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–
Empirical Evidence and Applications”

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Contents

List of Figures

List of Tables

1. Introduction

Since at least Adam Smith, human's attitude towards time has been of utmost importance to social scientists: Adam Smith himself thought that it determined the Wealth of Nations. Norbert Elias (1988) even went one step further and hypothesizes that a "linear and homogeneous" notion of time is a central pillar of none other than the "civilizing process" of modern man.

Building on Smith's insights, economists John Rae (1834), Eugen von Böhm-Bawerk (1884) and Irving Fisher (1930) sought to analyze the motives that influenced intertemporal decisions. They attributed a central role to *impatience* - "the marginal preference for present over future goods" (Fisher, 1930, part II, chapter 4). Impatience, in turn was said to be the product of numerous factors – objective or subjective in nature. Irving Fisher (1930, part II, chapter 4) names the following six subjective factors ("characteristics") that influence a person's impatience in the obvious directions:

1. Foresight
2. Self-control
3. Habit
4. Expectation of life
5. Concern for the life of other persons
6. Fashion

However, after Paul Samuelson published his "A Note on the Measurement of Utility" in 1937, all these considerations where compressed into *the* discount factor. Although Samuelson himself raised concern against the overly simplistic formulation of time preferences in this very paper, the vast majority of economic models that involved intertemporal decisions adopted his approach, that became known as the "exponentially discounted utility model".

Although the model lacked an empirical or normative foundation, it was not until the 1990s that economists turned to alternative models of time preference in increasing numbers. And quite often, they reverted to the "(neo-)classics" mentioned above.

Part I of the present thesis tries to review some of this recent research in the area of time preferences. The structure is as follows:

Chapter 2 briefly treats the axiomatic derivation of time preferences over outcome-date pairs in general and discusses some of the most important characteristics of time preferences. The following chapter then shows how these can be adopted to model preferences over dated streams of outcomes. Chapter 4 presents a number of alternatives to the exponentially discounted utility model and surveys their empirical evidence.

Part II then adopts the least "drastic" among these alternatives (with regard to exponential time preferences) – quasi-hyperbolic time preferences – and discusses how economic analysis changes: as it turns out, this will have dramatic consequences on the structure of most intertemporal models and will even require a different solution concept.

Part I.

Time Preferences

> What then is time? If no one asks me, I know what it is. If I wish to explain it to him who asks, I do not know.
>
> *(Saint Augustine)*

2. Classification of Preferences over Outcome-Date Pairs

In this chapter we give an axiomatic derivation and a characterization of preferences over outcome-date pairs. We will do so along the lines of Ok and Masatlioglu (2007) who came up with a novel approach dubbed 'relative discounting'. Their framework allows for a comprehensive classification of a broad range of time preferences that have been introduced in the literature, while maintaining a certain degree of concreteness and cohesiveness. In particular it captures a certain class of non-transitive time preferences.

In section 2.1 we establish the notation and preliminaries, while section 2.2 states and discusses the assumptions necessary for the important theorem of relative discounting. In the last section of this chapter we then examine the implications of characteristics of time preferences, such as transitivity or stationarity. Preferences over streams will be treated separately in chapter 3.

2.1. Preliminaries

In economics, the standard way to analyze time preferences is to model them as binary relations, $\succeq$, over outcome-date pairs. A generic pair being (x, t), where x denotes a 'prize' that is to be obtained in period t. Prizes are undated and are elements of an unidimensional[1] outcome space $X \subseteq \mathbb{R}$. In this section it will be convenient to employ a continuous and infinite notion of time, i.e. $t \in T = [0, \infty)$. Therefore, preferences are binary relations over the outcome-date space $\mathcal{X} \equiv X \times T$.

As usual in this context, $(x, t) \succeq (y, s)$ means "not to prefer (y, s) over (x, t)", while $\sim$ denotes that both, $(x, t) \succeq (y, s)$ and $(y, s) \succeq (x, t)$ hold, which we interpret as "the decision maker is indifferent between (x, t) and (y, s)."

Moreover, let $\succeq_t$ denote the t^{th} (canonical) projection of $\succeq$, i.e. the ordering of outcomes that are both due at time t, which we can interpret as the **material tastes** for outcomes that will be obtained at t. Formally, $\succeq_t$ is defined as $x \succeq_t y \Leftrightarrow (x, t) \succeq (y, t)$. In this notation, $\succeq_0$ are for instance the material tastes at time 0.

Following Ok and Masatlioglu (2007, pp.217) we stress that the preferences, denoted by $\succeq$, are the *commitment preferences* of an agent, i.e. we interpret $(x, t) \succeq (y, s)$ as

[1]Note however, that all results brought forward in this section also generalize to multidimensional prizes

"if she could commit in period 0 to either 'consume' x in period t or y in period s, she would (weakly) prefer the former". Troughout part I of the thesis we will assume that the decision maker is indeed able to commit to an action in period 0. In part II we will then drop this assumption and discuss the implications. Additionally, we will facilitate the analysis by explicitly ruling out a dimension of risk or uncertainty, i.e. we say that if a the decision maker opts for (x, t) she receives the pair with certainty.

We are now ready to define time preferences in the following sense (Ok and Masatlioglu, 2007, p.218):

Definition For an outcome space X, a binary relation $\succeq$ on $\mathcal{X}$ is a **time preference** on $\mathcal{X}$, denoted by $(\mathcal{X}, \succeq)$ if

1. $\succeq$ is **complete**: For every (x, t) and (y, s) in $\mathcal{X}$ either $(x, t) \succeq (y, s)$ or $(y, s) \succeq (x, t)$ or both (which also implies reflexivity).

2. $\succeq$ is **continuous** in the following sense:[2] Let a_n and b_n be convergent sequences in $\mathcal{X}$ in the Eucledian Norm, s.t. $a_m \succeq b_n$, $\forall n$, then it we require that also $\lim a_n \succeq \lim b_n$ has to hold.

3. $\succeq_0$ is **complete and transitive**, where transitivity of $\succeq_0$ means that for $x, y, z \in X$: $x \succeq_0 y$ and $y \succeq_0 z \Rightarrow x \succeq_0 z$. So we assume that the ranking over the goods that are available right now is transitive and complete.

4. $\succeq_0 = \succeq_t$ for all t. Therefore the **material tastes are the same throughout time**. This restriction explicitly rules out, say, changing tastes as the decision maker becomes older. Likewise, this assumption does not allow for an increased "demand" of for a bottle champagne on New Year's Eve (in the realm of multi-dimensional prizes).

Note that transitivity of $\succeq$ does *not* follow from the transitivity of $\succeq_t$. So, the preferences may generate cycles like $(x, t) \succ (x, t+2) \sim (x, t+1) \sim (x, t)$ but not cycles like $(x, t) \succ (y, t) \sim (z, t) \sim (x, t)$. In other words, we restrict ourselves to cycles that "arise due to the passage of time" (Ok and Masatlioglu, 2007, p. 218). Moreover, the assumptions we make below only permit cycles that involve three or more time periods.

Furthermore, when $(\mathcal{X}, \succeq)$ is not transitive, we cannot represent the preferences by a utility function, say, $w(x, t)$, in the usual manner: $(x, t) \succeq (y, s) \Leftrightarrow w(x, t) \geq w(y, s)$: for preference relation to be represented by a utility function it necessary that it is transitive and complete (see e.g. Mas-Colell, Whinston and Green, 1995,

[2] This concept of continuity is often called "sequential continuity". There is, however, a variety of other concepts of continuity for binary relations. For an overview see e.g. Baroni and Bridges (2008).

p.9). However, as Ok and Masatlioglu showed, we are able to represent them in a different way, provided that some additional assumptions hold: Theorem 1 says that $(x,t) \geq (y,s) \Leftrightarrow u(x) \geq \eta(s,t)u(y)$.

2.2. The Relative Discounting Representation and its Axioms

The following assumptions on $(\mathcal{X}, \succeq)$ will allow us to represent the time preferences by a combination of a (static) outcome utility function and a *relative* discount function, given in 1.

Axiom RD1 (Time Sensitivity): For any $x, y \in X$ and $t \geq 0$ there exists an $s \geq 0$ s.t. $(x,t) \succeq (y,s)$

Roughly speaking, this assumption ensures that the decision maker will not prefer an outcome-date pair that is delayed by a sufficiently large amount of time. Implicitly, this assumption already rules out *negative time preferences* (which would be that the decision maker *always* favours delays): Suppose for example that $t = 0$ and that $y \succeq_0 x$, s.t. $(x,0) \succeq (y,0)$. Then, it does not follow from assumption RD1 that delaying y makes it more desireable. Note however, that although this assumption treats delaying and expediting asymmetrically, it does not say that a delay **always** results in a loss of "attractiveness".

Axiom RD2 (Outcome sensitivity): For any $x \in X$ and $s,t \geq 0$, there exist $y,z \in X \backslash \{x\}$ s.t. $(z,s) \succeq (x,t) \succeq (y,s)$

This assumption is the natural counterpart of axiom RD1 and rules out e.g. lexicographic preferences over outcome-date pairs. Intuitively, this axiom states that delay can always be compensated with higher outcomes.

Axiom RD3 (Monotonicity): For any $x,y,z \in X$ and $s,t,r \geq 0$, if $t \geq r$ and $z \succeq_0 x$ then $(x,t) \succeq (y,s) \Rightarrow (z,r) \succeq (y,s)$.

This assumption strengthens the antecedents in the way that it brings forward the obvious notion of positive time preferences (or impatience): attributed to human preferences since at least Irving Fisher's Theory of Interest (1930): People always prefer sooner pleasures to deferred ones. If we assume for a second that X is a space of monetary outcomes, then assumption RD3 may be bluntly interpreted as: more money is always good whereas delay is always bad.

The next two assumptions guarantee that we can separate the effects of outcomes on preferences from the effects of time in the certain sense of Theorem 1.

Axiom RD4 (Separability): For any $x,y,z,w \in X$ and $s_1,s_2,t_1,t_2 \geq 0$ if $(x,t_1) \sim (y,s_1), (z,t_1) \sim (w,s_1)$ and $(x,t_2) \sim (y,s_2) \Rightarrow (z,t_2) \sim (w,s_2)$

As Ok and Masatlioglu (2007, p.220) put it, axiom RD4 ensures that the premium for delay is separated from the particular reward:

For the sake of argument suppose $t_1 < s_1$, $t_2 < s_2$ and that the decision maker told us that she is indifferent between getting x at period t_1 and getting y at s_1. Therefore we can think of $y - x > 0$ a premium that is needed to compensate the decision maker for delaying x from t_1 to s_1. Similarly, she can be compensated by a premium $w - z$ for postponing z from t_1 to s_1. If we additionally know, that the compensation for delaying x from t_2 to s_2 is also $y - x$, then axiom RD4 ensures that the compensation of delaying z is the same as before: $w - z$.

If we adopt a multidimensional prize space, this assumption ensures that discounting for, say, one's physical health, is the same as discounting for cigarette puffs.

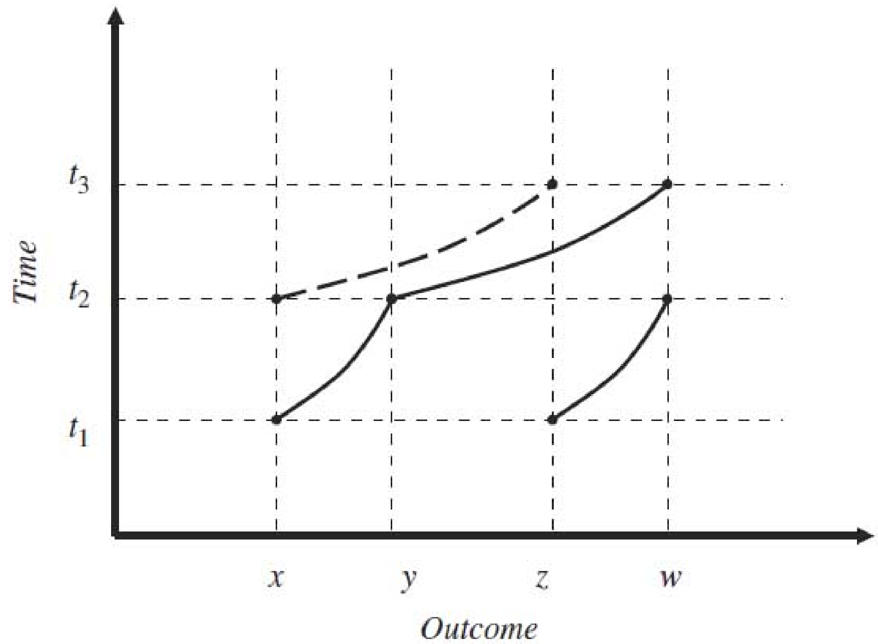


Figure 2.1.: An illustration of Axiom RD5 (Path Independence). Source: Ok and Masatlioglu (2007, p.221)

Axiom RD5 (Path Independence): For any $x, y, z, w \in X$ and $t_1, t_2, t_3 \geq 0$ if $(x, t_1) \sim (y, t_2), (z, t_1) \sim (w, t_2)$ and $(y, t_2) \sim (w, t_3) \Rightarrow (x, t_2) \sim (z, t_3)$

This path independence property ensures that the aggregate premium for delaying rewards are independent of their order (p.221): For the sake of illustration, suppose that $t_1 < t_2 < t_3$ and that $x < y < z < w$ (as depicted in figure 2.1). We will now "extract" premia in two different ways: First, let us assume that x is postponed twice:

from t_1 to t_2 and then again from t_2 to t_3. In order to make the decision maker willing to accept these two delays, we have to compensate her with a premium of at least $y-x$ for the first delay and then with $w-y$ for the second one. Therefore the aggregate premium involved is $w-x$.

Second, let consider a different order: We start with z and delay it from t_1 to t_2 requiring a premium of $w-z$. Next, we postpone x from t_2 to t_3. The associated aggregate premium would be $(w-z)+(\xi-x)$.

The aggregate premia are the same if $\xi = z$, which is precisely what axiom RD5 requires. Since this is quite an unintuitive assumption it is hoped that figure 2.1 brings some clarification.

Axiom RD6 (Monotonicity in prices): $\succeq_0$ is strictly increasing on X.

Axiom RD6 simply states that the elements of X are ordered according to their "attractiveness", which in a sense is a matter of convention.

We are now ready to formulate

Theorem 1 ***[Relative Discounting]*** *(Ok and Masatlioglu, 2007) Let X be an open interval and $\succeq$ a binary relation on $\mathcal{X}$. $\succeq$ is a time preference that satisfies axioms RD1-RD6, if and only if, there exists an homeomorphism $u: X \to \mathbb{R}_{++}$ and a continuous map $\eta: \mathbb{R}^2_+ \to \mathbb{R}_{++}$ such that, for all $x, y \in X$ and $s, t \geq 0$,*

$$(x,t) \succeq (y,s) \Leftrightarrow u(x) \geq \eta(s,t)u(y), \tag{2.1}$$

where

$$\eta(s,t) \text{ is strictly decreasing in its first argument and } \lim_{s\to\infty} \eta(s,t) = 0 \tag{2.2a}$$

$$\eta(t,s) = 1/\eta(s,t) \tag{2.2b}$$

$$u(x) \text{is increasing.} \tag{2.2c}$$

One important feature of this representation is that – given axioms RD1-RD6 hold – when evaluating the ranking of two outcome-date pairs, we can separate the "material ranking" from the timing: A static, undated utility $u(\cdot)$ and a *relative* discount factor $\eta(\cdot)$. In the light of this representation, $(x,t) \succ (y,s)$ is then interpreted as (Ok and Masatlioglu, 2007, pp.222-223):

> From the perspective of time 0, the worth at time t of the utility of y that is to be obtained at time s is strictly less than the worth at time t of the utility of x that is to be obtained at time t.

Again, we stress that these are *commitment preferences*, i.e. the decision maker commits to her decision in period 0.

Furthermore, we observe that by (2.2b) $\eta(t,t) = 1/\eta(t,t) \Rightarrow \eta(t,t) = 1$ since $\eta(s,t) > 0$, which corresponds to the assumption that time does not alter the material preferences.

Moreover, our assumption of positive time preferences, i.e. impatience at every period in time, is reflected by $0 < \eta(s,t) < 1 \Leftrightarrow s > t$: To see why this holds, suppose first that $0 \leq t < s < \infty$, then by the monotonicity of the discounting term: $0 < \eta(s,t) < \eta(t,t) = 1$ Conversely, suppose $0 < \eta(s,t) < 1 \overset{(2.2b)}{\Rightarrow} 0 < \eta(s,t) < \eta(t,t) \overset{(2.2a)}{\Rightarrow} s > t$.

Figure 4.7 (on page 59) shows six plots of the relative discount function, η, each one corresponding to a different kind of time preference.

2.3. Uniqueness

Theorem 2 ***[Uniqueness]*** *If a time preference $(\succeq, \mathcal{X})$ that satisfies RD1-RD6 is represented by (u, η) then it is also represented by (v, θ) if, and only if, $v = bu^a$ and $\theta = \eta^a$ for $a, b > 0$.*

This indicates that the structure of the preferences, imposed by RD1-RD6, restricts the permissible transformations up to simultaneous exponential transformations and multiplication with a positive constant. In other words, once we fixed the functional form of the relative discounting term, the static utility function is unique up to a proportional transformation. Therefore, we have to adopt a concept of "cardinal utiltiy" similar to von Neumann-Morgenstern utility in expected utility theory. But compared with von Neumann-Morgenstern utility we have one degree of freedom less when comparing outcome-date pairs.

2.4. Characteristics of Time Preferences

Theorem 1 is able to deal with a fairly broad class of time preferences. In this section we will discuss how characteristics of time preferences relate to the discount term, $\eta(\cdot,\cdot)$ in the relative discount representation.

2.4.1. Transitivity and Absolute Discounting

Up to now, we only assumed the material tastes $\succeq_t$ on X to be transitive. Now we will strengthen this assumption and discuss how "global" transitivity of $\succeq$ on $\mathcal{X}$ changes our analysis. Since "global" transitivity implies transitivity of $\succeq_t$ for all t, axioms RD1-RD6 still hold and we are therefore able to analyse such preferences within the framework of relative discounting. On top of that, we expect this assumption to facilitate our analysis and indeed, we observe that the transitivity of time preferences is interrelated with the relative discount function in the following way:

$$\eta(t,r) = \eta(t,s)\eta(s,r) \quad \forall r,s,t \geq 0 \tag{2.3}$$

We will show the "if" part of the proof for the case of indifference, "$\sim$":

$$\begin{gathered}(x,t) \sim (y,s), (y,s) \sim (z,r) \Rightarrow (x,t) \sim (z,r) \\ \text{which, by Theorem 1} \Longleftrightarrow \\ u(x) = u(y)\eta(s,t), u(y) = u(z)\eta(r,s) \Rightarrow u(x) = u(z)\eta(r,t) \\ \text{therefore, } \eta(r,t) = \eta(s,t)\eta(r,s) \\ \text{which, by part (2) of Theorem 1: } \Leftrightarrow \eta(t,r) = \eta(t,s)\eta(s,r)\end{gathered}$$

We can therefore exploit the transitivity in order to obtain the usual formulation of *absolute discounting*, where the time-perspective only enters in an *absolute* manner (Ok and Masatlioglu, 2007, p.224):

Theorem 3 ***[Absolute Discounting]*** *Let X be an open interval and $\succeq$ a transitive binary relation on $\mathcal{X}$. Then $\succeq$ is a* ***transitive time preference*** *that satisfies axioms RD1-RD6 if, and only if, there exist an increasing homemorphism $u : X \to \mathbb{R}_{++}$ and a decreasing and continuous map $\delta : \mathbb{R}_{+} \to (0,1]$ s.t. $\delta(0) = 1$, $\lim_{t\to\infty} \delta(t) = 0$ and*

$$(x,t) \succeq (y,s) \Leftrightarrow \delta(t)u(x) \geq \delta(s)u(y) \tag{2.4}$$

for all outcome-date pairs in $\mathcal{X}$.

Transitivity, as embodied in equation 2.3 allows us to separate the *relative* discounting function $\eta(s,t)$ into the quotient of two *absolute* discount functions: $\delta(s)/\delta(t)$. Put differently, we define a new function $\delta(t) \equiv \eta(t,0)$.

From another perspective, the assumption of transitivity together with the definition of time preferences and axioms RD1-RD3 ensure that the conditions of Debreu (1954)'s Theorem II are satisfied. Therefore, the ranking of outcome-date pairs can be represented by a utility function, say, $w(x,t)$ in the usual manner. The continuity assumptions of the time preferences even ensure that the utility function is continuous in both arguments. Furthermore, the separability assumptions RD4 and RD5 enable us to write $w(\cdot)$ as $\delta(t)u(x)$.[3]

The notion of absolute discounting (or just "discounting") also gives rise to a different interpretation of the preferences: We call the utility associated with an outcome-date pair, $\delta(t)u(x)$, the **present value** of x and say that $(x,t) \succ (y,s)$ whenever the present value of x exceeds the present value of y.

[3]In the presence of transitivity, one may derive this result with a somewhat simpler set of assumptions. In particular, one can impose a single separability condition that is less restrictive then RD4 and RD5 (see e.g. the "Thomsen Condition" (condition A6) presented in Fishburn and Rubinstein, 1982).

Virtually all economic models employ transitive time preferences, the most important being exponential discounting and (quasi-)hyperbolic disounting (see section 4).

2.4.2. The Discount Rate and the Discount Factor

For this class of transitive time preferences it is useful, to describe the shape of the (absolute) discount functions by two measures: the discount rate and the discount factor.

The discount rate describes the "rate of impatience" that is induced by the rate of decline of the discount function. It seems natural to capture this effect by its (negative) instantaneous growth rate, which for discount functions is usually refered to as the discount rate (e.g. Chabris, Laibson and Schuldt, 2008):

Definition In continuous time the **discount rate**, denoted by $\rho(t)$, of a differentiable discount function, $\delta(t)$, is given by

$$\rho(t) \equiv -\frac{\delta'(t)}{\delta(t)} \quad \forall t > 0 \tag{2.5}$$

Obviously, in the case of non-differentiable discount functions (see e.g. section 4.4) this is not a viable definition. Moreover, if we have a discrete notion of time, we are not interested in the rate of impatience for infinitesimal changes in time, but for the change from one period to another[4] (Laibson, 2003):

Definition In discrete time the **discount rate** of a discount function $\delta(t)$ is given by[5]

$$\rho(t) \equiv -\frac{\delta(t) - \delta(t-1)}{\delta(t)} = \frac{\delta(t-1)}{\delta(t)} - 1 \tag{2.6}$$

Another, perhaps more intuitive way to motivate the discount rate is the following: Suppose a decision maker can consume x in period 0, giving him a utility of $u(x)$. The discount rate, $\rho(1)$ tells us, how much more utility (in percentage terms) we have to offer her, so that she is just indifferent between consuming x now or in the next period. Put differently, $\rho(t)$ tells about the minimum compensation required for delaying a

[4]More general, we have to define the frequency of our observations first: suppose that time is measured in years. Then of course, we could decide to partition every year into twelve months. The length of the periods is then given by Δ (e.g. $\Delta = 1/12$ *years*). The discount rate is then given more generally by: $\rho(t) \equiv -\frac{(\delta(t)-\delta(t-\Delta))/\Delta}{\delta(t)}$. As time is measured finer and finer, i.e. $\Delta \to 0$, the continuous and the discrete formulation coincide. For simplicity, we use the definition where $\Delta = 1$.

[5]Note that some authors, e.g. Read (2003), give the alternative definition $\rho(t) \equiv -\frac{\delta(t)-\delta(t-1)}{\delta(t-1)}$, which captures the change of the discount function relative to the *previous* period.

prize from period $t-1$ to t. Therefore the discount rate can be interpreted as the "rate of impatience": the higher the discount rate the higher the impatience of the decision maker.

Note, that it follows from assumptions RD1 (time sensitivity) and RD3 (monotonicity) that the discount rate $\rho(t)$ is strictly positive – in other words the decision maker always needs a premium so that she is willing to postpone consumption.

By construction, the discount rate is independent of the size of the prize (it only depends on the discount function, which in turn has only time as its argument). This can also be seen as a direct consequence of assumption RD4 (separability), where we explicitly required that the compensation for delay can be seperated from the size of the prize that is to be delayed.

Another useful concept to capture the impatience that is induced by a discount function is the discount factor. Again, we provide the definitions for both, the continuous and the discrete case:

Definition In **continuous time**, the **discount factor** of a differentiable discount function is defined in the following way (Chabris, Laibson and Schuldt, 2008):

$$\phi(t) \equiv \lim_{h \to 0} \left(\frac{1}{1+\rho(t)h} \right)^{\frac{1}{h}} = [\lim_{g \to \infty} \left(1 + \frac{\rho(t)}{g} \right)^{g}]^{-1} = e^{-\rho(t)} \quad \forall t > 0 \tag{2.7}$$

Definition In **discrete time**, the **discount factor** is defined in the following way (Laibson, 2003)

$$\phi(t) \equiv \frac{1}{1+\rho(t)} = \frac{\delta(t)}{\delta(t-1)} \quad \text{for } t = 1, 2, \ldots \tag{2.8}$$

The interpretation of the discount factor in discrete time is straightforward: From the perspective of period 0, the discount factor for period t tells us, how much *additional discounting* is involved between period $t-1$ and period t.

Clearly, assumptions RD1 and RD3 require that $0 < \phi(t) < 1$, i.e. the decision maker is always sensitive to additional delay.

Furthermore, note that the value of the discount factor is inversely related to the discount rate: therefore, a high (close to 1) discount factor at period $t+1$ shows that delaying an outcome for one more period is not perceived as very harmful. On the other hand, a discount factor of close to zero implies that the decision maker does not care much about future satisfaction.

It might seem as a trivial observation, but it will proof to be very useful to note that in discrete time this allows us to write any (absolute) discount function in terms of discount factors: Starting at period 0 the discounting involved in waiting an additional period is

$$\phi(1) = \delta(1)/\delta(0) = \delta(1) \tag{2.9}$$

An additional delay of one period gives us

$$\underbrace{\frac{\delta(1)}{\delta(0)}}_{\phi(1)} \underbrace{\frac{\delta(2)}{\delta(1)}}_{\phi(2)} = \delta(2) \tag{2.10}$$

Iterating brings us to the result that we can write every discount function in terms of discount factors:

$$\delta(t) = \prod_{i=1}^{t} \phi(i) \quad t = 1, 2, \ldots \tag{2.11}$$

If it is the case that the sequence $\{\phi_i\}_{i=1}^{t}$is decreasing, i.e. $\phi(t+1) < \phi(t)$ then we can say that from the perspective of period 0 the decision maker perceives additional delays as increasingly harmful and we can therefore say that the rate of impatience is *increasing*. Conversely, if the sequence is increasing, then the decision maker is more patient for longer planning horizons.

If we plug this representation of the discount function into equation 2.4 of Theorem 3 we yield that for $s \geq t \geq 1$

$$(x,t) \succeq (y,s) \Leftrightarrow u(x) \geq u(y) \prod_{i=t+1}^{s} \phi(i) \tag{2.12}$$

Equation 2.12 is of course nothing else but a special case of the relative discount representation given in equation 2.1 for transitive time preferences: here, $\eta(s,t) = \prod\limits_{i=t+1}^{s} \phi(i)$. However, in the case of transitivity, we may interpret $\eta(s,t)$ as the **conditional** discount function:

$$\delta(s|s \geq t) \equiv \frac{\delta(s)}{\delta(t)} \tag{2.13}$$

Analogously, also in continuous time we can write (differentiable) discount functions in terms of discount factors: We take the definition of the discounting *rate* (equation 2.5) and solve the first order differential equation:

$$-\rho(t) \equiv \frac{\delta'(t)}{\delta(t)} \quad \forall t > 0 \tag{2.14}$$

We integrate both sides with respect to time:

$$-\int_0^t \rho(\tau)d\tau + c = \ln|\delta(t)| \tag{2.15}$$

$$C\exp\left(-\int_0^t \rho(\tau)d\tau\right) = \delta(t) \quad \forall t > 0 \tag{2.16}$$

normalizing $C = 1$ gives us the desired result. Again, we observe that an increasing function of discount factors implies a declining "rate of impatience".

As in the case of discrete time, we can use this identity to express the discount representation of preferences in the following way:

$$(x,t) \succeq (y,s) \Leftrightarrow u(x) \geq u(y)\exp\left(-\int_t^s \rho(\tau)d\tau\right) \quad \forall s > t > 0 \tag{2.17}$$

These observations may seem as tautologies but are clearly a direct consequence of the transitivity of the time preferences: The discounting that is "shared" by two outcome-date pairs, (or: takes place *up to* period t) does not play a role in the decision process. We stress that this is not to be confused with stationarity (see following section)!

2.4.3. Stationarity

Stationarity is a feature of time preferences that ensures a certain degree of temporal homogeneity, where the time effect is incorporated only by the **difference** between the dates on which prizes are obtained, formally

Definition (Fishburn and Rubinstein, 1982) A time preference on $\mathcal{X}$ is called **stationary** if

$$\begin{gathered}(x,t) \succ (y,s) \Leftrightarrow (x,t+\tau) \succ (y,s+\tau) \\ \forall (x,t),(y,s) \in \mathcal{X} \text{ and } \tau \in \mathbb{R} \text{ s.t } s+\tau, t+\tau \geq 0\end{gathered} \tag{2.18}$$

Within the framework of relative discounting this translates into

Theorem 4 ***[Stationarity]*** *$(\succeq, \mathcal{X})$ is a* ***stationary time preference*** *that satisfies axioms RD1-RD6 if, and only if, there exists an increasing homeomorphism $u : X \rightarrow$*

$\mathbb{R}_{++}$ *and a decreasing and continuous map* $\zeta : \mathbb{R} \to \mathbb{R}_{++}$ *s.t.*

$$(x,t) \succeq (y,s) \Leftrightarrow u(x) \geq \zeta(s-t)u(y) \quad \forall (x,t),(y,s) \in \mathcal{X} \tag{2.19}$$

with $\lim_{a\to\infty} \zeta(a) = 0$ *and* $\zeta(a) = 1/\zeta(-a),\ \forall a \geq 0$

Again, we stress that stationarity should not be confused with the result of equation (2.12): While stationarity means that the only way that timing influences the ranking of outcome-date pairs is by the difference of their receival times, transitivity merely requires that it does not matter how much the decision maker only focuses on the effect of the additional delay of the pair that is to be received later. This effect will in general be different across time.

Taken together with transitivity this poses enough structure on the time preferences so that the discount function $\delta(\cdot)$ is pinned down to an exponential function. To sketch why this is the case, note that by stationarity of $(\succeq, \mathcal{X})$, $(x,t) \succeq (y,s) \Leftrightarrow (x,0) \succeq (y, s-t)$. By transitivity and Theorem 3 this holds if, and only if $\delta(s)/\delta(t) = \delta(s-t)$. By defining $r \equiv s-t$ this can be rewritten to $\delta(r)\delta(t) = \delta(r+t)$, which gives rise to the conjecture that $\delta(t) = \delta^t$. Since we required $0 < \delta(t) \leq 1$ and that δ is decreasing in t, it must be the case that $0 < \delta < 1$. Moreover, it can be shown that this is the only continuous and decreasing function that satisfies the (exponential Cauchy) equation $f(s+t) = f(s)f(t)$.[6]

The consequences for discounting when both, transitivity and stationarity, *jointly* hold are summarized in the following Theorem (Ok and Masatlioglu, 2007):

Theorem 5 ***[Exponential Discounting]*** *Let X be an open interval and $\succeq$ be a binary relation on $\mathcal{X}$. Then $\succeq$ is a* ***transitive and stationary time preference*** *that satisfies axioms RD1-RD6 if and only if there exists an increasing homeomorphism $u : X \to \mathbb{R}_{++}$ and a $\delta \in (0,1)$ s.t.*

$$(x,t) \succeq (y,s) \Leftrightarrow \delta^t u(x) \geq \delta^s u(y) \quad \forall (x,t),(y,s) \in \mathcal{X} \tag{2.20}$$

One way to see, that only the difference of the receival-dates of the outcomes matters, divide both sides of the above equation by δ^t in order to obtain $u(x) \geq \delta^{(s-t)}u(y)$. Therefore in the case of transitivity the discounting function $\zeta(s-t)$ (see Theorem 4 equals $\delta^{(s-t)}$.

One important point is that due to the ambuiguity of the discount representation (see Theorem 2), δ can vary freely in the interval $(0,1)$ unless the discount function u is fixed (up to multiplication with a positive constant).

[6] The constant solution $f(t) \equiv 0$ is ruled out since we only look for decreasing functions.

This relatively simple representation of stationary and transitive preferences also gives rise to a different interpretation of the discounting term (Manzini and Mariotti, 2007, p. 4): First we rescale the present values by taking logs of the present values which gives

$$\log u(x) + t \log \delta \geq \log u(y) + s \log \delta$$

Dividing by $-\log \delta$ and defining $v(\cdot) \equiv -\log u(x) \log \delta /$ (whereby we exploit that the utility function is only unique up to multiplication with a positive constant) we obtain the form:

$$(x,t) \succeq (y,s) \Leftrightarrow v(x) - t \geq v(y) - s \quad \forall (x,t),(y,s) \in \mathcal{X} \tag{2.21}$$

2.4.4. Present Bias

One natural counterpart of stationarity is what economists came to call **present bias**: Special weight is attached to outcomes that are due today. One way[7] to formalize this idea is (Ok and Masatlioglu, 2007, p. 225)

Definition A time preference $(\succeq, \mathcal{X})$ exhibits **present bias** if

$$(x,t) \succeq (y,s) \;\Rightarrow\; (x,0) \succeq (y, s-t) \quad \forall x,y \in X,\; s > t \geq 0 \tag{2.22}$$

and if moreover, for any $s > t > 0$ there exist $x, y \in X$ such that

$$(x,t) \sim (y,s) \;\Rightarrow\; (x,0) \succ (y, s-t) \tag{2.23}$$

The framework of relative discounting can not only accomodate preferences of that kind, there is also the following connection between a present bias and the relative discounting term:

Theorem 6 ***[Present Bias]*** *(Ok and Masatlioglu, 2007) $(\succeq, \mathcal{X})$ is a time preference that satisfies axioms RD1-RD6 and has present bias if, and only if, $\succeq$ is represented by some (u, η) s.t. $\eta(s,t) \geq \eta(s-t, 0)$ holds whenever $s > t > 0$ for all with strict inequality ($>$) for some dates.*

When a time preference exhibits present bias, we can therefore say that the difference in the timing of two future outcomes is certainly not getting less important when both are speeded up so that the sooner one is can be obtained today. In the class of transitive time preferences that allow for an (absolute) discount representation, this translates to a lower discount factor at period 0 than at other points in time:

[7] see e.g. Hayashi (2003, p.348) for a slightly different definition

Present biased time preferences are one form of time preferences that allow for "preference reversals". For some outcome-date pairs (x, t) and (y, s) the preferences are reversed when both outcomes are delayed or speeded up. The classic example is due to Thaler (1981): Although a decison maker might choose one apple today over two apples tomorow, she as well might choose two apples in 101 days over one apple in 100 days. Stationarity on the other hand would require that she picked one apple in both choices: Applying the definion of stationarity as given in equation 2.18 yields

$$\begin{aligned} (\text{one apple}, 0 \text{ days}) \succeq (\text{two apples}, 1 \text{ day}) \Rightarrow \\ (\text{one apple}, 0 + 100 \text{ days}) \succeq (\text{two apples apple}, 1 + 100 \text{ days}) \end{aligned} \tag{2.24}$$

The simplest type time preferences that exhibit such a present bias is called "quasi-hyperbolic" time preference and is discussed in section 4.4.

3. Time Preferences over Infinite Streams of Outcomes

So far we discussed preferences over outcome-date pairs. In this chapter we will focus on preferences over dated *streams* of outcomes, which we will sometimes call "schedules". Also in this chapter, we will restrict ourselves to deterministic streams and assume that the decision maker has commitment power, i.e. once the decision maker has made a decision, she cannot alter it as time goes by. Therefore it suffices to analyze the decision makers preferences at time 0. From now on we will restrict ourselves to the case of discrete time, whenever possible.

We say that a **stream of outcomes** attributes an outcome $a_t \in A_t \subseteq \mathbb{R}^n$ to every point in time $t \in T$. We employ a discrete notion of time i.e. break down time into periods of equal length: $T = \mathbb{N}_0$. Furthermore, we require the outcome space to be the same in every period $A_s = A_t = A \quad \forall s, t \in T$. A stream of outcomes, or schedule, $a = (a_0, a_1, \ldots)$, is an element of $\prod_{t\geq 0} A \equiv \mathcal{A}$, where $\prod$ denotes the cartesian product. Sometimes we will denote parts of streams that start in period s (and end in period t) with ${}_sa$ (${}_sa_t$).

In economics, streams are for example, bundles of goods $(a_t)_{t\geq 0}$ where $a_t = (a_{t1}, a_{t2}, \ldots, a_{tn}) \in \mathbb{R}^n_+$. In a broad range of dynamic models economists conveniently assume that there is only a single consumption good $c_t \in \mathbb{R}_+$. Due to the infinite number of periods, the decision maker is often called "dynasty".

3.1. Aggregation of Present Values

Ever since its introduction by Paul Samuelson in 1937, the overwhelming majority of economics models implemented preferences over such infinite streams in the following way, which is often referred to as "additive discounted utility" or simply "discounted utility" (DU) representation of the preferences over streams:[1]

$$a \succeq b \Leftrightarrow \sum_{t\geq 0} \delta(t)u(a_t) \geq \sum_{t\geq 0} \delta(t)u(b_t) \quad \forall a, b \in \mathcal{A} \tag{3.1}$$

with $0 < \delta(t) < 1$ for $t > 0$ and $\delta(0) = 1$.

[1] see Weibull (1985) for the derivation of a "general discount representation" of preferences over streams, that also allow for negative discount rates

3.1.1. Intertemporal Noncomplementarity

As we will see, this is of course the most straightforward way to model time preferences over infinite streams of outcomes. But despite its popularity it was not until the 1960's that economists started to provide an axiomatic foundation, which indicated a highly restricted domain:

With respect to the classification in section 2.4 one assumes **transitivity over *date-outcomes pairs***, so that their rankings can be represented by their present value, $\delta(t)u(a_t)$, where $\delta(t)$ is for example one of the discounting functions, discussed in section 4 – Samuelson himself formulated the DU model with an exponential discounting function. In addition we postulate (Weibull, 1985):

Axiom S1 (strict intertemporal noncomplementarity): For every $a, b, c \in \mathcal{A}$ it holds that

$$a \succeq b \Leftrightarrow a + c \succeq b + c \tag{3.2}$$

Note that up to now, intertemporal complementarities were not an issue, since the decision maker received only single outcomes that were mutually exclusive: She *either* gets (x, t) *or* (y, s). In the realm of streams, however, there is room for this highly disputed assumption: As Koopmans (1960), who was probably the first to come forward with an axiom like this, put it, "[o]ne cannot claim a high degree of realism for such a postulate, because there is no clear reason why complementarity of goods could not extend over more than one time period." Samuelson (1952, p.674) made this point somewhat more crisply: "The amount of wine I drank yesterday and will drink tomorrow can be expected to have effects upon my today's indifference slope between wine and milk."

3.1.2. Critique: Habit Formation and Anticipated Utility

Concerns like these motivated economists (e.g. Constantinides, 1990) to study consumption under *habit formation*: Roughly speaking, habit formation tries to capture the effect that over time, people get used to a certain standard of consumption, a habit, therefore consumption in one period also directly influences the marginal utility of future consumption. Wathieu (1997) used a model model of habit formation in a finite time-framework to explain some of the anomalies of the exponential discounting model discussed in section 4.1.

Other authors adopted a similar approach but posited models where individuals also gain gratification from the *anticipated* utility (Frederick, Loewenstein and O'Donoghue, 2002, p.371). To indicate how this can be formulated formally, let $c \in \mathcal{A}$ denote, say, an infinite consumption stream and $v(c_t)$ be the utility derived from "actual" consumption in period t. Then the (total) utility that the individual perceives at period

t is given by $u({}_tc) = v(c_t) + \alpha(\gamma v(c_{t+1}) + \gamma^2 v(c_{t+2}) \ldots$ with $\gamma < 1$. This introduction of anticipated utility of course implies that people will sometime voluntarily in fact postpone gratifications into the future.

This is also consistent with the findings of Loewenstein and Sicherman (1991), who confronted subjects with several wage profiles, some of them decreasing, some of them increasing. The overwhelming majority of the respondents chose an increasing wage profile, *even after* being made aware of the fact that via appropriate saving the decreasing wage profiles could be converted into wage profiles that dominated the increasing ones. Unusual in the profession of economics, Loewenstein and Sicherman chose to report the respondent's reasons for choosing the increasing wage profiles. A large fraction reported either an "aversion of decrease", "inflation" or an (intrinsic?) "pleasure from increase".

3.1.3. Critique: Preference for Spreading Consumption

Loewenstein and Prelec (1993) argue that psychologically, individuals perceive choices over sequences of outcomes fundamentally different from choices over outcome-date pairs. They argue that decision makers have an intrinsic preference for the spread of consumption within a given period. They support this hypothesis by a series of mini-studies, two of which we will discuss here:

In one mini-study (Prelec and Loewenstein, 1991, p.95-96) they asked respondents the following question (original phrasing given):

> Suppose you were given two coupons for fancy dinners for two at the restaurant of your choice. The coupons are worth up to $100 each. When would you choose to use them? Please ignore considerations such as holidays, birthdays, etc.

The authors told one group that the coupons were valid for two years and another group that the coupons were valid for four years. Yet another group was given no constraint. On average, subjects that were given a constraint scheduled both dinners later than the control group (the one without an explicit constraint), confirming the hypothesis.

In another mini-study (Loewenstein and Prelec, 1993, example 5) presented subjects with the following pairs of questions (original phrasing given):

> Imagine that over the next five weekends you must decide how to spend your Saturday nights. From each pair of sequences of dinners below circle the one you would prefer. "Fancy French" [F] refers to dinner at a fancy French restaurant. "Fancy lobster" [L] refers to an exquisite lobster dinner at a four-star restaurant. Ignore scheduling considerations (e.g., your current plans).

The authors assumed that the subjects ate at home [H] on the remaining weekends. The results of this study are presented in table 3.1.

Option	Weekend 1	Weekend 2	Weekend 3	Weekend 4	Weekend 5	
A	F	H	H	H	H	(11%)
B	H	H	F	H	H	(89%)
C	F	H	H	H	L	(48%)
D	H	H	F	H	L	(51%)

Table 3.1.: Questions in Example 4 of Loewenstein and Prelec (1993) (answers in parentheses)

When asked to choose between options A and B the majority chose B, a result which the EDU model rationalizes with a negative rate of impatience, but can also be made sense of by insinuating a preference for spread. This hypothesis is confirmed when in addition to the fancy-french dinner, a lobster-dinner is scheduled on weekend 5: Suddenly the option where the french dinner is scheduled earlier becomes more attractive. The authors obviously interpreted the result in the way that adding the lobster dinner triggered a preference for spread of consumption and made some people chose the option with the french dinner on weekend 1.

The authors interpret both results as a straightforward violation of axiom S1 (strict intertemporal noncomplementarity) and suggest that the preferences over streams are in fact qualitatively different from preferences over outcome-date pairs in the sense that decision makers exhibit an intrinsic preference for spreading consumption.

Although these studies could easily be dismissed as circumstantial evidence and explained by entirely different factors (for instance the imputed income effect from adding a lobster dinner), the hypothesis is confirmed by our intuition.

3.2. Intergenerational Equity and Impatience

Axiom S1 makes it possible to aggregate the (static) utilities of every point in time and we can simply sum up over all present values the outcomes. For this reason, it is also admissible to speak of infinite utility streams (Diamond, 1965, see e.g.). This in itself brings about another issue: Since we sum over an infinite number of periods, it does not follow from any of the assumptions we made so far that

$$-\infty < \sum_{t\geq 0} \delta(t)u(a_t) < +\infty \tag{3.3}$$

So, if the limit does not exist, we cannot infer the ranking of two streams from simply comparing their accumulated present values. This motivated economists to employ a so called *overtaking criterion* to evaluate streams with infinite utility (Acemoglu, 2009, p.261): From a set of alternative streams, the decision maker is said to choose the stream that gives a higher payoff at all times from a certain (finite) period onwards.

This problem of "infinite utility" also has implications of a different kind: Suppose that the static utility of each period corresponds to the well-being of a generation. We then aggregate them into a function, which we interpret as a "social welfare function", W, of, say, a country, that maps infinite utility streams into the real numbers. We require social welfare functions to satisfy the following (innocuous) axioms:

- **(Weak) Pareto:** Whenever it holds for two *utility streams*, that all generations are strictly better off, i.e. obtain higher utility from one stream, a, than in the other b, it follows that $W(a) > W(b)$.

- **Intergenerational equity:** W does not discriminate between the generations in the sense that if two generations "swap" their levels of utility, it leaves the social welfare function unchanged.[2]

- **Continuity** in the sup-metric[3]

From Koopmans (1960) and Diamond (1965) onwards a number of studies showed that there is no social welfare function that jointly satisfies all three axioms: "Intuitively, the reason is that if there is in all circumstances a preference for postponing satisfaction-or even neutrality toward timing- then there is not enough room in the set of real numbers to accommodate and label numerically all the different satisfaction levels [...][of] an infinite future." (Koopmans, 1960, p. 288). In a more recent paper Basu and Mitra (2003) showed that the Pareto Axiom alone precludes an equal treatment of all generations.

3.3. An Alternative Approach

In section 3.1 we derived time preferences over infinite streams that were *induced* by time preferences over outcome-date pairs. We will now approch the issue of intertemporal utility of infinite streams from a different, perhaps more natural vantage point: We do not explicitly assume that axioms RD1-RD6 hold, so in a sense we will start from scratch and follow Koopmans (1960) in his axiomatic derivation of preferences over infinite utility streams. Our goal is to derive preferences over infinite streams directly (i.e. not as induced by preferences over outcome-date pairs). In addition to axiom S1 Koopmans makes the following assumptions:

[2]This particular notion of intergenerational equity is often refered to as the "anonymity axiom" (see e.g. Basu and Mitra, 2003, p.1559).

[3]Due to the infinite number of periods, streams are elements of an infinite dimensional vector space, which makes the concept of continuity sensitive to the metric employed.

K1 (Continuity): The preference relation $(\succeq, \mathcal{A})$ can be represented by a continuous (in the sup-metric) utility function

This implies transitivity, completeness and continuity of $\succeq$ in $\mathcal{X}$.

K2 (Sensitivity): The utility function is sensitive to changes in any period-utility. In the case of period 0 this requires that there exist c_0 and $c_0' \in A$ s.t. $(c_0,\ {}_1a) \succ (c_0',\ {}_1a)$ for all ${}_1a$. This assumption not only excludes the trivial case where the decision maker is indifferent between all streams, but also excludes the (somewhat pathological) case of a decision maker, who has the following "heroic" (Koopmans, 1960, p.291) preferences:

$$U(a) = \lim_{\tau \to \infty} \left(\sup_{t \geq \tau} (a_t) \right) \tag{3.4}$$

Further, we assume a form of temporal homogeneity in the form of

K3 (Stationarity of infinite streams): for all c_0 and all $a, b \in \mathcal{A}$ it holds that:

$$(c_0, a) \succeq (c_0, b) \Leftrightarrow a \succeq b \tag{3.5}$$

As in section 2.4.3 stationarity of streams says that the ranking of the alternatives does not change as both of them are speeded up or delayed by the same number of periods.

This allows us to write the utility function U in the form

$$U({}_0a) = V(u(a_0),\ U({}_1a)) \tag{3.6}$$

where $V(\cdot)$ is called the aggregator function that can be shown to be continuous and increasing in both of its arguments: u (instantaneous utility) and U (prospective utility). The crucial point is, that due to stationarity, neither u, nor U are dependent on time! This *recursive* form gives rise to the idea of dynamic programming, which we will make use of heavily in part II of this thesis.

Moreover, it follows that the decision maker has to exhibit a *sufficient* degree of impatience, in the sense that less weight is given to future utility as to immediate utility. Otherwise the previous axioms are incompatible with each other.

3.4. Uniqueness

In the case of streams the static utility function $u(\cdot)$ is comparable to von Neumann-Morgenstern utility functions in the sense that it is unique up to positive linear transformations. Formally, we say that the preferences over streams of outcomes, denoted by $(\succeq, \mathcal{A})$, that can be represented by the additive discount representation (δ, u):

$$a \succeq b \Leftrightarrow \sum_{t \geq 0} \delta(t) u(a_t) \geq \sum_{t \geq 0} \delta(t) u(b_t) \quad \forall a, b \in \mathcal{A} \tag{3.7}$$

can also be represented by an additive discount representation (δ, v) where $v = ku + d$ and $k > 0$. The main reason for this result is the linearity of the summation operator. In this case of infinite streams the discount function, $\delta(t)$, is unique.

4. Types of Time Preferences and their Empirical Evidence

In this chapter we discuss a number of time preferences that have been introduced in the literature. The first section will be devoted to the "top-dog": the exponentially discounted utility (EDU) model. Ever since its introduction by none other than the late Paul Samuelson it has been the most widely used model of intertemporal choice. But from the 1960s onwards, empirical evidence was mounting up against this model and the list of "anomalies" that the EDU model failed to explain became longer and longer. In section two we will review these findings and discuss in greater detail the issue of measuring discount rates empirically. The evidence led to a spate of papers that proposed to adopt different (absolute) discount functions that are hyperbolic in shape and that could explain most of these "anomalies". Two of these alternative discount functions will be discussed in sections three and four. As we will see, the empirical evidence does not point unequivocally in their direction if assessed critically. The following two sections will then provide us with models of time preferences that differ from the EDU model even more in the sense that they drop the assumption of transitivity, but still have a relative discount representation, i.e. they can be incorporated into the framework introduced in chapter 2. This will not be possible anymore in the case of the two types of time preferences discussed in sections seven and eight, which is why they therefore serve as examples of time preferences that violate one or more of the axioms of relative discounting.

4.1. Exponential Discounting

The exponential discount function is given by

$$\delta_E(t) \equiv \delta^t \quad \delta \in (0,1) \tag{4.1}$$

As we saw in chapter 2 the exponential discount function (EDF) is the only discount function that represents time preferences that are both, stationary and transitive. Due to the stationarity it exhibits a *constant* discount rate. Therefore, it is in a sense formulated in an analogy to a constant interest rate: In every period, the decision maker can be compensated for an additional delay of a prize x that gives static utility of $u(x)$ by an increase of $\rho u(x)$. Therefore the decision maker exhibits the same rate of impatience, independent of the planning horizon. Conversely, the discounted value or present value of a prize x that is obtained in period t is simply given by

$$\left(\frac{1}{1+\rho}\right)^t u(x) \tag{4.2}$$

The second measure of impatience introduced in section 2.4.2 was the discount factor, $\phi(t)$. The discount factor of the exponential discount function is given by

$$\phi_E(t) \equiv \frac{\delta_E(t)}{\delta_E(t-1)} = \frac{\delta^t}{\delta^{t-1}} = \delta \quad \forall t > 0 \tag{4.3}$$

that is, the discount factor is also constant across time. Moerover, it is relatively easy to show that the exponential discount function is the *only* discount function that exhibits a constant discount factor ϕ:

Proof

$$\begin{aligned} \phi(t) &\equiv \frac{\delta(t)}{\delta(t-1)} = \phi \\ \delta(t) &= \phi\delta(t-1) \end{aligned} \tag{4.4}$$

So $\delta(t)$ has to be a solution to the homogenous first-order linear difference equation with constant coefficients. The general solution of this difference equation is given by:

$$\delta(t) = b\phi^t \tag{4.5}$$

The intial value problem is $\delta(0) = 1$ and fixes $b = 1$, which gives us the solution $\delta(t) = \phi^t$. Moreover, since $\delta(t)$ is a discount function, we require $\delta(t)$ to be decreasing and nonnegative. These two conditions together imply that $\phi \equiv \delta \in (0, 1)$, which brings us to the exponential discount function and completes the proof.

Since the EDF represents transitive time preferences over outcome-date pairs, it may also be used to represent preferences over infinite streams. Plugging the exponential discount function into equation (3.1) gives us the **exponentially discounted utility model (EDU)**:

$$a \succeq b \Leftrightarrow \sum_{t\geq 0} \delta^t u(a_t) \geq \sum_{t\geq 0} \delta^t u(b_t) \quad \forall a, b \in \mathcal{A} \tag{4.6}$$

On as few as seven pages this model was introduced by Samuelson and given an axiomatic foundation by Koopmans (1960) (see section 3.3). Owing its popularity perhaps to its simplicity, the exponentially discounted utility model was adopted in virtually every model that involved intertemporal trade-offs: from project evaluation to micro-funded growth models. The manifold applications nonwithstanding, Samuelson himself saw the model merely as a starting point for further research and certainly did not endorse its widespread use as the following quote shows:

> In conclusion, any connection between utility as discussed here and any welfare concept is disavowed. The idea that the results of such a statistical investigation could have any influence upon ethical judgments of policy is one which deserves the impatience of modern economists. (Samuelson, 1937, p.161)

Moreover, neither Samuelson nor any other author ever adopted the exponential discounting function on empirical grounds or even pretended that it is a realistic model. On the contrary: in their axiomatic derivation of the exponential discounting representation Fishburn and Rubinstein say that "[they] know of no persuasive argument for stationarity as a psychologically viable assumption." (p.681).

In the next session we will assess how the exponentially discounted utility model holds up against reality.

4.2. On Eliciting Time Preferences

From about 1980 onwards economists and psychologists sought to infer discount rates from *human*[1] decisions. From hindsight, the first tentative steps to do so were not particularly successful in identifying the discount rates. But before we discuss in detail the issue of eliciting discount functions (see section 4.2.2) it may be useful to have a look at one of the pioneering studies, in particular since they are still quoted as a motivation for adopting discount functions that differ from the exponential one. We are therefore going to discuss the findings of Thaler (1981):

4.2.1. The first tentative steps

Thaler asked respondents the hypothetical question of how much money they had to be given in one month/one year/ten years in order to be indifferent to receiving \$15 now. In addition, he asked subjects, about the dollar amount that would make them indifferent to getting \$250 and to getting \$3000 in a year. The median responses and the implied discount rates are given in table 4.1.

Thaler calculated separately the **average annual discount rates** for each response given. He did so in the following way (Frederick, Loewenstein and O'Donoghue, 2002): In the case of (\$15, *now*) $\sim$ (\$50, *one year*) this means that $\$15 = \delta(\textit{one year})\$50$. Using the identity that we can express every discount function as a product of its discount factors (equation 2.17) the **average annual discount rate** is then given by the

[1] see section 4.3.2 for evidence of discount rates of pidgeons

Indifferent to		one month	one year	ten years
$15	Median Response	$20	$50	$100
	Annual Average Discount Rate	345%	120%	19%
	Annual Average Discount Factor	.03	.30	.83
$250	Median Response		$350	
	Annual Average Discount Rate		34%	
	Annual Average Discount Factor		.71	
$3000	Median Response		$4000	
	Annual Average Discount Rate		29%	
	Annual Average Discount Factor		.75	

Table 4.1.: Results of the experiment in Thaler (1981)

equation: $\$15 = \exp(-\rho\ 1)\$50$ So $\rho = 1.20$. Analogously, $\$15 = \exp(-3.45\ 1/12)\$20 = \exp(-.19\ 10)\$100$. That is we calculate an average discount rate *as if* the discount rate were constant in the interval for given time horizons: Therefore, $\int_0^t \rho(\tau)d\tau = \rho t$.

Alternatively we could calculate **average per period discount rates**: If we already now, that the average discount rate for a one month horizon is 345%, then the average discount rate in the period of between one month from now to one year from now is given by the equation $\$15 = \exp(-3.45\ 1/12)\exp(-\rho\ 11/12)\$50$. So ρ equals 100%. Likewise, a similar calculation yields that the average discount rate for the period starting at one year from now and ending ten years from now is about 7.7%.

These results suggest that discount rates decline in both, the planning horizon and the money involved.

4.2.2. Methodology

After having discussed the particular approach of Thaler (1981), it may be a good idea to lay out in a more general fashion, how one can measure time preferences. There is of course a large number of possibilities to elicit (average) discount rates from decisions: First of all, these decisions can be either observed *in real life* or in a *laboratory* environment.

Studies that base their estimations of discount rates on data from real life experience are often called field studies. One of the first studies to infer discount rates from real world decisions that involved intertemporal trade offs, was conducted by Hausman (1979) who collected data about the purchase of air conditioners: The intertemporal trade-off was given by the fact that the lower priced air conditioners had larger operating costs.

By the same token, the termination of about 66 000 military servicemen provided a cause for rejoicing for economists Warner and Pleeter: The employees faced the choice of either accepting a lump-sum payment of about \$25 000 or an annuity that - on the basis of a seven-percent interest rate - was "worth" about \$50 000. The overwhelming majority chose the lump-sum payment, which can only be rationalized within the EDU-model if the average discount rates were at least 17%. In nominal terms, this saved the government about \$1.7 billion in compensations.

One important drawback of field studies is that it may be difficult to isolate the effect of time preferences from other considerations: In the case of Hausman's study one could for instance argue that decision makers were not aware of the operating costs in the first place or that individuals simply found themselves liquidity constraint.

Therefore, the majority of studies tried to elicit discount rates in in experimental situations in order to control the decision environment and suppress the "background noise" of other economic considerations. These lab-experiments range from hypothetical "paper-and pencil" tests to experiments involving sizable monetary rewards. Frederick, Loewenstein and O'Donoghue (2002, p.386-389) distinguish between four different experimental procedures:

- **Choice Tasks:** In the case of choice tasks subjects are given the choice between two outcome-date pairs, where one outcome is smaller and due sooner than another one that is larger but due later. Suppose for instance that the smaller amount is \$100 due today, whereas the other amount is \$120 due in one year. If the subject choses the smaller, sooner amount, then the experimenter concludes that the discount rate is *at least* 20%. In order to narrow down the discount rate to a single number, subjects are often given a series of choices.[2] That in itself brings about the problem of the so called *anchoring effect*: suppose for instance that there are two test schedules, each consisting of two questions. One test schedule first gives the subject the choice between \$100 now vs. \$103 next year and then the choice between \$100 now vs. \$120 in one year. In the other schedule, the second question is the same, but the first question gives the choice between \$100 now vs. \$140 next year. The anchoring effect states that subjects tend to stick to the decision, "the anchor", they made in the first round and therefore a person is more likely to choose \$100 over \$120 in the first schedule than in the second.

- **Matching tasks** ask people for the corresponding value \$x that would make them indifferent between, say, \$100 now and \$x in one year. The study of Richard Thaler (1981) discussed above elicited discount rates in this manner. The advantage over choice tasks is that it gives one discount rate and excludes

[2] In a recent paper, Chabris, Laibson, Morris et al. (2008) also recorded the response time of subjects, i.e. the time it took subjects to answer questions. The authors surmise that longer response times indicate that subjects found it harder to give a ranking of the two options and therefore the two are perhaps perceived to be more similar.

the anchoring effect. The disadvantages are that subjects tend to give very crude responses that are just mulitples of the other outcome (here: $n * \$100$)

- In **rating tasks** subjects have to indicate the attractiveness of an outcome date-pair. In the case of a transitive model of time preferences this can be thought of as a proxy for the utility (the present value) of an outcome-date pair (i.e. $\delta(t)u(x)$).
- **Pricing tasks** are similar to rating tasks but ask subjects for their willingness to pay for a dated outcome.

Although it is not clear of wether it makes much of a difference (see e.g. Frederick, Loewenstein and O'Donoghue (2002) on this issue) to use either real or hypothetical rewards in experiments, it is certainly the case that studies that involve real rewards have to be incentive compatible: Just image matching tasks were people demand ridiculously huge sums in order to be compensated for a single day of delay. From a different perspective, the experimental procedures should be seen as mechanisms that should implement truthtelling. Therefore it seems only natural to resort to the best known mechanisms in economics: auctions. Manzini and Mariotti (2007) list three different types of auctions where truthtelling is a (weakly) dominant strategy:

- **Second Price Sealed Bid (Vickrey) auction:** The bidder that places the highest bid wins the prize, but pays the only the bid of the second highest bidder. It can be easily shown that truthtelling (here: stating the "correct" discount rate or indifference outcome) is a weakly dominant strategy. In addition, this auction format has the advantage that it is relatively easy to understand and "close" to a direct mechanism (i.e. an incentive compatible mechanism where the strategy space of the agents is identical to the type space)
- **English (ascending bid) auction:** The price of the good to be auctioned off increases steadily with time. Bidders may drop out at any time. When only one bidder is left, the auction stops and the remaining bidder gets the item at the last price. This auction is strategically equivalent to the Second Price Sealed Bid auction.
- **Becker-DeGroot-Marschak (BDM) procedure:** The decision maker "plays against" a uniform probability distribution. When the price drawn is lower than the price stated by the decision maker, she obtains the item for the price drawn, otherwise she gets nothing.

Note that while the first two auctions are robust to risk-aversion, the third one is not. Although experimenters usually put a lot of emphasis on explaining the procedures to the subjects, one might doubt wether this really implements the desired outcomes.

Furthermore, Manzini and Mariotti (2007, p.20-21) argue that "these elicitation methods suffer from serious incentive [problems] in the neighborhood of the truth telling [...] strategy: deviations may be 'cheap' enough".

4.2.3. Findings - Four "Anomalies" of the EDU model

A myriad of studies (Hausman, 1979; Benzion, Rapoport and Yagil, 1989; Kirby, 1997, just to mention a few) elicited discount rates in one of the ways described above. These studies were thought to document patterns that constitute "anomalies" of the (E)DU model. The most cited of these anomalies are the following (Loewenstein and Prelec, 1992):

Decreasing Discount Rates

Subsequent studies confirmed the findings of Thaler (1981): The *average* discount rates are strictly declining with time horizon. This implies that the discount rates themselves, $\rho(t)$ are also declining with time. Conversely, the discount factor, $\phi(t)$ is increasing with time. In other words, individuals are more patient for longer planning horizons. In their meta-study, Frederick, Loewenstein and O'Donoghue (2002) evaluate the reported discount rates of a plethora of studies. They then regressed the discount rates against the reported time horizon and found that across studies the discount factors increase with the time horizon (see figure 4.1).

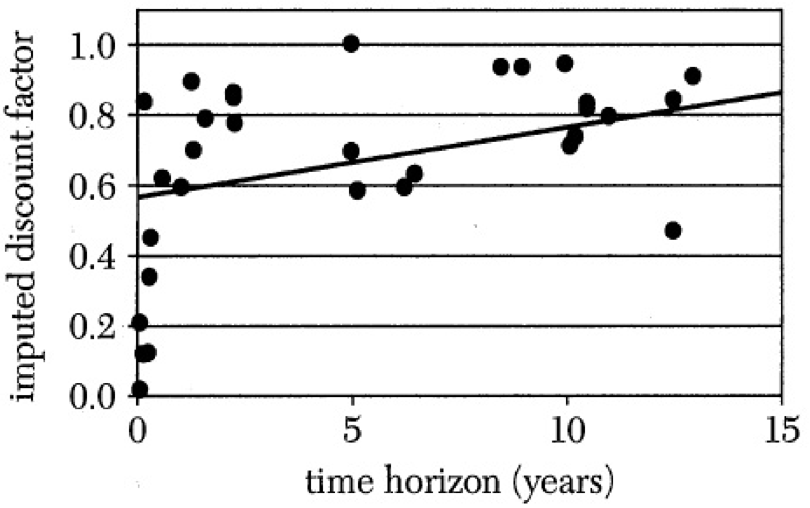


Figure 4.1.: Discount Factors and Time Horizons reported by empirical studies. The solid line is the least squares fit. Source: Frederick, Loewenstein and O'Donoghue (2002, p.362)

The (absolute) Magnitude Effect

The Magnitude Effect says that discount rates are lower for higher amounts of money relative the small amounts of money. As mentioned above, the magnitude effect can also be observed in the data presented by Thaler: The average one-year discount rates decline with the dollar amounts: Whereas the median annual discount rate for the \$15 prize was a whopping 120%, increasing the stakes to \$250 and \$3000 yielded by far lower discount rates of 34% and 29%, respectively. More elaborate studies (e.g. Benzion, Rapoport and Yagil, 1989) reproduced these results.

The Sign Effect

The Sign Effect or Gain-Loss Asymmetry captures the pattern found in experimental evidence that decision makers discount losses at a lower rate than gains. Sometimes this effect is so pronounced that researches reported negative discount rates for losses. Loewenstein and Prelec (1992, p.575) present evidence from earlier studies were respondents, on average, were indifferent between receiving \$10 now and reiceiving \$21 in one year, implying an average annual discount rate of 110%. In the case of losses on the other hand, individuals declared to be indifferent between losing \$10 now and losing \$15 one year later on average, implying a discount rate of only 50%.

The Delay-Speedup Asymmetry

Several studies (for instance Loewenstein, 1988) documented a framing effect that is present in intertemporal choice: In a typical study respondents are asked two questions: in question number one, they are asked for the minimum amount required to be willing to delay the receipt of, say, \$100 from period s to period t. Suppose that this premium is \$x. In question number two they are asked to speed up the receipt of \$100+x from period t to period s. As it turns out, the amount required for delay is by far larger (about three to four times) than the amount required for speeding up consumption. The choice pairs, however, are clearly the same: $(100, s)$ and $(100 + x, t)$, respectively.

4.2.4. Critique: Confounding Factors

Although most of these anomalies are robust findings across studies, it may be a good idea to pause for a second and assess what these studies are actually measuring. In recent years various scholars raised their concerns wether the empirical findigs really manage to identify time preferences, or if the findings are merely due to other, confounding factors. Frederick, Loewenstein and O'Donoghue (2002) mention the following points of critique:

Risk Aversion/Concave Utility Functions

The most important challenge to the findings is that almost all of the empirical studies explicitly or implicitly assumed risk neutrality of the agents. Within the context of the normative von Neumann-Morgenstern theory of decision under risk, an agent is risk neutral if and only if her preferences over *certain* monetary outcomes can be represented by a linear utility function. As mentioned in chapter 3, von Neumann-Morgenstern utility functions are only unique up to monotone affine transformations. Therefore, the utility function over static and certain monetary outcomes can be written as the identity function, $u(x) = x$, without loss of generality. In words, the decision maker values every additional euro the same. As a consequence, the wealth level of an economic agent can be ignored in the analysis of decision under risk: it does not matter if the first or the 1000th \$ is at stake. By the same token, the baseline consumption is also not important when evaluating the ranking of two outcome-date pairs, each one of which increasing consumption *on top* of the baseline consumption level. We will now discuss the possible implications on the measurement of discount rates, when the assumption of risk neutrality is violated.

Most studies try to measure the present value of outcome-date pairs in one of the following two ways (Loewenstein and Prelec, 1992, p.576).

They either try to elicit the *equivalent present value* of a delayed outcome, (x, t), which we denote by $\psi(x, t)$. It is usually defined implicitly as:

$$u(c + \psi) + \delta(t)u(c) = u(c) + \delta(t)u(c + x) \tag{4.7}$$

In words, the equivalent present value is the *increase* in immediate consumption that makes the decision maker indifferent to x *additional* units of consumption later. Explicitly, $\psi(x, t)$ is then given by

$$\psi(x, t) \equiv u^{-1}[(1 - \delta(t))u(c) + \delta(t)u(c + x)] - c \tag{4.8}$$

Alternatively, experimenters elicited the compensating present value, $\kappa(x, t)$, which is (implicitly) defined as:

$$u(c - \kappa) + \delta(t)u(c + x) = u(c) + \delta(t)u(c) \tag{4.9}$$

Let us focus on the equivalent present value, $\psi(x, t)$: As we saw in the case of Thaler (1981) experimenters then obtained their estimates of discounting by dividing the equivalent present value by x. Noor (2009a, p.871) refers to this function as the **money-discount function:**

$$D(x, t) \equiv \frac{\psi(x, t)}{x} \quad \forall x \neq 0 \tag{4.10}$$

It is easy to see, that in general, the money-discount function, $D(x,t)$ *does not coincide* with the discount function, $\delta(t)$, if $u(x) \neq x$, i.e. the decision maker is not risk neutral. Let us simplify things by assuming for a second that $c = 0$ and let us normalize $u(0) = 0$. Clearly, not even then do the money-discount function, $D(x,t)$ and the discount function, $\delta(t)$, coincide:

$$D(x,t) = \frac{u^{-1}[\delta(t)u(x)]}{x} \quad \forall x \neq 0 \tag{4.11}$$

From another perspective, what most empirical studies did is to work with a different underlying model of time preferences. Instead of working with the "usual" model of (absolute) discounting,

$$(x,t) \succeq (y,s) \Leftrightarrow \delta(t)u(x) \geq \delta(s)u(y) \tag{4.12}$$

which we introduced in section 2.4.1, the studies implicitly operated with the following representation

$$(x,t) \succeq (y,s) \Leftrightarrow D(x,t)x \geq D(y,s)y \tag{4.13}$$

To put it differently, this specification acts *as if* the decision maker was risk neutral. As Noor (2009a, p.874) demonstrates, in principle there is nothing "wrong" with this formulation in the sense that $D(x,t)x \geq D(y,s)s$ represents the same underlying preference relation. But then of course "discounting" is differently "defined" and $D(\cdot)$ clearly captures something entirely different than what one usually refers to as the "discount function", $\delta(t)$. Intuitively, the money-discount function has to incorporate all "non-linearities" of the outcome-evaluation.

So what happens to the discount-rate estimates if the utility function is not linear in x? For instance, if the decision maker is risk averse for every level of wealth, i.e. $u(x)$ is strictly concave? If this is the case, the "anomalies" of the (E)DU model, might be none at all and also the estimates of the discount rates are seriously biased in one direction:

- **Magnitude Effect**: By construction, the money discount function, D is dependent on x *per se*. Therefore the magnitude effect itself does not constitute an anomaly. In section 4.3.5 we will derive a condition on the utility function, that ensures that the magnitude effect holds (for money-discount functions).

- **Decreasing Impatience:** Likewise, there is also no reason to expect the money-discount function, $D(x,t)$, to be exponential in t, even if the underlying discount function, $\delta(t)$, in fact is. Therefore, decreasing impatience by itself does also not constitute an anomaly. However, as Noor (2009a) showes, the magnitude effect and decreasing impatience (of the money-discount function) are *jointly* incompatible with stationarity and therefore the exponential discount function.

- **Level of discount rates:** The strict concavity of the discount function means that additional units of money are ever less appreciated. In order to compensate for the postponement of a monetary reward, the decision maker will demand a certain increase in utility. This increase in utility on the other hand, corresponds to an increase of money that -in proportion- is higher than the increase in utility since every additional euro "buys" ever less units of utility. Formally,

$$\begin{aligned}D(x,t) &= \frac{u^{-1}[(1-\delta(t))u(c)+\delta(t)u(c+x)]-c}{x}\\ &< \frac{(1-\delta(t))c+\delta(t)(c+x)-c}{x}\\ &= \delta(t)\end{aligned} \tag{4.14}$$

  where we made use of Jensen's inequality[3]. Therefore, depending on the degree of risk aversion[4] the estimates of the discount factors will be seriously bias downwards (conversely, estimates of the discount rates will be upwards). Andersen, Harrison, Lau et al. (2008) *jointly* elicited discount rates and attitudes towards risk and conclude that on average the "true" discount rates (i.e. controlling for risk-aversion) are at about 10.1% per year. When assuming risk neutrality, however, the imputed discount rates are about 25.2%.

- **Non-constant baseline consumption:** Decision makers might anticipate higher future baseline consumption. This might exacerbate the upward bias of the discount rate estimates because the value of the compensation is even more diminished. Noor (2009b) conductes a calibration exercise and demonstrates that relatively "small" changes in anticipated baseline consumption have dramatic effects on the bias of discount rates that ignore these changes (see table 1 on page 2082). In the same way, a non-constant baseline consumption can "rationalize" virtually any discount function for any observed choice pattern, rendering the usual elicitation of discount rates meaningless if there the experimenter does not know about the anticipated baseline consumption.

  In a nutshell, the validity of the findings cited above hinges greatly on the assumption of risk neutrality.

Intertemporal Arbitrage and Consumption Reallocation

When calculating discount rates, experimenters usually assume that the outcomes are consumed on the very same day/in the very same year they are due. In theory, given perfect capital markets, the outcomes could be "shifted" through time at

[3] the assumed strict concavity of u implies strict convexity of u^{-1}

[4] measured by, say, the Arrow-Pratt index

the cost of the prevailing interest rate. One of the few studies that actually takes this effect into account is Harrison, Lau and Williams (2002). The authors argue that the discount rates might be censored by the market interest rate: Once again let us ignore risk aversion and assume that in a choice task (see above) the decision maker has to choose between ($\$100, now$) and ($\$103, one\ year$), further let the true annual discount rate of the decision maker be 2% and the (risk-free) interest rate 5%. Then, the decision maker will rationally choose to take \$100 now, invest them and consume \$105 in one year. But without controlling for this censoring effect, the experimenter would infer from the decision that the discount rate exceeds 3%.

Risk, Uncertainty and Hidden Costs

Although subjects in experiments are usually assured that they get postponed rewards for sure, they might have their doubts, which of course increase the attractiveness of immediate outcomes and bias the disount rates upwards. Note that this effect is not to be confused with risk aversion per se: If subjects think that they might not be paid the postponed rewards at all, they face the lottery $(p, x; 1-p, 0)$ with an expected value that is clearly smaller than x – therefore the results are also biased upwards for risk neutral decision makers.

Other authors argue that decision makers had to incure additional (mental or otherwise) cost when they pick up the postponed rewards that were guarenteed in experiments: Decision makers have to come again to the facility where the experiments took place or simply have to think about to pick up the rewards when they are due. In section 4.8 we will discuss a model of time preferences that explicitly incorporates such "fixed costs". For this reason, more recent studies (e.g. Harrison, Lau and Williams, 2002) tried to eliminate this potential factor by presenting the subjects with choices, where all alternatives where due in the future, so that the attractiveness of all alternatives is diminished by the fixed cost.

Inflation

If one assumes that decision makers do not have "money illusion", discount rates might be biased upwards when the experimenter does not account for the diminished real value of future outcomes. Moreover, the longer the planning horizon, the more pronounced this effect would be.

All confounding factors presented here suggested an upward bias of the estimated discount rates. Certainly, discount rates as high as 345% (as reported by Thaler (1981), see above) seem unrealistic.

Although these confounding factors by all means weaken the findings cited above, they should not be interpretated as a confirmation of the exponentially discounted utility model. Therefore, it should be worthwile to discuss alternatives to the EDU model, that were brought forward in the literature.

4.3. Hyperbolic Discounting

4.3.1. Definition

As the evidence against the exponentially discounted utility model was mounting up, economists and psychologists turned to different models of intertemporal choice: It seemed only natural to make the slightest possible change, i.e. retain the discounted utility model and simply replace the exponential discount function with another discount function. One of the functions that come to mind is a hyperbolic function, which in its most general specification is parameterized as:

$$\delta_H(t) = (1 + \alpha t)^{-\frac{\gamma}{\alpha}} \quad \alpha, \gamma > 0 \tag{4.15}$$

A plot of this function with parameters $\alpha = 4$ and $\gamma = 1$ can be found on page 52. Clearly, the function is a discount function as it is continuous on $\mathbb{R}_+$, is strictly decreasing and has domain $(0, 1]$.

We see that (in continuous time) the *discount rate* is given by:

$$\rho_H(t) \equiv -\frac{\delta'_H(t)}{\delta_H(t)} = \frac{\gamma}{\alpha(1 + \alpha t)} \tag{4.16}$$

That is, the discount rate of a hyperbolic discount function is decreasing with time and is therefore consistent with the findings discussed above, i.e. additional delays are seen less and less harmful. Therefore, in the case of continuous time (discrete time) the discount rate function (sequence) is strictly decreasing with time.

When we discussed the motivation for exponential discunting (section 4.1) we employed an analogy to the formulation of interest rates. A similar interpretation is possible in the case of hyperbolic discounting, when we set $\alpha = \gamma$, so that $\delta_H(t) \equiv (1+\alpha t)^{-1}$: Suppose that the decision maker is risk neutral, s.t. u(x)=x without loss of generality. Suppose further, that she can be made indifferent by an *absolute* increase of exactly αx for every period that the consumption of x is postponed to the future. Therefore

$$(x, 0) \sim ([1 + \alpha t]x, t) \Leftrightarrow x = \delta(t)[1 + \alpha t]x \tag{4.17}$$

We can then conclude that $\delta(t) = (1 + \alpha t)^{-1}$, which gives rise to an interpretation as "linear" discounting (as opposed to exponential) in the case of $\alpha = \gamma$.

Back to the generalized hyperbola, note that the parameter α determines how fast the discount rate goes to zero and we observe that for "extreme" values of α the hyperbola

1. **becomes the exponential discount function** ($\alpha \searrow 0$): $\lim_{\alpha \searrow 0}(1+\alpha t)^{-\frac{\gamma}{\alpha}} = \lim_{a\to\infty}(1+\frac{t}{a})^{-a\gamma} = [\lim_{a\to\infty}(1+\frac{t}{a})^{a}]^{-\gamma} = e^{-\gamma t} \equiv \delta_E(t)$ where we made the substitution $a \equiv 1/\alpha$.

2. or approximates a **step function:** $(\alpha, \gamma \to \infty)$[5]

$$\delta_H(t) = \begin{cases} 1 & \text{for } t = 0, \\ 0 < c(\alpha,\gamma) < 1 & \text{for } t > 0 \end{cases}$$

Apart from the empirical findings of decreasing discount rates, at least three other motivations or justifications for adopting a hyperbolic discount function are often brought forward in the literature: Herrnstein's Matching Law, Preference Reversals and Second-Order Stationarity.[6]

4.3.2. Herrnstein's Matching Law

Ever since its first formulation by Richard Herrnstein in 1961, the *Matching "Law"* has been one of the central paradigms in operant research, a branch of psychology. Roughly speaking it says that "[...] if an interval of time may be divided into more than one alternative activity [...], animals (nonhuman and human alike) will allocate their behavior to the activities in exact proportion to the value derived from each" (Herrnstein, 1997). To illustrate its content suppose that on a given day, a person can decide at the beginning of every hour if she wants to spend her time either eating or sleeping. Then the matching law says that after, say, one day the "average reinforcement rate of eating" will equal the "average reinforcement rate of sleeping". The "average reinforcement rate of eating" is the overall utility derived from eating divided by the number of hours spent eating.

Unfortunately, humans do not carry with them an apparatus that measures utility, like Edgeworth's hedonimeter. Therefore psychologists resorted to experiments with pigeons and equalized "utility" with food intake. Without going into the details of the experimental setups,[7] it should suffice to say that pigeons were given the choice to peck on one of two disks, each of which triggered the dispension of food after a random number of periods ("variable interval schedule"). Herrnstein documented that the

[5]To see this, note that, $\lim_{\alpha\to\infty} \delta'_H(t) = 0$ for $t > 0$. The value of the constant $c(\alpha,\gamma)$ then depends on the relative speed of convergence of α and β.

[6]For an "evolutionary" motivation of Hyperbolic Time Preferences that stems from uncertain payoffs see Dasgupta and Maskin (2005).

[7]For a detailed description of the experimental design and operant research in general, see Herrnstein (1997)

relative number of pecks (P/P') equaled the relative rate of reward from this schedule, i.e. (R/R') (Ainslie, 1992):

$$\frac{P}{P'} = \frac{R}{R'} \tag{4.18}$$

In a follow-up study[8] Chung and Herrnstein (1967) added a time dimension to the experiment and proposed the following relationship

$$\frac{P}{P'} = \frac{A}{A'}\frac{t'}{t} \tag{4.19}$$

where the As are the amounts of food delivered and ts are the delays. Ainslie (1975) was the first to note the relationship to discounting and was quick to state that therefore the "value" of a schedule is given by the following hyperbolic function:

$$V = \frac{A}{t} \tag{4.20}$$

A couple of years later evidence, again from experiments with pigeons, brought forward by Mazur and Herrnstein (1988) then showed that the equation

$$V = \frac{A}{1 + \alpha t} \tag{4.21}$$

fitted the data better. Clearly, equation 4.21 constitutes the special case of the (generalized) hyperbola given in equation 4.15 where $\alpha = \gamma$.

One could argue that these experiments with animals can only have very limited implications on how time preferences of humans can be seen or as Ariel Rubinstein (2001, p.1209) puts it: "the connection between findings on pigeons or even monkeys and the behavior of humans seems rather tenuous. We commonly believe that an animal does not understand the choice it is facing in the same way that a human being does." This critique nonwithstanding, economists and psychologists often refer to Herrstein's Matching Law and the findings discussed above as a justification for adopting hyperbolic discount functions.

4.3.3. Preference Reversals

One important assumption in the axiomatic derivation of the exponentially discounted utility model is stationarity (see section 2.4.3). A number of experiments showed that the assumption of stationarity does not hold up very well against reality. (Green, Firstoe and Myerson, 1994) presented 24 students with the choice of two outcome-date pairs: one smaller and sooner, the other larger, but later. The prices were such that the smaller sooner was preferred to the larger later. Stationarity requires that the ranking

[8] see Ainslie (1992, chapter 3) for a comprehensive discussion

is preserved when both outcomes are delayed or expedited by the same number of periods:

$$(x,t) \succ (y,s) \Rightarrow (x,t+\tau) \succ (y,s+\tau) \quad t<s,\ x<y,\ \forall \tau \ s.t.\ s+\tau,\ t+\tau \geq 0 \quad (4.22)$$

The researchers, however, observed that most students reversed the rankings for a large enough τ, a choice that exponential discounting fails to explain. Hyperbolic discounting, on the other hand is able to "rationalize" these choices:

First, assume discrete timing and recall that we can write every discount function in terms of the product of discount factors (equation (2.11)) and we therefore have

$$(x,t) \succeq (y,s) \Leftrightarrow u(x) \geq u(y) \prod_{i=t+1}^{s} \phi(i) \quad (4.23)$$

Now, while the sequence $\phi(i)$ is constant in the case of exponential discounting it is increasing in the case of hyperbolic discounting (the discount rates are decreasing with time, therefore the discount factors are increasing with time). Since $\lim_{t\to\infty} \phi(t) = 1$ and $u(x) < u(y)$ it has to hold that from a certain period, say t^*, onwards, the ranking is reversed: As both outcomes are delayed, their difference in timing is getting ever less important for the decision maker and she chooses the pair with the better outcome. Moreover, this reversal occurs exactly once.

Figure 4.2 illustrates such a preference reversal graphically: Hyperbolic functions are more "bowed" (determined by the value of the parameter α) than exponential functions, which allows for the present value curves to intersect.

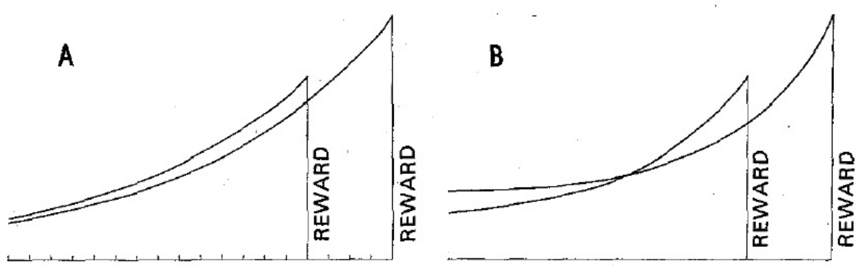


Figure 4.2.: "Preference reversals" are ruled out when time is discounted exponentially (subfigure A), but occur when time is discounted hyperbolically (subfigure B). Source: Ainslie (1975, p.471)

Moreover, hyperbolic discounting functions exhibit **present bias** (see section 2.4.4), which is just a special form of preference reversals.

4.3.4. Second-Order Stationarity

In a recent paper al Nowaihi and Dhami (2005) established an axiomatic derivation of the hyperbolic discounting function, largely building on the prior attempts of Loewenstein and Prelec (1992). As it turns out the crucial assumption for hyperbolic discounting is the following:

Assumption HD1: The time preference exhibits **second order stationarity**:

$$y \succ x, \text{ and}(x, 0) \sim (y, s) \Rightarrow (x, t) \sim (y, s + t + \alpha st), \quad \alpha > 0 \ \forall t > 0. \tag{4.24}$$

In order to be able to pin down $\delta(t)$ to a specific functional we introduce this (stronger) assumption about *how* stationarity is violated: The positive coefficent, α, of the interaction term, *st*, determines the departure from stationarity. Therefore we could nest the two special cases where (Loewenstein and Prelec, 1992, p.580):

1. $\alpha \searrow 0$ which brings us back to stationarity and therefore exponential discounting.
2. $\alpha \to \infty$: This means that whenever an outcome is delayed ($t > 0$) the decision maker is insensitive to *further* delays. It may be important to stress that this does in general not imply that the decision maker is myopic, i.e. only appreciates current satisfaction - it merely says that she exhibits "dichotomous" time preferences where all future satisfactions are discounted the same. Therefore the decision maker is infinitely impatient when $t > 0$. Note that this is of course ruled out by assumption RD1 (time sensitivity).

The following Theorem now states, that the hyperbolic discount function is to second-order stationarity what the exponential discount function is to "ordinary" stationarity:

Theorem 7 ***[Hyperbolic Discounting]****(al Nowaihi and Dhami, 2005) $(\succeq, \mathcal{X})$ is a time preference that satisfies axioms RD1-RD6 as well as axiom HD1 if, and only if, there exists an increasing homeomorphism $u : X \to \mathbb{R}_{++}$ and a decreasing and continuous map $\delta_H : \mathbb{R} \to \mathbb{R}_{++}$ s.t.*

$$(x, t) \succeq (y, s) \Leftrightarrow \delta_H(t)u(x) \geq \delta_H(s)u(y) \quad \forall (x, t), (y, s) \in \mathcal{X} \tag{4.25}$$

where $\delta_H(t) \equiv (1 + \alpha t)^{-\frac{\gamma}{\alpha}} \quad \alpha, \gamma > 0$

Summing up, the parameter α determines the departure from stationarity. A value close to one yields exponential discounting. Positive values result in a discount function that is more "bowed" than the exponential function. In the limit, as α approaches infinity, the discount function resembles a step function, where only delay per se and not further delay is perceived as harmful. For finite values of α, the other parameter, γ is closely related to the parameter ρ of the exponential discount function: Lower values result in higher rate of impatience and vice versa.

4.3.5. The Loewenstein and Prelec Model

Loewenstein and Prelec (1992) developed a model of intertemporal choice that is tailor-made for explaining the four "anomalies" of the EDU model (see section 4.2.3). In constrast to others, the authors acknowledge that a model of intertemporal choice consists of both a discount function **and** an outcome function. In a nutshell, their model weds a hyperbolic discount function with Kahneman and Tversky's *Prospect Theory*:

Assumptions

In *Prospect Theory* decision makers do not evaluate monetary outcomes according to their size - what is important is their *difference to the reference point* or a *status quo*: This can be thought of as a shift of the coordinate system, so that the reference point lies in the origin. Therefore, the utility function is not defined over monetary outcomes but rather over "relative changes to the reference point", which can either be perceived as gains or as losses. In order to emphasize the difference to a utility function, Kahneman and Tversky use the expression *value function* instead. Loewenstein and Prelec (1992, p.595) conjecture that individuals may "conserve on cognitive effort". This means that a decision maker is explicitly assumed not to incorporate preexisting plans into her evaluation of outcome-date pairs. The ranking of "new" outcome-date pairs is independent of, say, a baseline consumption c, which makes the model particularly suitable for analysis of decisions observed in lab experiments, where the baseline-consumption profile is seldomly known to the experimenter.

The key feature is that this value function is pieced together by two separate functions (connected at the reference point): one function for gains and one for losses. In the spirit of Kahneman and Tversky the authors impose some assumptions on the value function, v:

LP1 (Loss aversion): $v(x) < -v(-x) \quad \forall x > 0$. Moreover: for $0 < y < x$ and $t > 0$ it holds that $(y, 0) \sim (x, t) \Rightarrow (-y, 0) \succ (-x, t)$.

This assumption states that the decision maker attaches more weight to losses than to gains. This asymmetric treatment of gains and losses gives rise to the following situation: although the decision maker is indifferent between *receiving* x now and receiving $y > x$ in period s she is unwilling to *pay* x now in order to receive y in period s.

It is also useful to see that this assumption can be stated in terms of the elasticity of the value function or the relationship between the elasticities of the part for gains and the part for losses, respectively:

$$\epsilon_v(x) < \epsilon_v(-x) \quad \forall x > 0 \tag{4.26}$$

Proof To see this, let $(y, 0) \sim (x, t) \Leftrightarrow v(y) = \delta_H(t)v(x)$, i.e. the decision maker is indifferent between receiving y now and receiving $x > y$ later in time. Following the example above, assumption LP1 then predicts that in the case of losses $(-y, 0) > (-x, t) \Leftrightarrow v(-y) > \delta_H(t)v(-x)$, so the decision maker prefers to incur a small loss now to a larger, but discounted loss later.

Substituting for $\delta_H(t)$ and rearranging gives us (Loewenstein and Prelec, 1992, p.583):

$$\frac{v(y)}{v(x)} > \frac{v(-y)}{v(-x)} \quad \forall 0 < y < x \tag{4.27}$$

Taking logs on both sides and defining $v_g(x) \equiv -\log(v(x))$ and $v_l(x) \equiv -\log(-v(-x))$ gives

$$v_G(x) - v_G(y) > v_L(x) - v_L(y) \quad \forall 0 < y < x \tag{4.28}$$

Since $x > y$ we can divide both sides of the inequality by $x - y$ in order to obtain:

$$\frac{v_G(x) - v_G(y)}{x - y} > \frac{v_L(x) - v_L(y)}{x - y} \tag{4.29}$$

This inquality has to hold for arbitrarily small differences between x and y, so it also has to hold for $x - y \searrow 0$. We assume that the limits exists for both sides of the inequality and that they coincide with the limits as $x - y \nearrow 0$. In other words, we assume differentiability of v_G and v_L at x.

$$\begin{aligned} v_G'(x) < v_L'(x) &\Leftrightarrow \frac{-d\log(v(x))}{dx} > \frac{-d(\log(-v(-x))}{dx} \\ &\Leftrightarrow -\frac{v'(x)}{v(x)} > \frac{v'(-x)}{v(-x)} \\ &\Leftrightarrow \frac{xv'(x)}{v(x)} < \frac{-xv'(-x)}{v(-x)} \\ &\Leftrightarrow \epsilon_v(x) < \epsilon_v(-x) \quad \forall x > 0 \end{aligned} \tag{4.30}$$

Assumption LP2 (Subproportionality and the magnitude effect): For positive outcomes such that $0 < y < x$, $t > 0$ it holds that $(y, 0) \sim (x, t) \Rightarrow (\lambda y, 0) \prec (\lambda x, t)$

where $\lambda > 1$. Similarly, for negative outcomes such that $-x < -y < 0$. $t > 0$ it holds that $(-y, 0) \sim (-x, t) \Rightarrow (-\lambda y, 0) \prec (-\lambda x, t)$.

The magnitude effect ensures that money discounting is lower when stakes are increased (see above): When the absolute size of the outcome is increased by a factor $\lambda > 1$, less perceived discounting takes place. This assumption too, can be expressed in terms of elasticities:

$$\epsilon_v(y) < \epsilon_v(x) \quad \text{for } 0 < x < y \text{ or } y < x < 0 \tag{4.31}$$

In words, the elasticity of the value function is higher for outcomes that are larger in absolute terms.

Proof We will show the case of positive outcomes here, the proof for negative outcomes is analogous.

$$\begin{aligned} &(y, 0) \sim (x, t) \Rightarrow (\lambda y, 0) \prec (\lambda x, t) \quad \forall \lambda > 1 \\ \Leftrightarrow\ &v(y) = \delta_H(t) v(x) \Rightarrow v(\lambda y) < \delta_H(t) v(\lambda x) \end{aligned} \tag{4.32}$$

Combining these two equations and substituting for $\delta_H(t)$ yields

$$\frac{v(y)}{v(x)} > \frac{v(\lambda y)}{v(\lambda x)} \quad \forall 0 < y < x,\ \lambda > 1 \tag{4.33}$$

Functions that exhibit this characteristic are called *subproportional*. Taking logs on both sides and differentiating with respect to λ gives us:

$$\frac{y v'(\lambda y)}{v(\lambda y)} < \frac{x v'(\lambda x)}{v(\lambda x)} \tag{4.34}$$

Taking the limit of both sides as $\lambda \searrow 1$ establishes the desired result.

In a recent paper al Nowaihi and Dhami (2009) provide us with a class of continuous value functions that satisfy assumptions LP1 and LP2, dubbed simple increasing elasticity (SIE) value functions. Figure 4.3 plots this function for three different parameter configurations.

From the perspective of a *fixed* reference point, the preferences of the decision maker can then be represented in the following way:

$$(x, t) \succeq (y, s) \Leftrightarrow \delta_H(t) v(x) \geq \delta_H(s) v(y) \tag{4.35}$$

It is important to stress, however, that these are not preferences over outcome-date pairs, let alone *transitive* time preferences: The model merely captures preferences over

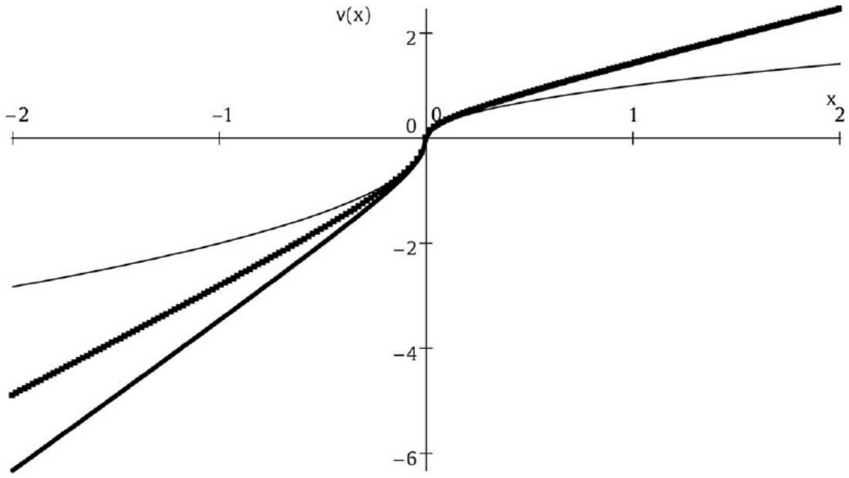


Figure 4.3.: Value functions for three different parameter configurations. The value function is steepest around the reference point and greater (in absolute terms) in the area of losses than in the area of gains. Source: al Nowaihi and Dhami (2009, p.227)

"change to the reference point-date" pairs. As the reference point itself is changing, also the valuations of outcomes do. This of course gives rise to preference cylces as the example above showed. In addition, it may be important to note that the intransitivity does not stem from the decision maker's attitude towards timing, but her attitude towards outcomes.

Implications and Predictions of the model

As indicated above, the Loewenstein and Prelec model can explain all four EDU anomalies: decreasing impatience, the magnitude effect, the delay-speedup asymmetry and the sign effect. While the line of argumentation in explaining the first three is rather straightforward (in a sense, the authors simply assumed that these anomalies hold), the last one may need some elaboration (Loewenstein and Prelec, 1992, pp.585):

The Sign Effect and the Aversion to Intertemporal Tradeoffs

Since only relative changes (to the reference point) matter, the money-discount function for equivalent present values is given by (in a way, we "set" $c = 0$):

$$D_i = \frac{\psi(x,t)}{x} = \frac{v^{-1}[\delta_H(t)v(x)]}{x} \quad i = G \ for \ x > 0; \ i = L \ for \ x < 0 \tag{4.36}$$

Since by assumption LP1 and LP2 gains and losses are "valued" asymmetrically, these calculations also lead to different money-discount factors for gains ($x > 0$) and losses ($x < 0$), which we may denote by D_G and D_L, respectively.

In a similar manner we can compute money-discount rates for borrowing (B) and saving (S). In this case we have to employ the compensating present value, i.e. we look for a κ such that: $v(-\kappa) + \delta_H(t)v(x) = 0$ (again, the reference point idea is taken into account by "setting" $c = 0$). The corresponding money-discount functions are given by

$$D_i = \frac{\kappa(x,t)}{x} = \frac{-v^{-1}[-\delta_H(t)v(x)]}{x} \quad i = S \ for \ x > 0; \ i = B \ for \ x < 0 \tag{4.37}$$

We can show, that the four money-discount rates are ordered in the following way: $0 < D_S < D_G < D_L < D_B$.

Proof We fix any $x > 0$, $t > 0$ and save on notation by defining $\delta \equiv \delta(t)$. In addition, recall that by continuity of $v(\cdot)$ we can apply the intermediate value theorem to ensure that there exists a y with $0 \leq y \leq x$ such that $v(y) = \delta v(x)$. We are now to show that -given assumptions LP1 and LP2 - the four inequalities hold:

- $0 < D_S$: immediate
- $D_S < D_G$:

$$\begin{aligned} \frac{-v^{-1}[-\delta v(x)]}{x} &< \frac{v^{-1}[\delta v(x)]}{x} \\ -v^{-1}[-\delta v(x)] &< y \\ -\delta v(x) &> v(-y) \\ \delta v(x) &< -v(-y) \\ \delta v(x) &= v(y) < -v(-y) \end{aligned}$$

  where the last inequality follows from assumption LP1.

- $D_G < D_L$:

$$\begin{aligned} \frac{v^{-1}[\delta v(x)]}{x} &< \frac{v^{-1}[\delta v(-x)]}{-x} \\ v^{-1}[v(y)] &< -v^{-1}[\delta v(-x)] \\ -y &> v^{-1}[\delta v(-x)] \\ v(-y) &> \delta v(-x)] \end{aligned}$$

  Since we defined y such that $v(y) = \delta v(x)$ it follos from assumption LP1 that $v(-y) > \delta v(-x)$.

- $D_L < D_B$:

$$\frac{v^{-1}[\delta v(-x)]}{-x} < \frac{-v^{-1}[-\delta v(-x)]}{-x}$$
$$v^{-1}[\delta v(-x)] > -v^{-1}[-\delta v(-x)]$$

Defining z implicitly by $v(-z) = \delta v(-x)$ yields

$$v(-z) < -\delta v(-x)$$
$$\delta v(x) < -\delta v(-x)$$
$$v(x) < -v(-x)$$

by assumption LP1.

If we furthermore assume that the value function is convex for negative changes and concave for positive changes, as Kahneman and Tversky (1979) and also Loewenstein and Prelec (1992) do, we even get the stronger result that $D_L < \delta$

- $D_L < \delta$

$$\frac{v^{-1}[\delta v(-x)]}{-x} < \delta$$
$$\delta v(-x) > v(-\delta x)$$
$$\delta v(-x) + (1-\delta)v(0) > v(-\delta x + (1-\delta)0) = v(-\delta 0)$$

The last line followed from the convexity of v for $x < 0$.

Since the choice of both, x and δ was arbitrary, the proof is complete.

Figure 4.4 illustrates the results in the case of $\delta_H(t) = .8$ and $x = \pm 1$.

If we strengthen assumption LP1 so that $v(x) < -\delta_H(t)v(-x)$ then the money discount rate for borrowing is even larger than one. In other words, the model predicts that the decision maker will not borrow at interest rates that are larger than a *negative* threshold. This assumption is of course most easily justified when $\delta_H(t)$ is close to one.

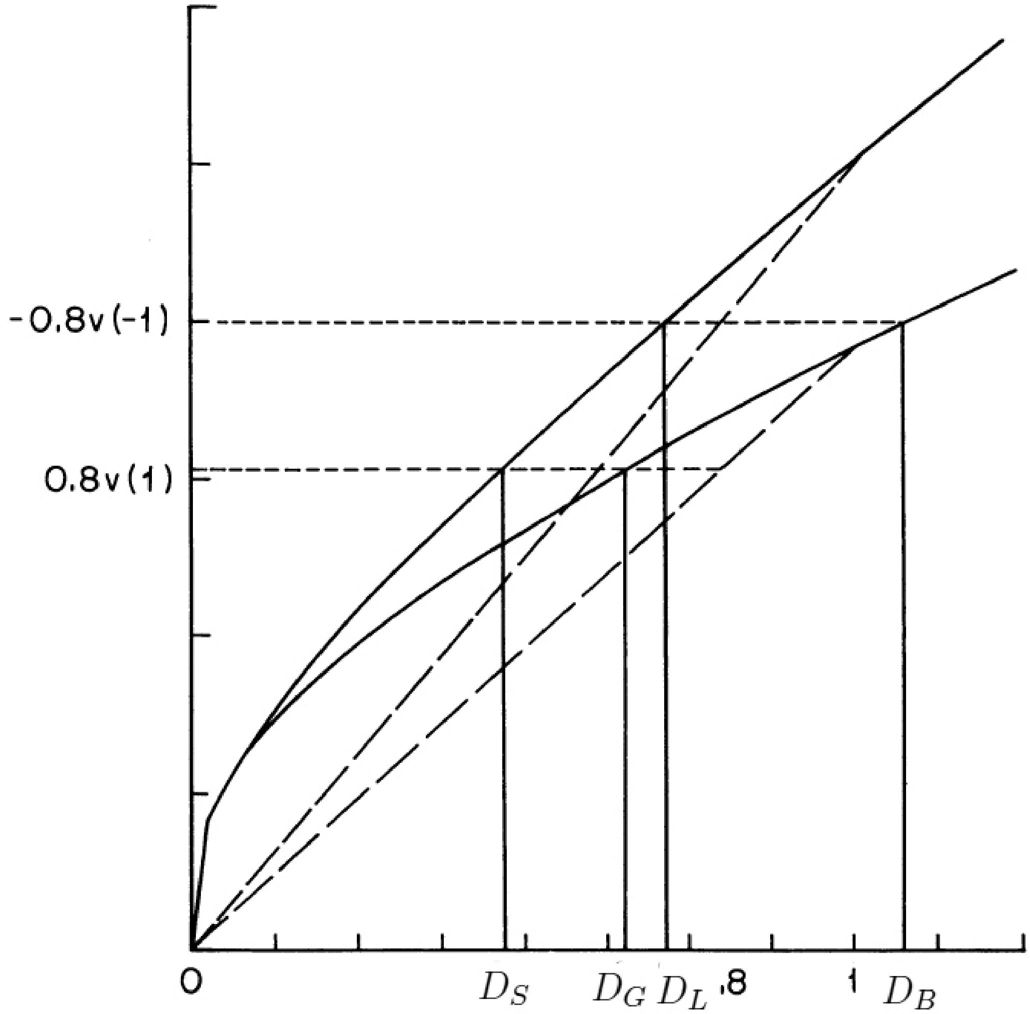


Figure 4.4.: Money-Discount Rates for Gains/Losses (equivalent present values) and Borrowing/Saving (compensating present values). The negative part of the value function was projected into the first quadrant. Source: Loewenstein and Prelec (1992, p.585)

4.4. Quasi-Hyperbolic Discounting

4.4.1. Definition

The quasi-hyperbolic discount function was introduced by Phelps and Pollak (1968). It is defined in discrete time, depicted in figure 4.6 and is given by:

$$\delta_Q(t) \equiv \begin{cases} 1 & \text{for } t = 0 \\ \beta\delta^t & \text{for } t = 1, 2, \ldots \quad \beta \in (0,1],\ \delta \in (0,1) \end{cases} \tag{4.38}$$

The parameter β can be interpreted as a measure of the *variable cost* associated with delaying outcomes into the future. This cost is constant with respect to time, i.e. it does not matter how far the outcome is delayed, future utility is lowered by a factor of $1-\beta$. Clearly, extreme values of β give us either exponential discounting ($\beta = 1$, i.e. no variable costs) or a myopic decision maker who cares only about immediate satisfaction and discounts the future at an infinitely high rate ($\beta \searrow 0$; $\beta = 0$ is of course ruled out by assumption RD1, time sensitivity). Quasi-hyperbolic time preferences are sometimes also refered to as "quasi-geometric" (invoking the connection to geometric=exponential discounting) or (β, δ)-time preferences.

If we again define $\delta \equiv 1/(1+\rho)$, the **discount rate** of a quasi-hyperbolic function is given by

$$\rho_Q(t) \equiv -\frac{\delta_Q(t) - \delta_Q(t-1)}{\delta_Q(t)} = \begin{cases} \frac{1}{\beta\delta} - 1 & \text{for } t = 1 \\ \frac{1}{\delta} - 1 \equiv \rho & \text{for } t = 2, 3, \ldots \end{cases} \tag{4.39}$$

In the same manner we can calculate the **discount factor**:

$$f_Q(t) \equiv \frac{\delta_Q(t)}{\delta_Q(t-1)} = \begin{cases} \beta\delta & \text{for } t = 1 \\ \delta & \text{for } t = 2, 3, \ldots \end{cases} \tag{4.40}$$

4.4.2. Quasi-stationarity and a Generalization

We see that for $\beta < 1$ the quasi-hyerbolic discount function mimicks the qualitative properties of the hyperbolic discount function in a special sense: It exhibits present bias, which can be easily confirmed by plugging the discount function into equation 2.23 of section 2.4.4, but it does not exhibit decreasing impatience elsewhere: The discount rate from period 0 to period 1 is higher than from period 1 to period 2 and so on. In other words, it "behaves" like an exponential discount function with respect to all delays that are greater than one unit of time. Hayashi (2003) dubbed this property **quasi-stationarity**:

Definition A time preference as said to be **quasi-stationary** if:

$$\begin{aligned} (x,t) \succ (y,s) \Leftrightarrow (x,t+\tau) \succ (y,s+\tau) \\ \forall (x,t),(y,s) \in \mathcal{X} \text{ with } s,t \geq 1,\ \tau \in \mathbb{Z} \text{ s.t } s+\tau, t+\tau \geq 1 \end{aligned} \tag{4.41}$$

Quasi-stationarity is weaker than stationarity, because it only requires the ranking to be preserved when *future* outcomes (i.e. $t \geq 1$) are shifted by a common number of time periods.

Following the work of Koopmans (1960) in the case of the exponential discount function (see section 3.3), Hayashi (2003) gives an axiomatic derivation of the additive utility model with quasi-hyperbolic discounting, which is given by

$$U({}_0a) = u(a_0) + \sum_{t=1}^{\infty} \beta\delta^t u(a_t) \tag{4.42}$$

The key assumptions seem to be that the time preferences *over streams* exhibit present bias and are quasi-stationary, but stationary in the case of constant streams.

Furthermore, Hayashi (2003) demonstrates that is it is also feasible to weaken quasi-stationarity further in order to derive a more general form of quasi-hyperbolic discount functions: he partitions time into *three* sections: present, near future and farther future and only assumes stationarity "within" farther future. The resulting discount function is then best written in the form of discount rates (see equation 2.12 in section 2.4.2)).

$$\delta_{Q^n}(t) \equiv \begin{cases} 1 & \text{for } t = 0 \text{ (present)} \\ \prod_{i=1}^{t} \beta_i & \text{for } 1 \leq t \leq n-1 \text{ (near future)} \\ \delta^{t-n+1} \prod_{i=1}^{n} \beta_i & \text{for } t \geq n \text{ (farther future)} \end{cases} \tag{4.43}$$

In the case of $n = 1$, there is no near future, and the generalized hyperbolic discount function coincides with the definition in equation 4.38.

4.4.3. Motivation and Empirical Evidence

Argueably, quasi-stationarity is one of the simplest deviations from stationarity, that also permits "preference reversals". Moreover, the functional form is relatively simple compared to the hyperbolic discount function, facilitating the tractability of more complex models. Therefore, the quasi-hyperbolic discount function is frequently used in applications, that analyze the dynamic behavior of decision makers (see part II).

With regard to empirical evidence, a number of studies posit that most, if not all, of the discounting of decision makers occurs from period 0 to period 1, and that hardly

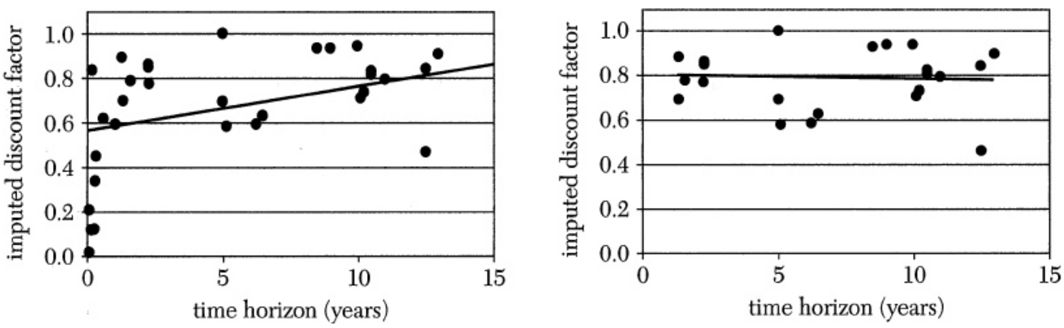
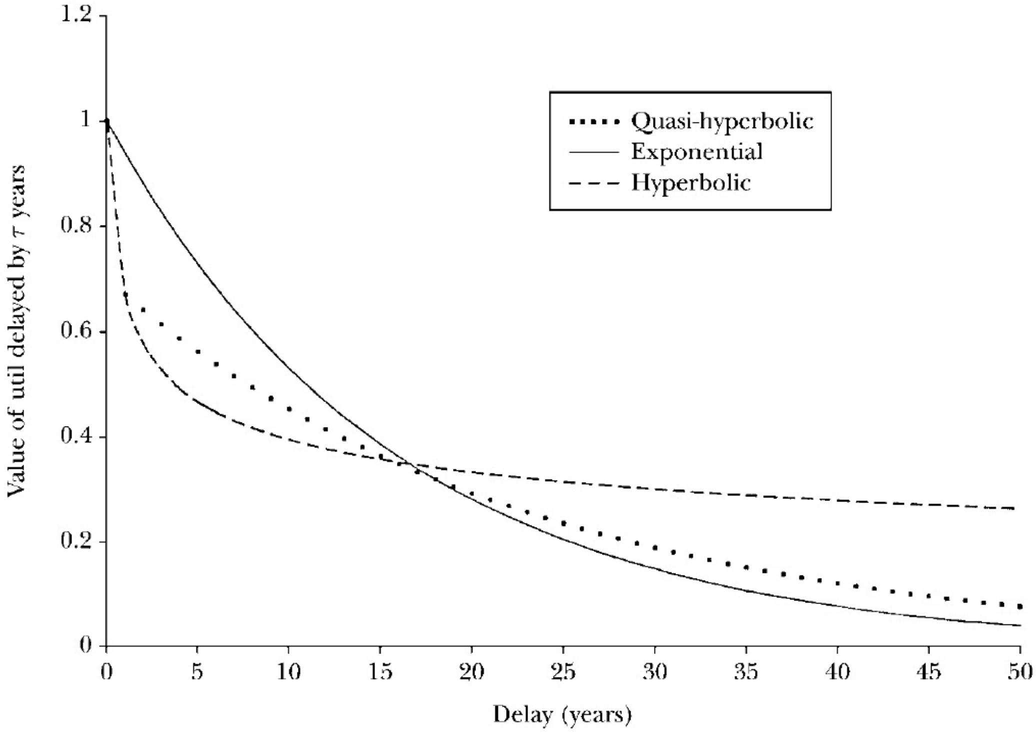


Figure 4.5.: Effect of omitting "short-run" discount factors on the LS estimation. Source Frederick, Loewenstein and O'Donoghue (2002, p.362)

any discounting takes place afterwards. In terms of discount factors, this means that $\phi(1)$ is significantly different from zero, whereas the consecutive discount factors may be not. Frederick, Loewenstein and O'Donoghue (2002) present evidence in support of this hypothesis: In section 4.2.1 we presented the results of their meta-analysis and saw that the imputed average discount factors increase with time, i.e. impatience is decreasing. As Frederick, Loewenstein and O'Donoghue (2002) argue, after excluding all observations (i.e. studies) with a time horizon of less than a year, the imputed discount factors are constant with respect to time. If we assume for a second that this really is the case, then the increase of the discount factors is merely due to an omitted variable bias (the omitted variable being a "one-year dummy") inflating the time-coefficient. Figure 4.5 depicts the change of the least-squares estimation.

The estimates of Frederick, Loewenstein and O'Donoghue (2002) suggest to calibrate the parameters as $\beta \cong .8$ and $\delta \cong 1$. In a more elaborate study, Angeletos, Laibson, Repetto et al. (2001) present similar, albeit somewhat lower estimates of $\beta \cong .7$ and $\delta \cong .957$

Up to now we presented three examples of transitive time preferences that can be incorporated into the framework of Relative Discounting. Due to their transitivity they even have the nice property of an absolute discount representation, i.e. their ranking can be represented by present values, depicted in 4.6. The remainder of this chapter is devoted to time preferences that are not transitive; the last two of them do not even have a Relative Discount representation.



Exponential: δ^{τ}, with $\delta = 0.944$; hyperbolic: $(1 + \alpha\tau)^{-\gamma/\alpha}$, with $\alpha = 4$ and $\gamma = 1$; and quasi-hyperbolic: $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots\}$, with $\beta = 0.7$ and $\delta = 0.957$.

Figure 4.6.: Plot of three absolute discount functions. Source: Angeletos, Laibson, Repetto et al. (2001, p.51)

4.5. Subadditive Discounting

4.5.1. Definition and Relation to Relative Discounting

In his critique of the hyperbolic discount model, (Read, 2001) proposed a different model in order to explain anomalies such as decreasing impatience (see above). He comes up with the interesting idea that the (average) discount rate in a given time interval increases with the number of subintervals in which the interval is partioned. Within the framework of relative discounting this reads as:

$$\eta(t,r) > \eta(t,s)\eta(s,r) \quad \forall\ r,s,t \text{ s.t. } 0 < r < s < t \tag{4.44}$$

which implies intransitivity of the generated time preferences. Applying Theorem 2.4, this prohibits a present value representation of the time preferences, which means strictly speaking one cannot speak of "discounting" as defined above.

Note that the time preferences discussed so far, all took *additive discounting* as given: Regardless of how many parts we divide an interval in, the "time effect" of the valuation of an outcome-date pair is the same. The intransitive *subadditive discounting* on the other hand allows for the following cycles

$$(x,t_0) \sim (y,t_1) \sim (z,t_2) \succ (x,t_0)$$

Ok and Masatlioglu argue that this type of time preferences satisfies axioms RD1-RD6 and therefore has a relative discounting representation. Moreover, they provide us with an example of an relative discounting term, that generates subadditive time preferences:

$$\eta(s,t) \equiv \begin{cases} e^{f(|s-t|)} & \text{whenever } s \le t \\ e^{-f(|s-t|)} & \text{otherwise} \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}_+$ is strictly convex with $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$

4.5.2. Explanations for Subadditive Time Preferences

Read (2001, pp.9) names two reasons why time preferences could be subadditive:

Drawing on results from Tversky and Koehler (1994) on decision under risk, Subadditive "discounting" might be an example of the *salience effect* in support theory: when an object or event is partitioned into parts, each of these parts gets more attention or appreciation then before. Frederick, Loewenstein and O'Donoghue (2002, p.361) provide us with the following illustrative example: People judge the probability of death by accident lower than the cumulative probability of death by fire, death by drowning and so on. In the case of intertemporal choice this means, that delay is emphasized in the eyes of decision makers when a long delay is divided into two or more shorter delays.

Another explanation is the presence of the so called *regression-to-the-mean* effect, that has been observed in experiments about decision under risk: Estimates are often biased towards the midpoint of the support of the probability distribution. So, in the case of a discrete random variable with values 0 and 1, the subjective estimates for the expected value will typically be biased towards .5. Read argues that a similar bias could be present in intertemporal choice: Suppose a decision maker is being asked the typical question of what amount y at time $s > t$ will make him indifferent to (x, t). If the length of the delay, $s - t$ is "small", then so will be the premium $y - x$. However, if we add some kind of risk in the judgment of how big the premium should be, then applying the "regression-to-the-mean" effect tells us that the decision makers estimate of the premium will be biased upwards, since the premium is left truncated (the decision maker would not put up with a negative premium). One could then argue that this effect becomes more pronounced as the number of subintervals increase.

4.5.3. Subadditivity and Decreasing Impatience

Subadditivity also gives rise to a different interpretation of the results of the experiments discussed in 4.3: When the realization of outcomes (in the form of real or imagined monetary prizes) is expedited, there are two kinds of effects: a *decrease in delay* and a *decrease in the length of the interval.* Previous studies omitted the second effect and their estimates could therefore exhibit some kind of "omitted variable bias" in the sense, that they attributed the observed decrease in impatience to the decrease in delay only.

4.5.4. Empirical Evidence

Read (2001) conducted a series of experiments that elicited discount rates for a two-year interval. Moreover he partitioned the interval into three equal sub-intervals, each 8 months long and elicited discount rates for these intervals as well. He found that the compounded discounting of the three sub-intervals was significantly higher than the (average) discount rate of the whole two-year interval, providing evidence for the case of subadditive discounting.

With respect to hyperbolic discounting he found no significant difference between the discount rates between the first, the second and the third eight-month periods, which essentially rejected the prediction of the hyperbolic discounting model. Therefore it is essential to stress, that the model of subadditive discounting should not be seen as a mere extension of hyperbolic discounting, but a challenge to it, providing us with a different psychological explanation for both, discounting and decreasing impatience.

4.6. Rubinstein's Procedural Approach

In his "'Economics and Psychology'? The Case of Hyperbolic Discounting" Rubinstein (2003) claims that while there is indeed a large body of evidence against exponential discounting, the widespread endorsement of hyperbolic discounting may be all too premature, because, he adds, "an infinite number of functional forms [are] consistent with the psychological findings that support hyperbolic discounting" (p. 1209). In other words, although critique of the exponential discounted utility model was long overdue, one should subject hyperbolic discounting to thorough analysis and not establish it as the "natural" successor of the EDU model.

Furthermore, it is worthwile not only to look for alternative functional forms, that give a better fit with the data collected in experiments, but to turn to introspection in order to open "the black box of decision making" (p. 1215). In this spirit Rubinstein proposes an alternative model of time preferences and intertemporal decision:

As Read in his explanation of "subadditive discounting" (see section 4.5) Rubinstein (2003) also tries to incorporate well-known psychological phenomena that provided "anomalies" to the normative theory of decision under risk. Building largely on his own work of decision theory he proposes a *procedural* approach, that incorporates - in a stylized way - the plausible point that a decision maker *preprocesses* her "problems": She uses a heuristic procedure that makes use of so called *similiarity relations*, which allow her to "simplifiy" the information underlying the intertemporal decision problems. Formally, similarity relations can be modeled as reflexive and symmetric binary relations, and we denote them by $\approx$.

According to Rubinstein's procedural approach, the choice for either (x, t) or (y, s) is made according to the following three-step procedure:

1. The decision maker looks for **dominance:** If $x \succeq_0 y$ and $t < s$ the decision maker readily chooses the former pair. So, in the case of positive time preferences, there is no tradeoff to consider when one pair is dominated by another.

2. If there was no dominance, the decision maker looks for **similiarities:** If there is similiarity in one dimension (i.e. either $x \approx y$ or $t \approx s$) then the decision maker can focus on the other dimension and choose accordingly.

3. If these stages were inconclusive, the decision maker turns to a different criterion, that remains unspecified by Rubinstein.

4.6.1. Experimental Evidence

Rubinstein provides us with evidence from a number of lab experiments where hyperbolic discounting could not explain the choice pattern of a significant proportion of the

Question 1:		You can receive the amounts of money indicated according to one of the two following schedules:			
	A	April 1st $1000	July 1st $1000	October 1st $1000	December 1st $1000
	B	March 1st $997	June 1st $997	September 1st $997	November 1st $997
Question 2:		You have to choose between			
	A	Receiving $1000 on December 1st			
	B	Receiving $997 on November 1st			

Table 4.2.: Questions I and II in Rubinstein (2003)

test takers. In one of these experiments, undergraduate students were asked to answer the following questions (original phrasing given in table 4.2):

According to hyperbolic discounting the individuals preferences over streams can be represented by:

$$U({}_0x) = u(x_0) + \sum_{t=1} \prod_{s=1}^{t} \phi_s u(x_t)$$

where ϕ_s is an increasing sequence of discount factors (see section 4.3.3). Therefore, if a decision maker is willing to "pay" $3 in order to speed up the receipt of a prize by one month from December to November, the nature of the decreasing impatience of the preferences implies that she is also willing to speed up consumption from October to September for the same or even smaller amount. Therefore no one ought to choose A in Q4 and B in Q3. However, in the experiment a significant share of the students did just that - contradicting the predicition of hyperbolic discounting.

From the vantage point of Rubinstein's procedural approach on the other hand, the behavior of the students who chose B in Q3 and A in Q4 can be explained in the following way: In the eyes of some students the stream ($\$1000, \$1000, \$1000, \$1000$) was less similiar to ($\$977, \$977, \$977, \$977$) than the single outcome $1000 was to $977. So, whereas the choice between $100 and $977 could be made according to step two of the procedural approach, the comparison of the streams was not, implying that another criterion was applied.

4.6.2. The Relation to Relative Discounting

Although it is not entirily in the spirit of Rubinstein, a somewhat "streamlined" variety of his procedural approach can in fact be incorporated into the framework of Relative Discounting, as Ok and Masatlioglu showed:

Clearly, the first step of Rubinstein's procedure is not an issue – we would consider unrealistic any model of time preferences that prescribes to choose a dominated alternative. With respect to similiarity, Relative Discounting of course requires to "quantify" similiarity in a certain sense: Ok and Masatlioglu (2007, p.228) distinguish between similiarity relations over outcomes and dates and define them analogously:

$$x \approx_X y \Leftrightarrow \frac{1}{\epsilon} \leq \frac{u(x)}{u(y)} \leq \epsilon$$
$$s \approx_T t \Leftrightarrow \frac{1}{\epsilon} \leq \frac{f(s)}{f(t)} \leq \epsilon \quad \text{for some } \epsilon > 1$$

where $f : \mathbb{R}_+ \to \mathbb{R}_{++}$, decreasing and $\lim_{t\to\infty} f(t) = 0$.

What is left to specify is step three, i.e. the evaluation when the first two steps remained inconclusive. In this particular case we make the (arbitrary) prediction that the decision maker evualuates an outcome-date pair (x, t) according to $w(x, t) \equiv f(t)u(x)$. As all cases have been covered, we are able to conclude that the time preferences generated by Rubinstein's procedural approach can be seen as a form of Relative discounting, where the relative discount function is given by[9]:

$$\eta(s,t) \equiv \begin{cases} 1 & \text{whenever } s \approx_T t \\ \frac{f(s)}{f(t)} & \text{otherwise} \end{cases}$$

With respect to the classification of time preferences given in 2.4, we can say that these time preferences belong to the class of intransitive time preferences, since in general it does not hold that $\eta(t, r) = \eta(t, s)\eta(s, r) \quad$ for all $r, s, t \geq 0$ - at least not for every $f(\cdot)$ defined above. Depending on the exact specification of the $f(\cdot)$ function, the time preferences may or may not exhibit (strong) present bias. However, Rubinstein itself may have endorsed a specification that allows for present bias. If we again employ the classic example of Thaler (see above), the decision maker might perceive the dates 100 days from now and 101 days from now as very similar, which activiated step two of the procedure and led her to choose two apples in 101 days over one apple in 100 days. When both outcomes are expedited by 100 days on the other hand, the decision maker may perceive the two dates as very different as well, which leads her to resort

[9] a plot of such a Relative Discount function can be found on page 59

to stage three of the procedure.

4.6.3. Discussion and Critique

Although Rubinstein comes forward with a novel approach that certainly helps to understand the choice pattern of decision maker in certain situations, there are some caveats and limitations: Rubinstein itself remains rather vague about what "similiarity" means, therefore virtually any "anomaly" can be explained by *perceived* similiarities. This ipso facto renders the model incontestable, therefore unscientific. Moreover, when one tries to "quantify" similiarity, i.e. to give a specific relation between objective "similiarity" in the outcome or date space and the similiarties as binary relations, as Ok and Masatlioglu did, one strips the model of a central feature: that similiarities are only perceived similiarities. With respect to the experiment discussed above, this means that this precludes perceiving \$1000 similar to \$1003 but not to \$997. Certainly, it would be possible to define similarities with more complex step-functions, but this would make the model overly complicated. Furthermore, one had to define similiarity relations over streams and not only outcome-date pairs, otherwise there would be no reason why $(\$1000, \$1000, \$1000, \$1000)$ is less similiar to $(\$977, \$977, \$977, \$977)$ than \$1000 is to \$997.

Up to now we only considered time preferences that could be incorporated into Ok and Masatlioglu's framework of Relative Discounting. The next two sections will be devoted to time preferences that do not satisfy axioms RD1-RD6 and hence cannot be represented by a Relative Discounting function.

4.7. Vague Time Preferences

4.7.1. Motivation and Representation

Bearing some resemblence to the approach of Rubinstein mentioned above, Manzini and Mariotti (2006) study time preferences that are generated by a particular procedural approach.

Their starting point is the (slightly overused) quote by Arthur Pigou who famously declared with respect to people's attitude towards the future, that "our telescopic faculty is defective, and we, therefore, see future pleasures, as it were, on a diminished scale." (Pigou, 1920, ch.2,\$3). Unlike other authors, Manzini and Mariotti take the telescope metaphor literally and come forward with the idea that decision makers perveive distant outcomes as blurred and are therefore only able to make a ranking if they differ "sufficiently" enough.

Formally, the decision problem of choosing one out of two outcome-date pairs (denoted by (x, t) and (y, s)) can be resolved right away, i.e. in the first step, whenever it

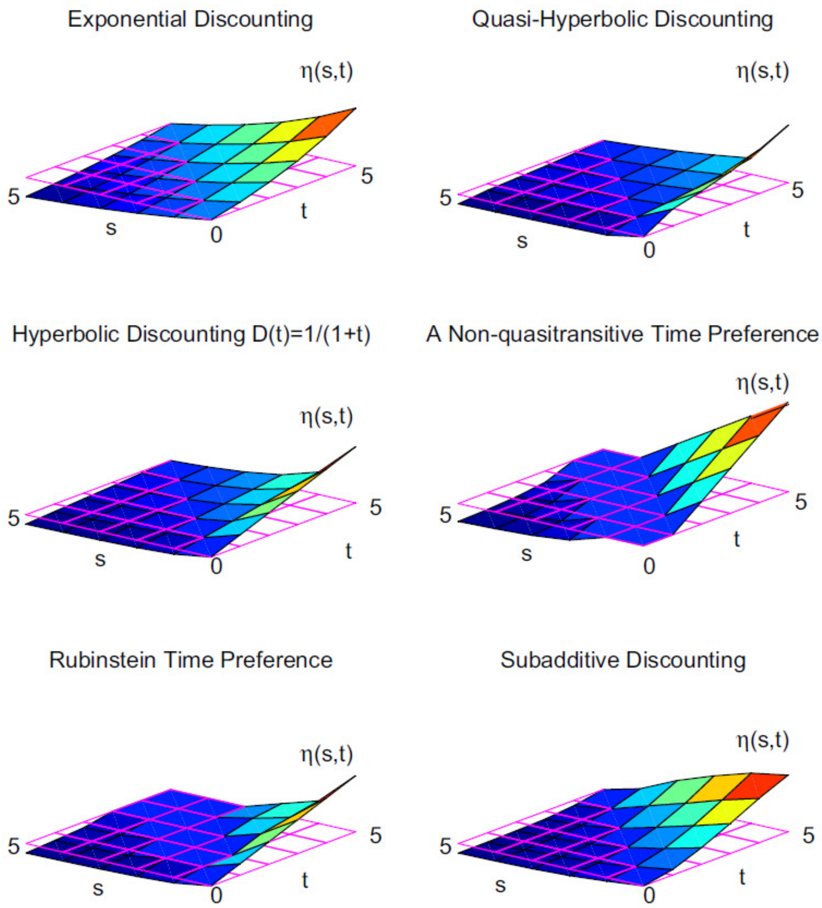


Figure 4.7.: Plots of the Relative Discouting Function $\eta(s,t)$: Subplots 1–3 represent the three transitive time preferences introduced above. The remainder depict intransitive time preferences, two of which are discussed in this thesis. Source Ok and Masatlioglu (2007, p.227)

holds that

$$w(x,t) > w(y,s) + \sigma(y,s) \tag{4.45}$$

where w is a "reasonable" utility function over the outcome-date space, e.g. one of the absolute discount representations of sections 4.1 – 4.4. $\sigma(\cdot) > 0$ is a "vagueness" factor that precludes the ranking of outcome-date pairs that are not different "enough" in terms of utils. This of course means that the ranking of alternatives is not complete in the first step.

Therefore, when the first step is not conclusive, the decision maker applies a different criterion: She will then compare the outcomes of the two pairs and only if she finds herself indifferent between them have a look at their respective receival times. This is what Manzini and Mariotti refer to as the "outcome prominence" version of their model. In the natural counterpart, the "time prominence" version, the orders are reversed: first the decision maker will look for a difference between the receival times and only when there is none, compare the outcomes. Therefore for all alternatives that are mutually "similar" (in terms of utility), the decision maker is said to have lexicographic preferences: In the outcome prominence version the outcome is the dominant dimension, in the time time prominence version it is the time dimension.

More formally, this heuristic procedure generates a two tier preference relation: The first tier of preferences are incomplete binary relations, $\succ$, that are defined as:

$$(x,t) \succ (y,s) \Leftrightarrow w(x,t) > w(y,s) + \sigma(y,s) \tag{4.46}$$

If neither $(x,t) \succ (y,s)$ nor $(y,s) \succ (x,t)$ holds we write $(x,t) \approx (y,s)$, i.e. the first step was not decisive.

In order to specify the second step, we suppose that the decision maker has complete and transitive preferences over the outcome space, which we denote by $\succeq^O$. Moreover, we assume that the lexicographic preferences are such that she prefers sooner to later outcomes. Therefore the preferences generated from the "outcome prominence" version of the model, $(\succ^*, \sim^*)$ are defined in the following way (Manzini and Mariotti, 2006, p.5):

- $(\mathbf{x}, \mathbf{t}) \succ^* (\mathbf{y}, \mathbf{s}) \Leftrightarrow$
    1. either $(x,t) \succ (y,s)$
    2. or
        a) $[(x,t) \approx (y,s),\ x \succ^O y]$ (Primary Criterion = Outcome)
        b) or $[(x,t) \approx (y,s),\ x \sim^O y,\ t < s]$ (Secondary Criterion = Timing)
- $(\mathbf{x}, \mathbf{t}) \sim^* (\mathbf{y}, \mathbf{s}) \Leftrightarrow [(x,t) \approx (y,s),\ x \sim^O y,\ t = s]$

The "time prominence" version of the model is defined analogously. In both cases the binary relations are complete. As we already saw in the Rubinstein model, heuristic procedures are prone to intransitivity and therefore also the "vague" time preferences will not be transitive in general.

4.7.2. The sigma-delta-Model

In the most parsimonious version of the model, there are only two parameters, $\sigma > 0$ and $\delta \in (0,1)$: The vagueness factor $\sigma(x,t)$ is assumed to be constant with respect to both, outcome and time. Moreover w, the utility function over the outcome-date space, is given by the exponentially discounted utility model, $\delta^t u(x)$. Furthermore, $u(x)$ is specified to be linear in outcome and without an intercept term – therefore the utility function can be also written as $u(x) = x$ without loss of generality. Taken together, the first step of the decision process is conclusive if, and only if:

$$(x,t) \succ (y,s) \Leftrightarrow \delta^t x > \delta^s y + \sigma \tag{4.47}$$

Even in this simple specification the model is able to generate "preference reversals". The intuition behind the result is that a constant vagueness term, σ, *relatively* gains in size as the alternatives are shifted backwards in time and therefore "lose in utility". So, if the alternatives are delayed beyond a critical point in time, the decision maker will not be able to distinguish the two alternatives and therefore activate step two of her decision procedure. Since the ranking of the two alternatives in the second step need not be the same as their ranking in the first step (up to the critical point) there is room for preference reversals, so the "vague" time preferences are also not stationary.

In addition, the (σ, δ)-model allows for preference cycles. The authors also provide us with sufficient conditions (in the form of parameter configurations) that ensure that such cycles occur.

4.7.3. Discussion

Manzini and Mariotti come forward with a novel approach that demonstrates how "anamolies" in intertemporal choice can be accounted for without adopting discount functions that differ from exponential discouting: qualitatively, even the parsimonious (σ, δ)-model is able to induce the same "preference reversals" and cycles as (quasi-)hyperbolic models.

Although the model also consists of a heuristic procedure, it differs from Rubinstein's procedural approach in a number of aspects: "similiarity" can only occur in the first step and in contrary to the Rubinstein model it can arise if two alternative differ in *both* dimensions. Furthermore, depending on the specification of the function $\sigma(\cdot)$ many decision problems can be resolved in the first nonheuristic step, whereas in Rubinstein's procedure, the ranking of alternatives might be unspecified.

Due to the two-tier preference structure of the model, in cannot be incorporated into Ok and Masatlioglu's framework of Relative Discounting. Moreover the lexicographic preferences in the second step of the decision procedure are a straightforward violation of axiom RD1 (time sensitivity) and RD2 (outcome sensitivity).

4.8. Time Preferences with Fixed Costs

4.8.1. Motivation for introducing fixed costs

In the discussion of quasi-hyperbolic time preferences (see section 4.4) one interpretation of the parameter β was a *variable* cost: The "cost" of deferring gratification to the future (regardless of the size of the parameter δ) is given by $(1-\beta)u$. In a recent paper Beinhabib, Bisin and Schotter take this interpretation as a starting point and introduce "mental" *fixed* costs[10] that decision makers occur for rewards that are to be obtained in the future –*in addition* to the "variable costs" and/or other forms of discounting: as we discussed in section 4.2.4 possible sources of fixed costs are that it might take subjects effort and time to obtain the deferred gratification, because they have to pick it up from the test center or that they simply have to think about having to pick up the reward later in time.

Let us assume that the decision maker is risk neutral and that the outcomes are positive monetary rewards, i.e. $X \equiv [0, \infty)$. Then, without loss of generality, the decision makers preferences over the elements of the outcome-date space can be represented in the following way:[11]

$$(x,t) \succeq (y,s) \Leftrightarrow \delta^*(x,t)x \geq \delta^*(y,t)y \tag{4.48}$$

$$\text{where} \quad \delta^*(x,t) = \begin{cases} 1 & \text{for } t = 0, \\ \beta\delta(t) - \frac{b}{x} & \text{for } t = 1, 2, \dots \end{cases} \tag{4.49}$$

The fixed costs term b may able to explain why in many studies the magnitude effect (declining discount rates as rewards increase) is observed: Higher prizes decrease the importance of the fixed costs and lead ipso facto to lower discounting. Note that in this specification the size of the monetary reward enters the discounting term. However, it is not to be confused with the money-discount function – here the discount function is money-dependent even under the assumption of risk neutrality.

[10]In a previous version of the paper the authors refered to it as a "contemplation cost".

[11]Equivalently, this representation could be written as $(x,t) \succeq (y,s) \Leftrightarrow \delta(t)x - c(t) \geq \delta(t)y - c(s)$ with $c(0) = 0$ and $c(t) > 0$.

4.8.2. Empirical Evidence

The authors restrict themselves to the cases where $\delta(t)$ is either the exponential discount function or the (generalized) hyperbolic discount function. As we saw in section 4.3 the generalized hyperbolic discount function is characterized by two parameters, α and γ and is able to nest the exponential function ($\alpha \to 0$). In order to eschew technical problems when conducting statistical tests, they used a different parametrization of the generalized hyperbola.[12] Therefore, Beinhabib, Bisin and Schotter obtained the following alternative parametrization of the discount term $\delta^*(x,t)$:

$$\delta^*(\beta,\pi,\theta,b,x,t) = \begin{cases} 1 & \text{for } t = 0, \\ \underbrace{\beta}_{VC}\underbrace{(1-(1-\theta)\pi t)^{\frac{1}{(1-\theta)}}}_{\delta_H} - \underbrace{\frac{b}{x}}_{FC} & \text{for } t = 1,2,\ldots \end{cases} \tag{4.50}$$

The authors elicited discount rates of 27 respondents using the Becker-DeGroot-Marschak procedure (see section 4.2). Using the collected data they estimated the parameters using non-linear least-squares estimation. They report a significant fixed cost (b) of about \$4 on average across respondents. The variable cost factor (β) is insignificant, suggesting that prior studies merely found the statistical artefact in the form of an omitted variable bias. Moreover, the authors performed robustness checks for either a possible framing effect or risk aversion (in the sense of a concave utility function) and found no significant difference between to the obtained estimates.

4.8.3. Discussion

The introduction of a fixed cost parameter is indeed able to explain the magnitude effect that had been documented in many studies before. Moreover, the findings of Beinhabib, Bisin and Schotter suggest that the present bias of decision makers stems from a different source: "mental" fixed costs. Although both, the experimental design of the study as well as the econometric specification seem sound, the low number of observations and a possible selection bias (all respondents were NYU students[13]) somehow call for further examination of their findings.

Note that this model cannot be incorporated into the framework of relative discounting: The presence of a fixed cost precludes the separation of the time and the outcome effects and are therefore a violation of the separability axioms RD4 and RD5 (see 2.2).

[12] Clearly, e.g. no *future* discounting $H_0 : \alpha = \infty$ in itself is hard to test. In their specification $\theta = \frac{\alpha}{\beta}+1$ and $\pi = \gamma$

[13] Although the authors do not explicitly say so, one can infer that from the fact that the respondent's rewards were sent to their campus mailboxes (p.7)

5. Summary of Part I

In the last three chapters, we saw that theories of intertemporal choice borrow heavily from results from theory of decision under risk and indeed some authors invoke the similarities between the discounted utility model on the hand and the expected utility model on the other hand. But it is important to stress that while the expected utility theory rests on a *normatively* solid foundation, the discounted utility theory does not: Clearly, independence between mutually exclusive events as in decision theory is something entirely different than intertemporal noncomplementarity. As Frederick, Loewenstein and O'Donoghue (2002) point out, people who do not decide according to the discounted utility model do not necessarily make a "mistake" from a normative point of view, this is of course different in the realm of decision under risk.

If there is one thing the studies cited above showed, then it is that we simply don't know how people make economic decisions that involve intertemporal trade offs. From a theoretical perspective, the problem is that even if we endorsed the discounted utility model, there are always *two* unknown functions that cannot be elicited separatly.

One thing, however, seems safe to say: the exponential discounted utility model is overly simplistic and condenses all the various dimensions of intertemporal choice into a single parameter. It can, therefore, serve as a benchmark model – at best. As mentioned before, the bulk of intertemporal economic models employed an exponential discount function. In part II of this thesis we will see how the predictions of these models change (provided that these models retain some degree of tractability) when we make the slightest possible modification: exchange the exponential discount function with a quasi-hyperbolic one.

Part II.

Applications of Hyperbolic Discounting

The future ain't what it used to be.

(Yogi Berra)

If you wake up at a different time, in a different place, could you wake up as a different person?

(Tyler Durden)

Througout part I of this thesis we analyzed the decision maker's preferences from the perspective of the present. So, in a way, we did not allow for time to pass. We are now going to jettison this assumption and examine the implications. Intuitively, this should make a huge difference, since plans that seem like a good idea today, might not be as enticing to follow tomorrow. Clearly, these considerations can be ruled out if we assume that the decision maker can *commit* to her actions - an assumption that we maintained througout part I of this thesis.

In chapter 6 we will define and illustrate the important concept of *dynamic consistency* and discuss the issue of self-control: the decision maker will incorporate into her considerations that she *herself* will not follow some plans.

Part II is organized as follows. In chapters 7 and 8 we will put our decision maker in ever more complex (economic) situations and compare the results of the situations under commitment with the results without commitment. We will do so by restricting ourselves to quasi-hyperbolic time preferences. In section 8.3 we also analyze the decision process of consumption/saving decisions under partial commitment and briefly elaborate on the ramifications of economic policy.

6. Dynamic Inconsistency and the Multiple Self

6.1. Definition of Dynamic Consistency

The starting point of our discussion of dynamic consistency is the classic example due to Richard Thaler, which we already encountered above twice. This time however, we modify the thought experiment in the following way:

> A decision maker prefers one apple today over two apples tomorrow. By the same token, she prefers two apples in 101 days over one apple in 100 days.

We will now argue that these preferences are prone to what we will call dynamically inconsistent planning: Suppose that the above cited decision maker now faces the option of either taking one apple in 100 days or taking two apples in 101 days. Clearly, she will *plan* to take the second option. However, we also know that after 100 days have passed she will want to *reconsider* her choice since she prefers one apple right now over two apples one day later. Therefore, her *plan* of opting for 2 apples that are to be obtained in 101 days will not be what we call *dynamically consistent*:

Definition A plan $(x_0^*, x_1^*, \ldots, x_T^*)$ is **dynamically consistent** if it holds that the path of choices it induces is also preferred over all feasible alternatives as time passes by. Formally:

$$(x_0^*, x_1^*, \ldots, x_T^*) \succeq (x_0, x_1, \ldots, x_T) \Rightarrow (x_\tau^*, x_{\tau+1}^*, \ldots, x_T^*) \succeq (x_\tau, x_{\tau+1}, \ldots, x_T) \quad (6.1)$$

$$\forall 0 \leq \tau < T - 1, \quad (6.2)$$

$$\forall (x_0, x_1, \ldots, x_T) \in \mathcal{A}, \quad (6.3)$$

$$\forall (x_\tau, x_{\tau+1}, \ldots, x_T) \in \prod_{\tau}^{T} A_t(x_0^*, x_1^*, \ldots, x_{\tau-1}^*) \quad (6.4)$$

In order to make sense of this definition, fix $\tau = 1$: Then dynamic consistency requires for a plan also to be a good idea to follow if one period passes by. In period

1, the decision maker will rethink her plan and only then follow it if there is no better (feasible) alternative.

Note that the set of feasible future alternatives may depend on the choices that have been made up to this point in time (i.e. the choice set is history dependent). On the extreme, this means that the choice set is a singleton at all time points in the future and the decision maker has therefore perfect self-control. This is the case if the decision maker has commitment power. Alternatively, there might be other (albeit less economically interesting situations) when we encounter this extreme case: Think of a decision maker who has to ration food for the next week. If she consumes her entire stock of food on the very first day, the set of alternatives tomorrow is effectively reduced to "starving".

In general, it will *not* be the case that plans that seem optimal from the perspective of the present are also dynamically consistent. There is, however, one notable exception: Inspection of this definition of dynamic consistency reveals a close relationship with our definition of stationarity. Apart from the notation, the first line in the definition of dynamic consistency coincides perfectly with the definition of stationarity (equation 3.5 in chapter 3.3): but whereas then we interpreted the variable τ as the number of periods that an outcome was speeded up or delayed, the interpretation here is that time itself passes by. Therefore, if the decison maker has stationary time preferences, then all plans that are optimal from the perspective of the present are also optimal in the future. In other words, stationary time preferences imply dynamically consistent plans. In the literature, this case is sometimes refered to as the "harmony" case. Recall that we showed in section 2.4.3 that among the class of transitive time preferences, stationarity brings us to exponential discounting.

Since at least Strotz (1955) it is well known among economists that therefore, modelling "exponential" and "non-exponential" decision makers is qualitatively different, which is also what we will see in the next chapters.

However, we will restrict ourselves on the case of quasi-hyperbolic time preferences, introduced in section 4.4. The main motivation why most, if not all of economic analysis of these kind is restricted to quasi-hyperbolic discounting is mostly that it allows for a certain (albeit limited) degree of tractability and that we can nest the behaviour of exponential discounting (by setting $\beta = 1$).

6.2. The Multiple Self

The considerations laid out in the preceeding section motivated economists and psychologists (Phelps and Pollak, 1968; Pollak, 1968; Goldman, 1980, just to name a few of them) to model intertemporal decisions of decision makers that have quasi-hyperbolic time preferences[1] as *intra*personal dynamic games. In this somewhat schizophrenic approach, every period in time, t, is populated by a decision maker, which is usually refered to as "self t". The different "selves" are usually thought of having both, identical material preferences and an identical attitude towards timing. However, they "evaluate" the discount function at different points in time: from the perspective of self 0, the present value of (x, t), $(t > 0)$ is $\beta\delta^t u(x)$; from the perspective of self $0 \leq \tau < t$ it is $\beta\delta^{t-\tau} u(x)$.

Splitting a decision maker into several multiple selves can also reconcile our thought experiment of choosing apples with the Weak Axiom of Revealed Preference: The preferences that were revealed are those of self 100 and not those of self 0 – therefore, it can very well be the case that self 0 prefers one apple in 100 days over two apples in 101 days.

[1] See Thaler and Shefrin (1981) for a different approach that bears much resemblence to the Freudian id/super-ego dichotomy: In their model the players are a far-sighted "planner" and T myopic "doers". The planner's utility is simply given by the sum of the utils of the doers.

7. Models of Procrastination

7.1. The Student's Curse

In order to illustrate the idea of a "mulitple self" further, let us consider the following thought experiment (Duffy, 2007):

Imagine a decison maker who has to hand in a term paper two weeks from now. She has the options of writing the paper either this week, next week or in two weeks. Suppose writing a term paper incures mental "costs" that are higher the later she writes her paper (because, say, she has one exam next week and a further two exams in two weeks). Let us fix these mental costs numerically as $1, \frac{3}{2}, \frac{5}{2}$ for writing the paper this week/one week from now/two weeks from now. Let us assume further that she discounts time quasi-hyperbolically with $\beta = \frac{1}{2}$ and $\delta = 1$: That is, there is no additional discounting from week one to week two.[1]

As a benchmark solution, let us as first assume that the decision maker has commitment power: She writes the paper with a friend who she has to schedule a time with. Since she does not want to stand her friend up, her friend effectively acts as a commitment device. Since she cannot reconsider her choices, all plans are dynamically consistent by default and she simply plans to write the paper on the time when the discounted costs are lowest, which is next week. Her costs under commitment are therefore given by $\frac{3}{4}$.

Let us now jettison this assumption of commitment power. Following O'Donoghue and Rabin (1999) there are now at least two ways of analyzing this situation. First, assume that the decision maker is "**naive**": She thinks that she has commitment power although she has not. Therefore, she will *plan* on writing the thesis next week. However, next week she (i.e. self 1) will want to procrastinate since the costs of writing the paper now $\left(\frac{3}{2}\right)$ exceed the (discounted) costs of writing the paper one week later $\left(\frac{1}{2} \times \frac{5}{2} = \frac{5}{4}\right)$. Therefore, self 0 ends up writing the thesis in week two - when (discounted) costs are *highest* $\frac{5}{4} > 1 > \frac{3}{4}$. Her ex-ante costs differ from her ex-post costs markedly.[2]

[1] Yes, this does violate axiom RD1, but go ahead anyway.

[2] Clearly it is somewhat strange to talk about ex-post costs of self 0, since she is not aware of that in period 0. To make this point more crisply, this notion of ex-post payoff would mean that self $t-1$ gains utility when self t experiences a windfall gain - e.g. a lottery win.

	This Week	One week from now	Two weeks from now
Costs for self 0	1	$\frac{3}{4}$	$\frac{5}{4}$
Costs for self 1	-	$\frac{3}{2}$	$\frac{5}{4}$

Table 7.1.: Writing a term paper: (Discounted) Costs for self 0 and self 1

The second way of analyzing this choice problem is to endow the decision maker with perfect rationality, which – in O'Donoghue and Rabin's terminology – makes her a "**sophisticate**": If this is the case, self 0 anticipates that writing the thesis is not a consistent plan and therefore not feasible. Effectively, she has the choice of writing the term paper either now or two weeks from now. Since writing it now is associated with smaller costs $1 < \frac{5}{4}$, she writes it right away. She realizes that she can only choose among the set of consistent plans and therefore she can only achive the "second best".

Since theoretical economists are usually not particularly interested in the behavior of irrational and "naive" decision makers, most of the literature deals with sophisticated agents.[3]

7.2. Costs of Self-Control

Recall that in the example discussed in the last section, self 0's costs under commitment was $\frac{3}{4}$ while without commitment, sophisticates had to incure (higher) costs of 1. Therefore, one could argue that the decision maker would sacrifice up to $\frac{1}{4}$ utils in order to get access to a commitment device. In other words, she is willing to pay for self-control.

This issue of self-control rationalizes certain real-world phenomena, economists could not make much sense of otherwise: Thaler and Shefrin (1981) interpret fat-farms (spas or resorts that specialize in weight loss) and smoker's clinics as examples for consistent planning. Moreover, christmas savings clubs seem to make a case for the need of commitment devices: In these savings clubs people deposit a pre-agreed amount every month but are allowed to withdraw their money only in the beginning of December. Since they usually pay *less* interest than normal checking accounts, their popularity can only be explained by some value-added, which some economists (e.g. Strotz, 1955) believe to be the possibility of self-control.

[3] A few notable exceptions include Akerlof (1991) and Tyson (2007).

	This Weekend	Weekend One	Weekend Two	Weekend Three
Self 0	3	$\frac{5}{2} = 2.5$	$\frac{8}{2} = 4$	$\frac{13}{2} = 6.5$
Self 1	-	5	$\frac{8}{2} = 4$	$\frac{13}{2} = 6.5$
Self 2	-	-	8	$\frac{13}{2} = 6.5$
Self 3	-	-	-	13

Table 7.2.: Payoffs of the multiple selves in the Fibonacci Cinema

7.3. Blessed are the Ignorant?

Can we conclude from the example in section 7.1 that sophisticates are *always* better off than naifs? As the title of this section suggest, the answer is "no". To see why, consider the following thought experiment (O'Donoghue and Rabin, 1999), Xavier Gabaix dubbed "The Fibonacci Cinema":

Suppose a cineastic decision maker faces the choice of watching a movie on one of the next four weekends. On every weekend the local cinema plays a different movie, granting the decision maker a different level of utility: On the first (this) weekend, the cinema shows an absolutely horrible movie, say, "Up in the Air", which gives the decision maker 3 utils. On weekend one there is a "mediocre" movie granting the decision maker 5 utils. One week later, on weekend two, the local cinema even shows a good movie (8 utils). The best movie, say, "Night on Earth" (13 utils) is being shown on the last weekend (weekend three). Furthermore, let us again assume that the decision maker discounts time quasi-hyperbolically with parameters $\beta = \frac{1}{2}$ and $\delta = 1$.

So when will the decision maker watch the movie?

Under Commitment the decision can simply choose the movie with the highest discounted utility, which is "Night on Earth" with 6.5 utils.

A naive decision maker falsely believes that she has commitment power and chooses not to watch the movie on the first weekend but on the last weekend. On weekend one, self 1 will also not watch the movie, since agrees with self 0 that watching the last movie is more enticing ($5 < 6.5$). Unfortunately, both, the plan of self 0 and and the plan of self 1 to watch the last movie, are not dynamically consistent because on weekend two, self 2 prefers not to wait one week ($8 > 6.5$) and watch the "good" movie, granting self 0 an ex-post payoff of $\frac{8}{2} = 4 < 6.5$.

A sophisticated cineastic self 1 will anticipate that self 2 will watch the movie anyway and therefore choose to watch it on "her" weekend ($5 > 4$). By the same token, self 0 will realize that she can either watch the movie now or next week -

since the former gives more utility ($3 > 2.5$) the sophisticated decision maker ends up watching the horrible "Up in the Air". In a sense, she successively "outsmarts" herself since she faces a series of Prisoner's Dilemmas (O'Donoghue and Rabin, 1999).

We stress that while Sophistication depends on the level of rationality, we endow the decision maker with, non-exponential (but transitive) discounting itself has nothing to do with rationality. However, non-exponential discount functions lead to argueably more difficult decision processes.

8. Consumption/Saving-Decisions with Quasi-Hyperbolic Discounting

In this chapter we will discuss how the presence of quasi-hyperbolic discounting alters consumption/saving-decisions of economic agents. In section 8.1 we provide some intuition with a three period model. We contrast the results in the case of commitment, which generalize to any T-period model, with the case where commitment is not an option.

In section 8.2 we define in closer detail the decision problem of the quasi-hyperbolic agent and derive a (quasi-) hyperbolic Euler equation.

In the last section we then discuss the issue of *partial* commitment and augment the model from the previous section with labor income and illiquid wealth.

8.1. An Introductory Example

In this section we motivate the isssue of dynamic consistency with a relatively simple three-period consumption-savings model. First, we will derive the solution under commitment and discuss its most important features. Then we drop the commitment assumption, which requires us to change our solution concept and contrast the results with the commitment case.

8.1.1. The Commitment Solution

Imagine that a decision maker is endowed with initial wealth $W_0 > 0$ which she can use for consumption in three periods $t = 0, 1, 2$. We denote the levels of consumption with c_t. Furthermore, assume that she can commit to the consumption plan in period 0. She also has access to a deposit account that pays a risk free interest rate of $0 < r < \infty$. In every period, she rationally only withdraws as much money as she needs for consumption in that very period. Defining the gross interest rate

$$R \equiv 1 + r \tag{8.1}$$

the (intertemporal) budget constraint is therefore given by

$$W_0 \geq c_0 + \frac{c_1}{R} + \frac{c_2}{R^2} \tag{8.2}$$

if we assume that the interest rate is constant. In words, the present value of the (finite) consumption stream (RHS) must not exceed the initial wealth (LHS).

The decision maker has quasi-hyperbolic time preferences and we assume that consumption in one period does not have any (direct) effects on the utility derived from consumption in another period. In other words, we assume that the decision maker evaluates the (finite) stream of outcomes (c_0, c_1, c_2) according to the additive discounted utility model (see section 3.1). That given, the preferences of self 0 can be represented by the following utility function:

$$U(c_0, c_1, c_2) = u(c_0) + \beta\delta u(c_1) + \beta\delta^2 u(c_2) \tag{8.3}$$

where $0 < \beta \leq 1$ (Note that a value of $\beta = 1$ gives exponential discounting). Moreover, we assume the static utility function over outcomes to be of the Constant Relative Risk Aversion (CRRA)-family:

$$u(c_t) = \begin{cases} \frac{c_t^{1-\sigma}}{1-\sigma} & \text{for } \sigma > 0 \text{ but } \sigma \neq 1 \\ \ln(c_t) & \text{for } \sigma = 1 \end{cases} \tag{8.4}$$

Moreover, note that the CRRA utility function satisfies the following (Inada) conditions:

- $u'(c)$ is strictly positive but decreasing
- $u(0) = 0$ for $\sigma \in (0, 1)$
- the utility function is C^2 on $\mathbb{R}^3_{++}$
- $\lim_{c \searrow 0} u'(c) = \infty$
- $\lim_{c \to \infty} u'(c) = 0$

The reason why we restrict our analysis to the case of CRRA utility functions is the following: If the utility function over the consumption of a single point in time (i.e. $u(c_t)$) is CRRA, the utility function over the entire consumption plan ($U(c_0, c_1, c_2)$) is of the constant elasticity of substitution (CES) familiy:

$$U(c_0, c_1, c_2) \equiv \begin{cases} A\left(w_0 c_0^q + w_1 c_1^q + w_2^q\right)^{1/q} & \text{for } q \in (\infty, 1] \text{ but } q \neq 0 \\ A c_0^{w_0} c_1^{w_1} c_2^{w_2} & \text{for } q = 1 \end{cases} \tag{8.5}$$

where $w_0 + w_1 + w_2 = 1$ and $w_0, w_1, w_2 \geq 0$

Proof In order to show this, we have to convert 8.3 into 8.5 by a series of order-preserving transformations.

First we divide by $(1+\delta_Q(1)+\delta_Q(2))$.This gives us the weights: $w_t \equiv \frac{\delta_Q(t)}{1+\delta_Q(1)+\delta_Q(2)}$. Note that the weights satisfy the conditions $w_t \geq 0$ and $\sum w_t = 1$.

$\sigma \neq 1$: Multiplying by $(1-\sigma)$ and taking the result to the power of $1/(1-\sigma)$ establishes the desired result[1] if we define $q \equiv 1-\sigma$.

$\sigma = 1$: In the logarithmic case we apply the operation $exp(\cdot)$ which gives us a Cobb-Douglas utility function with constant returns to scale.

Since the CES function exhibits homogeneity of degree one, the represented preferences are homothetic. That is, $x \sim y \Leftrightarrow \alpha x \sim \alpha y$, for $\alpha \geq 0$ (Mas-Colell, Whinston and Green, 1995, p.45). It is well known that in the case of homothetic preferences, the *fraction* of income that is devoted for a good is constant with respect to income. In the case of intertemporal choice this reads as: the fraction of the initial wealth that is to be consumed in a period is constant with respect to the level of the initial wealth. This fact greatly simplifies our analysis, since we are able to write the consumption in a given period as $\gamma_t W_t$, where γ_t is the fraction of income that is spent in this period.

Before we tackle the issue of finding the optimal consumption path, we have the following two observations concerning the set of feasible choices for (c_0, c_1, c_2):

Since the decision maker only cares for the present and the two periods to follow (because, say, the world will come to an end in period t=3), it is clear that in the optimum the consumption plan will be *ressource exhausting*, so the intertemporal budget constraint will bind in the optimum. That in mind, we can exchange the inequality sign with an equality sign in equation 8.2.

Moreover, note that $u(0)$ is not defined for $\sigma \in [1, +\infty)$. Therefore, we have to make the (technical) assumption that $c_t > 0$ for $t = 0, 1, 2$. This assumption, however, turns out not to be restrictive - not even for the case $\sigma \in (0, 1)$.

Taken together these imply that the intertemporal budget set of the decision maker is given by

$$\mathcal{B}^*(R, W_0) \equiv \mathbb{R}^3_{++} \cap \left\{(c_0, c_1, c_2) | c_0 - \frac{c_1}{R} - \frac{c_2}{R^2} = 0\right\} \tag{8.6}$$

We assume that the decision maker solves the (constrained) maximization problem:

$$\max_{(c_0, c_1, c_2) \in \mathcal{B}^*} U(c_0, c_1, c_2) \tag{8.7}$$

The presence of commitment power clearly strips the decision problem from its intertemporal aspect: From a technical point of view, problem 8.7 is a generic static utility maximization problem. We could simply think of the consumption plan as a vector of goods. Their (relative) prices are then given by the inverse of the gross interest rate, $1/R$ and $1/R^2$, respectively.

[1] for $\sigma > 1$ each of these two transformations is strictly order-*reversing*, but if we take these two together, the resulting single transformation is order-preserving

Furthermore, the problem is of a particular nice structure, because we have a concave utility function and a single linear equality constraint - tailor-made for solving by the Langrangian method.

The first order conditions of the Lagrangian,

$$\frac{u'(c_0)}{\beta\delta u'(c_1)} = R \tag{8.8}$$

$$\frac{u'(c_0)}{\beta\delta^2 u'(c_2)} = R^2 \tag{8.9}$$

$$\frac{\beta\delta u'(c_1)}{\beta\delta^2 u'(c_2)} = R \tag{8.10}$$

$$W_0 = c_0 + \frac{c_1}{R} + \frac{c_2}{R^2} \tag{8.11}$$

can be interpreted in the standard way: the LHS is simply the marginal rate of substitution, while the RHS is the price ratio. In intertemporal decision analysis however, these conditions are called Euler-equations. Rearranging gives us the typical Euler equation:

$$u'(c_{t-1}) = u'(c_t)R\phi_t \tag{8.12}$$

where ϕ_t is the discount factor at period t, which we defined as the additional disconting that takes on between period $t-1$ to period t. We already saw in chapter 4.4 that for quasi-hyperbolic discount functions, this discount factor takes the value $\beta\delta$ for $t = 1$ and δ at all other time periods.

In the case of CRRA-utility functions, these equations are linear in the consumption levels and can be written as

$$\frac{c_1}{c_0} = (R\beta\delta)^{1/\sigma} \tag{8.13}$$

$$\frac{c_2}{c_0} = \left(R^2\beta\delta^2\right)^{1/\sigma} \tag{8.14}$$

$$\frac{c_2}{c_1} = (R\delta)^{1/\sigma} \tag{8.15}$$

$$W_0 = c_0 + \frac{c_1}{R} + \frac{c_2}{R^2} \tag{8.16}$$

For a general class of transitive time preferences and under commitment (abstracting from the special case of quasi-hyperbolic discounting) the Euler-equations become

$$\frac{c_t}{c_{t-1}} = (R\phi_t)^{1/\sigma} \tag{8.17}$$

We see that consumption will increase from period $t-1$ to period t when it holds that $R\phi_t > 1$. If we plug in the definitions of R and ϕ (equations 8.1 and 2.8 respectively) this is the case if

$$\frac{1+r}{1+\rho_t} > 1 \tag{8.18}$$

$$r > \rho_t \tag{8.19}$$

If we interpret the interest rate as a measure of the collective or market rate of impatience, then we obtain the intuitive result, that the decision maker will have an increasing (decreasing) consumption profile if she is more (less) impatient than the market.

The exponent, $1/\sigma$, has an amplifying effect if $\sigma < 1$ and a dampening one if $\sigma > 1$. Moreover, if the decision maker has commitment power and CRRA-preferences, the exponent coincides with the elasticity of (intertemporal) substitution (EIS):

$$EIS(t) \equiv \frac{\partial \log\left(\frac{c_t}{c_{t-1}}\right)}{\partial \log(R)} = \frac{\partial \frac{1}{\sigma}(\log(R) + \log(\phi_t))}{\partial \log(R)} = \frac{1}{\sigma} \tag{8.20}$$

As usual the elasticity of substitution can be interpreted as the percentage change in relative demand of one good over the other (numerator) when the relative prices (denominator) are changed by 1%.[2] Constant relative risk aversion reads as a constant elasticity of substitution in the case of intertemporal choice. It is constant in both, the level of initial wealth, W_0 and time, t. This is what we expected, since we showed above that $U(c_0, c_1, c_2)$ is a CES function.

From another, perhaps more intuitive perspective there are two opposing forces at work in the considerations of how to allocate consumption over time. For the sake of illustration let us assume that the interest rate equals zero (so $R = 1$): On the one hand the decision maker will try to concentrate consumption on the very first period since every util derived from later consumption is diminished by discounting. On the other hand, the concavity of the static utility function suggests to equalize consumption over all periods.

More precisely, the degree of concavity of the utility function, determined by the parameter σ, then regulates how smooth the consumption path is. A parameter value for σ that is close to 0 results in an almost linear utility function u, which means a more or less one-to-one relationship between consumption and utility (without loss of generality). Therefore the utility gained from one *additional* unit of consumption does not depend on the *level* of consumption at that period of time. So, the decision

[2]Note that $\log(R) \equiv \log(1+r) \approx r$ for small r. This allows us to interpret the EIS as the change that is induced from raising the interest rate by 1 percentage *point*.

maker will react relatively much to small changes in the interest rate and adjust her consumption decisions accordingly. Conversely, a "large" value of σ causes much difference between the additional utility derived from, say, the tenth or the one hundredth unit of consumption. Therefore, the decision maker will smooth the consumption path.

But back to the case of quasi-hyperbolic discounting: If we solve the system of linear equations given in 8.13–8.16 we obtain the following solution in the commitment scenario, denoted by ${}_cc^* \equiv ({}_cc_0^*, {}_cc_1^*, {}_cc_2^*)$:

$$
\begin{aligned}
{}_cc_0^* &= \gamma_0 W_0 && (8.21)\\
{}_cc_1^* &= (R\beta\delta)^{1/\sigma}\,{}_cc_0^* && (8.22)\\
{}_cc_2^* &= (R^2\beta\delta^2)^{1/\sigma}\,{}_cc_0^* && (8.23)
\end{aligned}
$$

where $\gamma_0(\cdot)$ is the (marginal) propensity to consume in period zero, which is given by:

$$\gamma_0 = \left[1 + (R^{1-\sigma}\beta\delta)^{1/\sigma} + (R^{2(1-\sigma)}\beta\delta^2)^{1/\sigma}\right]^{-1} \tag{8.24}$$

As expected, the solutions are linear functions of the inital wealth.

Can we be sure that the consumption path given in 8.23 is really a solution to optimization problem 8.7? And if so, is it unique? The usual way to prove this is the following: First we check wether the critical point we found is a local maximum (i.e. we do so by veryfying that the bordered Hessian is negative definite at the critical point, which is the case here). Second, we check that the constraint qualification is met at all points that lie in the budget set (in our problem, there is only a single linear constraint, so this is trivially satisfied). In particular, the constraint qualification is met at the global maximum. Unfortunately, we cannot use the Weierstrass Theorem to prove the existence of this maximum since the budget set is not closed and therefore not compact. However, we may use the following Theorem to verify the existence of a maximum:

Theorem 8 *(Sundaram, 1996, p.213) Suppose $f : \mathcal{D} \to \mathbb{R}$ is strictly quasi-concave where $\mathcal{D} \subset \mathbb{R}^n$ is convex. Then, any local maximum of f on $\mathcal{D}$ is also a global maximum of f on $\mathcal{D}$. Moreover, the set* $\arg\max\{f(x)|x \in \mathcal{D}\}$ *of maximizers of f on $\mathcal{D}$ is either empty or a singleton.*

Our maximization problem satisfies the assumptions of Theorem 8: $U(\cdot)$ is clearly strictly concave, therefore it is also strictly quasi-conave - in particular on the budget set. The budget set is convex (a skewed simplex) and a subset of $\mathbb{R}^3$. Therefore, since our candidate consumption path is a local maximizer, we can conclude that it is also the unique global maximizer.

8.1.2. The Solution without Commitment

In this subsection we drop the assumption that the decision maker can commit to her actions and analyze the consequences. We assume that the decision maker rationally foresees that she might reconsider her choices later on. With respect to the classification discussed in section 7.1 this means that we focus on "sophisticate" decision makers.

We already argued above, that one way to analyze the decision problem when commitment is not possible, is via a dynamic intrapersonal game:[3] The players are the successive selves of the decision maker. In the case of our 3-period model, this means that there are three "players": self 0, self 1 and self 2. The players' actions are simply the levels of consumption they choose in "their" period. Their payoffs are determined by the consumption levels of the "their" present and future. Therefore, the payoff of player 0 is given by:

$$U_0(c_0, c_1, c_2) \equiv u(c_0) + \beta\delta u(c_1) + \beta\delta^2 u(c_2) \tag{8.25}$$

Analogously, the payoffs of players 1 and 2 are defined as

$$U_1(c_1, c_2) \equiv u(c_1) + \beta\delta u(c_2) \tag{8.26}$$

and

$$U_2(c_2) \equiv u(c_2) \tag{8.27}$$

It seems only natural to employ the solution concept of a subgame perfect Nash equilibrium (SPE): The reason why we use the refinement concept of subgame perfection is that we want to rule out equilibria of the following kind: Suppose self 2 threatens the preceeding selves by playing a "blackmailing" strategy: If she does not get exactly, say, 80% of the initial wealth, she will not consume anything and let her W_2 units of money go to waste. For an appropriate parameter configuration, it would then pay off for self 0 and self 1 to "give in". Therefore, we look for a nash equilibrium in every subgame of the game to rule out these uncredible threats: Clearly self 2 will simply consume what is left.

The subgames are the following: "After" every level of consumption, self 0 may choose, a new subgame opens up. Similiarly, after every feasible level of consumption, self 1 can choose, another subgame opens up. All in all, we therefore have a continuum of subgames, starting after every consumption decision.

The straightforward technique to solve a dynamic game like this is by backward induction - as we did implicitly in the case of the Fibonacci cinema example.

So, let us start by analyzing the behaviour of self 2:

[3] A proper definition of the game will follow in the next section

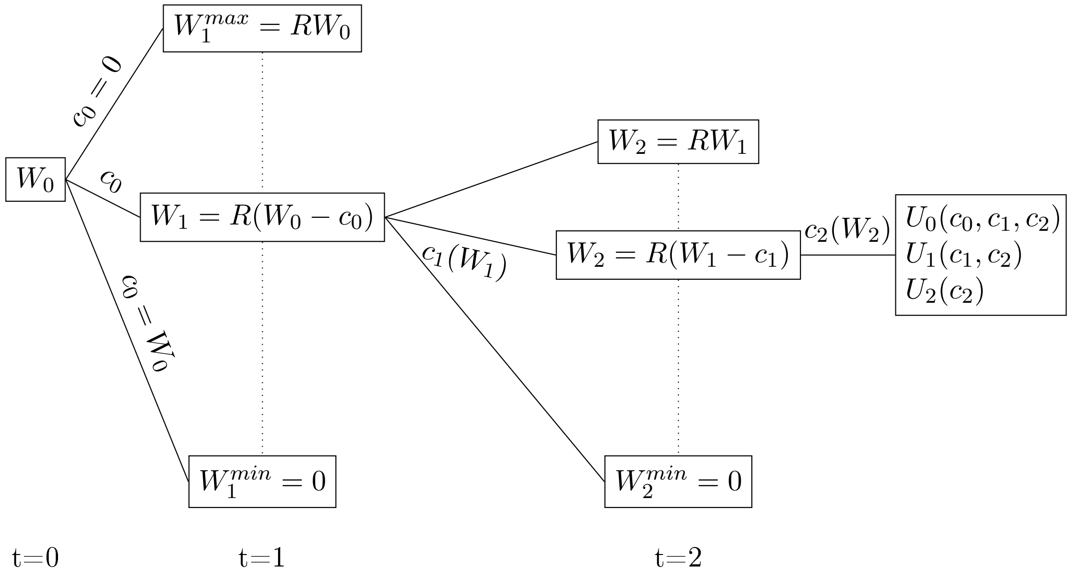


Figure 8.1.: Illustration of the Three-Period-Consumption/Saving game

Self 2 "inherits" wealth from self 1, which we denote by W_2. So in every subgame (i.e. for every level of wealth, W_2) her best response is given by:

$$_{nc}c_2^* = \arg \max_{0<c\leq W_2} u(c) \tag{8.28}$$

Since we assumed $u' > 0$ and since there are no future periods to distribute wealth over, her best response is simply given by $_{nc}c_2^* = W_2$.

Self 1 "inherits" wealth W_1 from self 0. She has rational expectations and anticipates self 2's choice. Therefore, her best response is given by:

$$_{nc}c_1^* \in \arg \max_{0<c\leq W_1} u(c) + \beta\delta u(\underbrace{R(W_1 - c)}_{c_2^*(c)}) \tag{8.29}$$

since the best response function of self 2 is linear in the choice of consumption of self 1, we can conclude that the objective function given 8.29 is concave. Moreover, the "budget" set $0 < c \leq W_1$ is clearly convex and we might therefore apply Theorem 8 in order to show that the set of best responses is either empty or a singleton. The Inada conditions of u then ensure that the latter holds. Therefore, we know that in every subgame (for every value of W_1), there is a unique Nash Equilibrium. Moreover, the utility function of self 1 is also of the CES familiy and therefore homothetic. As mentioned above, this means that we can write her choice of consumption as $\gamma_1 W_1$. That is, her strategy can be interpreted as choosing a consumption rate. The *level* of consumption is then determined by W_1.

Since we know that the maximizer lies in the interior of her budget set, the unique maximizer is characterized by the first order condition

$$u'(c_1^*) = \beta\delta R u'(c_2^*) \tag{8.30}$$

which is again an Euler equation. If we compare equation 8.30 with the corresponding Euler equation in the commitment case (equation 8.11) it is clear that these two are equal only if $\beta = 1$, that is exponential discounting. Therefore, a value of β that is strictly smaller than 1 results in overconsumption of self 1 from the perspective of self 0: the additional discounting that takes place between period 1 and 2 (that is the discount factor) differs between the two "players": the discount factor of self 0 is given by $\phi_2 \equiv \frac{\beta\delta^2}{\beta\delta} = \delta$ while the discount factor of self 1 is given by $\phi_1 \equiv \frac{\beta\delta}{1} = \beta\delta$ which is strictly smaller if $\beta < 1$.

Plugging in the specific functional form of the CRRA utility function, the response function of self 1 can then be written as a function of her initial wealth, W_1 (which is a linear function of the consumption decision of self 0: $W_1 = R(W_0 - c_0)$):

$$_{nc}c_1^*(W_1) = \gamma_1 W_1 \tag{8.31}$$

where

$$\gamma_1 = \left(1 + \left(\beta\delta R^{1-\sigma}\right)^{1/\sigma}\right)^{-1} \tag{8.32}$$

Clearly, it holds that $0 < \gamma_1 < 1$. Therefore, we can say that the strategic behaviour of self 1 can be summarized by the consumption function given in equation 8.31.

Self 0: Anticipating all this, self 0 chooses a level of consumption c_0 that maximizes her payoff, given in equation 8.25. Therefore her choice is given by

$$_{nc}c_0^* \in \arg\max_{0<c\leq W_0} \{u(c) + \beta\delta u(\underbrace{\gamma_1 R(W_0 - c)}_{c_1^*(c)}) + \beta\delta^2 u(\underbrace{R^2(w-c)(1-\gamma_1)}_{c_2^*(c_1^*(c))})\} \tag{8.33}$$

In other words, she chooses a subgame. Again, we observe that this is a strictly concave problem (a sum of concave functions that in turn are linear transformations of c) with a convex budget set, which ensures uniqueness of the solution by Theorem 8. The Inada conditions ensure the existence. Therefore the well-defined consumption level is given by

$$_{nc}c_0^* = \gamma_0 W_0 \tag{8.34}$$

where

$$\gamma_0 = \left[1 + \left(\beta\delta R^{1-\sigma}\left(\gamma_1^{1-\sigma} + \delta R^{1-\sigma}(1-\gamma_1)^{1-\sigma}\right)\right)^{1/\sigma}\right]^{-1} \tag{8.35}$$

Again, it holds that $0 < \gamma_0 < 1$, which is why we can say that the strategy of self 0 amounts to choosing an appropriate consumption *rate*. The consumption level is then determined by W_0, which is exogenously given.

From another perspective, the decision problem without commitment is similiar to the problem with commitment. However, we have to add the following "incentive compatibility contraint":

$$c_2 = \gamma_1(R(W_0 - c_1)) \tag{8.36}$$

Taken together with the other constraints self 0's problem is then the following:

$$\max_{(c_0,c_1,c_2)\in\mathcal{B}^{**}} U_0(c_0, c_1, c_2) \tag{8.37}$$

where

$$\mathcal{B}^{**}(R, W_0) \equiv B^* \cap \{(c_0, c_1, c_2) \mid c_2 = \gamma_1(R(W_0 - c_1))\} \tag{8.38}$$

Since this constraint effectively restricts the choice set, it is clear that self 0 cannot be better off than before. Moreover, we saw that in general the unique optimum of the ("unconstraint") problem with commitment will not satisfy the additional constraint, she will even be worse off. Note that $\beta = 1$ brings us back to the "harmony" case where the preferences of self 1 and self 0 are compatible with each other. In other words, this is the case when the optimum of the commitment-problem satisfies the additional constraint.

8.2. Consumption Decisions without Commitment

In this section we will generalize the insights we gained from the previous example for T+1 periods. We will do so along the lines of Laibson (1996). We start with a proper defininition of the dynamic game:

The "players" are clearly the temporal selves of the decision maker. We conveniently index them by their decision period: $t = 0, 1, 2, \ldots, T$.

Let $h_t \in H_t$ be the set of feasible histories at time t. Note that h_t is a $t+1$ period vector containing information about the initial wealth W_0 and the "moves", i.e. the consumption decisions, that took place up to (but excluding) period t: $h_t \equiv \left(W_0, c_0, c_1, \cdots, c_{t-1} \in \mathbb{R}^{t+1}_{++}\right)$. We already saw in the last section, that the way how the moves of previous selves affect the decisions of successive selves is a very particular one: what matters is only the wealth at the beginning of the decision period t, which is given by the following function:

$$W_t(h_t) = R^t W_0 - \sum_{i=0}^{t-1} R^{t-i} c_i \tag{8.39}$$

A player's strategy space is the set of feasible consumption decisions: Formally, that is:

$$S_t \equiv \{s_t \mid s_t : H_t \to \mathbb{R}_{++} \text{ and } 0 < s(h_t) \leq W(h_t) \quad \forall h_t \in H_t\} \tag{8.40}$$

The joint strategy space, S, is then given by the cartesian product over all player's strategy spaces: $S \equiv S_0 \times S_1 \times \ldots \times S_T$.

Finally, self t's payoff is given by the function

$$U_t(c_t, c_{t+1}1, \ldots, c_T) \equiv u(c_t) + \beta \sum_{i=1}^{T-t} \delta^i u(c_{t+i}) \tag{8.41}$$

where $u(\cdot)$ is of the CRRA family.

For this dynamic intrapersonal game, we have the following result:

Theorem 9 *(Laibson, 1996) For this finite-horizon game, there exists a unique subgame perfect equilibrium. This equilibrium is markov perfect and is characterized by time-dependent consumption rules which are linear in wealth.*

Proof We are first to show that the set of subgame perfect strategy-profiles, $S^P \subset S$, is a singleton. We will prove this result by induction. First, suppose that the T-period horizon game (i.e. T+1 period game) has a unique subgame perfect equilibrium. Now, we have to show, that that this implies that also the T+1 period horizon game has a unique subgame perfect equilibrium: Imagine that we already showed that the three-period example above has a unique subgame perfect equilibrium. Now we add another period *before* period 0.

As in the introductory example above, we suppose further that the equilibrium strategies of the T-period horizon game, which we denote by s_t^T are of the form $s_t^T(h_t) = \gamma_{T-t} W_t$ for $t = 0, 1, \ldots, T$.[4] Of course, it has to hold that $0 < \gamma_{T-t} < 1$.

The important point in this proof is the next one: We introduce a recursive element into the line of argumentation: the (continuation-) value function, which we define by

$$V(A, T+1) \equiv \beta\delta \sum_{t=0}^{T} \delta^t u(\gamma_{T-t} W_t) \tag{8.42}$$

where $A \equiv W_0$ and $W_{t+1} = R(1 - \gamma_{T-t}) W_t$, which is simple bookkeeping.

Given the assumptions we made so far, it holds for all $A \in R_{++}$ that

[4] For notational convenience the indizes of wealth and marginal consumption are running in opposite directions: Therefore, self T's strategy is for instance given by $\gamma_0 W_T$.

- $V_A(A, T+1) \equiv \frac{\partial V(A,T+1)}{\partial A} > 0$
- $V_{AA}(A, T+1) \equiv \frac{\partial^2 V(A,T+1)}{\partial A^2} < 0$
- $\lim\limits_{A \to 0} \frac{\partial V(A,T+1)}{\partial A} = \infty$

So the continuation payoff is concave in the wealth of the first continuation period. In other words, V "inherits" some properties of u.[5]

We are now ready to analyze the behaviour of self 0 in the T+1 period horizon game (i.e. T+2 periods). By choosing her level of consumption she directly determines the level of wealth in the next period, $W_1 = R(W_0 - c_0)$. By assumption, there is a unique subgame perfect equilibrium for every level of W_1. In the eyes of self 0, these subgame perfect equilibria have a (continuation) value of $V(R(W_0 - c), T+1)$. Therefore, her level of consumption has to satisfy:

$$c_0 \in \arg \max_{0 < c \leq W_0} u(c) + V(R(W_0 - c), T+1) \tag{8.43}$$

By the properties of V stated above, the solution to this problem is well-defined. Moreover, as argued above, the induced homotheticity of the CRRA utility function results in a solution that can be written as: $\gamma^* W_0$. Define $\gamma_{T+1} \equiv \gamma^*$.

Effectively, we have shown so far, if there exists a subgame perfect equilibrium in the T-period horizon game, there also exists one in the T+1-horizon game. In addition we showed, that if the strategies in the T-period horizon game are of the form $c_t = \gamma_{T-t} W_t$, then also the strategy of self 0 in a T+1-horizon game depends in this simple linear way on the initial wealth. In order to start the induction, consider a 0-period horizon game. Since there is only a single period, the decision maker will set $\gamma_0 = 1$, i.e. consume all the wealth.

Markov perfect equilibria are a subclass of subgame perfect equilibria, where the strategies have the following Markov property (Fudenberg and Tirole, 1991, p.501): The past influences the current play only through a state variable that summarizes the effect of all past actions. In our case this state variable is the level of wealth, W_t. Furthermore, we showed above that the strategies of the successive selves only depend on past level of consumptions, insofar as they influence the current level of wealth. Therefore, they have this desired Markov property. This completes the proof of Theorem 9.

[5] To see this, note that the stock of wealth at period t can be written explicitly as $W_t = W_0 \prod_{i=0}^{t-1} R(1 - \gamma_{T-i})$. Therefore, the argument of $u(\cdot)$ is a positive linear transformation of $A \equiv W_0$, which means that $u(\cdot)$ is concave in W_0 since $u(\cdot)$ itself is a concave function. $V(\cdot)$ in turn, is just a linear combination of the function u evaluated at different points and therefore also concave in W_0.

We are now going to give a closer characterization of the unique equilibrium consumption path of a finite consumption game: We take self 0's maximization problem stated in equation 8.43 as a starting point. The properties of $V(\cdot)$ ensure an inner solution, which is why the following first order condition has to hold with equality:

$$u'(c_0) = RV_A(W_1, T+1) \tag{8.44}$$

Since every period t-self faces the same kind of maximization problem, this relationship has to hold for all $t = 0, 1, \ldots, T$:

$$u'(c_t) = RV_A(W_{t+1}, T-t+1) \tag{8.45}$$

Using the definition of $V(\cdot)$ we can rewrite $V(W_{t+1}, T-t+1)$, as

$$\beta\delta u(c_{t+1}(W_{t+1}, T+1)) + \delta V(R(W_{t+1} - c_{t+1}(W_{t+1}, T+1)), T-t) \tag{8.46}$$

Therefore, the partial derivative, $V_A(W_{t+1}, T-t+1)$ can be written as

$$\beta\delta u'(c_{t+1})\frac{\partial c_{t+1}}{\partial W_{t+1}} + \delta RV_A(R(W_{t+1} - c_{t+1}), T-t)\left(1 - \frac{\partial c_{t+1}}{\partial W_{t+1}}\right) \tag{8.47}$$

Substituting $u'(c_{t+1})$ for $V_A(R(W_{t+1} - c_{t+1}), T-t)$ brings us to the following hyperbolic Euler equation:

$$u'(c_t) = Ru'(c_{t+1})\underbrace{\left[\beta\delta\frac{\partial c_{t+1}}{\partial W_{t+1}} + \delta\left(1 - \frac{\partial c_{t+1}}{\partial W_{t+1}}\right)\right]}_{\hat{\phi}_{t+1}} \tag{8.48}$$

Harris and Laibson (2001) dubbed the bracketed term, wich we denote henceforth by $\hat{\phi}_{t+1}$, the "*effective* discount factor".

In a sense it plays the same role as the (normal) discount factor in the case of commitment power (see equation 8.17 above), but whereas it is exogenous in the model with commitment, it is endogenous in the model without commitment.

The effective discount factor is a weighted sum of the short run discount rate, $\beta\delta$ and the long run discount rate, δ. The weights are composed of the marginal propensity to consume in period $t+1$. As we saw above, the CRRA utility function, result in consumption functions that are linear in wealth – therefore the marginal propensity to consume coincides with the consumption rate, which we denoted by γ_{t+1}. Put differently, in the case of CRRA utility, the weights are constant with respect to future wealth and are given by the consumption rate, γ, and the savings rate, $(1-\gamma)$, respectively. Since we know that $0 < \gamma_{t+1} < 1$, we can be sure that the effective discount factor is a convex combination of $\beta\delta$ and β.

The intuition behind the result is the following: First, note that for $\beta = 1$, we obtain the usual exponential Euler equation. For $\beta < 1$ however, we already saw in the introductory example that plans of self t and self t+1 are not aligned. That is, they will allocate a marginal euro differently between period t+1 (self t's tomorrow and self t+1's present) and period t+2 (self t's day after tomorrow and self (t+1)'s tomorrow). Moreover, the difference is such that self t would save more in period t+1 then self t+1 would. Figuring out how much more might not be that simple. However, we can be sure that if self t wants to reallocate a *marginal* euro from today to a point in the future, this point in the future would not be period t+1, but a later one. How much of this marginal euro "trickles down" to period t+2 or later is determined by the marginal propensity to consume of self t+1. If self t expects her tomorrow's self to consume relatively much (i.e. γ_{t+1} close to 1), most of self t's utility will derive from her self t+1's consumption and therefore the implied discount factor will be close to $\beta\delta$. If on the other hand self t+1's marginal consumption is close to zero, most of the marginal euro will be "passed on" to period t+2, which effectively increases the worth of a future marginal dollar to a value of close to $Ru'(c_{t+1})\delta$.

This intuition also holds when a variable labor income is introduced into the model. More precisely, if we assume that there is no asset market for labor income. Harris and Laibson (2001) showed that the hyperbolic Euler relation given in equation 8.48 also holds in a more general framework, when we allow for an infinite time horizon, stochastic income (in which case, we have to add the conditional expectations operator) and a much more general class of static utility functions.

However, the authors show that after introducing labor income, the hyperbolic Euler equation only holds when the consumption function is Lipschitz-continuous, which they prove is the case when β is in the neighborhood of 1. Therefore, Harris and Laibson (2001) derive a generalization of equation 8.48 which they dub "Weak Hyperbolic Euler Relation". Unfortunately, given its cumbersome structure, it seems safe to say that we will not see this generalization in applications.

8.3. Consumption/Saving-Decisions with Partial Commitment

Laibson (1997) supplements the model discussed in the previous section with a partial commitment technology in the form of illiquid wealth: The decision maker can either invest in liquid assets, x, or in illiquid assets, z. Liquidizing assets takes one period, which is why the decision maker at time t effectively only commands over the liquid portion of the asset stock and her income in that very period. Recall from section 8.2 that self t and self t+1 only disagree about the relative weights attached to periods t+1 and t+2 but are "in harmony" with respect to relative weights attached to periods that are farther in the future. Therefore, this commitment device is tailor-made for quasi-hyperbolic time preferences and if it was not for the labor income of self t+1 (that self t cannot control) even perfect commitment would be possible.

So let us consider in closer detail the decision process of the hyperbolic consumer endowed with perfect rationality in Laibson's "Golden Eggs and Hyperbolic Discounting":

8.3.1. Model Setup

In every period $t = 1, 2, \ldots, T$ the decison maker supplies one unit of labor that earns her an income of y_t (Note that income may vary in a *deterministic* way with time). Furthermore, she gets access to the liquid portion of her savings (chosen in $t-1$) which is given by Rx_{t-1} and she can choose a level of consumption

$$c_t \in (0, y_t + Rx_{t-1}) \tag{8.49}$$

Finally, she can decide upon the allocation of the remaining stock of wealth that effects her decision in the *next* period (we assume that it takes one period to liquidize wealth):

$$y_t + R(z_{t-1} + x_{t-1}) - c_t = z_t + x_t \tag{8.50}$$

$$x_t,\ z_t \geq 0 \tag{8.51}$$

(Laibson, 1997, p.448) justifies the nonnegativity constraint imposed on both asset stocks with the claim that U.S. courts would not enforce contracts that include forced saving. Moreover, if forced saving was possible, then self t could restrict self t+1's budget in any desirable way, which would strip the model of many interesting features. However, since this is not possible when we rule out forced saving, self t can only control the part t+1's budget that depends on her capital income - so self t+1's budget cannot be smaller than her labor income.

As in the previous sections we are now going to characterize the set of subgame perfect equilibria of this consumption/saving-game. In addition to her consumption decision, self t also has to choose her asset allocation, which is why the equilibrium path is now a sequence of triples:

$$\left\{x_0, z_0, (c_t, x_t, z_t)_{t=1}^{T}\right\} \tag{8.52}$$

We are now going to characterize this equilibrium path by four marginal conditions. As indicated in the previous section, unfortuantely it is not guaranteed that the equilibrium consumption strategy is differentiable at every point. Therefore, it is a priori not clear wether we really can hope to use marginal conditions. In order to ensure smoothness of the consumption function Laibson (1997) imposes the following restriction on the labor income sequence:

Assumption L1:

$$u'(y_t) \geq \beta\,(\delta R)^{\tau}\,u'(y_{t+\tau}) \quad \forall t,\tau \geq 1 \tag{8.53}$$

In the case of CRRA utility, this means that labor income has to increase with time and moreover, it has to increase "sufficiently" fast:

$$\frac{y_{t+\tau}}{y_t} \geq \beta^{1/\sigma}(\delta R)^{\tau/\sigma} \quad \forall t,\tau \geq 1 \tag{8.54}$$

Clearly, the closer β is to 0, the less growth is required in order for this assumption to be met.

8.3.2. Equilibrium Characterization

The following Theorem now ensures the existence and uniqueness of a subgame perfect equilibrium and gives four necessary conditions the equilibrium path has to fulfill:

Theorem 10 *(Laibson, 1997, p.453) In the T-period consumption/savings-game defined above that satisfies assumption L1, there exists a unique subgame perfect Nash equilibrium. The equilibrium path is resource exhausting and satisfies the following four conditions:*

$$u'(c_t) \geq \max_{\tau\in\{1,\dots,T-t\}} \beta(\delta R)^{\tau}u'(c_{t+\tau}) \tag{8.55a}$$

$$u'(c_t) > \max_{\tau\in\{1,\dots,T-t\}} \beta(\delta R)^{\tau}u'(c_{t+\tau}) \qquad \Rightarrow c_t = y_t + Rx_{t-1} \tag{8.55b}$$

$$u'(c_{t+1}) < \max_{\tau\in\{1,\dots,T-t-1\}} (\delta R)^{\tau}u'(c_{t+1+\tau}) \qquad \Rightarrow x_t = 0 \tag{8.55c}$$

$$u'(c_{t+1}) > \max_{\tau\in\{1,\dots,T-t-1\}} (\delta R)^{\tau}u'(c_{t+1+\tau}) \qquad \Rightarrow z_t = 0 \tag{8.55d}$$

each of which has to hold for all $t = 1, 2, \dots, T-2$.

The requirement of resource exhaustion has already been discussed in section 8.2: We saw that self T will simply consume the remaining wealth.

In order to guarentee that self T is able to do so in the game with partial commitment, self T-1 only passes on liquid assets, i.e. she sets $z_{T-1} = 0$.

The four marginal conditions are closely related to the Euler equations in the previous sections, we consider them in turn:

Condition 8.55a is a generalization of the Euler equation under liquidity constraints: Here the marginal utility of self t derived from the equilibrium consumption c_t can exceed to discounted marginal utility of self t+1.

If this is the case (8.55b), self t finds herself unable to reallocate more resources from self t+1 to t since she is liquidity constraint by her previous selves, i.e. $c_t = y_t + Rx_{t-1}$. Note that the reverse inequality cannot hold since she could simply choose to give up consumption in favour of one of her future selves.

The next two conditions relate to the decision maker's choice of x and z: Both of them concern tradeoffs of self t between periods $t+1$ and $t+\tau$. The respective discount factor is given by δ^{τ}, which is why β does not appear in the last two conditions. 8.55c now suggests that self t will try to implement the lowest possible consumption of self t+1 when she finds that self t+1 will overconsume (that is, from perspective of self t).

Conversely, 8.55d ensures that she will not restrict the consumption of self t+1 when she wants self t+1 to consume more.

8.3.3. Implications

Laibson (1997) conducts several calibration exercises and discusses the following implications of his model:

Consumption-Income Comovement

It seems to be a robust finding of many studies (for an overview see Thaler, 1990) that houshold consumption tracks income "too" closely: The standard (exponential) Permanent Income hypothesis states that (unexpected) changes[6] in current income will alter the consumption path of the (perfectly rational) decision maker only insofar, as they change the net present value of the decision maker's stock of wealth.

This effect will be most pronounced when the decision maker holds only a relatively small stock of wealth, since then she can only "absorb" shocks in a limited way.

However, this is not what empirical studies (see for example Bernheim, Skinner and Weinberg, 2001) usually find: Even though households hold a large enough level of wealth, their consumtion is highly correlated with their current income.

All in all, the strong empirical correlation between *current* income and consumption can not be rationalized within a exponential discounting consumption/saving setup.

In Laibson (1997) model, this empirical relationship is made sense of in the following way: It can be shown that on the equilibrium path, the decision maker will find herself cash-constrained (by her previous selves) at any point in time and she therefore consumes all the liquid assets available at time t, i.e. $c_t = y_t + Rx_{t-1}$. However, if β is sufficiently small[7] (so that the conflict of interest between self $t-1$ and self t is sufficiently high), self $t-1$ might not always be able to fully prevent her future self from "going on a consumption binge".

[6] As mentioned before, Laibson (1997) models the income stream to be deterministic. But even then, the decision maker finds herself unable to "smooth" consumption (relative to the exponential case).

[7] Given that the other model parameters are calibrated in a "reasonable" range.

Ricardian Equivalence

Closely related to the issue just discussed, is the question of wether Ricardian Equivalence still holds when decision makers discount time quasi-hyperbolically. Ricardian equivalence consitituted a major attack on the Keynsian school of thought since it states that rationality renders fiscal policy essentially ineffective under a wide range of economic situations. The line of arguments is as follows: Fiscal policy is seen as a simple reallocation of resources of one period in time to another. Given perfect capital markets this reallocation is always possible at an opportunity cost that equals the prevailing interest rate. Let us assume that the government wants to increase spending in period t_1 and therefore borrows money from the capital market. When the government pursues a long-term balanced budget policy, it is clear that it has to raise taxes at a point later in time in order to pay back it's creditors (say, in period t_2). The crucical point is that households are assumed to foresee that the government will do so (by e.g. raising taxes later on). If households face the same market interest rate as the governments, they will therefore simply offset the effect of the fiscal policy by reducing spending in period t_1 and increasing spending in t_2.

While a large body of critique concentrated around the assumption of perfect foresight of the households (see e.g. Akerlof, 2007), Laibson suggests that Ricardian Equivalence will also be violated when households are indeed perfectly rational.

To see why this is the case here, recall that at the equilibrium path, it holds that $c_t = y_t + Rx_{t-1}$. Therefore, fiscal policy that changes the income path,[8] will also have a similar effect on the (equilibrium) consumption path.

Mental Accounting

In his famous paper "Saving, Fungibility, and Mental Accounts" Thaler (1990) discussed the idea that decision makers use "rules of thumb" in their consumption/saving decision process (e.g. "save" 10% of their income). Thaler hypothesizes that decision makers will treat the various sources income (labor income, capital income, windfall gains) differently and will therefore exhibit different marginal propensities to consume across their "mental accounts". But while Thaler only endows the decision makers with bounded rationality[9] Laibson (1997) models them as perfectly rational.

[8] Without violating assumption L1

[9] "The modern theories of saving have made the representative consumer increasingly sophisticated. Expectations are taken to be the same as those which would be held by a sophisticated econometrician. The problem seems to be that while economists have gotten increasingly sophisticated and clever, consumers have remained decidedly human. This leaves open the question of whose behavior we are trying to model. Along these lines, at an NBER conference a couple years ago I explained the difference between my models and Robert Barro's by saying that he assumes the agents in his model are as smart as he is, while I portray people as being as dumb as I am. Barro agreed with this assessment." (Thaler, 1990, p.203)

In his model, the marginal propensities to consume (MPCs) out of the two types of assets too, differ markedly: While the MPC for the liquid asset equals 1, the MPC for the illiquid asset is 0.

The Costs of Financial Innovation

Laibson argues that the increase in available *instantaneous* credit has negative repercussions on consumer welfare. While standard (exponential) reasoning assumes that an increase of the options available cannot result in less welfare, the issue of self-control suggests otherwise.

Access to instanenous credit can be modelled by dropping our assumption that it takes the decision maker one period to liquidize her assets. Effectively, this brings us back to square one and section 8.2 where commitment was not possible. Laibson contrasts the case of partial commitment with the case of no commitment and finds that moving from the former to the latter (in a comparative static way) increases interest rates and decreases the capital/output ratio. These effects are of course more pronounced, when β is close to 0.

Laibson suggests that therefore, his model may help to explain the drop in the U.S. savings rate in the late 1980's/early 1990's: This drop was accompanied with an increase of instantaneous credit in the form of credit cards and ATMs.

Clearly, it is not straightforward to measure the welfare implications, since we are to analyze the effects of multiple *selves*. Although not undisputed (see e.g. Rubinstein, 2003, p.1208), economists seem to have converged in that they measure the change in welfare of self 0 (the planner).

Laibson does so in a way similar to section 7.2: He calculates the minumum (one-time) payment we had to pay self 0 to give up her access to a partial commitment device. He finds that for the empirically relevant values of β (somewhere between .6 and .8), this payment would amount to between 1.6% and 9% of the total output.

Implications Beyond the Golden Eggs Economy

In recent years a plethora of papers sought to adopt quasi-hyperbolic discount functions in order to explain economic phenomena, two of which we are discussing briefly:

In a sense, Duflo, Kremer and Robinson (2009) built on the work of none other than John Rae (1834) by trying to explain economic development with peoples' attitude towards time: The authors studied the demand for fertilizers of Kenyan farmers. Although only a small minority in their sample declared that they did not believe that using fertilizers would pay off, only 29% actually used them. Since almost all farmers *planned* to use fertilizers, the authors argued that the farmer's failure to buy fertilizer might stem from a lack of self control.

As a novel approach, Duflo, Kremer and Robinson (2009) conducted a randomized controlled trial:[10] The sample was partitioned in three subsets: The first subset received the following "treatment": They were granted free delivery of the fertilizers. The second subset was also granted free delivery but only if they had the fertilizers delivered early in the year (shortly before planting time). A third (control) group was given a 50% subsidy on the market prize of fertilizers.

The authors found that while the treatments for groups one and three did not raise the usage of fertilizer in a statistically significant way, the second one did. Most interestingly, subsidizing fertilizers, which is a lot more expensive than treatments one and two, respectively, seemed to have no effect.

Environmental Economists argue that dynamically inconsistent time preferences might be able to explain the inertia when it comes to tackling to issue of global warming: The problem is not so much that politicians do not care about the greenhouse effect and pollution, but that they naively expect them to make the right steps next time (when cheaper and more efficient technology is available, when there is no economic crises to get over with, when there are no elections in sight,...).

[10]Which would win Esther Duflo the John Bates Clark Medal in 2010.

9. Conclusion

In the second part of the present thesis we saw that even if we retain the basic structure of the discounted utility model and simply substitute the exponential discount function for a quasi-hyperbolic one, the predictions of most models are very likely to change fundamentally. The interrelated issue of dynamic consistency makes room for a benevolent social planner even in models without externalities or spillovers.

Therefore, a large number of results derived from "exponential" models is not robust with respect to the discount function used. Given the unsatisfactory empirical support for exponential discounting (see chapter 4) one should therefore be very careful about these results, especially when they serve as a basis for policy recommandations: Notable examples include Ricardian Equivalence and the effects of financial "innovation" (see chapter 8.3).

(Quasi-)Hyperbolic Discounting probably lends much of its popularity to the fact that it may rationalize observed economic behavior that could not be made much sense of otherwise: That is, establish it as an equilibrium outcome of the maximizing behavior of decision makers – ipso facto serving as an excuse to ignore the elephant called "irrationality" in the room called "economic analysis".

We stress that the quasi-hyperbolic discount function is probably the single least drastic deviation from exponential discounting and as we saw in chapter 4 there are plenty of other time preferences, some of which may even rest on a more solid empirical basis. However, as shown in chapter 8 already this small perturbation leads to an increase in complexity and makes the models prone to indeterminacy. Therefore, we can expect not to encounter one of the other five types of time preferences discussed in economic models any time soon. In particular, it is not even clear how one could extend non-transitive time preferences over outcome-date pairs (see chapters 4.5 – 4.8) in order to be able to represent preferences over streams of outcomes.

These points of critique nonwithstanding, the research on quasi-hyperbolic decision making not only helped to understand in a more comprehensive way the limitations of Samuelson's exponential discounted utility model, but also brought back into the discussion to mainstream economics the issue of self-control.

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A. Appendix

A.1. Abstract

Time preferences are at the very core of every intertemporal decision models. However, most economic models equate time preferences with *exponential discounting*. The first part of the present thesis gives an axiomatic derivation of time preferences in general, reviews selected types of alternatives to *exponential discounting* and assesses their empirical support. Of these alternatives, *quasi-hyperbolic discounting* constitutes the single least departure from *exponential discounting*. The second part of the thesis demonstrates that replacing *exponential discounting* with *quasi-hyperbolic discounting* even in relatively simple intertemporal decision models, not only yields numerically different results, but also changes the models qualitatively: Any deviation from *exponential discounting* raises the issues of dynamic consistency and self-control and therefore suggests to change the solution concept from a static game "against" nature to a dynamic game "against" *temporal selves*. Following Laibson (QJE, vol.112, p.443–477) potentially adverse effects of financial innovation and implications on Ricardian Equivalence as well as on economic policy are discussed.

A.2. Abstract in German

Zeitpräferenzen sind von zentraler Bedeutung für intertemporale Entscheidungsmodelle. In den meisten ökonomischen Modellen werden Zeitpräferenzen jedoch mit *exponential discounting* gleichgesetzt. Der erste Teil der vorliegenden Arbeit bespricht eine axiomatische Herleitung von Zeitpräferenzen im Allgemeinen, führt einige Alternativen zu *exponential discounting* an und diskutiert deren empirische Belege. Von diesen Alternativen stellt *quasi-hyperbolic discounting* wohl die geringste Abweichung von *exponential discounting* dar. Der zweite Teil demonstriert, dass sich beim Übergang von *exponential discounting* zu *quasi-hyperbolic discounting* schon bei vergleichsweise einfachen intertemporalen Modellen die Resultate nicht nur numerisch sondern auch qualitativ unterscheiden: Bei Abweichungen von "exponential discounting" müssen verschiedene Gesichtspunkte wie "dynamic consistency" und Selbstkontrolle berücksichtigt werden, weshalb auch das Lösungskonzept von einem statischen Spiel "gegen" die Natur zugunsten eines dynamischen Spiels "gegen" *temporal selves* aufgegeben werden sollte. Angelehnt an Laibson (QJE, vol.112, p.443–477) werden dann adverse

Effekte von Innovationen im Finanzsektor, sowie Auswirkungen auf "Ricardian Equivalence" und Wirtschaftspolitik diskutiert.

A.3. Curriculum Vitae

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