universität wien

DISSERTATION

Contents

Overview

The **main objective of this Ph.D. Thesis** is to study the occurring *phenomena in embedded resolution of singularities over algebraically closed fields of positive characteristic*. Within this investigation two goals were strived for:

(A) A new, systematic *approach to embedded surface resolution in positive characteristic*, which is so natural that it has at least the chance to be generalized to higher dimensions, should be given. Within this context, the behavior of resolution invariants from characteristic zero under blowups in positive characteristic should be explored. This knowledge should then be used, to modify the measures in order to get strictly decreasing resolution invariants for surface resolution in positive characteristic. Moreover the termination of the resolution algorithm in finitely many blowups should be proven.

(B) New ideas and attempts for the problem of resolution of *varieties with dimension larger than two*, especially threefolds, should be given. It was clear from the beginning, that a complete answer to this question would be too ambitious for a Ph.D. Thesis since many experts in the field have been trying to solve this problem for a long period of time. One approach which should be examined within this thesis is to extend the results from (A) to the case of threefolds. Further, those situations where the classical resolution invariant from characteristic zero increases when used in positive characteristic should be studied. Moreover new possible resolution invariants – both for characteristic zero and positive characteristic – should be investigated.

This Ph.D. Thesis is divided into the following chapters:

Chapter 1 – Surface resolution in positive characteristic

This chapter forms the mathematical core of this Ph.D. Thesis. It presents a new and systematic approach to embedded resolution of surface singularities defined over an algebraically closed field of positive characteristic. For this purpose two new resolution invariants are defined and investigated. Both are modifications of characteristic zero invariants and are so natural that they might be generalizable to higher dimensions.

The subsequent chapters 2 – 4 are all devoted to the study of resolution of threefolds and higher dimensional varieties over algebraically closed fields, especially of positive characteristic. Some attempts, new ideas and partial results for this problem are offered:

Chapter 2 – Threefold resolution in positive characteristic

In a first step to generalize the proof of surface resolution of chapter 1 to the threefold case, the effect of point blowups on the variety and its corresponding Newton polyhedron is studied. Several natural measures for the complexity of the singularities are defined and investigated. Further, some partial results, as for instance that the resolution invariant which is usually used in characteristic 0, may increase when used in positive characteristic, but at most by 1, are given. Moreover it is shown that this measure can't increase twice in two successive blowups.

Chapter 3 – Oblique Polynomials

In this chapter different characterization of oblique polynomials, i.e., those polynomials where the classical resolution invariant from characteristic 0 increases when used in positive characteristic, are given.

Chapter 4 – On Compact Facets and Normal Vectors of Newton Polyhedra

In this chapter a new measure for the complexity of the singularities of an algebraic hypersurface (both for characteristic zero and positive characteristic) – normal vectors of its corresponding Newton polyhedron – is examined. One goal is to study its behavior under the simple polyhedron game, originally introduced by Hironaka to describe the combinatorial part of the resolution problem. There it turns out that the main difficulties lie in Newton polyhedra without compact facets. In order to show that all Newton polyhedra can be transformed by finitely many point blowups into Newton polyhedra without compact facets, the second part of this chapter is devoted to the study of the locus of points on a hypersurface where the corresponding Newton polyhedron has a compact facet with respect to all possibly choices of local coordinates.

The last chapter addresses some basic constructions using étale neighborhoods. They can for instance be used to show that the resolution algorithm presented in chapter 1 terminates after a finite number of blowups.

Chapter 5 – Constructions using étale neighborhoods

In this chapter étale neighborhoods are used to show that the normal crossing locus of a variety over an algebraically closed field (of arbitrary characteristic) is Zariski-open. Similar questions are treated for other geometric notions.

Chapter 1

Surface resolution in positive characteristic

This chapter forms the mathematical core of the present Ph.D. Thesis. It presents a new approach, especially novel invariants, for the embedded surface resolution in positive characteristic. The content of this chapter will be submitted for publication soon.

Abstract

We introduce two new invariants for the inductive proof of the resolution of surfaces in positive characteristic by a sequence of blowups. The invariants are more systematic than the existing ones, and yield a quite transparent reasoning. This may facilitate to study the still unsolved case of the embedded resolution of three-folds in positive characteristic.

1.1   Introduction

In this chapter, two new invariants for the embedded resolution of two-dimensional hypersurface singularities in arbitrary characteristic are constructed. The first invariant is built on the now classical invariant from characteristic zero, consisting of a string of integers given by the local order of the defining equation and of the orders of the subsequent coefficient ideals (deleted by the exceptional factor). As hypersurfaces of maximal contact need not exist in positive characteristic, these orders have to be defined in a different way to make them intrinsic. The correct choice is the *maximum* of the order of the coefficient ideal over *all* choices of local regular hypersurfaces. The orders are thus well defined, i.e., independent of any choices.

By examples of Moh it is known that this invariant may increase under blowup with respect to the lexicographical order [38, 39]. Actually, its second component, the order of the first coefficient ideal, may increase at points where the first component has remained constant. The increase occurs at so called *kangaroo points* (in Hauser's terminology; they are called *metastatic points* by Hironaka). Moh was able to bound the possible increase from above, and Hauser gave a complete classification of kangaroo

points [20, 19].

Based on these results, we show in the present chapter (for purely inseparable two-dimensional hypersurfaces of order equal to the characteristic) that the sporadic increase of the invariant is dominated by larger decreases before or after the critical blowup. It thus decreases in the long run. Actually, to smooth the argument and to avoid considering packages of blowups, we subtract from the second component of the invariant in very specific situations a *bonus* (a rational number taking values 0, ε, δ or $1 + \delta$ with $0 < \varepsilon < \delta < 1$). This bonus is modeled so that the modified invariant decreases under *every* blowup (see theorem 1). It thus interpolates the "graph" of the original characteristic zero invariant by a monotonously decreasing function (see figure 1.1).
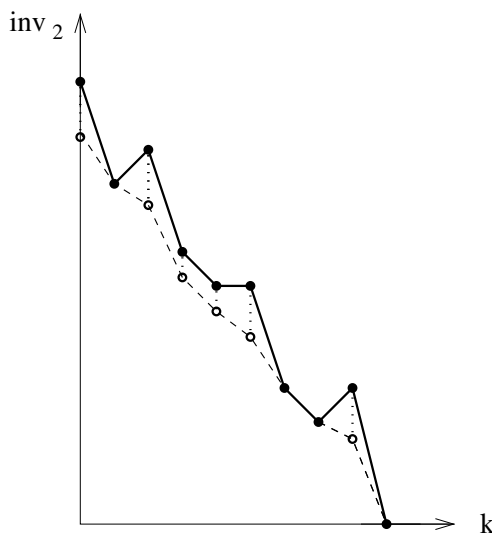
Figure 1.1: Modification of the classical invariant (solid line) by the bonus (dashed); vertically the 2nd component of the invariant, horizontally the number of blowups.

Our second invariant is built on a different measure, the *height*. This is a natural number which counts in an asymmetric way the distance of a hypersurface singularity from being a normal crossings divisor. The symmetry is broken by the consideration of local flags which accompany the resolution process. They allow to restrict the necessary coordinate changes to a "Borel" subgroup of the local formal automorphism group of the ambient scheme: the changes are *triangular* in a precise sense. This, in turn, allows to define the height as a minimum over all coordinate choices subordinate to the flag. Moreover, the local blowups given by choosing an arbitrary point in the exceptional divisor can be made monomial after applying at the base point below a suitable linear triangular coordinate change belonging to the subgroup. Combining these techniques one obtains an explicit control on the behavior of the height under blowup.

Experimentation shows that the height may also increase under blowup, as was the case for the order of the coefficient ideal. But Moh's bound applies again. In fact, the bonus which has to be subtracted to make the resulting invariant always drop is now

much easier to define than before. It is 0, ε or $1 + \delta$ according to the situation, with $0 < \varepsilon < \delta < 1$. As a consequence we can show quite directly that the vector of (modified) heights (of the subsequent coefficient ideals) drops lexicographically under blowup (again in the case of purely inseparable two-dimensional hypersurface singularities of order equal to the characteristic).

Both types of invariants as well as the respective definitions of the bonus provide a quite concise approach to the resolution of surface singularities. They thus form a substitute for Hironaka's invariant from the Bowdoin lectures [24], which is central in the recent works in positive characteristic of Cossart-Jannsen-Saito [10] on the embedded resolution of surfaces of arbitrary codimension and of Cutkosky [13] and Cossart-Piltant [11, 12] on the non-embedded resolution of three-dimensional varieties. All these results rely on Hironaka's invariant for surfaces.

It is appropriate to compare the new invariants with Hironaka's. All three can be defined through the Newton polyhedron of the singularity. They are made intrinsic by very subtle choices of local coordinates, and thus serve as genuine measures of the complexity of the singularity, not depending on any casual instance or choice.

Advantages of the new invariants: (1) They are very natural and easy to handle. (2) Their construction is systematic. This permits to investigate possible extensions to higher dimensions (though there are then various options of how to design them). (3) They do not increase even if the center was chosen too small (i.e., a point instead of a curve). This is not the case with Hironaka's invariant which requires to blow up in a center of maximal possible dimension. In contrast, for our invariants, the centers of blowup will always be a collection of isolated points, except if the first coefficient ideal is a monomial (the ν-quasi-ordinary case; this is a purely combinatorial situation). (4) The symmetry break in the definition of the second invariant may result fertile in the future. The proofs show that this is an efficient way to control blowups. It is built on the asymmetric decomposition of projective space (typically, the exceptional divisor of a point blowup) by affine spaces of decreasing dimensions. We thus partition the exceptional divisor by locally closed subsets instead of covering it by open affine subsets. The flags take into account this decomposition. (5) The bonus is based on a detailed analysis of the kangaroo phenomenon. The increase of the not yet modified invariant à la Moh under blowup can be shown to come along with a complementary improvement of the Newton polyhedron: It approaches a coordinate axis. Exploiting this incidence, first observed by Dominik Zeillinger in his thesis [50], the definition of the bonus comes quite automatically. (6) The proofs that the (modified) invariant drops are completely straight forward and thus – at least in principle – extendable to higher dimension.

Drawbacks of the new invariant: (7) The maximal order of the coefficient ideal over all coordinate changes, called here the *shade* of the singularity (which coincides in the purely inseparable case with the *residual order* of Hironaka), is not upper-semicontinuous when considering non-closed points. Hironaka calls this phenomenon *generic going up*. It causes technical complications in higher dimensions. These, however, seem not to be obstructive. (8) The introduction of the bonus is not completely satisfactory. It ensures that the modified invariant drops after each blowup, but its definition could be more conceptual (e.g., using differential operators). (9) The arguments of the present paper do not cover the case where the local order of the singularity is a p-th power p^k

with $k \geq 2$. In this case Moh's result bounds the maximal increase of the order of the coefficient ideal by p^{k-1}, which is much more delicate to recover by decreases when $k = 2$ or larger. (10) The extension of the results and techniques to the embedded resolution of threefolds – this is known to be the critical case for positive characteristic – is not obvious. There seem to appear additional complications which are not entirely understood.

The study of alternative invariants as for instance the two proposed in this paper will gradually deepen our understanding of resolution in positive characteristic. In this attempt one has to switch permanently between a close analysis of the specific phenomena and a remote perspective capturing the overall argument. In this spirit, our exposition is elementary and concrete while being as systematic and conceptual as possible.

1.2   Context

Hironaka's proof of resolution of singularities in characteristic zero in [28] is built on induction on the dimension of the ambient space. This descent in dimension persists as the key argument also in the later simplifications of Hironaka's proof by Villamayor, Bierstone-Milman, Encinas-Hauser, Bravo-Encinas-Villamayor, Włodarczyk, Kollár [47, 48, 6, 14, 7, 49, 33]: To an ideal sheaf $\mathcal{I}$ in an n-dimensional, smooth ambient scheme W one associates locally at each point a of W (or at least at each point of a suitable stratum of $\mathcal{I}$ in W, usually the top locus top$(\mathcal{I})$ of $\mathcal{I}$ consisting of those points where the local order of $\mathcal{I}$ attains its maximal value) a smooth hypersurface V of W through a and an ideal sheaf $\mathcal{J}$ in V, the coefficient ideal of $\mathcal{I}$ in V at a, which translates the resolution problem for $\mathcal{I}$ in W at a into a resolution problem of $\mathcal{J}$ in V. Once $\mathcal{J}$ is resolved – this can be assumed to be feasable by induction on the dimenson n – there is a relatively simple combinatorial procedure to also resolve $\mathcal{I}$.

Let us recall here that there exist various proofs for (embedded, respectively non-embedded) resolution of surfaces in arbitrary characteristic. Abhyankar's thesis [1] from 1956, Lipman's proof in [35] via pseudo-rational singularities for arbitrary 2-dimensional excellent schemes (but dispensing of embeddedness), and Hironaka's proof from his Bowdoin lectures [24], where an invariant is constructed from the Newton polyhedron of a hypersurface. This proof is used in in the recent work of Cossart-Jannsen-Saito [10] on embedded resolution of surfaces which are not hypersurfaces, Cutkosky's compact writeup [13] of Abhyankar's scattered proof of non-embedded resolution for threefolds in positive characteristic > 5 (hypersurface case), and the papers [11, 12] of Cossart and Piltant for the non-hypersurface case, where the result is established with considerably more effort for arbitrary reduced three-dimensional schemes defined over a field of positive characteristic which is differentially finite over a perfect subfield.

Moreover lately there have been new developments in the area of resolution of singularities of algebraic varieties of any dimension over fields of positive characteristic. For instance, several promising new approaches and programs have been presented during the conference "On the Resolution of Singularities" at RIMS Kyoto in December 2008: In [30, 25, 27] Hironaka studies differential operators in arbitrary characteristic in order

to construct generalizations of hypersurfaces of maximal contact. The main difficulty is thus reduced to the purely inseparable case and kangaroo/metastatic points. Hironaka then asserts that this type of singularities can be resolved directly [26]. There is no written proof of this available yet. Further Kawanoue and Matsuki have announced a program for arbitrary dimension and characteristic, which is partly already published [31, 32]. Again differential operators are used to define a suitable resolution invariant. The termination of the resulting algorithm seems not to be ensured yet. Additionally there is a novel approach to resolution by Villamayor and his collaborators Benito, Bravo and Encinas [15, 8, 5]. It is based on projections instead of restrictions for the descent in dimension. A substitute for coefficient ideals is constructed via Rees algebras and differential operators, called elimination algebras. It provides a new resolution invariant for characteristic p (which coincides with the classical one in characteristic zero). This allows to reduce to a so called monomial case (which, however, seems to be still unsolved, and could be much more involved than the classical monomial case).

In the course of Hironaka's reasoning of resolution of singularities in characteristic zero it is crucial that the local descent in dimension commutes with blowups in admissible centers (= smooth centers contained in $\text{top}(\mathcal{I})$) at all points of the exceptional divisor Y' where the local order of $\mathcal{I}$ has remained constant. More explicitly, this signifies that the coefficient ideal of the weak transform $\mathcal{I}^{\curlyvee}$ of $\mathcal{I}$ at a point a' of Y' where the order of $\mathcal{I}$ has remained constant equals the (controlled) transform of the coefficient ideal of $\mathcal{I}$ at a (for the involved notions of coefficient ideal, weak and controlled transforms, see [14]).

The commutativity of the local descent to coefficient ideals with blowups allows to prove – always in characteristic zero – that the order of the coefficient ideal $\mathcal{J}$ of $\mathcal{I}$ does not increase at points where the order of $\mathcal{I}$ has remained constant. (It is easy to see, using that the center is contained in $\text{top}(\mathcal{I})$, that the order of $\mathcal{I}$ itself cannot increase.) Therefore the pair $(\text{ord}_a(\mathcal{I}), \text{ord}_a(\mathcal{J}))$ does not increase under blowup when considered with respect to the lexicographic order.

The clue for this to work is the existence of hypersurfaces of maximal contact in characteristic zero. They are special choices of hypersurfaces V containing locally $\text{top}(\mathcal{I})$ at a and ensuring that the strict transform V^{st} of V contains again the top locus of the weak transform $\mathcal{I}^{\curlyvee}$ of $\mathcal{I}$, provided the maximum value of the local orders of $\mathcal{I}$ has remained constant. Moreover, it is required that this property persists for $\mathcal{I}^{\curlyvee}$ and V^{st} under any further admissible blowup. In particular, the various transforms of V contain all equiconstant points, i.e., points of the subsequent exceptional loci where the local order of the transforms has remained constant (at the other points, induction on the order applies).

This argument fails in positive characteristic. There are ideals in characteristic $p > 0$ (first given by Narasimhan in [42] and [43], then also studied by Mulay [40]), whose top locus is not locally contained in any smooth hypersurface. Consequently, when just taking any smooth hypersurface through the point a, its transforms under blowups eventually lose the equiconstant points of $\mathcal{I}$ (see [20] for the reason for this and a selection of examples). Hence the induction on the dimension breaks down in a first instance, because the descent in dimension does no longer commute with blowups in the above way.

In an attempt to overcome this flaw, one could choose after each blowup locally at

equiconstant points a' of the exceptional locus Y' a new local hypersurface V' (instead of the transform V^{st} of V) and try to compare the resulting coefficient ideal with the one below in V. In trying to do this, one has to choose carefully the hypersurfaces V and V'. The first should have transform V^{st} containing all equiconstant points a' in Y' (for reasons not apparent at the moment), so that only a *local* automorphism at a' is necessary to obtain V' from V^{st}. Moreover, V' should have the same property as V – but again only for the next blowup, not for all subsequent ones.

Additionally, a second condition is imposed on V. It is related to the construction of the resolution invariant. Usually, this invariant is a vector whose entries are the local orders of certain ideals: The first component is the order of $\mathcal{I}$ at a, the second the order of the coefficient ideal $\mathcal{J}$ of $\mathcal{I}$ at a in V (after having factored from it possible exceptional components). But this second order may depend on the choice of V, and we are better led to choose only such V for which the order of the coefficient ideal takes an intrinsic value.

In characteristic zero, another coincidence occurs. Hypersurfaces of maximal contact maximize the order of the coefficient ideal over all choices of local, smooth hypersurfaces. Thus, this order is intrinsic. In [49], Włodarczyk introduced a version of coefficient ideal whose analytic isomorphism class does not depend on V, so that its local order is automatically intrinsic. The maximality leads naturally to the notion of *weak maximal contact*, which was introduced in [14]: The local, smooth hypersurace V through a has weak maximal contact with $\mathcal{I}$ if the order of the coefficient ideal $\mathcal{J}$ of $\mathcal{I}$ in V is maximized over all smooth local hypersurfaces. This notion depends, of course, on the selected definition of coefficient ideal.

Maximality of orders can be traced back in many papers, and was especially for Abhyankar a decisive requirement [2]. He achieved it in characteristic zero by so called Tschirnhaus transformations, an algebraic construction of local coordinate changes yielding hypersurfaces slightly stronger than hypersurfaces of maximal contact (the resulting hypersurfaces are called *osculating* in [14]).

1.3   Results

The present chapter originates from all observations indicated in section 1.2. It exhibits, still for surfaces, but with the perspective of application to higher dimensional schemes, a characteristic free approach to hypersurfaces of weak maximal contact and their related coefficient ideals. It was observed by Moh in [38] and [39] that the order of the coefficient ideal of an ideal sheaf in a hypersurface of weak maximal contact may indeed increase in characteristic $p > 0$ – this was probably already clear to Abhyankar and Hironaka – but in addition he was able to bound the increase. And in fact, the increase is small. If $\mathcal{I}$ is a principal ideal of order p (the characteristic) at a given point, the increase of the order of the coefficient ideal is at most 1 (always considered at equiconstant points of $\mathcal{I}$ in Y', the only points of interest). This is not too bad, but, conversely, sufficient to destroy any kind of naive induction.

In the present chapter we investigate this increase closer in the case of surfaces. It is known from Hauser's work that an increase can occur only sporadically [20]. The

situations where an increase happens can be completely characterized. However, it cannot be excluded that the increase repeats an infinite number of times. This would not rule out the existence of resolution in positive characteristic, but it would show that the now classical characteristic zero resolution invariant formed by the orders of the successive coefficient ideals cannot be used directly in characteristic p.

The point is that, at least for surfaces, the same resolution invariant as in characteristic zero can be used also in characteristic p. It suffices to modify it slightly in some very specific circumstances to make it work again. The trick lies in subtracting occasionally a *bonus* from the invariant. This is a correction term (taking values $1 + \delta$, δ, ε or 0 for once chosen constants $0 < \varepsilon < \delta < 1$) which makes the modified invariant drop lexicographically after *each* blowup (with a few exceptions, so called quasi-monomials, where a direct resolution of the ideal sheaf can be given).

The classical resolution invariant – consisting of orders of successive coefficient ideals – and its modification will be treated in section 1.9 of this chapter. Instead we will define and work primarily with a new resolution invariant which was constructed in the thesis [50] of Zeillinger. As in the case of the classical characteristic zero resolution invariant, its components are related to the successive coefficient ideals. But instead of measuring the respective orders, we will associate to each of these ideals a certain "height". It measures in an asymmetric manner the distance of a hypersurface singularity from being a normal crossings divisor. We prefer this new resolution invariant because its correction term is easier to define and the induction argument becomes simpler.

The tough case in both Abhyankar's and Hironaka's approach is the purely inseparable equation

$$G = x^p + F(y, z)$$

with $\mathrm{ord}(G) = p$. The present chapter, therefore, concentrates on this situation. This eases the exposition, leaving mostly technical complications if one wants to extend the argument to arbitrary equations of surfaces (one would have to work with coefficient ideals as defined in [14], cf. also [13]). Coefficient ideals correspond geometrically to the projection of the Newton polyhedron of G from the point $x^{\mathrm{ord}(G)}$ onto the yz-coordinate plane (for more details we refer to [38, 20] and remark 4 in section 1.4) and yields a resolution problem which has exactly the same features as the purely inseparable equation.

The surfaces we are considering are embedded in a smooth three-dimensional algebraic variety over an algebraically closed field K. In general this variety does not allow a covering by open subsets isomorphic to open subsets of $\mathbb{A}^3_K$. To simplify the situation we are working in the completion of the local rings. This makes the construction of invariants easier and allows to restrict to the case that the completion of the local ring at a point is the quotient of a formal power series ring in three variables modulo a principal ideal. For simplicity of notation we will assume that this ideal is generated by a polynomial, i.e., that the surface is locally embedded in $\mathbb{A}^3_K$. The constructions in the general case are similar.

Therefore we will restrict to the case that F and G are elements of a polynomial ring R over an algebraically closed field of positive characteristic. Coordinate changes of the form $x \to x + a(y, z)$ allow to eliminate p-th powers from the polynomial F with-

out changing, up to isomorphism, the geometry of the algebraic variety defined by G. Therefore it is natural to work in the quotient $Q = R/R^p$ of R by the subring R^p of p-th powers, so that $Q = R/R^p$ consists of the equivalence classes of polynomials modulo p-th powers. Especially, resolution of G boils down to the monomialization of F modulo R^p. It appears to be very hard to extract substantial information on the complexity of the singularities of G from the knowledge of F up to p-th powers. In particular, any measure of complexity should not increase under blowup in smooth centers.

The invariant constructed by Hironaka in his Bowdoin lectures [24] is built on coordinate independent data extracted from the Newton polyhedron associated to the defining equation in local coordinates. It has the drawback that its improvement under blowup relies on the choice of an admissible center of *maximal possible dimension*. Said differently, when a smooth curve can be chosen as center (because it lies in the top locus), it has to be chosen, otherwise the invariant may go up under blowup. It is precisely this restriction which makes it very hard, if not impossible, to generalize the invariant and the induction argument of Hironaka to threefolds.

The new resolution invariants will also be constructed from the Newton polyhedron in a coordinate independent manner. The first is primarily based on the measure "height", which reflects in an asymmetric way the distance of the Newton polygon from being a quadrant. The second builds on the characteristic zero invariant. In very specific situations – according to special positions of the Newton polygon in the positive quadrant – these invariants are adjusted by subtracting a "bonus".

We shall give a precise formulation and a systematic proof of the following statement (cf. Theorem 2 in section 1.5, and section 1.6).

Theorem 1. *Let X be a singular surface in $\mathbb{A}^3$, defined over an algebraically closed field of characteristic $p > 0$ by a purely inseparable equation of the form*

$$G(x, y, z) = x^p + F(y, z)$$

with $\mathrm{ord}_0(F) \geq p$. Let $\tau : \widetilde{\mathbb{A}^3} \to \mathbb{A}^3$ be the point blowup of $\mathbb{A}^3$ with center the origin, and let $\pi : \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$ be the induced blowup of $\mathbb{A}^2 = 0 \times \mathbb{A}^2$ with exceptional divisor E. Let f be the residue class of F modulo p-th powers and assume that f is not a quasi-monomial at $a = 0$.

(i) There exists a local invariant $i_a(f)$ such that for any closed point a' in E one has

$$i_{a'}(f') < i_a(f),$$

where F' denotes the transform of F in $\widetilde{\mathbb{A}^2}$.

(ii) Finitely many point blowups transform f in any point of the exceptional divisor into a monomial or make $\mathrm{ord}(G)$ drop.

In section 1.7 it will be shown that the set of closed points $a \in \mathbb{A}_K^2$ in which $f \in Q = R/R^p$ is not monomial consists of at most finitely many points. Once f is monomial, there exists a simple combinatorial method to decrease the order of G by finitely many further point- and curve blowups (see section 1.8). Note that in contrast to Hironaka's invariant, which requires to choose in every step of the resolution algorithm a center of maximal possible dimension, we always blowup in a point until f is monomial. Only in this situation curve blowups are possibly needed in order to lower the order of G. Hence we achieve a new proof of the following result:

Corollary 1. *Finitely many blowups of points and smooth curves allow to decrease the order of any purely inseparable singular two-dimensional hypersurface whose maximum of local orders is less or equal to the characteristic of the ground field.*

It is known that the singularities of an arbitrary surface $X = V(G)$ in $\mathbb{A}_K^3$ with $\mathrm{ord}(G) < p$ can be resolved using the usual resolution algorithm from characteristic zero. Therefore, Theorem 1 and Corollary 1 imply (together with remark 9 in section 1.8) the following statement:

Corollary 2. *Finitely many blowups of points and smooth curves allow to construct an embedded resolution of a purely inseparable two-dimensional hypersurface X whose maximum of local orders is less or equal to the characteristic of the ground field (i.e., the total transform has become a normal crossings divisor.)*

1.4 The resolution invariant

In the last section we already indicated why resolution of the purely inseparable surface $G = x^p + F(y, z)$ with $\mathrm{ord}(G) = p$ boils down to the monomialization of F modulo R^p, where R denotes the coordinate ring of the affine plane $\mathbb{A}_K^2$ (polynomial ring in two variables) and R^p its subring of p-th powers. Therefore we will in the sequel restrict to the study of polynomials $F(y, z)$ modulo p-th powers.

Denote by R_a the localization of R at a closed point a of $\mathbb{A}^2$ and $\widehat{R}_a$ its completion with respect to the maximal ideal. A regular parameter system (y, z) of $\widehat{R}_a$ will be called *local coordinates* of R at a. Any choice of local coordinates (y, z) induces an isomorphism of $\widehat{R}_a$ with the formal power series ring $K[[y, z]]$ corresponding to the Taylor expansion of elements of R at a with respect to y and z. Therefore, for any residue class $f \in R/R^p$, there is a unique expansion $F = \sum_{\alpha\beta} c_{\alpha\beta} y^\alpha z^\beta$ of f in $K[[y, z]]$ with $(\alpha, \beta) \in \mathbb{N}^2 \setminus p \cdot \mathbb{N}^2$. This corresponds to considering $\mathbb{N}^2$ with "holes" at the points of $p \cdot \mathbb{N}^2$. We shall always distinguish carefully between elements f in R/R^p and their representatives F as expansions $F(y, z)$ in $K[[y, z]]$ without any p-th powers. The dependence of F on the coordinates y and z is always tacitly assumed without extra notation. The passage to the completion is necessary to dispose of a flexible notion of isomorphism.

A *local flag* $\mathcal{F}$ in $\mathbb{A}^2$ at a is a regular element h of $\widehat{R}_a$ (cf. [22]). Coordinates (y, z) are called *subordinate to the flag* if z and h generate the same ideal in $K[[y, z]]$. We denote by $\mathcal{C} = \mathcal{C}_{\mathcal{F}}$ the group of subordinate local coordinates. *Subordinate coordinate changes* are automorphisms of $K[[y, z]]$ of the form $(y, z) \rightarrow (y + a(y, z), z \cdot u(y, z))$ with series $a(y, z), u(y, z) \in K[[y, z]]$ satisfying $\partial_y a(y, 0) \neq -1$ and $u(0, 0) \neq 0$.

We will first define measures which reflect the distance of the expansion $F(y, z)$ of $f \in R/R^p$ with respect to fixed subordinate coordinates (y, z) from being a monomial up to units in $K[[y, z]]^*$. Afterwards they will be made coordinate independent in order to establish a resolution invariant $i_a(f)$ for residue classes $f \in R/R^p$.

The Newton polygon $N = N(F)$ of an element $F \in K[[y, z]]$ is the positive convex hull $\mathrm{conv}(\mathrm{supp}(F) + \mathbb{R}_+^2)$ of the support $\mathrm{supp}(F) = \{(\alpha, \beta) \in \mathbb{N}^2 \setminus p \cdot \mathbb{N}^2; c_{\alpha\beta} \neq 0\}$ of F. Newton polygons will be depicted in the positive quadrant of the real plane $\mathbb{R}^2$,
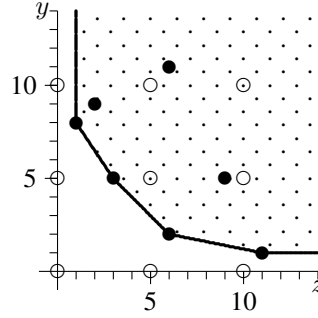
Figure 1.2: Newton polygon $N(F)$ for an element $F \in K[[y,z]]/K[[y,z]]^p$ with $p = 5$. The elements of $K[[y,z]]^p$ are indicated by "holes" $\circ$ at the points $p \cdot \mathbb{N}^2$.

the y-axis chosen vertically, the z-axis to the right (see figure 1.2).

Let $A \subset \mathbb{N}^2 \setminus p \cdot \mathbb{N}^2$ be the set of vertices of the Newton polygon N of F, i.e., the minimal set such that $N = \operatorname{conv}(A + \mathbb{R}_+^2)$. Then the *order* of F is defined as

$$\operatorname{ord}(F) = \min_{(\alpha,\beta) \in A} \alpha + \beta,$$

i.e., as the order of F as a power series. Note that $\operatorname{ord}(F)$ takes the same value for all coordinates $(y,z) \in \mathcal{C}$, it thus depends only on f and a. It will be called the *order* of $f \in R/R^p$, denoted by $\operatorname{ord}_a(f)$. The *initial form* f_d of f at a is the residue class of f modulo $\mathfrak{m}^{d+1}$, where $d = \operatorname{ord}_a(f)$ and $\mathfrak{m}$ denotes the maximal ideal of R at a. Given y and z it is induced by the homogeneous form F_d of lowest degree d of the expansion F of f, say $F = F_d + F_{d+1} + \ldots$, with $F_d \neq 0$. Furthermore denote by

$$\operatorname{ord}_y(F) = \min_{(\alpha,\beta) \in A} \alpha, \quad \deg_y(F) = \max_{(\alpha,\beta) \in A} \alpha,$$

the *order* and the *degree* of F *with respect to* y.

Remark 1. Let (α,β) be the vertex of N whose first component has the largest value among all vertices of A. Then the series $H(y,z) := z^{-\beta} \cdot F(y,z) \in K[[y,z]]$ is regular of order α with respect to the variable y. Due to the Weierstrass' Preparation Theorem one can hence assume that $H(y,z)$ is up to multiplication by a unit $U(y,z) \in K[[y,z]]^*$ a distinguished polynomial $P \in K[[z]][y]$ of degree α with respect to the variable y, i.e., $P(y,z) = U(y,z) \cdot H(y,z)$, where $P = y^\alpha + c_1(z)y^{\alpha-1} + \ldots + c_\alpha(z)$, $c_i \in K[[z]]$, denotes a polynomial of order α with respect to y. Hence, up to multiplication by a unit in $K[[y,z]]^*$, also $F(y,z) = z^\beta \cdot H(y,z) = U(y,z)^{-1} \cdot z^\beta \cdot P(y,z)$ is a polynomial of degree α with respect to the variable y. Therefore it is justified to call α the degree of F with respect to y.

We define the *height* of F as

$$\operatorname{height}(F) = \deg_y(F) - \operatorname{ord}_y(F).$$

This value clearly depends on the coordinates. It describes the vertical extension of the bounded edges of the Newton polygon (see figure 1.3) and will constitute (up to a correction term) the first component of our resolution invariant.

Analogously, we define the *width* of F as

$$\operatorname{width}(F) = \deg_z(F) - \operatorname{ord}_z(F),$$
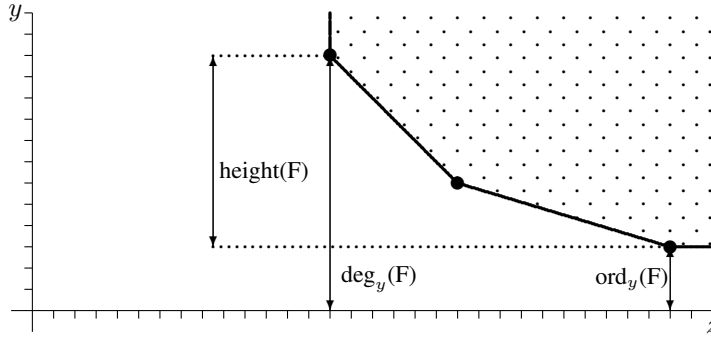
Figure 1.3: $\deg_y(F)$, $\operatorname{ord}_y(F)$ and height(F) of F.

where $\deg_z(F) = \max_{(\alpha,\beta) \in A} \beta$ and $\operatorname{ord}_z(F) = \min_{(\alpha,\beta) \in A} \beta$.

If N is a quadrant, we set the *slope* of F equal to slope$(F) = \infty$. Otherwise, we define it as

$$\text{slope}(F) = \frac{\alpha_1}{\alpha_1 - \alpha_2} \cdot (\beta_2 - \beta_1),$$

where (α_1, β_1) and (α_2, β_2) denote those elements of A whose first component have the highest respectively second highest value among all vertices of A (see figure 1.4). It is thus $-\alpha_1$ times the usual slope of the segment connecting the two points (α_1, β_1) and (α_2, β_2). It will be the second component of our resolution invariant.
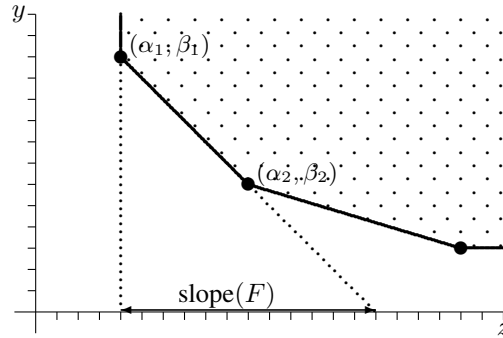
Figure 1.4: slope(F) of F.

As we will see in section 1.6.1, especially in Lemma 3 and in the example given in remark 8, the height can increase under blowup in some special situations. To correct this drawback, we will have to consider the position of the Newton polygon: Call F *adjacent* if $\operatorname{ord}_y(F) = 0$, *close* if $\operatorname{ord}_y(F) = 1$, and *distant* if $\operatorname{ord}_y(F) \geq 2$. The *bonus* of F is set equal to

$$\text{bonus}(F) = \begin{cases} 1 + \delta & \text{if } F \text{ adjacent,} \\ \varepsilon & \text{if } F \text{ close,} \\ 0 & \text{if } F \text{ distant,} \end{cases}$$

where δ and ε denote arbitrary constants $0 < \varepsilon < \delta < 1$. Note that all these definitions break the symmetry between y and z.

Then we define the *intricacy* of F as

$$\text{intricacy}(F) = \text{height}(F) - \text{bonus}(F).$$

We now associate these items in a coordinate independent way to residue classes f in R/R^p. For any choice of local coordinates (y, z) at $a \in \mathbb{A}^2$, take the unique expansion $F = \sum_{\alpha\beta} c_{\alpha\beta} y^\alpha z^\beta$ of f in $K[[y, z]]$ with $(\alpha, \beta) \in \mathbb{N}^2 \setminus p \cdot \mathbb{N}^2$. Let $\mathcal{F}$ be a local flag at a fixed throughout, and $\mathcal{C} = \mathcal{C}_{\mathcal{F}}$ the set of subordinate local coordinates (y, z) in R at a. Note that the highest vertex $c = (\alpha, \beta)$ of $N = N(F)$ does not depend on the choice of the subordinate coordinates, i.e., that any coordinate change subordinate to the flag $\mathcal{F}$ leaves this vertex invariant. Hence $\deg_y(F)$ takes the same value for all subordinate coordinates. For $f \in R/R^p$ with expansion $F = F(y, z)$ at a with respect to $(y, z) \in \mathcal{C}$ we set

$$\begin{aligned}
\text{height}_a(f) &= \min\{\text{height}(F); (y, z) \in \mathcal{C}\} \\
&= \deg_y(F) - \max\{\text{ord}_y(F); (y, z) \in \mathcal{C}\}
\end{aligned}$$

and call it the *height* of f at a. This number only depends on f, the point a and the chosen flag $\mathcal{F}$.

We say that f is *monomial* at a if there exists a (not necessarily subordinate) coordinate change transforming F into a monomial $y^\alpha z^\beta$ times a unit in $K[[y, z]]$. Note that this is in particular the case if $\text{height}_a(f) = 0$ (whereas the converse is not true).

Remark 2. A simple computation shows the following statement: If f is adjacent and not monomial at a, then $\text{height}_a(f)$ is at least equal to 2.

By definition, $\text{bonus}(F)$ takes the same value, $\text{bonus}_a(f)$, for all coordinates realizing $\text{height}_a(f)$, because $\text{ord}_y(F)$ does. We conclude that

$$\begin{aligned}
\text{intricacy}_a(f) &:= \text{height}_a(f) - \text{bonus}_a(f) \\
&= \min\{\text{height}(F) - \text{bonus}(F); (y, z) \in \mathcal{C}\}
\end{aligned}$$

only depends on $f \in R/R^p$, the point a and the chosen flag $\mathcal{F}$. This will be the first component of our local resolution invariant. It belongs to the well ordered set $\mathbb{N}_{\delta,\varepsilon} = \mathbb{N} - \delta \cdot \{0, 1\} - \varepsilon \cdot \{0, 1\}$. As we mostly consider fixed points we omit the reference to a and simply write

$$\text{intricacy}(f) = \text{height}(f) - \text{bonus}(f).$$

The second component of our local resolution invariant is given by

$$\text{slope}_a(f) := \max\{\text{slope}(F); (y, z) \in \mathcal{C} \text{ with } \text{height}(F) = \text{height}_a(f)\}.$$

It is called the *slope* of f and also only depends on $f \in R/R^p$, the point a and the chosen flag $\mathcal{F}$. Again we omit the reference to a and simply write $\text{slope}(f)$.

The local resolution invariant of $f \in R/R^p$ at a with respect to the chosen flag $\mathcal{F}$ is defined as

$$i_a(f) = (\text{intricacy}_a(f), \text{slope}_a(f)).$$

We consider this pair with respect to the lexicographic order with $(0, 1) < (1, 0)$, and call it the *adjusted height vector invariant* of f at a. Sometimes we shall write $i_a(f, \mathcal{F})$ in order to emphasize the dependence on the flag.

Remark 3. Note that $\text{height}_a(f)$ and $\text{slope}_a(f)$, which are the main ingredients of our local resolution invariant, and the primary measure $\text{ord}_b(G)$ are all of the same type (see figure 1.5): The order of $G(x, y, z)$ at a point b equals in the purely inseparable case $G = x^p + F(y, z)$ with $\text{ord}(F) \geq p$ the height of the Newton polyhedron $N(G) \subset \mathbb{N}^3$ with respect to the variable x. Furthermore the coefficient ideal of $G(x, y, z)$ with respect to $x = 0$ is generated by the polynomial $F(y, z)$. And $\text{height}(f)$ exactly measures the (minimal) height of the Newton polygon $N(F) \subset \mathbb{N}^2$ of the polynomial F with respect to the variable y. Finally the slope of F can be thought of as a certain height of the Newton polygon in $\mathbb{N}$ of the coefficient ideal of F in $y = 0$.
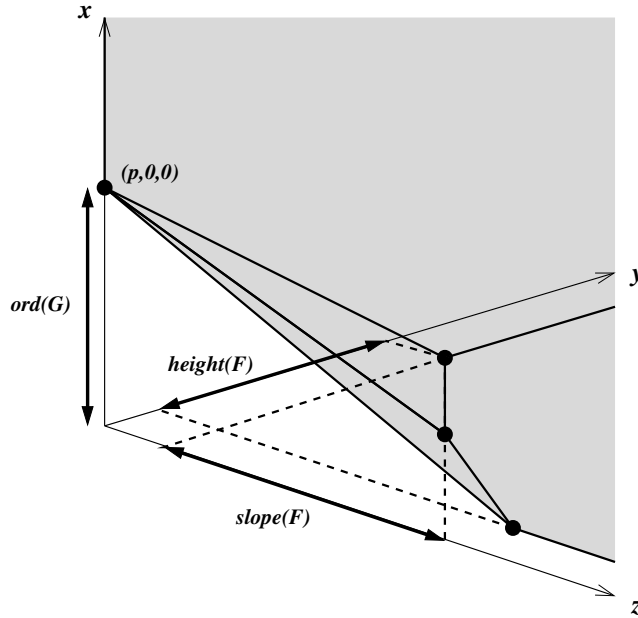
Figure 1.5: The measures $\text{ord}_b(G)$, $\text{height}(F)$ and $\text{slope}(F)$.

Remark 4. The expert reader will notice a similarity to the invariant of Hironaka, which we roughly describe now (for more details we refer to [24], see also [21]): Let g be an element of the coordinate ring S of $\mathbb{A}^3$ and $G = \sum_i c_i(y, z)x^i$ be the expansion of g with respect to any regular parameter system (x, y, z) of $\widehat{S}_b$ (where $\widehat{S}_b$ denotes the completion of the localization of S at the point $b \in \mathbb{A}^3$). Let d be the order of g at b. After a generic linear coordinate change we may assume that $c_d(0, 0) \neq 0$. Due to the Weierstrass Preparation Theorem there exists an invertible power series $u(x, y, z)$ such that $u \cdot g = x^d + \sum_{i<d} c'_i(y, z)x^i$. Now let $N_{yz}(G) \subseteq \mathbb{Q}^2_+$ be the projection with center $(d, 0, 0)$ of the Newton polyhedron $N(G)$ onto the yz-plane. Note that this projection allows to work with arbitrary surface equations and their associated first coefficient ideal instead of dealing only with purely inseparable surface equations. Let

$\alpha = (\alpha_y, \alpha_z)$ and $\beta = (\beta_y, \beta_z)$ denote those vertices of $N_{yz}(G)$ whose y-components have the largest respectively second largest value among all vertices of $N_{yz}(G)$ (in case that G is not monomial). Furthermore let $s_{\alpha\beta} = \frac{\alpha_y - \beta_y}{\alpha_z - \beta_z} \in \mathbb{Q}_-$ be the usual slope of the segment from α to β. Then Hironaka defines the following measure

$$J_{b,(x,y,z)}(G) = (\mathrm{ord}_a(g), \alpha_y, s_{\alpha\beta}, \alpha_y + \alpha_z).$$

To obtain a coordinate free definition, choose subordinate coordinates for the chosen flag, which maximize the vector $(\alpha_z, \alpha_y, s_{\alpha\beta}, \alpha_y + \alpha_z) \in \mathbb{Q}^4$ with respect to the lexicographic order on $\mathbb{Q}^4$. Now the resolution invariant is given as $i_b = i_{b,(x,y,z)} = (\mathrm{ord}_a(g), \alpha_y, s_{\alpha\beta}, \alpha_y + \alpha_z)$, where (x, y, z) denote such maximizing subordinate coordinates. Additionally to the difference in the definition of the invariants in the approach of Hironaka and ours, another crucial distinction lies in the choice of admissible centers. More precisely, whereas we always blow up in a point until G is of the form $G = x^p + y^m z^n A(y, z)$ with $A(0,0) \neq 0$ (and then also allow smooth curves as centers) or until $\mathrm{ord}(G)$ has dropped, Hironaka has to distinguish in each step whether the top locus of g contains a smooth curve or just consists of isolated points. In order to show the decrease of the invariant, he then has to choose the largest possible smooth center. This restriction makes it very difficult, to generalize the method and the invariant to higher dimensions.

1.5   Logical structure of proof of Theorem 1

We sketch in this section the reasons for the decrease of the adjusted height vector under point blowup, i.e., the proof of Theorem 1 (the details come in the next section). Due to the definition of the invariant, this will immediately imply the local monomialization of $F(y, z)$ modulo p-th powers, from which there is an easy combinatorial way to decrease the order of the purely inseparable surface equation $G = x^p + F(y, z)$ by finitely many further point and curve blowups (section 1.8). Together with the study of the non-monomial locus in section 1.7, this will also establish Corollary 1.

Before explaining the overall strategy we specify the statement of Theorem 1. Let a be a closed point of $\mathbb{A}^2$ and let $\mathcal{F}$ be a fixed local flag in $\mathbb{A}^2$ at a. Let $\pi : \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$ be the blowup with center a and exceptional divisor $E = \pi^{-1}(a)$. The flag $\mathcal{F}$ at a induces in a natural way a flag $\mathcal{G}$ at any closed point a' of E by setting

$$\mathcal{G} = \begin{cases} \mathcal{F}' & \text{if} \quad a' \in E \cap \mathcal{F}', \\ E & \text{if} \quad a' \notin E \cap \mathcal{F}', \end{cases}$$

where $\mathcal{F}'$ denotes the strict transform of $\mathcal{F}$ under π (for more details see [22]).

Denote by R' the respective Rees algebra of the coordinate ring R of $\mathbb{A}^2$, say $R' = \oplus_{k\geq 0} \mathfrak{m}^k$, where $\mathfrak{m}$ denotes the maximal ideal of R defining a. Denote by $f' \in R'/R'^p$ the strict transform of f under π, defined as the equivalence class of the strict transform F' of a representative F of f. It is a simple task to check that f' is well defined, i.e., does not depend on the various choices. Thus we dispose of the adjusted height vector $i_{a'}(f', \mathcal{F}')$ of f' at all points a' of E. Theorem 1 then reads as follows.

Theorem 2. *(i) Let F be a polynomial in two variables y, z over an algebraically closed field of characteristic $p > 0$. Denote by f be the residue class of F modulo p-th powers and assume that f is not a quasi-monomial at a given closed point a of $\mathbb{A}^2$. Fix a local flag $\mathfrak{F}$ in $\mathbb{A}^2$ at a. Let $\tau : \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$ be the point blowup with center a and exceptional divisor $E = \pi^{-1}(a)$. For any closed point a' in E, denoting by f' and $\mathfrak{G}$ the transforms of f and $\mathfrak{F}$ in $\widetilde{\mathbb{A}^2}$, the adjusted height vector $i_a(f, \mathfrak{F})$ of f at a with respect to $\mathfrak{F}$ satisfies*

$$i_{a'}(f', \mathfrak{G}) < i_a(f, \mathfrak{F}).$$

(ii) Let X be a reduced two-dimensional closed subscheme of a smooth three-dimensional ambient scheme W of finite type over an algebraically closed field of characteristic $p > 0$. Let b be a singular closed point of X of order p. Assume that X is defined in local coordinates of W at b by a purely inseparable equation of the form

$$G(x, y, z) = x^p + F(y, z).$$

Finitely many point blowups transform X into a scheme which, locally at any point of order p above b, can be defined by an equation $G(x, y, z) = x^p + F(y, z)$ with F a monomial.

Remark 5. It is easy to see that it doesn't make any difference in proving Theorem 1 if we work with the strict transform f^* or the total transform f' of f under the point blowup π, because their Newton polygons differ just in a displacement by p units in either the y- or the z-direction (depending on the point a' of E). The measure height is hence the same for both transforms. Moreover such a displacement may only increase the adjacency and consequently decreases the intricacy, i.e., intricacy$(f') \le$ intricacy(f^*). Furthermore the measure slope is, as we will see, only needed in the horizontal (see below) and in this situation slope$(f') =$ slope(f^*) holds. And since computations are simpler when using the total transform f^*, we will show that $i_{a'}(f^*) < i_a(f)$, which then immediately implies $i_{a'}(f') < i_a(f)$.

Remark 6. The transformation of the equation of our original surface $G(x, y, z) = x^p + F(y, z)$ under blowup of $\mathbb{A}^3_K$ in a point $b = (b_1, a)$, fulfilling $\operatorname{ord}_b(G) = p$, can be read off from the transformation rule of the point blowup π of $\mathbb{A}^2_K$ in a as follows: Let (y, z) and (x, y, z) be regular parameter systems of the rings $\widehat{R}_a$ and $\widehat{S}_b$. Furthermore let $F(y, z)$ respectively $G(x, y, z)$ be the expansions of $f \in R/R^p$ and $g \in S$ with respect to the chosen local coordinates. With g^* and $g' \in S'$ we denote the total respectively strict transform of $g \in S$, where S' denotes the Rees-algebra of S corresponding to the blowup of $\mathbb{A}^3_K$ in b. The chart-expressions for the total transform of G under the blowup $\tau : \widetilde{\mathbb{A}^3_K} \to \mathbb{A}^3_K$ with center $b = 0$ look as follows:

x-chart: $G^*(x, y, z) = x^p \cdot (1 + x^{-p}F(xy, xz))$,

y-chart: $G^*(x, y, z) = y^p \cdot (x^p + y^{-p}F(y, yz))$,

z-chart: $G^*(x, y, z) = z^p \cdot (x^p + z^{-p}F(yz, z))$.

The x-chart is not relevant since $\operatorname{ord}_b(G) = p$. In the y-chart (and symmetrically in the z-chart) either $\operatorname{ord}(g') < \operatorname{ord}(g) = p$ and we are done, or $\operatorname{ord}(g') = \operatorname{ord}(g) = p$, say $\operatorname{ord}(y^{-p}F(y, yz)) \ge p$, hence $G'(x, y, z) = x^p + y^{-p}F(y, yz)$ is of the same type as G. Since multiplying $F(y, yz)$ by y^{-p} again has only the effect of a displacement when regarding the corresponding Newton polygons, it is sufficient to study the total transform

of F under the blowup π of $\mathbb{A}^2_K$ with the two chart expressions $F^*(y, z) = F(y, yz)$ and $F^*(y, z) = F(yz, z)$.

Fix subordinate coordinates $(y, z) \in \mathcal{C}_{\mathcal{F}}$ at the closed point $a \in \mathbb{A}^2_K$ realizing the height of f, i.e., satisfying

$$\text{height}(F) = \text{height}(f),$$

where $F(y, z)$ denotes the expansion of $f \in R$ with respect to y and z. Let $a' \in E = \pi^{-1}(Z)$ be a point above a. There then exists a unique constant $t \in K$ such that the blowup $\widehat{R}_a \to \widehat{R}'_{a'}$ is given either by $(y, z) \to (yz + tz, z)$ or $(y, z) \to (y, yz)$. Accordingly, and distinguishing between $t = 0$ or not, f^* has expansion F^* in $\widehat{R}'_{a'} \cong K[[y, z]]$, where (y, z) now denote local coordinates subordinate to the induced flag $\mathcal{G}$ at a', given by the following formulas:

(A) Translational move: $F^*(y, z) = F(yz + tz, z), t \in K^*$,

(B) Horizontal move: $F^*(y, z) = F(yz, z)$,

(C) Vertical move: $F^*(y, z) = F(y, yz)$.

The naming of the moves (B) and (C) stems from the corresponding transformations of the Newton polygons. Note that there could be several different subordinate coordinates in $\mathcal{C}_{\mathcal{F}}$ realizing the height of f. If possible, we will choose among all these minimizing subordinate coordinates a pair $(y, z) \in \mathcal{C}_{\mathcal{F}}$ in which the blowup $\widehat{R}_a \to \widehat{R}'_{a'}$ is *monomial* (moves (B) and (C)).

The subtlety of the proof that the adjusted height vector drops under blowup for all points $a' \in E$ is due to the fact that the three moves change the Newton polygon in pretty different ways. The invariant has to drop lexicographically under *all* these moves. The key ingredients for this are the following:

Under translational moves, the height can at most increase by 1 (by Moh's bound), and if it does, the Newton polygon was not adjacent before the blowup (by Hauser's kangaroo description), but must be adjacent afterwards (by the definition of the height). Under horizontal moves, the height cannot increase (because the vertices of the Newton polygon move horizontally), the adjacency remains the same (for the same reason). Moreover, in a sequence of horizontal moves, the height must eventually drop (because the slope decreases in each move for which the height remains the same). Under vertical moves, the height decreases at least by 2 (by a simple computation, with the exception of quasi-monomials), and the polygon may quit being adjacent or close.

From these observations it is straightforward how to define the bonus (in function of the adjacency) so that one obtains a decrease of the adjusted invariant under each blowup: Take value 0 for f distant, ε for f close, $1 + \delta$ for f adjacent, with $\varepsilon < \delta$. This choice yields an adjusted height vector that interpolates the "graph" of the original height vector over a sequence of blowups by a strictly decreasing function. Induction applies!

Let us see this argument in more detail. Let a and a' be fixed. If there don't exist

subordinate coordinates at a realizing the height of f and so that the blowup is monomial (i.e., the translational move (A) is *forced*), then one always has

$$\text{intricacy}(f^*) < \text{intricacy}(f),$$

where f^* denotes the equivalence class of the transform $F^*(y, z) = F(yz + tz, z)$ with $t \neq 0$ and where the intricacy of an element $f \in R$ is defined as the difference $\text{intricacy}(f) = \text{height}(f) - \text{bonus}(f)$ (see section 1.4).

Next assume that one can choose subordinate coordinates (y, z) at a realizing the height of f such that a' is one of the two origins of $\widetilde{\mathbb{A}}^2$, say cases (B) or (C) given by monomial substitutions occur.

$$
\begin{aligned}
\text{intricacy}(f^*) = \text{height}(f^*) - \text{bonus}(f^*) \quad &\leq \quad \text{height}(F^*) - \text{bonus}(F^*) \\
&\leq \quad \text{height}(F) - \text{bonus}(F) \\
&= \quad \text{intricacy}(f),
\end{aligned}
$$

except for very special situations where f is a quasi-monomial (these can be resolved directly, see section 1.6.1). Moreover, excluding these exceptions, the inequality is strict for move (C). In case of equality

$$\text{intricacy}(f^*) = \text{intricacy}(f)$$

when applying move (B), we use the second component of the invariant and show first that

$$\text{slope}(F^*) < \text{slope}(F) \leq \text{slope}(f).$$

Realizing $\text{slope}(f^*)$ is by definition done by maximizing $\text{slope}(F^*)$ over all coordinate choices subordinate to the flag $\mathcal{G}$ at a'. It has to be shown that the necessary coordinate change φ' at a' stems from a coordinate change φ at a subordinate to $\mathcal{F}$ (see section 1.6.2). Or said differently, one has to prove that the following diagram commutes, where (y, z) denote local subordinate coordinates at a and where the blowup $\pi : \widehat{R}_a \to \widehat{R}'_{a'}$ is given by $(y, z) \to (yz, z)$ (inducing subordinate local coordinates to the flag $\mathcal{G}$ at a' on $R'_{a'}$)

$$
\begin{array}{ccc}
\widehat{R}'_{a'} & \xrightarrow{\varphi'} & \widehat{R}'_{a'} \\
\pi \uparrow & & \pi \uparrow \\
\widehat{R}_a & \xrightarrow{\varphi} & \widehat{R}_a
\end{array}
$$

and $\varphi'(y, z) = (y + A(z), z)$, $\varphi(y, z) = (y + A(z) \cdot z, z)$ with $A \in K[[z]]$.

The general behavior of the height is illustrated in figure 1.6: it may increase in one step (but only under translational moves), but decreases in the long run of the resolution process.
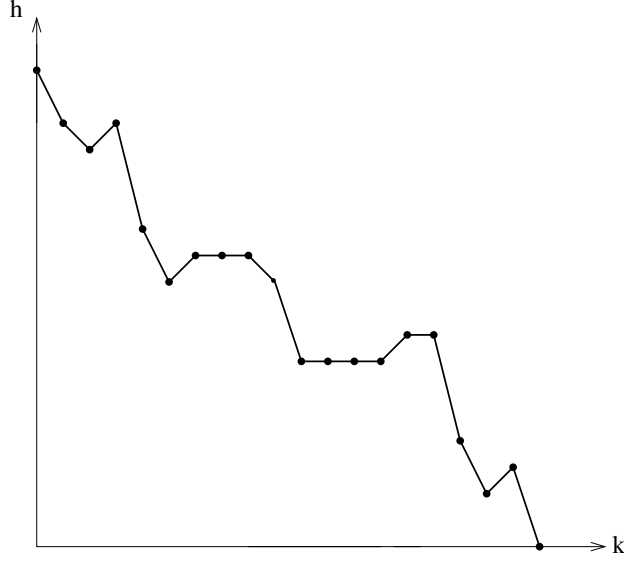
Figure 1.6: Possible behavior of the height under blowups (h denotes height(f), k the number of blowups); an increase can only occur under translational moves.

1.6   Proof of Theorem 1

We show in section 1.6.1 that the first component of the adjusted height vector $i_a(f) = (\text{intricacy}(f), \text{slope}(f))$ does not increase under point blowup (except for quasi-monomials). In section 1.6.2 it is shown that if the intricacy remains the same, the second component of $i_a(f)$ decreases.

1.6.1   Non-increase of the intricacy

The key argument in proving Theorem 1 is the following:

Proposition 1. *Let f be an element of $Q = R/R^p$, which is not a quasi-monomial at a given closed point a of $\mathbb{A}^2$. Fix a local flag $\mathcal{F}$ in $\mathbb{A}^2$ at a and denote by $\mathcal{G}$ the induced flag at $a' \in E$. Let $F \in K[[y, z]]$ be the expansion of f with respect to subordinate coordinates $(y, z) \in \mathcal{C}_{\mathcal{F}}$ realizing the height of f. Furthermore let $F^*(y, z)$ be one of the transformations $F^*(y, z) = F(yz + tz, z)$, with $t \in K$, or $F^*(y, z) = F(y, yz)$ and f^* the corresponding element in R'/R'^p. Then*

$$\text{intricacy}(f^*) \le \text{intricacy}(f).$$

Moreover, if either the translational move (A) is forced, or there exist subordinate coordinates realizing the height of f such that the blowup $\widehat{R}_a \to \widehat{R}'_{a'}$ is given by move (C), then

$$\text{intricacy}(f^*) < \text{intricacy}(f),$$

where $F^*(y, z) = F(yz + tz, z)$ with $t \ne 0$ respectively $F^*(y, z) = F(y, yz)$.

Adjacent series F with width(F) = 1 are called quasi-monomials. Quasi-monomials are not resolved directly, but if F is such, the order of G is decreased by line blowups.

Note that by the minimality of the height, there is no need to realize the height of f^* in R'/R'^p. The proof of Proposition 1 falls naturally into three parts corresponding to the three different moves (A), (B) and (C) defined in section 1.5.

(A) Translational moves

The goal of this paragraph is to show Proposition 1 for the translational move $F^*(y, z) = F(yz + tz, z)$ with $t \in K^*$. In particular we prove: the intricacy decreases if there don't exist minimizing subordinate coordinates such that $a' \in E$ is one of the origins of the two charts of the blowup. Since situations where a translational move is required are the most delicate ones, this section provides the main arguments for proving Theorem 1.

In the following $d = \mathrm{ord}(f)$ denotes the order of f in a and f_d its initial form. Furthermore the *parity* $\mathrm{par}(d)$ of d is set as 1 if $d \equiv 0 \bmod p$, and 0 otherwise.

In the sequel it will be assumed throughout that there don't exist subordinate coordinates at a realizing the height of f such that the blowup $\widehat{R}_a \to \widehat{R}'_{a'}$ is monomial. Or said differently, there don't exist minimizing subordinate coordinates such that $a' \in E$ is one of the origins of the two charts of the blowup. In this situation the total transform f^* of f under the blowup π is given as the equivalence class of the transform $F^*(y, z) = F(yz + tz, z)$, where $t \in K^*$, of a representative $F(y, z)$ of f with $\mathrm{height}(F) = \mathrm{height}(f)$. Fix such minimizing subordinate coordinates $(y, z) \in \mathcal{C}_{\mathcal{F}}$ and denote by $F(y, z)$ in the sequel always the expansion of f with respect to these chosen coordinates.

Remark 7. It can be easily verified that the situation $\mathrm{ord}_y(F^*) > \mathrm{ord}_y(F)$ cannot occur. This is due to the fact that the transformation $(y, z) \to (yz + tz, z)$ with $t \neq 0$ can be decomposed into a linear subordinate coordinate change $(y, z) \to (y + tz, z)$ followed by a horizontal move $(y, z) \to (yz, z)$. Due to the minimality of $\mathrm{height}(F)$, the first one does not increase the order with respect to the variable y. The second transformation clearly preserves it (see also section 1.6.1).

Moreover, in the case that $\mathrm{ord}_y(F^*) = \mathrm{ord}_y(F)$ the same argumentation shows that there exist subordinate coordinates realizing the height of f such that the blowup can be rendered monomial. By the assumption at the beginning of this section, one would thus choose these new minimizing coordinates and would hence be left with the examination of a horizontal move (see section 1.6.1).

Altogether this shows that for the study of translational moves it suffices to investigate the situations where $\mathrm{ord}_y(F^*) < \mathrm{ord}_y(F)$.

The proof of Proposition 1 in the case of translational moves is divided into a series of lemmata. Define the *adjacency* $\mathrm{adj}(F)$ of F as 2, 1 or 0 according to F being *adjacent*, $\mathrm{ord}_y(F) = 0$, *close*, $\mathrm{ord}_y(F) = 1$, or *distant*, $\mathrm{ord}_y(F) \geq 2$. By definition, $\mathrm{adj}(F)$ takes the same value, $\mathrm{adj}(f)$, for all coordinates realizing $\mathrm{height}(f)$, because $\mathrm{ord}_y(F)$ does.

Lemma 1. *Every* $f \in Q = R/R^p$ *satisfies*

$$\mathrm{height}(f) \leq \deg_y(F) - 2 + \mathrm{adj}(f).$$

Proof. This follows easily from the definition of the height of f, the adjacency of f and the degree of F with respect to y.

□

The next result is due to Moh (cf. Proposition 2, p. 989 in [38], or Theorem 3 in [20]):

Lemma 2. *Let F_d be homogenous of degree d. Set $F_d^+(y, z) = F_d(y + tz, z)$ with $t \neq 0$. Then*

$$\mathrm{ord}_y(F_d^+) \leq \mathrm{height}(F_d) + \mathrm{par}(d).$$

Proof. (a) First we consider the case $\mathrm{par}(d) = 1$. Let F_d have $\mathrm{height}(F_d) = k$ and represent it as

$$F_d(y, z) = \sum_{i=0}^{k} c_i y^{m-i} z^{n+i}$$

with $c_i \in K$, $c_0, c_k \neq 0$, $k, m, n \in \mathbb{N}$, $k \leq m$ and $m + n = d$. Set $v = \mathrm{ord}_y(F_d^+)$. First observation: The term $y^{m-k} z^n$ divides F_d, hence $F_d^+ \in z^n \langle y + tz \rangle^{m-k}$. By assumption $m + n \in p \cdot \mathbb{N}$ and $(m - k, n + k) \notin p \cdot \mathbb{N}^2$, which implies $m - k \notin p \cdot \mathbb{N}$. Therefore

$$\partial_y F_d^+ \in z^n \langle y + tz \rangle^{m-k-1}.$$

Second observation: There exists a polynomial R with $R(0, z) \neq 0$ and

$$F_d^+(y, z) = y^v \cdot R(y, z).$$

Since $v \notin p \cdot \mathbb{N}$ (otherwise the monomial $y^v z^{d-v}$ occurring in the expansion of F_d^+ would be a p-th power and thus $\mathrm{ord}_y(F_d^+) > v$), it follows that

$$\partial_y F_d^+ = v y^{v-1} R(y, z) + y^v \partial_y R(y, z) \neq 0,$$

and therefore

$$\partial_y F_d^+ \in \langle y \rangle^{v-1}.$$

Combining these two observations leads to

$$\partial_y F_d^+ \in z^n \langle y + tz \rangle^{m-k-1} \cap \langle y \rangle^{v-1}.$$

But $t \neq 0$ and thus

$$\partial_y F_d^+ \in z^n \langle y + tz \rangle^{m-k-1} \cdot \langle y \rangle^{v-1}.$$

Because of $\mathrm{ord}(F_d^+) = m + n$ and $\partial_y F_d^+ \neq 0$ it follows that

$$n + m - k - 1 + v - 1 \quad \leq \quad m + n - 1.$$

Hence $v \leq k + 1$ as required.

(b) In the same manner as in (a) one can see that in the case $\mathrm{par}(d) = 0$ one gets $F_d^+ \in z^n \langle y + tz \rangle^{m-k}$ and $F_d^+ \in \langle y \rangle^v$. Combining this and using $t \neq 0$ results in

$$F_d^+ \in z^n \langle y + tz \rangle^{m-k} \cdot \langle y \rangle^v.$$

From this it follows that $v \leq k$.

□

Lemma 3. *Let $F^*(y, z) = F(yz + tz, z)$ with $t \neq 0$ and $d = \mathrm{ord}(F)$. Then*

$$\mathrm{deg}_y(F^*) \leq \mathrm{height}(F_d) + \mathrm{par}(d).$$

Proof. Write F as $F(y,z) = F_d(y,z) + H(y,z)$ with $H \in K[[y,z]]$ and $\operatorname{ord}(H) > d$. Furthermore represent F_d as in the proof of Lemma 2. Since $t \neq 0$ one gets for F^*

$$
\begin{aligned}
F^*(y,z) &= \sum_{i=0}^{k} c_i (yz+tz)^{m-i} z^{n+i} + H(yz+tz, z) \\
&= \underbrace{z^d \cdot \sum_{j=0}^{m} c_j' y^j}_{=:A(y,z)} + \underbrace{H(yz+tz, z)}_{=:B(y,z)}.
\end{aligned}
$$

It is obvious that $\operatorname{ord}(A) \geq d = m + n$ and $\operatorname{ord}_z(B) > d$. Moreover the last lemma implies $\operatorname{ord}_y(A) \leq \operatorname{height}(F_d) + \operatorname{par}(d) = k + \operatorname{par}(d)$. Therefore there exists an integer $j \in \{0, 1, \ldots, k + \operatorname{par}(d)\}$ such that $c_j' \neq 0$. Let l be the smallest. Then A can be written as $A(y,z) = z^d \sum_{j=l}^{m} c_j' y^j$. It follows that

$$
\deg_y(F^*) = l \leq k + \operatorname{par}(F) = \operatorname{height}(F_d) + \operatorname{par}(d).
$$

$\square$

Remark 8. The inequality of the previous lemma is sharp! Take for example $p = 2$ and $F(y,z) = y^5 z + y^3 z^3 + y^3 z^8$. Then we have $\operatorname{height}(F_d) = 2$ and $F^*(y,z) = F(yz + 1 \cdot z, z)$ with $\deg_y(F^*) = 3$ (see figure 1.7).
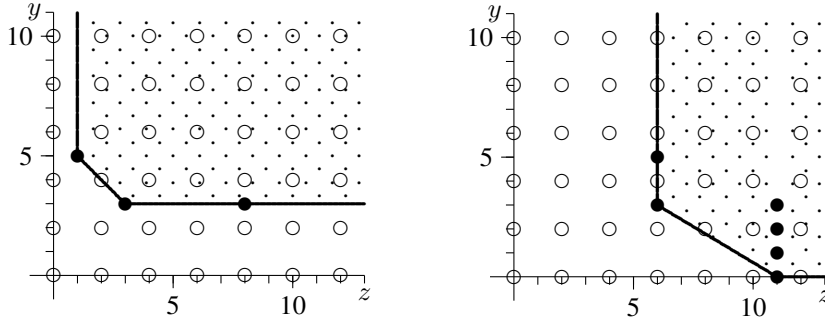
Figure 1.7: $F(y,z) = y^5 z + y^3 z^3 + y^3 z^8$ with $\operatorname{height}(F) = 2$ and $F^*(y,z) = F(yz + 1 \cdot z, z)$ with $\deg_y(F^*) = 3$.

Proposition 2. *Let f be an element of R/R^p. Suppose there don't exist subordinate coordinates realizing* $\operatorname{height}(f)$ *such that the blowup* $\widehat{R}_a \to \widehat{R}'_{a'}$ *is monomial (say* $F^*(y,z) = F(yz + tz, z)$ *with $t \neq 0$). Then*

$$
\operatorname{intricacy}_{a'}(f^*) < \operatorname{intricacy}_a(f).
$$

Proof. Due to remark 7 it is sufficient to show the result for the following situations:

F		F^*
distant	$\to$	distant
distant	$\to$	close
distant	$\to$	adjacent
close	$\to$	adjacent

Combining Lemmata 1 and 3 gives

$$
\begin{aligned}
\text{intricacy}(f^*) \quad &\leq \quad \text{height}(F^*) - \text{bonus}(F^*) \\
&\leq \quad \big(\deg_y(F^*) - 2 + \text{adj}(F^*)\big) - \text{bonus}(F^*) \\
&\leq \quad (\text{height}(F_d) + \text{par}(d)) - 2 + \text{adj}(F^*) - \text{bonus}(F^*) \\
&\leq \quad \text{height}(F) - (2 - \text{adj}(F^*) + \text{bonus}(F^*) - \text{par}(d)) =: (\triangle).
\end{aligned}
$$

Since by assumption $\varepsilon < \delta$, one can deduce that in the four situations described above

$$
(\triangle) \leq \text{height}(F) - \text{bonus}(F)
$$

holds. Consider for instance the situation where F is close and F^* is adjacent. In this case $(\triangle) = \text{height}(F) - (1 + \delta - \text{par}(d)) < \text{height}(F) - \varepsilon = \text{intricacy}(f)$. Altogether this proves the proposition. $\qquad\square$

(B) Horizontal moves

The goal of this section is to prove that the intricacy does not increase for the horizontal transform $F^*(y, z) = F(yz, z)$.

Throughout this section it will be assumed that $(y, z) \in \mathcal{C}_{\mathcal{F}}$ are chosen in a way such that $\text{height}(F) = \text{height}(f)$, where $F(y, z) \in K[[y, z]]$ denotes the expansion of f with respect to y and z, and such that the total transform f^* of f under the blowup π has expansion $F^*(y, z) = F(yz, z)$ in $\widehat{R}'_{a'} \cong K[[y, z]]$. It is obvious that

$$
\text{height}(F^*) \leq \text{height}(F)
$$

(with $\text{height}(F^*) < \text{height}(F)$ if $N(F)$ contains an edge whose angle with the horizontal line is bigger or equal than $45°$). And, clearly, by moving horizontally the adjacency and hence the bonus remain the same. This immediately implies that

$$
\begin{aligned}
\text{intricacy}(f^*) \quad &= \quad \text{height}(f^*) - \text{bonus}(f^*) \leq \text{height}(F^*) - \text{bonus}(F^*) \\
&\leq \quad \text{height}(F) - \text{bonus}(F) = \text{intricacy}(f)
\end{aligned}
$$

is fulfilled for all $f \in Q$.

(C) Vertical moves

In this section it will be shown that under vertical moves the non-monomial elements $f \in Q = R/R^p$ satisfy $\text{intricacy}(f') < \text{intricacy}(f)$. Assume that the subordinate coordinates $(y, z) \in \mathcal{C}_{\mathcal{F}}$ are chosen so that $\text{height}(F) = \text{height}(f)$, where $F(y, z) \in K[[y, z]]$ denotes the expansion of f. We may further assume that our reference point $a' \in E$ above a is the origin of the y-chart of the blowup. The total transform f^* of f is given as the equivalence class of the transform $F^*(y, z) = F(y, yz)$ of F. As the intricacy is a minimum it suffices to show that

$$
\text{intricacy}(F^*) < \text{intricacy}(F).
$$

Since F is non-monomial we know that $\text{height}(F) > 0$, from which a one-line computation yields

$$
\text{height}(F^*) \leq \text{height}(F) - 1.
$$

By the definition of the bonus it follows that $\text{intricacy}(F^*) < \text{intricacy}(F)$ except possibly if F is adjacent and $\text{height}(F^*) = \text{height}(F) - 1$. This equality only occurs if the width of F, i.e., $\text{width}(F) = \deg_z(F) - \text{ord}_z(F)$, equals 1. Recall that such series are named *quasi-monomials*. They also appear in Hironaka's program of resolution of singularities in characteristic $p > 0$ and any dimension [25].

In the case of width 1, we may assume, by prior line blowups with center the z-axis, that $\text{ord}_z(F) < p$. This combined with $\text{width}(F) = 1$ and F adjacent implies that F has a pure y-monomial y^m with $m \leq p$ (cf. fig. 8). But $m = p$ is not possible because F has its exponents in $\mathbb{N}^2 \setminus p \cdot \mathbb{N}^2$, and $m < p$ implies that the order of f (and hence G) has dropped below p. So quasi-monomials are handled by applying suitable line blowups. We conclude that under vertical moves either the order of G drops or $\text{intricacy}(f') < \text{intricacy}(f)$.
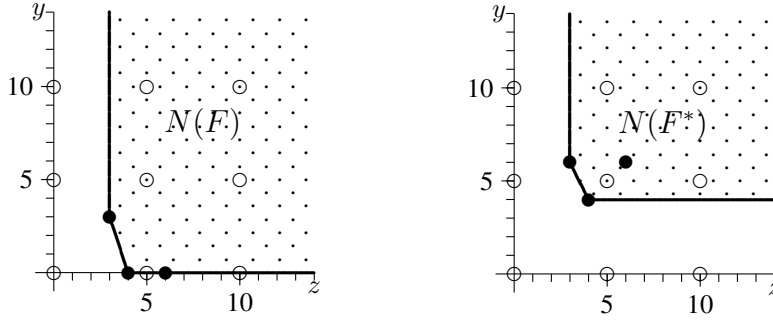
Figure 1.8: Configuration where the intricacy increases under blowup.

1.6.2 Decrease of the invariant

In order to prove Theorem 1 it remains, due to Proposition 2 of paragraph (A) in section 1.6.1, to show that all $f \in Q = R/R^p$ that are not quasi-monomials (which can be resolved directly, see paragraph (C) of section 1.6.1) fulfill

$$(\text{intricacy}(f^*), \text{slope}(f^*)) <_{lex} (\text{intricacy}(f), \text{slope}(f)), \quad (\square)$$

where f^* is given as the equivalence class of one of the transforms $F^*(y,z) = F(yz, z)$ or $F^*(y,z) = F(y, yz)$ of a representative $F(y,z)$ of f with $\text{height}(F) = \text{height}(f)$.

For the purpose of proving $(\square)$, fix throughout this section subordinate coordinates (y, z) at a realizing the height of f such that $a' \in E$ is one of the origins of the two charts of the blowup π. Then the total transform f^* of f under π is determined by one of the transforms $F^*(y,z) = F(yz, z)$ or $F^*(y,z) = F(y, yz)$ of $F(y,z)$.

Due to Proposition 1 of section 1.6.1, all elements $f \in Q$ which are not quasi-monomials satisfy $\text{intricacy}(f^*) \leq \text{intricacy}(f)$. Hence one is left with the case that

$$\text{intricacy}(f^*) = \text{intricacy}(f). \quad (\nabla)$$

Since the situation (∇) doesn't occur when applying translational or vertical moves, it suffices to consider the horizontal transform $F^*(y,z) = F(yz, z)$. It is obvious that

(∇) can only happen if the Newton polygon $N(F)$ of F consists just of edges whose angle with the horizontal line is smaller than $45°$. But in this case the vertices of $N(F)$ with the highest respectively second highest first component transform into vertices of the Newton polygon $N(F^*)$ of F^* with the same property. Moreover, then

$$\text{slope}(F^*) = \text{slope}(F) - \alpha_1 < \text{slope}(F),$$

where (α_1, β_1) denotes the vertex of $N(F)$ whose first component has the highest value among all vertices of A. Now assume that $\text{slope}(f^*) > \text{slope}(F^*)$. Then there exists a coordinate change φ' which is subordinate to the flag $\mathcal{G}$ at a' such that

$$\text{height}(\varphi'(F^*)) = \text{height}(F^*) \text{ and } \text{slope}(\varphi'(F^*)) > \text{slope}(F^*).$$

One can assume that φ' is of the form

$$\varphi' : (y, z) \to (y + A(z), z)$$

with $A \in K[[z]]$, $\text{ord}(A) \geq 1$. Let φ be the coordinate change subordinate to the flag $\mathcal{F}$ at a given by

$$\varphi : (y, z) \to (y + z \cdot A(z), z).$$

Then the computation

$$
\begin{aligned}
\varphi'\left(F^*(y, z)\right) &= \varphi'\left(F(yz, z)\right) \\
&= F\left((y + A(z))z, z\right) \\
&= \left(F\left(y + zA(z), z\right)\right)^* \\
&= \left(\varphi\left(F(y, z)\right)\right)^*
\end{aligned}
$$

shows that the necessary coordinate change φ' at a' stems from the coordinate change φ at a and that when applying φ and φ' the blowup remains monomial. In other words, if one realizes $\text{slope}(f^*)$ after applying the blowup by $\text{slope}(\varphi'(F^*))$, then $\text{slope}(\varphi(F))$ automatically realizes $\text{slope}(f)$. And consequently $\text{slope}(f^*) < \text{slope}(f)$.

1.7   Proof of Corollary 1

Recall that our strategy for improving the singularities of a purely inseparable two-dimensional hypersurface

$$G(x, y, z) = x^p + F(y, z)$$

of order equal to the characteristic and where $F(y, z)$ denotes the expansion of an element $f \in Q = R/R^p$ with respect to subordinate coordinates (y, z), is the following: As long as f is not monomial in a certain point $b = (b_1, a) \in V(G) \subset \mathbb{A}^3_K$ with $\text{ord}_b(G) = p$, we blow up $\mathbb{A}^3_K$ with center $Z = \{b\}$. Due to Theorem 1 this point blowup π improves the situation (except in the case that f is a quasi-monomial, which can be resolved directly, cf. section 1.6.1) in the sense that $i_{a'}(f') < i_a(f)$ for all points $a' \in E = \pi^{-1}(Z)$ above a, where $i_a(f) = (\text{height}_a(f) - \text{bonus}_a(f), \text{slope}_a(f))$ denotes the local resolution defined in section 1.4. Since $\text{height}_a(f) = 0$ especially implies that f is monomial, one can hence deduce by induction that f can be locally transformed by point blowups into a monomial. One is hence left with a combinatorial

situation. In section 1.8 it is shown that in this case the order of the surface can be decreased by finitely many further point- and curve blowups.

To ensure that *finitely* many point blowups suffice to transform f in *every* point $a \in V(G)$ into a monomial, it will be shown in this section that there are only finitely many closed points $b = (b_1, a)$ on $V(G)$ where f is not monomial in a (and $\mathrm{ord}_b(G) = p$). This establishes the termination of the algorithm described above.

The result will be proven in two steps: First it is shown – already for arbitrary dimensional purely inseparable hypersurfaces X with order equal to p – that the subset of X containing those points b where the coefficient ideal is not monomial (and the order of X in b is equal to p) is *Zariski-closed*. Afterwards this result will be used to prove that in the surface case there are only *finitely many* such points.

Proposition 3. *Let $G(x, y_1, \ldots, y_n) = x^p + F(y_1, \ldots, y_n)$ with $F \in K[y_1, \ldots, y_n]$ and where F is not a p-th power. Denote by y the n-tuple of variables $(y_1, \ldots, y_n)$. Then the set of closed points $b = (0, a_1, \ldots, a_n) \in \mathbb{A}_K^{1+n}$ such that there exist a local formal coordinate change ψ at b of the form $\psi : (x, y_1, \ldots, y_n) \rightarrow (x - H(y), \alpha_1(y), \ldots, \alpha_n(y))$, where $H \in K[[y]]$ and $\varphi : (y_1, \ldots, y_n) \rightarrow (\alpha_1(y), \ldots, \alpha_n(y))$ is an element of $\mathrm{Aut}(K[[y]])$, and a unit $u \in K[[y]]^*$ with the property that*

$$G(\psi(x, y) + b) = x^p + u(y) \cdot y^\beta$$

for some vector $\beta \in \mathbb{N}^n \setminus p \cdot \mathbb{N}^n$, is Zariski-open in $\{0\} \times \mathbb{A}_K^n$.

Proof. The assertion of the lemma is clearly equivalent to the statement that the following set is Zariski-open in $\mathbb{A}_K^n$:

$$\begin{aligned} \mathrm{mon}(F) := \{a \in \mathbb{A}_K^n; \exists\, \varphi \in \mathrm{Aut}(K[[y]])\, \exists\, H \in K[[y]]\, \exists\, u \in K[[y]]^* \\ \text{such that for some } \beta \in \mathbb{N}^n \setminus p \cdot \mathbb{N}^n \\ F(\varphi(y) + a) = u(y) \cdot y^\beta + H(y)^p\}. \end{aligned}$$

Note that if a series $A \in K[[y]]$ factors into a monomial times a unit $U \in K[[y]]^*$, i.e.,

$$A(y) = U(y) \cdot y^\gamma,$$

where at least one of the components of γ is not a multiple of the characteristic p of the ground field K, then there exists a coordinate change $\tau \in \mathrm{Aut}(K[[y]])$ such that

$$A(\tau(y)) = y^\gamma.$$

This is due to the fact that a unit $U \in K[[y]]^*$ has a r-th root $U^{1/r}$ in $K[[y]]^*$ if $(r, p) = 1$ (and can for example be deduced from Lemma 4.2 in [**?**]). Since the image of a p-th power under an automorphism $\tau \in \mathrm{Aut}(K[[y]])$ is again a p-th power, the set $\mathrm{mon}(F)$ can be rewritten as

$$\begin{aligned} \mathrm{mon}(F) = \{a \in \mathbb{A}_K^n; \exists\, \varphi \in \mathrm{Aut}(K[[y]])\, \exists\, H \in K[[y]] \\ \text{such that for some } \beta \in \mathbb{N}^n \setminus p \cdot \mathbb{N}^n \\ F(\varphi(y) + a) = y^\beta + H(y)^p\}. \end{aligned}$$

We will prove that this set is Zariski-open in $\mathbb{A}_K^n$ by following a construction which will be explained in detail in the forthcoming article [9] (cf. also chapter 5 of this thesis): Consider for a fixed point $a \in \mathbb{A}^n$ the equation

$$F(\varphi(y) + a) = y^\beta + H(y)^p. \quad (\star)$$

By Artin's Approximation Theorem [3] it follows that if for some vector $\beta \in \mathbb{N}^n \setminus p \cdot \mathbb{N}^n$ there exist solutions $\overline{\varphi}(y) = (\overline{\alpha_1}(y), \dots, \overline{\alpha_n}(y))$ and $\overline{H}(y)$ of $(\star)$ in the ring $K[[y]]$ of formal power series, then there already exist solutions $\varphi(y) = (\alpha_1(y), \dots, \alpha_n(y))$ and $H(y)$ of $(\star)$ in the henselisation of $K[y]$, i.e., in the ring $K\langle\langle y \rangle\rangle$ of algebraic power series in n variables, such that both solutions agree modulo $\langle y \rangle^c$ for a chosen constant $c \in \mathbb{N}$. Note that if one chooses $c = 2$, then the property for $\overline{\varphi}$ to be an automorphism is also ensured for φ. Since H and the components α_i of φ are elements of $K\langle\langle y \rangle\rangle$, they are regular functions on an étale neighborhood $\theta_a : (V, v) \to (\mathbb{A}_K^n, a)$ of $a = \theta_a(v)$. Now consider the monomial locus $\mathrm{mon}(Q, a)$ of

$$Q(y) := F(\varphi(y) + a) - H(y)^p$$

in V, i.e., the set of points $v' \in V$ such that there exist local coordinates $w = (w_1, \dots, w_n)$ at v' with $Q(w + v') = w^\gamma$ in $\widehat{\mathcal{O}}_{V, v'} = K[[w]]$ for some $\gamma \in \mathbb{N}^n$. In [9] it is proven that $\mathrm{mon}(Q, a)$ is a Zariski-open subset of V. Due to $\widehat{\mathcal{O}}_{V, v'} = \widehat{\mathcal{O}}_{\mathbb{A}_K^n, \theta_a(v')}$, $v' \in \mathrm{mon}(Q, a)$ implies that $F(w + \theta_a(v')) = w^\gamma + H(w + \theta_a(v'))^p$. Note that at first sight it seems to be possible that $\gamma \in p \cdot \mathbb{N}^n$, and in this case $\theta_a(v')$ wouldn't be an element of $\mathrm{mon}(F)$. But if all components of γ are multiples of p then $F(w + \theta_a(v')) = w^\gamma + H(w + \theta_a(v'))^p$ would be a p-th power, which contradicts our assumption (since $F(w) \in K[w]$ is a p-th power if and only if $F(\phi(w) + c)$ is for all $\phi \in \mathrm{Aut}(K[[w]])$ and all $c \in \mathbb{A}_K^n$). Consequently $\theta_a(v')$ is contained in $\mathrm{mon}(F)$. By the openness of étale morphisms it follows that $\theta_a(\mathrm{mon}(Q, a))$ is an open subset of $\mathrm{mon}(F)$. This procedure can be carried out for all points $a \in \mathrm{mon}(F)$. Then the set

$$\bigcup_{a \in \mathrm{mon}(F)} \theta_a(\mathrm{mon}(Q, a))$$

clearly equals $\mathrm{mon}(F)$ and is as a union of Zariski-open sets itself Zariski-open. $\qquad \square$

Proposition 4. *Let f be an element of R which is not a p-th power. Then the closed points $a \in V(f) \subset \mathbb{A}_K^2$ in which f has order $\mathrm{ord}_a(f) \geq p$ and in which f is, when considered as an element of R/R^p, not monomial, are isolated (in particular, finite in number).*

Proof. Note that the set of closed points $a \in \mathbb{A}_K^2$ in which $f \in R/R^p \setminus \{0\}$ is monomial, is equal to the set $\mathrm{mon}(F)$ (with $n = 2$) introduced in the proof of the last theorem, which which was shown to be Zariski-open. Its complement in $\mathbb{A}_K^2$ – which equals the set of points of $\mathbb{A}_K^2$ in which f is not monomial – is hence algebraically closed. We are only interested in those points $a \in \mathbb{A}_K^2 \setminus F_{mon}$ in which the order of $f \in R$ is bigger or equal to p (which clearly implies that $a \in V(f)$), thus in the points of the intersection

$$(\triangle) := \left(\mathbb{A}_K^2 \setminus F_{mon} \right) \cap \{a \in \mathbb{A}_K^2 ; \mathrm{ord}_a(f) \geq p\}.$$

By the upper-semicontinuity of the order function it is clear that also the second of these two sets is a Zariski-closed subset of $\mathbb{A}_K^2$. Consequently, the points a in $(\triangle)$ form an algebraic subset of $\mathbb{A}^2$. Moreover, the set $(\triangle)$ is a subset of the singular locus $\mathrm{Sing}(X)$ of $X = V(f) \subset \mathbb{A}^2$. And since any algebraic curve has only finitely many singular points, the set $(\triangle)$ consists of at most finitely many points. $\qquad \square$

1.8 Monomial Case

The goal of this section is to decrease the order of the purely inseparable equation

$$G = x^p + F(y, z)$$

with $\operatorname{ord}_0(G) = p$ in every point of the singular surface $X = V(G) \subset \mathbb{A}_K^3$ by a finite sequence of blowups to a value which is smaller than p. In section 1.5, especially in remark 6, we explained why a point blowup of such a surface can be reduced to a point blowup of the plane curve $F(y, z) = 0$ modulo p-th powers. Moreover, in section 1.6 it was shown that a finite number of point blowups transforms F in every point b of X with $\operatorname{ord}_b(G) = p$ into a monomial times a unit (or makes the order of G drop). This is done by using a local resolution invariant associated to F. To decrease the order of G one can therefore assume that G is of form

$$G(x, y, z) = x^p + y^m z^n A(y, z)$$

with $(m, n) \in \mathbb{N}^2 \setminus p \cdot \mathbb{N}^2$, $m + n \geq p$ and $A(0, 0) \neq 0$. After a formal coordinate change one can furthermore assume that $A(y, z) = 1$ (for a detailed argumentation of this, see the proof of Lemma 3 in section 1.5). Once F is monomial, there is an immediate combinatorial way to lower the order of G, which will be described in the sequel.

Let (y, z) and (x, y, z) be regular parameter systems of $\widehat{R}_a$ and $\widehat{S}_b$, where $\widehat{R}_a$ and $\widehat{S}_b$ denote the completion of the localization of the coordinate ring R of $\mathbb{A}_K^2$ at the point a respectively the coordinate ring S of $\mathbb{A}_K^3$ at $b = (b_1, a)$. Furthermore let $F(y, z)$ and $G(x, y, z)$ be the expansions of $f \in R/R^P$ and $g \in S$ with respect to the chosen local coordinates.

The center of the next blowup is defined by means of the top locus $\operatorname{top}(G)$ of X. Recall that $\operatorname{top}(G)$ consists of those points $b \in X$ where the local order of G attains its maximal value. Thus

$$\operatorname{top}(G) = \{b \in X; \operatorname{ord}_b(G) = p\}.$$

We may assume that the top locus has no self intersections (otherwise further point blowups have to be applied to ensure this condition).

Then there are three different cases according to the values of m and n (for a geometric description see figure 1.9):

(1) Case $m \geq p$: This implies that $G \in \langle x, y \rangle^p$ and hence the z-axis is included in the top locus of $V(G)$. In this case we choose locally the z-axis as the center of the blowup. This yields in the x-chart a variety which is smooth in all of its points and in the y-chart $G^*(x, y, z) = y^p \cdot (x^p + y^{m-p} z^n)$ with $m - p < m$. Hence induction can be applied until $m < p$.

(2) Case $n \geq p$: Symmetrically, we choose locally the y-axis as center and apply induction until $n < p$.

Iterate this process until $m, n < p$!

(3) Case $m < p$ and $n < p$: In this situation we choose as center the origin of $\mathbb{A}_K^3$,
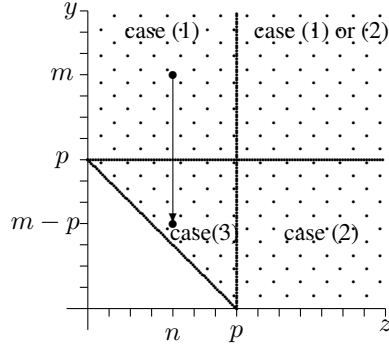
Figure 1.9: Choice of center in the monomial case.

which is in this case the only element of the top locus of $V(G)$. This yields in the x-chart a variety which is smooth in all of its points. In the y-chart, and analogously in the z-chart, one gets $G^*(x, y, z) = y^p(x^p + y^{m+n-p}z^n)$ with $m + n - p < m$, and therefore induction on (m, n) works.

Altogether this yields that g is after finitely many blowups, where the centers have to be chosen in the manner described above, locally in every (singular) point of $V(G)$ given by

$$G(x, y, z) = x^p + F(y, z)$$

with $\mathrm{ord}(F) < p$.

Remark 9. In order to achieve an embedded resolution of the purely inseparable two-dimensional hypersurface X, it is necessary that in every step of the resolution algorithm the chosen center is transversal to the already existing exceptional divisor. In this section it was shown that the only higher dimensional centers which are possibly required during our algorithm, are the y- and the z-axis of $\mathbb{A}_K^3$. If the already existing exceptional divisor is not yet transversal to one of the chosen axis, then one first has to apply point blowups in order to achieve transversality.

1.9   A second resolution invariant

In this section we will define a second local resolution invariant which also works for surfaces in characteristic p. It is a modification of the classical resolution invariant used in characteristic zero. Furthermore we will prove that this invariant also drops lexicographically under point blowups (except for a specific quasi-monomial, which can be resolved directly) and hence can be used alternatively to prove Theorem 1 and Corollary 1.

1.9.1   Definition of the second invariant

Let R be the coordinate ring of the affine plane $\mathbb{A}_K^2$ over an algebraically closed field K of characteristic p. Furthermore let R_a be the localization of R at a closed point a of

$\mathbb{A}_K^2$ and $\widehat{R}_a$ its completion. We fix for the entire section a local flag $\mathcal{F}$ in $\mathbb{A}_K^2$ at a. By (y, z) we denote local coordinates subordinate to $\mathcal{F}$ and by $F = F(y, z)$ the expansion of an element $f \in Q = R/R^p$ in $K[[y, z]]$. Moreover let $N = N(F)$ be the Newton polygon of F and $A \subset \mathbb{N}^2$ its set of vertices.

Denote by $\mathrm{ord}_z(F) = \min_{(\alpha, \beta) \in A} \beta$ the *order of F with respect to z* (see figure 1.10). Then the *shade* of F is defined as

$$\mathrm{shade}(F) = \mathrm{ord}(F) - \mathrm{ord}_y(F) - \mathrm{ord}_z(F).$$

It is thus the maximal side length of all equilateral axes-parallel triangles which can be inscribed in $((\mathrm{ord}_y(F), \mathrm{ord}_z(F)) + \mathbb{R}_+^2) \setminus N(F)$ (see figure 1.10). Or in other words, if $y^m z^n$ is the maximal monomial which can be factored from $F(y, z)$ and $H(y, z) = y^{-m} z^{-n} F(y, z)$, then $\mathrm{shade}(F) = \mathrm{ord}(H)$. The shade thus measures the distance of F from being a monomial up to units. It will constitute together with a correction term the first component of our new resolution invariant.
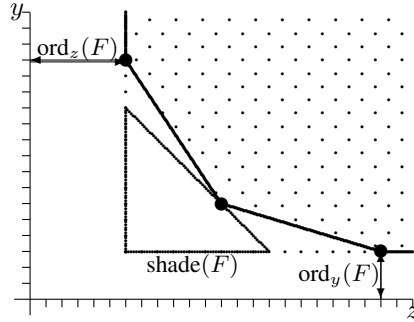

Figure 1.10: $\mathrm{ord}_y(F)$, $\mathrm{ord}_z(F)$ and $\mathrm{shade}(F)$ of F.

The second component of our new resolution invariant will be defined as follows: If N is not a quadrant, we set the *dent* of F as the vector

$$\mathrm{dent}(F) = (\alpha_1 - \alpha_2, \beta_2 - \beta_1),$$

where (α_1, β_1) and (α_2, β_2) denote those elements of A whose first component have the highest respectively second highest value among all vertices of A. The first respectively second component of this vector will be denoted by $\mathrm{updent}(F)$ and $\mathrm{indent}(F)$ and called the *updent* respectively *indent* of F (see figure 1.11).

It is clear that $\mathrm{height}(F_d) = \mathrm{shade}(F_d)$. Therefore Lemma 3 of section 1.6.1 tells us that also the shade can increase in characteristic $p > 0$ under blowup at most by 1. But the modification of the measure $\mathrm{shade}(F)$ in order to get a deceasing resolution invariant is more involved. Recall that the *adjacency* $\mathrm{adj}(F)$ of F is equal to 2, 1 or 0 according to F being *adjacent*, $\mathrm{ord}_y(F) = 0$, *close*, $\mathrm{ord}_y(F) = 1$, or *distant*, $\mathrm{ord}_y(F) \geq 2$. The *defect* of F is defined as follows:
If $\mathrm{shade}(F) = \deg_y(F) - \mathrm{ord}_y(F)$, the *defect* of F is defined to be $1 + \delta$ for F being adjacent, ε for F being close and 0 otherwise. If $\mathrm{shade}(F) = \deg_y(F) - \mathrm{ord}_y(f) - 1$, the *defect* of F is set equal to δ for F being adjacent and 0 otherwise. And if $\mathrm{shade}(F) \leq$
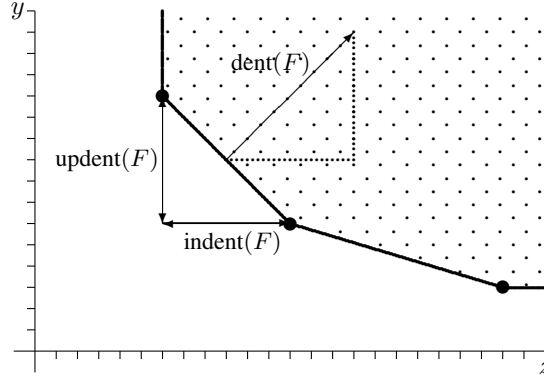
Figure 1.11: The measures $\mathrm{indent}(F)$, $\mathrm{updent}(F)$ and $\mathrm{dent}(F)$ of F.

$\deg_y(F) - \mathrm{ord}_y(f) - 2$, the *defect* of F is defined as 0. In all cases ε, δ denote arbitrarily chosen positive numbers between 0 and 1 with $\varepsilon < \delta$.

This defect is a correction term that takes additionally to the position of the Newton polygon $N(F)$ with respect to the z-axis (as the correction term bonus defined earlier does) also the occurrence of edges in $N(F)$ whose angle with the horizontal line is bigger than $45°$ into account. Note that this definition breaks the symmetry between y and z. In figure 1.12 some possible configurations of $N(F)$ and the corresponding values of $\mathrm{defect}(F)$ are illustrated.
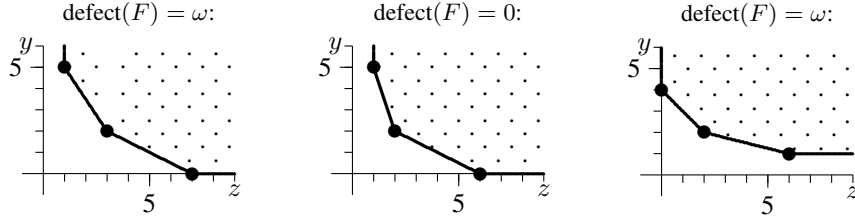
Figure 1.12: Some examples for $\mathrm{defect}(F)$.

Now these measures will be associate in a coordinate independent way to residue classes f in R/R^p. Denote by $\mathcal{C} = \mathcal{C}_{\mathcal{F}}$ as usual the set of subordinate local coordinates (y, z) in $\widehat{R}_a$. Since the highest vertex $c = (\alpha, \beta)$ of $N = N(F)$ does not depend on the choice of the subordinate coordinates, $\deg_y(F)$ and $\mathrm{ord}_z(F)$ take the same value for all elements in $\mathcal{C}$. Recall that also the value $\mathrm{ord}(F)$ is independent of the choice of subordinate coordinates and is called the *order* of $f \in R/R^p$.

For $f \in R/R^p$ with expansion $F = F(y, z)$ at a with respect to $(y, z) \in \mathcal{C}$ we set

$$
\begin{aligned}
\mathrm{shade}_a(f) &= \min\{\mathrm{shade}(F); (y, z) \in \mathcal{C}\} \\
&= \mathrm{ord}(F) - \mathrm{ord}_z(F) - \max\{\mathrm{ord}_y(F); (y, z) \in \mathcal{C}\}
\end{aligned}
$$

and call it the *shade* of f. This number only depends on f, the point a and the chosen

flag $\mathcal{F}$.

We say that f is *monomial* at a if there exists a local (not necessarily subordinate) coordinate change transforming F into a monomial $y^\alpha z^\beta$ times a unit in $K[[y, z]]$. Note that this is in particular the case if $\text{shade}_a(f) = 0$ (which is equivalent to $\text{height}_a(f) = 0$).

Since $\text{adj}(F)$ takes the same value, say $\text{adj}_a(f)$, for all coordinates realizing $\text{shade}(f)$, it is a simple matter to check that also $\text{defect}(F)$ takes the same value, say $\text{defect}_a(f)$, for all these coordinates. Therefore the *complicacy*

$$\begin{aligned} \text{complicacy}_a(f) \quad &:= \quad \text{shade}_a(f) - \text{defect}_a(f) \\ &= \quad \min\{\text{shade}(F) - \text{defect}(F); (y, z) \in \mathcal{C}\} \end{aligned}$$

only depends on $f \in R/R^p$, the point a and the chosen flag $\mathcal{F}$. This will be the first component of our new local resolution invariant. We will leave out the reference to the point a when a is fixed and simply write

$$\text{complicacy}(f) = \text{shade}(f) - \text{defect}(f).$$

The second component of our new local resolution invariant will be

$$\text{dent}_a(f) := (\text{updent}_a(f), \text{indent}_a(f)),$$

where $\text{updent}(F)$ is minimized and afterwords $\text{indent}(F)$ is maximized over all subordinate coordinates $(y, z) \in \mathcal{C}$ for which the expansion $F(y, z)$ fulfills $\text{shade}(F) = \text{shade}_a(f)$. It also only depends on $f \in R/R^p$, the point a and the chosen flag $\mathcal{F}$. Again we omit the reference to a and simply write $\text{dent}(f)$.

The new local resolution invariant of $f \in R/R^p$ at a with respect to $\mathcal{F}$ is then defined as

$$j_a(f) = (\text{complicacy}_a(f), \text{dent}_a(f)),$$

considered with respect to the lexicographic order with $(0, 1) < (1, 0)$. Note that $j_a(f)$ is an element of a well-ordered set.

1.9.2 Non-increase of the complicacy under blowup

In order to prove Theorem 1 for the resolution invariant defined in section 1.9.1, we start by showing the following theorem:

Proposition 5. *Let f be an element of R/R^p, which is not a (specific) quasi-monomial, and $F \in K[[y, z]]$ its expansion with respect to subordinate coordinates $(y, z) \in \mathcal{C}_{\mathcal{F}}$ realizing the shade of f. Furthermore let $F^*(y, z)$ be one of the transformations $F^*(y, z) = F(yz + tz, z)$ or $F^*(y, z) = F(y, yz)$ and f^* the corresponding element in R'/R'^p. Then it holds that*

$$\text{complicacy}(f^*) \leq \text{complicacy}(f). \quad (\square)$$

Moreover, if either the translational move (A) is forced, or there exist subordinate coordinates realizing the height of f such that the blowup $\widehat{R}_a \to \widehat{R}'_{a'}$ is given by move (C), then

$$\text{complicacy}(f^*) < \text{complicacy}(f),$$

where $F^*(y, z) = F(yz + tz, z)$ with $t \neq 0$.

The theorem above will again be proven separately for the three different moves (A), (B) and (C) defined in section 1.5.

(A) Translational moves

Assume that there don't exist subordinate coordinates at a realizing the shade of f such that the blowup is monomial. In this situation the total transform f^* of f under the blowup π is given as the equivalence class of the total transform $F^*(y, z) = F(yz + tz, z)$ where $t \in K^*$, of a representative $F(y, z)$ of f with $\mathrm{shade}(F) = \mathrm{shade}(f)$. Fix such minimizing subordinate coordinates $(y, z) \in \mathcal{C}_{\mathcal{F}}$ and denote by $F(y, z)$ in the sequel always the expansion of f with respect to these chosen coordinates.

Denote by d the order of f and by f_d its tangent cone. The *parity* $\mathrm{par}(d)$ of d is defined as in section 1.6.1, i.e., set equal to 1 if $d \equiv 0 \bmod p$, and 0 otherwise.

Since $\mathrm{height}(F_d) = \mathrm{shade}(F_d)$, Lemmata 2 and 3 of section 1.6.1 can be immediately applied to the shade of F_d respectively f_d. One hence gets

$$\deg_y(F^*) \leq \mathrm{shade}(F_d) + \mathrm{par}(d).$$

Remark 10. Note that the above inequality nevertheless only implies a possible increase of the shade if the Newton polygon $N(F^*)$ of F^* consists only of edges whose angle with the horizontal line is smaller or equal than $45°$, i.e., if $\mathrm{height}(F^*) = \mathrm{shade}(F^*)$ (see figures 1.7 and 1.13). And moreover, it for sure decreases in the case that $N(F^*)$ contains edges with slope smaller than -2, i.e., if $\mathrm{height}(F^*) - \mathrm{shade}(F^*) > 1$.
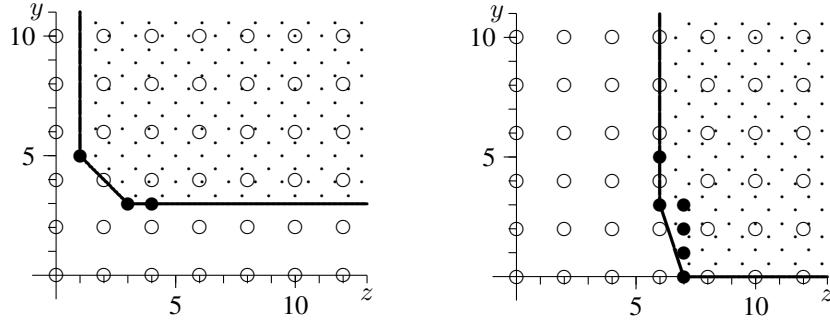
Figure 1.13: $F(y, z)$ with $\mathrm{shade}(F) = 2$ and $F^*(y, z) = F(yz + 1 \cdot z, z)$ with $\deg_y(F^*) = 3$, but $\mathrm{shade}(F^*) = 1 < 2 = \mathrm{shade}(F)$.

Due to remark 7 of section 1.6.1 it follows that it remains to consider the following situations

F		F^*
distant	$\longrightarrow$	distant
distant	$\longrightarrow$	close
distant	$\longrightarrow$	adjacent
close	$\longrightarrow$	adjacent

Investigating these four cases in detail, one can show similarly as in the proof of Proposition 2 in section 1.6.1 that

$$\text{complicacy}(f^*) < \text{complicacy}(f).$$

(B, C) Horizontal and vertical moves

The goal of this section is to prove Proposition 5 for the two monomial transformations $F^*(y,z) = F(yz,z)$ and $F^*(y,z) = F(y,yz)$.

Assume for this purpose throughout this section that (y,z) are subordinate coordinates realizing $\text{shade}(F) = \text{shade}(f)$ such that the total transform f^* of f under the blowup $\widetilde{R}_a \to R'_{a'}$ is given as the equivalence class of one of the transforms $F^*(y,z) = F(yz,z)$ respectively $F^*(y,z) = F(y,yz)$ of F.

In section 1.6.1 we already proved the analogous statement for the measure intricacy defined in section 1.4. And since the argumentation runs here quite similar, we will skip some computational parts of the proof of Proposition 5.

First note that for both, the horizontal and the vertical move, the inequality $\text{shade}(F^*) \leq \text{shade}(F)$ holds for all series $F \in K[[y,z]]$. We start by establishing Proposition 5 for the horizontal move. It is not too hard to check that if $N(F)$ contains at least one edge whose angle with the horizontal line is bigger than $45°$, i.e., if $(\deg_y(F) - \text{ord}_y(F)) - \text{shade}(F) \geq 1$, then $\text{defect}(F) \in \{0, \omega\}$ and $\text{shade}(F^*) \leq \text{shade}(F) - 1$. This immediately implies

$$\text{shade}(F^*) - \text{defect}(F^*) < \text{shade}(f) - \text{defect}(f) = \text{complicacy}(f).$$

We are hence left with series F whose Newton polygon consists only of edges whose angles with the horizontal line are smaller or equal than $45°$. Some further, but easy, considerations show that in this case the inequality

$$\text{shade}(F^*) - \text{defect}(F^*) \leq \text{shade}(f) - \text{defect}(f) = \text{complicacy}(f)$$

is always fulfilled. And since all coordinate changes subordinate to the flag $\mathcal{F}$ leave the highest vertex of $N(F^*)$ fixed, the inequality $(\square)$ follows.
Now we will turn to the vertical move $F^*(y,z) = F(y,yz)$. We will assume that $N(F)$ contains at least one edge whose angle with the horizontal line is bigger than $45°$ (otherwise $N(F^*)$ is already a quadrant). It can be seen easily that then $\text{defect}(F)$ is either 0 or ω. In the case that $\text{defect}(F) = 0$, the inequality $(\square)$ follows immediately. Therefore, let $\text{defect}(F) = 1$. This implies that F is adjacent and $\text{shade}(F) = (\deg_y(F) - \text{ord}_y(F)) - 1$. Furthermore it is a simple matter to check that if $N(F)$ contains an edge whose angle with the horizontal line is smaller or equal than $45°$, then $\text{shade}(F^*) \leq \text{shade}(F) - 1$, hence no increase of the complicacy can happen. So we are left with the case that $N(F)$ contains only edges whose angels with the horizontal line are bigger than $45°$. It is a simple matter to check that then an increase of the complicacy is only possible if F is of the form

$$F(y,z) = z^m \cdot (cy^2 + dz) + H(y,z)$$

with $m \in \mathbb{N}$, $c,d \in K^*$ and $H \in K[[y,z]]$ with $N(H) \subset N(F) \setminus A$. Obviously this series is a special quasi-monomial (see section 1.6.1) and hence can be transformed

into a monomial times a unit by a finite number of further blowups, indeed here only one further blowup (and possibly a subsequent coordinate change) is necessary.

Together with the investigation of translational moves in the last section this proves Proposition 5.

1.9.3   Decrease of the invariant

In order to show that the invariant $j(f) = (\text{complicacy}(f), \text{dent}(f))$ decreases for all $f \in Q = R/R^p$ which are not quasi-monomials, it remains due to Proposition 5 to prove the inequality

$$(\text{complicacy}(f^*), \text{dent}(f^*)) <_{lex} (\text{complicacy}(f), \text{dent}(f)),$$

where f^* corresponds to one of the transforms $F^*(y, z) = F(yz, z)$ or $F^*(y, z) = F(y, yz)$ of a representative $F(y, z)$ of f with $\text{shade}(F) = \text{shade}(f)$ and $\text{updent}(F) = \text{updent}(f)$, in the case that

$$\text{complicacy}(f^*) = \text{complicacy}(f)   (\triangle).$$

Fix for this purpose subordinate coordinates $(y, z) \in \mathcal{C}_\mathcal{F}$ with $\text{shade}(F) = \text{shade}(f)$ and $\text{updent}(F) = \text{updent}(f)$ such that the transform f^* of f is given as the equivalence class of one of the series $F^*(y, z) = F(yz, z)$ or $F^*(y, z) = F(y, yz)$.

We will first concentrate on the horizontal transformation $F^*(y, z) = F(yz, z)$. In this case one can show similarly as in section 1.6.2 that under the assumption $(\triangle)$ the inequality

$$\text{dent}(f^*) < \text{dent}(f)$$

holds. Now we will treat the vertical move $F^*(y, z) = F(y, yz)$. It is easy to see that $(\triangle)$ can only occur if the Newton polygon $N(F)$ consists just of edges whose angle with the horizontal line is bigger than $45°$. But in this case the vertices of $N(F)$ with the highest respectively second highest first component transform into vertices of the Newton polygon $N(F^*)$ of F^* with the same property. Hence it follows easily that $\text{updent}(f^*) \leq \text{updent}(F^*) < \text{updent}(F)$.

1.10   Alternative approach for surface resolution in positive characteristic

In this section we will indicate an alternative approach for resolution of surfaces which are defined by purely inseparable equations over an algebraically closed field K of positive characteristic. It is based on a theorem which characterizes in any dimension completely the shape of the tangent cone of those purely inseparable polynomials for which the shade increases under a translational blowup (see Thm. 1, sec. 5 and Thm. 2, sec. 12 in [20]). We will briefly recall the theorem without giving its proof:

Theorem 3. *Let $\pi : (W, q') \to (W, q)$ be a local point blowup of $W = \mathbb{A}^{1+m}$ with center $Z = \{q\}$ the origin. Let $(x, w_m, \ldots, w_1)$ be local coordinates at q such that*

$$G(x, w) = x^p + w^r \cdot \hat{F}(w) \in \widehat{\mathcal{O}}_{W,q}$$

has order p and $\mathrm{shade}_q(w^r \cdot \hat{F}) = \mathrm{ord}_q(\hat{F})$ at q with exceptional divisor $w^r = 0$. Let G' and F' be the strict transforms of G respectively $F = w^r \cdot \hat{F}(w)$ at $q' \in E = \pi^{-1}(Z)$. Then, for a $q' \in \pi^{-1}(q)$ to be a kangaroo point *for G, i.e., fulfilling*

$$\mathrm{ord}_{q'}(G') = \mathrm{ord}_q(G) \quad \text{and} \quad \mathrm{shade}_{q'}(F') > \mathrm{shade}_q(F),$$

the following conditions must hold at q:

(1) The order $\mathrm{ord}(F) = |r| + \mathrm{ord}_q(\hat{F})$ is a multiple of p.

(2) The exceptional multiplicities r_i at q satisfy

$$\overline{r_m} + \ldots + \overline{r_1} \leq (\phi_p(r) - 1) \cdot p,$$

where $0 \leq \overline{r_i} < p$ denote the residues of the components r_i of $r = (r_m, \ldots, r_1)$ modulo p and $\phi_p(r) := \#\{i \leq m; r_i \not\equiv 0 \bmod p\}$.

(3) The point q' is determined by the expansion of G at q. It lies on none of the strict transforms of the exceptional components $w_i = 0$ for which r_i is not a multiple of p.

(4) The tangent cone of $\hat{F}$ equals, up to linear coordinate changes and multiplication by p-th powers, a specific homogenous polynomial, which is unique for each choice of p, r and degree.

The point q prior to a kangaroo point will be called antelope point.

Note that for surfaces $(m = 2)$ condition (2) of the last theorem can be reformulates as

$$r_1, r_2 \not\equiv 0 \bmod p \quad \text{and} \quad \overline{r_1} + \overline{r_2} \leq p.$$

Consequently, condition (3) implies that the point q has to leave both exceptional components in order to arrive at a kangaroo point. Together this yields that an increase of the shade can only occur when applying a translational move subsequent to at least one horizontal and one vertical move. Therefore we will analyze how the shade changes under such moves prior to the jump at the kangaroo point:

Suppose that before this increase of the shade at the kangaroo point, already u horizontal and v vertical moves (in a specific order), where $u, v > 1$, have taken place. Assume for sake of simplicity further that $F(y, z)$ has at the very beginning of these series of blowups been a binomial, i.e., has been of the form,

$$F(y, z) = y^r z^s \cdot (cy^k + dz^l) \in K[[y, z]]/K[[y^p, z^p]]$$

with $r, s \in \mathbb{N}$, $k, l \in \mathbb{N}_{>0}$ and $c, d \in K$. Clearly a series of u horizontal and v vertical moves contains at least one subsequence where a horizontal move is followed by a vertical one or the other way round. Denote by $F^{(c)}$ the transform of F under the

moves prior to the first of these subsequences, where $0 \leq c \leq u + v$. Note that $F^{(c)}$ is of the form

$$F^{(c)}(y, z) = y^{r'} z^{s'} \cdot (c' y^{k'} + d' z^{l'})$$

with $r', s', k', l' \in \mathbb{N}$ and $c', d' \in K$. Since we are considering moves prior to an increase at the kangaroo point, it follows that $\text{shade}(F^{(c)}) = \min(k', l') > 0$. Without loss of generality assume further that afterwards first an horizontal move and then a subsequent vertical move is applied to $F^{(c)}$ (clearly the case of applying the moves in the reverse order works symmetrically). Now consider the transforms of $F^{(c)}$ under these two moves, i.e., $F^{(c+1)}(y, z) = F^{(c)}(yz, z)$ and $F^{(c+2)}(y, z) = F^{(c+1)}(y, yz)$ (see figure 1.14).
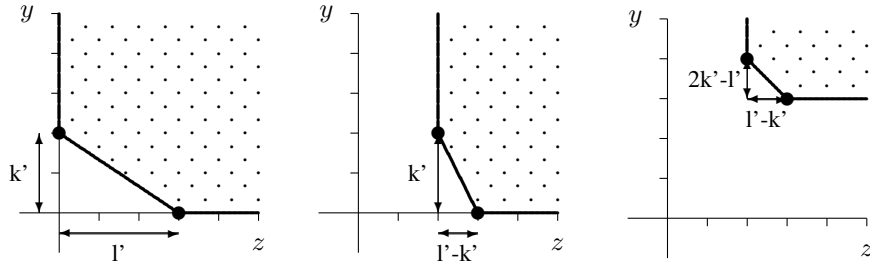

Figure 1.14: The transforms of $F^{(c)}$ under a horizontal and a vertical move.

In the case that $N(F^{(c+2)})$ is not a quadrant, which especially presumes that

$$\text{shade}\left(F^{(c)}\right) = k' < l' \text{ and } l' - k' < k', \qquad (\star)$$

the shade of $F^{(c+2)}$ is given by $\text{shade}(F^{(c+2)}) = \min(2k' - l', l' - k')$. But due to $(\star)$ it follows easily (see figure 1.15) that

$$\text{shade}\left(F^{(c+2)}\right) \leq \frac{k'}{2} = \frac{1}{2} \cdot \text{shade}\left(F^{(c)}\right).$$
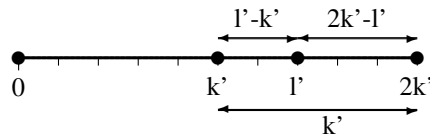
Figure 1.15: Illustration of the inequalities $(\star)$ and the value of $\text{shade}(F^{(c+2)})$.

In the case that $\text{shade}(F^{(c)})$ has already been smaller or equal to the half of $\text{shade}(F)$, i.e., $\text{shade}(F^{(c)}) \leq \frac{1}{2} \cdot \text{shade}(F)$, we are already done since it is known that the shade can't increase under monomial moves and it thus immediately follows that

$$\text{shade}\left(F^{(u+v)}\right) \leq \text{shade}\left(F^{(c)}\right) \leq \frac{1}{2} \cdot \text{shade}(F),$$

where $F^{(u+v)}$ denotes the transform after the u horizontal and the v vertical moves prior to the increase at the kangaroo point. So it remains to consider the case $\text{shade}(F^{(c)}) >$

$\frac{1}{2} \cdot \operatorname{shade}(F)$. But in this situation one has

$$\operatorname{shade}\left(F^{(u+v)}\right) \le \operatorname{shade}\left(F^{(c+2)}\right) \le \frac{1}{2} \cdot \operatorname{shade}\left(F^{(c)}\right) \le \frac{1}{2} \cdot \operatorname{shade}(F),$$

since clearly $\operatorname{shade}(F^{(c)}) \le \operatorname{shade}(F)$.

It is not hard to see that the previous inequalities also hold for an arbitrary series $F(y, z)$. This proves the following proposition, which is also already indicated in [20]:

Proposition 6. *Let $\pi : (\widetilde{\mathbb{A}}^3, b') \to (\mathbb{A}^3, b)$ be a local point blowup with center $Z = \{b\}$ and (x, y, z) local coordinates at b such that $G(x, y, z) = x^p + F(y, z)$ has order p at b. Let b' be a kangaroo point for G and b its antelope point. Further let be given a sequence of point blowups prior to π in a three dimensional ambient space for which the subsequent centers are equiconstant points (i.e., points of the subsequent exceptional loci where the shade of the transforms has remained constant). Call the last point $b°$ below the antelope point b where none of the exceptional components through b has appeared yet, **oasis point**. Then, the shade has dropped between the oasis point $b°$ and the antelope point b of the kangaroo point b' at least to its half.*

The increase at the kangaroo point by 1 is therefore, except in the case that the shade at the oasis point is equal to 1 or 2, in the long run dominated by the decrease of the shade in the prior blowups. By $(\star)$, one immediately sees that in the first case no increase of the shade is possible. If the shade at the oasis point is equal to 2, this is not possible either. This can be checked by an easy computation using the special shape of F in this case.

Chapter 2

Threefold resolution in positive characteristic – some Studies

After having obtained new resolution invariants for the embedded resolution of purely inseparable surface singularities defined over an algebraically closed field of positive characteristic (see last chapter), the idea was to generalize these invariants to the purely inseparable threefold case. However, it turned out that the threefold situation is much more complicated. Therefore no complete solution to this problem can be given in this thesis. Nevertheless some ideas and phenomena are presented in this chapter. Especially a couple of reasonable invariants are investigated. Among these, the classical resolution invariant used in characteristic 0 seems to be the most promising one. But it is not completely clear how to modify it by some correction term in order to get a strictly decreasing resolution invariant.

2.1 Introduction

For a detailed outline of the history of resolution of singularities and the occurring problems and phenomena in positive characteristic we refer to section 1.2 of chapter 1.

We want to recall once more that the only existing complete proofs for resolution of threefolds in positive characteristic are up to the present Cutkosky's paper [13] on Abhyankar's proof of non-embedded resolution for threefolds (hypersurface case) and the papers [11, 12] of Cossart and Piltant for the non-hypersurface case.

As in the case of surfaces, the key problem consists of the resolution of purely inseparable equations (cf. section 1.3 of chapter 1), i.e., equations of the form

$$G(w, x, y, z) = w^p + F(x, y, z) = 0$$

with $\mathrm{ord}_0(F) \geq p$ and where F is an element of a polynomial ring over an algebraically closed field K of characteristic $p > 0$. Therefore ee will restrict our study to this situation. Again coordinate changes of the form $(w, x, y, z) \rightarrow (w + r(x, y, z), x, y, z)$, where $r(x, y, z) \in K[x, y, z]$, allow to eliminate all p-th powers from the expansion of F without changing, up to isomorphism, the geometry of the algebraic variety

$X = V(G)$. Therefore it is reasonable to work in the quotient $Q = R/R^p$, where R^p denotes the subring of R containing all p-th powers of elements of R. This implies that resolution of the threefold $X = V(G)$ can be reduced to resolution of the surface $V(F)$ modulo p-th powers.

In the present chapter I will define and examine several different measures for the complexity of the singularities of F (see sections 2.3, 2.5 and 2.6) gained from its associated Newton polyhedron. The most promising measure is the (modified) classical resolution invariant used in characteristic 0, which is treated in detail in section 2.6. The final goal would be to give a complete proof of the following statement (compare to Theorems 1 and 2 in chapter 1):

Conjecture 1. *Let X be a singular threefold in $\mathbb{A}_K^4$, defined over an algebraically closed field K of characteristic $p > 0$ by a purely inseparable equation of the form*

$$G(w, x, y, z) = w^p + F(x, y, z)$$

with $\mathrm{ord}_0(F) \geq p$. Denote by f the residue class of F modulo p-th powers and assume that f is not a (quasi-)monomial. Then finitely many blowups of points and isolated curves transform f in any point of the exceptional divisor into a monomial or make $\mathrm{ord}(G)$ drop.

As in the last chapter it is intended to prove the above conjecture by defining a local resolution invariant associated to f (possibly a modification of the classical resolution invariant used in characteristic 0). Unfortunately it is not completely clear how to modify the classical resolution invariant from characteristic zero in order to obtain an invariant which drops in every step of the resolution process. Nevertheless some partial results are presented in section 2.6 of this chapter.

Once f is monomial, there is – similar to the surface case (see section 1.8 in chapter 1) – an easy combinatorial way to decrease the order of G (see section 2.7) by further blowups in points, smooth curves and smooth surfaces. If one could prove Conjecture 1, it would hence provide a proof of the following statement:

Corollary 3. *The order of any purely inseparable singular three-dimensional hypersurface in $\mathbb{A}_K^4$ whose maximum of local orders is less or equal to the characteristic of the ground field can by decreased by blowups.*

2.2 Basic definitions

In this section we will generalize the definitions of sections 1.4 and 1.9.1 of chapter 1 to the present situation of purely inseparable threefold equations. For this purpose let R be the coordinate ring of $\mathbb{A}_K^3$, where K denotes an algebraically closed field K of characteristic p. Further let R_a be the localization of R at a closed point a of $\mathbb{A}_K^3$ and $\widehat{R}_a$ its completion with respect to the maximal ideal. A system of regular parameters (x, y, z) of $\widehat{R}_a$ will be called *local coordinates* of R at a. Any choice of local coordinates (x, y, z) induces an isomorphism of $\widehat{R}_a$ with the ring $K[[x, y, z]]$ of formal power series corresponding to the Taylor expansion of elements of R at a with respect to x, y and z. For any residue class $f \in R/R^p$, there is hence a unique expansion

$F = \sum_{\alpha\beta\gamma} c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma$ of f in $K[[x, y, z]]$ with $(\alpha, \beta, \gamma) \notin p \cdot \mathbb{N}^3$. This corresponds to regarding $\mathbb{N}^3$ with holes at the points $p \cdot \mathbb{N}^3$. We will distinguish again carefully between elements f in R/R^p and their representatives F as expansions $F(x, y, z)$ in $K[[x, y, z]]$ without p-th powers monomials.

A *local flag* $\mathcal{F}$ in $\mathbb{A}^3_K$ at a is a sequence $\mathcal{F} : M_3 = 0 \subset M_2 \subset M_1 \subset M_0$ of regular ideals M_i of height $3 - i$ of R (cf. [22]). Local coordinates (x, y, z) of R at a are called *subordinate to the flag* $\mathcal{F}$ if $M_0 = \langle x, y, z \rangle$, $M_1 = \langle y, z \rangle$ and $M_2 = \langle z \rangle$ in $K[[x, y, z]]$. By $\mathcal{C} = \mathcal{C}_\mathcal{F}$ we denote the set of subordinate local coordinates. Note that *subordinate coordinate changes* are automorphisms of $K[[x, y, z]]$ of the form

$$(x, y, z) \rightarrow (x + c(x, y, z), y + d(y, z), z \cdot u(x, y, z))$$

with series $c(x, y, z)$, $d(y, z)$ and $u(x, y, z)$ satisfying $\partial_x c(x, 0, 0) \neq -1$, $\partial_y d(y, 0) \neq -1$ and $u(0, 0, 0) \neq 0$.

The *Newton polyhedron* $N = N(F)$ of an element $F \in K[[x, y, z]]$ is defined as the positive convex hull $\text{conv}(\text{supp}(F) + \mathbb{R}^3_+)$ of the support $\text{supp}(F) = \{(\alpha, \beta, \gamma) \in \mathbb{N}^3 \setminus p \cdot \mathbb{N}^3; c_{\alpha\beta\gamma} \neq 0\}$ of F. Newton polyhedrons will be depicted in the positive octant of $\mathbb{R}^3$, the x-axis chosen vertically and the y- and the z-axis in the horizontal plane (see figure 2.1).
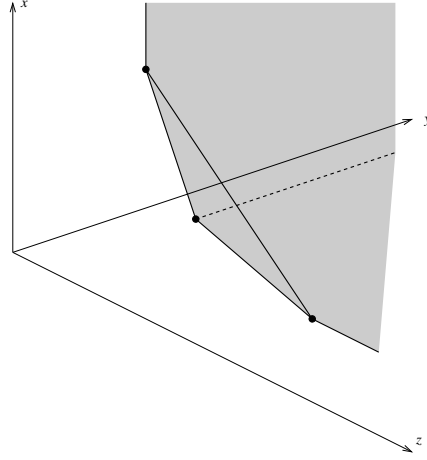
Figure 2.1: Newton polyhedron $N = N(x^5 y^2 z + y^3 z + y^2 z^6)$.

Denote by $A \subset \mathbb{N}^3 \setminus p \cdot \mathbb{N}^3$ the set of vertices of the Newton polyhedron N of F, i.e., the minimal set such that $N = \text{conv}(A + \mathbb{R}^3_+)$. The *order* of F is defined as

$$\text{ord}(F) = \min_{(\alpha, \beta, \gamma) \in A} \alpha + \beta + \gamma.$$

It takes the same value, say $d = \text{ord}_a(f)$, for all coordinates $(x, y, z) \in \mathcal{C}$ and is said to be the *order* of $f \in R/R^p$. Then the *initial form* f_d of f is the residue class of f modulo $\mathfrak{m}^{d+1}$, where $\mathfrak{m}$ denotes the maximal ideal of R. Given local subordinate coordinates x, y and z it is induced by the homogeneous form F_d of lowest degree d of

the expansion F of f, say $F = F_d + F_{d+1} + \ldots$, with $F_d \neq 0$.

Further denote by

$$\mathrm{ord}_x(F) = \min_{(\alpha,\beta,\gamma)\in A} \alpha, \quad \mathrm{ord}_y(F) = \min_{(\alpha,\beta,\gamma)\in A} \beta, \quad \mathrm{ord}_z(F) = \min_{(\alpha,\beta,\gamma)\in A} \gamma,$$

the order of F *with respect to x, y* respectively z. Moreover we call

$$\mathrm{ord}_{xy}(F) = \min_{(\alpha,\beta,\gamma)\in A} \alpha + \beta \quad \text{and} \quad \deg_{xy}(F) = \max_{(\alpha,\beta,\gamma)\in A} \alpha + \beta$$

the *order* respectively the *degree* of F *with respect to x and y*.

We define the *shade* of F as

$$\mathrm{shade}(F) = \mathrm{ord}(F) - \mathrm{ord}_x(F) - \mathrm{ord}_y(F) - \mathrm{ord}_z(F).$$

If the Newton polyhedron $N(F)$ of F contains a compact facet, it is thus the maximal side length of all equilateral axes-parallel tetrahedra which can be inscribed in $\big((\mathrm{ord}_x(F), \mathrm{ord}_y(F), \mathrm{ord}_z(F)) + \mathbb{R}^3_+\big) \setminus N(F)$ (see figure 2.2). The shade can also be described differently: If $x^r y^s z^t$ is the maximal monomial which can be factored from F and $H(x,y,z) = x^{-r}y^{-s}z^{-t}F(y,z)$, then $\mathrm{shade}(F) = \mathrm{ord}(H)$.
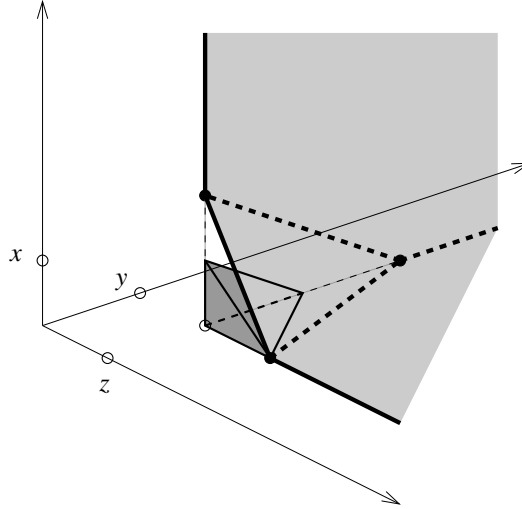
Figure 2.2: Illustration of the shade of $F = yz(x^2 + y^2 + z)$.

Now these measures will be defined in a coordinate independent manner for residue classes f in R/R^p. Let $\mathcal{F}$ be a fixed local flag at the closed point $a \in \mathbb{A}^2_K$ and $\mathcal{C} = \mathcal{C}_{\mathcal{F}}$ the set of subordinate local coordinates in $\widehat{R}_a$. Note that the vertex $c = (\alpha, \beta, \gamma)$ of A with the highest first component stays invariant under all coordinate changes subordinate to the flag $\mathcal{F}$. Therefore $\mathrm{ord}_z(F)$ takes the same value for all subordinate coordinates.

For an element $f \in R/R^p$ with expansion $F = F(x, y, z)$ at a with respect to subordinate coordinates $(x, y, z) \in \mathcal{C}$ we define

$$
\begin{aligned}
\text{shade}_a(f) &= \min\{\text{shade}(F); (x, y, z) \in \mathcal{C}\} \\
&= d - \text{ord}_z(F) - \max\{\text{ord}_x(F) + \text{ord}_y(F); (x, y, z) \in \mathcal{C}\}
\end{aligned}
$$

and call it the *shade* of f. Its value depends only on f, the point a and the chosen flag $\mathcal{F}$.

An element $f \in R/R^p$ is said to be *monomial* at a if there exists a (not necessarily subordinate) coordinate change transforming F into a monomial $x^\alpha y^\beta z^\gamma$ times a unit in $K[[x, y, z]]$. Note that this is especially the case if $\text{shade}_a(f) = 0$, whereas the converse is not true.

2.3   Some first observations for threefold resolution

Let $G(x, y, z) \in K[[x, y, z]]$ be a series in fixed local coordinates (x, y, z), $N = N(F)$ the associated Newton polyhedron in $\mathbb{R}^3$ and $A \subset \mathbb{N}^3 \setminus p \cdot \mathbb{N}^3$ its set of vertices.

Assumption $(*)$**:** N contains a compact facet $\mathfrak{F}$, i.e., a 2-dimensional facet whose area is bounded (or equivalently, whose normal vector is an element of $+\mathbb{N}_{>0}^3$ or $-\mathbb{N}_{>0}^3$).

Observations

(1) Normal vector – last component of the local resolution invariant

Under a monomial blowup the normal vector $\mathfrak{n}(\mathfrak{F})$ on $\mathfrak{F}$ decreases with respect to the lexicographical ordering (see part I of chapter 4 where the behavior of normal vectors onto facets of the Newton polyhedron under monomial blowups is investigated in detail), if the facet "survives" (i.e., doesn't flip inside the transformed polyhedron).

But one has to specify one single compact facet (for example the compact facet $\mathfrak{F}$ of $N(F)$ for which the normal vector $\mathfrak{n}(\mathfrak{F})$ is maximal with respect to the lexicographical ordering among all normal vectors $\mathfrak{n}(\mathfrak{G})$ on compact facets $\mathfrak{G}$ of $N(F)$) to use its normal vector as a measure for the complexity of $N(F)$. If one achieves to make this definition coordinate independent, by for example maximizing over all subordinate coordinate changes, the normal vector would be a good candidate for the *last component of the* (not yet defined) *local resolution invariant* (compare to the measure $\text{dent}(F)$ defined in section 1.9.1 of chapter 1).

(2) Preceding components of the local resolution invariant

The components of the local resolution invariant which are preceding the normal vector have to take care of the following situations:

- Monomial blowups, where the chosen facet doesn't "survive".

- Translational blowups (see next section).

(3) Translational moves with translations in two directions

The most complicated transformation of the Newton polyhedron $N(F)$ under a point blowup is the translational move $(x, y, z) \to (xz + sz, yz + tz, z)$ with $s, t \in K^*$ (see next section). In analogy to the proof of resolution of surfaces, one goal is to exploit the following phenomena occurring when applying translational moves:

- In order to measure the complexity of the singularities of the total transform F^* of F under a translational blowup, the initial form F_d of F plays the main role. More precisely, the "size" of the initial form F_d of F bounds in some sense the "size" of the transform F^* of F.

- It is known that the usual resolution invariants from characteristic zero may increase under translational moves when used in positive characteristic. Nevertheless one should examine whether after such an increase the Newton polyhedron $N(F^*)$ of the total transform F^* of F is in special position in the positive octant, for instance always adjacent to the z-axis, i.e., satisfies $\mathrm{ord}_{xy}(F^*) = 0$ (compare to Lemma 10 of chapter 1). Such a behavior would indicate to define a correction term "bonus" in dependence of the value of $\mathrm{ord}_{xy}(F)$ in order to obtain a monotonously decreasing resolution invariant (compare to section 1.4 in chapter 1).

In section 2.6 these two phenomena are treated in detail.

One problem which arises in this context is how to measure the "sizes" of F_d, F and F^* in a convenient way. In sections 2.5 and 2.6 some reasonable measures are discussed.

2.4   First steps towards point blowups of threefolds

Let R be the coordinate ring of $\mathbb{A}^3_K$ over an algebraically closed field K of characteristic p and f a residue class in R/R^p. Let $\pi : \widetilde{\mathbb{A}}^3_K \to \mathbb{A}^3_K$ be the blowup with center $Z = V(\mathfrak{I})$, where $\mathfrak{I} \subset R$ denotes some regular ideal (for instance $\mathfrak{I} = \langle x, y, z \rangle$ with $Z = V(\mathfrak{I}) = \{0\}$), and denote by $E = \pi^{-1}(Z)$ its exceptional divisor. Furthermore we denote by R' the respective Rees algebra of R, i.e., $R' = \oplus_{k \geq 0} \mathfrak{I}^k$. Let $f^* \in R'/R'^p$ and $f' \in R'/R'^p$ be the total respectively strict transform of $f \in R/R^p$ under π. It is easy to see that the flag $\mathcal{F}$ at a induces in a natural way a flag $\mathcal{G} : N'_3 = 0 \subset N'_2 \subset N'_1 \subset N'_0$ at any closed point a' of E. In the case of a point blowup, i.e., $\mathfrak{I} = \langle x, y, z \rangle$, this is for example

$$N'_2 := \begin{cases} M'_2 & \text{if} \quad a' \in E \cap M'_2, \\ E & \text{if} \quad a' \notin E \cap M'_2, \end{cases}$$

$$N'_1 := \begin{cases} M'_1 & \text{if} \quad a' \in E \cap M'_1, \\ M'_2 \cap E & \text{if} \quad a' \in E \cap M'_2 \setminus M'_1, \\ \text{projective line in } E \text{ through } a' \text{ and } M'_1 \cap E & \text{if} \quad a' \notin M'_2, \end{cases}$$

where M'_1 and M'_2 denote the strict transforms of M_1 respectively M_2 under π (for details and the induced flag $\mathcal{G}$ in the case of a positive dimensional center we refer to [22]). Having chosen a closed point a in $\mathbb{A}^3_K$ and a local flag $\mathcal{F}$ in $\mathbb{A}^3_K$ at a, the shade is thus defined at all points a' of E.

The main step in proving Conjecture 1, i.e., the monomialization of f modulo p-th powers, is to find a local resolution invariant i_a which decreases in every step of the resolution algorithm in any point of the exceptional divisor of the blowup. Note that furthermore the strategy of choosing the center of the next blowup (i.e., choosing either a single point, a (smooth) curve or a (smooth) surface) has to be given. The goal would be to prove the following result:

Aim. *Assume that $f \in R/R^p$ is not (quasi-)monomial. Let $\pi : \widetilde{\mathbb{A}}_K^3 \rightarrow \mathbb{A}_K^3$ be the blowup with center Z given by a specified strategy and exceptional divisor E. Then one has for any point a' in E above $a \in Z$*

$$i_{a'}(f') < i_a(f).$$

Unfortunately it is not clear up to now how to define a right strategy for choosing the next center. Therefore we will in the sequel *restrict to the examination of point blowups* and try to find a measure which drops under these kind of blowups. We will see that this task is already very challenging!

Remark 11. Note that under the assumption that one only works with invariants which do not depend on the position of the Newton polyhedron in the positive octant, one can use instead of the strict transform f' the total transform f^* of f as well (see also remark 5 in chapter 1). And since computations are simpler when using the total transform f^*, we will in the sequel work with this transform of f.

Moreover the transformation of our original purely inseparable threefold defined by $G(w, x, y, z) = w^p + F(x, y, z)$ under blowup of $\mathbb{A}_K^4$ in a point $b = (b_1, a)$ can be deduced from the transformation of $V(F)$ under the point blowup π of $\mathbb{A}_K^3$ in a (see analogous remark 6 in chapter 1).

Let us give a brief *outline of the search for a resolution invariant which behaves well under point blowups* (compare to section 1.5 of chapter 1). Let a be a closed point of $\mathbb{A}_K^3$ which satisfies $\mathrm{ord}_a(f) \geq p$. Furthermore let $\mathcal{F}$ be a fixed local flag in $\mathbb{A}_K^3$ at a. Fix subordinate coordinates $(x, y, z) \in \mathcal{C}_{\mathcal{F}}$ at a realizing $i_a(f)$ (which is not yet defined). Let $a' \in E = \pi^{-1}(Z)$ be a point above a. After localization and completion of R and R' at a respectively a', there then exist constants $r, s, t \in K$ such that $\widehat{R}_a \rightarrow \widehat{R}'_{a'}$ is given either by $(x, y, z) \rightarrow (xz + sz, yz + tz, z)$, $(x, y, z) \rightarrow (xy + ry, y, yz)$ or $(x, y, z) \rightarrow (x, xy, xz)$. Depending on the point $a' \in E$ (see figure 2.3 for the corresponding partition of the exceptional divisor $E \cong \mathbb{P}^2$ of the blowup) and distinguishing between different values of r, s and t, the expansion F^* of f^* in $\widehat{R}'_{a'} \cong K[[x, y, z]]$, where (x, y, z) now denote local coordinates subordinate to the induced flag $\mathcal{G}$ at a', is thus given by one of the following formulas:

    (A) Translational move with translations in two directions:
    $F^*(x, y, z) = F(xz + sz, yz + tz, z)$, $s, t \in K^*$,

    (B) Translational move with translation in x-direction 1:
    $F^*(x, y, z) = F(xz + sz, yz, z)$, $s \in K^*$,

    (C) Translational move with translation in y-direction:
    $F^*(x, y, z) = F(xz, yz + tz, z)$, $t \in K^*$,

    (D) Horizontal move in z-direction:
    $F^*(x, y, z) = F(xz, yz, z)$,

(E) Translational move with translation in x-direction 2:
$F^*(x, y, z) = F(xy + ry, y, yz), r \in K^*,$

(F) Horizontal move in y-direction:
$F^*(x, y, z) = F(xy, y, yz),$

(G) Vertical move:
$F^*(x, y, z) = F(x, xy, xz).$

The naming of the moves arise from the corresponding transformations of the Newton polyhedra. The formulas are compatible with the flags $\mathcal{F}$ and $\mathcal{G}$ at a respectively a' which is shown in more generality in [22].
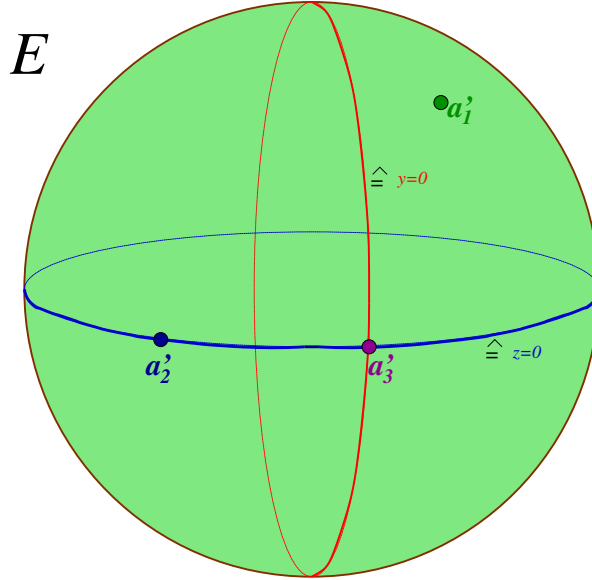

Figure 2.3: Partition of the exceptional divisor $E \cong \mathbb{P}^2$ into the three locally closed subsets $\{z \neq 0\}$, $\{z = 0, y \neq 0\}$ and $\{z = y = 0\}$.

The difficulty in finding a local resolution invariant associated to f and in proving that it drops under a (point) blowup in all points $a' \in E$ lies in the fact that the seven moves (and moves coming from blowups in larger centers) transform the Newton polyhedron $N(F)$ in very different manners. Therefore it is a very challenging task to control them with *one* (vector of) measure(s).

Consider the three affine charts on E induced by the coordinates (x, y, z) at a in $\mathbb{A}^3_K$: The origins of these charts are the intersection points of E with the strict transforms of the coordinate hyperplanes $x = 0$, $y = 0$ and $z = 0$ in $\widetilde{\mathbb{A}}^3_K$. The situations where $a' \in E$ is not one of these origins correspond to the translational moves (A)-(D). Among these, move (A) is the most complicated one, since here the Newton polyhedron transforms in the most complicated manner. It is hence natural to consider this case first and try to find a measure which *always decreases* under move (A) and doesn't

increase under the remaining moves. Afterwards this component has to be complemented by an additional measure, which decreases in all cases where the first measure has remained constant under moves (B)-(G).

It is well-known that the classical resolution invariant used in characteristic zero – order of ideals and their associated coefficient ideals – can possibly increase when considered in positive characteristic (see for instance [38] or [20]). In the next section some reasonable measures are defined and their behavior under blowup is examined. In section 2.6 the behavior of the previously defined shade under point blowup is studied in detail, since it is up to now the most promising measure. One reason for this is that a result of Moh (cf. [38] and [20]) implies that the shade can increase under translational moves at most by 1.

A detailed study of this phenomena should detect whether after such an increase the Newton polyhedron of F^* is of a certain special shape or in a certain special position in the positive octant, which could be used to guarantee that such an increase can't happen infinitely many times during the resolution process and that a certain measure decreases in the long run.

After having found a candidate for a measure (possibly the shade subtracted by some not yet specified correction term) which always decreases under move (A), one has to show that this measure doesn't increase under the remaining 6 moves. Then one should search for the second component of the resolution invariant and so on.

2.5 Measures which have been studied up to now

In this section I will state the most reasonable measures which I have studied up to now. Moreover I will give for all of them, with the exception of the shade (see below), examples which show that these measures are inappropriate to define a local resolution invariant.

Measure 1: $\deg_{xy} - \operatorname{ord}_{xy}$

In analogy to the invariant "height" used in the surface case (where $\operatorname{height}(F) = \deg_y(F) - \operatorname{ord}_y(F)$, see section 1.4 of chapter 1), a first guess for a possible measure in the threefold case is

$$\deg_{xy}(F) - \operatorname{ord}_{xy}(F).$$

But one can easily see that this measure behaves more or less arbitrarily under monomial moves:

Example 1. Let $p = 2$ and

$$F(x, y, z) = x^8 + y^7 + z$$

with $\deg_{xy}(F) - \operatorname{ord}_{xy}(F) = 8 - 0 = 8$. Now consider the transform of F under the monomial move in x-direction, i.e.,

$$F^*(x, y, z) = F(x, xy, xz) = x^8 + x^7 y^7 + xz.$$

Then $\deg_{xy}(F^*) - \operatorname{ord}_{xy}(F^*) = 14 - 1 = 13$. The measure has hence increased by 5!

Measure 2: $\deg_x + \deg_y - \mathrm{ord}_x - \mathrm{ord}_y$

Another reasonable measure would be

$$\deg_x + \deg_y - \mathrm{ord}_x - \mathrm{ord}_y.$$

Nevertheless the next example shows that it doesn't behave nice under monomial moves:

Example 2. Let $p = 3$ and

$$F(x, y, z) = x^4 y^3 + y^4 + z.$$

Then $\deg_x(F) + \deg_y(F) - \mathrm{ord}_x(F) - \mathrm{ord}_y(F) = 4 + 4 - 0 - 0 = 8$. Now consider the transform of F under the monomial move $(x, y, z) \to (x, xy, xz)$, i.e.,

$$F^*(x, y, z) = F(x, xy, xz) = x^7 y^3 + x^4 y^4 + xz.$$

Thus $\deg_x(F^*) + \deg_y(F^*) - \mathrm{ord}_x(F^*) - \mathrm{ord}_y(F^*) = 7 + 4 - 1 - 0 = 10$.

Measure 3: $\deg_{xy} - \mathrm{ord}_x - \mathrm{ord}_y$

Next we considered the similar, but a little bit more refined measure

$$\mathrm{ord}_{xy}(F) - \mathrm{ord}_x(F) - \mathrm{ord}_y(F).$$

But in the following example it can be seen that it is not suited as a resolution invariant either:

Example 3. Let $p = 2$ and

$$F(x, y, z) = xyz(x + y + z) + xyz^2(x^2 + y^2 + xy).$$

Then $\deg_{xy}(F) - \mathrm{ord}_x(F) - \mathrm{ord}_y(F) = 3 - 1 - 1 = 1$ and $F^* = F(xz + z, yz + z, z)$ equals

$$F^* = z^4(x^2 y + xy^2 + xy) + z^6(x^3 + x^2 y + xy^2 + y^3 + x^3 y + xy^3).$$

Therefore $\deg_{xy}(F^*) - \mathrm{ord}_x(F^*) - \mathrm{ord}_y(F^*) = 3 - 0 - 0 = 3$.

One reason for this behavior under translational blowups is that it is impossible to control $\deg_{xy}$ after blowup (see for instance Proposition 9 in section 2.3).

Note that this measure also increases under monomial moves (see for instance example 1 above).

Measure 4: $\mathrm{ord}_{xy} - \mathrm{ord}_x - \mathrm{ord}_y$

Since $\deg_{xy}$ is not controllable under translational blowups, whereas ord_{xy} is (see Proposition 9 in section 2.3), it is reasonable to consider the measure

$$\mathrm{ord}_{xy}(F) - \mathrm{ord}_x(F) - \mathrm{ord}_y(F).$$

But it still isn't able to measure the improvement of F under translational blowups: In the last example one would have

$$
\begin{aligned}
\mathrm{ord}_{xy}(F) - \mathrm{ord}_x(F) - \mathrm{ord}_y(F) &= 2 - 1 - 1 = 0, \\
\mathrm{ord}_{xy}(F^*) - \mathrm{ord}_x(F^*) - \mathrm{ord}_y(F^*) &= 2 - 0 - 0 = 2.
\end{aligned}
$$

Measure 5: shade $= \mathrm{ord} - \mathrm{ord}_x - \mathrm{ord}_y - \mathrm{ord}_z$

In analogy to the local resolution invariant used for the proof of resolution of singularities over fields of characteristic zero, I started to examine this measure also in the case of characteristic $p > 0$. In a first step I examined the purely inseparable surface case (see section 1.9 of chapter 1). In this situation I was able to find, besides the new invariant "height–bonus", an appropriate "correction" of the measure "shade " which yields a strictly decreasing resolution invariant. My next goal was to generalize this measure to threefolds, i.e., set

$$\mathrm{shade}(F) = \mathrm{ord}(F) - \mathrm{ord}_x(F) - \mathrm{ord}_y(F) - \mathrm{ord}_z(F),$$

and study its behavior (see figure 2.2 for an illustration of the shade of F). This treatment is done in detail in section 2.6. Though I was up to now not able to find a "correction term" for it which yields a strictly decreasing invariant, I still believe that it is the most promising measure which I have considered so far. One reason for this is that the shade behaves well under monomial moves. A second reason is that a result of Moh (cf. [38] and [20]) implies that the shade can increase under translational blowups at most by 1 (see Propositions 9 and Corollary 5 respectively Proposition 11 and Corollary 6). But one possibly needs further insight in the situation after such an increase and a good idea to define an appropriate "correction term" which ensures the decreasing of the invariant.

2.6   Behavior of the shade under point blowups

The goal of this section is to study the behavior of the shade under point blowups. Let for this purpose a be a closed point of $\mathbb{A}^3_K$ and $\mathcal{F}$ a fixed local flag in $\mathbb{A}^3_K$ at a. Denote by $\pi : \widetilde{\mathbb{A}}^3_K \to \mathbb{A}^3$ the blowup of $\mathbb{A}^3_K$ with center $Z = \{a\}$ and exceptional divisor $E = \pi^{-1}(Z) \cong \mathbb{P}^2$. Furthermore denote by R' the Rees algebra of the coordinate ring R of $\mathbb{A}^3_K$ and by $f^* \in R'/R'^p$ the total transform of an element $f \in Q = R/R^p$ under π, which is defined as the equivalence class of the total transform F^* of a representative F of f.

Let f in the sequel be an element of $Q = R/R^p$. Denote by d the order of f and by f_d its initial form. Further the *parity* $\mathrm{par}(e)$ of an element $e \in \mathbb{N}$ is defined as 1, if $e \equiv 0 \bmod p$, and 0 otherwise.

Fix throughout this section subordinate coordinates $(x, y, z) \in \mathcal{C}_\mathcal{F}$ at a realizing the shade of f, i.e., satisfying

$$\mathrm{shade}(F) = \mathrm{shade}(f),$$

where $F(x, y, z)$ denotes the expansion of $f \in Q$ with respect to x, y and z. Let $a' \in E$ be a point above a. Depending on the point $a' \in E$ and $s, t = 0$ or not, the expansion of the total transform F^* of f^* in $\widehat{R}'_{a'} \cong K[[x, y, z]]$ is given by one of the moves (A)–(G) defined in the last section of this chapter.

Remark 12. Note that there could be several different subordinate coordinates in $\mathcal{C}_\mathcal{F}$ realizing the shade of f. For sake of simplicity, we will always choose among these minimizing coordinates a triple (x, y, z) in which the blowup $\widehat{R}_a \to \widehat{R}'_{a'}$ can be described with as few translations as possible. This assumption implies for instance, that

under move (A) the situation $\operatorname{ord}_x(F^*) + \operatorname{ord}_y(F^*) \geq \operatorname{ord}_x(F) + \operatorname{ord}_y(F)$ can not occur (compare to remark 7 in chapter 1). This is due to the fact that the transformation $(x, y, z) \rightarrow (xz + sz, yz + tz, z)$ with $s, t \neq 0$ can be regarded as the composition of the translation $\varphi : (x, y, z) \rightarrow (x + sz, y + tz, z)$ and the monomial move $(D) : (x, y, z) \rightarrow (xz, yz, z)$. By the minimality of shade(F), the first one satisfies $\operatorname{ord}_x(\varphi(F)) + \operatorname{ord}_y(\varphi(F)) \leq \operatorname{ord}_x(F) + \operatorname{ord}_y(F)$ and the second clearly leaves the order with respect to the variable x respectively y invariant. Furthermore the same arguing shows that if $\operatorname{ord}_x(\varphi(F)) + \operatorname{ord}_y(\varphi(F)) = \operatorname{ord}_x(F) + \operatorname{ord}_y(F)$, then there exist subordinate coordinates realizing the shade of f such that the blowup is monomial. Similar considerations for the remaining translational moves yield that under the transformations (A), (B), (C) and (E) only the following situations have to be considered:

$$
\begin{array}{llll}
\text{Move (A):} & \operatorname{ord}_x(F^*) + \operatorname{ord}_y(F^*) & < & \operatorname{ord}_x(F) + \operatorname{ord}_y(F), \\
\text{Move (B):} & \operatorname{ord}_x(F^*) & < & \operatorname{ord}_x(F), \\
\text{Move (C):} & \operatorname{ord}_y(F^*) & < & \operatorname{ord}_y(F), \\
\text{Move (E):} & \operatorname{ord}_x(F^*) & < & \operatorname{ord}_x(F).
\end{array}
$$

2.6.1 Result of Moh & Consequences – General Case

Before studying the moves (A)–(G) separately in detail, we will first apply a result of Moh (see Proposition 2, page 989 in [38], or Theorem 3 in [20]) to the general situation of a homogenous polynomial in $m + n$ variables and a translation in $m - 1$ directions:

Proposition 7. *Let $F_d(\underline{x}, \underline{y})$ be a homogenous polynomial of degree d in the $m + n$ variables $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_m)$. Further let $\underline{w} = (y_1, \ldots, y_{m-1})$ and $\underline{t} = (t_1, \ldots, t_{m-1}, 0) \in (K^*)^{m-1} \times \{0\}$. Denote by F_d^+ the polynomial*

$$
F_d^+(\underline{x}, \underline{y}) = F_d(\underline{x}, \underline{y} + \underline{t} \cdot y_m).
$$

Then

$$
\operatorname{ord}_{\underline{w}}(F_d^+) \leq \text{shade}(F_d) + \text{par}(d),
$$

where for an element $H \in K[[\underline{x}, \underline{y}]]$ the shade of H is defined as $\text{shade}(H) := \operatorname{ord}(H) - \sum_{i=1}^n \operatorname{ord}_{x_i}(H) - \sum_{j=1}^m \operatorname{ord}_{y_j}(H)$.

Proof. Write the homogenous polynomial F_d as

$$
F_d(\underline{x}, \underline{y}) = \sum_{i_1 = o_{x_1}}^d \cdots \sum_{i_n = o_{x_n}}^d \sum_{j_1 = o_{y_1}}^d \cdots \sum_{j_{m-1} = o_{y_{m-1}}}^d c_{\underline{ij}} \cdot \underline{x}^{\underline{i}} \cdot \underline{w}^{\underline{j}} \cdot y_m^{d - |\underline{i}| - |\underline{j}|}
$$

with $\underline{i} = (i_1, \ldots, i_n)$, $\underline{j} = (j_1, \ldots, j_{m-1})$, $|\underline{i}| = i_1 + \ldots + i_n$, $|\underline{j}| = j_1 + \ldots + j_{m-1}$, $c_{\underline{ij}} \in K$ and where o_{x_i} respectively o_{y_j} denote $\operatorname{ord}_{x_i}(F_d)$ respectively $\operatorname{ord}_{y_j}(F_d)$. Now decompose F_d in the following way: $F_d = \sum_{i_1 = o_{x_1}}^d \cdots \sum_{i_n = o_{x_n}}^d F_{d, \underline{i}}$ with

$$
F_{d, \underline{i}} = \underline{x}^{\underline{i}} \cdot \sum_{j_1 = o_{y_1}}^d \cdots \sum_{j_{m-1} = o_{y_{m-1}}}^d c_{\underline{ij}} \cdot \underline{w}^{\underline{j}} \cdot y_m^{d - |\underline{i}| - |\underline{j}|}.
$$

We will prove the following assertion, which immediately implies the inequality of the proposition (for illustrations in the situations $n = 1, m = 2$ respectively $n = 0, m = 3$ see figures 2.4 and 2.7):

Claim ($\star$). Set $F_{d,\underline{i}}^+(x, y, z) = F_{d,\underline{i}}(\underline{x}, \underline{y} + \underline{t} \cdot y_m)$. Then all polynomials $F_{d,\underline{i}} \neq 0$ in the expansion of F_d satisfy

$$\operatorname{ord}_{\underline{w}}(F_{d,\underline{i}}^+) \leq \operatorname{shade}(F_d) - \left(|\underline{i}| - \sum_{i=1}^n \operatorname{ord}_{x_i}(F_d)\right) + \operatorname{par}(\underline{i}) \cdot \operatorname{par}(d),$$

where $\operatorname{par}(\underline{i}) := \operatorname{par}(i_1) \cdots \operatorname{par}(i_n)$.
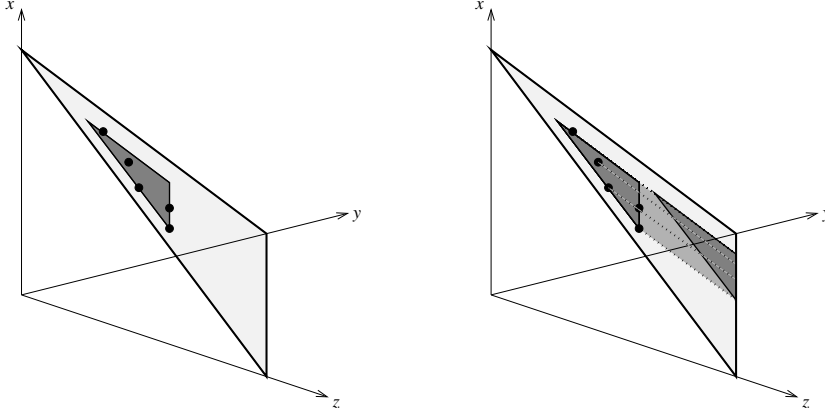

Figure 2.4: Geometric illustration of the claim ($\star$) appearing in the proof of Proposition 7, specialized to the case $n = 1, m = 2$.

Set $u := \operatorname{ord}_{\underline{w}}(F_{d,\underline{i}}^+)$.

(1) Let us first examine the case $\operatorname{par}(\underline{i}) \cdot \operatorname{par}(d) = 1$, i.e., that $i_1, \ldots, i_n, d \equiv 0 \bmod p$.

(a) Note first that the term $y_1^{o_{y_1}} \cdots y_{m-1}^{o_{y_{m-1}}} \cdot y_m^{o_{y_m}}$ divides $F_{d,\underline{i}}$. Consequently $F_{d,\underline{i}}^+ \in y_m^{o_{y_m}} \langle y_1 + t_1 y_m \rangle^{o_{y_1}} \cdots \langle y_{m-1} + t_{m-1} y_m \rangle^{o_{y_{m-1}}}$. Thus for $1 \leq j < m$

$$\partial_{y_j} F_{d,\underline{i}}^+ \in y_m^{o_{y_m}} \langle y_1 + t_1 y_m \rangle^{o_{y_1}} \cdots \langle y_j + t_j y_m \rangle^{o_{y_j} - 1} \cdots \langle y_{m-1} + t_{m-1} y_m \rangle^{o_{y_{m-1}}}.$$

(b) By definition of $u = \operatorname{ord}_{\underline{w}}(F_{d,\underline{i}}^+)$, there exist polynomials $R_\alpha(\underline{w}, y_m)$ such that

$$F_{d,\underline{i}}^+(\underline{x}, \underline{y}) = \underline{x}^{\underline{i}} \sum_{|\alpha|=u} R_\alpha(\underline{w}, y_m) \underline{w}^\alpha$$

with $\alpha = (\alpha_1, \ldots, \alpha_{m-1})$ and where at least one tuple $\alpha \in \mathbb{N}^{m-1}$ with $|\alpha| = u$ satisfies $R_\alpha(0, y_m) \neq 0$ and $\alpha \notin p \cdot \mathbb{N}^{m-1}$. Choose such an α. Assume w.l.o.g. that $\alpha_1 \notin p \cdot \mathbb{N}$. Then $\partial_{y_1} F_{d,\underline{i}}^+ \neq 0$ and

$$\partial_{y_1} F_{d,\underline{i}}^+ \in \langle y_1 \rangle^{\alpha_1 - 1} \cdot \langle y_2 \rangle^{\alpha_2} \cdots \langle y_{m-1} \rangle^{\alpha_{m-1}}.$$

Together observations (a) und (b) yield

$$\partial_{y_1} F_{d,\underline{i}}^+ \in y_m^{o_{y_m}} \langle y_1 + t_1 y_m \rangle^{o_{y_1} - 1} \cdots \langle y_{m-1} + t_{m-1} y_m \rangle^{o_{y_{m-1}}} \cap \langle \underline{w} \rangle^{u-1}.$$

And since by assumption $t_j \neq 0$ for all $1 \leq j < m$, this yields

$$\partial_{y_1} F_{d,\underline{i}}^+ \in y_m^{o_{y_m}} \langle y_1 + t_1 y_m \rangle^{o_{y_1} - 1} \cdots \langle y_{m-1} + t_{m-1} y_m \rangle^{o_{y_{m-1}}} \cdot \langle \underline{w} \rangle^{u-1}.$$

Due to $\operatorname{ord}(F_{d,\underline{i}}^{+}) = \operatorname{ord}(F_{d,\underline{i}}) = d - |\underline{i}|$, it follows that

$$\underbrace{(o_{x_1} + \ldots + o_{x_n}) + (o_{y_1} + \ldots + o_{y_m} - 1)}_{=d-\operatorname{shade}(F_d)-1} + u - 1 - (o_{x_1} + \ldots + o_{x_n}) \le d - |\underline{i}| - 1.$$

Hence $\operatorname{ord}_{\underline{w}}(F_{d,\underline{i}}^{+}) = u \le \operatorname{shade}(F_d) - (|\underline{i}| - o_{x_1} - \ldots - o_{x_n}) + 1$ as asserted.

(2) In the same way as in (1) one can deduce that in the case $\operatorname{par}(\underline{i}) \cdot \operatorname{par}(d) = 0$ one has

$$F_{d,\underline{i}}^{+} \in y_m^{o_{y_m}} \langle y_1 + t_1 y_m \rangle^{o_{y_1}} \cdots \langle y_{m-1} + t_{m-1} y_m \rangle^{o_{y_{m-1}}} \cdot \langle \underline{w} \rangle^{u}$$

This implies $\operatorname{ord}_{\underline{w}}(F_{d,\underline{i}}^{+}) = u \le \operatorname{shade}(F_d) - (|\underline{i}| - o_{x_1} - \ldots - o_{x_n})$.

$\square$

The last proposition can be used to prove the following relation between the shade of F respectively F^* (compare to Lemma 10 in chapter 1):

Proposition 8. *Let $F(\underline{x}, \underline{y}) \in K[[\underline{x}, \underline{y}]]$ with $\underline{x} = (x_1, \ldots, x_n)$, $\underline{y} = (y_1, \ldots, y_m)$ and $d = \operatorname{ord}(F)$. Further let $\underline{t} = (t_1, \ldots, t_{m-1}, 0) \in (K^*)^{m-1} \times \{0\}$. Set $F^*(\underline{x}, \underline{y}) = F(\underline{x} \cdot y_m, \underline{y} \cdot y_m + \underline{t} \cdot y_m)$. Then*

$$\operatorname{shade}(F^*) + \sum_{j=1}^{m-1} \operatorname{ord}_{y_j}(F^*) \le \operatorname{shade}(F) + \operatorname{par}(d).$$

Proof. Decompose F similarly as in the proof of Proposition 7 into

$$
\begin{aligned}
F(\underline{x}, \underline{y}) &= F_d(\underline{x}, \underline{y}) + H(\underline{x}, \underline{y}) \\
&= \sum_{i_1=o_{x_1}}^{d} \cdots \sum_{i_n=o_{x_n}}^{d} F_{d,\underline{i}}(\underline{x}, \underline{y}) + H(\underline{x}, \underline{y})
\end{aligned}
$$

with

$$F_{d,\underline{i}}(\underline{x}, \underline{y}) := \underline{x}^{\underline{i}} \cdot \sum_{j_1=o_{y_1}}^{d} \cdots \sum_{j_{m-1}=o_{y_{m-1}}}^{d} c_{\underline{i}\underline{j}} \cdot \underline{w}^{\underline{j}} \cdot y_m^{d-|\underline{i}|-|\underline{j}|},$$

where $\underline{w} = (y_1, \ldots, y_{m-1})$ and $\operatorname{ord}(H) > d$. Applying the transformation $(\underline{x}, \underline{y}) \to (\underline{x} \cdot y_m, \underline{y} \cdot y_m + \underline{t} \cdot y_m)$ yields

$$
\begin{aligned}
F^*(\underline{x}, \underline{y}) &= F(\underline{x} \cdot y_m, \underline{y} \cdot y_m + \underline{t} \cdot y_m) \\
&= \sum_{i_1=o_{x_1}}^{d} \cdots \sum_{i_n=o_{x_n}}^{d} F_{d,\underline{i}}^*(\underline{x}, \underline{y}) + H(\underline{x} \cdot y_m, \underline{y} \cdot y_m + \underline{t} \cdot y_m)
\end{aligned}
$$

with

$$F_{d,\underline{i}}^*(\underline{x}, \underline{y}) = \underline{x}^{\underline{i}} \cdot y_m^d \cdot \sum_{j_1=o_{y_1}}^{d} \cdots \sum_{j_{m-1}=o_{y_{m-1}}}^{d} c_{\underline{i}\underline{j}} \cdot (\underline{w} + \underline{t}')^{\underline{j}},$$

where $\underline{t}' := (t_1, \ldots, t_{m-1})$. It is obvious that $\operatorname{ord}_{y_m}(F_d(\underline{x} \cdot y_m, \underline{y} \cdot y_m + \underline{t} \cdot y_m)) = d$ and $\operatorname{ord}_z(H(\underline{x} \cdot y_m, \underline{y} \cdot y_m + \underline{t} \cdot y_m)) \ge d+1$. Moreover one can deduce from the proof of Proposition 7 that all $F_{d,\underline{i}}(\underline{x}, \underline{y}) \ne 0$ in the expansion of F_d satisfy

$$\operatorname{ord}_{\underline{w}}(F_{d,\underline{i}}^*) \le \operatorname{shade}(F_d) - \left(|\underline{i}| - \sum_{i=1}^{n} \operatorname{ord}_{x_i}(F_d)\right) + \operatorname{par}(\underline{i}) \cdot \operatorname{par}(d).$$

This immediately implies

$$
\begin{aligned}
\mathrm{ord}(F^*_{d,\underline{i}}(\underline{x},\underline{y})) \;\leq\;& |\underline{i}| + d + \left(\mathrm{shade}(F_d) - |\underline{i}| + \sum_{i=1}^{n} \mathrm{ord}_{x_i}(F_d) + \mathrm{par}(\underline{i}) \cdot \mathrm{par}(d) \right) \\
=\;& 2d - \sum_{j=1}^{m} \mathrm{ord}_{y_j}(F_d) + \mathrm{par}(\underline{i}) \cdot \mathrm{par}(d).
\end{aligned}
$$

Hence $\mathrm{ord}(F^*) \leq 2d - \sum_{j=1}^{m} \mathrm{ord}_{y_j}(F_d) + \mathrm{par}(d)$. Note further that $\mathrm{ord}_{y_m}(F^*) = d$ and $\mathrm{ord}_{x_i}(F^*) = \mathrm{ord}_{x_i}(F)$ for $1 \leq i \leq n$. Consequently

$$
\begin{aligned}
\mathrm{shade}(F^*) + \sum_{j=1}^{m-1} \mathrm{ord}_{y_j}(F^*) \;=\;& \mathrm{ord}(F^*) - \sum_{i=1}^{n} \mathrm{ord}_{x_i}(F^*) - \mathrm{ord}_{y_m}(F^*) \\
\leq\;& d - \sum_{i=1}^{n} \mathrm{ord}_{x_i}(F) - \sum_{j=1}^{m} \mathrm{ord}_{y_j}(F_d) + \mathrm{par}(d) \\
\leq\;& d - \sum_{i=1}^{n} \mathrm{ord}_{x_i}(F) - \sum_{j=1}^{m} \mathrm{ord}_{y_j}(F) + \mathrm{par}(d) \\
=\;& \mathrm{shade}(F) + \mathrm{par}(d).
\end{aligned}
$$

$\square$

The last proposition immediately implies the following corollary:

Corollary 4. *Let be the setting as in the last proposition. Then*

$$
\mathrm{shade}(F^*) \leq \mathrm{shade}(F) + \mathrm{par}(d).
$$

2.6.2 Translational moves with translations in two directions

In this section we want to study the behavior of the shade under translation moves of the form $F^*(x, y, z) = F(xz + sz, yz + tz, z)$ with $s, t \in K^*$ (compare to the similar treatment in the case of one dimension less in part (A) of section 1.6.1 in chapter 1).

Result of Moh for move (A)

Applying Proposition 7 of section 2.6.1 to the current situation yields:

Proposition 9. *Let F_d be a homogenous polynomial of degree d and $F_d^+(x, y, z) = F_d(x + sz, y + tz, z)$ with $s, t \neq 0$. Then*

$$
\mathrm{ord}_{xy}(F_d^+) \leq \mathrm{shade}(F_d) + \mathrm{par}(d).
$$

Remark 13. In the case of $\mathrm{par}(d) = 0$ one can prove the statement of the last proposition also in a more geometric manner, which we will indicate in the sequel. Note that in this case none of the exponents (α, β, γ) of terms $c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma$ occurring in the expansion of F_d and F_d^+ are elements of $p \cdot \mathbb{N}^3$. Therefore one doesn't have to take care about "holes" in the Newton polyhedron.

For a better understanding consider first the situation of a homogenous polynomial F_d of degree d in only *two variables*, i.e.,

$$F_d(y, z) = \sum_{i=0}^{a} c_i y^{n-i} z^{m+i}$$

with $c_i \in K$, $c_0, c_a \neq 0$. Now consider the transformation

$$\varphi : (y, z) \rightarrow (y + tz, z),$$
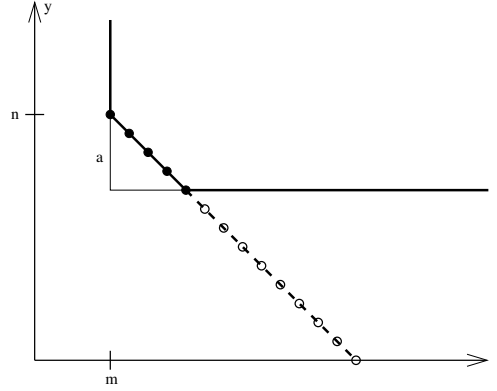
where $t \in K^*$ (see figure 2.5).

Figure 2.5: The transformation of F_d under $\varphi : (y, z) \rightarrow (y + tz, z)$, $t \neq 0$.

This yields

$$\varphi(F) = \sum_{i=0}^{a} c_i (y + tz)^{n-i} z^{m+i} = \sum_{i=0}^{a} \sum_{j=0}^{n-i} c_i \binom{n-i}{j} t^j y^{n-i-j} z^{m+i+j}$$

$$= \sum_{i=0}^{n} c_i' y^i z^{m+n-i},$$

where the new coefficients c_i' of $\varphi(F)$ are for $i = 0, \ldots, a$ given by the following formula:

$$\begin{pmatrix} c_0' \cdot t^0 \\ c_1' \cdot t^1 \\ \vdots \\ c_a' \cdot t^a \end{pmatrix} = t^n \cdot \underbrace{\begin{pmatrix} \binom{n}{0} & \binom{n-1}{0} & \cdots & \binom{n-a-1}{0} & \binom{n-a}{0} \\ \binom{n}{1} & \binom{n-1}{1} & \cdots & \binom{n-a-1}{1} & \binom{n-a}{1} \\ \vdots & \cdots & \vdots & \cdots & \cdots \\ \binom{n}{a} & \binom{n-1}{a} & \cdots & \binom{n-a-1}{a} & \binom{n-a}{a} \end{pmatrix}}_{=:M} \cdot \begin{pmatrix} c_0 \cdot t^{-0} \\ c_1 \cdot t^{-1} \\ \vdots \\ c_a \cdot t^{-a} \end{pmatrix}$$

It can be checked that the matrix M is invertible (see [50]). Or formulated differently, the relation between the coefficients $c_0, \ldots, c_a$ of F_d and the coefficients $c_0', \ldots, c_a'$ of F_d^+ is bijective. Since by assumption $c_0, c_a \neq 0$, it follows that at least one of the coefficients $c_0', \ldots, c_a'$ is not equal to 0. This implies in the case $d \notin p \cdot \mathbb{N}$ immediately that $\text{ord}_y(F_d^+) \leq a$.

Consider now – as in the last proposition – a homogenous polynomial F_d of degree d in *three variables* (see figure 2.6), i.e.,

$$F_d(x, y, z) = \sum_{i=o_x}^{d-o_y-o_z} \sum_{j=o_y}^{d-o_z-i} c_{ij} x^i y^j z^{d-i-j},$$

where o_x, o_y and o_z denote $\mathrm{ord}_x(F_d)$, $\mathrm{ord}_y(F_d)$ respectively $\mathrm{ord}_z(F_d)$, and its transform

$$F_d^+(x, y, z) = F_d(x + sz, y + tz, z) = \sum_{k=0}^{d} \sum_{l=0}^{d} c_{kl}' x^k y^l z^{d-k-l},$$

where $s, t \neq 0$. Recall that we are only considering the case $\mathrm{par}(d) = 0$.
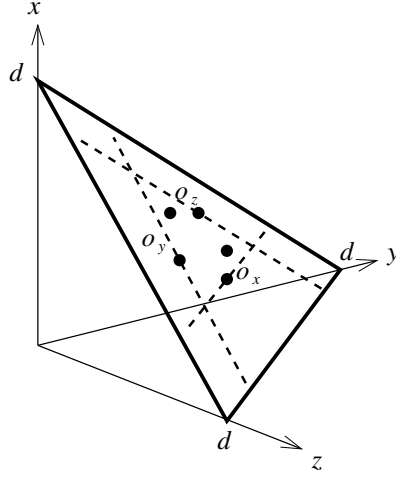

Figure 2.6: Illustration of a homogenous series F_d with degree d.

Then the following statements hold (cf. chapter 11 of [20] for more details): The transformation matrix $M = (M_{i,j;k,l})$ from the coefficients c_{ij} of F_d to the coefficients c_{kl}' of F_d^+ is given by

$$M_{i,j;k,l} = c_{ij} s^{i-k} t^{j-l} \binom{i}{k} \binom{j}{l}.$$

Furthermore the matrix

$$\left(M_{o_x+i, o_y+j; k, l} \right)_{0 \leq i+j, k+l \leq \mathrm{shade}(F_d)}$$

is invertible. This implies the assertion of the last proposition, i.e., the inequality $\mathrm{ord}_{xy} \leq \mathrm{shade}(F_d)$.

Note that the *main reason for considering the shade* instead of a different measure is that then the relation between the coefficients c_{ij} of F_d and the coefficients c_{kl}' of F_d^+ can be described by the invertible matrix $B_{i,j;k,l}$. If one uses a different measure, it is not clear at all how to find a bijective relation between the coefficients of F_d and F_d^+.

Geometrically the last proposition can be described in the following way (see figure 2.7): Consider the smallest equilateral triangle which circumscribes the support

supp(F_d) of F_d. Its vertices are given by $(d - o_y - o_z, o_y, o_z)$, $(o_x, d - o_x - o_z, o_z)$ and $(o_x, o_y, d - o_x - o_y)$. Then the triangle generated by the three vertices (shade$(F_d), 0, d -$ shade$(F_d))$, $(0, \text{shade}(F_d), d - \text{shade}(F_d))$ and $(0, 0, d)$ – which has the same size as the original one – contains at least one point $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ such that the corresponding monomial $x^\alpha y^\beta z^\gamma$ occurs in the expansion of F_d^+ and has a non vanishing coefficient.
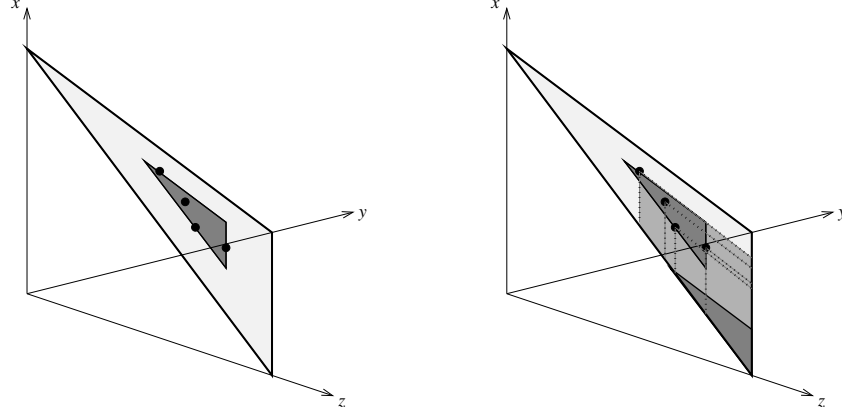

Figure 2.7: Geometric description of the phenomenon of Proposition 9.

Examples. In the sequel we will give some examples which illustrate that the bound of the previous inequality is sharp. Note that series F_d for which the inequality

$$\text{ord}_{xy}(F_d^+) = \text{shade}(F_d) + 1 \quad (\triangle)$$

holds, must have a very special form. Before we state some examples fulfilling $(\triangle)$, we will indicate two necessary conditions for such series F_d. Write F_d as

$$F_d(x, y, z) = x^k y^l z^m \cdot A(x, y, z)$$

with $k, l, m \in \mathbb{N}$, $A \in K[x, y, z]$ with $\deg(A) = e$. Assume for sake of simplicity that $k, l, m < p$. Then the invertibility of the transformation matrix $B_{i,j;k,l}$ (see last remark) implies that $(\triangle)$ can only occur if the following conditions are satisfied (for more details we refer to chapter 16 in [20] respectively Theorem 3 in section 1.10 of chapter 1):

(1) the degree $d = k + l + m + e$ of F_d is a multiple of p,

(2) $\overline{k} + \overline{l} + \overline{m} \le (\varphi_p(k, l, m) - 1) \cdot p$,

where $\varphi_p(n_1, \dots, n_m) := \#\{i; n_i \notin p \cdot \mathbb{N}\}$ and $\overline{k}, \overline{l}, \overline{m}$ denote the residues of k, l, m modulo p.

Example 4. Let $p = 2$ and

$$F_d(x, y, z) = xyz(x + y + z).$$

Consider the transform $F_d^+(x, y, z) = F_d(x + 1 \cdot z, y + 1 \cdot z, z)$ in R/R^p, i.e.,

$$F_d^+(x, y, z) = x^2 yz + xy^2 z + xyz^2.$$

Then: $\operatorname{ord}_{xy}(F_d^+) = 2 = 1 + 1 = \operatorname{shade}(F_d) + 1!$

Example 5. Let $p = 2$, $s, t = 1$ and

$$F_d(x, y, z) = xy(x^2 + y^2)$$

with $\operatorname{shade}(F_d) = 2$. Then

$$F_d^+(x, y, z) = x^3 y + xy^3 + x^3 z + y^3 z + x^2 yz + xy^2 z$$

with $\operatorname{ord}_{xy}(F_d^+) = 3$.

Example 6. Let $p = 3$ and

$$F_d(x, y, z) = x^2 y^2 z (x + y - z).$$

Consider its transform $F_d^+(x, y, z) = F_d(x + z, y + z, z)$, i.e.,

$$F_d^+(x, y, z) = xy \cdot (x^2 yz + xy^2 z - x^2 z^2 - xyz^2 - y^2 z^2 + xz^3 + yz^3 - z^4).$$

Then: $\operatorname{ord}_{xy}(F_d^+) = 2 = 1 + 1 = \operatorname{shade}(F_d) + 1.$

Example 7. Let $p = 3$, $s = 1$ and $t = 2$. Furthermore let

$$F_d(x, y, z) = xy(x - y)$$

with $\operatorname{shade}(F_d) = 1$. Then

$$F_d^+(x, y, z) = x^2 y + 2xy^2 + xyz + 2x^2 z + 2y^2 z$$

with $\operatorname{ord}_{xy}(F_d^+) = 2$.

Examples. Now we will consider the case $\operatorname{par}(d) = 0$, i.e., $d \notin p \cdot \mathbb{N}$, and state an example, where the inequality of the last proposition is strict.

Example 8. Let $p = 5$ and

$$F_d(x, y, z) = x^2 - 2xy + y^2$$

with $\operatorname{shade}(F_d) = \operatorname{ord}(F_d) - \operatorname{ord}_x(F_d) - \operatorname{ord}_y(F_d) - \operatorname{ord}_z(F_d) = 2 - 0 - 0 - 0 = 2$.
Now consider its transform $F_d^+(x, y, z) = F_d(x + 1 \cdot z, y + 1 \cdot z, z)$ in R/R^p, i.e.,

$$F_d^+(x, y, z) = x^2 - 2xy + y^2 = F_d(x, y, z).$$

Then: $\operatorname{ord}_{xy}(F_d^+) = 2 = \operatorname{shade}(F_d)!$

Behaviour of the shade under translational moves of type (A)

The next proposition is a specialization of Proposition 8 of section 2.6.1 and establishes the relation between the shade of F_d and F^* under move (A).

Proposition 10. *Let $F^*(x, y, z) = F(xz + sz, yz + tz, z)$ with $s, t \neq 0$ and $d = \mathrm{ord}(F)$. Then*

$$\mathrm{shade}(F^*) + \mathrm{ord}_x(F^*) + \mathrm{ord}_y(F^*) \leq \mathrm{shade}(F_d) + \mathrm{par}(d).$$

Remark 14. The last proposition shows that the shade may, as in the case of one dimension less, increase at most by 1! Moreover it suggests to define the "correction term" for an element $F \in K[[x, y, z]]$ according to the adjacency of F to the yz-plane and the xz-plane, i.e., in accordance with the values of $\mathrm{ord}_x(F)$ and $\mathrm{ord}_y(F)$ (see also remark 16). Further it is worth mentioning that this correction term is only of importance if $\mathrm{ord}(F^*)$ is given by the order of $(F_d)^*$ (cf. proof of Proposition 8 in section 2.6.1).

Proposition 10 immediately implies the following corollary:

Corollary 5. *Let $F^*(x, y, z) = F(xz + sz, yz + tz, z)$ with $s, t \neq 0$ and $d = \mathrm{ord}(F)$. Then*

$$\mathrm{shade}(F^*) \leq \mathrm{shade}(F) + \mathrm{par}(d).$$

Examples. The bounds of the previous corollary are sharp too:

Example 9. Let $p = 2$ and $s, t = 1$. Consider

$$F(x, y, z) = xyz(x + y + z) + xyz^2(x^2 + y^2 + z^2)$$

with $\mathrm{shade}(F_d) = 1$. Then an easy computation yields that

$$F^* = z^4(xy + x^2y + xy^2) + z^6(y + x^3 + x^2y + xy^2 + y^3 + x^3y^2 + x^2y^3 + xy^4).$$

Consequently

$$\begin{aligned}
\mathrm{shade}(F^*) &= \mathrm{ord}(F^*) - \mathrm{ord}_x(F^*) - \mathrm{ord}_y(F^*) - \mathrm{ord}_z(F^*) \\
&= 6 - 0 - 0 - 4 = 2.
\end{aligned}$$

Example 10. Let again be $p = 2$ and $s, t = 1$. Furthermore let

$$F(x, y, z) = xyz(x + y + z) + xyz^2(x^2 + xy).$$

Then $\mathrm{shade}(F_d) = 1$ and

$$F^* = z^4(xy + x^2y + xy^2) + z^6(x + y + xy + x^3 + x^2y + x^3y).$$

Thus $\mathrm{shade}(F^*) = 6 - 0 - 0 - 4 = 2$ (see figure 2.8).

Example 11. Let $p = 2$, $s, t = 1$ and

$$F(x, y, z) = xy(x^2 + y^2 + xy) + xy^3(x^2z^2 + z^2 + x^2y^2).$$

Then a simple computation shows that $\mathrm{shade}(F^*) = 7 - 0 - 0 - 4 = 3 = 2 + 1 = \mathrm{shade}(F_d) + 1$.
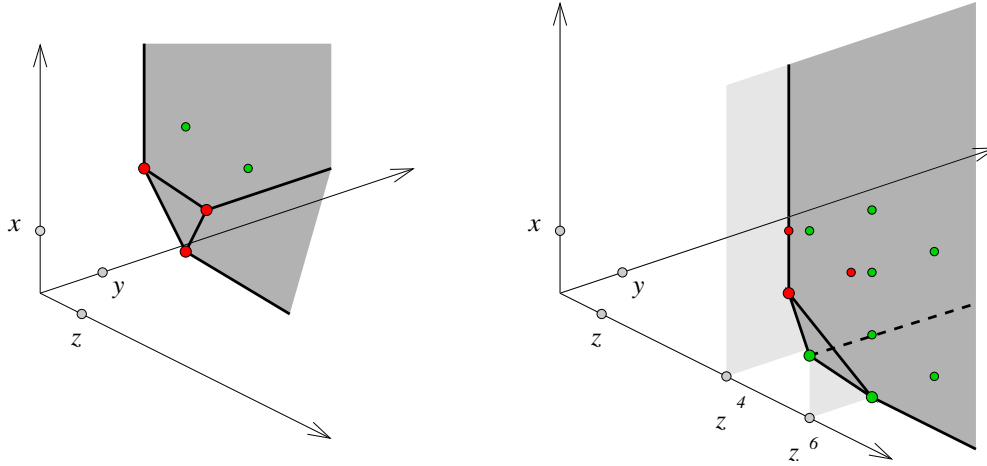
Figure 2.8: A series $F(x, y, z)$ with $\mathrm{shade}(F_d) = 1$ and $\mathrm{shade}(F^*) = 2$.

Example 12. Let $p = 5$ and $s, t = 1$. Consider

$$F(x, y, z) = x^2 - 2xy + y^2 + x^3 - 3x^2y + 3xy^2 - y^3$$

with $\mathrm{shade}(F_d) = 2$. Then

$$F^*(x, y, z) = z^2(x^2 - 2xy + y^2) + z^3(x^3 - 3x^2y + 3xy^2 - y^3)$$

and $\mathrm{shade}(F^*) = \mathrm{ord}(F^*) - \mathrm{ord}_x(F^*) - \mathrm{ord}_y(F^*) - \mathrm{ord}_z(F^*) = 4 - 0 - 0 - 2 = 2$.

Remark 15. Consider again the last example, where $p = 5$ and $F(x, y, z) = (x - y)^2 + (x - y)^3$. It is simple matter to check that successive translation moves of type (A) in the fixed local coordinates (x, y, z) yield a sequence of transforms $F^* = F(xz + 1 \cdot z, yz, z)$, $F^{**} = F^*(xz + 1 \cdot z, yz, z)$, ..., which all have the same shade as F:

$$
\begin{array}{ll}
F(x, y, z) = (x - y)^2 + (x - y)^3 & \mathrm{shade}(F) = 2 - 0 - 0 - 0 = 2, \\
F^*(x, y, z) = z^2(x - y)^2 + z^3(x - y)^3 & \mathrm{shade}(F^*) = 4 - 0 - 0 - 2 = 2, \\
F^{**}(x, y, z) = z^4(x - y)^2 + z^6(x - y)^3 & \mathrm{shade}(F^{**}) = 6 - 0 - 0 - 4 = 2, \\
\quad\quad\quad \vdots & \quad\quad\quad \vdots \\
F^{(i)}(x, y, z) = z^{2i}(x - y)^2 + z^{3i}(x - y)^3 & \mathrm{shade}(F^{(i)}) = (2i + 2) - 0 - 2i = 2.
\end{array}
$$

But note that the coordinate change $\varphi : (x, y, z) \to (x + y, y, z)$ would yield

$$\varphi(F) = x^2 + x^3 = x^2(1 + x)$$

and $\mathrm{shade}(\varphi(F)) = 2 - 2 - 0 - 0 = 0$. Thus f is already monomial!

Remark 16. As already indicated in remark 14, due to Proposition 10 a first guess for a "correction term" bonus(F) for the measure shade would be the following: Set bonus(F) equal to $1+\delta$ if $\mathrm{ord}_x(F) = 0 = \mathrm{ord}_y(F)$, equal to ε if $\mathrm{ord}_x(F)+\mathrm{ord}_y(F) = 1$ and equal to 0 otherwise, where $\delta, \varepsilon \in K^*$ with $0 < \varepsilon < \delta < 1$. Note that then the measure shade(F) $-$ bonus(F) decreases in all examples (except in example 12, where a linear coordinate change φ would actually yield shade($\varphi(F)$) $= 0$) stated in this section. But it is not clear, if this is just a coincidence. Moreover the correction term has to control the possibly increase under move (A) as well as potentially increases under the translational moves (B), (C) and (E).

Examination of initial forms without a specific monomial

During intensive investigation of the behavior of the shade under translational moves of type (A), the following question arised:

> **Question** ($\diamondsuit$)**.** Assume that the vertex $(\mathrm{ord}_x(F_d), \mathrm{ord}_y(F_d), d - \mathrm{ord}_x(F_d) - \mathrm{ord}_y(F_d))$ is not contained in the support $\mathrm{supp}(F_d)$ of F_d, where $d = \mathrm{ord}(F)$ and F_d denotes the initial form of F. Is then the bound of Proposition 9 not anymore sharp?

If the answer to the question would be positive, the situations where an increase of shade happens, could be restricted to series F whose initial form F_d contain the monomial corresponding to the vertex$(\mathrm{ord}_x(F_d), \mathrm{ord}_y(F_d), d - \mathrm{ord}_x(F_d) - \mathrm{ord}_y(F_d)) \in N(F)$. But unfortunately the answer to the question posed above is negative, which we will see in the sequel.

To tackle question ($\diamondsuit$), we will first to compute the transformation matrix between the coefficients c_{ij} of F_d and the coefficients c'_{kl} of F_d^+ for small values of shade (see remark 13) explicitly.

For this purpose let $F_d(x, y, z)$ be a homogenous polynomial in three variables of degree d and with shade(F_d) $= a$, i.e.,

$$F_d(x, y, z) = \sum_{i=0}^{a} \sum_{j=0}^{a-i} c_{ij} x^{m+i} y^{n+j} z^{d-m-i-n-j},$$

where $c_{ij} \in K$.

We want to investigate the behavior of F_d under a translational move $\varphi : (x, y, z) \rightarrow (x + sz, y + tz, z)$, where $s, t \in K^*$, under the assumption that the corner vertex $(m, n, d - m - n)$ is not contained in $\mathrm{supp}(F_d)$. We will hence assume that

$$c_{00} = 0.$$

Furthermore we will restrict to the case that

$$d \equiv 0 \bmod p,$$

since it is the more delicate one.
Is it possible that in this special configuration the bound of the inequality of Proposition 9 is not sharp, i.e., that

$$\mathrm{ord}_{xy}(\varphi(F_d)) < \mathrm{shade}(F_d) + 1$$

is fulfilled? To simplify the notation we will dehomogenize the problem by setting $z = 1$ and pose the same question for the polynomial

$$\widetilde{F}_d(x, y) = \sum_{i=0}^{a} \sum_{j=0}^{a-i} c_{ij} x^{m+i} y^{n+j}$$

with $c_{00} = 0$ and shade equal to a (see figure 2.9) and the translation

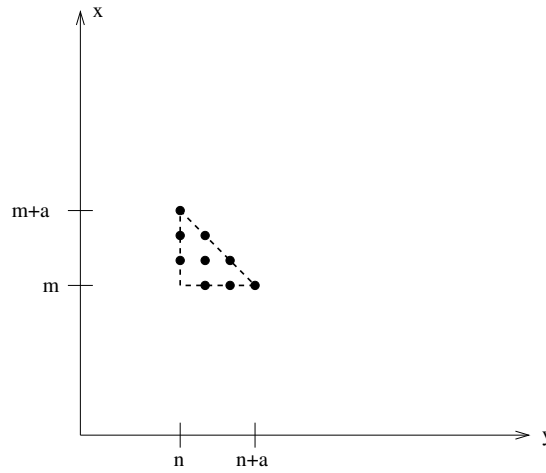$$\phi : (x, y) \rightarrow (x + s, y + t),$$

$s, t \in K^*$.

Figure 2.9: The support of an initial form F_d whose corner vertex is missing.

Applying ϕ yields

$$
\begin{aligned}
\phi(\widetilde{F}_d) \quad &= \quad \sum_{i=0}^{a} \sum_{j=0}^{a-i} c_{ij} (x + s)^{m+i} (y + t)^{n+j} \\
&= \quad \sum_{k=0}^{m} \sum_{l=0}^{n-i} c'_{kl} x^i y^j
\end{aligned}
$$

with coefficients $c'_{kl} \in K$. Note that the assumption $d \equiv 0 \bmod p$ implies that the coefficient c'_{00} (which corresponds in the homogenized situation to the monomial z^d) is equal to 0.

Case shade(F_d) = 1:

Let the shade of $F_d(x, y, z)$ be equal to 1, i.e., let $\widetilde{F}_d$ be of the form

$$\widetilde{F}_d(x, y) = x^m y^n \cdot (c_{10} x + c_{01} y).$$

Then the first two coefficients c'_{10} and c'_{01} of $\phi(\widetilde{F}_d))$ are given by the following matrix:

$$\begin{pmatrix} c'_{10} \cdot t^{-1} \\ c'_{01} \cdot s^{-1} \end{pmatrix} = s^{m-1}t^{n-1} \cdot \underbrace{\begin{pmatrix} 1 + \binom{m}{1} & \binom{m}{1} \\ \binom{n}{1} & 1 + \binom{n}{1} \end{pmatrix}}_{=:M} \cdot \begin{pmatrix} c_{10} \cdot s \\ c_{01} \cdot t \end{pmatrix}$$

This immediately implies that

$$\det(M) = 1 + m + n.$$

Therefore M is in general not an invertible matrix! Hence the the bound of Proposition 9 might still be strict. The following example illustrates that this is really the case:

Example 13. Let $p, d = 5$, $s, t = 1$ and $\widetilde{F}_d(x, y) = x^2 y^2 (x + y)$ with shade$(F_d) = 1$. Then

$$\phi(\widetilde{F}_d) = -x^2 + 2xy - y^2 + \text{ terms of higher order}.$$

One can easily check that $\operatorname{ord}_{xy}(F_d(x+1 \cdot z, y+1 \cdot z, z) = \operatorname{ord}_{xy}(\widetilde{F}_d(x+1, y+1)) = 2$, so the bound of Proposition 9 is strict!

Case shade$(F_d) = 2$:

Let the shade of $F(x, y, z)$ be equal to 2, i.e.,

$$\widetilde{F}_d(x, y) = x^m y^n \cdot (c_{10}x + c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{01}y).$$

Then the coefficients c'_{kl} of $\phi(\widetilde{F}_d)$ with $0 \le k + l \le 2$ are given by the following formula:

$$\begin{pmatrix} c'_{10} \cdot s^{-1}t^{-2} \\ c'_{20} \cdot t^{-2} \\ c'_{11} \cdot s^{-1}t^{-1} \\ c'_{02} \cdot s^{-2} \\ c'_{01} \cdot s^{-2}t^{-1} \end{pmatrix} = s^{m-2}t^{n-2} \cdot M \cdot \begin{pmatrix} c_{10} \cdot s^1 \\ c_{20} \cdot s^2 \\ c_{11} \cdot s^1 t^1 \\ c_{02} \cdot t^2 \\ c_{01} \cdot t^1 \end{pmatrix}$$

with M equal to

$$\begin{pmatrix} 1 + \binom{m}{1} & 2 + \binom{m}{1} & 1 + \binom{m}{1} & \binom{m}{1} & \binom{m}{1} \\ \binom{m}{1} + \binom{m}{2} & 1 + 2\binom{m}{1} + \binom{m}{2} & \binom{m}{1} + \binom{m}{2} & \binom{m}{2} & \binom{m}{2} \\ \binom{n}{1} + \binom{m}{1}\binom{n}{1} & \binom{m}{1}\binom{n}{1} + 2\binom{n}{1} & 1 + \binom{m}{1} + \binom{n}{1} + \binom{m}{1}\binom{n}{1} & 2\binom{m}{1} + \binom{m}{1}\binom{n}{1} & \binom{m}{1} + \binom{m}{1}\binom{n}{1} \\ \binom{n}{2} & \binom{n}{2} & \binom{n}{1} + \binom{n}{2} & 1 + 2\binom{n}{1} + \binom{n}{2} & \binom{n}{1} + \binom{n}{2} \\ \binom{n}{1} & \binom{n}{1} & 1 + \binom{n}{1} & 2 + \binom{n}{1} & 1 + \binom{n}{1} \end{pmatrix}$$

Since the determinant of this matrix is given by

$$\det(M) = \frac{1}{2} \cdot (m^2 + n^2) + mn + \frac{3}{2} \cdot (m + n) + 1 = \frac{1}{2} \cdot (m + n + 1) \cdot (m + n + 2)$$

it is easy to see that the matrix M is in general not invertible! Here is an example where the bound of Proposition 9 is really sharp:

Example 14. Let $p = 3$, $d = 6$, $s, t = 1$ and $\widetilde{F}_d(x, y) = xy \cdot (2x + x^2 + xy + y^2 + 2y)$ with shade$(F_d) = 2$. Then

$$\phi(\widetilde{F}_d) = xy(x + y + x^2 + xy + y^2).$$

Thus $\operatorname{ord}_{xy}(F_d(x + 1 \cdot z, y + 1 \cdot z, z)) = \operatorname{ord}_{xy}(\widetilde{F}_d(x + 1, y + 1)) = 3$.

Answer. Examples 13 and 14 show that the question ($\Diamond$) posed in the beginning of this paragraph can unfortunately not be answered positively.

2.6.3 Translational moves with translations in one direction

We now want to examine the behavior of the shade under the three translational moves (B), (C) and (E), which involve only translations in one direction:

(B) $F^*(x,y,z) = F(xz + sz, yz, z), s \in K^*,$

(C) $F^*(x,y,z) = F(xz, yz + tz, z), t \in K^*,$

(E) $F^*(x,y,z) = F(xy + ry, y, yz), r \in K^*.$

We use here the same notations as in section 2.6.2, i.e., f denotes an element of $Q = R/R^p$ and $F \in K[[x,y,z]]$ its expansion with respect to subordinate local coordinates (x,y,z). Moreover we again denote by $d = \mathrm{ord}(f)$ and by f_d be the initial form of f. As before, we define the *parity* $\mathrm{par}(e)$ of $e \in \mathbb{N}$ as 1, if $e \equiv 0 \bmod p$, and 0 otherwise.

Result of Moh for moves (B), (C) and (E)

Specialization of Proposition 7 of section 2.6.1 to the moves (B), (C) and (E) results in:

Proposition 11. *Let F_d be a homogenous polynomial of degree d. Set $F_{d,(B)}^+(x,y,z) = F_d(x + sz, y, z)$, $F_{d,(C)}^+(x,y,z) = F_d(x, y + tz, z)$ and $F_{d,(E)}^+(x,y,z) = F_d(x + ry, y, z)$ with $r, s, t \neq 0$. Then*

$$\text{Move (B):} \qquad \mathrm{ord}_x\left(F_{d,(B)}^+\right) \leq \mathrm{shade}(F_d) + \mathrm{par}(d),$$
$$\text{Move (C):} \qquad \mathrm{ord}_y\left(F_{d,(C)}^+\right) \leq \mathrm{shade}(F_d) + \mathrm{par}(d),$$
$$\text{Move (E):} \qquad \mathrm{ord}_x\left(F_{d,(E)}^+\right) \leq \mathrm{shade}(F_d) + \mathrm{par}(d).$$

Remark 17. Due to the proof of Proposition 7 in section 2.6.1 it follows moreover, that the bounds of the inequalities in Proposition 11 are only sharp, if the homogenous polynomial $F_d(x,y,z)$ is of very special shape, i.e.,

$$\text{Move (B):} \qquad F_d(x,y,z) = y^{o_y} \cdot \sum_{i=o_x}^{d-o_y-o_z} c_i x^i z^{d-o_y-i} \quad \text{with} \quad o_y, d \in p \cdot \mathbb{N},$$

$$\text{Move (C):} \qquad F_d(x,y,z) = x^{o_x} \cdot \sum_{i=o_y}^{d-o_x-o_z} c_i y^i z^{d-o_x-i} \quad \text{with} \quad o_x, d \in p \cdot \mathbb{N},$$

$$\text{Move (E):} \qquad F_d(x,y,z) = z^{o_z} \cdot \sum_{i=o_x}^{d-o_y-o_z} c_i x^i y^{d-o_z-i} \quad \text{with} \quad o_z, d \in p \cdot \mathbb{N},$$

where o_x, o_y, o_z denote $\mathrm{ord}_x(F_d)$, $\mathrm{ord}_y(F_d)$ respectively $\mathrm{ord}_z(F_d)$.

Behavior of the shade under the translational moves (B), (C) and (E)

Applying Proposition 8 of section 2.6.1 to the current moves yields the following proposition (compare to Proposition 10 of section 2.6.2).

Proposition 12. *Let $F \in K[[x, y, z]]$ with $d = \operatorname{ord}(F)$. Further denote by $F^*_{(B)}(x, y, z) = F(xz + sz, yz, z)$, $F^*_{(C)}(x, y, z) = F(xz, yz + tz, z)$ and $F^*_{(E)}(x, y, z) = F(xy + ry, y, yz)$ with $r, s, t \neq 0$. Then*

$$\textit{Move (B):} \qquad \operatorname{shade}(F^*_{(B)}) + \operatorname{ord}_x\left(F^*_{(B)}\right) \leq \operatorname{shade}(F) + \operatorname{par}(d),$$

$$\textit{Move (C):} \qquad \operatorname{shade}(F^*_{(C)}) + \operatorname{ord}_y\left(F^*_{(C)}\right) \leq \operatorname{shade}(F) + \operatorname{par}(d),$$

$$\textit{Move (E):} \qquad \operatorname{shade}(F^*_{(E)}) + \operatorname{ord}_x\left(F^*_{(E)}\right) \leq \operatorname{shade}(F) + \operatorname{par}(d).$$

Remark 18. The proposition above shows that the shade may increase under the translational moves (B), (C) and (E) also at most by 1 (compare to Proposition 10 and remark 14 in section 2.6.2). Further, similarly to the case of move (A), Proposition 12 indicates to define the "correction term" to the measure shade for an element $F \in K[[x, y, z]]$ in accordance to values of $\operatorname{ord}_x(F)$ respectively $\operatorname{ord}_y(F)$, i.e., according to the adjacency of F to the yz- respectively xz-plane. Nevertheless it is not clear at all how to define for elements $F \in K[[x, y, z]]$ *one* common "correction term" $\operatorname{bonus}(F)$ to the measure $\operatorname{shade}(F)$ in order to get an invariant which (strictly) decreases for all translational moves (A), (B), (C) and (E). Note that the correction term defined in remark 16 is for instance not suitable to adjust increases of the shade under moves (B), (C) and (E). One possibly needs further insight in the situation after such an increase and a good idea to define an appropriate "correction term" which ensures the decreasing of the invariant.

The next statement is an immediate corollary to the last proposition:

Corollary 6. *Let $F^*(x, y, z)$ be the transform of $F \in K[[x, y, z]]$ under one of the translational moves (B), (C) or (E) and $d = \operatorname{ord}(F)$. Then*

$$\operatorname{shade}(F^*) \leq \operatorname{shade}(F) + \operatorname{par}(d).$$

2.6.4 Monomial moves

In this section we will investigate the behavior of the shade under the remaining three moves

(D) $F^*(x, y, z) = F(xz, yz, z)$.

(F) $F^*(x, y, z) = F(xy, y, yz)$,

(G) $F^*(x, y, z) = F(x, xy, xz)$,

We will show that the shade can not increase under these transformations.

First note that in these three moves the positive characteristic of the field K does not

play an important role: Consider for this purpose for example move (D), i.e., the monomial transformation $\psi : (x, y, z) \rightarrow (xz, yz, z)$. Then the vertices (α, β, γ) of the Newton polyhedron $N(F)$ of F transform via ψ in the following way:

$$(\alpha, \beta, \gamma) \in N(F) \rightarrow (\alpha, \beta, \alpha + \beta + \gamma) \in N(F^*).$$

It is a simple matter to check that if the vertex $(\alpha, \beta, \gamma) \in N(F)$ is not an element of $p \cdot \mathbb{N}^3$, then neither is $(\alpha, \beta, \alpha + \beta + \gamma) \in N(F^*)$. Thus under the monomial moves there doesn't occur the phenomena, as one has in the case of moves (A), (B), (C) and (E), that certain monomials might vanish when applying the transformation yielding F^*. Therefore monomial moves are much simpler to handle than translational moves.

The following proposition shows that the shade can not increase under monomial moves:

Proposition 13. *Let F^* be one of the three monomial moves (D), (F) and (G), then*

$$\text{shade}(F^*) \leq \text{shade}(F).$$

Proof. We will prove the result in the case of move (D), i.e., let

$$F^*(x, y, z) = F(xz, yz, z).$$

The proof in the two remaining moves runs completely analogously.
Choose subordinate coordinates (x, y, z) such that $\text{shade}(F) = \text{shade}(f)$. It is obvious that $\text{ord}_x(F^*) = \text{ord}_x(F)$ and $\text{ord}_y(F^*) = \text{ord}_y(F)$. Further one can easily see that $\text{ord}_z(F^*) = \text{ord}(F)$. Moreover it is a simple matter to check that

$$\text{ord}(F^*) \leq 2 \cdot \text{ord}(F) - \text{ord}_z(F).$$

Altogether this yields

$$
\begin{aligned}
\text{shade}(F^*) &= \text{ord}(F^*) - \text{ord}_x(F^*) - \text{ord}_y(F^*) - \text{ord}_z(F^*) \\
&\leq 2 \cdot \text{ord}(F) - \text{ord}_z(F) - \text{ord}_x(F) - \text{ord}_y(F) - \text{ord}(F) \\
&= \text{shade}(F).
\end{aligned}
$$

Hence $\text{shade}(f^*) \leq \text{shade}(F^*) \leq \text{shade}(F) = \text{shade}(f)$.

$\square$

2.6.5 Behavior of the shade under two successive blowups

The goal of this section is to discuss the following question:

Is it possible that the shade increases twice in two successive blowups?

The answer to this question is for arbitrary threefolds up to my knowledge not known! The next proposition will give a partial answer to this question. More precisely, we will prove in the sequel that this is not possible when applying two successive point

blowups.

Proposition 14. *Let f be an element of $Q = R/R^p$ and $F \in K[[x, y, z]]$ its expansion with respect to subordinate coordinates $(x, y, z) \in \mathcal{C}_{\mathcal{F}}$ realizing* shade(f). *Let $F^*(x, y, z)$ be one of the transforms (A)–(G) and f^* the corresponding element in R'/R'^p. Let $(x', y', z') \in \mathcal{C}_{\mathcal{F}}$ be subordinate coordinates which satisfy* shade$(F^*) =$ shade(f^*), *where $F^*(x, y, z)$ denotes the expansion of f^* with respect to x',, y' and z'. Further let $F^{**}(x', y', z')$ be one of the transforms (A)–(G) and f^{**} its equivalence class. Then*

$$\text{shade}(f^{**}) \leq \text{shade}(f) + 1.$$

Especially: It is not possible that the shade increases twice in two successive point blowups.

Proof. Since an increase of the shade can only occur under translational moves, we are left with the consideration of moves (A), (B), (C) and (E). Recall that under these transformations only certain situations have to be considered (see remark 12 at the beginning of this section).
Assume that the shade has increased in the first blowup, i.e., that

$$\text{shade}(f^*) = \text{shade}(F) + 1 \quad (\triangle)$$

(otherwise the statement of the result follows immediately from corollaries 5 and 6). This especially implies that the order d of f has to be a multiple of p.
The rest of the proof falls naturally into four parts corresponding to the different types of translational moves.
(A) Let $F^*(x, y, z) = F(xz + sz, yz + tz, z)$ with $s, t \in K^*$. Then it follows by the assumption $(\triangle)$ and Proposition 10 of section 2.6.2 that

$$\text{ord}_x(F^*) = \text{ord}_y(F^*) = 0.$$

Due to remark 12 this immediately implies that there exist subordinate coordinates (x', y', z') realizing shade(f^*) such that the second blowup is given by one of the monomial moves (D), (F) or (G).
(B) Assume that $F^*(x, y, z) = F(xz + sz, yz, z)$ with $s \neq 0$. Due to Proposition 12 of section 2.6.3, the assumption $(\triangle)$ implies that

$$\text{ord}_x(F^*) = 0.$$

Moreover the proof of proposition 8 in section 2.6.1 applied to move (B) implies that if shade$(F^*) = \text{shade}(F) + 1$, then there exists at least one monomial

$$M := c \cdot x^{\text{shade}(F_d) - (j - \text{ord}_y(F_d)) + 1} \cdot y^j \cdot z^d,$$

where $c \in K^*$ and $\text{ord}_y(F_d) \leq j \leq d - \text{ord}_x(F_d) - \text{ord}_z(F_d)$, in the expansion of F_d^* which is not a p-th power and satisfies $j \in p \cdot \mathbb{N}$. Moreover, by the assumption $(\triangle)$ this monomial fulfills $\text{ord}(F^*) = \text{ord}(M)$. Hence

$$
\begin{aligned}
d^* := \text{ord}(F^*) &= (\text{shade}(F_d) - (j - \text{ord}_y(F_d)) + 1) + j + d \\
&= \text{shade}(F_d) + d + \text{ord}_y(F_d) + 1.
\end{aligned}
$$

Since M is not a p-th power, i.e., $(\text{shade}(F_d) - j + \text{ord}_y(F_d) + 1, j, d) \notin p \cdot \mathbb{N}^3$, and $d, j \equiv 0 \bmod p$, it follows that $\text{shade}(F_d) - j + \text{ord}_y(F_d) + 1 \notin p \cdot \mathbb{N}$. Hence

$$d^* = d + \text{shade}(F_d) + \text{ord}_y(F_d) + 1 \notin p \cdot \mathbb{N}$$

and thus $\mathrm{par}(d^*) = 0$. Therefore a further increase of the shade under a subsequent blowup is not possible.

(C,E) Analogously to (B), one can show that after the translational moves (C) and (E) the order d^* of f^* is not a multiple p and hence no further increase of the shade can occur when applying a subsequent translational move.

<div align="right">□</div>

Remark 19. The last proposition shows that it is not possible that the shade increases twice in two successive point blowups. But unfortunately it is not clear if it decreases in the long run of the resolution process.

2.7 Monomial Case

Once $F(x, y, z)$ is monomial, there is an easy way to lower the order of G (compare to section 1.8 in chapter 1). We will describe this strategy here more generally for a purely inseparable equation in $n + 1$ variables, i.e., for

$$G(w, x_1, \ldots, x_n) = w^p + F(x_1, \ldots, x_n)$$

with $\mathrm{ord}(G) = p$. By assumption $F(x_1, \ldots, x_n) = x_1^{i_1} \cdots x_n^{i_n} \cdot U(x_1, \ldots, x_n)$, where $(i_1, \ldots, i_n) \in \mathbb{N}^n \setminus p \cdot \mathbb{N}^n$ and $U \in K[[x_1, \ldots, x_n]]^*$. After a formal coordinate change one can further assume that G is of the form

$$G(w, x_1, \ldots, x_n) = w^p + x_1^{i_1} \cdots x_n^{i_n}$$

with $i_1 + \ldots + i_n \geq p$.

Now the center of the next blowup is defined in the following way: Choose a subset $J \subseteq \{1, \ldots, n\}$ such that

$$\sum_{j \in J} i_j \geq p \quad \text{and} \quad \sum_{j \in J \setminus \{j_0\}} i_j < p \text{ for all } j_0 \in J.$$

Note that the first restriction ensures that the center is chosen inside the top locus of F.

Then the center of the blowup is set equal to

$$Z := V(w, x_j; j \in J).$$

In the w-chart one hence gets

$$G^*(w, x, y, z) = w^p \left(1 + x_1^{i_1} \cdots x_n^{i_n} w^{\sum_{j \in J} i_j - p} \right).$$

We thus get a variety which is smooth along the exceptional divisor $w = 0$. Now consider the remaining charts of the blowup. Without loss of generality we will assume that $1 \in J$. Then the total transform of G in the x_1-chart of the blowup is given by

$$G^*(w, x_1, \ldots, x_n) = x_1^p \left(w^p + x_1^{\sum_{j \in J} i_j - p} x_2^{i_2} \cdots x_n^{i_n} \right).$$

By the choice of the set J it follows that

$$\sum_{j \in J} i_j - p = i_1 + \sum_{j \in J \setminus \{1\}} i_j - p < i_1.$$

Therefore induction can be applied until $i_1 + \ldots + i_n < p$, or in other words until $\text{ord}(G) < p$!

Chapter 3

Oblique Polynomials

In this chapter we look more closely at those purely inseparable hypersurface equations for which the classical resolution invariant "shade" of characteristic zero, which was introduced in chapters 1.9 and 2.2 in the case of surfaces respectively threefolds, might increase under point blowups when used in characteristic $p > 0$. Especially, we are interested in finding characterizations of these equations. First we will give an explicit description in the surface case. Then we will establish a depiction for the purely inseparable hypersurface case in arbitrary dimension. Finally several examples will be stated.

3.1 Definition of oblique polynomials

Let
$$G(x, y_n, \ldots, y_1) = x^p + F(y_n, \ldots, y_1)$$
be a purely inseparable equation in $n + 1$ variables such that $F(y_n, \ldots, y_1) \notin R^p$ and $d := \mathrm{ord}_0(F) \geq p$. Set $\underline{y} = (y_n, \ldots, y_1)$, $l = n - 1$ and $\underline{w} = (y_l, \ldots, y_1)$. Denote by

$$F_d(\underline{y}) = \underline{y}^{\underline{r}} \cdot \widetilde{F}_d(\underline{y})$$

the tangent cone of $F(\underline{y})$, where $\underline{y}^{\underline{r}} = y_1^{r_1} \cdots y_n^{r_n}$ denotes the maximal monomial which can be factored from F_d, and by $e = d - |\underline{r}| = d - (r_1 + \ldots + r_n)$ the degree of $\widetilde{F}_d$.

Remark 20. It is worth pointing out that a possible increase of the invariant shade does only depend on the tangent cone $F_d(\underline{y})$ of $F(\underline{y})$. This can, for instance, be seen in Lemma 3 of chapter 1.6.1, Propositions 10 and 12 of chapter 2.6 respectively Theorem 1 in section 5 and Theorem 2 in section 12 of [20] (which is also recalled in section 1.10 of chapter 1).

We call $F_d(\underline{y})$ an *oblique polynomial with parameters* $(p, \underline{r}, e)$, if there exists a vector $\underline{t} = (0, t_l, \ldots, t_1)$ with $t_l, \ldots, t_1 \in K^*$ such that

$$F_d^+(\underline{y}) = (\underline{y} + \underline{t} \cdot y_n)^{\underline{r}} \cdot \widetilde{F}_d(\underline{y} + \underline{t} \cdot y_n) \in R/R^p$$

has order $e + 1$ with respect to the variables $\underline{w} = (y_l, \ldots, y_1)$, i.e.,

$$\mathrm{ord}_{\underline{w}}(F_d^+) = e + 1.$$

Remark 21. An important point to note here is that an oblique polynomial is up to multiplication with p-th powers unique for each choice of p, $\underline{r}$ and e. Furthermore all oblique polynomials satisfy that their order $d = (r_1 + \ldots + r_n) + e$ is a multiple of p. These two facts are proven in Theorem 1 of section 5 and Theorem 2 of section 12 in [20].

Example 15. One of the easiest examples of an oblique polynomial is

$$F_d(z, y) = y^3 z^3 (y^2 + z^2) = y^3 z^3 \cdot \widetilde{F}_d,$$

where the characteristic p of the ground field is equal to 2. Setting $t = (0, 1)$ yields $F_d^+(y, z) = F_d(y + z, z) = y^5 z^3 + y^4 z^4 + y^3 z^5 + y^2 z^6$ and thus after deleting all p-th powers,

$$F_d^+(y, z) = y^3 z^3 (y^2 + z^2).$$

Hence: $\mathrm{ord}_y(F_d^+) = 3 = \deg(\widetilde{F}_d) + 1$.

3.2 Oblique polynomials in the surface case

There are various ways to obtain an explicit description of oblique polynomials in 2 variables. One such method is explained in detail in section 5.3 of the thesis [50] of Zeillinger using similarly as in remark 13 of chapter 2.6 the *invertible binomial matrix* between the coefficients of F_d and the $|\underline{r}| + e + 1$ coefficients of lowest degree in the variable y of F_d^+. In [50] oblique polynomials are called "hydra polynomials".

Another formula for oblique polynomials is explored in section 12 of [20], where such polynomials are named "hybrid". There, first the following candidate for oblique polynomials with parameters $(p, (r, s), e)$ is given

$$Q(y, z) = y^r z^s \mathbb{H}_r^e(y, tz - y) = y^r z^s \sum_{i=0}^{e} \binom{e + r}{r + i} y^i (tz - y)^{e-i}.$$

Afterwards it is checked that $Q(y, z)$ is an oblique polynomial. By the uniqueness of oblique polynomials (up to multiplication with p-th powers) it then follows that all oblique polynomials are of the type above. Note that this formula is only valid if $\binom{e+r}{e+1}$ is not a multiple of p. The reason for this will be explained in remark 22 below. Furthermore in section 12 of [20] it is indicated that $\mathbb{H}_r^e(y + tz, tz - y)$ equals after dehomogenization by setting $z = 1$ the expansion of $t^{e+r}(t + y)^{-r}$ truncated at degree e in y. This yields a further explicit description for oblique polynomials in 2 variables:

$$F_d(y, z) = y^r z^s \cdot \sum_{i=0}^{e} \binom{-r}{i} (y - tz)^i (tz)^{e-i}.$$

A similar formula will be deduced for oblique polynomials in n variables in section 3.3.

Here we want to deduce an explicit characterization of oblique polynomials in 2 variables by *using derivatives*. Write F_d as

$$F_d(y, z) = y^r z^s \sum_{i=0}^{e} c_i y^i z^{e-i}$$

with $c_i \in K$ and $c_0, c_e \neq 0$. Recall that the order of any oblique polynomial is divisible by p (see remark 21 in the last section respectively [20]). This implies that differentiating F_d with respect to the variable y eliminates only monomials in the expansion of F_d which are p-th powers and no further ones. Moreover dehomogenizing F_d with respect to the variable z preserves p-th powers. Therefore we will in the sequel concentrate on the dehomogenization $P(y)$ of $F_d(y, z)$, i.e.,

$$P(z) := F_d(y, 1) = \sum_{i=0}^{e} c_i y^{r+i},$$

from which F_d can easily be reconstructed. Now consider the first derivative of P with respect to y, i.e.,

$$P' := \partial_y P(y) = \sum_{i=0}^{e} (r + i) c_i y^{r+i-1}.$$

Clearly $\mathrm{ord}_0(P') = r - 1$, or in other words, 0 is a zero of multiplicity $r - 1$ of P'. Applying the transformation

$$\psi : y \to y + t,$$

which corresponds to the linear transformation $\varphi : (y, z) \to (y + tz, z)$ for $F_d(y, z)$, to P' yields

$$\psi(P') = \sum_{i=0}^{e} (r + i) c_i (y + t)^{r+i-1}.$$

Note that $\psi(P') = (\psi(P))'$. Furthermore the condition for $F_d(y, z)$ of being oblique is equivalent to demand that the order of $\psi(P')$ is equal to e. Or said differently, that $t \in K^*$ is a zero of multiplicity e of P'. Since P' is a polynomial of degree $r + e - 1$ in one variable, this altogether implies that P' has to be of the form

$$P'(y) = C \cdot y^{r-1} \cdot (y - t)^e,$$

where C denotes a constant in K^*. By integration with respect to the variable y one gets the following expansion for P

$$
\begin{aligned}
P(y) &= \int C y^{r-1} (y - t)^e \, dy \\
&= C \cdot \int \sum_{i=0}^{e} \binom{e}{i} (-t)^{e-i} y^{r+i-1} \, dy \\
&= C \cdot \sum_{i=0}^{e} \binom{e}{i} \frac{1}{r+i} (-t)^{e-i} y^{r+i} \; + \; D,
\end{aligned}
$$

where D denotes a not yet determined integration constant.
Homogenization of the expression above leads to the following description of $F_d(y, z)$:

$$F_d(y, z) = y^r \cdot z^s \cdot C \cdot \sum_{i=0}^{e} \binom{e}{i} \frac{1}{r+i} y^i z^{e-i} \; + \; D \cdot z^{r+s+e}$$

with $C, D \in K$. But due to our assumption that $\widetilde{F}_d(y, z) = y^{-r} z^{-s} F_d(y, z)$ is a homogenous polynomial of order e it follows that $D = 0$. Altogether this proves the following proposition:

Proposition 15. *An oblique polynomial $F_d(y, z)$ with parameters $(p, (r, s), e)$ is up to multiplication with p-th powers of the form*

$$F_d(y, z) = C \cdot y^r \cdot z^s \cdot \left(\sum_{i=0}^{e} \binom{e}{i} \frac{1}{r+i} (-t)^{e-i} y^i z^{e-i} \right) \quad (\nabla)$$

with $C \in K$ and $t \in K^*$, where the expansion above is considered in R/R^p.

Note that the denominator $r + i$ in (∇) only vanishes if the respective monomial $y^{r+i} z^{e-i}$ is a p-th power. Therefore, the description (∇) of oblique polynomials is well-defined.

Remark 22. Computation of examples show that the expansions $F_d(y, z)$ of Proposition 15 and $Q(y, z) = y^r z^s \mathbb{H}(y, tz - y)$ from [20] only coincide in the case that

$$\binom{e+r}{e+1} \notin p \cdot \mathbb{N}.$$

Comparing both expansions over a field with characteristic zero and setting $t = 1$ would yield the following relation

$$F_d(y, z) = \frac{(-1)^e}{e+1} \cdot \frac{1}{\binom{e+r}{e+1}} \cdot Q(y, z).$$

One sees that here also the "mysterious" factor $\binom{e+r}{e+1}$ appears. The reason for the difference in the two formulas lies in the fact that the coefficient of $Q(y + tz, z)$ of the monomial of order $e + 1$ in the variable y is equal to $\binom{e+r}{e+1}(-1)^e t^{r-1}$ and thus might vanish for certain values of e and r. This would yield that $\mathrm{ord}_y(F_d^+) \geq e + 2$, which is a contradiction to the lemma of Moh (cf. Lemma 2 in chapter 1.6.1). Therefore, in the case that $\binom{e+r}{e+1} \in p \cdot \mathbb{N}$, the expansion $Q(y, z)$ doesn't yield an oblique polynomial and one can further show that then $Q(y, z)$ is itself a p-th power of a polynomial.

3.3   Oblique polynomials in dimension n

This section provides a exposition of those purely inseparable polynomials in $n+1$ variables for which the classical resolution invariant might increase under point blowups. It is adapted from [18].

Consider again a purely inseparable equation

$$G(x, y_n, \ldots, y_1) = x^p + F(y_n, \ldots, y_1) \in R/R^p$$

in $n + 1$ variables such that $d := \mathrm{ord}_0(F) \geq p$. Set $\underline{y} = (y_n, \ldots, y_1)$ and denote by

$$F_d(\underline{y}) = \underline{y}^{\underline{r}} \cdot \widetilde{F}_d(\underline{y})$$

the tangent cone of $F(\underline{y})$ and by e the degree of $\widetilde{F}_d$.

First dehomogenize – as in section 3.2 – the tangent cone $F_d(\underline{y})$ with respect to y_n. Using remark 21 it can be easily checked that this preserves p-th powers and produces

no new p-th powers. As in the surface case, also oblique polynomials in $n+1$ variables are thus completely determined by its dehomogenization. Let $\underline{t} = (0, t_{n-1}, \ldots, t_1)$ with $t_{n-1}, \ldots, t_1 \in K^*$. Set $l = n - 1$, $\underline{w} = (y_l, \ldots, y_1)$, $\underline{r}' = (r_l, \ldots, r_1)$ and $\underline{t}' = (t_l, \ldots, t_1)$. Then dehomogenizing $F_d(y_n, \underline{w})$ by setting $y_n = 1$ yields

$$P(\underline{w}) := F_d(1, \underline{w}) = \underline{w}^{\underline{r}'} \cdot \widetilde{F}_d(1, \underline{w}).$$

Applying the translation

$$\psi : \underline{w} \to \underline{w} + \underline{t}'$$

corresponding to the transformation $\varphi : \underline{y} \to \underline{y} + \underline{t} \cdot y_n$ on F_d results in

$$\psi(P) = (\underline{w} + \underline{t}')^{\underline{r}'} \cdot Q(\underline{w} + \underline{t}')$$

with $Q(\underline{w}) := \widetilde{F}_d(1, \underline{w})$. Again $\psi(P)$ is considered modulo p-th powers. The condition $\mathrm{ord}_{\underline{w}}(\varphi(F_d)) \geq e + 1$ can now be reformulated as

$$\psi(P) \in \langle y_l, \ldots, y_1 \rangle^{e+1} + K[w^p].$$

Or written differently,

$$(\underline{w} + \underline{t}')^{\underline{r}'} \cdot Q(\underline{w} + \underline{t}') - H(\underline{w})^p \in \langle y_l, \ldots, y_1 \rangle^{e+1}$$

for a polynomial $H \in K[\underline{w}]$. Note that $\deg(Q) \leq e$ implies that $H \neq 0$. Moreover it is worth pointing out that the condition $\mathrm{ord}_{\underline{w}}(\varphi(F_d)) \geq e + 1$ is stable under multiplication with p-th powers of homogenous polynomials $R(\underline{w})$. More precisely,

$$\mathrm{ord}_{\underline{w}}(R^p \cdot \varphi(F_d)) \geq e + 1 + p \cdot \deg(R).$$

Since by assumption $\underline{t}' \in (K^*)^l$ it immediately follows that the polynomial $(\underline{w} + \underline{t}')^{\underline{r}'}$ is invertible in $K[[\underline{w}]]$. This provides the following formula for $Q(\underline{w} + \underline{t}')$:

$$Q(\underline{w} + \underline{t}') = \lfloor (\underline{w} + \underline{t}')^{-\underline{r}'} \cdot H(\underline{w})^p \rfloor_e,$$

where $\lfloor u(\underline{w}) \rfloor_e$ denotes the e-jet (i.e., its power series expansion up to degree e) of the formal power series $u(\underline{w})$.

Remark 23. Due to the lemma of Moh (cf. [38] and [20]) we know that $(\underline{w} + \underline{t}') \cdot Q(\underline{w} + \underline{t}') - H(\underline{w})^p \notin \langle y_l, \ldots, y_1 \rangle^{e+2}$. This implies that if $H(\underline{w})$ is a constant, then the homogenous part of degree $e+1$ of $(\underline{w} + \underline{t}')^{-\underline{r}'}$, which is given by $\sum_{\alpha \in \mathbb{N}^l, |\alpha| = e+1} \binom{-\underline{r}'}{\alpha} \underline{w}^{\alpha}$, must be different from 0. And further, if all $\binom{-\underline{r}'}{\alpha}$ with $|\alpha| = e + 1$ are zero in K, then $H(\underline{w})$ hasn't been a constant. This occurs for instance for $n = 2$, $\underline{t} = (0, 1)$ and the parameters $(p, \underline{r}, e) = (2, (3, 3), 2)$.

By inverting the translation $\psi : \underline{w} \to \underline{w} + \underline{t}'$ one gets the following description for the dehomogenized tangent cone $F_d(1, \underline{w})$:

$$F_d(1, \underline{w}) = \underline{w}^{\underline{r}'} \cdot Q(\underline{w}) = \underline{w}^{\underline{r}'} \cdot \psi^{-1} \left(\lfloor (\underline{w} + \underline{t}')^{-\underline{r}'} \cdot H(\underline{w})^p \rfloor_e \right).$$

Homogenization of this polynomial with respect to the variable y_n and multiplication with $y_n^{r_n}$ finally yields the oblique polynomial $F_d(\underline{y}) = \underline{y}^{\underline{r}} \cdot \widetilde{F}_d(\underline{y})$ with parameters $(p, \underline{r}, e)$.

3.4 Examples of oblique polynomials

The intention of this section is to present collected examples of oblique polynomials in 2 and 3 variables. They are partially taken from chapter 5.3 of [50] and chapter 17 of [20] and complemented with examples which were computed in chapter 1 and 2.

Examples of oblique polynomials in 2 variables

$(p, (r, s), e)$	$F_d(y, z) \in R/R^p$	t'
$(2, (3, 1), 2)$	$y^3 z(y^2 + z^2)$	1
$(2, (3, 3), 2)$	$y^3 z^3(y^2 + z^2)$	1
$(2, (3, 7), 2)$	$y^3 z^7(y^2 + z^2)$	1
$(2, (5, 3), 4)$	$y^5 z^3(y^4 + z^4)$	1
$(2, (11, 1), 6)$	$y^{11} z(y^6 + y^4 z^2 + y^2 z^4 + z^6)$	1
$(2, (13, 5), 8)$	$y^{13} z^5(y^8 + z^8)$	1
$(3, (1, 1), 1)$	$yz(2z - y)$	1
$(3, (1, 2), 3)$	$yz^2(y^3 - z^3)$	1
$(3, (5, 1), 3)$	$y^5 z(y^3 + 2z^3)$	1
$(3, (1, 1), 4)$	$yz(y^4 + y^3 z - yz^3 - z^4)$	1
$(3, (1, 2), 6)$	$yz^2(y^6 + y^3 z^3 + z^6)$	1
$(3, (4, 2), 6)$	$y^4 z^2(y^6 - z^3 y^3 + z^6)$	2
$(5, (1, 2), 2)$	$yz^2(y^2 + 2yz + 3z^2)$	1
$(5, (1, 3), 1)$	$yz^3(2z - y)$	1
$(5, (2, 2), 1)$	$y^2 z^2(3y + 3z)$	1
$(5, (4, 1), 5)$	$y^4 z(y^5 + z^5)$	4
$(5, (2, 3), 5)$	$y^2 z^3(z^5 - y^5)$	1
$(5, (2, 3), 10)$	$y^2 z^3(y^{10} + y^5 z^5 + 4z^{10})$	2
$(7, (1, 5), 1)$	$yz^5(2z - y)$	1
$(7, (2, 4), 1)$	$y^2 z^4(5y + 3z)$	1
$(7, (1, 4), 2)$	$yz^4(y^2 + 4yz + 3z^2)$	1
$(7, (1, 3), 3)$	$yz^3(6y^3 + 4y^2 z + yz^2 + 4z^3)$	1
$(7, (1, 2), 4)$	$yz^2(y^4 + 2y^3 z + 3y^2 z^2 + 4yz^3 + 5z^4)$	1
$(7, (6, 1), 7)$	$y^6 z(y^7 + z^7)$	6

Examples of oblique polynomials in 3 variables

$(p, \underline{r}, e)$	$F_d(z, y, x) \in R/R^p$	$\underline{t}'$
$(2, (1, 1, 1), 1)$	$xyz(x + y + z)$	$(1, 1)$
$(2, (0, 1, 1), 1)$	$xy(x^2 + y^2)$	$(1, 1)$
$(3, (0, 1, 1), 1)$	$xy(x - y)$	$(2, 1)$
$(3, (1, 2, 2), 1)$	$x^2 y^2 z(x + y + 2z)$	$(1, 1)$
$(3, (2, 1, 1), 2)$	$xyz^2(2xz + x^2 + xy + y^2 + 2yz)$	$(1, 1)$
$(5, (0, 2, 2), 1)$	$x^2 y^2(x + y)$	$(1, 1)$

Chapter 4

On Compact Facets and Normal Vectors of Newton Polyhedra

This chapter is devoted to the study of normal vectors to facets of Newton polyhedrons associated to algebraic hypersurfaces. First the behavior of these vectors under Hironaka's simple polyhedron game (where the characteristic of the underlying field K is arbitrary) is examined. It will turn out that the main obstacle for using normal vectors as measure for the complexity of the singularities of the associated algebraic varieties is given by Newton polyhedra without compact facets. In the second part of the chapter the locus of points on an algebraic variety where the associated Newton polyhedron has a compact facet with respect to *all* local coordinates is investigated. For the case of $K = \mathbb{C}$ it will be proven that under the assumption that this set is Zariski–closed, it consists of at most finitely many points.

This chapter consists mainly of observations and ideas which could be the topic of further investigations.

Part I – A new approach to the simple polyhedron game

In the simple polyhedron game the combinatorial part of blowing up singularities of algebraic varieties over fields of arbitrary characteristic is reformulated in terms of polyhedra. It was originally introduced by Hironaka in 1970 (cf. [29]). In 1983 Spivakovsky gave in [45] a solution to the game. We will here first indicate a different solution which was established in the thesis [50] of Zeillinger. Afterwards a new approach to the game using normal vectors to facets of the Newton polyhedron corresponding to a given algebraic variety is examined. The reason why the search for a new solution to the simple polyhedron game is of interest, lies in the fact that such a solution could possibly lead to a different solution to the hard polyhedron game (for its description see for instance [50] and part II of this chapter), which reflects all details of resolution of singularities.

4.1   Description of Hironaka's simple polyhedron game

The simple polyhedron game reflects the combinatorial part of blowing singularities of algebraic varieties. Therefore, before giving its description, I want to recall briefly the process of embedded resolution of singularities using blowups by means of an example:

Let $X \subset \mathbb{A}_K^3$ be the algebraic variety defined by $F(x, y, z) = x^2 - yz = 0$, i.e., a cone. The blowup of X with center $Z = \{0\}$, the origin, yields the following total transforms F^* in the respective charts of the blowup:

$$
\begin{array}{ll}
x\text{-chart:} & F^*(x, y, z) = F(x, xy, xz) = x^2(1 - yz), \\
y\text{-chart:} & F^*(x, y, z) = F(xy, y, yz) = y^2(x^2 - z), \\
z\text{-chart:} & F^*(x, y, z) = F(xz, yz, z) = z^2(x^2 - y).
\end{array}
$$

This process is now repeated until the total transform F^* is in all charts of the form

$$
F^*(x, y, z) = x^a y^b z^c U(x, y, z)
$$

for some $a, b, c \in \mathbb{N}$ and $U \in K[x, y, z]$ with $U(0) \neq 0$.
In the example above, this is only the case in the x-chart, so further blowups would be necessary.

This procedure can now be reformulated as a game: In every step of the resolution process *player 1* has to choose a *center Z* for the next blowup. Combinatorially, this center can be described (locally) as a subset J of $\{1, \ldots, n\}$, where n denotes the numbers of variables of the polynomial $F(x_1, \ldots, x_n)$ defining the given algebraic variety $X = V(F) \subset \mathbb{A}_K^n$. Afterwards *player 2* chooses *one of the charts* of the blowup with center Z, or said combinatorially, an element $j \in J$. The goal of the resolution problem is to obtain after finitely many repetitions of this procedure in every chart a monomial times a unit. Described in terms of the game, player 1 wins the game if after finitely many rounds the obtained total transform is – regardless to the choices of player 2 – a monomial times a unit $U(x, y, z) \in K[[x, y, z]]$. The combinatorial part of the resolution problem for X is thus equivalent to the question whether there exists a winning strategy for player 1.

Note further that when applying a blowup, only the exponents of monomials occurring in $F = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$ change. It is hence sufficient to consider the support $\operatorname{supp}(F) = \{\alpha; c_\alpha \neq 0\}$ of the polynomial $F(x)$. In the example above one can see that the support of F^* can easily be deduced from the support of F. In the y-chart of the blowup for instance

$$
\operatorname{supp}(F^*) = \{(a + b, b, b + c); (a, b, c) \in \operatorname{supp}(F)\}.
$$

Altogether these observations lead to the following description of the simple polyhedron game:

Rules of the game

Let $A \subset \mathbb{N}^n$ be a finite subset and denote by $N(A) = \operatorname{conv}(A + \mathbb{R}_+^n)$ the positive convex hull of A. Furthermore define for a set $\emptyset \neq J \subset \{1, \ldots, n\}$ and an alement

$j \in J$ the map $\tau_{J,j}$ as

$$\tau_{J,j} : \mathbb{N}^n \longrightarrow \mathbb{N}^n, p = (p_1, \ldots, p_n) \rightarrow p' = (p_1, \ldots, p_{j-1}, \sum_{k \in J} p_k, p_{j+1}, \ldots, p_n).$$

Course of the game

(1) Player 1 chooses, taking into account the set A (respectively $N(A)$), a subset

$$\emptyset \neq J \subset \{1, \ldots, n\}.$$

(2) Player 2 chooses in consideration of the sets A and J an element

$$j \in J$$

and substitutes $A = \{a_1, \ldots, a_s\}$ by the set

$$A' := \tau_{J,j}(A) = \{\tau_{J,j}(a_1), \ldots, \tau_{J,j}(a_s)\}$$

(and $N(A)$ by $N(A')$).

This procedure is now repeated. Denote by $(A^i)_{i \in \mathbb{N}}$ the resulting subsets of $\mathbb{Z}^n$, where $A^0 := A$, $A^1 := A'$ and $A^{n+1} := (A^n)'$.

End of the game

Player 1 wins the game if after finitely many repetitions, say m, the Newton polyhedron $N(A^m)$ is a quadrant, i.e., if there exists an $\alpha \in \mathbb{N}^n$ such that

$$N(A^m) = \alpha + \mathbb{R}_+^n.$$

Problem

Show that there exists a winning strategy for player 1!

4.2 One possible solution to the simple polyhedron game

In his thesis [50], Zeillinger developed a new solution to Hironaka's simple polyhedron game based on a solution to the easier "vector game", which will be recalled briefly in the sequel.

4.2.1 Definition of the vector game

The **vector game** corresponds to the simple polyhedron game with only two vertices α and β. Since the important information for solving the problem, is contained in the relative position of these two points, is suffices to investigate only the vector $\alpha - \beta$ connecting the points α and β. The rules of the game can thus be reformulated as:

Let v be a vector in $\mathbb{Z}^n$ and the map $\tau_{J,j}$ defined as in 4.1.

(1) Player 1 chooses, taking into account the vector v, a subset $\emptyset \neq J \subset \{1, \ldots, n\}$.
(2) Player 2 chooses in consideration with v and J an element $j \in J$ and substitutes

the vector v by $v' := \tau_{J,j}(v)$.

Denote by $(v^i)_{i \in \mathbb{N}}$ the resulting sequence of vectors when repeating the procedure (where $v^0 := v$ and $v^{n+1} = (v^n)'$).

Player 1 wins the game if there exists an $m \in \mathbb{N}$ such that

$$v^m \in \mathbb{N}^n \text{ or } v^m \in -\mathbb{N}^n.$$

Problem: Show that there exists a winning strategy for player 1!

4.2.2 A solution to the vector game

First we will describe a possible solution to the vector game in the case of $n = 2$. Afterwards we will indicate how this winning strategy can easily be generalized to the case $n > 2$. For details we refer to [50].

Proposition 16. *Let $v \in \mathbb{Z}^2$ and $v \notin \pm\mathbb{N}^2$ (otherwise the game would already be over). If player 1 chooses throughout the whole game the set $J = \{1, 2\}$, then he will win the vector game after finitely many rounds.*

Proof. The idea is to define a map $b : \mathbb{Z}^2 \to \mathbb{N}$, which measures "how far the vector $v \in \mathbb{Z}^2$ is away from being an element of $\pm\mathbb{N}^2$". An element $w \in \pm\mathbb{N}^2$ should thus fulfill $b(w) = 0$. The map b should moreover have the following property: Let $v \notin \pm\mathbb{N}^2$ and v' the vector after one round of the game, then it should hold that $b(v') < b(v)$.
Since $\mathbb{N}$ is a well-ordered set, these properties immediately imply that the sequence $(v^n)_{n \in \mathbb{N}}$ becomes stationary, i.e., that there exists an $m \in \mathbb{N}$ such that $v^k \in \pm\mathbb{N}^2$ for all $k \geq m$. In this situation player 1 would win the game.
We will check that the following map satisfies the required conditions:

$$b : \mathbb{Z}^2 \longrightarrow \mathbb{N}, v = (v_1, v_2) \to b(v) := \begin{cases} 0 & \text{if } v \in \pm\mathbb{N}^2, \\ |v_1| + |v_2| & \text{otherwise.} \end{cases}$$

Let $w \in \mathbb{Z}^2 \setminus \pm\mathbb{N}^2$. Without loss of generality one can assume that

$$w = (c, -d) \text{ with } c, d \in \mathbb{N} \setminus \{0\}.$$

Hence $b(w) = c + d$. Further assume that player 2 chooses $j = 1$. Then

$$w' = (c - d, -d).$$

Now there are two different cases. Either $c - d \leq 0$, thus $w' \in -\mathbb{N}^2$ and player 1 wins the game. Or $c - d > 0$, and consequently $b(w') = (c - d) + d = c < c + d = b(w)$. Analogous argumentation for $j = 2$ yields the assertion of the proposition. $\qquad\square$

In the case that the dimension $n \in \mathbb{N}$ is arbitrary, the following proposition shows that the right choice of subsets $J \subset \{1, \ldots, n\}$ with only two elements ensures that player 1 wins the game:

Proposition 17. *Let $v \in \mathbb{Z}^n$ and $v \notin \pm\mathbb{N}^n$. Further let k and l such that*

$$v_k = \min_{1 \leq i \leq n} v_i \quad \text{and} \quad v_l = \max_{1 \leq i \leq n} v_i.$$

Then choosing the set $J = \{k, l\}$ yields a winning strategy for player 1.

The proof of this proposition can be found in [50]. Here we will only state the map $b : \mathbb{Z}^n \to \mathbb{N}^2$ which measures – as in the proof of the last proposition – "how far a vector $v \in \mathbb{Z}^n$ is away from being an element of $\mathbb{N}^n$" and fulfills $b(v') <_{lex} b(v)$ for the described strategy of choosing the set J (where v' denotes the vector after one round of the game):

$$b : \mathbb{Z}^n \longrightarrow \mathbb{N}^2, v \to b(v) := \begin{cases} (0,0) & \text{if } v \in \pm\mathbb{N}^n, \\ (L(v), n - V(v)) & \text{otherwise.} \end{cases}$$

The measures $L(v)$ and $V(v)$ are defined as

$$L(v) := \max_{1 \le i \le n} v_i - \min_{1 \le i \le n} v_i \quad \text{and} \quad V(v) := \#\{j; \min_{1 \le i \le n} v_i < v_j < \max_{1 \le i \le n} v_i\}$$

and called the *length* and *volume* of the vector v.

4.2.3 One possible solution to the simple polyhedron game

The solution to the vector game described in section 4.2.2 can now be used to deduce a winning strategy for the simple polyhedron game. We will here only indicate one method for player 1 of choosing the set J in order to win the game. The complete proof that this really yields a winning strategy for player 1 can be found in [50].

Let $A = \{a_1, \ldots, a_s\} \subset \mathbb{N}^n$ a finite subset and $N = N(A)$ the associated Newton polyhedron. Without loss of generality one can assume that A consists only of vertices of $N(A)$. Now consider the following set of vectors

$$V := \{\alpha - \beta; \alpha, \beta \in A\}.$$

Let $v_A \in V$ such that $(L(v_A), n - V(v_A))$ is minimal with respect to the lexicographical ordering among all choices of vectors $v \in V$, where $L(v)$ and $V(v)$ are defined as in section 4.2.2.
Now let k and l be, as in the vector game, so that

$$(v_A)_k = \min_{1 \le i \le n} (v_A)_i \quad \text{and} \quad (v_A)_l = \max_{1 \le i \le n} (v_A)_i.$$

Then choosing $J = \{k, l\}$ leads to a winning strategy for player 1!

Instead of giving the whole proof of this winning strategy, we will only state the map $b : \mathbb{N}^n \to \mathbb{N}^3$ which measures similarly as in the vector game "how far the Newton polyhedron $N(A)$ is away from being a quadrant" and whose value decreases with respect to the lexicographical order in every round of the game if the set J is chosen as described above:

$$b(N(A)) := \begin{cases} (0,0,0) & \text{if } N(A) \text{ is a quadrant,} \\ (\#A, L(v_A), n - V(v_A)) & \text{otherwise.} \end{cases}$$

4.3 A new approach to the simple polyhedron game using normal vectors

Since – as already indicated in the beginning of this chapter – a differnt solution to the simple polyhedron game could possibly lead to a new solution of the hard polyhedron

game, which reformulates the problem of resolution of singularities of algebraic varieties $X = V(F) \subset \mathbb{A}_K^n$ over fields K of arbitrary characteristic, it is interesting to study the behavior of new measures under the simple polyhedron. In this section we will in particular investigate the transformation of normal vectors to facets of the Newton polyhedron $N(A)$ associated to X under the map $\tau_{J,j}$, which represents the transformation of $F(x_1, \ldots, x_n)$ under the blowup of $\mathbb{A}_K^n$ with center $Z = V(x_i; i \in J)$ in the x_j-chart. Moreover, it will be illustrated that in the case that the Newton polyhedron $N(A)$ associated to $X = V(F)$ doesn't contain a compact facet, certain phenomena occur.

Definition. Let $N(A)$ be the Newton polyhedron of a finite set $A \subset \mathbb{N}^n$. Denote by $\{e_1, \ldots, e_n\} := \{(1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, 1)^T\}$ the *standard basis* of $\mathbb{R}^n$. Each $(n-1)$–dimensional facet $\mathfrak{F}$ of $N(A)$ satisfies an equation $\sum_{i=1}^n c_i x_i = d$ with $c_i, d \in \mathbb{N}$ and $\gcd(c_1, \ldots, c_n) = 1$. The *normal vector* $\mathfrak{n}(\mathfrak{F})$ to the facet $\mathfrak{F}$ is defined as

$$\mathfrak{n}(\mathfrak{F}) := (c_1, \ldots, c_n)^T \in \mathbb{N}^n.$$

Clearly the vectors $e_1, \ldots, e_n$ are normal vectors to any Newton polyhedron. Moreover, $N(A)$ is an octant if and only if the set of normal vectors to $N(A)$ consists only of the standard basis vectors $e_1, \ldots, e_n$.

4.3.1   Case: The Newton polyhedron contains a compact facet

The goal of this paragraph is to show that under the assumption that the Newton polyhedron $N(A)$ of $A \subset \mathbb{N}^n$ contains at least one compact facet, point blowups suffice to transform $N(A)$ into a Newton polyhedron without compact facets. Moreover, the improvement under blowup can be measured by means of the set of normal vectors to the compact facets of $N(A)$.

Assumption.

We will assume in this paragraph that the Newton polyhedron $N(A)$ contains at least one $(n-1)$–dimensional facet $\mathfrak{F}$ that is compact, i.e., whose area is bounded, or equivalently, whose normal vector $\mathfrak{n}(\mathfrak{F})$ is an element of $\mathbb{N}_{>0}^n$.

Transformation of normal vectors under point blowups

First we will develop a transformation rule for normal vectors $\mathfrak{n}(\mathfrak{F})$ to compact facets $\mathfrak{F}$ of $N(A)$ under point blowups, i.e., under the choice $J = \{1, \ldots, n\}$ of player 1:

Recall the definition of the map $\tau_{J,j}$, which was introduced in section 4.1 and corresponds to the transformation of F under blowup with center $Z = V(x_i; i \in J)$ in the x_j-th chart: $\tau_{J,j} : \mathbb{N}^n \longrightarrow \mathbb{N}^n, p = (p_1, \ldots, p_n) \to p' = (p'_1, \ldots, p'_n)$, where

$$p'_i := \begin{cases} \sum_{k \in J} p_k & \text{for } i = j, \\ p_i & \text{otherwise.} \end{cases}$$

Denote by $N(A')$ the Newton polyhedron of $A' = \tau_{J,j}(A)$. Further let in the sequel J always be equal to $\{1, \ldots, n\}$. Then it is easy to see that the map $\tau_{J,j}$ can be rewritten in matrix form as:

$$\tau_{J,j} : \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \in N(A) \longrightarrow \begin{pmatrix} p'_1 \\ \vdots \\ p'_n \end{pmatrix} = M \cdot \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \in N(A')$$

with

$$M = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

where every entry in the j-th row is equal to 1. Note that M is an invertible matrix.

Now let $\mathfrak{n} = (\mathfrak{n}_1, \ldots, \mathfrak{n}_n)^T$ be a vector in $\mathbb{R}^n$, for instance the normal vector $\mathfrak{n}(\mathfrak{F})$ to a compact facet $\mathfrak{F}$ of $N(A)$, defining an $(n-1)$–dimensional subspace of $\mathbb{R}^n$ by:

$$\mathfrak{n}_1 \cdot \alpha_1 + \ldots + \mathfrak{n}_n \cdot \alpha_n = 0.$$

It is well known that $\mathfrak{n}$ transforms under $\tau_{J,j}$ by the dual map of $\tau_{J,j}$, i.e., as follows:

$$\tau_{J,j}^{dual} : \mathfrak{n} \to \mathfrak{n}' = (M^T)^{-1} \cdot \mathfrak{n},$$

where

$$(M^T)^{-1} = \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & -1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & -1 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & -1 & 0 & \ldots & 1 \end{pmatrix}.$$

Or written differently,

$$\tau_{J,j}^{dual} : \mathfrak{n} \longrightarrow \mathfrak{n}' = (\mathfrak{n}_1 - \mathfrak{n}_j, \ldots, \mathfrak{n}_{j-1} - \mathfrak{n}_j, \mathfrak{n}_j, \mathfrak{n}_{j+1} - \mathfrak{n}_j, \ldots, \mathfrak{n}_n - \mathfrak{n}_j)^T.$$

Remark 24. From this calculation one can see, that the normal vector $\mathfrak{n}(\mathfrak{F})$ to a compact facet $\mathfrak{F}$ of $N(A)$ decreases with respect to the lexicographical ordering under point blowup. Therefore it seems to be natural to use it in order to measure the improvement of singularities under point blowups. Note that there arises a certain inconvenience if the transform $\mathfrak{F}'$ of the facet $\mathfrak{F} \subset N(A)$ isn't any more a (compact) facet of $N(A')$ since then $(\mathfrak{n}(\mathfrak{F}))'$ is not any more a normal vector of $N(A')$.

Remark 25. Due to the fact that the normal vector $\mathfrak{n}(\mathfrak{F})$ is dual to $\mathfrak{F}$, the following statement easily follows: The transform $\mathfrak{F}' = \tau_{J,j}(\mathfrak{F})$ is a compact facet of $N(A')$ if and only if $\tau_{J,j}^{dual}(\mathfrak{n}(\mathfrak{F})) \in \mathbb{N}_{>0}^n$. In this situation $\tau_{J,j}^{dual}(\mathfrak{n}(\mathfrak{F}'))$ is the normal vector to $\mathfrak{F}'$, i.e.,

$$\tau_{J,j}^{dual}(\mathfrak{n}(\mathfrak{F})) = \mathfrak{n}(\mathfrak{F}').$$

A first step towards a solution to the simple polyhedron game

We will now prove the following proposition, which constitutes a first step towards a solution to the simple polyhedron game, i.e., in monomializing the Newton polyhedron $N(A)$ by finitely many blowups:

Proposition 18. *Let $A \subset \mathbb{N}^n$ be a finite subset and $N(A)$ its Newton polyhedron. Assume that $N(A)$ contains at least one compact facet. Then repeatedly choosing the set $J = \{1, \ldots, n\}$ yields after finitely steps a Newton polyhedron without compact facets.*

Proof. In order to proof the assertion of the proposition above, we will define a map $b : \mathbb{N}^n \to \mathbb{N} \times (\mathbb{N}^n_{>0})^c$, which measures "how far the Newton polyhedron $N(A)$ is away from being a polyhedron without compact facets". The number $c \in \mathbb{N}^n$ will be specified below. We will show moreover that $b(A') <_{lex} b(A)$, where A' denotes the transform of A under the map $\tau_{J,j}$ with $J = \{1, \ldots, n\}$.

These properties of the map b immediately imply that the sequence $(A^i)_{i \in \mathbb{N}}$ of resulting Newton polyhedra becomes stationary after finitely many rounds of the game. Due to the definition of b it will follow that in this situation the Newton polyhedron of $N(A^i)$ doesn't contain a compact facet.

In order to define the map b, denote by $\mathfrak{CF} = \{\mathfrak{F}; \mathfrak{F} \text{ compact facet of } N(A)\}$ the set of compact facets of $N(A)$. Further let $\mathfrak{n}(\mathfrak{CF}) = \{\mathfrak{n}(\mathfrak{F}); \mathfrak{F} \in \mathfrak{CF}\}$. Then the map b is defined as follows:

$$b : \mathbb{N}^n \longrightarrow \mathbb{N} \times (\mathbb{N}^n_{>0})^c, N(A) \to b(N(A)) := \begin{cases} (0, 0) & \text{if } \mathfrak{CF} = \emptyset, \\ (\#\mathfrak{CF}, \mathfrak{n}(\mathfrak{CF})) & \text{otherwise,} \end{cases}$$

where $c = \#\mathfrak{CF}$. Due to the transformation rule of normal vectors under the map $\tau_{J,j}$ determined above and remark 25, one can easily deduce that the transform $A' = \tau_{J,j}(A)$ of A satisfies the inequality

$$b(N(A')) <_{lex} b(N(A)),$$

where in the case $\#\mathfrak{CF}' = \#\mathfrak{CF}$, the set $\mathfrak{n}(\mathfrak{CF}') = \{\mathfrak{n}(\mathfrak{F}'); \mathfrak{F}' \text{ compact facet of } A'\}$ has to be ordered in the same manner as $\mathfrak{n}(\mathfrak{CF})$, i.e., $(\mathfrak{n}(\mathfrak{CF}'))_i = \tau_{J,j}^{dual}((\mathfrak{n}(\mathfrak{CF})_i)$ for $1 \leq i \leq \#\mathfrak{CF} = \#\mathfrak{CF}'$. This implies the assertion of the proposition. $\qquad \square$

4.3.2   Case: The Newton polyhedron doesn't contain compact facets

In the last paragraph we have seen that a Newton polyhedron $N(A)$ containing compact facets can be transformed by finitely many point blowups into a Newton polyhedron without compact facets and during this procedure the set of normal vectors of $N(A)$ gives a measure for the improvement. We are hence left with the case that the Newton polyhedron $N(A)$ has no compact facet. Surprisingly, in this situation strange phenomena occur, which will be described in the sequel, and make any kind of induction via normal vectors in order to solve the simple polyhedron game very challenging.

Phenomenon 1: Transformation rule of section 4.3.1 fails

A first difficulty in solving the simple polyhedron game for Newton polyhedra without compact facets using normal vectors lies in the fact, that normal vectors to non–compact facets are in general transformed differently to the rule determined in section 4.3.1 of this chapter. This can for instance be seen in the following example:

Example 16. Let $A = \{(2,0,0),(0,2,1)\} \subset \mathbb{N}^3$, which corresponds to the Whitney umbrella $x^2 - y^2 z = 0 \subset \mathbb{A}_K^3$. It is a simple matter to check that the Newton polyhedron $N(A)$ doesn't contain a compact facet and that its normal vectors are given by

$$e_1 = (1,0,0)^T, e_2 = (0,1,0)^T, e_3 = (0,0,1)^T, \mathfrak{n}_1 = (1,0,2)^T, \mathfrak{n}_2 = (1,1,0)^T$$
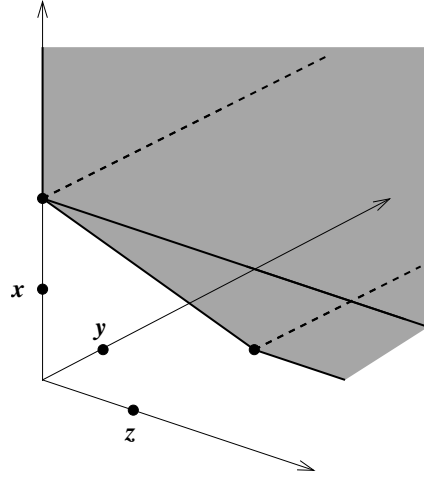
(see figure 4.1).

Figure 4.1: $N(A)$ for $A = \{(2,0,0),(0,2,1)\}$.

Now we will investigate the transformation of A and e_1, e_2, e_3, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ under a point blowup, i.e., for the choice $J = \{1,2,3\}$ of player 1. Consider especially the z-chart of the blowup, i.e., to the choice $j = 3$ of player 2. The map $\tau_{\{1,2,3\},3}$ then yields

$$A' = \{(2,0,2),(0,2,3)\}.$$

Easy considerations show that the set of normal vectors to $N(A')$ is the same as for $N(A)$. But these don't coincide with the transforms of the e_1, e_2, e_3, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ under the map $\tau_{\{1,2,3\},3}^{dual}$ dual to $\tau_{\{1,2,3\},3}$:

$$
\begin{aligned}
\tau_{\{1,2,3\},3}^{dual}(e_1) &= (1-0,0-0,0)^T = (1,0,0)^T &&= (1,0,0)^T = e_1 \\
\tau_{\{1,2,3\},3}^{dual}(e_2) &= (0-0,1-0,0)^T = (0,1,0)^T &&= (0,1,0)^T = e_2 \\
\tau_{\{1,2,3\},3}^{dual}(e_3) &= (0-1,0-1,1-0)^T = (-1,-1,1)^T &&\neq (0,0,1)^T = e_3 \\
\tau_{\{1,2,3\},3}^{dual}(\mathfrak{n}_1) &= (1-2,0-2,2)^T = (-1,-2,2)^T &&\neq (1,0,2)^T = \mathfrak{n}_1 \\
\tau_{\{1,2,3\},3}^{dual}(\mathfrak{n}_2) &= (1-0,1-0,0)^T = (1,1,0)^T &&= (1,1,0)^T = \mathfrak{n}_2
\end{aligned}
$$

Remark 26. Note further that in the example above the Newton polyhedra $N(A)$ and $N(A')$ differ only in a displacement by 2 units in the z-direction. For this reason the choice $J = \{1,2,3\}$ didn't improve the Newton polyhedron. At first sight it is not

clear from the shape of the Newton polyhedron $N(A)$ which choice of a larger center (for instance $J = \{1,2\}$ or $J = \{1,3\}$) would advance the situation. Computing the singular locus of $X : x^2 - y^2z = 0$ results in $\mathrm{Sing}(X) = \{z\text{–axis}\}$ and one can easily check that choosing $J = \{1,2\}$ yields for both choices of player 2 an improvement: For $j = 1$ the obtained Newton polyhedron $N(A')$ is already a quadrant; for $j = 2$ the polyhedron $N(A') = N(\{(2,2,0),(0,2,1)\})$ is not yet a quadrant, but now only has one normal vector different from the standard basis vectors. The choice $J = \{1,3\}$ will be treated in the next example.

Phenomenon 2: New normal vectors arise

A further difficulty in measuring the complexity of $N(A)$ by its normal vectors stems from the fact that not all normal vectors of the transformed Newton polyhedron $N(A')$ originate from normal vectors of $N(A)$. This can already be seen in example 16 above, where the normal vector $(1,0,2)^T$ of $N(A')$ doesn't correspond to one of the transforms of the normal vectors of $N(A)$. In order to show that this phenomenon can also occur when choosing a different set J (respectively a different center of the blowup), a second example is stated:

Example 17. Let again $A = \{(2,0,0),(0,2,1)\}$. Then the Newton polyhedron $N(A)$ has two facets which are not parallel to one of the three coordinate planes in $\mathbb{R}^3$. They are parallel to the y– respectively the z–axis. Now consider the transform of A for $J = \{1,3\}$ and $j = 3$:

$$A' = \{(2,0,2),(0,2,1)\}.$$
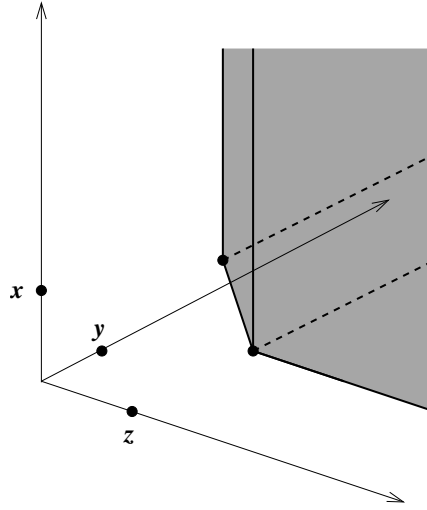
Figure 4.2: $N(A')$ for $A' = \{(2,0,2),(0,2,1)\}$.

One can check that the Newton polyhedron $N(A')$ has again two facets which are not parallel to one of the coordinate planes and that its normal vectors are given by

$$\mathfrak{n}_1' = (0,1,2)^T, \mathfrak{n}_2' = (1,1,0).$$

But these facets are now parallel to the x– and the y–axis (see figure 4.2). Further the normal vector $\mathfrak{n}_1'$ doesn't stem from a normal vector of $N(A)$. The example thus shows

that the direction of the non–compact facets of the Newton polyhedron may change during the simple polyhedron game. These phenomena make any kind of induction very challenging.

4.3.3 Summary & Ideas for further investigations

In section 4.3.1 we have proven that any Newton polyhedron containing at least one compact facet can be transformed by the repeated choice $J = \{1, \ldots, n\} \subset \mathbb{N}^n$ of player 1 – independently of the choice of player 2 – in finitely many rounds of the simple polyhedron game into a Newton polyhedron without compact facets.

Therefore it remains to study Newton polyhedra $N(A)$ without compact facets. But in section 4.3.2 we have seen that in this situation the behavior of normal vectors under the simple polyhedron game is more involved. Hence also the choice of a suitable set $J \subset \{1, \ldots, n\}$ according to the normal vectors of $N(A)$ is much more complicated and is not completely clear up to now. This could be examined further in the future. For this purpose it will be possibly of great help to explore the relation between the singular locus of an algebraic variety and the facets of its corresponding Newton polyhedron in more details.

A *different strategy* for solving the simple polyhedron game could be the following: Let $A \subset \mathbb{N}^n$ and $N(A)$ its Newton polyhedron. Then the first goal for player 1 could be to transform $N(A)$ by finitely many (point) blowups into a *cartesian product* in the sense that all vertices of $N(A)$ are contained in a plane which is parallel to one of the coordinate planes. Note that this condition is stronger than not containing compact facets. If one achieves to transform $N(A)$ into a polyhedron $N(A')$ satisfying this condition, then $N(A')$ can be seen as a Newton polyhedron in one dimension less and the problem could be solved by induction on the dimension n. Unfortunately it is up to now not clear how to transform a Newton polyhedron into a Newton polyhedron with a cartesian product structure.

Part II – Locus of points where the Newton polyhedron has a compact facet w.r.t. all choices of local coordinates

In addition to introducing the simple polyhedron game, Hironaka also established rules for the hard polyhedron game in $\mathbb{N}^n$, which is intended to not only describe the combinatorial part of blowups, but all its details of resolution of singularities is arbitrary characteristic. Especially, its winning strategy should not depend on any local coordinates. Historically one should mention that the description of the hard polyhedron game by Hironaka is so general that it doesn't have a winning strategy (Spivakovsky could give in [46] a counterexample). Therefore we will give in the sequel a definition of the hard polyhedron game which includes all details of resolution of singularities via blowups but where the possibilities of player 2 are limited in such a way that there exists a winning strategy for player 1.

Over fields of characteristic zero the hard polyhedron game is well–understood and there exist various solutions to it, as for instance in [50]. There one main ingredient is the so–called Tschirnhaus transformation, which has the lack of breaking down in positive characteristic. In the case of positive characteristic there exist only solutions for small values of the dimension n of the ambient affine space. For $n = 2$ there are various methods to the prove the result (see for instance [23] and [50]). In chapter 1 of this thesis a new approach for purely inseparable equations in $\mathbb{A}_K^3$ is given. Some ideas and the occurring phenomena in resolution of purely inseparable equations in $\mathbb{A}_K^4$ are indicated in chapter 2 of this thesis.

Recall that a new solution to the simple polyhedron game – for instance the method of using normal vectors examined in section 4.3 of this chapter – could lead to a novel winning strategy for player 1 in the hard polyhedron game and hence would yield a new strategy in resolution of singularities of algebraic varieties.

The fact that there are up to now no complete proofs for a solution to the hard polyhedron game for $n > 4$, although many experts in the field are working on a strategy for resolution of singularities in positive characteristic, shows that this task is very challenging. Therefore we will here only deal with a question affecting the termination of a resolution algorithm using the strategy for the simple polyhedron game developed in section 4.3 of this chapter. The exact formulation of the problem will be given in section 4.4. But before posing the question and describing its connection to the hard polyhedron game in more details, we will state for sake of completeness rules for the hard polyhedron game. For more details we refer to [50].

The hard polyhedron game

Let K be an algebraically closed field of arbitrary characteristic and $F(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in K[[x_1, \ldots, x_n]]$.

Course of the game

(1) Player 1 chooses, taking into account the series F, a subset $J \subset \{1, \ldots, n\}$.

(2) Player 2 chooses in consideration of the series F and the set J an element $j \in J$ and parameters $t_i \in K$, where $i \in J \setminus \{j\}$. Afterwards he substitutes F by $F' := \varphi(F)$ with

$$\varphi(x_i) := \begin{cases} x_i x_j + t_i x_j & \text{for } i \in J \setminus \{j\}, \\ x_i & \text{otherwise.} \end{cases}$$

This routine is now repeated, yielding a sequence $(F^m)_{m \in \mathbb{N}}$ of series F^m where $F^0 := F$ and $F^{m+1} := (F^m)'$.

End of the game

Player 1 wins the game if after finitely many, say l, rounds of the game the resulting series F^l is up to a formal coordinate change a monomial.

Problem

Show that there exists a winnings strategy for player 1.

Remark 27. Note that the simple polyhedron game introduced in part I of this chapter can be reformulated in the language of power series in the above way as well. It is easy to see that it corresponds to the hard polyhedron game, where player 2 doesn't have the possibility of choosing additionally to the element $j \in J$ also parameters $t_i \in K$, but where $t_i = 0$ for all i throughout the game.

4.4 Formulation of the problem

In section 4.3 the behavior of normal vectors to facets of the Newton polyhedron $N(A)$ of given algebraic hypersurface X was investigated. Further it was shown that under the assumption that the Newton polyhedron $N(A)$ contains at least one compact facet, the set of normal vectors to compact facets of $N(A)$ serves as a good measure for the complexity of the singularities of X. Moreover, it was illustrated that the main obstacle in the simple polyhedron game is given by Newton polyhedra without compact facets. Note that the occurrence of compact facets in the Newton polyhedron $N(A)$ of $X : F(x_1, \ldots, x_n) = 0$ depends on the choice of the coordinates $(x_1, \ldots, x_n)$ of $\mathbb{A}_K^n$. A sub–ordinate target to a solution of the hard polyhedron game could be – similarly as in Proposition 18 in section 4.3 – to achieve that the Newton polyhedron $N_a(A)$ does in all points $a \in X$ not contain a compact facet (with respect to at least one regular system of parameters $(x_1, \ldots, x_n)$). According to Proposition 18 of section 4.3 this can be realized by successive point blowups. But in order to show that this goal can be achieved by *finitely many* point blowups, it is of interest "how many" closed points $a \in X$ there exist where the Newton polyhedron $N_a(A)$ has with respect to all local coordinates at a compact facets. Such points will be called *blunt points*. The intention of this section is thus to examine the locus X_{blunt} of closed points $a \in X$ where the corresponding Newton polyhedron $N_a(A)$ does contain with respect to all choices of regular parameter systems $(x_1, \ldots, x_n)$ in a at least one compact facet. We are especially interested in the questions whether the set X_{blunt} is Zariski–closed and/or consists only of finitely many points. (Partial) answers to these two questions will be given in sections 4.5 and 4.6.

Setting

Let R be the coordinate ring of $\mathbb{A}^n$ over an algebraically closed field K of arbitrary characteristic. Denote by R_a the localization of R at a closed point a of $\mathbb{A}^n$ and by $\widehat{R}_a$ its completion with respect to the maximal ideal. Then any regular system of parameters $(x_1, \ldots, x_n)$ of $\widehat{R}_a$ will be called *local coordinates* of R at a. Any choice of such coordinates induces an isomorphism $\widehat{R}_a \cong K[[x_1, \ldots, x_n]]$ corresponding to the Taylor expansion of elements $g \in R$ and their expansion $G = \sum c_{\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in K[[x_1, \ldots, x_n]]$.

Let in the sequel f be an irreducible element of R. Further let $X = V(f)$ be the hypersurface defined by $f = 0$, a a closed point of X and $(x_1, \ldots, x_n)$ local coordinates of R at a. Denote by $N_a(F)$ the Newton polyhedron of the expansion $F \in K[[x_1, \ldots, x_n]]$ of f with respect to the local coordinates $(x_1, \ldots, x_n)$ at a. Furthermore denote by $A_a(F)$ its set of vertices, i.e., the minimal set $A_a(F)$ such that $N_a(F) = \mathrm{conv}(A_a(F) + \mathbb{R}_+^n)$.

Definitions

An element $g \in R$ is said to be *monomial* at a if there exists a formal coordinate change transforming G into a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ times a unit in $K[[x_1, \ldots, x_n]]$.

A closed point $a \in X = V(f)$ is called a *blunt point of X* if for all local coordinates $(x_1, \ldots, x_n)$ at a the Newton polyhedron $N_a(F)$ contains a compact facet. We will denote the set of blunt points of X by X_{blunt}.

Questions

The goal of the next two sections is to find (at least partial) answers to the following two questions:

(a) Do the blunt points of X form a Zariski–closed subset?

(b) Are there only finitely many blunt points on X?

Since finding an answer to question (b) is easier, we will start with the treatment of this problem. This is done in the section 4.5. First it will be indicated that X_{blunt} is a subset of the singular locus of X. From this it immediately follows that plane algebraic curves contain only finitely many blunt points. Afterwards the set of blunt points of algebraic varieties in $\mathbb{A}_{\mathbb{C}}^3$ is examined in concrete examples. Finally it will be proven that in the case of $K = \mathbb{C}$, and under the assumption that question (a) could be answered positively, $X \subset \mathbb{A}_{\mathbb{C}}^n$ contains only finitely many blunt points.

Section 4.6 is devoted to the study of the algebraicity of the set X_{blunt}. Unfortunately I don't have a definite answer to this problem up to now. Therefore only some remarks and ideas will be stated.

Remark 28. Note that question (a) above is quite similar to the question whether the monomial locus X_{mon} of X, which will be studied in chapter 5 and consists of those closed points $a \in X$ such that there exist local coordinates $x = (x_1, \ldots, x_n)$ at a with $F(x) = x^\alpha$ for some $\alpha \in \mathbb{N}^n$, is Zariski–open. In chapter 5 it will be shown by means of constructions using étale neighborhoods that X_{mon} is Zariski–open. Obviously for $n = 2$ the property for a point $a \in X$ to be blunt is equivalent to not being contained in X_{mon}. For $n > 2$ this is not anymore the case; being a blunt point is a much stronger condition than being not contained in X_{mon}. Therefore it is much harder to prove that the set X_{blunt} is Zariski–closed.

4.5   Are there only finitely many blunt points on X?

4.5.1   Blunt points are contained in the singular locus of X

Let a be a smooth point of $X = V(f)$. Then the linear part $F_1(x_1, \ldots, x_n)$ of the expansion $F(x_1, \ldots, x_n)$ of f with respect to any local coordinates $(x_1, \ldots, x_n)$ at a is not identically zero. Due to Weierstrass' Preparation Theorem one can thus assume that F is (up to multiplication by a unit in $K[[x_1, \ldots, x_n]]^*$) a distinguished polynomial of degree 1 in x_n, i.e., is of the form

$$F(x_1, \ldots, x_n) = x_n + H(x_1, \ldots, x_{n-1})$$

with $H \in K[[x_1, \ldots, x_{n-1}]]$ of order $\mathrm{ord}(H) \geq 1$. Now consider the linear coordinate change $\phi : (x_1, \ldots, x_n) \to (x_1, \ldots, x_{n-1}, x_n - H(x_1, \ldots, x_{n-1}))$. Then the Newton polyhedron of $\varphi(F(x_1, \ldots, x_n)) = x_n$ clearly doesn't contain a compact facet. This shows that smooth points $a \in X$ are not contained in X_{blunt}. More precisely, f is in every smooth point a of X monomial, i.e., in such points there always exist local coordinates $(x_1, \ldots, x_n)$ such that the Newton polyhedron of the expansion $F(x_1, \ldots, x_n)$ of f in a is a quadrant.

4.5.2 Case of algebraic curves in $\mathbb{A}^n$

Due to the last section blunt points of X are necessarily singular. In the case of algebraic curves this immediately implies that there are at most finitely many blunt points on X.

4.5.3 Some examples in the case of surfaces in $\mathbb{A}_{\mathbb{C}}^3$

In the sequel some examples of surfaces in $\mathbb{A}_{\mathbb{C}}^3$ are treated. They indicate that blunt points $a \in X$ are quite rare. Nevertheless the last two examples would suggest that blunt points are not isolated on X.

Dingdong

Let $X = V(f) \subset \mathbb{A}_{\mathbb{C}}^3$, where the Taylor expansion of f with respect to local coordinates (x, y, z) of $\mathbb{A}^3$ at the origin is given by

$$F(x, y, z) = x^2 + y^2 - z^2 + z^3$$

(see figure 4.3 for a picture of Dingdong over the real numbers).
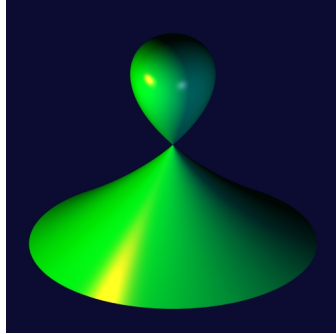
Figure 4.3: $X = V(x^2 + y^2 - z^2 + z^3)$.

Clearly, the singular locus of X consists only of the origin. This is hence the only candidate for a blunt point on X. The Newton polyhedron corresponding to X in $(0, 0, 0)$ is given by

$$N_0(F) = \mathrm{conv}\left(\{(2, 0, 0), (0, 2, 0), (0, 0, 2)\} + \mathbb{R}_+^3\right),$$

thus contains a compact facet. But applying the coordinate change $\varphi : (x, y, z) \to (x + y, i \cdot (y - x), z)$, where $i = \sqrt{-1}$, yields $\varphi(F(x, y, z)) = 4xy - z^2(1 - z)$.

Clearly the corresponding Newton polyhedron $N_0(\varphi(F))$ does not anymore contain a compact facet.

Summary: Dingdong doesn't contain a blunt point!

Whitney umbrella

Now consider a surface whose singular locus contains not only isolated points. Take for instance the Whitney-umbrella $X = V(F) \subset \mathbb{A}^3_{\mathbb{C}}$ (see figure 4.4) with

$$F(x, y, z) = x^2 - y^2 z.$$

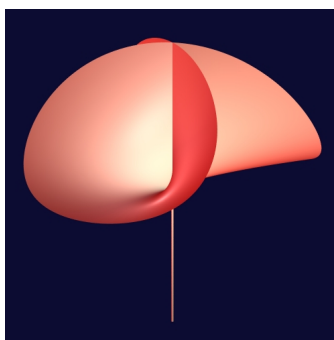The singular locus of X consists of a smooth curve, more precisely $\text{Sing}(X) = \{z -$

Figure 4.4: $X = V(x^2 - y^2 z)$.

axis$\}$. First consider the Newton polyhedron $N_0(F)$ corresponding to X in the origin: Since its set of vertices contains only two points, $N_0(F)$ clearly doesn't contain a compact facet. Now consider a different point $a' = (0, 0, t)$, $t \neq 0$, of $\text{Sing}(X)$. The Taylor expansion of F in a' is given by

$$F(x, y, z + t) = x^2 - y^2 z - y^2 t.$$

Clearly also the Newton polyhedron $N_{a'}(F)$ corresponding to X in the point $a' = (0, 0, t)$ doesn't contain a compact facet.

Summary: The Whitney umbrella has no blunt points!

Calyx

Let $X = V(F) \subset \mathbb{A}^3_{\mathbb{C}}$ with

$$F(x, y, z) = x^2 + y^2 z^3 - z^4$$

(see figure 4.5). Then the singular locus of X is given by $\text{Sing}(X) = \{y - \text{axis}\}$. The Newton polyhedron corresponding to X at the origin is

$$N_0(F) = \text{conv}\left(\{(2, 0, 0), (0, 2, 3), (0, 0, 4)\} + \mathbb{R}^3_+\right).$$

Note that the vertices $(2, 0, 0)$, $(0, 2, 3)$ and $(0, 0, 4)$ define a compact facet of $N_0(F)$ since its normal vector is equal to $(4, 1, 2)$. Nevertheless it is not clear if there exists
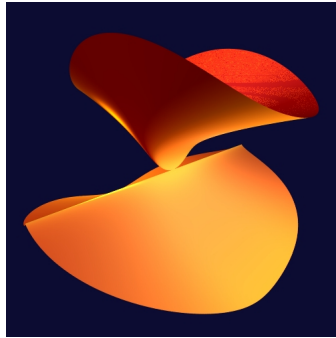
Figure 4.5: $X = V(x^2 + y^2 z^3 - z^4)$.

a coordinate change which transforms $N_0(F)$ into a Newton polyhedron without compact facets. Now consider the Newton polyhedron corresponding to X in a different point $a' = (0, t, 0)$, $t \neq 0$, of the y-axis. The Taylor expansion of F in a' is given by

$$F(x, y + t, z) = x^2 + y^2 z^3 + 2tyz^3 + t^2 z^3 - z^4.$$

Since by assumption $t \neq 0$, the set $A_{a'}(F)$ of vertices of the Newton polyhedron $N_{a'}(F)$ is equal to $\{(2, 0, 0), (0, 0, 3)\}$, consequently $N_{a'}(F)$ doesn't contain a compact facet.

Summary: The only point of Calyx which is possibly a blunt point is the origin!

Plop

Consider the algebraic variety $X = V(F) \subset \mathbb{A}_{\mathbb{C}}^3$ with

$$F(x, y, z) = x^2 + (z + y^2)^3$$

(see figure 4.6).
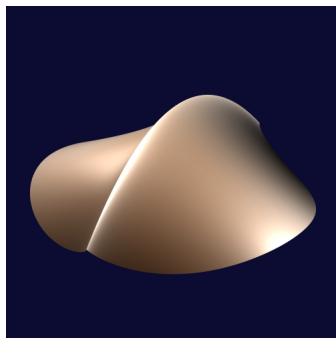
Figure 4.6: $X = V(x^2 + (z + y^2)^3)$.

Its singular locus is given by the curve $C = V(x, z + y^2) \subset X$. The set of vertices of the Newton polyhedron $N_0(F)$ corresponding to X in the origin is equal to

$$A_0(F) = \{(2, 0, 0), (0, 6, 0), (0, 0, 3)\}.$$

It of course has a compact facet. But the coordinate change

$$\varphi : (x, y, z) \rightarrow (x, y, z - y^2)$$

yields $\varphi(F) = x^2 + z^3$. It thus transforms the Newton polyhedron $N_0(F)$ into a Newton polyhedron without a compact facet. Now let a' be a different point of $\text{Sing}(X) = \{(0, t, -t^2), t \in K\}$. Then the Taylor expansion of F in a' is equal to $F(x, y + t, z - t^2) = x^2 + \left(z + y^2 + 2ty\right)^3$. The Newton polyhedron $N_{a'}(F)$ corresponding to X in $a' = (0, t, -t^2)$ with $t \neq 0$ is hence given by

$$N_{a'}(F) = \text{conv}\left(\{(2, 0, 0), (0, 3, 0), (0, 0, 3)\} + \mathbb{R}_+^3\right)$$

and thus contains a compact facet. But there again exists a coordinate change ψ at a', namely $\psi : (x, y, z) \rightarrow (x, y, z - y^2 + 2yt)$, such that $N_{a'}(\psi(F))$ has no compact facet.

Summary: Plop doesn't contain a blunt point!

Daisy

Now consider the more complicated algebraic surface Daisy (see figure 4.7), defined by $X = V(F) \subset \mathbb{A}_{\mathbb{C}}^3$ with

$$F(x, y, z) = (x^2 - y^3)^2 - (z^2 - y^2)^3.$$
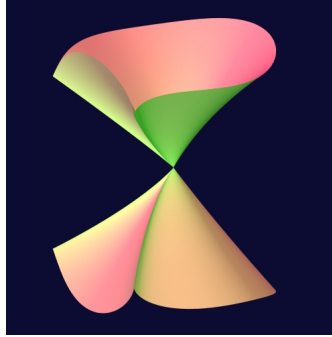
Figure 4.7: $X = V\left((x^2 - y^3)^2 - (z^2 - y^2)^3\right)$.

It can be checked that the singular locus of X is given by

$$\text{Sing}(X) = V(x^2 - y^3, y^2 - z^2),$$

thus consists of the two singular curves $C_1 = V(x^2 - y^3, y - z)$ and $C_2 = V(x^2 - y^3, y + z)$. Consider first the Newton polyhedron $N_0(F)$ corresponding to X in the origin. It is simple task to check that its set of vertices is equal to $A_0(F) = \{(4, 0, 0), (0, 6, 0), (0, 0, 6)\}$. Since the normal vector of the facet $\mathfrak{F}$ defined by the three elements of $A_0(F)$ is an element of $\mathbb{N}_{>0}^3$, $\mathfrak{F}$ is a compact facet of $N_0(F)$. Now look at the Newton polyhedron $N_{a'}(F)$ corresponding to X at a different point a' of

the curve $C_1 = \{(t^3, t^2, t^2), t \in K\}$ (the treatment of points on the curve C_2 works analogously). Then the Taylor expansion of F in a' is given by

$$\begin{aligned}
F_{a'}(x, y, z) &= F(x + t^3, y + t^2, z + t^2) \\
&= \left(x^2 + 2t^3 x - y^3 - 3t^2 y^2 - 3t^4 y\right)^2 - \left(z^2 + 2t^2 z - y^2 - 2t^2 y\right)^3.
\end{aligned}$$

It can be checked that the set of vertices $A_{a'}(F)$ of the Newton polyhedron $N_{a'}(F)$ is for all points $a' \in C_1$ equal to

$$A_{a'}(F) = \{(2, 0, 0), (0, 2, 0), (0, 0, 3)\}.$$

Thus $N_{a'}(F)$ contains for all $a' \in C_1$ a compact facet! It is not clear at all if there exists a coordinate change transforming $N_{a'}(F)$ into a Newton polyhedron without compact facets. Consider for example the point $a' = (1, 1, 1) \in X$. Applying the formal coordinate change

$$\varphi : (x, y, z) \rightarrow \left(-1 + \sqrt{1 + x}, -1 + \sqrt{1 + y}, -1 + \sqrt{1 + y + z}\right)$$

at a' yields $\varphi(F_{a'}) = \left(1 + x - \sqrt{1 + y} - y\sqrt{1 + y}\right)^2 - z^3$. The corresponding Newton polyhedron $N_{a'}(\varphi(F))$ still has a compact facet, but now its set of vertices is, with the exception of one element, entirely contained in the x-y-plane. Nevertheless it is still unclear if there exists a further coordinate change at a' which transforms $\varphi(F_{a'})$ into a series whose Newton polygon has no compact facet.

Summary: At all points a' of the two curves $C_1 \cup C_2 = V(x^2 - y^3, y^2 - z^2) \subset X$ there exist local coordinates such that the Newton polyhedron corresponding to X at a' has a compact facet. It is not clear if all these points are contained in X_{blunt}.

Spitz

Let $X = V(F) \subset \mathbb{A}_{\mathbb{C}}^3$ with

$$F(x, y, z) = (y^3 - x^2 - z^2)^3 - 27x^2 y^3 z^2$$

(see figure 4.8). A simple computation yields that the singular locus of X is

$$\mathrm{Sing}(X) = V(x, y^3 - z^2) \cup V(y, x^2 + z^2) \cup V(x^2 - y^3, z) \cup V(x^2 - z^2, y^3 + z^2).$$

The Newton polyhedron $N_0(F)$ corresponding to X in the origin is given by

$$N_0(F) = \mathrm{conv}\left(\{(6, 0, 0), (0, 9, 0), (0, 0, 6)\} + \mathbb{R}_+^3\right),$$

hence contains a compact facet. Now consider for instance a point a' on the curve $C = V(x^2 - y^3, z) = \{(t^3, t^2, 0), t \in K\} \subset \mathrm{Sing}(X)$. Then the Taylor expansion of F in a' is

$$\begin{aligned}
F_{a'}(x, y, z) &= F(x + t^3, y + t^2, z) \\
&= \left((y + t^2)^3 - (x + t^3)^2 - z^2\right)^3 - 27(x + t^3)^2 (y + t^2)^3 z^2.
\end{aligned}$$

One can check that for all $a' \in C \setminus \{(0, 0, 0)\}$ the set of vertices $A_{a'}(F)$ of $N_{a'}(F)$ is equal to

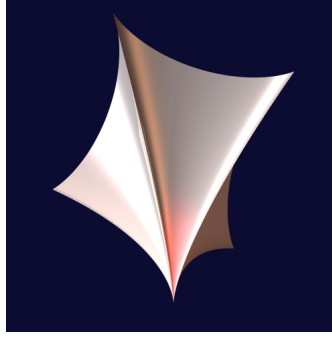$$A_{a'}(F) = \{(3, 0, 0), (0, 3, 0), (0, 0, 2)\}.$$

Figure 4.8: $X = V\left((y^3 - x^2 - z^2)^3 - 27x^2y^3z^2\right)$.

Consequently $N_{a'}(F)$ does for all $a' \in C$ contain a compact facet! Again it is not clear if there exists a coordinate change transforming $N_{a'}(F)$ into a Newton polyhedron without compact facets.

Summary: X contains at least one curve C with the following property: At all points $a' \in C$ there exist local coordinates such that the Newton polyhedron $N_{a'}(F)$ does contain a compact facet. Again it is not clear if all these points are contained in X_{blunt}.

Remark 29. It is conspicuous that the only examples (among a lot of examples, which were computed) in which possibly non isolated blunt points occur, are examples where the singular locus contains (cusp–like) singular curves.

4.5.4   Case $K = \mathbb{C}$: X **contains only finitely many blunt points**

Although the last two examples would indicate that blunt points are not isolated on X, we will prove in this section that in the case $K = \mathbb{C}$ the opposite is true if one assumes that the set X_{blunt} of blunt points in X is Zariski–closed. The limitation of the proof given below to the case $K = \mathbb{C}$ is due to the fact that it uses the Identity Theorem for analytic power series.

In the proof of the Proposition below we will suppose that the set of blunt points of X satisfies the following condition:

Assumption $(*)$. The blunt points of X form an algebraic subset of X.

Remark 30. The question whether the set X_{blunt} of blunt points of X is algebraic is treated in section 4.6. Unfortunately the answer to this question is not clear up to now, but some ideas will be indicated.

The rest of this section is devoted to the proof of the following statement:

Theorem 4. *Suppose that the assumption* $(*)$ *holds. Then any algebraic hypersurface* $X = V(f) \subset \mathbb{A}_{\mathbb{C}}^n$ *contains only finitely many blunt points.*

Proof. We begin by proving that each (not necessarily smooth) algebraic curve on $X = V(f)$ contains at most isolated blunt points. Suppose the opposite, i.e., that there exists an irreducible algebraic curve $C \subset X$ containing non–isolated blunt points. Due to our assumption $(*)$, this implies that each point on C is a blunt point of X. Let $a \in C$ be a smooth point of C. Then there exist local analytic coordinates $(x_1, \ldots, x_n)$ at a such that C is in an euclidean neighborhood U_a of a given by $x_1 = \ldots = x_{n-1} = 0$. Since a is a blunt point of X, the Newton polyhedron $N_a(F)$ of the expansion F_a of f with respect to $(x_1, \ldots, x_n)$ does contain a compact facet. Now consider the expansion $F_{a'}$ of f in a different point a' of $C \cap U_a$ (after possibly shrinking U_a in order to ensure that all series converge). It is given by

$$F_{a'}(x_1, \ldots, x_n) = F_a(x_1, \ldots, x_{n-1}, x_n + t)$$

with $t \in K^*$. Note that the corresponding Newton polyhedron $N_{a'}(F)$ consists of points of the Newton polyhedron $N_a(F)$ and of several additional points in the negative x_n–direction from all the points of $N_a(F)$. See figures 4.9 and 4.10 for illustrations of $N_a(F)$ and $N_{a'}(F)$ in the cases $n = 2$ and $n = 3$.
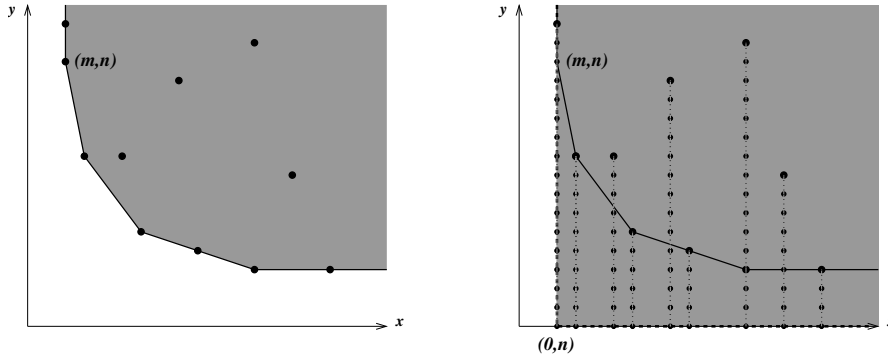
Figure 4.9: $N_a(F)$ and $N_{a'}(F)$ in the case $n = 2$.

Let $c_{\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be the monomial corresponding to a vertex $(\alpha_1, \ldots, \alpha_n)$ contained in the set of vertices $A_a(F)$ of the Newton polyhedron $N_a(F)$. Then the coefficient of the monomial $x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^0$ occurring in the expansion of $F_{a'}$ is a analytic power series $P(t) \neq 0$ in the variable t. The Identity Theorem for analytic functions implies that if the analytic power series $P(t)$ has an accumulation point in $C \cap U_a$, then P would be identically zero, which is a contradiction. It thus follows that $P(t)$ vanishes at most in isolated points of $C \cap U_a$.

Denote the vertices in $A_a(F)$ by

$$v_1 = (\alpha_{1,1}, \ldots, \alpha_{1,n}), \ldots, v_m = (\alpha_{m,1}, \ldots, \alpha_{m,n}).$$

Then the set of points of $C \cap U_a$ such that one of the coefficients of the monomials $x_1^{\alpha_{1,1}} \cdots x_{n-1}^{\alpha_{1,n-1}}, \ldots, x_1^{\alpha_{m,1}} \cdots x_{n-1}^{\alpha_{m,n-1}}$ occurring in the expansion of $F_{a'}$ vanishes, consists only of isolated points. In all other cases the set of vertices $A_{a'}(F)$ of $N_{a'}(F)$ is equal to a subset of

$$A_{a'}(F) = \{(\alpha_{1,1}, \ldots, \alpha_{1,n-1}, 0), \ldots, (\alpha_{m,1}, \ldots, \alpha_{m,n-1}, 0)\}.$$

In this situation $N_{a'}(F)$ clearly doesn't contain a compact facet! This shows that an euclidean neighborhood U_a of a smooth point $a \in C$ contains at most isolated blunt points $a' \in C \cap U_a$. And since every curve does only have at most finitely many singular points it follows that the blunt points on the curve C are isolated.

We are now in the position to show that there are only finitely many blunt points on X. By the observation above it follows that X_{blunt} does not contain any algebraic curves. Since X_{blunt} is by assumption $(*)$ an algebraic subset of X, this immediately implies that it consists of at most finitely many blunt points.
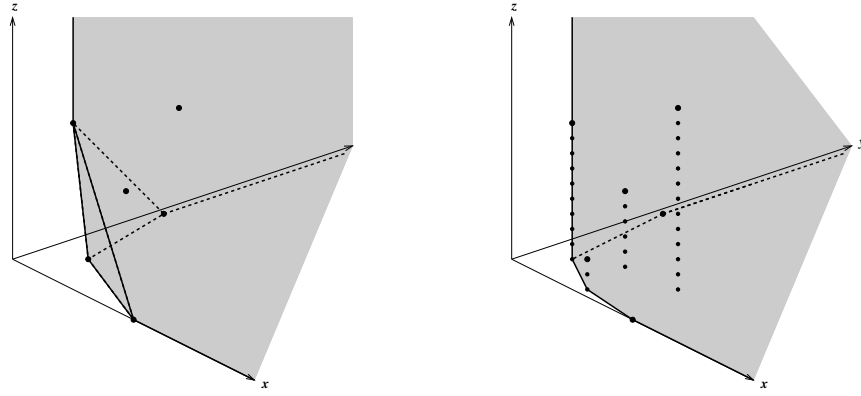
Figure 4.10: $N_a(F)$ and $N_{a'}(F)$ in the case $n = 3$.

□

Remark 31. Note that the proof of the last theorem actually shows more, namely that the following stronger statement is valid:

Theorem 5. *Suppose that the assumption $(*)$ holds. Consider the subset Y of points $a \in X$ such that for all local coordinates $(x_1, \ldots, x_n)$ at a, the set of vertices of the Newton polyhedron $N_a(F)$ is not entirely contained in one of the coordinate hyperplanes. Or said differently, the subset Y containing all points a in which there don't exist local coordinates $(x_1, \ldots, x_n)$ such that the set $A_a(F)$ of vertices of the Newton polyhedron $N_a(F)$ is entirely contained in one of the coordinate hyperplanes. Then the set $Y \subset \mathbb{A}_{\mathbb{C}}^n$ consists of at most finitely many points.*

Since $X_{blunt} \subset Y$, it is obvious that the proof of this theorem immediately implies theorem 4. Maybe it is reasonable to modify the definition of blunt points in the direction of the last theorem. Note that Newton polyhedrons with "product structure" already appeared in section 4.3.3 of part I of this chapter, where a new strategy towards a solution to the simple polyhedron game was suggested. Nevertheless also with this modified definition of blunt points, it is not even in concrete examples (see last section, especially the examples Daisy and Spitz) clear how to find a coordinate change which transforms a given Newton polyhedron with a compact facet into one whose set of vertices is entirely contained in a coordinate hyperplane.

4.6   Is the locus of blunt points a Zariski–closed subset?

Unfortunately I haven't found an answer to this quite difficult question up to to now. Therefore I will only indicate some ideas and difficulties related to the answer of this question.

The main problem is that the definition of blunt points is not very handy. Recall that a closed point a of $X = V(f)$ is called a blunt point, if the Newton polyhedron $N_a(F)$ contains for all expansions F of f with respect to different local coordinates at a a compact facet.

To tackle the problem one could start by the investigation of the following questions: Fix a Newton polyhedron $N_a(F)$ with (at least) one compact facet $\mathfrak{F}$. Now apply different local coordinate changes φ at a to F and explore the transform $N_a(\varphi(F))$ of the initial Newton polyhedron $N_a(F)$ and especially the transform $\varphi(\mathfrak{F})$ of the facet $\mathfrak{F}$. Which facets $\varphi(\mathfrak{F})$ do appear when considering all possible local coordinate changes? In particular, is it possible that there occur only finitely many different facets? Furthermore, is there an algebraic condition on the set of coordinate changes φ which yield again the same compact facet, i.e., coordinate changes φ such that $\varphi(\mathfrak{F}) = \mathfrak{F}$? Another question would be: Under which circumstances does $N_a(\varphi(F))$ not contain anymore a compact facet?

Perhaps it is reasonable to change the definition of blunt points in the direction indicated in remark 31. More precisely, study Newton polyhedra for which there doesn't exist a local coordinate change such that the set of vertices of the transformed Newton polyhedron is entirely contained in one of the coordinate hyperplanes. Maybe this condition is more natural and easier to handle.

Note that the question whether the set of blunt points of X is Zariski–closed is quite similar to the questions investigated in chapter 5 of this thesis. There it is shown by means of constructions in the étale topology that the (algebraic) normal crossing locus and the monomial locus (for the definitions of these subsets of X we refer to section 5.1 respectively 5.3 of chapter 5) are Zariski–open in X. It is planned to examine more closely whether this method could also be used to show that the set X_{blunt} of blunt points of X (respectively the set Y introduced in remark 31 of the last section) is algebraic.

Chapter 5

Constructions using étale neighborhoods

In algebraic geometry it is often of special interest whether a particular subset of a given algebraic variety has specific topological properties. For example one might be interested in the singular locus or the normal crossings locus and ask whether it is Zariski–open, Zariski–closed or constructible. For the finiteness of the surface resolution algorithm described in chapter 1, it was necessary to answer this question for the set of points $a \in \mathbb{A}_K^2$ in which an element $f \in R$ is a monomial (modulo p-th powers and up to multiplication by a unit in $K[[y, z]]$). In Proposition 3 of section 1.7 in chapter 1 it is shown that this set is Zariski–closed. The proof is based on general constructions using étale neighborhoods, which will be described in detail in this chapter. As applications of these constructions, it will be shown that the (algebraic) normal crossings locus and the monomial locus of an algebraic variety are Zariski–closed. Further the property of being a "mikado" point is examined. The content of this chapter was developed together with C. Bruschek and is planned to be published as a survey article. Note that the result on the (algebraic) normal crossings locus is well-known to experts in the field, but we couldn't find any references and hence proved the result ourselves.

Setting

Let K be an algebraically closed field of arbitrary characteristic. Denote by $K[x]$ the polynomial ring in the variables $x = (x_1, \ldots, x_n)$ over K. Its completion with respect to the maximal ideal $\langle x \rangle$ is the ring of formal power series $K[[x]]$. For an ideal $I \subseteq K[x]$ we will denote by $V(I)$ the algebraic set defined by I, i.e., the set of points $p \in \mathbb{A}^n$ such that $f(p) = 0$ for all $f \in I$.

5.1   Global Constructions

Let X be a finite union of algebraic varieties over K (i.e., integral separated schemes of finite type over K). All properties which will be studied here are local, hence we may assume that all varieties are affine. In fact, we will consider mainly subvarieties

of some $\mathbb{A}_K^n$, $n \in \mathbb{N}$. We say that X *is algebraic normal crossings* (in short *an*) *at a point* $p \in X$ if there are local coordinates $y_1, \ldots, y_n$ at p such that X is locally at p given by $y_1 \cdots y_e = 0$ with $e \leq n$ (in the literature this property is also referred to as *simple* or *strict normal crossings*). By *"local coordinates"* we mean a regular system of parameters for the local ring $\mathcal{O}_{\mathbb{A}^n, p}$. We say that X *is normal crossings* (in short *nc*) *at* p if p is an algebraic normal crossings point for $\widehat{X}_p$, i.e., if the formal germ of X at p is defined by $y_1 \cdots y_e = 0$, where $y_1, \ldots, y_n$ is a formal coordinate system at p. A *formal coordinate system* is a regular system of parameters for $\widehat{\mathcal{O}}_{\mathbb{A}^n, p}$. The locus of points in which X is algebraic normal crossings (resp. normal crossings) is called the *algebraic normal crossings locus of X (resp. normal crossings locus of X)* and will be denoted by X_{anc} (resp. X_{nc}).

Example 18. The hypersurface $X = \operatorname{Spec} K[x, y, z]/\langle x^2 - y^2 z^2 \rangle \subseteq \mathbb{A}^3$ is algebraic normal crossings, thus also normal crossings, at all points except the origin. The origin is not a normal crossings point. In contrast, the hypersurface $Y = \operatorname{Spec} K[x, y, z]/\langle x^2 - y^2 z \rangle$ is irreducible, thus has no algebraic normal crossings points. Its normal crossings locus is $Y_{nc} = Y \setminus \{0\}$ (see figure 5.1), which is open in Y.
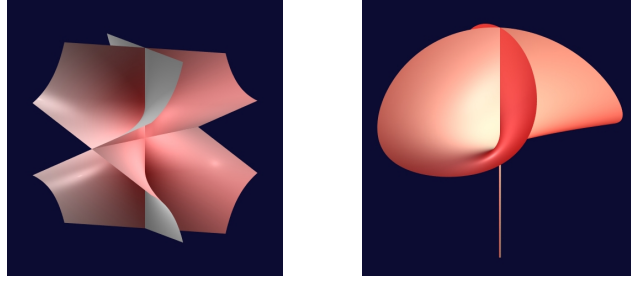
Figure 5.1: $X = V(x^2 - y^2 z^2)$ and $Y = V(x^2 - y^2 z)$.

Let $f \in K[x]$ and $p \in \mathbb{A}^n$. We denote by $\mathfrak{m}_p$ the maximal ideal of $\mathcal{O}_{\mathbb{A}^n, p}$ and by $df(p)$ *the differential of f at p*, i.e., the class $\bar{f} \in \mathfrak{m}_p/\mathfrak{m}_p^2$.

Lemma 4. *Let X be a union of hypersurfaces $X_i = \operatorname{Spec} K[x]/\langle f_i \rangle$, $1 \leq i \leq s$, $s \in \mathbb{N}$. Then X is anc at $p \in \mathbb{A}^n$ if and only if $p \notin (X_i)_{\text{sing}}$ for all i and the $\{df_i(p); f_i(p) = 0\}$ are K-linearly independent.*

Proof. The "if" part is obvious. For the other direction let X and p fulfill the conditions above. Since the $df_i(p)$ are K-linearly independent, the f_i vanishing at p are part of a regular system of parameters of $\mathcal{O}_{\mathbb{A}^n, p}$ (see [36], Thm. 14.2) from which the assertion follows immediately. □

Proposition 19. *The algebraic normal crossings locus of a finite union of hypersurfaces is open.*

Proof. Let $X = \cup_{1 \leq i \leq s} X_i$ with X_i be hypersurfaces defined by $f_i \in K[x]$. If p lies in the intersection $X_i \cap X_j$ of two hypersurfaces and is not an anc point of $X_i \cup X_j$,

then $p \in X_i \cap X_j \cap X_l$ is not an anc point for any X_l of $X_i \cup X_j \cup X_l$. Define the following (Zariski–) closed subsets of X. For $q \geq 1$:

$$A_q = \cup_{(i_1,\ldots,i_q)} \left(X_{i_1} \cap \cdots \cap X_{i_q} \cap V(M_q(df_{i_1},\ldots,df_{i_q})) \right),$$

where $M_q(df_{i_1},\ldots,df_{i_q})$ denotes the ideal generated by all q-minors of the matrix $(df_{i_1},\ldots,df_{i_q})$ and the union is taken over all q-tuples $(i_1,\ldots,i_q)$ with distinct entries $i_j \in \{1,\ldots,s\}$. Finally, set $A = \cup_{q=1}^s A_q$. Clearly A is closed and contains by Lemma 4 all non-anc points. Conversely, no point of A is an anc point. Thus $X_{anc} = X \setminus A$ is open. $\qquad\square$

Example 19. Figure 5.2 illustrates Proposition 19. Here X is the union of three plane curves.
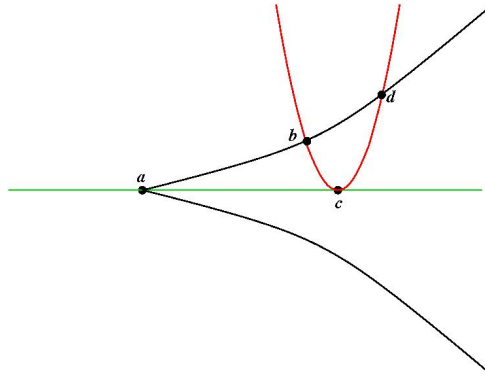
Figure 5.2: The points a and c are not anc, but b and d are.

5.2 Étale and Formal Neighbourhoods

Let X be a Noetherian scheme and Y a closed subscheme of X given by a sheaf of ideals $\mathcal{I}$. The *formal neighbourhood* $(\widehat{X}, \mathcal{O}_{\widehat{X}})$ *of X in Y* is the ringed space defined by the topological space Y and the sheaf of rings

$$\mathcal{O}_{\widehat{X}} = \varprojlim_n \mathcal{O}_X/\mathcal{I}^n$$

(see [17], II.9). If $X = \operatorname{Spec} A$ is affine and $p \in X$ is a closed point, then the formal neighbourhood of X in p is the one point space $\{p\}$ together with the sheaf of rings given by $\widehat{A}_p$, the completion of the local ring A_p with respect to its maximal ideal.

Example 20. Let $X = \operatorname{Spec} K[x,y]/\langle y^2 - x^2 - x^3 \rangle$ and $p = 0$. The structure sheaf of the formal neighbourhood $\widehat{X}$ of X in 0 is given by

$$K[[x,y]]/\langle y^2 - x^2 - x^3 \rangle,$$

which is not an integral domain. Indeed, $y^2 - x^2 - x^3$ factors in $K[[x,y]]$ as

$$y^2 - x^2 - x^3 = (y + x\sqrt{1+x})(y - x\sqrt{1+x}).$$

Therefore the formal neighbourhood of X in 0 is reducible.

Let X and Y be varieties over K. For a point $p \in X$ we denote by $C_p(X)$ the *tangent cone of X at p*. It is given by the associated graded algebra of the local ring of X at p:

$$\operatorname{gr}(\mathcal{O}_{X,p}) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1},$$

where $\mathfrak{m} \subseteq \mathcal{O}_{X,p}$ denotes the maximal ideal and $\mathfrak{m}^0 = \mathcal{O}_{X,p}$. More explicit, we may define $C_0(X)$ for an affine variety $X \subseteq \mathbb{A}^n$ given by an ideal $I \subseteq K[x_1, \dots, x_n]$ as follows: Denote by I_* the ideal of initial forms of elements of I, where the initial form of an element $f \in K[x]$ is its homogenous part of lowest degree. Then $C_0(X) \cong K[x]/I_*$.

For a point x of a variety (or scheme) X we denote by $\kappa(x)$ its residue field, i.e., $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Let $\varphi \colon X \to Y$ be a morphism of varieties such that the induced map $\varphi_p : \kappa(\varphi(p)) \to \kappa(p)$ is an isomorphism. We call φ *étale at $p \in X$* if the induced map on the tangent cones $d_p\varphi : C_p(X) \to C_{\varphi(p)}(Y)$ is an isomorphism. If $p \in X$ is regular then the tangent cone agrees with the tangent space, and an étale morphism is a morphism whose tangent map is an isomorphism. Especially if φ is étale we know that $p \in X_{\mathrm{reg}}$ if and only if $\varphi(p) \in Y_{\mathrm{reg}}$.

Example 21. Consider the following simple example. Let $X = \operatorname{Spec} K[x,y]/\langle x - y^2 \rangle$ and $Y = \operatorname{Spec} K[x]$ with $\varphi \colon X \to Y$ induced by the inclusion $K[x] \hookrightarrow K[x,y]$. Clearly for any $p = (p_x, p_y) \in X \setminus \{0\}$ the map $d_p\varphi$ gives an isomorphism between $C_{p_x}(Y)$ and $C_p(X)$, thus defines an étale map. Note that this is not the case for the point $p = 0$. Note further that in the case of $K = \mathbb{C}$ we may consider X and Y as manifolds over $\mathbb{C}$. Then the implicit function theorem is applicable, and states that X can (Euclidean–) locally at $p \in X \setminus \{0\}$ be parametrized by a neighbourhood of $p_x \in Y$.

For arbitrary schemes X and Y a morphism $\varphi \colon X \to Y$ is called *étale* if it is flat and unramified. It is called *étale at p* if the induced morphism of local schemes $\varphi \colon X_p \to Y_{\varphi(p)}$ is étale. For the convenience of the reader we summarize some properties of étale (resp. flat and unramified) morphisms in the next Proposition. Details, especially proofs, can be found for example in the excellent sources [17], [34] or [37].

Proposition 20. *Let X, Y be schemes. For a point $x \in X$ we denote by $\kappa(x)$ its residue field, i.e., $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.*

1. *A flat morphism of finite type between Noetherian schemes is open.*

2. *If $f \colon X \to Y$ is locally of finite type, then f is unramified if and only if the sheaf of relative differentials vanishes, i.e., $\Omega^1_{X/Y} = 0$.*

3. *Open immersions, compositions of étale morphisms and any base changes of étale morphisms are étale.*

4. *Let $f : X \to Y$ be of finite type, Y locally Noetherian, $x \in X$ and $y = f(x)$ so that $\kappa(x) = \kappa(y)$. Moreover, let $\theta : \widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ be the canonical morphism. Then f is étale if and only if θ is an isomorphism.*

The last observation especially implies that $\dim_x X = \dim_y Y$. Étale morphisms of varieties over $\mathbb{C}$ are morphisms which are local isomorphisms in the analytic sense, see example 21. Locally all étale morphisms are of the form

$$F : \operatorname{Spec} R[T_1, \ldots, T_n]/(f_1, \ldots, f_n) \to \operatorname{Spec} R$$

with $\det \frac{\partial F}{\partial T}$ a unit in $R[T]/(f_1, \ldots, f_n)$ for some ring R. This is equivalent to F being flat and unramified (see for example [37] and [41]).

An *étale neighbourhood of a point* $x \in X$ is a pair (U, u) of a scheme U and a point $u \in U$ with an étale morphism $\varphi : U \to X$, $\varphi(u) = x$.

Example 22. Let X be the node in the plane with coordinate ring $A = K[x, y]/\langle y^2 - x^2 - x^3 \rangle$. Clearly X is irreducible and so is $\operatorname{Spec} A_{(x,y)}$. In the formal neighbourhood of the origin the germ of X is reducible (see example 20), since

$$y^2 - x^2 - x^3 = (y + x\sqrt{1 + x})(y - x\sqrt{1 + x}).$$

In the case of $K = \mathbb{C}$ this factorization holds in an Euclidean neighbourhood of the origin, since the factors are algebraic power series, thus convergent (see [44], p. 106). Though we cannot obtain this decomposition in a Zariski-open neighbourhood of 0, it will be possible in an étale neighbourhood. Consider

$$U = \operatorname{Spec} A_a[T]/\langle T^2 - a \rangle$$

where $a = 1 + x$ and the canonical map $\varphi \colon U \to X$ given by

$$A_a \to A_a[T]/\langle T^2 - a \rangle.$$

Let us denote the coordinate ring of U by B, and set $f = T^2 - a$. By the remark above φ is (standard) étale if and only if $\partial f / \partial T = 2T$ is a unit in B. But $2a$ is a unit in B, hence $2T(2a)^{-1}T = 1$. Thus φ is étale. On U the polynomial $y^2 - x^2 - x^3$ factors into $y - Tx$ and $y + Tx$. Therefore U is an étale neighbourhood which is reducible with two smooth branches intersecting transversally.

The connected étale neighbourhoods of $x \in X$ form a filtered system. The *local ring* $\mathcal{O}_{X,\bar{x}}$ *of X at x with respect to the étale topology* is defined as

$$\mathcal{O}_{X,\bar{x}} = \varinjlim_{(U,u)} \Gamma(U, \mathcal{O}_U),$$

where the limit is taken over the system of connected étale neighbourhoods (U, u) of x. Let $(A, \mathfrak{m})$ be a local ring. It is called *Henselian* with respect to $\mathfrak{m}$ if it has the following property: If $F \in A[T]$ with $F(0) \in \mathfrak{m}$ and $F'(0) \in (A/\mathfrak{m})^\times$, then there exists an $a \in \mathfrak{m}$ with $F(a) = 0$. As usual, F' denotes here the derivative of F with respect to T. The *Henselization A^h of A* is defined to be the smallest Henselian ring containing A. More precisely this means: The ring A^h is Henselian, there is a local homomorphism $i : A \to A^h$, and any other local homomorphism $\theta : A \to B$, with B Henselian, factors through i. Important examples of Henselian rings are complete local rings. Especially $K[[x]]$, the completion of $K[x]$ with respect to $\langle x \rangle$, is Henselian. But it is not the smallest Henselian ring containing $K[x]_{\langle x \rangle}$. In fact, $K[x]^h_{\langle x \rangle}$ equals $K\langle\langle x \rangle\rangle$ the ring of *algebraic power series*. Recall that a power series $f \in K[[x]]$ is called

algebraic if there exists a non-zero polynomial $P(x,t) \in K[x,t]$ with $P(x,f) = 0$. The local ring of a scheme X at x with respect to the étale topology is the Henselization (with respect to $\mathfrak{m}_x$) of the local ring with respect to the Zariski-topology:

$$\mathcal{O}_{X,\bar{x}} = \mathcal{O}_{X,x}^h.$$

For further details, including proofs, see [37].

Remark 32. Note that the étale neighbourhoods are not the open sets of a topology, but take their part in a *Grothendieck topology* (see for example [16], I.2). Although it is not a "true" topology on X it is still enough to allow analogous constructions (like cohomology theories).

Properties of formal and étale neighbourhoods are strongly related via the powerful Artin Approximation Theorem:

Theorem 6. *(see [4], Thm. 1.10) Let K be a field or an excellent discrete valuation ring, and let A^h be the Henselization of a K-algebra of finite type at a prime ideal. Let I be a proper ideal of A^h. Given an arbitrary system of polynomial equations in $Y = (Y_1, \ldots, Y_N)$,*

$$f(Y) = 0,$$

with coefficients in A^h, a solution $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)$ in the I-adic completion $\widehat{A}$ of A, and an integer c, there exists a solution $y = (y_1, \ldots, y_N) \in A^h$ with

$$y_i = \bar{y}_i \mod \mathfrak{m}^c.$$

5.3 Applications

In this section we give some applications of étale neighbourhoods. The prototype of questions we study will be the following: Let X be a variety, $p \in X$ a closed point. Moreover, let $\mathcal{P}$ be a property of the formal neighbourhood $\widehat{\mathcal{O}}_{X,p}$, e.g., normal crossings, reducible, ... Is the set of points $q \in X$ for which $\widehat{\mathcal{O}}_{X,q}$ has property $\mathcal{P}$ open (resp. closed or locally closed) in the Zariski-topology?

Theorem 7. *The normal crossings locus X_{nc} of a hypersurface $X \subseteq \mathbb{A}^n$ is open in X.*

Proof. (i) We first show that if X is normal crossings at p, then there exists an étale neighbourhood $\varphi \colon (U, u) \to \mathbb{A}^n$ of p such that u is an algebraic normal crossings point of $\varphi^{-1}(X)$. Without loss of generality we may assume $p = 0$. Since $p \in X_{nc}$ there exist $\bar{g}_1, \ldots, \bar{g}_m \in K[[x]]$, $m \leq n$, building part of a regular system of parameters of $\widehat{\mathcal{O}}_{\mathbb{A}^n,0}$ such that

$$f = \bar{g}_1 \cdots \bar{g}_m.$$

By Theorem 6 there exists an étale neighbourhood $\varphi \colon (U, u) \to \mathbb{A}^n$ of $p \in \mathbb{A}^n$ with $\varphi(u) = p$,

$$\varphi^*(f) = g_1 \cdots g_m,$$

on U and $g_i = \bar{g}_i \mod \langle x \rangle^c$. Note that $g_1, \ldots, g_m$ are regular on U. By choosing the constant c of Theorem 6 equal to 2 we can assure that the g_i are part of a regular

system of parameters of $\mathcal{O}_{U,u}$. Thus $\varphi^{-1}(X)$ is algebraic normal crossings at u.

(ii) If $w \in U$ is an algebraic normal crossings point or normal crossings point, then $\varphi(w) \in X$ is a normal crossings point. This follows immediately from Proposition 20, (4).

(iii) By (i) every point $p \in X_{nc}$ has an étale neighbourhood

$$\varphi_p \colon (U_p, u_p) \to X$$

so that $\varphi_p^{-1}(X)$ is anc at u_p. By Proposition 19 we see that $(U_p)_{anc} \subseteq U_p$ is open. Openness of étale maps (see Proposition 20, (1)) implies that $\varphi_p((U_p)_{anc}) \subseteq X$ is open, and so is

$$\bigcup_{p \in X} \varphi_p((U_p)_{anc}) \subseteq X.$$

$\square$

Let $X = \operatorname{Spec} K[x]/\langle f \rangle$ be a scheme defined by a not necessarily reduced polynomial $f \in K[x]$. Analogous to the normal crossings locus of X we ask for the *monomial locus X_{mon} of X*. This is the locus of points $p \in X$ so that there exist formal coordinates $y_1, \ldots, y_n$ with $f = y^\alpha$ for some element $\alpha \in \mathbb{N}^n$. Denote by X_{red} the reduced scheme associated to X. Then $X_{mon} = (X_{red})_{nc}$. Indeed, let $f_1, \ldots, f_s$ be the distinct irreducible factors of f in $\mathcal{O}_{\mathbb{A}^n, p}$ defining hypersurfaces X_i. By definition $\mathcal{O}_{X_{red}, p}$ is reduced. Since it is essentially of finite type (see [36], p. 232, 260), we conclude that $\widehat{\mathcal{O}}_{X_{red}, p}$ is reduced. The same is true for $\widehat{\mathcal{O}}_{X_i, p}$. If $p \in X_{mon}$ then each irreducible component of $f_i \in \widehat{\mathcal{O}}_{\mathbb{A}^n, p}$ corresponds to one of the $y_1, \ldots, y_n$ after a change of coordinates. Therefore:

$$p \in X_{mon} \Leftrightarrow p \in \cap_i (X_i)_{nc} \Leftrightarrow p \in (X_{red})_{nc}.$$

The last theorem thus implies:

Corollary 7. *Let $X = \operatorname{Spec} K[x]/\langle f \rangle$ be a hypersurface in $\mathbb{A}^n$ defined by a not necessarily reduced polynomial $f \in K[x]$. Then the monomial locus X_{mon} of X is open in X.*

5.4   Ideas for further Applications

As already mentioned at the very beginning of this chapter, its content was developed together with C. Bruschek. In this section two further interesting subsets of algebraic varieties will be defined and briefly investigated.

5.4.1   Mikado Schemes

Let X be an excellent scheme. Denote by $X_p^1, \ldots, X_p^N$, $N = N(p) \in \mathbb{N}$, the components of X passing through p. Then X is said to be *mikado at p* if $p \in (X_p^i)_{\mathrm{reg}}$ for all $1 \leq i \leq N$ and $p \in (Z_p)_{\mathrm{reg}}$, where $Z_p = X_p^1 \cap \cdots \cap X_p^N$ is the scheme-theoretic intersection of the components at p. The locus of all mikado points of X will be denoted by X_{mik}.

Example 23. Let $X = \operatorname{Spec} K[x, y]/\langle y(y - x^2)\rangle$ and $Y = \operatorname{Spec} K[x, y]/\langle xy(y - x^2)\rangle$. Clearly, X is not mikado at 0, but Y is; neither of them is normal crossings at the origin.

In order to successfully apply the étale construction to the locus of "formally mikado points" completely analogous to the proof for the normal crossings locus X_{nc}, it would be necessary that X_{mik} is open in X. The next example gives a counterexample: $X \setminus X_{\mathrm{mik}}$ is locally closed but not open. Especially in this way we can construct examples of schemes/varieties which have only constructible X_{mik}, which is – in the algebraic category – the worst possible behavior. Therefore the proof of Theorem 7 can't be applied directly to the situation of X_{mik}.

Example 24. Consider $X = \operatorname{Spec} K[x, y, z]/\langle yz(x - z)(y - x^2)\rangle$, which is a union of four hypersurfaces in $\mathbb{A}^3$. It's easy to see that $X \setminus X_{\mathrm{mik}} = V(x, y) \setminus V(x, y, z)$, which is locally closed in X (especially not closed!).

5.4.2   Formally Irreducible

Let $X = \cup_{1 \leq i \leq s} X_i$ be a union of algebraic varieties (not necessarily hypersurfaces). We say that X is *irreducible (resp. formally irreducible) at* $p \in X$ if $\mathcal{O}_{X,p}$ (resp. $\widehat{\mathcal{O}}_{X,p}$) is an integral domain. Otherwise X is called *reducible (resp. formally reducible) at* p. We denote the locus of reducible resp. formally reducible $p \in X$ by X_{re} resp. X_{fre}.

Proposition 21. *Let X be a finite union of algebraic varieties. Then $X_{re} \subseteq X$ is closed.*

Proof. This follows from the simple fact that

$$X_{re} = \cup_{i \neq j} \left(X_i \cap X_j \right).$$

$\square$

One further application of the constructions described at the beginning of this chapter would be to determine whether the locus X_{fre} of formally reducible points $p \in X$ is an algebraically closed subset of X. Recall that the proof that the normal crossings locus X_{nc} is open in X (cf. Theorem 7) was based on the following two facts: (1) the algebraic normal crossings locus X_{anc} is open, (2) étale morphisms are open. These where used to construct an open cover of X_{nc} by images of étale neighborhoods. Here in contrast the subset X_{re} is Zariski-closed in X. Therefore the method from before can not be applied analogously to the set X_{fre}. It is not clear how to work with $X \setminus X_{re}$ since we don't know a characterization of this set in the completion. Moreover, it is not clear whether it is true, that X_{fre} is an algebraically closed subset of X.

Bibliography

[1] S. Abhyankar. Local uniformization of algebraic surfaces over ground fields of characteristic $p \neq 0$. *Ann. of Math. (2)*, 63:491–526, 1956.

[2] S. Abhyankar. Desingularization of plane curves. In *Singularities, Part 1 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 1–45. Amer. Math. Soc., Providence, RI, 1983.

[3] M. Artin. Algebraic approximation of structures over complete local rings. *Inst. Hautes Études Sci. Publ. Math.*, (36):23–58, 1969.

[4] M. Artin. Algebraic approximation of structures over complete local rings. *Inst. Hautes Études Sci. Publ. Math.*, (36):23–58, 1969.

[5] A. Benito and O. Villamayor. Singularities in positive characteristic: elimination and and monoidal transformations. arXiv:math/0811.4148, 2008.

[6] E. Bierstone and P. D. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.*, 128(2):207–302, 1997.

[7] A. Bravo, S. Encinas, and O. Villamayor. A simplified proof of desingularization and applications. *Rev. Mat. Iberoamericana*, 21(2):349–458, 2005.

[8] A. Bravo and O. Villamayor. Hypersurface singularities in positive characteristic and stratification of the singular locus. arXiv:math/0807.4308, 2008.

[9] C. Bruschek and D. Wagner. Some constructions in the étale topology. In preparation.

[10] V. Cossart, U. Jannsen, and S. Saito. Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes. Preprint.

[11] V. Cossart and O. Piltant. Resolution of singularities of threefolds in positive characteristic I. 2007. Preprint available on http://hal.archives-ouvertes.fr/hal-00139124/fr/.

[12] V. Cossart and O. Piltant. Resolution of singularities of threefolds in positive characteristic II. 2007. Preprint available on http://hal.archives-ouvertes.fr/hal-00139445/fr/.

[13] S. D. Cutkosky. Resolution of Singularities for 3-folds in positive characteristic. arXiv:math/0606530, 2006.

[14] S. Encinas and H. Hauser. Strong resolution of singularities in characteristic zero. *Comment. Math. Helv.*, 77(4):821–845, 2002.

[15] S. Encinas and O. Villamayor. Rees algebras and resolution of singularities. *Rev. Mat. Iberoamericana. Proceedings XVI-Coloquio Latinoamericano de « Algebra 2006.*

[16] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, and A. Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[17] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[18] H. Hauser. Gochumoku onegaishimasu: kangaroo points. Notes from lectures at RISC Kyoto, December 2008.

[19] H. Hauser. On the problem of resolution of singularities in positive characteristic (or: A proof that we are still waiting for). To appear in Bull. Amer. Math. Soc.

[20] H. Hauser. Why the characteristic zero proof of resolution fails in positive characteristic. Manuscript, available on www.hh.hauser.cc.

[21] H. Hauser. Excellent surfaces and their taut resolution. In *Resolution of singularities (Obergurgl, 1997)*, volume 181 of *Progr. Math.*, pages 341–373. Birkhäuser, Basel, 2000.

[22] H. Hauser. Three power series techniques. *Proc. London Math. Soc. (3)*, 89(1):1–24, 2004.

[23] H. Hauser and G. Regensburger. Explizite Auflösung von ebenen Kurvensingularitäten in beliebiger Charakteristik. *Enseign. Math. (2)*, 50(3-4):305–353, 2004.

[24] H. Hironaka. Desingularization of excellent surfaces. Notes by B. Bennett at the Conference on Algebraic Geometry, Bowdoin 1967. Reprinted in: Cossart, V., Giraud, J., Orbanz, U.: Resolution of surface singularities. Lecture Notes in Mathematics 1101, Springer 1984.

[25] H. Hironaka. A program for resolution of singularities, in all characteristics $p > 0$ and in all dimensions. Lecture notes from the summer school and conference on resolution of singularities, ICTP Trieste, June 2006.

[26] H. Hironaka. Program for resolution of singularities in characteristic $p > 0$. Notes from lectures at RIMS Kyoto, December 2008.

[27] H. Hironaka. Program for resolution of singularities in characteristic $p > 0$. Notes from lectures at the Clay Mathematics Institute, September 2008.

[28] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[29] H. Hironaka. Characteristic polyhedra of singularities. *J. Math. Kyoto Univ.*, 7:251–293, 1967.

[30] H. Hironaka. Theory of infinitely near singular points. *J. Korean Math. Soc.*, 40(5):901–920, 2003.

[31] H. Kawanoue. Toward resolution of singularities over a field of positive characteristic. I. Foundation; the language of the idealistic filtration. *Publ. Res. Inst. Math. Sci.*, 43(3):819–909, 2007.

[32] H. Kawanoue and K. Matsuki. Toward resolution of singularities over a field of positive characteristic. Part II. Basic invariants associated to the idealistic filtration and their properties. arXiv:math/0612008v3, 2009.

[33] J. Kollár. *Lectures on resolution of singularities*, volume 166 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, 2007.

[34] H. Kurke, G. Pfister, and M. Roczen. *Henselsche Ringe und algebraische Geometrie*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1975. Mathematische Monographien, Band II.

[35] J. Lipman. Desingularization of two-dimensional schemes. *Ann. Math. (2)*, 107(1):151–207, 1978.

[36] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[37] J. S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[38] T. T. Moh. On a stability theorem for local uniformization in characteristic p. *Publ. Res. Inst. Math. Sci.*, 23(6):965–973, 1987.

[39] T. T. Moh. On a Newton polygon approach to the uniformization of singularities of characteristic p. In *Algebraic geometry and singularities (La Rábida, 1991)*, volume 134 of *Progr. Math.*, pages 49–93. Birkhäuser, Basel, 1996.

[40] S. B. Mulay. Equimultiplicity and hyperplanarity. *Proc. Amer. Math. Soc.*, 89(3):407–413, 1983.

[41] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[42] R. Narasimhan. Hyperplanarity of the equimultiple locus. *Proc. Amer. Math. Soc.*, 87(3):403–408, 1983.

[43] R. Narasimhan. Monomial equimultiple curves in positive characteristic. *Proc. Amer. Math. Soc.*, 89(3):402–406, 1983.

[44] J. M. Ruiz. *The basic theory of power series*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1993.

[45] M. Spivakovsky. A solution to Hironaka's polyhedra game. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 419–432. Birkhäuser Boston, Boston, MA, 1983.

[46] Mark Spivakovsky. A counterexample to Hironaka's "hard" polyhedra game. *Publ. Res. Inst. Math. Sci.*, 18(3):1009–1012, 1982.

[47] O. Villamayor. Constructiveness of Hironaka's resolution. *Ann. Sci. École Norm. Sup. (4)*, 22(1):1–32, 1989.

[48] O. Villamayor. Patching local uniformizations. *Ann. Sci. École Norm. Sup. (4)*, 25(6):629–677, 1992.

[49] J. Włodarczyk. Simple Hironaka resolution in characteristic zero. *J. Amer. Math. Soc.*, 18(4):779–822 (electronic), 2005.

[50] D. Zeillinger. Polyederspiele und Auflösen von Singularitäten. PhD Thesis, Universität Innsbruck, 2005.

Acknowledgement

Kurzfassung englisch

The main objective of this Ph.D. Thesis is to study the occurring phenomena in embedded resolution of singularities over fields of positive characteristic.

In the first part a new and systematic approach to embedded surface resolution in positive characteristic, which is so natural that it has the chance to be generalized to higher dimensions, is given. This is achieved by introducing two new invariants for the inductive proof of the resolution by a sequence of blowups. The invariants are more systematic than the existing ones, and yield a transparent reasoning. This may facilitate to study the still unsolved case of embedded resolution of threefolds in positive characteristic. Moreover the termination of the described algorithm for the embedded resolution of purely inseparable two-dimensional hypersurfaces of order equal to the characteristic in finitely many blowups is proven.

In the second part new ideas and attempts for resolution of varieties with dimension larger than two, especially threefolds, are given. One approach that is examined within this thesis is to extend the results from the first part to the case of threefolds. Further, those situations where the classical resolution invariant from characteristic zero increases when used in positive characteristic are studied. Moreover new possible resolution invariants – both for characteristic zero and positive characteristic – are investigated.

The last part addresses some basic constructions using étale neighborhoods. They are for instance used to prove the termination of the resolution algorithm presented in the first part.

Kurzfassung deutsch

Das Hauptanliegen dieser Dissertation ist es, die auftretenden Phänomene bei der eingebetteten Auflösung von Singularitäten über Körpern mit positiver Charakteristik zu untersuchen.

Im ersten Abschnitt wird ein neuer und systematischer Ansatz für die eingebettete Flächenauflösung in positiver Charakteristik entwickelt, welcher derart natürlich ist, dass zumindest die Chance besteht ihn auf höhere Dimensionen verallgemeinern zu können. Dies wird durch die Einführung zweier neuer Invarianten für den induktiven Beweis der Auflösung durch eine Folge von Explosionen bewerkstelligt. Die Invarianten sind systematischer als die bereits bestehenden und ermöglichen eine transparente Argumentation. Dies könnte dazu dienen den noch immer ungelösten Fall der eingebetteten Auflösung von Dreivarietäten in positiver Charakteristik zu untersuchen. Weiters wird die Termination des beschriebenen Algorithmus zur eingebetteten Auflösung von rein inseparablen zwei-dimensionalen Hyperflächen mit Ordnung gleich der Charakteristik durch endlich viele Explosionen bewiesen.

Im zweiten Abschnitt werden neue Ideen und Herangehensweisen für die Auflösung von Varietäten mit Dimension größer als zwei, speziell Dreivarietäten, behandelt. Ein Ansatz, der hierzu untersucht wird, ist die Verallgemeinerung der Resultate aus dem ersten Abschnitt auf den Fall von Dreivarietäten. Ferner werden jene Situationen, in denen die klassische Auflösungsinvariante von Charakteristik Null bei Anwendung in positiver Charakteristik fehlschlägt, studiert. Weiters werden neue Invarianten – für Charaketristik Null sowie positive Charakteristik – untersucht.

Der letzte Abschnitt befasst sich mit grundlegenden Konstruktionen mittels étaler Umgebungen. Diese werden zum Beispiel verwendet um die Termination des im ersten Abschnitt dargestellten Auflösungsalgorithmus zu zeigen.

Dominique Wagner

Curriculum Vitae

Engerthstraße 138/9
1200 Wien
Österreich
📱 *+43 699 10183511*
✉ *dominique.wagner@univie.ac.at*

PERSONAL INFORMATION

Date of Birth	May 13, 1982
Place of Birth	Kufstein, Austria
Nationality	Austria
Marital Status	Single

EDUCATION

2007 – now **Ph.D. Student in Mathematics** at the Department of Mathematics, University of Vienna. Continuation of my *Ph.D. Thesis*.

2005 – 2007 **Ph.D. Student in Mathematics** at the Department of Mathematics, University of Innsbruck. Starting my *Ph.D. Thesis* "Studies in Resolution of Singularities is positive characteristic: A new invariant for the surface case & Approaches towards resolution in higher dimension."

2000 – 2005 Study of **Technical Mathematics** at the Department of Mathematics, University of Innsbruck.
Title of Diploma Thesis: "Algebraic Power Series".
Graduated with *High Honor*, examination fields "Algebra" and "Probability Theory and Statistics"; Degree: Diplom-Ingenieur.

1992 – 2000 **High school** education at the BRG Wörgl.
Best female participant of the Tyrolean *Physics Olympiad* in 1997; among the best fifteen participants of the Austrian Physics Olympiad in 1999 and 2000.
Matura with excellence in June 2000.

1988 – 1992 **Elementary school** in Bad Häring

Employment History

Oct. '07 – now	**Ph.D. research assistant** at the Department of Mathematics, *University of Vienna*, in the framework of the FWF project "Solving algebraic equations".
May '06 – Sept. '07	**Ph.D. research assistant** at the Department of Mathematics, *University of Innsbruck*, in the framework of the FWF project "Solving algebraic equations".
Mar. '06 – Jan. '07	**Teaching Assistant** for "Proseminar Algorithmische Mathematik 2" and "Proseminar Lineare Algebra" at the Department of Mathematics respectively Informatics, *University of Innsbruck*.
Oct. '04 – Dec. '05	**Research grant** ("Forschungsbeihilfe") in the framework of the FWF project "Resolution of Singularities", *University of Innsbruck*.
2001 – 2004	Employment as **private tutor** (Nachhilfelehrer) in Mathematics and Physics at the *Schülerhilfe Innsbruck*.
Summer '00, '01	Working for *FB Ketten Kufstein*.
Summer 1997	Working in the *Tourist Office Bad Häring*.

Scientific Activities

2008	Research stay in Valladolid, *Spain*.
	Research stay at RISC Linz, *Austria*; Talk: "Blunt points on a variety".
	Participation at the conference "On the resolution of singularities" in Kyoto, *Japan*.
	Research stay at the Tokyo University of Technology, *Japan*; Talk: "Some phenomena in resolution of singularities in positive characteristic".
	Organizer of the "Vienna Geometry Day" in Wien, *Austria*.
2007	Research stay at RISC Linz, *Austria*; Talk: "Resolution of plane Quings."
	Participant of the "YMIS winter school – Algebra and Topology of Singularities" in Sedano, *Spain*.
	Participant of the conference "Singularities, Computing and Visualization" in Segovia, *Spain*.
	Participant of the workshop "Algebraic Geometry" in Nové Hrady, *Czech Republic*; Talk: "Polyhedra games & resolution of singularities".
	Participation of the conference "MEGA" in Strobl, *Austria*.
	Organiser of the "First α–ω–Conference in Algebraic Geometry" in Obergurgl, *Austria*.
2006	Research stay in Valladolid, *Spain*.
	Research stay in Madrid, *Spain*.
	Participant of summer school "Resolution of singularities" in Trieste, *Italy*.
	Participant of the "CIMPA summer school – new trends in singularity theory" in Madrid, *Spain*.
	Participant of the "International Congress of Mathematicians" in Madrid, *Spain*.
	Organizer of the workshop "Singularities" in Obergurgl, *Austria*.
2005	Participant at the winter school "Singularities" in Marseille, *France*.
	Participant of the workshop "Algebraic geometry & Singularities" in Aschau, *Austria*; Talk: "Algebraic power series".

Academic Honors and Awards

2008	"**For Women in Science**" Award of L'Oréal Austria, the Austrian commission for UNESCO and the Austrian Academy of Sciences.
2006 – 2007	Research fellowship for writing my Ph.D. Thesis (**"Doktoratsstipendium"**) from the University of Innsbruck.
2006 and 2007	Scholarship for writing scientific papers ("**Förderungsstipendium"**) from the University of Innsbruck in the years 2006 and 2007.
2002 and 2003	Scholarships for extraordinary achievements ("**Leistungsstipendium**") from the University of Innsbruck in the years 2002 and 2003.
2001 – 2004	Scholarships of the **Julius-Raab-Stiftung** (Linz) in the years 2001, 2002, 2003 and 2004.
2000	Fifth prize at the **Austrian Physics Olympiad**.

Publications

[1] C. Bruschek, S. Gann, H. Hauser, D. Wagner, D. Zeillinger: UFOs – Unidentified figurative objects, a geometric challenge. arXiv:math/0512160v1, 2005, 1–14.

[2] D. Wagner: Algebraic Power Series. Diploma Thesis, Universität Innsbruck, 2005.

[3] Ch. Niederegger, C. Bruschek, M. Koppi, H. P. Schröcker, D. Wagner: Verbesserung von Frisch– und Festbetoneigenschaften durch Mischung der Haufwerksporosität von Bindemitteln mittels Approximation der Fuller–Kurve durch Mischen von Kornfraktionen. Beton vol. 5 (2007), 220–222.

[4] C. Bruschek, D. Wagner: Ansichtssache Algebra, to appear in Bildwelten des Wissens, 2009.