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"L^0–convex Analysis and Conditional Risk Measures"

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Nicolas Vogelpoth

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Betreuer:	Prof. Dr. Damir Filipović
Zweitbetreuer:	Prof. Dr. Michael Kupper
Gutachter:	Prof. Dr. Freddy Delbaen
Gutachter:	Prof. Dr. Walter Schachermayer

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Contents

Abstract

Motivated by financial applications, we study convex analysis for modules over the ordered ring L^0 of random variables. We establish a module analogue of locally convex vector spaces, namely locally L^0–convex modules. In this context, we prove hyperplane separation theorems.

We investigate continuity, subdifferentiability and Fenchel–Moreau type dual representations for L^0–convex functions from L^0–modules into L^0.

We introduce topological L^0–modules of L^p and Orlicz type. We investigate completeness and we compute the topological dual L^0–module of the L^p type L^0–module. Applications in terms of risk measures are given.

Further, we establish automatic continuity and subdifferentiability results for monotone convex functions from L^0–modules into L^0. The results are generalizations of classical results on automatic continuity and subdifferentiability of monotone convex functions and convex risk functions.

We present and compare two different approaches to conditional risk measures. One approach draws from vector space based convex analysis and presents conditional risk measures as functions on L^p spaces while the other approach utilizes module based convex analysis as presented in this thesis where conditional risk measures are defined on L^p type L^0–modules. Both approaches utilize general duality theory for vector valued convex functions in contrast to the current literature.

By presenting several applications such as monotone and (sub)cash invariant hulls with corresponding examples we illustrate that module based convex analysis is well suited to the concept of conditional risk measures.

Zusammenfassung

Motiviert durch Anwendungen aus der Finanzmathematik wird in der vorliegenden Dissertation konvexe Analysis für Moduln über dem geordneten Ring L^0 aller Zufallsvariablen studiert. Dabei werden L^0–Moduln erarbeitet, die als Pendant zu lokal konvexen Vektorräumen begriffen werden können, sogenannte lokal L^0–konvexe Moduln. Für solche L^0–Moduln werden Hyperebenen Trennungssätze bewiesen.

Des weiteren werden Stetigkeits– und Subdifferenzierbarkeitseigenschaften sowie duale Darstellungen von Fenchel–Moreau Typ für L^0–konvexe Funktionen untersucht, welche L^0–Moduln nach L^0 abbilden.

Als Beispiele für lokal L^0–konvexe Moduln werden L^0–Moduln analog zu L^p und Orlicz Räumen präsentiert. Hierbei wird vor allem topologische Vollständigkeit der L^0–Moduln untersucht und im Falle des L^0–Moduls von L^p Typ wird zusätzlich das duale L^0–Modul charakterisiert. Anwendungen für Risikomaße werden beispielhaft demonstriert.

Des weiteren werden Resultate bezüglich automatischer Stetigkeit und Subdifferenzierbarkeit monotoner konvexer Funktionen, welche L^0–Moduln nach L^0 abbilden, präsentiert. Diese Resultate stellen Verallgemeinerungen von klassischen Resultaten über automatische Stetigkeit und Subdifferenzierbarkeit monotoner konvexer Funktionen und konvexer Risikomaße dar.

Als zentrale Motivation der vorliegenden Arbeit werden zwei unterschiedliche Zugänge zu bedingten Risikomaßen vorgestellt und verglichen. Während dem einen vektorraumbasierte konvexe Analysis zugrunde liegt, innerhalb dessen bedingte Risikomaße als Funktionen auf L^p Räumen verstanden werden, liegt dem anderen Zugang modulbasierte konvexe Analysis, wie in dieser Dissertation erarbeitet, zugrunde. Bei letzterem werden bedingte Risikomaße als Funktionen auf L^0–Moduln von L^p Typ verstanden.

Durch verschiedene Anwendungen, wie zum Beispiel montone (sub)cash invariante Hüllen, die im Rahmen zahlreicher Beispiele dargestellt werden, wird aufgezeigt, dass modulbasierte konvexe Analysis viele nützliche Resultate für das Konzept bedingter Risikomaße bereitstellt.

Introduction

When in 1999 Artzner et al. introduced the notion of monetary risk measures they combined two fields of study: a practitioners axiomatic standard for sound risk assessment and the mathematical discipline of convex analysis, [**ADEH99, ADEH02**]. From this interchange, a lively and rich discussion originated resulting in a number of articles in both the field of stochastic finance and convex analysis. As a central theme, the relevant articles address the question to what extent dual representation and subdifferentiability results for general convex functions can be rendered more precisely in the specific context of risk measures.

In the course of this discussion, there have been many contributions to extend the one period setup first to contingent initial data and then eventually to a multi period framework. The significance and popularity of dynamic financial models and the delicate matter of applying convex analysis within such models, stand in contrast to the literature, which neither provides a comprehensive and tailor made convex analysis toolkit for dynamic financial models nor does it present its subsequent applications.

To this effect, the aim of the present thesis is twofold: in part one, we establish a number of standard results from convex analysis in the context of $L^0(\mathcal{F})$–modules which are perfectly suited for multi period frameworks. While in this part, we only exploit the prominent structure of convexity, part two is dedicated to an application in terms of risk measures for which we explore additional structures that are economically motivated. Part two should be understood as one possible application of part one, the latter of which we believe can be beneficial to concepts in stochastic finance far beyond risk measures.

Why $L^0(\mathcal{F})$–modules? A number of fundamental results in mathematical finance such as arbitrage theory and duality of risk and utility functions draw from the fundamental Hahn–Banach extension theorem and its consequences for hyperplane separation in locally convex vector spaces, cf. [**DS06, FS04**].

The simplest situation is a one period setup:

$$\mathbb{R} \overset{\pi,\rho,u}{\underset{p}{\leftleftarrows}} E \tag{1}$$

$$0 \text{ ——— } T$$

Random future (date T) payments are modeled as elements of a locally convex vector space E endowed with semi norms p. Price, risk or utility assessments π, ρ, or u, map E linearly, convexly, or concavely, into the real line $\mathbb{R}$, respectively.

However, the ideas of hedging and maximizing random future payments with respect to risk constraints or indifference and equilibrium pricing develop their full power in a multi period setting. Therefore the initial data is randomized, and $\pi = \pi(\omega,\cdot)$, $\rho = \rho(\omega,\cdot)$, or $u = u(\omega,\cdot)$, become ω dependent, where $\omega \in \Omega$ denotes the initial states modeled by a probability space $(\Omega,\mathcal{F},P)$. Here $\mathcal{F}$ is understood as the information available at some future initial date $t < T$.

While classical convex analysis perfectly applies in the one period model (1), its application in a multi period framework is rather delicate. Take, for instance, the convexity properties of the risk measure ρ. These properties have to be extended to ω wise convexity properties of $\rho(\omega,\cdot)$ for almost all $\omega \in \Omega$. But ω wise convex duality correspondences for $\rho(\omega,\cdot)$ have to be made $\mathcal{F}$–measurable in ω to assert intertemporal consistency in a recursive multi period setup. This would require heavy measurable selection criteria.

We propose instead to consider $\pi = \pi(\omega,\cdot)$, $\rho = \rho(\omega,\cdot)$, or $u = u(\omega,\cdot)$, as maps into $L^0(\mathcal{F}) = L^0(\Omega,\mathcal{F},P)$, the ordered ring of (equivalence classes of) $\mathcal{F}$–measurable random variables:

$$\begin{tikzcd} \mathbb{R} & L^0(\mathcal{F}) & E \arrow[l, "{\pi,\rho,u}"'] \arrow[ll, bend left, "p"] \end{tikzcd} \tag{2}$$

$$0 \longrightarrow t \longrightarrow T$$

The space E, in turn, is considered as module over $L^0(\mathcal{F})$.

This requires hyperplane separation and convex duality results on topological $L^0(\mathcal{F})$–modules, which seem to be new in the literature. Part one of this thesis provides a comprehensive treatment of convex analysis for topological $L^0(\mathcal{F})$–modules. While the emphasis is on financial applications, the results in this part are of theoretical nature. We illustrate the scope of applications that can be covered by the results in part two of this thesis.

From static to conditional risk. Throughout the recent literature there has been a significant effort to establish Fenchel–Moreau type dual representation and subdifferentiability results for (static) convex risk measures, cf. [**CL08, CL09, CK07, Del00, Del02, Del06, FS08b, FS02a, FS02b, FS04, FRG02, KR09, RS06, Web06**]. In the one period model (1) E is equipped with a partial order and a convex risk measure is a function $\rho : E \to (-\infty,+\infty]$ that satisfies

(i) $\rho(yX+(1-y)X') \le y\rho(X)+(1-y)\rho(X')$ for all $X, X' \in E$ and $y \in [0,1]$,
(ii) $\rho(X) \le \rho(X')$ for all $X, X' \in E$ with $X \ge X'$ and
(iii) $\rho(X+y) = \rho(X) - y$ for all $X \in E$ and $y \in \mathbb{R}$.

The economic interpretation and reasonability of this set of axioms as defining framework for financial risk measures have been widely discussed in the relevant literature and there are a number of recent publications that weaken or modify this set of axioms. Among the most prominent modifications are the concepts of subcash invariance and quasi convexity; we refer to [**EKR08**] and [**CVMMM08**].

Dual representation results and subdifferentiability of ρ are strongly related to hyperplane separation of the convex epigraph $\mathrm{epi}\rho = \{(X,c) \in E \times \mathbb{R} \mid f(X) \le c\}$ or the convex level sets $\{X \in E \mid \rho(X) \le c\}$, $c \in \mathbb{R}$, of ρ. Continuity properties of ρ are crucial to meet the assumptions on closedness or openness of these sets so that hyperplane separation theorems from convex analysis become applicable. It is therefore a natural question to what extent the set of axioms, with which a risk measure is defined, implicitly guarantees sufficient continuity properties.

This question is nicely addressed by Ruszczyński and Shapiro in [**RS06**] who present continuity and subdifferentiability results for static convex risk measures which draw from results that go back to the Namioka–Klee theorem on automatic continuity of monotone linear (or even monotone convex) functions on partially ordered linear topological spaces [**Nam57**] and which were further generalized by [**Bor87**]. The main statement of Ruszczyński and Shapiro is that if E is a Banach lattice the convex risk measure ρ is automatically continuous and subdifferentiable throughout the interior of its effective domain.

In the meanwhile, many articles have introduced the notion of conditional convex risk measures which extends the notion of convex risk measures to a multi period framework, cf. [**ADE$^+$07, BN04, CDK04, CDK05, CDK06, DS05, FP06, KS07, Rie04**].

As already presented in (2) we assume contingent initial data and randomize the convex risk measure $\rho = \rho(\omega, \cdot)$. Then $\rho(\omega, X)$ is understood as the risk assigned to the uncertain payoff X given the event $\{\omega\}$. Keeping a tree model in mind provides us with a nice intuition.

(3)
$$
\begin{tikzcd}
 & & \vdots \\
 & \vdots & X(\omega_{n+2}) \\
 & \rho(\omega_{n+2}, X) = \rho(\omega_{n+1}, X) \arrow[ur, dash] \arrow[dr, dash] & \\
\vdots & & X(\omega_{n+1}) \\
\bullet \arrow[uur, dash] \arrow[ddr, dash] & & \\
\vdots & & X(\omega_n) \\
 & \rho(\omega_n, X) = \rho(\omega_{n-1}, X) \arrow[ur, dash] \arrow[dr, dash] & \\
 & \vdots & X(\omega_{n-1}) \\
 & & \vdots
\end{tikzcd}
$$

Formally, we fix a stochastic basis $(\Omega, \mathcal{E}, P)$ and the $L^0(\mathcal{F})$–module E is now a subset of $L^0(\mathcal{E})$. The time t information $\mathcal{F}$ is a sub σ–algebra of $\mathcal{E}$. In the tree model (3) above the singletons $\{\omega_{n-1}\}$, $\{\omega_n\}$, $\{\omega_{n+1}\}$ and $\{\omega_{n+2}\}$ are $\mathcal{E}$–atoms and $\{\omega_{n-1}, \omega_n\}, \{\omega_{n+1}, \omega_{n+2}\}$ are $\mathcal{F}$–atoms.

By $\bar{L}^0(\mathcal{F})$ we denote the class of all $\mathcal{F}$–measurable random variables which take values in $[-\infty, +\infty]$. A function $\rho : E \to \bar{L}^0(\mathcal{F})$ is proper if $\rho(X) > -\infty$ for all $X \in E$ and if there is at least one $X \in E$ so that $\rho(X) < +\infty$, where equalities and (strict) inequalities are understood in the almost sure sense. Led by the tree picture (3) we call a proper function $\rho : E \to \bar{L}^0(\mathcal{F})$ a conditional convex risk measure if it satisfies

(i) $\rho(YX + (1-Y)X') \leq Y\rho(X) + (1-Y)\rho(X')$ for all $X, X' \in E$ and $Y \in L^0(\mathcal{F})$ with $0 \leq Y \leq 1$,
(ii) $\rho(X) \leq \rho(X')$ for all $X, X' \in E$ with $X \geq X'$ and
(iii) $\rho(X + Y) = \rho(X) - Y$ for all $X \in E$ and $Y \in L^0(\mathcal{F})$,

with the conventions $0 \cdot \infty = 0, \infty + \infty = \infty$. As suggested in (2) the randomization of the risk assessment is now given through the $L^0(\mathcal{F})$–valued function ρ.

As in the one period case, continuity properties are crucial to dual representation results and subdifferentiability of conditional convex risk measures. A conditional convex risk measure, however, is defined with a set of economically motivated axioms. Mathematically yielding properties, such as continuity are a priori not addressed by this set of axioms, nevertheless they remain crucial for convex analysis.

Therefore, the first aim of part two of this thesis is to establish automatic continuity and subdifferentiabiliy results for conditional convex risk measures that generalize the results of [**Nam57**] and [**RS06**]. Automatic continuity results in the theory of static risk measures have first been applied in [**CDK04**] and further developed in [**RS06, KR09**] and [**CL09**].

As the results are again of theoretical nature but motivated by financial applications the second aim of part two is to illustrate these results by means of conditional convex risk measures.

Conditional convex risk measures suggest modules. The above tree model (3) is not only a good picture to be kept in mind when reading this thesis. In fact, it provides us with an additional motivation for the module approach we follow. This motivation is entirely driven by the axiomatic and intrinsic approach to conditional convex risk measures.

The reason for this is that the $\mathcal{F}$–atoms $A_- = \{\omega_{n-1}, \omega_n\}$ and $A_+ = \{\omega_{n+1}, \omega_{n+2}\}$, which represent parts of the available non trivial information $\mathcal{F}$, reveal a local property as a very natural underlying structure of conditional convex risk assessment.

More precisely, given the information A_- the risk of the position X should be assessed irrespective of the values X takes on A_+. Conversely, a sound risk assessment of the position X given the information A_+ should be independent from its values at A_-. Formally, this is captured by the following local property

$$1_A \rho(X) = 1_A \rho(1_A X) \text{ for all } A \in \mathcal{F}.$$

It is good news, as we will see below, that conditional convex risk measures share this local property.

Even though the tree model (3) is a simplification of the general structure of a filtered probability space, the significance of the local property as underlying structure of conditional convex risk assessment in general filtered probability spaces must not be underestimated.

The significance of the local property is due to the fact that the theory of static as well as conditional risk measures heavily draws from duality techniques of convex analysis for which continuous linear functions play a very prominent role. As mentioned above, one may think of the Hahn–Banach extension theorem, hyperplane separation theorems and Fenchel–Moreau type dual representation theorems.

The recent literature has shown that for the popular model space $L^\infty(\mathcal{E})$, the space of all essentially bounded $\mathcal{E}$–measurable random variables, those continuous linear functions $\mu : L^\infty(\mathcal{E}) \to L^\infty(\mathcal{F})$ are of interest which are of the form $E[\mu(\cdot)] = E[Z\cdot]$ for some $Z \in L^1(\mathcal{E})$, where $L^p(\mathcal{E})$ denotes the space of p–integrable random variables, $p \in [1, \infty)$. Moreover, the local property of a conditional convex risk measure implies that only those μ which are local as well are relevant. By definition, any linear local μ with $E[\mu(\cdot)] = E[Z\cdot]$ has to be a conditional expectation $\mu = E[Z\cdot \mid \mathcal{F}]$. But any such conditional expectation is in fact a module homomorphism of the $L^\infty(\mathcal{F})$–module $L^\infty(\mathcal{E})$ into the underlying ring $L^\infty(\mathcal{F})$.

Therefore, the local property suggests to view conditional convex risk measures as ring valued functions defined on modules rather than as vector valued functions defined on vector spaces.

Outline. Part one of this thesis is devoted to convex analysis in topological $L^0(\mathcal{F})$–modules. We establish the corresponding Hahn–Banach extension theorem, hyperplane separation theorems and we investigate Fenchel–Moreau type dual representation results as well as subdifferentiability. We further present important examples of locally $L^0(\mathcal{F})$–convex modules. This part comprises of [**FKV09b**] and of parts of [**KV09**].

In part two of this thesis we provide an application of convex analysis in $L^0(\mathcal{F})$–modules. We establish automatic continuity and subdifferentiability results for monotone convex functions and we present and compare two different approaches to the concept of conditional convex risk measures. This part comprises of parts of [**KV09**] and of [**FKV09a**].

The present thesis as it stands does not bring together the ideas of only one single person:

First and foremost, I owe thanks to Damir Filipović. His supervision was impeccable at all times and with each and every single discussion, comment, suggestion and correction he understood to motivate me, to make the best of our ideas and eventually to bring out the best in me.

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To all of you, may only the best encounter you in all your endeavors and may our paths cross again in the future.

Setup and notational conventions

Throughout this thesis we let $(\Omega, \mathcal{E}, P)$ be a probability space and consider a sub σ–algebra $\mathcal{F} \subset \mathcal{E}$. We denote by $L^0(\mathcal{G})$ the ring of real valued $\mathcal{G}$–measurable random variables for a generic sub σ–algebra $\mathcal{G} \subset \mathcal{E}$. Random variables and sets which coincide almost surely are identified. Recall that $L^0(\mathcal{G})$ equipped with the order of almost sure dominance is a lattice ordered ring. Throughout, the strict inequality $X > Y$ between two random variables is to be understood as point–wise almost surely (in other texts, "$X > Y$" is sometimes interpreted as "$X \geq Y$ and $X \neq Y$"). We further define $L^0_+(\mathcal{G}) = \{Y \in L^0(\mathcal{G}) \mid Y \geq 0\}$ and $L^0_{++}(\mathcal{G}) = \{Y \in L^0(\mathcal{G}) \mid Y > 0\}$. By $\bar{L}^0(\mathcal{G})$ we denote the space of all $\mathcal{G}$–measurable random variables which take values in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and we define $\bar{L}^0_+(\mathcal{G}) = \{Y \in \bar{L}^0(\mathcal{G}) \mid Y \geq 0\}$. Throughout, we follow the convention $0 \cdot (\pm\infty) = 0$.

By $L^k(\mathcal{G}) = L^k(\Omega, \mathcal{G}, P)$ we denote the space of $\mathcal{G}$–measurable functions with finite kth moments, that is,

$$L^k(\mathcal{G}) = \{X \in L^0(\mathcal{G}) \mid E[|X|^k] < +\infty\}$$

where $k \in [1, +\infty)$. $L^\infty(\mathcal{G}) = L^\infty(\Omega, \mathcal{G}, P)$ denotes the space of essentially bounded $\mathcal{G}$–measurable random variables.

We further introduce some basic topological concepts and their notation. Let $\mathcal{T}$ be a topology on some set E. Then $K \subset E$ is closed if $K^c \in \mathcal{T}$. The interior, boundary and closure of K are denoted by $\overset{\circ}{K}$, ∂K, $\bar{K}$, respectively. Moreover, $\overset{\circ}{K} \cap \partial K = \emptyset$, K is open if and only if $K = \overset{\circ}{K}$, and K is closed if and only if $K = \bar{K}$. An element $X \in \overset{\circ}{K}, \partial K, \bar{K}$ is an interior, boundary, closure point of K, respectively.

Part 1

Separation and duality

CHAPTER 1

Separation in locally $L^0(\mathcal{F})$–convex modules

In this chapter we state the main results on locally $L^0(\mathcal{F})$–convex topologies and hyperplane separation in locally $L^0(\mathcal{F})$–convex modules. We will consequently suppress the notational dependence of $L^0 = L^0(\mathcal{F})$ on $\mathcal{F}$ as we consider one fixed σ–algebra $\mathcal{F}$.

For the sake of readability, the main results are collected in Section 1.1 while the proofs are postponed to the subsequent respective sections. In Section 1.2 we prove a Hahn–Banach type extension theorem in the context of L^0–modules. Instead of sublinear and linear functions on a vector space we study L^0–sublinear and L^0–linear functions on an L^0–module. In Section 1.3 we characterize a class of topological L^0–modules, namely locally L^0–convex modules. An important feature of a locally L^0–convex module E is that the neighborhoods of 0 absorb E over L^0. This is the key difference to the notion of a locally convex module which is endowed with a linear topology and therefore absorbent over the real line, cf. [**Har64, OT72, Wel64**]. In [**Guo09**] further examples of linear topologies on L^0–modules are introduced and a relation is given to locally L^0–convex topologies which are not linear in general. In fact, in [**Guo09**] it is clarified under which conditions closed sets of the respective topologies coincide. The neighborhood base of a locally L^0–convex module is constructed by means of L^0–semi norms. Such vector valued, or vectorial, norms go back to [**Kan39**]. In Section 1.4 we establish some preliminary results for L^0–valued gauge functions. In Section 1.5 we prove the hyperplane separation theorems in locally L^0–convex modules. We separate a non empty open L^0–convex set from an L^0–convex set and we strictly separate a point from a non empty closed L^0–convex set by means of continuous L^0–linear functions.

1.1. Main results

The order of almost sure dominance allows us to define the following topology on L^0. We let

$$B_\varepsilon = \{Y \in L^0 \mid |Y| \leq \varepsilon\}$$

denote the ball of radius $\varepsilon \in L^0_{++}$ centered at $0 \in L^0$. A set $V \subset L^0$ is a neighborhood of $Y \in L^0$ if there is $\varepsilon \in L^0_{++}$ such that $Y + B_\varepsilon \subset V$. A set $V \subset L^0$ is open if it is a neighborhood of all $Y \in V$. Inspection shows that the collection of all open sets is a topology on L^0, which is referred to as topology induced by $|\cdot|$. By construction, $\mathcal{U} = \{B_\varepsilon \mid \varepsilon \in L^0_{++}\}$ is a neighborhood base of $0 \in L^0$. Throughout, we make the convention that $L^0 = (L^0, |\cdot|)$ is endowed with this topology.

Notice that $(L^0, |\cdot|)$ is not a real topological vector space, in general. Indeed, suppose $(\Omega, \mathcal{F}, P)$ is atom–less. Then the scalar multiplication $\mathbb{R} \to L^0$, $\alpha \mapsto \alpha \cdot 1$ is not continuous at $\alpha = 0$. The topology on L^0 induced by $|\cdot|$ is finer than the topology of convergence in probability, which is often used in convex analysis on L^0, such as in [**BS99**]. For example, L^0_{++} is open in $(L^0, |\cdot|)$ but not in the topology of convergence in probability.

However, it follows from Theorem 1.1.4 below that $(L^0, |\cdot|)$ is a topological ring or, equivalently, a topological L^0–module in the following sense:

Definition 1.1.1. *A topological L^0–module $(E, \mathcal{T})$ is an L^0–module E endowed with a topology $\mathcal{T}$ such that the module operations*

(i) $(E, \mathcal{T}) \times (E, \mathcal{T}) \to (E, \mathcal{T}), (X_1, X_2) \mapsto X_1 + X_2$ *and*
(ii) $(L^0, |\cdot|) \times (E, \mathcal{T}) \to (E, \mathcal{T}), (Y, X) \mapsto YX$

are continuous w.r.t. the corresponding product topologies.

Locally convex topologies in our framework are defined as follows:

Definition 1.1.2. *A topology $\mathcal{T}$ on E is locally L^0–convex if $(E, \mathcal{T})$ is a topological L^0–module and there is a neighborhood base $\mathcal{U}$ of $0 \in E$ for which each $U \in \mathcal{U}$ is*

(i) *L^0–convex: $YX_1 + (1 - Y)X_2 \in U$ for all $X_1, X_2 \in U$ and $Y \in L^0$ with $0 \leq Y \leq 1$,*
(ii) *L^0–absorbent: for all $X \in E$ there is $Y \in L^0_{++}$ such that $X \in YU$,*
(iii) *L^0–balanced: $YX \in U$ for all $X \in U$ and $Y \in L^0$ with $|Y| \leq 1$.*

In this case, $(E, \mathcal{T})$ is a locally L^0–convex module.

Note that an L^0–convex set $K \subset E$ with $0 \in K$ satisfies $YK \subset K$ for all $Y \in L^0$ with $0 \leq Y \leq 1$; in particular, $1_A K \subset K$ for all $A \in \mathcal{F}$.

Next we show how to construct, and actually characterize all, locally L^0–convex modules. Let E be an L^0–module.

Definition 1.1.3. *A function $\|\cdot\| : E \to L^0_+$ is an L^0–semi norm on E if:*

(i) *$\|YX\| = |Y|\|X\|$ for all $Y \in L^0$ and $X \in E$,*
(ii) *$\|X_1 + X_2\| \leq \|X_1\| + \|X_2\|$ for all $X_1, X_2 \in E$.*

If, moreover,

(iii) *$\|X\| = 0$ implies $X = 0$,*

then $\|\cdot\|$ is an L^0–norm on E.

Any family $\mathcal{P}$ of L^0–semi norms on E induces a topology in the following way. For finite $\mathcal{Q} \subset \mathcal{P}$ and $\varepsilon \in L^0_{++}$ we define

$$U_{\mathcal{Q},\varepsilon} = \left\{ X \in E \mid \sup_{\|\cdot\| \in \mathcal{Q}} \|X\| \leq \varepsilon \right\}$$

and

$$\mathcal{U} = \{U_{\mathcal{Q},\varepsilon} \mid \mathcal{Q} \subset \mathcal{P} \text{ finite and } \varepsilon \in L^0_{++}\}. \tag{4}$$

We then proceed as for $(L^0, |\cdot|)$ above and define a topology, referred to as topology induced by $\mathcal{P}$, on E with neighborhood base $\mathcal{U}$ of 0. We thus obtain a locally L^0-convex module, as the following theorem states:

Theorem 1.1.4. *A topological L^0–module $(E, \mathcal{T})$ is locally L^0–convex if and only if $\mathcal{T}$ is induced by a family of L^0–semi norms.*

Proof. This follows from Lemma 1.3.1 and Corollary 1.4.4. $\square$

By convention, an L^0–normed module $(E, \|\cdot\|)$ is always endowed with the locally L^0–convex topology induced by $\|\cdot\|$. Notice that any L^0–norm $\|\cdot\|$ on $E = L^0$ satisfies $\|1\| > 0$ and $\|\cdot\| = \|1\| \, |\cdot|$.

An important L^0–normed module is given in the following example which will be discussed in more detail in Section 3.2 of Chapter 3. Recall that a function $\mu : E \to L^0$ is L^0–linear if $\mu(Y_1X_1 + Y_2X_2) = Y_1\mu(X_1) + Y_2\mu(X_2)$ for all $X_1, X_2 \in E$ and $Y_1, Y_2 \in L^0$.

EXAMPLE 1.1.5. *Let $p \in [1, +\infty]$. We define the function $\|\cdot\|_p : \bar{L}^0(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ by*

$$\|X\|_p = \begin{cases} \lim_{n\to\infty} E[|X|^p \wedge n \mid \mathcal{F}]^{1/p} & \text{if } p < +\infty, \\ \text{ess.inf}\{Y \in \bar{L}^0(\mathcal{F}) \mid Y \geq |X|\} & \text{if } p = +\infty, \end{cases} \tag{5}$$

and denote

$$L^p_{\mathcal{F}}(\mathcal{E}) = \left\{X \in L^0(\mathcal{E}) \mid \|X\|_p \in L^0(\mathcal{F})\right\}.$$

In Section 3.2 of Chapter 3, it is shown that $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$ is an $L^0(\mathcal{F})$–normed module, which is complete in the sense that any Cauchy net in $L^p_{\mathcal{F}}(\mathcal{E})$ has a limit in $L^p_{\mathcal{F}}(\mathcal{E})$. Moreover, for $p < \infty$, the $L^0(\mathcal{F})$–module of all continuous $L^0(\mathcal{F})$–linear functions $\mu : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ can be identified with $L^q_{\mathcal{F}}(\mathcal{E})$, where $q = p/(p-1)$ ($= +\infty$ if $p = 1$).

Since $X/\|X\|_p \in L^p(\mathcal{E})$ (with the convention $0/0 = 0$) for $X \in L^p_{\mathcal{F}}(\mathcal{E})$, we conclude that $L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F}) \cdot L^p(\mathcal{E})$ as sets, cf. Proposition 3.2.1. In particular, for $\mathcal{F} = \{\emptyset, \Omega\}$ the function $\|\cdot\|_p$ can be identified with the classical L^p–norm. In turn $L^p_{\{\emptyset,\Omega\}}(\mathcal{E})$ can be identified with the classical L^p space $L^p(\mathcal{E})$. In fact, whenever $\mathcal{F} = \sigma(A_1, \ldots, A_n)$ is finitely generated, we can identify $L^p_{\sigma(A_1,\ldots,A_n)}(\mathcal{E})$ with $L^p(\mathcal{E})$. This relation between $L^p_{\mathcal{F}}(\mathcal{E})$ and $L^p(\mathcal{E})$ is further discussed and illustrated in Example 6.1.1.

Hahn–Banach type extension theorems for modules appear already in the fifties. This started with [**Ghi50**], where modules over totally ordered rings were considered. Modules over rings which are algebraically and topologically isomorphic to the space of essentially bounded measurable functions on a finite measure space were considered in [**Har65, VS67, Orh69**]. Nowadays, it is well known, cf. [**BS77, Vuz82**], that a Hahn–Banach type extension theorem for modules over more general ordered rings can be established. In particular, this is the case for L^0–modules.

However, to our knowledge, the following hyperplane separation theorems for L^0–modules are new in the literature. The proofs are given in Section 1.5 below.

THEOREM 1.1.6 (Hyperplane Separation I). *Let E be a locally L^0–convex module and let $K, M \subset E$ be L^0–convex, K open and non empty. If $1_A M \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$ then there is a continuous L^0–linear function $\mu : E \to L^0$ such that*

$$\mu Y < \mu Z \text{ for all } Y \in K \text{ and } Z \in M.$$

For the second hyperplane separation theorem we need to impose some technical assumption on the topology.

DEFINITION 1.1.7. *A topological L^0–module E has the countable concatenation property if for every countable collection (U_n) of neighborhoods of $0 \in E$ and for* *every countable partition* $(A_n) \subset \mathcal{F}$ *(*$A_n \cap A_m = \emptyset$ *for* $n \neq m$ *and* $\bigcup_{n\in\mathbb{N}} A_n = \Omega$*) the set*

$$\sum_{n\in\mathbb{N}} 1_{A_n} U_n$$

again is a neighborhood of $0 \in E$.

Notice that any L^0–normed module has the countable concatenation property.

Theorem 1.1.8 (Hyperplane Separation II). *Let* E *be a locally* L^0*–convex module that has the countable concatenation property and let* $K \subset E$ *be closed* L^0*–convex and non empty. If* $X \in E$ *satisfies* $1_A\{X\} \cap 1_A K = \emptyset$ *for all* $A \in \mathcal{F}$ *with* $P[A] > 0$ *then there is* $\varepsilon \in L^0_{++}$ *and a continuous* L^0*–linear function* $\mu : E \to L^0$ *such that*

$$\mu Y + \varepsilon < \mu X \text{ for all } Y \in K.$$

1.2. Hahn–Banach extension theorem

In this section, we establish a Hahn–Banach type extension theorem. We recall that the main result of this section, Theorem 1.2.6, is already contained in [**BS77, Vuz82**]. Nevertheless, for the sake of completeness, we provide a self contained proof which is tailored to our setup. The fact that not all elements in L^0 possess a multiplicative inverse leads to difficulties in showing that the "one step extension" from the proof of the classical Hahn–Banach theorem is well defined in our framework. For this reason, we derive some preliminary results first.

The following lemma recalls that $\mathcal{F}$ is a complete lattice w.r.t. the partial order of almost sure set inclusion.

Lemma 1.2.1. *Every non empty collection* $\mathcal{D} \subset \mathcal{F}$ *has a supremum denoted by* ess.sup$\mathcal{D}$ *and called essential supremum of* $\mathcal{D}$*. Further, if* $\mathcal{D}$ *is directed upwards (*$A \cup B \in \mathcal{D}$ *for all* $A, B \in \mathcal{D}$*) there is an increasing sequence* (A_n) *in* $\mathcal{D}$ *such that* ess.sup$\mathcal{D} = \bigcup_{n\in\mathbb{N}} A_n$.

If $\mathcal{D} \subset \mathcal{F}$ is empty we set ess.sup$\mathcal{D} = \emptyset$.

Proof. For a countable set $\mathcal{C} \subset \mathcal{D}$ define $A_{\mathcal{C}} = \bigcup_{A\in\mathcal{C}} A$. Then $A_{\mathcal{C}} \in \mathcal{F}$ and the upper bound

$$c = \sup\{P[A_{\mathcal{C}}] \mid \mathcal{C} \subset \mathcal{D} \text{ countable}\}$$

is attained by some $\mathcal{C}_{\sup}$; indeed, take a sequence $(\mathcal{C}_n)$ in $\mathcal{D}$ with $P[A_{\mathcal{C}_n}] \to c$ and $\mathcal{C}_{\sup} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$. Then, $\mathcal{C}_{\sup} \in \mathcal{F}$ and $P[A_{\mathcal{C}_{\sup}}] = c$. We conclude that ess.sup$\mathcal{D} = A_{\mathcal{C}_{\sup}}$ is as required. Indeed, ess.sup$\mathcal{D}$ is an upper bound of $\mathcal{D}$, otherwise there would be $A \in \mathcal{D}$ with $P[A \setminus \text{ess.sup}\mathcal{D}] > 0$ and in turn $P[A_{\mathcal{C}_{\sup}\cup\{A\}}] > P[A_{\mathcal{C}_{\sup}}] = c$. To see that ess.sup$\mathcal{D}$ is a least upper bound, observe ess.sup$\mathcal{D} \subset A'$ whenever $A' \in \mathcal{F}$ with $A \subset A'$ for all $A \in \mathcal{D}$. By construction, there is an increasing sequence approximating ess.sup$\mathcal{D}$ if $\mathcal{D}$ is directed upwards. □

Let E be an L^0–module. For a set $C \subset E$, we define the map $M(\cdot \mid C) : E \to \mathcal{F}$,

$$M(Z \mid C) = \text{ess.sup}\{A \in \mathcal{F} \mid 1_A Z \in C\}. \tag{6}$$

If C is an L^0–submodule of E the collection $\{A \in \mathcal{F} \mid 1_A Z \in C\}$ is directed upwards for all $Z \in E$ and hence there exists an increasing sequence $(M_n) \subset \mathcal{F}$ such that

$$M(Z \mid C) = \bigcup_{n\in\mathbb{N}} M_n. \tag{7}$$

DEFINITION 1.2.2. *A set $C \subset E$ has the closure property if*

$$1_{M(Z|C)}Z \in C \text{ for all } Z \in E. \tag{8}$$

By $\hat{C}$ we denote the smallest subset of E that has the closure property and contains C.

Note that $\hat{C}$ is given by

$$\hat{C} = \{1_{M(Z|C)}Z \mid Z \in E\}$$

and therefore $\hat{C}$ always exists and is well defined. By definition, the closure property is a property in reference to E. In particular, E has the closure property.

LEMMA 1.2.3. *Let $C \subset E$ be an L^0–submodule. Then $\hat{C}$ is again an L^0–submodule.*

PROOF. Let $X \in \hat{C}$ and $Y \in L^0$. Denote $Z = YX$. By definition, there exists some $X' \in E$ with $X = 1_{M(X'|C)}X'$. Since C is an L^0–submodule of E there exist an increasing sequence $(M_n) \subset \mathcal{F}$ with $M_n \nearrow M(X' \mid C)$ such that $1_{M_n}X' \in C$. Hence $1_{M_n}Z = Y1_{M_n}X' \in C$, and thus $M_n \subset M(Z \mid C)$, for all $n \in \mathbb{N}$. We conclude that $M(X' \mid C) \subset M(Z \mid C)$ and thus

$$YX = Y1_{M(X'|C)}X' = 1_{M(Z|C)}Z \in \hat{C}.$$

Now let $X = 1_A X'$, $Y = 1_B Y' \in \hat{C}$ where $A := M(X' \mid C)$ and $B := M(Y' \mid C)$, for some X', $Y' \in E$. Denote

$$Z = X + Y = 1_{A\setminus B}X + 1_{A\cap B}(X+Y) + 1_{B\setminus A}Y.$$

As above there exist increasing sequences (A_n), $(B_n) \subset \mathcal{F}$ with $A_n \nearrow A$ and $B_n \nearrow B$ such that $1_{A_n}X'$, $1_{B_n}Y' \in C$ and thus

$$\begin{aligned} 1_{A_n\setminus B}X &= 1_{A\setminus B}1_{A_n}X' \in C \\ 1_{A_n\cap B_n}(X+Y) &= 1_{B_n}1_{A_n}X' + 1_{A_n}1_{B_n}Y' \in C \\ 1_{B_n\setminus A}Y &= 1_{B\setminus A}1_{B_n}Y' \in C. \end{aligned}$$

Define the disjoint union $M_n = (A_n \setminus B) \cup (A_n \cap B_n) \cup (B_n \setminus A)$. We obtain

$$1_{M_n}Z = 1_{A_n\setminus B}X + 1_{A_n\cap B_n}(X+Y) + 1_{B_n\setminus A}Y \in C,$$

and thus $M_n \subset M(Z \mid C)$, for all $n \in \mathbb{N}$. Since $M_n \nearrow A \cup B$, we conclude that $A \cup B \subset M(Z \mid C)$ and thus

$$X + Y = 1_{M(Z|C)}Z \in \hat{C}.$$

Hence the lemma is proved. □

For a set $C \subset E$ we denote by

$$\mathrm{span}_{L^0}(C) = \left\{ \sum_{i=1}^{n} Y_i X_i \mid X_i \in C, Y_i \in L^0, 0 \le i \le n, n \in \mathbb{N} \right\}$$

the L^0–submodule of E generated by C. The next example illustrates the situation where an L^0–submodule C of E does not have the closure property.

EXAMPLE 1.2.4. *Consider the probability space $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$ the Borel σ–algebra and P the Lebesgue measure on $[0,1]$. Let $E = L^0$, and define*

$$C = \operatorname{span}_{L^0}\{1_{[1-2^{-(n-1)},1-2^{-n}]} \mid n \in \mathbb{N}\}.$$

Then, $1 \notin C$ but $1 \in \hat{C}$.

PROPOSITION 1.2.5. *Let $C \subset E$ be an L^0–submodule of E, $Z' \in E$ and $Z = 1_{M(Z'|C)^c} Z'$. Then*

(i) *$M(Z' \mid C) = M(Z \mid C)$,*
(ii) *$X = X'$ and $Y = Y'$ on $M(Z \mid C)^c$ whenever $X + YZ = X' + Y'Z$ for $X, X' \in C$ and $Y, Y' \in L^0$, and*
(iii) *for $W \in 1_{M(Z|C)^c} L^0$ and an L^0–linear function $\mu : C \to L^0$*

$$\bar{\mu}(X + YZ) = \mu X + YW \text{ for all } X \in C \text{ and } Y \in L^0 \tag{9}$$

defines the unique L^0–linear extension of μ to $\operatorname{span}_{L^0}(C, Z)$ which satisfies $\bar{\mu}Z = W$.

If in addition to this C has the closure property,

(iv) *$\operatorname{span}_{L^0}(C, Z') = \operatorname{span}_{L^0}(C, Z)$.*

PROOF. (i) By definition of Z, $M(Z' \mid C) \subset M(Z \mid C)$, and since $P[M(Z \mid C) \setminus M(Z' \mid C)] > 0$ would contradict the definition of $M(Z' \mid C)$ we have $M(Z' \mid C) = M(Z \mid C)$.

(ii) $X + YZ = X' + Y'Z$ is equivalent to $X - X' = (Y' - Y)Z$. If $B = \{Y' - Y \neq 0\} \cap M(Z \mid C)^c$ had positive measure then on B, $Z = (X - X')/(Y' - Y) \in C$ in contradiction to the definition of $M(Z \mid C)$. Hence $Y = Y'$ and in turn $X = X'$ on $M(Z \mid C)^c$.

(iii) This is an immediate consequence of (ii).

(iv) By definition of Z, $\operatorname{span}_{L^0}(C, Z) \subset \operatorname{span}_{L^0}(C, Z')$. Since C has the closure property, $1_{M(Z'|C)} Z' \in C$ and hence $\operatorname{span}_{L^0}(C, Z) = \operatorname{span}_{L^0}(C, Z')$. □

A function $p : E \to L^0$ is L^0–sublinear if $p(YX) = Yp(X)$ for all $X \in E$ and $Y \in L^0_+$ and $p(X_1 + X_2) \le p(X_1) + p(X_2)$ for all $X_1, X_2 \in E$. We can now state and prove the main result of this section.

THEOREM 1.2.6 (Hahn–Banach). *Consider an L^0–sublinear function $p : E \to L^0$, an L^0–submodule C of E and an L^0–linear function $\mu : C \to L^0$ such that*

$$\mu X \le p(X) \text{ for all } X \in C.$$

Then μ extends to an L^0–linear function $\bar{\mu} : E \to L^0$ such that $\bar{\mu}X \le p(X)$ for all $X \in E$.

PROOF. **Step 1:** In view of Lemma 1.2.7 below we can assume that C has the closure property and that there exists $Z' \in E \setminus C$. Then $Z = 1_{M(Z'|C)^c} Z' \notin C$ and $Z \neq 0$. We will show that μ extends L^0–linearly to $\bar{\mu} : \operatorname{span}_{L^0}(C, Z) \to C$, such that

$$\bar{\mu}X \le p(X) \text{ for all } X \in \operatorname{span}_{L^0}(C, Z). \tag{10}$$

More precisely, we claim that

$$W = 1_{M(Z|C)^c} \operatorname*{ess.sup}_{X \in C}(\mu X - p(X - Z))$$

and $\bar{\mu}$ defined as in (9) satisfies

$$\mu X + YW \leq p(X + YZ) \text{ for all } X \in C \text{ and } Y \in L^0 \tag{11}$$

which, apparently, is equivalent to (10). To verify this claim, let $X, X' \in C$ and observe

$$\begin{aligned} \mu X + \mu X' &= \mu(X + X') \\ &\leq p(X + X') \\ &= p(X' + Z + X - Z) \\ &\leq p(X' + Z) + p(X - Z). \end{aligned}$$

Hence,

$$\mu X - p(X - Z) \leq p(X' + Z) - \mu X' \text{ for all } X, X' \in C. \tag{12}$$

Since $Z = 0$ on $M(Z \mid C)$ we have $\mu X - p(X - Z) \leq 0$ on $M(Z \mid C)$ as well as $p(X' + Z) - \mu X' \geq 0$ on $M(Z \mid C)$ for all $X, X' \in C$. Hence, (12) implies

$$\mu X - p(X - Z) \leq W \leq p(X' + Z) - \mu X' \text{ for all } X, X' \in C \tag{13}$$

and in turn

$$\mu X \pm W \leq p(X \pm Z) \text{ for all } X \in C.$$

From this we derive

$$1_A(\mu X + W) \leq 1_A p(X + Z) = 1_A p(X + 1_A Z) \tag{14}$$

$$1_{A^c}(\mu X - W) \leq 1_{A^c} p(X - Z) = 1_{A^c} p(X - 1_{A^c} Z) \tag{15}$$

for all $A \in \mathcal{F}$. Adding up the inequalities in (14) and (15) yields

$$\mu X + (1_A - 1_{A^c})W \leq p(X + (1_A - 1_{A^c})Z) \text{ for all } X \in C \text{ and } A \in \mathcal{F}. \tag{16}$$

Further, for all $Y \in L^0$ with $P[Y \neq 0] = 1$ we have $Y/|Y| = 1_A - 1_{A^c}$, where $A = \{Y > 0\} \in \mathcal{F}$. Thus, (16) implies

$$|Y| \left(\mu \left(\frac{X}{|Y|} \right) + \frac{Y}{|Y|} W \right) \leq |Y| p \left(\frac{X}{|Y|} + \frac{Y}{|Y|} Z \right).$$

for all $X \in C$ and $Y \in L^0$ with $P[Y \neq 0] = 1$. From this we derive

$$\mu X + YW \leq p(X + YZ) \text{ for all } X \in C \text{ and } Y \in L^0 \text{ with } P[Y \neq 0] = 1. \tag{17}$$

But this already implies the required inequality in (11). Indeed, for $X \in C$ and arbitrary $Y \in L^0$ we define $Y' = Y 1_A + 1_{A^c}$, where $A = \{Y \neq 0\}$, and derive from (17)

$$1_A(\mu X + YW) = 1_A(\mu X + Y'W) \leq 1_A p(X + Y'Z) = 1_A p(X + YZ) \tag{18}$$

$$1_{A^c}(\mu X + YW) = 1_{A^c}(\mu X) \leq 1_{A^c} p(X) = 1_{A^c} p(X + YZ). \tag{19}$$

Adding up (18) and (19), we see that (17) implies (11) and complete this step.

Step 2: The set

$$\mathcal{I} = \left\{ (D, \bar{\mu}) \mid \begin{array}{c} C \subset D \overset{L^0\text{–linear}}{\subset} E, D \text{ has the closure property} \\ \bar{\mu} : D \overset{L^0\text{–linear}}{\to} L^0, \bar{\mu}|_C = \mu \text{ and } \bar{\mu} X \leq p(X) \text{ for all } X \in D \end{array} \right\}$$

is partially ordered by

$$(D, \bar{\mu}) \leq (D', \bar{\mu}') \text{ if and only if } D \subset D' \text{ and } \bar{\mu}'|_D = \bar{\mu}.$$

We will show that a totally ordered subset $\{(D_i, \bar{\mu}_i), i \in I\}$ of $\mathcal{I}$ (that is, for all i, j either $(D_i, \bar{\mu}_i) \leq (D_j, \bar{\mu}_j)$ or $(D_i, \bar{\mu}_i) \geq (D_j, \bar{\mu}_j)$) has an upper bound and then we will apply Zorn's lemma. To this end, observe that D given by

$$C \subset D = \bigcup_{i \in I} D_i \subset E$$

is an L^0–module since $\{(D_i, \bar{\mu}_i), i \in I\}$ is totally ordered. $\bar{\mu} : D \to L^0$ given by $\bar{\mu}|_{D_i} = \bar{\mu}_i$ is L^0–linear, dominated by p on all of D and $\bar{\mu}|_C = \mu$. Further, in view of Lemma 1.2.7 below, we can assume that D has the closure property. Hence, $(D, \bar{\mu}) \in \mathcal{I}$ is an upper bound for $\{(D_i, \bar{\mu}_i), i \in I\}$ and Zorn's lemma yields the existence of a maximal element $(D_{\max}, \bar{\mu}_{\max}) \in \mathcal{I}$, i.e.

$$(D_{\max}, \bar{\mu}_{\max}) \leq (D, \bar{\mu}) \in \mathcal{I} \text{ implies } (D_{\max}, \bar{\mu}_{\max}) = (D, \bar{\mu}).$$

Assume that $D_{\max} \neq E$. Then, by the first step of this proof, $\bar{\mu}_{\max}$ extends to

$$\bar{\mu}'_{\max} : \operatorname{span}_{L^0}(D_{\max}, Z) \to L^0,$$

where $Z \in E \setminus D_{\max}$, which contradicts the maximality of $(D_{\max}, \bar{\mu}_{\max})$. Hence, $D_{\max} = E$ and $\bar{\mu}_{\max}$ is as desired. $\square$

LEMMA 1.2.7. *Let C, μ, p be as in Theorem 1.2.6. Then μ extends uniquely to an L^0–linear function $\hat{\mu} : \hat{C} \to L^0$ such that $\hat{\mu}X \leq p(X)$ for all $X \in \hat{C}$.*

PROOF. For $Z \in E$, let

$$\hat{\mu}(1_{M(Z|C)}Z) = \lim_{n \to \infty} \mu(1_{M_n}Z), \tag{20}$$

where $M(Z \mid C) = \bigcup_{n \in \mathbb{N}} M_n$ as in (7). Since for all $n \leq m$

$$\mu(1_{M_n}Z) = \mu(1_{M_m}Z) \text{ on } M_n$$

(20) uniquely and unambiguously defines the L^0–linear extension $\hat{\mu} : \hat{C} \to L^0$ of μ to $\hat{C}$. Further, (20) guarantees that $\hat{\mu}X \leq p(X)$ for all $X \in \hat{C}$. $\square$

1.3. Locally L^0–convex modules

In this section we establish some facts about locally L^0–convex modules. For more background on general topological spaces we refer to the comprehensive Chapter 2 of [**AB06**].

Now let E be an L^0–module and $\mathcal{T}$ the topology induced by some family $\mathcal{P}$ of L^0–semi norms on E, see Definition 1.1.3 and below. The following result gives one direction in the proof of Theorem 1.1.4. The converse direction is proved in Corollary 1.4.4 below.

LEMMA 1.3.1. *$(E, \mathcal{T})$ is a locally L^0–convex module.*

PROOF. Let $\mathcal{U}$ denote the neighborhood base given in (4). It follows by inspection that each $U \in \mathcal{U}$ is L^0–convex, L^0–absorbent and L^0–balanced as in Definition 1.1.2. To establish (i) and (ii) of Definition 1.1.1, let $O \in \mathcal{T}$.

(i) We show that $\widetilde{O} = \{(X, Y) \in E \times E \mid X + Y \in O\}$ is open. Let $(X, Y) \in \widetilde{O}$ and $U = U_{\mathcal{Q},\varepsilon} \in \mathcal{U}$ such that $X + Y + U \subset O$. Then $V = U_{\mathcal{Q},\varepsilon/2}$ satisfies $V + V \subset U$ and hence $(X + V) \times (Y + V) \subset \widetilde{O}$. This means that (X, Y) is an interior point of $\widetilde{O}$ and (i) follows.

(ii) We show that $\widetilde{O} = \{(X,Y) \in E \times L^0 \mid XY \in O\}$ is open. Consider $(X,Y) \in \widetilde{O}$ and $U = U_{\mathcal{Q},\varepsilon} \in \mathcal{U}$ such that $XY + U \subset O$. We find $\varepsilon \in L^0_{++}$ and $W \in \mathcal{U}$ such that

$$W \times \{Z \in L^0 \mid |Z - Y| \leq \varepsilon\} \subset \widetilde{O}$$

as follows. As in the proof of (i) let $V \in \mathcal{U}$ be such that $V + V \subset U$ and let $\varepsilon \in L^0_{++}$ be such that $\varepsilon X \in V$, which is possible since V is L^0–absorbing. Further, since V is L^0–balanced,

$$(Z - Y)X \in V \text{ if } |Z - Y| \leq \varepsilon.$$

V is of the form $V = U_{\mathcal{Q},\delta}$, hence $W = U_{\mathcal{Q},\delta/(\varepsilon+|Y|)}$ satisfies $(\varepsilon + |Y|)W \subset V$ and since W is L^0–balanced

$$ZW \subset V \text{ for all } Z \in L^0 \text{ with } |Z| \leq \varepsilon + |Y|.$$

Finally, for $|Z - Y| \leq \varepsilon$ and $X' \in W$ we derive

$$Z(X + X') - YX = (Z - Y)X + ZX' \in V + V \subset U$$

and the assertion is proved. □

Here is a trivial example.

Example 1.3.2 (Chaos Topology). *The locally L^0–convex topology $\mathcal{T}$ induced by the trivial L^0–semi norm $\|\cdot\| \equiv 0$ on L^0 consists of the sets $\emptyset$ and L^0. $\mathcal{T}$ is called chaos topology and it is an example for a locally L^0–convex topology which is not Hausdorff. Note that $\mathcal{T}$ is locally convex and locally L^0–convex at the same time.*

1.3.1. The countable concatenation property. A technicality we encounter is a certain concatenation property. This concatenation property is crucial in the context of hyperplane separation, cf. Lemma 1.5.3, Theorem 1.1.8 and the Examples 1.5.4 and 1.5.5 in Section 1.5 below.

The following result motivates the subsequent definition.

Lemma 1.3.3. *Let $\mathcal{P}$ be a family of L^0–semi norms inducing a locally L^0–convex topology $\mathcal{T}$ on E.*

(i) *For $A \in \mathcal{F}$ and $\|\cdot\| \in \mathcal{P}$, $1_A\|\cdot\|$ is an L^0–semi norm.*
(ii) *For a finite collection $\|\cdot\|_1, \dots, \|\cdot\|_n \in \mathcal{P}$, $\sup_{i=1,\dots,n} \|\cdot\|_i$ is an L^0–semi norm.*
(iii) *Define*

$$\mathcal{P}' = \mathcal{P} \cup \{1_A\|\cdot\| \mid A \in \mathcal{F}, \|\cdot\| \in \mathcal{P}\}$$
$$\mathcal{P}'' = \mathcal{P}' \cup \left\{ \sup_{\|\cdot\| \in \mathcal{Q}} \|\cdot\| \mid \mathcal{Q} \subset \mathcal{P}' \text{ finite} \right\}$$

and denote $\mathcal{T}'$ and $\mathcal{T}''$ the induced locally L^0–convex topologies, respectively. Then $\mathcal{T} = \mathcal{T}' = \mathcal{T}''$; in other words, we may always assume that, with every $\|\cdot\| \in \mathcal{P}$, $\mathcal{P}$ contains $1_A\|\cdot\|$ for all $A \in \mathcal{F}$ and that $\mathcal{P}$ is closed under finite suprema.

Proof. (i) and (ii) follow from the properties of L^0–semi norms.

(iii) Since $\mathcal{P} \subset \mathcal{P}' \subset \mathcal{P}''$ we have $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}''$. The inclusion $\mathcal{T}'' \subset \mathcal{T}$ follows from the fact that for all $\varepsilon \in L^0_{++}$,

$$U_{\{\|\cdot\|\},\varepsilon} \subset U_{\{1_A\|\cdot\|\},\varepsilon} \text{ for all } \|\cdot\| \in \mathcal{P} \text{ and } A \in \mathcal{F} \text{ and}$$
$$U_{\{\|\cdot\|_1,\dots,\|\cdot\|_n\},\varepsilon} = U_{\{\sup_{i=1,\dots,n}\|\cdot\|_i\},\varepsilon} \text{ for all } \|\cdot\|_1,\dots,\|\cdot\|_n \in \mathcal{P}.$$

□

For a finite collection $U_{\mathcal{Q}_1,\varepsilon_1},\dots,U_{\mathcal{Q}_n,\varepsilon_n}$ and a finite collection of pairwise disjoint sets $A_1,\dots,A_n \in \mathcal{F}$ ($A_i \cap A_j = \emptyset$ for $i \neq j$), the preceding lemma shows that $\sum_{i=1}^n 1_{A_i} U_{\mathcal{Q}_i,\varepsilon_i}$ is a neighborhood of $0 \in E$. Indeed, let

$$\|\cdot\| = \sum_{i=1}^n 1_{A_i} \sup_{\|\cdot\| \in \mathcal{Q}_i} \|\cdot\| = \sup_{i=1,\dots,n} 1_{A_i} \sup_{\|\cdot\| \in \mathcal{Q}_i} \|\cdot\|$$

and $\varepsilon = \sum_{i=1}^n 1_{A_i}\varepsilon_i$. Then, $\sum_{i=1}^n 1_{A_i} U_{\mathcal{Q}_i,\varepsilon_i} = U_{\{\|\cdot\|\},\varepsilon}$.

In the case of a countably infinite sequence $(U_{\mathcal{Q}_n,\varepsilon_n})$ and a pairwise disjoint sequence $(A_n) \subset \mathcal{F}$ ($A_i \cap A_j = \emptyset$ for $i \neq j$) the next example illustrates that the above reasoning does not apply, as the L^0–semi norm given by

$$\|\cdot\| = \sum_{n \in \mathbb{N}} 1_{A_n} \sup_{\|\cdot\| \in \mathcal{Q}_n} \|\cdot\| = \sup_{n \in \mathbb{N}} 1_{A_n} \sup_{\|\cdot\| \in \mathcal{Q}_n} \|\cdot\|$$

cannot be assumed to belong to $\mathcal{P}$ in general.

Example 1.3.4. *Consider the probability space $\Omega = [0,1]$, $\mathcal{F} = \sigma(A_n \mid n \in \mathbb{N})$ the σ-algebra generated by the sets $A_n = [1-2^{-(n-1)}, 1-2^{-n}]$, and P the Lebesgue measure. Define $B_n = \cup_{m \le n} A_m$, and let $E = L^0$. For the family $\mathcal{P}$ of L^0–semi norms $|\cdot|_n = 1_{A_n}|\cdot|$, $n \in \mathbb{N}$, we subsequently derive the following:*

(i) $|\cdot| = \sum_{n \in \mathbb{N}} |\cdot|_n \notin \mathcal{P}$.
(ii) *For all $\varepsilon \in L^0_{++}$, $U_{\{|\cdot|\},\varepsilon} = \sum_{n \in \mathbb{N}} 1_{A_n} U_{\{|\cdot|_n\},\varepsilon}$ is not a neighborhood of the origin in the locally L^0–convex topology induced by $\mathcal{P}$.*
(iii) *The sequence $(1_{B_n}\frac{1}{n} + 1_{B_n^c})_{n \in \mathbb{N}}$ converges to 0 w.r.t. the locally L^0–convex topology induced by $\mathcal{P}$ but it does not converge to 0 in the locally L^0–convex topology induced by $\mathcal{P} \cup \{|\cdot|\}$.*

This leads us to the following definition.

Definition 1.3.5. *A family $\mathcal{P}$ of L^0–semi norms has the countable concatenation property if*

$$\sum_{n \in \mathbb{N}} 1_{A_n} \|\cdot\|_n \in \mathcal{P},$$

for every pairwise disjoint sequence $(A_n) \subset \mathcal{F}$ and for every sequence of L^0–semi norms $(\|\cdot\|_n)$ in $\mathcal{P}$.

If $\mathcal{P}$ is a family of L^0–semi norms which has the countable concatenation property then $(E,\mathcal{T})$ has the countable concatenation property in the sense of Definition 1.1.7. Conversely, if $(E,\mathcal{T})$ is a topological L^0–module which has the countable concatenation property, where $\mathcal{T}$ is induced by a family $\mathcal{P}$ of L^0–semi norms, we can always assume that $\mathcal{P}$ has the countable concatenation property. Indeed, inspection shows that

$$\left\{ \sum_{n \in \mathbb{N}} 1_{A_n} \|\cdot\|_n \mid (A_n) \subset \mathcal{F} \text{ pairwise disjoint}, (\|\cdot\|_n) \subset \mathcal{P} \right\}$$

also induces $\mathcal{T}$.

In view of Lemma 1.3.3 we can always assume that a finite family of L^0–semi norms has the countable concatenation property.

1.3.2. The index set of nets. The neighborhood base $\mathcal{U}$ of $0 \in E$ given in (4) is indexed with the collection of all finite subsets of $\mathcal{P}$ and L^0_{++}. We introduce a direction " $\geq$ " on this index set as follows:

$$(\mathcal{R}_2, \alpha_2) \geq (\mathcal{R}_1, \alpha_1) \text{ if and only if } \mathcal{R}_2 \subset \mathcal{R}_1 \text{ and } \alpha_1 \leq \alpha_2 \tag{21}$$

for all finite $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{P}$ and $\alpha_1, \alpha_2 \in L^0_{++}$. We denote nets w.r.t. this index set by $(X_{\mathcal{R},\alpha})$. If E is a topological L^0–module, not necessarily locally L^0–convex, nets are denoted by $(X_\alpha)_{\alpha \in \mathcal{D}}$ or (X_α) for corresponding index set $\mathcal{D}$.

1.4. The gauge function

Let E be an L^0–module.

Definition 1.4.1. *The gauge function $p_K : E \to \bar{L}^0_+$ of a set $K \subset E$ is defined by*

$$p_K(X) = \text{ess.inf}\{Y \in L^0_+ \mid X \in YK\}. \tag{22}$$

The gauge function p_K of an L^0–absorbent set $K \subset E$ maps E into L^0_+. Moreover:

Proposition 1.4.2. *The gauge function p_K of an L^0–absorbent set $K \subset E$ satisfies:*

(i) *$p_K(X) \leq 1$ for all $X \in K$.*
(ii) *$1_A p_K(1_A X) \geq 1_A p_K(X)$ for all $X \in E$ and $A \in \mathcal{F}$.*
(iii) *$Y p_K(1_{\{Y>0\}} X) = p_K(YX)$ for all $X \in E$ and $Y \in L^0_+$; in particular, $Y p_K(X) = p_K(YX)$ if $Y \in L^0_{++}$.*

Proof. (i) This assertion follows immediately from the definition of p_K.

(ii) Let $X \in E$ and $A \in \mathcal{F}$. We have

$$\begin{aligned} 1_A \operatorname*{ess.inf}_{X \in ZK} Z &= 1_A \operatorname*{ess.inf}_{X \in ZK} 1_A Z \\ &\geq 1_A \operatorname*{ess.inf}_{1_A X \in 1_A ZK} 1_A Z \\ &= 1_A \operatorname*{ess.inf}_{1_A X \in ZK} 1_A Z = 1_A \operatorname*{ess.inf}_{1_A X \in ZK} Z, \end{aligned} \tag{23}$$

where the inequality in (23) follows since $X \in ZK$ implies $1_A X \in 1_A ZK$. Hence, $1_A p_K(X) \geq 1_A p_K(1_A X)$.

(iii) Let $X \in E$, $Y \in L^0_+$ and define $A = \{Y > 0\}$. We have

$$\begin{aligned} Y \operatorname*{ess.inf}_{1_A X \in ZK} Z &= \operatorname*{ess.inf}_{1_A X \in ZK} YZ \\ &\overset{Z' = YZ}{=} \operatorname*{ess.inf}_{1_A XY \in 1_A Z' K} Z' \\ &= \operatorname*{ess.inf}_{1_A XY \in ZK} Z = \operatorname*{ess.inf}_{XY \in ZK} Z, \end{aligned}$$

and hence $Y p_K(1_A X) = p_K(YX)$. $\square$

A non empty L^0–absorbent L^0–convex set $K \subset E$ always contains the origin; indeed, let $X \in E$ and $Y_1, Y_2 \in L^0_{++}$ be such that $X/Y_1, -X/Y_2 \in K$. Then, since K is L^0–convex,

$$\frac{Y_1}{Y_1+Y_2}\frac{X}{Y_1} + \frac{Y_2}{Y_1+Y_2}\frac{-X}{Y_2} = \frac{X-X}{Y_1+Y_2} = 0 \in K. \tag{24}$$

Depending on the choice of $K \subset E$, the gauge function p_K can be L^0–sublinear or an L^0–semi norm.

PROPOSITION 1.4.3. *The gauge function p_K of an L^0–absorbent L^0–convex set $K \subset E$ satisfies:*

(i) $p_K(X) = \text{ess.inf}\{Y \in L^0_{++} \mid X \in YK\}$ *for all* $X \in E$.
(ii) $Yp_K(X) = p_K(YX)$ *for all* $Y \in L^0_+$ *and* $X \in E$.
(iii) $p_K(X+Y) \le p_K(X) + p_K(Y)$ *for all* $X, Y \in E$.
(iv) *For all $X \in E$ there exists a sequence (Z_n) in L^0 such that*

$$Z_n \searrow p_K(X) \ a.s. \tag{25}$$

In particular, since $0 \in K$ (cf. (24)), p_K is L^0–sublinear.

If in addition to this K is L^0–balanced then p_K satisfies:

(v) $p_K(YX) = |Y|p_K(X)$ *for all* $Y \in L^0$ *and for all* $X \in E$.

In particular, p_K is an L^0–semi norm.

PROOF. (i) As "$\le$" follows from the definition of p_K we only prove the reverse inequality. To this end, let $Y \in L^0_+$ with $X = YZ$ for some $Z \in K$. Then $\{Y = 0\} \subset \{X = 0\}$ and in turn $A = \{Y > 0\} \supset \{X \neq 0\}$. Thus, with $Y_\varepsilon = 1_A Y + 1_{A^c}\varepsilon$ for $\varepsilon \in L^0_{++}$ we have

$$X = 1_A X = Y 1_A Z = Y_\varepsilon 1_A Z \in Y_\varepsilon 1_A K \subset Y_\varepsilon K.$$

The claim now follows since $\text{ess.inf}_{\varepsilon \in L^0_{++}} Y_\varepsilon = Y$.

(ii) To prove this assertion we first show that

$$1_A p_K(1_A X) = 1_A p_K(X) \text{ for all } X \in E \text{ and } A \in \mathcal{F}. \tag{26}$$

(ii) then follows from (iii) of Proposition 1.4.2 together with (26).

To establish (26), we only have to prove the reverse inequality in (23). To this end, let $Y_1, Y_2 \in L^0_+$ with $1_A X = 1_A Y_1 Z_1$, $X = Y_2 Z_2$ for $Z_1, Z_2 \in K$ and $A \in \mathcal{F}$. In particular, $1_{A^c} X = 1_{A^c} Y_2 Z_2$. We have

$$X = 1_A Y_1 Z_1 + 1_{A^c} Y_2 Z_2 = (1_A Y_1 + 1_{A^c} Y_2)(1_A Z_1 + 1_{A^c} Z_2)$$

and since L^0–convexity of K implies that $1_A Z_1 + 1_{A^c} Z_2 = 1_A Z_1 + (1 - 1_A) Z_2 \in K$ the required inequality follows.

(iii) Let $X_1, X_2 \in E$ and $Y_1, Y_2 \in L^0_{++}$ such that $X_1/Y_1, X_2/Y_2 \in K$. Since K is L^0–convex

$$\frac{Y_1}{Y_1+Y_2}\frac{X_1}{Y_1} + \frac{Y_2}{Y_1+Y_2}\frac{X_2}{Y_2} = \frac{X_1+X_2}{Y_1+Y_2} \in K.$$

Thus, $p_K(\frac{X_1+X_2}{Y_1+Y_2}) \le 1$, and hence $p_K(X_1+X_2) \le Y_1 + Y_2$. Since Y_1 and Y_2 are arbitrary, we may take the essential infimum over all such pairs Y_1, Y_2 and – in view of (i) – we derive

$$p_K(X_1 + X_2) \le p_K(X_1) + p_K(X_2).$$

(iv) As in the proof of (26), L^0–convexity of K implies that the set

$$\{Y \in L^0_+ \mid X \in YK\}$$

is directed downwards (and upwards) for all $X \in E$.

(v) Let $X \in E, Y \in L^0$ and $A = \{Y \geq 0\}$. Then (26) and (ii) imply

$$p_K(YX) = 1_A|Y|p_K(X) + 1_{A^c}|Y|p_K(-X),$$

and hence it remains to prove that $p_K(-X) = p_K(X)$. But since K is L^0–balanced we have $-K = K$ and hence

$$p_K(-X) = p_{-K}(-X) = p_K(X).$$

□

As a consequence of Proposition 1.4.3, we can now complete the proof of Theorem 1.1.4:

Corollary 1.4.4. *Any locally L^0–convex topology $\mathcal{T}$ on E is induced by a family of L^0–semi norms.*

Proof. Let $\mathcal{U}$ be a neighborhood base of $0 \in E$ such that every $U \in \mathcal{U}$ is L^0–absorbent, L^0–convex and L^0–balanced. Then, the family of gauge functions

$$\mathcal{P} = \{p_U \mid U \in \mathcal{U}\},$$

by Proposition 1.4.3, is a family of L^0–semi norms and the topology induced by $\mathcal{P}$ coincides with $\mathcal{T}$. □

Proposition 1.4.5. *The gauge function p_K of an L^0–absorbent L^0–convex set $K \subset E$ (recall that $0 \in K$, cf. (24)) satisfies:*

(i) $p_K(X) \geq 1$ *for all* $X \in E$ *with* $1_A X \notin 1_A K$ *for all* $A \in \mathcal{F}$ *with* $P[A] > 0$.

If in addition to this, E is a locally L^0–convex module, then p_K satisfies:

(ii) $p_K(X) < 1$ *for all* $X \in \overset{\circ}{K}$.

Proof. To prove (i) let us assume that $\{p_K(X) < 1\}$ has positive P–measure for some $X \in E$ with $X1_A \notin K$ for all $A \in \mathcal{F}$ with $P[A] > 0$. With (iv) of Proposition 1.4.3 we know that there is $Y \in L^0_+$ such that $B = \{Y < 1\}$ has positive P–measure and

$$X \in YK.$$

But this is a contradiction as we derive

$$X1_B \in Y1_B K \subset 1_B K,$$

where the last inclusion follows from the L^0–convexity of $1_B K$. (Note that $0 \in K$.)

(ii) Let $X \in \overset{\circ}{K}$. Then there exists a neighborhood $U_{\mathcal{Q},\varepsilon}$ ($\mathcal{Q} \subset \mathcal{P}$ finite and $\varepsilon \in L^0_{++}$) of $0 \in E$ such that $X + U_{\mathcal{Q},\varepsilon} \subset K$. In view of Proposition 1.3.3 we can assume that $\mathcal{P}$ is closed under finite suprema and that $U_{\mathcal{Q},\varepsilon} = U_{\{\|\cdot\|_{\sup}\},\varepsilon}$, where $\|\cdot\|_{\sup} = \sup_{\|\cdot\| \in \mathcal{Q}} \|\cdot\|$. Then, for all $\delta \in L^0_{++}$,

$$\|X - X(1+\delta)\|_{\sup} = \delta \|X\|_{\sup}.$$

Thus, choosing δ such that $\delta\|X\|_{\sup} \leq \varepsilon$, we derive $X(1+\delta) \in K$ and hence $p_K(X) \leq 1/(1+\delta) < 1$. □

1.5. Hyperplane separation

Let E be a locally L^0–convex module.

Let $X \in E$ be such that there is an L^0–linear bijection $\mu : \operatorname{span}_{L^0}(X) \to L^0$. Then, necessarily

$$\mu(YX) = Y\mu X \text{ for all } Y \in L^0 \tag{27}$$

and $\mu^{-1} : L^0 \to \operatorname{span}_{L^0}(X)$ is L^0–linear as well. Since μ is a surjection we derive from (27) that $P[\mu X \neq 0] = 1$. Further,

$$Y = \mu(\mu^{-1}(Y)) = \mu(\bar{Y}X) = \bar{Y}\mu X$$

for all $Y \in L^0$. Hence, $\bar{Y} = Y/\mu X$ and in turn $\mu^{-1}(Y) = YX/\mu X$. On replacing μ by $\mu/(\mu X)$, we can always assume that $\mu X = 1$. In this case, $\mu(YX) = Y$ and $\mu^{-1}Y = YX$ for all $Y \in L^0$.

Lemma 1.5.1. *Let $K, M \subset E$ be L^0–convex, K open and non empty. If $1_A M \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$, then there is an L^0–linear function $\mu : E \to L^0$ such that*

$$\mu Y < \mu Z \textit{ for all } Y \in K \textit{ and } Z \in M. \tag{28}$$

Proof. We can assume that M is non empty.

Step 1: Suppose first that $M = \{X\}$ is a singleton.

Without loss of generality, we may assume that $0 \in K$. Indeed, if $0 \notin K$ replace X by $X - Y$ and K by $K - Y$ for some $Y \in K$ which is possible since $K \neq \emptyset$. Note that $\{X - Y\}, K - Y$ remain L^0–convex, that $K - Y$ remains open non empty and that an L^0–linear function $\mu : E \to L^0$ separates $\{X\}$ from K – in the sense of (28) – if and only if it separates $\{X - Y\}$ from $K - Y$.

Thus, let K be L^0–convex open non empty and $0 \in K$. (Note that K is L^0–absorbent.) By assumption, $1_A X \notin K$ for all $A \in \mathcal{F}$ with $P[A] > 0$. In particular, $1_A X \neq 0$ for all $A \in \mathcal{F}$ with $P[A] > 0$. Hence, $YX = Y'X$ implies $Y = Y'$ for all $Y, Y' \in L^0$ and $\mu : \operatorname{span}_{L^0}(X) \to L^0$,

$$\mu(YX) = Y \text{ for all } Y \in L^0, \tag{29}$$

is a well-defined L^0–linear bijection of $\operatorname{span}_{L^0}(X)$ into L^0. By Proposition 1.4.3, the gauge function $p_K : E \to L^0$ is L^0–sublinear. We show $p_K(Z) \geq \mu Z$ for all $Z \in \operatorname{span}_{L^0}(X)$. For $Z \in \operatorname{span}_{L^0}(X)$ let $Y \in L^0$ be the unique element with $Z = YX$. From (26) in the proof of Proposition 1.4.3 we derive

$$p_K(YX) = 1_A p_K(1_A YX) + 1_{A^c} p_K(1_{A^c} YX) \tag{30}$$

for $A = \{Y \geq 0\}$. Further, with (ii) of Proposition 1.4.3 and (i) of Proposition 1.4.5 we know that

$$1_A p_K(1_A YX) = 1_A Y p_K(X) \geq 1_A Y = 1_A \mu(YX) \tag{31}$$

and since $p_K \geq 0$

$$1_{A^c} p_K(1_{A^c} YX) \geq 1_{A^c} Y = 1_{A^c} \mu(YX). \tag{32}$$

Adding up (31) and (32), together with (30), yield

$$p_K(YX) \geq \mu(YX).$$

Hence, $p_K(Z) \geq \mu Z$ for all $Z \in \operatorname{span}_{L^0}(X)$ and therefore μ extends by the Hahn–Banach Theorem 1.2.6 to $\mu : E \to L^0$ such that

$$\mu Y \leq p_K(Y) \text{ for all } Y \in E.$$

In particular, for all $Y \in K$

$$\mu Y \leq p_K(Y) < 1 = \mu X,$$

where the strict inequality follows from (ii) of Proposition 1.4.5 and the equality follows from (29).

Step 2: Now let M be as in the lemma. Then, $K - M$ is L^0–convex open non empty and $1_A\{0\} \cap 1_A(K - M) = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. Thus, from the first step of this proof, there is an L^0–linear function $\mu : E \to L^0$ with

$$\mu(Y - Z) < 0 \text{ for all } Y \in K \text{ and } Z \in M$$

and the assertion is proved. $\square$

Lemma 1.5.2. *Let $K \subset E$ be open L^0–convex with $0 \in K$. If $\mu : E \to L^0$ is L^0–linear such that*

$$\mu(X) \leq p_K(X) \text{ for all } X \in E$$

then μ is continuous.

Proof. It suffices to show that $\mu^{-1}B_\varepsilon$ is a neighborhood of $0 \in E$ for each ball B_ε centered at $0 \in L^0$. Thus, let $\varepsilon \in L^0_{++}$. The set $U = \varepsilon K \cap -\varepsilon K$ is a neighborhood of $0 \in E$. (Indeed, let $V = U_{\mathcal{Q},\delta} \subset K$, be a neighborhood of $0 \in E$, which exists since K is open and $0 \in K$. Then, $\varepsilon V = U_{\mathcal{Q},\varepsilon\delta}$ is an L^0–balanced neighborhood of $0 \in E$. Further, $\varepsilon V \subset \varepsilon K$, $-\varepsilon V \subset -\varepsilon K$ and since εV is L^0–balanced $\varepsilon V = -\varepsilon V$ and in turn $\varepsilon V \subset \varepsilon K \cap -\varepsilon K$.) Further, for all $X \in U$ we have

$$\mu(X) \leq p_K(X) \leq \varepsilon \text{ and}$$
$$-\mu(X) = \mu(-X) \leq p_K(-X) \leq \varepsilon.$$

Thus, $|\mu(X)| \leq \varepsilon$ and hence $U \subset \mu^{-1}B_\varepsilon$. $\square$

We can now prove Theorem 1.1.6.

Proof. We can assume that M is non empty. Define $L = K - M$. For $X \in L$, the set $L - X$ is L^0–convex open and $0 \in L - X$. By assumption, $0 \notin 1_A L$ for all $A \in \mathcal{F}$ with $P[A] > 0$ and so $1_A(-X) \notin 1_A(L - X)$. From the first step of the proof of Lemma 1.5.1 we know that there is an L^0–linear function $\mu : E \to L^0$ such that

$$\mu Y \leq p_{L-X}(Y) \text{ for all } Y \in E.$$

By Lemma 1.5.2, μ is continuous. Further,

$$\mu Y < \mu(-X) \text{ for all } Y \in L - X.$$

and Theorem 1.1.6 is proved. $\square$

Lemma 1.5.3. *Let $\mathcal{P}$ be a family of L^0–semi norms inducing a locally L^0–convex topology on E and let $K \subset E$ be closed with $1_A X + 1_{A^c} X' \in K$ for all $A \in \mathcal{F}$ and $X, X' \in K$. If $\mathcal{P}$ has the countable concatenation property and $X \in E$ satisfies $1_A\{X\} \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$, then there is an L^0–convex, L^0–absorbent and L^0–balanced neighborhood U of $0 \in E$ such that*

$$1_A(X + U) \cap 1_A(K + U) = \emptyset$$

for all $A \in \mathcal{F}$ with $P[A] > 0$.

Proof. We can assume that $K \neq \emptyset$. Via translation by X, we can assume that $0 \notin 1_A K$ for all $A \in \mathcal{F}$ with $P[A] > 0$. Thus, it suffices to construct an L^0–convex, L^0–absorbent and L^0–balanced neighborhood U of $0 \in E$ such that

$$1_A U \cap 1_A(K+U) = \emptyset$$

for all $A \in \mathcal{F}$ with $P[A] > 0$.

Step 1: In this step we construct an L^0–convex, L^0–absorbent, L^0–balanced neighborhood U of $0 \in E$ such that $1_A U \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. To this end, define

$$\varepsilon^* = 1 \wedge \operatorname*{ess.sup}_{\mathcal{Q} \subset \mathcal{P} \text{ finite}} \text{ess.inf}\{\varepsilon \in L^0_{++} \mid U_{\mathcal{Q},\varepsilon} \cap K \neq \emptyset\}$$

(Note that for all $\mathcal{Q} \subset \mathcal{P}$ finite there is $\varepsilon \in L^0_{++}$ such that $U_{\mathcal{Q},\varepsilon} \cap K \neq \emptyset$ since all neighborhoods of $0 \in E$ are L^0–absorbent.) Successively we show that ε^* satisfies:

(i) $\varepsilon^* \in L^0_{++}$.

(ii) There is an L^0–semi norm $\|\cdot\|^* \in \mathcal{P}$ such that

$$\frac{\varepsilon^*}{2} < \text{ess.inf}\{\varepsilon \in L^0_{++} \mid U_{\{\|\cdot\|^*\},\varepsilon} \cap K \neq \emptyset\}.$$

(iii) $1_A U_{\{\|\cdot\|^*\},\varepsilon^*/2} \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. (Note that $U_{\{\|\cdot\|^*\},\varepsilon^*/2}$ is L^0–convex, L^0–absorbent, L^0–balanced and closed.)

(i) Suppose $P[A] > 0$, $A = \{\varepsilon^* = 0\}$. Then for all $\mathcal{Q} \subset \mathcal{P}$ finite and for all $\alpha \in L^0_{++}$ there is $X_{\mathcal{Q},\alpha} \in K$ such that

$$1_A X_{\mathcal{Q},\alpha} \in U_{\mathcal{Q},1/\alpha} \cap 1_A K.$$

Hence, for $X \in K$ the net $(1_A X_{\mathcal{Q},\alpha} + 1_{A^c} X)$ converges to $1_{A^c} X$ and $1_A X_{\mathcal{Q},\alpha} + 1_{A^c} X \in K$ for all $\mathcal{Q} \subset \mathcal{P}$ finite and for all $\alpha \in L^0_{++}$. Since K is closed, we derive $1_{A^c} X \in K$, which is impossible as it would imply $0 = 1_A 1_{A^c} X \in 1_A K$.

(ii) For all finite $\mathcal{Q} \subset \mathcal{P}$, let

$$\varepsilon_{\mathcal{Q}} = \text{ess.inf}\{\varepsilon \in L^0_{++} \mid U_{\mathcal{Q},\varepsilon} \cap K \neq \emptyset\}.$$

For finite $\mathcal{Q}, \mathcal{Q}' \subset \mathcal{P}$, $U_{\mathcal{Q}\cup\mathcal{Q}',\varepsilon} \subset U_{\mathcal{Q},\varepsilon}, U_{\mathcal{Q}',\varepsilon}$. Thus, the collection $\{\varepsilon_{\mathcal{Q}} \mid \mathcal{Q} \subset \mathcal{P} \text{ finite}\}$ is directed upwards and hence there is an increasing sequence $(\varepsilon_{\mathcal{Q}_n})$ with $1 \wedge \varepsilon_{\mathcal{Q}_n} \nearrow \varepsilon^*$ a.s. Let

$$\begin{aligned} A_1 &= \{\varepsilon_{\mathcal{Q}_1} > \varepsilon^*/2\}, \\ A_n &= \{\varepsilon_{\mathcal{Q}_n} > \varepsilon^*/2\} \setminus A_{n-1} \text{ for all } n \geq 2. \end{aligned}$$

Then, $\bigcup_{n\in\mathbb{N}} A_n \nearrow \Omega$ since $\varepsilon^* > \varepsilon^*/2$. Further, the L^0–semi norm

$$\|\cdot\|^* = \sum_{n\in\mathbb{N}} 1_{A_n} \sup_{\|\cdot\|\in\mathcal{Q}_n} \|\cdot\|$$

is an element of $\mathcal{P}$ since $\mathcal{P}$ has the countable concatenation property and $\|\cdot\|^*$ is as required.

(iii) Finally, assume there is $A \in \mathcal{F}$, $P[A] > 0$, and $X \in K$ such that $1_A X \in 1_A U_{\{\|\cdot\|^*\},\varepsilon^*/2}$. Then

$$1_A \text{ess.inf}\{\varepsilon \in L^0_{++} \mid U_{\{\|\cdot\|^*\},\varepsilon} \cap K \neq \emptyset\} \leq 1_A \frac{\varepsilon^*}{2},$$

in contradiction to the statement in (ii).

Step 2: From the first step we have $\|\cdot\| \in \mathcal{P}$ and $\varepsilon \in L^0_{++}$ such that $1_A U_{\{\|\cdot\|\},\varepsilon} \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. This implies $1_A U_{\{\|\cdot\|\},\varepsilon/2} \cap 1_A(K + U_{\{\|\cdot\|\},\varepsilon/2}) = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$ and the assertion follows. $\square$

The next example illustrates, that the countable concatenation property, as an assumption on $\mathcal{P}$ in Lemma 1.5.3, cannot be omitted.

EXAMPLE 1.5.4. *Let $(\Omega, \mathcal{F}, P)$, A_n, and the family $\mathcal{P}$ of L^0–semi norms on $E = L^0$ be as in Example 1.3.4. From Example 1.3.4 we know that $\mathcal{P}$ does not have the countable concatenation property. We now further derive the following:*

(i) *The set $K = \{X \in E \mid X \geq 1\}$ is closed with respect to the locally L^0–convex topology on E induced by $\mathcal{P}$.*
Indeed, if $X \notin K$ then there is $n \in \mathbb{N}$ such that $0 < 1 - X = c \in \mathbb{R}$ on A_n. But then $X + U_{\{1_{A_n}|\cdot|\},c/2}$ defines a neighborhood of X which is disjoint of K. Hence K^c is open.

(ii) *$1_A K \cap \{0\} = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$.*
This follows as $1_{A_n} K \cap \{0\} = \emptyset$, for all atoms $A_n, n \in \mathbb{N}$.

(iii) *For every neighborhood U of $0 \in E$ there exists $A \in \mathcal{F}$ with $P[A] > 0$ such that $1_A K \cap U \neq \emptyset$.*
Indeed, for every neighborhood U of $0 \in E$ there is $n \in \mathbb{N}$ and $\varepsilon \in L^0_{++}$ such that $U_{\{1_{B_n}|\cdot|\},\varepsilon} \subset U$. Note that $P[B_n] < 1$. But now, $1_{B_n^c} K \subset 1_{B_n^c} E = 1_{B_n^c} U_{\{1_{B_n}|\cdot|\},\varepsilon} \subset U$.

We can now prove Theorem 1.1.8.

PROOF. Recall we can assume a family $\mathcal{P}$ of L^0–semi norms induces the locally L^0–convex topology on E and that $\mathcal{P}$ inherits the countable concatenation property from E.

By Lemma 1.5.3, there is an L^0–convex, L^0–absorbent and L^0–balanced neighborhood U of $0 \in E$ such that

$$1_A(X + U) \cap 1_A(K + U) = \emptyset$$

for all $A \in \mathcal{F}$ with $P[A] > 0$. Since $K + \mathring{U}, X + \mathring{U}$ are L^0–convex open and $K + \mathring{U}$ is non empty Theorem 1.1.6 yields a continuous L^0–linear function $\mu : E \to L^0$ such that

$$\mu Y < \mu Z \text{ for all } Y \in K + \mathring{U} \text{ and } Z \in X + \mathring{U}.$$

Further, from the first step of the proof of Lemma 1.5.1 we know that there is $X_0 \in E$ such that

$$\mu(Y X_0) = Y \text{ for all } Y \in L^0.$$

Since $\mathring{U}$ is L^0–absorbent and L^0–balanced there is $\varepsilon \in L^0_{++}$ such that $-\varepsilon X_0 \in \mathring{U}$. Thus,

$$\mu Y < \mu(X - \varepsilon X_0) = \mu X - \varepsilon \text{ for all } Y \in K + \mathring{U}.$$

In particular,

$$\mu Y + \varepsilon < \mu X \text{ for all } Y \in K,$$

whence Theorem 1.1.8 is proved. $\square$

We provide an example which illustrates that the countable concatenation property, as an assumption on $\mathcal{P}$ in Theorem 1.1.8, cannot be omitted.

EXAMPLE 1.5.5. *Let* $(\Omega, \mathcal{F}, P)$, A_n, *and the family* $\mathcal{P}$ *of* L^0*–semi norms on* $E = L^0$ *be as in Example 1.5.4. Then the closed subset* $K = \{X \in E \mid X \geq 1\}$ *of* E *cannot be separated from* 0 *by a continuous* L^0*–linear function.*

Indeed, as every L^0*–linear function* $\mu : E \to L^0$ *is of the form*

$$\mu X = \sum_{n \in \mathbb{N}} 1_{A_n} a_n X \text{ for all } X \in E,$$

for some sequence $(a_n) \subset \mathbb{R}$, *we conclude that* $a_n > 0$ *for all* $n \in \mathbb{N}$ *if* μ *separates* 0 *from* K*. Such* μ*, however, is not continuous at* 0*. To see this, let* $Z = \sum_{n \in \mathbb{N}} 1_{A_n} a_n$, $\varepsilon \in L^0_{++}$ *and observe that*

$$\mu^{-1}\{Y \in L^0 \mid |Y| \leq \varepsilon\} = \{X \in E \mid |X/Z| \leq \varepsilon\}$$

is not a neighborhood of $0 \in E$.

CHAPTER 2

Duality in locally $L^0(\mathcal{F})$–convex modules

In this chapter we state the main Fenchel–Moreau type duality results in locally L^0–convex modules. (As in the preceding chapter, we will consequently suppress the notational dependence of L^0 on the σ–algebra $\mathcal{F}$.)

As in the previous chapter, the main results are presented in Section 2.1. Section 2.2 illustrates the scope of financial applications. As in the previous chapter, all proofs are postponed to the subsequent respective sections. In Section 2.3 we prove that L^0–convex functions share a certain local property. In Section 2.4 we characterize lower semi continuous functions. In Section 2.5 we establish continuity results for L^0–convex functions. For instance, under topological assumptions on the L^0–module E, proper L^0–convex functions are automatically continuous on the interior of their effective domain. In Section 2.6 we prove that proper lower semi continuous L^0–convex functions are subdifferentiable on the interior of their effective domain. In Section 2.7 we prove our Fenchel–Moreau type dual representation for proper lower semi continuous L^0–convex functions.

2.1. Main results

We first recall and introduce some terminology. Let E be an L^0–module. A function $f : E \to \bar{L}^0$ is proper if $f(X) > -\infty$ for all $X \in E$ and if there is at least one $X \in E$ such that $f(X) < +\infty$. The effective domain of a proper function $f : E \to \bar{L}^0$ is defined by $\mathrm{dom} f = \{X \in E \mid f(X) < +\infty\}$. The epigraph of f is denoted by $\mathrm{epi} f = \{(X, Y) \in E \times L^0 \mid f(X) \leq Y\}$.

Definition 2.1.1. *Let E be an L^0–module and $f : E \to \bar{L}^0$ a proper function.*

(i) *f is L^0–convex if $f(YX_1 + (1-Y)X_2) \leq Yf(X_1) + (1-Y)f(X_2)$ for all $X_1, X_2 \in E$ and $Y \in L^0$ with $0 \leq Y \leq 1$.*

(ii) *f has the local property if $1_A f(X) = 1_A f(1_A X)$ for all $X \in E$ and $A \in \mathcal{F}$. In this case, f is also called local.*

As a first result in this chapter, we obtain that L^0–convexity enforces the local property. The proof is given in Section 2.3 below.

Theorem 2.1.2. *Let E be an L^0–module. A proper function $f : E \to \bar{L}^0$ is L^0–convex if and only if f has the local property and* $\mathrm{epi} f$ *is L^0–convex.*

We now address some topological properties of L^0–convex functions.

Definition 2.1.3. *Let E be a topological L^0–module. A function $f : E \to \bar{L}^0$ is lower semi continuous if for all $Y \in L^0$ the level set $\{X \in E \mid f(X) \leq Y\}$ is closed.*

As one expects from the real case, lower semi continuity of an L^0–convex function can also be characterized in terms of its epigraph. In fact, the following result is proved in Section 2.4.

PROPOSITION 2.1.4. *Let E be a locally L^0–convex module that has the countable concatenation property. A proper function $f : E \to \bar{L}^0$ that has the local property is lower semi continuous if and only if* epi$f$ *is closed.*

A subset B of a topological L^0–module E is an L^0–barrel if it is L^0–convex, L^0–absorbent, L^0–balanced and closed. A locally L^0–convex module E is an L^0–barreled module if every L^0–barrel is a neighborhood of $0 \in E$. It follows by inspection that L^0–normed modules are L^0–barreled. The following result is proved in Section 2.5.

PROPOSITION 2.1.5. *Let E be an L^0–barreled module. A proper lower semi continuous L^0–convex function $f : E \to \bar{L}^0$ is continuous on* $\overset{\circ}{\mathrm{dom}} f$.

We now turn to our main, Fenchel–Moreau type, duality results. Let E be a topological L^0–module, and denote by $\mathcal{L}(E, L^0)$ the L^0–module of continuous L^0–linear functions $\mu : E \to L^0$. The conjugate $f^* : \mathcal{L}(E, L^0) \to \bar{L}^0$ of a function $f : E \to \bar{L}^0$ is defined by

$$f^*(\mu) = \operatorname*{ess.sup}_{X \in E}(\mu X - f(X)). \tag{33}$$

Further, the conjugate $f^{**} : E \to \bar{L}^0$ of f^* is defined by

$$f^{**}(X) = \operatorname*{ess.sup}_{\mu \in \mathcal{L}(E, L^0)}(\mu X - f^*(\mu)). \tag{34}$$

DEFINITION 2.1.6. *Let E be a topological L^0–module. An element μ of $\mathcal{L}(E, L^0)$ is a subgradient of a proper function $f : E \to \bar{L}^0$ at $X_0 \in \mathrm{dom} f$ if*

$$\mu(X - X_0) \le f(X) - f(X_0), \text{ for all } X \in E.$$

The set of all subgradients of f at X_0 is denoted by $\partial f(X_0)$.

A pre stage of Theorem 2.1.7 below, which we will prove in Section 2.6, is given in Kutateladze [**Kut79, Kut80, Kut81**]. However, Kutateladze entirely remains within an algebraic scope as he does not address topological aspects such as continuity. More precisely, he provides necessary and sufficient conditions for the existence of algebraic subgradients of L^0–sublinear functions in terms of the underlying ring. Further, Kutateladze only covers the case of L^0–sublinear functions which take values in L^0 adjoint $+\infty$, that is, $L^0 \cup \{+\infty\}$ rather than functions which take values in $\bar{L}^0$.

THEOREM 2.1.7. *Let E be an L^0–barreled module that has the countable concatenation property. Let $f : E \to \bar{L}^0$ be a proper lower semi continuous L^0–convex function. Then,*

$$\partial f(X) \neq \emptyset \text{ for all } X \in \overset{\circ}{\mathrm{dom}} f.$$

Here is the generalized Fenchel–Moreau duality theorem, the proof of which is given in Section 2.7.

THEOREM 2.1.8. *Let E be a locally L^0–convex module that has the countable concatenation property. Let $f : E \to \bar{L}^0$ be a proper lower semi continuous L^0–convex function. Then,*

$$f = f^{**}.$$

2.2. Financial applications

In this section we illustrate the scope of applications that can be covered by our results. The entropic risk measure $\rho_0 : \bar{L}^0 \to [-\infty, +\infty]$ is defined as

$$\rho_0(X) = \log E[\exp(-X)].$$

Its restriction to the locally convex vector space L^p, $p \in [1, +\infty]$, is proper convex lower semi continuous. Classical convex analysis yields the dual representation

$$\rho_0(X) = \sup_{Z \in L^q} (E[ZX] - \rho_0^*(Z))$$

with conjugate function

$$\rho_0^*(Z) = \sup_{X \in L^p} (E[ZX] - \rho_0(X))$$

(which equals $E[-Z \log(-Z)]$ if defined and $+\infty$ otherwise) where $q = p/(p-1)$ ($= +\infty$ if $p = 1$), cf. [**FS08b**]. For $p = +\infty$, in particular, ρ_0 is continuous and subdifferentiable on $\overset{\circ}{\mathrm{dom}}\rho_0 = L^\infty$ with unique subgradient $-\exp(-X)/E[\exp(-X)]$ at $X \in L^\infty$.

Market models in stochastic finance involve filtrations which represent the flow of information provided by the market. Let $(\Omega, \mathcal{E}, P)$ be endowed with a filtration $(\mathcal{F}_t)_{t \in \mathbb{N}}$. We shall now write $L^0(\mathcal{E})$, $L^0(\mathcal{F}_t)$, etc. to express the respective reference σ-algebra. The $[-\infty, +\infty]$–valued entropic risk measure ρ_0 can be made contingent on the information available at t by modifying it to $\rho_t : \bar{L}^0(\mathcal{E}) \to \bar{L}^0(\mathcal{F}_t)$,

$$\rho_t(X) = \log E[\exp(-X) \mid \mathcal{F}_t].$$

As in the deterministic case, subdifferentiability and dual representation of ρ_t are important aspects in risk management applications. For this reason, ρ_t must be restricted to a space which allows for convex analysis.

The restriction ρ_t to bounded risks, that is $L^\infty(\mathcal{E})$, has been analyzed in [**BN04, CDK06, DS05, FP06**]. It turns out that ρ_t maps $L^\infty(\mathcal{E})$ into $L^\infty(\mathcal{F}_t)$. Convex analysis of ρ_t can then be carried out by means of scalarization, an idea which goes back to [**Har65, Orh69, VS67**].

However, $L^\infty(\mathcal{E})$ is a too narrow model space for financial risks. For instance, it does not contain normal distributed random variables. The space $L^p(\mathcal{E})$, for $p \in [1, +\infty)$, is larger and already sufficient for many applications. But ρ_t restricted to $L^p(\mathcal{E})$ takes values in $\bar{L}^0(\mathcal{F}_t)$ and the scalarization method used in the previous literature would have to be adapted accordingly.

Exploiting our results, we thus propose to view ρ_t as a function on the $L^0(\mathcal{F}_t)$–module $L^p_{\mathcal{F}_t}(\mathcal{E})$, defined in Example 1.1.5, which in fact is much larger than $L^p(\mathcal{E})$ and thus even better apt for applications. The function $\rho_t : L^p_{\mathcal{F}_t}(\mathcal{E}) \to \bar{L}^0(\mathcal{F}_t)$ is proper L^0–convex. Fatou's generalized lemma and Lemma 2.4.2 show that ρ_t is lower semi continuous. Moreover, from Theorem 2.1.8 we know that the following dual representation applies

$$\begin{aligned} \rho_t(X) &= \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}_t}(\mathcal{E})} (E[ZX \mid \mathcal{F}_t] - \rho_t^*(Z)) \\ &= \operatorname*{ess.sup}_{Y \in L^0(\mathcal{F}_t), Z' \in L^q(\mathcal{E})} (Y E[Z'X \mid \mathcal{F}_t] - \rho_t^*(YZ')). \end{aligned}$$

For time-consistent dynamic risk assessment, compositions of the form $\rho_t \circ (-\rho_{t+1})$ are another important aspect, cf. [**CDK06, FP06**]. For the entropic risk measure we derive in an ad hoc manner that $\rho_t\circ(-\rho_{t+1}) = \rho_t$ on $\bar{L}^0(\mathcal{E})$. Hence, our results immediately apply to the dynamic risk assessment by means of the entropic risk measure.

2.3. Proof of Theorem 2.1.2

To prove the if statement, let $X_1, X_2 \in E$ and $Y \in L^0$, $0 \leq Y \leq 1$. The inequality

$$f(YX_1 + (1-Y)X_2) \leq Yf(X_1) + (1-Y)f(X_2) \tag{35}$$

is trivially valid on $\{f(X_1) = +\infty\}\cup\{f(X_2) = +\infty\}$. Since f is proper there is $X \in E$ such that $f(X) < +\infty$. Since f has the local property we derive $f(X_1'), f(X_2') < +\infty$ for

$$X_1' = 1_{\{f(X_1)<+\infty\}}X_1 + 1_{\{f(X_1)=+\infty\}}X \text{ and}$$
$$X_2' = 1_{\{f(X_2)<+\infty\}}X_2 + 1_{\{f(X_2)=+\infty\}}X.$$

From L^0–convexity of epi$f$ we derive

$$f(YX_1' + (1-Y)X_2') \leq Yf(X_1') + (1-Y)f(X_2'). \tag{36}$$

The local property of f together with (35) and (36) yields

$$f(YX_1 + (1-Y)X_2) \leq Yf(X_1) + (1-Y)f(X_2),$$

that is, f is L^0–convex.

To establish the only if statement, observe that epi$f$ is L^0–convex if f is L^0–convex. Thus, it suffices to prove that f has the local property. This, however, follows from the inequalities

$$\begin{aligned} f(1_AX) &= f(1_AX + 1_{A^c}0) \leq 1_Af(X) + 1_{A^c}f(0) \\ &= 1_Af(1_A(1_AX) + 1_{A^c}X) + 1_{A^c}f(0) \\ &\leq 1_Af(1_AX) + 1_{A^c}f(0) \end{aligned}$$

which become equalities if multiplied with 1_A.

2.4. Lower semi continuous functions

LEMMA 2.4.1. *Let E be a topological L^0–module. The essential supremum of a family of lower semi continuous functions $f_i : E \to \bar{L}^0$, $i \in I$, I an arbitrary index set, is lower semi continuous.*

PROOF. The assertion follows from the identity

$$\{X \mid X \in E \text{ and } \operatorname*{ess.sup}_{i\in I} f_i(X) \leq Y\} = \bigcap_{i\in I}\{X \mid X \in E \text{ and } f_i(X) \leq Y\}$$

for all $Y \in L^0$. □

The essential limit inferior ess.liminf$_\alpha X_\alpha$ of a net $(X_\alpha) \subset L^0$ is defined by

$$\operatorname*{ess.liminf}_{\alpha} X_\alpha = \operatorname*{ess.sup}_{\alpha}\operatorname*{ess.inf}_{\beta\geq\alpha} X_\beta$$

LEMMA 2.4.2. *Let E be a locally L^0–convex module that has the countable concatenation property. A proper function $f : E \to \bar{L}^0$ that has the local property is lower semi continuous if and only if*

$$\operatorname*{ess.liminf}_{\alpha} f(X_\alpha) \geq f(X) \tag{37}$$

for all nets $(X_\alpha) \subset E$ with $X_\alpha \to X$ for some $X \in E$.

PROOF. Assume that f has the local property, is lower semi continuous and let $(X_\alpha) \subset E$ be such that $X_\alpha \to X$ for some $X \in E$. Let $Y \in L^0$ be such that $Y < f(X)$ which is possible since f is proper. By lower semi continuity of f, the set $V = \{Z \in E \mid f(Z) \leq Y\}$ is closed and by the local property we have $1_A X' + 1_{A^c} X'' \in V$ for all $A \in \mathcal{F}$ and $X', X'' \in V$. Further,

$$1_A X \notin 1_A V$$

for all $A \in \mathcal{F}$ with $P[A] > 0$. By Lemma 1.5.3 there is a neighborhood U of $0 \in E$ such that $1_A(X + U) \cap 1_A V = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. Since $X_\alpha \to X$ there is α_0 such that $X_\beta \in X + U$ for all $\beta \geq \alpha_0$. Due to the local property, $1_A X_\beta \notin 1_A V$ for all $\beta \geq \alpha_0$ and $A \in \mathcal{F}$ with $P[A] > 0$. Hence, $f(X_\beta) > Y$ for all $\beta \geq \alpha_0$ and in turn

$$\begin{aligned} \operatorname*{ess.liminf}_{\alpha} f(X_\alpha) &= \operatorname*{ess.sup}_{\alpha} \operatorname*{ess.inf}_{\beta \geq \alpha} f(X_\beta) \\ &\geq \operatorname*{ess.inf}_{\beta \geq \alpha_0} f(X_\beta) \geq Y. \end{aligned}$$

Since Y was arbitrary, we deduce (37).

Now assume (37) and let $Y \in L^0$. We have to show that the set

$$V = \{Z \in E \mid f(Z) \leq Y\}$$

is closed. To this end, let $(X_\alpha) \subset V$ and $X \in E$ with $X_\alpha \to X$ for some $X \in E$. Then, from the inequality $f(X_\alpha) \leq Y$ for each α, we obtain

$$f(X) \leq \operatorname*{ess.liminf}_{\alpha} f(X_\alpha) \leq Y,$$

so $X \in V$. That is, V is closed, and hence f is lower semi continuous. □

Next, we prove Proposition 2.1.4.

PROOF. Define $\phi : E \times L^0 \to \bar{L}^0$ by

$$\phi(X, Y) = f(X) - Y.$$

From Lemma 2.4.2 and the definition of the product topology we derive that lower semi continuity of f on E is equivalent to lower semi continuity of ϕ on $E \times L^0$. For all $Z \in L^0$ we have

$$\{(X, Y) \in E \times L^0 \mid \phi(X, Y) \leq Z\} = \text{epi}f - (0, Z).$$

Since $E \times L^0$ is a topological L^0–module we derive that $\{(X, Y) \in E \times L^0 \mid \phi(X, Y) \leq Z\}$ is closed if and only if epi$f$ is closed. This proves Proposition 2.1.4. □

2.5. Lower semi continuous L^0–convex functions

LEMMA 2.5.1. *Let E be a topological L^0–module. If in the neighborhood of $X_0 \in E$ a proper L^0–convex function $f : E \to \bar{L}^0$ is bounded above by $Y_0 \in L^0$ then f is continuous at X_0.*

PROOF. On replacing f by $f(\cdot + X_0) - f(X_0)$, we assume that $X_0 = f(X_0) = 0$. Let $\delta \in L^0_{++}$ and $f(X) \leq Y_0$ for all X in a neighborhood V of $0 \in E$. We have to show that there is a neighborhood W_δ of $0 \in E$ such that $|f(X)| \leq \delta$ for all $X \in W_\delta$.

Without loss of generality we can assume that Y_0 is such that $\varepsilon = \delta/Y_0 > 0$ is well defined and $\varepsilon < 1$. Since E is a topological L^0–module $W = V \cap -V$ is a symmetric ($W = -W$) neighborhood of $0 \in E$. We will show that the neighborhood $W_\delta = \varepsilon W$ is as required. Indeed, for all $X \in \varepsilon W$ we have $\pm X/\varepsilon \in V$ and hence L^0–convexity of f implies

$$\begin{aligned} f(X) &\leq (1-\varepsilon)f(0) + \varepsilon f(X/\varepsilon) \leq \varepsilon Y_0 = \delta \\ \text{and } f(X) &\geq (1+\varepsilon)f(0) - \varepsilon f(-X/\varepsilon) \geq -\varepsilon Y_0 = -\delta. \end{aligned}$$

Thus, $|f(X)| \leq \delta$ for all $X \in W_\delta$, whence the required continuity follows. □

PROPOSITION 2.5.2. *Let E be a topological L^0–module. Let $f : E \to \bar{L}^0$ be a proper L^0–convex function. The following statements are equivalent:*

(i) *There is a non empty open set $O \subset E$ on which f is bounded above by $Y_0 \in L^0$.*

(ii) *f is continuous on $\overset{\circ}{\mathrm{dom}} f$ and $\overset{\circ}{\mathrm{dom}} f \neq \emptyset$.*

PROOF. (ii) implies (i) since for every $X_0 \in \overset{\circ}{\mathrm{dom}} f$ and for every $\delta \in L^0_{++}(\mathcal{F})$ there is a neighborhood V of X_0 such that $f(X_0) - \delta \leq f(X) \leq f(X_0) + \delta$ for all $X \in V$. $O = \overset{\circ}{V}$ and $Y_0 = f(X_0) + \delta$ are then as required.

Conversely, let O and Y_0 be as in (i) and take $X_0 \in O$. Then, $X_0 \in \overset{\circ}{\mathrm{dom}} f$, whence $\overset{\circ}{\mathrm{dom}} f \neq \emptyset$. To see that f is continuous on $\overset{\circ}{\mathrm{dom}} f$, let $X_1 \in \overset{\circ}{\mathrm{dom}} f$. Observe that there is $Y_1 \in L^0_{++}$, $Y_1 > 1$, such that $X_2 = X_0 + Y_1(X_1 - X_0) \in \overset{\circ}{\mathrm{dom}} f$. Since E is a topological L^0–module the map $H : E \to E$ given by

$$H(X) = X_2 - \frac{Y_1 - 1}{Y_1}(X_2 - X) \text{ for all } X \in E$$

is continuous and has continuous inverse H^{-1}. As H transforms X_0 into X_1, it transforms O into an open set $H(O)$ containing X_1. By L^0–convexity of f, we have for all $X \in H(O)$

$$\begin{aligned} f(X) &= f\left(\frac{Y_1 - 1}{Y_1} H^{-1}(X) + \frac{1}{Y_1} X_2\right) \\ &\leq \frac{Y_1 - 1}{Y_1} f(H^{-1}(X)) + \frac{1}{Y_1} f(X_2) \\ &\leq \frac{Y_1 - 1}{Y_1} Y_0 + \frac{1}{Y_1} f(X_2). \end{aligned}$$

In other words, for every $X_1 \in \overset{\circ}{\mathrm{dom}} f$ there is a neighborhood of X_1 on which f is bounded above by an element of L^0. By Lemma 2.5.1, f is continuous at X_1. □

Corollary 2.5.3. *Let E be a topological L^0–module and $X \in E$. Every proper L^0–convex function $f : \operatorname{span}_{L^0}(X) \to \bar{L}^0$ is continuous (with respect to the trace topology) on* $\overset{\circ}{\operatorname{dom}} f$.

Proof. Without loss of generality we assume that $0 \in \overset{\circ}{\operatorname{dom}} f$, else translate. Then there is a neighborhood U of $0 \in \operatorname{span}_{L^0}(X)$ and $Y \in L^0_{++}$ such that $\widetilde{X} = YX \in U \subset \overset{\circ}{\operatorname{dom}} f$. From L^0–convexity it follows that f is bounded above by $\sup(f(0), f(\widetilde{X}))$ on the open set

$$\{\lambda \widetilde{X} \mid 0 < \lambda < 1, \lambda \in L^0\}$$

and hence, by Proposition 2.5.2, f is continuous on $\overset{\circ}{\operatorname{dom}} f$. □

We can now prove Proposition 2.1.5.

Proof. Assume that there is $X_0 \in \overset{\circ}{\operatorname{dom}} f$. By translation, we may assume $X_0 = 0$. Take $Y_0 \in L^0$ such that $f(0) < Y_0$. By assumption, the level set $C = \{X \in E \mid f(X) \leq Y_0\}$ is closed. Further, for all $X \in E$ the net $(X/Y)_{Y \in L^0_{++}}$ converges to $0 \in E$. By Corollary 2.5.3, the restriction of f to $\operatorname{span}_{L^0}(X)$ is continuous at 0, hence $f(X/Y) < Y_0$ for large Y which implies that C is L^0–absorbent. Hence, $C \cap -C$ is an L^0–barrel and in turn a neighborhood of $0 \in E$. Thus, C is a neighborhood of $0 \in E$ and since f is bounded above by Y_0 on all of C it is continuous at 0. This proves Proposition 2.1.5. □

2.6. Subdifferentiability

Let E be a topological L^0–module and $f : E \to \bar{L}^0$ be a proper function. Recall the Definitions (33) and (34) of the conjugates f^* and f^{**} of f and f^*, respectively. If f^* is proper its effective domain is given by the set

$$\left\{\mu \in \mathcal{L}(E, L^0) \mid \exists Y \in L^0 : \operatorname*{ess.sup}_{X \in E}(\mu X - f(X)) \leq Y\right\}.$$

Since f is proper f^* maps its effective domain into L^0 and f^* is L^0–convex if f is so. If f^{**} is proper its effective domain is given by the set

$$\left\{X \in E \mid \exists Y \in L^0 : \operatorname*{ess.sup}_{\mu \in \mathcal{L}(E, L^0)}(\mu X - f^*(\mu)) \leq Y\right\}.$$

Again, if f^* is proper f^{**} maps its effective domain into L^0 and f^{**} is L^0–convex if f^* is so. Since for all $X \in E$ and $\mu \in \mathcal{L}(E, L^0)$,

$$f^*(\mu) \geq \mu X - f(X) \tag{38}$$

we have for all $X \in E$

$$f(X) \geq f^{**}(X). \tag{39}$$

For $\mu \in \mathcal{L}(E, L^0)$ and $X_0 \in \operatorname{dom} f$ we have

$$\mu \in \partial f(X_0) \text{ if and only if } f(X_0) = \mu X_0 - f^*(\mu). \tag{40}$$

Indeed, $\mu \in \partial f(X_0)$ by definition means

$$f(X_0) \leq \mu X_0 - (\mu X - f(X)) \text{ for all } X \in E.$$

This is equivalent to

$$f(X_0) \leq \mu X_0 - \underset{X\in E}{\text{ess.sup}}(\mu X - f(X)) = \mu X_0 - f^*(\mu)$$

which, by (38), is equivalent to $f(X_0) = \mu X_0 - f^*(\mu)$.

With (39) and (40) we know that $\mu \in \partial f(X_0)$ maximizes (34) at X_0, i.e.

$$f^{**}(X_0) = \mu X_0 - f^*(\mu).$$

LEMMA 2.6.1. *Let E be an L^0–barreled module that has the countable concatenation property. Let $f : E \to \bar{L}^0$ be a proper lower semi continuous function that has the local property. Equivalent are:*

(i) $\overset{\circ}{\text{dom}} f \neq \emptyset$.

(ii) $\overset{\circ}{\text{epi}} f \neq \emptyset$.

Further, for all $X \in \text{dom} f$, $(X, f(X)) \in \partial\text{epi} f$ and $1_A(X, f(X)) \notin 1_A\overset{\circ}{\text{epi}} f$ for all $A \in \mathcal{F}$ with $P[A] > 0$.

PROOF. To prove that (i) implies (ii), let $\varepsilon \in L^0_{++}$ and $X \in \overset{\circ}{\text{dom}} f$. We claim

$$(X, f(X) + \varepsilon) \in \overset{\circ}{\text{epi}} f. \tag{41}$$

To verify this, we show that there is a neighborhood U of $(X, f(X) + \varepsilon)$ such that $U \subset \text{epi} f$. By Proposition 2.1.5, f is continuous at X. Hence, there is a neighborhood U_E of X such that

$$f(X) + \varepsilon/3 \geq f(X') \text{ for all } X' \in U_E.$$

This implies

$$(X, f(X) + \varepsilon) \in U_E \times U_{L^0} \subset \text{epi} f,$$

where

$$U_{L^0} = \{Y \in L^0 \mid |f(X) + \varepsilon - Y| \leq \varepsilon/3\}.$$

$U = U_E \times U_{L^0}$ is as required and (41) is proved.

Conversely, to prove that (ii) implies (i), let $(X, Y) \in \overset{\circ}{\text{epi}} f$. Then there are neighborhoods U_E and U_{L^0} of X and Y respectively such that $U = U_E \times U_{L^0} \subset \text{epi} f$. In particular, $f(X') < +\infty$ for all $X' \in U_E$ and hence $X \in \overset{\circ}{\text{dom}} f$.

Next, let $X \in \text{dom} f$. To prove $(X, f(X)) \in \partial\text{epi} f$ we show that every $U \subset E \times L^0$ of the form

$$U = U_E \times \{Y \in L^0 \mid |f(X) - Y| \leq \varepsilon\},$$

$U_E \subset E$ a neighborhood of X, satisfies

$$U \cap \text{epi} f \neq \emptyset \neq U \cap \text{epi} f^c.$$

Observe $(X, f(X) - \varepsilon/2), (X, f(X) + \varepsilon/2) \in U$ and $(X, f(X) - \varepsilon/2) \notin \text{epi} f$ and $(X, f(X) + \varepsilon/2) \in \text{epi} f$, which proves $(X, f(X)) \in \partial\text{epi} f$. For fixed $A \in \mathcal{F}$ with $P[A] > 0$, we show in a similar way that $1_A(X, f(X)) \notin 1_A\overset{\circ}{\text{epi}} f$. Observe that every $U \subset E \times L^0$ of the form

$$U = U_E \times \{Y \in L^0 \mid |1_A f(X) - Y| \leq \varepsilon\},$$

$U_E \subset E$ a neighborhood of $1_A X$, satisfies

$$U \cap \text{epi} f^c \neq \emptyset.$$

Indeed, $1_A(X, f(X) - \varepsilon/2) \in U$ and yet $1_A(X, f(X) - \varepsilon/2) \notin 1_A\text{epi}f$ by the local property of f. This proves $1_A(X, f(X)) \notin 1_A\overset{\circ}{\text{epi}}f$. $\square$

Next, we prove Theorem 2.1.7.

PROOF. Let $X_0 \in \overset{\circ}{\text{dom}}f$. We separate $(X_0, f(X_0))$ from $\overset{\circ}{\text{epi}}f$ by means of Theorem 1.1.6. By Lemma 2.6.1, $\overset{\circ}{\text{epi}}f$ is non empty, $(X_0, f(X_0)) \in \partial\text{epi}f$ and

$$1_A\{(X_0, f(X_0))\} \cap 1_A\overset{\circ}{\text{epi}}f = \emptyset \text{ for all } A \in \mathcal{F} \text{ with } P[A] > 0.$$

Hence, there are continuous L^0–linear functions $\mu_1 : E \to L^0$ and $\mu_2 : L^0 \to L^0$ such that

$$\mu_1 X + \mu_2 Y < \mu_1 X_0 + \mu_2 f(X_0) \text{ for all } (X, Y) \in \overset{\circ}{\text{epi}}f. \tag{42}$$

From (42) together with the fact that $\mu_2 Y = Y\mu_2 1$ for all $Y \in L^0$ we derive that $\mu_2 1 < 0$. We will show that $-\mu_1/\mu_2 1 \in \partial f(X_0)$. To this end, let $X \in E$, $A = \{f(X) = +\infty\}$ and $\widetilde{X} = 1_A X_0 + 1_{A^c} X$. Then, $\widetilde{X} \in \overset{\circ}{\text{dom}}f$ and in turn $(\widetilde{X}, f(\widetilde{X})) \in \partial\text{epi}f$. Thus, there is a net $(X_{\mathcal{R},\alpha}, Y_{\mathcal{R},\alpha}) \subset \overset{\circ}{\text{epi}}f$ which converges to $(\widetilde{X}, f(\widetilde{X}))$ and for which

$$\mu_1 X_{\mathcal{R},\alpha} + Y_{\mathcal{R},\alpha}\mu_2 1 < \mu_1 X_0 + \mu_2 f(X_0) \text{ for all } \mathcal{R}, \alpha. \tag{43}$$

Since μ_1 is continuous we may pass to limits in (43) yielding

$$\frac{-\mu_1(\widetilde{X} - X_0)}{\mu_2 1} \leq f(\widetilde{X}) - f(X_0).$$

Finally, from the local property of f and μ_1 we derive

$$\frac{-\mu_1(X - X_0)}{\mu_2 1} \leq f(X) - f(X_0)$$

and since $X \in E$ was arbitrary we conclude that $-\mu_1/\mu_2 1$ indeed is a subgradient of f at X_0. This proves Theorem 2.1.7. $\square$

2.7. Proof of the Fenchel–Moreau duality theorem 2.1.8

In this section, we prove Theorem 2.1.8. The proof follows a known pattern, cf. Proposition A.6 in [**FS04**]; however, it contains certain subtleties due to our L^0–convex framework.

We fix $X_0 \in E$, and proceed in two steps.

Step 1: Let $\beta \in L^0$ with $\beta < f(X_0)$. In this step, we show there is a continuous function $h : E \to L^0$ of the form

$$h(X) = \mu X + Z, \tag{44}$$

where $\mu : E \to L^0$ is continuous L^0–linear and $Z \in L^0$, such that $h(X_0) = \beta$ and $h(X) \leq f(X)$ for all $X \in E$. To this end, we separate (X_0, β) from $\text{epi}f$ by means of Theorem 1.1.8. It applies since $\beta < f(X_0)$ and the local property of f imply

$$1_A\{(X_0, \beta)\} \cap 1_A\text{epi}f = \emptyset \text{ for all } A \in \mathcal{F} \text{ with } P[A] > 0.$$

(Note, $\text{epi}f$ is closed by Proposition 2.1.4.) Hence, there are continuous L^0–linear functions $\mu_1 : E \to L^0$ and $\mu_2 : L^0 \to L^0$ such that

$$\delta = \operatorname*{ess.sup}_{(X,Y) \in \text{epi}f} \mu_1 X + \mu_2 Y < \mu_1 X_0 + \mu_2 \beta. \tag{45}$$

This has two consequences:

(i) $\mu_2 1 \leq 0$.
Indeed, $\mu_2 Y = Y\mu_2 1$ for all $Y \in L^0$. Further, $(X,Y) \in \mathrm{epi}f$ for arbitrarily large Y as long as $f(X) \leq Y$. Hence, for large $Y \in L^0$, $\mu_1 X + \mu_2 Y$ is large on $\{\mu_2 1 > 0\}$ and yet bounded above by $\mu_1 X_0 + \mu_2 \beta$. This implies $P[\mu_2 1 > 0] = 0$.

(ii) $\{f(X_0) < +\infty\} \subset \{\mu_2 1 < 0\}$.
Indeed, define $\widetilde{X}_0 = 1_{\{f(X_0)<+\infty\}}X_0 + 1_{\{f(X_0)=+\infty\}}X$ for some $X \in \mathrm{dom}f$. (f is proper by assumption.) By L^0–convexity of f, $\widetilde{X}_0 \in \mathrm{dom}f$. Local property of f and (45) imply on $\{f(X_0) < +\infty\}$

$$\mu_1 X_0 + \mu_2 f(X_0) = \mu_1 \widetilde{X}_0 + \mu_2 f(\widetilde{X}_0) < \mu_1 X_0 + \mu_2 \beta.$$

Hence, $f(X_0)\mu_2 1 = \mu_2 f(X_0) < \mu_2 \beta = \beta\mu_2 1$ on $\{f(X_0) < +\infty\}$ and so $\mu_2 1 < 0$ on $\{f(X_0) < +\infty\}$.

We distinguish the cases $X_0 \in \mathrm{dom}f$ and $X_0 \notin \mathrm{dom}f$.

Case 1. Assume $X_0 \in \mathrm{dom}f$. By (ii), $\mu_2 1 < 0$. Thus, define h by

$$h(X) = -\frac{\mu_1(X - X_0)}{\mu_2 1} + \beta \text{ for all } X \in E$$

which is as required. Indeed, $h(X) \leq f(X)$ for all $X \in \mathrm{dom}f$ as a consequence of (45). If $X \notin \mathrm{dom}f$ we have

$$1_B h(X) = 1_B h(X') \leq 1_B f(X') = 1_B f(X), \tag{46}$$

where $X' = 1_B X + 1_{B^c} X''$ for some $X'' \in \mathrm{dom}f$ and $B = \{f(X) < +\infty\}$. Hence, $h(X) \leq f(X)$ for all $X \in E$.

Case 2. Assume $X_0 \notin \mathrm{dom}f$. Then chose any $X_0' \in \mathrm{dom}f$ and let h' be the corresponding L^0–affine minorant as constructed in case 1 above. Define $A_1 = \{\mu_2 1 < 0\}$, $A_2 = A_1^c$ and $h_1, h_2 : E \to L^0$,

$$h_1(X) = 1_{A_1}\left(-\frac{\mu_1(X - X_0)}{\mu_2 1} + \beta\right),$$
$$h_2(X) = \begin{cases} 1_{A_2}\left(h'(X) + \beta - h'(X_0)\right) & \text{on } \{h'(X_0) \geq \beta\} \\ 1_{A_2}\left(h'(X) + \frac{\beta - h'(X_0)}{\widetilde{h}(X_0)}\widetilde{h}(X)\right) & \text{on } \{h'(X_0) < \beta\} \end{cases},$$

with the convention $0/0 = 0$, where $\widetilde{h} : E \to L^0$,

$$\widetilde{h}(X) = \delta - \mu_1 X.$$

Note that on $\{\mu_2 1 = 0\}$, $\widetilde{h}(X_0) < 0$ and $\widetilde{h}(X) \geq 0$ for all $X \in \mathrm{dom}f$. It follows that

$$h = h_1 + h_2$$

is as required. (As in (46) we see $h(X) \leq f(X)$ for all $X \in E$.)

Step 2: Recall $f \geq f^{**}$, cf. (39). By way of contradiction, assume $f(X_0) > f^{**}(X_0)$ on a set of positive measure. Then there is $\beta \in L^0$ with $\beta > f^{**}(X_0)$ on a set of positive measure and $\beta < f(X_0)$. The first step of this proof yields $h : E \to L^0$,

$$h(X) = \mu X + Z \text{ for all } X \in E,$$

for continuous L^0–linear $\mu : E \to L^0$ and $Z \in L^0$, such that $h(X_0) = \beta$ and $h(X) \leq f(X)$ for all $X \in E$. We derive a contradiction as

$$\begin{aligned} f^{**}(X_0) &\geq \mu X_0 - f^*(\mu) \\ &= \mu X_0 - \operatorname*{ess.sup}_{X \in E}(\mu X - f(X)) \\ &\geq \mu X_0 - \operatorname*{ess.sup}_{X \in E}(\mu X - h(X)) = \beta \end{aligned}$$

negates $\beta > f^{**}(X_0)$ on a set of positive measure. This finishes the proof of Theorem 2.1.8.

CHAPTER 3

Examples

In this chapter we present important examples of $L^0(\mathcal{F})$–normed modules with a focus on $L^0(\mathcal{F})$–modules of $L^p(\mathcal{E})$ and Orlicz type.

Section 3.1 deals with the free $L^0(\mathcal{F})$–module $(L^0(\mathcal{F}))^d$, the d–fold cartesian product of $L^0(\mathcal{F})$. This subsection does not contain any explicit financial applications. Nevertheless, it serves to illustrate the idea of how to prove completeness in the context of $L^p(\mathcal{E})$ and Orlicz type $L^0(\mathcal{F})$–modules. In Section 3.2 and Section 3.3 we present $L^p(\mathcal{E})$ and Orlicz type $L^0(\mathcal{F})$–modules. We show that these modules are complete and in the $L^p(\mathcal{E})$ type case we find the $L^0(\mathcal{F})$–module of continuous $L^0(\mathcal{F})$–linear functions. A discussion of selected conditional risk measures defined on the respective $L^0(\mathcal{F})$–modules is presented in Section 4.2.

Lemma 3.0.1. *Let $(E, \|\cdot\|)$ be an $L^0(\mathcal{F})$–normed module. For an $L^0(\mathcal{F})$–linear function $\mu : (E, \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ the following statements are equivalent:*

(i) *μ is continuous.*

(ii) *μ is continuous at 0.*

(iii) *There is $\varepsilon \in L^0_{++}(\mathcal{F})$ such that*

$$|\mu X| \leq \varepsilon \|X\| \text{ for all } X \in E.$$

Proof. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are immediate; indeed, (iii) implies (i) as

$$|\mu X - \mu X_0| = |\mu(X - X_0)| \leq \varepsilon \|X - X_0\| \text{ for all } X, X_0 \in E.$$

(ii) $\Rightarrow$ (iii) Suppose (iii) is not valid. Then, for every $\varepsilon \in L^0_{++}(\mathcal{F})$ there is $X_\varepsilon \in E$ such that $|\mu X_\varepsilon| > \varepsilon \|X_\varepsilon\|$ on a set of positive measure. Let

$$Y_\varepsilon = \begin{cases} \frac{X_\varepsilon}{\varepsilon \|X_\varepsilon\|} & \text{if } \|X_\varepsilon\| \neq 0 \\ 0 & \text{else} \end{cases}.$$

Then, $|\mu Y_\varepsilon| = (\mu X_\varepsilon)/(\varepsilon \|X_\varepsilon\|) > 1$ on a set of positive measure and $\|Y_\varepsilon\| \leq 1/\varepsilon$ implies that $Y_\varepsilon \to 0$. But this contradicts the continuity of μ at 0 as to which $|\mu Y_\varepsilon| \leq 1$ for all $\varepsilon \in L^0_{++}(\mathcal{F})$ with ε sufficiently large. □

3.1. The free $L^0(\mathcal{F})$–module $(L^0(\mathcal{F}))^d$

The $L^0(\mathcal{F})$–module $E = (L^0(\mathcal{F}))^d$ is a free $L^0(\mathcal{F})$–module of rank $d \in \mathbb{N}$. The function $\|\cdot\| : (L^0(\mathcal{F}))^d \to L^0_+(\mathcal{F})$,

$$\|X\| = \left(\sum_{i=1}^{d} X_i^2 \right)^{1/2} \quad \text{for all } X = (X_1, \dots, X_d) \in (L^0(\mathcal{F}))^d, \tag{47}$$

defines an $L^0(\mathcal{F})$–norm on $(L^0(\mathcal{F}))^d$. $(L^0(\mathcal{F}))^d$ is finitely generated over $L^0(\mathcal{F})$ by the elements $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, $1 \leq i \leq d$. The elements e_i, $1 \leq i \leq d$, form a basis, that is, $\sum_{i=1}^d Y_i e_i = 0$ implies $Y_1, \dots, Y_d = 0$ for all $Y_1, \dots, Y_d \in L^0(\mathcal{F})$.

PROPOSITION 3.1.1. *A function* $\mu : (L^0(\mathcal{F}))^d \to L^0(\mathcal{F})$ *is* $L^0(\mathcal{F})$*–linear if and only if there exists* $Z = (Z_1, \dots Z_d) \in (L^0(\mathcal{F}))^d$ *such that* $\mu X = \sum_{i=1}^d Z_i X_i$ *for all* $X = (X_1, \dots, X_d) \in (L^0(\mathcal{F}))^d$*. Moreover, any* $L^0(\mathcal{F})$*–linear function* $\mu : ((L^0(\mathcal{F}))^d, \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ *is continuous.*

PROOF. Any $X = (X_1, \dots, X_d) \in (L^0(\mathcal{F}))^d$ is of the form $\sum_{i=1}^d X_i e_i$. By $L^0(\mathcal{F})$–linearity

$$\mu X = \mu(\sum_{i=1}^d X_i e_i) = \sum_{i=1}^d X_i \mu e_i = \sum_{i=1}^d X_i Z_i, \text{ where } Z_i = \mu e_i.$$

Conversely, for $(Z_1, \dots Z_d) \in (L^0(\mathcal{F}))^d$, $(L^0(\mathcal{F}))^d \to L^0(\mathcal{F}), (X_1, \dots, X_d) \mapsto \sum_{i=1}^d X_i Z_i$, defines an $L^0(\mathcal{F})$–linear mapping, which is continuous, due to

$$|\mu X|^2 \le d \max\{Z_1^2, \dots, Z_d^2\} \sum_{i=1}^d X_i^2$$

for all $X = (X_1, \dots, X_d) \in (L^0(\mathcal{F}))^d$. □

THEOREM 3.1.2. $((L^0(\mathcal{F}))^d, \|\cdot\|)$ *is complete for all* $d \in \mathbb{N}$.

PROOF. A net (X_ε) is Cauchy (convergent) in $((L^0(\mathcal{F}))^d, \|\cdot\|)$ if and only if all its components are Cauchy (convergent) in $(L^0(\mathcal{F}), |\cdot|)$. Therefore, it suffices to prove $(L^0(\mathcal{F}), |\cdot|)$ is complete.

Let (X_ε) be Cauchy in $(L^0(\mathcal{F}), |\cdot|)$. Throughout, $\varepsilon, \varepsilon_0, \delta, \delta_0$ denote elements of $L^0_{++}(\mathcal{F})$.

Step 1. In this step, we prove there is $Y \in L^0_{++}(\mathcal{F})$ and $X_0 \in L^1(\mathcal{F})$ such that

(i) $\frac{X_\varepsilon}{Y} \to X_0$ in $L^1(\mathcal{F})$ for sufficiently large ε and
(ii) $(\frac{X_\varepsilon}{Y})$ is Cauchy in $(L^0(\mathcal{F}), |\cdot|)$.

Since (X_ε) is Cauchy in $(L^0(\mathcal{F}), |\cdot|)$, there is ε_0 such that

$$|X_\varepsilon - X_\delta| \le 1 \text{ for all } \varepsilon, \delta \ge \varepsilon_0.$$

In particular, $|X_\varepsilon| \le 1 + |X_{\varepsilon_0}|$ for all $\varepsilon \ge \varepsilon_0$. Hence, $Y = 1 + |X_{\varepsilon_0}|$ is as required. Indeed, we have $X_\varepsilon / Y \in L^1(\mathcal{F})$ for all $\varepsilon \ge \varepsilon_0$. Further, (X_ε/Y) is still Cauchy in $(L^0(\mathcal{F}), |\cdot|)$ which implies that (X_ε/Y) is Cauchy in the Banach space $L^1(\mathcal{F})$, whence (X_ε/Y) converges to some X_0 in $L^1(\mathcal{F})$, which proves (i) and (ii).

Step 2. In this step we prove that $X_\varepsilon/Y \to X_0$ in $(L^0(\mathcal{F}), |\cdot|)$ which implies that $X_\varepsilon \to Y X_0$ in $(L^0(\mathcal{F}), |\cdot|)$.

By way of contradiction, assume (X_ε/Y) does not converge to X_0 in $(L^0(\mathcal{F}), |\cdot|)$. Then, $\tilde{X}_\varepsilon = X_\varepsilon/Y - X_0$ satisfies

$$\widetilde{X}_\varepsilon \to 0 \text{ in } L^1(\mathcal{F}) \text{ for sufficiently large } \varepsilon \text{ and} \tag{48}$$

$$(\widetilde{X}_\varepsilon) \text{ is Cauchy in } (L^0(\mathcal{F}), |\cdot|) \text{ whereas} \tag{49}$$

$$(\widetilde{X}_\varepsilon) \text{ does not converge to } 0 \text{ in } (L^0(\mathcal{F}), |\cdot|). \tag{50}$$

(50) implies

$$\exists \varepsilon_0 \forall \delta_0 \exists \delta \ge \delta_0 : |\widetilde{X}_\delta| > \frac{1}{\varepsilon_0} \text{ on } A_\delta, \tag{51}$$

where $A_\delta \in \mathcal{F}$ with $P[A_\delta] > 0$. Further, from (49) we know that

$$\exists \delta_0 \forall \delta, \varepsilon \geq \delta_0 : |\widetilde{X}_\delta - \widetilde{X}_\varepsilon| \leq \frac{1}{2\varepsilon_0}, \tag{52}$$

which together with (51) yields

$$\exists \delta \geq \delta_0 : |\widetilde{X}_\delta| > \frac{1}{\varepsilon_0} \text{ on } A_\delta,$$

where $A_\delta \in \mathcal{F}$ with $P[A_\delta] > 0$. Since $\delta \geq \delta_0$ we know from (52) that

$$\forall \varepsilon \geq \delta : |\widetilde{X}_\delta - \widetilde{X}_\varepsilon| \leq \frac{1}{2\varepsilon_0}$$

and hence

$$\forall \varepsilon \geq \delta : |\widetilde{X}_\delta| - |\widetilde{X}_\varepsilon| \leq \frac{1}{2\varepsilon_0}.$$

From this we derive that $|\tilde{X}_\varepsilon| > 1/(2\varepsilon_0)$ on A_δ for all $\varepsilon \geq \delta$, in contradiction to (48). □

3.2. $L^0(\mathcal{F})$–modules of $L^p(\mathcal{E})$ type

In this section we provide a detailed discussion of the $L^p(\mathcal{E})$ type $L^0(\mathcal{F})$–module which was first introduced with Example 1.1.5.

We recall that the classical conditional expectation $E[\cdot \mid \mathcal{F}] : L^1(\mathcal{E}) \to L^1(\mathcal{F})$ extends to the (generalized) conditional expectation $E[\cdot \mid \mathcal{F}] : L^0_+(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ by

$$E[X \mid \mathcal{F}] = \lim_{n\to\infty} E[X \wedge n \mid \mathcal{F}]. \tag{53}$$

Beppo–Levi's monotone convergence theorem yields for all $X, X' \in L^0_+(\mathcal{E})$ and $Y \in L^0_+(\mathcal{F})$

(i) $YE[X \mid \mathcal{F}] = E[YX \mid \mathcal{F}]$,
(ii) $E[X + X' \mid \mathcal{F}] = E[X \mid \mathcal{F}] + E[X' \mid \mathcal{F}]$,
(iii) $E[X] = E[E[X \mid \mathcal{F}]]$,

where the last expectation might equal ∞, however, as such, it remains well defined.

For $p \in [1, \infty]$ we recall the definition in (5) of $\|\cdot\|_p : L^0(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$,

$$\|X\|_p = \begin{cases} E[|X|^p \mid \mathcal{F}]^{1/p} & \text{if } p \in [1,\infty) \\ \text{ess.inf}\{Y \in \bar{L}^0_+(\mathcal{F}) \mid Y \geq |X|\} & \text{if } p = \infty \end{cases}, \tag{54}$$

and

$$L^p_{\mathcal{F}}(\mathcal{E}) = \{X \mid X \in L^0(\mathcal{E}), \|X\|_p \in L^0(\mathcal{F})\}.$$

The properties of the (generalized) conditional expectation guarantee that $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$ is an $L^0(\mathcal{F})$–normed module.

Proposition 3.2.1. *$L^p_{\mathcal{F}}(\mathcal{E})$ has the product structure*

$$L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F})L^p(\mathcal{E}) = \{YX \mid Y \in L^0(\mathcal{F}),\, X \in L^p(\mathcal{E})\}. \tag{55}$$

Proof. $L^p(\mathcal{E})$ is a subspace of $L^p_{\mathcal{F}}(\mathcal{E})$. Hence, $\supset$ in (55) follows from property (i) in Definition 1.1.3 of $\|\cdot\|_p$. The reverse inclusion follows as every $X \in L^p_{\mathcal{F}}(\mathcal{E})$ is of the form

$$X = (1 + \|X\|_p)\frac{X}{1 + \|X\|_p} \in L^0(\mathcal{F})L^p(\mathcal{E}).$$

□

(55) suggests an alternative view of $L^p_{\mathcal{F}}(\mathcal{E})$, namely $L^p_{\mathcal{F}}(\mathcal{E})$ is the smallest $L^0(\mathcal{F})$–module containing the classical $L^p(\mathcal{E})$ space. That is, $L^p(\mathcal{E})$ generates $L^p_{\mathcal{F}}(\mathcal{E})$.

We work with the convention that the (generalized) conditional expectation $E[\cdot \mid \mathcal{F}] : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is understood as

$$E[X \mid \mathcal{F}] = E[X^+ \mid \mathcal{F}] - E[X^- \mid \mathcal{F}], \tag{56}$$

the right hand side of which is understood as in (53). Note that for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$, both $E[X^+ \mid \mathcal{F}]$ and $E[X^- \mid \mathcal{F}]$ are in $L^0(\mathcal{F})$ and consequently $E[X \mid \mathcal{F}] \in L^0(\mathcal{F})$. Further, note that $E[\cdot \mid \mathcal{F}] : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is $L^0(\mathcal{F})$–linear.

For $p \in (1,\infty)$ let $q = p/(p-1)$, if $p = 1$ let $q = \infty$ and if $p = \infty$ let $q = 1$. Then, for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $X' \in L^q_{\mathcal{F}}(\mathcal{E})$

$$\|XX'\|_1 \leq \|X\|_p \|X'\|_q. \tag{57}$$

Indeed, with the classical Hölder inequality for conditional expectations we know that

$$\|(X \wedge n)(X' \wedge n)\|_1 \leq \|X \wedge n\|_p \|X' \wedge n\|_q$$

for all $n \in \mathbb{N}$ and Beppo–Levi's monotone convergence theorem yields the assertion. Monotone convergence shows that (57) even holds for $p \in \bar{L}^0_+(\mathcal{F})$, $p \geq 1$, with $q = p/(p-1)$ and the conventions $q = 1$ and $q = \infty$ on the sets where $p = \infty$ and $p = 1$, respectively.

Theorem 3.2.2. *Let $p \in [1,\infty)$ and $q = p/(p-1)$ with $q = \infty$ for $p = 1$. Every continuous $L^0(\mathcal{F})$–linear function $\mu : (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ is of the form*

$$\mu X = E[ZX \mid \mathcal{F}] \tag{58}$$

for some $Z \in L^q_{\mathcal{F}}(\mathcal{E})$. Conversely, every function $\mu : (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ of the form (58) is continuous $L^0(\mathcal{F})$–linear.

Proof. For $Z \in L^q_{\mathcal{F}}(\mathcal{E})$, $E[Z\cdot \mid \mathcal{F}] : (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ is $L^0(\mathcal{F})$–linear and (57) guarantees that $E[ZX \mid \mathcal{F}] \in L^0(\mathcal{F})$ for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ as well as the required continuity.

Conversely, let $\mu : (L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|) \to (L^0(\mathcal{F}), |\cdot|)$ be continuous $L^0(\mathcal{F})$–linear. From Lemma 3.0.1 we know that there is $\varepsilon \in L^0_{++}(\mathcal{F})$ such that

$$|\mu X| \leq \varepsilon \|X\|_p \text{ for all } X \in L^p_{\mathcal{F}}(\mathcal{E})$$

and hence we can define

$$\widetilde{\mu} X = \frac{\mu X}{\varepsilon} \text{ for all } X \in L^p_{\mathcal{F}}(\mathcal{E}).$$

Then, for all $X' \in L^p(\mathcal{E})$, we have

$$E[|\widetilde{\mu} X'|^p] \leq E[E[|X'|^p \mid \mathcal{F}]].$$

Jensen's inequality further implies

$$|E[\widetilde{\mu} X']| \leq E[|X'|^p]^{1/p}$$

which means that $E \circ \widetilde{\mu} : L^p(\mathcal{E}) \to \mathbb{R}$ is continuous. The topological dual of $L^p(\mathcal{E})$ can be identified with $L^q(\mathcal{E})$. Hence, there is $Z' \in L^q(\mathcal{E})$ such that

$$E[\widetilde{\mu} X'] = E[Z'X'] \text{ for all } X' \in L^p(\mathcal{E}).$$

From $L^0(\mathcal{F})$–linearity of $\widetilde{\mu}$ we derive

$$\widetilde{\mu} X' = E[Z'X' \mid \mathcal{F}] \text{ for all } X' \in L^p(\mathcal{E})$$

which is equivalent to

$$\mu X' = E[Z'\varepsilon X' \mid \mathcal{F}] \text{ for all } X' \in L^p(\mathcal{E}).$$

(55) guarantees that $Z = Z'\varepsilon \in L^q_{\mathcal{F}}(\mathcal{E})$ and that every $X \in L^p_{\mathcal{F}}(\mathcal{E})$ is of the form $X = YX'$ for some $Y \in L^0(\mathcal{F})$, $X' \in L^p(\mathcal{E})$. From this we conclude

$$\mu X = Y\mu X' = YE[ZX' \mid \mathcal{F}] = E[ZX \mid \mathcal{F}].$$

□

THEOREM 3.2.3. *$(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|)$ is complete for all $p \in [1, \infty]$.*

PROOF. The proof is similar to that of Theorem 3.1.2. We sketch the main steps. Throughout, $\varepsilon, \varepsilon_0, \delta, \delta_0$ denote elements of $L^0_{++}(\mathcal{F})$. Let (X_ε) be Cauchy in $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$ for some $p \in [1, \infty]$. Since (X_ε) is Cauchy there is ε_0 such that $\|X_\varepsilon\|_p \leq 1 + \|X_{\varepsilon_0}\|_p = Y$ for all $\varepsilon \geq \varepsilon_0$. We derive $X_\varepsilon / Y \in L^p(\mathcal{E})$ for all $\varepsilon \geq \varepsilon_0$,

(i) $\frac{X_\varepsilon}{Y} \to X_0$ in $L^p(\mathcal{E})$ for sufficiently large ε and

(ii) $\left(\frac{X_\varepsilon}{Y}\right)$ is Cauchy in $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$.

We prove (i) and (ii) in case $p \in [1, \infty)$. The case of $p = \infty$ is analogue, only the notation is different. For every ε_0 there is δ_0 such that

$$\forall \varepsilon, \delta \geq \delta_0 : \| \frac{X_\varepsilon}{Y} - \frac{X_\delta}{Y} \|_p \leq \frac{1}{\varepsilon_0}$$

which implies that

$$\forall \varepsilon, \delta \geq \delta_0 : E\left[|\frac{X_\varepsilon}{Y} - \frac{X_\delta}{Y}|^p \right]^{1/p} \leq E\left[\left(\frac{1}{\varepsilon_0}\right)^p \right]^{1/p}.$$

Hence (X_ε / Y) is Cauchy in the Banach space $L^p(\mathcal{E})$ and converges to some X_0 in $L^p(\mathcal{E})$, which proves (i) and (ii).

In a second step one shows by way of contradiction that (X_ε / Y) converges to X_0 in $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|)$. The proof is identical to the respective part of the proof of Theorem 3.1.2. □

3.3. $L^0(\mathcal{F})$–modules of Orlicz type

A number of recent articles have a focus on convex risk measures defined on Orlicz spaces, cf. [**CL08, CL09**]. Orlicz spaces and Orlicz hearts share some useful properties with $L^p(\mathcal{E})$ spaces. For instance, they are Banach spaces and admit nice duality. However, they are technically more involving than the familiar $L^p(\mathcal{E})$ spaces and therefore require some motivation.

The reason to employ Orlicz theory is that the interior of the effective domain of many examples of convex risk measures defined on $L^p(\mathcal{E})$ spaces is empty. While it is still possible to establish Fenchel–Moreau type dual representation results for such risk measures, subdifferentiabilty results do not apply.

For example, the entropic risk measure is a lower semi continuous convex risk measure on $L^1(\mathcal{E})$, Fenchel–Moreau's theorem therefore applies, but the interior of its effective domain is empty. However, if the $L^p(\mathcal{E})$ space topology is replaced by a suitable and finer Orlicz space topology the interior of its effective domain is no longer empty. Subdifferentiability results then apply. A respective discussion is provided in Section 4.2.

The aim of this section is to present $L^0(\mathcal{F})$–modules of Orlizc type. We adopt the setup and notation of the previous section.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing convex function with $\varphi(0) = 0$. The respective Orlicz space is

$$L^{\varphi}(\mathcal{E}) = \{X \in L^0(\mathcal{E}) \mid \exists \lambda \in (0, \infty) :\ E[\varphi(|X/\lambda|)] < \infty\}.$$

Recall that $L^{\varphi}(\mathcal{E})$ endowed with the Luxemburg–norm $\|\cdot\| : L^{\varphi}(\mathcal{E}) \to \mathbb{R}_+$,

$$\|X\| = \inf\{\lambda \in (0, \infty) \mid E[\varphi(|X/\lambda|)] \le 1\}$$

is a Banach space.

We define a module analogue of the Orlicz space by

$$L^{\varphi}_{\mathcal{F}}(\mathcal{E}) = \{X \in L^0(\mathcal{E}) \mid \exists Y \in L^0_{++}(\mathcal{F}) : E[\varphi(|X/Y|) \mid \mathcal{F}] \in L^0(\mathcal{F})\}$$

and the map $\|\cdot\|_{\varphi} : L^0(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$,

$$\|X\|_{\varphi} = \text{ess.inf}\{Y \in L^0_{++}(\mathcal{F}) \mid E[\varphi(|X/Y|) \mid \mathcal{F}] \le 1\}$$

which generalizes the Luxemburg–norm. Then

$$L^{\varphi}_{\mathcal{F}}(\mathcal{E}) = \{X \in L^0(\mathcal{E}) \mid \|X\|_{\varphi} \in L^0(\mathcal{F})\}. \tag{59}$$

Indeed, the inclusion $\supset$ in (59) follows by definition. As to the reverse inclusion, let $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0_{++}(\mathcal{F})$ with $E[\varphi(|X/Y|) \mid \mathcal{F}] \in L^0(\mathcal{F})$. Then there is $Y' \in L^0(\mathcal{F})$, $Y' \ge 1$, such that $E[\varphi(|X/Y|) \mid \mathcal{F}]/Y' \le 1$. By convexity of φ,

$$E[\varphi(|X/(YY')|) \mid \mathcal{F}] \le E[\varphi(|X/Y|) \mid \mathcal{F}]/Y' \le 1$$

so that "$\subset$" in (59) follows.

Proposition 3.3.1. *$(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$ is an $L^0(\mathcal{F})$–normed module.*

Proof. First, we establish that $\|\cdot\|_{\varphi}$ is an $L^0(\mathcal{F})$–norm. To this end, observe that (59) implies that $\|\cdot\|_{\varphi}$ maps $L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ into $L^0_+(\mathcal{F})$.

To prove (iii) of Definition 1.1.3, let $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ and suppose $\|X\|_{\varphi} = 0$. Convexity of $\varphi(|\cdot|)$ yields

$$nE[\varphi(|X|) \mid \mathcal{F}] \le E[\varphi(n|X|) \mid \mathcal{F}] \le 1 \quad \text{for all } n \in \mathbb{N}$$

which implies $\varphi(|X|) = 0$, whence $X = 0$.

To prove (i) of Definition 1.1.3, let $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E}), Y' \in L^0(\mathcal{F})$. Then

$$\begin{aligned}
\|Y'X\|_{\varphi} &= \text{ess.inf}\{Y \in L^0_{++}(\mathcal{F}) \mid E[\varphi(|Y'X/Y|) \mid \mathcal{F}] \le 1\} \\
&= 1_{\{|Y'|>0\}}\text{ess.inf}\{|Y'|Y \mid Y \in L^0_{++}(\mathcal{F}), E[\varphi(|Y'X/(Y'Y)|) \mid \mathcal{F}] \le 1\} \\
&= |Y'|\|X\|_{\varphi}.
\end{aligned}$$

To prove (ii) of Definition 1.1.3, let $X_1, X_2 \in L^{\varphi}_{\mathcal{F}}(\mathcal{E}), Y_1, Y_2 \in L^0_{++}(\mathcal{F})$ with $E[\varphi(|X_1/Y_1|) \mid \mathcal{F}] \le 1$ and $E[\varphi(|X_2/Y_2|) \mid \mathcal{F}] \le 1$. Convexity of $\varphi(|\cdot|)$ yields

$$E\left[\varphi\left(|\frac{Y_1}{Y_1+Y_2}\frac{X_1}{Y_1} + \frac{Y_2}{Y_1+Y_2}\frac{X_2}{Y_2}|\right) \mid \mathcal{F}\right] \le 1,$$

whence $\|X_1 + X_2\|_{\varphi} \le Y_1 + Y_2$.

Finally, (59) and properties (i) and (ii) of $\|\cdot\|_{\varphi}$ imply that $L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ is an $L^0(\mathcal{F})$–module. $\square$

Proposition 3.3.2. *$L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ has the product structure $L^{\varphi}_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F})L^{\varphi}(\mathcal{E})$.*

PROOF. To show that $L^{\varphi}_{\mathcal{F}}(\mathcal{E}) \subset L^0(\mathcal{F})L^{\varphi}(\mathcal{E})$, let $X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0_{++}(\mathcal{F})$ such that $E[\varphi(X/Y) \mid \mathcal{F}] \in L^0(\mathcal{F})$. By convexity of $\varphi(|\cdot|)$,

$$E\left[\varphi\left(|\frac{X/Y}{1+E[\varphi(|X/Y|) \mid \mathcal{F}]}|\right)\right] \leq E\left[\frac{E[\varphi(|X/Y|) \mid \mathcal{F}]}{1+E[\varphi(|X/Y|) \mid \mathcal{F}]}\right] \leq 1. \tag{60}$$

This implies

$$X = (1+E[\varphi(|X/Y|) \mid \mathcal{F}])Y\frac{X/Y}{1+E[\varphi(|X/Y|) \mid \mathcal{F}]} \in L^0(\mathcal{F})L^{\varphi}(\mathcal{E}).$$

To establish the reverse inclusion, let $X' \in L^{\varphi}(\mathcal{E})$, $Y \in L^0(\mathcal{F})$ and $X = YX'$. By definition, there is $\lambda \in (0,\infty)$ such that $E[\varphi(|X'/\lambda|)] \leq 1$. Hence, $E[\varphi(|X'/\lambda|) \mid \mathcal{F}] \in L^0_+(\mathcal{F})$ and therefore by convexity of φ

$$E\left[\varphi\left(|\frac{X'/\lambda}{1+E[\varphi(|X'/\lambda|)]}|\right) \mid \mathcal{F}\right] \leq \frac{E[\varphi(|X'/\lambda|) \mid \mathcal{F}]}{1+E[\varphi(|X'/\lambda|] \mid \mathcal{F}]} \leq 1.$$

This shows that

$$\frac{1}{\lambda(1+E[\varphi(|X/\lambda|) \mid \mathcal{F}])}X' \in L^{\varphi}_{\mathcal{F}}(\mathcal{E}).$$

Since $L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ is an $L^0(\mathcal{F})$–module we derive $X' \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$ and in turn $YX' = X \in L^{\varphi}_{\mathcal{F}}(\mathcal{E})$. □

THEOREM 3.3.3. *$(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$ is complete.*

PROOF. Again, the proof is similar to that of Theorem 3.1.2 and we sketch the main steps. Throughout, $\varepsilon, \varepsilon_0, \delta, \delta_0$ denote elements of $L^0_{++}(\mathcal{F})$. Let (X_ε) be Cauchy in $(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$. Since (X_ε) is Cauchy there is ε_0 such that $\|X_\varepsilon\|_{\varphi} \leq 1+\|X_{\varepsilon_0}\|_{\varphi} = Y$ for all $\varepsilon \geq \varepsilon_0$. (60) yields $X_\varepsilon/Y \in L^{\varphi}(\mathcal{E})$ for all $\varepsilon \geq \varepsilon_0$. Moreover,

(i) $\frac{X_\varepsilon}{Y} \to X_0$ in $L^{\varphi}(\mathcal{E})$ for sufficiently large ε and
(ii) $\left(\frac{X_\varepsilon}{Y}\right)$ is Cauchy in $(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$.

Indeed, for every $\varepsilon_0 \in L^0_{++}(\mathcal{F})$ (in particular for $\varepsilon \equiv n$, $n \in \varepsilon$) there is δ_0 such that

$$\forall \varepsilon, \delta \geq \delta_0 : \|\frac{X_\varepsilon}{Y} - \frac{X_\delta}{Y}\|_{\varphi} \leq \frac{1}{\varepsilon_0}.$$

Hence, by definition of $\|\cdot\|_{\varphi}$,

$$\forall \varepsilon, \delta \geq \delta_0 : E\left[\varphi\left(|\frac{X_\varepsilon}{Y} - \frac{X_\delta}{Y}|\varepsilon_0\right)\right] \leq 1$$

Thus, (X_ε/Y) is Cauchy in the Banach space $L^{\varphi}(\mathcal{E})$ and converges to some $X_0 \in L^{\varphi}(\mathcal{E})$, which proves (i) and (ii).

Again, by way of contradiction one shows that (X_ε/Y) converges to X_0 in $(L^{\varphi}_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_{\varphi})$. The proof is identical to the respective part of the proof of Theorem 3.1.2. □

Part 2

Conditional risk measures

CHAPTER 4

Automatic continuity and subdifferentiability

In this chapter we present our results on automatic continutity and subdifferentiabilty of monotone $L^0(\mathcal{F})$–convex functions. These can be summarized as follows: if an $L^0(\mathcal{F})$–normed module E is complete and also a module lattice, then any proper monotone $L^0(\mathcal{F})$–convex function $f : E \to \bar{L}^0(\mathcal{F})$ is continuous and subdifferentiable throughout the interior of its effective domain.

The main results are presented in Section 4.1. The results can be viewed as module variants of the results in [**Nam57**] Section 5 and of [**RS06**] Proposition 3.1 and are suited to conditional risk measures as illustrated with Section 4.2. A crucial aspect in the proof of our results is that proper $L^0(\mathcal{F})$–convex functions $f : E \to \bar{L}^0(\mathcal{F})$ are local. This allows us to characterize continuity in terms of a specific class of nets rather than in terms of general abstract nets. This specific class of nets admits a one to one relation to sequences and therefore we can draw from classical results on continuity of monotone convex functions as established in [**Nam57**] and [**Bor87**]. This is outlined in Sections 4.3 and 4.4, where we establish the main results.

4.1. Main results

Let $(E, \|\cdot\|)$ be an $L^0(\mathcal{F})$–normed module. We define $\mathbb{N}(\mathcal{F}) = \{N \in L^0(\mathcal{F}) \mid N(\Omega) \subset \mathbb{N}\}$. The ball of radius $1/N$, $N \in \mathbb{N}(\mathcal{F})$, centered at $0 \in E$ is given by

$$B_{1/N} = \{X \in E \mid \|X\| \leq 1/N\}. \tag{61}$$

Further, we let $\mathcal{U} = \{B_{1/N} \mid N \in \mathbb{N}(\mathcal{F})\}$ be the collection of all these balls.

As in Section 1.1, we define a set $V \subset E$ to be a neighborhood of $X \in E$ if there is $U \in \mathcal{U}$ such that $X + U \subset V$. A set $V \subset E$ is open if it is a neighborhood of all $X \in V$. Inspection shows that the collection of all open subsets of E forms a topology on E.

Each $U \in \mathcal{U}$ is

$L^0(\mathcal{F})$–convex: $YX + (1-Y)X' \in U$ for all $X, X' \in U$ and $Y \in L^0(\mathcal{F}), 0 \leq Y \leq 1$,

$L^0(\mathcal{F})$–absorbent: for all $X \in E$ there exists $Y \in L^0_{++}(\mathcal{F})$ such that $X \in YU$,

$L^0(\mathcal{F})$–balanced: $YX \in U$ for all $X \in U$ and $Y \in L^0(\mathcal{F})$ with $|Y| \leq 1$.

In other words, the choice of the index set $\mathbb{N}(\mathcal{F})$ leads to a locally $L^0(\mathcal{F})$–convex topology on E. In fact, since for each $\varepsilon \in L^0_{++}(\mathcal{F})$ there is $N \in \mathbb{N}(\mathcal{F})$ such that $1/N \leq \varepsilon$ and vice versa, the locally $L^0(\mathcal{F})$–convex topology on E induced by the balls $(B_{1/N})$ is the same as the locally $L^0(\mathcal{F})$–convex topology on E induced by $\|\cdot\|$ as in Section 1.1. In the sequel we will always assume that $E = (E, \|\cdot\|)$ is endowed with this topology.

We recall the notation $\bar{K}$ and $\overset{\circ}{K}$ for the closure and interior of a set $K \subset E$ respectively. Unless stated otherwise, nets in E are denoted by (X_N), where N is an element of the index set $\mathbb{N}(\mathcal{F})$ which is directed by the order of almost sure dominance.

A net (X_N) is Cauchy in $(E, \|\cdot\|)$ if for all $N \in \mathbb{N}(\mathcal{F})$ there is $N_0 \in \mathbb{N}(\mathcal{F})$ such that for $M \geq N_0$, $\|X_M - X_{N_0}\| \leq 1/N$. $(E, \|\cdot\|)$ is complete if every Cauchy net has a limit.

$(E, \|\cdot\|)$ is Hausdorff. Indeed, let $X, X' \in E$ with $X \neq X'$. The neighborhood basis $\mathcal{U}$ of $0 \in E$ satisfies $\bigcap_{U \in \mathcal{U}} U = \{0\}$ so let $U \in \mathcal{U}$ such that $X - X' \notin U$. Since $(X, X') \mapsto X - X'$ is continuous, there exist $V, W \in \mathcal{U}$ such that $V - W \subset U$. Then $(X + V) \cap (X' + W) = \emptyset$, whence E is Hausdorff.

The $L^0(\mathcal{F})$–module E is referred to as ordered module if it is equipped with a partial order $\geq$ that is compatible with its algebraic structure. For $X, X' \in E$ we use the notation $X \leq X'$ in place of $X' \geq X$. Further, E is referred to as module lattice if it is an ordered module that is also a lattice.

The lattice operations are denoted by $X^+ = \sup\{X, 0\}$, $X^- = \sup\{-X, 0\}$, $|X| = X^+ + X^-$ so that $X = X^+ - X^-$ for all $X \in E$. Further, we let $E_+ = \{X \in E \mid X \geq 0\}$. The least upper bound, if it exists, of a set $C \subset E$ is denoted by $\sup_{X \in C} X$.

Recall the definition of the effective domain $\mathrm{dom} f = \{X \in E \mid f(X) < +\infty\} = \{X \in E \mid f(X) \in L^0(\mathcal{F})\}$ of a proper function $f : E \to \bar{L}^0(\mathcal{F})$. The function f is monotone if $X \leq X'$ in the module lattice E implies $f(X) \leq f(X')$ in the lattice ordered ring $L^0(\mathcal{F})$ or equivalently in the almost sure sense.

Definition 4.1.1. *Let $(E, \|\cdot\|)$ be an $L^0(\mathcal{F})$–normed module lattice. $\|\cdot\|$ is a lattice $L^0(\mathcal{F})$–norm if*

(i) *$\|X\| = \||X|\|$ for all $X \in E$,*
(ii) *$0 \leq X \leq X'$ implies $\|X\| \leq \|X'\|$ for all $X, X' \in E_+$.*

Remark 4.1.2. *If $(E, \|\cdot\|)$ is an $L^0(\mathcal{F})$–normed module lattice with lattice $L^0(\mathcal{F})$–norm $\|\cdot\|$, then the lattice operations $(E, \|\cdot\|) \to (E, \|\cdot\|), X \mapsto X^+, X \mapsto X^-, X \mapsto |X|$ are uniformly continuous. Indeed, observe for instance $|X^+ - X'^+| \leq |X - X'|$ for all $X, X' \in E$.*

We are now able to present the following two theorems as the main results of this chapter. The corresponding proofs are postponed to Sections 4.3 and 4.4.

The next theorem is a module variant of the main result of [**Nam57**] Section 5.

Theorem 4.1.3. *Let $(E, \|\cdot\|)$ be a complete $L^0(\mathcal{F})$–normed module lattice with lattice $L^0(\mathcal{F})$–norm $\|\cdot\|$. Any monotone convex local function $f : E \to L^0(\mathcal{F})$ is continuous.*

The next theorem is a generalization of [**RS06**] Proposition 3.1.

Theorem 4.1.4. *Let $(E, \|\cdot\|)$ be a complete $L^0(\mathcal{F})$–normed module lattice with lattice $L^0(\mathcal{F})$–norm $\|\cdot\|$. Any proper monotone $L^0(\mathcal{F})$–convex function $f : E \to \bar{L}^0(\mathcal{F})$ is continuous and subdifferentiable throughout $\overset{\circ}{\mathrm{dom}} f$.*

Remark 4.1.5. *In fact, in the proofs of Theorem 4.1.3 and Theorem 4.1.4 we establish a little more: let $d : E \times E \to L^0_+(\mathcal{F})$ satisfy* (i) *$d(X, X') = d(X', X)$ for all $X, X' \in E$,* (ii) *$d(X, X') = 0$ if and only if $X = X'$ and* (iii) *$d(X, Y) \leq$*

$d(X,Z)+d(Z,Y)$ for all $X,Y,Z\in E$. As in (61) we can define balls $\{X\in E\mid d(X,0)\le 1/N\}$, $N\in\mathbb{N}(\mathcal{F})$, in turn obtain a topological module which we denote by (E,d) and call (E,d) complete if every Cauchy net w.r.t. d has a limit. If (E,d) is complete, has a solid neighborhood base of 0, if $1_A d(X,X')=1_A d(1_A X,1_A X')$ for all $X,X'\in E$ and $A\in\mathcal{F}$ and if (E,d) is also a module lattice, then Theorem 4.1.3 and Theorem 4.1.4 remain valid on replacing $(E,\|\cdot\|)$ with (E,d). Such (E,d) can be considered a module analogue of a Fréchet lattice.

4.2. Examples

4.2.1. Conditional expected shortfall risk measure. In this subsection we consider the complete $L^0(\mathcal{F})$–normed module $(L^1_{\mathcal{F}}(\mathcal{E}),\|\cdot\|_1)$ as presented in Section 3.2.

If equipped with the order of almost sure dominance, $(L^1_{\mathcal{F}}(\mathcal{E}),\|\cdot\|_1)$ is an $L^0(\mathcal{F})$–normed module lattice. From its definition, we derive that $\|\cdot\|_1$ is a lattice $L^0(\mathcal{F})$–norm and hence the results of Section 4.1 apply.

Let $\lambda\in L^0(\mathcal{F})$, $0<\lambda<1$. The conditional expected shortfall $\rho: L^1_{\mathcal{F}}(\mathcal{E})\to L^0(\mathcal{F})$ at level λ is defined by

$$\rho(X)=\operatorname*{ess.sup}_{Z\in\mathcal{P}} E[ZX\mid\mathcal{F}], \tag{62}$$

where

$$\mathcal{P}=\left\{Z\in L^0(\mathcal{F})\mid Z\le 0, |Z|\le\frac{1}{\lambda}, E[Z\mid\mathcal{F}]=-1\right\}.$$

The conditional expected shortfall is antitone ($\rho(X)\le\rho(X')$ for all $X,X'\in L^1_{\mathcal{F}}(\mathcal{E})$ with $X\ge X'$) and $L^0(\mathcal{F})$–convex. In particular, ρ is convex local. By Theorem 4.1.3 (note that $\widetilde{\rho}(\cdot)=\rho(-\cdot)$ satisfies the respective assumptions), ρ is continuous. Further, by Theorem 4.1.4, $\partial\rho(X)\neq\emptyset$ for all $X\in L^1_{\mathcal{F}}(\mathcal{E})$.

LEMMA 4.2.1. *Let $Y_0\in L^0(\mathcal{F})$ and $f:(L^0(\mathcal{F}),|\cdot|)\to(L^0(\mathcal{F}),|\cdot|)$ be a continuous local function. If there are $Y_1,Y_2\in L^0(\mathcal{F})$ with $f(Y_1)<Y_0<f(Y_2)$, then there is $Y^*\in L^0(\mathcal{F})$ with $f(Y^*)=Y_0$.*

PROOF. Define

$$Y^*=\operatorname{ess.inf}\{Y\in L^0(\mathcal{F})\mid f(Y)\ge Y_0, Y\ge Y_1\}.$$

Then $Y^*\in L^0(\mathcal{F})$ and we claim that Y^* is as required. Indeed, assume by way of contradiction that $f(Y^*)>Y_0$ with positive probability and let $Y_N=(Y^*-1/N)\vee Y_1$, $N\in\mathbb{N}(\mathcal{F})$. Then $Y_N\to Y^*$. Further, $f(Y_N)<Y_0$; else, since f is local, we would derive a contradiction to the minimality of Y^*. Altogether, we derive $f(Y_N)<Y_0<f(Y^*)$ with positive probability in contradiction to the continuity of f as to which $Y_N\to Y^*$ implies $f(Y_N)\to f(Y^*)$. □

DEFINITION 4.2.2. *A random variable $X\in L^0(\mathcal{E})$ is conditionally continuously distributed if the map*

$$(L^0(\mathcal{F}),|\cdot|)\to(L^0(\mathcal{F}),|\cdot|), Y\mapsto E[1_{\{X\le Y\}}\mid\mathcal{F}],$$

is continuous.

If $X \in L^0(\mathcal{E})$ is conditionally continuously distributed, then Lemma 4.2.1 implies that there is $Y^* \in L^0(\mathcal{F})$ with $E[1_{\{X \leq Y^*\}} \mid \mathcal{F}] = \lambda$. Indeed, since

$$\lambda < 1 = \operatorname*{ess.sup}_{Y \in L^0(\mathcal{F})} E[1_{\{X \leq Y\}} \mid \mathcal{F}]$$
$$\lambda > 0 = \operatorname*{ess.inf}_{Y \in L^0(\mathcal{F})} E[1_{\{X \leq Y\}} \mid \mathcal{F}]$$

there are nets $(Y_N^-), (Y_N^+)$ with

$$E[1_{\{X \leq Y_N^-\}} \mid \mathcal{F}] < \lambda < E[1_{\{X \leq Y_N^+\}} \mid \mathcal{F}]$$

for sufficiently large $N \in \mathbb{N}(\mathcal{F})$ and therefore the assumptions of Lemma 4.2.1 are met. Such Y^* can be viewed as a conditional λ–quantile of X.

Proposition 4.2.3. *For all conditionally continuously distributed $X \in L^1_{\mathcal{F}}(\mathcal{E})$ and $\lambda \in L^0(\mathcal{F})$ with $0 < \lambda < 1$,*

$$-\frac{1}{\lambda} 1_{\{X \leq Y^*\}} \in \partial \rho(X),$$

where $Y^* \in L^0(\mathcal{F})$ is such that $E[1_{\{X \leq Y^*\}} \mid \mathcal{F}] = \lambda$.

Proof. Let $Y^* \in L^0(\mathcal{F})$ with $E[1_{\{X \leq Y^*\}} \mid \mathcal{F}] = \lambda$ and $Z^* = -(1/\lambda)1_{\{X \leq Y^*\}}$. Then,

$$E[Z^* \mid \mathcal{F}] = -(1/\lambda)E[1_{\{X \leq Y^*\}} \mid \mathcal{F}] = -1$$

and hence $Z \in \mathcal{P}$. Further, for all $Z \in \mathcal{P}$ we have $E[(Z^* - Z)Y^*] = 0$. Thus,

$$\begin{aligned} E[(Z^* - Z)X \mid \mathcal{F}] \quad &= \quad E[\underbrace{1_{\{X \leq Y^*\}}(Z^* - Z)(X - Y^*)}_{\geq 0} \mid \mathcal{F}] \\ &\quad + E[\underbrace{1_{\{X > Y^*\}}(Z^* - Z)(X - Y^*)}_{\geq 0} \mid \mathcal{F}] \geq 0. \end{aligned}$$

Hence, Z^* optimizes (62) and is therefore a subgradient of ρ at X. □

4.2.2. Conditional entropic risk measure. In this subsection we consider the complete $L^0(\mathcal{F})$–normed module $(L^\varphi_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_\varphi)$ as presented in Section 3.3, where we let $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(x) = \exp(x) - 1$.

If equipped with the order of almost sure dominance, $(L^\varphi_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_\varphi)$ is an $L^0(\mathcal{F})$–normed module lattice. From its definition, we derive that $\|\cdot\|_\varphi$ is a lattice $L^0(\mathcal{F})$–norm and hence the results of Section 4.1 apply.

We recall the definition of the entropic risk measure $\rho : L^\varphi_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ with risk aversion coefficient $\gamma \in (0, \infty)$,

$$\rho(X) = \frac{1}{\gamma} \log E[\exp(-\gamma X) \mid \mathcal{F}].$$

We have already seen in Section 2.2 that ρ admits Fenchel–Moreau type dual representations if considered as a function on $L^p_{\mathcal{F}}(\mathcal{E})$, $p \in [1, \infty]$.

To establish subdifferentiability results, we consider ρ as a function on the Orlicz type module $L^\varphi_{\mathcal{F}}(\mathcal{E})$ and define

$$\chi = \{X \in L^\varphi_{\mathcal{F}}(\mathcal{E}) \mid \exists \varepsilon \in L^0_{++}(\mathcal{F}) : E[\exp((1 + \varepsilon)\gamma X^-) \mid \mathcal{F}] \in L^0(\mathcal{F})\}.$$

A variant of the following lemma for static convex risk measures is established in [**BF07**].

Lemma 4.2.4. $\chi \subset \overset{\circ}{\operatorname{dom}}\, \rho$.

Proof. Let $X \in \chi$ with $E[\exp((1+\varepsilon)\gamma X^-) \mid \mathcal{F}] \in L^0(\mathcal{F})$ for some $\varepsilon \in L^0_{++}(\mathcal{F})$. Let $p = (1+\varepsilon)/(1+\varepsilon/2)$ with conjugate $q = p/(p-1) = 2(1+\varepsilon)/\varepsilon$. Since there is $N \in \mathbb{N}(\mathcal{F})$ with $1/N \leq 1/((1+\varepsilon/2)\gamma q)$ it suffices to show that $X + B_{1/((1+\varepsilon/2)\gamma q)} \subset \chi$, where $B_{1/((1+\varepsilon/2)\gamma q)} = \{Y \in L^\varphi_{\mathcal{F}}(\mathcal{E}) \mid \|Y\|_\varphi \leq 1/((1+\varepsilon/2)\gamma q)\}$.

To this end, fix $Y \in B_{1/((1+\varepsilon/2)\gamma q)}$. By definition,

$$E[\exp((1+\varepsilon/2)\gamma q|Y|) \mid \mathcal{F}] \leq E[\exp(|Y|/\|Y\|_\varphi) \mid \mathcal{F}] \leq 1.$$

Hence, Hölder's inequality yields

$$\begin{aligned} & E[\exp((1+\varepsilon/2)\gamma(X+Y)^-) \mid \mathcal{F}] \\ \leq\ & E[\exp((1+\varepsilon/2)\gamma X^-)\exp((1+\varepsilon/2)\gamma Y^-) \mid \mathcal{F}] \\ \leq\ & E[\exp((1+\varepsilon/2)\gamma X^-)^p \mid \mathcal{F}]^{1/p} E[\exp((1+\varepsilon/2)\gamma Y^-)^q \mid \mathcal{F}]^{1/q} \\ \leq\ & E[\exp((1+\varepsilon)\gamma X^-) \mid \mathcal{F}]^{1/p} E[\exp((1+\varepsilon/2)\gamma q|Y|) \mid \mathcal{F}]^{1/q} \\ \leq\ & E[\exp((1+\varepsilon)\gamma X^-) \mid \mathcal{F}]^{1/p} \in L^0(\mathcal{F}). \end{aligned}$$

This shows that χ is open in $L^\varphi_{\mathcal{F}}(\mathcal{E})$. Moreover, for all $X \in \chi$,

$$E[\exp(-\gamma X) \mid \mathcal{F}] \leq 1 + E[\exp(\gamma X^-) \mid \mathcal{F}] \in L^0(\mathcal{F}).$$

Hence, $\chi \subset \operatorname{dom}\rho$. We conclude $\chi \subset \mathring{\chi} \subset \mathring{\operatorname{dom}}\rho$. □

The entropic risk measure ρ is proper antitone $L^0(\mathcal{F})$–convex and hence continuous and subdifferentiable throughout $\mathring{\operatorname{dom}}\rho$, in particular throughout χ, by Theorem 4.1.4.

Lemma 4.2.5. *Let $X_0 \in \chi$ and $Z_0 = \exp(-\gamma X_0)/E[\exp(-\gamma X_0) \mid \mathcal{F}]$. Then, $\mu : L^\varphi_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$, $\mu(X) = E[(-Z_0)X \mid \mathcal{F}]$, defines a continuous $L^0(\mathcal{F})$–linear function.*

Proof. First, we show that μ maps $L^\varphi_{\mathcal{F}}(\mathcal{E})$ into $L^0(\mathcal{F})$. To this end, let $\varepsilon \in L^0_{++}(\mathcal{F})$ such that $E[\exp(-\gamma(1+\varepsilon)X_0^-) \mid \mathcal{F}] \in L^0(\mathcal{F})$ which exists since $X_0 \in \chi$. Define $p = 1+\varepsilon$ with conjugate $q = (1+\varepsilon)/\varepsilon$. There exists $c \in \mathbb{N}(\mathcal{F})$ such that $X^q \leq c(1+\exp(X))$ for all $X \in L^0_+(\mathcal{E})$ which shows that $L^\varphi_{\mathcal{F}}(\mathcal{E}) \subset L^q_{\mathcal{F}}(\mathcal{E})$. Hence, Hölder's inequality yields for all $X \in L^\varphi_{\mathcal{F}}(\mathcal{E})$

$$|\mu(X)| \leq (1 + E[\exp(-\gamma(1+\varepsilon)X_0^-) \mid \mathcal{F}]^{1/p}) E[|X|^q \mid \mathcal{F}]^{1/q} \in L^0(\mathcal{F}).$$

Further, μ is antitone $L^0(\mathcal{F})$–linear by definition and hence, in particular, antitone convex local. Thus, μ is continuous $L^0(\mathcal{F})$–linear by Theorem 4.1.3. □

Proposition 4.2.6. *Let X_0, Z_0 and μ be as in Lemma 4.2.5. Then, $\mu \in \partial\rho(X_0)$.*

Proof. On adapting the proof of Lemma 3.29 in [**FS04**], we derive for all $X \in \operatorname{dom}\rho$

$$\begin{aligned} \frac{1}{\gamma}\log E[\exp(-\gamma X) \mid \mathcal{F}] &\geq E[-Z_0 X \mid \mathcal{F}] - \frac{1}{\gamma}E[Z_0 \log Z_0 \mid \mathcal{F}] \\ \frac{1}{\gamma}\log E[\exp(-\gamma X_0) \mid \mathcal{F}] &= E[-Z_0 X_0 \mid \mathcal{F}] - \frac{1}{\gamma}E[Z_0 \log Z_0 \mid \mathcal{F}]. \end{aligned}$$

Hence, for any $X \in L^\varphi_{\mathcal{F}}(\mathcal{E})$

$$\frac{1}{\gamma}\log E[\exp(-\gamma X) \mid \mathcal{F}] - \frac{1}{\gamma}\log E[\exp(-\gamma X_0) \mid \mathcal{F}] \geq E[(-Z_0)(X - X_0) \mid \mathcal{F}].$$

Thus, $\mu \in \partial\rho(X_0)$. $\square$

4.3. Proof of Theorem 4.1.3

Let E be an $L^0(\mathcal{F})$–module and (X_N) be a net in E. (X_N) is directed if

$$1_A X_N + 1_{A^c} X_{N'} = X_{1_A N + 1_{A^c} N'} \tag{63}$$

for all $N, N' \in \mathbb{N}(\mathcal{F})$ and $A \in \mathcal{F}$. If (X_N) is directed so is every subnet. We denote by (X_n) the sequence obtained via $X_n = X_N$ for the constant map $N \equiv n$, $n \in \mathbb{N}$. With (X_n) we define a net $(\widetilde{X}_N)$ by

$$\widetilde{X}_N(\omega) = X_{N(\omega)}(\omega) \text{ for all } \omega \in \Omega \text{ and } N \in \mathbb{N}(\mathcal{F}). \tag{64}$$

If (X_N) is directed then it is recovered by (64), that is, $X_N = \widetilde{X}_N$ for all $N \in \mathbb{N}(\mathcal{F})$. In this sense, directed nets correspond to sequences.

Recall the definition in (6) of $M(\cdot \mid C) : E \to \mathcal{F}$,

$$M(X \mid C) = \text{ess.sup}\{A \in \mathcal{F} \mid 1_A X \in C\}.$$

Lemma 4.3.1. *Let $(E, \|\cdot\|)$ be an $L^0(\mathcal{F})$–normed module, $f : E \to \bar{L}^0(\mathcal{F})$ a local function and $X_0 \in \text{dom} f$. Equivalent are:*

(i) *f is continuous (w.r.t. $\|\cdot\|$ and $|\cdot|$) at X_0.*
(ii) *For all directed nets (X_N) in E, $X_N \to X_0$ implies $f(X_N) \to f(X_0)$.*

Proof. We only have to prove that (ii) implies (i). On replacing f by $f(\cdot + X_0) - f(X_0)$ we may assume that $X_0 = f(X_0) = 0$ (leaving hypotheses and conclusion unchanged). Let $N_0 \in \mathbb{N}(\mathcal{F})$. We have to show that $f^{-1}(B_{1/N_0})$ is a neighborhood of $0 \in E$. Define

$$\varepsilon^* = \text{ess.sup}\{\varepsilon \in L^0_+(\mathcal{F}) \mid B_\varepsilon \subset f^{-1}(B_{1/N_0})\} \wedge 1,$$

where

$$B_\varepsilon = \{X \in E \mid \|X\| \le \varepsilon\}.$$

It suffices to show that $\varepsilon^* \in L^0_{++}(\mathcal{F})$. Indeed, since f is local, $\mathcal{G} = \{\varepsilon \in L^0_+(\mathcal{F}) \mid B_\varepsilon \subset f^{-1}(B_{1/N_0})\}$ is directed upwards. Hence, there is an increasing sequence $(\varepsilon_n) \subset \mathcal{G}$ with $\varepsilon_n \nearrow \varepsilon^*$ a.s. Thus, $A_1 = \{\varepsilon_1 > 0\}$, $A_n = \{\varepsilon_n > 0\} \setminus A_{n-1}$, $n \ge 2$, satisfies $\bigcup_{n\in\mathbb{N}} A_n = \Omega$ as $\varepsilon^* > 0$. For $N \in \mathbb{N}(\mathcal{F})$ such that

$$\frac{1}{N} \le \sum_{n\in\mathbb{N}} 1_{A_n}\varepsilon_n$$

we see that $B_{1/N} \subset f^{-1}(B_{1/N_0})$ since f is local. Thus, f is continuous at $0 \in E$ if $\varepsilon^* > 0$.

By way of contradiction let us assume that $P[A] > 0$, where $A = \{\varepsilon^* = 0\}$. Fix $n_0 \in \mathbb{N}$. We will show that there is $\bar{X}_{n_0} \in E$ with $\bar{X}_{n_0} \in B_{1/n_0}$ and

$$1_B \bar{X}_{n_0} \notin 1_B f^{-1}(B_{1/N_0}) \text{ for all } B \in \mathcal{F} \text{ with } B \subset A \text{ and } P[B] > 0. \tag{65}$$

To this end, consider the collection $\mathcal{D}$ of all sets $B \in \mathcal{F}$ such that there is $X \in E$ with $X \in B_{1/n_0}$ and

$$1_{B'} X \notin 1_{B'} f^{-1}(B_{1/N_0}) \text{ for all } B' \in \mathcal{F} \text{ with } B' \subset B \text{ and } P[B'] > 0. \tag{66}$$

We claim that $A \subset \text{ess.sup}\mathcal{D}$. Indeed, assume that $P[C] > 0$, $C = A \setminus \text{ess.sup}\mathcal{D}$. Since $C \subset A$ and in turn $\varepsilon^* = 0$ on C we derive from the maximality of ε^* that $1_C B_{1/n_0}$ cannot be a subset of $1_C f^{-1}(B_{1/N_0})$. Hence, take $Z \in 1_C B_{1/n_0} \setminus$

$1_C f^{-1}(B_{1/N_0})$ and observe that $P[C'] > 0$, $C' = C \setminus M(Z \mid f^{-1}(B_{1/N_0}))$, since f is local. But this contradicts the maximality of ess.sup$\mathcal{D}$ since

$$1_{B'} Z \notin 1_{B'} f^{-1}(B_{1/N_0}) \text{ for all } B' \in \mathcal{F} \text{ with } B' \subset C' \text{ and } P[B'] > 0.$$

Thus, $A \subset$ ess.sup$\mathcal{D}$. Since the collection $\mathcal{D}$ is directed upwards there exists an increasing sequence (C_n) in $\mathcal{D}$ with $C_n \nearrow$ ess.sup$\mathcal{D}$ and a corresponding sequence (X_n) satisfying (66). Then

$$\bar{X}_{n_0} = \sum_{n \in \mathbb{N}} 1_{C_n} X_n$$

is an element of B_{1/n_0} and satisfies (65), that is, $\bar{X}_{n_0}$ is as required.

Finally, we proceed in the same manner as in (64) and construct a directed net $(\bar{X}_N)$ by means of the sequence $(\bar{X}_n)$ we just constructed and we observe that

$$\bar{X}_N \to 0 \text{ but still}$$
$$1_B f(X_N) \notin 1_B B_{1/N_0}(Y)$$

for all $N \in \mathbb{N}(\mathcal{F})$ and $B \in \mathcal{F}$ with $B \subset A$ and $P[B] > 0$ which contradicts (ii) and concludes the proof. $\square$

The next lemma is a module variant of Theorem 8.43 in [**AB06**].

LEMMA 4.3.2. *Let $(E, \|\cdot\|)$ be an $L^0(\mathcal{F})$–normed module lattice with lattice $L^0(\mathcal{F})$–norm $\|\cdot\|$.*

(i) *E_+ is closed in $(E, \|\cdot\|)$.*
(ii) *If (X_N) is a net in E with $X_N \le X_M$ for all $N \le M$ and $X_N \to X$ for some $X \in E$, then $\sup_{N \in \mathbb{N}(\mathcal{F})} X_N = X$.*

PROOF. (i) Since $X = X^+ - X^-$ for all $X \in E$, we see that $E_+ = \{X \in E \mid X^- = 0\}$. In other words, E_+ is the pre–image of $\{0\}$ under the continuous lattice operation $(E, \|\cdot\|) \to (E, \|\cdot\|), X \mapsto X^-$, cf. Remark 4.1.2. Since $(E, \|\cdot\|)$ is Hausdorff $\{0\}$ is closed, and the assertion follows.

(ii) Throughout, N, M denote elements of $\mathbb{N}(\mathcal{F})$. Let (X_N) in E with $X_N \le X_M$ for all $N \le M$ and $X_N \to X$ for some $X \in E$. Since $X_M - X_N \in E_+$ for all $M \ge N$, we see that for all N the net $(X_M - X_N)_{M \ge N}$ in E_+ satisfies $X_M - X_N \to X - X_N$. By (i), E_+ is closed, hence $X - X_N \in E_+$ for all N. Thus, X is an upper bound of the net (X_N). To see that X is the least upper bound of (X_N) take $X' \in E$ with $X' \ge X_N$ for all N. Then, $X' - X_N \in E_+$ for all N and $X' - X_N \to X' - X$ imply $X' - X \in E_+$, whence $X' \ge X$. $\square$

We can now prove Theorem 4.1.3. The proof follows a known pattern; cf. Theorem 9.6 in [**AB06**].

PROOF. Let $f : E \to L^0(\mathcal{F})$ be a monotone convex local function and $X_0 \in E$. On replacing f by $f(\cdot + X_0) - f(X_0)$ we may assume that $X_0 = 0$ and $f(X_0) = 0$. By way of contradiction, assume that there exists a directed (cf. Lemma 4.3.1) net (X_N) with $X_N \to 0$ in $(E, \|\cdot\|)$ such that $f(X_N)$ does not converge to 0 in $(L^0(\mathcal{F}), |\cdot|)$. By passing to a subnet of (X_N), we can assume that $f(X_N) \notin W$ for all $N \in \mathbb{N}(\mathcal{F})$ and some neighborhood W of 0 in $(L^0(\mathcal{F}), |\cdot|)$. Consider the neighborhood base of 0 in $(E, \|\cdot\|)$ consisting of all neighborhoods of the form

$$V_N = B_{1/2^N}, \text{ where } N \in \mathbb{N}(\mathcal{F}).$$

Then $V_{N+1}+V_{N+1}\subset V_N$ for all $N\in\mathbb{N}(\mathcal{F})$. Again, by passing to a subnet of (X_N) we can assume that $NX_N\in V_N$ for all $N\in\mathbb{N}(\mathcal{F})$. Next, for each $N\in\mathbb{N}(\mathcal{F})$ define

$$Y_N=\sum_{n\in\mathbb{N}}1_{\{N=n\}}\sum_{i=1}^{n}i|X_i|,$$

where X_i denotes X_N for $N\equiv i$, $i\in\mathbb{N}$. For all $N,M\in\mathbb{N}(\mathcal{F})$, $A=\{N=n,M=m\}$ and $n,m\in\mathbb{N}$ we observe

$$1_A(Y_{N+M}-Y_N)=1_A\sum_{i=n+1}^{n+m}i|X_i|\in 1_A(V_{n+1}+V_{n+2}+\cdots+V_{n+m})\subset 1_AV_n.$$

Hence, $Y_{N+M}-Y_N\in V_N$ for all $N,M\in\mathbb{N}(\mathcal{F})$, that is (Y_N) is Cauchy and so $Y_N\to Y$ for some $Y\in E$. By construction, $0\le Y_N\le Y_M$ for all $N\le M$, $N,M\in\mathbb{N}(\mathcal{F})$. Hence, (ii) of Lemma 4.3.2 implies $Y=\sup_{N\in\mathbb{N}(\mathcal{F})}Y_N$. Monotonicity of f, convexity of f and $f(0)=0$ show for all $n\in\mathbb{N}$

$$|f(X_n)|\le f(|X_n|)\le\frac{1}{n}f(n|X_n|)\le\frac{1}{n}f(Y_n)\le\frac{1}{n}f(Y).$$

Moreover, since f is local we derive for all $N\in\mathbb{N}(\mathcal{F})$

$$|f(X_N)|\le\frac{1}{N}f(Y).$$

This shows that $f(X_N)\to 0$ in $(L^0(\mathcal{F}),|\cdot|)$ in contradiction to $f(X_N)\notin W$, whence the required continuity. $\square$

4.4. Proof of Theorem 4.1.4

To establish Theorem 4.1.4 we follow the ideas of [**RS06**]; however, the proofs are more technical as we work with $L^0(\mathcal{F})$–modules and we have to establish a variety of preliminary results on the way.

Let E be an $L^0(\mathcal{F})$–module. We recall that a function $p:E\to L^0(\mathcal{F})$ is $L^0(\mathcal{F})$–positively homogeneous if $p(YX)=Yp(X)$ for all $X\in E$ and $Y\in L^0_+(\mathcal{F})$. p is subadditive if $p(X+X')\le p(X)+p(X')$ for all $X,X'\in E$. p is $L^0(\mathcal{F})$–sublinear if it is $L^0(\mathcal{F})$–positive homogeneous and subadditive.

LEMMA 4.4.1. *Let E be an $L^0(\mathcal{F})$–normed module, $f:E\to\bar{L}^0(\mathcal{F})$ be a proper $L^0(\mathcal{F})$–convex function and $X_0\in\overset{\circ}{\mathrm{dom}}f$. Then, the directional derivative $Df(X_0):E\to\bar{L}^0(\mathcal{F})$,*

$$Df(X_0)(X)=\underset{Y\in L^0_{++}(\mathcal{F})}{\mathrm{ess.inf}}\frac{f(X_0+YX)-f(X_0)}{Y},\tag{67}$$

of f at X_0 satisfies:

(i) *For all $X\in E$ and $Y,Y'\in L^0_{++}(\mathcal{F})$ with $Y\le Y'$*
$$\frac{f(X_0+YX)-f(X_0)}{Y}\le\frac{f(X_0+Y'X)-f(X_0)}{Y'}.$$
In particular, the essential infimum in (67) can be taken over all $Y\in L^0_{++}(\mathcal{F})$ with $Y\le 1$.

(ii) *$Df(X_0)$ is finite valued, that is $Df(X_0)(X)\in L^0(\mathcal{F})$ for all $X\in E$.*

(iii) *$Df(X_0)$ is $L^0(\mathcal{F})$–convex.*

(iv) *$Df(X_0)$ is $L^0(\mathcal{F})$–positively homogeneous.*

(v) *$Df(X_0)$ satisfies the subgradient inequality, that is, $Df(X_0)(X-X_0) \leq f(X)-f(X_0)$ for all $X \in E$.*

In particular, $Df(X_0): E \to L^0(\mathcal{F})$ is $L^0(\mathcal{F})$–sublinear.

Proof. Throughout, Y, Y' denote elements of $L^0_{++}(\mathcal{F})$ and $X, X' \in E$.

(i) We have $X_0 + YX = \frac{Y}{Y'}(X_0 + Y'X) + \left(1 - \frac{Y}{Y'}\right) X_0$ so that for $L^0(\mathcal{F})$–convex f

$$f\left(X_0 + YX\right) \leq \frac{Y}{Y'} f\left(X_0 + Y'X\right) + \left(1 - \frac{Y}{Y'}\right) f(X_0)$$

for all $Y \leq Y'$. Now, divide by Y and rearrange.

(ii) In the same manner as in the proof of Lemma 5.41 in [**AB06**] we derive for $L^0(\mathcal{F})$–convex f and all $Y \leq 1$

$$|f(X_0 + YZ) - f(X_0)| \leq Y \max\{f(X_0 + Z) - f(X_0), f(X_0 - Z) - f(X_0)\}.$$

whenever $Z \in E$ is such that $X_0 + Z, X_0 - Z \in \operatorname{dom} f$. Since $X_0 \in \overset{\circ}{\operatorname{dom}} f$ there is Y' such that $X_0 + Y'X, X_0 - Y'X \in \operatorname{dom} f$. The assertion now follows from (i).

(iii) By $L^0(\mathcal{F})$–convexity of f

$$\begin{aligned} & \frac{f(X_0 + Y(ZX + (1-Z)X')) - f(X_0)}{Y} \\ \leq \quad & Z \frac{f(X_0 + YX) - f(X_0)}{Y} + (1-Z) \frac{f(X_0 + YX') - f(X_0)}{Y} \end{aligned}$$

for all Y and $Z \in L^0_+(\mathcal{F})$ with $0 \leq Z \leq 1$.

(iv) Let $Z \in L^0_+(\mathcal{F})$. Since f is $L^0(\mathcal{F})$–convex f is local. With the convention $0/0 = 0$ we derive for local f and for all Y

$$\frac{f(X_0 + YZX) - f(X_0)}{Y} = Z \frac{f(X_0 + YZX) - f(X_0)}{ZY}.$$

(v) For all $Y \leq 1$ we have $X_0 + Y(X - X_0) = YX + (1-Y)X_0$. Thus, by $L^0(\mathcal{F})$–convexity

$$\frac{f(X_0 + Y(X - X_0)) - f(X_0)}{Y} \leq f(X) - f(X_0).$$

In view of (i), this yields the assertion. □

The next lemma is a module variant of Lemma 1.1 in [**Lev85**]. A pre stage of Lemma 4.4.2 which, however, only addresses algebraic subdifferentiability of $L^0(\mathcal{F})$–sublinear functions is established in [**Kut79, Kut80, Kut81**].

Lemma 4.4.2. *Let E be an $L^0(\mathcal{F})$–normed module. Any proper $L^0(\mathcal{F})$–convex function $f: E \to \bar{L}^0(\mathcal{F})$ is algebraically subdifferentiable throughout $\overset{\circ}{\operatorname{dom}} f$, that is, for each $X_0 \in \overset{\circ}{\operatorname{dom}} f$ there is an $L^0(\mathcal{F})$–linear function $\mu: E \to L^0(\mathcal{F})$ such that*

$$\mu(X - X_0) \leq f(X) - f(X_0) \text{ for all } X \in E.$$

Proof. Let $X_0 \in \overset{\circ}{\operatorname{dom}} f$ and denote $p = Df(X_0)$ the directional derivative of f at X_0. By Lemma 4.4.1, $p: E \to L^0(\mathcal{F})$ is $L^0(\mathcal{F})$–sublinear. We can assume that there is $0 \neq X \in E$. The function $\mu: \operatorname{span}_{L^0(\mathcal{F})}(X) \to L^0(\mathcal{F})$, $\mu(YX) = Yp(X)$ is well defined. Indeed let $Y, Y' \in L^0(\mathcal{F})$ with $YX = Y'X$ and define $A = \{\|X\| = 0\}$. Then, $(Y - Y')X = 0$ and hence

$$0 = \|(Y - Y')X\| = |Y - Y'|\|X\|.$$

Thus, $Y = Y'$ on A^c. Further, $0 = 1_A\|X\| = \|1_A X\|$ and hence $1_A X = 0$. Altogether this yields $Yp(X) = Y'p(X)$ as p is local.

Moreover, $\mu(YX) \leq p(YX)$ for all $Y \in L^0(\mathcal{F})$. Indeed, for $Y \in L^0_+(\mathcal{F})$ this follows from $L^0(\mathcal{F})$–positive homogeneity of p and the definition of μ. For arbitrary $Y \in L^0(\mathcal{F})$ this follows as p is local and $-p(-X') \leq p(X')$ for all $X' \in E$. Thus, $L^0(\mathcal{F})$–sublinear p dominates μ on the $L^0(\mathcal{F})$–submodule generated by X in E and by the Hahn–Banach extension theorem, cf. Theorem 1.2.6, there is an $L^0(\mathcal{F})$–linear function $\bar{\mu} : E \to L^0(\mathcal{F})$ with $\bar{\mu}(X') \leq p(X')$ for all $X' \in E$. By (v) of Lemma 4.4.1, $\bar{\mu}$ is as required. $\square$

Lemma 4.4.3 below is a module variant of Theorem 3.3 in [**Phe89**]. Its proof draws from results of Chapter 2. Therefore, we adopt the respective notation and denote by (X_α) arbitrary nets in $L^0(\mathcal{F})$. Further, we recall the definition of the essential limit inferior $\text{ess.liminf}_\alpha X_\alpha$ of a net (X_α) in $L^0(\mathcal{F})$,

$$\operatorname*{ess.liminf}_{\alpha} X_\alpha = \operatorname*{ess.sup}_{\alpha} \operatorname*{ess.inf}_{\beta \geq \alpha} X_\beta.$$

Lemma 4.4.3. *Let E be an $L^0(\mathcal{F})$–normed module. If a proper $L^0(\mathcal{F})$–convex function $f : E \to \bar{L}^0(\mathcal{F})$ satisfies*

$$f(X) \leq \operatorname*{ess.liminf}_{\alpha} f(X_\alpha) \tag{68}$$

for all nets (X_α) with $X_\alpha \to X$ for some $X \in \overset{\circ}{\text{dom}} f$, then f is continuous throughout $\overset{\circ}{\text{dom}} f$.

Proof. Let $X_0 \in \overset{\circ}{\text{dom}} f$. On replacing f by $f(\cdot + X_0)$ we can assume that $X_0 = 0$. We will show that f is continuous at 0.

Since $0 \in \overset{\circ}{\text{dom}} f$ let $N \in \mathbb{N}(\mathcal{F})$ such that $B_{1/N} \subset \overset{\circ}{\text{dom}} f$. We define $\widetilde{f} : E \to \bar{L}^0(\mathcal{F})$,

$$\widetilde{f}(X) = f(X) + \infty 1_{\{\|X\| > 1/N\}}.$$

Note that $1_{\{\|X\| \leq 1/N\}} X \in B_{1/N}$. Then, $\widetilde{f}$ coincides with f on the neighborhood $B_{1/N}$ of $0 \in E$. Therefore, it suffices to show that $\widetilde{f}$ is continuous at 0. Indeed, if (X_α) is a net in E with $X_\alpha \to 0$, then eventually $X_\alpha \in B_{1/N}$ for large α. Continuity of $\widetilde{f}$ then shows $f(X_\alpha) = \widetilde{f}(X_\alpha) \to \widetilde{f}(0) = f(0)$.

To establish continuity of $\widetilde{f}$ we show that $\widetilde{f}$ is proper $L^0(\mathcal{F})$–convex (hence local) and satisfies

$$\widetilde{f}(X) \leq \operatorname*{ess.liminf}_{\alpha} \widetilde{f}(X_\alpha) \tag{69}$$

for all nets (X_α) with $X_\alpha \to X$ for some $X \in E$. We then apply results of Chapter 2. Properness follows from that of f. For $L^0(\mathcal{F})$–convexity, let $X, X' \in E$, $Y \in L^0_+(\mathcal{F})$, $0 \leq Y \leq 1$ and observe on $A = \{\|X\| > 1/N\} \cup \{\|X'\| > 1/N\}$

$$\widetilde{f}(YX + (1-Y)X') \leq Y\widetilde{f}(X) + (1-Y)\widetilde{f}(X'),$$

as the righthand side is ∞ on A. Since $A^c = \{\|X\| \leq 1/N\} \cap \{\|X'\| \leq 1/N\} \subset \{\|YX + (1-Y)X'\| \leq 1/N\}$ the required inequality on A^c follows from $L^0(\mathcal{F})$–convexity of f. To establish (69) let $(X_\alpha) \subset E$ with $X_\alpha \to X$, $X \in E$. Define

$\widetilde{X}_\alpha = 1_{\{\|X\|\leq 1/N\}} X_\alpha$. Then $\widetilde{X}_\alpha \to 1_{\{\|X\|\leq 1/N\}} X \in B_{1/N} \subset \overset{\circ}{\operatorname{dom}} f$. Hence,

$$\operatorname{ess.liminf}_\alpha \widetilde{f}(\widetilde{X}_\alpha) \geq \operatorname{ess.liminf}_\alpha f(\widetilde{X}_\alpha) \tag{70}$$

$$\geq f(1_{\{\|X\|\leq 1/N\}} X), \tag{71}$$

where (70) is by definition of $\widetilde{f}$ and (71) is by (68). Multiplying by $1_{\{\|X\|\leq B_{1/N}\}}$ yields (for local $\widetilde{f}$)

$$1_{\{\|X\|\leq 1/N\}} \operatorname{ess.liminf}_\alpha \widetilde{f}(\widetilde{X}_\alpha) \geq 1_{\{\|X\|\leq 1/N\}} \widetilde{f}(X).$$

Since $X_\alpha \to X$ we have $\|X_\alpha\| > 1/N$ on $\{\|X\| > 1/N\}$ for sufficiently large α. Thus, $\widetilde{f}(X_\alpha) = \infty = \widetilde{f}(X)$ on $\{\|X\| > 1/N\}$ for sufficiently large α, that is, we established (69).

Altogether, $\widetilde{f} : E \to \bar{L}^0(\mathcal{F})$ is proper $L^0(\mathcal{F})$–convex and satisfies (69). By Lemma (i), $\widetilde{f}$ is continuous throughout $\overset{\circ}{\operatorname{dom}} \widetilde{f}$. In particular, $\widetilde{f}$ is continuous at 0. Thus, f is so. □

We can now prove Theorem 4.1.4.

Proof. Let $X_0 \in \overset{\circ}{\operatorname{dom}} f$. By Lemma 4.4.2, there is $L^0(\mathcal{F})$–linear $\mu : E \to L^0(\mathcal{F})$ such that

$$\mu(X - X_0) \leq f(X) - f(X_0) \text{ for all } X \in E. \tag{72}$$

μ is monotone as f is so. Indeed, assume, by way of contradiction, there is $X \in E_+$ such that $P[\mu X < 0] > 0$. Then, by (72), $f(X_0 - X) \geq f(X_0) - \mu X$. Hence, $P[f(X_0 - X) > f(X_0)] > 0$ in contradiction to monotonicity of f. Consequently, μ is monotone. In particular, μ is $L^0(\mathcal{F})$–convex and so it is continuous by Theorem 4.1.3. Thus, μ is a subgradient of f at X_0.

To establish continuity of f at X_0 observe that (72) together with the continuity of μ implies that f satisfies

$$\operatorname{ess.limsup}_\alpha f(X) - f(X_\alpha) \leq \operatorname{ess.limsup}_\alpha \mu(X - X_\alpha) = 0,$$

and whence (68). Thus, f is continuous throughout $\overset{\circ}{\operatorname{dom}} f$ due to Lemma 4.4.3. □

CHAPTER 5

Conditional risk measures on vector spaces

In this chapter we present conditional risk measures on $L^p(\mathcal{E})$ spaces. The approach we follow is vector space based and it complements the current literature where conditional risk measures are studied on the smaller Banach space $L^\infty(\mathcal{E})$. The results draw from a general vector space duality result, established by Zowe in [**Zow75**]. This result forms the base of our observations from which we will subsequently derive more specific results for conditional risk measures. This approach can be regarded as a top down approach as it originates from a dual representation result for general vector valued convex functions and then reveals how additional properties of the represented functions translate to properties of the representing continuous linear functions. This translation is of particular interest in the context of conditional risk measures as it clarifies under which conditions the represented convex function admits the well known interpretation as the maximum of expected losses under different scenarios subject to penalization.

In the preliminary section, Section 5.1, we introduce and recall some basic definitions and notational conventions and we present Zowe's convex duality relation. In Section 5.2 we investigate continuous linear functions on $L^p(\mathcal{E})$ spaces. Contraction, projection, monotonicity and local properties play an important role. Section 5.3 deals with convex functions on $L^p(\mathcal{E})$ with corresponding properties subcash invariance, cash invariance, monotonicity and local property. In Section 5.4 we prove Zowe's convex duality relation.

5.1. Preliminaries

For all of this chapter we fix $1 \leq r \leq p < \infty$. We denote by s and q the respective duals of r and p. That is, $s = r/(r-1)$, $q = p/(p-1)$ with the convention $s, q = \infty$ if $r, p = 1$.

In this chapter we do not cover the case of $p = +\infty$. The reason for this is that numerous articles from the literature on financial risk measures deal with conditional risk measures on $L^\infty(\mathcal{E})$; we refer to [**ADE+07, BN04, CDK06, DS05, FP06, Rie04**] and the references therein.

Definition 5.1.1. *A function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is*

(i) *monotone if $f(X) \leq f(X')$ for all $X, X' \in L^p(\mathcal{E})$ with $X \geq X'$,*
(ii) *subcash invariant if $f(X+Y) \geq f(X) - Y$ for all $X \in L^p(\mathcal{E})$ and $Y \in L^r(\mathcal{F})$ with $Y \geq 0$,*
(iii) *cash invariant if $f(X+Y) = f(X) - Y$ for all $X \in L^p(\mathcal{E})$ and $Y \in L^r(\mathcal{F})$.*

Recall that a function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is convex if $f(\alpha X + (1-\alpha)X') \leq \alpha f(X) + (1-\alpha) f(X')$ for all $X, X' \in L^p(\mathcal{E})$ and $\alpha \in [0,1]$. As in Definition 2.1.1 we call f is local if

$$1_A f(X) = 1_A f(1_A X) \text{ for all } X \in L^p(\mathcal{E}) \text{ and } A \in \mathcal{F}. \tag{73}$$

As outlined in the introduction, a convex function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ which is monotone and cash invariant is referred to as conditional convex risk measure. The reason for this is the economic interpretation of $f(X)$ as a capital requirement a financial institution has to meet on assuming the uncertain profit or loss $X \in L^p(\mathcal{E})$ adherent to a financial position.

By the Fisher-Riesz theorem any continuous linear function $\mu : L^p(\mathcal{E}) \to \mathbb{R}$ is of the form

$$\mu X = E[ZX]$$

for some $Z \in L^q(\mathcal{E})$. Further, any proper lower semi continuous (l.s.c.) convex function $f : L^p(\mathcal{E}) \to (-\infty, +\infty]$ admits the Fenchel–Moreau dual representation

$$f(X) = \sup_{Z \in L^q(\mathcal{E})} (E[ZX] - f^*(Z)), \tag{74}$$

where $f^*(Z) = \sup_{X \in L^p(\mathcal{E})}(E[ZX] - f(X))$ denotes the conjugate function of f.

Dual representations as in (74) and subdifferentiability are of distinct interest in various contexts such as optimal investment problems with respect to robust utility functionals [**SW05, Sch07**], portfolio optimization under risk constraints [**GW07, GW08**], risk sharing [**BEK05, BR06, Acc07, BR08, FS08a, FK08b, JST08, LR08, Che09**], indifference and equilibrium pricing [**KS07, FK08a**], efficient hedging [**FL00, Rud07, Rud09, Che09, İJS09**] and many more.

Moreover, such representations provide us with a plausible interpretation of the subjective risk assessment of an economic agent. More precisely, let us assume an agent faces the uncertain payoff $X \in L^p(\mathcal{E})$. Dual representations of the form (74) suggest that the agent computes the expected payoff $E[ZX]$ within the specific model $Z \in L^q(\mathcal{E})$ he selected from a variety of probabilistic models. In addition to this, the agent takes into account a certain model ambiguity as the possible outcome subject to model Z is penalized by $-f^*(Z)$. The higher $f^*(Z)$ the less plausible the agent views model Z. In evaluating the capital requirement $f(X)$ for the uncertain payoff X the agent then takes a worst case approach.

For these reasons, the question arises to what extent representations of the form (74) are preserved in the context of conditional risk measures when $\mathbb{R}$ is replaced by $L^r(\mathcal{F})$.

To address this question, we introduce some terminology suited to the vector space approach followed in this chapter. We denote by $\mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$ the space of all continuous linear functions from $L^p(\mathcal{E})$ into $L^r(\mathcal{F})$ and consider a function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$. We define $f^* : \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F})) \to \bar{L}^0(\mathcal{F})$ by

$$f^*(\mu) = \operatorname*{ess.sup}_{X \in L^p(\mathcal{E})} (\mu X - f(X))$$

and $\operatorname{dom} f^* = \{\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F})) \mid f^*(\mu) \in L^r(\mathcal{F})\}$. We recall the convention, that the essential supremum of an empty family of random variables is $-\infty$. Further, we define $f^{**} : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ by

$$f^{**}(X) = \operatorname*{ess.sup}_{\mu \in \operatorname{dom} f^*} (\mu X - f^*(\mu)).$$

An element $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$ is a subgradient of f at $X_0 \in L^p(\mathcal{E})$ if

$$\mu(X - X_0) \leq f(X) - f(X_0)$$

for all $X \in L^p(\mathcal{E})$.

The set of all subgradients of f at X_0 is called the subdifferential of f at X_0 and denoted by $\partial f(X_0)$. By definition of the subdifferential $\partial f(X_0)$ we have the well known relation

$$\mu_0 \in \partial f(X_0) \text{ if and only if } \mu_0 \in \mathrm{dom} f^* \text{ and } f(X_0) = \mu_0 X_0 - f^*(\mu_0). \tag{75}$$

Reference should be made that in part one of this thesis and that in Chapter 6 below we have and will encounter a slightly different notion of conjugate functions, effective domains and subdifferentials. Nevertheless, there will be no source of ambiguity as this chapter is entirely self contained.

EXAMPLE 5.1.2. *Let us assume that $\mathcal{F} = \sigma(A_n)$ is generated by a countable partition (A_n) of Ω. In this case, we can identify $L^r(\mathcal{F})$ with $l^r(\mathcal{F})$, the space of all real valued sequences (x_n) with $\sum_{n=1}^\infty p_n |x_n|^r < \infty$, where $p_n = P[A_n]$ for all $n \geq 1$. Hence, any function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is of the form*

$$f = (f_1, f_2, f_3, \ldots)$$

with a sequence of functions $f_n : L^p(\mathcal{E}) \to \mathbb{R}$, $n \in \mathbb{N}$, such that $\sum_{n=1}^\infty p_n |f_n(X)|^r < \infty$ for all $X \in L^p(\mathcal{E})$.

Localness of the function f is now reflected by the intuitive relation

$$(\underbrace{0, \ldots, 0}_{n-1-times}, f_n(X), 0, \ldots) = f(\underbrace{0, \ldots, 0}_{n-1-times}, f_n(1_{A_n} X), 0, \ldots) \text{ for all } n \in \mathbb{N},$$

that is, the nth component f_n of f only depends on the coordinate spanned by the vector 1_{A_n}.

EXAMPLE 5.1.3. *The local structure of Example 5.1.2 becomes even more apparent if $\mathcal{E}$ is generated by a finite partition $B_1, \ldots, B_n$ of Ω. In this case, $A_j = \bigcup_{i \in I_j} B_i$, where $\{1, \ldots, n\} = \bigcup_{1 \leq j \leq m} I_j$ so that $L^p(\mathcal{E}) = L^0(\mathcal{E}) \cong \mathbb{R}^n$ as well as $L^r(\mathcal{F}) = L^0(\mathcal{F}) \cong \mathbb{R}^m$.*

The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is now of the form $f = (f_1, \ldots, f_m)$ with arbitrary functions $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$. Localness of f now means that for each $1 \leq j \leq m$ the function f_j only depends on the coordinates I_j. We abuse notation and identify f_j with its restriction to the coordinates I_j. In other words, $f = (f_1, \ldots, f_m)$ for functions $f_1 : \mathbb{R}^{I_1} :\to \mathbb{R}, \ldots, f_m : \mathbb{R}^{I_m} \to \mathbb{R}$ (after rearranging the coordinates $1, \ldots, n$ suitably).

Moreover, if f is $C^1(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\begin{aligned} \mathrm{D}f(X) &= \begin{pmatrix} \partial_{I_1} f_1(X) & 0 & \cdots & 0 \\ 0 & \partial_{I_2} f_2(X) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \partial_{I_m} f_m(X) \end{pmatrix} \\ &= \begin{pmatrix} Df_1(X_{I_1}) & 0 & \cdots & 0 \\ 0 & Df_2(X_{I_2}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Df_m(X_{I_m}) \end{pmatrix} \end{aligned}$$

for all $X \in \mathbb{R}^n$. (Note that the zeroes in the above matrices are understood as generic vector zeroes possibly differing in their dimensions.)

Zowe proves in [**Zow75**] the following dual representation result which, in fact, he establishes in a more general setup.

THEOREM 5.1.4. *Let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be a convex function. If f is continuous at $X_0 \in L^p(\mathcal{E})$ then $\partial f(X_0) \neq \emptyset$ and*

$$f(X_0) = f^{**}(X_0). \tag{76}$$

For the sake of completeness, we provide a self contained proof in Section 5.4, tailored to our setup.

The preluding questions can now be specified as follows. Which linear $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is of the form

$$\mu X = E[ZX \mid \mathcal{F}] \tag{77}$$

for some $Z \in L^q(\mathcal{E})$? And further, for which convex $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is each $\mu \in \mathrm{dom} f^*$ of the form (77) so that

$$f(X) = \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} (E[ZX \mid \mathcal{F}] - f^*(Z)), \tag{78}$$

where $f^*(Z)$ is understood as $f^*(E[Z\cdot \mid \mathcal{F}])$?

5.2. Linear functions on $L^p(\mathcal{E})$

In this section we study representation results and corresponding continuity properties of linear functions from $L^p(\mathcal{E})$ to $L^r(\mathcal{F})$. The results are of preliminary nature for the following section on convex functions.

PROPOSITION 5.2.1. *A function $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is*

(i) *continuous linear and*
(ii) *local*

if and only if it is of the the form (77) for some unique representing $Z \in L^q(\mathcal{E})$ which satisfies the integrability condition $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})$, where $r(p-1)/(p-r)$ is understood as $+\infty$ if $p = r$.

PROOF. To prove the if statement, let $\mu = E[Z\cdot \mid \mathcal{F}]$, $Z \in L^q(\mathcal{E})$ with $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{(p-r)}}(\mathcal{F})$. Inspection shows that μ is linear and local. To establish continuity we assume $1 < r < p$, the other cases work analogously. By Hölder's inequality

$$\begin{aligned} E\left[|E[ZX \mid \mathcal{F}]|^r\right] &\leq E\left[E[|Z|^q \mid \mathcal{F}]^{\frac{r}{q}} E[|X|^p \mid \mathcal{F}]^{\frac{r}{p}}\right] \\ &\leq E\left[E[|Z|^q \mid \mathcal{F}]^{\frac{pr}{q(p-r)}}\right]^{\frac{p-r}{p}} E\left[|X|^p\right]^{\frac{r}{p}}. \end{aligned}$$

Since

$$E\left[|Z|^q \mid \mathcal{F}\right]^{\frac{pr}{q(p-r)}} = E\left[|Z|^q \mid \mathcal{F}\right]^{\frac{r(p-1)}{p-r}} \in L^1(\mathcal{F}),$$

we deduce $\|E[ZX \mid \mathcal{F}]\|_r \leq c\|Z\|_p$ for some $c \in \mathbb{R}_+$. Hence, μ is continuous.

Conversely, if $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is a continuous linear function then so is $E \circ \mu : L^p(\mathcal{E}) \to \mathbb{R}$ and by the Fisher–Riesz theorem there is $Z \in L^q(\mathcal{E})$ such that $E[\mu X] = E[ZX]$ for all $X \in L^p(\mathcal{E})$. Since μ is local we derive $E[1_A \mu X] = E[\mu(1_A X)] = E[Z 1_A X]$ for all $A \in \mathcal{F}$ and $X \in L^p(\mathcal{E})$. Thus, $\mu X = E[ZX \mid \mathcal{F}]$ for all $X \in L^p(\mathcal{E})$. It remains to show that $E[|Z|^q \mid \mathcal{F}] \in L^{r(p-1)/(p-r)}(\mathcal{F})$. We distinguish between two different cases. If $r = 1$ then $E[|Z|^q \mid \mathcal{F}] \in L^1(\mathcal{F})$ as $E[|E[|Z|^q \mid \mathcal{F}]|] = E[|Z|^q] \in \mathbb{R}$. It remains to show the case $1 < r \leq p$. To this end, consider the adjoint $\mu' : L^s(\mathcal{F}) \to L^q(\mathcal{E})$ of μ. By definition,

$$(\mu' Y)(X) = E[Y E[ZX \mid \mathcal{F}]] = E[YZX], \quad X \in L^p(\mathcal{E}). \tag{79}$$

Since $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is continuous so is $\mu' : L^s(\mathcal{F}) \to L^q(\mathcal{E})$, that is, for all $Y \in L^s(\mathcal{F})$

$$\|\mu' Y\|_q \leq c\|Y\|_s,$$

for some real constant c. Since the L^q–norm coincides with the corresponding operator norm we derive for all $Y \in L^s(\mathcal{F})$

$$\sup_{X \in L^p(\mathcal{E}), \|X\|_p \leq 1} |(\mu' Y)(X)| = \sup_{X \in L^p(\mathcal{E}), \|X\|_p \leq 1} E[YZX] \leq cE[|Y|^s]^{\frac{1}{s}}. \tag{80}$$

With equation (79) we know that $E[YZ\cdot]$ is a continuous linear function from $L^p(\mathcal{E})$ to $\mathbb{R}$. Since the topological dual of $L^p(\mathcal{E})$ can be identified with $L^q(\mathcal{E})$ we derive that necessarily $YZ \in L^q(\mathcal{E})$. Therefore, we can define

$$X_Y = \operatorname{sign}(YZ) \times |YZ|^{\frac{1}{(p-1)}} / E[|YZ|^q]^{\frac{1}{p}}$$

(with the convention $0/0 = 0$). Then $X_Y \in L^p(\mathcal{E})$ and $\|X_Y\|_p \leq 1$ for all $Y \in L^s(\mathcal{F})$. Hence, we derive from (80) that for all $Y \in L^s(\mathcal{F})$

$$E[|YZ|^q]^{\frac{1}{q}} = E[YZX_Y] \leq cE[|Y|^s]^{\frac{1}{s}}.$$

In particular, $Y \mapsto E[E[|Z|^q \mid \mathcal{F}]Y]$ is a linear, continuous function from $L^{\frac{s}{q}}(\mathcal{F})$ to $\mathbb{R}$. Again, since the topological dual of $L^{\frac{s}{q}}(\mathcal{F})$ can be identified with $L^{\frac{r(p-1)}{p-r}}(\mathcal{F})$ we derive that necessarily $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})$. $\square$

The next proposition provides a different set of conditions that are sufficient for μ to be of the form (77). These conditions spotlight the emphasis on conditional risk measures.

Proposition 5.2.2. *A function $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is*

(i) *continuous linear,*
(ii) *$\mu Y \geq -Y$ for all $Y \in L^r(\mathcal{F})$ with $Y \geq 0$, and*
(iii) *monotone, $\mu X \leq 0$ for all $X \in L^p(\mathcal{E})$, $X \geq 0$,*

if and only if it is of the the form (77) for some representing $Z \in L^q(\mathcal{E})$ with $E[Z \mid \mathcal{F}] \geq -1$ and $Z \leq 0$ and which satisfies the integrability condition $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{(p-r)}}(\mathcal{F})$.

Proof. The if statement follows by inspection, where continuity follows as in Proposition 5.2.1.

As to the only if statement we show that (ii) and (iii) imply that μ is local. To this end, let $X \in L^p(\mathcal{E})$ be essentially bounded in a first step. Then $X \leq 1_A X + \|X - 1_A X\|_\infty$, where for $X' \in L^p(\mathcal{E})$,

$$\|X'\|_\infty = \operatorname{ess.inf}\{Y \in L^0(\mathcal{F}) \mid Y \geq |X'|\}.$$

Since μ is positive and $\mu Y \geq -Y$ for all $Y \in L^r_+(\mathcal{F})$ we derive

$$\mu X \geq \mu(1_A X + \|X - 1_A X\|_\infty) \geq \mu(1_A X) - \|X - 1_A X\|_\infty.$$

On exchanging X and $1_A X$ we derive

$$|1_A \mu X - 1_A \mu(1_A X)| = 1_A|\mu X - \mu(1_A X)| \leq 1_A \|X - 1_A X\|_\infty = 0.$$

Thus, μ is local for all essentially bounded X. By a standard truncation and approximation argument we derive that μ is local for all $X \in L^p(\mathcal{E})$. Thus, we established that μ is continuous linear local and hence of the form (77) for some representing $Z \in L^q(\mathcal{E})$. The integrability condition $E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{(p-r)}}(\mathcal{F})$

follows as in the proof of Proposition 5.2.1. Further, (ii) and (iii) imply $E[Z \mid \mathcal{F}] \geq -1$ and $Z \leq 0$. $\square$

REMARK 5.2.3. *In Proposition 5.2.2 above one can replace $\mu Y \geq -Y$ for all $Y \in L^r(\mathcal{F})$ with $Y \geq 0$ by the projection property $\mu Y = -Y$ for all $Y \in L^r(\mathcal{F})$ with the result that $E[Z \mid \mathcal{F}] = -1$ in place of $E[Z \mid \mathcal{F}] \geq -1$.*

EXAMPLE 5.2.4. *Positivity* (iii) *is needed in Proposition 5.2.2, as the following example shows. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{E} = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\})$, $P[\omega_1] = 1/2$, $P[\omega_2] = P[\omega_3] = 1/4$ and $\mathcal{F} = \sigma(A_1, A_2)$ with $A_1 = \{\omega_1\}$ and $A_2 = \{\omega_2, \omega_3\}$. Define the random variables*

$$Z_1 = (-2, 1, -1), \quad Z_2 = (0, -2, -2)$$

and the linear map $\mu : L^0(\mathcal{E}) \to L^0(\mathcal{F})$ by

$$\mu(X) = \sum_{i=1}^{2} E[Z_i X] 1_{A_i}.$$

Then μ satisfies (i) *and* (ii) *of Proposition 5.2.2, but not* (iii) *since $\mu(0, 4, 0) = (1, -2, -2)$.*

Now suppose μ were of the form (77) *for some (not necessarily positive) $Z \in L^0(\mathcal{E})$. This implies, in particular, that*

$$E[1_{A_1} \mu X] = E[1_{A_1} Z X]$$

for all $X \in L^0(\mathcal{E})$. But for $X = (0, 4, 0)$ we obtain zero on the right hand side and $1/2$ on the left hand side, which is absurd. Hence μ cannot be of the form (77).

5.3. Monotone (sub)cash invariant convex functions on $L^p(\mathcal{E})$

Given the results of the preceding section we now turn our attention to convex functions.

LEMMA 5.3.1. *Let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be a function.*

(i) *If f is local then every $\mu \in \mathrm{dom} f^*$ is local.*
(ii) *If f is monotone then μ is monotone for each $\mu \in \mathrm{dom} f^*$.*
(iii) *If f is subcash invariant then $\mu Y \geq -Y$ for all $Y \in L^r(\mathcal{F})$ with $Y \geq 0$ for each $\mu \in \mathrm{dom} f^*$.*
(iv) *If f is cash invariant then $-\mu$ is a projection for each $\mu \in \mathrm{dom} f^*$.*

PROOF. (i) Take non local $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$. Then there are $X \in L^p(\mathcal{E})$, $A, B \in \mathcal{F}$, $A \subset B$, with $\mu(1_B X) < \mu X$ on A. Then $\mu(-1_{B^c} X) = \mu(1_B X - X) < 0$ on A or, equivalently, $\mu(1_{B^c} X) > 0$ on A. This implies for all $n \in \mathbb{N}$

$$\mu(1_{B^c} n X) - f(1_{B^c} n X) = n \mu(1_{B^c} X) - f(0)$$

on A. As n tends to ∞, we conclude $\mu \notin \mathrm{dom} f^*$.

(ii) Let $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$ and suppose there is $X \geq 0$ such that $\mu X > 0$ with positive positive probability. By monotonicity of f, $f(nX) \leq f(X)$ for all $n \in \mathbb{N}$. Hence,

$$f^*(\mu) \geq \mu(nX) - f(nX) \geq n \mu X - f(X)$$

for all $n \in \mathbb{N}$. This implies $\mu \notin \mathrm{dom} f^*$.

(iii) Let $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$. By subcash invariance of f we have

$$\begin{aligned} f^*(\mu) &\geq \operatorname*{ess.sup}_{X \in L^p(\mathcal{E})} (\mu(X) - f(X + nY) - nY) \\ &\overset{X'=X+nY}{=} \operatorname*{ess.sup}_{X' \in L^p(\mathcal{E})} (\mu(X' - nY) - f(X') - nY) \\ &= \operatorname*{ess.sup}_{X' \in L^p(\mathcal{E})} (\mu(X') - f(X') + n(-\mu Y - Y)) \\ &= f^*(\mu) + n(-\mu Y - Y) \end{aligned}$$

for all $Y \in L^r(\mathcal{F})$ with $Y \geq 0$ and $n \in \mathbb{N}$. Hence, $\mu \notin \mathrm{dom} f^*$ if $\mu Y < -Y$ with positive probability.

(iv) Let $\mu \in \mathcal{L}(L^p(\mathcal{E}), L^r(\mathcal{F}))$. Since f is cash invariant we derive for all $Y \in L^r(\mathcal{F})$

$$f^*(\mu) \geq \mu Y - f(Y) = \mu Y + Y - f(0).$$

This implies that $\mu \in \mathrm{dom} f^*$ only if $\mu Y = -Y$ for all $Y \in L^r(\mathcal{F})$; whence $-\mu$ is a projection. $\square$

In view of Lemma 5.3.1 (i) we derive a convex variant of Proposition 5.2.1.

Proposition 5.3.2. *A function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is*

(i) *continuous convex and*
(ii) *local*

if and only if $\mathrm{dom} f^* \subset \{Z \in L^q(\mathcal{E}) \mid E[|Z|^q \mid \mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}\}$. *Moreover, in this case*

$$f(X) = \operatorname*{ess.sup}_{Z \in L^q(\mathcal{E}), E[|Z|^q | \mathcal{F}] \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})} (E[ZX \mid \mathcal{F}] - f^*(Z)). \tag{81}$$

In the same manner, we derive from Lemma 5.3.1 (ii) and (iii) a convex analogue of Proposition 5.2.2.

Proposition 5.3.3. *A function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is*

(i) *continuous convex,*
(ii) *monotone and*
(iii) *subcash invariant*

if and only if $\mathrm{dom} f^* \subset \mathcal{C} = \{Z \in L^q(\mathcal{E}) \mid E[Z \mid \mathcal{F}] \geq -1, Z \leq 0, E[|Z|^q \mid \mathcal{F}] \in L^{r(p-1)/(p-r)}(\mathcal{F})\}$. *Moreover, in this case*

$$f(X) = \operatorname*{ess.sup}_{Z \in \mathcal{C}} (E[ZX \mid \mathcal{F}] - f^*(Z)). \tag{82}$$

Remark 5.3.4. *We obtain the convex variant of Remark 5.2.3; that is, we can replace subcash invariance by cash invariance in Proposition 5.3.3 with the result that $E[Z \mid \mathcal{F}] = -1$ in place of $E[Z \mid \mathcal{F}] \geq -1$.*

To draw a conclusion, standard vector space based convex analysis is applicable to a selected class of conditional risk measures. This class contains risk measures which map $L^p(\mathcal{E})$ into $L^r(\mathcal{F})$.

5.4. Proof of Theorem 5.1.4

In this section we provide a proof of Zowe's convex duality result in the form of Theorem 5.1.4. The setup and notation is as in Section 5.1. We first present a topological lemma.

LEMMA 5.4.1. *There exists a base of neighborhoods V of $0 \in L^k(\mathcal{G})$ such that*

$$V = (V + L^k_+(\mathcal{G})) \cap (V - L^k_+(\mathcal{G})), \tag{83}$$

where $L^k_+(\mathcal{G}) = \{X \in L^k(\mathcal{G}) \mid X \geq 0\}$, $k \in [1, +\infty]$ and $\mathcal{G} \subset \mathcal{E}$ denotes a generic sub σ–algebra of $\mathcal{E}$.

PROOF. For each $n \in \mathbb{N}$ we denote by $B_{1/n}$ the ball of radius $1/n$ centered at $0 \in L^k(\mathcal{G})$. The collection $(B_{1/n})$ is the canonical base of neighborhoods in $L^k(\mathcal{G})$. We claim that $V_{1/n} = (B_{1/n} + L^k_+(\mathcal{G})) \cap (B_{1/n} - L^k_+(\mathcal{G}))$, $n \in \mathbb{N}$, defines a neighborhood base as required. Indeed, each $V_{1/n}$ satisfies (83) and $B_{1/n} \subset V_{1/n}$ for each $n \in \mathbb{N}$ by construction. To show that $(V_{1/n})$ is a base of neighborhoods it remains to prove $V_{1/2n} \subset B_{1/n}$ for each $n \in \mathbb{N}$. To this end, let $X \in V_{1/2n}$. Then there are $Y, Y' \in B_{1/2n}$ and $Z, Z' \in L^k_+(\mathcal{G})$ such that $X = Y + Z = Y' - Z'$. We derive that $Y \leq X \leq Y'$ and in turn $|X| \leq \sup\{|Y|, |Y'|\}$. The triangle inequality now yields the assertion. □

The epigraph epi$f$ of a function $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ is understood as $\{(X, Y) \in L^p(\mathcal{E}) \times L^r(\mathcal{F}) \mid f(X) \leq Y\}$. The next lemma proves the first assertion of Theorem 5.1.4.

LEMMA 5.4.2. *Let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be a convex function. If f is continuous at $X_0 \in L^p(\mathcal{E})$ then f has a subgradient at X_0.*

PROOF. The set

$$A = \text{epi} f - \{(X_0, Y) \in \{X_0\} \times L^r(\mathcal{F}) \mid Y \leq f(X_0)\}$$

is nonempty and convex. Thus,

$$B = \bigcup_{\lambda \in [0, +\infty)} \lambda A$$

is a convex cone in $L^p(\mathcal{E}) \times L^r(\mathcal{F})$, that is $B + B \subset B$ and $\lambda B \subset B$ for all $\lambda \in [0, +\infty)$. By means of B we will construct a sublinear mapping $p : L^p(\mathcal{E}) \to L^r(\mathcal{F})$, that is, p is subadditive $p(X_1 + X_2) \leq p(X_1) + p(X_2)$ and positively homogeneous $p(\lambda X_1) = \lambda p(X_1)$ for all $X_1, X_2 \in L^p(\mathcal{E})$ and $\lambda \in [0, +\infty)$. To this end, we define

$$S_X = \{Y \in L^r(\mathcal{F}) \mid (X, Y) \in B\},$$

for all $X \in L^p(\mathcal{E})$. We will show that S_X is nonempty and bounded from below for all $X \in L^p(\mathcal{E})$.

Since B is a convex cone we observe first that

$$S_{X_1} + S_{X_2} \subset S_{X_1 + X_2}, \text{ for all } X_1, X_2 \in L^p(\mathcal{E}). \tag{84}$$

For $X \in L^p(\mathcal{E})$ we have

$$(X, f(X_0 + X) - f(X_0)) = (X_0 + X, f(X_0 + X)) - (X_0, f(X_0)) \in A,$$

and hence $(X, f(X_0 + X) - f(X_0)) \in B$. Thus,

$$S_X \neq \emptyset \text{ for all } X \in L^p(\mathcal{E}). \tag{85}$$

Let $(0, Y) \in B$, $Y \neq 0$. Then $(0, Y) = \lambda((X_1, Y_1) - (X_2, Y_2))$ for some $\lambda \in (0, +\infty)$, $X_1 = X_2 = X_0$ and $Y_1 \geq f(X_0) \geq Y_2$. Thus, $Y = \lambda(Y_1 - Y_2) \geq 0$, and hence

$$S_0 \subset L_+^r(\mathcal{F}). \tag{86}$$

For $X \in L^p(\mathcal{E})$ take $Y \in S_{-X}$ which is possible due to (85). From (84) and (86) we derive for all $Z \in S_X$

$$Z + Y \in S_X + S_{-X} \subset S_0 \subset L_+^r(\mathcal{F}).$$

Hence $-Y$ is a lower bound for S_X. Since $L^r(\mathcal{F})$ is order complete the mapping $p : L^p(\mathcal{E}) \to L^r(\mathcal{F})$,

$$p(X) = \text{ess.inf}\{Y \mid Y \in S_X\}$$

is well defined. Next, we show that p is sublinear.

For $\lambda \in (0, +\infty)$ we have $\lambda B = B$, and hence $\lambda p(X) = \text{ess.inf}\{\lambda Y \mid (X, Y) \in B\} = \text{ess.inf}\{\lambda Y \mid (\lambda X, \lambda Y) \in B\} = p(\lambda X)$. Since $p(0) = 0$ it follows that p is positively homogeneous. Further, from (84) we derive for all $X_1, X_2 \in L^p(\mathcal{E})$

$$p(X_1 + X_2) \leq Y_1 + Y_2, \text{ for all } Y_1 \in S_{X_1}, Y_2 \in S_{X_2}.$$

Thus, $p(X_1 + X_2) \leq p(X_1) + p(X_2)$. Hence, p is subadditive and in turn sublinear. By the Hahn–Banach extension theorem in the form of Theorem 8.30 in [**AB06**] there exists a linear mapping $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ such that $\mu X \leq p(X)$ for all $X \in L^p(\mathcal{E})$. Since $f(X) - f(X_0) \in S_{X - X_0}$ for all $X \in L^p(\mathcal{E})$ we have

$$\mu(X - X_0) \leq p(X - X_0) \leq f(X) - f(X_0). \tag{87}$$

for all $X \in L^p(\mathcal{E})$. Thus, μ is a subgradient of f at X_0 if we can show that μ is continuous.

To this end, let V be a neighborhood of $0 \in L^r(\mathcal{F})$. We can assume that $V = -V$ and, due to Lemma 5.4.1, $V = (V + L_+^r(\mathcal{F})) \cap (V - L_+^r(\mathcal{F}))$. Since f is continuous at X_0 there exists a symmetric neighborhood W ($W = -W$) of $0 \in L^p(\mathcal{E})$ such that

$$f(X_0 + W) \subset f(X_0) + V.$$

Hence, $f(X_0 + W) - f(X_0) \subset V$ and therefore

$$f(X_0 + X) - f(X_0) \in V \text{ for all } X \in W.$$

From (87) we derive for all $X \in L^p(\mathcal{E})$ that $\mu X = \mu(X_0 + X - X_0) \leq f(X_0 + X) - f(X_0)$. Hence for all $X \in W = -W$

$$\mu X \in f(X_0 + X) - f(X_0) - L_+^r(\mathcal{F}) \subset V - L_+^r(\mathcal{F})$$

and

$$\mu X \in -(f(X_0 - X) - f(X_0) - L_+^r(\mathcal{F})) \subset -V + L_+^r(\mathcal{F}) = V + L_+^r(\mathcal{F}).$$

We conclude that $\mu(W) \subset (V + L_+^r(\mathcal{F})) \cap (V - L_+^r(\mathcal{F})) = V$ and continuity of μ follows at $0 \in L^p(\mathcal{E})$. Linearity of μ yields continuity on all of $L^p(\mathcal{E})$. $\square$

The second assertion of Theorem 5.1.4 can be proved as follows. We let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be a convex function which is continuous at $X_0 \in L^p(\mathcal{E})$. We define $\text{dom} f^{**} = \{X \in L^p(\mathcal{E}) \mid f^{**}(X) \in L^r(\mathcal{F})\}$. Lemma 5.4.2 together with (75) yields $\text{dom} f^* \neq \emptyset$ and we get

$$\mu X_0 - f^*(\mu) \leq \mu X_0 - (\mu X_0 - f(X_0)) = f(X_0), \text{ for all } \mu \in \text{dom} f^*.$$

Hence, $X_0 \in \mathrm{dom} f^{**}$ and $f^{**}(X_0) \leq f(X_0)$. The reverse inequality follows from the observation that again Lemma 5.4.2 together with (75) yields the existence of μ_0 such that $f(X_0) = \mu_0 X_0 - f^*(\mu_0)$ which concludes the proof.

CHAPTER 6

Conditional risk measures on modules

In this chapter, we present a module based approach to duality theory of conditional convex risk measures. In contrast to Chapter 5 the spirit of this approach can be referred to as bottom up. The reason for this is that from the beginning on we establish that continuous module homomorphisms, which now take the place of continuous linear functions, are necessarily conditional expectations. As a consequence, dual representations of conditional convex risk measures can immediately be interpreted as the maximum of expected losses subject to penalization. It is due to this approach that the discussion of Section 5.2 becomes obsolete to a large extent. Nevertheless, this comes at the cost of module based convex analysis which is a technically involving matter. The main advantage of this approach, however, is that the derived duality theory for conditional risk measures is very similar to that of static risk measures.

In Section 6.1 we recall and adapt some terminology and definitions of Chapter 2 to not necessarily proper functions. A detailed motivation is given below. Section 6.2 provides dual characterizations of monotonicity and (sub)cash invariance of $L^0(\mathcal{F})$–convex functions. In Section 6.3 indicator and support functions are introduced and some technical lemmas for the following section are presented. The aim of Section 6.4 is to approximate convex functions by means of monotone and (sub)cash invariant functions. Duality theory is utilized to find a monotone and (sub)cash invariant function "closest", expressed in dual terms, to a given function. The idea of this duality based construction principle is already presented in [**FK07**] which, however, only covers the static case.

6.1. Preliminaries

Throughout this section, unless stated otherwise, we let $p \in [1, +\infty]$ and q dual to p as in the preceding Chapter 5. We consider the $L^0(\mathcal{F})$–module $L^p_{\mathcal{F}}(\mathcal{E})$ as introduced in Section 3.2. As in (56) we will work with the convention that the conditional expectation $E[\cdot \mid \mathcal{F}] : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is defined by $E[X \mid \mathcal{F}] = E[X^+ \mid \mathcal{F}] - E[X^- \mid \mathcal{F}]$, the right hand side of which is understood as in (53).

Example 6.1.1. *Let us assume that $\mathcal{F} = \sigma(A_1, \dots, A_m)$ is generated by a finite partition $A_1, \dots, A_m$ of Ω.*

The local structure, formerly a property in reference to the functions we studied, now also appears as a property of the model spaces $L^p_{\mathcal{F}}(\mathcal{E})$ in the sense that on each $\mathcal{F}$–atom A_i, $1 \leq i \leq m$, we consider a classical L^p space, namely $L^p(\mathcal{E} \cap A_i) = L^p(\Omega \cap A_i, \mathcal{E} \cap A_i, P_i)$, where P_i denotes $P[\cdot \mid A_i]$. Over all of $\mathcal{E}$, these spaces are "pasted" together to become

$$L^p_{\mathcal{F}}(\mathcal{E}) = \sum_{i=1}^{m} 1_{A_i} L^p(\mathcal{E} \cap A_i).$$

Consequently, if $\mathcal{F}$ is finitely generated, $L^p(\mathcal{E}) = L^p_{\mathcal{F}}(\mathcal{E})$ and no additional structure is provided.

However, if $\mathcal{F}$ is generated by a countable partition (A_n) of Ω then $L^p_{\mathcal{F}}(\mathcal{E})$ becomes

$$L^p_{\mathcal{F}}(\mathcal{E}) = \sum_{n\in\mathbb{N}} 1_{A_n} L^p(\mathcal{E}\cap A_n)$$

which in fact is an $L^0(\mathcal{F})$–module significantly larger than $L^p(\mathcal{E})$. Indeed, it is not hard to see that $X_n \in L^p(\mathcal{E}\cap A_n)$ for all $n\in\mathbb{N}$ is not sufficient for $\sum_{n\in\mathbb{N}} 1_{A_n} X_n \in L^p(\mathcal{E})$ in general.

In contrast to Chapter 2, we are now considering functions on the specific $L^0(\mathcal{F})$–module $L^p_{\mathcal{F}}(\mathcal{E})$ for which (if $p\in[1,+\infty)$) we explicitly know its topological dual $L^0(\mathcal{F})$–module of continuous $L^0(\mathcal{F})$–linear functions. For example, in the setup of Chapter 2 we have $(f^*)^* \neq f^{**}$ in general. Indeed, the $L^0(\mathcal{F})$–module on which $(f^*)^*$ is defined may be significantly larger than E, the formerly given $L^0(\mathcal{F})$–module on which f^{**} is defined. However, due to the "reflexivity" of $L^p_{\mathcal{F}}(\mathcal{E})$ and $L^q_{\mathcal{F}}(\mathcal{E})$ in this chapter, we have $(f^*)^* = f^{**}$.

For convenience, we will exploit this reflexivity by constantly interchanging the primal and dual perspective throughout this chapter. To this end, we have to adapt some definitions of Chapter 2 to not necessarily proper functions. For instance, the conjugate function of a proper function need not be proper in general. As we have defined $L^0(\mathcal{F})$–convexity, effective domain, subdifferentials, etc. for proper functions only it would be useless to consider the not proper conjugate function from a primal perspective. We note, that the following terminological adaption is fully compatible with Chapter 2.

We recall that a function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is proper if $f(X) > -\infty$ for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and if there is at least one $X \in L^p_{\mathcal{F}}(\mathcal{E})$ such that $f(X) < +\infty$. In Definition 2.1.1 we have already defined localness for proper functions $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$. We adapt this to general functions $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ with the convention $0\cdot(\pm\infty) = 0$.

Further, we introduce for a function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,

$$\begin{aligned} PI(f) &= \text{ess.sup}\{A\in\mathcal{F} \mid 1_A f = 1_A(+\infty)\} \\ MI(f) &= \text{ess.sup}\{A\in\mathcal{F} \mid \exists X\in L^p_{\mathcal{F}}(\mathcal{E}) : 1_A f(X) = 1_A(-\infty)\} \\ P(f) &= (PI(f)\cup MI(f))^c \end{aligned}$$

so that f is proper on $P(f)$, $f\equiv +\infty$ on $PI(f)$ and f may take value $-\infty$ on $MI(f)$.

The effective domain $\text{dom} f$ of f is defined by

$$\text{dom} f = \{X\in L^p_{\mathcal{F}}(\mathcal{E}) \mid 1_{PI(f)^c} f(X) < +\infty\}. \tag{88}$$

Trivially, $PI(f)\cap MI(f)\cap P(f) = \emptyset$ so that f is proper only if $PI(f) = MI(f) = \emptyset$. If f is local then we even have "if and only if".

In Definition 2.1.1 $L^0(\mathcal{F})$–convexity is only defined for proper functions. For the purposes of Section 6.4 below in which we use dual techniques to construct hulls of proper $L^0(\mathcal{F})$–convex functions, we have to extend this definition in a consistent way to not proper functions.

In vector space theory one agrees on the convention that $-\infty + \infty = +\infty$ and defines a function $f : V \to [-\infty,+\infty]$ on a real vector space V to be convex if

$f(\alpha v + (1-\alpha)w) \leq \alpha f(v) + (1-\alpha) f(w)$ for all $v, w \in V$, $\alpha \in [0,1]$. In line with this, we set $-\infty + \infty = +\infty$ and define $L^0(\mathcal{F})$–convexity as follows.

DEFINITION 6.1.2. *A function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is $L^0(\mathcal{F})$–convex if*

$$f(YX + (1-Y)X') \leq Y f(X) + (1-Y) f(X')$$

for all $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0(\mathcal{F})$ with $0 \leq Y \leq 1$. (Recall the convention $0 \cdot (\pm\infty) = 0$.)

REMARK 6.1.3. *Inspection shows that a function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is $L^0(\mathcal{F})$–convex if and only if for all $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0(\mathcal{F})$ with $0 \leq Y \leq 1$,*

$$f(YX + (1-Y)X') \leq Y f(X) + (1-Y) f(X') \tag{89}$$

on the set $(\{f(X) = -\infty, f(X') = +\infty\} \cup \{f(X) = +\infty, f(X') = -\infty\})^c$.

LEMMA 6.1.4. *Any $L^0(\mathcal{F})$–convex function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is local.*

PROOF. Let $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $A \in \mathcal{F}$. Then, we derive the inequalities

$$\begin{aligned} f(1_A X) &\leq 1_A f(X) + 1_{A^c} f(0) \\ &= 1_A f(1_A(1_A X) + 1_{A^c} X) + 1_{A^c} f(0) \\ &\leq 1_A f(1_A X) + 1_{A^c} f(0) \end{aligned}$$

which become equalities on multiplying with 1_A. □

Consider a local function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$. As in Chapter 2, we call f lower semi continuous (l.s.c.) if for any convergent net $X_\alpha \to X$ in $L^p_{\mathcal{F}}(\mathcal{E})$ we have

$$\operatorname{ess.liminf}_{\alpha} f(X_\alpha) \geq f(X),$$

where we recall that $\text{ess.liminf} Y_\alpha = \text{ess.sup}_\alpha \text{ess.inf}_{\beta \geq \alpha} Y_\beta$ for a net (Y_α) in $L^p_{\mathcal{F}}(\mathcal{E})$.

DEFINITION 6.1.5. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a local function. The closure $\mathrm{cl}(f) : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ of f is given by*

$$\mathrm{cl}(f) = 1_{MI(f)^c} g + 1_{MI(f)}(-\infty),$$

where g is the greatest l.s.c. $L^0(\mathcal{F})$–convex function majorized by $1_{MI(f)^c} f$. The function f is closed if $f = \mathrm{cl}(f)$.

By definition, $\mathrm{cl}(f)$ is l.s.c. $L^0(\mathcal{F})$–convex and in particular local. Thus, a closed function is l.s.c. $L^0(\mathcal{F})$–convex.

For $p \in [1, +\infty)$ we recall the analogy to (74): any continuous $L^0(\mathcal{F})$–linear function $\mu : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is of the form

$$\mu X = E[ZX \mid \mathcal{F}] \tag{90}$$

for some $Z \in L^q_{\mathcal{F}}(\mathcal{E})$, q dual to p as usual, cf. Theorem 3.2.2. Consequently, the conjugate function $f^* : L^q_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ of a local function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is given by

$$f^*(Z) = \operatorname*{ess.sup}_{X \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f(X)) = \operatorname*{ess.sup}_{X \in \mathrm{dom} f} (E[ZX \mid \mathcal{F}] - f(X))$$

and the conjugate $f^{**} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ of f^* is given by

$$f^{**}(X) = \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f^*(Z)) = \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} (E[ZX \mid \mathcal{F}] - f^*(Z)), \tag{91}$$

where the second equalities follow from the definition of the effective domain in (88). The next theorem presents an $L^0(\mathcal{F})$–convex duality relation which slightly generalizes the Fenchel–Moreau type dual representation of Theorem 2.1.8 in the specific context of $L^p_{\mathcal{F}}(\mathcal{E})$.

THEOREM 6.1.6. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a local function. Then,*

$$f^{**} = \mathrm{cl}(f).$$

In particular, if f is proper l.s.c. $L^0(\mathcal{F})$–convex then $f = f^{**}$.

PROOF. We first prove the auxiliary claim that an $L^0(\mathcal{F})$–convex l.s.c. function $g : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ with $g > -\infty$ satisfies the $L^0(\mathcal{F})$–convex duality relation

$$g = g^{**} \tag{92}$$

which proves the second statement. Indeed, let $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and define $A = \{g(X) < +\infty\}$. Then, on A^c the relation (92) is trivially valid for X. To see that (92) is also valid for X on A it suffices to observe that $1_A g$ is a proper $L^0(\mathcal{F})$–convex l.s.c. function and to apply Theorem 2.1.8 by which $1_A g = (1_A g)^{**}$. Since g is local by $L^0(\mathcal{F})$–convexity we conclude

$$g(X) = 1_A g(X) + 1_{A^c} g(X) = (1_A g)^{**}(X) + (1_{A^c} g)^{**}(X) = g^{**}(X)$$

which proves the auxiliary claim.

Next, define $f_1 = 1_{MI(f)^c} f$ and $f_2 = 1_{MI(f)} f$. We show separately that

$$f_1^{**} = \mathrm{cl}(f_1) \text{ and } f_2^{**} = \mathrm{cl}(f_2)$$

which by localness of f^{**} and $\mathrm{cl}(f)$ yields the assertion.

To see that $f_1^{**} = \mathrm{cl}(f_1)$ observe that by definition f_1^{**} is $L^0(\mathcal{F})$–convex l.s.c. and $-\infty < f_1^{**} \le f_1$. Further, from

$$\mathrm{cl}(f_1) \le f_1 \text{ implies } \mathrm{cl}(f_1)^* \ge f_1^* \text{ implies } \mathrm{cl}(f_1) = \mathrm{cl}(f_1)^{**} \le f_1^{**}$$

we derive $f_1^{**} = \mathrm{cl}(f_1)$.

To establish $f_2^{**} = \mathrm{cl}(f_2)$ we show that there is some $X_{-\infty} \in L^p_{\mathcal{F}}(\mathcal{E})$ with $f_2(X_{-\infty}) = 1_{MI(f)}(-\infty)$. Indeed, since f is local the collection

$$\mathcal{S} = \{A \in \mathcal{F} \mid \exists X \in L^p_{\mathcal{F}}(\mathcal{E}) : f(X) = -\infty \text{ on } A\}$$

is directed upwards and by definition we have $\mathrm{ess.sup}\mathcal{S} = MI(f)$. Hence, there exists an increasing sequence $(A_n) \subset \mathcal{F}$ and a corresponding sequence (X_n) in $L^p_{\mathcal{F}}(\mathcal{E})$ with $A_n \nearrow M_{-\infty}$ and $f(X_n) = -\infty$ on A_n for each $n \in \mathbb{N}$. Since f is local

$$X_{-\infty} = \sum_{i=1}^{\infty} 1_{A_i \setminus \bigcup_{j=1}^{i-1} A_j} X_i$$

is as required with $A_0 = \emptyset$. We conclude that

$$\begin{aligned} f_2^* &= \operatorname*{ess.sup}_{X \in L^p_{\mathcal{F}}(\mathcal{E})} \left(E[\cdot X \mid \mathcal{F}] - f_2(X)\right) \\ &\ge E[\cdot X_{-\infty} \mid \mathcal{F}] - f_2(X_{-\infty}) \ge 1_{MI(f)}(+\infty). \end{aligned}$$

This together with (92) and localness of f implies $f_2^{**} = 1_{MI(f)}(-\infty) = \mathrm{cl}(f_2)$. (Note, that $MI(f) = MI(f_2)$.) $\square$

REMARK 6.1.7. *The epigraph* $\mathrm{epi}f = \{(X,Y) \in L^p_{\mathcal{F}}(\mathcal{E}) \times L^0(\mathcal{F}) \mid f(X) \leq Y\}$ *of a closed function* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *is closed w.r.t. the product topology. To see this, it suffices to observe that* $1_{MI(f)^c}\mathrm{epi}f_1$ *is closed cf. Proposition2.1.4 and that* $1_{MI(f)}\mathrm{epi}f_2 = 1_{MI(f)}(L^p_{\mathcal{F}}(\mathcal{E}) \times L^0(\mathcal{F}))$ *is closed as well;* f_1 *and* f_2 *are understood as in the above proof. Since* $MI(f)$ *and* $MI(f)^c$ *are disjoint the sum of the two* $1_{MI(f)^c}\mathrm{epi}f_1 + 1_{MI(f)}(L^p_{\mathcal{F}}(\mathcal{E}) \times L^0(\mathcal{F})) = \mathrm{epi}f$ *is also closed.*

LEMMA 6.1.8. *Let* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *be a local function. Then,*

$$PI(f) \subset MI(f^*) \text{ and } MI(f) \subset PI(f^*). \tag{93}$$

If f *is closed* $L^0(\mathcal{F})$*–convex we have equalities.*

PROOF. Since f is local (93) follows from the definitions of $PI(\cdot), MI(\cdot)$ and f^*. On replacing f with f^* the reverse inclusions follow as for closed $L^0(\mathcal{F})$–convex f we have $f = f^{**}$, cf. Theorem 6.1.6. □

The preceding lemma reveals in particular that for a closed $L^0(\mathcal{F})$–convex function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ we have the following decompositions

$$f^* = 1_{PI(f)}(-\infty) + 1_{MI(f)}(+\infty) + 1_{R(f)}f^* \tag{94}$$

$$f = f^{**} = 1_{PI(f)}(+\infty) + 1_{MI(f)}(-\infty) + 1_{R(f)}f^{**}. \tag{95}$$

Subgradients in the context of $L^p_{\mathcal{F}}(\mathcal{E})$ are given by the following definition.

DEFINITION 6.1.9. *Let* $p \in [1, +\infty)$*,* q *be as above and* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *be a proper function. An element* $Z \in L^q_{\mathcal{F}}(\mathcal{E})$ *is a subgradient of* f *at* $X_0 \in \mathrm{dom}f$ *if*

$$E[Z(X - X_0) \mid \mathcal{F}] \leq f(X) - f(X_0), \text{ for all } X \in L^p_{\mathcal{F}}(\mathcal{E}).$$

The set of all subgradients of f *at* X_0 *is denoted by* $\partial f(X_0)$*.*

EXAMPLE 6.1.10. *Let* $\mathcal{F} = \sigma(A_1, A_2, A_3)$ *be finitely generated, where* $(A_i)_{1 \leq i \leq 3} \subset \mathcal{E}$ *is a partition of* Ω*. We consider a function* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *and we identify* $\bar{L}^0(\mathcal{F})$ *with* $\bar{\mathbb{R}}^3$ *so that* $f = (f_1, f_2, f_3)$ *for three functions* $f_1, f_2, f_3 : L^p_{\mathcal{F}}(\mathcal{E}) \to [-\infty, +\infty]$*. Let us further assume that* $f_1 \equiv +\infty$*,* f_2 *is proper and there exists* $X \in L^p_{\mathcal{F}}(\mathcal{E})$ *such that* $f_3(X) = -\infty$*.*

Then $PI(f) = A_1$ *and* $MI(f) = A_3$*. Further,* $X \in \mathrm{dom}f$ *if and only if* $f_2(X), f_3(X) < +\infty$ *irrespectively of the fact that* $f_1(X) = +\infty$*. The function* f *would be proper if and only if* f_1*,* f_2 *and* f_3 *were proper at the same time. Thus,* $1_{A_2}f$ *is proper while* f *is not. In the same way we see that* f *is* $L^0(\mathcal{F})$*–convex if and only if each* f_i *is convex,* $1 \leq i \leq 3$*.*

If in addition f *is local then we can identify* f *with three functions* $f_1, f_2, f_3 : L^p(\mathcal{E} \cap A_i) \to [-\infty, +\infty]$ *defined on classical* L^p *spaces. Then* f *is l.s.c. if and only if each* f_i *is l.s.c.,* $1 \leq i \leq 3$*, and its closure is given by*

$$\mathrm{cl}(f) = (\mathrm{cl}(f_1), \mathrm{cl}(f_2), \mathrm{cl}(f_3)) = (+\infty, f_2^{**}, -\infty).$$

The main advantage of the module approach over the vector space approach from Chapter 5 is the fact that we can consider conditional risk measures on $L^p_{\mathcal{F}}(\mathcal{E})$ which is a much larger model space than $L^p(\mathcal{E})$. Furthermore, within the module approach, duality results are applicable to functions which may take values in $\bar{L}^0(\mathcal{F})$. As a consequence, examples such as the entropic risk measure are fully covered.

Further, within the vector space approach, continuous linear functions $\mu : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ are not necessarily conditional expectations. One has to employ the results of Section 5.2 to derive that only those continuous linear functions which are conditional expectations are relevant for conditional risk measures.

In contrast to this, continuous $L^0(\mathcal{F})$–linear functions of $L^p_{\mathcal{F}}(\mathcal{E})$ into $L^0(\mathcal{F})$ are conditional expectations as stated in (90). Results analogue to Proposition 5.2.1, Proposition 5.2.2 and Remark 5.2.3 as presented in Section 5.2 are not required. In this sense, the module approach a priori provides us with an interpretation of (91) in terms of expected losses under different scenarios which, by virtue of f^*, are taken more or less seriously.

6.2. Monotone (sub)cash invariant $L^0(\mathcal{F})$–convex functions on $L^p_{\mathcal{F}}(\mathcal{E})$

Throughout this section, we fix $p \in [1, +\infty)$ and define q dual to p, as usual. The next definition is similar to that of 5.1.1. However, as we work in a module setup, a few amendments are needed.

Definition 6.2.1. *A function* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$

(i) *is monotone if* $f(X) \leq f(X')$ *for all* $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$ *with* $X \geq X'$,
(ii) *is subcash invariant if* $f(X + Y) \geq f(X) - Y$ *for all* $X \in L^p_{\mathcal{F}}(\mathcal{E})$ *and* $Y \in L^0_+(\mathcal{F})$,
(iii) *is cash invariant if* $f(X + Y) = f(X) - Y$ *for all* $X \in L^p_{\mathcal{F}}(\mathcal{E})$ *and* $Y \in L^0(\mathcal{F})$.

Recall that a set $\mathcal{P} \subset L^0(\mathcal{E})$ is $L^0(\mathcal{F})$–convex if $YX + (1-Y)X' \in \mathcal{P}$ whenever $X, X' \in \mathcal{P}$ and $Y \in L^0(\mathcal{F})$ with $0 \leq Y \leq 1$. $\mathcal{P}$ is an $L^0(\mathcal{F})$–cone if $YX \in \mathcal{P}$ for all $X \in \mathcal{P}$ and $Y \in L^0_+(\mathcal{F})$.

From now on, let $\mathcal{P} = \{X \in L^p_{\mathcal{F}}(\mathcal{E}) \mid X \geq 0\}$ be the closed $L^0(\mathcal{F})$–convex $L^0(\mathcal{F})$–cone. $\mathcal{P}$ induces the partial order of almost sure dominance on $L^p_{\mathcal{F}}(\mathcal{E})$ via

$$X \geq X' \Leftrightarrow X - X' \in \mathcal{P}.$$

We recall that $(L^p_{\mathcal{F}}(\mathcal{E}), \geq)$ is an ordered module. The polar $L^0(\mathcal{F})$–cone $\mathcal{P}^\circ$ of $\mathcal{P}$ is

$$\mathcal{P}^\circ = \{Z \in L^q_{\mathcal{F}}(\mathcal{E}) \mid \forall X \in \mathcal{P} : E[ZX \mid \mathcal{F}] \leq 0\}.$$

Inspection shows that $\mathcal{P}^\circ = \{Z \in L^q_{\mathcal{F}}(\mathcal{E}) \mid Z \leq 0\}$ by definition of $\mathcal{P}$. Further, define

$$\begin{aligned} s\mathcal{D} =& \{Z \in L^q_{\mathcal{F}}(\mathcal{E}) \mid E[Z \mid \mathcal{F}] \geq -1\} \\ \mathcal{D} =& \{Z \in L^q_{\mathcal{F}}(\mathcal{E}) \mid E[Z \mid \mathcal{F}] = -1\}. \end{aligned}$$

Note that if $Z \in s\mathcal{D}$ then $E[-ZY \mid \mathcal{F}] \leq Y$ and if $Z \in \mathcal{D}$ then $E[-ZY \mid \mathcal{F}] = Y$ for all $Y \in L^0(\mathcal{F})$.

Proposition 6.2.2. *Let* $X, X' \in L^p_{\mathcal{F}}(\mathcal{E})$. *Then* $X \geq X'$ *if and only if* $E[Z(X - X') \mid \mathcal{F}] \leq 0$ *for all* $Z \in \mathcal{P}^\circ$.

Proof. This follows from the corresponding definitions. □

Lemma 6.2.3. *Let* $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *be a closed* $L^0(\mathcal{F})$*–convex function.*

(i) f *is monotone if and only if* $1_{P(f)}\mathrm{dom} f^* \subset 1_{P(f)}\mathcal{P}^\circ$.
(ii) f *is subcash–invariant if and only if* $1_{P(f)}\mathrm{dom} f^* \subset 1_{P(f)}s\mathcal{D}$.
(iii) f *is cash–invariant if and only if* $1_{P(f)}\mathrm{dom} f^* \subset 1_{P(f)}\mathcal{D}$.

PROOF. Let $X_0 \in L^p_{\mathcal{F}}(\mathcal{E})$ be such that $f(X_0) \in L^0(\mathcal{F})$ on $P(f)$.

(i) To prove the only if statement, assume by way of contradiction that there is $Z \in \mathrm{dom} f^*$ with $P[\{Z > 0\} \cap P(f)] > 0$. By monotonicity of f we have $f(X_0 + n) \le f(X_0)$ for all $n \in \mathbb{N}$. Thus,

$$f^*(Z) \ge E[Z(X_0+n) \mid \mathcal{F}] - f(X_0+n) \ge nE[Z \mid \mathcal{F}] + E[ZX_0 \mid \mathcal{F}] - f(X_0)$$

which contradicts $f^*(Z) < +\infty$ on $P(f)$. To establish the if statement, recall the decompositions (94) and (95). Thus, $1_{P(f)}\mathrm{dom} f^* \subset 1_{P(f)}\mathcal{P}^\circ$ implies

$$\begin{aligned} f(X) &= \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} (E[ZX \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} (E[ZX \mid \mathcal{F}] - f^*(Z)), \end{aligned}$$

for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$. Hence, by Proposition 6.2.2, f is monotone.

(ii) To prove the only if statement, let $Z \in \mathrm{dom} f^*$ and assume that $P[\{E[Z \mid \mathcal{F}] < -1\} \cap P(f)] > 0$. By subcash invariance of f,

$$\begin{aligned} f^*(Z) &\ge \operatorname*{ess.sup}_{X \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f(X+nY) - nY) \\ &\overset{X'=X+nY}{=} \operatorname*{ess.sup}_{X' \in L^p_{\mathcal{F}}(\mathcal{E})} (E[Z(X'-nY) \mid \mathcal{F}] - f(X') - nY) \\ &= \operatorname*{ess.sup}_{X' \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX' \mid \mathcal{F}] - f(X') - nY(E[Z \mid \mathcal{F}] + 1)) \\ &= f^*(Z) - nY(E[Z \mid \mathcal{F}] + 1) \end{aligned}$$

for all $Y \in L^0_+(\mathcal{F})$ and $n \in \mathbb{N}$ which contradicts $f^*(Z) < +\infty$ on $P(f)$. To establish the if statement, observe that the decompositions in (94) and (95) together with $1_{P(f)}\mathrm{dom} f^* \subset 1_{P(f)} s\mathcal{D}$ imply

$$\begin{aligned} f(X+Y) &= \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}}(\mathcal{E})} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in s\mathcal{D}} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \ge f(X) - Y \end{aligned}$$

for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0_+(\mathcal{F})$.

(iii) To prove the only if statement, assume that there is $Z \in \mathrm{dom} f^*$ with $P[\{E[Z \mid \mathcal{F}] \neq -1\} \cap P(f)] > 0$. Since f is cash invariant we derive for all $Y \in L^0(\mathcal{F})$

$$f^*(Z) \ge E[Z(X_0+Y) \mid \mathcal{F}] - f(X_0+Y) = Y(E[Z \mid \mathcal{F}] + 1) + E[ZX_0 \mid \mathcal{F}] - f(X_0).$$

This contradicts $f^*(Z) < +\infty$ on $P(f)$. Conversely, to establish the if statement, let $X \in L^p_{\mathcal{F}}(\mathcal{E})$ and $Y \in L^0(\mathcal{F})$. From the decompositions (94) and (95) together with $1_{P(f)}\mathrm{dom} f^* \subset 1_{P(f)}\mathcal{D}$ we derive

$$\begin{aligned} f(X+Y) &= \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}}(\mathcal{E})} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathrm{dom} f^*} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) \\ &= \operatorname*{ess.sup}_{Z \in \mathcal{D}} (E[Z(X+Y) \mid \mathcal{F}] - f^*(Z)) = f(X) - Y. \end{aligned}$$

□

Two immediate consequences are the following representation results for monotone subcash invariant $L^0(\mathcal{F})$–convex functions and conditional convex risk measures.

COROLLARY 6.2.4. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0$ be proper l.s.c. $L^0(\mathcal{F})$–convex.*

(i) *If f is monotone and subcash invariant, then for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$*

$$f(X) = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ \cap s\mathcal{D}} (E[ZX \mid \mathcal{F}] - f^*(Z)). \tag{96}$$

(ii) *If f is monotone and cash invariant, then for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$*

$$f(X) = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ \cap \mathcal{D}} (E[ZX \mid \mathcal{F}] - f^*(Z)). \tag{97}$$

Elements of $\mathcal{P}^\circ \cap \mathcal{D}$ can be viewed as transition densities which serve as probabilistic models relative to the initial information $\mathcal{F}$ and uncertain future events $\mathcal{E}$. In this sense, the economic interpretation of static risk measures is preserved under assuming non trivial initial information.

6.3. Indicator and support functions

Let $C \subset L^p_{\mathcal{F}}(\mathcal{E})$ be an $L^0(\mathcal{F})$–convex set. We define the mapping $M(\cdot \mid C) : L^p_{\mathcal{F}}(\mathcal{E}) \to \mathcal{F}$ slightly more general than its former definition in (6)

$$M(X \mid C) = \operatorname{ess.sup}\{A \in \mathcal{F} \mid 1_A X \in 1_A C\}.$$

The reason for this generalization is that here we consider $L^0(\mathcal{F})$–convex sets $C \subset L^p_{\mathcal{F}}(\mathcal{E})$ which do not necessarily contain 0. (Together with $L^0(\mathcal{F})$–convexity this would imply $1_A C \subset C$ for all $A \in \mathcal{F}$). Further, we recall that set C has the closure property if for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$

$$1_{M(X|C)} X \in 1_{M(X|C)} C,$$

We note that the closure property should not be seen as a property in reference to the topology of $L^p_{\mathcal{F}}(\mathcal{E})$. In fact, if $0 \in C$ the closure property is closely related to order completeness as it states that a family $(1_A X)_A \subset C$ has a least upper bound in C, namely $\operatorname{ess.sup}_A 1_A X = 1_{M(X|C)} X$.

From now on we assume that C has the closure property. The indicator function $\delta(\cdot \mid C) : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ of C is defined by

$$\delta(X \mid C) = \begin{cases} 0 & \text{on } M(X \mid C) \\ +\infty & \text{on } M(X \mid C)^c \end{cases}.$$

By the closure property of C, $\operatorname{epi}\delta(\cdot \mid C) = C \times L^0_+(\mathcal{F})$. Since a proper local function is l.s.c. if and only if its epigraph is closed, cf. Proposition 2.1.4 we derive that $\delta(\cdot \mid C)$ is l.s.c. if and only if C is closed.

The support function $\delta^*(\cdot \mid C) : L^q_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ of C is defined by

$$\delta^*(Z \mid C) = \operatorname*{ess.sup}_{X \in C} E[ZX \mid \mathcal{F}].$$

Since C is $L^0(\mathcal{F})$–convex (in particular $1_A X + 1_{A^c} X' \in C$ for all $A \in \mathcal{F}$ whenever $X, X' \in C$) the support function of C coincides with the conjugate of the indicator function $\delta(\cdot \mid C)$, i.e. for all $Z \in L^q_{\mathcal{F}}(\mathcal{E})$

$$\operatorname*{ess.sup}_{X \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - \delta(X \mid C)) = \operatorname*{ess.sup}_{X \in C} E[ZX \mid \mathcal{F}]. \tag{98}$$

Note that this is also the case if $C = \emptyset$. (98) justifies the notation $\delta^*(\cdot \mid C)$ of the support function.

We define $\delta^{**}(\cdot \mid C) : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ as the conjugate of the support function, i.e.

$$\delta^{**}(X \mid C) = \operatorname*{ess.sup}_{Z \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - \delta^*(Z \mid C)).$$

If C is closed, we have

$$\delta(\cdot \mid C) = \delta^{**}(\cdot \mid C). \tag{99}$$

Lemma 6.3.1. *Let $\mathcal{P} = \{X \in L^p_{\mathcal{F}}(\mathcal{E}) \mid X \geq 0\}$ be the order inducing $L^0(\mathcal{F})$–cone and $\mathcal{P}^\circ$ its polar $L^0(\mathcal{F})$–cone. Then*

$$\delta(\cdot \mid \mathcal{P}) = \delta^*(\cdot \mid \mathcal{P}^\circ) \tag{100}$$

$$\delta^*(\cdot \mid \mathcal{P}) = \delta(\cdot \mid \mathcal{P}^\circ) \tag{101}$$

$$\delta^*(X \mid \mathcal{D}) = \begin{cases} -X & \text{on } M(X \mid L^0(\mathcal{F})) \\ \infty & \text{on } M(X \mid L^0(\mathcal{F}))^c \end{cases} \quad \text{for all } X \in L^p_{\mathcal{F}}(\mathcal{E}). \tag{102}$$

Proof. To see (100), recall that $\delta^*(X \mid \mathcal{P}^\circ) = \operatorname{ess.sup}_{Z \in \mathcal{P}^\circ} E[ZX \mid \mathcal{F}]$. Further, $1_{M(X|\mathcal{P})} X \geq 0$ implies $1_{M(X|\mathcal{P})} ZX \leq 0$ for all $Z \in \mathcal{P}^\circ$. Since $M(X \mid \mathcal{P}) \in \mathcal{F}$ and since $\mathcal{P}^\circ$ is an $L^0(\mathcal{F})$–cone we derive

$$1_{M(X|\mathcal{P})} \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} E[ZX \mid \mathcal{F}] = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} E[1_{M(X|\mathcal{P})} ZX \mid \mathcal{F}] = 0.$$

This proves (100) on $M(X \mid \mathcal{P})$.

By definition of $M(X \mid \mathcal{P})$, $1_A X \notin \mathcal{P}$ for all $A \in \mathcal{F}$ with $P[A] > 0$ and $A \subset M(X \mid \mathcal{P})^c$. Since $\mathcal{P}$ is closed $L^0(\mathcal{F})$–convex Theorem 1.1.8 implies that there exists $Z_0' \in L^q_{\mathcal{F}}(\mathcal{E})$ and $\varepsilon \in L^0_{++}(\mathcal{F})$ with

$$E[Z_0' X' \mid \mathcal{F}] + \varepsilon \leq E[Z_0' X \mid \mathcal{F}] \tag{103}$$

on $M(X \mid \mathcal{P})^c$ for all $X' \in \mathcal{P}$. The same is true if Z_0' is replaced by $Z_0 = 1_{M(X|\mathcal{P})} Z_0'$. Since $\mathcal{P}$ is an $L^0(\mathcal{F})$–cone we derive that $E[Z_0 X' \mid \mathcal{F}] \leq 0$ for all $X' \in \mathcal{P}$; whence $Z_0 \in \mathcal{P}^\circ$. Further, since $0 \in \mathcal{P}^\circ$ we derive from (103) that $E[Z_0 X \mid \mathcal{F}] > 0$ on $M(X \mid \mathcal{F})^c$. Thus,

$$1_{M(X|\mathcal{P})^c} \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ} E[ZX \mid \mathcal{F}] \geq 1_{M(X|\mathcal{P})^c} \operatorname*{ess.sup}_{Y \in L^0_+(\mathcal{F})} Y E[Z_0 X \mid \mathcal{F}] = 1_{M(X|\mathcal{P})^c}(+\infty)$$

as $\mathcal{P}^\circ$ is an $L^0(\mathcal{F})$–cone. This proves (100) on all of Ω.

The identity (101) follows by a dual argument as in (99).

To prove (102) we define $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,

$$f(X) = -1_{M(X|L^0(\mathcal{F}))} X + \infty 1_{M(X|L^0(\mathcal{F}))^c}$$

and show that $f^* = \delta(\cdot \mid \mathcal{D})$. (Note that f is the function on the right hand side of (102).) The identity in (102) then follows from a dual argument since $\mathcal{D}$ has the closure property and is $L^0(\mathcal{F})$–convex closed. By definition of f, we have

$$\begin{aligned} f^*(Z) &= \operatorname*{ess.sup}_{X \in L^p_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f(X)) \\ &= \operatorname*{ess.sup}_{X \in L^0(\mathcal{F})} (X E[Z \mid \mathcal{F}] + X) \\ &= \operatorname*{ess.sup}_{X \in L^0(\mathcal{F})} X(E[Z \mid \mathcal{F}] + 1) \end{aligned}$$

for all $Z \in L^q_{\mathcal{F}}(\mathcal{E})$. The equality $f^* = \delta(\cdot \mid \mathcal{D})$ now follows from the observation that $M(Z \mid \mathcal{D}) = \{E[Z \mid \mathcal{F}] = -1\}$ for all $Z \in L^q_{\mathcal{F}}(\mathcal{E})$. □

6.4. Monotone and (sub)cash invariant hulls

PROPOSITION 6.4.1. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a proper $L^0(\mathcal{F})$–convex function.*

(i) *The greatest monotone closed $L^0(\mathcal{F})$–convex function majorized by f is given by $f_{\mathcal{P}^\circ} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,*
$$f_{\mathcal{P}^\circ}(X) = \operatorname*{ess.sup}_{Z \in \mathcal{P}^\circ}(E[ZX \mid \mathcal{F}] - f^*(Z)).$$

(ii) *The greatest (sub)cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f is given by $f_{(s)\mathcal{D}} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,*
$$f_{(s)\mathcal{D}}(X) = \operatorname*{ess.sup}_{Z \in s\mathcal{D}}(E[ZX \mid \mathcal{F}] - f^*(Z)).$$

(iii) *The greatest monotone (sub)cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f is given by $f_{\mathcal{P}^\circ,(s)\mathcal{D}} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$,*
$$f_{\mathcal{P}^\circ,(s)\mathcal{D}}(X) = \operatorname*{ess.sup}_{\mu \in \mathcal{P}^\circ \cap s\mathcal{D}}(\mu X - f^*(\mu)).$$

Accordingly, we call $f_{\mathcal{P}^\circ}$, $f_{(s)\mathcal{D}}$ and $f_{\mathcal{P}^\circ,(s)\mathcal{D}}$ the monotone, (sub)cash invariant and monotone (sub)cash invariant hull of f, respectively.

PROOF. (i) Monotonicity of $f_{\mathcal{P}^\circ}$ follows from Lemma 6.2.3 (i) and closedness follows from its definition. Further, $f_{\mathcal{P}^\circ} \leq f^{**} \leq f$. Now let $g : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a monotone closed $L^0(\mathcal{F})$–convex function with $g \leq f$. By Lemma 6.2.3 (i), $1_{P(g)}\mathrm{dom} g^* \subset 1_{P(g)}\mathcal{P}^\circ$. Thus, $g^* = g^* + \delta(\cdot \mid \mathcal{P}^\circ) \geq f^* + \delta(\cdot \mid \mathcal{P}^\circ)$. Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a proper $L^0(\mathcal{F})$–convex function. Since $\mathcal{P}^\circ$ is closed $L^0(\mathcal{F})$–convex and has the closure property $\delta(\cdot \mid \mathcal{P}^\circ)$ is l.s.c. $L^0(\mathcal{F})$–convex and hence

$$(f_{\mathcal{P}^\circ})^* = f^* + \delta(\cdot \mid \mathcal{P}^\circ). \tag{104}$$

Hence, $g = g^{**} \leq f_{\mathcal{P}^\circ}$.

(ii) follows similarly.

(iii) As in (104), one checks that $(f_{\mathcal{P}^\circ,(s)\mathcal{D}})^* = f^* + \delta(\cdot \mid \mathcal{P}^\circ \cap (s)\mathcal{D})$. Now the assertion follows as in (i). □

The next remark provides us with an interpretation of monotone and cash invariant hulls.

REMARK 6.4.2. *Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be proper $L^0(\mathcal{F})$–convex.*

(i) *Define $g : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ by*
$$g(X) = \operatorname*{ess.inf}_{X' \in L^p_{\mathcal{F}}(\mathcal{E}), X' \leq X} f(X').$$

Note that g need not be proper. For instance, take $f = E[\cdot \mid \mathcal{F}] : L^1_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$, then $g \equiv -\infty$. Nevertheless, g is $L^0(\mathcal{F})$–convex and monotone with $g \leq f$, and $g = f$ if and only if f is monotone. Moreover, if g is closed then $g = g^{**} = f_{\mathcal{P}^\circ}$ is the greatest monotone closed $L^0(\mathcal{F})$–convex function majorized by f. Indeed, for all $X \in L^p_{\mathcal{F}}(\mathcal{E})$
$$g(X) = \operatorname*{ess.inf}_{X_1, X_2 \in L^p_{\mathcal{F}}(\mathcal{E}), X_1 + X_2 = X}(f(X_1) + \delta(X_2 \mid \mathcal{P})).$$

With (101) *of Lemma 6.3.1 one checks that the conjugate of the right hand side equals* $f^* + \delta(\cdot \mid \mathcal{P}^\circ)$. *Hence,* $g^* = (f_{\mathcal{P}^\circ})^*$ *by* (104) *and in turn* $g^{**} = f_{\mathcal{P}^\circ}$.

(ii) *Define* $h : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ *by*

$$h(X) = \operatorname*{ess.inf}_{Y \in L^0(\mathcal{F})} (f(X - Y) - Y).$$

Then h *is* $L^0(\mathcal{F})$*–convex and cash invariant with* $h \leq f$, *and* $h = f$ *if and only if* f *is cash invariant. Moreover, if* h *is closed then* $h = h^{**} = f_{\mathcal{D}}$ *is the greatest cash invariant closed* $L^0(\mathcal{F})$*–convex function majorized by* f. *Indeed, by Lemma 6.3.1* (102) *we have for all* $X \in L^p_{\mathcal{F}}(\mathcal{E})$

$$h(X) = \operatorname*{ess.inf}_{X_1, X_2 \in L^p_{\mathcal{F}}(\mathcal{E}), X_1 + X_2 = X} (f(X_1) + \delta^*(X_2 \mid \mathcal{D})).$$

Inspection shows that the dual of the right hand side equals $f^* + \delta(\cdot \mid \mathcal{D})$. *As in* (104) *we have* $(f_{\mathcal{D}})^* = f^* + \delta(\cdot \mid \mathcal{D})$. *Hence,* $h^* = (f_{\mathcal{D}})^*$ *and in turn* $h^{**} = f_{\mathcal{D}}$.

Let $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a proper $L^0(\mathcal{F})$–convex function. Since

$$\delta(\cdot \mid \mathcal{P}^\circ) + \delta(\cdot \mid (s)\mathcal{D}) = \delta(\cdot \mid \mathcal{P}^\circ \cap (s)\mathcal{D})$$

we derive

$$f_{\mathcal{P}^\circ,(s)\mathcal{D}} = (f_{\mathcal{P}^\circ})_{(s)\mathcal{D}} = (f_{(s)\mathcal{D}})_{\mathcal{P}^\circ}$$

Further, note that if for instance f is (sub)cash invariant then $f_{\mathcal{P}^\circ,(s)\mathcal{D}} = f_{\mathcal{P}^\circ}$. However, if f is monotone (sub)cash invariant we only have $f_{\mathcal{P}^\circ,(s)\mathcal{D}} = f^{**} \leq f$ as f need not be closed in general.

CHAPTER 7

Examples

In this chapter we present examples of $L^0(\mathcal{F})$–convex functions and show how to obtain conditional convex risk measures from them via monotone and (sub)cash invariant hulls. The purpose of this chapter is to illustrate the theory that lead the way.

In Section 7.1 we illustrate that the conditional mean variance can be derived as the cash invariant hull of the semi deviation risk measure. In Section 7.2 we define the conditional monotone mean variance as monotone hull of the conditional mean variance. In both cases we provide a detailed discussion of dual representations and subdifferentials.

7.1. Conditional mean variance as cash invariant hull

In this section, we consider the L^2 type module $L^2_{\mathcal{F}}(\mathcal{E})$ and fix $\beta \in \mathbb{R}$, $\beta > 0$. We define a conditional variant $f : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ of the $L^2(\mathcal{E})$–(semi)–deviation risk measure by

$$f(X) = E[-X \mid \mathcal{F}] + \frac{\beta}{2} E[X^2 \mid \mathcal{F}].$$

One checks that f is proper $L^0(\mathcal{F})$–convex and by Hölder's inequality in the form of (57) f is continuous. Next, we consider the mapping $h : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$

(105)

$$h(X) = \operatorname*{ess.sup}_{Y \in L^0(\mathcal{F})} (f(X-Y) - Y) = \operatorname*{ess.sup}_{Y \in L^0(\mathcal{F})} \left(E[-X \mid \mathcal{F}] - \frac{\beta}{2} E[(X-Y)^2 \mid \mathcal{F}] \right).$$

An element $Y' \in L^0(\mathcal{F})$ which satisfies the first order condition

$$\beta(E[X \mid \mathcal{F}] - Y') = 0$$

is necessarily a maximizer of the integrands $E[-X \mid \mathcal{F}] - \frac{\beta}{2} E[(X-Y)^2 \mid \mathcal{F}]$ of the righthand side of (105). Thus, plugging in the maximizer $Y^* = E[X \mid \mathcal{F}]$ we derive that h is of the form

$$h(X) = E[-X \mid \mathcal{F}] + \frac{\beta}{2} Var[X \mid \mathcal{F}],$$

where $Var[X \mid \mathcal{F}] = E[X^2 \mid \mathcal{F}] - E[X \mid \mathcal{F}]^2$ denotes the (generalized) conditional variance of $X \in L^2_{\mathcal{F}}(\mathcal{E})$. From this we derive that h is proper $L^0(\mathcal{F})$–convex continuous and in particular closed. By Remark 6.4.2 (ii) we therefore know that $h = f_{\mathcal{D}}$ is the greatest cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f.

In line with the relevant literature we refer to $f_{\mathcal{D}}$ as conditional mean variance. Since $f_{\mathcal{D}}$ is continuous Theorem 2.1.7 implies that $\partial f_{\mathcal{D}}(X) \neq \emptyset$ for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$. In particular, for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$ $f_{\mathcal{D}}$ admits a representation of the form

$$f_{\mathcal{D}}(X) = \operatorname*{ess.sup}_{Z \in L^2_{\mathcal{F}}(\mathcal{E})} (E[ZX \mid \mathcal{F}] - f^*(Z)).$$

In what follows we will construct a subgradient of $f_{\mathcal{D}}$ by means of the following lemmas.

LEMMA 7.1.1. *Let* $g : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ *be a function. If* $Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ *satisfies*

$$g(X) = E[Z^* X \mid \mathcal{F}] - g^*(Z^*),$$

then $Z^* \in \partial g(X)$.

PROOF. By definition,

$$g^*(Z) \geq E[ZX \mid \mathcal{F}] - g(X) \tag{106}$$

for all $X, Z \in L^2_{\mathcal{F}}(\mathcal{E})$. Now, let $X, Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ and assume $g(X) = E[Z^* X \mid \mathcal{F}] - g^*(Z^*)$. Then, (106) implies $g(X) \leq E[Z^* X \mid \mathcal{F}] - E[Z^* X' \mid \mathcal{F}] + g(X')$ for all $X' \in L^2_{\mathcal{F}}(\mathcal{E})$, hence $Z^* \in \partial g(X)$. □

LEMMA 7.1.2. *Let* $f_{\mathcal{D}} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ *denote the conditional mean variance. Then,*

$$\mathrm{dom} f^*_{\mathcal{D}} = \{Z \in L^2_{\mathcal{F}}(\mathcal{E}) \mid E[Z \mid \mathcal{F}] = -1\}. \tag{107}$$

Moreover, for all $Z \in \mathrm{dom} f^*_{\mathcal{D}}$

$$f^*_{\mathcal{D}}(Z) = \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}]$$

and, in particular, $(1+Z)/\beta \in \partial f^*_{\mathcal{D}}(Z)$.

PROOF. The conditional mean variance is cash invariant closed $L^0(\mathcal{F})$–convex and $P(f_{\mathcal{D}}) = \Omega$. Hence, Lemma 6.2.3 (iii) yields the inclusion "$\subset$" in (107).

To prove the reverse inclusion in (107), let $Z \in L^2_{\mathcal{F}}(\mathcal{E})$ with $E[Z \mid \mathcal{F}] = -1$. We will show that $f^*(Z) = \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}]$. To this end, observe

$$\begin{aligned}
f^*(Z) &= \operatorname*{ess.sup}_{X \in L^2_{\mathcal{F}}(\mathcal{E})} \left(E[ZX \mid \mathcal{F}] - f(X)\right) \\
&= \operatorname*{ess.sup}_{X \in L^2_{\mathcal{F}}(\mathcal{E})} \left(E[(1+Z)X \mid \mathcal{F}] - \frac{\beta}{2} Var[X \mid \mathcal{F}]\right) \\
&= \operatorname*{ess.sup}_{X \in L^2_{\mathcal{F}}(\mathcal{E}), E[X|\mathcal{F}]=0} \left(E[(1+Z)X \mid \mathcal{F}] - \frac{\beta}{2} Var[X \mid \mathcal{F}]\right) \\
&= \operatorname*{ess.sup}_{X \in L^2_{\mathcal{F}}(\mathcal{E}), E[X|\mathcal{F}]=0} E\left[(1+Z)X - \frac{\beta}{2} X^2 \mid \mathcal{F}\right]. \qquad (108)
\end{aligned}$$

An element $X' \in L^2_{\mathcal{F}}(\mathcal{E})$ which satisfies the first order condition

$$1 + Z - \beta X^* = 0 \tag{109}$$

is necessarily a point wise maximizer of the integrands $(1+Z)X - \frac{\beta}{2}X^2$ in (108) (maximized over all of $L^2_{\mathcal{F}}(\mathcal{E})$). In view of (109) we therefore define the maximizer $X^* = (1+Z)/\beta$; fortunately, $E[X^* \mid \mathcal{F}] = 0$. Plugging X^* into (108) yields the assertion. □

Combining lemmas 7.1.1 and 7.1.2 we conclude: if $Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ maximizes

$$\begin{aligned}
f_{\mathcal{D}}(X) &= \operatorname*{ess.sup}_{Z \in L^2_{\mathcal{F}}(\mathcal{E})} \left(E[ZX \mid \mathcal{F}] - f^*_{\mathcal{D}}(Z)\right) \\
&= \operatorname*{ess.sup}_{Z \in L^2_{\mathcal{F}}(\mathcal{E}), E[Z|\mathcal{F}]=-1} \left(E[ZX \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}]\right) \qquad (110)
\end{aligned}$$

that is

$$f_{\mathcal{D}}(X) = E[Z^* X \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z^*)^2 \mid \mathcal{F}]$$

for some $X \in L^2_{\mathcal{F}}(\mathcal{E})$, then $Z^* \in \partial f_{\mathcal{D}}(X)$.

THEOREM 7.1.3. *Let $f_{\mathcal{D}} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ denote the conditional mean variance. Then, for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$*

$$\beta(X - E[X \mid \mathcal{F}]) - 1 \in \partial f_{\mathcal{D}}(X).$$

PROOF. Let $X \in L^2_{\mathcal{F}}(\mathcal{E})$. Since $f(X - E[X \mid \mathcal{F}]) = f(X) + E[X \mid \mathcal{F}]$ we have $\partial f(X - E[X \mid \mathcal{F}]) = \partial f(X)$. If $Z' \in L^2_{\mathcal{F}}(\mathcal{E})$ satisfies the first order condition

$$X - E[X \mid \mathcal{F}] - \frac{1}{\beta}(1+Z^*) = 0 \tag{111}$$

then Z' is necessarily a point wise maximizer of the integrands

$$Z(X - E[X \mid \mathcal{F}]) - \frac{1}{2\beta}(1+Z)^2$$

in (110) (adjusted for $-E[X \mid \mathcal{F}]$ and maximized over all of $L^2_{\mathcal{F}}(\mathcal{E})$). In view of (111) we define the maximizer $Z^* = \beta(X - E[X \mid \mathcal{F}]) - 1$; fortunately $E[Z^* \mid \mathcal{F}] = -1$ which means that Z^* maximizes (110). $\square$

EXAMPLE 7.1.4. *If we let $\mathcal{F} = \sigma(A_n)$ as in Example 6.1.1 we can nicely relate the preceding results to the static case results presented in* [**FK07**]. *More precisely, we can identify $f : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ with a sequence of static $L^2(\mathcal{E})$–(semi)–deviation risk measures $f = (f_1, f_2, f_3, \ldots)$, where $f_n : L^2(\mathcal{E} \cap A_i) \to \mathbb{R}$ is given by*

$$f_n(X) = E_{P_i}[-X] + \frac{\beta}{2} E_{P_i}[X^2],$$

where $E_{P_i}[\cdot]$ denotes the expectation with respect to the probability measure P_i. As derived above, the greatest cash invariant closed $L^0(\mathcal{F})$–convex function majorized by f is given by the conditional mean variance $f_{\mathcal{D}} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ which we can also identify with a sequence of static conditional mean variances $f_{\mathcal{D}} = (f_{1,\mathcal{D}}, f_{2,\mathcal{D}}, f_{3,\mathcal{D}}, \ldots)$, where $f_{n,\mathcal{D}} : L^2(\mathcal{E} \cap A_n) \to \mathbb{R}$ is given by

$$f_{n,\mathcal{D}}(X) = E_{P_n}[-X] + \frac{\beta}{2} Var_{P_n}[X]$$

where $Var_{P_n}[\cdot]$ denotes the variance w.r.t. the probability measure P_n, $n \in \mathbb{N}$. Further, by Theorem 7.1.3 we know that for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$

$$(\beta(X_1 - E_{P_1}[X_1]) - 1, \beta(X_2 - E_{P_2}[X_2]) - 1, \beta(X_3 - E_{P_3}[X_3]) - 1, \ldots) \in \partial f_{\mathcal{D}}(X),$$

where X_n denotes the restriction of X to $\Omega \cap A_n$ which lies in $L^2(\mathcal{E} \cap A_n)$, $n \in \mathbb{N}$.

Alternatively, we could apply the results of Section 5.3 in [**FK07**]. *According to* [**FK07**] *the greatest cash invariant closed convex function majorized by f_n is given by the classical mean variance $f_{n,\mathcal{D}}$ for each $n \in \mathbb{N}$. Consequently, the greatest cash invariant closed $L^0(\mathcal{F})$–convex function majorized by $f = (f_1, f_2, f_3, \ldots)$ must be $f_{\mathcal{D}} = (f_{1,\mathcal{D}}, f_{2,\mathcal{D}}, f_{3,\mathcal{D}}, \ldots)$. In the same way, one could proceed with the subgradient, which however is not computed in* [**FK07**].

7.2. Conditional monotone mean variance as monotone hull

As in the previous section we consider the L^2 type module $L^2_{\mathcal{F}}(\mathcal{E})$ and fix $\beta \in \mathbb{R}$, $\beta > 0$. To ease notation we denote by $f : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ (in place of $f_{\mathcal{D}}$) the conditional mean variance as introduced in the previous section. In line with Proposition 6.4.1 we define the conditional monotone mean variance $f_{\mathcal{P}^\circ} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ as the greatest monotone (cash invariant) closed $L^0(\mathcal{F})$–convex function majorized by f. That is,

$$\begin{aligned} f_{\mathcal{P}^\circ}(X) &= \operatorname*{ess.sup}_{Z\in\mathcal{P}^\circ} \left(E[ZX \mid \mathcal{F}] - f^*(Z)\right) \\ &= \operatorname*{ess.sup}_{Z\in\mathcal{P}^\circ\cap\mathcal{D}} \left(E[ZX \mid \mathcal{F}] - \frac{1}{2\beta}E[(1+Z)^2 \mid \mathcal{F}]\right) \qquad (112) \end{aligned}$$

By the automatic continuity and subdifferentiability result in the form of Theorem 4.1.4 the conditional monotone mean variance $f_{\mathcal{P}^\circ}$ is continuous and $\partial f_{\mathcal{P}^\circ}(X) \neq \emptyset$ for all $X \in L^2_{\mathcal{F}}(\mathcal{E})$. Again, in what follows, we explicitly construct a subgradient.

Lemma 7.2.1. *Let $f : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ and $\alpha : L^2_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be functions such that α represents f in the sense that $f = \operatorname{ess.sup}_{Z\in L^2_{\mathcal{F}}(\mathcal{E})}(E[Z\cdot \mid \mathcal{F}] - \alpha(Z))$. If $Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ satisfies*

$$f(X) = E[Z^*X \mid \mathcal{F}] - \alpha(Z^*)$$

then $Z^* \in \partial f(X)$.

Proof. Since α represents f we have

$$\alpha(Z) \geq E[ZX \mid \mathcal{F}] - f(X). \qquad (113)$$

Now, let $X, Z^* \in L^2_{\mathcal{F}}(\mathcal{E})$ and assume $f(X) = E[Z^*X \mid \mathcal{F}] - \alpha(Z^*)$. Then, (113) implies $f(X) \leq E[Z^*X \mid \mathcal{F}] - E[Z^*X' \mid \mathcal{F}] + f(X')$ for all $X' \in L^2_{\mathcal{F}}(\mathcal{E})$, hence $Z^* \in \partial f(X)$. □

Lemma 7.2.2. *For all $X \in L^2_{\mathcal{F}}(\mathcal{E})$ and $Z \in L^0_+(\mathcal{F})$ there exists $Y \in L^0(\mathcal{F})$ such that*

$$E[(X+Y)^- \mid \mathcal{F}] = Z.$$

Proof. Let $X \in L^2_{\mathcal{F}}(\mathcal{E})$, $Z \in L^0_+(\mathcal{F})$ and define

$$Y = \operatorname{ess.sup}\{Y' \in L^0(\mathcal{F}) \mid E[(X+Y')^- \mid \mathcal{F}] \geq Z\}.$$

Then Y is as required. Indeed, observe that the function $L^0(\mathcal{F}) \to L^0_+(\mathcal{F}), Y \mapsto E[(X+Y)^- \mid \mathcal{F}]$, is antitone, that is $E[(X+Y_1)^- \mid \mathcal{F}] \geq E[(X+Y_2)^- \mid \mathcal{F}]$ whenever $Y_1 \leq Y_2$. Further,

$$E[(X-n)^- \mid \mathcal{F}] \nearrow +\infty \text{ a.s.}$$

as n tends to $+\infty$. Thus, there exists $Y' \in L^0(\mathcal{F})$ with $E[(X+Y')^- \mid \mathcal{F}] \geq Z$. Hence $Y \in L^0(\mathcal{F})$ and by construction $E[(X+Y)^- \mid \mathcal{F}] \geq Z$. By way of contradiction, assume that $P[A > 0]$, $A = \{E[(X+Y)^- \mid \mathcal{F}] > Z\}$. Let $Y_n = Y + 1/n$, $n \in \mathbb{N}$. Then

$$E[(X+Y_n)^- \mid \mathcal{F}] \nearrow E[(X+Y)^- \mid \mathcal{F}] \text{ a.s.}$$

Hence, $A_n = \{E[(X+Y_n)^- \mid \mathcal{F}] > Z\} \nearrow A$. Thus, there exists $n_0 \in \mathbb{N}$ with $P[A_{n_0}] > 0$. But then,

$$E[(X + 1_{A^c_{n_0}} Y + 1_{A_{n_0}} Y_{n_0})^- \mid \mathcal{F}] \geq Z$$

and $1_{A^c_{n_0}} Y + 1_{A_{n_0}} Y_{n_0} > Y$ on A_{n_0} in contradiction to the maximality of Y. Thus, $E[(X+Y)^- \mid \mathcal{F}] = Z$. □

THEOREM 7.2.3. *Let $f_{\mathcal{P}^\circ} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ denote the conditional monotone mean variance. For $X \in L^2_{\mathcal{F}}(\mathcal{E})$ let $Y \in L^0(\mathcal{F})$ be such that $E[-\beta(X+Y)^- \mid \mathcal{F}] = -1$. Then*

$$-\beta(X+Y)^- \in \partial f_{\mathcal{P}^\circ}(X).$$

(Due to Lemma 7.2.2, such Y exists.)

PROOF. Let $X \in L^2_{\mathcal{F}}(\mathcal{E})$. In view of Lemma 7.2.1, it suffices to show that $Z^* = -\beta(X+Y)^-$ maximizes (112).

Step 1. Due to $f(X+Y) = f(X) + Y$ for all $Y \in L^0(\mathcal{F})$ an element $Z^* \in \mathcal{P}$ maximizes

$$\operatorname*{ess.sup}_{Z \in \mathcal{P}} \left(E[ZX \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}] \right) \tag{114}$$

if and only if it maximizes

$$\operatorname*{ess.sup}_{Z \in \mathcal{P}} \left(E[Z(X+Y) \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}] \right).$$

Thus, we can assume that $E[-\beta X^- \mid \mathcal{F}] = -1$ since else we could replace X by $X+Y$ for the unique $Y \in L^0(\mathcal{F})$ with $E[-\beta(X+Y)^- \mid \mathcal{F}] = -1$.

Step 2. For all $Z \in \mathcal{P}$

$$E[ZX \mid \mathcal{F}] - \frac{1}{2\beta} E[(1+Z)^2 \mid \mathcal{F}] = E\left[ZX - \frac{1}{2\beta} Z^2 \mid \mathcal{F}\right] - \frac{3}{2\beta}.$$

Hence, $Z^* \in \mathcal{P}$ maximizes (114) if and only if it maximizes

$$\operatorname*{ess.sup}_{Z \in \mathcal{P}} E\left[ZX - \frac{1}{2\beta} Z^2 \mid \mathcal{F}\right].$$

For $Z^* \in \mathcal{P}$ the following statements are equivalent:

(i)
$$E\left[Z^*X - \frac{1}{2\beta} Z^{*2} \mid \mathcal{F}\right] = \operatorname*{ess.sup}_{Z \in \mathcal{P}} E\left[ZX - \frac{1}{2\beta} Z^2 \mid \mathcal{F}\right].$$

(ii) For all $Z \in \mathcal{P}$ and $\varepsilon \in [0,1]$,
$$E\left[Z^*X - \frac{1}{2\beta} Z^{*2} \mid \mathcal{F}\right] \geq E\left[Z_\varepsilon X - \frac{1}{2\beta} Z_\varepsilon^2 \mid \mathcal{F}\right],$$
where $Z_\varepsilon = \varepsilon Z + (1-\varepsilon) Z^*$. (Note that $Z_\varepsilon \in \mathcal{P}$ for all $Z \in \mathcal{P}$.)

(iii) For all $Z \in \mathcal{P}$,
$$\frac{d}{d\varepsilon} E\left[Z_\varepsilon X - \frac{1}{2\beta} Z_\varepsilon^2 \mid \mathcal{F}\right] |_{\varepsilon=0} \leq 0.$$

Indeed, for all $Z \in \mathcal{P}$ and $\varepsilon \in [0,1]$
$$E\left[Z_\varepsilon X - \frac{1}{2\beta} Z_\varepsilon^2 \mid \mathcal{F}\right] = \varepsilon Y_1 - \frac{\varepsilon^2}{2\beta} E[(Z - Z^*)^2 \mid \mathcal{F}] + Y_2$$
for some $Y_1 = Y_1(Z, Z^*), Y_2 = Y_2(Z, Z^*) \in F$. In particular, $\varepsilon \mapsto \varepsilon Y_1 - \frac{\varepsilon^2}{2\beta} E[(Z-Z^*)^2 \mid \mathcal{F}] + Y_2$ is point wise concave on $[0,1]$ and hence (iii) implies (ii).

(iv) For all $Z \in \mathcal{P}$, $E\left[(Z - Z^*)\left(X - \frac{1}{\beta}Z^*\right) \mid \mathcal{F}\right] \le 0$.

Hence, $Z^* = -\beta X^- \in \mathcal{P}$ maximizes (114). $\square$

EXAMPLE 7.2.4. *Again we employ the results of Section 5.3 in* [**FK07**] *to derive the above results for the specific case of $\mathcal{F} = \sigma(A_n)$, cf. Example 7.1.4. We identify the conditional mean variance, this time simply denoted by f, with its corresponding sequence of static mean variances $f = (f_1, f_2, f_3, \ldots)$.*

According to the above results, the greatest monotone closed $L^0(\mathcal{F})$–convex function majorized by f is given by $f_{\mathcal{P}^\circ} : L^2_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ identified with $f_{\mathcal{P}^\circ} = (f_{1,\mathcal{P}^\circ}, f_{2,\mathcal{P}^\circ}, f_{3,\mathcal{P}^\circ}, \ldots)$, where $f_{n,\mathcal{P}^\circ} : L^2(\mathcal{E} \cap A_n) \to \mathbb{R}$ is given by

$$f_{n,\mathcal{P}^\circ}(X) = \sup_{Z \in L^2(\mathcal{E}\cap A_n), Z \le 0, E_{P_n}[Z] = -1} \left(E_{P_n}[ZX] - \frac{1}{2\beta} E_{P_n}[(1+Z)^2] \right),$$

for all $n \in \mathbb{N}$.

Alternatively, due to Section 5.3 in [**FK07**] *the greatest monotone closed convex function majorized by f_n is given by the static monotone mean variance $f_{n,\mathcal{P}^\circ}$ for each $n \in \mathbb{N}$. Consequently, the greatest monotone closed $L^0(\mathcal{F})$–convex function majorized by $f = (f_1, f_2, f_3, \ldots)$ must be $f_{\mathcal{P}^\circ} = (f_{1,\mathcal{P}^\circ}, f_{2,\mathcal{P}^\circ}, f_{3,\mathcal{P}^\circ}, \ldots)$.*

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Curriculum Vitae

Personal Data

Name:	Nicolas Vogelpoth
Date of Birth:	08.12.1981
Nationality:	German

Education

Oct 2007 – Oct 2009:	Ph.D. student in Mathematical Finance at University of Vienna Supervisor: Prof. Dr. Damir Filipović Basic Subject: Risk Measures
Jul 2006 – Jun 2007:	Ph.D. student in Mathematical Finance at University of Munich Supervisor: Prof. Dr. Damir Filipović Basic Subject: Risk Measures
Jun 2006:	Diploma in Mathematics, Financial and Insurance Mathematics, at University of Munich
Apr 2002 – Jun 2006:	Studies in Mathematics, Financial and Insurance Mathematics, at University of Munich and studies in Philosophy at Hochschule für Philosophie München
Jun 2001:	Abitur, Otto–Pankok–Schule

Employment

Oct 2007 – Oct 2009:	Junior Scientist, Vienna Institute of Finance
Jul 2007 – Sep 2007:	Summer intern, Morgan Stanley, Global Capital Markets
Mar 2007:	Freelancer, Munich Re
Jun 2006 – Oct 2006:	Freelancer, Munich Re

Research Interests

· Mathematical Finance, in particular Risk Measures
· Convex Analysis
· Stochastic Calculus