# DISSERTATION

Titel der Dissertation

"$q, t$-Fuß-Catalan numbers for finite reflection groups"

"$q, t$-Fuß-Catalan Zahlen für endliche Spiegelungsgruppen"

Verfasser

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English Abstract. In type $A$, the $q,t$-Catalan numbers and the $q,t$-Fuß-Catalan numbers can be defined as a bigraded Hilbert series of a module associated to the symmetric group $\mathcal{S}_n$. We generalize this construction to (finite) complex reflection groups and exhibit many nice conjectured algebraic and combinatorial properties of these symmetric polynomials in $q$ and $t$ with non-negative integer coefficients.
We combinatorially define $q$-Fuß-Catalan numbers as the generating function of a statistic on the extended Shi arrangement which seem to describe the specialization $t = 1$ in the $q,t$-Fuß-Catalan numbers.
The exhibited statistic yields a definition of Catalan paths of type $B$ of which we further investigate several properties. In particular, we define for types $A$ and $B$ bijections between Catalan paths, non-crossing partitions and Coxeter sortable elements which transfer arising questions concerning the $q,t$-Catalan numbers from Catalan paths to non-crossing partitions and to Coxeter sortable elements.
Finally, we present several ideas how the $q,t$-Fuß-Catalan numbers could be related to some graded Hilbert series of modules arising in the context of rational Cherednik algebras and thereby generalize known connections.

Deutsche Zusammenfassung. Im Typ $A$ können die $q,t$-Catalan Zahlen und die $q,t$-Fuß-Catalan Zahlen als bigraduierte Hilbertreihe eines Moduls über der symmetrischen Gruppe $\mathcal{S}_n$ definiert werden. Wir verallgemeinern diese Konstruktion auf (endliche) komplexe Spiegelungsgruppen und beschreiben einige vermutete algebraische und kombinatorische Eigenschaften dieser symmetrischen Polynome in $q$ und $t$ mit nicht-negativen ganzzahligen Koeffizienten.
Weiterhin definieren wir $q$-Fuß-Catalan Zahlen kombinatorisch als Erzeugendenfunktion einer Statistik auf dem verallgemeinerten Shi-Gefüge. Diese scheinen die Spezialisierung $t = 1$ der $q,t$-Fuß-Catalan Zahlen zu beschreiben.
Diese neue Statistik führt zu einer Definition von Catalan-Pfaden im Typ $B$, welche wir auf weitere Eigenschaften hin untersuchen. Unter anderem finden wir für die Typen $A$ und $B$ Bijektionen zwischen Catalan-Pfaden, nicht-kreuzenden Partitionen und Coxeter-sortierbaren Elementen, die Fragestellungen bzgl. der $q,t$-Catalan Zahlen von Catalan-Pfaden zu nicht-keuzenden Partitionen und Coxeter-sortierbaren Elementen transferiert.
Schließlich präsentieren wir einige Ideen, wie die $q,t$-Fuß-Catalan Zahlen mit Moduln, die im Kontext von rationalen Cherednik-Algebren auftreten, in Beziehung stehen könnten. Dabei verallgemeinern wir bereits untersuchte Verbindungen.


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Christian Stump

# Contents

# Introduction

One of the most famous and most studied integer sequences in combinatorics is the sequence of Catalan numbers. These are defined as

$$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$$

and are named after the Belgian mathematician Eugène Charles Catalan who discovered them while considering an elegant solution of the problem

*"In how many ways is it possible to dissect a convex $(n+2)$-gon into triangles?"*.

In his book "Enumerative Combinatorics Vol. 2"[**95**], R.P. Stanley lists more than 66 combinatorial interpretations of these numbers.

Several authors studied combinatorially defined $q$-extensions of the Catalan numbers. The two we mostly deal with are given by the generating function for the *major index*, which was introduced by P.A. MacMahon in [**81**], and by the generating function for the *area statistic*, which was considered by J. Fürlinger and J. Hofbauer in [**51**]. Both statistics are defined on Catalan paths of length $n$ which are lattice paths consisting of $2n$ steps from $(0,0)$ to $(n,n)$ that stay above the diagonal $x = y$. These paths are well-known to be counted by the $n$-th Catalan number $\mathrm{Cat}_n$.

The notion of Catalan paths can be generalized to $m$-Catalan paths by considering instead lattice paths from $(0,0)$ to $(mn,n)$ that stay above the diagonal $x = my$. These paths are counted by a generalization of the Catalan numbers which we call Fuß-Catalan numbers and which are defined as

$$\mathrm{Cat}_n^{(m)} := \frac{1}{mn+1}\binom{(m+1)n}{n}.$$

The notion for a Catalan path to have a certain area naturally generalizes to $m$-Catalan paths. Unfortunately, no such generalization is known for the major index, but MacMahon's $q$-Catalan numbers factorize nicely and this factorization gives rise to a natural generalization of this generating function.

A. Postnikov connected Catalan paths in a very intuitive way to the reflection group of type $A$ and generalized them to all irreducible reflection groups by introducing *non-nesting partitions* [**86**, Remark 2]. V. Reiner then observed that there exists a type-independent expression counting these non-nesting partitions, see [**86**]. C.A. Athanasiadis later also generalized the notion of $m$-Catalan paths as well as of Reiner's uniform expression by considering *filtered chains of non-nesting partitions*, see [**9**]. The Fuß-Catalan numbers $\mathrm{Cat}^{(m)}(W)$ associated to a reflection group $W$ are defined as this uniform expression,

$$\mathrm{Cat}^{(m)}(W) := \prod_{i=1}^{l} \frac{d_i + mh}{d_i}.$$

Here, $l$ is the *rank* of $W$, $d_1 \leq \ldots \leq d_l$ are its *degrees* and $h$ is its *Coxeter number.* They reduce in type $A$ to the classical Fuß-Catalan numbers,

$$\mathrm{Cat}^{(m)}(A_{n-1}) = \mathrm{Cat}_n^{(m)}.$$

This notion makes even sense for any (finite) complex reflection group $W$ but if $W$ is not well-generated, $\mathrm{Cat}^{(m)}(W)$ may fail to be an integer. In full generality of well-generated complex reflection groups, this product was first considered by D. Bessis in [**20**] where he constructed objects called *chains in the non-crossing partition lattice* and where he showed that these are counted by $\mathrm{Cat}^{(m)}(W)$. His work generalizes the work of T. Brady and C. Watt [**28, 30**] for the classical groups.

The work on non-nesting partitions on the one hand and on non-crossing partitions – which are certain elements in the reflection group – on the other hand suggests that both are, at least in a numerical sense, deeply connected. These connections were nicely described in D. Armstrong's PhD thesis, see [**7**, Chapter 5.1.3]. Unfortunately, no bijections between both are known so far, except for type $A$, and these connections are still mysterious and not well understood.

One contribution of this thesis is to generalize the notion of the area statistic on $m$-Catalan paths to all filtered chains of non-nesting partitions, or equivalently to *positive regions in the extended Shi arrangement* (Definition 2.18), and to exhibit bijections between non-nesting and non-crossing partitions for types $A$ and $B$ which translate this generalized statistic on non-nesting partitions to the length function (in the sense of reflection groups) on non-crossing partitions (Theorem 2.48).

Recently, N. Reading introduced the notion of *Coxeter sortable elements* [**83**]. These are also elements in the reflection group satisfying conditions different from non-crossing partitions but which are in bijection with these. We also exhibit bijections between non-nesting partitions and Coxeter sortable elements in types $A$ and $B$ having the same properties as the bijections between non-nesting and non-crossing partitions (Theorem 2.63). As Coxeter sortable elements in type $A$ can be considered as 231-avoiding permutations, this bijection leads to a new perception of bijections between Catalan paths and 231-avoiding permutations, which were widely studied by several authors [**14, 33, 73, 84**].

The second main contribution of this thesis is more of algebraic nature. It was actually discovered first and the combinatorial part of the thesis should also be seen as efforts in the combinatorial understanding of the algebraic objects which arise.

In type $A$, the *$q,t$-Catalan numbers* and their generalization, which we call *$q,t$-Fuß-Catalan numbers*, appeared first in two seemingly unrelated fields of mathematics:

M. Haiman defined them as bigraded Hilbert series of certain modules arising in the representation theory of *diagonal coinvariant rings* [**63**], and A. Garsia and Haiman defined them shortly later as a complicated rational function in $q$ and $t$ which arose in the context of *modified Macdonald polynomials* [**55**]. Using the second definition, they were able to prove that this rational function, which is by construction symmetric in $q$ and $t$, defines a $q,t$-extension of the Fuß-Catalan numbers in the sense that specializing the variables $q$ and $t$ to 1 reduces this rational function to $\mathrm{Cat}_n^{(m)}$. Furthermore, they were able to show that specializing $t = q^{-1}$ and multiplying by the highest power of $q$ recovers MacMahon's $q$-Fuß-Catalan numbers and specializing $t = 1$ recovers Fürlinger and Hofbauer's $q$-Fuß-Catalan numbers.
Finally, after 15 years of intensive research, Haiman proved the equivalence of both definitions in a series of papers in which he proved the $n!$-*conjecture* and the $(n+1)^{n-1}$-*conjecture* [**63, 64, 65, 67**].

The definition as a bigraded Hilbert series implies that the $q,t$-Catalan numbers and the $q,t$-Fuß-Catalan numbers are in fact polynomials with non-negative integer coefficients. This yields the problem of finding a second statistic on Catalan paths (which, by symmetry, has to be equally distributed with area) such that these polynomials can be completely described combinatorially. In [**60**], J. Haglund found such a statistic, which he called *bounce*, and together with Garsia, he proved in [**55**] for the $q,t$-Catalan numbers that the complicated rational function is in fact equal to the generating function for the bistatistic $(area, bounce)$ on Catalan paths. In [**77**], N. Loehr generalized the definition of the bounce statistic and conjectured that the bistatistic $(area, bounce)$ also describes the $q,t$-Fuß-Catalan numbers. This conjecture is still open.

This first definition of the $q,t$-Catalan numbers describes them as the bigraded Hilbert series of the alternating component of the diagonal coinvariant ring $DR_n$ which is defined as

$$DR_n := \mathbb{C}[\mathbf{x},\mathbf{y}]/\mathcal{I},$$

where $\mathbb{C}[\mathbf{x},\mathbf{y}] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ is the polynomial ring in two sets of variables on which the symmetric group $\mathcal{S}_n$ acts diagonally and where $\mathcal{I}$ is the ideal generated by all *invariant polynomials without constant term.*
Haiman proved that the dimension of the diagonal coinvariant ring is equal to $(n+1)^{n-1}$, see [**65**]. This expression can be written in terms of the reflection group of type $A_{n-1}$ as $(h+1)^l$ where $h = n$ is the Coxeter number of type $A_{n-1}$ and $l = n-1$ is its rank. Furthermore, he considered the diagonal coinvariant ring for the other classical reflection groups but their dimensions seemed to be "too big", see Figure 31 on page 88 for their actual dimensions. This phenomenon was solved for crystallographic reflection groups by I. Gordon. In [**57**], he showed that there exists a surjection

$$DR(W) \otimes \epsilon \twoheadrightarrow \mathrm{gr}(L), \tag{1}$$

where $L$ is a module over the *rational Cherednik algebra* depending on a parameter $c = 1 + \frac{1}{h}$ and where $\mathrm{gr}(L)$ is its associated graded module.

By work of Y. Berest, P. Etingof and V. Ginzburg [**15**], this module has, for the more general parameter $c = m + \frac{1}{h}$, dimension equal to $(mh+1)^l$ and its Hilbert series is given by

$$\mathcal{H}(\mathrm{gr}(L); q) = ([mh+1]_q)^l.$$

Furthermore, its invariant component has dimension $\mathrm{Cat}^{(m)}(W)$ and the Hilbert series equals MacMahon's $q$-Fuß-Catalan number,

$$\mathcal{H}(\mathbf{e}\,\mathrm{gr}(L); q) = \text{diag-}\mathrm{Cat}^{(m)}(W; q) := \prod_{i=1}^{l} \frac{[d_i + mh]_q}{[d_i]_q}.$$

As the surjection in (1) is *not* a bijection in general, this seemed to be the end of the story concerning the diagonal coinvariant ring for other reflection groups.

The starting point of this thesis were several computer experiments with the computer algebra systems `singular` [**92**] and `Macaulay 2` [**78**] which were inspired by [**66**, Problem 1.13(b)].

We considered only the alternating component of the diagonal coinvariant ring,

$$M := \mathcal{A}/\langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A},$$

as well its generalization

$$M^{(m)} := \mathcal{A}^m/\langle \mathbf{x}, \mathbf{y} \rangle \mathcal{A}^m,$$

which is equal to the alternating component of the generalized coinvariant ring

$$DR^{(m)}(W) := \left(\mathcal{A}^{m-1}/\mathcal{A}^{m-1}\mathcal{I}\right) \otimes \epsilon^{\otimes(m-1)},$$

where $\mathcal{A}$ is the ideal generated by all *alternating polynomials*.

The computations suggested that this alternating part still has dimension $\mathrm{Cat}^{(m)}(W)$ and this leads to the main definition of this thesis (Definition 4.6): the $q,t$-Fuß-Catalan numbers $\mathrm{Cat}^{(m)}(W; q, t)$ are defined as the bigraded Hilbert series of $M^{(m)}$,

$$\mathrm{Cat}^{(m)}(W; q, t) := \mathcal{H}(M^{(m)}; q, t).$$

Based on the computations, we conjecture that $\mathrm{Cat}^{(m)}(W; q, t)$ reduces in fact for $q = t = 1$ to the Fuß-Catalan numbers and the specialization $t = q^{-1}$ in $\mathrm{Cat}^{(m)}(W; q, t)$ yields, up to a power of $q$, MacMahon's $q$-Fuß-Catalan numbers $\text{diag-}\mathrm{Cat}^{(m)}(W; q)$ (Conjecture 4.7).

Following work of J. Alfano and E. Reiner [**63**], we prove both conjectures for the dihedral groups (Corollary 4.12) and give an explicit expression for the $q,t$-Fuß-Catalan numbers in this case (Theorem 4.13).

Using descriptions of $\mathrm{gr}(L)$ from [**15**], we generalize Gordon's surjection to all real reflection groups and also to surjections between the generalized diagonal coinvariant ring and $\mathrm{gr}(L)$ for the parameter $c = m + \frac{1}{h}$ (Theorem 5.7). This implies that for real reflection groups and for arbitrary $m$, both conjectures would be implied by (and in fact are *both* equivalent to) the conjecture that the kernel of the constructed surjection does not contain a copy of the trivial representation (Conjecture 5.9).

Like in type $A$, the definition of the $q,t$-Fuß-Catalan numbers implies that they are symmetric polynomials in $q$ and $t$ with non-negative integer coefficients. This yields again the problem of finding a combinatorial description. We also made some progress concerning the first statistic but again we are only able to state a conjecture: the specialization $t = 1$ in $\mathrm{Cat}^{(m)}(W; q, t)$ can be combinatorially described by the generalization of the area statistic on filtered chains of non-nesting partitions or on positive regions in the extended Shi arrangement which was mentioned above (Conjecture 4.8).

This thesis is organized as follows:

In Chapter 1, we give an overview of the theory of reflection groups. First, we introduce real reflection groups and then we generalize the notions we need to complex reflection groups. At the end of this chapter, we briefly introduce some basic representation theory in general and in particular some representation theory concerning reflection groups.

In Chapter 2, we first recall some classical combinatorial constructions, namely Catalan paths, set partitions and pattern-avoiding permutations and their connections. In Section 2.1, we introduce Fuß-Catalan numbers for well-generated complex reflection groups and as well a $q$-extension generalizing MacMahon's $q$-Fuß-Catalan numbers and show how these constructions reduce to the classical constructions in type $A$. In Sections 2.2 and 2.3, we define non-nesting partitions and the closely related extended Shi arrangement which generalize the notion of Catalan paths, and in Section 2.4, we define the coheight statistic on regions in the extended Shi arrangement and show that this statistic generalizes the area statistic on Catalan paths. In particular, this yields a definition of Catalan paths of type $B$ which can also be seen as lattice paths satisfying certain conditions, and we prove a recurrence and a generating function identity for those. In Sections 2.5 and 2.6, we define non-crossing partitions and recall several of their properties and we give a bijection between non-nesting and non-crossing partitions in types $A$ and $B$ which simultaneously describes the area and the major index on the appropriate Catalan paths in terms of non-crossing partitions. Finally, in Sections 2.7–2.9, we define Coxeter sortable elements, construct in types $A$ and $B$ a bijection between non-nesting partitions and Coxeter sortable elements with the same properties, and we describe this bijection in type $A$ in terms of known bijections.

In Chapter 3, we describe the history of the $q,t$-Catalan numbers and the $q,t$-Fuß-Catalan numbers in type $A$. In Section 3.1, we introduce the ring of symmetric functions, its most important bases including *Schur functions* and *Macdonald polynomials*, some combinatorics arising in this context, the *nabla operator* and *plethystic notation*. In Sections 3.2 and 3.3, we define the *Garsia-Haiman module* and the *diagonal coinvariant ring* and finally in Section 3.4, we define the $q,t$-Fuß-Catalan numbers in type $A$ and recall its combinatorial properties.

In Chapter 4, we generalize the definition of $q,t$(-Fuß)-Catalan numbers to all complex reflection groups and exhibit several beautiful algebraic and combinatorial conjectures. In particular, we prove the conjectures for the dihedral groups.

In Chapter 5, we describe the connection between $q,t$-Fuß-Catalan numbers and rational Cherednik algebras.

In Appendix A, we list all our computer experiments. First we list the computations of the dimension of $M^{(m)}$ for the classical groups of lower rank and for small $m$ as well as for several exceptional reflection groups using `singular` [**92**], second we list its bigraded Hilbert series using `Macaulay 2` [**78**]. Finally we list the computations of the coheight statistic on positive regions in the extended Shi arrangement. For those, we used `GAP` [**52**].

CHAPTER 1

# Reflection groups

In this first chapter, we provide an introduction to the theory of *finite reflection groups*.

We mainly focus on *real reflection groups* as we mostly deal with these. They are discussed in the first part of this chapter mainly following the book "Reflection Groups and Coxeter Groups" by J.E. Humphreys [**70**]. Further brilliant references for Coxeter groups are [**26, 68**] and for Coxeter's original approach see [**37**].

The second part of this chapter deals with *complex reflection groups* which generalize the notion of real reflection groups. We briefly state the facts we need in this more general context. Complex reflection groups were classified by G.C. Shephard and J.A. Todd in the famous paper "Finite unitary reflection groups" [**90**]. For further information about complex reflection groups see e.g. [**32, 36, 82, 89, 90**].

We only deal with *finite* reflection groups, therefore we suppress the term *finite*. For further information on infinite (real) reflection groups, called *affine reflection groups*, we again refer to [**70**].

At the end of this chapter, we give a brief overview of the representation theoretical concepts we need. They are adapted from the book "Representation Theory, A first course" by W. Fulton and J. Harris [**50**].

## 1.1. Real reflection groups

The main tool for studying finite groups generated by reflections in real vector spaces of finite dimension is a well-chosen set of vectors called *roots* which are orthogonal to reflecting hyperplanes. A subset of *simple roots* yields an efficient generating set for the group, leading eventually to a very simple presentation by generators and relations as a *Coxeter group*.

**1.1.1. Reflections.** Let $V$ be a finite dimensional real vector space endowed with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$. A reflection in $V$ is a linear operator $s_\alpha$ on $V$ which sends some nonzero vector $\alpha$ to its negative while fixing pointwise the hyperplane $H_\alpha$ orthogonal to $\alpha$. There is a simple formula for this linear operator:

$$s_\alpha(\lambda) = \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\, \alpha.$$

It is easy to see that this formula is correct: it is obviously correct for $\lambda = \alpha$ and for $\lambda \in H_\alpha$ and $V$ can be written as $V = \mathbb{R}\alpha \oplus H_\alpha$.

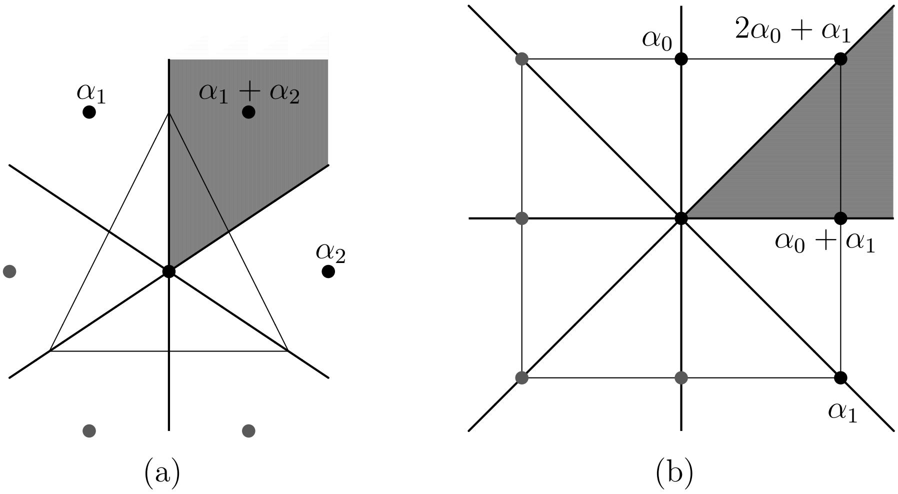



FIGURE 1. The reflection groups of types $I_2(3) = A_2$ and $I_2(4) = B_2$.

The bilinearity of $\langle \cdot, \cdot \rangle$ implies that $s_\alpha$ is an orthogonal transformation, i.e.,

$$\langle s_\alpha(\lambda), s_\alpha(\mu) \rangle = \langle \lambda, \mu \rangle,$$

and as $s_\alpha^2 = 1$, $s_\alpha$ has order 2 in the group $\mathrm{O}(V) \subseteq \mathrm{GL}(V)$ of all orthogonal transformations of $V$.

DEFINITION 1.1. A reflection group is a finite subgroup of $\mathrm{O}(V)$ generated by reflections.

REMARK. Mostly, we denote reflection groups by the letter $W$. This letter is used as "most" reflection groups turn out to be *Weyl groups* associated with *semisimple Lie algebras*, see e.g. [**27**], [**69**] and [**70**].

The following examples of reflection groups are labelled by *types* according to the classification of real reflection groups which will be carried out in Section 1.1.7:

EXAMPLE 1.2 ($I_2(k), k \geq 3$). The first reflection group we deal with is the dihedral group $\mathcal{D}_k$. It is the group of order $2k$ consisting of all orthogonal transformations which preserve the *regular $k$-gon*. $\mathcal{D}_k$ contains $k$ rotations (through multiples of $2\pi/k$) and $k$ reflections (about the *diagonals* of the regular $k$-gon, i.e. a line bisecting the $k$-gon, joining two vertices or the midpoints of opposite sides if $k$ is even, or joining a vertex to the midpoint of the opposite side if $k$ is odd). This group is generated by reflections as a rotation through $2\pi/k$ can be achieved as a product of two reflections relative to a pair of adjacent diagonals which meet at an angle of $\pi/k$. The regular 3- and 4-gons as well as the appropriate reflecting diagonals are shown in Figure 1.

EXAMPLE 1.3 ($A_{n-1}, n \geq 2$). The second reflection group we want to look at is the symmetric group $\mathcal{S}_n$. Group-theoretically, this is the group of all bijections of the set $[n] := \{1, \ldots, n\}$. Its elements are called permutations. $\mathcal{S}_n$ can be thought of as a subgroup of $\mathrm{O}(V)$ for $V = \mathbb{R}^n$ by permuting the standard basis vectors $\epsilon_1, \ldots, \epsilon_n$. Observe that the transposition $t_{ij}$ which interchanges $i$ and $j$ then acts as the reflection $s_{\epsilon_j - \epsilon_i}$. Since $\mathcal{S}_n$ is generated by its transpositions, it is a reflection group and as it is minimally generated by its simple transpositions $t_{i,i+1}$, we denote this minimal set of

generators of the reflection group by $s_i$ for $1 \leq i < n$.
The action of $\mathcal{S}_n$ on $\mathbb{R}^n$ fixes pointwise the line spanned by $\epsilon_1 + \ldots + \epsilon_n$ and these are clearly the only fixed points. Furthermore, it leaves stable the orthogonal complement which consists of the points with coordinates summing up to 0. Thus $\mathcal{S}_n$ also acts on an $(n-1)$-dimensional vector space as a group generated by reflections, fixing no point except the origin. This accounts for the subscript in the label $A_{n-1}$. In Figure 1(a), the reflection group of type $A_2$ is shown (with the projection of $\mathbb{R}^3$ on the subspace of points with coordinates summing up to 0).

Example 1.4 ($B_n, n \geq 2$). Again, let the symmetric group $\mathcal{S}_n$ act on $\mathbb{R}^n$ as above. Define further reflections $t_i := s_{\epsilon_i}$ for $1 \leq i \leq n$ and $t_{i,-j} := s_{\epsilon_i + \epsilon_j}$. These reflections together with the transpositions $t_{ij}$ generate the reflection group of type $B_n$. As a group, it can be considered as the group of all bijections of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying $\phi(-i) = -\phi(i)$. It is called group of signed permutations. In Figure 1(b), the reflection group of type $B_2$ is shown. Observe that this reflection group is minimally generated by the reflections $s_0 := t_1$ and $s_i = t_{i,i+1}$ for $1 \leq i < n$.

Example 1.5 ($D_n, n \geq 4$). When we allow only those signed permutations that change an even number of signs, we get another reflection group acting on $\mathbb{R}^n$ which is a subgroup of index 2 in the reflection group of type $B_n$. This group is called group of even signed permutations. The reflections in this group are given by all transpositions $t_{ij}$ and $t_{i,-j}$ and it is minimally generated by the reflections $s_0 := t_{1,-2}$ and $s_i = t_{i,i+1}$ for $1 \leq i < n$.

**1.1.2. Roots.** To understand the group structure of a reflection group $W$, we first explore the way in which $W$ acts on the vector space $V$: each reflection $s_\alpha \in W$ determines a reflecting hyperplane $H_\alpha$ and a line $\mathbb{R}\alpha$ orthogonal to $H_\alpha$. As for $\omega \in W$,

$$s_{\omega\alpha} = \omega s_\alpha \omega^{-1} \in W,$$

$W$ permutes the collection of all lines $\mathbb{R}\alpha$ where $s_\alpha$ ranges over the set of reflections contained in $W$. For example, the dihedral group $\mathcal{D}_4$ preserves the collection of lines through the following eight vectors in $\mathbb{R}^2$, see Figure 1(b):

$$\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1).$$

Definition 1.6. A root system $\Phi$ is a finite set of nonzero vectors in $V$ satisfying

(i) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Phi$,
(ii) the associated reflection group $W(\Phi) := \langle s_\alpha : \alpha \in \Phi \rangle$ permutes $\Phi$ among itself, i.e.,

$$s_\alpha(\beta) \in \Phi$$

for all $\alpha, \beta \in \Phi$.

The elements in $\Phi$ are called roots.

Remark. The term "root" comes again from the historical connection with semisimple Lie algebras.

The group $W(\Phi)$ is, by definition, generated by reflections and as one can embed $W(\Phi)$ into the permutation group of $\Phi$, it is finite. This shows that for any root system $\Phi$, $W(\Phi)$ is a reflection group. On the other hand, the above discussion shows that every reflection group is equal to $W(\Phi)$ for some root system $\Phi$.

**1.1.3. Positive roots and simple roots.** Fix a root system $\Phi$ and the associated reflection group $W = W(\Phi)$. While $W$ is completely determined by the geometric configuration of $\Phi$, the size of $\Phi$ can be extremely large compared with the dimension of $V$. For example, when $W$ is the dihedral group, $\Phi$ may have just as many elements as $W$, even though the dimension of $V$ is 2.
This leads us to the notion of simple roots from which $\Phi$ can somehow be reconstituted:

DEFINITION 1.7. A simple system in $\Phi$ is a subset $\Delta \subseteq \Phi$, such that

(i) $\Delta$ forms a vector space basis of the $\mathbb{R}$-span of $\Phi$ in $V$ and
(ii) any $\alpha \in \Phi$ is a linear combination of $\Delta$ with coefficients all of the same sign.

For the fact that every root system contains a simple system, see [**70**, Theorem 1.3]. The elements of the simple system are called simple roots and the associated reflections are called simple reflections. A simple system yields a partition of $\Phi$ into positive roots and negative roots:

DEFINITION 1.8. A positive system in $\Phi$ is a subset $\Phi^+ \subseteq \Phi$ consisting of all $\alpha \in \Phi$ which are positive linear combinations of a simple system $\Delta$, its elements are called positive roots.

REMARK. One could have started also with a positive system by saying that the positive roots are these lying on one side of a generic hyperplane in $V$ (i.e. a hyperplane that does not contain a root) and then showing that every positive system contains a unique simple system.

EXAMPLE 1.9. In Figure 1(a), one possible choice of a simple and a positive system is

$$\Delta = \{\alpha_1, \alpha_2\} \quad , \quad \Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}.$$

and in Figure 1(b), one possible choice of a simple and positive system is

$$\Delta = \{\alpha_0, \alpha_1\} \quad , \quad \Phi^+ = \{\alpha_0, \alpha_1, \alpha_0 + \alpha_1, 2\alpha_0 + \alpha_1\}.$$

From the previous remark one can deduce the following corollary:

COROLLARY 1.10. *Any two positive systems respectively simple systems in $\Phi$ are conjugate under the action of $W$.*

As we have seen, a given choice of simple and positive roots is as good as another. In types $A$, $B$ and $D$, we fix the "natural" choices explored in Examples 1.3 – 1.5: in type $A_{n-1}$, we have seen that the reflections are given by the transpositions $t_{ij} = s_{\epsilon_j - \epsilon_i}$ and that it is minimally generated by $s_i = t_{i,i+1}$ for $1 \leq i < n$. This yields the following choice of simple and positive roots:

$$\Delta = \big\{\epsilon_{i+1} - \epsilon_i : 1 \leq i < n\big\} \quad , \quad \Phi^+ = \big\{\epsilon_j - \epsilon_i : 1 \leq i < j \leq n\big\}.$$

In type $B_n$, the reflections are given by the transpositions $t_{ij}$, $t_{i,-j}$ and $t_i$ and the group of signed permutations is minimally generated by $s_0 = t_1$ and $s_i = t_{i,i+1}$ for $1 \leq i < n$, the associated choice of simple and positive roots is

$$\Delta = \big\{\epsilon_{i+1} - \epsilon_i, \epsilon_1 : 1 \leq i < n\big\} \quad , \quad \Phi^+ = \big\{\epsilon_j \pm \epsilon_i : 1 \leq i < j \leq n\big\} \cup \big\{\epsilon_i : 1 \leq i \leq n\big\},$$

and in type $D_n$, the reflections are given by the transpositions $t_{ij}$ and $t_{i,-j}$ and it is minimally generated by $s_0 = t_{1,-2}$ and $s_i = t_{i,i+1}$ for $1 \leq i < n$, the associated choice of simple and positive roots is

$$\Delta = \{\epsilon_{i+1} - \epsilon_i, \epsilon_1 + \epsilon_2 : 1 \leq i < n\} \quad , \quad \Phi^+ = \{\epsilon_j \pm \epsilon_i : 1 \leq i < j \leq n\}.$$

Observe that the cardinality of any simple system equals the dimension of the $\mathbb{R}$-span of $\Phi$ in $V$, it is called rank of $\Phi$ and $W$ and is denoted by $l = l(\Phi) = l(W)$. For example, the rank of the dihedral group $\mathcal{D}_m$ is 2 and the rank of the symmetric group $\mathcal{S}_n$ is $n-1$. Furthermore, the number of positive roots is denoted by $N = N(\Phi) = N(W)$.

The following theorem is classical, for a proof see [**70**, Theorem 1.5]:

THEOREM 1.11. *Let $\Phi$ be a root system and let $\Delta \subseteq \Phi$ be a simple system. Then $W(\Phi)$ is generated by the simple reflections in $\Delta$,*

$$W(\Phi) = \langle s_\alpha : \alpha \in \Delta \rangle.$$

**1.1.4. Generators and relations.** Now that we have seen that $W$ can be generated by relatively few reflections, we now want to describe it as an abstract group in terms of these generators subject to suitable relations. Obviously, we have the following relations for the product of two simple reflections:

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1,$$

where $m(\alpha, \beta)$ denotes the order of $s_\alpha s_\beta$ in $W$. It turns out that these relations completely determine $W$. For the sake of completeness, we state this presentation as a theorem:

THEOREM 1.12. *Let $\Delta \subseteq \Phi$ be a simple system. Then $W(\Phi)$ is generated by $\{s_\alpha : \alpha \in \Delta\}$, subject only to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1, \;\; \alpha, \beta \in \Delta.$$

This presentation leads to the following definition:

DEFINITION 1.13. A finite group $G$ generated by $S \subseteq G$ subject only to the relations
(i) $s^2 = 1$ for $s \in S$ and
(ii) $(st)^{m(s,t)} = 1$ for $s, t \in S$ with $s \neq t$ and for some $m(s, t) \geq 2$

is called (finite) Coxeter group.

REMARK. One could allow also infinite groups generated by a finite set $S$ having such a presentation with possibly $m(s, t) = \infty$.

We have seen that finite reflection groups are (finite) Coxeter groups and it turns out that the reverse also holds. The notion of reflection groups focuses on its action on the vector space, while the notion of Coxeter groups rather focuses on its simple presentation in terms of generators and relations.

**1.1.5. The length function.** By Theorem 1.11, every $\omega \in W$ can be written as a product of simple reflections. This yields the definition of the length of elements in a Coxeter group:

DEFINITION 1.14. Let $\Delta \subseteq \Phi$ be a simple system and let $\omega \in W$. The length of $\omega$, denoted by $\mathrm{l}_S(\omega)$, is the smallest $k$ for which $\omega$ can be written as a product of $k$ simple reflections. Such a shortest expression of $\omega$ in terms of simple reflections is called reduced word for $\omega$.

REMARK. The subscript $S$ is used to indicate that $\mathrm{l}_S(\omega)$ is the length of $\omega$ when $\omega$ is expressed as a product of simple reflections. It is used to distinguish $\mathrm{l}_S$ from $\mathrm{l}_T$, the *absolute length* which we will introduce in Section 2.5.1.

Clearly, $\mathrm{l}_S(\omega) = 1$ if and only if $\omega$ is a simple reflection. As every simple reflection has, considered as a linear operator, determinant $-1$, the product formula for determinants implies

$$\det(\omega) = (-1)^{\mathrm{l}_S(\omega)}.$$

The length function can also be expressed in terms of the action of $W$ on the root system: it is equal to the number of positive roots sent by $\omega$ to negative roots,

$$\mathrm{l}_S(\omega) = |\{\Phi^+ \cap \omega^{-1}(-\Phi^+)\}|.$$

**1.1.6. The weak order on $W$.** Next, we use the length function defined in the previous section to obtain a partial order on $W$:

DEFINITION 1.15. Define the (right) weak order on $W$ by setting

$$\omega \leq_S \tau :\Leftrightarrow \mathrm{l}_S(\tau) = \mathrm{l}_S(\omega) + \mathrm{l}_S(\omega^{-1}\tau)$$

for all $\omega, \tau \in W$, and denote this poset by $\mathsf{Weak}(W)$.

REMARK. The right weak order and the analogously defined left weak order are isomorphic but they do not coincide. The Bruhat order, also called *strong order*, removes this "sidedness" by defining $\omega \leq_B \tau$ whenever $\omega$ is an arbitrary subword (not necessarily a prefix) of a reduced expression for $\tau$. See [**25**] for further information.

The poset $\mathsf{Weak}(W)$ is in fact a lattice, i.e., for any two elements in $\mathsf{Weak}(W)$ the *meet* and *join* exist, see e.g. the book "Combinatorics of Coxeter groups" by A. Björner and F. Brenti [**26**, Chapter 3.2]. There exists also a nice geometric interpretation of $\mathsf{Weak}(W)$ as the 1-*skeleton* of the *permutahedron*, for definitions and further properties see [**7**, Section 2.3]

**1.1.7. Classification.** In this section we want to state the classification theorem for reflection groups. The classification is done in terms of Coxeter graphs (or Dynkin diagrams) which are labelled graphs and are obtained from a Coxeter group as follows: the vertex set is given by the set of simple roots and two simple roots $\alpha$ and $\beta$ are joined by an edge, which is labelled by $m(\alpha, \beta)$, if $m(\alpha, \beta) \geq 3$. For more readability, the label $m(\alpha, \beta) = 3$ will be suppressed.

NOTE. Corollary 1.10 ensures that the Coxeter graph does not depend on the specific choice of the simple system.

DEFINITION 1.16. A reflection group $W$ is called irreducible if one of the two following equivalent statements holds:

(i) The Coxeter graph of $W$ is connected,
(ii) $W$ cannot be written as the product of two proper subgroups which are reflection groups themselves.

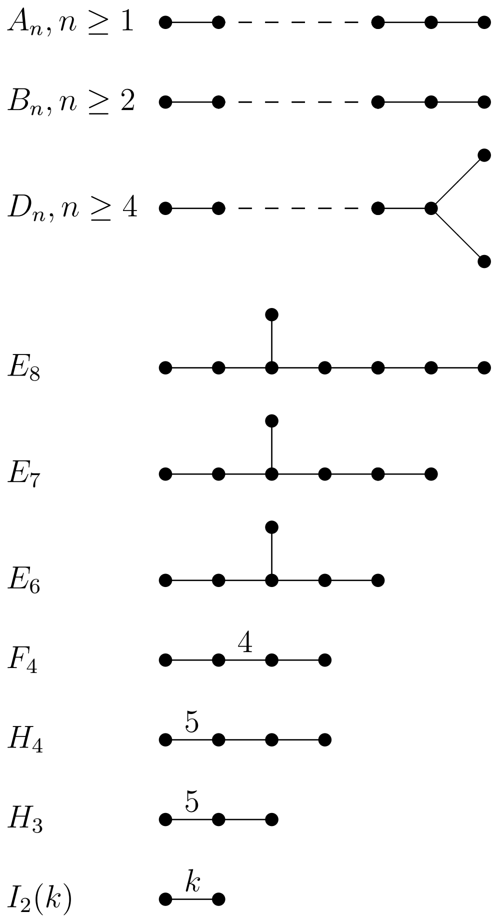



FIGURE 2. All connected Coxeter graphs.

THEOREM 1.17. *Every connected graph that is the Coxeter graph of some reflection group is one of those shown in Figure 2 on page 13. To refer to the associated reflection group, we label them by types indexed by the rank.*

The proof of the classification theorem of irreducible reflection groups is done by answering the question which labelled graphs can occur as Coxeter graphs.

REMARK. We often denote an irreducible reflection group by its type, for example $\mathcal{D}_k = I_2(k)$ and $\mathcal{S}_n = A_{n-1}$. The three infinite families of reflection groups which vary in rank are $A_n, B_n$ and $D_n$. They are often called classical reflection groups while the other are called exceptional reflection groups.

REMARK. In Section 1.1.11, we will discuss *crystallographic reflection groups*. In the classification of semisimple Lie algebras, one studies crystallographic root systems and it turns out that there exist two different crystallographic root systems denoted by types $B_n$ and $C_n$ which yield the same reflection group $B_n$. This is the reason why a type $C_n$ is missing in the classification of reflection groups.

**1.1.8. Polynomial invariants.** The action of $W$ on $V$ – as of any subgroup of $\mathrm{GL}(V)$ – induces a natural action of $W$ on the ring of polynomial functions on $V$ in the following way: define the contragredient action of $W$ on $V^* = \mathrm{Hom}(V, \mathbb{R})$ as

$$\omega(\rho) := \rho \circ \omega^{-1}.$$

This gives an action of $W$ on the symmetric algebra $S(V^*)$ which is the algebra of polynomial functions in $V$. After fixing a basis for $V$, $S(V^*)$ can be identified with

$$\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \ldots, x_n]$$

where the $x_i$'s are the coordinate functions.

EXAMPLE 1.18. In type $A_{n-1}$, this action is the well-known action of the symmetric group on the coordinate functions,

$$\sigma(x_i) = x_{\sigma(i)} \text{ for } \sigma \in \mathcal{S}_n,$$

and in type $B$, this action is given by

$$\sigma(x_i) = \begin{cases} x_{\sigma(i)} & ; \quad \sigma(i) > 0 \\ -x_{-\sigma(i)} & ; \quad \sigma(i) < 0 \end{cases} \text{ for } \sigma \in B_n.$$

The following definition can be stated for any subgroup $G \subseteq \mathrm{GL}(V)$:

DEFINITION 1.19. A polynomial $p \in S(V^*)$ is called invariant if

$$g(p) = p \text{ for all } g \in G.$$

All such polynomials form a subring of $S(V^*)$ which is called ring of polynomial invariants of $G$ or invariant ring of $G$,

$$S^G := \big\{p \in S(V^*) : g(p) = p \text{ for all } g \in G\big\}.$$

A famous result by Shephard and Todd [**90**] and by C. Chevalley [**36**] from the 50's states that for a reflection group $W$ of rank $l$, the ring of polynomial invariants $S^W$ is itself a polynomial ring with homogeneous generators $f_1, \ldots, f_l$, called fundamental invariants, of uniquely determined degrees $d_1, \ldots, d_l$,

$$S^W = \mathbb{R}[f_1, \ldots, f_l].$$

This result leads to the following definition:

DEFINITION 1.20. The degrees $d_1, \ldots, d_l$ of the fundamental invariants are called degrees of $W$. Usually, the degrees are written in increasing order,

$$d_1 \leq \ldots \leq d_l.$$

One property of the degrees is that the order of a reflection group $W$ can be computed as the product of its degrees $d_1, \ldots, d_l$,

$$|W| = d_1 \cdots d_l.$$

REMARK. In Figure 3, the degrees of all irreducible reflection groups are shown.

In the next section, we will present a way to determine the degrees of a reflection group by computing the eigenvalues of a well-chosen element called *Coxeter element*. Later, in Sections 1.1.11 and 1.1.12, we will present the notion for a reflection group to be *crystallographic* and a way to compute the degrees for crystallographic reflection groups combinatorially.

| Type | $d_1, \ldots, d_l$ |
|---|---|
| $A_{n-1}$ | $2, 3, \ldots, n$ |
| $B_n$ | $2, 4, \ldots, 2n$ |
| $D_n$ | $2, 4, \ldots, 2(n-1), n$ |
| $E_8$ | $2, 8, 12, 14, 18, 20, 24, 30$ |
| $E_7$ | $2, 6, 8, 10, 12, 14, 18$ |
| $E_6$ | $2, 5, 6, 8, 9, 12$ |
| $F_4$ | $2, 6, 8, 12$ |
| $H_4$ | $2, 12, 20, 30$ |
| $H_3$ | $2, 6, 10$ |
| $I_2(k)$ | $2, k$ |

FIGURE 3. The degrees of the irreducible real reflection groups.

### 1.1.9. Coxeter elements and the Coxeter number.

DEFINITION 1.21. For any set $\{s_1, \ldots, s_l\}$ of simple reflections in a reflection group $W$ and for any permutation $\sigma$ of $\{1, \ldots, l\}$, the product $s_{\sigma(1)} \cdots s_{\sigma(l)}$ is called standard Coxeter element of $W$ and any element that is conjugated to a standard Coxeter element is called Coxeter element.

The fact that all Coxeter elements are conjugate in $W$ follows from the fact that all simple systems are conjugate (see Corollary 1.10) and that all reorderings of some product of simple reflections are conjugate in $W$.

This leads to the second definition of this section:

DEFINITION 1.22. The Coxeter number of $W$ is defined to be the common order of all Coxeter elements. We denote this number by $h = h(\Phi) = h(W)$.

EXAMPLE 1.23. For the dihedral group $\mathcal{D}_k$, this number is very easy to compute as the product of the two simple roots is a rotation through $2\pi/k$ and hence $h = k$. For the symmetric group $\mathcal{S}_n$, the product of the simple transpositions $(i, i+1), 1 \leq i < n$ is an $n$-cycle implying that the Coxeter number is equal to $n$.

The fact that all Coxeter elements are conjugate ensures that they have the same characteristic polynomials and eigenvalues. If $\zeta$ is a primitive $h$-th root of unity in $\mathbb{C}$, these eigenvalues are of the form $\zeta^k$ for $0 \leq k < h$. These $k$ for which $\zeta^k$ is in fact an eigenvalue are called exponents of $W$, usually they are denoted in increasing order by

$$e_1 \leq \ldots \leq e_l.$$

One can show that the smallest exponent is equal to 1 and that the largest is equal to $h - 1$ and furthermore that they sum up to the number of positive roots,

$$e_1 + \ldots + e_l = N.$$

Another property of the exponents is their connection to the degrees:

$$e_1 = d_1 - 1, \ldots, e_l = d_l - 1.$$

REMARK. In Figure 4, the numbers of positive roots and the Coxeter numbers of all irreducible real reflection groups are shown.

| $A_{n-1}$ | $B_n$ | $D_n$ | $E_8$ | $E_7$ | $E_6$ | $F_4$ | $H_4$ | $H_3$ | $I_2(k)$ |
|---|---|---|---|---|---|---|---|---|---|
| $n(n-1)/2$ | $n^2$ | $n(n-1)$ | 120 | 63 | 36 | 24 | 60 | 15 | $k$ |
| $n$ | $2n$ | $2(n-1)$ | 30 | 18 | 12 | 12 | 30 | 10 | $k$ |

FIGURE 4. The numbers of positive roots and the Coxeter numbers of the irreducible real reflection groups.

**1.1.10. The Coxeter arrangement.** Let $\Phi$ be a root system and let $W$ be the associated reflection group acting on the underlying vector space $V$. The collection of the reflecting hyperplanes as well as the open sets into which $V$ is dissected by these hyperplanes are of interest themselves.

DEFINITION 1.24. The Coxeter arrangement is the collection of the reflecting hyperplanes in $W$,

$$\mathcal{C}(W) := \{H_\alpha : \alpha \in \Phi^+\}.$$

Furthermore, a chamber in $\mathcal{C}(W)$ is a connected component of $V \setminus \bigcup_{\alpha\in\Phi^+} H_\alpha$.

The chambers in $\mathcal{C}(W)$ are in one-to-one correspondence with the possible choices of a simple system $\Delta$: after fixing a simple system $\Delta$, the associated chamber is given by

$$\langle x, \alpha\rangle > 0, \text{ for all } \alpha \in \Delta$$

and is called fundamental chamber.

EXAMPLE 1.25. In Figure 1 on page 8, the fundamental chamber associated to the given choice of the simple system $\Delta$ is shaded.

The Coxeter arrangement provides a geometric interpretation of the length function on $W$. We have seen in Section 1.1.5 that the length $l_S(\omega)$ for some $\omega \in W$ is equal to the number of positive roots sent by $\omega$ to negatives. The equivalent statement in the Coxeter arrangement setting is

$$l_S(\omega) = \text{height}(\omega(C^0)),$$

where for any chamber $C$, the height of $C$ is defined to be the number of hyperplanes in $\mathcal{C}(W)$ separating $C$ from the fundamental chamber $C^0$.

REMARK. In Section 2.3, we introduce another arrangement containing the Coxeter arrangement. The height statistic on that arrangement will later play an important role.

**1.1.11. Crystallographic reflection groups.** This notion is adapted from the notion for arbitrary subgroups of the general linear group $\mathrm{GL}(V)$: a subgroup $G \subseteq \mathrm{GL}(V)$ is called crystallographic if it stabilizes the $\mathbb{Z}$-span of a basis of $V$. As we are interested in a very special class of subgroups of $\mathrm{GL}(V)$, we give an equivalent definition in terms of reflection groups respectively Coxeter groups:

DEFINITION 1.26. Let $\Phi$ be a root system and let $W = W(\Phi)$ be the associated reflection group. Then both $\Phi$ and $W$ are called crystallographic if the following equivalent statements hold for simple roots $\alpha \neq \beta$:

(i) $2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$,
(ii) $m(\alpha, \beta) \in \{2, 3, 4, 6\}$.

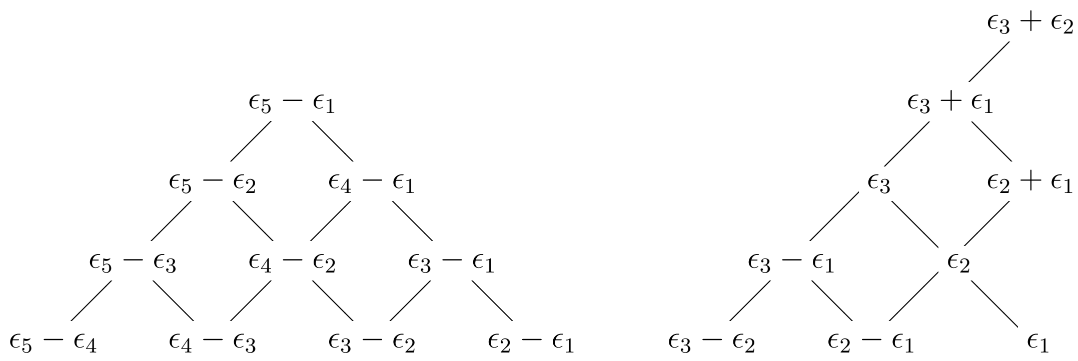



FIGURE 5. The root posets of types $A_4$ and $B_3$.

As mentioned in the beginning of Section 1.1.2, "most" reflection groups are Weyl groups associated with semisimple Lie algebras. In fact, a reflection group is a Weyl group if and only if it is crystallographic. As one can see in Figure 2, an irreducible reflection group $W$ is crystallographic if and only if $W$ is of one of the following types:

$$A_n, B_n, D_n, E_6, E_7, E_8, F_4 \text{ and } I_2(6).$$

REMARK. In the classification of semisimple Lie algebras, the reflection group $I_2(6)$ is mostly denoted by $G_2$.

For a crystallographic reflection group $W$ with associated root system $\Phi$, define the root lattice $Q = Q(\Phi)$ as the $\mathbb{Z}$-span of $\Phi$ in $V$. Note that this lattice is $W$-stable by construction.

**1.1.12. The root poset.** For a crystallographic reflection group $W$, there exists a very interesting combinatorial object called root poset:

DEFINITION 1.27. Let $\Phi$ be a crystallographic root system and let $\Delta \subseteq \Phi^+ \subseteq \Phi$ be a simple respectively positive system in $\Phi$. Define a covering relation $\prec$ on $\Phi^+$ by

$$\alpha \prec \beta :\Leftrightarrow \beta - \alpha \in \Delta.$$

This covering relation turns $\Phi^+$ into a poset, the root poset associated to $W$ and $\Phi$.

EXAMPLE 1.28. In Figure 5, the root posets of type $A_4$ and of type $B_3$ are shown, recall the given choice of simple and positive roots from Section 1.1.3.

The root poset is graded and the rank of a positive root is the sum of its coefficients when expressed as an integer linear combination of simple roots. Much of the combinatorics of a crystallographic reflection group is reflected in its root poset:
If $k_i$ denotes the number of positive roots of rank $i$, it turns out that $k_i \geq k_{i+1}$ and $(k_1, \ldots, k_{h-1})$ defines a partition of $N$. One can express the degrees of $W = W(\Phi)$ in terms of this partition: the conjugate partition $(k_1, \ldots, k_l) \vdash N$ equals $(e_1, \ldots, e_l)$, where the $e_i$'s are the exponents of $W$ which were shown to be by one smaller than the degrees. For a definition of partitions as well as of their conjugates, see Section 3.1.1.

A second combinatorial property which can be found in the root poset was proved by F. Chapoton in [**35**, Proposition 1.1]:

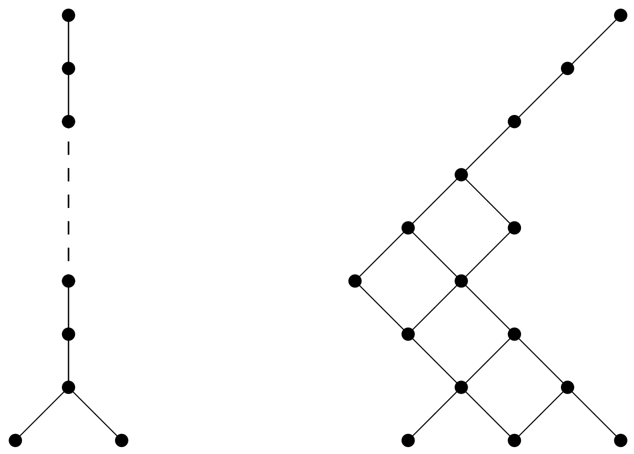


FIGURE 6. Armstrong's suggestions for root posets of types $I_2(k)$ and $H_3$.

THEOREM 1.29 (Chapoton). *Let $W$ be a reflection group with exponents $e_1, \dots, e_l$. Then the number of reflections in $W$ which do not occur in any reflection subgroup generated by a subset of the simple reflections in $W$ is equal to $M(W)$, where*

$$M(W) := \frac{nh}{|W|} \prod_{i=2}^{l} (e_i - 1).$$

REMARK. The reflection subgroups occurring in the previous theorem are called standard parabolic subgroups.

In terms of the root poset, the theorem can be rephrased in the crystallographic case:

COROLLARY 1.30. *The number of positive roots $\alpha \in \Phi^+$ for which all simple roots are contained in its simple root expansion is equal to $M(W)$.*

Beside the properties of $W$ that can be combinatorially described in terms of the root poset, we will see in Section 2.2 that the root poset carries even more combinatorial information of a crystallographic reflection group.

**1.1.13. On root posets for some non-crystallographic reflection groups.** Unfortunately it turns out that the analogous definition of a poset for the non-crystallographic reflection groups does not have those nice properties and so far, no definition for such a poset is known.

In his thesis [**7**] Armstrong suggests, how these root posets *should* look like in types $I_2(k)$ and $H_3$, such that the root poset has the discussed properties. We reproduce these posets in Figure 6.

In Section 4.3, we will present another conjectured property of the root poset and see that this conjecture also supports Armstrong's suggestion.

**1.1.14. Some statistics on classical reflection groups.** In the last part of this section, we want to introduce certain statistics on the classical reflection groups presented in Examples 1.3–1.5.

REMARK. The term statistic does not have a mathematically strict meaning, it is widely used in the literature when one associates to each element in a set $S$ a non-negative integer weight by a "combinatorial rule". An example of a statistic on a reflection group $W$ is the length function $\mathrm{l}_S$ introduced in Section 1.1.5.

Define a statistic called inversion number on finite words in a totally ordered alphabet by

$$\mathrm{inv}(w) := \big|\big\{i < j : w_i > w_j\big\}\big|$$

for any word $w = w_1 w_2 \cdots w_k$.

The inversion number can be used to compute the length function on permutations, signed permutations and even signed permutations. To do so, we introduce the one-line notation of a given $\omega$: identify $\omega$ with the sequence $[\omega_1, \ldots, \omega_n]$, where $\omega_i := \omega(i)$. Then we have

$$\begin{aligned} A_{n-1} : \mathrm{l}_S(\omega) &= \mathrm{inv}([\omega_1, \ldots, \omega_n]), \\ B_n : \mathrm{l}_S(\omega) &= \mathrm{inv}([\omega_1, \ldots, \omega_n]) - \sum_{i \in \mathrm{Neg}(\omega)} \omega_i, \\ D_n : \mathrm{l}_S(\omega) &= \mathrm{inv}([\omega_1, \ldots, \omega_n]) - \sum_{i \in \mathrm{Neg}(\omega)} \omega_i - \mathrm{neg}(\omega), \end{aligned}$$

where $\mathrm{Neg}(\omega) := \{i \in [n] : \omega_i < 0\}$ and $\mathrm{neg}(\omega) := |\,\mathrm{Neg}(\omega)|$. Type $A$ is classical and was proved by MacMahon, see e.g. [**81**], whereas types $B$ and $D$ were proved in [**31**] by Brenti.

Next, we define the major index on words in a totally ordered alphabet. Let $w = w_1 w_2 \cdots w_k$ be a finite word, its descent set $\mathrm{Des}(w)$ is the set of all integers $i$ such that $w_i > w_{i+1}$, $\mathrm{des}(w)$ denotes the cardinality of $\mathrm{Des}(w)$, and the major index of $w$ is defined as

$$\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i.$$

REMARK. The term "major index" was first used by D. Foata to indicate its origin, as MacMahon was a major in the British Army in the early 20th century, as well as the idea of counting positions of certain "major" elements. MacMahon himself used the term *greater index*.

We want to use this statistic to define a major index on the classical reflection groups:

DEFINITION 1.31. Let $\omega$ be an element in the reflection group of a classical type with one-line notation $[\omega_1, \ldots, \omega_n]$. The major index of $\omega$ is then defined as

$$\begin{aligned} A_{n-1} : \mathrm{maj}(\omega) &:= \mathrm{maj}([\omega_1, \ldots, \omega_n]), \\ B_n : \mathrm{maj}(\omega) &:= 2\,\mathrm{maj}([\omega_1, \ldots, \omega_n]) + \mathrm{neg}(\omega), \\ D_n : \mathrm{maj}(\omega) &:= \mathrm{maj}([\omega_1, \ldots, \omega_n]) - \sum_{i \in \mathrm{Neg}(\omega)} \omega_i - \mathrm{neg}(\omega). \end{aligned}$$

REMARK. For permutations, this definition appeared in [**81**]. For signed permutations, the major index was introduced by R.M. Adin and Y. Roichman in [**1**]. For even signed permutations, the major index was introduced in a slightly different way by R. Biagioli in [**23**], the definition we use was introduced by Biagioli and F. Caselli in [**24**]. In [**1**] and [**23**], the major index for types $B$ and $D$ was called *f-major index* and in [**24**], the major index for type $D$ was called *d-major index*. This was done to distinguish between different "major-like" statistics they were studying.

In [**1, 24, 81**], the following result was proved for classical reflection groups:

THEOREM 1.32. *Let $W$ be one of the reflection groups $A_n$, $B_n$ or $D_n$. The major index on $W$ is equally distributed with the length function $\mathrm{l}_S$,*

$$\sum_{\omega \in W} q^{\mathrm{maj}(\omega)} = \sum_{\omega \in W} q^{\mathrm{l}_S(\omega)}.$$

For later convenience, we define also

$$\mathrm{iDes}(\omega) := \mathrm{Des}(\omega^{-1}), \mathrm{ides}(\omega) := \mathrm{des}(\omega^{-1}) \text{ and } \mathrm{imaj}(\omega) := \mathrm{maj}(\omega^{-1}),$$

the later is called inverse major index. In type $A$, the inverse major index is discussed in detail in [**43**, Chapters 11 and 12] by Foata and G.-N. Han. They proved many equalities including the major index and the inverse major index. For example, they gave a bijective proof of the following equality:

$$\sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{imaj}(\sigma)} t^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{imaj}(\sigma)} t^{\mathrm{maj}(\sigma)}.$$

## 1.2. Complex reflection groups

As mentioned at the beginning of this chapter, we now want to generalize the concept of reflection groups to a certain class of finite subgroups of $\mathrm{O}(V)$, where now $V$ is a complex vector space.

**1.2.1. Reflections.** A complex reflection in a finite dimensional complex vector space $V$ is a linear operator $s \in \mathrm{O}(V)$ which has finite order and where its fixed-point space has codimension 1. In the literature also the terms *pseudo-reflection*, *unitary reflection* or just *reflection* are used.

DEFINITION 1.33. A complex reflection group is a finite subgroup of $\mathrm{O}(V)$ generated by complex reflections.

NOTE. The complexification of a real vector space turns all real reflection groups into complex reflection groups.

**1.2.2. Classification.** The classification of complex reflection groups was done in the famous paper "Finite unitary reflection groups" by Shephard and Todd [**90**].

As for real reflection groups, see Definition 1.16, a complex reflection group $W$ is called irreducible if $W$ cannot be written as the product of two proper subgroups which are both complex reflection groups.

THEOREM 1.34 (Shephard, Todd). *Let $W$ be an irreducible complex reflection group. Then $W$ is either equal to $G(m,p,n)$ with $p$ dividing $m$ or one of 34 exceptional types. Here, the group $G(m,p,n)$ is the symmetry group of the complex polytope whose vertices are of the form $(\zeta^{a_1},\ldots,\zeta^{a_n})$, where $\zeta$ is a primitive $m$-th root of unity and the exponents sum up to an integer multiple of $p$. It has order $m^n n!/p$.*

As we will not use this construction, we do not go into details here, nor do we present all the 34 exceptional types. For more information see [**90**].

REMARK. The real reflection groups $A_{n-1}, B_n, D_n$ and $I_2(k)$ appear in the classification of complex reflection groups as follows:

- $G(1,1,n) = A_{n-1} = \mathcal{S}_n$,
- $G(2,1,n) = B_n$,
- $G(2,2,n) = D_n$ and
- $G(k,k,2) = I_2(k)$.

**1.2.3. Polynomial invariants.** As in Definition 1.19, we define in the complex case the ring of polynomial invariants, $S^G$ for a subgroup $G \subseteq \mathrm{GL}(V)$.

In [**90**], Shephard and Todd proved the following famous result using the classification of complex reflection groups. Soon afterwards, Chevalley gave a uniform proof [**36**].

THEOREM 1.35 (Shephard, Todd; Chevalley). *Let $G \subseteq \mathrm{GL}(V)$ be a finite subgroup of the general linear group. Then $G$ is a complex reflection group if and only if its ring of invariants is a polynomial ring,*

$$S^G = \mathbb{C}[f_1,\ldots,f_l],$$

*where $f_1,\ldots,f_l$ are the* fundamental invariants *of uniquely determined homogeneous degrees $d_1,\ldots,d_l$ which are called* degrees *of $W$.*

**1.2.4. Degrees and codegrees.** It turns out that there is an equivalent way to define the degrees using the coinvariant algebra $S/\langle \mathbf{f}\rangle$, where $\langle \mathbf{f}\rangle = \langle f_1,\ldots,f_l\rangle = S_+^W$ is the ideal in $S$ generated by all invariants without constant term. Both Chevalley [**36**] and Shephard and Todd [**90**] showed that $S/\langle \mathbf{f}\rangle$ carries the *regular representation of $W$*, see Section 1.3 for the definition of the regular representation and the representation theory needed. By [**50**, Corollary 2.18], $S/\langle \mathbf{f}\rangle$ contains exactly $k$ copies of any irreducible $W$-representation $U$ of dimension $k$. In particular $S/\langle \mathbf{f}\rangle$ contains $l$ copies of $V$. The $U$-exponents $e_1(U),\ldots,e_k(U)$ are the degrees of the homogeneous components of $S/\langle \mathbf{f}\rangle$ in which these $k$ copies of $U$ occur.
It is known, see e.g. [**82**], that the degrees of $W$ are uniquely determined by saying that the $V$-exponents are equal to $d_1-1,\ldots,d_l-1$.

This characterization has the advantage that one can define also codegrees of $W$,

$$d_1^* \geq \ldots \geq d_l^*,$$

by saying that the $V^*$-exponents are $d_1^*+1,\ldots,d_l^*+1$.

**1.2.5. Well-generated complex reflection group.** By case-by-case inspection it can be observed that every complex reflection group can be generated either by $l$ or by $l+1$ complex reflections. This observation together with the notion of degrees and codegrees in the previous section leads to the following definition:

DEFINITION 1.36. A complex reflection group $W$ is well-generated if it satisfies the following equivalent conditions:

(i) $W$ can be generated by $l$ reflections,
(ii) the degrees $d_1 \leq \ldots \leq d_l$ and the codegrees $d_1^* \geq \ldots \geq d_l^*$ of $W$ satisfy

$$d_i + d_i^* = d_n.$$

The fact that all real reflection groups are well-generated follows directly from Theorem 1.11. To verify the second equivalent condition, observe that in this case, $V = V^*$ and therefore $d_i^* = d_{l+1-i} - 2$. From Figure 3 on page 15 one can deduce that

$$d_i + d_i^* = d_i + d_{l+1-i} - 2 = d_l.$$

As we will see in the next section, the notion of well-generation allows us to construct a *Coxeter element* for all those groups.

**1.2.6. Coxeter elements and the Coxeter number.** An element $c$ in a well-generated complex reflection group $W$ is called regular if it has an eigenvector lying in the complement $V^{reg}$ of the reflecting hyperplanes of $W$ and furthermore $\xi$-regular if this eigenvector may be taken to have eigenvalue $\xi$. In this case, the multiplicative order $d$ of $\xi$ is called a regular number for $W$. G.I. Lehrer and T.A. Springer [**76**] first observed the following theorem using the Shephard-Todd classification, it was later proven uniformly by Lehrer and J. Michel [**75**].

THEOREM 1.37 (Lehrer, Springer; Lehrer, Michel). *For any complex reflection group, $d$ is a regular number if and only if $d$ divides as many degrees as it divides codegrees.*

Together with the definition of well-generation, it follows immediately that for a well-generated reflection group, $h := d_n$ is always a regular number. This means that there exists a regular element $c$ with eigenvalue $\xi_h$, a primitive $h$-th root of unity. In [**93**], Springer showed that for any $\xi$, all $\xi$-regular elements are $W$-conjugate and hence $c$ is unique up to conjugacy. Any element in this conjugacy class is called Coxeter element.

## 1.3. Basic representation theory

In this section, we want to give a brief introduction to the representation theoretical concepts we need in the context of reflection groups. For a more detailed introduction to representation theory of finite groups, see [**50**].

A representation of a finite group $G$ is an action of $G$ on a finite-dimensional vector space $V$, i.e. a homomorphism $\rho : G \to \mathrm{GL}(V)$ of $G$ into the group of automorphisms of $V$. This action turns $V$ into a $G$-module. If there is little ambiguity about the action, $V$ itself is called a representation of $G$ and we identify $g \in G$ and $\rho(g) : V \to V$.

EXAMPLE 1.38 (Trivial representation). The most basic representation is the trivial representation $\mathbb{C}$ of $G$ defined by

$$gz := z \text{ for all } z \in \mathbb{C}.$$

EXAMPLE 1.39 (Reflection representation). For any reflection group $W$, the reflection representation is the representation of $W$ on its underlying vector space $V$.

EXAMPLE 1.40 (Sign representation). For the reflection representation of a reflection group $W$, there exists another one-dimensional representation of $W$ which is called sign representation and which we denote by $\epsilon$. It is defined by

$$\omega z := \det(\omega) z \text{ for all } z \in \epsilon.$$

If $V' \subseteq V$ is fixed by the action of $G$ than the action of $G$ on $V'$ is called subrepresentation of the action of $G$ on $V$. A representation $V$ is called irreducible if $V$ has no proper subrepresentations.

If $V$ and $V'$ are representations of $G$, the *direct sum* $V \oplus V'$ and the *tensor product* $V \otimes V'$ are also representations, where for the later,

$$g(v \otimes w) := gv \otimes gw.$$

Sometimes, we write $V^{\otimes k}$ for the $k$-th tensor product of $V$.

The following theorem is taken from [**50**, Corollary 1.6]:

THEOREM 1.41. *Any representation is a direct sum of irreducible representations.*

NOTE. In the case of $W$ being a real reflection group, we have seen in Section 1.1.5 that

$$\det(\omega) = (-1)^{l_S(\omega)}.$$

In particular, this implies that the square of the sign representation, $\epsilon^{\otimes 2} = \epsilon \otimes \epsilon$, equals the trivial representation.

Let $V^* := \mathrm{Hom}(V, \mathbb{C})$ be the *dual* of $V$. The dual representation is defined by the action $\rho^* : G \to \mathrm{GL}(V^*)$ given by

$$\rho^*(g) := {}^t\rho(g^{-1}) : V^* \to V^*.$$

For a reflection group $W$, this action on the dual representation is the contragredient action defined in Section 1.1.8. This induces an action of $W$ on the space of polynomial functions on $V$ which we also call reflection representation.

The regular representation of $G$ is the group algebra $\mathbb{C}[G]$, the space of complex-valued functions on $G$, where an element $g \in G$ acts on the function $v$ by

$$gv(h) := v(g^{-1}h) \text{ for } h \in G.$$

As we have used it in the previous section, we state the following fact about the regular representation, see [**50**, Corollary 2.18]:

THEOREM 1.42. *Every irreducible representation $V$ of $G$ appears in the regular representation* $\dim(V)$ *times.*

### 1.3.1. Characters.

Definition 1.43. Let $V$ be a representation of $G$. The character of $V$ is the complex-valued function $\chi_V$ on $G$ defined as

$$\chi_V(g) := \mathrm{Tr}(g),$$

the *trace* of $g$ on $V$.

As $\chi_V(hgh^{-1}) = \chi_V(g)$, $\chi_V$ is constant on the conjugacy classes of $G$, such functions are called class functions. Denote the set of all class functions by $\mathbb{C}_{class}(G)$.

Note. The value of $\chi_V$ on the identity equals the dimension of $V$, $\chi_V(1) = \dim V$, and furthermore,

$$\chi_{V\oplus V'} = \chi_V + \chi_{V'} \quad , \quad \chi_{V\otimes V'} = \chi_V \cdot \chi_{V'}.$$

Define an Hermitian inner product on $\mathbb{C}_{class}(G)$ by

$$\langle \alpha, \beta\rangle := \frac{1}{|G|}\sum_{g\in G} \overline{\alpha(g)}\beta(g).$$

The following theorem is taken from [**50**, Theorem 2.12]:

Theorem 1.44. *With respect to this inner product, the set*

$$\{\chi_V : V \textit{ irreducible}\}$$

*forms an orthonormal basis of $\mathbb{C}_{class}(G)$, i.e., for two irreducible representations $V$ and $V'$, we have*

$$\langle \chi_V, \chi_{V'}\rangle = \begin{cases} 1 & ; \quad V \cong V' \\ 0 & ; \quad \textit{otherwise.} \end{cases}$$

This implies that the multiplicity of an irreducible representation $V^\lambda$ in any representation $V$ is equal to $\langle \chi_{V^\lambda}, \chi_V\rangle$. Denote this multiplicity by $\mathrm{mult}(V^\lambda, V)$.

Remark. Theorem 1.44 is equivalent to the fact that the number of irreducible representations of $G$ is equal to the number of conjugacy classes in $G$. For $G = \mathcal{S}_n$, there exists an explicit bijection between conjugacy classes of the symmetric group, which are indexed by partitions, and irreducible representations using *Young tableaux*. We will study the representation theory of the symmetric group in the context of symmetric functions in Section 3.1.

**1.3.2. Idempotents.** Let $W$ be a reflection group and let $S = S(V^*)$ be the reflection representation. Define the trivial idempotent $\mathbf{e}$ by

$$\mathbf{e} := \frac{1}{|W|}\sum_{\omega\in W} \omega \quad \in \mathrm{End}(S)$$

and the sign idempotent $\mathbf{e}_\epsilon$ by

$$\mathbf{e}_\epsilon := \frac{1}{|W|}\sum_{\omega\in W} \det(\omega)\omega \quad \in \mathrm{End}(S).$$

Then $\mathbf{e}$ is a projection from $S$ to $S^W$ where $S^W$ is the ring of polynomial invariants defined in Section 1.1.8.

Analogously, define a polynomial $p \in S$ to be alternating if

$$\det(\omega)\omega(p) = p \text{ for all } \omega \in W$$

and the ring of alternating polynomials by

$$S^{\epsilon} := \left\{ p \in S : \det(\omega)\omega(p) = p \text{ for all } \omega \in W \right\}.$$

Then $\mathbf{e}_{\epsilon}$ is a projection from $S$ to $S^{\epsilon}$.

# CHAPTER 2

## Fuß-Catalan numbers and combinatorics

As defined in the introduction, the $n$-th Catalan number is given by

$$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}.$$

Simple computations show that the Catalan numbers satisfy the recurrence relation

$$\mathrm{Cat}_{n+1} = \sum_{k=0}^{n} \mathrm{Cat}_k \mathrm{Cat}_{n-k}, \qquad \mathrm{Cat}_0 = 1.$$

In Example 2.1, we describe certain lattice paths which are counted by the Ballot numbers

$$\frac{k-n}{n+k}\binom{k+n}{n}.$$

For the special case $k = n+1$, these lattice paths reduce to the Catalan paths defined in the introduction.

In this chapter, we want to see how the Catalan numbers $\mathrm{Cat}_n$ – as well as the Fuß-Catalan numbers $\mathrm{Cat}_n^{(m)}$ – are attached to the reflection group $\mathcal{S}_n = A_{n-1}$ and how they generalize to other reflection groups. Furthermore, we want to present several classes of objects related to Fuß-Catalan numbers and some, partially new, interesting combinatorics concerning them. Namely we will study *non-crossing partitions*, *Coxeter sortable elements*, *non-nesting partitions* (and the closely related *Shi arrangement*).

We start with 4 basic occurrences of the Catalan numbers. They can be found in [**95**, 6.19 (h), (pp), (uu), (ff)]:

Example 2.1 (Catalan paths). The so-called “Ballot problem” was one of the first (if not the first) lattice path enumeration problem. It was posed by M.J. Bertrand in the late 19th century when he asked for the number of lattice paths from $(1,0)$ to $(k,n)$, where $k$ and $n$ are non-negative integers with $k > n$, that never touch the line $x = y$ [**19**]. In the same journal, D. André offered a direct proof using a method which was later enhanced and became famous as *André's reflection principle* [**3**]. A nice survey of the history of this principle was described in detail by M. Renault in [**88**]. The special case $k = n+1$ yields Catalan paths of length $n$ which were defined to be lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(n,n)$ consisting of $n$ north steps of the form $(0,1)$ and $n$ east steps of the form $(1,0)$ with the additional property that the path never goes below the line $x = y$. Denote the set of all Catalan paths of length $n$ by $\mathcal{D}_n$ (the notion should not cause confusion with the dihedral group which was denoted by the same letter).

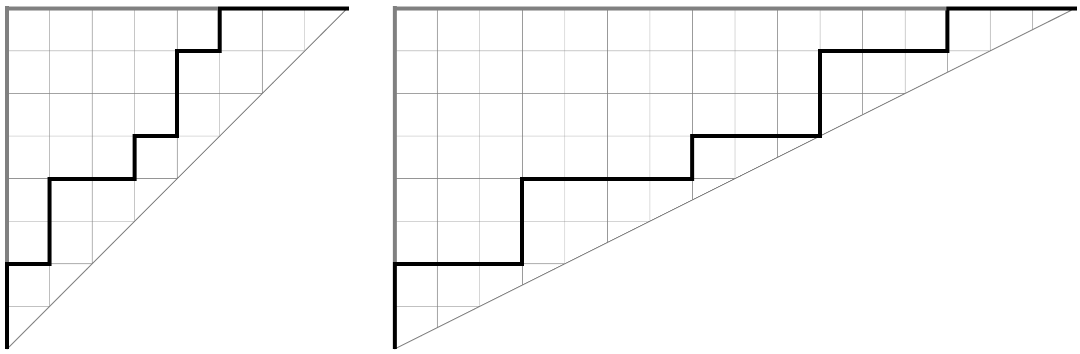



FIGURE 7. A Catalan path and a 2-Catalan path, both of length 8.

As we need them later, we give three alternative descriptions of Catalan paths:

- a Catalan path can be encoded by a word $D$ consisting of $n$ $N$'s and $n$ $E$'s where any prefix of $D$ does not contain more $E$'s than $N$'s,
- a Catalan path can be identified with a partition, i.e. a weakly decreasing sequence of non-negative integers, fitting inside the partition $(n-1,\dots,2,1,0)$ (see Section 3.1.1 for the definition) and
- a Catalan path can be identified with a sequence $(a_1,\dots,a_n)$ of non-negative integers with $a_1=0$ and $a_{i+1}\le a_i+1$.

In Figure 7(a), a Catalan path of length 8 is shown. It is encoded by the word

$$NNENNEENENNENEEE,$$

the associated partition is given by

$$\lambda=(5,4,4,3,1,1,0)\subseteq(7,6,5,4,3,2,1,0)$$

and the associated sequence by

$$a=(0,1,1,2,1,1,2,2).$$

Observe that $\lambda$ and $a$ are complementary, i.e., for $0\le i<n$ we have $a_{i+1}+\lambda_{n-i}=i$.

From the solution of the Ballot problem, it follows immediately that Catalan paths of length $n$ are counted by the $n$-th Catalan number,

$$|\mathcal{D}_n|=\mathrm{Cat}_n\,.$$

Recall the definition of $m$-Catalan paths of length $n$ from the introduction and denote the set of all $m$-Catalan paths of length $n$ by $\mathcal{D}_n^{(m)}$. As Catalan paths, $m$-Catalan paths can also be described as words respectively as partitions and sequences:

- an $m$-Catalan path can be encoded by a word $D$ consisting of $n$ $N$'s and $nm$ $E$'s where the number of $E$'s in any prefix of $D$ is smaller than or equal to $m$ times the number of $N$'s,
- an $m$-Catalan path can be identified with a partition fitting inside the partition $((n-1)m,\dots,2m,m)$ and
- an $m$-Catalan path can be identified with a sequence $(a_1,\dots,a_n)$ of non-negative integers with $a_1=0$ and $a_{i+1}\le a_i+m$.

In Figure 7(b), a 2-Catalan path of length 8 is shown. It is encoded by the word

$$NNEEENNEEEENEEENNEEENEEE,$$

the associated partition is given by

$$\lambda = (13, 10, 10, 7, 3, 3, 0, 0) \subseteq (14, 12, 10, 8, 6, 4, 2, 0)$$

and the associated sequence by

$$a = (0, 2, 1, 3, 1, 0, 2, 1).$$

For $m$-Catalan paths, $\lambda$ and $a$ are complementary in the sense that $a_{i+1} + \lambda_{n-i} = mi$.

The number of $m$-Catalan paths is given by the Fuß-Catalan numbers $\mathrm{Cat}_n^{(m)}$,

$$\left|\mathcal{D}_n^{(m)}\right| = \mathrm{Cat}_n^{(m)} := \frac{1}{mn+1}\binom{(m+1)n}{n}.$$

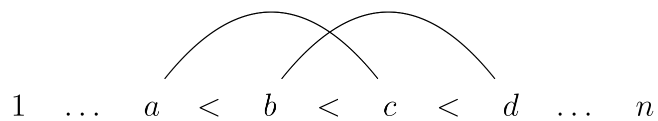
This can for example be shown using the *cycle lemma* introduced by A. Dvoretzky and T. Motzkin in [**39**].

Remark. Often, Catalan paths are called *Dyck paths* – this is why we denote the set of Catalan paths of length $n$ by $\mathcal{D}_n$ – but as we will generalize them in a "Catalan way", we prefer to use the former term.

Example 2.2 (Non-crossing set partitions). Let $[n]$ be the set of the first $n$ integers. A set partition of $[n]$ is a partition of $[n]$ into non-empty pairwise disjoint subsets, called blocks, $B_1, \ldots, B_k$ with $\bigcup B_i = [n]$. A set partition $\{B_1, \ldots, B_k\}$ is non-crossing if

$$a < b < c < d \text{ with } a, c \in B_i \text{ and } b, d \in B_j \text{ implies } B_i = B_j.$$

The set of all non-crossing set partitions of $[n]$ is denoted by $NC(n)$. Often, a set partition $\mathcal{B}$ is visualized by drawing the numbers 1 to $n$ in a row and then drawing arcs on top of two adjacent elements in a block of $\mathcal{B}$. Then the condition for a set partition to have two crossing blocks is visualized as follows:

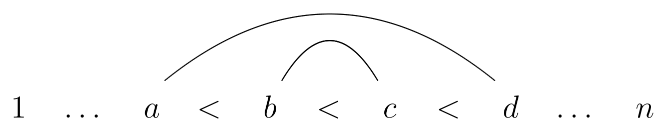



There exists a nice and simple bijection between non-crossing set partitions and Catalan paths via non-nesting set partitions:

Example 2.3 (Non-nesting set partitions). A set partition $\{B_1, \ldots, B_k\}$ is non-nesting if

$$a < b < c < d \text{ with } a, d \in B_i \text{ and } b, c \in B_j \text{ implies } B_i = B_j.$$

The set of all non-nesting set partitions of $[n]$ is denoted by $NN(n)$. The condition for a set partition to have two nesting blocks is visualized as follows:



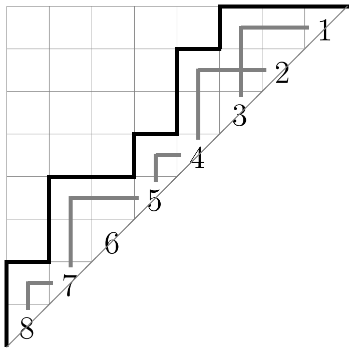



FIGURE 8. The bijection between Catalan paths and non-nesting set partitions. The shown path is mapped to the partition $\{\{1,3\},\{2,4,5,7,8\},\{6\}\}$

The intuitive map that locally converts each nesting into a crossing defines a bijection between non-crossing and non-nesting set partitions and the map indicated in Figure 8 is a bijection between Catalan paths and non-nesting set partitions (see e.g. [**7**, Section 5.1.2]). We will see in Note 2.2 that the later bijection has a very simple description in terms of *root posets*.

NOTE. The notion of non-crossing set partitions depends only on the cyclic order on $[n]$, i.e., the permutation given by the long cycle $(1,2,\ldots,n)$ acts on the set of non-crossing partitions. The same does *not* hold for non-nesting partitions.

EXAMPLE 2.4 (3-pattern-avoiding permutations). Recall that for a permutation $\sigma \in \mathcal{S}_n$, the one-line notation of $\sigma$ is the presentation of $\sigma$ as the list $[\sigma_1,\ldots,\sigma_n]$ where $\sigma_i := \sigma(i)$. A subword of $\sigma$ is a subsequence $[\sigma_{i_1},\ldots,\sigma_{i_k}]$ with $i_1 < \ldots < i_k$ of $\sigma$. For $\tau \in \mathcal{S}_k$, $\sigma$ is called $\tau$-avoiding if $\sigma$ does not contain a subword of length $k$ having the same relative order as $\tau$. By $\mathcal{S}_n(\tau)$, we denote the set of of $\tau$-avoiding permutations in $\mathcal{S}_n$. In [**71**], D.E. Knuth proved for any $\tau \in \mathcal{S}_3$ that the number of $\tau$-avoiding permutations in $\mathcal{S}_n$ is equal to $\mathrm{Cat}_n$. Is Section 2.9, we will give a bijection between non-crossing set partitions and 3-pattern avoiding permutations, i.e. $\tau$-pattern-avoiding permutations for any $\tau \in \mathcal{S}_3$.

Later in this chapter, we will see that the four examples can be seen as the type $A$ instances of more general constructions called *non-nesting partitions*, *non-crossing partitions* and *Coxeter sortable elements* which can be attached to certain classes of reflection groups. To be precise, we will define non-nesting partitions, as well as *Shi arrangements* which are closely related, for all crystallographic reflection groups, non-crossing partitions for all well-generated complex reflection group and Coxeter sortable elements for all real reflection groups. Furthermore, we will present existing generalizations of those constructions.

Before getting to the objects mentioned, we define Fuß-Catalan numbers for well-generated complex reflection groups and present a first $q$-extension of these numbers.

| $A_{n-1}$ | $B_n$ | $D_n$ | $I_2(k)$ | $H_3$ | $H_4$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{n+1}\binom{2n}{n}$ | $\binom{2n}{n}$ | $\binom{2n}{n}-\binom{2n-2}{n-1}$ | $k+2$ | 32 | 280 | 105 | 833 | 4160 | 25080 |

FIGURE 9. $\mathrm{Cat}(W)$ for all irreducible real reflection groups.

### 2.1. Fuß-Catalan numbers for well-generated complex reflection groups

Recall the definition of a well-generated complex reflection group as well as of its rank, its degrees and its Coxeter number from Section 1.2.

DEFINITION 2.5. Let $W$ be a well-generated complex reflection group and let $m$ be a non-negative integer. Define the **Fuß-Catalan numbers** $\mathrm{Cat}^{(m)}(W)$ as

$$\mathrm{Cat}^{(m)}(W) := \prod_{i=1}^{l} \frac{d_i + mh}{d_i} = \frac{1}{|W|}\prod_{i=1}^{l}(d_i + mh),$$

where $l$ is the rank of $W$, $d_1 \leq \ldots \leq d_l$ are its degrees and $h$ is its Coxeter number. The second equality follows from the fact that $d_1 \cdots d_l = W$ which was discussed in Section 1.1.8.

REMARK. Of course, $\mathrm{Cat}^{(m)}(W)$ could be defined in the same way for any complex reflection group $W$ but if $W$ is not well-generated, $\mathrm{Cat}^{(m)}(W)$ may fail to be an integer, see [**20**].

As we will often refer to the case $m = 1$, we define $\mathrm{Cat}(W)$ by

$$\mathrm{Cat}(W) := \mathrm{Cat}^{(1)}(W)$$

and refer to $\mathrm{Cat}(W)$ as the **Catalan numbers** associated to $W$. In Figure 2.1, the Catalan numbers $\mathrm{Cat}(W)$ are shown for all irreducible real reflection groups.

REMARK. We actually use the term *Fuß-Catalan* to refer to the additional parameter $m$ as it has commonly been used in the literature for $m$-Catalan numbers of general type, see e.g. [**7**], [**44**], [**45**]. In the literature concerning only type $A$, the names *higher* or *generalized* Catalan numbers were more usual.

NOTE. As shown in Figure 3 on page 15, the degrees of $W = A_{n-1}$ are given by $\{2, 3, \ldots, n\}$ and therefore,

$$\mathrm{Cat}^{(m)}(A_{n-1}) = \mathrm{Cat}^{(m)}_n .$$

The definition of the Catalan numbers $\mathrm{Cat}(W)$ attached to reflection groups appeared first in [**86**] where Reiner proved for the classical reflection groups case-by-case that the number of non-crossing partitions (see Section 2.5) is equal to the number of non-nesting partitions (see Section 2.2) and equal to the number of regions in the positive chamber of the Shi arrangement (see Section 2.3). The later were counted case-by-case by Athanasiadis in [**9**] and Reiner found this uniform formula for all types. The first object which was defined for any well-generated complex reflection group and is counted by $\mathrm{Cat}^{(m)}(W)$ is the number of chains in the non-crossing partition lattice. In full generality, these were introduced by Bessis in [**20**], where he called them *weak chains in the dual braid monoid.*

**2.1.1. A first $q$-extension of $\mathrm{Cat}^{(m)}(W)$.** In type $A$, the first $q$-extension of $\mathrm{Cat}_n^{(m)}$ we want to draw attention to is given by

$$\text{diag-Cat}_n^{(m)}(q) := \frac{1}{[mn+1]_q} \begin{bmatrix} (m+1)n \\ n \end{bmatrix}_q,$$

where $[k]_q := 1 + q + \ldots + q^{k-1}$ is the usual $q$-extension of an integer $k$, $[k]_q! := [1]_q[2]_q \cdots [k]_q$ is the $q$-factorial of $k$ and $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q := [k]_q!/[l]_q![k-l]_q!$ is the $q$-binomial coefficient.

Remark. The reason why we choose the term "diag-Cat" for this $q$-extension of the Fuß-Catalan numbers will become clear in Section 4.3.

For $m = 1$, MacMahon showed in [**81**] that this generalization is equal to the generating function for the major index on Catalan paths: recall from Example 2.1, that a Catalan path can be encoded by a word in the alphabet $\{N, E\}$ and set $N < E$. For a Catalan path $D$ of length $n$, the major index of $D$ is defined as

$$\mathrm{maj}(D) := \sum_{i \in \mathrm{Des}(D)} (2n - i).$$

Then

$$\sum_{D \in \mathcal{D}_n} q^{\mathrm{maj}(D)} = \text{diag-Cat}_n^{(1)}(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

Remark. When only dealing with Catalan paths, the major index of a given path $D$ is usually defined as $\sum_{i \in \mathrm{Des}(D)} i$. The involution $\mathbf{c}$ on Catalan paths sending a path to the path obtained by reversing the associated word in $\{N, E\}$ and then interchanging the $N$-th and $E$-th gives the same generating function. As $\mathbf{c}$ can equivalently described by the involution which conjugates the associated partition, we call $\mathbf{c}(D)$ the conjugate of $D$. In Section 2.4.2, we will introduce Catalan paths of type $B_n$ and we will see that in this context, the given definition is more convenient.

A valley of a Catalan path $D$ is a lattice point $(i, j)$ in $D$ for which the previous step is an east step and the next step is a north step. By definition, the descents of a Catalan word are in one-to-one correspondence with the valleys in the Catalan path. We denote the number of valleys of a Catalan path $D$ by $\mathrm{des}(D)$ and furthermore, we define the sets $\mathrm{Set}_X(D)$ and $\mathrm{Set}_Y(D)$ to be the set of $x$-coordinates and the set of $y$-coordinates of the valleys of $D$.

Example 2.6. Let $D := NE\ NNE\ NEE\ NE\ NE \in \mathcal{D}_6$. As indicated by the blanks, the descent set of $D$ is given by $\mathrm{Des}(D) = \{2, 5, 8, 10\}$ and its major index is given by $(12-2) + (12-5) + (12-8) + (12-10) = 10 + 7 + 4 + 2 = 23$. Its associated lattice path is shown in Figure 10 on page 33. Furthermore, the valleys of $D$ - indicated in the picture by dots - have coordinates $(1, 1), (2, 3), (4, 4)$ and $(5, 5)$ and

$$\mathrm{Set}_X(D) = \{1, 2, 4, 5\}\ ,\ \mathrm{Set}_Y(D) = \{1, 3, 4, 5\}.$$

Remark. D. Callan called these sets the "ascent-descent code" of $D$, see [**33**, Section 3].

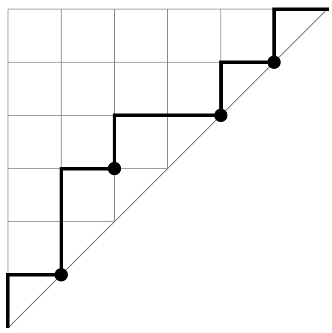


FIGURE 10. A Catalan path of length 6 with descent set $\{2,5,8,10\}$ and major index $(12-2)+(12-5)+(12-8)+(12-10) = 10+7+4+2 = 23$.

We will often use the following obvious proposition:

PROPOSITION 2.7. *A Catalan path $D$ is uniquely determined by* $\mathrm{Set}_X(D)$ *and* $\mathrm{Set}_Y(D)$ *and furthermore*

$$\mathrm{maj}(D) = \sum_{i\in\mathrm{Set}_X(D)} (n-i) + \sum_{j\in\mathrm{Set}_Y(D)} (n-j).$$

Using the involution $\mathbf{c}$ on Catalan paths, we derive the following proposition:

PROPOSITION 2.8. *The sequence of coefficients of the $q$-Catalan numbers* $\text{diag-Cat}_n^{(1)}(q)$ *is symmetric, i.e.,*

$$\text{diag-Cat}_n^{(1)}(q) = q^{n(n-1)}\,\text{diag-Cat}_n^{(1)}(q^{-1}).$$

For $m > 1$, no statistic on $\mathcal{D}_n^{(m)}$ that describes this $q$-extension of the Fuß-Catalan numbers is known, this yields the following open problem:

OPEN PROBLEM. Find a statistic maj on $\mathcal{D}_n^{(m)}$ such that

$$\sum_{D\in\mathcal{D}_n^{(m)}} q^{\mathrm{maj}(D)} = \text{diag-Cat}_n^{(m)}(q).$$

At the workshop "Braid groups, clusters and free probability" which was held at the American Institute of Mathematics in 2005, Athanasiadis suggested to generalize this $q$-extension of $\mathrm{Cat}_n^{(m)}$ as

$$\text{diag-Cat}^{(m)}(W;q) := \prod_{i=1}^{l} \frac{[d_i+mh]_q}{[d_i]_q}.$$

Furthermore, he, and independently S. Garoufalidis, conjectured for well-generated complex reflection groups that it is a polynomial with non-negative coefficients, see [**6**, Problem 2.1].

For real reflection groups, these extensions of the Fuß-Catalan numbers seem to have first appeared in a paper by Berest, Etingof and Ginzburg [**15**] where they are obtained as certain Hilbert series. Their work implies that for real reflection groups this extension is in fact a polynomial with non-negative integer coefficients. For well-generated complex reflection groups, this is still true, but so far it has only been verified by appeal to the classification. In Section 4.3, we will present a conjecture which would imply the non-negativity in general and in Chapter 5, we will present the connection of the conjecture to the work of Berest, Etingof and Ginzburg.

## 2.2. Non-nesting partitions

First, we observe that a Catalan path $D$ can be identified with the collection of cells $b_{i,j}$ which lie strictly below $D$ and strictly above the line $x = y$, where for $0 \leq i < j < n$ the cell $b_{i,j} \subseteq \mathbb{R}^2$ is given by

$$b_{i,j} := \big\{(x, y) \in \mathbb{R}^2 : i < x < i + 1, j < y < j + 1\big\}.$$

Example 2.9. The Catalan path shown in Figure 10 on page 33 can be identified with the collection of cells given by $\{b_{12}, b_{23}\}$.

To see how this construction can be interpreted in terms of the root poset $\Phi^+$ of type $A_{n-1}$, observe that for $1 \leq i < j \leq n$, the map sending the positive root $\epsilon_j - \epsilon_i \in \Phi^+$ to the cell $b_{n-j,n-i}$ defines a bijection between order ideals in $\Phi^+$ and Catalan paths of length $n$, where an order ideal is a subset $I \subseteq \Phi^+$ with the property that $\beta \leq \alpha \in I$ implies $\beta \in I$. It is denoted by $I \trianglelefteq \Phi^+$.

Example (continued) 2.10. The order ideal associated to the Catalan path in Figure 10 on page 33 is given by $\{\epsilon_5 - \epsilon_4, \epsilon_4 - \epsilon_3\}$.

We use this interpretation of Catalan paths as the motivation to define non-nesting partitions as follows:

Definition 2.11. Let $W$ be a crystallographic reflection group with root poset $\Phi^+$. The non-nesting partition lattice $NN(W) := \{I \trianglelefteq \Phi^+\}$ is the collection of all order ideals in $\Phi^+$ ordered by inclusion.

Remark. Originally, $NN(W)$ was defined by Postnikov as *antichains* in the root poset, i.e. subsets of pairwise non-comparable elements, see [**86**, Remark 2]. Sending an order ideal to its maximal elements gives a natural bijection between order ideals and antichains.

Note. The bijection between non-nesting set partitions of $[n]$ and Catalan paths of length $n$ described in Example 2.3 can be formulated in terms of the root poset $\Phi^+$ of type $A_{n-1}$ as putting the integers $i$ and $j$ into the same block if the root $\alpha_j - \alpha_i$ in contained in the associated antichain in $\Phi^+$. In the previous examples, we have seen the Catalan path shown in Figure 10 on page 33 is associated to the antichain $\{\epsilon_5-\epsilon_4, \epsilon_4-\epsilon_3\}$ and is thereby mapped to the non-nesting set partition $\{\{1\}, \{2\}, \{3, 4, 5\}, \{6\}\}$ of $[6]$.

The following theorem is due to Postnikov, see [**86**, Remark 2]:

Theorem 2.12 (Postnikov). *Let $W$ be a crystallographic reflection group. Then*

$$|NN(W)| = \mathrm{Cat}(W).$$

In [**12**], Athanasiadis generalized the construction of non-nesting partitions as follows: let $\mathcal{I}$ be an increasing chain of order ideals

$$I_1 \subseteq \ldots \subseteq I_m \subseteq \Phi^+.$$

We call $\mathcal{I}$ a filtered chain of length $m$ if

$$(I_i + I_j) \cap \Phi^+ \subseteq I_{i+j}$$

holds for all $i, j \geq 1$ with $i + j \leq m$, and

$$(J_i + J_j) \cap \Phi^+ \subseteq J_{i+j}$$

holds for all $i, j \geq 1$, where $J_i = \Phi^+ \setminus I_i$ and $J_i = J_m$ for $i > m$.

Definition 2.13. Let $W$ be a crystallographic reflection group with root poset $\Phi^+$. The lattice of filtered chains of non-nesting partitions is defined as

$$NN^{(m)}(W) := \{\mathcal{I} : \mathcal{I} \text{ is a filtered chain of length } m \text{ in } \Phi^+\},$$

ordered componentwise by inclusion.

Question. If we identify an order ideal $I$ with the root space spanned by $I$, this definition somehow seems to be a *filtration*. Does it have any algebraic meaning?

This construction was introduced to generalize Theorem 2.12 to Fuß-Catalan numbers:

Theorem 2.14 (Athanasiadis). *The number of filtered chains in $\Phi^+$ is equal to the $m$'s Fuß-Catalan number of the given type,*

$$\left|NN^{(m)}(W)\right| = \mathrm{Cat}^{(m)}(W).$$

In the next section, we will introduce the extended Shi arrangement, which is closely related to $NN^{(m)}(W)$ and which is actually the key to understand the notion of filtered chains defined by Athanasiadis on $NN(W)$.

## 2.3. The extended Shi arrangement

Recall from Section 1.1.10 the definition of the Coxeter arrangement. J.-Y. Shi [**91**] constructed a hyperplane arrangement which contains the Coxeter arrangement. The following definition generalizes this construction and is due to Athanasiadis [**11**].

Definition 2.15. Let $W$ be a crystallographic reflection group. The extended Shi arrangement $\mathsf{Shi}^{(m)}(W)$ is the collection of hyperplanes in the underlying vector space $V$ given by the affine equations $(\alpha, x) = k$ for $\alpha \in \Phi^+$ and integers $k$ with

$$-m < k \leq m.$$

Furthermore, a connected component of the complement of the hyperplanes in $\mathsf{Shi}^{(m)}(W)$ is called region of $\mathsf{Shi}^{(m)}(W)$ and a positive region is a region which lies in the fundamental chamber of the Coxeter arrangement.

The following result concerning the total number of regions of $\mathsf{Shi}^{(m)}(W)$ as well as the number of bounded regions of $\mathsf{Shi}^{(m)}(W)$ had been conjectured by P. Edelman and Reiner [**40**, Conjecture 3.3], and by Athanasiadis [**10**, Question 6.2] and was proved uniformly by M. Yoshinaga in [**97**, Theorem 1.2].

THEOREM 2.16 (Yoshinaga). *Let $W$ be a crystallographic reflection group and let $m$ be a positive integer. Then the number of regions of $\mathsf{Shi}^{(m)}(W)$ is equal to*

$$(mh+1)^l,$$

*where $l$ is the rank of $W$ and $h$ is the Coxeter number. Furthermore, the number of bounded regions of $\mathsf{Shi}^{(m)}(W)$ is equal to*

$$(mh-1)^l.$$

In [**11**], Athanasiadis counted to number of positive regions by showing that these are in one-to-one correspondence with filtered chains in $\Phi^+$: define a map from positive regions of $\mathsf{Shi}^{(m)}(W)$ to filtered chains in $\Phi^+$ by mapping some positive region $R$ to the filtered chain $I_1 \subseteq \ldots \subseteq I_m \subseteq \Phi^+$ such that for $x \in R$,

$$\begin{aligned} (\alpha, x) &< i, && \text{if } \alpha \in I_i \\ (\alpha, x) &> i, && \text{if } \alpha \in J_i = \Phi^+ \setminus I_i. \end{aligned}$$

Furthermore, he was also able to count the number of bounded positive regions by considering the characteristic polynomial of $\mathsf{Shi}^{(m)}(W)$, see [**11**].

THEOREM 2.17 (Athanasiadis). *The map described above is a bijection between positive regions of $\mathsf{Shi}^{(m)}(W)$ and filtered chains in $\Phi^+$. In particular,*

$$\left|\left\{\textit{positive regions of } \mathsf{Shi}^{(m)}(W)\right\}\right| = \mathrm{Cat}^{(m)}(W).$$

*Furthermore, the number of bounded positive regions of $\mathsf{Shi}^{(m)}(W)$ is given by*

$$\left|\left\{\textit{bounded positive regions of } \mathsf{Shi}^{(m)}(W)\right\}\right| = \prod_{i=1}^{l} \frac{d_i^* + mh}{d_i^*},$$

*where, as usual, $l$ is the rank of $W$ and $h$ is its Coxeter number, but now $d_1^* \leq \ldots \leq d_l^*$ are its codegrees introduced in Section 1.2.4.*

REMARK. The formula counting bounded positive regions in $\mathsf{Shi}^{(m)}(W)$ is of interest itself and appears different contexts, see [**7**, Section 3.7]. There, Armstrong called this product positive Fuß-Catalan numbers and denoted it by $\mathrm{Cat}_+^{(m)}(W)$.

### 2.4. $q$-Fuß-Catalan numbers for crystallographic reflection groups

In this section, we want to introduce another $q$-extension of $\mathrm{Cat}^{(m)}(W)$. This is done in terms of a statistic on regions in the extended Shi arrangement. We will see, how this definition generalizes the well-understood case of $W = A_{n-1}$. After that, we will also see, how the situation looks like in type $B$. Beside the fact that this construction generalizes type $A$ in a nice way, the more sophisticated reason for our definition will come up in Section 4.3.

Let $W$ be a crystallographic reflection group. Fix the positive region $R^0$ to be the region given by

$$\left\{x : 0 < (\alpha, x) < 1 \text{ for all } \alpha \in \Phi^+\right\}.$$

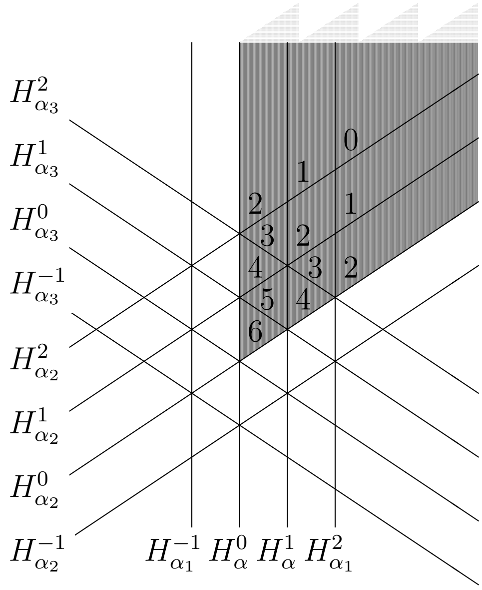



FIGURE 11. The extended Shi arrangement $\mathsf{Shi}^{(2)}(A_2)$.

The height of a region of $\mathsf{Shi}^{(m)}(W)$ is defined to be the number of hyperplanes in $\mathsf{Shi}^{(m)}(\Phi)$ that separate $R$ from $R^0$ and the coheight of a region $R$, denoted by $\mathrm{coh}(R)$, is defined as

$$\mathrm{coh}(R) := mN - \mathrm{height}(R),$$

where $N$ is the number of positive roots.

REMARK. In terms of *affine reflection groups*, $R^0$ is called *fundamental alcove.*

DEFINITION 2.18. The $q$-Fuß-Catalan number associated to $W$ is defined to be the generating function for the coheight statistic on positive regions in $\mathsf{Shi}^{(m)}(\Phi)$,

$$\mathrm{Cat}^{(m)}(W;q) := \sum_R q^{\mathrm{coh}(R)}.$$

REMARK. The poset of all regions defined by the covering relation $R \prec R'$ if and only if $\mathrm{height}(R') = \mathrm{height}(R) + 1$ and both share an edge is isomorphic to the poset $NN^{(k)}(W)$ and we could equivalently define coh as the rank-function on $NN^{(k)}(W)$.

EXAMPLE 2.19. Let $W = A_2$ and $m = 2$. In Figure 11 on page 37, the extended Shi arrangement of the given type is shown. The positive roots are denoted by $\alpha_1$, $\alpha_2$ and $\alpha_3 := \alpha_1 + \alpha_2$, the fundamental chamber is shaded and we labelled every region in the fundamental chamber by its coheight. This gives

$$\mathrm{Cat}^{(2)}(W;q) = 1 + 2q + 3q^3 + 2q^3 + 2q^4 + q^5 + q^6.$$

PROPOSITION 2.20. *Let $I_1 \subseteq \ldots \subseteq I_m \subseteq \Phi^+$ be a filtered chain and let $R$ be the associated region. Then the bijection given above implies that*

$$\mathrm{coh}(R) = \sum_{1 \le i \le m} |I_i|.$$

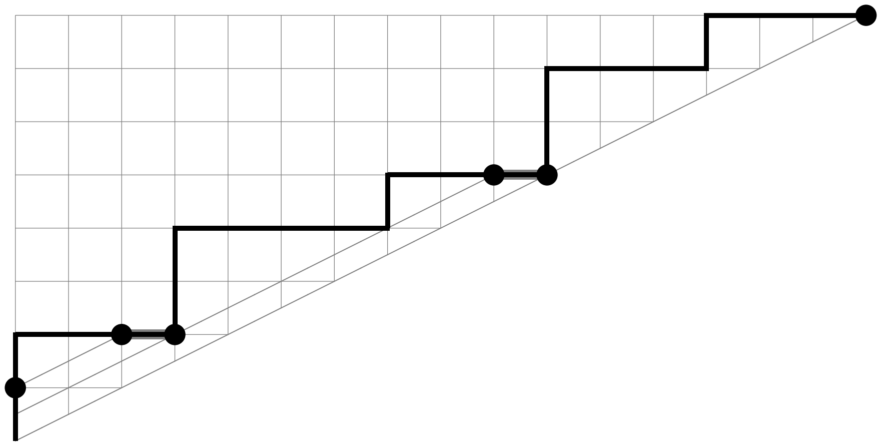


FIGURE 12. The dissection of a 2-Catalan path of length 8 into three 2-Catalan paths of lengths 1, 3 and 3 respectively.

**2.4.1. Type $A$.** We now turn to the case of $W = A_{n-1}$. By Proposition 2.20 and the discussion in Section 2.2, the coheight statistic on a non-nesting partition $I \trianglelefteq \Phi^+$ is equal to the number of cells $b_{ij}$ which lie below the Coxeter path associated to $I$. This statistic was first studied by Fürlinger and Hofbauer in [**51**] and is probably the most studied statistic on Catalan paths. It is called area statistic.

The goal of this section is to shows that the coheight statistic also generalizes the area statistic on $\mathcal{D}_n^{(m)}$, which is analogously defined to be the number of cells which lie below a given $m$-Catalan path and strictly above the line $x = my$, where now $m \geq 1$. Before stating the theorem, we define $q$-Fuß-Catalan number of type $A_{n-1}$ as

$$\mathrm{Cat}_n^{(m)}(q) := \sum_{D \in \mathcal{D}_n^{(m)}} q^{\mathrm{area}(D)}$$

and derive a recurrence for $\mathrm{Cat}_n^{(m)}(q)$.

C. Krattenthaler studied lattice paths with linear boundary with respect to a statistic which can be reformulated in terms of the area statistic, see [**72**].

THEOREM 2.21.

$$\mathrm{Cat}_{n+1}^{(m)}(q) = \sum_{k_1+\ldots+k_{m+1}=n} q^{n(\mathbf{k})}\,\mathrm{Cat}_{k_1}^{(m)}(q)\ldots\mathrm{Cat}_{k_{m+1}(q)}^{(m)}, \qquad \mathrm{Cat}_0^{(m)}(q) = 1,$$

*where* $n(\mathbf{k}) = n(k_1,\ldots,k_{m+1}) := \sum (m+1-i)k_i$.

PROOF. Dissect an $m$-Catalan path $D$ of length $n+1$ into $(m+1)$ $m$-Catalan paths $D_0,\ldots,D_m$, with lengths summing up to $n$, at every first east step of $D$ from level $i$ to level $i-1$ for $1 \leq i \leq m$, where the level of a lattice point is meant to be the horizontal distance to the diagonal $x = my$. Furthermore, delete the very first north step of $D$. See Figure 12 for an example. This gives the proposed recurrence. $\square$

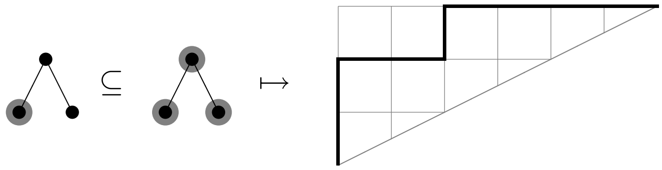


FIGURE 13. A filtered chain of order ideals in the root poset and the associated 2-Catalan path.

EXAMPLE 2.22. Consider the 2-Catalan path $D$ of length 8 in Figure 12. The east steps of $D$ from level 1 to level 0 and from level 2 to level " are highlighted and the start and end points of the three 2-Catalan paths are marked. This gives

$$D_0 = NEE,\ D_1 = NNEEEENEE \text{ and } D_2 = NNEEENEEE.$$

THEOREM 2.23. *Let* $W = A_{n-1}$*. Then*

$$\mathrm{Cat}^{(m)}(W;q) = \mathrm{Cat}_n^{(m)}(q)$$

PROOF. Recall that an $m$-Catalan path of length $n$ can be encoded as a sequence $(a_1, \ldots, a_n)$ of integers such that $a_1 = 0$ and $a_{i+1} \leq a_i + m$. Define the sum of an $m$-Catalan path and an $m'$-Catalan path both of length $n$ to be the $m+m'$-Catalan path obtained by adding the associated sequences componentwise,

$$(a_1, \ldots, a_n) + (a'_1, \ldots, a'_n) = (a_1 + a'_1, \ldots, a_n + a'_n).$$

This yields a map from filtered chains of length $m$ in $\Phi^+$ to $m$-Catalan paths which sends the coheight of a given filtered chain to the area of the associated $m$-Catalan path by just summing the Catalan paths associated to the ideals in the filtered chain, see Figure 13 for an example. To show that this map is in fact a bijection between filtered chains of length $m$ and $m$-Catalan paths take two filtered chains $I_1 \subseteq \ldots \subseteq I_m$ and $I'_1 \subseteq \ldots \subseteq I'_m$ which map to the same path and assume they are not equal, i.e., there exists an $\epsilon_j - \epsilon_i$ contained in $I_i$ and not contained in $I'_i$. Therefore there exists $\epsilon_i - \epsilon_{i'}$ and $j > 0$, such that $(\epsilon_j - \epsilon_i) + (\epsilon_i - \epsilon_{i'}) = \epsilon_j - \epsilon_{i'}$ is contained in $I'_{i+j}$ and is not contained in $I_{i+j}$. As both chains are filtered, we get $\epsilon_i - \epsilon_{i'} \in I'_j$ but $\epsilon_i - \epsilon_{i'} \notin I_j$. This gives rise to an infinite sequence $(\epsilon_j - \epsilon_i, \epsilon_i - \epsilon_{i_1}, \epsilon_{i_1} - \epsilon_{i_2}, \ldots)$ of positive roots, which is a contradiction.
As it is known that both sets have the same cardinality, the statement follows. □

**2.4.2. Type** $B$**.** Following the idea of identifying order ideals in the root poset with lattice paths in $\mathbb{Z}^2$ such that the number of elements in an order ideal matches somehow with the number of cells confined by the path together with other restrictions, we want to define Catalan paths of type $B$ together with an area-statistic and moreover, we want to establish an analogous recurrence as obtained for Catalan paths of type $A$. Furthermore, we will describe why we were not able to construct neither type $B$ Catalan paths for higher $m$, nor type $D$ Catalan paths.

DEFINITION 2.24. A Catalan path of type $B_n$ is a lattice paths of $2n$ steps, either north or east, that starts at $(0,0)$ and stays above the diagonal $x = y$. For such a path $D$, we define area$(D)$ to be the number of cells $b_{ij}$ as defined above which lie below $D$, but now with the additional property that $1 \leq i < j \leq 2n + 1 - i$.

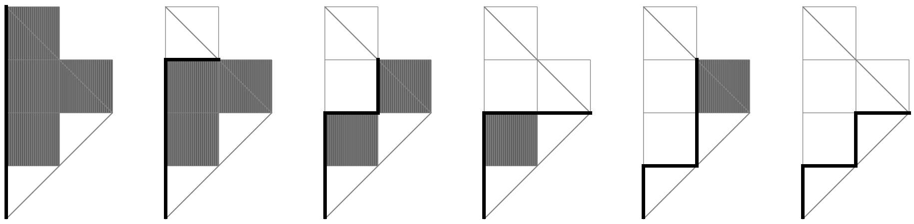


FIGURE 14. All type $B$ Catalan paths of length 2.

In analogy to type $A$, we define $q$-Catalan numbers for type $B$ in the following way:

DEFINITION 2.25.

$$\mathrm{Cat}_{B_n}(q) := \sum q^{\mathrm{area}(D)},$$

where the sum ranges over all Catalan paths of type $B_n$.

EXAMPLE 2.26. In Figure 14, we list all Catalan paths of type $B_2$. The cells which contribute to the area are shaded. Therefore, we have

$$\mathrm{Cat}_{B_2}(q) = q^4 + q^3 + q^2 + 2q + 1.$$

COROLLARY 2.27. *For $W = B_n$, the construction implies*

$$\mathrm{Cat}^{(1)}(W, q) = \mathrm{Cat}_{B_n}(q).$$

$q$-Catalan numbers of type $B$ satisfy the following recurrence involving $q$-Catalan numbers of type $A$:

THEOREM 2.28.

$$\mathrm{Cat}_{B_n}(q) = \mathrm{Cat}_n(q) + \sum_{k=0}^{n-1} q^{2k+1}\,\mathrm{Cat}_{B_k}(q)\,\mathrm{Cat}_{n-k}(q), \qquad \mathrm{Cat}_{B_0}(q) = 1.$$

PROOF. Let $D$ be a Catalan path of type $B_n$. Then either $D$ has as many east as north steps, which means it is equal to a type $A$ Catalan path of length $n$, or there exists a last point $(k, k+1)$ where the path touches the diagonal $x + 1 = y$ and stays strictly above afterwards. Now, we have an initial type $A$ Catalan path of length $k+1$, except that the last step is a north step instead of an east step, see Figure 15 for an example. After this north step, a Catalan path of type $B_{n-k-1}$ starts. This gives

$$\begin{aligned}
\mathrm{Cat}_{B_n}(q) &= \mathrm{Cat}_n(q) + \sum_{k=0}^{n-1} q\,\mathrm{Cat}_{k+1}(q) q^{2(n-k-1)}\,\mathrm{Cat}_{B_{n-k-1}}(q)\\
&= \mathrm{Cat}_n(q) + \sum_{k=0}^{n-1} q^{2k+1}\,\mathrm{Cat}_{B_k}(q)\,\mathrm{Cat}_{n-k}(q)
\end{aligned}$$

□

EXAMPLE 2.29. Figure 15 shows a Catalan path of type $B_6$. It starts with a type $A$ like Catalan path $D$ of length 3, followed by a Catalan path $D'$ of type $B_3$,

$$D = NENNEN, \quad D' = NNENNE.$$

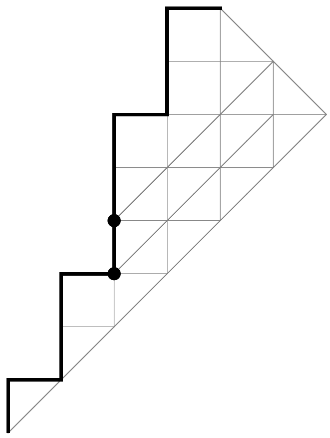


FIGURE 15. A Catalan path of type $B_6$.

COROLLARY 2.30. $\mathrm{Cat}_{B_n}(q)$ *satisfies the following generating function identity:*

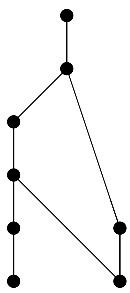
$$\sum_{n\geq 0} \frac{x^n q^{-n(n-1)}(1-qx)}{(-x;q^{-1})_{2n+1}} \mathrm{Cat}_{B_n}(q) = 1.$$

PROOF. The recurrence in Theorem 2.28 can be written as

$$(1+q^{2n+1})\,\mathrm{Cat}_{B_n}(q) - \sum_{k=0}^{n} q^{2k+1}\,\mathrm{Cat}_{B_k}(q)\,\mathrm{Cat}_{n-k}(q) = \mathrm{Cat}_n(q).$$

Multiplying both sides of the equality by $x^n q^{-n(n-1)}/(-x;q^{-1})_{2n+1}$ and summing over all $n$ gives the proposed generating function identity. $\square$

REMARK. It is not possible to construct Catalan paths of type $B$ for higher $m$ as lattice paths, at least not in the manner of defining an area generating function equal to the specialization $t = 1$ of $q,t$-Fuß-Catalan numbers which we will define in Section 4.2: if one wants to define lattice paths consisting of north and east steps with a certain boundary (like in type $A$ the linear boundary $x = y$) for which the *area generating function* describes the $t = 1$ specialization in $\mathrm{Cat}^{(m)}(W;q,t)$, one can always equivalently transform this problem into the problem of counting order ideals by cardinality in certain posets. Beside other properties, these posets have to be locally graded, i.e., for any elements $a \leq b$, all paths from $a$ to $b$ have to have the same length. Now consider the following poset:



This poset is the unique poset $P$ such that $\mathrm{Cat}^{(2)}(B_2;q,1) = \sum_{I \trianglelefteq P} q^{|I|}$. As this poset is *not* locally graded, the construction fails in this case. This is likely to be the reason why we were – so far – not able to find a recurrence in type $B$ for higher $m$.

Obviously, these posets also have to be *planar*. As the root poset of type $D_n$ is *not* planar for $n \geq 4$, this construction also fails for type $D$.

In the remaining part of this section, we want to exhibit a major index on Catalan paths of type $B_n$ for which, in analogy to type $A$, the generating function is equal to diag-Cat$^{(1)}(B_n; q)$.
It is well-known that for the major index on lattice paths consisting on $n$ north and $n$ east steps without any further restrictions we have

$$\sum q^{\mathrm{maj}(L)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

where the major index is defined with respect to the order $E < N$ by

$$\mathrm{maj}(L) := \sum_{i \in Des(L)} (2n - i),$$

see e.g. [**4**] and [**81**].

We now define a major index of a Catalan path $D$ of type $B_n$ in the following way:

$$\mathrm{maj}(D) := 2 \cdot \big(\mathrm{neg}(D) + \sum_{i \in \mathrm{Des}(D)} (2n - i)\big).$$

Here, $\mathrm{neg}(D)$ is defined to be the number of east steps in $D$ and $\mathrm{Des}(D)$ is, as for Catalan paths of type $A$, the descent set with respect to the order $N < E$.

Example 2.31. The Catalan path $D$ of type $B_6$ shown in Figure 15 is encoded by the word

$$NE\ NNE\ NNNE\ NNE.$$

This gives $\mathrm{neg}(D) = 4$, $\mathrm{Des}(D) = \{2, 5, 9\}$ and therefore,

$$\mathrm{maj}(D) = 2(4 + (12 - 2) + (12 - 5) + (12 - 9)) = 2(4 + 10 + 7 + 3) = 48.$$

Note. Equivalently, we could have defined the major index on Catalan paths of type $B$ as $\mathrm{maj}(D) := 2\,\mathrm{maj}(w)$, where $w$ is the reverse of the Catalan word of $D$. In the previous example, we would have $w = E\ NNE\ NNNE\ NNE\ N$ and therefore, $\mathrm{Des}(w) = \{1, 4, 8, 11\}$, $2\,\mathrm{maj}(w) = 2(1 + 4 + 8 + 11) = 48 = \mathrm{maj}(D)$.

Definition 2.32.

$$\text{diag-Cat}_{B_n}(q) := \sum q^{\mathrm{maj}(D)},$$

where the sum ranges over all Catalan paths of type $B_n$.

Proposition 2.33. *The generating function for the major index on Catalan paths of type $B_n$ is equal to* diag-Cat$^{(1)}(B_n; q)$*,*

$$\text{diag-Cat}_{B_n}(q) = \text{diag-Cat}^{(1)}(B_n; q).$$

Proof. Define a bijection between lattice paths form $(0, 0)$ to $(n, n)$ to Catalan paths of type $B_n$ by replacing the first east step from level $i$ to level $i - 1$ by a north step for all $i < 0$ for which such an east step exists. For example, the lattice path shown in Figure 16 is mapped to the Catalan path shown in Figure 15. This transformation

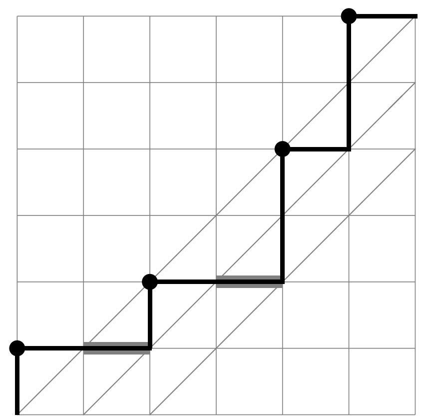


FIGURE 16. The lattice path from $(0,0)$ to $(6,6)$ which is mapped to to Catalan path of type $B_6$ shown in Figure 15.

does not affect the major index with respect to the ordering $E < N$. For a lattice path $L$ and its image $D$, we then have $\mathrm{maj}(D) = 2\,\mathrm{maj}(L)$ and therefore

$$\text{diag-Cat}_{B_n}(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}.$$

This can easily be seen to be equal to $\text{diag-Cat}^{(1)}(W;q)$. $\square$

EXAMPLE (CONTINUED) 2.34. The major index of the lattice path $L$ shown in Figure 16 is given by

$$\mathrm{maj}(L) = (12-1) + (12-4) + (12-8) + (12-11) = 11+8+4+1 = 24.$$

As we have seen in Example 2.31, this gives $\mathrm{maj}(D) = 2\,\mathrm{maj}(L)$.

The following proposition is the analogue of Proposition 2.8:

PROPOSITION 2.35. *The sequence of coefficients of the $q$-Catalan numbers* $\text{diag-Cat}_{B_n}(q)$ *is symmetric, i.e.,*

$$\text{diag-Cat}_{B_n}(q) = q^{2n^2}\,\text{diag-Cat}_{B_n}(q^{-1}).$$

PROOF. As we have seen, we have $\text{diag-Cat}_{B_n}(q) = \sum q^{2\,\mathrm{maj}(D)}$, where the sum ranges over all lattice paths from $(0,0)$ to $(n,n)$ without any restrictions. The statement follows by applying the same involution as in type $A$ to such a lattice path. $\square$

## 2.5. Non-crossing partitions

We start with the case of $W$ being a real reflection group. In Section 1.1.5, the length of an element $\omega$ in $W$ was defined as the length of a reduced word of $\omega$ expressed in terms of simple reflections and in Section 1.1.6 this length function was used to define a partial order on $W$ that turns $W$ into a lattice. In order to introduce the notion of

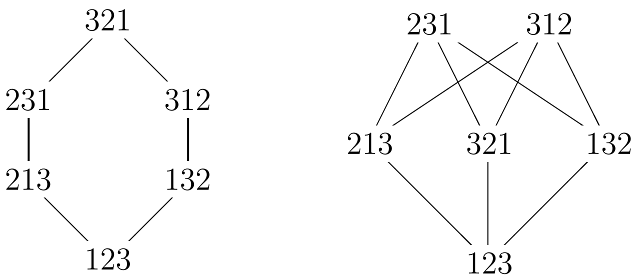



FIGURE 17. $\mathsf{Weak}(A_2)$ and $\mathsf{Abs}(A_2)$.

non-crossing partitions, we first present another length function on $W$ and the analogous partial order on $W$.

We basically follow Armstrong's PhD thesis [**7**] which contains a very comprehensive outline of what is known on Fuß-Catalan combinatorics.

**2.5.1. The absolute order on $W$.** The definition of the absolute length is adapted from the analogous definition of the ordinary length of some element in a real reflection group:

DEFINITION 2.36. Let $\Phi$ be a root system and let

$$T := \{s_\alpha : \alpha \in \Phi^+\} \subseteq W = W(\Phi)$$

be the set of reflections in $W$. The absolute length of a given $\omega \in W$, denoted by $\mathrm{l}_T(\omega)$, is the smallest $k$ for which $\omega$ can be written as a product of $k$ reflections.

As $S$, the set of simple reflections, is contained in $T$, it immediately follows

$$\mathrm{l}_T(\omega) \leq \mathrm{l}_S(\omega) \text{ for all } \omega \in W.$$

Using $\mathrm{l}_T$, define again a partial order on $W$ by

$$\omega \leq_T \tau :\Leftrightarrow \mathrm{l}_T(\tau) = \mathrm{l}_T(\omega) + \mathrm{l}_T(\omega^{-1}\tau),$$

and denote the resulting poset by $\mathsf{Abs}(W)$. As $\mathsf{Weak}(W)$, this poset is graded with rank function $\mathrm{l}_T$ and unique minimal element $1 \in W$ but it is *not* a lattice as in general it does *not* have a unique maximal element. Figure 17 shows the Hasse diagrams of $\mathsf{Weak}(A_2)$ and $\mathsf{Abs}(A_2)$.

NOTE. By construction, the Coxeter elements of $W$ have maximal absolute order or, equivalently, all Coxeter elements are among the top elements in $\mathsf{Abs}(W)$.

For the classical reflection groups, the absolute length can be combinatorially computed using the cycle notation: a cycle $(i_1, i_2, \ldots, i_k)$ is a list of integers in $\{\pm 1, \ldots, \pm n\}$ with distinct absolute values except for $i_k$ which can possibly be equal to $-i_1$. We can regard a cycle as a signed permutation as follows: if $i_k \neq -i_1$ we obtain a signed permutation consisting of the cycles $i_1 \mapsto i_2 \mapsto \ldots \mapsto i_k \mapsto i_1$ and its negative analogue. If $i_k = -i_1$ we obtain a signed permutation consisting of the cycle $i_1 \mapsto i_2 \mapsto \ldots \mapsto i_k \mapsto -i_1 \mapsto -i_2 \mapsto \ldots \mapsto -i_k \mapsto i_1$. In both cases, all remaining integers are fixed points. Any signed permutation is a product of cycles and expressing a given $\sigma$ in this way is called cycle notation of $\sigma$. For even signed permutations and ordinary permutations, the cycle notation is defined analogously. Often the cycles consisting only of single elements are not written. We illustrate the cycle notation by some examples:

$$\begin{aligned}
[4,2,6,5,1,3] &= (1,4,5)(3,6), \\
[4,2,-6,5,1,-3] &= (1,4,5)(3,-6), \\
[4,2,-6,5,1,3] &= (1,4,5)(3,-6,-3) = (1,4,5)(6,3,-6), \\
[4,2,6,5,-1,-3] &= (1,4,5,-1)(3,6,-3).
\end{aligned}$$

Define the length of a cycle $c = (i_1, \dots, i_k)$ to be $k-1$ and denote it by $\mathrm{l}(c)$. Then the absolute length of a signed permutation $\sigma$ is equal to the sum of the lengths of the cycles in its cycle notation $\sigma = c_1 \cdots c_k$, in symbols

$$\mathrm{l}_T(\sigma) = \mathrm{l}(c_1) + \ldots + \mathrm{l}(c_k).$$

EXAMPLE 2.37. As the absolute length is by definition equal to the rank function in the Hasse diagram, the equality can be seen for type $A_3$ in Figure 18.

**2.5.2. Fixed space and moved space.** Following Brady and Watt [**28**], define the fixed space and the moved space of an element $\omega \in W$ as

$$\mathsf{Fix}(\omega) := \ker(\omega - \mathrm{id}) \quad \text{and} \quad \mathsf{Mov}(\omega) := \mathrm{im}(\omega - \mathrm{id}).$$

They proved the following statement which gives a geometric interpretation of the absolute length $\mathrm{l}_T$ and of the poset $\mathsf{Abs}(W)$:

THEOREM 2.38 (Brady, Watt). *Let $\omega, \tau \in W$ such that $\omega$ and $\tau$ have a common larger element. Then*

- $\mathrm{l}_T(\omega) = \dim \mathsf{Mov}(\omega)$,
- $\omega \leq_T \tau \Leftrightarrow \mathsf{Mov}(\omega) \subseteq \mathsf{Mov}(\tau)$.

This theorem implies that every interval in $\mathsf{Abs}(W)$ can be interpreted via the map $\omega \mapsto \mathsf{Mov}(\omega)$ as a poset of subspaces of $V$.

**2.5.3. Non-crossing partitions.** Together with the fact that the Coxeter elements of $W$ all have maximal absolute length, Theorem 2.38 yields the definition of non-crossing partitions. It goes back mainly to Brady and Watt [**28, 29**].

DEFINITION 2.39. Let $c$ be a Coxeter element in $W$. The non-crossing partition lattice $NC(W,c)$ is defined as the interval in $\mathsf{Abs}(W)$ between 1 and $c$,

$$NC(W,c) := [1,c]_T = \big\{\omega \in W : 1 \leq_T \omega \leq_T c\big\}.$$

EXAMPLE 2.40. In Figure 18, $\mathsf{Abs}(A_3)$ is shown with the interval $[1,c]_T$ for $c$ being the long cycle $(1,2,3,4)$ highlighted.

This definition seems to depend on the choice of the Coxeter element $c$, but as all Coxeter elements form a conjugacy class in $W$ and since conjugation by a group element is an automorphism on $\mathsf{Abs}(W)$ it follows that for Coxeter elements $c$ and $c'$,

$$NC(W,c) \cong NC(W,c').$$

In [**30**], Brady and Watt proved that $NC(W,c)$ is in fact a lattice. Furthermore, Armstrong showed in [**7**] that it is self-dual and locally self-dual.

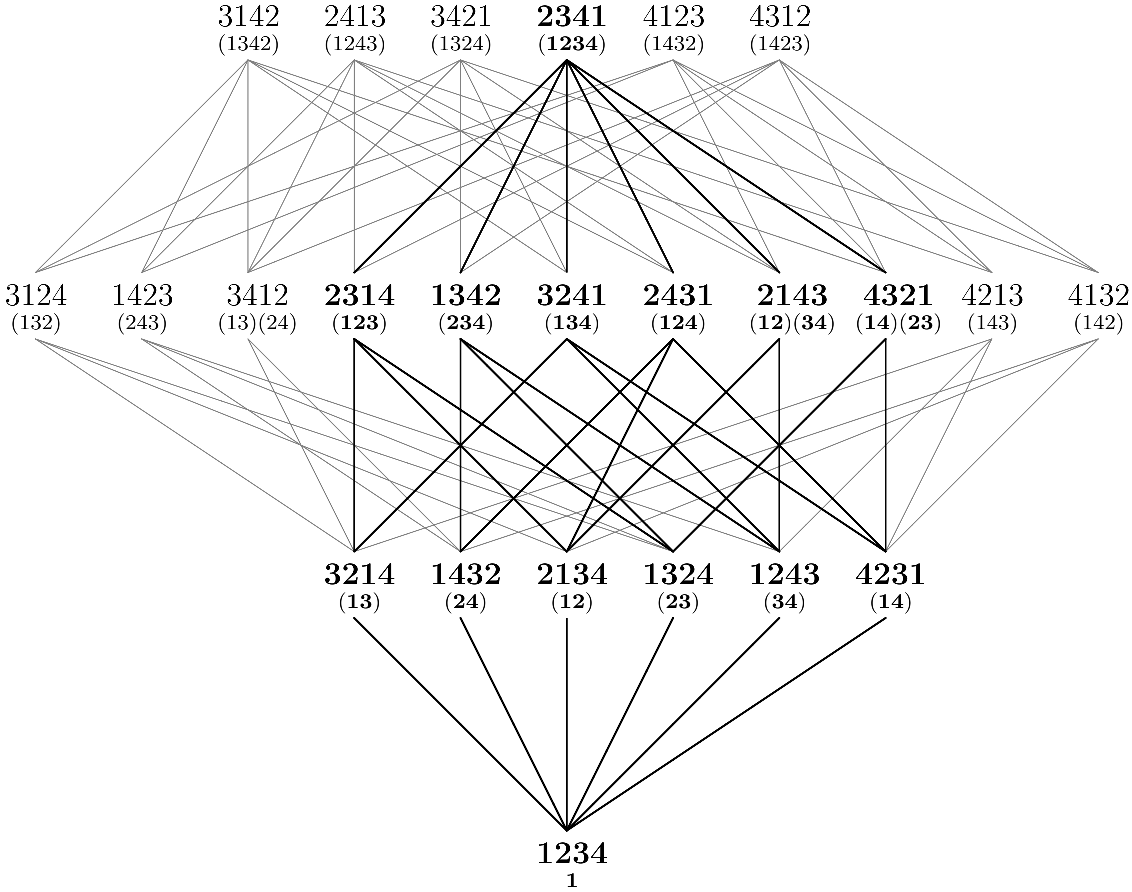



FIGURE 18. $\mathsf{Abs}(A_3)$ with $NC(A_3, (1, 2, 3, 4))$ highlighted.

**2.5.4. Non-crossing partitions of type $A$.** We now want to describe, how non-crossing set partitions can be seen as the type $A$ instance of non-crossing partitions. Write $NC(A_{n-1})$ for $NC(A_{n-1}, c)$ where $c = (1, 2, \ldots, n)$. Then

$$\sigma \leq_T c \Leftrightarrow \text{all cycles in } \sigma \text{ are increasing and pairwise non-crossing,}$$

where a cycle $(i_1, \ldots, i_k)$ is called increasing if, after a possible cyclic shift, $i_1 < \ldots < i_k$, and where two cycles $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_{k'})$ are called non-crossing if the associated blocks $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{k'}\}$ do not cross in the sense of Example 2.2. This equivalence yields a bijection between non-crossing set partitions and $NC(A_{n-1})$: map some $\{B_1, \ldots, B_k\}$ to the permutation having a cycle for each $B_i$ with the given elements in increasing order. Furthermore, the ordering of non-crossing set partitions by refinement turns $NC(n)$ into a lattice with $\hat{1} = [n]$ and $\hat{0} = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and we have

$$NC(n) \cong NC(A_{n-1}).$$

**2.5.5. Non-crossing partitions of types $B$ and $D$.** There exists also a description of non-crossing partitions of types $B$ and $D$ in terms of set partitions. They were introduced in type $B$ by Reiner in [**86**] and in type $D$ by Athanasiadis and Reiner in [**13**]. As we only need the description in type $B$, we relax the description in type $D$.

DEFINITION 2.41. A type $B_n$ set partition is a set partition $\mathcal{B}$ of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying the following two conditions:

(i) if $B$ is a block in $\mathcal{B}$ then $-B$ is also a block in $\mathcal{B}$,
(ii) there exists at most one block $B$ in $\mathcal{B}$ for which $B = -B$.

Order the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ by

$$-1 < -2 < \ldots < -n < 1 < 2 < \ldots < n.$$

A type $B_n$ set partition $\{B_1, \ldots, B_k\}$ is called non-crossing if

$$a < b < c < d \text{ such that } a, c \in B_i, b, d \in B_j \text{ implies } i = j.$$

The lattice of all non-crossing set partitions of type $B_n$ ordered by refinement is denoted by $NC_B(n)$.

NOTE. One can visualize non-crossing set partitions of type $B_n$ in the same way as non-crossing set partitions of type $A_{n-1}$. This visualization shows immediately that the property of being non-crossing implies (ii) in the definition of type $B_n$ set partitions.

This yields a bijection between $NC_B(n)$ and $NC(B_n)$, where $NC(B_n)$ is defined to be $NC(B_n, c)$ for $c = (1, 2, \ldots, n, -1)$. Map some $\{B_1, \ldots, B_k\}$ to the permutation having a cycle for each $B_i$ with the given elements in increasing order. We have

$$NC_B(n) \cong NC(B_n).$$

**2.5.6. Non-crossing partitions for well-generated complex reflection groups.** Recently, Bessis observed that one can generalize non-crossing partitions to any well-generated complex reflection group $W$ [**20, 21**]. One can define the total order on $W$ in exactly the same way as for real reflection groups,

$$\omega \leq_T \tau :\Leftrightarrow \mathrm{l}_T(\tau) = \mathrm{l}_T(\omega) + \mathrm{l}_T(\omega^{-1}\tau),$$

and, as discussed in Section 1.2, the Coxeter elements are among the top elements in the resulting poset $\mathsf{Abs}(W)$.

DEFINITION 2.42. Let $W$ be a well-generated complex reflection group. Then the non-crossing partition lattice $NC(W)$ is defined as the interval in $\mathsf{Abs}(W)$ between 1 and $c$,

$$NC(W, c) := [1, c]_T = \big\{\omega \in W : 1 \leq_T \omega \leq_T c\big\},$$

where $c \in W$ is any Coxeter element.

In [**21**], Bessis proved the following theorem:

THEOREM 2.43 (Bessis). *Let $W$ be a well-generated complex reflection group and let $c$ be a Coxeter element. Then $NC(W, c)$ is a bounded, graded lattice which is self-dual and locally self-dual.*

REMARK. The Coxeter element $c$ acts on $NC(W, c)$ by conjugation. This action generalizes the action of the long cycle $(1, 2, \ldots, n)$ on $NC(n)$ described in the beginning of this chapter.

**2.5.7. Generalized non-crossing partitions.** The notion of non-crossing partitions can be generalized in the following sense. This was done by Armstrong for real reflection groups and in full generality by Bessis in [**21**]:

Definition 2.44. Let $W$ be a well-generated complex reflection group, $c$ be a Coxeter element and $m$ be a non-negative integer. The generalized non-crossing partition lattice $NC^{(m)}(W,c)$ is defined as the set of all chains in $NC(W,c)$ having $m$ elements,

$$NC^{(m)}(W,c) := \big\{(\omega_1 \leq_T \ldots \leq_T \omega_m) : \omega_i \in NC(W,c)\big\},$$

and which are ordered by

$$(\omega_1 \leq_T \ldots \leq_T \omega_m) \leq (\tau_1 \leq_T \ldots \leq_T \tau_m) :\Leftrightarrow \omega_i^{-1}\omega_{i+1} \geq_T \tau_i^{-1}\tau_{i+1} \text{ for } 0 \leq i \leq m,$$

where $\omega_{m+1}$ and $\tau_{m+1}$ are both set to be equal to $c$.

Remark. For real reflection groups, the generalized non-crossing partition lattice was introduced by Armstrong in [**7**], where he gave them the name *$m$-divisible non-crossing partitions.*

**2.5.8. Enumeration of non-crossing partitions.** The generalized non-crossing partitions were counted by Armstrong for real reflection groups in [**7**] and for well-generated complex reflection groups by Bessis in [**21**]:

Theorem 2.45 (Armstrong; Bessis). *Let $W$ be a well-generated complex reflection group and let $c$ be a Coxeter element. Then the number of generalized non-crossing partitions equals the Fuß-Catalan number,*

$$\big|NC^{(m)}(W,c)\big| = \mathrm{Cat}^{(m)}(W).$$

**2.5.9. The cyclic sieving phenomenon.** In [**87**], Reiner, D. Stanton and D. White described the following enumerative phenomenon:

Definition 2.46. Let $C$ be a cyclic group of order $n$ acting on a finite set $X$ and let $X(q)$ be a polynomial in $\mathbb{Z}[q]$. Then the triple $\big(X, X(q), C\big)$ exhibits the cyclic sieving phenomenon (or short CSP) if for every $a \in C$ and any $\zeta$ being a complex root of unity having the same multiplicative order as $a$, one has

$$X(\zeta) = \big|\big\{x \in X : a(x) = x\big\}\big|.$$

In particular, the definition implies that $|X| = X(1)$. So one can think of $X(q)$ as a generating function for the set $X$.

In [**22**], Bessis and Reiner proved a cyclic sieving phenomenon for non-crossing partitions:

Theorem 2.47 (Bessis, Reiner). *Let $W$ be a well-generated complex reflection group and let $c$ be a Coxeter element in $W$ acting on $NC(W,c)$ by conjugation. Then the triple*

$$\big(NC(W,c), \text{diag-Cat}^{(1)}(W), \langle c\rangle\big)$$

*exhibits the cyclic sieving phenomenon.*

This theorem was conjectured in [**7**] by Armstrong. Indeed, he constructed a natural cyclic group action of order $mh$ on $NC^{(m)}$ and conjectured that the triple

$$\big(NC^{(m)}(W,c), \text{diag-Cat}^{(m)}(W), \langle c\rangle\big)$$

exhibits the cyclic sieving phenomenon. This conjecture is still open.

### 2.6. $q$-Catalan numbers for non-crossing partitions in types $A$ and $B$

In this section, we want to describe the $q$-Catalan numbers $\mathrm{Cat}^{(1)}(W;q)$ as well as the $q$-Catalan numbers diag-$\mathrm{Cat}^{(1)}(W;q)$ in terms of non-crossing partitions, when $W$ is either of type $A$ or of type $B$. We have already seen in the previous sections that

$$\begin{aligned}\mathrm{Cat}^{(1)}(W;q) &= \sum q^{\mathrm{area}(D)},\\ \text{diag-}\mathrm{Cat}^{(1)}(W;q) &= \sum q^{\mathrm{maj}(D)},\end{aligned}$$

where the sums range over all Catalan paths of the given type and where area and maj are the appropriate statistics.

Let rev be the involution on signed permutations which reverses the negative elements in the one-line notation, e.g. $\mathrm{rev}([2,-4,3,-1]) = [2,-1,3,-4]$. Fix $NC(W)$ as described for type $A$ in Section 2.5.4 and for type $B$ in Section 2.5.5. We will bijectively prove the following theorem:

Theorem 2.48. *Let $W$ be the reflection group $A_{n-1}$ or $B_n$. Then the $q$-Catalan numbers $\mathrm{Cat}^{(1)}(W;q)$ and diag-$\mathrm{Cat}^{(1)}(W;q)$ can be interpreted in terms of non-crossing partitions as follows:*

$$\mathrm{Cat}^{(1)}(W;q) = \sum_{\sigma\in\mathrm{rev}(NC(W))} q^{\mathrm{ls}(\sigma)}, \tag{2}$$

$$\text{diag-}\mathrm{Cat}^{(1)}(W;q) = \sum_{\sigma\in\mathrm{rev}(NC(W))} q^{\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma)}. \tag{3}$$

Remark. The analogous statement is false in type $D$: by computer experiments it is easy to show that for any Coxeter element $c \in D_4$,

$$\mathrm{Cat}^{(1)}(D_4;q) \neq \sum_{\sigma\in\mathrm{rev}(NC(D_4,c))} q^{\mathrm{ls}(\sigma)}.$$

**2.6.1. The bijection in type $A$.** Define a map $\phi_n : NN(A_{n-1}) \to NC(A_{n-1})$ as indicated in Figure 19 on page 50: write the numbers 1 to $n$ below the root poset of type $A_{n-1}$ from right to left and then associate to a given order ideal $I \trianglelefteq \Phi^+$ the permutation obtained by the shown "shelling" of $I$. In terms of the root poset $\Phi^+$, $\phi_n$ can be described as follows: for $1 \leq i < j \leq n$, set $[i,j]$ to be the positive root $\epsilon_j - \epsilon_i$. For an order ideal $I = \{[i_1,j_1],\ldots,[i_{max},j_{max}]\} \trianglelefteq \Phi^+$, let $[a_1,b_1],\ldots,[a_k,b_k]$ with $a_1 < \ldots < a_k$ be the maximal elements in $I$. The list of maximal elements in $I$ decompose into blocks, such that $[a_{i-1},b_{i-1}]$ is the last element in one block and $[a_i,b_i]$ is the first element in the next block if and only if $b_{i-1} \leq a_i$. To $I$, we associate a permutation $\sigma(I)$ having a cycle for each block, where the cycle starts with the first $a_i$ in the block followed by all $a_i$'s that are equal to $b_{i-1}$'s and which ends with the last $b_i$ in the block.
Furthermore, set $I' \trianglelefteq \Phi^+$ to be the order ideal given by

$$I' := \big\{[i+1,j-1] : [i,j] \in I \text{ and } j-i > 2\big\}.$$

Then $\phi_n$ can be described as

$$\phi_n(I) := \sigma(I) \circ \phi_n(I').$$

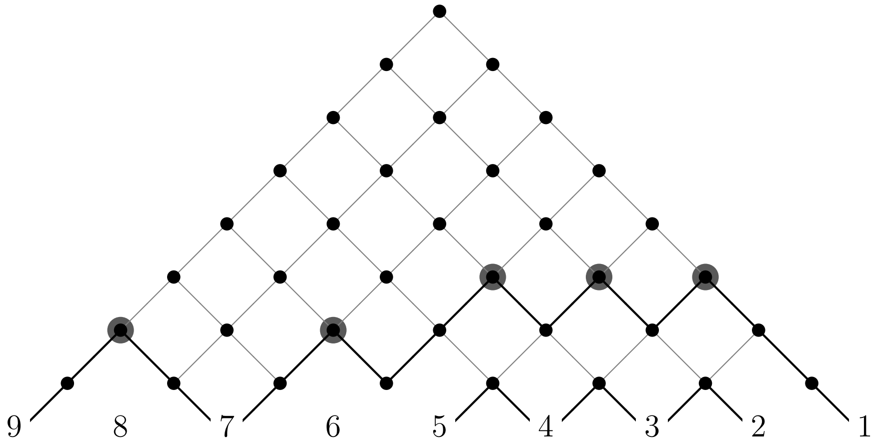



FIGURE 19. The bijection $\phi_9$ sending the shown non-nesting partition to the non-crossing partition $\sigma = (1,7,9)(2,3,4,5) = [7,3,4,5,2,6,9,8,1]$.

EXAMPLE 2.49. The ideal shown in Figure 19 is given by

$$I = \left\{ \begin{array}{c} [1,2],[2,3],[3,4],[4,5],[5,6],[6,7],[7,8],[8,9], \\ [1,3],[2,4],[3,5],[4,6],[5,7],[7,9],[1,4],[2,5],[3,6] \end{array} \right\},$$

its maximal elements are $\{[1,4],[2,5],[3,6],[5,7],[7,9]\}$ and $I' = \{[2,3],[3,4],[4,5]\}$. This gives $\phi_9(I) = (1,7,9) \circ \phi_9(I')$ and as all elements in $I'$ are minimal, we have $\phi_9(I') = (2,3,4,5)$ and thereby

$$\phi_9(I) = (1,7,9)(2,3,4,5).$$

From the construction, it is clear that $\phi_n$ is in fact a bijection between $NN(A_{n-1})$ and $NC(A_{n-1})$.

As we have seen in Section 1.1.14, $l_S(\sigma) = \mathrm{inv}(\sigma)$ for $\sigma \in \mathcal{S}_n$. Therefore, the following proposition proves Eq. (2) in Theorem 2.48 for type $A$:

PROPOSITION 2.50. *$\phi_n$ maps the coheight statistic on $NN(n)$ to the inversion number on $NC(A_{n-1})$, i.e., for $I \in NN(A_{n-1})$, we have*

$$|I| = \mathrm{inv}([\sigma_1\sigma_2\cdots\sigma_n]),$$

*where $[\sigma_1\sigma_2\cdots\sigma_n]$ is the one-line notation of $\phi_n(I)$.*

PROOF. For simplicity, we assume that a given $I$ contains only one "shell" or, equivalently, $\phi_n(I)$ is a 1-cycle, say $(i_1,\ldots,i_k)$, the general case is solved by applying the same argument several times. The number of elements in $I$ is equal to $2(i_k - i_1) - 1 - (k-2) = 2(i_k - i_1) - k + 1$. It is easy to see that this is also equal to the inversion number of the cycle $(i_1,\ldots,i_k)$. $\square$

EXAMPLE (CONTINUED) 2.51. Let $I$ and $\sigma = \phi_9(I)$ as in Example 2.49. Then the "first shell" contains 14 elements which is equal to the inversion number of the associated cycle $(1,7,9) = [7,2,3,4,5,6,9,8,1]$, the "second shell" contains 3 elements which is equal to the inversion number of the associated cycle $(2,3,4,5) = [1,3,4,5,2,6,7,8,9]$.

The following theorem together with Proposition 2.8 proves Eq. (3) in Theorem 2.48 for type $A$:

THEOREM 2.52. *Let $I \trianglelefteq \Phi^+$ be an order ideal in the root poset of type $A_{n-1}$ and let $N = \binom{n}{2}$ be the number of positive roots. Then*

$$\mathrm{maj}(I) + \mathrm{maj}(\phi_n(I)) + \mathrm{imaj}(\phi_n(I)) = 2N = n(n-1).$$

Before proving the theorem, we get back to the ongoing example:

EXAMPLE (CONTINUED) 2.53. The descent set of the ideal $I$ considered in Example 2.49 and the descent set and the inverse descent set of $\sigma = \phi_9(I)$ are given by

$$\mathrm{Des}(I) = \{5, 8, 11, 13\},\ \mathrm{Des}(\sigma) = \{1, 4, 7, 8\},\ \mathrm{iDes}(\sigma) = \{1, 2, 6, 8\}$$

and therefore,

$$\begin{aligned} \mathrm{maj}(I) &= (18-5) + (18-8) + (18-11) + (18-13) = 35, \\ \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) &= (1+4+7+8) + (1+2+6+8) = 20 + 17, \\ \mathrm{maj}(I) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) &= 35 + 37 = 72 = 9 \cdot 8 = n(n-1). \end{aligned}$$

We prove the theorem in several steps:

LEMMA 2.54. *Let $\sigma \in NC(n)$. Then* $\mathrm{des}(\sigma) = \mathrm{ides}(\sigma)$.

PROOF. We first prove the lemma for the case that $\sigma$ has only one cycle: let $\sigma = (i_1, \ldots, i_k)$. As $\sigma$ is in $NC(n)$, we have $i_1 < \ldots < i_k$. Therefore, we can describe the descent set and the inverse descent set of $\sigma$:

$$\begin{aligned} \mathrm{Des}(\sigma) &= \{i_l : l < k, i_l + 1 < i_{l+1}\} \cup \{i_k - 1\}, \\ \mathrm{iDes}(\sigma) &= \{i_l - 1 : 1 < l, i_{l-1} + 1 < i_l\} \cup \{i_1\}. \end{aligned}$$

The case that $\sigma$ has more than one cycle follows from the fact that $\sigma$ is non-crossing and therefore the descent set and the inverse descent set of $\sigma$ are given by the above rule for each cycle. ☐

LEMMA 2.55. *Let $I \in NN(n)$. Then*

$$\mathrm{des}(I) + \mathrm{des}(\phi_n(I)) = n - 1.$$

PROOF. Let $\sigma := \phi_n(I)$ and let $min$ respectively $max$ be the minimal respectively maximal element not mapped by $\sigma$ to itself. Set $a$ to be the number of valleys of $I$ between positions $min$ and $max$. Then the proof of the previous lemma implies $\mathrm{des}(\sigma) = max - min - a$. By definition, $\mathrm{des}(I)$ equals the total number of valleys of $I$ and therefore, $\mathrm{des}(I) = a + min - 1 + n - max$. This completes the proof. ☐

Define a lifting $\Delta$ from $NN(n)$ to $NN(n+1)$ by taking an order ideal $I \in NN(n)$ and embed it into $NN(n+1)$ by adding the whole "bottom row", see Figure 20 for an example.

LEMMA 2.56. *Let $I \in NN(n)$ and $\sigma := \phi_n(I)$. Furthermore, set $I' := \Delta(I)$ and $\sigma' := \phi_{n+1}(\Delta(I))$. Then*

$$\mathrm{maj}(I) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) + 2n = \mathrm{maj}(I') + \mathrm{maj}(\sigma') + \mathrm{imaj}(\sigma').$$

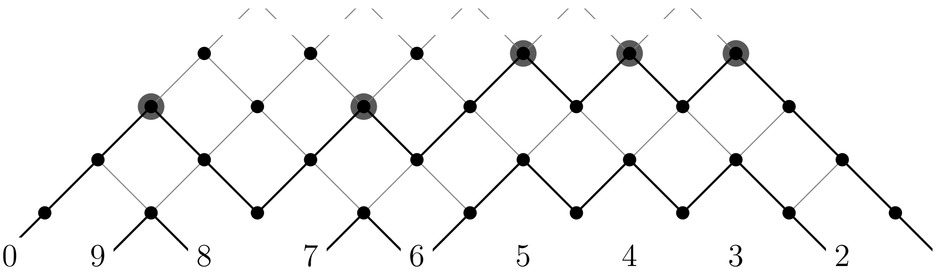



FIGURE 20. The lifting $\Delta(I)$ of the non-nesting partition $I$ shown in Figure 19 and its image $\phi_{10}(\Delta(I)) = (1,10)(2,6,7)(8,9) = [10,6,3,4,5,7,2,9,8,1]$.

PROOF. Observe that

$$\mathrm{Des}(\sigma') = \mathrm{iDes}(\sigma) \cup \{n\} \quad , \quad \mathrm{iDes}(\sigma') = \{i+1 : i \in \mathrm{Des}(\sigma)\} \cup \{1\}$$

and therefore,

$$\mathrm{maj}(\sigma') - \mathrm{imaj}(\sigma) = n \quad , \quad \mathrm{imaj}(\sigma') - \mathrm{maj}(\sigma) = \mathrm{des}(\sigma) + 1. \tag{4}$$

On the other hand, we have

$$\mathrm{maj}(I') - \mathrm{maj}(I) = \mathrm{des}(I) = \mathrm{des}(I'). \tag{5}$$

The Lemma follows by (4), (5) and Lemma 2.55. $\square$

EXAMPLE (CONTINUED) 2.57. In our ongoing example, we have already seen that

$$\mathrm{maj}(I) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) + 2n = 72 + 18 = 90 = n(n+1).$$

On the other hand, we have

$$\mathrm{Des}(I') = \{6,9,12,14\}, \mathrm{Des}(\sigma') = \{1,2,6,8,9\}, \mathrm{iDes}(\sigma) = \{1,2,5,8,9\}.$$

This gives

$$\begin{aligned} \mathrm{maj}(I') &= (20-6) + (20-9) + (20-12) + (20-14) = 39, \\ \mathrm{maj}(\sigma') + \mathrm{imaj}(\sigma') &= (1+2+6+8+9) + (1+2+5+8+9) = 26 + 25, \\ \mathrm{maj}(I') + \mathrm{maj}(\sigma') + \mathrm{imaj}(\sigma') &= 39 + 51 = 90 = n(n+1). \end{aligned}$$

PROOF OF THEOREM 2.52. We prove the Theorem by induction on $n$. Let $I \in NN(n)$ and $I' \in NN(n')$ and let $D$ and $D'$ be the associated Catalan paths in $\mathcal{D}_n$ and $\mathcal{D}_{n'}$ respectively. Then the concatenation of $I$ and $I'$ is given by the concatenation $DD' \in \mathcal{D}_{n+n'}$. The proof consists of two parts:

(i) first, we prove that if the theorem holds for elements $I \in NN(n)$ and $I' \in NN(n')$ then it holds for the concatenation $II'$ of $I$ and $I'$ which lies in $NN(n+n')$, and
(ii) second, we prove that if the theorem holds for $I \in NN(n)$ then it holds also for $\Delta(I) \in NN(n+1)$.

As the case $n = 1$ is obvious, the theorem then follows.

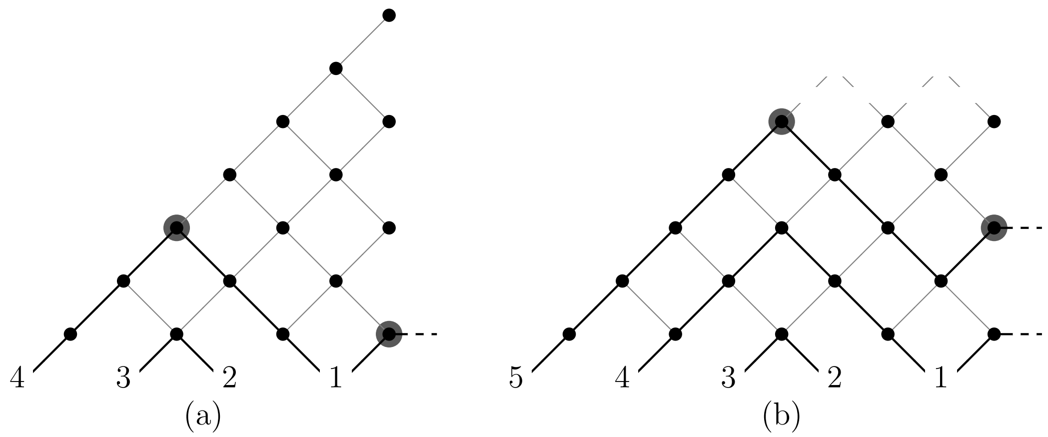



FIGURE 21. (a) shows the bijection $\phi_4$ sending the given non-nesting partition $I$ to the non-crossing partition $\phi_4(I) = (1,4,-1)(2,3) = [4,3,2,-1]$, (b) shows its lift $\Delta(I)$, as shown, the image of this lift is $\phi_5(\Delta(I)) = (1,4,-1)(2,3)(5,-5) = [4,3,2,-1,-5]$.

(i) set $\sigma := \phi_n(I), \sigma' := \phi_{n'}(I')$ and $\tau := \phi_{n+n'}(II')$. Then we have

$$\begin{aligned} \mathrm{maj}(II') + \mathrm{maj}(\tau) + \mathrm{imaj}(\tau) &= \mathrm{maj}(I) + \mathrm{maj}(I') + 2n(\mathrm{des}(I') + 1) \\ &+ \mathrm{maj}(\sigma) + \mathrm{maj}(\sigma') + n\,\mathrm{des}(\sigma') \\ &+ \mathrm{imaj}(\sigma) + \mathrm{imaj}(\sigma') + n\,\mathrm{ides}(\sigma'). \end{aligned}$$

By Lemma 2.54 and Lemma 2.55, the right-hand side of this equation is equal to $\mathrm{maj}(I)+\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma)+\mathrm{maj}(I')+\mathrm{maj}(\sigma')+\mathrm{imaj}(\sigma')+2nn'$. By induction, this reduces to $n(n-1) + n'(n'-1) + 2nn' = (n+n')(n+n'-1)$.

(ii) this is an immediate consequence of Lemma 2.56.

□

**2.6.2. The bijection in type $B$.** The bijection $\phi_n : NN(A_{n-1}) \longrightarrow NC(A_{n-1})$ can be adapted to type $B_n$ as follows: write the numbers 1 to $n$ below the root poset of type $B_n$ from right to left as shown in Figure 21 and map a given order ideal $I \trianglelefteq \Phi^+$ in the same way to a signed permutation as in type $A_{n-1}$ with the additional rule that if a "shell" ends at the "right boundary" of $\Phi^+$ then add the negative of the first element of the given cycle to its end. This map defines a bijection between $NN(B_n)$ and $\mathrm{rev}(NC(B_n))$. To prove Theorem 2.48 in type $B$ we only have to modify the lifting $\Delta$ from $NN(B_n)$ to $NN(B_{n+1})$ which is now defined by adding *two* "bottom rows", see Figure 21 for an example, and slightly different induction steps.

Recall from Section 1.1.14 that in type $B_n$, $\mathrm{l}_S(\sigma) = \mathrm{inv}([\sigma_1\sigma_2\cdots\sigma_n]) - \sum_{i\in\mathrm{Neg}} \sigma(i)$. Therefore, the following proposition proves Eq. (2) in Theorem 2.48 for type $B$:

PROPOSITION 2.58. *For $I \in NN(B_n)$, we have*

$$|I| = \mathrm{inv}([\sigma_1\sigma_2\cdots\sigma_n]) - \sum_{i\in\mathrm{Neg}(\sigma)} \sigma_i,$$

*where $\sigma := \phi_n(I)$ and $[\sigma_1\sigma_2\cdots\sigma_n]$ is its one-line notation.*

Proof. The proof follows exactly the same idea as the proof in type $A$. $\square$

The following theorem together with Proposition 2.35 proves Eq. (3) in Theorem 2.48 for type $B$:

Theorem 2.59. *Let $I \trianglelefteq \Phi^+$ be an order ideal in the root poset of type $B_n$ and let $N = n^2$ be the number of positive roots. Then*

$$\operatorname{maj}(I) + \operatorname{maj}(\phi_n(I)) + \operatorname{imaj}(\phi_n(I)) = 2N = 2n^2.$$

Proof. We prove the theorem as in type $A$ by induction. As in type $A$, the case $n = 1$ is obvious, therefore the theorem follows by proving the following 3 cases:

(i) first, we prove that if the theorem holds for elements $I \in NN(A_{n-1})$ and $I' \in NN(B_{n'})$ then it holds for the concatenation $II' \in NN(B_{n+n'})$,
(ii) second, we prove that if the theorem holds for elements $I \in NN(A_{n-1})$ and $I' \in NN(B_{n'})$ then it holds for the order ideal $J \in NN(B_{n+n'})$ obtained by replacing the last east step in the Catalan word associated to $I$ by a north step and then concatenating it with $I'$, and
(iii) third, we prove that if the theorem holds for $I \in NN(B_n)$ then it holds also for $\Delta(I) \in NN(B_{n+1})$.

Set $\sigma := \phi_n(I), \sigma' := \phi_{n'}(I')$ and $\tau := \phi_{n+n'}(II')$.

(i) The proof of (i) is the same as in type $A_{n-1}$ with $n$ replaced by $2n$:

$$\begin{aligned}\operatorname{maj}(II') + \operatorname{maj}(\tau) + \operatorname{imaj}(\tau) &= \operatorname{maj}(I) + \operatorname{maj}(I') + 4n(\operatorname{des}(I') + 1)\\ &+ \operatorname{maj}(\sigma) + \operatorname{maj}(\sigma') + 2n \operatorname{des}(\sigma')\\ &+ \operatorname{imaj}(\sigma) + \operatorname{imaj}(\sigma') + 2n \operatorname{ides}(\sigma').\end{aligned}$$

By Lemma 2.54 and Lemma 2.55, the right-hand side of this equation is equal to $\operatorname{maj}(I)+\operatorname{maj}(\sigma)+\operatorname{imaj}(\sigma)+\operatorname{maj}(I')+\operatorname{maj}(\sigma')+\operatorname{imaj}(\sigma')+4nn'$. By induction, this reduces to $2n^2 + 2n'^2 + 4nn' = 2(n + n')^2$.

(ii)

$$\begin{aligned}\operatorname{maj}(II') + \operatorname{maj}(\tau) + \operatorname{imaj}(\tau) &= \operatorname{maj}(I) + \operatorname{maj}(I') + 4n(\operatorname{des}(I') + 1)\\ &+ \operatorname{maj}(\sigma) + \operatorname{maj}(\sigma') + 2n \operatorname{des}(\sigma') + 1\\ &+ \operatorname{imaj}(\sigma) + \operatorname{imaj}(\sigma') + 2n \operatorname{ides}(\sigma') + 1 - 2.\end{aligned}$$

Again, by Lemma 2.54 and Lemma 2.55, the right-hand side of this equation is equal to $\operatorname{maj}(I) + \operatorname{maj}(\sigma) + \operatorname{imaj}(\sigma) + \operatorname{maj}(I') + \operatorname{maj}(\sigma') + \operatorname{imaj}(\sigma') + 4nn' = 2n^2 + 2n'^2 + 4nn' = 2(n + n')^2$.

(iii) We have $\phi_{n+1}(\Delta(I)) = \sigma \circ (n+1, -n-1)$. As $\operatorname{maj}(\Delta(I)) = \operatorname{maj}(I)$, this gives

$$\operatorname{maj}(\Delta(I))+\operatorname{maj}(\phi_{n+1}(\Delta(I)))+\operatorname{imaj}(\phi_{n+1}(\Delta(I))) = \operatorname{maj}(I)+\operatorname{maj}(\sigma)+\operatorname{imaj}(\sigma)+4n+2,$$

and the right-hand side is by induction equal to $2n^2 + 4n + 2 = 2(n + 1)^2$.

$\square$

Remark. At the beginning of this chapter, we mentioned a bijection between non-crossing and non-nesting set partitions that locally converts each nesting into a crossing. When generalizing this bijection to type $B$ in the canonical way, one obtains – as we do – a bijection between $NN(B_n)$ and $\operatorname{rev}(NC(B_n))$.

## 2.7. Coxeter sortable elements

In [**83**], Reading introduced another subset of a real reflection group $W$ which he called Coxeter sortable elements and gave a bijection to non-crossing partitions. In the following sections, we want to present his construction and explore – as for non-crossing partitions – an interesting connection to non-nesting partitions in types $A$ and $B$.

Let $c$ be a Coxeter element in $W$ and fix a reduced word for $c$, say $c = s_1 s_2 \cdots s_l$. For $\omega \in W$, the $c$-sorting word of $\omega$ is defined to be the lexicographically first reduced expression for $\omega$ when expressed as a subword of the half infinite word

$$c^\infty := s_1 s_2 \cdots s_l | s_1 s_2 \cdots s_l | s_1 s_2 \cdots s_l | \cdots ,$$

where the divider $|$ is introduced just to distinguish between different occurrences of $s_1 s_2 \cdots s_l$.

Note. The $c$-sorting word can be interpreted as a sequence of subsets of the simple reflections: the subsets in this sequence are the sets of letters of the $c$-sorting word which occur between adjacent dividers.

Definition 2.60. An element $\omega \in W$ is called $c$-sortable if its $c$-sorting word defines a sequence of subsets which is decreasing under inclusion. Furthermore, define $\mathrm{Cox}_c(W)$ as the set of all $c$-sortable elements in $W$,

$$\mathrm{Cox}_c(W) := \big\{\omega \in W : \omega \ c\text{-sortable}\big\}.$$

Note. The definition of $c$-sortable does not depend on the specific choice of the reduced word for $c$ as different reduced words are related by commutations of letters with no commutations across dividers.

Example 2.61. Consider the case of the symmetric group $\mathcal{S}_n$. Let $c$ be the long cycle $(n, \ldots, 2, 1) = s_{n-1} \cdots s_2 s_1$. Then we have for $\sigma \in \mathcal{S}_n$,

$$\sigma \text{ is } c\text{-sortable} \Leftrightarrow \sigma \text{ is 231-avoiding}.$$

We consider only the case $n = 3$, the case $n > 3$ is similar. When expressing the permutations in $\mathcal{S}_3$ by their $c$-sorting words, we have

$$\mathcal{S}_3 = \big\{1, s_2, s_2 s_1, s_2 s_1 | s_2, s_1, s_1 | s_2\big\}.$$

The only element which is *not* Coxeter sortable is $s_1 | s_2 = [2, 3, 1]$.

As already mentioned, Reading proved the following theorem bijectively, see [**83**, Section 6]:

Theorem 2.62 (Reading). *Let $W$ be a real reflection group and let $c$ be a Coxeter element in $W$. Then*

$$|\mathrm{Cox}_c(W)| = \mathrm{Cat}(W).$$

Remark. In [**83**], Reading showed moreover that Coxeter sortable elements provide a connection between non-crossing partitions and *clusters* (facets) in the *generalized cluster complex*. This simplicial complex was constructed by S. Fomin and A. Zelevinsky in the context of *cluster algebras*, see [**46, 47, 44, 45**], which arose within the last years in more and more contexts in various fields of mathematics.

## 2.8. $q$-Catalan numbers for Coxeter sortable elements in type $A$ and $B$

Now, we want to describe the $q$-Catalan numbers $\operatorname{Cat}^{(1)}(W;q)$ and diag-$\operatorname{Cat}^{(1)}(W;q)$ in terms of Coxeter sortable elements for $W$ being of type $A$ or $B$.

Surprisingly, for Coxeter sortable elements we obtain almost the same as for non-crossing partitions with the advantage that we do not need to reverse the negative elements, compare Theorem 2.48.

THEOREM 2.63. *Let $W$ be the reflection group $A_{n-1}$ or $B_n$ and set $c = s_{n-1}\cdots s_2 s_1$ or $c = s_{n-1}\cdots s_2 s_1 s_0$ respectively. Then the $q$-Catalan numbers* $\operatorname{Cat}^{(1)}(W;q)$ *and* diag-$\operatorname{Cat}^{(1)}(W;q)$ *can be interpreted in terms of $c$-sortable elements as*

$$\operatorname{Cat}^{(1)}(W;q) = \sum_{\operatorname{Cox}_c(W)} q^{\mathrm{l}_S(\sigma)}, \tag{6}$$

$$\text{diag-}\operatorname{Cat}^{(1)}(W;q) = \sum_{\operatorname{Cox}_c(W)} q^{\operatorname{maj}(\sigma)+\operatorname{imaj}(\sigma)}, \tag{7}$$

*where the sums range over all $c$-sortable elements of the given type and where* maj *and* imaj *are the appropriate statistics.*

REMARK. As for non-crossing partitions, the analogous statement is false in type $D$: for any Coxeter element $c \in D_4$,

$$\operatorname{Cat}^{(1)}(D_4;q) \neq \sum_{\sigma\in\operatorname{Cox}(D_4,c)} q^{\mathrm{l}_S(\sigma)}.$$

### 2.8.1. The bijection in type $A$.

In Example 2.61, we have seen that for $W = A_{n-1}$ and $c = (n,\ldots,2,1)$, being $c$-sortable is the same as being 231-avoiding. Bijections between 3-pattern-avoiding permutations and Catalan paths are very well studied, e.g. see [**14, 33, 73, 84**], we will discuss these connections in detail in Section 2.9. Here we only define the bijection we need and state the proposition from which Theorem 2.63 follows in type $A$. For a proof of the proposition itself, we refer to Section 2.9.

Let $D$ be a Catalan path of length $n$ and identify $D$ with the set $\{b_{ij}\}$ of cells below $D$ as described in the beginning of Section 2.2. Label every cell $b_{ij}$ by $s_{n-1-i}$. The bijection between Catalan paths and $c$-sortable elements is then defined by mapping $D \in \mathcal{D}_n$ to the $c$-sorting word $\sigma := \prod s_{n-1-i}$, where the product ranges over all cells $b_{ij}$ in the order as indicated in Figure 22. By construction, $\sigma$ is $c$-sortable.

EXAMPLE 2.64. The Catalan path shown in Figure 22 is mapped to the $c$-sortable element

$$s_5s_4s_3s_2s_1|s_5s_4s_2|s_5 = [6,2,1,5,4,3].$$

Theorem 2.63 follows in type $A$ from the following proposition:

PROPOSITION 2.65. *Let $D$ be a Catalan path and let $\sigma$ be the image of $D$ under the bijection just defined. Then* $\operatorname{area}(D) = \mathrm{l}_S(\sigma)$ *and furthermore,*

$$\begin{aligned}\operatorname{Des}(\sigma) &= [n-1] \setminus \{n-i : i \in \operatorname{Set}_X(D)\},\\ \operatorname{iDes}(\sigma) &= [n-1] \setminus \{n-i : i \in \operatorname{Set}_Y(D)\},\end{aligned}$$

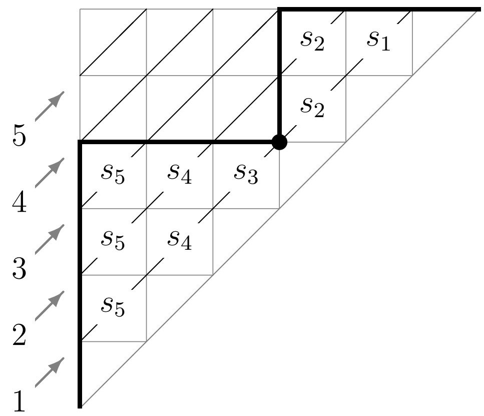



FIGURE 22. A Catalan path of length 6 with cells labelled by simple transpositions.

*in particular,*

$$\mathrm{maj}(D) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) = n(n-1).$$

PROOF. The fact that $\mathrm{area}(D) = \mathrm{l}_S(\sigma)$ follows directly from the construction. We will prove the second statement in Section 2.9.4, it is equivalent to Corollary 2.105. $\square$

The given bijection also preserves another statistic on Catalan paths, namely the length of the "last descent", we will use this fact for constructing the bijection in type $B$:

PROPOSITION 2.66. *Let $D$ be a Catalan path of length $n$ and let $k$ be the number of east steps after the last north step. Then $\sigma(k) = 1$ for $\sigma$ being the image of $D$ and furthermore, $\{1, \ldots, k-1\} \subseteq \mathrm{Des}(\sigma)$.*

PROOF. Let $S_1|S_2|\cdots|S_k$ be the initial segment of the the $c$-sorting word for $\sigma$, with $S_k$ possibly empty. Then, by construction, the last simple reflection in $S_i$ is $s_i$ for $i < k$ and $s_k$ is *not* contained in $S_k$. Therefore, $k$ is mapped by $\sigma$ to 1 and, as $\sigma$ is 231-avoiding, it follows immediately that $\{1, \ldots, k-1\} \subseteq \mathrm{Des}(\sigma)$ (of course, the later can also be obtained directly). $\square$

**2.8.2. The bijection in type** $B$**.** As in type $A$, let $D$ be a Catalan path of type $B_n$ and identify $D$ with the set $\{b_{ij}\}$ of cells below $D$ as described in Definition 2.24. Label every cell $b_{ij}$ with $j < n$ by $s_{n-1-i}$ and $b_{ij}$ with $j \geq n$ by $s_{2(n-1)-(i+j)}$. The bijection between Catalan paths of type $B_n$ and $c$-sortable elements is then defined by mapping $D \in \mathcal{D}_n$ to the $c$-sorting word $\sigma$ which is the product of the simple transpositions in the cells $b_{ij}$ in the order as indicated in Figure 23.

EXAMPLE 2.67. The Catalan path shown in Figure 22 is mapped to the $c$-sortable element

$$s_5s_4s_3s_2s_1s_0|s_5s_4s_2s_1s_0|s_5s_2s_1 = [1, -2, -6, 5, 4, 3].$$

To see that the image $\sigma = S_1|S_2|\cdots|S_k$ of a given $D$ is in fact $c$-sortable, we only have to show that $S_1|S_2|\cdots|S_k$ is a reduced expression for $\sigma$ as the inclusion property $S_1 \supseteq S_2 \supseteq \ldots \supseteq S_k$ is given by construction.

PROPOSITION 2.68. *$S_1|S_2|\cdots|S_k$ is a reduced expression for $\sigma$.*

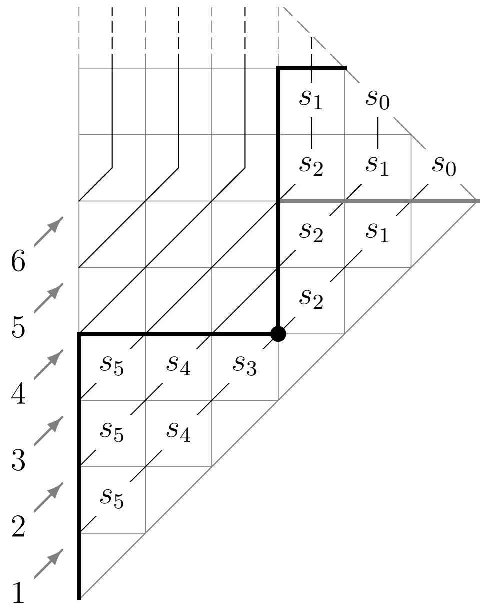



FIGURE 23. A Catalan path of type $B_6$ with cells labelled by simple transpositions.

PROOF. If $s_i s_{i-1}$ occurs in $S_j$ and in $S_{j+1}$ for some $i$ and $j$ then $s_{i-2}$ occurs also in $S_j$ except for the case $i = 1$. But if $s_1 s_0$ occurs in $S_j$ and in $S_{j+1}$ and furthermore, $s_1$ occurs in $S_{j+2}$ then $s_2$ occurs in $S_{j+2}$ left of $s_1$. The proposition follows. □

This proposition immediately implies the following corollary which proves Eq. (6) in Theorem 2.63 for type $B$:

COROLLARY 2.69. *Let $D$ be a Catalan path of type $B_n$ and let $\sigma$ be its image under the above bijection. Then*

$$\mathrm{area}(D) = \mathrm{l}_S(\sigma).$$

To prove Eq. (7) in Theorem 2.63 for type $B$, we use the fact that a Catalan path $D$ of type $B_n$ consists of a "lower part" $D_1$ which is a Catalan path of type $A_{n-1}$, and an "upper part" $D_2$. $D_1$ is obtained from $D$ by replacing all north steps after the $n$-th north step by east steps and $D_2$ is obtained as the suffix of $D$ after the $n$-th north step. For example, the Catalan path of type $B_6$ in Figure 23 consists of a lower part which is the Catalan path shown in Figure 22 and an upper part given by the word $NNE$.

As $s_i$ and $s_j$ commute for $|i-j| > 1$, we can write the image of $D$ as the image of $D_1$ followed by the product of the cells below $D_2$ row by row from bottom to top and from right to left. Set $\sigma, \sigma_1$ and $\sigma_2$ to be the signed permutations associated to $D, D_1$ and $D_2$. For example,

$$\begin{aligned} \sigma &= s_5s_4s_3s_2s_1s_0|s_5s_4s_2s_1s_0|s_5s_2s_1 \\ &= \sigma_1 \cdot \sigma_2 \\ &= s_5s_4s_3s_2s_1|s_5s_4s_2|s_5 \cdot s_0s_1s_2|s_0s_1. \end{aligned}$$

As we have seen in the previous section, we have, when considering $D_1$ in type $A_{n-1}$,

$$\mathrm{maj}(D_1) + \mathrm{maj}(\sigma_1) + \mathrm{imaj}(\sigma_1) = n(n-1).$$

This gives, when considered in type $B_n$,

$$\mathrm{maj}(D_1) + \mathrm{maj}(\sigma_1) + \mathrm{imaj}(\sigma_1) = 2n(n-1) + 2n = 2n^2.$$

We will use this fact and are going to show that

$$\mathrm{maj}(D) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) \quad = \quad \mathrm{maj}(D_1) + \mathrm{maj}(\sigma_1) + \mathrm{imaj}(\sigma_1), \tag{8}$$

Eq. (7) in Theorem 2.63 for type $B$ then follows.

We prove Eq. (8) in several steps:

LEMMA 2.70. *$\mathrm{neg}(D) + \mathrm{neg}(\sigma) = n$.*

PROOF. By definition, $\mathrm{neg}(D)$ is given by the number of east steps in $D$ and by construction of $\sigma$, the number of $s_0$'s in the $c$-sorting word for $\sigma$ in $n - \mathrm{neg}(D)$. Therefore,

$$\mathrm{neg}(\sigma) = n - \mathrm{neg}(D).$$

□

To keep the notation simple, we set

$$\mathrm{maj}(D_2) := \sum_{i \in \mathrm{Des}(D_2)} 2(k-i),$$

where $k$ is the number of steps in $D_2$.

LEMMA 2.71. *Let $k$ be the number of steps in $D_2$ or, equivalently, let $k$ be the number of east steps in $D_1$ after the last north step. Then*

$$\mathrm{maj}(D) = \mathrm{maj}(D_1) + \mathrm{maj}(D_2) - 2(n - \mathrm{neg}(D)).$$

PROOF. By definition, $\mathrm{neg}(D)$ and $\mathrm{neg}(D_1)$ are the number of east steps in $D$ and in $D_1$, in particular, $\mathrm{neg}(D_1) = n$, and we have

$$\mathrm{maj}(D) = 2 \cdot \big(\mathrm{neg}(D) + \sum_{i \in \mathrm{Des}(D)} (2n-i)\big) \quad , \quad \mathrm{maj}(D_1) = 2 \cdot \big(n + \sum_{\substack{i \in \mathrm{Des}(D) \\ i < 2n-k}} (2n-i)\big).$$

The lemma follows. □

LEMMA 2.72. *Let $S_1|S_2|\cdots|S_k$ be the expression for $\sigma_2$. Then*

$$\mathrm{Neg}(\sigma_2) = \{j+1 : s_j \text{ is the rightmost simple reflection in } S_i \text{ for some } i\}$$

*and the images of $\mathrm{Neg}(\sigma_2)$ under $\sigma_2$ are the negatives of the first $\mathrm{neg}(\sigma_2)$ integers in increasing order and the image of the complement of $\mathrm{Neg}(\sigma_2)$ are the last $k - \mathrm{neg}(\sigma_2)$ integers also in increasing order.*

PROOF. This can be seen immediately from the expression $\sigma_2 = S_1|S_2|\cdots|S_k$. □

EXAMPLE 2.73. Let $\sigma_2 = S_1|S_2 = s_0s_1s_2|s_0s_1$ as above. Then $\mathrm{Neg}(\sigma_2) = \{3, 2\}$ and

$$\sigma_2(3) = -1, \sigma_2(2) = -2 \quad , \quad \sigma_2(1) = 3, \sigma_2(4) = 4.$$

LEMMA 2.74. *Let $k$ be the number of steps in $D_2$. Then*

$$\mathrm{Des}(D_2) = \big\{i < k : i \in \mathrm{Neg}(\sigma_2) \text{ and } i+1 \notin \mathrm{Neg}(\sigma_2)\big\}.$$

PROOF. Let $S_1|S_2|\cdots|S_k$ be the expression for $\sigma_2$ as described above. Using Lemma 2.72, we get

$$\begin{aligned} i \in \mathrm{Des}(D_2) \;&\Leftrightarrow\; s_{i-1} \in S_j, s_i \notin S_j \text{ and } s_{i+1} \in S_{j-1} \text{ for some } j \\ &\Leftrightarrow\; \sigma_2(i) \in \mathrm{Neg}(\sigma_2) \text{ and } i+1 \notin \mathrm{Neg}(\sigma_2). \end{aligned}$$

□

LEMMA 2.75. *Let $k$ be the number of steps in $D_2$. Then*

$$\mathrm{Des}(\sigma) = \mathrm{Des}(\sigma_1) \setminus \big\{ i < k : i \in \mathrm{Neg}(\sigma_2) \text{ and } i+1 \notin \mathrm{Neg}(\sigma_2) \big\}.$$

PROOF. First, observe that $\mathrm{Neg}(\sigma_2) = \mathrm{Neg}(\sigma)$ and second, observe that the descents of $\sigma$ and the descents of $\sigma_1$ which are larger than $k$ coincide and that $k$ is neither a descent of $\sigma$ nor a descent of $\sigma_1$. By Lemma 2.66, we have to show that the descents of $\sigma$ which are smaller than $k$ are given by

$$\big\{ i < k : i \notin \mathrm{Neg}(\sigma) \text{ or } i+1 \in \mathrm{Neg}(\sigma) \big\}$$

and this can be deduced from Lemma 2.72. □

LEMMA 2.76. $\mathrm{imaj}(\sigma) = \mathrm{imaj}(\sigma_1) + \mathrm{neg}(\sigma)$.

PROOF. As $\sigma = \sigma_1\sigma_2$ and $\mathrm{iDes}(\sigma_2) = \emptyset$, we have

$$\mathrm{iDes}(\sigma) = \mathrm{iDes}(\sigma_1).$$

The lemma follows with the fact that $\mathrm{neg}(\sigma_1) = 0$. □

PROOF OF EQ. (8): Lemma 2.70 and Lemma 2.71 imply

$$\mathrm{maj}(D) = \mathrm{maj}(D_1) + \mathrm{maj}(D_2) - 2\,\mathrm{neg}(\sigma)$$

and by Lemma 2.74 and Lemma 2.75, we have

$$\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma_1) - \mathrm{maj}(D_2) + \mathrm{neg}(\sigma).$$

Together with Lemma 2.76, Eq. (8) follows. □

NOTE. As we use the results of type $A$ to prove type $B$, so far everything is only proved under the assumption that Proposition 2.65 holds.

## 2.9. Bijections between Catalan paths and 3-pattern avoiding permutations

In the first part of this section, we construct a bijection $\Phi$ between Catalan paths and 231-avoiding permutations with the additional property

$$\mathrm{maj}(\Phi(\sigma)) = \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma).$$

Later, we will use this bijection to deduce Proposition 2.65, see Corollary 2.105.

REMARK. There exist plenty of bijections between pattern-avoiding permutations and Catalan paths. To mention one, J. Bandlow and K. Killpatrick introduced an interesting bijection which sends the inversion statistic of a 312-avoiding permutation to the area statistic of the corresponding Catalan path [**14**]. We will discuss this and other bijections in more detail later, see Section 2.9.4.

As already mentioned in Remark 2.1.1, in type $A$, the major index of a Catalan path $D$ is usually defined as

$$\operatorname{maj}(D) = \sum_{i \in \operatorname{Des}(D)} i.$$

For the sake of simplicity, we use this definition in this section.

The ascent set of a permutation $\sigma$ is defined to be the complement of the descent set,

$$\operatorname{Asc}(\sigma) := [n] \setminus \operatorname{Des}(\sigma) = \big\{ i < n : \sigma_i < \sigma_{i+1} \big\} \cup \{n\}.$$

As we count $n$ as an ascent, this differs from the usual definition. By $\operatorname{asc}(\sigma)$ denote the number of ascents, and, as for descents, set $\operatorname{iAsc}(\sigma) := \operatorname{Asc}(\sigma^{-1})$.

We will need following two elementary involutions on $\mathcal{S}_n$: define $\rho$ to be the involution sending $[\sigma_1, \ldots, \sigma_n]$ to $[\sigma_n, \ldots, \sigma_1]$ and $\sigma \mapsto \hat{\sigma}$ to be the involution sending a permutation to its inverse. We can easily describe the descent set and the inverse descent set of the images of those involutions: sending a permutation to its inverse interchanges Des and iDes and for $\rho$, we have

$$\begin{aligned}
\operatorname{Des}(\rho(\sigma)) &= [n-1] \setminus \{ n - i : i \in \operatorname{Des}(\sigma) \}, \\
\operatorname{iDes}(\rho(\sigma)) &= [n-1] \setminus \operatorname{iDes}(\sigma).
\end{aligned}$$

**2.9.1. The bijection.** We now construct the proposed bijection $\Phi$ between $\mathcal{S}_n(231)$ and $\mathcal{D}_n$ having the property that

$$\operatorname{maj}(\Phi(\sigma)) = \operatorname{maj}(\sigma) + \operatorname{imaj}(\sigma).$$

LEMMA 2.77. *Let $\sigma \in \mathcal{S}_n(231)$ with* $\operatorname{Des}(\sigma) = \{i_1, \ldots, i_k\}$, $\operatorname{Asc}(\sigma) = \{j_1, \ldots, j_{n-k}\}$. *Then*

$$\begin{aligned}
\operatorname{iDes}(\sigma) &= \{ \sigma_{i_1} - 1, \ldots, \sigma_{i_k} - 1 \}, \\
\operatorname{iAsc}(\sigma) &= \big( \{ \sigma_{j_1} - 1, \ldots, \sigma_{j_{n-k}} - 1 \} \setminus \{0\} \big) \cup \{n\}
\end{aligned}$$

PROOF. Let $i$ be a descent of $\sigma$. As $\sigma$ is 231-avoiding, $[\sigma_i - 1, \sigma_i]$ cannot be a subword of $\sigma$. In other words,

$$\hat{\sigma}(\sigma_i - 1) > \hat{\sigma}(\sigma_i) = i,$$

which proves that $\sigma_i - 1$ is a descent of $\hat{\sigma}$.
On the other hand let $i'$ be a descent of $\hat{\sigma}$, which is 312-avoiding. The same argument as above yields $\sigma(\hat{\sigma}_{i'+1}) > \sigma(\hat{\sigma}_{i'+1} + 1)$ or, equivalently, $\hat{\sigma}_{i'+1}$ is a descent of $\sigma$. This implies

$$\operatorname{iDes}(\sigma) = \{ \sigma_{i_1} - 1, \ldots, \sigma_{i_k} - 1 \}.$$

As $\operatorname{Asc}(\sigma) = [n] \setminus \operatorname{Des}(\sigma)$, the statement about $\operatorname{iAsc}(\sigma)$ follows. □

Recall the definition of the involution $\rho$ on $\mathcal{S}_n$ and its properties. Together with Lemma 2.77, this implies the following corollary:

COROLLARY 2.78. *Let $\sigma$ be $\tau$-avoiding for $\tau \in \{132, 231, 312, 213\}$. Then*

$$\operatorname{des}(\sigma) = \operatorname{ides}(\sigma).$$

NOTE. For $\tau \in \{123, 321\}$, the analogous statement of the previous corollary is false: for example, $\sigma = [2, 4, 1, 3]$ is 123-avoiding and

$$\mathrm{Des}(\sigma) = \{2\} \quad , \quad \mathrm{iDes}(\sigma) = \{1, 3\}.$$

LEMMA 2.79. *Let $j$ be an ascent of a 231-avoiding permutation $\sigma$ and let $k > j$. Then*

$$\sigma_k \geq \sigma_j.$$

PROOF. This follows immediately from the fact that $\sigma$ is 231-avoiding. □

For the remaining part of this subsection, set $\sigma$ to be a 231-avoiding permutation. Our next goal is to show that $\sigma$ is uniquely determined by its ascent set

$$\mathrm{Asc}(\sigma) = \{j_1, \ldots, j_{n-k}\}$$

and the image

$$\sigma(\mathrm{Asc}(\sigma)) = \{\sigma_{j_1}, \ldots, \sigma_{j_{n-k}}\}$$

of its ascent set. By the previous lemma, $\sigma_{j_1} < \ldots < \sigma_{j_{n-k}}$ and thereby $\sigma$ is determined on its ascent set.
To determine $\sigma$ on its descent set, we determine $\sigma$ on its descent blocks, the maximal sequences of consecutive descents. On all descent blocks, $\sigma$ is decreasing and bounded from below by the image on the ascent following the sequence (recall that our definition of the ascent set implies that every descent block is followed by an ascent). Together with the property of being 231-avoiding, this determines $\sigma$ on its descents from right to left.

REMARK. In [**84**, Proposition 2.4] and the following discussion, A. Reifegerste obtained an analogous result for 132-avoiding permutations that can be translated by the involution $\rho$.

By Lemma 2.77 and the above discussion, we conclude the following:

COROLLARY 2.80. *$\sigma$ is uniquely determined by its descent set and the descent set of its inverse.*

EXAMPLE 2.81. Let $\sigma \in \mathcal{S}_6(231)$ such that

$$\mathrm{Des}(\sigma) = \{1, 2, 4, 5\} \quad , \quad \mathrm{iDes}(\sigma) = \{1, 3, 4, 5\}.$$

This implies

$$\mathrm{Asc}(\sigma) = \{3, 6\} \quad , \quad \mathrm{iAsc}(\sigma) = \{2, 6\}.$$

By Lemma 2.77, we have

$$\sigma(\mathrm{Des}(\sigma)) = \{2, 4, 5, 6\} \text{ and } \sigma(\mathrm{Asc}(\sigma)) = \{1, 3\}.$$

As described in the previous discussion we first determine $\sigma$ on its ascent set,

$$\sigma_3 = 1 < \sigma_6 = 3,$$

and then, we determine $\sigma$ on its descent blocks, starting from right to left: the last block is $\{4, 5\}$,

$$\sigma_6 = 3 < \sigma_5 = 4 < \sigma_4 = 5,$$

and the next (and first) block is $\{1, 2\}$,

$$\sigma_3 = 1 < \sigma_2 = 2 < \sigma_1 = 6.$$

Our next goal is to construct a bijection between $\mathrm{Asc}(\sigma)$ and $\mathrm{iAsc}(\sigma)$ such that the image of any ascent $j$ is less than or equal to $j$.

Let $j$ be an ascent of $\sigma$. By $\tau(j)$, we denote the size of the descent block immediately left of $j$, or equivalently,

$$\tau(j) := j - 1 - j',$$

with $j'$ being the largest ascent such that $j' < j$ (respectively 0 if $j$ is the first ascent).

LEMMA 2.82. *Let $j$ be an ascent of $\sigma$. Then*

$$j \geq \sigma_j + \tau(j).$$

PROOF. Lemma 2.79 implies $\sigma_k > \sigma_j$ for all $k$ such that $k > j$ and for all $k$ such that $j > k > j - \tau(j)$. Therefore, $n - j \leq n - \sigma(j) - \tau(j)$, which is equivalent to the statement. □

COROLLARY 2.83. *Let $\mathrm{Asc}(\sigma) = \{j_1, \ldots, j_{n-k}\}$. Then, for any $1 \leq l < n-k$, we have*

$$j_l \geq \sigma_{j_{l+1}} - 1.$$

PROOF. By Lemma 2.82, $j_{l+1}$ is greater than or equal to $\sigma_{j_{l+1}} + \tau(j_{l+1})$, which, by definition, is equal to $\sigma_{j_{l+1}} + j_{l+1} - 1 - j_l$. This proves the corollary. □

By Lemma 2.77 and Corollary 2.83, we can define a bijection between $\mathrm{Asc}(\sigma)$ and $\mathrm{iAsc}(\sigma)$, which has the desired property that the image of an ascent $j$ is less than or equal to $j$ in the following way:

$$\begin{aligned} j_l &\mapsto \sigma_{j_{l+1}} - 1 \quad \text{for } 1 \leq l < n-k, \\ j_{n-k} &\mapsto n \end{aligned}$$

This implies the following corollary concerning the descent sets of $\sigma$ and $\hat{\sigma}$:

COROLLARY 2.84. *Let $\mathrm{Des}(\sigma) = \{i_1, \ldots, i_k\}$ and let $\mathrm{iDes}(\sigma) = \{i'_1, \ldots, i'_k\}$ such that $i_l < i_{l+1}$ and $i'_l < i'_{l+1}$. Then*

$$i_l \leq i'_l.$$

Now we define the proposed bijection:

DEFINITION 2.85. Define a map $\Phi$ from $\mathcal{S}_n(231)$ to $\mathcal{D}_n$ as follows: Let $\sigma \in \mathcal{S}_n(231)$. Then $\Phi(\sigma)$ is given by

$$\mathrm{Set}_X(\Phi(\sigma)) = \mathrm{Des}(\sigma)\ ,\ \mathrm{Set}_Y(\Phi(\sigma)) = \mathrm{iDes}(\sigma).$$

THEOREM 2.86. *The map $\Phi$ from the previous definition is well-defined, bijective and* $\mathrm{maj}(\Phi(\sigma)) = \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma)$.

PROOF. By Corollary 2.78, Proposition 2.84 and Proposition 2.7, $\Phi$ is well-defined and by Proposition 2.80 and it is injective and therefore bijective.
The equality $\mathrm{maj}(\Phi(\sigma)) = \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma)$ follows immediately from the definitions. □

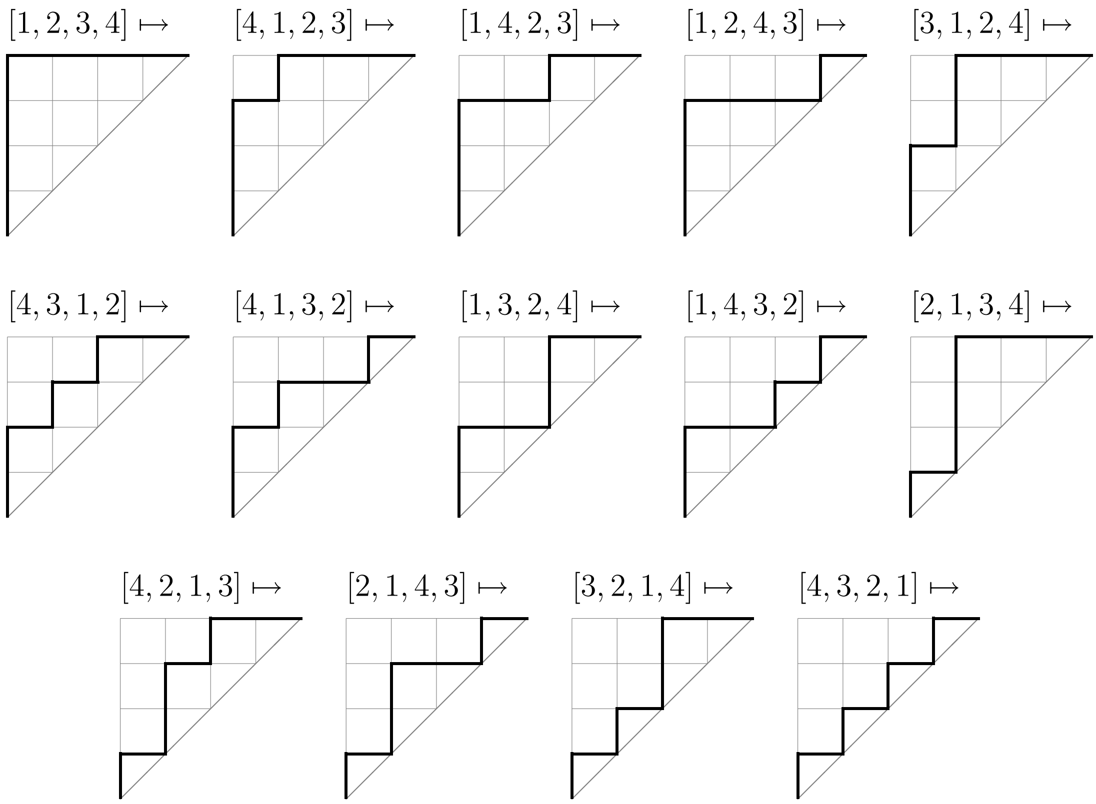



FIGURE 24. All 231-avoiding permutations of $\{1,2,3,4\}$ and its associated Catalan paths.

EXAMPLE 2.87. Let $\sigma$ be the permutation defined in Example 2.81. As the descent set of $\sigma$ is $\{1,2,4,5\}$ and the descent set of its inverse is $\{1,3,4,5\}$, the coordinates of the descents of $\Phi(\sigma)$ are $(1,1),(2,3),(4,4)$ and $(5,5)$ and therefore

$$\Phi(\sigma) \;=\; NE\ NNE\ NEE\ NE\ NE$$

which is the Catalan path described in Example 2.6. Furthermore,

$$\begin{aligned}
\mathrm{maj}(\sigma) &= 1+2+4+5 \;= 12,\\
\mathrm{imaj}(\sigma) &= 1+3+4+5 \;= 13,\\
\mathrm{maj}(\Phi(\sigma)) &= 2+5+8+10 = 25.
\end{aligned}$$

EXAMPLE 2.88. In Figure 24 on page 64 all 231-avoiding permutations of the set $\{1,2,3,4\}$ and their associated Catalan paths are shown.

**2.9.2. A bistatistic on 231-avoiding parmutations.** Define the polynomial $\mathcal{A}_n(q,t)$ by

$$\mathcal{A}_n(q,t) := \sum_{\sigma\in\mathcal{S}_n(231)} q^{\mathrm{maj}(\sigma)} t^{\binom{n}{2}-\mathrm{imaj}(\sigma)}.$$

Obviously, $\mathcal{A}_n(q,t)$ reduces for $q=t=1$ to $\mathrm{Cat}_n$ and the bijection $\Phi$ defined in the previous section shows that $\mathcal{A}_n(q,t)$ can also be described in terms of Catalan paths:

| $n$ | $\mathcal{A}_n(q,t)$ |
|---|---|
| 1 | $1$ |
| 2 | $q+t$ |
| 3 | $q^3+q^2t+qt^2+t^3+qt$ |
| 4 | $q^6+q^5t+q^4t^2+2q^3t^3+q^2t^4+qt^5+t^6+q^4t+q^3t^2+q^2t^3+qt^4+q^3t+qt^3$ |

FIGURE 25. $\mathcal{A}_n(q,t)$ for $n \leq 4$.

define two statistics $\mathrm{maj}_0$ and $\mathrm{maj}_1$ on a Catalan path $D$ by

$$\mathrm{maj}_0(D) := \sum_{i \in \mathrm{Des}(D)} \left|\{j \leq i : D_j = 0\}\right| \quad , \quad \mathrm{maj}_1(D) := \sum_{i \in \mathrm{Des}(D)} \left|\{j \leq i : D_j = 1\}\right|.$$

COROLLARY 2.89. *Let $\Phi$ be the bijection defined in Theorem 2.86 and let $\sigma$ be a 231-avoiding permutation. Then*

$$\mathrm{maj}_1(\Phi(\sigma)) = \mathrm{maj}(\sigma) \quad , \quad \mathrm{maj}_0(\Phi(\sigma)) = \mathrm{imaj}(\sigma)$$

*and furthermore*

$$\mathcal{A}_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{maj}_1(D)} t^{\binom{n}{2} - \mathrm{maj}_0(D)}.$$

EXAMPLE 2.90. In Figure 25, $\mathcal{A}_n(q,t)$ is shown for $n \leq 4$.

As $\mathrm{maj}_1(D) + \mathrm{maj}_0(D) = \mathrm{maj}(D)$ we get

$$q^{\binom{n}{2}} \mathcal{A}_n(q, q^{-1}) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{maj}(D)}.$$

Together with an identity proved by Fürlinger and Hofbauer in [**51**] we obtain the following generating function identity, which gives an equivalent definition of $\mathcal{A}_n(q,t)$:

THEOREM 2.91.

$$\sum_{n \geq 0} \frac{\mathcal{A}_n(q,t)\, z^n}{(1+qz)\cdots(1+q^{n+1}z)(1+tz)\cdots(1+t^{n+1}z)} = 1.$$

PROOF. Setting $x = 1, a = q^{-1}$ and $b = q$ in [**51**, Theorem 5] gives the proposed identity. □

The following corollary follows immediately from the fact that the generating function identity proved in Theorem 2.91 is symmetric in $q$ and $t$:

COROLLARY 2.92.

$$\mathcal{A}_n(q,t) = \mathcal{A}_n(t,q).$$

Now, we want to provide bijective proof of a refinement of this symmetry:

THEOREM 2.93.

$$\sum_{\sigma \in \mathcal{S}_n(231)} a^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{imaj}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(312)} a^{n-1-\mathrm{des}(\sigma)} q^{\binom{n}{2} - \mathrm{maj}(\sigma)} t^{\binom{n}{2} - \mathrm{imaj}(\sigma)}.$$

PROOF. Let $\Psi$ be the involution on $\mathcal{S}_n(231)$ defined by the rule that a given $\sigma$ is mapped to the unique $\sigma'$ with $\mathrm{Des}(\sigma') := [n-1] \backslash \mathrm{iDes}(\sigma)$ and $\mathrm{iDes}(\sigma') := [n-1] \backslash \mathrm{Des}(\sigma)$. This implies $\mathrm{des}(\sigma) = n - 1 - \mathrm{des}(\Psi(\sigma))$ and furthermore

$$\mathrm{maj}(\sigma) = \binom{n}{2} - \mathrm{imaj}(\Psi(\sigma)) \quad , \quad \mathrm{imaj}(\sigma) = \binom{n}{2} - \mathrm{maj}(\Psi(\sigma)).$$

As mapping a permutation to its inverse interchanges $\mathrm{Des}(\sigma)$ and $\mathrm{iDes}(\sigma)$, the statement follows. □

REMARK. Equivalently, we could have defined the bijection $\Psi$ in the proof of Theorem 2.93 in terms of Catalan paths by the rule that a given Catalan path $D$ is mapped to the unique Catalan path $D'$ with

$$\mathrm{Set}_X(D') = [n-1] \setminus \mathrm{Set}_Y(D) \quad , \quad \mathrm{Set}_Y(D') = [n-1] \setminus \mathrm{Set}_X(D).$$

**2.9.3. Another description of $\Phi$.** The bijection $\Phi$ defined in Definition 2.85 is closely connected to other bijections from pattern-avoiding permutations to Catalan paths. In [**72**], Krattenthaler constructed two bijections from 132- respectively 123-avoiding permutations to Catalan paths which were recently related to others by Callan in [**33**]. In this section, we express the bijection $\Phi$ in terms of Krattenthaler's bijection from 132-avoiding permutations to Catalan paths, which we denote by $\kappa$.

First, we recall the definition of $\kappa$ from [**72**, Section 2] and give an example. For any Catalan path $D$, the height of $D$ at position $i$ is the number of north steps minus the number of east steps until and including position $i$. Now, let $\sigma = [\sigma_1, \ldots, \sigma_n]$ be a 132-avoiding permutation and let $h_i$ denote the number of $j$'s larger than $i$ such that $\sigma_j > \sigma_i$. Read $\sigma$ from left to right and successively generate a Catalan path. When $\sigma_i$ is read, then in the path we adjoin as many north steps as necessary, followed by an east step from height $h_i + 1$ to height $h_i$.

EXAMPLE 2.94. Let $\sigma = [3, 4, 5, 1, 2, 6]$. Then $(h_1, \ldots, h_6) = (3, 2, 1, 2, 1, 0)$ which gives the heights of the east steps of the corresponding Catalan paths. Therefore,

$$\kappa(\sigma) = NNNNEEE\ NNEEE.$$

Our next goal is to express $\kappa$ in terms of the descent set of $\sigma$ and the descent set of $\hat{\sigma}$.

LEMMA 2.95. *Let $\sigma$ be any permutation. Then*

- $h_{i+1} < h_i$ *if and only if* $i \in \mathrm{Asc}(\sigma)$,
- $\sigma \in \mathcal{S}_n(132)$ *if and only if* $h_{i+1} \geq h_i - 1$ *for all* $1 \leq i < n$.

PROOF. The first statement is obvious as is the fact that $\sigma \in \mathcal{S}_n(132)$ implies $h_{i+1} \geq h_i - 1$ for all $1 \leq i < n$. For the reverse statement we use that $h_{i+1} \geq h_i - 1$ implies for an ascent $i$ that there exists no $k > i$ with $\sigma_i < \sigma_k < \sigma_{i+1}$. □

PROPOSITION 2.96. *Let $\sigma$ be a 132-avoiding permutation. Then $\kappa(\sigma)$ is the Catalan path $D$ given by*

$$\mathrm{Set}_X(D) = \mathrm{Des}(\sigma) \quad \textit{and} \quad \mathrm{Set}_Y(D) = \{i + h_i : i \in \mathrm{Des}(\sigma)\}.$$

PROOF. The statement follows from the first part of the previous lemma and the definition of $\kappa$. □

PROPOSITION 2.97. *Let $\sigma$ be a 132-avoiding permutation. Then*

$$\mathrm{iDes}(\sigma) = \big\{n - i - h_i : i \in \mathrm{Des}(\sigma)\big\}.$$

PROOF. Let $i$ be a descent of $\sigma$. The number of $\sigma_j$'s right of $\sigma_i$ that are smaller than $\sigma_i$ is equal to $n - i - h_i$. As $\sigma$ is 132-avoiding, we know that the set of these $\sigma_j$'s is equal to $\{1, \ldots, n - i - h_i\}$. This implies that $\hat{\sigma}_{n-i-h_i} > \hat{\sigma}_{n-i-h_i+1}$ and therefore $n - i - h_i$ is a descent of $\hat{\sigma}$.
Now let $i$ be a descent of $\sigma$ and let $j > i$. The previous lemma implies that $i+h_i \neq j+h_j$ as otherwise $\{i, i+1, \ldots, j-1\} \subseteq \mathrm{Asc}(\sigma)$, a contradiction to $i \in \mathrm{Des}(\sigma)$. The proposition follows from Corollary 2.78. □

These two propositions characterize $\kappa$ in terms of the descent set of $\sigma$ and the descent set of $\hat{\sigma}$: let $\sigma \in \mathcal{S}_n(132)$. Then

$$\mathrm{Set}_X(\kappa(\sigma)) = \mathrm{Des}(\sigma) \quad \text{and} \quad \mathrm{Set}_Y(\kappa(\sigma)) = \{n - j : j \in \mathrm{iDes}(\sigma)\}.$$

EXAMPLE 2.98. Continuing with the previous example, we have that $\mathrm{Des}(\sigma)$ and $\mathrm{iDes}(\sigma)$ are given by $\{3\}$ and $\{2\}$ respectively. On the other hand,

$$\mathrm{Set}_X(\kappa(\sigma)) = \{3\} = \mathrm{Des}(\sigma) \quad \text{and} \quad \mathrm{Set}_Y(\kappa(\sigma)) = \{4\} = \{n - j : j \in \mathrm{iDes}(\sigma)\}.$$

Let $D$ be a Catalan path with valleys $\{(i_1, j_1), \ldots, (i_k, j_k)\}$. The last map we need is the involution $\Psi$ as described in the Remark at the end of Section 2.9.2 followed by the involution $\mathbf{c}$ on Catalan paths sending $D$ to its conjugate Catalan path: set $\psi(D)$ to be the unique Catalan path with

$$\begin{aligned}\mathrm{Set}_X(\psi(D)) &= [n-1] \setminus \big\{n - i : i \in \mathrm{Set}_X\big\} \\ \mathrm{Set}_Y(\psi(D)) &= [n-1] \setminus \big\{n - i : i \in \mathrm{Set}_Y\big\}\end{aligned}$$

or, equivalently,

$$\psi = \mathbf{c} \circ \Psi.$$

Now we can describe the relation of $\Phi$ and $\kappa$ using the involutions $\rho$ and $\psi$:

THEOREM 2.99. $\kappa = \psi \circ \Phi \circ \rho$.

PROOF. Let $\sigma$ be a 132-avoiding permutation. Then

$$\begin{aligned}\mathrm{Set}_X(\Phi \circ \rho(\sigma)) &= \mathrm{Des}(\rho(\sigma)) &&= [n-1] \setminus \{n - i : i \in \mathrm{Des}(\sigma)\}, \\ \mathrm{Set}_Y(\Phi \circ \rho(\sigma)) &= \mathrm{iDes}(\rho(\sigma)) &&= [n-1] \setminus \mathrm{iDes}(\sigma).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{Set}_X(\psi(D)) &= \{n - i : i \in [n-1] \setminus \mathrm{Set}_X(D)\}, \\ \mathrm{Set}_Y(\psi(D)) &= \{n - i : i \in [n-1] \setminus \mathrm{Set}_Y(D)\},\end{aligned}$$

for a Catalan path $D$. So, in total, we have

$$\begin{aligned}\mathrm{Set}_X(\psi \circ \Phi \circ \rho(\sigma)) &= \mathrm{Des}(\sigma), \\ \mathrm{Set}_Y(\psi \circ \Phi \circ \rho(\sigma)) &= \{n - j : j \in \mathrm{iDes}(\sigma)\}.\end{aligned}$$

Together with the above discussion, this implies the theorem. □

EXAMPLE 2.100. Set $\sigma := [6, 2, 1, 5, 4, 3]$. As shown in Example 2.87, $\Phi(\sigma) = NE\ NNE\ NEE\ NE\ NE$ and therefore $\psi(\Phi(\sigma)) = NNNNEEE\ NNEEE$. As shown in Example 2.94, this equals $\kappa(\rho(\sigma))$.

| | $\{(\operatorname{maj}(\sigma), \operatorname{imaj}(\sigma)) : \sigma \in \mathcal{S}_4(\tau)\}$ |
|---|---|
| (231) | $(0,0),(3,3),(2,2),(2,3),(5,5),(1,1),(4,4),$ $(1,2),(3,3),(1,3),(4,5),(3,4),(3,5),(6,6)$ |
| (132) | $(0,0),(1,1),(2,1),(3,1),(1,2),(3,3),(4,3),$ $(2,2),(5,3),(1,3),(3,4),(4,4),(3,5),(6,6)$ |
| (213) | $(0,0),(3,3),(3,2),(2,3),(5,5),(3,1),(5,4),$ $(2,2),(5,3),(1,3),(4,5),(4,4),(3,5),(6,6)$ |
| (123) | $(5,5),(2,4),(4,4),(5,4),(4,2),(3,3),(4,3),$ $(2,2),(5,3),(4,5),(3,4),(4,4),(3,5),(6,6)$ |
| (321) | $(0,0),(3,3),(2,2),(3,2),(2,3),(1,1),(4,4),$ $(2,1),(3,1),(2,4),(1,2),(4,2),(2,2),(1,3)$ |

FIGURE 26. The bistatistic $(\operatorname{maj}(\sigma), \operatorname{imaj}(\sigma))$ on 3-pattern-avoiding permutations.

Next, we want to use the involutions $\Psi, \rho$ and $\sigma \mapsto \hat{\sigma}$ (we denote the later by $\mathbf{i}$) to prove an analogue of Theorem 2.93 for $\mathcal{S}_n(132)$ and $\mathcal{S}_n(213)$:

THEOREM 2.101.
$$\sum_{\sigma \in \mathcal{S}_n(132)} a^{\operatorname{des}(\sigma)} q^{\operatorname{maj}(\sigma)} t^{\operatorname{imaj}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(213)} a^{n-1-\operatorname{des}(\sigma)} q^{\binom{n}{2}-\operatorname{maj}(\sigma)} t^{\binom{n}{2}-\operatorname{imaj}(\sigma)}.$$

PROOF. The bijection between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(213)$ defined as $\rho \circ \mathbf{i} \circ \Psi \circ \rho$, where $\Psi$ is meant as described in the proof of Theorem 2.93, sends the tristatistic $(\operatorname{des}, \operatorname{maj}, \operatorname{imaj})$ to the tristatistic $(n-1-\operatorname{des}, \binom{n}{2}-\operatorname{maj}, \binom{n}{2}-\operatorname{imaj})$:
$$\begin{aligned}
(\operatorname{des}, \operatorname{maj}, \operatorname{imaj}) &\overset{\rho}{\mapsto} (n-1-\operatorname{des}, \binom{n}{2} - n\operatorname{des} + \operatorname{maj}, \binom{n}{2} - \operatorname{imaj})\\
&\overset{\Psi}{\mapsto} (\operatorname{des}, \operatorname{imaj}, n\operatorname{des} - \operatorname{maj})\\
&\overset{\mathbf{i}}{\mapsto} (\operatorname{des}, n\operatorname{des} - \operatorname{maj}, \operatorname{imaj})\\
&\overset{\rho}{\mapsto} (n-1-\operatorname{des}, \binom{n}{2} - \operatorname{maj}, \binom{n}{2} - \operatorname{imaj}).
\end{aligned}$$
□

COROLLARY 2.102.
$$\sum_{\sigma \in \mathcal{S}_n(132)} q^{\operatorname{maj}(\sigma)} t^{\operatorname{imaj}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(213)} q^{\binom{n}{2}-\operatorname{maj}(\sigma)} t^{\binom{n}{2}-\operatorname{imaj}(\sigma)}.$$

EXAMPLE 2.103. In Figure 26, the bistatistic $(\operatorname{maj}(\sigma), \operatorname{imaj}(\sigma))$ is shown for 3-pattern avoiding permutations of $\{1, 2, 3, 4\}$.

Computations suggest that the same statement is true when summing over $\mathcal{S}_n(123)$ respectively $\mathcal{S}_n(321)$. It would be implied by the following conjecture:

CONJECTURE 2.104. There exists a bijection from $\mathcal{S}_n(321)$ to itself which leaves Des invariant and maps iDes to $\{n - j : j \in \operatorname{iDes}\}$.

**2.9.4. The proof of Proposition 2.65.** We describe the connection between $\Phi$ and Bandlow-Killpatrick's bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ to see how the area statistic on Catalan paths can be represented in $\mathcal{S}_n(231)$ via $\Phi$ and to develop the connection to the bijection we constructed in Section 2.8. Let us denote this bijection for the moment by $\gamma$.

As already mentioned, Bandlow and Killpatrick defined in [**14**] a bijection $\beta$ between $\mathcal{S}_n(231)$ and $\mathcal{D}_n$ which can be described in terms of $\gamma$ as

$$\beta = \mathbf{c} \circ \gamma^{-1},$$

in other words, $\beta^{-1}$ maps a given Catalan path $D$ to the image of the conjugate of $D$ under $\gamma$. They actually defined $\beta$ in a slightly different way but it is easy to see that both definition are in fact equivalent.

M. Fulmek and Krattenthaler described the connection between $\beta$ and $\kappa$ in a comment on Bandlow and Killpatrick's paper [**48**]:

$$\beta = \Psi \circ \Phi.$$

Together with the connection between $\beta$ and $\gamma$, the following corollary finally proves Proposition 2.65:

Corollary 2.105. *$\beta^{-1}$ maps the area statistic on Catalan paths to the inversion number on 231-avoiding permutations and furthermore,*

$$\mathrm{Des}(\sigma) = [n-1] \setminus \mathrm{Set}_Y(D) \quad , \quad \mathrm{iDes}(\sigma) = [n-1] \setminus \mathrm{Set}_X(D),$$

*for a Catalan path $D$ and its image $\sigma = \beta^{-1}(D)$. In particular,*

$$\mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) + \mathrm{maj}(D) = n(n-1).$$

Proof. The corollary follows immediately from the definitions of $\Psi$ and $\Phi$. $\square$

Example 2.106. As we have seen in Example 2.64, the Catalan path

$$D = NNNNEEE\ NNEEE$$

shown in Figure 22 is mapped by $\gamma$ to the permutation $\sigma := \gamma(D) = [6,2,1,5,4,3]$. On the other hand, we can deduce from Example 2.100 that

$$\Psi \circ \Phi(\sigma) = \beta(\sigma) = \mathbf{c}(D).$$

This gives

$$\begin{aligned} \mathrm{maj}(\mathbf{c}(D)) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) &= 5 + (1,2,4,5) + (1,3,4,5) \\ &= 30 = 6 \cdot 5 = n(n-1). \end{aligned}$$

# CHAPTER 3

## $q,t$-Fuß-Catalan numbers of type $A$

The $q,t$-Catalan numbers and later the $q,t$-Fuß-Catalan numbers arose within the last 15 years in more and more contexts in different areas of mathematics, namely in *symmetric functions theory*, *algebraic* and *enumerative combinatorics*, *representation theory* and *algebraic geometry*. They first appeared in a paper by Haiman as the Hilbert series of the alternating component of the *diagonal coinvariant ring* [**63**]. In [**55**], Garsia and Haiman defined them as a rational function in the context of *modified Macdonald polynomials*. Later, in his work on the $n!$– and on the $(n+1)^{n-1}$–*conjecture*, Haiman showed that both definitions coincide [**65**]. Haglund [**60**] found a very interesting combinatorial interpretation of the $q,t$-Catalan numbers which he proved together with Garsia in [**53**]. In [**77**], Loehr conjectured a generalization of this combinatorial interpretation for $q,t$-Fuß-Catalan numbers. This conjecture is still open.

The $q,t$-Fuß-Catalan numbers have many interesting algebraic and combinatorial properties. To mention some: they are symmetric functions in $q$ and $t$ with non-negative integer coefficients and specialize for $q = t = 1$ to $\mathrm{Cat}_n^{(m)}$. Furthermore, specializing $t = 1$ reduces them to the $q$-Fuß-Catalan numbers introduced by Fürlinger and Hofbauer in [**51**] and specializing $t = q^{-1}$ and multiplying by the highest power of $q$ reduces them to the $q$-Fuß-Catalan numbers introduced in the classical case by MacMahon in [**81**].

In this chapter, we want to discuss several appearances of the $q,t$-Fuß-Catalan numbers, their connections to modified Macdonald polynomials, their interpretation in terms of the diagonal coinvariant ring and finally their combinatorial interpretation.

It turns out that the interpretation in terms of the diagonal coinvariant ring is attached to the reflection group of type $A$ whereas the other interpretations can – so far – not be generalized to other reflection groups. In Chapter 4, we will define the diagonal coinvariant ring and will thereby give a definition of $q,t$-Fuß-Catalan numbers for complex reflection groups. Moreover, we will explore their properties in this generalized context.

The standard reference for symmetric functions and Macdonald polynomials is Macdonald's book "Symmetric functions and Hall polynomials" [**80**]. For a more compact introduction as well as a generalization of Macdonald polynomials to crystallographic reflection groups, see [**79**]. Recently, Haglund published the book "The $q,t$-Catalan numbers and the space of diagonal harmonics" [**61**]. The theory presented in this chapter is mainly based on these references.

## 3.1. Symmetric functions

When introducing symmetric functions, one should mention that the term "symmetric function" is a little misleading. When we refer to symmetric functions – and mostly when people do – we mean symmetric polynomials or symmetric formal power series rather than just functions.

Fix a set $\mathbf{x} := \{x_1, \ldots, x_n\}$ of indeterminants. A polynomial or formal power series $p$ in $\mathbf{x}$ is symmetric if for any permutation $\sigma \in \mathcal{S}_n$,

$$p(x_1, \ldots, x_n) = p(x_{\sigma_1}, \ldots, x_{\sigma_n}).$$

NOTE. As we have seen in Example 1.18, the notion of symmetric polynomials coincides with the notion of being invariant with respect to the reflection group $A_{n-1} = \mathcal{S}_n$.

By $\Lambda_n$, we denote the ring of symmetric polynomials,

$$\Lambda_n = \mathbb{Z}[\mathbf{x}]^{\mathcal{S}_n}.$$

For any $n$, we have a surjective homomorphism from $\Lambda_{n+1} \to \Lambda_n$ defined by setting $x_{n+1} = 0$. Therefore, we can define $\Lambda$ as its inverse limit and we have a surjective homomorphism $\Lambda \to \Lambda_n$ defined by setting $x_m = 0$ for all $m > n$. $\Lambda$ is obviously a graded ring and it is called ring of symmetric functions. For any commutative ring $R$, we write

$$\Lambda_R = \Lambda \otimes_{\mathbb{Z}} R, \ \ \Lambda_{n,R} = \Lambda_n \otimes_{\mathbb{Z}} R.$$

The ring $\Lambda$ has – as a vector space – some remarkable bases which are indexed by combinatorial objects called partitions. In order to present these bases, we first introduce partitions and some of its combinatorial properties.

For a detailed introduction see the book "Young tableaux" by Fulton [**49**].

**3.1.1. Partitions.** A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of a non-negative integer $n$, written $\lambda \vdash n$, is a finite, weakly decreasing sequence of non-negative integers such that the $\lambda_i$'s sum up to $n$. In order to keep the notation simple, we identify a partition $\lambda$ with the infinite sequence $(\lambda_1, \ldots, \lambda_l, 0, \ldots)$ and may write expressions like $\sum_{i \geq 1} \lambda_i = n$. The positive $\lambda_i$'s are called parts of $\lambda$ and the number of parts is denoted by $l(\lambda)$. If $\lambda_i$ occurs $k$ times in $\lambda$ then we sometimes write $\lambda_i^k$ for short, e.g.

$$(4, 2, 2, 1, 1, 1) = (4, 2^2, 1^3) \vdash 11.$$

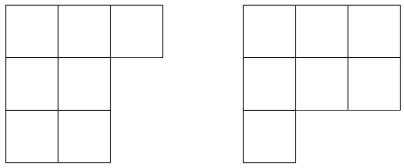
NOTE. When the blocks in a set partition $\{B_1, \ldots, B_k\}$ of $[n]$ are ordered decreasingly by cardinality, then

$$(|B_1|, \ldots, |B_k|) \vdash n$$

is a partition of $n$.

The shape of $\lambda$ (also called *Ferrer's shape* or *Young diagram*) is defined to be the array of unit squares, called cells, with $\lambda_i$ cells in the $i$-th row from the top, with the first cell in each row left-justified.

EXAMPLE 3.1. In Figure 27(a), the Ferrer's shape of the partition $\lambda = (3, 2^2) \vdash 7$ is shown.



$(a)$ $(b)$

FIGURE 27. The shape of the partition $\lambda = (3, 2^2) \vdash 7$ and of its conjugate partition $\lambda' = (3^2, 1) \vdash 7$.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 4 \\ \cline{1-2} 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 5 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array}$$

FIGURE 28. The 5 possible SSYT of shape $(3, 2^2) \vdash 7$ and weight $(2^2, 1^3) \vdash 7$.

REMARK. There exist several ways to define the shape of a partition. The definition we gave is called *English notation.* For example, if one flips the English notation horizontally, i.e., if one builds the rows of cells from bottom to top, one gets the *French notation.*

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, the conjugate partition $\lambda' \vdash n$ is obtained from $\lambda$ by transposing its shape, i.e. $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ with

$$\lambda'_i := \bigl|\{\lambda_k : \lambda_k \geq i\}\bigr|.$$

EXAMPLE (CONTINUED) 3.2. The conjugate of the partition $\lambda = (3, 2^2)$ is given by $\lambda' = (3^2, 1)$ whose shape is shown in Figure 27(b).

**3.1.2. Semi-standard Young tableaux.** Given two partitions $\lambda, \mu \vdash n$, a semi-standard Young tableau (or short SSYT) of shape $\lambda$ and weight $\mu$ is a filling of the cells of $\lambda$ with the elements of the multiset $\{1^{\mu_1}, 2^{\mu_2}, \ldots\}$ such that the numbers *weakly increase* along rows from left to right and *strictly increase* along columns from top to bottom.

The number of SSYT of shape $\lambda$ and weight $\mu$ and a certain $q, t$-generalization of this number play a central role in the theory of symmetric functions, in combinatorics of tableaux and in the related representation theory. This number is known as the Kostka number $K_{\lambda,\mu}$. In Section 3.1.9, we will present the mentioned generalization of the Kostka numbers, the *$q, t$-Kostka numbers.*

NOTE. As there is only one SSYT with the same shape and weight, we have

$$K_{\lambda,\lambda} = 1 \text{ for all } \lambda.$$

In the next section, we will describe a criterion for the Kostka numbers to be positive.

EXAMPLE 3.3. In Figure 28, the 5 semi-standard Young tableaux of shape $\lambda = (3, 2^2) \vdash 7$ and weight $\mu = (2^2, 1^3) \vdash 7$ are shown.

**3.1.3. The dominance order.** The dominance order on the set of partitions of $n$ is the partial order defined as

$$\mu \leq \lambda :\Leftrightarrow \mu_1 + \ldots + \mu_i \leq \lambda_1 + \ldots + \lambda_i \text{ for all } i.$$

for $\lambda, \mu \vdash n$.

It is easy to deduce from this definition that $\lambda \leq \mu$ if and only if there exists a SSYT of shape $\lambda$ and weight $\mu$,

$$\mu \leq \lambda \Leftrightarrow K_{\lambda,\mu} > 0.$$

EXAMPLE 3.4. For $\lambda = (3, 2^2) \vdash 7$, the set of partitions that are smaller than or equal to $\lambda$ in the dominance order are

$$(3, 2^2), (3, 2, 1^2), (3, 1^4), (2^3, 1), (2^2, 1^3), (2, 1^5) \text{ and } (1^7).$$

The dominance order and the Kostka numbers $K_{\lambda,\mu}$ will later appear in the definition of Schur functions which form a basis of $\Lambda$, see Section 3.1.5. We will also see how the $q,t$-Kostka numbers $K_{\lambda,\mu}(q,t)$ appear in a generalization of the Schur functions called *Macdonald polynomials* which form a basis of $\Lambda_{\mathbb{Q}(q,t)}$, see Section 3.1.9.

**3.1.4. Some bases of $\Lambda$.** The most basic symmetric functions are the monomial symmetric functions $m_\lambda$ which are indexed by partitions. To a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ associate the monomial

$$x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l},$$

and define $m_\lambda$ as the sum of all distinct monomials obtained from this by permuting the variables. Obviously, the set $\{m_\lambda\}$ forms a basis of $\Lambda$.

From this basis one obtains immediately three other basis as follows: the power sum symmetric functions $p_\lambda$ are defined in terms of monomial symmetric functions as

$$p_k := m_{(k)}, \quad p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l},$$

the elementary symmetric functions $e_\lambda$ are defined as

$$e_k := m_{(1^k)}, \quad e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l},$$

and the complete homogeneous symmetric functions $h_\lambda$ are defined as

$$h_k := \sum_{\lambda \vdash k} m_\lambda, \quad h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}.$$

EXAMPLE 3.5.

$$\begin{aligned}
m_{(2)} &= \sum_i x_i^2 &&= p_{(2)}, \\
m_{(1,1)} &= \sum_{i<j} x_i x_j &&= e_{(2)}, \\
m_{(2,1)} &= \sum_{i \neq j} x_i^2 x_j, \\
p_{(2,1)} &= \sum_{i,j} x_i^2 x_j &&= m_{(2)} m_{(1)}, \\
e_{(2,1)} &= \sum_{i<j,k} x_i x_j x_k &&= m_{(1,1)} m_{(1)}, \\
h_{(2)} &= m_{(1,1)} + m_{(2)} &&= e_{(2)} + p_{(2)}.
\end{aligned}$$

One can compute the $p_\lambda$, $e_\lambda$ and $h_\lambda$ from each other by the formulas

$$\begin{aligned}
e_k &= \sum_{\lambda \vdash k} p_\lambda \frac{(-1)^{k-l(\lambda)}}{z_\lambda} \text{ and} \\
h_k &= \sum_{\lambda \vdash k} p_\lambda \frac{1}{z_\lambda},
\end{aligned}$$

where $z_\lambda := \prod_i i^{n_i} n_i!$ and $n_i = n_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$.

**3.1.5. Schur functions.** The next basis of $\Lambda$ we want to describe is the basis of Schur functions, which are fundamental to the theory of symmetric functions. There are many ways to define a Schur function $s_\lambda$; we present a definition that leads to a definition of Macdonald polynomials which we will introduce later.

DEFINITION 3.6. Let $\lambda$ be a partition. The Schur function $s_\lambda$ is defined as

$$s_\lambda := \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu.$$

As we have seen in Section 3.1.3, one can restrict the sum on the right-hand side in the definition to these $\mu$ that are smaller than or equal to $\lambda$,

$$s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda,\mu} m_\mu.$$

EXAMPLE 3.7. It is easy to see that for $\lambda = (2^2, 1)$,

$$\mu \leq \lambda \Leftrightarrow \mu \in \{(2^2,1), (2,1^3), (1^4)\}$$

and that

$$K_{\lambda,(2^2,1)} = 1,\ K_{\lambda,(2,1^3)} = 2,\ K_{\lambda,(1^4)} = 5.$$

This gives

$$s_{(2^2,1)} = m_{(2^2,1)} + 2m_{(2,1^3)} + 5m_{(1^4)}.$$

NOTE. For any partition $\mu \vdash k$, we have

$$(1^k) \leq \mu \leq (k)$$

in the dominance order and $K_{(k),\mu} = 1$. This gives

$$s_{(1^k)} = e_k \quad , \quad s_{(k)} = h_k.$$

**3.1.6. The Hall inner product on $\Lambda$.** The Hall inner product $\langle \cdot, \cdot \rangle$ on the ring $\Lambda$ of symmetric functions is defined such that the power sum symmetric functions are orthogonal and $\langle p_\lambda, p_\lambda \rangle = z_\lambda$, with $z_\lambda$ as defined in Section 3.1.4. In symbols,

$$\langle p_\lambda, p_\mu \rangle := \chi_{(\lambda=\mu)} \, z_\lambda.$$

With respect to this inner product, the complete homogeneous symmetric functions and the monomial symmetric functions are dual to each other,

$$\langle h_\lambda, m_\mu \rangle = \chi_{(\lambda=\mu)},$$

and the Schur functions are orthonormal,

$$\langle s_\lambda, s_\mu \rangle = \chi_{(\lambda=\mu)}.$$

The reason to define the inner product on the power sum symmetric functions and then showing that the Schur functions are orthonormal with respect to this inner product is that one could have defined the Schur functions also as the unique family $\{s_\lambda\}$ of symmetric functions with rational coefficients indexed by partitions which are unitriangular when expressed in terms of the monomial symmetric functions and which are orthogonal with respect to the Hall inner product,

(i) $s_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu$ for suitable coefficients $c_{\lambda,\mu}$ and
(ii) $\langle s_\lambda, s_\mu \rangle = 0$ if $\lambda \neq \mu$.

Of course, these two conditions overdetermine the family $\{s_\lambda\}$ and one has some work to do to prove that such a family exists. This can be done by setting $c_{\lambda,\mu} = K_{\lambda,\mu}$ and showing that they have the desired properties. In Section 3.1.9, we will define Macdonald polynomials analogously such that the Schur functions appear as a certain specialization.

**3.1.7. Representation theory of the symmetric group.** As mentioned in Section 1.3.1, the irreducible representations of the symmetric group are in one-to-one correspondence with conjugacy classes of $\mathcal{S}_n$: for a partition $\lambda \vdash n$, $\mathcal{S}_n$ acts on (the shape of) $\lambda$ by permuting the cells. Define $P_\lambda$ and $Q_\lambda$ by

$$\begin{aligned} P_\lambda &:= \{\sigma \in \mathcal{S}_n : \sigma \text{ leaves the rows of } \lambda \text{ invariant}\}, \\ Q_\lambda &:= \{\sigma \in \mathcal{S}_n : \sigma \text{ leaves the columns of } \lambda \text{ invariant}\}. \end{aligned}$$

Furthermore, let $a_\lambda := \sum_{\sigma \in P_\lambda} \sigma$ and $b_\lambda := \sum_{\sigma \in Q_\lambda} \sigma$. Then the irreducible representations of $\mathcal{S}_n$ are given by

$$\big\{\mathbb{C}[G] a_\lambda b_\lambda : \lambda \vdash n\big\}.$$

For $\chi^\lambda$ being the character of the irreducible representation $\mathbb{C}[G]a_\lambda b_\lambda$, a classical result by F.G. Frobenius is that

$$\chi^\lambda(\sigma) = \langle s_\lambda, p_{\tau(\sigma)} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Hall inner product and $\tau(\sigma)$ is the cycle type of the permutation $\sigma$. This equality can equivalently be stated as

$$s_\lambda = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \chi^\lambda(\sigma) p_{\tau(\sigma)}.$$

This yields the following definition:

DEFINITION 3.8. The Frobenius character $\mathcal{F}$ is the map from $\mathbb{C}_{class}(G)$ to $\Lambda_{n,\mathbb{C}}$ that sends $\chi^\lambda$ to $s_\lambda$. In symbols,

$$\begin{aligned} \mathcal{F}(\chi) &= \sum_{\lambda \vdash n} \mathrm{mult}(\chi^\lambda, \chi) s_\lambda \\ &= \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) p_{\tau(\sigma)}. \end{aligned}$$

NOTE. As $\{\chi^\lambda\}$ and $\{s_\lambda\}$ are orthonormal bases of $\mathbb{C}_{class}(G)$ and of $\Lambda_{n,\mathbb{C}}$ respectively, $\mathcal{F}$ is an *isometry*.

**3.1.8. Plethystic notation.** Let $E(t_1, t_2, \ldots)$ be a formal power series of rational functions in the parameters $t_1, t_2, \ldots$. Define the plethystic substitution of $E$ into $p_k$, denoted $p_k[E]$, by

$$p_k[E] := E(t_1^k, t_2^k, \ldots).$$

The square "plethystic" brackets around $E$ are introduced to distinguish $p_k[E]$ from the ordinary $k$-th power sum in a set of indeterminants $E$ which are defined as $p_k(E)$. When a set $X = \{x_1, x_2, \ldots\}$ of indeterminants is written in plethystic brackets, then by convention $X$ is meant as $p_1(X) = x_1 + x_2 + \ldots$. Note that it is also important to distinguish between indeterminants and parameters, see the following example.

EXAMPLE 3.9. In the case just mentioned, we have $p_k[X] = p_k(X)$. Let $Y$ be another set of indeterminants and let $t, q$ be real parameters. Then

$$\begin{aligned} p_k[tX] &= t^k p_k[X], \\ p_k[X(1-t)] &= \sum_i x_i^k (1-t^k), \\ p_k[X-Y] &= \sum_i (x_i^k - y_i^k) = p_k[X] - p_k[Y], \\ p_k\left[\frac{X(1-t)}{1-q}\right] &= \sum_i \frac{x_i^k(1-q^k)}{1-q^k}. \end{aligned}$$

**3.1.9. Macdonald polynomials.** The polynomials we present in this section form a basis for $\Lambda_{\mathbb{Q}(q,t)}$, the space of symmetric polynomials with coefficients being rational functions in $q$ and $t$. For $t = q$, they specialize to the Schur functions.

In order to introduce these polynomials, we extend the Hall inner product to $\Lambda_{\mathbb{Q}(q,t)}$ by

$$\langle p_\lambda, p_\mu \rangle_{q,t} := \chi_{(\lambda=\mu)}\ z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

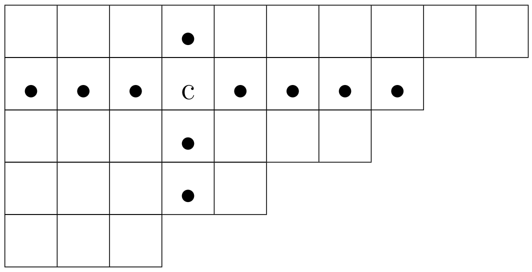


FIGURE 29. A cell $c$ in the shape of the partition $\lambda = (10, 8, 7, 5, 3) \vdash 33$ with $a(c) = 4, a'(c) = 3, l(c) = 2$ and $l'(c) = 1$.

The following two conditions uniquely determine a family of symmetric functions, $\{P_\lambda(X; q, t)\}$, indexed by partitions with coefficients in $\mathbb{Q}(q, t)$: they are unitriangular when expressed in terms of the monomial symmetric functions and they are orthogonal with respect to the Hall inner product,

(i) $P_\lambda(X; q, t) = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu(X)$, with $c_{\lambda,\mu} \in \mathbb{Q}(q, t)$,
(ii) $\langle P_\lambda(X; q, t), P_\mu(X; q, t)\rangle_{q,t} = 0$ if $\lambda \neq \mu$.

As for Schur functions, these two conditions overdetermine the family $\{P_\lambda(X; q, t)\}$ and it needs some sophisticated work to prove the existence of such a family. The uniqueness follows easily. The idea of the proof of the existence is to construct a certain operator on $\Lambda_{\mathbb{Q}(q,t)}$ and to show that the eigenfunctions of this operator have in fact the desired properties. For a clear and self-contained proof see [**79**, Chapter 1].

NOTE. Since the $q, t$ extension of the Hall inner product reduces to the ordinary Hall inner product when $q = t$, it follows that $P_\lambda(X; q, q) = s_\lambda(X)$.

In [**79**, Chapter 2], Macdonald generalized the construction to all crystallographic reflection groups by introducing another scalar product for which the $P_\lambda$'s are also orthogonal and which can be generalized to other reflection groups.

**3.1.10. Modified Macdonald polynomials.** In order to define $q, t$-Fuß-Catalan numbers in terms of Macdonald polynomials, we have to deal with a modified version of the $P_\lambda(X; q, t)$'s. These are obtained by a certain plethystic substitution and so far it is totally unclear, how this substitution could be translated into the language of reflection groups.

The first modification is done by introducing *integral forms* of Macdonald polynomials. To define these, set arm $a(c)$, coarm $a'(c)$, leg $l(c)$ and coleg $l'(c)$ of a cell $c$ in a partition $\lambda$ to be the number of cells right, left, below and above the cell $c$ in the shape of $\lambda$. See Figure 29 for an example.

Define a modification of $P_\lambda(X; q, t)$ by

$$J_\lambda(X; q, t) := \prod_{c\in\lambda} (1 - q^{a(c)} t^{l(c)+1}) P_\lambda(X; q, t).$$

Macdonald showed that expanding $J_\lambda$ in terms of $\{s_\mu[X(1-t)]\}$ gives

$$J_\lambda(X;q,t) := s_\lambda[X(1-t)] + \sum_{\mu<\lambda} K_{\lambda,\mu}(q,t) s_\mu[X(1-t)],$$

for suitable $K_{\lambda,\mu}(q,t) \in \mathbb{Q}(q,t)$ satisfying $K_{\lambda,\mu}(1,1) = K_{\lambda,\mu}$. Furthermore, he conjectured that $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

NOTE. The coefficients $K_{\lambda,\mu}(q,t)$ are the $q,t$-Kostka numbers. The conjecture that $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$ became a famous problem in combinatorics known as *Macdonald's positivity conjecture*.

A specialization of the conjecture was proved by Lascoux and Schützenberger in [**74**] by introducing a statistic on tableaux called *charge* such that the specialization $q=0$ in $K_{\lambda,\mu}(q,t)$ has the following combinatorial interpretation:

$$K_{\lambda,\mu}(0,t) = \sum t^{charge(T)},$$

where the sum ranges over all SSYT of shape $\lambda$ and weight $\mu$.

Beside this combinatorial interpretation, there are a few known ways to prove the positivity of $K_{\lambda,\mu}(0,t)$ by interpreting them representation theoretically or geometrically. Such an interpretation was conjectured also for $K_{\lambda,\mu}(q,t)$ by Garsia and Haiman in [**55**] and was finally proved by Haiman as a corollary of his proof of the $n!$-*conjecture* [**65**].

As we will see in the next section, from the representation theoretical point of view, it is more convenient to work with the polynomials

$$\widetilde{K}_{\lambda,\mu}(q,t) := t^{n(\mu)} K_{\lambda,\mu}(q,t^{-1}),$$

where $n(\mu) := \sum_{c\in\mu} l(c) = \sum (i-1)\mu_i$.

DEFINITION 3.10. The modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ are defined as

$$\widetilde{H}_\mu(X;q,t) := \sum_{\lambda\leq\mu} \widetilde{K}_{\lambda,\mu} s_\lambda(X).$$

REMARK. The work on the charge statistic implies that Macdonald's positivity conjecture holds for $K_{\lambda,\mu}(q,t)$ if and only if it holds for $\widetilde{K}_{\lambda,\mu}(q,t)$.

**3.1.11. The nabla operator.** The nabla operator $\nabla$ is the linear operator on $\Lambda_{\mathbb{Q}(q,t)}$ defined on the modified Macdonald polynomials as

$$\nabla \widetilde{H}_\mu(X;q,t) := q^{n(\mu')} t^{n(\mu)} \widetilde{H}_\mu(X;q,t).$$

Some of the basic properties of $\nabla$ were first obtained by F. Bergeron in [**16**] and more enhanced applications followed in a series of papers by Bergeron, Garsia, Haiman and G. Tesler [**17**], [**18**] and [**56**].

The following proposition was first proved in [**55**] by expressing the image of an elementary symmetric function under $\nabla$ in terms of modified Macdonald polynomials:

PROPOSITION 3.11.

$$\nabla e_n = \sum_{\mu \vdash n} \frac{q^{n(\mu')}t^{n(\mu)}(1-q)(1-t)B_\mu \Pi_\mu \widetilde{H}_\mu}{\prod_{c \in \mu}\left(q^{a(c)} - t^{l(c)+1}\right)\left(t^{l(c)} - q^{a(c)+1}\right)},$$

*where* $B_\mu = \sum_{c \in \mu} q^{a'(c)}t^{l'(c)}$ *and* $\Pi_\mu = \prod_{c \in \mu \setminus (0,0)} (1 - q^{a'(c)}t^{l'(c)})$.

REMARK. So far, it is totally unclear how the nabla operator could extend to other reflection groups.

## 3.2. The Garsia-Haiman module

Before we introduce a module which was constructed by Garsia and Haiman to prove Macdonald's positivity conjecture, we first want to recall the definition of *Hilbert series* and *Frobenius series*. To keep things as simple as possible, we will introduce them not in full generality but only in the context of the Garsia-Haiman module.

The objects we are looking at are subrings of the polynomial ring

$$\mathbb{C}[\mathbf{x},\mathbf{y}] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n].$$

By a slight modification of the action introduced in Section 1.1.8, $\mathbb{C}[\mathbf{x},\mathbf{y}]$ comes equipped with an additional *diagonal action* of the symmetric group $\mathcal{S}_n$,

$$\sigma(x_i) = x_{\sigma(i)}, \sigma(y_i) = y_{\sigma(i)} \text{ for } \sigma \in \mathcal{S}_n.$$

Note that $\mathbb{C}[\mathbf{x},\mathbf{y}]$ is bigraded by degree in $\mathbf{x}$ and degree in $\mathbf{y}$,

$$\mathbb{C}[\mathbf{x},\mathbf{y}] = \bigoplus_{i,j \geq 0} \mathbb{C}[\mathbf{x},\mathbf{y}]_{i,j},$$

where $\mathbb{C}[\mathbf{x},\mathbf{y}]_{i,j}$ denotes the set of all polynomials which are *bihomogeneous* of $\mathbf{x}$-degree $i$ and $\mathbf{y}$-degree $j$ and furthermore note that the diagonal action preserves this grading.

As we will deal with subspaces of $\mathbb{C}[\mathbf{x},\mathbf{y}]$, we have to carry over the notion of being bigraded to those: a submodule $M \subseteq \mathbb{C}[\mathbf{x},\mathbf{y}]$ is said to be bigraded if it contains all of the bihomogeneous components of each of its elements, i.e., if $p \in M$ with $p = \sum_{i,j} p_{i,j}$ and $p_{i,j} \in \mathbb{C}[\mathbf{x},\mathbf{y}]_{i,j}$, we have $p_{i,j} \in M$ for all $i, j$.

**3.2.1. The Hilbert series and the Frobenius series.** Let $M$ be a bigraded subspace of $\mathbb{C}[\mathbf{x},\mathbf{y}]$,

$$M = \bigoplus_{i,j \geq 0} M_{i,j}.$$

Define the bigraded Hilbert series of $M$ as the generating function for the dimensions of its bigraded components,

$$\mathcal{H}(M; q, t) := \sum_{i,j \geq 0} \dim(M_{i,j}) q^i t^j.$$

Furthermore, when $M$ is invariant under the diagonal action we can define the bigraded Frobenius series of $M$ as the generating function for the Frobenius characters of its bigraded components,

$$\mathcal{F}(M;q,t) := \sum_{i,j\geq 0} \mathcal{F}(\chi_{i,j}) q^i t^j,$$

where $\chi_{i,j} := \chi_{M_{i,j}}$ is the character of $M_{i,j}$.

**3.2.2. The bivariate Vandermonde determinant and the $n!$–conjecture.** The Vandermonde determinant $\Delta(\mathbf{x})$ is defined as

$$\Delta(\mathbf{x}) := \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix} = \prod_{i<j} (x_j - x_i).$$

This determinant can be generalized as follows: for $X = \big\{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\big\} \subseteq \mathbb{N} \times \mathbb{N}$ define the polynomial $\Delta_X$ by

$$\Delta_X(\mathbf{x},\mathbf{y}) := \det \begin{pmatrix} x_1^{\alpha_1} y_1^{\beta_1} & \dots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & \dots & x_n^{\alpha_n} y_n^{\beta_n} \end{pmatrix},$$

and for a partition $\lambda \vdash n$ set $\Delta_\lambda := \Delta_{X_\lambda}$ where

$$X_\lambda = \{(i,j) \in \mathbb{N}^2 : 0 \leq i < l(\lambda), 0 \leq j < \lambda_i\}.$$

As $\Delta_X(\mathbf{x},\mathbf{y})$ reduces for $X = X_{(1^n)} = \big\{(0,0), \dots, (n-1,0)\big\}$ to $\Delta(\mathbf{x})$ and symmetrically for $X = X_{(n)}$ to $\Delta(\mathbf{y})$, it is a bivariate analogue of the Vandermonde determinant.

Definition 3.12. Fix a partition $\mu \vdash n$. The module $V(\mu)$ is defined to be the linear span of all partial derivatives of $\Delta_\mu(\mathbf{x},\mathbf{y})$,

$$V(\mu) := \mathcal{L}[\partial_{\mathbf{x}}^p \partial_{\mathbf{y}}^q \Delta_\mu(\mathbf{x},\mathbf{y})].$$

The module $V(\mu)$ was introduced in [**55**] by Garsia and Haiman where they conjectured that the bigraded Frobenius series of $V(\mu)$ is equal to the modified Macdonald polynomials. As this conjecture would imply that $\dim(V(\mu)) = n!$, it became known as the *$n!$-conjecture*. As this conjecture states that $\widetilde{K}_{\lambda,\mu}(q,t)$ can be realized as a certain character, it implies Macdonald's positivity conjecture; in [**64**], Haiman proved the surprising result that both conjectures are in fact equivalent. In [**65**], Haiman finally proved them after almost 10 years of intensive research.

Theorem 3.13 (Haiman). *Let $\mu \vdash n$ be a partition of $n$. Then*

$$\mathcal{F}(V(\mu);q,t) = \widetilde{H}_\mu(X;q,t).$$

*In particular, $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.*

Remark. In her PhD thesis [**8**], S. Assaf was able to prove the non-negativity of $\widetilde{K}_{\lambda,\mu}(q,t)$ combinatorially.

Corollary 3.14 (Haiman).

$$\dim(V(\mu)) = n!.$$

## 3.3. Diagonal coinvariants

Define the polarized power sum symmetric functions $p_{k,l}(\mathbf{x},\mathbf{y})$ as

$$p_{i,j}(\mathbf{x},\mathbf{y}) := \sum_{k=1}^{n} x_k^i y_k^j.$$

In [**96**], H. Weyl observed that the ring of diagonal invariants, $\mathbb{C}[\mathbf{x},\mathbf{y}]^{\mathcal{S}_n}$, is generated by $\{p_{i,j}(\mathbf{x},\mathbf{y})\}$.

Definition 3.15. The diagonal coinvariant ring $DR_n$ is given by

$$DR_n := \mathbb{C}[\mathbf{x},\mathbf{y}]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal in $\mathbb{C}[\mathbf{x},\mathbf{y}]$ generated by all polarized power sums of positive degree, or equivalently, the ideal generated by all invariant polynomials without constant term.

The study of the space of diagonal coinvariants is closely related to the study of the space $V(\mu)$ defined in the previous section. They are connected via the space of diagonal harmonics, which is defined as

$$DH_n := \{f \in \mathbb{C}[\mathbf{x},\mathbf{y}] : p_{i,j}(\partial\mathbf{x},\partial\mathbf{y})f = 0 \text{ for all } i+j>0\}.$$

Here, $p_{i,j}(\partial\mathbf{x},\partial\mathbf{y})$ is the operator given by the polarized power sum in the linear operators $\partial x_1, \partial y_1, \ldots, \partial x_n, \partial y_n$.

Remark. It is easy to see that $DR_n$ and $DH_n$ are isomorphic as vector spaces, see [**63**, Section 1.3], and that $DH_n$ is closed under partial differentiation and contains $\Delta_\mu$. This implies that $V(\mu)$ is an $\mathcal{S}_n$-submodule of $DH_n$ for any $\mu \vdash n$.

In [**63**], Haiman conjectured that $DR_n$ and $DH_n$ have dimension $(n+1)^{n-1}$. He derived this result as a corollary of a formula for the Frobenius series of $DH_n$ which was conjectured by Garsia and him in [**55**]. He was finally able to prove this formula using the techniques developed to prove the $n!$–conjecture, see Theorem 3.16.

Remark. The number $(n+1)^{n-1}$ is known to count the number of *parking functions* on $n$ cars, for definitions see [**94**]. Haglund and Loehr found a conjectured combinatorial interpretation of $\mathcal{H}(DR_n;q,t)$, the Hilbert series of the diagonal coinvariants, in terms of statistics on parking functions [**62**].

The diagonal coinvariants $DR_n$ have a closely related extension for each integer $m \geq 1$: as above let $\mathcal{I}$ be the ideal generated by all polarized power sums of positive degree and let $\mathcal{A}$ be the ideal generated by all alternating polynomials. Then the space $DR_n^{(m)}$ was defined by Garsia and Haiman in [**55**] as

$$DR_n^{(m)} := \left(\mathcal{A}^{m-1}/\mathcal{A}^{m-1}\mathcal{I}\right) \otimes \epsilon^{\otimes(m-1)}.$$

As $DR_n^{(m)}$ is an $m$-extension of the diagonal coinvariant ring, we call it generalized diagonal coinvariant ring.

Theorem 3.16 (Haiman).

$$\mathcal{F}(DR_n^{(m)};q,t) = \nabla^m e_n.$$

Corollary 3.17 (Haiman).

$$\dim DR_n^{(m)} = (mn+1)^{n-1}.$$

**3.3.1. The alternating component of $DR_n^{(m)}$.** Observe that the natural $\mathcal{S}_n$-action is twisted by the $(m-1)$-st power of the alternating representation such that the generators of this module, which are the minimal generators of $\mathcal{A}^{m-1}$, become invariant. One can show that the alternating component of $DR_n^{(m)}$ is, except for the sign-twist, naturally isomorphic to the *minimal generating space* of $\mathcal{A}^m$. Define $M^{(m)}$ to be this alternating component of $DR_n^{(m)}$,

$$\begin{aligned} M^{(m)} &:= \mathbf{e}_\epsilon(DR_n^{(m)}) \\ &= \big(\mathcal{A}^m/\langle \mathbf{x},\mathbf{y}\rangle\mathcal{A}^m\big)\otimes \epsilon^{\otimes(m-1)}, \end{aligned}$$

where $\langle \mathbf{x},\mathbf{y}\rangle = \langle x_1,y_1,\ldots,x_n,y_n\rangle$ is the ideal of all polynomials without constant term and where $\mathbf{e}_\epsilon$ is the sign idempotent as defined in Section 1.3.2.

Remark. The name *minimal generating space* comes from the fact that – as a vector space – $M^{(m)}$ is isomorphic to the complex vector space with basis in one-to-one correspondence to any homogeneous minimal generating set of $\mathcal{A}^m$.

As a vector space, the space $\mathbb{C}[\mathbf{x},\mathbf{y}]^\epsilon$ of all alternating polynomials has a well-known basis given by

$$\mathcal{B} = \big\{\Delta_X : X \subseteq \mathbb{N}\times\mathbb{N}, |X| = n\big\},$$

in particular the ideal generated by these elements equals $\mathcal{A}$. Together with the previous remark, this implies that $M^{(1)}$ is isomorphic to the vector space with basis in one-to-one correspondence to any maximal algebraically independent subset of $\mathcal{B}$. Unfortunately, so far no one was able to construct such a maximal algebraically independent subset in general.

## 3.4. $q,t$-Fuß-Catalan numbers

The $q,t$-Catalan numbers $\mathrm{Cat}_n(q,t)$ appeared first in [**63**] where Haiman defined them as

$$\mathrm{Cat}_n(q,t) := \langle \mathcal{F}(DR_n;q,t), s_{(1^n)}\rangle, \tag{9}$$

and conjectured that $\mathrm{Cat}_n(q,t)$ is in fact a $q,t$-extension of the Catalan numbers, i.e., $\mathrm{Cat}_n(1,1) = \mathrm{Cat}_n$. As seen in the last section, the right-hand side of Eq. (9) is equal to the Hilbert series of $M^{(1)}$,

$$\mathrm{Cat}_n(q,t) = \mathcal{H}(M^{(1)};q,t).$$

The second appearance was in [**55**] where Garsia and Haiman defined $\mathrm{Cat}_n(q,t)$ together with its generalization $\mathrm{Cat}_n^{(m)}(q,t)$ as

$$\mathrm{Cat}_n^{(m)}(q,t) := \sum_{\mu\vdash n} \frac{q^{(m+1)n(\mu')}t^{(m+1)n(\mu)}(1-q)(1-t)B_\mu\Pi_\mu}{\prod_{c\in\mu}\big(q^{a(c)}-t^{l(c)+1}\big)\big(t^{l(c)}-q^{a(c)+1}\big)}.$$

With Corollary 3.11 and the fact that $\langle \widetilde{H}_\mu, s_{1^n}\rangle = q^{n(\mu')}t^{n(\mu)}$, this complicated rational function can be nicely described in terms of the nabla operator introduced in Section 3.1.11:

$$\mathrm{Cat}_n^{(m)}(q,t) = \langle \nabla^m e_n, e_n\rangle.$$

At that point, it was still a long way to show that $\mathrm{Cat}_n^{(m)}(q,t)$ reduces to $\mathrm{Cat}_n(q,t)$ for $m = 1$, which is an immediate consequence of Theorem 3.16:

COROLLARY 3.18.

$$\begin{aligned} \mathrm{Cat}_n^{(m)}(q,t) &= \langle \mathcal{F}(DR_n^{(m)};q,t), s_{(1^n)} \rangle \\ &= \mathcal{H}(M^{(m)};q,t). \end{aligned}$$

**3.4.1. The specializations $q = t = 1$, $t = 1$ and $t = q^{-1}$.** Using their definition as a rational function, Garsia and Haiman were able to show that specializing $q = t = 1$ gives in fact the Fuß-Catalan numbers,

$$\mathrm{Cat}_n^{(m)}(1,1) = \mathrm{Cat}_n^{(m)},$$

and furthermore they proved the specializations $t = 1$ and $t = q^{-1}$, which were already conjectured for $\mathrm{Cat}_n(q,t)$ in [**63**]. Both specializations are equal to well-known $q$-extensions of the Catalan numbers, namely the generating function for the area statistic on $m$-Catalan paths defined in Section 2.4.1,

$$\mathrm{Cat}_n^{(m)}(q,1) = \mathrm{Cat}_n^{(m)}(q) = \sum_{D \in \mathcal{D}_n^{(m)}} q^{\mathrm{area}(D)},$$

and MacMahon's $q$-Catalan numbers $\mathrm{Cat}_n^{(m)}$ defined in Section 2.1.1,

$$q^{m\binom{n}{2}} \mathrm{Cat}_n^{(m)}(q,q^{-1}) = \text{diag-}\mathrm{Cat}_n^{(m)}(q) = \frac{1}{[mn+1]_q} \begin{bmatrix} (m+1)n \\ n \end{bmatrix}_q .$$

For $m = 1$ we have seen in Sections 2.6 and 2.8 that there exists a bijection between Catalan paths and non-crossing partitions and a bijection between Catalan paths and Coxeter sortable elements that send the area statistic to the length function and simultaneously the major index to the sum of the major and the inverse major indices. This implies the following corollary:

COROLLARY 3.19. *Let $NC(A_{n-1})$ the the non-crossing partition lattice of type $A_{n-1}$ and let $\mathrm{Cox}_c(A_{n-1})$ be the set of Coxeter sortable elements as given in Theorem 2.48 and in Theorem 2.63 respectively. Then*

$$\begin{aligned} \mathrm{Cat}_n(q,1) &= \sum_{\sigma \in NC(A_{n-1})} q^{\mathrm{l}_S(\sigma)} &&= \sum_{\sigma \in \mathrm{Cox}_c(A_{n-1})} q^{\mathrm{l}_S(\sigma)}, \\ q^{\binom{n}{2}} \mathrm{Cat}_n(q,q^{-1}) &= \sum_{\sigma \in NC(A_{n-1})} q^{\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma)} &&= \sum_{\sigma \in \mathrm{Cox}_c(A_{n-1})} q^{\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma)}. \end{aligned}$$

**3.4.2. Combinatorial descriptions of $\mathrm{Cat}_n^{(m)}(q,t)$.** The – at that time conjectured – description of $\mathrm{Cat}_n^{(m)}(q,t)$ as a bigraded Hilbert series would imply that $\mathrm{Cat}_n^{(m)}(q,t) \in \mathbb{N}[q,t]$. Together with the combinatorial description of the specialization $t = 1$ this was the starting point of an intensive search for a second statistic tstat on $\mathcal{D}_n^{(m)}$ such that

$$\mathrm{Cat}_n^{(m)}(q,t) = \sum_{D \in \mathcal{D}_n^{(m)}} q^{\mathrm{area}(D)} t^{\mathrm{tstat}(D)}.$$

In [**60**], Haglund came up with a statistic he called bounce on Catalan paths and conjectured that the $q,t$-Catalan numbers are described as the bistatistic $(\mathrm{area}, \mathrm{bounce})$

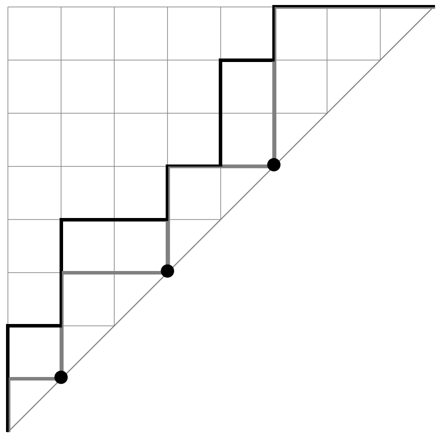


FIGURE 30. A Catalan path $D \in \mathcal{D}_8$ and its bounce path $\mathrm{bp}(D)$.

on Catalan paths. The bounce statistic is defined as follows: recall the definition of a valley of $D \in \mathcal{D}_n$ from Section 2.1.1. $D$ is called *bounce path* if all its valleys are at the diagonal $x = y$, i.e. all valleys of $D$ are of the form $(i, i)$. To any Catalan path $D \in \mathcal{D}_n$, define the bounce path of $D$ to be the path described by the following algorithm:

*Start at $(n, n)$ and travel west along $D$ until you encounter a north step.*
*Turn south and travel straight until you hit the diagonal $x = y$.*
*Turn west and travel again until you encounter a north step.*
*Continue this procedure until you reach the point $(0, 0)$.*

Notice that the resulting path is by construction a bounce path, denote it by $\mathrm{bp}(D)$. The bounce statistic $\mathrm{bounce}(D)$ is now defined as

$$\mathrm{bounce}(D) := \sum_{(i,i) \text{ valley of } \mathrm{bp}(D)} i.$$

Often, the valleys of a bounce path are called bounce points.

EXAMPLE 3.20. In Figure 30, a Catalan path $D \in \mathcal{D}_8$ and its associated bounce paths $\mathrm{bp}(D)$ is shown. As indicated by the dots, $\mathrm{bp}(D)$ has valleys $(1, 1), (3, 3)$ and $(5, 5)$. This gives

$$\mathrm{bounce}(D) = 1 + 3 + 5 = 9.$$

The conjecture mentioned above was shortly later proved in [**53, 54**] by Garsia and Haglund:

THEOREM 3.21 (Garsia, Haglund).

$$\mathrm{Cat}_n^{(1)}(q, t) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)}.$$

REMARK. The bounce path occurred independently in a paper by G.E. Andrews, Krattenthaler, L. Orsina and P. Papi [**5**] (see also [**34**]) in which they computed the number of ad-nilpotent ideals in sl(n). They prove that the ad-nilpotent ideals are in one-to-one correspondence with order ideals in the root poset and, in type $A$, that the degree of nilpotence is equal the number of bounce points of the associated bounce path. This occurrence of the bounce path is very remarkable as it discloses a possible connection which no one – so far – was able to develop.

From the definition it is easy to see that $\mathrm{Cat}_n^{(m)}(q,t)$ is symmetric in $q$ and $t$ since arm and leg length of some partition $\mu \vdash n$ equal leg and arm length of $\mu'$ respectively. This gives

$$\sum_{D\in\mathcal{D}_n} q^{\mathrm{area}(D)}t^{\mathrm{bounce}(D)} = \sum_{D\in\mathcal{D}_n} q^{\mathrm{bounce}(D)}t^{\mathrm{area}(D)}.$$

So far no bijective proof of this fact is known:

Open Problem. Find a bijection on Catalan paths which interchanges area and bounce.

Haiman discovered another closely related statistic dinv on Catalan paths called diagonal inversion number. For $D \in \mathcal{D}_n$, let $\lambda \subseteq (n-1,\ldots,2,1,0)$ be the associated partition as described in Example 2.1. Then

$$\mathrm{dinv}(D) := \big|\big\{c \in \lambda : a(c) - l(c) \in \{0,1\}\big\}\big|.$$

Haglund constructed a bijection on $\mathcal{D}_n$ which sends the bistatistic (area, bounce) to the bistatistic (dinv, area), see [**61**, Theorem 3.15]. This implies immediately that

$$\mathrm{Cat}_n^{(1)}(q,t) = \sum_{D\in\mathcal{D}_n} q^{\mathrm{area}(D)}t^{\mathrm{dinv}(D)}.$$

In [**77**], Loehr generalized the definitions of bounce and dinv to $\mathcal{D}_n^{(m)}$. As the generalization of bounce is a little more difficult, we content ourselves with the generalized definition of the diagonal inversion number: as before, fix $D \in \mathcal{D}_n^{(m)}$ and set $\lambda \subseteq (m(n-1),\ldots,2m,m,0)$ to be the associated partition. Then

$$\mathrm{dinv}(D) := \big|\big\{c \in \lambda : a(c) - ml(c) \in \{0,1,\ldots,m\}\big\}\big|.$$

Loehr conjectured that the bistatistics (area, bounce) and (area, dinv) describes the $q,t$-Fuß-Catalan numbers combinatorially:

Conjecture 3.22 (Loehr).

$$\mathrm{Cat}_n^{(m)}(q,t) = \sum_{D\in\mathcal{D}_n^{(m)}} q^{\mathrm{area}(D)}t^{\mathrm{bounce}(D)} = \sum_{D\in\mathcal{D}_n^{(m)}} q^{\mathrm{area}(D)}t^{\mathrm{dinv}(D)}.$$

This conjecture is still open, only the specializations $q=1$ and $t=q^{-1}$ are proved in [**77**].

CHAPTER 4

# $q, t$-Fuß-Catalan numbers for complex reflection groups

In this chapter, we generalize the definition of $q, t$-Fuß-Catalan numbers to arbitrary (finite) complex reflection groups.

## 4.1. Diagonal coinvariants

In Section 3.3, the generalized diagonal coinvariant ring $DR_n^{(m)}$ was defined and in Corollary 3.17, we have seen that

$$\dim DR_n^{(m)} = (mn+1)^{n-1}. \tag{10}$$

The definition of the generalized coinvariant ring makes sense for any complex reflection group:

DEFINITION 4.1. Let $W$ be a complex reflection group of rank $l$ acting diagonally on a polynomial ring in two sets of variables, $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \ldots, x_l, y_l]$. Furthermore, let $\mathcal{I} \trianglelefteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$ be the ideal generated by all invariant polynomials without constant term and let $\mathcal{A} \trianglelefteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$ be the ideal generated by all alternating polynomials. For any positive integer $m$, define the generalized diagonal coinvariant ring $DR^{(m)}(W)$ as

$$DR^{(m)}(W) := \left(\mathcal{A}^{m-1}/\mathcal{A}^{m-1}\mathcal{I}\right) \otimes \epsilon^{\otimes(m-1)}.$$

REMARK. For $m = 1$, this definition can be found in [**63**, Section 7].

In type $A$, the dimension of $DR_n^{(m)}$, see Eq. (10), can be expressed in terms of the reflection group $A_{n-1}$ as

$$\dim DR^{(m)}(A_{n-1}) = (mh+1)^l,$$

where $h = n$ is the Coxeter number of $A_{n-1}$ and $l = n-1$ is its rank.

We reproduce the following incorrect guess from [**63**, Guess 7.1.1]:

GUESS (incorrect, see below). Let $W$ be a (crystallographic) reflection group of rank $l$ having Coxeter number $h$. Then

$$\dim DR^{(1)}(W) = (h+1)^l.$$

As we have seen, the guess is correct for type $A$. Furthermore, it follows from computations by Alfano [**2**] and Reiner [**85**] that it is correct for the dihedral groups:

THEOREM 4.2 (Alfano, Reiner). *Let $W = I_2(k)$ be a dihedral group. Then*

$$\dim DR^{(1)}(W) = (k+1)^2.$$

| | dim | $(h+1)^l$ |
|---|---|---|
| $B_4$ | $9^4+1$ | $9^4$ |
| $B_5$ | $11^5+33$ | $11^5$ |
| $D_4$ | $7^4+40$ | $7^4$ |

FIGURE 31. The actual dimension of $DR^{(1)}(W)$ for the reflection groups $B_4, B_5$ and $D_4$.

Haiman computed the actual dimension of $DR^{(1)}(W)$ for the reflection groups $B_4, B_5$ and $D_4$. The results can be found in Figure 31.

These "counterexamples" led him to the following conjecture [**63**, Conjecture 7.1.2]:

CONJECTURE 4.3 (Haiman). For each (crystallographic) reflection group $W$, there exists a "natural" quotient ring $R_W$ of $\mathbb{C}[\mathbf{x},\mathbf{y}]$ by some homogeneous ideal containing $\mathcal{I}$ such that

$$\dim R_W = (h+1)^l.$$

This conjecture was proved by Gordon in [**57**] using subtle results from the theory of *rational Cherednik algebras*:

THEOREM 4.4 (Gordon). *Let $W$ be the crystallographic reflection group. There exists a graded $W$-stable quotient ring $R_W$ of $DR^{(1)}(W)$ such that*

(i) $\dim(R_W) = (h+1)^l$ *and furthermore*
(ii) $q^N \mathcal{H}(R_W;q) = [h+1]_q^l$,
(iii) $R_W \otimes \epsilon$ *is isomorphic to the permutation representation of $W$ on*

$$Q/(h+1)Q,$$

*where $Q$ is the root lattice associated to $W$ defined in Section 1.1.11.*

In Chapter 5, we will show that (i) and (ii) in Corollary 4.4 hold in fact for all real reflection groups and for arbitrary non-negative $m$, see Corollary 5.8.

## 4.2. $q,t$-Fuß-Catalan numbers

Haiman's computations of the dimension of the diagonal coinvariants in types $B_4$, $B_5$ and $D_4$ seemed to be the end of the story, but computations of the dimension of the alternating component of the generalized coinvariant ring $DR^{(m)}(W)$ suggest the following conjecture:

CONJECTURE 4.5. Let $W$ be a well-generated complex reflection group. Then

$$\dim \mathbf{e}_\epsilon(DR^{(m)}(W)) = \mathrm{Cat}(W).$$

REMARK. In Appendix A.1, the computations of the dimensions for which the conjecture holds are shown. We used the computer algebra system `Singular` [**92**] for these computations, in particular, we used the following two commands:

```
module M = modulo(I^m,J*I^m);
dim(std(M));
```

As in type $A$, define the $W$-module $M^{(m)}(W)$ as the alternating component of $DR^{(m)}(W)$,

$$\begin{aligned} M^{(m)}(W) &:= \mathbf{e}_\epsilon(DR^{(m)}(W)) \\ &= \big(\mathcal{A}^m/\langle \mathbf{x},\mathbf{y}\rangle\mathcal{A}^m\big)\otimes \epsilon^{\otimes(m-1)}. \end{aligned}$$

The space $\mathbb{C}[\mathbf{x},\mathbf{y}]^{\epsilon,W}$ of all alternating polynomials has a well-known basis which is given by

$$\mathcal{B}_W := \big\{\mathbf{e}_\epsilon(\mathrm{m}(\mathbf{x},\mathbf{y})) : m(\mathbf{x},\mathbf{y}) \text{ monomial in } \mathbf{x},\mathbf{y} \text{ with } \mathbf{e}_\epsilon(m(\mathbf{x},\mathbf{y}))\neq 0\big\}.$$

This basis reduces in type $A$ to the basis $\mathcal{B}$ defined in Section 3.3.1. For the other classical types, $\mathcal{B}_W$ can also be described using the bivariate analogue of the Vandermonde determinant. In type $B$, it reduces to

$$\mathcal{B}_{B_n} = \big\{\Delta_X : X = \{(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)\}\subseteq \mathbb{N}\times\mathbb{N}, |X| = n, \alpha_i+\beta_i\equiv 1 \bmod 2\big\}$$

and in type $D$, it reduces to

$$\mathcal{B}_{D_n} = \big\{\Delta_X : X = \{(\alpha_i,\beta_i),\dots,(\alpha_n,\beta_n)\}\subseteq \mathbb{N}\times\mathbb{N}, |X| = n, \alpha_i+\beta_i\equiv \alpha_j+\beta_j \bmod 2\big\}.$$

Conjecture 4.5 leads to the following definition:

Definition 4.6. Let $W$ be a complex reflection group and let $M^{(m)}(W)$ be the alternating component of the diagonal coinvariants $DR^{(m)}(W)$. Define $q,t$-Fuß-Catalan numbers $\mathrm{Cat}^{(m)}(W;q,t)$ as

$$\begin{aligned} \mathrm{Cat}^{(m)}(W;q,t) &= \langle \mathcal{F}(DR^{(m)}(W);q,t), s_{(1^n)}\rangle \\ &= \mathcal{H}(M^{(m)}(W);q,t). \end{aligned}$$

Note. By definition, the $q,t$-Fuß-Catalan numbers $\mathrm{Cat}^{(m)}(W;q,t)$ are symmetric polynomials in $q$ and $t$ with non-negative integer coefficients. Conjecture 4.5 would imply for a well-generated complex reflection group $W$, that

$$\mathrm{Cat}^{(m)}(W;1,1) = \mathrm{Cat}^{(m)}(W).$$

In the remaining part of this chapter, we want to present several beautiful conjectured properties of $\mathrm{Cat}^{(m)}(W;q,t)$ which are based on further computations.

## 4.3. Conjectured properties of the $q,t$-Fuß-Catalan numbers

In addition to the computations of the dimension of $M^{(m)}(W)$, see Remark 4.2, we computed its bigraded Hilbert series $\mathrm{Cat}^{(m)}(W;q,t)$. The following routines of the computer algebra system `Macaulay 2` [**78**] were used:

```
M = mingens(I^m);
<< for i from 0 to numgens source M - 1 list degree M_i;;
```

The computations are listed in Appendix A.2. The following conjectures are based on these computations.

The following conjecture, which is obviously stronger than Conjecture 4.5, would generalize the specialization $t = q^{-1}$ presented in Section 3.4 and would thereby give a new answer to a question of C. Kriloff and Reiner in [**6**, Problem 2.2]:

CONJECTURE 4.7. Let $W$ be a well-generated complex reflection group of rank $l$. Then

$$q^{mN}\operatorname{Cat}^{(m)}(W;q,q^{-1}) = \text{diag-Cat}^{(m)}(W;q),$$

where $N := \sum(d_i - 1)$ and where diag-$\operatorname{Cat}^{(m)}(W;q)$ is defined as in Section 2.1.1.

REMARK. The $(d_i-1)$'s appearing in the conjecture are the $V$-exponents of $W$ and for real reflection groups, $N$ is equal to the total number of reflections.

As we will see in Chapter 5, this conjecture can be stated equivalently in the context of rational Cherednik algebras.

By definition, the $q,t$-Fuß-Catalan numbers are polynomials in $\mathbb{N}[q,t]$. As in Section 3.4.2 for type $A$, this leads to the natural question of a combinatorial description of $\operatorname{Cat}^{(m)}(W;q,t)$:

OPEN PROBLEM. Are there statistics *qstat* and *tstat* on objects counted by $\operatorname{Cat}^{(m)}(W)$ which generalize the *area* and the *bounce* statistics on Catalan paths $\mathcal{D}_n^{(m)}$ such that

$$\operatorname{Cat}^{(m)}(W;q,t) = \sum_D q^{\operatorname{qstat}(D)} t^{\operatorname{tstat}(D)}?$$

The next conjecture we want to present concerns the specialization $t = 1$ of this open problem. In Section 4, we have defined combinatorially $q$-Fuß-Catalan numbers as the generating function for the coheight statistic on the extended Shi arrangement,

$$\operatorname{Cat}^{(m)}(W;q) = \sum_R q^{\operatorname{coh}(R)},$$

where the sum ranges over all positive regions in the extended Shi arrangement $\mathsf{Shi}^{(m)}(W)$.

The following conjecture would generalize the statement for type $A$ from Section 3.4 and would partially answer the open problem:

CONJECTURE 4.8. Let $W$ be a crystallographic reflection group. Then the $q,t$-Fuß-Catalan numbers reduce to the $q$-Fuss-Catalan numbers for the specialization $t = 1$,

$$\operatorname{Cat}^{(m)}(W;q,1) = \operatorname{Cat}^{(m)}(W;q).$$

**4.3.1. The conjectures in type $B$.** Analogously to type $A$, we have seen for $m = 1$ in Sections 2.6 and 2.8 that there exists a bijection between Catalan paths and non-crossing partitions with reversed negative elements and a bijection between Catalan paths and Coxeter sortable elements that send the area statistic to the length function and simultaneously the major index to the sum of the major and the inverse major indices. Therefore, Conjecture 4.7 and Conjecture 4.8 would imply the following corollary:

COROLLARY 4.9. *Let $NC(B_n)$ the the non-crossing partition lattice of type $B_n$ and let $\mathrm{Cox}_c(B_n)$ be the set of Coxeter sortable elements as given in Theorem 2.48 and in Theorem 2.63 respectively. If Conjectures 4.7 and 4.8 hold in type $B$, then*

$$\begin{aligned}\mathrm{Cat}(B_n,q,1) &= \sum_{\sigma\in\mathrm{rev}(NC(B_n))} q^{\mathrm{l}_S(\sigma)} = \sum_{\sigma\in\mathrm{Cox}_c(B_n)} q^{\mathrm{l}_S(\sigma)},\\ q^{n^2}\,\mathrm{Cat}(B_n;q,q^{-1}) &= \sum_{\sigma\in\mathrm{rev}(NC(B_n))} q^{\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma)} = \sum_{\sigma\in\mathrm{Cox}_c(B_n)} q^{\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma)}.\end{aligned}$$

## 4.4. $q,t$-Fuß-Catalan numbers for the dihedral groups

In unpublished work in the context of their PhD-theses [**2, 85**], Alfano and Reiner were able to describe uniformly the coinvariant ring $DR^{(1)}(I_2(k))$ for the dihedral groups. For the sake of more readability, we introduce the $q,t$-extension $[n]_{q,t}$ of the integer $n$ which we define by

$$[n]_{q,t} := \frac{q^n-t^n}{q-t} = q^{n-1}+q^{n-1}t+\ldots+qt^{n-2}+t^{n-1}.$$

Then $[n]_{q1}=[n]_q$ and $[n]_{1t}=[n]_t$ are the well-known $q$- and $t$-extensions of the integer $n$ defined at the beginning of Section 2.1.1.

The following description is taken from [**63**, Section 7.5]:

THEOREM 4.10 (Alfano, Reiner). *Let $W=I_2(k)$ be a dihedral group. Then*

$$\mathcal{H}(DR^{(1)}(W);q,t) = 1+[k+1]_{q,t}+qt+2\sum_{i=1}^{k-1}[i+1]_{q,t}.$$

Even more, they obtained an exact description of $DR^{(1)}(W)$:

(i) the first 1 belongs to the unique copy of the trivial representation,
(ii) the string

$$[k+1]_{q,t}+qt = q^k+q^{k-1}t+\ldots+qt^{k-1}+t^k+qt$$

belongs to copies of the sign representation which are generated by

$$D,\Delta(D),\ldots,\Delta^k(D) \text{ and } x_1y_2-x_2y_1,$$

where

$$D(x_1,x_2) := 2k\prod_{i=0}^{k-1}(\sin(\pi i/k)x_1+\cos(\pi i/k)x_2)$$

is the discriminant of $W$ and $\Delta$ is the operator defined as $\Delta := \partial_{x_1}\cdot y_1+\partial_{x_2}\cdot y_2$,
(iii) all other strings belong to copies of the permutation representation.

REMARK. The description of the alternating component was obtained by proving that Haiman's *operator conjecture* [**63**, Conjecture 5.1.1] holds for the dihedral groups.

By Theorem 4.10 and the discussion, we can immediately compute the $q,t$-Fuß-Catalan numbers for the dihedral groups:

COROLLARY 4.11. *Let $W = I_2(k)$. Then*

$$\begin{aligned}\mathrm{Cat}^{(1)}(W;q,t) &= [k+1]_{q,t} + qt \\ &= q^k + q^{k-1}t + \ldots + qt^{k-1} + t^k + qt.\end{aligned}$$

This corollary supports the suggested root poset of type $I_2(k)$ introduced in Section 1.1.13:

COROLLARY 4.12. *Let $W = I_2(k)$ be a dihedral group and let $\Phi^+$ be the* root poset *for $I_2(K)$ suggested by Armstrong, see Figure 6 on page 18. Then*

$$\mathrm{Cat}^{(1)}(W;q,1) = \sum_{I \trianglelefteq \Phi^+} q^{\mathrm{coh}(I)}.$$

From Theorem 4.10, one can also deduce the $q,t$-Fuß-Catalan numbers $\mathrm{Cat}^{(m)}(W;q,t)$.

THEOREM 4.13. *The $q,t$-Fuß-Catalan numbers for the dihedral group $I_2(k)$ are given by*

$$\mathrm{Cat}^{(m)}(I_2(k);q,t) = \sum_{j=0}^{m} q^{m-j}t^{m-j}[jk+1]_{q,t}.$$

To prove the theorem, we need the following lemma:

LEMMA 4.14. *Fix the monomial $o := x_1^a x_2^b$. Let $m$ be a positive integer and let $i_1,\ldots,i_m$, $j_1,\ldots,j_m$ be two sequences of non-negative integers such that $\sum i_l = \sum j_l$. Then*

$$\Delta^{i_1}(o)\cdots\Delta^{i_m}(o) \equiv \Delta^{j_1}(o)\cdots\Delta^{j_m}(o) \mod \langle I^{m-1}\cdot(x_1y_2 - x_2y_1)\rangle,$$

*where $I \trianglelefteq \mathbb{C}(\mathbf{x},\mathbf{y})$ is the ideal generated by $\{o, \Delta(o), \ldots, \Delta^{a+b}(o), x_1y_2 - x_2y_1\}$.*

PROOF. By definition, $\Delta^i(o)$ is given for any $i$ by

$$\Delta^i(o) = i!\sum_{l=0}^{i}\binom{a}{l}\binom{b}{i-l}x_1^{a-l}y_1^l x_2^{b-i+l}y_2^{i-l}.$$

This implies

$$\Delta^i(o) \equiv \Delta^i_{\mathrm{mod}}(o) := \left[i!\cdot\sum_{l=0}^{i}\binom{a}{l}\binom{b}{i-l}\right]x_1^a x_2^{b-i}y_2^i \mod \langle x_1y_2 - x_2y_1\rangle, \tag{11}$$

and thereby

$$\Delta^{i_1}(o)\cdots\Delta^{i_m}(o) \equiv \Delta^{i_1}(o)\cdots\Delta^{i_{m-1}}(o)\Delta^{i_m}_{\mathrm{mod}}(o) \mod \langle I^{m-1}\cdot(x_1y_2 - x_2y_1)\rangle.$$

With $\Delta^i(o)$ and $x_1y_2 - x_2y_1$, $\Delta^i_{\mathrm{mod}}(o)$ is also contained in $I$ and therefore,

$$\Delta^{i_1}(o)\cdots\Delta^{i_m}(o) \equiv \Delta^{i_1}_{\mathrm{mod}}(o)\cdots\Delta^{i_m}_{\mathrm{mod}}(o) \mod \langle I^{m-1}\cdot(x_1y_2 - x_2y_1)\rangle. \tag{12}$$

If follows from Eq. (11) that the right-hand side of Eq. (12) does *not* depend on the specific $i_l$'s but only on their sum $\sum i_l$. The lemma follows. □

PROOF OF THEOREM 4.13. By definition, the ideal $\mathcal{A}^m \trianglelefteq \mathbb{C}[\mathbf{x},\mathbf{y}]$ is generated by all products of $m$ generators of the different sign representation. As seen above, these generators are given by $D, \Delta(D), \ldots, \Delta^k(D)$ and $x_1y_2 - x_2y_1$.
For a given $0 \le j \le m$, the previous lemma implies that any minimal generating set

of $I^m$ contains one and only one generator for each $0 \leq i \leq k(m-j)$ where the factor $(x_1y_2 - x_2y_1)$ appears exactly $j$ times. It is of the form

$$\Delta^{i_1}(D)\cdots\Delta^{i_{m-j}}(D)(x_1y_2 - x_2y_1)^j,$$

where the $i_l$'s sum up to $i$. □

This theorem immediately implies the following recurrence relation:

Corollary 4.15. *The $q,t$-Fuß-Catalan numbers for the dihedral group $I_2(k)$ satisfy the recurrence relation*

$$\mathrm{Cat}^{(m)}(W;q,t) = [mk+1]_{q,t} + qt\,\mathrm{Cat}^{(m-1)}(W;q,t).$$

We can also deduce Conjecture 4.5 and Conjecture 4.7 for the dihedral groups:

Corollary 4.16. *Conjectures 4.5 and 4.7 hold for the dihedral groups: let $W = I_2(k)$, then*

$$q^{mk}\,\mathrm{Cat}^{(m)}(W;q,q^{-1}) = \frac{[2+mk]_q[k+mk]_q}{[2]_q[k]_q}.$$

Proof. By Theorem 4.13, we have

$$q^{mk}\,\mathrm{Cat}^{(m)}(W;q,q^{-1}) \;=\; q^{mk}\sum_{j=0}^{m}[jk+1]_{q,q^{-1}} = \sum_{j=0}^{m} q^{(m-j)k}\frac{[2jk+2]_q}{[2]_q}.$$

Therefore, it remains to show that

$$\sum_{j=0}^{m} q^{(m-j)k}[2jk+2]_q \;=\; \frac{[2+mk]_q[k+mk]_q}{[k]_q}. \tag{13}$$

To prove this equality observe that on the left-hand side of (13), the terms for $j$ and $m-j$ sum up to $[mk+2]_q(q^{jk}+q^{(m-j)k})$. This gives

$$\sum_{j=0}^{m} q^{(m-j)k}[2jk+2]_q \;=\; [mk+2]_q(1+q^k+\ldots+q^{mk}),$$

the corollary follows. □

## CHAPTER 5

# Connections to rational Cherednik algebras

In this chapter, we want to investigate (conjectured) connections between the extended coinvariant ring $DR^{(m)}(W)$ and a module that naturally arises in the context of rational Cherednik algebras. These algebras were introduced by Etingof and Ginzburg in [**42**]. The notion is adapted from this reference as well as from a paper by Berest, Etingof and Ginzburg [**15**] from which most of the facts about rational Cherednik algebras are taken.

The work by Berest, Etingof and Ginzburg deals only with real reflection groups but in [**59**], S. Griffeth partially generalized the work to complex reflection groups. We expect that the conjectured connection of the $q,t$-Fuß-Catalan numbers, see Corollary 5.10, should also hold in this generalized context.

In this chapter, we fix $W$ to be a real reflection group acting on a complex vector space $\mathfrak{h}$ (which is by construction the complexification of a real vector space) and fix $T \subseteq W$ to be the set of reflections in $W$.

### 5.1. The rational Cherednik algebra

In [**42**], Etingof and Ginzburg defined the rational Cherednik algebra as follows:

Definition 5.1. Let $c : T \to \mathbb{C}, t \mapsto c_t$ be a $W$-invariant function on the set of reflections. The rational Cherednik algebra $\mathsf{H}_c = \mathsf{H}_c(W)$ is the associative algebra generated by the vector spaces $\mathfrak{h}, \mathfrak{h}^*$ and the set $W$ subject to the defining relations

$$\begin{aligned} \omega x \omega^{-1} = \omega(x), \omega y \omega^{-1} = \omega(y) &\quad \text{for all} \quad y \in \mathfrak{h}, x \in \mathfrak{h}^*, \omega \in W, \\ x_1 x_2 = x_2 x_1, y_1 y_2 = y_2 y_1 &\quad \text{for all} \quad y_1, y_2 \in \mathfrak{h}, x_1, x_2 \in \mathfrak{h}^*, \\ yx - xy = \langle y, x\rangle - \sum_{t \in T} c_t \langle y, \alpha_t\rangle \langle \alpha_t^\vee, x\rangle t &\quad \text{for all} \quad y \in \mathfrak{h}, x \in \mathfrak{h}^*. \end{aligned}$$

The polynomial ring $\mathbb{C}[\mathfrak{h}]$ sits inside $\mathsf{H}_c$ as the subalgebra generated by $\mathfrak{h}^*$ and the polynomial ring $\mathbb{C}[\mathfrak{h}^*]$ as the subalgebra generated by $\mathfrak{h}$. Furthermore, the elements in $W$ span a copy of the group algebra $\mathbb{C}[W]$ sitting naturally inside $\mathsf{H}_c$.

By [**42**, Theorem 1.3], there is a *Poincaré-Birkhoff-Witt isomorphism*

$$\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}^*] \xrightarrow{\sim} \mathsf{H}_c.$$

**5.1.1. An induced grading.** Define the element $\mathbf{h} \in \mathsf{H}_c$ by

$$\mathbf{h} := \frac{1}{2} \sum_i (x_i y_i + y_i x_i),$$

where $\{x_i\}$ and $\{y_i\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. It is easy to see that

$$\mathbf{h} \cdot x = x, \mathbf{h} \cdot y = -y \text{ and } \mathbf{h} \cdot \omega = 0 \text{ for } x \in \mathfrak{h}^*, y \in \mathfrak{h}, \omega \in W.$$

This gives an induced grading on $\mathsf{H}_c$ by

$$\deg x = 1, \deg y = -1, \deg \omega = 0.$$

**5.1.2. A natural filtration.** Beside the discussed grading, there exists a natural filtration on $\mathsf{H}_c$ which is given by

$$\deg x = \deg y = 1, \deg \omega = 0 \text{ for } x \in \mathfrak{h}^*, y \in \mathfrak{h}, \omega \in W.$$

By the Poincaré-Birkhoff-Witt isomorphism, the associated graded module $\mathrm{gr}(\mathsf{H}_c)$ is given by

$$\mathrm{gr}(\mathsf{H}_c) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W. \tag{14}$$

For a definition of filtrations as well as for a definition associated graded modules, we refer to in [**41**, Chapter 5].

## 5.2. A module over the rational Cherednik algebra

Define for any $\mathsf{H}_c$-representation $\tau$ an $\mathsf{H}_c$-module $M(\tau)$, called standard module, to be the induced module

$$M_c(\tau) := \mathsf{H}_c \otimes_{\mathbb{C}[\mathfrak{h}] \rtimes W} \tau,$$

where $\mathbb{C}[\mathfrak{h}] \rtimes W$ acts on $\tau$ by $p\omega \cdot a := p(0)(\omega(a))$ for $p \in \mathbb{C}[\mathfrak{h}], \omega \in W$ and $a \in \tau$.

For our purposes it is enough to restrict to the case where $\tau$ is the trivial representation and where the parameter $c = c_t$ is a rational constant. Denote this module by $M := M_c(\mathbb{C})$. Many nice properties of $M_c$ even hold in the more general context of $\tau$ being any $\mathsf{H}_c$-representation and for arbitrary parameter $c$.

C.F. Dunkl and E. Opdam showed in [**38**] that $M$ has a unique simple quotient, which we denote by $L = L_c(\mathbb{C})$. In [**15**], Berest, Etingof and Ginzburg investigated this module $L$ and showed in [**15**, Theorem 1.4] for $c = \frac{1}{h} + m$ (where $h$ is the Coxeter number of $W$) that $L$ is the only finite dimensional irreducible $\mathsf{H}_c$-module. They also showed that $L$ is graded and they computed the graded Hilbert series of $L$. See [**15**, Section 1.7] for the property of being graded and [**15**, Theorem 1.6] for the Hilbert series:

Theorem 5.2 (Berest, Etingof, Ginzburg). *Let $c = \frac{1}{h} + m$ as before. Then*

$$\mathcal{H}(L; q) = q^{-mN}([mh+1]_q)^l,$$

*where $l$ is the rank of $W$ and $N$ is the number of positive roots. In particular,*

$$\dim(L) = (mh+1)^l.$$

Consider the graded module associated to $L$: since $\mathbf{h}$ induces a grading on both $L$ and on $\mathrm{gr}(L)$, we see that the Hilbert series of $L$ and $\mathrm{gr}(L)$ with respect to this grading coincide.

Corollary 5.3.

$$\mathcal{H}(\mathrm{gr}(L); q) = q^{-mN}([mh+1]_q)^l.$$

## 5.3. The spherical subalgebra

Define the spherical subalgebra of $\mathsf{H}_c$ as $\mathbf{e}\mathsf{H}_c\mathbf{e} \subseteq \mathsf{H}_c$ where $\mathbf{e}$ is the trivial idempotent defined in Section 1.3.2. For any $\mathsf{H}_c$-module $V$, $\mathbf{e}V$ has a natural $\mathbf{e}\mathsf{H}_c\mathbf{e}$-module structure and the element $\mathbf{h}$ preserves $\mathbf{e}V \subseteq V$, which implies that $\mathbf{h}$ also induces a grading on $\mathbf{e}V$. For the following theorem see [**15**, Theorem 1.10]:

THEOREM 5.4 (Berest, Etingof, Ginzburg). *$\mathbf{e}L$ is the only finite dimensional simple $\mathbf{e}\mathsf{H}_c\mathbf{e}$-module for the parameter $c = \frac{1}{h} + m$. The Hilbert series of $\mathbf{e}L$ is given by*

$$\mathcal{H}(\mathbf{e}L; q) = q^{-mN} \prod_{i=1}^{l} \frac{[d_i + mh]_q}{[d_i]_q},$$

*where $l$ is the rank of $W$, $d_1 \leq \ldots \leq d_l$ are the degrees and $h$ is the Coxeter number.*

The same argument as above implies

COROLLARY 5.5. *The Hilbert series of the associated graded module $\mathbf{e}(\mathrm{gr}(L))$ is also given by*

$$\mathcal{H}(\mathbf{e}(\mathrm{gr}(L)); q) = q^{-mN} \prod_{i=1}^{l} \frac{[d_i + mh]_q}{[d_i]_q}.$$

REMARK. Berest, Etingof and Ginzburg proved Theorem 5.2 and Theorem 5.4 by constructing a certain homogeneous system of parameters (hsop) inside the $p$-th component of $S = \mathrm{Sym}(V^*)$ such that its span is $W$-stable and carries a $W$-representation isomorphic to $V^*$. In [**22**, Section 4], Bessis and Reiner described how such a construction would imply both theorems for arbitrary complex reflection groups and conjectured in [**22**, Conjecture 4.3] that such hsop exist at least for well-generated groups. Furthermore, they described how the positive Fuß-Catalan numbers $\mathrm{Cat}_+(W)$ come into play in this algebraic context and they constructed a hsop for any group $G(m, p, n)$. This implies Theorems 5.2 and 5.4 in this slightly more general context.

**5.3.1. A decomposition of $L$ and $\mathrm{gr}(L)$.** There exist nice and important decompositions of the modules in question, which were explored in [**15**].

Set $\mathsf{H}^{(m)} := \mathsf{H}_c$ for $c = \frac{1}{h} + m$. Then there exists a filtered algebra isomorphism

$$\mathbf{e}\mathsf{H}^{(m)}\mathbf{e} \xrightarrow{\sim} \mathbf{e}_\epsilon \mathsf{H}^{(m+1)} \mathbf{e}_\epsilon,$$

see [**15**, Proposition 4.6]. Applying this isomorphism iteratively gives the following decomposition of the $\mathsf{H}^{(m)}$-module $L$,

$$L = \mathsf{H}^{(m)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(m-1)}\mathbf{e}} \mathbf{e}\mathsf{H}^{(m-1)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(m-2)}\mathbf{e}} \cdots \otimes_{\mathbf{e}\mathsf{H}^{(1)}\mathbf{e}} \mathbf{e}\mathsf{H}^{(1)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(0)}\mathbf{e}} \mathbb{C}, \tag{15}$$

and also a decomposition of the $\mathbf{e}\mathsf{H}^{(m)}\mathbf{e}$-module $\mathbf{e}L$ as

$$\mathbf{e}L = \mathbf{e}\mathsf{H}^{(m)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(m-1)}\mathbf{e}} \mathbf{e}\mathsf{H}^{(m-1)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(m-2)}\mathbf{e}} \cdots \otimes_{\mathbf{e}\mathsf{H}^{(1)}\mathbf{e}} \mathbf{e}\mathsf{H}^{(1)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(0)}\mathbf{e}} \mathbb{C}. \tag{16}$$

**5.3.2. A bigrading on $\mathrm{gr}(L)$.** We have already seen that $\mathrm{gr}(L)$ carries the same grading as $L$ which is induced by $\mathbf{h}$. On the other hand, it is by construction equipped with the grading induced by the filtration $\deg x = \deg y = 1$ and $\deg \omega = 0$. This allows to define a bigrading on $\mathrm{gr}(L)$, see [**15**, Formula 7.9]:

$$\mathrm{gr}(L) = \oplus_{p,q \in \mathbb{Z}} L_{p,q}, \text{ where } \mathbf{h}|_{L_{p,q}} = (p - q)\mathsf{Id}_{L_{p,q}}.$$

Similar considerations and definitions apply to the $\mathbf{e}\mathsf{H}_c\mathbf{e}$-module $\mathbf{e}L$.

**5.3.3.** $\mathrm{gr}(L)$ **and the extended coinvariant ring.** As indicated in Theorem 4.4, Gordon connected the coinvariant ring with the $\mathsf{H}^{(0)}$-module $L$. Using the above decompositions, we prove his theorem in a slightly more general context using the same argument. The following well-known lemma is taken from [**58**, Lemma 6.7 (2)]:

LEMMA 5.6. *Let $R = \bigcup F^i R$ be a filtered $\mathbb{C}$-algebra, $A = \bigcup F^i A$ a filtered right $R$-module and $B = \bigcup F^i B$ a filtered left $R$-module. Filter $A \otimes_R B$ by the* tensor product filtration

$$F^i(A \otimes B) = \sum_j F^j A \otimes F^{n-j} B.$$

*Then there is a natural surjection*

$$\mathrm{gr}_F A \otimes_{\mathrm{gr}(R)} \mathrm{gr}_F B \twoheadrightarrow \mathrm{gr}_F(A \otimes_R B).$$

THEOREM 5.7. *Let $W$ be a real reflection group, let $DR^{(m)}(W)$ be the generalized coinvariant ring and let $\mathrm{gr}(L)$ be bigraded as described in Section 5.3.2. Then there exists a natural surjection of bigraded $W$-modules,*

$$DR^{(m)}(W) \otimes \epsilon \twoheadrightarrow \mathrm{gr}(L).$$

PROOF. Consider the decomposition (15),

$$L = \mathsf{H}^{(m)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(m-1)}\mathbf{e}} \mathbf{e}\mathsf{H}^{(m-1)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(m-2)}\mathbf{e}} \cdots \otimes_{\mathbf{e}\mathsf{H}^{(1)}\mathbf{e}} \mathbf{e}\mathsf{H}^{(1)}\mathbf{e}_\epsilon \otimes_{\mathbf{e}\mathsf{H}^{(0)}\mathbf{e}} \mathbb{C}.$$

The iterative application of the tensor product filtration to the right hand side defines a filtration on $L$. To see that the associated graded module equals $\mathrm{gr}(L)$, observe that for both, the $n$-th degree consists of these elements of which the total degree equals $n$. By Lemma 5.6 and the identities

$$\mathrm{gr}(\mathbf{e}\mathsf{H}_c\mathbf{e}) \cong \mathbf{e}\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \quad , \quad \mathrm{gr}(\mathbf{e}\mathsf{H}_c\mathbf{e}_\epsilon) \cong \mathbf{e}_\epsilon\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*],$$

which can be found in [**15**, Eq. (7.8)], we have the surjection

$$(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \otimes \epsilon) \otimes_{\mathbf{e}\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]} \mathbf{e}_\epsilon\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \otimes_{\mathbf{e}\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]} \cdots \otimes_{\mathbf{e}\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]} \mathbf{e}_\epsilon\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \otimes_{\mathbf{e}\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]} \mathbb{C} \twoheadrightarrow \mathrm{gr}(L).$$

As the left-hand side equals $DR^{(m)}(W) \otimes \epsilon$, the theorem follows.

□

REMARK. Work of Griffeth [**59**] suggests that Theorem 5.7 holds also for the complex reflection group $G(m, p, n)$ defined in Theorem 1.34.

Theorem 5.7 partially generalizes Theorem 4.4:

COROLLARY 5.8. *Let $W$ be the real reflection group. There exists a graded $W$-stable quotient ring $R_W$ of $DR^{(m)}(W)$ such that*

(i) $\dim(R_W) = (mh + 1)^l$ *and*
(ii) $q^{mN}\mathcal{H}(R_W; q) = [mh + 1]_q^l$.

For real reflection groups, Theorem 5.4 and Theorem 5.7 imply that Conjecture 4.5 and Conjecture 4.7 would follow from (and are in fact equivalent to) the following conjecture:

CONJECTURE 5.9. The kernel of the surjection defined in Theorem 5.7 does not contain a copy of the trivial representation.

This conjecture would show that the module $M^{(m)}$, and thereby the $q,t$-Fuß-Catalan numbers $\mathrm{Cat}^{(m)}(W;q,t)$, can be described in terms of the module $L$ over the rational Cherednik algebra:

Corollary 5.10. *Let $W$ be a real reflection group, let $M^{(m)}$ be the bigraded $W$-module defined in Section 4.2 and let $\mathrm{gr}(L)$ be the bigraded $W$-module defined in Section 5.3.2. If Conjecture 5.9 holds, then*

$$M^{(m)} \otimes \epsilon \cong \mathbf{e}(\mathrm{gr}(L))$$

*as bigraded $W$-modules.*

APPENDIX A

# Computations with Singular, Macaulay2 and GAP

## A.1. Computations of $\dim M^{(m)}$

To compute the dimensions of the module $M^{(m)}$ defined in Section 4.2, we used the computer algebra system `Singuar` [**92**].

TABLE 1. $\dim M^{(m)}$ for the classical types $A$, $B$ and $D$:

| $m$ | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n=1$ | 1 | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 1 | 1 | 1 | 1 |
| $n=2$ | 2 | 3 | 4 | 5 | 6 | 15 | 28 | 45 | 4 | 9 | 16 | 25 |
| $n=3$ | 5 | 12 | 22 | 35 | 20 | 84 | | | 14 | 55 | 140 | 285 |
| $n=4$ | 14 | 55 | 140 | | 70 | 495 | | | 50 | 336 | | |
| $n=5$ | 42 | | | | | | | | | | | |

## A.2. Computations of $\mathrm{Cat}^{(m)}(W,q,t)$

For the computations of the bigraded Hilbert series of $M^{(m)}$ we used the computer algebra system `Macaulay 2` [**78**]. We write $[n]$ for $[n]_{qt}$.

TABLE 2. $\mathrm{Cat}^{(m)}(W,q,t)$ for $W=W(A_{n-1})$:

| | |
|---|---|
| $n=2, m=1$ | $[2]$ |
| $m=2$ | $[3]$ |
| $m=3$ | $[4]$ |
| $n=3, m=1$ | $[4]+qt[1]$ |
| $m=2$ | $[7]+qt[4]+q^2t^2[1]$ |
| $m=3$ | $[10]+qt[7]+q^2t^2[4]+q^3t^3[1]$ |
| $n=4, m=1$ | $[7]+qt[4]+qt[3]$ |
| $m=2$ | $[13]+qt[10]+qt[9]+q^2t^2[7]+q^2t^2[6]+q^2t^2[5]+q^3t^3[4]+q^4t^4[1]$ |
| $m=3$ | $[19]+qt[16]+qt[15]+q^2t^2[13]+q^2t^2[12]+q^3t^3[10]+q^2t^2[11]+$ $q^3t^3[9]+q^4t^4[7]+q^3t^3[8]+q^4t^4[6]+q^5t^5[4]+q^3t^3[7]+q^5t^5[3]$ |

TABLE 3. $\mathrm{Cat}^{(m)}(W,q,t)$ for $W=W(B_n)$:

| | |
|---|---|
| $n=1, m=1$ | $[2]$ |
| $m=2$ | $[3]$ |
| $m=3$ | $[4]$ |
| $n=2, m=1$ | $[5]+qt[1]$ |
| $m=2$ | $[9]+qt[5]+q^2t^2[1]$ |
| $m=3$ | $[13]+qt[9]+q^2t^2[5]+q^3t^3[1]$ |
| $n=3, m=1$ | $[10]+qt[6]+qt[4]$ |
| $m=2$ | $[19]+qt[15]+qt[13]+q^2t^2[11]+q^2t^2[9]+q^3t^3[7]+q^2t^2[7]+q^4t^4[3]$ |
| $m=3$ | $[28]+qt[24]+qt[22]+q^2t^2[20]+q^2t^2[18]+q^3t^3[16]+$ $q^2t^2[16]+q^3t^3[14]+q^4t^4[12]+q^3t^3[12]+q^4t^4[10]+$ $q^5t^5[8]+q^3t^3[10]+q^5t^5[6]+q^6t^6[4]$ |
| $n=4, m=1$ | $[17]+qt[13]+qt[11]+q^2t^2[9]+qt[9]+q^3t^3[5]+q^2t^2[5]+q^4t^4[1]$ |
| $m=2$ | $[33]+qt[29]+qt[27]+q^2t^2[25]+qt[25]+q^2t^2[23]+q^3t^3[21]+$ $2q^2t^2[21]+q^3t^2[19]+q^4t^4[17]+q^2t^2[19]+2q^3t^3[17]+q^4t^4[15]+$ $q^5t^5[13]+q^2t^2[17]+q^3t^3[15]+2q^4t^4[13]+q^5t^5[11]+q^6t^6[9]+$ $q^3t^3[13]+q^4t^4[11]+2q^5t^5[9]+q^6t^6[7]+q^7t^7[5]+q^9t^9[1]+$ $q^4t^4[9]+2q^6t^6[5]+q^8t^8[1]$ |

TABLE 4. $\mathrm{Cat}^{(m)}(W,q,t)$ for $W=W(D_n)$:

| | |
|---|---|
| $n=2, m=1$ | $[3]+qt[1]$ |
| $m=2$ | $[5]+qt[3]+q^2t^2[1]$ |
| $m=3$ | $[7]+qt[5]+q^2t^2[3]+q^3t^3[1]$ |
| $n=3, m=1$ | $[7]+qt[4]+qt[3]$ |
| $m=2$ | $[13]+qt[10]+qt[9]+q^2t^2[7]+q^2t^2[6]+q^2t^2[5]+q^3t^3[4]+q^4t^4[1]$ |
| $m=3$ | $[19]+qt[16]+qt[15]+q^2t^2[13]+q^2t^2[12]+q^3t^3[10]+q^2t^2[11]+$ $q^3t^3[9]+q^4t^4[7]+q^3t^3[8]+q^4t^4[6]+q^5t^5[4]+q^3t^3[7]+q^5t^5[3]$ |
| $n=4, m=1$ | $[13]+2qt[9]+qt[7]+2q^2t^2[5]+q^4t^4[1]+q^3t^3[1]$ |
| $m=2$ | $[25]+2qt[21]+qt[19]+3q^2t^2[17]+2q^2t^2[15]+$ $4q^3t^3[13]+q^2t^2[13]+2q^3t^3[11]+5q^4t^4[9]+$ $q^5t^5[7]+2q^6t^6[5]+q^8t^8[1]+q^4t^4[7]+$ $2q^5t^5[5]+q^7t^7[1]+q^6t^6[1]$ |

TABLE 5. $\mathrm{Cat}^{(m)}(W,q,t)$ for some exceptional groups:

| | | |
|---|---|---|
| $I_2(k), k\in\{6,10,12\}$ : | $m=1$ | $[k+1]+qt[1]$ |
| | $2\le m\le 4$ | $[km+1]+qt\,\mathrm{Cat}^{(m-1)}(W,q,t)$ |
| $H_3$ : | $m=1$ | $[16]+qt[10]+qt[6]$ – insecure |
| $G(m,1,1)$ : | $m\le 10$ | $\mathrm{Cat}^{(m)}(W_{A_1},q,t)$ |
| $G(4,2,2)$ : | $m\le 4$ | $\mathrm{Cat}^{(m)}(W_{G_2},q,t)$ |

## A.3. Computations of $\mathrm{Cat}^{(m)}(W, q)$

For the computations of the the $q$-Fuß-Catalan numbers defined in Section 2.4, we used the programming language `GAP` [**52**]. For simplicity, we list only the coefficients of the polynomials, e.g. $1 + 2q + q^2 + q^3 + q^4$ is listed as $1, 2, 1, 1, 1$.

TABLE 6. List of the coefficients of $\mathrm{Cat}^{(m)}(W_{B_n}, q)$:

| | |
|---|---|
| $n = 2, m = 1$ | 1, 2, 1, 1, 1 |
| $m = 2$ | 1, 2, 3, 2, 2, 2, 1, 1, 1 |
| $m = 3$ | 1, 2, 3, 4, 3, 3, 3, 2, 2, 2, 1, 1, 1 |
| $n = 3, m = 1$ | 1, 3, 3, 3, 3, 2, 2, 1, 1, 1 |
| $m = 2$ | 1, 3, 6, 7, 8, 8, 8, 7, 7, 6, 5, 4, 4, 3, 2, 2, 1, 1, 1 |
| $m = 3$ | 1, 3, 6, 10, 12, 14, 15, 15, 15, 15, 14, 13, 13, 11, 10, 9, 8, 7, 6, 5, 4, 4, 3, 2, 2, 1, 1, 1 |
| $n = 4, m = 1$ | 1, 4, 6, 7, 8, 7, 7, 6, 5, 5, 4, 3, 2, 2, 1, 1, 1 |
| $m = 2$ | 1, 4, 10, 16, 22, 26, 30, 31, 32, 32, 31, 29, 28, 26, 24, 22, 20, 18, 16, 14, 12, 10, 9, 7, 6, 5, 4, 3, 2, 2, 1, 1, 1 |
| $m = 3$ | 1, 4, 10, 20, 31, 43, 54, 63, 71, 77, 81, 83, 85, 84, 83, 81, 78, 75, 72, 68, 64, 61, 57, 53, 49, 45, 41, 38, 34, 30, 27, 24, 21, 19, 16, 14, 12, 10, 9, 7, 6, 5, 4, 3, 2, 2, 1, 1, 1 |
| $n = 5, m = 1$ | 1, 5, 10, 14, 18, 19, 20, 20, 18, 18, 17, 15, 13, 12, 10, 9, 8, 6, 5, 4, 3, 2, 2, 1, 1, 1 |
| $m = 2$ | 1, 5, 15, 30, 49, 68, 88, 104, 118, 128, 136, 139, 141, 140, 139, 134, 131, 126, 120, 114, 108, 100, 94, 87, 80, 74, 68, 61, 55, 50, 44, 39, 35, 30, 26, 23, 19, 16, 14, 11, 10, 8, 6, 5, 4, 3, 2, 2, 1, 1, 1 |
| $m = 3$ | 1, 5, 15, 35, 65, 105, 152, 202, 254, 304, 351, 392, 429, 458, 482, 500, 511, 518, 521, 519, 514, 508, 498, 488, 476, 461, 446, 431, 413, 396, 378, 359, 341, 324, 305, 288, 271, 253, 237, 221, 205, 190, 176, 161, 148, 136, 123, 112, 102, 91, 82, 74, 65, 58, 52, 45, 40, 35, 30, 26, 23, 19, 16, 14, 11, 10, 8, 6, 5, 4, 3, 2, 2, 1, 1, 1 |

TABLE 7. List of the coefficients of $\mathrm{Cat}^{(m)}(W_{D_n}, q)$:

| | |
|---|---|
| $n = 4, m = 1$ | 1, 4, 6, 7, 7, 6, 6, 4, 3, 3, 1, 1, 1 |
| $m = 2$ | 1, 4, 10, 16, 22, 25, 28, 28, 28, 27, 25, 22, 21, 16, 14, 13, 9, 7, 7, 4, 3, 3, 1, 1, 1 |
| $m = 3$ | 1, 4, 10, 20, 31, 43, 53, 61, 67, 71, 73, 73, 73, 70, 67, 64, 58, 53, 50, 42, 37, 35, 28, 24, 22, 17, 14, 13, 9, 7, 7, 4, 3, 3, 1, 1, 1 |
| $n = 5, m = 1$ | 1, 5, 10, 14, 17, 17, 18, 17, 15, 14, 12, 10, 8, 7, 5, 4, 3, 2, 1, 1, 1 |
| $m = 2$ | 1, 5, 15, 30, 49, 67, 85, 98, 109, 115, 120, 120, 121, 116, 112, 106, 100, 91, 85, 76, 69, 61, 54, 47, 41, 35, 30, 25, 21, 17, 14, 11, 9, 7, 5, 4, 3, 2, 1, 1, 1 |
| $m = 3$ | 1, 5, 15, 35, 65, 105, 151, 199, 247, 291, 331, 363, 391, 410, 425, 435, 439, 438, 436, 426, 414, 402, 385, 367, 349, 329, 308, 289, 267, 247, 228, 208, 189, 173, 155, 140, 125, 111, 98, 87, 76, 66, 57, 49, 42, 36, 30, 25, 21, 17, 14, 11, 9, 7, 5, 4, 3, 2, 1, 1, 1 |

TABLE 8. List of the coefficients of $\mathrm{Cat}^{(m)}(W_{G_2}, q)$:

| | |
|---|---|
| $m = 1$ | 1, 2, 1, 1, 1, 1, 1 |
| $m = 2$ | 1, 2, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1 |
| $m = 3$ | 1, 2, 3, 4, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1 |
| $m = 4$ | 1, 2, 3, 4, 5, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1 |

TABLE 9. List of the coefficients of $\mathrm{Cat}^{(m)}(W_{F_4}, q)$:

| | |
|---|---|
| $m = 1$ | 1, 4, 6, 7, 8, 7, 8, 7, 7, 7, 6, 5, 5, 5, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1 |
| $m = 2$ | 1, 4, 10, 16, 22, 26, 30, 31, 33, 33, 34, 33, 34, 33, 32, 31, 30, 29, 28 25, |
| | 24, 23, 22, 20, 18, 17, 17, 15, 13, 12, 11, 10, 9, 8, 7, 6, 6, 5, 4, 3, 3, |
| | 3, 2, 2, 1, 1, 1, 1, 1 |
| $m = 3$ | 1, 4, 10, 20, 31, 43, 54, 63, 71, 77, 82, 84, 88, 88, 90, 90, 91, 90, 90, 88, |
| | 87, 86, 84, 80, 79, 76, 74, 71, 67, 64, 63, 60, 56, 53, 51, 49, 46, 43, 40, |
| | 39, 36, 33, 31, 29, 27, 25, 23, 21, 19, 18, 17, 15, 13, 12, 11, 10, 9, 8, 7, |
| | 6, 6, 5, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1 |

TABLE 10. List of the coefficients of $\mathrm{Cat}^{(m)}(W_{E_6}, q)$:

| | |
|---|---|
| $m = 1$ | 1, 6, 15, 25, 35, 41, 46, 50, 50, 51, 50, 49, 45, 44, 40, 38, 34, |
| | 31, 27, 25, 22, 19, 16, 14, 12, 10, 8, 7, 6, 4, 3, 3, 2, 1, 1, 1, 1 |
| $m = 2$ | 1, 6, 21, 50, 95, 151, 216, 281, 345, 400, 451, 491, 528, 553, 575, |
| | 587, 598, 598, 599, 590, 583, 568, 555, 535, 518, 496, 477, 452, 431, |
| | 406, 385, 360, 339, 315, 294, 272, 253, 232, 213, 195, 180, 162, 147, |
| | 133, 121, 107, 96, 86, 77, 67, 59, 52, 46, 39, 33, 29, 26, 21, 17, 15, |
| | 13, 10, 8, 7, 6, 4, 3, 3, 2, 1, 1, 1, 1 |

TABLE 11. List of the coefficients of $\mathrm{Cat}^{(m)}(W_{E_7}, q)$:

| | |
|---|---|
| $m = 1$ | 1, 7, 21, 41, 65, 87, 106, 124, 135, 145, 152, 157, 159, 160, 159, |
| | 158, 156, 151, 147, 142, 138, 132, 126, 119, 114, 108, 101, 95, 90, |
| | 83, 77, 72, 66, 61, 56, 51, 47, 43, 38, 34, 31, 28, 25, 22, 19, 17, |
| | 15, 13, 11, 10, 8, 7, 6, 5, 4, 3, 3, 2, 2, 1, 1, 1, 1, 1 |

TABLE 12. List of the coefficients of $\mathrm{Cat}^{(m)}(W_{E_8}, q)$:

| | |
|---|---|
| $m = 1$ | 1, 8, 28, 63, 112, 168, 224, 281, 330, 373, 411, 442, 469, 489, 507, 520, |
| | 534, 539, 546, 547, 551, 550, 550, 544, 542, 536, 531, 523, 518, 508, 499, |
| | 490, 481, 470, 460, 449, 439, 428, 416, 404, 393, 382, 370, 359, 347, 335, |
| | 324, 313, 301, 291, 279, 268, 258, 248, 237, 227, 217, 207, 198, 189, 179, |
| | 171, 163, 155, 147, 139, 131, 124, 118, 111, 104, 98, 92, 87, 82, 76, 71, |
| | 66, 62, 58, 54, 50, 46, 43, 40, 37, 34, 31, 28, 26, 24, 22, 20, 18, 17, 15, |
| | 14, 12, 11, 10, 9, 8, 7, 6, 5, 5, 5, 4, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1 |

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# Curriculum Vitae

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|---|---|
| **Date of birth** | October 1, 1980 |
| **Academic education** | |
| **Oct 2006 – Sep 2008** | PhD studies at the University of Vienna. |
| **Mar 2008** | Participation in the theme semester<br>"Combinatorial Representation Theory"<br>at the Mathematics Sciences Research Institute in Berkeley, USA. |
| **Sep 2007** | Participation in the workshop<br>"Applications of Macdonald Polynomials"<br>at the Banff International Research Station in Banff, Canada. |
| **Apr – Jun 2007** | Participation in the theme semester<br>"Recent Advances in Combinatorics"<br>at the Centre de Recherches Mathématiques of the Université de Montréal in Montreal, Canada. |
| **Apr – Jun 2006** | Participation in the predoc course<br>"Optimization Methods in Discrete Geometry"<br>at the Berlin Institute of Technology organized by Prof. Dr. Günter Ziegler und Prof. Dr. Thorsten Theobald. |
| **Oct 2000 – Apr 2005** | Studies at the Faculty of Mathematics at Philipps-University Marburg, Germany. Diploma advisor was Prof. Dr. Volkmar Welker, the title of the diploma thesis was<br>"Polytopes and Hilbert series".<br>Diploma (M.A. equivalent) in April 2005 (with honors). |
| **Employment history** | |
| **Since Jul 2008** | FWF research assistant in the FWF project S-9600<br>"Analytic Combinatorics and Probabilistic Number Theory"<br>supervised by Prof. Dr. Christian Krattenthaler. |
| **Jul 2006 – Jun 2008** | FWF research assistant in the FWF project P17563<br>"Macdonald polynomials and $q$-hypergeometric series"<br>supervised by Dr. Michael Schlosser. |
| **Oct 2003 – Sep 2004** | Teaching assistant at the Faculty of Mathematics, Philipps-University Marburg. |
| **Mar 2002 – Sep 2002** | Position at the Hospital of the Philipps-University Marburg as a software engineer. |