsets of* 2^ω *such that* $\sum_i \mu(S_i) = \mu(\bigcup_i S_i) = 1$*. In particular,* $A_n = \{[S_{n,0}], [S_{n,1}]\}$ *where* $S_{n,i} = \{f \in 2^\omega : f(n) = i\}$ *is a maximal antichain, since* $\mu(S_{n,1}) = 1/2 > 0$ *and* $\mu(S_{n,0} \cup S_{n,1}) = 1$*. Condition* $[S_{n,i}]$ *forces the random real to take value* i *at* n*. Thus,* $(A_n, f_n)_{n<\omega}$*, where* $f_n([S_{n,i}]) = i$*, gives a normal name for the random real.*

Mathias forcing relative to an ultrafilter

Let $\mathcal{U}$ be an ultrafilter and consider the preorder $\mathbb{P} = \mathbb{M}_\mathcal{U}$ [7] (which is a variation of [17]) of pairs (a, A), where a is a finite subset of ω and $A \in \mathcal{U}$ such that every element in a is smaller than every element in A. The order relation is given by (a, A) extends (b, B) if (1) $a \supset b$, (2) $A \subset B$ and (3) $a \backslash b \subset B$. We will show that Mathias forcing is c.c.c. and introduces a real almost contained in every $A \in \mathcal{U}$. We can obtain normal names for Mathias reals as in Lemma 26.

Example 41 (Canonical names). $\mathbb{P} = \mathbb{M}_\mathcal{U}$ *consists of pairs* (a, A)*, for* $a \in [\omega]^{<\omega}$ *and* $A \in [\omega]^\omega$ *such that* $\max a < \min A$*. We will work with non-principal ultrafilters* $\mathcal{U}$*, which contain all cofinite sets* $\omega - n$*. For each* $s \subset n$*, condition* $(s, \omega - n)$ *forces that the intersection of the Mathias real with* n *is precisely* s*. Thus,* $W_n = \{(s, \omega - n) : s \subset n\}$ *is a maximal antichain, since any condition* (a, A) *is compatible with exactly one* $(a \cap n, \omega - n) \in W_n$*. Thus, the assignment* $f_n((s, \omega - n)) = 1$ *when* $n \in s$ *and* 0 *otherwise, gives a normal name* $(W_n, f_n)_{n<\omega}$ *for the Mathias real.*

Claim 42. *Mathias forcing is c.c.c.*

Proof. Notice that (a, A) and (a, B) are always compatible, since $(a, A \cap B)$ is a common extension because $A \cap B \in \mathcal{U}$. Thus, an antichain can not contain two elements with the same finite set $a \in [\omega]^{<\omega}$ as first component. There are countably many finite subsets, thus every antichain is countable. □

Claim 43. *Mathias forcing introduces a Mathias real.*

Proof. Let G be a $\mathbb{M}_\mathcal{U}$-generic filter over the ground model V. Then $s_G = \bigcup\{a : (a, A) \in G\} \in [\omega]^\omega$ defines a real as a subset of ω. Moreover, G can be recovered from s_G. Indeed, the filter $H = \{(b, B) \in \mathbb{M}_\mathcal{U} : b \subset s_G\}$, contains all conditions in G. Conversely, take a condition (b, B) such that $b \subset s_G$.

There must be a condition $(a, A) \in G$ such that $b \subset a$ by genericity. Then $(a, A \cap B) \in G$ extends (b, B), thus $(b, B) \in G$, since G is a filter. □

Thus we can write $V[G] = V[s_G]$ for the generic extension.

Claim 44. *Mathias real is almost contained in every $A \in \mathcal{U}$.*

Proof. A $\mathbb{M}_{\mathcal{U}}$-generic ideal G gives a real number $s_G = \bigcup\{a : (a, A) \in G\}$. Each condition $(a, A) \in G$ forces s to contain a and be almost contained in A. For each $A \in \mathcal{U}$, the set $D_A = \{(b, B) \in \mathbb{M}_{\mathcal{U}} : B \subset A\}$ is dense, since for any $(b, B) \in \mathbb{M}_{\mathcal{U}}$ we have the condition $(b, A \cap B) \leq (b, B)$ in D_A. Thus, G contains some condition $(a, A \cap B)$ that forces s to be almost contained in A. □

The next claim from [7] will be used in the main theorem of Section 3.1.

Claim 45. *A Mathias real partitions ω into complementary sets A and $\omega - A$ that do not contain an infinite set from the ground model V.*

Proof. Let s be the Mathias real introduced with an ultrafilter $\mathcal{U}$ over the ground model V. We obtain a partition of ω using even-numbered blocks of s, $A = \{n < \omega : |s \cap n| \text{ is even}\}$. We show that neither A nor $\omega - A$ contain any infinite set $B \in [\omega]^\omega$ in V. Imagine there is a condition (c, C) in $\mathbb{M}_{\mathcal{U}}$ forcing "$B \subset A$" or "$B \subset \omega - A$". Let $k = \min C$ and $b \in B$ larger than k. Then, the two extensions $(c, C - b)$ and $(c, (C - b) \cup \{k\})$ force "$s \cap b = c$" and "$s \cap b = c \cup \{k\}$". The first one, because no extension of $(c, C - b)$ can introduce an element in b, the last one because $(c \cup \{k\}, C - b)$ extends $(c, (C - b) \cup \{k\})$ introducing k. Therefore, one of the two extensions of (c, C) forces "$b \in A$" while the other forces "$b \in \omega - A$", contradicting the assumption on (c, C). □

To iterate Mathias forcing we need to specify ultrafilters on the intermediate models. Choosing them carefully we can grant some desirable properties in the extensions. In particular, the following result is the main lemma from [7], which shows that we can add Mathias reals that do not dominate specific families of unbounded functions on the ground model.

Lemma 46. *Let $M \subset N$ be models of ZFC and $\mathcal{U}$ an ultrafilter in M. Imagine that $g \in \omega^\omega \cap N$ is unbounded over M. Then there exists an ultrafilter $\mathcal{V} \supset \mathcal{U}$ in N such that:*

(i) every maximal antichain of $\mathbb{M}_{\mathcal{U}}$ in M remains a maximal antichain of $\mathbb{M}_{\mathcal{V}}$ in N; and
(ii) for every $\mathbb{M}_{\mathcal{U}}$-name $\mathring{f}$ in M for a real, $\Vdash_{\mathbb{M}_{\mathcal{V}}}$ "$\mathring{f}$ does not dominate $\check{g}$".

Notice that $\mathcal{U} \subset \mathcal{V}$ implies that $\mathbb{M}_{\mathcal{U}} \subset \mathbb{M}_{\mathcal{V}}$ and incompatible elements in $\mathbb{M}_{\mathcal{U}}$ remain incompatible in $\mathbb{M}_{\mathcal{V}}$. Condition (i) tells that maximality of antichains is additionally preserved, thus $\mathbb{M}_{\mathcal{U}} \lessdot \mathbb{M}_{\mathcal{V}}$ [Definition 60]. Then, a $\mathbb{M}_{\mathcal{U}}$-name for a real in M is also a $\mathbb{M}_{\mathcal{V}}$-name for a real in N. Thus, condition (ii) is well posed. Moreover, it then follows from Lemma 3 that if H is $\mathbb{M}_{\mathcal{V}}$-generic over N, then $G = H \cap \mathbb{M}_{\mathcal{U}}$ is $\mathbb{M}_{\mathcal{U}}$-generic over M, and both evaluate the same real. In this sense, $\mathcal{U}$ and $\mathcal{V}$ introduce the same Mathias real (see Corollary 49).

Definition 47 (Forbidden sets)**.** *A violation of* (i) *is given by a maximal antichain* L *of* $\mathbb{M}_{\mathcal{U}}$ *and a condition* $(b, B) \in \mathbb{M}_{\mathcal{V}}$ *incompatible with all elements of* L*. We say such* B *is* forbidden *by* L *and* b*. We can replace in* (ii) *dominate by totally dominate. A violation is then given by a name for a real* $\mathring{f} = (W_n, f_n)_{n<\omega}$ *in* M *and a condition* $(c, C) \in \mathbb{M}_{\mathcal{V}}$ *such that* $(c, C) \Vdash$ "$\forall n(\mathring{f}(n) \geq \check{g}(n))$"*. Equivalently,* (c, C) *is incompatible with every* $p \in W_n$ *such that* $f_n(p) < g(n)$*. We say such* C *is* forbidden *by* $\mathring{f}$ *and* c*.*

Proving Lemma 46 amounts to showing that $\mathcal{U}$ can be extended to an ultrafilter in N that avoids all forbidden sets. By Lemma 8, it is enough to show that no set in $\mathcal{U}$ is covered by finitely many forbidden sets (in N). Imagine $Z \in \mathcal{U}$ is covered by $B_1, \ldots, B_k, C_1, \ldots, C_k$ where B_i is forbidden by L_i and b_i and C_i is forbidden by $\mathring{f}$ and c_i for each $i \leq k$. We can assume the same $\mathring{f}$ (taking the maximum) and also the sets $B_1, \ldots, C_k$ to be disjoint (subsets of forbidden sets are forbidden). Also $Z \subset \omega - n_0$, where n_0 is the smallest number greater than all elements in each b_i and c_i. We will use the following intermediate result from [7].

Claim 48. *There exists* $h \in \omega^\omega \cap M$ *with* $h(n) > n$ *such that* $Z \cap [n, h(n))$ *is partitioned into the subsets* $B_i \cap [n, h(n))$ *and* $C_i \cap [n, h(n))$*, and at least one of them, call it* P*, verifies that*

(i) for each $i \leq k$ *there is a finite* $e \subset P$ *such that* $b_i \cup e$ *is permitted by* L_i*,*

(ii) for each $i \leq k$ *there is a finite* $e \subset P$ *such that* $c_i \cup e$ *is permitted by some* $p \in W_n$ *with* $f_n(p) < h(n)$*.*

Proof. Imagine that for some $n < \omega$, no matter how large we take $h(n)$, none of the pieces of $Z \cap [n, h(n))$ has such properties. Then, by König's Lemma[1], none of the pieces of $Z - n$ has such properties. Since $\mathcal{U}$ is an ultrafilter in

[1]Every finitely branching tree has an infinite branch [18, Lemma IX.2.17]. In particular, the tree of initial segments of $Z - n$ such that no piece satisfies the claim.

M, there is a piece P of $Z - n$ that belongs to $\mathcal{U}$. Take $i \le k$, L_i is a maximal antichain in $\mathbb{M}_\mathcal{U}$ so there is $p \in L_i$ compatible with (b_i, P). That is, there is an extension $(b_i \cup e, P')$ compatible with p for some finite $e \subset P$, thus p permits $b_i \cup e$. Similarly, W_n is a maximal antichain in $\mathbb{M}_\mathcal{U}$, so there is an extension $(c_i \cup e, P')$ for some finite $e \subset P$ permitted by some $p \in W_n$. Thus, the claim holds provided that we take $h(n) > f_n(p)$, contrary to our initial assumption. □

Using Claim 48 we can finally prove Lemma 46 [7, Main Lemma].

Proof of Lemma 46. Take $h\colon \omega \to \omega$ and P from Claim 48. If $P = B_i \cap [n, h(n))$, there exists a finite $e \subset P \subset B_i$ such that $b_i \cup e$ is permitted by some $p \in L_i$. However, (b_i, B_i) also permits $b_i \cup e$, which contradicts B_i being forbidden by b_i and L_i. Thus, $P = C_i \cap [n, h(n))$. In this case, there exists a finite $e \subset P \subset C_i$ such that $c_i \cup e$ is permitted by some $p \in W_n$ with $f_n(p) < h(n)$. Notice that $c_i \cup e$ is also permitted by (c_i, C_i), which is incompatible with every $q \in W_n$ such that $f_n(q) \le g(n)$. Thus, $g(n) < f_n(p) < h(n)$ for each $n \in \omega$, contradicting g being unbounded over M. □

Corollary 49. *Let $\mathcal{U} \subset \mathcal{V}$ be ultrafilters as in Lemma 46. The Mathias real s introduced with $\mathcal{U}$ over M coincides with the Mathias real s' introduced with $\mathcal{V}$ over N.*

Proof. For each condition $(a, A) \in \mathbb{M}_\mathcal{V}$ forcing $a \subset s'$, $(a, \omega - \min A) \in \mathbb{M}_\mathcal{U}$ forces $a \subset s$. Conversely, every condition $(a, A) \in \mathbb{M}_\mathcal{U}$ forcing $a \subset s$ is already in $\mathbb{M}_\mathcal{V}$ forcing $a \subset s'$. Therefore, the $\mathbb{M}_\mathcal{U}$-generic filter H over N gives the same Mathias real as the $\mathbb{M}_\mathcal{U}$-generic filter $G = H \cap \mathbb{M}_\mathcal{U}$ over M. □

Hachler dominating forcing

Hechler dominating forcing $\mathbb{D}$ [13, §11.6] is used in the constructions showing the consistency of $\mathfrak{b} = \mathfrak{a} < \mathfrak{d}$ in Section 3.2 from [9], and $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$ in Section 3.3 from [8]. We will show that Hechler dominating forcing has c.c.c. and indeed introduces a dominating real.

Definition 50 (Hechler dominating forcing)**.** $\mathbb{P} = \mathbb{D}$ *consists of pairs $(s, f) \in \omega^{<\omega} \times \omega^\omega$, with order $(s_1, f_1) \le (s_2, f_2)$ if $s_2 \subset s_1$ is an initial segment, $f_2 \le f_1$ and $s_1(n) > f_2(n)$ for each $n \in \operatorname{dom} s_1 - \operatorname{dom} s_2$.*

Claim 51. $\mathbb{D}$ *is c.c.c.*

Proof. Two conditions (s, f_1) and (s, f_2) are compatible, since we can take $f(n) = \max\{f_1(n), f_2(n)\}$ for each $n < \omega$, for which $(s, f) \leq (s, f_1), (s, f_2)$. Thus, an antichain contains at most one condition (s, f) for each $s \in \omega^{<\omega}$, and there are countably many. □

Claim 52. *Hechler forcing $\mathbb{D}$ introduces a dominating real.*

Proof. A $\mathbb{D}$-generic filter G over V defines a real $d = \bigcup\{s : (s, f) \in \mathbb{D}\} \in \omega^\omega$, since conditions (s, f) with $n \in \operatorname{dom} s$ form a dense subset of $\mathbb{D}$ for each $n < \omega$. Conversely, we show that we can recover G from d. Clearly, $H = \{(s, f) \in \mathbb{D} : s \subset d, f \leq d\} \supset G$, since a condition $(s, f) \in G$ forces d to contain s as initial segment and totally dominate f. Conversely, for any $(s, f) \in H$, we have that $(s, d) \leq (s, f)$ and $(s, d) \in G$ because the conditions with domain containing $\operatorname{dom} s$ form a dense subset of $\mathbb{D}$, thus $(s, f) \in G$.

Last, d dominates any function $g \in \omega^\omega$ in the ground model V. It is enough to show that conditions (s, g) for $s \in \omega^{<\omega}$ form a predense subset of $\mathbb{D}$, since G is a generic filter. To see this, for $(s, f) \in \mathbb{D}$ take $h(n) = \max\{f(n), g(n)\}$ for each $n < \omega$ and notice that $(s, h) \leq (s, f)$ and $(s, h) \leq (s, g)$. Thus, every condition in $\mathbb{D}$ is compatible with one of the form (s, g). □

Thus, we can write $V[G] = V[d]$ for the $\mathbb{D}$-generic real added by G. The same argument shows that Hechler forcing $\mathbb{D}_F = \{(s, f) \in \mathbb{D} : f \in F\}$ restricted to a family of reals $F \subset [\omega]^\omega$ that is closed under taking the maximum of two reals, introduces a real that dominates F.

Hechler forcing for adding an almost disjoint family

We also review one last forcing notion, also called Hechler forcing $\mathbb{H}$, for adding maximal almost disjoint families [19, Theorem 3.2] which is used in the constructions from [8, 9] discussed in Sections 3.2 and 3.3 . We will show that Hechler forcing has c.c.c. and indeed introduces an almost disjoint family.

Definition 53 (Hechler forcing $\mathbb{H}$)**.** $\mathbb{P} = \mathbb{H}_\kappa$ *consists of functions* $p\colon F_p \times n_p \to 2$*, where* $F_p \in [\kappa]^{<\omega}$ *and* $n_p < \omega$*. The order is given by* $q \leq p$ *when* $q \supset p$ *and* $q(a, i) = 1$ *for at most one* $a \in F_q$ *when* $i \in n_q - n_p$*.*

Claim 54. *Hechler forcing $\mathbb{H}_\kappa$ is c.c.c.*

Proof. Two conditions p and q with $n_p = n_q$ are compatible, since an extension $q \leq p$ is given by $q \supset p$. Then, $p \cup q$ gives a common extension of p and q. There are countably many possible $n_p < \omega$, thus, antichains are countable. □

Hechler forcing adds maximal almost disjoint families [19, Theorem 3.2].

Claim 55. *Hechler forcing $\mathbb{H}_\kappa$ introduces an almost disjoint family.*

Proof. Let G be a $\mathbb{H}_\kappa$-generic filter over the ground model V. Take $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$, where $A_\alpha = \{i \in \omega : \exists p \in G(p(\alpha, i) = 1)\} \in [\omega]^\omega$. Imagine $i \in A_\alpha \cap A_\beta$, witnessed by $p(\alpha, i) = 1 = q(\beta, i)$. Since p and q are finite functions, they can witness at most finitely many elements in the intersection. A common extension r of p and q can not add new elements to the intersection $A_\alpha \cap A_\beta$, since at most one of the assignments $r(\alpha, k) = 1$ or $r(\beta, k) = 1$ is allowed by the ordering in $\mathbb{H}_\kappa$. Thus, $A_\alpha \cap A_\beta$ is finite for each α and β in κ distinct. □

Claim 56. *The almost disjoint family $\mathcal{A}$ is maximal when κ is uncountable.*

Proof. Take $X \in [\omega]^\omega$ in the generic extension $V[G]$. Since $\mathbb{H}_\kappa$ is c.c.c. there is a $\mathbb{H}_\kappa$-name for X with countably many conditions p, by Lemma 26. Each condition is a finite partial function $p\colon F_p \times n_p \to 2$ and κ is uncountable, so we can take $\beta \in \kappa - \bigcup_p F_p$. We show that $D_{p,m} = \{q \le p : \exists k > m\,(q(\beta, k) = 1)\}$ is dense above p for each $m < \omega$. Thus, any such p forces that $|X \cap A_\beta|$ is infinite, which proves that the disjoint family $\mathcal{A}$ is maximal. Indeed, for any $r \le p$ there is an extension $q \le r$ with $q(\beta, k) = 1$ for each $k > \max\{m, n_r\}$. □

2.3 Iterated forcing and coherent systems

Let $\mathbb{P}$ be a forcing notion and $\mathring{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing notion. Let G be a $\mathbb{P}$-generic filter over V and H a $\mathring{\mathbb{Q}}[G]$-generic filter over $V[G]$. The iteration $V[G][H]$ can be performed in one step $V[G * H]$ [12, Theorem II.1.4] with an equivalent forcing notion $\mathbb{P} * \mathring{\mathbb{Q}}$, called *forcing iteration* [12, Definition II.1.2]. Partial evaluation of the iteration is well defined by [12, Lemma II.1.5].

Definition 57 (Forcing iteration)**.** *Let $\mathbb{P}$ be a forcing notion and $\mathring{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing notion. The* forcing iteration *is the forcing notion*

$$\mathbb{P} * \mathring{\mathbb{Q}} = \{\langle p, \mathring{q}\rangle \in \mathbb{P} \times V^{\mathbb{P}} : \Vdash_{\mathbb{P}} \text{“}\mathring{q} \in \mathring{\mathbb{Q}}\text{”}\}, \tag{2.1}$$

where $\langle p_1, \mathring{q}_1\rangle \le \langle p_2, \mathring{q}_2\rangle$ if $p_1 \le p_2$ and $p_1 \Vdash_{\mathbb{P}}$ “$\mathring{q}_1 \le \mathring{q}_2$”. For generic $G \subset \mathbb{P}$ and $H \subset \mathring{\mathbb{Q}}[G]$, denote $G * H = \{\langle p, \mathring{q}\rangle \in \mathbb{P} * \mathring{\mathbb{Q}} : p \in G \wedge \mathring{q}[G] \in H\} \subset \mathbb{P} * \mathring{\mathbb{Q}}$, which is generic. The partial evaluation *of a $\mathbb{P} * \mathring{\mathbb{Q}}$-name $\mathring{\tau}$ on a generic $G \subset \mathbb{P}$ is a $\mathring{\mathbb{Q}}[G]$-name $\mathring{\tau}/G$ such that for each generic $H \subset \mathring{\mathbb{Q}}[G]$, $\mathring{\tau}/G[H] = \mathring{\tau}[G * H]$.*

We can extend this idea to κ iterations, $\langle \mathring{\mathbb{Q}}_\alpha\rangle_{\alpha<\kappa}$. In order for $\mathring{\mathbb{Q}}_\alpha$ to use previously introduced generic objects, we consider a hierarchy of forcing notions $\mathbb{P}_\alpha$ that keeps track of what has been introduced at each step. In particular, we consider a special class of well-behaved iterations that introduce no new conditions at limit steps [12, Definition II.2.2].

Definition 58 (Finite support iteration)**.** *Let* $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha \rangle_{\alpha<\kappa}$ *be a sequence where*

(i) $\mathring{\mathbb{Q}}_\alpha$ *is a* $\mathbb{P}_\alpha$*-name for a forcing notion and*
(ii) $\mathbb{P}_\alpha$ *is a forcing notion recursively defined as the partial order of finite functions* f *with domain contained in* α *such that* $f(\beta)$ *is* $\mathbb{P}_\beta$*-name for an element of* $\mathring{Q}_\beta$*.*

The direct limit $\mathbb{P}_\kappa = \varinjlim \mathbb{P}_\alpha$ *is called the* finite support iteration *of* $\langle \mathring{\mathbb{Q}}_\alpha \rangle_{\alpha<\kappa}$*.*

One feature of finite support iterations, of which we need to be aware for the constructions in Chapter 3, is that these always introduce Cohen reals at limit steps of countable cofinality.

Lemma 59. *Let* γ *be an ordinal with* $\mathrm{cf}(\gamma) = \omega$ *and* $\mathbb{P}_\gamma$ *the finite support iteration of non-trivial forcing notions* $\langle \mathring{\mathbb{Q}}_\alpha \rangle_{\alpha<\gamma}$*. Then* $\mathbb{P}_\gamma$ *introduces Cohen reals not introduced by* $\mathbb{P}_\alpha$ *with* $\alpha < \gamma$*.*

Proof. Let $\langle \alpha_n \rangle_{n<\omega}$ be a cofinal sequence in γ. For each α_n take a $\mathbb{P}_{\alpha_n}$-name $\mathring{a}_n$ for an element of $\mathring{\mathbb{Q}}_{\alpha_n}$, and for a generic $G \subset \mathbb{P}_\gamma$, let $G_n = G \cap \mathbb{P}_{\alpha_n}$. Then, $a = \{n : \mathring{a}_n[G_n] \in G\} \in [\omega]^\omega$ gives a Cohen real in $V[G]$. It is not introduced with a previous $\mathbb{P}_\alpha$ with $\alpha < \gamma$ because Cohen reals are not in the ground model and $\langle \alpha_n \rangle_{n<\omega}$ is cofinal in γ. □

Finite forcing iterations preserve c.c.c. That is, when $\mathbb{P}$ is c.c.c. and $\Vdash_{\mathbb{P}}$ "$\mathring{\mathbb{Q}}$ is c.c.c.", then $\mathbb{P} * \mathring{\mathbb{Q}}$ is c.c.c. [12, Theorem II.2.6]. Importantly, c.c.c. is also preserved under finite support iterations [12, Theorem II.2.8]. Forcing iterations split into smaller steps. That is, for $\beta < \alpha$ there exists a $\mathbb{P}_\beta$-name $\mathring{\mathbb{P}}_{\beta,\alpha}$ for a forcing notion such that $\mathbb{P}_\alpha = \mathbb{P}_\beta * \mathring{\mathbb{P}}_{\beta,\alpha}$ [12, Theorem II.2.4]. In such a case, $\mathbb{P}_\beta$ completely embeds into $\mathbb{P}_\alpha$ [11, Lemma V.3.4] in the sense of the following definition [12, Definition I.5.3].

Definition 60 (Complete embedding)**.** *A* complete embedding *between forcing notions* $\mathbb{P}$ *and* $\mathbb{Q}$ *is a function* $\psi : \mathbb{P} \to \mathbb{Q}$ *preserving extensions and maximal antichains. That is,*

(i) $\psi(p) \leq \psi(q)$ *when* $p \leq q$*; and*
(ii) $\psi(A)$ *is a maximal* $\mathbb{Q}$*-antichain when* A *is a maximal* $\mathbb{P}$*-antichain.*

We write $\mathbb{P} \lessdot \mathbb{Q}$ *if there exists an injective complete embedding from* $\mathbb{P}$ *to* $\mathbb{Q}$*.*

More general systems of iterations that produce complete embeddings were introduced in [9, Definition 3.2], generalizing *matrix iterations* introduced in [7] and further explored in [8].

Definition 61 (Coherent system)**.** *Let κ be an ordinal and I a partial order. A* coherent system *of finite support iterations is a sequence $\langle \mathbb{P}_{i,\alpha}, \mathring{\mathbb{Q}}_{i,\alpha} \rangle_{i \in I, \alpha \le \kappa}$, where each $\mathbb{P}_{i,\alpha}$ is a forcing notion, and each $\mathring{\mathbb{Q}}_{i,\alpha}$ is a $\mathbb{P}_{i,\alpha}$-name for a forcing notion such that:*

(i) $\langle \mathbb{P}_{i,\alpha}, \mathring{\mathbb{Q}}_{i,\alpha} \rangle_{\alpha < \kappa}$ *is a finite support iteration for each $i \in I$,*
(ii) $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$ *when $i \le j$ for each $\alpha \le \kappa$,*
(iii) $\mathbb{P}_{j,\alpha}$ *forces $\mathring{\mathbb{Q}}_{i,\alpha} \lessdot \mathring{\mathbb{Q}}_{j,\alpha}$ whenever $i \le j$.*

If $G_{i,\alpha}$ is a $\mathbb{P}_{i,\alpha}$-generic filter over V, we write $V_{i,\alpha} = V[G_{i,\alpha}]$ for the generic extension. We say that the coherent system $\langle \mathbb{P}_{i,\alpha}, \mathring{\mathbb{Q}}_{i,\alpha} \rangle_{i \in I, \alpha \le \kappa}$ is c.c.c when each $\mathbb{P}_{i,\alpha}$ is c.c.c. and forces $\mathring{\mathbb{Q}}_{i,\alpha}$ to be c.c.c.

Notice that it is enough for (ii) to hold at $\alpha = 0$ whenever $\mathbb{P}_{j,\alpha}$ forces that $\mathring{\mathbb{Q}}_{i,\alpha} \lessdot \mathring{\mathbb{Q}}_{j,\alpha}$ for each $\alpha \le \lambda$ when $i < j$ [9, Lemma 3.6]. Indeed, at successor stages we use the following result [8, Lemma 13].

Lemma 62. *If $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$ and $\Vdash_{\mathbb{P}_{j,\alpha}} \mathring{\mathbb{Q}}_{i,\alpha} \lessdot \mathring{\mathbb{Q}}_{j,\alpha}$, then $\mathbb{P}_{i,\alpha} * \mathring{\mathbb{Q}}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha} * \mathring{\mathbb{Q}}_{j,\alpha}$. That is, $\mathbb{P}_{i,\alpha+1} \lessdot \mathbb{P}_{j,\alpha+1}$.*

Proof. It is clear that $\mathbb{P}_{i,\alpha} * \mathring{\mathbb{Q}}_{i,\alpha} \subset \mathbb{P}_{j,\alpha} * \mathring{\mathbb{Q}}_{j,\alpha}$. Let $(p_1, \mathring{q}_1) \le (p_2, \mathring{q}_2)$ in $\mathbb{P}_{i,\alpha} * \mathring{\mathbb{Q}}_{i,\alpha}$. That is, $p_1 \le p_2$ in $\mathbb{P}_{i,\alpha}$ and $p_1 \Vdash \mathring{q}_1 \le \mathring{q}_2$ in $\mathring{\mathbb{Q}}_{i,\alpha}$, which means that $p_1 \le p_2$ in $\mathbb{P}_{j,\alpha}$ since $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$; and $p_1 \Vdash \mathring{q}_1 < \mathring{q}_2$ in $\mathring{\mathbb{Q}}_{j,\alpha}$ since $\mathbb{P}_{j,\alpha}$ forces that $\mathring{\mathbb{Q}}_{i,\alpha} \lessdot \mathring{\mathbb{Q}}_{j,\alpha}$. Then, $(p_1, \mathring{q}_1) \le (p_2, \mathring{q}_2)$ in $\mathbb{P}_{j,\alpha} * \mathring{\mathbb{Q}}_{j,\alpha}$, so the order is preserved.

Let $(p_1, \mathring{q}_1) \perp (p_2, \mathring{p}_2)$ be incompatible conditions in $\mathbb{P}_{i,\alpha} * \mathring{\mathbb{Q}}_{i,\alpha}$. That is, $p_1 \perp p_2$ in $\mathbb{P}_{i,\alpha}$ or $p' \Vdash \mathring{q}_1 \perp \mathring{q}_2$ in $\mathring{\mathbb{Q}}_{i,\alpha}$ for any common extension $p' \le p_1, p_2$. Imagine $(p', \mathring{q}') \le (p_1, \mathring{q}_1), (p_2, \mathring{q}_2)$ in $\mathbb{P}_{j,\alpha} * \mathring{\mathbb{Q}}_{j,\alpha}$, that is, $p' \le p_1, p_2$ in $\mathbb{P}_{j,\alpha}$ and $p' \Vdash \mathring{q}' \le \mathring{q}_1, \mathring{q}_2$ in $\mathring{\mathbb{Q}}_{j,\alpha}$. Then, $p_1 \not\perp p_2$ in $\mathbb{P}_{i,\alpha}$ since $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$, and $p' \Vdash \mathring{q}_1 \not\perp \mathring{q}_2$ in $\mathring{\mathbb{Q}}_{i,\alpha}$ since $p' \Vdash \mathring{\mathbb{Q}}_{i,\alpha} \lessdot \mathring{\mathbb{Q}}_{j,\alpha}$, which contradicts $(p_1, \mathring{q}_1) \perp (p_2, \mathring{q}_2)$. Thus, incompatibility is preserved.

Let $\mathcal{A} = \{(p_\beta, \mathring{q}_\beta)\}_{\beta < \kappa}$ be a maximal antichain in $\mathbb{P}_{i,\alpha} * \mathring{\mathbb{Q}}_{i,\alpha}$, and imagine there is some $(p, \mathring{q})$ incompatible with $\mathcal{A}$ in $\mathbb{P}_{j,\alpha} * \mathring{\mathbb{Q}}_{j,\alpha}$. If $p \perp p_\beta$ for each $\beta < \kappa$, then $\{p_\beta\}_{\beta < \kappa}$ is a maximal antichain in $\mathbb{P}_{i,\alpha}$ but not in $\mathbb{P}_{j,\alpha}$, contradicting $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$. Thus, $p \not\perp p_\beta$ for some β and $p' \Vdash \mathring{q} \perp \mathring{q}_\beta$ in $\mathring{\mathbb{Q}}_{j,\alpha}$ for each common extension $p' \le p, p_\beta$. Let B be the collection of such β, then $\{\mathring{q}_\beta\}_{\beta \in B}$ is a $\mathbb{P}_{j,\alpha}$-name for a maximal antichain in $\mathring{\mathbb{Q}}_{i,\alpha}$ which is not a maximal antichain in $\mathring{\mathbb{Q}}_{j,\alpha}$, which contradicts the assumption that $\mathbb{P}_{j,\alpha}$ forces $\mathring{\mathbb{Q}}_{i,\alpha} \lessdot \mathring{\mathbb{Q}}_{j,\alpha}$. □

At limit stages, we use the following result [8, Lemma 10].

Lemma 63. *Let $\langle \mathbb{P}_{i,\alpha}, \mathring{\mathbb{Q}}_{i,\alpha} \rangle_{\alpha \le \lambda}$ and $\langle \mathbb{P}_{j,\alpha}, \mathring{\mathbb{Q}}_{j,\alpha} \rangle_{\alpha \le \lambda}$ be finite support iterations. If $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$ for each $\alpha < \lambda$, then $\mathbb{P}_{i,\lambda} \lessdot \mathbb{P}_{j,\lambda}$.*

$$\begin{array}{ccccccccc}
\mathbb{P}_{0,0} & \xrightarrow{\mathring{\mathbb{Q}}_{0,0}} & \mathbb{P}_{0,1} & \xrightarrow{\mathring{\mathbb{Q}}_{0,1}} \cdots \longrightarrow & \mathbb{P}_{0,\alpha} & \xrightarrow{\mathring{\mathbb{Q}}_{0,\alpha}} \cdots \longrightarrow & \mathbb{P}_{0,\kappa} \\
\rotatebox[origin=c]{-90}{$\lessdot$} & & \rotatebox[origin=c]{-90}{$\lessdot$} & & \rotatebox[origin=c]{-90}{$\lessdot$} & & \rotatebox[origin=c]{-90}{$\lessdot$} \\
\mathbb{P}_{1,0} & \xrightarrow{\mathring{\mathbb{Q}}_{1,0}} & \mathbb{P}_{1,1} & \xrightarrow{\mathring{\mathbb{Q}}_{1,1}} \cdots \longrightarrow & \mathbb{P}_{1,\alpha} & \xrightarrow{\mathring{\mathbb{Q}}_{1,\alpha}} \cdots \longrightarrow & \mathbb{P}_{1,\kappa}
\end{array}$$

Figure 2.1: **Coherent pair of finite support iterations.** Each row corresponds with a finite support iteration $\langle \mathbb{P}_{i,\alpha}, \mathring{\mathbb{Q}}_{i,\alpha} \rangle_{\alpha<\kappa+1}$ of length $\kappa+1$ over the ground model $V_{0,0}$, such that $\mathbb{P}_{0,\alpha} \lessdot \mathbb{P}_{1,\alpha}$ for each $\alpha \leq \kappa$.

Proof. Clearly, $\mathbb{P}_{i,\lambda} \subset \mathbb{P}_{j,\lambda}$ by the properties of the finite support iteration and $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$ for each $\alpha < \lambda$. Let $p, q \in \mathbb{P}_{i,\lambda}$ such that $p \leq q$. Then, $p, q \in \mathbb{P}_{i,\alpha}$ for some $\alpha < \lambda$, which implies $p \leq q$ in $\mathbb{P}_{j,\alpha}$ and hence in $\mathbb{P}_{j,\lambda}$. Thus, the order is preserved. Incompatibility is also preserved. Any p and q incompatible in $\mathbb{P}_{i,\lambda}$ are already in $\mathbb{P}_{i,\alpha}$ for some $\alpha < \lambda$. Imagine some $r \in \mathbb{P}_{j,\lambda}$ extends p and q. By increasing α, we can assume $r \in \mathbb{P}_{j,\alpha}$, which contradicts $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$.

Now let $\mathcal{A} = \{p_\beta\}_{\beta<\kappa}$ be a maximal antichain in $\mathbb{P}_{i,\lambda}$ and take p in $\mathbb{P}_{j,\lambda}$. Then, $p \in \mathbb{P}_{j,\alpha}$ for some $\alpha < \lambda$ and there is some $q \in \mathbb{P}_{i,\alpha}$ such that $q \leq p$ in $\mathbb{P}_{j,\alpha}$, since $\mathbb{P}_{i,\alpha} \lessdot \mathbb{P}_{j,\alpha}$. Since $\mathcal{A}$ is a maximal antichain in $\mathbb{P}_{i,\lambda}$, q must be compatible with some p_β and so must p. □

We review some special cases [9, §3]. When $I = \{\bullet\}$, we recover finite support iterations. When I is a linear order, we obtain a *matrix iteration* as introduced in [7, 8] and depicted in Figure 3.1. In particular, when $I = \{0, 1\}$, we obtain a *coherent pair* of finite support iterations (see Figure 2.1).

Example 64 (Matrix iteration). *Let $I = \delta$ be an ordinal and consider a coherent system $\langle \mathbb{P}_{\alpha,\beta}, \mathring{\mathbb{Q}}_{\alpha,\beta} \rangle_{\alpha<\delta,\beta\leq\kappa}$. For each $\alpha < \delta$, $\langle \mathbb{P}_{\alpha,\beta}, \mathring{\mathbb{Q}}_{\alpha,\beta} \rangle_{\beta<\kappa+1}$ is a finite support iteration of length $\delta + 1$, thus $\mathbb{P}_{\alpha,\beta_1} \lessdot \mathbb{P}_{\alpha,\beta_2}$ whenever $\beta_1 \leq \beta_2$. Furthermore, for each $\beta \leq \kappa$, $\mathbb{P}_{\alpha_1,\beta} \lessdot \mathbb{P}_{\alpha_2,\beta}$ whenever $\alpha_1 \leq \alpha_2$. That is, we have complete embeddings between the forcing posets under both indices.*

In Chapter 3 we present some matrix iterations introduced in [7, 9, 8] to obtain the announced consistency results between certain cardinal invariants of the continuum. These constructions are depicted in Figures 3.1, 3.2 and 3.3. The key feature of matrix iterations is that they produce complete embeddings between forcing posets with respect to two indices simultaneously, allowing much finer control on the generic extensions. Interestingly, coherent systems cover more general iteration schemes that produce complete embeddings under more than two indices, although applications for these will not be addressed.

Example 65 (3D iteration)**.** *For ordinals μ and ν, take $I = \mu \times \nu$ ordered as $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ when $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$ and consider the coherent system $\langle \mathbb{P}_{\alpha,\beta,\gamma}, \mathring{\mathbb{Q}}_{\alpha,\beta,\gamma} \rangle_{\alpha<\mu,\beta<\nu,\gamma\leq\kappa}$, where $\mathbb{P}_{\alpha,\beta,\gamma} = \mathbb{P}_{(\alpha,\beta),\gamma}$. Then, $\langle \mathbb{P}_{\alpha,\beta,\gamma}, \mathring{\mathbb{Q}}_{\alpha,\beta,\gamma} \rangle_{\gamma<\kappa+1}$ is a finite support iteration of length $\kappa + 1$, thus $\mathbb{P}_{\alpha,\beta,\gamma_1} \lessdot \mathbb{P}_{\alpha,\beta,\gamma_2}$ whenever $\gamma_1 \leq \gamma_2$. Furthermore, $\mathbb{P}_{\alpha_1,\beta_1,\gamma} \lessdot \mathbb{P}_{\alpha_2,\beta_2,\gamma}$ whenever $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. Thus, we have complete embeddings between the forcing posets under all three indices simultaneously.*

Chapter 3

Applications

In this chapter, we finally present the matrix iterations introduced in [7, 8, 9] to obtain the announced consistency results between certain cardinal invariants of the continuum.

3.1 Small ultrafilter number

We begin chronologically with the consistency result from [7] regarding the ultrafilter $\mathfrak{u}$ [Definition 9] and dominating $\mathfrak{d}$ [Definition 14] numbers.

Theorem 66. *Let ν and δ be uncountable regular cardinals in a model of ZFC + GCH. There is a c.c.c. forcing extension such that $\mathfrak{u} = \nu$ and $\mathfrak{d} = \delta$.*

For the case $\delta \leq \nu$, a two step iteration of c.c.c. posets is enough.

Proposition 67. *There is a c.c.c. forcing extension such that $\mathfrak{d} = \delta \leq \nu = \mathfrak{u}$.*

Proof. We start adding δ Cohen reals $\langle r_\alpha \rangle_{\alpha<\delta}$. As we saw in Claim 32, Cohen reals are unbounded so the dominating number in the extension is $\delta \leq \mathfrak{d} \leq \mathfrak{c}$. Counting names as in Lemma 27 gives $\mathfrak{c} = \delta = \mathfrak{d}$ in the extension. Then, we add ν random reals $\langle a_\xi \rangle_{\xi<\nu}$. We saw in Claim 38 that random forcing does not introduce unbounded reals. Thus, the ground reals in the Cohen extension form a dominating family and the dominating number in the iterated extension is $\mathfrak{d} = \delta$. We showed in Claim 30 and 36 that both forcing notions are c.c.c. thus, the iteration is c.c.c. Finally, $\nu \leq \mathfrak{u} \leq \mathfrak{c}$ in the final extension, since we showed in Claim 39 that neither the random reals nor their complements contain infinite sets from the ground model. Counting names in the final extension as in Lemma 27 gives $\mathfrak{c} = \nu = \mathfrak{u}$. □

For $\nu < \delta$, we construct a finite support matrix iteration $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{Q}}_{\alpha,\xi} \rangle_{\alpha<\delta,\xi<\nu}$, with $I = \delta + 1$ and $\kappa = \nu$. Namely, $\mathbb{P}_{\alpha,0} = \mathrm{Fn}(\alpha \times \omega, 2)$ is Cohen forcing adding α reals and each $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{Q}}_{\alpha,\xi} \rangle_{\xi<\nu}$ is a finite support iteration of size δ where $\mathring{\mathbb{Q}}_{\alpha,\xi} = \mathring{\mathbb{M}}_{\alpha,\xi}$ is a $\mathbb{P}_{\alpha,\xi}$ name for a forcing notion adding a Mathias real s_ξ for an ultrafilter $\mathcal{U}_{\alpha,\xi}$ on $V_{\alpha,\xi}$ chosen according to Lemma 69. The iteration is depicted in Figure 3.1. This is indeed a matrix iteration, since $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi}$ for each $\xi \le \nu$ and $\alpha \le \beta \le \delta$. This is clear for $\xi = 0$ and extends to successor stages by Lemma 62 and to limit stages by Lemma 63. As we will see in the proof of Lemma 69, this is because the ultrafilters at successor steps are chosen according to Lemma 46, thus $\mathbb{P}_{\beta,\xi} \Vdash$ "$\mathring{\mathbb{M}}_{\alpha,\xi} \lessdot \mathring{\mathbb{M}}_{\beta,\xi}$" when $\alpha \le \beta \le \delta$. We will additionally show that this iteration has two special properties [9, §3]:

(1) For $\alpha < \delta$ there is a real $r_\alpha \in V_{\alpha,0}$ unbounded in $V_{\alpha,\xi}$ [8, Lemmas 10–12].
(2) Any real in $V_{\alpha,\nu}$ is already a real in $V_{\alpha,\xi}$ for some $\xi < \nu$ [8, Lemma 15].

We use Lemma 46 to ensure (1) recursively by suitably choosing the ultrafilters at successor stages according to Lemma 69, and using Lemma 70 at limit stages of countable cofinality. Limit stages of uncountable cofinality in finite support iterations do not introduce reals, so there is nothing to prove. We show that such a construction additionally satisfies (2) in Lemma 72.

Proposition 68. *There is a c.c.c. forcing extension such that* $\mathfrak{u} = \nu < \delta = \mathfrak{d}$.

Proof. Cohen and Mathias forcing notions are c.c.c. (see Claim 30 and Claim 42), thus the final extension of the finite support matrix iteration described above is c.c.c. Moreover, from Claim 32 Cohen reals are unbounded and from Lemma 69 Mathias reals do not introduce a small dominating family so $\delta \le \mathfrak{d} \le \mathfrak{c}$. Since $\delta > \nu$, counting names using Lemma 27 gives $\mathfrak{c} = \delta$ in the final extension, so $\mathfrak{d} = \delta$.

From Lemma 69 and Claim 44 we can assume Mathias reals form an almost decreasing sequence $\langle s_\xi \rangle_{\xi<\nu}$, thus it follows from Lemma 7 that $\mathfrak{u} \le \nu$. To see the equality, take a family $\mathcal{G}$ of less than ν infinite subsets of ω. By Lemma 72, $\mathcal{G}$ lies in $V_{\delta,\xi}$ for some $\xi < \nu$. Thus, neither the set $X \subset \omega$ from Claim 45 nor its complement $\omega - X$ contain any infinite set $Y \subset \omega$ in $V_{\delta,\xi}$. □

Choosing the ultrafilters $\mathcal{U}_\xi$ such that Mathias forcing does not introduce a small dominating family is the most difficult step of the proof, which requires a fine control of the intermediate models $V_{\alpha,\xi}$, as depicted in Figure 3.1.

Lemma 69. *We can choose ultrafilters* $\langle \mathcal{U}_\xi \rangle_{\xi<\nu}$ *such that:*

(i) $s_\xi \in \mathcal{U}_{\xi+1}$,
(ii) no family of less than δ *functions in* $V_{\delta,\nu}$ *dominate all Cohen reals.*

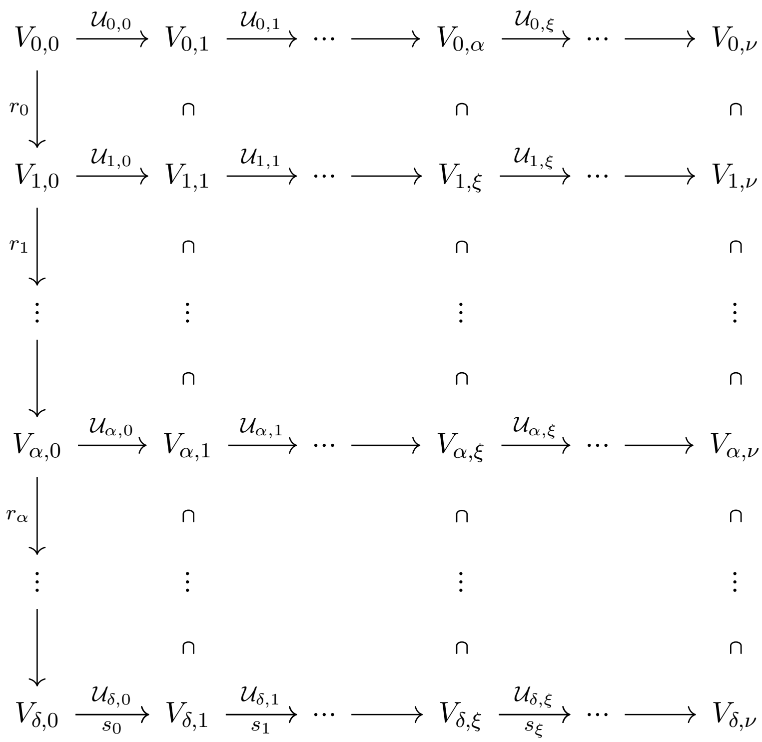

Figure 3.1: **Matrix iteration forcing for Theorem 66.** We depict all intermediate models obtained with the matrix iteration $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{M}}_{\alpha,\xi} \rangle_{\alpha<\delta,\xi<\nu}$. Here, $\mathbb{P}_{\delta,0} = \mathrm{Fn}(\delta \times \omega, 2)$ is Cohen forcing adding δ reals $\langle r_\alpha \rangle_{\alpha<\delta}$ over the ground model $V_{0,0}$, which can be realized as a finite support iteration of length δ corresponding to the first column in the diagram. For each $\alpha \leq \delta$, corresponding to row α in the diagram, we have a finite support iteration $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{M}}_{\alpha,\xi} \rangle_{\xi<\nu}$ of length ν, where $\mathring{\mathbb{M}}_{\alpha,\xi}$ is a $\mathbb{P}_{\alpha,\xi}$-name for Mathias forcing $\mathbb{M}_{\mathcal{U}_{\alpha,\xi}}$ with respect to an ultrafilter $\mathcal{U}_{\alpha,\xi}$ over the previous model $V_{\alpha,\xi}$ chosen according to Lemma 46. For each $\xi \leq \nu$, the complete embedding $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi}$ between the forcing posets when $\alpha \leq \beta \leq \delta$ gives the inclusion $V_{\alpha,\xi} \subset V_{\beta,\xi}$ between the generic extensions.

Proof. We proceed inductively, starting from any ultrafilter $\mathcal{U}_{0,0}$ in the ground model $V_{0,0}$. At successor stages apply Lemma 46 with $M = V_{\alpha,0}$ for $\mathcal{U} = \mathcal{U}_{\alpha,0}$ and $g = r_\alpha$ to extend the ultrafilter to $\mathcal{U}_{\alpha+1,0}$ in $V_{\alpha+1,0}$. At limit stages of uncountable cofinality, take $\mathcal{U}_{\lambda,0} = \bigcup_{\alpha<\lambda} \mathcal{U}_{\alpha,0}$. This is an ultrafilter since $\mathcal{U}_{\alpha,0}$ are increasing and no real is introduced in stages of uncountable cofinality. Moreover, it clearly satisfies all three conditions in Lemma 46. At limit stages of countable cofinality we need to extend $\bigcup_{\alpha<\lambda} \mathcal{U}_{\alpha,0}$ to an ultrafilter $\mathcal{U}_{\lambda,0}$ satisfying the conditions in Lemma 46. This is again done following the argument in the proof of Lemma 46, by showing that no set in a previous $\mathcal{U}_{\alpha,0}$ can be covered by finitely many *forbidden* sets. Then, take $\mathcal{U}_0 = \mathcal{U}_{\delta,0}$ and s_0 the Mathias real introduced with $\mathcal{U}_{\delta,0}$ over $V_{\delta,0}$, which coincides by Corollary 49 with the real introduced with $\mathcal{U}_{\alpha,0}$ over $V_{\alpha,0}$ for each $\alpha < \delta$.

By Lemma 46, no real in $V_{\alpha,1} = V_{\alpha,0}[s_0]$ dominates r_α. Let $\mathcal{U}_{1,0}$ be any ultrafilter in $V_{0,1} = V_{0,0}[s_0]$ containing s_0. Extend it as before to get $\mathcal{U}_{\alpha,1}$, for each $\alpha < \delta$ with similar properties. Then, take $\mathcal{U}_1 = \mathcal{U}_{\delta,1}$ and s_1 the Mathias real introduced with $\mathcal{U}_1$ over $V_{\delta,1}$. We can iterate this process for adjoining n Mathias reals, and state the iteration as the forcing notion $\mathbb{P}_{\delta,n}$ over $V_{\delta,0}$. Thus we get ultrafilters $\mathcal{U}_{\alpha,n}$ for each $\alpha \leq \delta$, $n < \omega$ and $\mathcal{U}_n = \mathcal{U}_{\delta,n}$. We extend the iteration with finite support to limit stages and obtain $\mathbb{P}_{\alpha,\omega}$ for $\alpha \leq \delta$. Since $\mathbb{P}_{\beta,n} \Vdash$ "$\mathring{\mathbb{M}}_{\alpha,n} \lessdot \mathring{\mathbb{M}}_{\beta,n}$" when $\alpha \leq \beta$ for each $n < \omega$ by our choice of ultrafilters from Lemma 46, it follows from Lemma 63 that $\mathbb{P}_{\alpha,\omega} \lessdot \mathbb{P}_{\beta,\omega}$ when $\alpha \leq \beta \leq \delta$. Lemma 70 shows that no real in $V_{\alpha,\omega}$ dominates r_α, which serve as an analogue of Lemma 46 for limit stages in the iteration.

This allows to analogously construct all intermediate $\mathcal{U}_{\alpha,\xi}$ over the models $V_{\alpha,\xi}$, preserving the crucial property that no real in $V_{\alpha,\xi}$ dominates the corresponding Cohen real r_α. To finish the proof it is enough to notice that since δ is regular, any family of less than δ reals in $V_{\delta,\nu}$ is already included in $V_{\alpha,\nu}$ for some $\alpha < \delta$, by Lemma 72, and thus fails to dominate r_α. □

We are left to show that properties (1) and (2) of the iteration extend to limit steps on second index, as we claimed in the proof of Lemma 69.

Lemma 70. *No real in $V_{\alpha,\omega}$ dominates r_α.*

We already know from Lemma 46 that no real in $V_{\alpha,n}$ dominates r_α. However, finite support iterations of non-trivial forcing notions always introduce new Cohen reals at limit steps of countable cofinality, by Lemma 59. Lemma 70 shows that these reals do not dominate r_α.

Proof. We work in $V_{\delta,\omega}$ and show that no real in $V_{\alpha,\omega}$ *totally* dominates r_α. For any $\beta \leq \delta$ and $\xi \leq \omega$, the model $V_{\beta,\xi} = V[r_\alpha]_{\alpha<\beta}[s_\eta]_{\eta<\xi}$ can be obtained from V by a generic subset $G_{\beta,\xi}$ of the forcing notion $\mathbb{P}_{\beta,\xi}$ adding β Cohen reals and ξ Mathias reals. This splits as $\mathbb{P}_{\beta,\omega} = \mathbb{P}_{\beta,k} * \mathring{\mathbb{R}}_{\beta,k}$ for $k < \omega$, where $\mathbb{P}_{\beta,k}$ adds β Cohen reals and the first k Mathias reals and $\mathring{\mathbb{R}}_{\beta,k}$ is a $\mathbb{P}_{\beta,k}$-name for a forcing notion that adds the remaining of the first ω Mathias reals with finite support iteration. A condition in $\mathring{\mathbb{R}}_{\beta,k}$ is an ω-sequence, where the j-th term is forced by the preceding ones to belong to $\mathbb{M}_{\mathcal{U}_{\beta,k+j}}$. That is, to be (a, A) for some finite $a \subset \omega$, and $A \in \mathcal{U}_{\beta,k+j}$ such that the elements in a are smaller than those in A.

Imagine there is a real in $V_{\alpha,\omega}$ totally dominating r_α, fix a $\mathbb{P}_{\alpha,\omega}$-name $\mathring{f}$ in V for it, and a condition $p \in G_{\alpha,\omega}$ forcing "$\mathring{f}$ totally dominates $\check{r}_\alpha$". Then $p \in G_{\alpha,k}$ for some $k < \omega$, since the iteration has finite support. Let us denote $\mathbb{R}_{\alpha,k} = \mathring{\mathbb{R}}_{\alpha,k}[G_{\alpha,k}]$ and $f' = \mathring{f}/G_{\alpha,k}$ the partial evaluation of $\mathring{f}$ with respect to $G_{\alpha,k}$. That is, a $\mathbb{R}_{\alpha,k}$-name in $V_{\alpha,k}$ such that $f'[H] = \mathring{f}[G_{\alpha,k} * H]$ for any generic $H \subset \mathbb{R}_{\alpha,k}$. It has normal form $f' = (W_n, f'_n)_{n<\omega}$, and we can define the function $g{:}\,\omega \to \omega$ in $V_{\alpha,k}$ by $g(n) = \min\{f'_n(q) : q \in W_n\}$. Given $n < \omega$, fix $q \in W_n \subset \mathbb{R}_{\alpha,k} \subset \mathbb{R}_{\delta,k}$ such that $f'_n(q) = g(n)$. Let H' be $V_{\delta,k}$-generic subset of $\mathbb{R}_{\delta,k}$ containing q, and $H = H' \cap \mathbb{R}_{\alpha,k}$. We have

$$\mathring{f}[G_{\delta,k} * H'](n) = \mathring{f}[G_{\alpha,k} * H](n) = f'[H](n) = g(n)\,. \tag{3.1}$$

At the same time, it follows from Lemma 46 that $G_{\delta,k}$ contains a condition p forcing "$\mathring{f}$ totally dominates $\check{r}_\alpha$", thus $g(n) = \mathring{f}[G_{\delta,k} * H'](n) \geq r_\alpha(n)$. Therefore, r_α is dominated by the function g in $V_{\alpha,k}$, which contradicts Lemma 46. □

Lemma 70 extends to arbitrary limit steps of countable cofinality $\gamma < \nu$. Combined with Lemma 69, these ensure the matrix iteration satisfies (1).

Corollary 71. *For $\alpha < \delta$ there is a real $r_\alpha \in V_{\alpha,0}$ unbounded over each $V_{\alpha,\xi}$*

Last, we show that the matrix iteration we have constructed also satisfies property (2). This property is common to all three construction we will consider in this Chapter. The following result appears in [7, Lemma p.270] (also [8, Lemma 15] and [9, Corollary 3.9]).

Lemma 72. *For any $\xi \leq \nu$,*

(i) every condition in $\mathbb{P}_{\delta,\xi}$ is already in $\mathbb{P}_{\alpha,\xi}$ for some $\alpha < \delta$, and
(ii) any $\mathbb{P}_{\delta,\xi}$-name for a real is already a $\mathbb{P}_{\alpha,\xi}$-name for some $\alpha < \delta$.

Proof. First notice that (i) already implies (ii), since δ is regular and uncountable and $\mathbb{P}_{\delta,\xi}$ is c.c.c. Thus, by Lemma 26 a name for a real can be described with countably many conditions, which cannot be cofinal in δ. We prove (i) by induction on ξ. If $\xi = 0$, it is clear by the definition of Cohen forcing. If ξ is a limit, it is also clear since the iteration has finite support. If $\xi = \eta + 1$, a condition $p \in \mathbb{P}_{\delta,\xi}$ is a condition $q \in \mathbb{P}_{\delta,\eta}$ followed by a $\mathbb{P}_{\delta,\eta}$-name for a condition in $\mathring{\mathbb{M}}_{\mathcal{U}_\eta}$, (a, A). By induction, $q \in \mathbb{P}_{\alpha_1,\eta}$ and (a, A) is a $\mathbb{P}_{\alpha_2,\eta}$-name for some $\alpha_1, \alpha_2 < \delta$. Thus $p \in \mathbb{P}_{\alpha,\xi}$ for $\alpha = \max\{\alpha_1, \alpha_2\}$. □

This concludes the proof of Proposition 68 and Theorem 66.

3.2 Almost disjoint families

In this section we introduce a matrix iteration to obtain a consistency result between the bounding $\mathfrak{b}$, almost disjoint $\mathfrak{a}$ and dominating $\mathfrak{d}$ numbers. The following theorem is a partial result contained in [9, Theorem 5.1].

Theorem 73. *Let $\kappa < \lambda$ be regular uncountable cardinals in a model of ZFC + GCH. There is a c.c.c. forcing extension such that $\mathfrak{b} = \mathfrak{a} = \kappa$ and $\mathfrak{d} = \mathfrak{c} = \lambda$.*

We take a coherent system of finite support iterations $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{Q}}_{\alpha,\xi} \rangle_{\alpha \le \kappa, \xi \le \lambda}$ with $I = \kappa + 1$. More concretely, for each $\alpha \le \kappa$ we take $\mathbb{P}_{\alpha,0} = \mathbb{H}_\alpha$ Hechler forcing in Definition 50 adding an almost disjoint family of size α, and $\mathring{\mathbb{Q}}_{\alpha,\xi} = \mathring{\mathbb{D}}_{\alpha,\xi}$ a $\mathbb{P}_{\alpha,\xi}$-name for a Hechler dominating forcing in Definition 50 with respect to a suitable family of reals $F_{\alpha,\xi} \subset [\omega]^\omega \cap V_{\alpha,0}$, where $V_{\alpha,\xi}$ denotes the generic extension for $\mathbb{P}_{\alpha,\xi}$. For the last row, we take $F_{\kappa,\xi} \subset [\omega]^\omega \cap V_{\kappa,0}$ chosen according to Lemma 81. For $\alpha < \kappa$, we take $F_{\alpha,\xi} = F_{\kappa,\xi}$ when $F_{\kappa,\xi} \subset V_{\alpha,0}$, and $F_{\alpha,\xi} = \varnothing$ otherwise. Thus, in either case $\mathbb{P}_{\beta,\xi} \Vdash$ "$\mathring{\mathbb{Q}}_{\alpha,\xi} \lessdot \mathring{\mathbb{Q}}_{\beta,\xi}$" when $\alpha \le \beta$. Henceforth, this defines indeed a matrix iteration, i.e. $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi}$ when $\alpha \le \beta \le \kappa$ and $\xi \le \lambda$. For $\xi = 0$, this follows by considering Hechler focring as a two step iteration, so the first poset completely embeds into the iteration. Moreover, it holds at successor stages by Lemma 62 and at limit stages by Lemma 63. The important properties of this matrix iteration, depicted in Figure 3.2, are:

(1) The mad family introduced with $\mathbb{P}_{\kappa,0}$ is preserved [9, Theorem 4.17].
(2) Any real in $V_{\kappa,\xi}$ is already a real in $V_{\alpha,\xi}$ for some $\alpha < \kappa$ [8, Lemma 15].

Showing the preservation property (1), stated in Theorem 74 as a special case of [9, Theorem 4.17], will require some auxiliary results. Property (2) is shown in Lemma 79, with similar arguments as those used in Lemma 72.

Theorem 74. *Let κ be an uncountable regular cardinal. The maximal almost disjoint family introduced with $\mathbb{H}_\kappa$ is preserved after forcing with finite support iterations $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha \rangle_{\alpha < \lambda}$, where each step is a c.c.c. forcing notion of size $< \kappa$.*

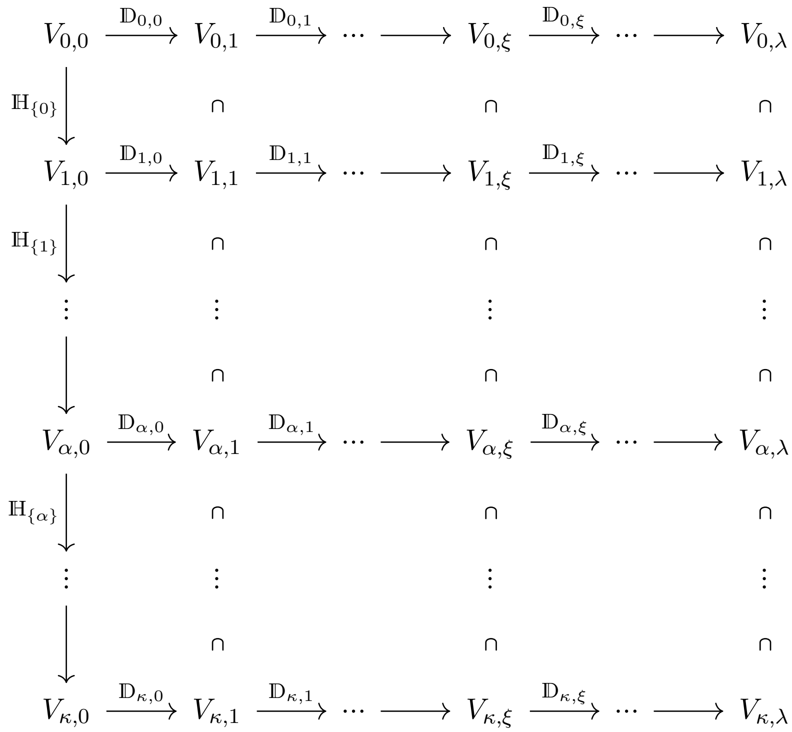

Figure 3.2: **Matrix iteration forcing for Theorem 73.** We depict all intermediate models obtained with the matrix iteration $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{D}}_{\alpha,\xi} \rangle_{\alpha\le\kappa,\xi\le\lambda}$. First column corresponds with Hechler forcing $\mathbb{P}_{\kappa,0} = \mathbb{H}_\kappa$ introducing a maximal almost disjoint family $\mathcal{A} = \langle A_\alpha \rangle_{\alpha<\kappa}$ over the ground model $V_{0,0}$, considered as a finite support iteration of length κ. For each $\alpha \le \kappa$, corresponding to row α in the diagram, we have a finite support iteration $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{D}}_{\alpha,\xi} \rangle_{\xi\le\lambda}$ of length λ, where $\mathring{\mathbb{D}}_{\alpha,\xi}$ is a $\mathbb{P}_{\alpha,\xi}$-name for Hechler forcing $\mathbb{D}_F$ with respect to a small family $F_{\alpha,\xi}$ over the intermediate model $V_{\alpha,\xi}$ chosen according to Lemma 81. More concretely, $F_{\kappa,\xi} \subset [\omega]^\omega \cap V_{\kappa,0}$ is chosen from Lemma 81, and $F_{\alpha,\xi} = F_{\kappa,\xi}$ whenever $F_{\kappa,\xi} \subset V_{\alpha,\xi}$, or $F_{\alpha,\xi} = \varnothing$ otherwise (and $\mathring{\mathbb{D}}_{\alpha,\xi}$ is a name for the trivial poset $\mathbf{1}$). In either case, $\mathbb{P}_{\beta,\xi} \Vdash$ "$\mathring{\mathbb{Q}}_{\alpha,\xi} \lessdot \mathring{\mathbb{Q}}_{\beta,\xi}$" when $\alpha \le \beta$, which ensures that the iteration is indeed a matrix iteration. For $\xi \le \lambda$, the complete embedding $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi}$ between the forcing posets when $\alpha \le \beta \le \kappa$ gives the inclusion $V_{\alpha,\xi} \subset V_{\beta,\xi}$ between the corresponding generic extensions.

In order to prove this preservation result, we introduce a technical notion from [9, Definition 4.2], previously considered in [8, Definition 2].

Definition 75 (Diagonalization). *Let $\mathcal{A} \subset [\omega]^\omega$ be a family of size κ in the ground model V and $A \in [\omega]^\omega$ not necessarily in V. We say that A* diagonalizes V outside $\mathcal{A}$ *if for each $h{:}\,\omega \times [\kappa]^{<\omega} \to \omega$ and each $m < \omega$ there exist (i) some $m < i < \omega$ and (ii) some $F \in [\kappa]^{<\omega}$ such that $[i, h(i, F)) - \bigcup \mathcal{A}|_F \subset A$.*

This property guarantees that A has infinite intersection with any $B \in [\omega]^\omega$ not almost covered by finitely many elements of $\mathcal{A}$ [8, Lemma 2].

Lemma 76. *If A diagonalizes V outside $\mathcal{A}$ and $B \in [\omega]^\omega$ in V is not almost contained in any finite union of elements of $\mathcal{A}$, then $|A \cap B| = \omega$.*

Proof. Imagine $A \cap B \subset n$. For any finite $F \subset \kappa$ and $m > n$, pick $b(m, F) < \omega$ such that $m \leq b(m, F) \in B - \bigcup \mathcal{A}|_F$. Then set $h(m, F) = 1 + b(m, F)$ when $m > n$ and $h(m, F) = 0$ when $m \leq n$. For any $m > n$, we have that $b(m, F) \in [m, h(m, F)) - \bigcup \mathcal{A}|_F$ but $b(m, F) \notin A$ since $b(m, F) \in B$ and $A \cap B \subset n$. □

Hechler forcing $\mathbb{H}_\kappa$ can be considered a finite support iteration forcing at each step $\alpha < \kappa$ a set $A_\alpha \in [\omega]^\omega$ that is almost disjoint with A_β for each $\beta < \alpha$. Next lemma shows that such A_α diagonalizes $V[A_\beta]_{\beta<\alpha}$ outside the family $\mathcal{A}|_\alpha = \{A_\beta : \beta < \alpha\}$ [8, Lemma 4].

Lemma 77. *Let $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ be the almost disjoint family introduced with Hechler forcing $\mathbb{H}_\kappa$ and $\alpha < \kappa$. Then, $\mathbb{H}_\kappa$ forces that A_α diagonalizes $V^{\mathbb{H}_{\kappa-\{\alpha\}}}$ outside $\mathcal{A} - \{A_\alpha\}$.*

Proof. We can implement Hechler forcing in two steps $\mathbb{H}_\kappa = \mathbb{P} * \mathring{\mathbb{Q}}$, where $\mathbb{P}$ contains conditions for $\kappa - \{\alpha\}$ and $\mathring{\mathbb{Q}}$ contains $\mathbb{P}$-names for conditions for $\{\alpha\}$. Let G and H be respectively $\mathbb{P}$-generic and $\mathring{\mathbb{Q}}[G]$-generic filters. Recall that H introduces the subset $A_\alpha = \{i \in \omega : \exists p \in H(p(i, \alpha) = 1)\}$. Let $h{:}\,\omega \times [\kappa]^{<\omega} \to \omega$ in $V[G]$ and $m < \omega$. For a condition $(p, F) \in \mathring{\mathbb{Q}}[G]$ with $\operatorname{dom} p = n_p \times \{\alpha\}$, pick $n > \max\{m, n_p\}$ and set $n_q = h(n, F)$. We obtain $q \supset p$ with $\operatorname{dom} q = n_q \times \{\alpha\}$ by setting $q(i, \alpha) = 1$ if $n \leq i < n_q$ and $i \notin \bigcup \mathcal{A}|_F$, and $q(i, \alpha) = 0$ otherwise. Then, $(q, F) \leq (p, F)$ and $(q, F) \Vdash$ "$[n, \check{h}(n, F)) - \bigcup \check{\mathcal{A}}|_F \subset \check{A}_\alpha$". □

That is, if we consider $\mathbb{H}_\kappa = \langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha \rangle_{\alpha<\kappa}$ as a finite support iteration of size κ, each step introduces A_α that diagonalizes V_α outside $\mathcal{A}|_\alpha = \{A_\beta\}_{\beta<\alpha}$. This means that Hechler forcing satisfies the conditions in the following corollary [9, Corollary 4.5].

Corollary 78. *Let $\langle V_\alpha \rangle_{\alpha<\kappa}$ be an increasing sequence of transitive models such that $[\omega]^\omega \cap V_\kappa = \bigcup_{\alpha<\kappa}[\omega]^\omega \cap V_\alpha$, and let $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa} \subset [\omega]^\omega$ in V_κ such that for each $\alpha < \kappa$, $\mathcal{A}|_\alpha$ is in V_α and $A_\alpha \in V_{\alpha+1}$ diagonalizes V_α outside $\mathcal{A}|_\alpha$. Then, for each $B \in [\omega]^\omega \cap V_\kappa$ there is some $\alpha < \kappa$ such that $|A_\beta \cap B| = \omega$. Thus, $\mathcal{A}$ is a maximal almost disjoint family in V_κ.*

Proof. Any $B \in [\omega]^\omega$ in V_κ is already in V_α for some $\alpha < \kappa$ by assumption. If B can be almost covered with finitely many elements of $\mathcal{A}|_\alpha$, then there must be some $\beta < \alpha$ with $|A_\beta \cap B| = \omega$. On the other hand, if B can not be covered by finitely many such elements Lemma 76 implies $|A_\alpha \cap B| = \omega$. □

The following observation is the analogue of Lemma 72 [8, Lemma 15], that shows that the matrix iteration also satisfies property (2).

Lemma 79. *For any $\xi \leq \lambda$,*

(i) every condition in $\mathbb{P}_{\kappa,\xi}$ is already in $\mathbb{P}_{\alpha,\xi}$ for some $\alpha < \kappa$,
(ii) every $\mathbb{P}_{\kappa,\xi}$-name for a real is already a $\mathbb{P}_{\alpha,\xi}$-name for a real for some $\alpha < \kappa$.

Proof. First notice that (i) already implies (ii), since κ is regular and uncountable and $\mathbb{P}_{\kappa,\xi}$ is c.c.c. Thus, by Lemma 26 a name for a real can be described with countably many conditions, which cannot be cofinal in κ. We show (i) by induction on ξ. For $\xi = 0$, it holds because we have a finite support iteration of Hechler forcing $\langle \mathbb{H}_\alpha, \mathring{\mathbb{H}}_{\{\alpha\}} \rangle_{\alpha\leq\kappa}$. For ξ limit, there is again nothing to prove because we are iterating with finite support. For $\xi = \eta + 1$, a condition $p \in \mathbb{P}_{\kappa,\eta+1}$ consists of a condition $p_0 \in \mathbb{P}_{\kappa,\eta}$ followed by a $\mathbb{P}_{\kappa,\eta}$-name for a condition in $\mathring{\mathbb{Q}}_{\kappa,\xi} = \mathring{\mathbb{D}}_F$ for some family $\mathring{F}$. That is, a pair $(s, \mathring{f})$ of $s \in \omega^{<\omega}$ and $\mathring{f}$ a $\mathbb{P}_{\kappa,\eta}$-name of a real in $\mathring{F}$. By induction hypothesis, we can find $\alpha < \kappa$ such that $p_0 \in \mathbb{P}_{\alpha,\eta}$ and $\mathring{f}$ is a $\mathbb{P}_{\alpha,\eta}$-name for a real, so that $p \in \mathbb{P}_{\alpha,\xi}$. □

Combining these results, we can finally prove Theorem 74.

Proof of Theorem 74. Consider the increasing sequence of models $\langle V_{\alpha,\lambda} \rangle_{\alpha<\kappa}$ obtained from our matrix iteration, described in the paragraph above Theorem 74 (see Figure 3.2). By Lemma 77 and Lemma 79 these models satisfy the hypothesis in Corollary 78 taking $\mathcal{A}$ to be the almost disjoint family introduced with Hechler forcing $\mathbb{P}_{\kappa,0} = \mathbb{H}_\kappa$. Thus, $\mathcal{A}$ is a maximal almost disjoint family in the final extension $V_{\kappa,\lambda}$. □

This already provides the first part of Theorem 73.

Proposition 80. $\mathfrak{b} = \mathfrak{a} = \kappa$.

Proof. Recall that $\mathbb{P}_{\kappa,0} = \mathbb{H}_\kappa$ introduces a mad family of size κ over the ground model $V_{0,0}$ by Claim 56, which remains a mad family in the final extension $V_{\kappa,\lambda}$ by Theorem 74. Thus, $\mathfrak{a} \leq \kappa$. Now, from Lemma 81 bellow, it follows that for every $\mathbb{P}_{\kappa,\lambda}$-name for a subset of ω^ω with size smaller than κ, the real introduced by $\mathring{\mathbb{Q}}_{\kappa,\xi}$ for some $\xi < \lambda$ dominates the family. Thus, $\kappa \leq \mathfrak{b}$. Finally, by Proposition 19 $\mathfrak{b} \leq \mathfrak{a}$, thus $\mathfrak{b} = \mathfrak{a} = \kappa$. □

Lemma 81. *Let $\mathring{F}$ be a $\mathbb{P}_{\kappa,\xi}$-name for a family of reals in $[\omega^\omega]^{<\kappa}$. We can choose $\mathring{\mathbb{Q}}_{\alpha,\xi}$ such that the real introduced with $\mathring{\mathbb{Q}}_{\kappa,\eta}$ dominates every real in the family for some $\eta \geq \xi$.*

Proof. This is a book-keeping argument. For each $F \in [\omega^\omega]^{<\kappa} \cap V_{\kappa,0}$, consider Hechler dominating forcing $\mathbb{D}_F$ relative to F. That is, whose conditions are pairs $(s, f) \in \omega^{<\omega} \times F$. We can assume without changing its cardinality, that F is closed under taking maxima. That is, for each $f, g \in F$, also the function $\max\{f, g\}(n) = \max\{f(n), g(n)\}$ belongs to F. The real introduced by $\mathbb{D}_F$ dominates the family F by Claim 52. Since $\kappa < \lambda$, there are less than λ many families of reals with size smaller than κ. Thus, we can take care of each of them in an intermediate step of the forcing iteration. □

Second part of Theorem 73 follows from Lemma 81 and counting names.

Proposition 82. $\mathfrak{d} = \mathfrak{c} = \lambda$.

Proof. Since $\mathbb{P}_{\kappa,\lambda}$ is a c.c.c. forcing notion with size $|\mathbb{P}_{\kappa,\lambda}| \leq \lambda$ over the ground model, $\mathfrak{c} \leq \lambda$ by counting names as in Lemma 27. Moreover, each $\mathbb{P}_{\kappa,\xi} \lessdot \mathbb{P}_{\kappa,\lambda}$ with $\xi < \lambda$ introduces a real over $V_{\kappa,\xi}$, thus $\lambda \leq \mathfrak{c}$. A similar argument shows that $\lambda \leq \mathfrak{d}$. Indeed, any family F of reals in $V_{\kappa,\lambda}$ with size less than λ is already introduced in some intermediate step $V_{\kappa,\xi}$, with $\xi < \lambda$. Then, by Lemma 81 the real introduced with $\mathring{\mathbb{Q}}_{\kappa,\xi} = \mathring{\mathbb{D}}_F$ dominates all reals in F. All families with size less than λ are not dominating, thus $\lambda \leq \mathfrak{d}$. □

The results from this section can be alternatively derived using a linear forcing iteration that first adds the maximal almost disjoint family forcing with $\mathbb{H}_\kappa$ and then adds the reals dominating small families forcing with $\langle \mathring{\mathbb{D}}_{\kappa,\xi} \rangle_{\xi \leq \lambda}$ with finite support. However, on the one hand, the preservation of the maximal almost disjoint family from Theorem 74 holds for more general matrix iterations [9, Theorem 4.14], and the original result [9, Theorem 5.1] discusses some additional cardinal invariants that were not addressed here. On the other hand, the matrix iteration in Section 3.3 combines the constructions from this and the previous section. Thus, some of the constructions and results obtained here will be used later.

3.3 Splitting number

The following result from [8, Theorem 17] deciding the splitting number can be considered as a strengthening of Theorem 73, since $\mathfrak{s} \le \mathfrak{d}$ by Theorem 22. It is obtained slightly adapting the model constructed in the Section 3.2, although the construction in [8] chronologically precedes the one from [9]. We will adapt the arguments and lemmas obtained for the matrix iterations in Sections 3.1 and 3.2 to the matrix iteration we will introduce here.

Theorem 83. *Let $\kappa < \lambda$ be regular uncountable cardinals in a model of ZFC + GCH. There is a c.c.c. forcing extension such that $\mathfrak{b} = \mathfrak{a} = \kappa$ and $\mathfrak{s} = \mathfrak{c} = \lambda$.*

We take the system of matrix iterations $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{Q}}_{\alpha,\xi} \rangle_{\alpha\le\kappa,\xi\le\lambda}$ from [8, §4], with $I = \kappa + 1$. For each $\alpha < \kappa$, $\mathbb{P}_{\alpha,0} = \mathbb{H}_\alpha$ is a Hechler forcing as in Definition 53 adding an almost disjoint family of size α, and $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{Q}}_{\alpha,\xi} \rangle_{\alpha<\lambda}$ is a finite support iteration of length λ that combines Matthias forcing $\mathring{\mathbb{M}}_{\alpha,\xi}$ with respect to some suitably chosen ultrafilter $\mathcal{U}_{\alpha,\xi}$, and Hechler dominating forcing $\mathring{\mathbb{D}}_{\alpha,\xi}$ with respect to a suitably chosen family $F_{\alpha,\xi}$ (see Figure 3.3). More concretely, take a surjection $f\colon \{\eta < \lambda : \eta = 0 \mod 2\} \to \kappa$ that takes each value $\alpha < \kappa$ cofinally often. Then, recursively define $\mathbb{P}_{\alpha,\xi} = \mathbb{P}_{\alpha,\eta} * \mathring{\mathbb{Q}}_{\alpha,\eta}$ at successor stages $\xi = \eta + 1$ as

$$\mathring{\mathbb{Q}}_{\alpha,\eta} = \begin{cases} \mathring{\mathbb{M}}_{\alpha,\eta} & \eta = 1 \mod 2, \\ \mathring{\mathbb{D}}_{\alpha,\eta} & \eta = 0 \mod 2, \end{cases} \tag{3.2}$$

where $F_{\alpha,\eta} = \varnothing$ when $\alpha \le f(\eta)$, thus $\mathring{\mathbb{D}}_{\alpha,\eta}$ would be a name for the trivial forcing poset $\mathbb{1}$, and $F_{\alpha,\eta} = [\omega]^\omega \cap V_{f(\eta),\eta}$ when $\alpha > f(\eta)$, thus $\mathring{\mathbb{D}}_{\alpha,\eta}$ introduces a real dominating all reals in $V_{f(\eta),\eta}$. In either case, we will show that $\mathbb{P}_{\beta,\xi} \Vdash$ "$\mathring{\mathbb{Q}}_{\alpha,\xi} \lessdot \mathring{\mathbb{Q}}_{\beta,\xi}$" when $\alpha \le \beta$ for each $\xi \le \lambda$. As in Section 3.1, we will recursively choose the ultrafilters $\mathcal{U}_{\alpha,\xi}$ to ensure the following key properties for any $\xi \le \lambda$:

(1) $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi}$ when $\alpha < \beta$,
(2) A_α diagonalizes $V_{\alpha,\xi}$ outside $\mathcal{A}|_\alpha$,
(3) Any real in $V_{\kappa,\xi}$ is already a real in $V_{\alpha,\xi}$ for some $\alpha < \kappa$ [8, Lemma 15].

Most results for this section will follow by combining the analogues for Mathias forcing in Blass-Shelah construction (Section 3.1) and those for Hechler dominating forcing in our partial Fischer-Friedman-Mejía-Montoya construction (Section 3.2). Property (1) is clear for $\xi = 0$ and odd successor

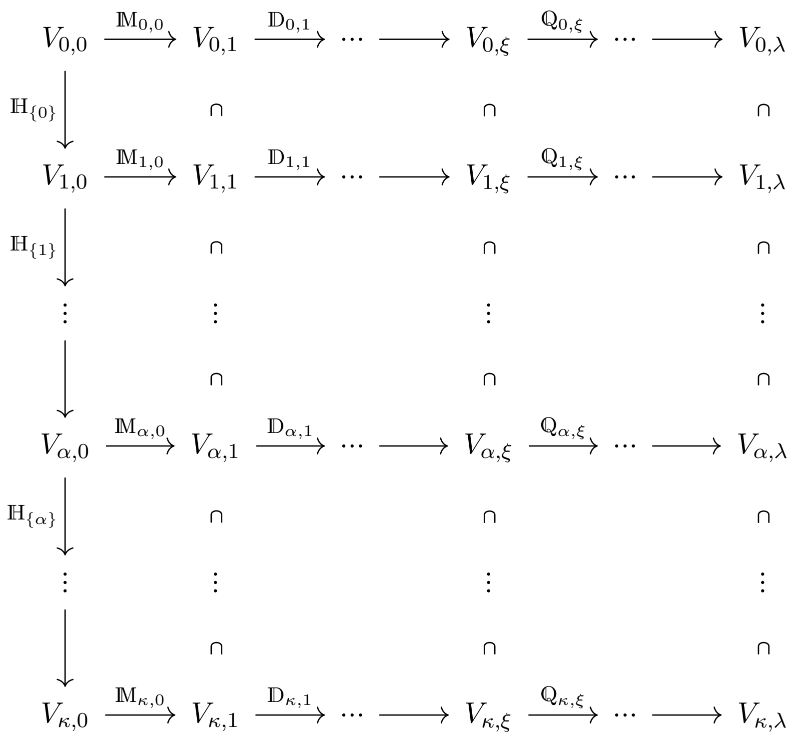

Figure 3.3: **Matrix iteration forcing for Theorem 83.** We depict all intermediate models obtained with the matrix iteration $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{Q}}_{\alpha,\xi} \rangle_{\alpha \le \kappa, \xi \le \lambda}$. First column corresponds with Hechler forcing $\mathbb{P}_{\kappa,0} = \mathbb{H}_\kappa$ introducing a maximal almost disjoint family $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ over the ground model $V_{0,0}$, considered as a finite support iteration of length κ. For each $\alpha \le \kappa$, corresponding with row α in the diagram, we have a finite support iteration $\langle \mathbb{P}_{\alpha,\xi}, \mathring{\mathbb{Q}}_{\alpha,\xi} \rangle_{\xi \le \lambda}$ of length λ combining Mathias forcing and Hechler dominating forcing. More concretely, we fix a surjection $f\colon \{\xi < \lambda : \xi = 0 \mod 2\} \to \kappa$ that takes each value $\alpha < \kappa$ cofinally often. At successor stages $\xi = \eta + 1$, we have $\mathbb{P}_{\alpha,\xi} = \mathbb{P}_{\alpha,\eta} * \mathring{\mathbb{Q}}_{\alpha,\eta}$. When $\eta = 1 \mod 2$, we take $\mathring{\mathbb{Q}}_{\alpha,\eta}$ to be a $\mathbb{P}_{\alpha,\eta}$-name for Mathias forcing $\mathbb{M}_{\mathcal{U}_{\alpha,\eta}}$ with respect to an ultrafilter $\mathcal{U}_{\alpha,\eta}$ chosen according to Lemma 84. When $\eta = 0 \mod 2$ and $\alpha > f(\eta)$, we take $\mathring{\mathbb{Q}}_{\alpha,\eta}$ to be a $\mathbb{P}_{\alpha,\eta}$-name for Hechler dominating forcing $\mathbb{D}_{F_{\alpha,\eta}}$ with respect to the family $F_{\alpha,\eta} = [\omega]^\omega \cap V_{f(\eta),\eta}$. When $\alpha \le f(\eta)$, we take $\mathring{\mathbb{Q}}_{\alpha,\eta}$ to be a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing poset. In either case, $\mathbb{P}_{\beta,\xi} \Vdash$ "$\mathring{\mathbb{Q}}_{\alpha,\xi} \lessdot \mathring{\mathbb{Q}}_{\beta,\xi}$" when $\alpha \le \beta$ for each $\xi \le \lambda$. The complete embedding $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi}$ between the forcing notions when $\alpha \le \beta \le \kappa$, give the inclusion $V_{\alpha,\xi} \subset V_{\beta,\xi}$ between the corresponding generic extensions.

stages, it will follow for even successor stages from property (i) in Lemma 84 using Lemma 62, and it also holds at limit stages by Lemma 63. Thus, the iteration is indeed a matrix iteration. Property (2) will follow from Lemma 87 at odd successor stages, from property (ii) of Lemma 84 at even successor stages, and from Lemma 88 at limit stages. Last, property (3) will follow from Lemma 89.

The following result [8, Lemma 7] can be seen as the analogue of Lemma 46 with respect to the property of *diagonalizing outside*.

Lemma 84. *Let $M \subset N$ be models of set theory, $\mathcal{A} \subset [\omega]^\omega \cap M$ and $A \in [\omega]^\omega \cap N$ such that $|\mathcal{A}| = \alpha$ and A diagonalizes M outside $\mathcal{A}$. Let $\mathcal{U} \in M$ be an ultrafilter. There exists an ultrafilter $\mathcal{V} \in N$ extending $\mathcal{U}$ such that:*

- *(i) every maximal antichain of $\mathbb{M}_\mathcal{U}$ in M remains a maximal antichain of $\mathbb{M}_\mathcal{V}$ in N,*
- *(ii) A diagonalizes $M[G]$ outside $\mathcal{A}$, when G is a $\mathbb{M}_\mathcal{V}$-generic filter over N.*

Recall that $\mathcal{U} \subset \mathcal{V}$ implies that $\mathbb{M}_\mathcal{U} \subset \mathbb{M}_\mathcal{V}$ and condition (i) tells that $\mathbb{M}_\mathcal{U} \lessdot \mathbb{M}_\mathcal{V}$. Thus, it follows from Lemma 3 that if G is $\mathbb{M}_\mathcal{V}$-generic over N, then $G \cap \mathbb{M}_\mathcal{U}$ is $\mathbb{M}_\mathcal{U}$-generic over M, and both evaluate the same real. In this sense, $\mathcal{U}$ and $\mathcal{V}$ introduce the same Mathias real (see Corollary 49). Following the lines of Lemma 46, the construction of such an ultrafilter relies on avoiding certain undesirable sets as in Definition 47.

Definition 85 (Forbidden set)**.** *A violation of (i) is given by a maximal antichain L of $\mathbb{M}_\mathcal{U}$ and a condition $(b, B) \in \mathbb{M}_\mathcal{V}$ incompatible with all elements of L. We say that such B is* forbidden *by L and b. Let $\mathring{h}$ be a $\mathbb{M}_\mathcal{U}$-name in M for a function $\omega \times [\alpha]^{<\omega} \to \omega$, with normal name $\langle W_{n,F}, f_{n,F} \rangle_{n<\omega, F\in[\alpha]^{<\omega}}$ (see Lemma 26). That is, for each $n < \omega$ and $F \subset \alpha$ finite, $W_{n,F}$ is a maximal antichain and $f_{n,F} \colon W_{n,F} \to \omega$ such that $p \Vdash$ "$\mathring{h}(n, F) = f_{n,F}(p)$" for each $p \in W_{n,F}$. A violation of (ii) would be a condition $(c, C) \in \mathbb{M}_\mathcal{V}$ forcing that "$[n, \mathring{h}(n, F)) - \bigcup \check{\mathcal{A}}|_F \notin \check{A}$" for all $n < \omega$ and $F \in [\alpha]^{<\omega}$. That is, $[n, f_{n,F}(p)) - \bigcup \mathcal{A}|_F \notin A$ for each $p \in W_{n,F}$ compatible with (c, C). We say that such C is* forbidden *by $\mathring{h}$ and c.*

Proving Lemma 84 amounts to showing that $\mathcal{U}$ can be extended to an ultrafilter in N that avoids all forbidden sets. By Lemma 8, it is enough to show that no set in $\mathcal{U}$ is covered by finitely many forbidden sets in N. Imagine $Z \in \mathcal{U}$ is covered by $B_1, \dots, B_k, C_1, \dots, C_k$ where B_i is forbidden by L_i and b_i and C_i is forbidden by $\mathring{h}_i$ and c_i for each $i \le k$. Also $Z \subset \omega - n_0$, where n_0 is the smallest number greater than all elements in each b_i and c_i. We use the following result [8, Claim 9], analogue to Claim 48.

Claim 86. *There exists $h: \omega \times [\alpha]^{<\omega} \to \omega$ in M with $h(n, F) \geq n$ such that $Z \cap [n, h(n, F))$, which can be partitioned into the subsets $B_i \cap [n, h(n, F))$ and $C_i \cap [n, h(n, F))$ with $i \leq k$, has at least one part, call it P, that verifies*

(i) for each $i \leq k$ there is a finite $e \subset P$ such that $b_i \cup e$ is permitted by L_i,
(ii) for each $i \leq k$ there is some finite $e \subset P$ such that $c_i \cup e$ is permitted by some $p \in W_{n,F}$ with $f_{n,F}(p) < h(n, F)$.

Proof. Imagine that for some $n < \omega$ and $F \in [\alpha]^{<\omega}$, none of the pieces of $Z \cap [n, h(n, F)]$ has such properties for any value of $h(n, F)$. Then, by König's Lemma, none of the pieces of $Z - n$ has such properties. Since $\mathcal{U}$ is an ultrafilter in M, there is a piece $P \in \mathcal{U}$. Take $i \leq k$, L_i is a maximal antichain in $\mathbb{M}_{\mathcal{U}}$ so there is $p \in L_i$ compatible with (b_i, P). That is, there is an extension $(b_i \cup e, P')$ compatible with p for some finite $e \subset P$, thus p permits $b_i \cup e$. Similarly, $W_{n,F}$ is a maximal antichain in $\mathbb{M}_{\mathcal{U}}$, so there is an extension $(c_i \cup e, P')$ for some finite $e \subset P$ permitted by some $p \in W_{n,F}$. Thus, such P has the desired properties provided that we take $h(n, F) > f_{n,F}(p)$, contrary to our assumption. □

Proof of Lemma 84. For any $n < \omega$ and $F \in [\alpha]^{<\omega}$ take $h: \omega \times [\alpha]^{<\omega} \to \omega$ in M and P from Claim 86. If $P = B_i \cap [n, h(n))$, there exists a finite $e \subset P \subset B_i$ such that $a_i \cup e$ is permitted by some $p \in L_i$. However, (b_i, B_i) also permits $b_i \cup e$, which contradicts B_i being forbidden by b_i and L_i. Thus, $P = C_i \cap [n, h(n))$. In this case, there exists a finite $e \subset P \subset C_i$ such that $c_i \cup e$ is permitted by some $p \in W_{n,F}$ with $f_{n,F}(p) < h(n, F)$. Notice that $c_i \cup e$ is also permitted by (c_i, C_i), which is incompatible with every $q \in W_{n,F}$ such that $[n, f_{n,F}(q)) - \bigcup \mathcal{A}|_F \subset A$. Thus, $[n, f_{n,F}(p)) - \bigcup \mathcal{A}|_F \not\subset A$ and hence $[n, h(n, F)) - \bigcup \mathcal{A}|_F \not\subset A$, contradicting that A diagonalizes M outside $\mathcal{A}$. □

Lemma 84 allows to recursively choose an ultrafilter $\mathcal{U}_{\alpha,\eta+1}$ at even successor stages satisfying property (2) above. Odd successor stages correspond with Hechler dominating forcing, for which we need the following result [8, Lemma 11].

Lemma 87. *Let $\mathbb{D} \in M$ be a forcing poset such that $\mathbb{D} \subset M$ and G a $\mathbb{D}$-generic filter over M. If A diagonalizes M outside $\mathcal{A}$, then A diagonalizes $M[G]$ outside $\mathcal{A}$.*

Proof. Imagine there is some $h:\omega\times[\alpha]^{<\omega}\to\omega$ in $M[G]$ and $m<\omega$ such that $[n,h(n,F))-\bigcup\mathcal{A}|_F\notin A$ for each $m<n<\omega$ and $F\in[\alpha]^{<\omega}$. Then, there is a $\mathbb{D}$-name $\mathring{h}$ and $p\in G$ forcing that $\mathring{h}$ witnesses that A does not diagonalize $M[G]$ outside $\mathcal{A}$. For each $m<n<\omega$ and $F\in[\alpha]^{<\omega}$ pick an extension $p_{n,F}\le p$ deciding the corresponding value of $h(n,F)$, that is, $p_{n,F}\Vdash$ "$\mathring{h}(n,H)=k_{n,F}$" for some $k_{n,F}<\omega$. This defines a function $g:\omega\times[\alpha]^{<\omega}\to\omega$ in M such that $[n,g(n,F))-\bigcup\mathcal{A}|_F\notin A$ for each $m<n<\omega$ and $F\in[\alpha]^{<\omega}$, which contradicts A diagonalizing M outside $\mathcal{A}$. □

Recall that limit stages of countable cofinality in finite support iterations always introduce Cohen reals by Lemma 59. For these, we use the following result [8, Lemma 12], analogue to Lemma 70 from Section 3.1, to ensure that property (2) holds.

Lemma 88. *If A diagonalizes $V_{\alpha,n}$ outside $\mathcal{A}$ for each $n<\omega$, then A diagonalizes $V_{\alpha,\omega}$ outside $\mathcal{A}$.*

Proof. Let $V_{\alpha,\omega}=V_{0,0}[G_{\alpha,\omega}]$ for some $\mathbb{P}_{\alpha,\omega}$-generic filter $G_{\alpha,\omega}$. Imagine that there is a function $h:\omega\times[\alpha]^{<\omega}\to\omega$ in $V_{\alpha,\omega}$ such that $[n,h(n,F))-\bigcup\mathcal{A}|_F\notin A$ for each $m<n<\omega$ and $F\in[\alpha]^{<\omega}$. Then, there is a $\mathbb{P}_{\alpha,\omega}$-name $\mathring{h}$ and a condition $p\in G_{\alpha,\omega}$ forcing it. Since the iteration has finite support, $p\in G_{\alpha,k}$ for some $k<\omega$, where $G_{\alpha,\omega}=G_{\alpha,k}*H$ for some $\mathbb{P}_{\alpha,k}$-generic filter $G_{\alpha,k}$ and $\mathbb{R}_{\alpha,k}$-generic filter H. Here, we have split the iteration as $\mathbb{P}_{\alpha,\omega}=\mathbb{P}_{\alpha,k}*\mathring{\mathbb{R}}_{\alpha,k}$ and partially evaluated $\mathbb{R}_{\alpha,k}=\mathring{\mathbb{R}}_{\alpha,k}[G_{\alpha,k}]$. Let $h'=\mathring{h}/G_{\alpha,k}$ be the partial evaluation of $\mathring{h}$. That is, a $\mathbb{R}_{\alpha,k}$-name such that $\mathring{h}[G_{\alpha,k}*H]=h'[H]$.

For each $m<n<\omega$ and $F\in[\alpha]^{<\omega}$, pick $p_{n,F}\in\mathbb{R}_{\alpha,k}$ and $c_{n,F}<\omega$ such that $p_{n,F}\Vdash$ "$h'(n,F)=c_{n,F}$". Set $c_{n,F}=0$ when $n\le m$ and define $h_0:\omega\times[\alpha]^{<\omega}\to\omega$ in $V_{\alpha,k}$ via $h_0(n,F)=c_{n,F}$, which then satisfies that $[n,h_0(n,F))-\bigcup\mathcal{A}|_F\notin A$ for each $m<n<\omega$ and $F\in[\alpha]^{<\omega}$. This contradicts the assumption that A diagonalizes $V_{\alpha,k}$ outside $\mathcal{A}$. Therefore, A diagonalizes $V_{\alpha,\omega}$ outside $\mathcal{A}$. □

With Lemmas 84, 87 and 88, we finish the recursive construction of the matrix iteration $\langle\mathbb{P}_{\alpha,\xi},\mathring{\mathbb{Q}}_{\alpha,\xi}\rangle_{\alpha\le\kappa,\xi\le\lambda}$ where the intermediate models $V_{\alpha,\xi}$ satisfy properties (1) and (2). Property (3) follows from the lemma below, which is analogue to Lemma 72 and Lemma 79 from previous sections.

Lemma 89. *For any $\xi\le\lambda$:*

(i) every condition in $\mathbb{P}_{\kappa,\xi}$ is already in $\mathbb{P}_{\alpha,\xi}$ for some $\alpha<\kappa$,
(ii) every $\mathbb{P}_{\kappa,\xi}$-name for a real is already a $\mathbb{P}_{\alpha,\xi}$-name for a real for some $\alpha<\kappa$.

Proof. First notice that (i) already implies (ii), since κ is regular and uncountable and $\mathbb{P}_{\kappa,\xi}$ is c.c.c. Thus, by Lemma 26 a name for a real can be described with countably many conditions, which cannot be cofinal in κ. The proof of (i) goes by induction. It is clear for $\xi = 0$, since $\langle \mathbb{P}_{\alpha,0} \rangle_{\alpha \leq \kappa}$ is a finite support iteration. At even stages in the iteration, which correspond with Matthias forcing, use the proof in Lemma 72. At (non-trivial) odd stages in the iteration, which correspond with Hechler dominating forcing, use the proof in Lemma 79. □

As in the previous section, we show that the maximal almost disjoint family introduced with Hechler forcing $\mathbb{P}_{\kappa,0} = \mathbb{H}_\kappa$ is preserved through the iteration.

Lemma 90. *The maximal almost disjoint family $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ introduced by Hechler forcing $\mathbb{H}_\kappa$ over the ground model $V_{0,0}$ remains a maximal almost disjoint family in the final extension $V_{\kappa,\lambda}$.*

Proof. For any $B \in [\omega]^\omega \cap V_{\kappa,\lambda}$, there is $\alpha < \kappa$ such that $B \in V_{\alpha,\lambda}$ by Lemma 89. By property (2) of the matrix iteration, $A_{\alpha+1}$ diagonalizes $V_{\alpha,\lambda}$ outside $\mathcal{A}|_{\alpha+1} = \{A_\beta\}_{\beta\leq\alpha}$. By Lemma 76, this implies that either $|A_{\alpha+1} \cap B| = \omega$ or B is almost contained in some A_β with $\beta \leq \alpha$. In both cases, $|A_\beta \cap B| = \omega$ for some $\beta < \kappa$. Thus, $\mathcal{A}$ is a maximal almost disjoint family in the final extension $V_{\kappa,\lambda}$. □

This already provides the first part of Theorem 83.

Proposition 91. $\mathfrak{b} = \mathfrak{a} = \kappa$.

Proof. The maximal almost disjoint family introduced by Hechler forcing $\mathbb{P}_{\kappa,0} = \mathbb{H}_\kappa$ over the ground model $V_{0,0}$ remains a maximal almost disjoint family in the final extension $V_{\kappa,\lambda}$, by Lemma 90. Thus, $\mathfrak{a} \leq \kappa$.

For any $B \in [\omega]^\omega \cap V_{\kappa,\lambda}$, there is $\alpha < \kappa$ such that $B \in V_{\alpha,\lambda}$ by Lemma 89. Thus, for any $\mathcal{B} \subset [\omega]^\omega \cap V_{\kappa,\lambda}$ with $|\mathcal{B}| < \kappa$, there are $\alpha < \kappa$ and $\xi < \lambda$ such that $\mathcal{B} \subset V_{\alpha,\xi}$, since $\kappa < \lambda$ and λ is regular. Since f takes the value α cofinally often, pick an even $\eta > \xi$ with $f(\eta) = \alpha$. Then, $\mathring{Q}_{\alpha,\eta}$ is a name for Hechler forcing with respect to $F_{\alpha,\eta} = V_{\alpha,\eta}$. By Claim 52, $\mathbb{P}_{\alpha,\eta+1}$ introduces a real dominating all reals in $V_{\alpha,\eta}$, which contains $V_{\alpha,\xi} \supset \mathcal{B}$. This implies that $\kappa \leq \mathfrak{b}$. Finally, $\mathfrak{b} \leq \mathfrak{a}$ from Proposition 19, thus $\mathfrak{b} = \mathfrak{a} = \kappa$. □

Second part of Theorem 83 readily follows from the properties of Mathias forcing established for the construction in Section 3.1 and counting names.

Proposition 92. $\mathfrak{s} = \mathfrak{c} = \lambda$.

Proof. Since $\mathbb{P}_{\kappa,\lambda}$ is a c.c.c. forcing notion with size $|\mathbb{P}_{\kappa,\lambda}| \leq \lambda$ over the ground model, $\mathfrak{c} \leq \lambda$ by counting names as in Lemma 27. Moreover, each $\mathbb{P}_{\kappa,\xi} \lessdot \mathbb{P}_{\kappa,\lambda}$ with $\xi < \lambda$ introduces a real over $V_{\kappa,\xi}$, thus $\lambda \leq \mathfrak{c}$. To see that $\lambda \leq \mathfrak{s}$, take a family $\mathcal{S} \subset [\omega]^\omega$ in $V_{\kappa,\lambda}$ with $|\mathcal{S}| < \lambda$. There is some $\xi = \eta + 1 = 0 \mod 2$ such that $\mathcal{S} \subset V_{\kappa,\eta}$, since the iteration has finite support. By Claim 44, the Mathias real s introduced by $\mathbb{M}_{\kappa,\eta}$ is almost contained in every $A \in \mathcal{U}_{\kappa,\eta}$, thus can not split by $\mathcal{S}$. Indeed, for each $B \in S$ either $B \in \mathcal{U}_{\kappa,\eta}$ and $s \subset^* B$, or $(\omega - B) \in \mathcal{U}_{\kappa,\eta}$ and $s \subset^* \omega - B$. In both cases, either $|s \cap (\omega - B)|$ or $|s \cap B|$ is finite. □

This concludes the proof of Theorem 83.

3.4 Filter combinatorics

Unrestricted Matthias forcing $\mathbb{M}$ introduces a dominating real. The question of whether Matthias forcing $\mathbb{M}_{\mathcal{U}}$ with respect to some ultrafilter $\mathcal{U}$ does not introduce a dominating real was posed and answer in the positive by Canjar [10]. The construction from [7] forcing with matrix iterations to show the consistency of $\mathfrak{u} = \nu < \delta = \mathfrak{d} = \mathfrak{c}$, discussed in Section 3.1 and sketched in Figure 3.1, provides an example of an ultrafilter $\mathcal{U}_{\delta,\xi}$ in $V_{\delta,\xi}$ chosen according to Lemma 46 with the property that $\mathbb{M}_{\mathcal{U}_{\delta,\xi}}$ does not introduce dominating reals. Similarly, the construction from [8], discussed in Section 3.3 and sketched in Figure 3.3, to show the consistency of $\mathfrak{b} = \mathfrak{a} = \kappa < \lambda = \mathfrak{s}$ produces an ultrafilter $\mathcal{U}_{\kappa,\nu}$ in $V_{\kappa,\nu}$ which has the property that $\mathbb{M}_{\mathcal{U}_{\kappa,\nu}}$ preserves the unboundedness of a small family of reals. Namely, the family $F_{\kappa,0}$ of reals of $V_{\kappa,0}$, where $|F_{\kappa,0}| = \kappa$ and $\mathfrak{c} = \nu$ in $V_{\kappa,\nu}$ for each $\kappa^+ < \nu < \lambda$. There is a line of research in set theory around the preservation of small families of reals. This construction, to the best knowledge of the author, is the most general instance of such phenomena.

The following result [10, Theorem 10] generalizes the construction in [7].

Theorem 93. *Assume $\mathfrak{d} = \mathfrak{c}$. Then there exists an ultrafilter $\mathcal{U}$ with the property that $\mathbb{M}_{\mathcal{U}}$ does not introduce a dominating real.*

One limitation of this result is that it can not be iterated. Results from [10] were further strengthened and generalized in [20]. In particular, the theorem below [20, Theorem 2.5] can be used within an iterated forcing construction to obtain a model of $\mathfrak{b} = \kappa < \kappa^+ = \mathfrak{s}$.

Theorem 94. *Let κ be a regular cardinal such that $2^\lambda \leq \kappa$ for each $\lambda < \kappa$ and assume $\mathrm{cov}(\mathcal{M}) = \kappa$. Then there is an ultrafilter $\mathcal{U}$ such that $\mathbb{M}_{\mathcal{U}}$ does not introduce a dominating real.*

The ultrafilters that appear in the matrix iteration construction from [8] leading to the consistency of $\mathfrak{b} = \kappa < \lambda = \mathfrak{s}$, revisited in Section 3.3 and depicted in Figure 3.3, can be seen as a generalization of Theorem 93 and Theorem 94. At the same time, these results can be seen as generalizations of the construction from [7] leading to the consistency of $\mathfrak{u} = \nu < \delta = \mathfrak{d}$, revisited in Section 3.1 and depicted in Figure 3.1. This small discussion illustrates further applications of forcing techniques with matrix iterations in other current research areas of set theory such as filter combinatorics.

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