universität wien

# DIPLOMARBEIT

Titel der Diplomarbeit

# On Units of Central Division Algebras over Algebraic Number Fields

angestrebter akademischer Grad

**Magistra der Naturwissenschaften (Mag. rer. nat.)**


| | |
|---|---|
| Verfasser: | Sophie Koch |
| Matrikel-Nummer: | 0607764 |
| Studienrichtung: | Mathematik |
| Betreuer: | Univ.Prof. Dr. Joachim Schwermer |


Wien, am 03.02.2011

# Contents

# 0 Introduction

In algebraic number theory one studies the structure of the integral closure of $\mathbb{Z}$ in an algebraic number field $k$. The integral closure of $\mathbb{Z}$ in $k$ is a ring denoted by $\mathfrak{o}_k$. It consists of all elements $x \in k$ satisfying an equation of the form $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with coefficients $a_0, \ldots, a_{n-1} \in \mathbb{Z}$. We know from basic number theory that every integer can be written uniquely, up to ordering and sign, as a product of prime numbers. One can prove something similar for the ideals of $\mathfrak{o}_k$.

First one generalizes the notion of an ideal to define fractional ideals of $k$. A fractional ideal is an $\mathfrak{o}_k$-submodule $M$ of $k$ such that there exists an element $x \in \mathfrak{o}_k$, $x \neq 0$ with $xM \subseteq \mathfrak{o}_k$. Of course any ideal of $\mathfrak{o}_k$ defined as usual is fractional. One can show that every fractional ideal $M$ can be written uniquely (up to units and ordering) as a product of the form $M = \prod\limits_{\mathfrak{p}} \mathfrak{p}^{m_\mathfrak{p}}$. The product is taken over all prime ideals $\mathfrak{p} \neq 0$ of $\mathfrak{o}_k$. Furthermore it satisfies $m_\mathfrak{p} \in \mathbb{Z}$ for all $\mathfrak{p}$ and $m_\mathfrak{p} = 0$ for all but a finite number of prime ideals $\mathfrak{p}$ of $\mathfrak{o}_k$. Now one can easily see that every nonzero fractional ideal is invertible, which means that for every fractional ideal $M$ there exists a fractional ideal $N$ such that $MN = \mathfrak{o}_k$.

Hence there exists a group structure on the set of nonzero fractional ideals where the group operation is defined via multiplication. We denote this group by $J_{\mathfrak{o}_k}$. The discussion above shows that $J_{\mathfrak{o}_k}$ is the free abelian group generated by the set of nonzero prime ideals $\mathfrak{p}$ of $\mathfrak{o}_k$. Now for every element $a \in k^*$ we can define the fractional ideal $(a)$ in $J_{\mathfrak{o}_k}$, hence we get a homomorphism of $k^*$ into $J_{\mathfrak{o}_k}$. The image of this homomorphism is denoted by $P_{\mathfrak{o}_k}$. So one can define the factor group $J_{\mathfrak{o}_k}/P_{\mathfrak{o}_k}$, which is called the ideal class group of $\mathfrak{o}_k$. The following result is fundamental.

**Theorem** *The ideal class group of $\mathfrak{o}_k$ is finite.*

The number of elements of this group, denoted by $h(k)$, is an important invariant of the algebraic number field $k$.

A second basic result is Dirichlets Unit Theorem. This theorem deals with the structure of the unit group $\mathfrak{o}_k^*$ of $\mathfrak{o}_k$. Let $\mu_k$ be the set of all roots of unity contained in $k$. Then $\mu_k$ is contained in $\mathfrak{o}_k^*$. Let $n$ be the degree of $k$ over $\mathbb{Q}$. Then one has $n$ embeddings of $k$ into $\mathbb{C}$. Let $r_1$ be the number of real embeddings and $2r_2$ the number of strictly complex embeddings of $k$. The number of strictly complex embeddings has to be even, as for every complex embedding $\sigma$, its conjugate $\overline{\sigma}$ is another different embedding. Dirichlets Unit Theorem states:

**Theorem** *The group $\mathfrak{o}_k^*$ is a finitely generated $\mathbb{Z}$-module. Furthermore $\mathfrak{o}_k^* \cong \mu_k \times \mathbb{Z}^{r_1+r_2-1}$.*

This is usually proved by using Minkovskis theorem on convex bodies. However one can use a different approach, which is what I do in the first part of my thesis. To begin with I introduce completions and valuations, which are essential tools for dealing with algebraic number fields. The aim is to find out more about the algebraic number field $k$ by studying its completions $k_\nu$ with respect to its inequivalent valuations $\nu$. Now the natural language to deal with the completions of algebraic number fields simultaneously, without loosing important properties like the existence of a unique Haar measure, is that of adeles and ideles. The ring of adeles $\mathbb{A}_k$ is defined to be the restricted direct product of the set of fields $\{k_\nu\}_\nu$ with respect to the set $\{\mathfrak{o}_\nu\}_\nu$, where $\mathfrak{o}_\nu$ denotes the closure of $\mathfrak{o}_k$ in $k_\nu$ for all finite places $\nu$. There is a natural embedding of $k$ into $\mathbb{A}_k$, so one may view $k$ as a discrete subring of $\mathbb{A}_k$. Doing that one gets quite naturally that $\mathbb{A}_k/k$, both viewed as additive groups, is compact in the quotient topology.

Now we define the group of idele $\mathbb{I}_k$ as $\mathbb{I}_k = \mathbb{A}_k^*$. However $\mathbb{I}_k$ is not necessarily a topological group under multiplication as inversion may not be continuous. Therefore one has to refine the topology of $\mathbb{I}_k$ such that it becomes a locally compact topological group under multiplication. Again one may embed $k^*$ into $\mathbb{I}_k$ as a discrete subgroup. However one has to be careful as the analogue of the compactness assertion for adeles is not valid. One has to restrict to the group of ideles of norm one, denoted by $\mathbb{I}_k^1$. When defining that norm one uses the Haar measure of $\mathbb{I}_k$.

**Theorem** *The group $\mathbb{I}_k^1/k^*$ is compact.*

Using this result one can easily deduce the finiteness of the class number of $k$ as well as Dirichlets Unit Theorem.

The second part of my thesis generalizes the methodological approach introduced in the first part. I start with the definition of adeles and ideles of finite dimensional central algebras $A$ over algebraic number fields $k$. The ring of adeles $A_\mathbb{A}$ of an algebra is the restricted direct product of the family of rings $\{A \otimes_k k_\nu\}_\nu$ with respect to the family of $\mathfrak{o}_\nu$-modules generated by a basis of $A$ over $k$ in $A \otimes_k k_\nu$ for all finite places $\nu$ of $k$. This construction is independent of the choice of basis. Again, as in the case when $A$ is a field, there is a natural embedding of $A$ into $A_\mathbb{A}$, hence one can identify $A$ with its image in the adeles over $A$. So $A$ may be viewed as a discrete subring of $A_\mathbb{A}$. Furthermore one has that the adele ring and idele group are locally compact. The results concerning the compactness of the additive group remain valid.

Given a central division algebra $D$ over $k$, the notion of a maximal order in $D$ naturally generalizes the notion of the ring $\mathfrak{o}_k$ of algebraic integers in $k$. One can use maximal orders in the definition of the ring of adeles and the group of ideles. Using that the group of ideles

$D^*_{\mathbb{A}}$ of a division algebra $D$ remains locally compact one can define a Haar measure on $D^*_{\mathbb{A}}$. Doing so one can specify the norm one ideles to be the set of ideles such that the (left) multiplication with this element has modulus one. Denoting this group with $D_1$, one has

**Theorem** *Given a finite dimensional division algebra $D$ over an algebraic number field and let $D_1 = SL_1(D)_{\mathbb{A}}$ be the group of norm one elements of $D$ over the adeles. Then $D_1$ contains $D^*$ as a discrete subgroup and the quotient $D_1/D^*$ is compact.*

This theorem is an adelic formulation of a result given by Käte HEY in her doctoral thesis *"Analytische Zahlentheorie in Systemen hyperkomplexer Zahlen"*. [1] In her thesis Hey extended the known functional equation of zeta-functions of algebraic number fields to division algebras of finite dimension over algebraic number fields. Even more generally G. FUJISAKI extends in [2] the functional equation constructed by her for finite dimensional central simple algebras over algebraic number fields.

Using the theorem stated above one can define a Haar measure on the ideles over $D$ via the canonical multiplicative measure. This measure is defined as a product of the measure of $D_{\mathbb{A}}/D_1$, which is isomorphic to $\mathbb{R}^+$ with its natural measure, the measure $\mu$ on $D_1/D^*$ satisfying $\mu(D_1/D^*) = 1$ and the measure on $D^*$ with measure 1 at each point. This leads to a functional equation of the zeta function $\zeta_D(s)$ over a division algebra. This is done for instance in [2] and [13]. Using this functional equation one can show that $\zeta_D$ is regular, except for the values $s = 0$ and $s = 1$ where $\zeta_D(s)$ has poles of first order. Knowing this one can prove

**Theorem** *Let $D$ be a finite dimensional division algebra over an algebraic number field $k$ such that $D \neq k$. Then there exists a prime ideal $\mathfrak{p}$ of $k$, that is ramified in $D$.*

However this (respectively an even stronger result) was proven in a famous paper written by BRAUER-HASSE-NOETHER in 1932 and became known as the *Main Theorem on Algebras*. Thus if any of these mathematicians had known about Heys thesis, the proof of the *Main Theorem* could have been completed earlier. As the second Ph.D. student of Artin, Max ZORN pointed out:

> The theory of the $\zeta$-function of a skew field has been developed in detail by Miss K. Hey in her dissertation (Hamburg 1929). In the present note I would like to draw attention to the arithmetic consequences which are derived there, so that

[1] Hey was the first Ph.D. student of the mathematician Emil ARTIN. She finished her thesis 1927 and it was printed 1929. Afterwards she became a school teacher for Mathematics and Physics. This was very likely due to financial reasons. Her two sisters should study at university as well. On the other hand, it was generally very common among female scientists to start teaching at school after finishing their studies at university. Hey had four children and died 1990. More details concerning Heys life and her thesis can be found in [9]. Kleinert has published a proof of Heys result in [6].

> after some correction and streamlining they are recognized as a new proof of a main theorem on algebras and of the general quadratic reciprocity law. [2]

In the case where $D$ is an algebraic number field, one can deduce Dirichlets Unit Theorem from the compactness of $\mathbb{I}_k/k^*$. So one might expect something similar for division algebras. The natural generalization of the ring of integers $\mathfrak{o}_k$ in a finite dimensional division algebras $D$ is an order in $D$. One can prove that every order is contained in a ring consisting of all elements of $D$ that are integral over $k$. Now the compactness of $D_1/D^*$ leads to the proof, that the unit group of this ring is finitely generated. Hence the compactness of $D_1/D^*$ leads to a generalization of Dirichlets Unit Theorem concerning the units of orders of finite dimensional division algebras over algebraic number fields. This generalization states that the unit group of an order of a division algebra over an algebraic number field is finitely generated.

[2] Zorn wrote this at the beginning of a manuscript of a paper, which he sent to the editor of the *Hamburger Abhandlungen* one year after the Brauer-Hasse-Noether Paper was published. More details about the Brauer-Hasse-Noether Theorem can be found in [12].

# Part I

# 1 Algebraic Number Fields

In this chapter we are going to introduce the basic concepts that are required to study algebraic number fields. We are closely following [1], [7] and [13].

## 1.1 Valuations

**Definition 1.1.** *An absolute value or valuation* $|\,|$ *on a field* $k$ *is a function defined on* $k$ *with values in the non-negative real numbers having the following properties:*

1. $|0| = 0$ *and* $|x| = 0$ *if and only if* $x = 0$
2. $|xy| = |x| \cdot |y|$
3. *There exists a constant* $C > 0$ *such that* $|x + y| \leq C \max\{|x|, |y|\}$ *for all* $x, y \in k$.

If (3) holds for $C = 1$, then $|\,|$ is said to be *non-archimedean* or *finite*. Otherwise $|\,|$ is said to be *archimedean* or *infinite*. The trivial valuation of $k$ is that for which $|x| = 1$ for all $x \neq 0$. We will exclude it from our discussion.

**Definition 1.2.** *Two valuations* $|\,|_1$, $|\,|_2$ *on the same field* $k$ *are called equivalent, if there exists a constant* $s > 0$ *such that* $|x|_1 = |x|_2^s$. *A place is an equivalence class of valuations.*

Note that if we replace the valuation $|\,|$ satisfying (3) for some constant $C$ by $|\,|^s$ for some $s > 0$, then $C$ is replaced by $C^s$. So every valuation is equivalent to one with $C = 2$. For such a valuation it can be shown, that $|x + y| \leq |x| + |y|$ (the triangle-inequality) holds.

**Example** If $k$ is a number field and $\mathfrak{p}$ a prime ideal, then we associate to $\mathfrak{p}$ an absolute value $|\,|_{\mathfrak{p}}$ as follows: For $x \in k$ we have $(x) = \mathfrak{p}^{ord_{\mathfrak{p}}}\mathfrak{a}$ where $\mathfrak{a}$ is a fractional ideal satisfying $(\mathfrak{a}, \mathfrak{p}) = 1$. Then we define $|x|_{\mathfrak{p}} = c^{-ord_{\mathfrak{p}}}$ for some constant $c > 0$. This is a non-archimedean valuation.

**Example** In case $k = \mathbb{Q}$ there is one non-archimedean place for every prime $p$, the *p-adic valuation* $|\,|_p$ defined by $|p^a \frac{u}{v}|_p = p^{-a}$, where $u$ and $v$ are not divisible by $p$.

**Theorem 1.1.** *(Ostrowski) The only non-trivial places on* $\mathbb{Q}$ *are those equivalent to the* $|\,|_p$ *or the ordinary absolute value* $|\,|_\infty$.

A proof of this theorem can be found in [1] on the pages 45 and 46.

**Example** Archimedian absolute values on arbitrary algebraic number fields $k$ can be constructed as follows: Let $x \to x^{(i)}$ $(1 \leq i \leq n)$ be an embedding of $k$ into $\mathbb{C}$. Define for $x \in k$ an infinite place by $|x|_{\infty,i} = |x^{(i)}|$, where $|\,|$ denotes the ordinary absolute value on $\mathbb{C}$. In this way we obtain $n$ archimedean valuations on $k$. But not all of them are distinct, since complex conjugate embeddings give rise to identical places. We arrange the embeddings in a way such that the first $r_1$ map $k$ into $\mathbb{R}$. The remaining $2r_2$ map $k$ strictly into $\mathbb{C}$ and we enumerate them so that the $j$th and the $r_2 + j$th component form a conjugate pair. Then $n = r_1 + 2r_2$ by construction and we have formed $r_1 + r_2$ inequivalent valuations.

## 1.2 Completions

Any valuation $|\,|$ on a field $k$ defines a nonnegative function $d$ on $k \times k$ by $d(x, y) = |x - y|$. This function can be taken as a metric on $k$ and therefore induces a topology in which a basis for the neighborhoods of $x$ are the sets $\{\, y \in k \mid |y - x| < \delta\}$ for $\delta > 0$. Equivalent valuations induce the same topology. It is clear from the definition of a valuation:

**Lemma 1.1.** *A field with the topology induced by a valuation is a topological field, i.e. the operations sum, product, reciprocal are continuous.*

**Definition 1.3.** *A field $k$ is complete with respect to a valuation $|\,|$, if it is complete as a metric space with respect to the metric induced by $d$. More specifically if given any sequence $(x_n)$ with*

$$|x_n - x_m| \to 0 \;\; (m, n \to \infty, \infty)$$

*there is an $x \in k$ such that $|x_n - x| \to 0$.*

**Remark** Let $k$ be an algebraic number field. Then the completion of $k$ with respect to a valuation $|\,|$ is called a local field.

Every field $k$ with valuation $|\,|$ can be embedded in a complete field $K$ with a valuation $|\,|$ extending the original one in such a way, that $K$ is the closure of $k$ with respect to $|\,|$. Further $K$ is unique up to isomorphism, see, for example [1], page 47.

**Example** Let $|\,|$ be an archimedean valuation on an algebraic number field $k$, corresponding to the embedding $x \mapsto x^{(i)}$ $(1 \leq i \leq r_1 + r_2)$ of $k$ in $\mathbb{C}$. Then the completion of $k$ with respect to $|\,|$ ist just the closure of $k^{(i)}$ in $\mathbb{C}$. Either $k^{(i)} \subseteq \mathbb{R}$, then the completion of $k$ equals $\mathbb{R}$ (for it contains the closure of $\mathbb{Q}$ in $\mathbb{R}$, which is just $\mathbb{R}$). In case $(r_1 + 1 \leq i \leq r_1 + r_2)$, the completion of $k$ is $\mathbb{C}$. In the first case we call the valuation *real* and in the second case *complex*.

**Notation**: We will usually denote a valuation (finite or infinite) by $\nu$. If we apply $\nu$ to an element $x$ of $k$ we will write $|x|_\nu$. In the case of a finite valuation, $\nu$ corresponds to a prime ideal $\mathfrak{p}$ of $k$. If we want to emphasize the prime ideal $\mathfrak{p}$, which induces a certain valuation $\nu$, we will also write $|\,|_{\mathfrak{p}}$.

The completion of $k$ with respect to a valuation $\nu$ will be denoted by $k_\nu$. Furthermore we shall denote the extension of the valuation $|\,|_\nu$ to $k_\nu$ also by $|\,|_\nu$. A valuation $\nu$ is non-archimedean on $k$ if and only if it is non-archimedean on its completion of $k$.

## 1.3 Normalizations of Valuations

Let $k$ be an algebraic number field, let $\mathfrak{p}$ be any integral $k$-ideal. Let $\mathfrak{o}_k$ be the integral closure of $\mathbb{Z}$ in $k$. Then $\mathfrak{o}_k/\mathfrak{p}$ is finite. We define the Norm $N$ of a prime ideal $\mathfrak{p}$ in $\mathfrak{o}_k$ as the number of elements of the ring $\mathfrak{o}_k/\mathfrak{p}$ and denote it by $N\mathfrak{p}$. The norm is multiplicative, so if $\mathfrak{p}_1$, $\mathfrak{p}_2$ are integral $k$-ideals, $N(\mathfrak{p}_1\mathfrak{p}_2) = N(\mathfrak{p}_1)N(\mathfrak{p}_2)$.

We introduce normalized valuations as follows: If $\nu$ is a valuation induced by the prime ideal $\mathfrak{p}$, set

$$|a|_\nu = (N\mathfrak{p})^{-ord_{\mathfrak{p}}(a)}$$

If $\nu$ is a (infinite) real valuation, set

$$|a|_\nu = |a^{(i)}|$$

If $\nu$ is a (infinite) complex valuation, set

$$|a|_\nu = |a^{(i)}|^2$$

for $a$ in $k$. For $a \neq 0$ consider the map $x \mapsto ax$ from $k_\nu^+$ to $k_\nu^+$, where $k_\nu^+$ denotes the additive group of $k_\nu$, which is obtained by forgetting about the multiplication. The group $k_\nu^+$ is locally compact, therefore it admits a unique (up to a scalar multiple) Haar measure $\mu$. Let $X$ be a measurable subset of $k_\nu^+$ and let $\alpha$ be an automorphism. Then $\alpha(X)$ remains measurable, so one can define another Haar measure on $k_\nu^+$ by setting $\mu^*(X) := \mu(\alpha(X))$. As $\mu$ is unique up to a scalar multiple, there exists a constant $\mathrm{mod}_{k_\nu}(\alpha)$ such that $\mu^*(X) = \mathrm{mod}_{k_\nu}(\alpha)\mu(X)$. This constant is called the modulus of the automorphism $\alpha$. This is explained in more detail in the Appendix, chapter B.

Let $\mathrm{mod}_{k_\nu}(a)$ be the modulus of the automorphism $x \mapsto ax$ and let $\nu$ be a normalized valuation. Then we have that

$$\mathrm{mod}_{k_\nu}(a) = |a|_\nu$$

This characterizes the normalized valuation among equivalent ones. If not stated otherwise, a valuation will always be considered as being normalized. When a statement involves a place $\nu$ of $k$ we will always consider the normalized valuation in this place.

Let $k$ be a number field with non-archimedean valuation $|\,|_\nu$ induced by a prime ideal $\mathfrak{p}$. Define

$$\mathfrak{o}_\nu = \{x \in k_\nu \,:\, |x|_{\mathfrak{p}} \leq 1\}$$

$$\tilde{\mathfrak{p}} = \{x \in k_\nu \,:\, |x|_\mathfrak{p} < 1\}$$

$$\mathfrak{u}_\nu = \{x \in k_\nu \,:\, |x|_\mathfrak{p} = 1\}$$

We see that $\mathfrak{o}_\nu$ is a closed subring of $k_\nu$, using the fact that $|\,|_\nu$ is non-archimedean. It is called the *ring of* $\mathfrak{p}$*-adic integers*. The set $\tilde{\mathfrak{p}}$ is a maximal ideal of $\mathfrak{o}_\nu$, in fact, the unique maximal ideal. Indeed let $\mathfrak{q}$ be any maximal ideal of $\mathfrak{o}_\nu$, and $a \in \mathfrak{q}$. If $|a|_\mathfrak{p} < 1$, then $a \in \tilde{\mathfrak{p}}$. If $|a| = 1$, then $|a^{-1}| = 1$ and so $a$ is a unit in $\mathfrak{o}_\nu$. Hence $\mathfrak{q} = \tilde{\mathfrak{p}}$. The same argument shows that $\mathfrak{u}_\nu$ is the group of units in $\mathfrak{o}_\nu$.

Choose $\pi \in \mathfrak{o}_\nu$ such that $|\pi|_\mathfrak{p} = (N\mathfrak{p})^{-1}$. Then $\pi$ is called a *local uniformizing parameter* at $\mathfrak{p}$. Any two local uniformizing parameters at $\mathfrak{p}$ differ from one another by multiplication of an element of $\mathfrak{u}_\nu$.

**Theorem 1.2.** *Every ideal of $\mathfrak{o}_\nu$ is of the form $\tilde{\mathfrak{p}}^r$ $(r > 0)$. Moreover $\tilde{\mathfrak{p}}^r = (\pi^r)$, so that $\mathfrak{o}_\nu$ is a principal ideal domain. The ring $\mathfrak{o}_\nu$ (respectively the ideal $\tilde{\mathfrak{p}}$) is the closure of $\mathfrak{o}_k$ (respectively $\mathfrak{p}$) in $k_\nu$.*

A proof of this is found in [3] on page 26.

## 1.4 Extensions of Valuations

**Definition 1.4.** *Let $k$ be a field with valuation $\nu$ and let $E$ be a vector space of finite dimension over $k$. A real valued function $\|\,\|$ on $E$ is called a norm, if*

1. $\|v\| > 0$ *for* $v \in E$, $v \neq 0$
2. $\|v + w\| \leq \|v\| + \|w\|$
3. $\|av\| = |a|_\nu \|v\|$ *for* $v \in E$ *and* $a \in k$

*Two norms $\|\,\|_1$, $\|\,\|_2$ on $E$ are called equivalent if there exist constants $c_1$, $c_2$ such that*

$$\|v\|_1 \leq c_1 \|v\|_2 \qquad \|v\|_2 \leq c_2 \|v\|_1$$

**Lemma 1.2.** *Let $K$ be a complete field with respect to a valuation $|\,|$ and $E$ be a finite dimensional vector space over $K$. Then any two norms on $E$ are equivalent.*

*Proof.* Let $v_1, \ldots v_n$ be a basis of $E$ over $K$. We define a norm $\|\,\|_0$ on $E$ by

$$\|\sum_i \xi_i v_i\|_0 = \max_i |\xi_i|$$

Clearly the statement is proved, if we show, that any norm $\| \, \|$ is equivalent to $\| \, \|_0$. Let $v = \sum_i \xi_i v_i$. Then we have

$$\|v\| = \|\sum_i \xi_i v_i\| \leq \sum_i \|\xi_i v_i\| = \sum_i |\xi_i| \|v_i\| \leq \max_i |\xi_i| \sum_i \|v_i\| = c_1 \|v\|_0$$

for $c_1 = \sum_i \|v_i\|$. Suppose there is no constant $c_2$ such that $\|v\|_0 \leq c_2 \|v\|$. Then for any $\varepsilon > 0$ there exist $\xi_1, \ldots, \xi_n$ in $K$ such that

$$0 < \|\sum_i \xi_i v_i\| < \varepsilon \max_i |\xi_i|$$

Otherwise there would be an $\varepsilon$ such that for all $\xi_1, \ldots, \xi_n$ we have $\|\sum_i \xi_i v_i\| \geq \varepsilon \max_i |\xi_i|$, which contradicts our assumption.

Without loss of generality we may assume, that $\max_i |\xi_i| = |\xi_n|$ and then by homogeneity that $\xi_n = 1$. For $m = 1, 2 \ldots$ we can therefore find $\xi_{m,i}$ $(1 \leq i \leq n)$ such that

$$\|\sum_{i=1}^{n-1} \xi_{m,i} v_i + v_n\| \to 0 \qquad \text{for } m \to \infty$$

so

$$\|\sum_{i=1}^{n-1} (\xi_{m,i} - \xi_{l,i}) v_i\| \to 0 \qquad \text{for } m, l \to \infty, \infty$$

The lemma is trivial for $n = 1$, so we use induction and suppose, that it is true for the $n-1$-dimensional space spanned by $v_1, \ldots v_{n-1}$.

In that space we have $|\xi_{m,i} - \xi_{l,i}| \to 0$ for $m, l \to \infty, \infty$ for $1 \leq i \leq n-1$. The field $K$ is complete, so there exists an element $\xi_i^*$ in $K$ sucht that

$$|\xi_{m,i} - \xi_i^*| \to 0 \qquad \text{for } m \to 0$$

Then

$$\|\sum_{i=1}^{n-1} \xi_i^* v_i + v_n\| \leq \|\sum_{i=1}^{n-1} \xi_{m,i} v_i + v_n\| + \sum_{i=1}^{n-1} |\xi_i^* - \xi_{m,i}| \, \|v_i\| \to 0$$

which contradicts the definition of a norm.

□

**Definition 1.5.** *Let $k$ be an algebraic number field and $L$ a finite algebraic extension of $k$. Let $\nu$, $\omega$ be valuations of $k$, $L$ respectively. We say that $\omega$ lies above $\nu$ if $|x|_\nu = |x|_\omega$ for all $x \in k$. This will be denoted by $\omega|\nu$.*

**Theorem 1.3.** *Let $k$ be complete with respect to a (not necessarily normalized) valuation $\nu$ and let $L$ be a finite algebraic extension of $k$ with $[L : k] = n < \infty$. Then there is precisely* *one extension of* $\nu$ *to* $L$ *namely*

$$|x|_\omega = |N_{L/k}x|_\nu^{1/n}$$

*If* $\nu$ *is a normalized valuation of* $k$ *the normalized valuation* $\omega'$ *of* $L$*, which is equivalent to the unique extension of* $\nu$ *to* $L$ *is given by*

$$|x|_{\omega'} = |N_{L/k}x|_\nu$$

These statements are proved in [1], page 56 and 59.

# 2 The Adele Ring of an Algebraic Number Field

Our aim is to find out more about an algebraic number field $k$ by studying all its local fields (i.e. its completions with respect to its places) simultaneously. In particular we will see, how we can collect all the local fields in one object, such that we retain the arithmetic information possessed by these fields, but still have a locally compact set, on which we can use the tools introduced in the first chapter. The statements and proofs can be found in [1] and [10].

## 2.1 Restricted Direct Product

Let $\{\nu\}$ be a possibly infinite set of indices. For each $\nu$ let $G_\nu$ be a locally compact topological group. For all but finitely many $\nu$, let $H_\nu$ be a compact open subgroup of $G_\nu$. We define the *restricted direct product*, denoted $G = \prod_\nu (G_\nu : H_\nu)$, of the family $\{G_\nu\}$ with respect to $\{H_\nu\}$ as follows: It is the group of all $(\alpha_\nu) \in \prod_\nu G_\nu$ such that $\alpha_\nu \in H_\nu$ for almost all $\nu$. We want to topologize the restricted direct product as a subspace of the direct product, such that it remains locally compact. As this is not the case in general, if we take the subspace topology, we must proceed with some care. Let $S$ be any finite subset of $\{\nu\}$, containing all $\nu$, such that $H_\nu$ is not defined. We give the restricted direct product a topology so that

$$G(S) = \prod_{\nu \in S} G_\nu \times \prod_{\nu \notin S} H_\nu$$

is an open subgroup for all $S$ considered. We define such a topology on $G$ by specifying a neighborhood base of the identity consisting of sets of the form $\prod_\nu N_\nu$, where $N_\nu$ is a neighborhood of the identity in $G_\nu$ and $N_\nu = H_\nu$ for almost all $\nu$. This topology is not the product topology, though, as we shall see later, it induces the product topology on $G(S)$. The discussion above shows:

**Theorem 2.1.** *The restricted direct product $G$ of a family $\{G_\nu\}$ with respect to the family of open compact subgroups $\{H_\nu\}$ with $H_\nu \subset G_\nu$ for almost all $\nu$ is locally compact.*

## 2.2 Topology of the Adele Ring

Let $k$ be a finite extension of $\mathbb{Q}$ and $|\,|_\nu$ a normalized valuation of $k$. We denote the completion of $k$ with respect to $|\,|_\nu$ by $k_\nu$. If $|\,|_\nu$ is non-archimedean, denote by $\mathfrak{o}_\nu$ the ring of integers of $k_\nu$, i.e. $\mathfrak{o}_\nu = \{x \in k_\nu \,|\, |x|_\nu \leq 1\}$. Then $\mathfrak{o}_\nu$ is an open, compact subring of $k_\nu$. The *adele ring* $\mathbb{A}_k$ of $k$ is the restricted direct product of the additive groups $k_\nu$ with respect to $\mathfrak{o}_\nu$. Define $S_\infty$ to be the set of all infinite valuations. Let $S \supset S_\infty$ be a finite set of valuations, we call

$$\mathbb{A}_k(S) = \prod_{\nu \in S} k_\nu \times \prod_{\nu \notin S} \mathfrak{o}_\nu$$

the *$S$-adeles*. By definition of the topology as above, for every set $S$ (defined as above) the $S$-adeles form an open subgroup of $\mathbb{A}_k$. Addition and multiplication are defined componentwise, so $(xy)_\nu = x_\nu y_\nu$ and $(x+y)_\nu = x_\nu + y_\nu$ for all $x, y \in \mathbb{A}_k$. As it can easily be verified, this definition makes sense, so if $x, y \in \mathbb{A}_k$ then $xy, x+y \in \mathbb{A}_k$. The adele ring $\mathbb{A}_k$ is locally compact, since $k_\nu$ is locally compact for all $\nu$ and the $\mathfrak{o}_\nu$ are compact. In particular

**Theorem 2.2.** *Let $\nu$ run through the set of all places of $k$, including all infinite ones. Then the restricted direct product of $k_\nu$ with respect to $\mathfrak{o}_\nu$ is locally compact, and hence it admits a unique (up to scalar multiple) Haar measure.*

We can make a different approach when defining the adeles. Let $S$ be again a finite set containing $S_\infty$ and $\mathbb{A}_k(S) = \prod_{\nu \in S} k_\nu \times \prod_{\nu \notin S} \mathfrak{o}_\nu$. Then the set $\mathbb{A}_k(S)$ with the product topology is locally compact since the $k_\nu$ are locally compact and for all finite $\nu$ the ring $\mathfrak{o}_\nu$ is compact. We can put a ring structure on $\mathbb{A}_k(S)$ by defining addition and multiplication componentwise. With this topology $\mathbb{A}_k(S)$ becomes a topological ring. If $S'$ is another finite set containing $S$ then we have

$$\mathbb{A}_k(S) = \prod_{\nu \in S} k_\nu \times \prod_{\nu \notin S} \mathfrak{o}_\nu \subset \prod_{\nu \in S'} k_\nu \times \prod_{\nu \notin S'} \mathfrak{o}_\nu = \mathbb{A}_k(S')$$

Since $\mathfrak{o}_\nu$ is open in $k_\nu$ for all finite places $\nu$ we can conclude that $\mathbb{A}_k(S)$ is an open subring of $\mathbb{A}_k(S')$. Moreover the topology of $\mathbb{A}_k(S)$ is that induced by $\mathbb{A}_k(S')$.

The ring of adeles $\mathbb{A}_k$ of $k$ can defined to be the union of the sets $\mathbb{A}_k(S)$. This construction coincides with the direct limit of the sets $\mathbb{A}_k(S)$. In other words, the set $\mathbb{A}_k$ consists of the elements $(x_\nu)$ in $\prod_\nu k_\nu$ such that $|x_\nu|_\nu \leq 1$ for almost all $\nu$. We define the topology on $\mathbb{A}_k$ by prescribing, that each set $\mathbb{A}_k(S)$ is an open subring of $\mathbb{A}_k$. We get a fundamental system of neighborhoods of 0 by taking a system of neighborhoods in any of the sets $\mathbb{A}_k(S)$. This definition induces the same topology as the restricted direct product.

**Remark** One can generalize this definition by setting $S_\infty$ to be the set of all $\nu$ such that $H_\nu$ does not exist. Then the restricted direct product of the set $\{G_\nu\}$ with respect to the set

$\{H_\nu\}$ is the direct limit of the sets $G(S) = \prod_{\nu \in S} G_\nu \times \prod_{\nu \notin S} H_\nu$, where $S$ is a finite set containing $S_\infty$. Once again, the topologies coincide.

## 2.3 Properties of the Adele Ring

To proceed further we need the following:

**Theorem 2.3.** *Let $k$ be an algebraic number field and $x$ any element of $k$. Then $|x|_\nu \leq 1$ for almost all places $\nu$ of $k$.*

*Proof.* First we consider the case $k = \mathbb{Q}$. Here $x = a/b$ for some integers $a, b \in \mathbb{Z}$ and $b \neq 0$. Then $|x|_\nu \leq 1$ for all primes, which do not divide $b$. Now we consider a finite extension $k$ of $\mathbb{Q}$. Then $x$ satisfies the equation

$$x^n + a_{n-1}x^{n-1} + \ldots a_0 = 0$$

with coefficients $a_i \in \mathbb{Q}$ for $0 \leq i \leq n-1$. Let $S$ be a finite set of places of $\mathbb{Q}$, consisting of the infinite place and all the places induced by the primes, which occur in the denominators of the $a_i$. By Theorem 1.3 the set $S'$ of the places of $k$ which lie above the places of $\mathbb{Q}$ belonging to $S$ is finite. For any place $\nu$ of $k$ with $\nu \notin S'$ the place $p$ of $\mathbb{Q}$, which lies below $\nu$ is not in $S$. Hence $|a_i|_\nu \leq 1$ for $1 \leq i \leq n-1$. Therefore $x$ is integral over $\mathbb{Z}_p$ for almost all $p$ which is equivalent to $|x|_\nu \leq 1$ for almost all places $\nu$. □

This theorem shows, that we can naturally embed $k$ into $\mathbb{A}_k$. For an arbitrary element $x \in k$ we have $x \in k_\nu$ for all $\nu$ and $x \in \mathfrak{o}_\nu$ for almost all $\nu$. Thus we can embed $k$ in $\mathbb{A}_k$ via the diagonal map

$$x \mapsto (x, x, x, \ldots) \in \mathbb{A}_k$$

The map is an injection, because the map of $k$ into $k_\nu$ is an injection for all $\nu$. We will identify $k$ with its image in $\mathbb{A}_k$, so we may speak of $k$ as a subring of $\mathbb{A}_k$.

**Theorem 2.4.** *Let $L$ be an algebraic extension of the algebraic number field $k$ of degree $[L : k] = N < \infty$. Then there are at most $N$ extensions of a valuation $\nu$ of $k$ to $L$, say $\omega_i$ ($1 \leq i \leq J$) for $J \leq N$. Let $k_\nu$ be the completion of $k$ with respect to $\nu$ and $L_i$ the completion of $L$ with respect to $\omega_i$. Then*

$$k_\nu \otimes_k L = \bigoplus_{1 \leq i \leq J} L_i$$

*algebraically and topologically, where the right hand side is given the product topology.*

**Lemma 2.1.** *Let $k$ be a algebraic number field with valuation $\nu$ and $L$ a finite extension of $k$. Then we know that $k_\nu \otimes_k L = \bigoplus_{i=1}^{J} L_i$. Let $a_1, \ldots, a_n$ be a basis of $K/k$. Then*

$$a_1\mathfrak{o} \oplus \ldots \oplus a_n\mathfrak{o} = \mathfrak{O}_1 \oplus \ldots \oplus \mathfrak{O}_J$$

*where $\mathfrak{o} = \mathfrak{o}_\nu$ is the ring of integers of $k$ with respect to $\nu$ and $\mathfrak{O}_i \subset L_i$ is the ring of integers in $L_i$ with respect to $\omega_i$ $(1 \leq i \leq J)$.*

The proofs of the two statements above can be found in [1] on the pages 57 and 61.

**Theorem 2.5.** *Let $L$ be a finite extension of the algebraic number field $k$. Then there is a unique isomorphism*

$$\mathbb{A}_k \otimes_k L \cong \mathbb{A}_L$$

*algebraically and topologically. In this correspondence $k \otimes_k L = L \subset \mathbb{A}_k \otimes_k L$ where $k \subset \mathbb{A}_k$ is mapped on $L \subset \mathbb{A}_L$.*

*Proof.* We start with establishing an isomorphism of the two sides, where we see $\mathbb{A}_k \otimes_k L$ and $\mathbb{A}_L$ as topological spaces. So let $a_1, \ldots a_n$ (where $[L : k] = n$) be a basis of $L/k$ and let $\nu$ run through the set of normalized valuations of $k$. The left hand side is given the tensor product topology. But this is just the restricted direct product of $k_\nu \otimes_k L = k_\nu a_1 \oplus \ldots \oplus k_\nu a_n$ with respect to $\mathfrak{o}_\nu a_1 \oplus \ldots \oplus \mathfrak{o}_\nu a_n$. On the other hand we know from Theorem 2.4 that $k_\nu \otimes_k L$ is isomorphic to the direct product of the form $\bigoplus_{1 \leq i \leq J} L_i$ where $L_i$ is the completion of $L$ with respect to $\omega_i$, and the $\omega_i$ $(1 \leq i \leq J)$ are the normalized extensions of $\nu$ from $k$ to $L$. Let $\mathfrak{O}_i$ be the ring of integers in $L_i$ with respect to $\omega_i$ $(1 \leq i \leq J)$. Then we know from Lemma 2.1 that $\mathfrak{o}_\nu a_1 \oplus \ldots \oplus \mathfrak{o}_\nu a_n = \mathfrak{O}_1 \oplus \ldots \oplus \mathfrak{O}_J$ for almost all $\nu$. Hence the restricted direct product of $k_\nu \otimes_k L$ with respect to $\mathfrak{o}_\nu a_1 \oplus \ldots \oplus \mathfrak{o}_\nu a_n$ equals the restricted direct product of $L_1 \oplus \ldots \oplus L_J$ with respect to $\mathfrak{O}_1 \oplus \ldots \oplus \mathfrak{O}_J$. This is clearly the same thing as the restricted direct product of $L_\omega$ with respect to $\mathfrak{O}_\omega$ where $\omega$ runs through all normalized valuations of $L$. This establishes the isomorphism of the two sides as topological spaces. As all the isomorphisms used to establish that topological isomorphism are algebraic, it is easy to see, that it is also an algebraic isomorphism. $\square$

**Corollary 2.1.** *Let $\mathbb{A}_k^+$ denote the topological group obtained from $\mathbb{A}_k$ by forgetting about multiplication, $L$ an algebraic extension of the algebraic number field $k$ such that $[L : k] = n$. Then*

$$\mathbb{A}_L^+ = \underbrace{\mathbb{A}_k^+ \oplus \ldots \oplus \mathbb{A}_k^+}_{n \text{ summands}}$$

*This is an isomophism of (additive) topological groups.*

*Proof.* Let $a \in L$ be any nonzero element. Then clearly $a\mathbb{A}_k^+ \subset \mathbb{A}_L^+$ is isomorphic to $\mathbb{A}_k^+$ as a topological group. So we have the isomorphism

$$\mathbb{A}_L^+ = \mathbb{A}_k^+ \otimes_k L = a_1\mathbb{A}_k^+ \oplus \ldots \oplus a_n\mathbb{A}_k^+ = \mathbb{A}_k^+ \oplus \ldots \oplus \mathbb{A}_k^+$$

which proves our claim. $\square$

**Theorem 2.6.** *The field $k$ (considered as a subspace of $\mathbb{A}_k$ via the diagonal map) is a discrete subring of $\mathbb{A}_k$.*

*Proof.* Take $\mathbb{Q}$ instead of $k$ and $k$ instead of $L$ in the preceding corollary. This shows, that it is enough to verify the statement for $\mathbb{Q}$. Because of the group structure of $\mathbb{Q}$, we only need to show, that there exists a neighborhood $U$ of 0 in $\mathbb{A}_{\mathbb{Q}}^{+}$, which contains no other elements of $\mathbb{Q}$. This proves that $\mathbb{Q}^{+}$ is discrete in $\mathbb{A}_{\mathbb{Q}}^{+}$. We take for $U$ the set of $x \in \mathbb{A}_{\mathbb{Q}}^{+}$ with

$$\begin{aligned} |x_\infty|_\infty &< 1 \\ |x_p|_p &\leq 1 \end{aligned}$$

where $|\,|_\infty$ denotes the absolute value, and $|\,|_p$ the $p$-adic valuations on $\mathbb{Q}$. For $x \in \mathbb{Q} \cap U$, we have $|x|_p \leq 1$ for all prime numbers $p$ in $\mathbb{Z}$ and hence, $x$ must be an integer. But $|x|_\infty < 1$, and so $x = 0$ and $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}_{\mathbb{Q}}$. □

**Theorem 2.7.** *The factor group $\mathbb{A}_k^{+}/k^{+}$ is compact in the quotient topology.*

*Proof.* The field $k$ is a finite extension of $\mathbb{Q}$ of degree $N$. Using Corollary 2.1 we see that $\mathbb{A}_k^{+}/k^{+} \cong (\mathbb{A}_{\mathbb{Q}}^{+}/\mathbb{Q}^{+})^N$. Hence it is compact if and only if $\mathbb{A}_{\mathbb{Q}}^{+}/\mathbb{Q}^{+}$ is compact. Let $\infty$, respectively $|\,|_\infty$, denote the ordinary absolute value of $\mathbb{Q}$, which is the only infinite place. Define the subset $W \subset \mathbb{A}_k^{+}$ by

$$W = \{\, \boldsymbol{\alpha} \in \mathbb{A}_k \,|\, |\alpha_\infty|_\infty \leq 1/2 \text{ and } |\alpha_p|_p \leq 1 \,\forall\, p \in \mathbb{P}\}$$

where $\mathbb{P}$ denotes the set of prime numbers. Then $W$ is compact. We have already shown that $W \cap \mathbb{Q} = \{0\}$ to prove Theorem 2.6. Now what remains is to prove that $\mathbb{A}_{\mathbb{Q}} = W + \mathbb{Q}$.

Let $\boldsymbol{\alpha}$ be any adele. We claim that we may choose $x \in \mathbb{Q}$ such that $|\alpha_p - x|_p \leq 1$ for all prime numbers $p \in \mathbb{Z}$ where $|\,|_p$ is the $p$-adic valuation. Since $\boldsymbol{\alpha}$ is an adele, we only need to worry about a finite set $S$ of places. For each $p \in S$ we can find an element $\mu(p) \in \mathbb{Q}$ such that $|\alpha_p - \mu(p)|_p \leq 1$ and $\mu(p)$ is integral at all finite places different from $p$. To see this note that for any prime number $p \in \mathbb{Z}$, $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}_p$ relatively prime to $p$ we can always find a third element $c \in \mathbb{Z}_p$ such that $\frac{a}{bp^n} - \frac{c}{p^n} \in \mathbb{Z}_p$, because the congruence $a \equiv bc(\mod p^n)$ is clearly solvable for $c$. The element $\mu = \sum_{p \in S} \mu(p) \in \mathbb{Q}$ is well definied and $|\alpha_p - \mu|_p \leq 1$ for all $p$. Now we choose $s \in \mathbb{Z}$ such that $|\alpha_\infty - \mu - s|_\infty \leq 1/2$. By construction $\boldsymbol{\alpha} - \mu$ and $s$ are elements of $\mathbb{Z}_p$ for all finite $p$. Thus

$$|\alpha_p - \mu - s|_p \leq 1 \text{ and } |\alpha_\infty - \mu - s|_\infty|_\infty \leq 1/2$$

Hence $\boldsymbol{\beta} := \boldsymbol{\alpha} - \mu - s \in W$ and $\boldsymbol{\alpha} = \boldsymbol{\beta} + \mu + s \in W + \mathbb{Q}$.

Therefore the continuous map $W \to \mathbb{A}_{\mathbb{Q}}^{+}/\mathbb{Q}^{+}$ induced by the quotient map $\mathbb{A}_{\mathbb{Q}} \to \mathbb{A}_{\mathbb{Q}}^{+}/\mathbb{Q}^{+}$ is surjective. But $W$ is compact which completes the proof.

□

# 3 The Idele Group of an Algebraic Number Field

In the previous chapter we studied the additive structure of local fields. One might also be interested in the multiplicative structure. This chapter is mainly following [1], [3] and [10]. The proofs from this chapter can be found there.

## 3.1 Topology of the Idele Group

Consider the set $R^*$ of invertible elements of a topological ring $R$. It forms a group under multiplication. In general $R^*$ is not a topological group, since inversion need not be continuous in the subspace topology. So we give $R^*$ the following topology: First we embed $R^* \hookrightarrow R \times R$ via the map $x \mapsto (x, x^{-1})$. Then we give $R^*$ the corresponding subset topology. With this topology, $R^*$ is a topological group and the embedding $R^* \to R$ is continuous.

**Definition 3.1.** *The idele group $\mathbb{I}_k$ of $k$ is the group $\mathbb{A}_k^*$ of invertible elements of the adele ring $\mathbb{A}_k$ with the topology defined above.*

**Lemma 3.1.** *Let $k$ be an algebraic number field, $| \, |_\nu$ a normalized valuation of $k$. Then $\mathfrak{o}_\nu^*$ is a compact subgroup of $k_\nu^*$. The restricted direct product of $k_\nu^*$ with respect to $\mathfrak{o}_\nu^*$ is algebraically and topologically isomorphic to $\mathbb{I}_k$.*

We are going to prove this more generally in chapter 6, Theorem 5.1.

**Example** Let $k = \mathbb{Q}$ and $S$ be any finite set of places (denoted by $p$) induced by primes including the infinite place $\nu = \infty$. Let $N_p$ be any neighborhood of 1 in $\mathbb{Q}_p$ for $p \in S$. Then

$$(\prod_{p \in S} N_p \times \prod_{p \notin S} \mathbb{Z}_p) \cap \mathbb{I}_k \nsubseteq \mathbb{R}^* \times \prod_{p \neq \infty} \mathbb{Z}_p^*$$

as we can construct an idele $x = (x_\nu)_\nu \in \mathbb{I}_k$ such that for some $p \in S$ we have that $x_p \in \mathbb{Z}_p$ but $x \notin \mathbb{Z}_p^*$. Then $x$ is still an element of $\mathbb{I}_k$ but it is not contained in the set $\mathbb{R}^* \times \prod_{p \neq \infty} \mathbb{Z}_p^*$. Hence the neighborhood base of the relative topology of $\mathbb{I}_k$ induced by $\mathbb{A}_k$ cannot in general contain the open sets in the idele topology.

## 3.2 Properties of the Idele Group

We have a natural embedding of $k^*$ into $\mathbb{I}_k$ given by $x \mapsto (x, x, x, \ldots) \in \mathbb{I}_k$, which is clearly well defined.

**Lemma 3.2.** *The group $k^*$ of invertible elements of $k$ is a discrete subgroup of $\mathbb{I}_k$.*

*Proof.* The group $k^*$ is canonically embedded in $\mathbb{I}_k$ via the diagonal map. Define $f : \mathbb{A}_k \times \mathbb{A}_k \to \mathbb{A}_k$ by $f(\alpha, \beta) = \alpha\beta$. Then $k^*$ is homeomorphic to $f^{-1}(\{1\})$. Therefore an element $x \in k^*$ is mapped via $x \mapsto (x, x^{-1})$ onto the intersection of $f^{-1}(\{1\})$ and the discrete subset $k \times k$ of $\mathbb{A}_k \times \mathbb{A}_k$. Hence it is a discrete subgroup of $\mathbb{I}_k$. □

Since $\mathbb{A}_k/k$ (as an additive group) is compact, one could expect something similar for $\mathbb{I}_k/k^*$. However this turns out not to be true.

**Definition 3.2.** *For $\boldsymbol{\alpha} = (\alpha_\nu)$ in $\mathbb{I}_k$ define $\|\boldsymbol{\alpha}\| = \prod_\nu |\alpha_\nu|_\nu$ to be the content (sometimes also called volume) of $\boldsymbol{\alpha}$.*

**Theorem 3.1.** *(Product Formula) Let $x \in k^*$. Then*

$$\|x\| = \prod_\nu |x|_\nu = 1$$

*Proof.* Let $x \in k^*$. Define $\tilde{x} : \mathbb{A}_k \to \mathbb{A}_k$ to be the automorphism of the additive group of $\mathbb{A}_k$, that maps $\alpha \in \mathbb{A}_k$ to $x\alpha$. By abuse of notation we will write $x$ for that automorphism. Then clearly $x(k) = k$ and $k$ is a closed normal subgroup of $\mathbb{A}_k$. The additive group $\mathbb{A}_k$ is locally compact, hence it admits a unique (up to a scalar multiple) Haar measure. Let $\operatorname{mod}_{\mathbb{A}_k}(x)$ be the modulus of the automorphism $\alpha \mapsto x\alpha$. Let $\mu$ be a Haar measure on $\mathbb{A}_k$. Then one can uniquely define a Haar measure $\mu'$ on $\mathbb{A}_k/k$, so that the following equality holds

$$\operatorname{mod}_{\mathbb{A}_k}(x) = \operatorname{mod}_k(x) \cdot \operatorname{mod}_{\mathbb{A}_k/k}(x)$$

This is done in some more detail in the Appendix. Since $k$ is discrete and $\mathbb{A}_k/k$ is compact, the two factors on the right hand side must equal 1. Therefore it remains to show, that

$$\operatorname{mod}_{\mathbb{A}_k}(x) = \prod_\nu |x|_\nu$$

We already showed that $\operatorname{mod}_{k_\nu}(x) = |x|_\nu$, as $\nu$ is normalized. Furthermore we obviously have $\operatorname{mod}_{\mathbb{A}_k}(x) = \prod_\nu \operatorname{mod}_{k_\nu}(x)$, which completes the proof.

□

**Lemma 3.3.** *The map $\| \, \| : \mathbb{I}_k \to \mathbb{R}_+$ is a continuous homomorphism of the topological group $\mathbb{I}_k$ into the multiplicative group of the (strictly) positive real numbers.*

*Proof.* The map $\mathrm{mod}_{k_\nu}$ from $k_\nu^*$ to $\mathbb{R}_+$ is continuous for all places $\nu$. Therefore $\| \, \|$ is continuous. Further it is obviously a homomorphism. $\square$

For $\boldsymbol{\alpha}$ in $\mathbb{I}_k$ the map $\tilde{\boldsymbol{\alpha}}$ defines an automorphism of $\mathbb{A}_k$ via $\tilde{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \boldsymbol{\alpha}\boldsymbol{\beta}$. Now by the constructions used in the proof above it is clear that:

**Lemma 3.4.** *One has that* $\mathrm{mod}_{\mathbb{A}_k}(\boldsymbol{\alpha}) = \|\boldsymbol{\alpha}\|$ *for all* $\boldsymbol{\alpha} = (\alpha_\nu) \in \mathbb{I}_k$.

We can map $\mathbb{R}_+$ into $\mathbb{I}_k$ by the mapping $j : t \mapsto (\underbrace{t^{1/n}, \ldots, t^{1/n}}_{\text{infinite places}}, 1, 1, \ldots)$. Then $\|j(t)\| = |t^{r_1/n + 2r_2/n}| = t$ since all the valuations are normalized. Hence $\| \, \|$ is surjective. Let $\mathbb{I}_k^1$ be the kernel of $\| \, \|$, so $\mathbb{I}_k^1 = \{\boldsymbol{\alpha} \in \mathbb{I}_k \mid \|\boldsymbol{\alpha}\| = 1\}$. Then we have the exact sequence

$$(1) \longrightarrow \mathbb{I}_k^1 \longrightarrow \mathbb{I}_k \longrightarrow \mathbb{R}_+ \longrightarrow (1)$$

where all maps are continuous group homomorphisms.

**Theorem 3.2.** *(Minkowski-Chevalley-Weil) There exists a constant $\delta > 0$, depending only on the algebraic number field $k$ with the following property: Let $\boldsymbol{\alpha} = (\alpha_\nu) \in \mathbb{I}_k$ be such that $\|\boldsymbol{\alpha}\| > \delta$. Then there exists $\beta \in k^* \subset \mathbb{I}_k$ such that $|\beta|_\nu \leq |\alpha_\nu|_\nu$ for all places $\nu$.*

*Proof.* Let $\mu$ be the Haar-measure of $\mathbb{A}_k$. Let $\mu'$ be the quotient measure (defined as in the Appendix) induced by $\mu$ on $\mathbb{A}_k^+/k^+$. If there is no risk of confusion we will write $\mu$ instead of $\mu'$. Let $c_0 = \mu(\mathbb{A}_k^+/k^+)$, which is finite as $\mathbb{A}_k^+/k^+$ is compact. Let $c_1$ be the Haar-measure of the set defined by

$$C = \prod_{\nu \in S_\infty} N_\nu \times \prod_{\nu \notin S_\infty} \mathfrak{o}_\nu$$

where $N_\nu = \{x \in k_\nu | \, |x| \leq \frac{1}{10}\}$. Similarly since $C$ is compact, $c_1$ is finite.

We want to show that the constant $\delta = c_0/c_1$ has the required properties.

Define the set $T$ of $\boldsymbol{\tau} \in \mathbb{A}_k$ with $|\tau_\nu|_\nu \leq \frac{1}{10}|\alpha_\nu|_\nu$ if $\nu$ is archimedean, and $|\tau_\nu|_\nu \leq |\alpha_\nu|_\nu$ if $\nu$ is non-archimedean. Then $T$ has measure $\mu(T) = \mu(C) \cdot \|\boldsymbol{\alpha}\| = c_1 \prod_\nu |\alpha_\nu|_\nu > c_1\delta = c_0$. Hence in the natural quotient map $\mathbb{A}_k^+ \to \mathbb{A}_k^+/k^+$ there must be a pair of distinct points of $T$, which have the same image in $\mathbb{A}_k^+/k^+$. Lets say these elements are $\boldsymbol{\tau}'$ and $\boldsymbol{\tau}''$ in $T$. Then $\boldsymbol{\tau}' - \boldsymbol{\tau}'' = \beta$ for some element $\beta \in k^+$ since they have the same image in $\mathbb{A}_k^+/k^+$. Then

$$|\beta|_\nu = |\tau'_\nu - \tau''_\nu| \leq |\alpha_\nu|_\nu$$

for all $\nu$, which completes the proof. $\square$

**Remark** This theorem can be reformulated as follows: Let $\Pi_1 = \{\xi \in \mathbb{I}_k | \, |\xi_\nu|_\nu \leq 1 \, \forall \nu\}$. There exists a positive constant $\delta$ such that if $\|\alpha\| > \delta$ then $\alpha \Pi_1 \cap k \neq \{0\}$.

**Theorem 3.3.** *The group $\mathbb{I}_k^1$ is closed both as a subset of $\mathbb{A}_k$ and as a subset of $\mathbb{I}_k$. The two induced topologies on it coincide.*

*Proof.* The set $\mathbb{I}_k^1$ is clearly closed in $\mathbb{I}_k$, since it is the preimage of the closed subset (1) of $\mathbb{R}_+$ under the continuous map $\| \, \|$. Let $\boldsymbol{\alpha} \in \mathbb{A}_k$, $\boldsymbol{\alpha} \notin \mathbb{I}_k^1$. We want to show that for every such $\boldsymbol{\alpha}$ there exists an $\mathbb{A}_k$-neighborhood $W$, sucht that $W \cap \mathbb{I}_k^1 = \emptyset$. This would prove, that $\mathbb{A}_k \setminus \mathbb{I}_k^1$ is open an hence $\mathbb{I}_k^1$ is a closed subset of $\mathbb{A}_k$.

*1st Case:* $\prod_\nu |\alpha_\nu|_\nu < 1$. Then there is a finite set $S$ of places $\nu$ such that

1. $S$ contains all $\nu$ such that $|\alpha_\nu|_\nu > 1$
2. $\prod_{\nu \in S} |\alpha_\nu|_\nu < 1$

Then we define the neighborhood $W$ to be the set of all $\boldsymbol{\xi} = (\xi_\nu) \in \mathbb{A}_k$ such that

$$\begin{array}{rcll} |\xi_\nu - \alpha_\nu|_\nu & < & \varepsilon & \nu \in S \\ |\xi_\nu|_\nu & \leq & 1 & \nu \notin S \end{array}$$

Then by definition of $S$ we have $\boldsymbol{\alpha} \in W$. For $\boldsymbol{\xi} \in W$ we have

$$\prod_\nu |\xi_\nu|_\nu \leq \prod_{\nu \in S} |\xi_\nu|_\nu \leq \prod_{\nu \in S} (|\xi_\nu - \alpha_\nu|_\nu + |\alpha_\nu|_\nu)$$

the first inequality holds since $|\xi_\nu|_\nu \leq 1$ for $\nu \notin S$ and the second uses the triangle inequality, which holds for every normalized (archimedean and nonarchimedean) valuation. The set $S$ is finite, therefore

$$\prod_{\nu \in S} (|\xi_\nu - \alpha_\nu|_\nu + |\alpha_\nu|_\nu) = \prod_{\nu \in S} |\alpha_\nu|_\nu + O(\varepsilon)$$

Since $\prod_{\nu \in S} |\alpha_\nu|_\nu < 1$ by definition of $S$, we have that $\prod_\nu |\xi_\nu|_\nu < 1$ for sufficiently small $\varepsilon$.

*2nd Case:* $\prod_\nu |\alpha_\nu|_\nu > 1$, say $\prod_\nu |\alpha_\nu|_\nu = C$. Then there is a set $S$ of places $\nu$ defined by

1. $S$ contains all infinite places $\nu$
2. if $\nu \notin S$ then $|\xi_\nu|_\nu < 1$ implies that $|\xi_\nu|_\nu < \frac{1}{2C}$
3. $1 < \prod_{\nu \in S} |\alpha_\nu|_\nu < 2C$

The fact that $\prod_\nu |\alpha_\nu|_\nu$ converges implies that $|\alpha_\nu|_\nu = 1$ for almost all $\nu$. Hence the set $S$ is finite. Let $|\xi_\nu - \alpha_\nu| < \varepsilon$. We have

$$1 < \prod_\nu |\alpha_\nu|_\nu \leq \prod_{\nu \in S} |\alpha_\nu|_\nu \leq \prod_{\nu \in S} (|\alpha_\nu - \xi_\nu|_\nu + |\xi_\nu|_\nu) \leq \prod_{\nu \in S} |\xi_\nu|_\nu + O(\varepsilon)$$

and

$$\prod_{\nu \in S} |\xi_\nu|_\nu \leq \prod_{\nu \in S} (|\xi_\nu - \alpha_\nu|_\nu + |\alpha_\nu|_\nu) \leq \prod_{\nu \in S} |\alpha_\nu|_\nu + O(\varepsilon) < 2C + O(\varepsilon)$$

Then one can choose $\varepsilon$ sufficiently small such that $|\xi_\nu - \alpha_\nu| < \varepsilon$ implies $1 < \prod_{\nu \in S} |\xi_\nu|_\nu < 2C$.

We may define the neighborhood $W$ to be the set of all $\boldsymbol{\xi} = (\xi_\nu) \in \mathbb{A}_k$, such that

$$\begin{aligned} |\xi_\nu - \alpha_\nu|_\nu &< \varepsilon \qquad \nu \in S \\ |\xi_\nu|_\nu &\leq 1 \qquad \nu \notin S \end{aligned}$$

Then of course $\boldsymbol{\alpha} \in W$. Let $\boldsymbol{\xi} \in W$. If there exists a place $\nu \notin S$ such that $|\xi_\nu|_\nu < 1$ then we have

$$\prod_\nu |\xi_\nu|_\nu = \prod_{\nu \in S} |\xi_\nu|_\nu \prod_{\nu \notin S} |\xi_\nu|_\nu < 2C \prod_{\nu \notin S} |\xi_\nu|_\nu < 2C \prod_{\substack{\nu \notin S \\ |\xi_\nu|_\nu < 1}} |\xi_\nu|_\nu \leq \frac{2C}{2C} = 1$$

If for all $\nu \notin S$ we have $|\xi_\nu|_\nu \geq 1$ then

$$\prod_\nu |\xi_\nu|_\nu = \prod_{\nu \in S} |\xi_\nu|_\nu \prod_{\nu \notin S} |\xi_\nu|_\nu > \prod_{\nu \in S} |\xi_\nu|_\nu > 1$$

holds. In particular, every idele in $W$ has content not equal to 1 and hence $W \cap \mathbb{I}_k^1 = \emptyset$. This completes the proof that $\mathbb{I}_k^1$ is a closed subset of $\mathbb{A}_k$.

Now we turn to the proof, that the $\mathbb{I}_k$- and $\mathbb{A}_k$-topologies on $\mathbb{I}_k^1$ are the same. For every $\boldsymbol{\alpha} \in \mathbb{I}_k^1$ we must show that every $\mathbb{I}_k$-neighborhood of $\boldsymbol{\alpha}$ contains an $\mathbb{A}_k$-neighborhood and vice versa.

So let $W \subset \mathbb{I}_k^1$ be an $\mathbb{A}_k$-neighborhood of $\boldsymbol{\alpha}$. Then there exists a finite set $S$, containing at least all archimedean places, such that $W$ contains a neighborhood of the form

$$N = \prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} \mathfrak{o}_\nu$$

where $N_\nu$ is a neighborhood of $\alpha_\nu$ in $k_\nu$. We may define $N_\nu$ to be $\{\, \xi_\nu \in k_\nu |\, |\xi_\nu - \alpha_\nu|_\nu < \varepsilon\}$ for some given $\varepsilon > 0$. Then the neighborhood $N$ contains the $\mathbb{I}_k$-neighborhood $\prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} \mathfrak{o}_\nu^*$. This part of the proof is merely saying that the embedding of $\mathbb{I}_k$ into $\mathbb{A}_k$ is continuous.

Now let $H \subset \mathbb{I}_k^1$ be a $\mathbb{I}_k$-neighborhood. Then $H$ contains an $\mathbb{I}_k$-neighborhood of the type $M = \prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} \mathfrak{o}_\nu^*$, where $S$ is a finite set of places which contains all archimedean places and $N_\nu = \{\, \xi_\nu \in k_\nu |\, |\xi_\nu - \alpha_\nu|_\nu < \varepsilon\}$. Since $\prod_\nu |\alpha_\nu|_\nu = 1$ we may choose $\varepsilon$ sufficiently small such that for all $\boldsymbol{\xi} \in H$

$$\prod_\nu |\xi_\nu|_\nu = \prod_{\nu \in S} |\xi_\nu|_\nu \leq \prod_{\nu \in S} (|\xi_\nu - \alpha_\nu|_\nu + |\alpha_\nu|_\nu) \leq 1 + O(\varepsilon) \leq 2$$

holds. Then the intersection of the (previously defined) neighborhood $N$ with $\mathbb{I}_k^1$ is the same as that of $M$ with $\mathbb{I}_k^1$. Hence $M$ defines an $\mathbb{A}_k$-neighborhood, which completes the proof.

□

**Theorem 3.4.** *The factor group $\mathbb{I}_k^1/k^*$ with the quotient topology is compact.*

*Proof.* Using the Theorem 3.3, it is enough, to find a set $W$, which is compact in $\mathbb{A}_k$, such that the map $W \cap \mathbb{I}_k^1 \to \mathbb{I}_k^1/k^*$ is surjective. Let $\delta$ be a constant, which fulfills the requirements of Theorem 3.2. Let $\boldsymbol{\alpha}$ be any idele with $\|\boldsymbol{\alpha}\| > \delta$. We take for $W$ the set of all $\boldsymbol{\xi}$ with $|\xi_\nu|_\nu \leq |\alpha_\nu|_\nu$ for all places $\nu$. Let $\boldsymbol{\eta} \in \mathbb{I}_k^1$. Since $\|\boldsymbol{\eta}\| = 1$ and $\|\boldsymbol{\alpha}\| > \delta$ also $\|\boldsymbol{\eta}^{-1}\alpha\| > \delta$. Then by Theorem 3.2 there is an element $\beta \in k^*$ such that

$$|\beta|_\nu \leq |\eta_\nu^{-1}\alpha_\nu|_\nu$$

holds. Hence $\beta\boldsymbol{\eta} \in W$ and the mapping above is surjective. □

## 3.3 Consequences

From the last theorem we can easily deduce the finiteness of the class number and Dirichlets Unit Theorem. We start with proving the finiteness of the class number.

### 3.3.1 The Finiteness of the Class Number

First we need the following theorem

**Theorem 3.5.** *Let $\nu$ be a non-archimedean valuation. The set of values assumed by $|\,|_\nu$ on $k_\nu$ coincides with the set of values taken by $|\,|_\nu$ on $k$.*

This statement is proved in [3], page 25.

Now let $I_k$ denote the group of all $k$-ideals. Let $P_k$ denote the group of all principal $k$-ideals. Let $\nu$ be a finite place of $k$. Then $\nu$ is induced by some prime ideal $\mathfrak{p}$ of $\mathfrak{o}_k$. For the rest of this section we will write $\nu_\mathfrak{p}$ respectively $|\,|_\mathfrak{p}$ to emphasize the corresponding prime ideal $\mathfrak{p}$. Let $x_{\nu_\mathfrak{p}} \in k_{\nu_\mathfrak{p}}$, then $|x_{\nu_\mathfrak{p}}|_\mathfrak{p} = |y|_\mathfrak{p}$ for some element $y \in k$ by the theorem above. Hence $|x_{\nu_\mathfrak{p}}|_\mathfrak{p} = (N\mathfrak{p})^{ord_\mathfrak{p}(y)}$. Set $ord_\mathfrak{p}(x_{\nu_\mathfrak{p}}) := ord_\mathfrak{p}(y)$ for some suitably chosen element $y \in k$. Hence one can map $x_{\nu_\mathfrak{p}}$ onto the fractional ideal $\mathfrak{p}^{ord_\mathfrak{p}(x_{\nu_\mathfrak{p}})}$.

We define $\phi : \mathbb{I}_k \to I_k$ by

$$(x_{\nu_\mathfrak{p}}) \mapsto \prod_{\nu_\mathfrak{p} \, finite} \mathfrak{p}^{ord_\mathfrak{p}(x_{\nu_\mathfrak{p}})}$$

Then $\phi$ is well defined, as $|x_{\nu_\mathfrak{p}}|_\mathfrak{p} = 1$ for almost all places of $k$, hence $ord_\mathfrak{p}(x_{\nu_\mathfrak{p}}) = 0$ for almost all prime ideals $\mathfrak{p}$ of $\mathfrak{o}_k$. Furthermore this map is continuous if $I_k$ is endowed with the discrete topology. The image of $k^*$ under this map is the group of principal ideals $P_k$.

**Theorem 3.6.** *The group of ideal classes, i.e. $I_k$ modulo $P_k$, is finite.*

*Proof.* The map $\phi$ restricted to $\mathbb{I}_k^1$ is clearly surjective. Furthermore $I_k$ is discrete. So $I_k/P_k$ is the image of the compact set $\mathbb{I}_k^1/k^*$ under the continuous map $\phi$. Hence it is both discrete and compact and therefore $I_k/P_k$ is finite. □

### 3.3.2 Dirichlets Unit Theorem

Now we are going to prove Dirichlets Unit Theorem by using the compactness of $\mathbb{I}_k^1/k^*$. We start with some definitions. Throughout the rest of this chapter let $S$ be a finite set of places containing all infinite places.

**Definition 3.3.** *Let $k_S$ be the group of all $x \in k$ such that $|x|_\nu = 1$ for all $\nu \notin S$. We call $k_S$ the group of $S$-units.*

**Remark** For $S = S_\infty$ we obviously have $k_S = \mathfrak{o}_k^*$, the group of units of $\mathfrak{o}_k$.

We need the following two easy lemmas:

**Lemma 3.5.** *Fix $c, C$ in $\mathbb{R}$ such that $0 < c < C < \infty$. Then the set of $S$-units $x$ satisfying $c \leq |x_\nu|_\nu \leq C$ for all $\nu \in S$ is finite.*

*Proof.* Define $W$ to be the set of ideles $(x_\nu)$ satisfying $c \leq |x_\nu|_\nu \leq C$ for $\nu \in S$ and $x_\nu \in \mathfrak{o}_\nu$ for $\nu \notin S$. Then $W$ is compact, as it is the product of compact sets with the product topology. Then set of $S$-units $x$ satisfying $c \leq |x_\nu|_\nu \leq C$ for all $\nu \in S$ is just the intersection of the discrete subset $k$ with the compact set $W$. Hence it is finite.

□

**Lemma 3.6.** *There are only finitely many $\xi$ in $k$ such that $|\xi|_\nu = 1$ for all (finite and infinite) places $\nu$ of $k$. They are precisely the roots of unity in $k$.*

*Proof.* Let $\xi$ be a root of unity in $k$. Then clearly $|\xi|_\nu = 1$ for all places $\nu$ of $k$. By the lemma above the set $W$ of elements of $k$ satisfying $|\xi|_\nu = 1$ for all places $\nu$ is finite. It forms a group under multiplication. Suppose there is an element $x \in k$ with $|x|_\nu = 1$ for all places $\nu$. Then $x^i$ is contained in $W$ for all $i \geq 1$. But $W$ is finite, hence there exists a positive integer $m \geq 1$ with $x^m = 1$. So $x$ is a root of unity. □

Now we are ready to prove

**Theorem 3.7.** *The group $k_S$ is the direct sum of a finite cyclic group and a free abelian group of rank $s-1$, where $s = |S|$.*

*Proof.* We define $\mathbb{I}_k^1(S) = \mathbb{I}_k(S) \cap \mathbb{I}_k^1$, where $\mathbb{I}_k(S) = \prod_{\nu \in S} k_\nu^* \times \prod_{\nu \notin S} \mathfrak{o}_\nu^*$. Then $\mathbb{I}_k^1(S)$ is an open subgroup of $\mathbb{I}_k^1$, as $\mathbb{I}_k(S)$ is an open subgroup of $\mathbb{I}_k$ by definition and the topology on $\mathbb{I}_k^1$ is induced by the topology on $\mathbb{I}_k$. Clearly we have that $k_S = k^* \cap \mathbb{I}_k(S) \cap \mathbb{I}_k^1 = k^* \cap \mathbb{I}_k(S)$. Therefore

$$\mathbb{I}_k^1(S)/k_S = \mathbb{I}_k^1(S)/(k^* \cap \mathbb{I}_k(S))$$

is an open subgroup of $\mathbb{I}_k^1/k^*$. As $\mathbb{I}_k^1(S)/k_S$ is a subgroup of $\mathbb{I}_k^1/k^*$, it is also closed in $\mathbb{I}_k^1/k^*$, and hence it is compact.

Now we consider the mapping

$$\gamma : \mathbb{I}_k(S) \to \bigoplus_{i=1}^{s} \mathbb{R}^+$$

where $\gamma$ is defined by

$$\zeta_{\nu_1} \times \ldots \times \zeta_{\nu_s} \times \prod_{\nu \notin S} \zeta_\nu \mapsto (\log(|\zeta_{\nu_1}|_{\nu_1}), \ldots, \log(|\zeta_{\nu_s}|_{\nu_s}))$$

for an element $(\zeta_\nu)_\nu \in \mathbb{I}_k(S)$. The set $S$ consists of the valuations $\nu_1, \ldots, \nu_s$. Then $\gamma$ is continuous, as $\log(|\,|_{\nu_i})$ is continuous on each $k_{\nu_i}$, and surjective for the same reason. Let $\gamma_{k_S}$ denote the restriction of $\gamma$ to $k_S$. Then the kernel of $\gamma_{k_S}$ consists of the elements $x$ in $k$, such that $|x|_\nu = 1$ for all places $\nu$ of $k$, and therefore by the lemma above, it is a finite cyclic group.

Let $\Gamma = \gamma(k_S)$. By the first lemma above, there are only finitely many $x \in k$ such that $\frac{1}{2} \leq |x|_{\nu_i} \leq 2$ for $1 \leq i \leq s$. Therefore $\Gamma$ is discrete in $\bigoplus_{i=1}^{s} \mathbb{R}^+$.

Let $\Delta = \gamma(\mathbb{I}_k^1(S))$. Then $\Delta$ generates the hyperplane $H^{s-1}$ determined by the equation $\sum_{i=1}^{s} x_i = 0$. Indeed we can choose $s-1$ coordinates arbitrary and adjust the last, as $S_\infty \subset S$. Let $W$ be the subspace generated by $\Gamma$ in $\mathbb{R}^s$. Then we have a homomorphism

$$\mathbb{I}_k^1/k_S \to H^{s-1}/W.$$

We know that $\mathbb{I}_k^1(S)/k_S$ is compact and hence $\Delta/\Gamma$ is compact, as it is the image of a compact set under a continuous map. Therefore $\Gamma$ generates $H^{s-1}$ and has rank $s-1$. Hence $\Gamma$ is free of $s-1$ generators as asserted.

□

In particular

**Theorem 3.8.** *The group $\mathfrak{o}_k^*$ is a finitely generated $\mathbb{Z}$-module. Let $[k : \mathbb{Q}] = n = r_1 + 2r_2$, where $r_1$ is the number of real places and $r_2$ the number of complex places. Let $\mu_k$ denote the set of roots of unity in $k$. Then $\mathfrak{o}_k \cong \mu_k \times \mathbb{Z}^{r_1+r_2-1}$.*

# Part II

# 4 The Adele Ring of an Algebra

Now we turn our attention to the construction of adeles of algebras. The first two sections and the proofs can be found in [13]. The last section of this chapter brings together the definition for adele of division algebras used by [13] and [2], which will be needed in the next chapter.

Throughout this chapter let $k$ be an algebraic number field, $\nu$ an archimedean or non-archimedean normalized valuation of $k$ and let $k_\nu$ be the completion of $k$ with respect to $\nu$. For $\nu$ non-archimedean let $\mathfrak{o}_\nu = \{x \in k_\nu \,|\, |x|_\nu \leq 1\}$ denote the closure of $\mathfrak{o}_k$ in $k_\nu$. An algebra $A$ over $k$ is always understood to be central, that means, that the center of $A$ coincides with $k$.

## 4.1 Construction of the Adele Ring of an Algebra

Now we will define adeles for finite dimensional algebras (especially division algebras) over $k$. As we are mostly concerned with the additive structure of adeles, we need not restrict ourselves to algebras, but may start with vector spaces. So let $E$ be a vector space of finite dimension $n$ over $k$. We define for each place $\nu$ an analogue for the completion of $k$ with respect to $\nu$ by $E_\nu := E \otimes_k k_\nu$. We can naturally embed $E$ into $E_\nu$ via the map $e \mapsto e \otimes 1_{k_\nu}$. As we can embed $k_\nu$ into $\mathbb{A}_k$, we see that there is a canonical injection of $E_\nu = E \otimes_k k_\nu \hookrightarrow E \otimes_k \mathbb{A}_k$. So we define $E_\mathbb{A} := E \otimes_k \mathbb{A}_k$ and get the natural embedding of $E$ into $E_\mathbb{A}$ via $e \mapsto e \otimes 1_{\mathbb{A}_k}$, where $1_{\mathbb{A}_k} = (1, 1, 1, \ldots)$.

Let $\varepsilon$ be a basis of $E$ over $k$. With this basis we can find an isomorphism of $E \cong k^n$. This induces an isomorphism of $\mathbb{A}_k^n$ onto $E_\mathbb{A}$, since

$$E_\mathbb{A} = E \otimes_k \mathbb{A}_k \cong k^n \otimes_k \mathbb{A}_k = \mathbb{A}_k^n$$

So we define the topology on $E_\mathbb{A}$ to be the topology which is obtained by transferring the product topology of $\mathbb{A}_k^n$ to $E_\mathbb{A}$. This topology does not depend on the choice of a basis $\varepsilon$.

There is also an alternative probably more natural approach. In the definition of adeles of number fields $k$, we noticed that $\mathbb{A}_k$ is the union of all sets $\mathbb{A}_k(S)$, where $S$ is a finite set of places containing the infinite places. We can do something similar for vector spaces. Let $\varepsilon$ be a basis of $E$ over $k$ and for all finite places $\nu$ call $\varepsilon_\nu$ the $\mathfrak{o}_\nu$-module generated by $\varepsilon$ in $E_\nu$.

Again let $S$ be a finite set of places containing all infinite places and define

$$E_{\mathbb{A}}(S,\varepsilon) := \prod_{\nu\in S} E_\nu \times \prod_{\nu\notin S} \varepsilon_\nu$$

Then this set is endowed with the product topology and we can prove, that $E_{\mathbb{A}}$ is the union of these sets and that the two topologies coincide. We start with the following lemma, in order to prove an even stronger result of what we just claimed.

**Lemma 4.1.** *Let $E$ and $E_\nu$ be as above. Let $\varepsilon$, $\varepsilon'$ be two finite subsets of $E$, both containing a basis of $E$ over $k$. Let $\varepsilon_\nu$, $\varepsilon'_\nu$ denote the $\mathfrak{o}_\nu$-modules generated by $\varepsilon$, $\varepsilon'$ in $E_\nu$. Then for almost all $\nu$ we have $\varepsilon_\nu = \varepsilon'_\nu$.*

*Proof.* Set $\varepsilon = \{e_1, \dots, e_r\}$ and $\varepsilon' = \{e'_1, \dots, e'_s\}$. As $\varepsilon$ contains a basis of $E$ over $k$ we can express (perhaps not uniquely) each $e'_j$ (for $1 \le j \le s$) in terms of the $e_i$, ($1 \le i \le r$), i.e. $e'_j = \sum_{i=1}^{s} c_{ji} e_i$ with coefficients $c_{ji}$ in $k$. In case $|c_{ji}|_\nu \le 1$ for all $1 \le j \le s$, $1 \le i \le r$ we have $\varepsilon'_\nu \subset \varepsilon_\nu$. Therefore $\varepsilon'_\nu \subset \varepsilon_\nu$ for almost all $\nu$, since there are only finitely many coefficients and for all $x \in k$ we have proved, that $|x|_\nu \le 1$ for almost all $\nu$. Interchanging $\varepsilon$ and $\varepsilon'$ completes the proof. □

Now we can show:

**Theorem 4.1.** *Let $E$ be a vector space of finite dimension $n$ over $k$. Let $\varepsilon$ be a finite subset of $E$ containing a basis of $E$ over $k$. For each finite place $\nu$ of $k$ call $\varepsilon_\nu$ the $\mathfrak{o}_\nu$-module generated by $\varepsilon$ in $E_\nu$. For each finite set $S$ of places of $k$ containing all infinite places write*

$$E_{\mathbb{A}}(S,\varepsilon) = \prod_{\nu\in S} E_\nu \times \prod_{\nu\notin S} \varepsilon_\nu$$

*Then each $E_{\mathbb{A}}(S,\varepsilon)$ is an open subgroup of $E_{\mathbb{A}}$, and $E_{\mathbb{A}}$ is the union of these subgroups.*

**Remark** When we choose a basis $\varepsilon'$ of $E$ over $k$ we see that $\varepsilon'_\nu \cong \mathfrak{o}_\nu^n$ and hence it is compact and open in $E_\nu$.

*Proof.* Let $S'$ be a finite set containing $S$. Since $\varepsilon_\nu$ is open in $E_\nu$, it follows that $E_{\mathbb{A}}(S,\varepsilon)$ is an open subgroup of $E_{\mathbb{A}}(S',\varepsilon)$ as both sets are endowed with the product topology. Let $n$ be the dimension of $E$ over $k$ and let $\gamma$ be a basis. The choice of basis induces an isomorphism of $E$ onto $k^n$ and therefore we get the isomorphism $E_{\mathbb{A}}$ onto $(\mathbb{A}_k)^n$. The adele group $\mathbb{A}_k$ is the union of all sets $\mathbb{A}_k(S) = \prod_{\nu\in S} k_\nu \times \prod_{\nu\notin S} \mathfrak{o}_\nu$, where $S$ is a finite set of places containing $S_\infty$. We also have

$$E \otimes_k \mathbb{A}_k(S) = E \otimes_k \big(\prod_{\nu\in S} k_\nu \times \prod_{\nu\notin S} \mathfrak{o}_\nu\big) = \prod_{\nu\in S} E \otimes_k k_\nu \times \prod_{\nu\notin S} \gamma_\nu = \prod_{\nu\in S} E_\nu \times \prod_{\nu\notin S} \gamma_\nu$$

where $\gamma_\nu$ is the $\mathfrak{o}_\nu$-module generated by $\gamma$ in $E_\nu$. This shows that $E_\mathbb{A}$ is the union of all sets $E_\mathbb{A}(S,\gamma)$

The lemma above shows, that there is a finite set of places $S_0$ containing $S_\infty$ such that $\gamma_\nu = \varepsilon_\nu$ for all $\nu \notin S_0$. Hence $E_\mathbb{A}$ is the union of the sets $E_\mathbb{A}(S,\varepsilon)$. For each finite set $S' \supset S_0 \cup S$ we have that $E_\mathbb{A}(S,\varepsilon)$ is an open subring of $E_\mathbb{A}(S',\gamma)$, and hence an open subring of $E_\mathbb{A}$. This finishes our proof. □

**Remark** The construction above, including its topology, coincides with the restricted direct product of $E_\nu$ with respect to $\varepsilon'_\nu$ as additive groups.

One can prove:

**Theorem 4.2.** *Let $E$ be a finite dimensional vector space over a locally compact field $k$. Then $E$ is locally compact.*

This is proved in [13], page 5.

Therefore $E_\mathbb{A}$ is locally compact, as it is the restricted direct product of the family $\{E_\nu\}$ with respect to the family $\{\varepsilon_\nu\}$.

Now let $A$ be a finite dimensional algebra over $k$. We can extend the multiplication law of $A$ to $A_\mathbb{A} = A \otimes_k \mathbb{A}_k$ by defining the multiplication componentwise.

**Lemma 4.2.** *Let $A$ be a finite dimensional algebra over the field $k$. Let $\alpha$ be a finite subset of $A$ containing a basis of $A$ over $k$. For each finite place $\nu$ of $k$ call $\alpha_\nu$ the $\mathfrak{o}_\nu$-module generated by $\alpha$ in $A_\nu = A \otimes_k k_\nu$. Then for almost all $\nu$, $\alpha_\nu$ is a compact subring of $A_\nu$.*

*Proof.* Set $\alpha = \{a_1, \ldots, a_r\}$ and $\alpha' = \{1, a_1, \ldots, a_r\}$. As $\alpha$ contains a basis of $A$ over $k$ we can write $a_i a_j = \sum_{h=1}^{r} c_{ijh} a_h$ $(1 \le i,j \le r)$ for some $c_{ijh}$ in $k$. If for some finite $\nu$ we have $|c_{ijh}|_\nu \le 1$ for $1 \le i,j,h \le r$ then $\alpha'_\nu$ is a subring of $A_\nu$, which is compact. Furthermore Lemma 4.1 shows that $\alpha_\nu = \alpha'_\nu$ for almost all $\nu$ which completes the proof. □

Now we obviously have

**Corollary 4.1.** *Let $A$, $k$, $\alpha$ and $\alpha_\nu$ be as above. Let $S$ be a finite set of places containing $S_\infty$, the set of infinite places. Define*

$$A_\mathbb{A}(S,\alpha) = \prod_{\nu \in S} A_\nu \times \prod_{\nu \notin S} \alpha_\nu$$

*There is a set $S_0$ with the property, that $A_\mathbb{A}(S,\alpha)$ is an open subring of $A_\mathbb{A}$ whenever $S \supset S_0$. Furthermore $A_\mathbb{A}$ is the union of these subrings.*

## 4.2 Properties of the Adele Ring of an Algebra

We are going to adopt some of the properties of adeles of number fields $k$. The aim of this section is to show, that $E$ is a discrete subgroup of $E_{\mathbb{A}}$ and that $E_{\mathbb{A}}/E$ is compact. Due to Theorem 2.5, which stated that for a finite dimensional field extension $L$ of $k$, the rings $\mathbb{A}_k \otimes L$ and $\mathbb{A}_L$ are isomorphic. We get the following result:

**Theorem 4.3.** *Let $k$ be an algebraic number field and $E$ a vector space of finite dimension over $k$. Then $E$ is discrete in $E_{\mathbb{A}}$ and $E_{\mathbb{A}}/E$ is compact in the quotient topology.*

*Proof.* If $n$ is the dimension of $E$ over $k$, then $E \cong k^n$ and $E_{\mathbb{A}} \cong \mathbb{A}_k^n$. The assertion is already proved for the case $E = k$, so it must be true in general. □

## 4.3 Alternative Approach

In the construction of the ring of adeles of finite dimensional algebras $A$ over $k$ we tried to generalize the definition of $\mathfrak{o}_\nu$. Let us consider the case where $A$ is a field $K$ (so $K$ is a finite algebraic extension of $k$). Theorem 2.5 states, that if we view $K$ as an algebra over $k$, then the construction of the adeles (as an algebra) conincides with the usual adeles when $K$ is seen as an algebraic number field. So in our notation

$$\mathbb{A}_K = K_{\mathbb{A}}$$

We used different methods when defining these rings. On the left hand side we defined the adele ring as the restricted direct product of the family $\{K_\omega\}$ with respect to the family $\{\mathfrak{O}_\omega\}$ where $\{\omega\}_\omega$ is the set of places of $K$ and $\mathfrak{O}_\omega = \{x \in K_\omega | \, |x|_\omega \leq 1\}$. On the right hand side we have to choose a basis $\alpha = \{a_1, \ldots, a_n\}$ of $K$ over $k$ and set $\alpha_\nu = a_1\mathfrak{o}_\nu \oplus \ldots \oplus a_n\mathfrak{o}_\nu$, where $\nu$ is a place of $k$. In fact one can take an arbitrary finite set $\beta$ containing a basis, as we have proved that $\alpha_\nu = \beta_\nu$ for almost all $\nu$. By choosing $\alpha \subset \mathfrak{O}_K$, where $\mathfrak{O}_K$ is the ring of integers in $K$, one immediately has

$$\alpha_\nu = a_1\mathfrak{o}_\nu \oplus \ldots \oplus a_n\mathfrak{o}_\nu = \mathfrak{O}_K \otimes_{\mathfrak{o}_k} \mathfrak{o}_\nu$$

We would like to generalize this concept.

Now let us turn to the case of a division algebra $D$ of finite dimension over $k$. We are going to need the following definitions for algebraic number fields only:

**Definition 4.1.** *Let $V$ be a finite dimensional vector space over $k_\nu$ and let $\mathfrak{o}_\nu$ be defined as usual. A full $\mathfrak{o}_\nu$-lattice in $V$ is a finitely generated $\mathfrak{o}_\nu$-submodule $M$ in $V$ such that $k_\nu \cdot M = V$. An $\mathfrak{o}_\nu$-order in the algebra $A_\nu$ is a subring $\Delta_\nu$ of $A_\nu$ having the same unity element as $A_\nu$, and such that $\Delta_\nu$ is a full $\mathfrak{o}_\nu$-lattice in $A_\nu$. A maximal $\mathfrak{o}_\nu$-order in $A_\nu$ is an $\mathfrak{o}_\nu$-order which is not properly contained in any other $\mathfrak{o}_\nu$-order in $A_\nu$. An $\mathfrak{o}_k$-order in $A$ is defined similarly.*

**Remark** If there is no risk of confusion we will write order instead of $\mathfrak{o}_\nu$-order respectively $\mathfrak{o}_k$-order.

Let $D$ be a finite dimensional division algebra over $k$. Let $D_\nu = D \otimes_k k_\nu$ for a (finite or infinite) place $\nu$ of $k$. Let $N_{D_\nu/k_\nu}$ denote the norm of the regular representation of $D_\nu$, i.e. $N_{D_\nu/k_\nu}(d)$ is the determinant of the map $a \mapsto da$ for $a, d \in D_\nu$ where $D_\nu$ is viewed as a vectorspace over $k_\nu$. Define for all finite places $\nu$ the set

$$\Delta_\nu = \{d \in D_\nu | \, |N_{D_\nu/k_\nu}(d)|_\nu \leq 1\}$$

Then we have the following theorem

**Theorem 4.4.** *Notation as above. Then $\Delta_\nu$ is the unique maximal order in $D_\nu$. Furthermore it is the integral closure of $\mathfrak{o}_\nu$ in $D$.*

A proof of this theorem can be found in [11] on page 137.

Now let us turn to the construction of adeles of a division algebra $D$. We can choose a maximal order in $D$, say $\Delta$. Let $\alpha = \{a_1, \ldots, a_n\}$ be a basis of $D$ over $k$ such that $a_i \in \Delta$ for $1 \leq i \leq n$. Then we have that

$$D_\nu = D \otimes_k k_\nu = k_\nu a_1 \oplus \ldots \oplus k_\nu a_n$$

for all finite places $\nu$ of $k$.

Now we need the following theorem:

**Theorem 4.5.** *Let $\mathfrak{o}_k$, $\mathfrak{o}_\nu$, $D$ and $\Delta$ be defined as above and set $\Delta'_\nu = \mathfrak{o}_\nu \otimes_{\mathfrak{o}_k} \Delta$. Then $\Delta'_\nu$ is a maximal $\mathfrak{o}_\nu$-order in $D_\nu$.*

This is proved more generally in [11], page 133.

As $\Delta_\nu$ is the unique maximal order of $D \otimes_k k_\nu$ we have that $\Delta'_\nu = \Delta_\nu$ for all finite places $\nu$. On the other hand we have that

$$\mathfrak{o}_\nu \otimes_{\mathfrak{o}_k} \Delta = \mathfrak{o}_\nu a_1 \oplus \ldots \oplus \mathfrak{o}_\nu a_n$$

as $\alpha \subset \Delta$. The right hand side equals $\alpha_\nu$, as it is usually defined. For an arbitrary finite set $\alpha'$ containing a basis of $D$ over $k$, we know that $\alpha_\nu = \alpha'_\nu$ for almost all $\nu$.

This shows that there are two (algebraically and topologically) equivalent constructions of the adeles of division algebras, both generalizing the construction for number fields, namely

- Let $\alpha$ be a finite subset of $D$ containing a basis of $D$ over $k$. Define $\alpha_\nu$ as usual. Then $(x_\nu)_\nu \in D_{\mathbb{A}}$ if and only if $x_\nu \in \alpha_\nu$ for almost all places $\nu$.

- Define $\Delta_\nu = \{a \in D_\nu | \, |N_{D_\nu/k_\nu}(a)|_\nu \leq 1\}$, for all finite places $\nu$. Then $(x_\nu)_\nu \in D_{\mathbb{A}}$ if and only if $x_\nu \in \Delta_\nu$ for almost all places $\nu$.

Using this we get the following theorem:

**Theorem 4.6.** *Let $D$ be a finite dimensional division algebra over $k$. For each finite place $\nu$ of $k$ let $\Delta_\nu = \{a \in D_\nu | \, |N_{D_\nu/k_\nu}(a)|_\nu \leq 1\}$. Let $S$ be a finite set of places of $k$ containing $S_\infty$ and define*

$$D_{\mathbb{A}}(S) = \prod_{\nu \in S} D_\nu \times \prod_{\nu \notin S} \Delta_\nu$$

*Then $D_{\mathbb{A}}(S)$ is an open subgroup of $D_{\mathbb{A}}$. The topology induced on it by $D_{\mathbb{A}}$ is the same as the product topology for the right-hand side. Furthermore $D_{\mathbb{A}}$ is the union of these sets.*

# 5 The Idele Group of an Algebra

In this chapter we are interested in the multiplicative structure of the adele introduced in the previous section. We can develop similar ideas as we have done for ideles of algebraic number fields. In this chapter we are mainly following [2] and [13]. Most results are generalizations of statements proved in [1] and [4].

## 5.1 Topology of the Idele Group of an Algebra

Let $A$ be a finite dimensional algebra over $k$ and let $k$ be an algebraic number field. Let $\rho$ be the regular representation of $A$, i.e. $\rho : A \to End(A)$ where $\rho(a)$ maps an element $x \in A$ onto $ax$. Then we have the following lemma

**Lemma 5.1.** *An element $a \in A$ is invertible if and only if $\rho(a)$ is in $Aut(A)$.*

*Proof.* Let $a$ be in $A^*$, where $A^*$ is the group of invertible elements of $A$. Then $\rho(a)$ has the inverse $\rho(a^{-1})$. Now if $\rho(a) \in Aut(A)$ for some $a \in A$, then $\rho(a)$ is per definition an endomorphism of $A$, viewed as a vectorspace. As it is an automorphism, it is surjective, so there exists $b \in A$ such that $\rho(a)(b) = ab = 1$, hence $a$ is invertible. $\square$

This lemma shows, that the group $A^*$ is determined by $N_{A/k}(a) := \det(\rho(a)) \neq 0$. So in the case that $k$ is a topological field, $A^*$ is an open subgroup of $A$. Moreover $\rho$ is a topological isomorphism of $A$ into a subalgebra of $End(A)$. The group $A^*$ is then a topological group for the topology induced on it by that of $A$.

Now let $A^*_{\mathbb{A}}$ be the group of invertible elements of the ring $A_{\mathbb{A}}$. As we have already seen in case $A = k$, the map $x \mapsto x^{-1}$ need not be continuous on that group for the topology induced on it by $A_{\mathbb{A}}$. We need the topology to satisfy the following properties:

1. The injection $A^*_{\mathbb{A}}$ into $A_{\mathbb{A}}$ is continuous (this immediately implies that the addition and multiplication are continuous)
2. The map $x \mapsto x^{-1}$ is continuous

We can reformulate these conditions:

**Definition 5.1.** *Let $A$ be an algebra of finite dimension over the algebraic number field $k$. Then we denote by $A^*_{\mathbb{A}}$ the group of invertible elements of $A_{\mathbb{A}}$ with the topology for which $x \mapsto (x, x^{-1})$ is a homeomorphism of $A^*_{\mathbb{A}}$ onto its image in $A_{\mathbb{A}} \times A_{\mathbb{A}}$.*

The group $A^*_{\mathbb{A}}$ with this topology is called *the idele group of* $A$ and its elements are called *the ideles of* $A$. Let $\phi$ be the homeomorphism $A^*_{\mathbb{A}}$ onto $A_{\mathbb{A}} \times A_{\mathbb{A}}$ as in the definition above. Then we can rewrite the map $x \mapsto x^{-1}$ as

$$x \xrightarrow{\phi} (x, x^{-1}) \xrightarrow{proj._2} x^{-1}$$

where $proj._2$ denotes the projection of $A_{\mathbb{A}} \times A_{\mathbb{A}}$ onto the second factor, which is trivially continuous. Obviously the mapping $(x, y) \mapsto xy$ is continuous on $A^*_{\mathbb{A}}$.

Let $f$ be the map $A \times A \to A$ defined by $(x, y) \mapsto xy$ and denote its natural extension to $A_{\mathbb{A}} \times A_{\mathbb{A}}$ with $f$ as well. Then the composition

$$A^*_{\mathbb{A}} \xrightarrow{\phi} A_{\mathbb{A}} \times A_{\mathbb{A}} \xrightarrow{f} A_{\mathbb{A}}$$

shows, that $A^*_{\mathbb{A}}$ is homeomorphic to $f^{-1}(\{1\})$. As $\phi$ and $f$ are continuous, $A^*_{\mathbb{A}}$ is closed as a subset of $A_{\mathbb{A}}$. As $A_{\mathbb{A}}$ is locally compact, $A^*_{\mathbb{A}}$ is locally compact.

**Lemma 5.2.** *The set $A^*$ of invertible elements of $A$ is a discrete subgroup of $A^*_{\mathbb{A}}$.*

*Proof.* The algebra $A^*$ is canonically embedded in $A^*_{\mathbb{A}}$ via the diagonal map. Therefore $x \mapsto (x, x^{-1})$ is mapped onto the intersection of $f^{-1}(\{1\})$ and the discrete subset $A \times A$ of $A_{\mathbb{A}} \times A_{\mathbb{A}}$. Hence it is a discrete subgroup of $A^*_{\mathbb{A}}$. $\square$

In the case when $A$ is a field we have seen, that we can also define the topology on $A^*_{\mathbb{A}}$ using the restricted direct product, respectively defining $A^*_{\mathbb{A}}$ as an infinite union of certain sets.

Let $\alpha$ be a finite subset of $A$ containing a basis of $A$ over $k$ and call $\alpha_\nu$ the $\mathfrak{o}_\nu$-module generated by $\alpha$ in $A_\nu = A \otimes_k \mathbb{A}_k$. Using Lemma 4.2 there exists a finite set $S_0$ of places of $k$ such that $\alpha_\nu$ is a compact subring of $A_\nu$ for all $\nu$ not in $S_0$.

It is clear that for all $\nu$ the set $k^*_\nu$ is open in $k_\nu$ and from above that $A^*$ is open in $A$. Hence $A^*_\nu$ is open in $A_\nu$. Furthermore $x \mapsto x^{-1}$ is continuous on $A^*_\nu$. Indeed $x \mapsto x^{-1}$ is continuous on $k_\nu$ as $k_\nu$ is a topological field and the map $x \mapsto x^{-1}$ is continuous on $A^*$ with the topology defined as above. Therefore the map $x \mapsto (x, x^{-1})$ is a homeomorphism from $A^*_\nu$ onto its image in $A_\nu \times A_\nu$. For $\nu$ not in $S_0$ the set $\alpha^*_\nu$ is by definition the set of elements of $A^*_\nu$, which is mapped into $\alpha_\nu \times \alpha_\nu$ by $x \mapsto (x, x^{-1})$. Therefore $\alpha^*_\nu$ is an open compact subgroup of $A^*_\nu$.

**Theorem 5.1.** *Let $A$, $\alpha$, $\alpha_\nu$ and $S_0$ be as explained above. Let $S$ be any finite set of places containing $S_0$. Then the group*

$$A^*_{\mathbb{A}}(S, \alpha) = \prod_{\nu \in S} A^*_\nu \times \prod_{\nu \notin S} \alpha^*_\nu$$

*is an open subgroup of $A_{\mathbb{A}}^*$. The topologies induced on it by those of $A_{\mathbb{A}}^*$ and of $A_{\mathbb{A}}$ are both the same as the product topology for the right-hand side. Furthermore $A_{\mathbb{A}}^*$ is the union of these groups.*

*Proof.* Let $A_{\mathbb{A}}(S,\alpha) = \prod_{\nu \in S} A_\nu \times \prod_{\nu \notin S} \alpha_\nu$. The topology, that $A_{\mathbb{A}}$ induces on $A_{\mathbb{A}}^*(S,\alpha)$ is the same, as the topology that $A_{\mathbb{A}}(S,\alpha)$ induces on $A_{\mathbb{A}}^*(S,\alpha)$. But the topology of $A_{\mathbb{A}}(S,\alpha)$ is the product topology of $\prod_{\nu \in S} A_\nu \times \prod_{\nu \notin S} \alpha_\nu$ so this clearly induces $\prod_{\nu \in S} A_\nu^* \times \prod_{\nu \notin S} \alpha_\nu^*$ on $A_{\mathbb{A}}(P,\alpha)^*$.

For each $\nu$, as we have seen above, $A_\nu^*$ is open in $A_\nu$ and the map $x \mapsto x^{-1}$ is continuous on $A_\nu^*$. Therefore $x \mapsto x^{-1}$ is continuous on $A_{\mathbb{A}}^*(S,\alpha)$, endowed with the product topology. This implies, that $x \mapsto x^{-1}$ is a homeomorphism of $A_{\mathbb{A}}^*(S,\alpha)$ onto $A_{\mathbb{A}} \times A_{\mathbb{A}}$. Therefore the product topology on $A_{\mathbb{A}}^*(S,\alpha)$ coincides with the topology induced by $A_{\mathbb{A}}^*$.

The set $A_{\mathbb{A}}^*(S,\alpha)$ is mapped onto $A_{\mathbb{A}}(S,\alpha) \times A_{\mathbb{A}}(S,\alpha)$ via $x \mapsto (x, x^{-1})$. But $A_{\mathbb{A}}(S,\alpha)$ is an open subgroup of $A_{\mathbb{A}}$, therefore $A_{\mathbb{A}}^*(S,\alpha)$ is open in $A_{\mathbb{A}}^*$. We also know that $A_{\mathbb{A}}$ is the union of the sets $A_{\mathbb{A}}(S,\alpha)$, and so is $A_{\mathbb{A}}^*$. This completes the proof. □

## 5.2 Properties of the Idele Group of an Algebra

We are going to investigate, whether the idele group of $A$ modulo the algebra (as an embedding into the idele group) is compact. However, as the case $A = k$ showed, we need to restrict ourselves to a closed subgroup. So we start with the following:

**Lemma 5.3.** *Let $E$ be a vectorspace of finite dimension $n$ over $k$. Put $A = End(E)$, let $\alpha$ be a finite subset of $A$ containing a basis of $A$ over $k$ and set $\alpha_\nu$ the $\mathfrak{o}_\nu$-module generated by $\alpha$ in $A_\nu$. Then there exists a finite set of places $S_0$ such that for any finite place $\nu$ of $k$ with $\nu \notin S_0$ we have:*

$$a_\nu \in \alpha_\nu \textit{ is in } \alpha_\nu^* \textit{ if and only if } |\det(a_\nu)|_\nu = 1$$

*Proof.* Let $\varepsilon$ be a basis of $E$ over $k$. Then we can identify $E$ with $k^n$ and $A$ with $M_n(k)$. Define the matrices $a_{ij}$ for $(1 \le i,j \le n)$, where $a_{ij}$ is the matrix $(x_{\lambda\mu})$ with $x_{\lambda\mu} = 1$ for $(\lambda,\mu) = (i,j)$ and $x_{\lambda\mu} = 0$ for $(\lambda,\mu) \neq (i,j)$. Then the set $\alpha' = \{a_{ij} \,|\, 1 \le i,j \le n\}$ forms a basis of $A$ over $k$. Now we have

$$A_\nu = A \otimes_k k_\nu = M_n(k) \otimes_k k_\nu \cong M_n(k_\nu)$$

The fact that $M_n(k) \otimes_k k_\nu \cong M_n(k_\nu)$ holds is proved in [5], page 308. An element $a_\nu \in M_n(k_\nu)$ is invertible if and only if $\det(a_\nu) \neq 0$. An element $a_\nu \in \alpha'_\nu = M_n(\mathfrak{o}_\nu)$ invertible in $M_n(\mathfrak{o}_\nu)$ if and only if $\det(a_\nu)$ is invertible in $\mathfrak{o}_\nu$. Indeed if $\det(a_\nu) \in \mathfrak{o}_\nu^*$ one can compute $a_\nu^{-1}$ using the adjoint matrix, which shows that $a_\nu^{-1} \in M_n(\mathfrak{o}_\nu)$. The other direction is obvious. We have that $x \in \mathfrak{o}_\nu^*$ if and only if $|x|_\nu = 1$, hence $a_\nu \in M_n(\mathfrak{o}_\nu)$ is invertible if and only if $|\det(a_\nu)|_\nu = 1$. Then Corollary 4.2 shows, that there is a finite set of places $S_0$ such that $\alpha_\nu = \alpha'_\nu$ for all $\nu$ not in $S_0$. □

**Remark** We can apply this lemma to the underlying vectorspace $E$ of $A$ and get the result that for almost all $\nu$ we have $a_\nu \in \alpha_\nu^*$ if and only if $|N_{A_\nu/k_\nu}(a_\nu)|_\nu = 1$, where $N_{A/k}$ denotes the norm of the regular representation of $A$ and $\alpha$ is some finite subset of $A$ containing a basis of $A$ over $k$.

**Definition 5.2.** *For every element $a = (a_\nu)$ in $A_{\mathbb{A}}^*$ we define*

$$\|a\| = \prod_\nu |N_{A_\nu/k_\nu}(a_\nu)|_\nu$$

*Then $\|a\|$ is called the volume or content of $a$.*

**Theorem 5.2.** *(Product formula) For all $a \in A^*$ we have*

$$\|a\| = \prod_\nu |N_{A_\nu/k_\nu}(a)|_\nu = 1$$

*Proof.* Let $\alpha$ be a basis of $A$ over $k$. Then $\alpha$ remains a basis of $A_\nu = A \otimes_k k_\nu$ over $k_\nu$. Therefore we can identify for an element $a \in A$ the map $N_{A/k}$ with $N_{A_\nu/k_\nu}$, as when using the same basis, the map $x \mapsto ax$ has the same determinant in $A$ and in $A_\nu$. For any $\xi \in A_{\mathbb{A}}^*$ we have that $(N_{A_\nu/k_\nu}(\xi_\nu))_\nu$ is an element of $\mathbb{A}_k^*$, which is clear from the remark above. Then from the Product Formula (Theorem 3.1) for $k$ we obtain

$$\|a\| = \prod_\nu |N_{A_\nu/k_\nu}(a)|_\nu = \prod_\nu |N_{A/k}(a)|_\nu = 1$$

which completes the proof. □

**Theorem 5.3.** *Let $E$ be a vectorspace of finite dimension $n$ over $k$. Put $A = End(E)$ and let $a = (a_\nu)$ be an element of $A_{\mathbb{A}}$. Then the following assertions are equivalent:*

1. *$a$ is in $A_{\mathbb{A}}^*$*
2. *$\det(a) = (\det(a_\nu))$ is in $\mathbb{A}_k^*$*
3. *the map $e \mapsto ae$ for $e \in E_{\mathbb{A}}$ is an automorphism of $E_{\mathbb{A}}$*

*When that is so, the module of the latter automorphism is $\|\det(a)\|$. Moreover, the mappings $a \mapsto \det(a)$ and $a \mapsto \|\det(a)\|$ are morphisms of $A_{\mathbb{A}}^*$ into $\mathbb{A}_k^*$ and into $\mathbb{R}_+^*$, respectively.*

*Proof.* Lemma 5.3 shows the equivalence of the first two statements. It also shows, that the map $a \mapsto \det(a)$ of $A_{\mathbb{A}}^*$ into $\mathbb{A}_k^*$ is continuous. Indeed det is continuous for every $A_\nu^*$ and hence on every $A_{\mathbb{A}}^*(S, \alpha)$ with the product topology (for a finite set $S$ of places containing $S_\infty$), where $\alpha$ is finite set in $A$ containing a basis of $A$ over $k$. As $A_{\mathbb{A}}^*$ is the union of the $A_{\mathbb{A}}^*(S, \alpha)$ we have that det is continuous on $A_{\mathbb{A}}^*$. The same argument shows that for all $\boldsymbol{\xi} \in \mathbb{I}_k$ the map $\boldsymbol{\xi} \mapsto \|\boldsymbol{\xi}\|$ is continuous.

Let $a = (a_\nu)$ be an element of $A^*_{\mathbb{A}}$. Then there exists $a^{-1} \in A^*_{\mathbb{A}}$ such that the map $e \mapsto ae$ has inverse $e \mapsto a^{-1}e$, hence it is an automorphism. We also have (using Corollary 4.1 ) that $a_\nu$ is an element of $M_n(k_\nu)$ for all $\nu$ and that $a_\nu \in M_n(\mathfrak{o}_\nu)$ for almost all $\nu$. On the other hand Theorem 4.1 applied to $E$ with basis $\varepsilon$ shows that a fundamental system of neighborhoods of 0 is given by the sets

$$U(S) = \prod_{\nu \in S} U_\nu \times \prod_{\nu \notin S} (\mathfrak{o}_\nu)^n$$

where $S$ is a finite set of places containing $S_\infty$ and $U_\nu$ is a neighborhood of 0 in $E_\nu = (k_\nu)^n$. If $a \in A^*_{\mathbb{A}}$, the map $e \mapsto ae$ is an automorphism of $E_{\mathbb{A}}$ and therefore it must map any neighborhood of 0 onto a neighborhood of 0. This shows, that $a_\nu$ is an invertible element of $M_n(k_\nu)$ for all $\nu$, as $a_\nu$ must have full rank.

For almost all $\nu$ the image of $(\mathfrak{o}_\nu)^n$ under $a_\nu$ contains $(\mathfrak{o}_\nu)^n$. The map $a_\nu^{-1}$ exists as $a_\nu \in M_n(k_\nu)^*$ and hence $a_\nu^{-1}(\mathfrak{o}_\nu^n) \subset \mathfrak{o}_\nu^n$. This shows that $a_\nu^{-1}$ is an element of $M_n(\mathfrak{o}_\nu)$ for almost all $\nu$. The same computations for the automorphism $e \mapsto a^{-1}e$ shows, that $a_\nu \in M_n(\mathfrak{o}_\nu)$ for almost all $\nu$. Hence $a \in M_n(k_\nu)^*$ for all $\nu$ and $a \in M_n(\mathfrak{o}_\nu)^*$ for almost all $\nu$. Using Theorem 5.1 shows, that $a \in A^*_{\mathbb{A}}$. This shows that the first and the third assertions are equivalent.

Define $S$ to be the set of all places $\nu$ of $k$ sucht that $a_\nu \in \alpha_\nu^*$ for all $\nu \notin S$. The set $E_{\mathbb{A}}(S, \varepsilon)$ is open in $E_{\mathbb{A}}$ by definition of the topology of $E_{\mathbb{A}}$. Furthermore it is invariant under the automorphism $e \mapsto ae$ by construction. We have that $\mathrm{mod}_{E_{\mathbb{A}}} = \mathrm{mod}_{E_{\mathbb{A}}(S,\varepsilon)}$, as the set $E_{\mathbb{A}}/E_{\mathbb{A}}(S, \varepsilon)$ is discrete. By definition $E_{\mathbb{A}}(S, \varepsilon) = \prod_{\nu \in S} E_\nu \times \prod_{\nu \notin S} \varepsilon_\nu$, so the module of the automorphism $e \mapsto ae$ is the product of the modules $e_\nu \mapsto a_\nu e_\nu$. It is well known that if $V$ is a finite dimensional vector space over a field $K$ and $M$ is an endomorphism of $V$ then $\mathrm{mod}_V(M) = \mathrm{mod}_K(\det(M))$. Applying this to $E_\nu$ and the automorphism $e_\nu \mapsto a_\nu e_\nu$ one has

$$\mathrm{mod}_{E_{\mathbb{A}}}(e \mapsto ae) = \prod_\nu \mathrm{mod}_{k_\nu}(\det(a_\nu))$$

We know that $\det(a_\nu) \in k_\nu$ for all $\nu$ so using Lemma 3.4, we get

$$\prod_\nu \mathrm{mod}_{k_\nu}(\det(a_\nu)) = \prod_\nu |\det(a_\nu)|_\nu = \|\det(a_\nu)\|$$

as required. $\square$

**Corollary 5.1.** *Let $A$ be an algebra of finite dimension over $k$, and let $a = (a_\nu)$ be an element of $A_{\mathbb{A}}$. Then the following assertions are equivalent*

1. *$a$ is in $A^*_{\mathbb{A}}$*
2. *$N_{A/k}(a) := (N_{A_\nu/k_\nu}(a_\nu))$ is in $\mathbb{A}^*_k$*
3. *the map $x \mapsto ax$ is an automorphism of the additive group of $A_{\mathbb{A}}$*

*When that is so, the module of that automorphism is* $\|N_{A/k}(a)\|$. *Moreover,* $a \mapsto N_{A/k}(a)$ *and* $a \mapsto \|N_{A/k}(a)\|$ *are morphisms of* $A^*_{\mathbb{A}}$ *into* $\mathbb{A}^*_k$ *and into* $\mathbb{R}^*_+$, *respectively.*

**Remark** This corollary is obvious in the case where $A$ is a division algebra. Using this and the approach used in the previous chapter, one gets the following:

**Corollary 5.2.** *Let* $D$ *be a division algebra of finite dimension over* $k$. *For each finite place* $\nu$ *let* $\Delta^*_\nu = \{a \in D_\nu \,|\, |N_{D_\nu/k_\nu}(a)|_\nu = 1\}$. *Let* $S$ *be a finite set of places of* $k$ *and define*

$$D_{\mathbb{A}}(S)^* = \prod_{\nu \in S} D^*_\nu \times \prod_{\nu \notin S} \Delta^*_\nu$$

*Then* $D_{\mathbb{A}}(S)^*$ *is an open subgroup of* $D_{\mathbb{A}}$. *The topology induced on it by* $D^*_{\mathbb{A}}$ *is the same as the product topology for the right-hand side. Furthermore* $D^*_{\mathbb{A}}$ *is the union of these groups.*

## 5.3 The Idele Group of a Division Algebra

All that can be said about the endomorphism $x \mapsto ax$ of an algebra $A$ holds for $x \mapsto xa$ as well. The determinant of the latter endomorphism will be called the *coregular norm*, denoted by $N'_{A/k}(a)$. We will write $N'(a)$ instead of $N'_{A/k}$ if the reference to $A$ and $k$ is clear. This is again a polynomial function of degree equal to the dimension of $A$ over $k$. As in Corollary 5.1 one can prove that the module of the automorphism $x \mapsto xa$ of $A_{\mathbb{A}}$, for $a \in A^*_{\mathbb{A}}$ is equal to $\|N'(a)\|$.

**Definition 5.3.** *For every real number* $\lambda \geq 1$, *call* $D_\lambda$ *the set of the elements* $d$ *of* $D^*_{\mathbb{A}}$ *such that the modules of the automorphisms* $x \mapsto dx$ *and* $x \mapsto xd$ *of* $D_{\mathbb{A}}$ *are respectively* $\leq \lambda$ *and* $\geq \lambda^{-1}$.

Using Theorem 5.3 we have

$$D_\lambda = \{d \in D^*_{\mathbb{A}} \,|\, N(d) \leq \lambda \text{ and } N'(d) \geq \lambda^{-1}\}$$

where $N(d) = N_{D/k}(d) = (N_{D_\nu/k_\nu}(d_\nu))$ in $\mathbb{A}_k$ denotes the regular norm, and $N'(d) = (N'_{D_\nu/k_\nu}(d_\nu))$ denotes the coregular norm of an element $d$ in $D^*_{\mathbb{A}}$. One can show that for a finite dimensional central simple algebra the norm of the regular and the coregular representation coincide. This will be done in the Appendix, chapter A. Therefore, in particular for division algebras, one gets

$$D_\lambda = \{d \in D^*_{\mathbb{A}} \,|\, \lambda^{-1} \leq N(d) \leq \lambda\}$$

In particular taking $\lambda = 1$ we can write

$$D_1 = \{d \in D^*_{\mathbb{A}} \,|\, N(d) = 1\}$$

Now we prove:

**Theorem 5.4.** *Let $D$ be a division algebra of finite dimension over $k$. Then $D_\lambda$ is a closed subset of $D_{\mathbb{A}}^*$ whose image in $D_{\mathbb{A}}^*/D^*$ is compact.*

*Proof.* Theorem 5.3 shows, that the map $d \mapsto \|N(d)\|$ is continuous. One can prove the same for $d \mapsto \|N'(d)\|$. Therefore $D_\lambda$, which is the intersection of the closed sets $\{d \in D_{\mathbb{A}}^* \,|\, \|N(d)\| \leq \lambda\}$ and $\{d \in D_{\mathbb{A}}^* \,|\, \|N'(d)\| \geq \lambda^{-1}\}$ is closed in $D_{\mathbb{A}}^*$.

Theorem 4.3 implies, that $D$ is discrete in $D_{\mathbb{A}}$ and that $D_{\mathbb{A}}/D$ is compact. So there is a Haar measure $\mu$ on $D_{\mathbb{A}}$ such that $\mu(D_{\mathbb{A}}/D) = 1$.

As $D_{\mathbb{A}}$ is not compact and so unbounded, it is possible to choose a compact subset $C$ of $D_{\mathbb{A}}$ such that $\mu(C) > \lambda$. Now we define two subsets of $D_{\mathbb{A}}$. Let $C_-$ be the image of $C \times C$ under the map $(x, y) \mapsto x - y$. So $C_- = \{x - y \in D_{\mathbb{A}} \,|\, x, y \in C\}$. As the map $(x, y) \mapsto x - y$ is continuous on $D_{\mathbb{A}} \times D_{\mathbb{A}} \to D_{\mathbb{A}}$, the set $C_-$ is compact.

Let $C_-^\times$ be the image of $C_- \times C_-$ under the map $(x, y) \mapsto xy$. For equal reasons as above $C_-^\times$ is compact.

Let $d$ be an arbitrary element of $D_\lambda$. Then $\operatorname{mod}_{D_{\mathbb{A}}}(x \mapsto xd) = N'(d) \geq \lambda^{-1}$. The automorphism $x \mapsto xd$ maps $C$ onto $Cd$. The measure of $Cd$ is given by

$$\mu(Cd) = \mu(C) \operatorname{mod}_{D_{\mathbb{A}}}(x \mapsto xd) = \mu(C)N'(d) > \lambda N'(d) \geq \lambda\lambda^{-1} = 1$$

Now let $G$ be an arbitrary locally compact group with Haar measure $\mu$ and discrete subgroup $\Gamma$ such that $G/\Gamma$ is compact. Then we can prove that for a measureable subset $X$ of $G$ with $\mu(X) > \mu(G/\Gamma)$ there exist two distinct elements $x, x' \in X$ such that $x^{-1}x' \in \Gamma$. This is proved in the Appendix, Lemma B.1. Now $D$ is a discrete subgroup of the locally compact group (written additively) $D_{\mathbb{A}}$, $D_{\mathbb{A}}/D$ is compact and $Cd$ is a compact (hence measurable) subset of $D_{\mathbb{A}}$ with measure $\mu(Cd) > \mu(D_{\mathbb{A}}/D) = 1$. Then Lemma B.1 states, that there are two elements $x, y$ in $C$ such that $xd - yd$ is in $D$ with $xd - yd \neq 0$. Hence $xd - yd$ is invertible. Write $c_1 = x - y$ and $\delta_1 = c_1 d$. Then $c_1 \in C_-$ and $\delta_1 = c_1 d = xd - yd$ is in $D^*$.

Similarly

$$\mu(d^{-1}C) = \operatorname{mod}_{D_{\mathbb{A}}}(x \mapsto d^{-1}x)\mu(C) > N(d^{-1})\mu(C) > N(d^{-1})\lambda \geq 1$$

as $N(d) \leq \lambda$. Hence we can find elements $x', y'$ in $C$ such that $d^{-1}x' - d^{-1}y' \in D^*$. Again set $c_2 = x' - y'$ and $\delta_2 = d^{-1}c_2$. Then $c_2 \in C_-$ and $\delta_2 \in D^*$.

Then $\delta_1\delta_2 = c_1 d d^{-1} c_2 = c_1 c_2$. But $\delta_1, \delta_2 \in D^*$ and $c_1, c_2 \in C_-$ so $c_1 c_2 \in C_-^\times$. Therefore $\delta_1\delta_2 \in C_-^\times \cap D^*$. We know that $D$ is a discrete subgroup and $C_-^\times$ is a compact subset of $D_{\mathbb{A}}$. Hence $D^* \cap C_-^\times$ is finite. Call $\{\gamma_1, \ldots, \gamma_s\}$ the distinct elements of $D^* \cap C_-^\times$. Then $c_1 c_2 = \delta_1\delta_2$ is equal to one of them, say $c_1 c_2 = \gamma_i$. Then $\gamma_i^{-1} c_1 c_2 = 1$, which shows, that $c_2$ is invertible in $D_{\mathbb{A}}$ and has the inverse $c_2^{-1} = \gamma_i^{-1} c_1$.

Call $X$ the subset of $D_{\mathbb{A}}^*$, that is defined by

$$X = \{x \in D_{\mathbb{A}}^* \,|\, (x, x^{-1}) \in \bigcup_{i=1}^{s} (C_- \times \gamma_i^{-1} C_-)\} = \{x \in D_{\mathbb{A}}^* \,|\, x \in C_- \text{ and } x^{-1} \in \bigcup_{i=1}^{s} \gamma_i^{-1} C_-\}$$

But we know that $c_2 = d\delta_2 \in C_-$ and $c_2^{-1} = \gamma_i^{-1} c_1$, where $c_1 \in C_-$, hence $c_2 \in X$. By the definition of the topology on $D_{\mathbb{A}}^*$, the set $X$ is compact in $D_{\mathbb{A}}^*$.

So for any $d \in D_\lambda$, we have found elements $c_2 \in X$ and $\delta_2$ in $D^*$ such that $d\delta_2 = c_2$. It follows at once that $D_\lambda \subset X \cdot D^*$. So the image of $D_\lambda$ in $D_{\mathbb{A}}^*/D^*$ is contained in the image of $X$ in $D_{\mathbb{A}}^*/D^*$. As $X$ is compact and $D_\lambda$ is closed, it follows, that $D_\lambda/D^*$ is compact. □

Obviously in the case that $A$ is commutative, one has that $N = N'$. However this turns out to be true for central simple algebras. This is proved in Corollary A.1. So in particular for division algebras one can reformulate Theorem 5.4 as follows

**Theorem 5.5.** *Let $D$ be a division algebra of finite dimension over $k$. Let $D_1$ be the set of all $d$ in $D_{\mathbb{A}}^*$ such that $\|d\| = \prod_{\nu} |N_{D_\nu/k_\nu}(d_\nu)|_\nu = 1$. Then $D_1$ is a closed subset of $D_{\mathbb{A}}^*$ and $D_1/D^*$ is compact in the quotient topology.*

## 5.4 Units of Orders of Division Algebras

Let $D$ be a finite dimensional division algebra over an algebraic number field $k$ and let $\mathfrak{o}_k$ denote the ring of integers in $k$. Furthermore for all finite places $\nu$ let $\Delta_\nu$ be the unique maximal order in $D_\nu$ defined by $\Delta_\nu = \{d \in D_\nu| \, |N_{D_\nu/k_\nu}(d)|_\nu \leq 1\}$. We define the set $I_C = \bigcap_{\nu \text{ finite}} \Delta_\nu$. Then $I_C$ is the integral closure of $\mathfrak{o}_k$ in $D$ and $I_C$ is clearly a subring of $D$, as it is the intersection of rings.

**Definition 5.4.** *Let $S$ be a finite set of valuations containing $S_\infty$. Then we define*

$$D(S) = \{d \in D| \, d \in \Delta_\nu^* \; \forall \nu \notin S\} = D \cap D_{\mathbb{A}}^*(S)$$

*where $D(S)$ is called the group of $S$-units of $D$. For $S = S_\infty$ one has that $D(S)$ is the group of units in the ring $I_C$.*

Then one can prove

**Theorem 5.6.** *The group $D(S)$ is finitely generated.*

*Proof.* We define $D_1(S) = D_1 \cap D_{\mathbb{A}}^*(S)$. Then one has, as the same argument as in the case $D = k$ shows, that $D_1(S)/D(S)$ is compact. Define $\gamma : D_{\mathbb{A}}^*(S) \to \bigoplus_{i=1}^{|S|} \mathbb{R}^+$ via $(\xi_\nu)_\nu \mapsto (\log |N_{D_\nu/k_\nu}(\xi_\nu)|_\nu)_{\nu \in S}$. Then $\gamma$ is continuous and surjective. Furthermore $\gamma(D(S))$ is discrete. Indeed for arbitrary positive numbers $c, C$ one has that the set of $d \in D(S)$

satisfying $c \leq |N_{D_\nu/k_\nu}(d)|_\nu \leq C$ is finite, as it it the intersection of the compact set $\prod_{\nu\in S}\{x \in D_\nu | c \leq |N_{D_\nu/k_\nu}(d)|_\nu \leq C\} \times \prod_{\nu\notin s} \Delta_\nu^*$ with the discrete set $D^*$.

Furthermore the kernel of $\gamma$ restricted to $D(S)$ is finite (take $c = C = 1$). One has that $\gamma(D_1(S))$ is a $|S|-1$ dimensional hyperplane and $\gamma(D_1(S)/D(S))$ is compact. Hence $\gamma(D(S))$ is a free abelian subgroup of rank $|S| - 1$. Therefore $D(S)$ is finitely generated. □

One can show:

**Theorem 5.7.** *Every element of an $\mathfrak{o}_k$-order $\Delta$ in $D$ is integral over $\mathfrak{o}_k$. Furthermore for $d \in \Delta$ the minimal polynomial and the characteristic polynomial of $d$ are in $\mathfrak{o}_k[X]$.*

This is proved in [11], page 110.

This theorem shows, that every order of $D$ is contained in $I_C$. In particular if $I_C$ is an order, it is the unique maximal order of $D$. Furthermore the unit group of an order $\Delta$ of $D$ is contained in the unit group of $I_C$. This proves:

**Theorem 5.8.** *The unit group of an order of a division algebra $D$ is finitely generated.*

# A Central Simple Algebras

The proofs and definitions of this paragraphs are in [13] and [11].

Let $k$ be a field and $A$ a finite dimensional algebra over $k$. Let $a$ be an element of $A$ and consider the endomorphism $\rho(a)$ which is defined by $x \mapsto ax$ when $A$ is viewed as a vector space over $A$. Call $End(A)$ the $k$-algebra of all endomorphisms of $A$ as a vectorspace. Then the map $\rho : A \to End(A)$ defined by $a \mapsto \rho(a)$ is a representation of $A$. It is called the regular representation. The trace and the determinant of $\rho$ are known as the regular trace and the regular norm taken in $A$ over $k$. They will be denoted by $T_{A/k}$ and $N_{A/k}$

Throughout the rest of this chapter let $A$ be a central simple algebra of finite dimension $n_0$ over a field $k$. Then $n_0 = n^2$ for an element $n \in \mathbb{N}$. This is proved in [11], page 92 and 97. Let $a \in A$ be an arbitrary element, then the characteristic polynomial of $a$ is defined as the characteristic polynomial of the regular representation of $a$. Note that

$$p_{A/k}(a) = X^m - T_{A/k}(a)X^{m-1} + \dots (-1)^m N_{A/k}(a)$$

If there is no danger of confusion, we will write $p(a)$, $N(a)$ and $T(a)$ instead of $p_{A/k}(a)$, $N_{A/k}(a)$ and $T_{A/k}(a)$. We are trying to generalize the definition of the regular representation, namely by introducing the regular norm and regular trace. The algebra $A$ is central and simple, hence there exists an extension field $E$ of $k$ which splits $A$. So there is an isomorphism

$$E \otimes_k A \cong M_n(E).$$

The characteristic polynomial of an element of $E \otimes_k A$ does not depend on the choice of this isomorphism, as all automorphisms of $M_n(E)$ are inner. Indeed if we have two isomorphisms $h, g$ from $E \otimes_k A$ to $M_n(E)$, then $h \cdot g^{-1}$ is an automorphism of $M_n(E)$, and therefore there exists an invertible element $t \in M_n(E)$, such that $h(u) = tg(u)t^{-1}$ for all $u \in E \otimes_k A$. Therefore we can define the *reduced characteristic polynomial* as

$$\text{red } p(a) := p(h(1 \otimes a)).$$

**Theorem A.1.** *For each $a \in A$, red $p(a)$ is an element of $k[X]$. It is independent of the choice of the splitting field $E$ of $A$ used to define red $p$.*

A proof of this statement can be found in [11], page 113

**Theorem A.2.** *Let $A$ be a central simple algebra of dimension $n^2$ over a field $k$. Then*

$$p(a) = (red\ p(a))^n$$

*for all $a \in A$.*

This is proved in [11], page 115.

We define $N_{red}(a)$ and $T_{red}(a)$ in the obvious way. Then we get:

**Corollary A.1.** *Let $A$ be a central simple algebra of dimension $n^2$ over a field $k$. Then for every $a \in A$ the endomorphisms $x \mapsto ax$ and $x \mapsto xa$ of the underlying vectorspace of $A$ over $k$ have both the determinant $N_{A/k}(a) = N_{red}(a)^n$.*

*Proof.* By the above theorem it is clearly enough to verify this for $E \otimes_k A$, for an suitable splitting field $E$ of $A$. But then $E \otimes_k A$ is isomorphic to $M_n(E)$. In the case that $x \mapsto ax$ (respectively $x \mapsto xa$) is an automorphism, one gets from the Skolem-Noether Theorem that these are both inner automorphisms, and so they are conjugate and have the same characteristic polynomial. If $x \mapsto ax$ (respectively $x \mapsto xa$) is not an automorphism, then both have norm 0, which proves the statement. $\square$

# B Haar measure

The first two parts of this section are due to [10] and [8]. The last part can be found in [13], page 35 and 36.

## Definition

**Definition B.1.** *A topological group $G$ is called locally compact, if it is Hausdorff and every point admits a compact neighborhood.*

A family $\mathcal{A}$ of subsets of $G$ is called a $\sigma$-algebra, if it satisfies the following conditions

1. $G \in \mathcal{A}$
2. If $A \in \mathcal{A}$ then $G \setminus A \in \mathcal{A}$
3. If $(A_n)_n \in \mathcal{A}$ then $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{A}$

A set $G$ together with such a family $\mathcal{A}$ is called a *measurable set*. Suppose that $G$ is a topological space and $\mathcal{B}$ is the smallest $\sigma$-algebra containing all open sets of $G$. Then the elements of $\mathcal{B}$ are called *Borel subsets* of $G$. A *positive measure* $\mu$ on a measurable space $G$ with $\sigma$-algebra $\mathcal{A}$ is a function $\mu : \mathcal{A} \to \mathbb{R}_+ \cup \{\infty\}$ such that $\mu\left(\bigcup\limits_{i=0}^{n} A_n\right) = \sum\limits_{i=0}^{n} \mu(A_n)$ for any family of disjoint sets with $A_i \in \mathcal{A}$ for $1 \leq i \leq n$. A positive measure defined on a Borel set is called a *Borel measure*.

Let $\mu$ be a Borel measure on a locally compact Hausdorff space $G$, let $B$ be any Borel subset and let $C$ be any open subset of $G$. If

$$\mu(B) = \inf\{\mu(U) \,|\, U \supseteq B, U \text{ open}\}$$

and

$$\mu(C) = \sup\{\mu(K) \,|\, K \subseteq C,\, K \text{ compact}\}$$

and furthermore $\mu(D)$ is finite for any compact subset $D$ of $G$, then $\mu$ is called a *Radon measure*.

Let $G$ be a locally compact topological group and let $\mu$ be a Borel measure. We say that $\mu$ is *left translation invariant* if for all Borel subsets $B$ of $G$

$$\mu(gB) = \mu(B)$$

for all $g \in G$. We can define *right translation invariance* similarly.

A left (respectively right) *Haar measure* on $G$ is a nonzero Radon measure $\mu$ on $G$ that is left (respectively, right) translation invariant.

**Theorem B.1.** *Let $G$ be a locally compact group. Then $G$ admits a left (hence right) Haar measure. Moreover this measure is unique up to a scalar multiple.*

A detailed proof of this statement can be found in [8], page 115 and 116.

**Example** For $G = \mathbb{R}$ the Haar measure $\mu$ is the usual Lebesgue measure restricted to the Borel subsets of $\mathbb{R}$. In particular for any interval $[a, b]$ with $-\infty < a < b < \infty$ one has $\mu([a, b]) = b - a$.

**Example** For $G = \mathbb{Q}_p$ one has that $\mathbb{Z}_p$ is compact. Hence one can normalize the Haar measure $\mu$ such that $\mu(\mathbb{Z}_p) = 1$. It is also clear that for arbitrary $n \geq 0$ one has that $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is finite, as it is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Hence $\mathbb{Z}_p$ is the disjoint union of the cosets of $p^n\mathbb{Z}_p$ and therefore $\mu(p^n\mathbb{Z}_p) = p^{-n}$. Furthermore $p^n\mathbb{Z}_p \cdot p^{-n}\mathbb{Z}_p = \mathbb{Z}_p$ and so $\mu(p^{-n}\mathbb{Z}_p) = p^n$. So one has for arbitrary $m \in \mathbb{Z}$ that

$$\mu(p^m\mathbb{Z}_p) = p^{-m}.$$

## The Module of an Automorphism

Let $G$ be a locally compact group with Haar-measure $\mu$ and consider an algebraic and topological isomorphism $\alpha$ of $G$. If $A \subseteq G$ is $\mu$-measurable, then $\alpha A$ is also. So we may define a new measure $\mu'$ on $G$ by

$$\mu'(A) = \mu(\alpha A)$$

Then $\mu'$ is again a left-Haar-measure on $G$ which must differ from $\mu$ by a positive factor. We will call this positive factor the *modulus of the automorphism* $\alpha$ and denote it by $\mathrm{mod}_G(\alpha)$ (respectively $\mathrm{mod}(\alpha)$ if the reference to $G$ is clear). So by definition we have

$$\mu(\alpha A) = \mathrm{mod}(\alpha)\mu(A)$$

for all $\mu$-measurable sets $A$ of $G$. The modulus is independent of the original choice of $\mu$ and it is multiplicative, in the sense that $\mathrm{mod}_G(\alpha\beta) = \mathrm{mod}_G(\alpha)\mathrm{mod}_G(\beta)$ for all automorphisms $\alpha$ and $\beta$. Suppose that $H \subseteq G$ is a normal, closed subgroup and $\alpha$ an automorphism such that $\alpha(H) = H$. Then $\alpha_1 = \alpha|_H$ is an automorphism of $H$. Also the map induced by $\alpha$ on $G/H$, we shall call it $\alpha_2$, is an automorphism of $G/H$.

**Theorem B.2.** *Notation as above. Then* $\mathrm{mod}_G(\alpha) = \mathrm{mod}_H(\alpha_1)\ \mathrm{mod}_{G/H}(\alpha_2)$.

If $G$ is compact, then $G$ is $\mu$-measurable by definition of the Haar-measure. On the other hand, $\alpha(G) = G$ for every automorphism $\alpha$. Thus, if $G$ is compact, $\mathrm{mod}_G(\alpha) = 1$ for all automorphisms $\alpha$. If $G$ is discrete, then $\{e\}$ (the identity in $G$ ) is measurable and similarly $\alpha(e) = e$ for all automorphisms, so $\mathrm{mod}_G(\alpha) = 1$.

For the rest of this chapter assume that $k$ is a locally compact field with Haar measure $\mu$. Then one has

**Theorem B.3.** *The function* $\mathrm{mod}_k : k \to \mathbb{R}_+$ *is continuous.*

This is proved in [8], page 117.

Assume that for all $x, y \in k$ the inequality

$$\mathrm{mod}_k(x + y) \leq \max\{\mathrm{mod}_k(x), \mathrm{mod}_k(y)\}$$

holds. Then we can define the following subsets of $k$

$$A = \{\, x \in k \,|\, \mathrm{mod}_k(x) \leq 1\}$$

$$A^* = \{\, x \in k \,|\, \mathrm{mod}_k(x) = 1\}$$

$$P = \{\, x \in k \,|\, \mathrm{mod}_k(x) < 1\}$$

**Theorem B.4.** *The set $A$ is the unique maximal compact subring of $k$. The subset $A^*$ of $A$ is the group of invertible elements of $A$. Then $P$ is the unique two sided ideal of $A$. The residual field $A/P$ is finite. The uniformizing parameter $\pi$ is given as any element in $k^*$ such that $\gamma = \mathrm{mod}_k(\pi)$ is the maximal element of $\mathrm{mod}_k(k)$ less than 1.*

A proof of this can be found in [13], page 13.

Let $\nu$ be a normalized (archimedean or non-archimedean) valuation of $k$ and let $k_\nu$ be the completion of $k$ with respect to $\nu$. For $x \neq 0$ in $k_\nu$ consider the map $\tilde{x}(a) = xa$. By abuse of notation we will write $\mathrm{mod}_{k_\nu}(x)$ for $\mathrm{mod}_{k_\nu}(\tilde{x})$. Then we have by definition of a normalized valuation

$$\mathrm{mod}_{k_\nu}(x) = |x|_\nu$$

In fact this characterizes the normalized valuation among equivalent ones. It also shows, that for all finite places $\nu$ the set $\mathfrak{o}_\nu$ is a maximal compact subring with unique two-sided ideal $\mathfrak{p}$.

## Discrete Subgroups

Let $G$ be a locally compact topological group with Haar measure $\mu$ and let $\Gamma$ be a discrete subgroup of $G$. Let $\phi$ be the canonical map from $G$ to $G/\Gamma$. Let $X \subset G$ be a measurable subset of $G$. Define

$$X^{-1} \cdot X = \{x^{-1}y | x, y \in X\}$$

If a subset $X$ satisfies $(X^{-1} \cdot X) \cap \Gamma = \{e\}$, where $e$ denotes the neutral element of $G$, then $\phi$ maps $X$ homeomorphic onto its image in $G$. Furthermore $gX$ is homeomorphic to $\phi(gX)$ for every $g \in G$. Indeed fix $g$ in $G$ and take elements $x, y \in X$, such that $\phi(gx) = \phi(gy)$. This is equivalent to saying that there is an element $\gamma$ in $\Gamma$ such that $gx\gamma = gy$. But that is the same as $(gx)^{-1}(gy) = \gamma$, hence $x^{-1}y \in \Gamma$, and of course by definition $x^{-1}y \in X^{-1} \cdot X$, so $x^{-1}y \in (X^{-1} \cdot X) \cap \Gamma = \{e\}$ and therefore $x = y$ and $\phi$ is injective. The fact that $\phi$ is surjective, continuous and that $\phi^{-1}$ is continuous follows directly from the defintion of $\phi$.

There exists a unique Haar measure $\mu'$ on $G/\Gamma$ such that whenever $X \subset G$ is mapped homeomorphic to $\phi(X)$, then $\mu(X)$ is equal to $\mu'(\phi(X))$. If there is no risk of confusion we will write $\mu$ instead of $\mu'$.

**Lemma B.1.** *Let $G$ be a locally compact group with Haar measure $\mu$. Let $\Gamma$ be a discrete subgroup of $G$ such that $G/\Gamma$ is compact. Let $X$ be a measurable subset of $G$ such that $\mu(X) > \mu(G/\Gamma)$. Then there are two distinct elements $x$, $x'$ of $X$ such that $x^{-1}x' \in \Gamma$.*

*Proof.* Let $X$ be a measurable subset of $G$ satisfying $(X^{-1} \cdot X) \cap \Gamma = \{e\}$. Then by the explanations above, $X$ is homeomorphic to $\phi(X)$. But then we have

$$\mu(X) = \mu(\phi(X)) = \mu(X/\Gamma) \leq \mu(G/\Gamma)$$

as $X/\Gamma \subset G/\Gamma$. But this contradicts the assumption that $\mu(X) > \mu(G/\Gamma)$.

□

Note that this is a generalization of Lemma 3.2.

# Bibliography


[1] J.W.CASSELS, A. FRÖHLICH (Editors), *Algebraic Number Theory,* Academic Press, London 1967

[2] G. FUJISAKI, *On the Zeta-Function of the Simple Algebra over the Field of Rational Numbers,* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 7 1958, 567-604

[3] L.J.GOLDSTEIN, *Analytic Number Theory,* Prentice Hall, Englewood Cliffs, N.J. 1971

[4] J.E. HUMPHREYS, *Arithmetic Groups,* Lecture Notes in Mathematics, Springer, Berlin-Heidelberg-New York 1980

[5] J.C.JANTZEN, J.SCHWERMER, *Algebra*, Springer, Heidelberg 2006

[6] E.KLEINERT, *Units in Skew Fields,* Birkhäuser, Basel-Boston-Berlin 2000

[7] S.LANG, *Algebraic Number Theory,* Springer, New York 1986

[8] L.H.LOOMIS, *An Introduction to Abstract Harmonic Analysis,* D. Van Nostrand Company, Toronto-New York-London 1953

[9] F.LORENZ, *Käte Hey und der "Hauptsatz der Algebrentheorie",* Mitteilungen der Math. Gesellschaft Hamburg 23.2, 2004, 75 - 92.

[10] D.RAMAKRISHNAN, R.J.VALENZA, *Fourier Analysis on Number Fields,* Springer, New York 1999

[11] I.REINER, *Maximal Orders,* Academic Press, London 1975

[12] P.ROQUETTE, *The Brauer-Hasse-Noether Theorem in Historical Perspective,* Schriften der Math.-naturw. Klasse der Heidelberger Akad. Wiss. 15 2004.

[13] A. WEIL, *Basic Number Theory,* Springer, Second Edition, New York 1973

# Abstract

In algebraic number theory one studies the structure of the ring of algebraic integers $\mathfrak{o}_k$ of an algebraic number field $k$. That ring is defined to be the set of elements satisfying an equation of the form $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with coefficients $a_0, \ldots a_{n-1} \in \mathbb{Z}$. Some properties of $\mathbb{Z}$ can be formulated in terms of ideals of $\mathfrak{o}_k$. First one generalizes the definition of ideals to get fractional ideals. Then one can show that the nonzero fractional ideals form a group $J_{\mathfrak{o}_k}$ under multiplication. Further this group is generated by the prime ideals. One can embed $k^*$ into the group via the map $a \mapsto (a)$. So one can form the factor group $C_k$ of $J_{\mathfrak{o}_k}$ modulo the image of $k^*$ under this mapping, which is called the ideal class group of $\mathfrak{o}_k$. One important result in algebraic number theory states that $C_k$ is finite, the number of elements in $C_k$ is an important invariant of the algebraic number field $k$ and is denoted by $h(k)$, the class number of $k$. Another important result is Dirichlets Unit Theorem. Let $\mu_k$ be the set of roots of unity in $k$. Let $r_1$ be the number of real, $2r_2$ the number of complex embeddings of $k$ into $\mathbb{C}$. Then Dirichlet Unit Theorem states that the unit group $\mathfrak{o}_k^*$ of $\mathfrak{o}_k$ is a finitely generated $\mathbb{Z}$ module. Furthemore $\mathfrak{o}_k^* \cong \mu_k \times \mathbb{Z}^{r_1+r_2-1}$.

The first part of my thesis is dedicated to the study of algebraic number fields. In doing that one uses the local-global principle, i.e. one wants to find out more about the global field $k$ by studying its local fields, where the local fields are the completions of $k$ with respect to a valuation. The natural language to do that are adeles $\mathbb{A}_k$ and ideles $\mathbb{I}_k$. There is a natural embedding of $k$ into $\mathbb{A}_k$, so one may view $k$ as a discrete subring of $\mathbb{A}_k$. When dealing with the ring of adles one can show that $\mathbb{A}_k/k$ is compact. One might expect a similar result for the group of ideles. However this turns out to be wrong. One has to restrict to the norm one ideles $\mathbb{I}_k^1$ to get $\mathbb{I}_k^1/k^*$ is compact. The compactness of $\mathbb{I}_k^1/k^*$ leads to a proof of Dirichlets Unit Theorem and one can also easily prove the finiteness of the class number of $k$.

In the second part I generalize the ideas developed in the first part to study adeles and ideles over central algebras of finite dimension over an algebraic number field. As in the case of a field, one has that the adele ring of an algebra $A$ modulo the algebra itself, viewed as a discrete subring of the adele ring, is compact. This assertion holds for arbitrary finite dimensional algebras over algebraic number fields. However when turning our attention to the group of ideles, one has to restrict to the case where $A$ is a division algebra. In doing so one can define the norm one idele $D_1$ of a division algebra $D$ and show that $D_1/D^*$ is compact. Using this result one can generalize Dirichlets Unit Theorem given and show that the unit group of an order $\Delta$ of $D$ is finitely generated.

# Zusammenfassung

Die algebraische Zahlentheorie beschäftigt sich mit der Struktur des Ringes der ganzen Zahlen $\mathfrak{o}_k$ in einem algebraischen Zahlkörper $k$. Dieser Ring ist dadurch definiert, dass er alle Zahlen $x$ aus $k$ enthält, die Nullstellen eines normierten Polynomes in $\mathbb{Z}[X]$ sind. Des weiteren verallgemeinert man die Definition von Idealen und erhält die gebrochenen Ideale. Die Menge der gebrochenen Ideale ungleich 0 bildet eine Gruppe $J_{\mathfrak{o}_k}$ bezügliche Multiplikation. Diese Gruppe ist erzeugt von den Primidealen von $\mathfrak{o}_k$, da jedes (gebrochene) Ideal eindeutig als Produkt von endlich vielen Primidealen darstellbar ist. Außerdem kann man den Zahlkörper $k$ in diese Gruppe einbetten, und zwar mithilfe der Abbildung $a \mapsto (a)$. Damit lässt sich die Faktorgruppe $C_k$ von $J_{\mathfrak{o}_k}$ modulo dem Bild von $k^*$ unter dieser Abbildung bilden. Ein wichtiges Resultat der algebraischen Zahlentheorie besagt, dass die Gruppe $C_k$ endlich ist. Die Anzahl der Elemente $C_k$ nennt man die Klassenzahl von $k$. Ein weiteres Resultat ist der Dirichletsche Einheitensatz. Es sei $r_1$ die Anzahl der reellen und $2r_2$ die Anzahl der komplexen Einbettungen von $k$ nach $\mathbb{C}$. Des weiteren bezeichnet man mit $\mu_k$ die Menge der Einheitswurzeln, die in $k$ enthalten sind. Dann besagt der Dirichletsche Einheitensatz, dass die Einheitengruppe von $\mathfrak{o}_k$ ein endlich erzeugtes $\mathbb{Z}$-Modul ist und weiteres, dass $\mathfrak{o}_k^* \cong \mu_k \times \mathbb{Z}^{r_1+r_2-1}$ gilt.

Der erste Teil meiner Diplomarbeit beschäftigt sich mit algebraischen Zahlkörpern. Um diese zu behandeln nutzt man das sogenannte Lokal-Global-Prinzip. Das heißt, man betrachtet alle Vervollständigungen eines algebraischen Zahlkörpers gleichzeitig um dabei mehr über den zugrundeliegenden Körper herauszufinden. Eine geeignete Methode hierfür sind Adele $\mathbb{A}_k$ und deren Einheitengruppe die Idele $\mathbb{I}_k$. Man kann den Körper $k$ kanonisch als diskreten Unterring der Adele über $k$ auffassen und erhält des weiteren dass $\mathbb{A}_k/k$ kompakt ist. Ebenso kann man $k^*$ als diskreten Unterring von $\mathbb{I}_k$ auffassen. Daher würden man vielleicht auch erwarten, dass $\mathbb{I}_k/k^*$ kompakt ist. Dies ist aber nicht der Fall. Um ein Kompaktheitsresultat zu erhalten, muss man sich auf die Idele mit Norm eins $\mathbb{I}_k^1$ beschränken. Dann gilt $\mathbb{I}_k^1/k^*$ ist kompakt. Aus diesem Satz lassen sich auch der Dirichletsche Einheitensatz und die Endlichkeit der Klassenzahl beweisen.

Im zweiten Teil werden die Ideen des ersten Teiles auf den Fall von endlich dimensionalen zentralen Algebren über algebraischen Zahlkörpern übertragen. Wie im Fall des Zahlkörpers erhält man, dass die Algebra $A$ ein diskreter Unterring der Adele über $A$ ist und außerdem, dass die Adele modulo der Algebra kompakt ist. Im Fall der Gruppe der Idele muss man sich, um ein analoges Kompaktheitsresultat zu erhalten, auf Divisionsalgebren $D$ beschränken.

Wiederum muss man sich auf die Idele mit Norm eins $D_1$ beschränken und erhält so dass $D_1/D^*$ kompakt ist. Damit lässt sich eine Verallgemeinerung des Dirichletschen Einheitensatzes beweisen, nämlich dass die Einheitengruppe einer Ordnung $\Delta$ einer Divisionsalgebra $D$ endlich erzeugt ist.

# Curriculum Vitae

## Personal Information

| | |
|---|---|
| Name: | Sophie Koch |
| Date of Birth: | 23.07.1988 |
| Place of Birth: | Neunkirchen |
| Citizenship: | Austria |

## Education

| | |
|---|---|
| 1994 – 1998 | Elementary School, Wölfnitz |
| 1998 – 2006 | Secondary School (Bundesrealgymnasium), Viktring |
| 7/2004 – 12/2004 | Student exchange, Pemberton, Australia |
| since 2006 | Study of Mathematics at the University of Vienna |
| 8/2008 – 1/2009 | Student exchange at the Luleå University of Technology, Sweden |
| 10/2009 – 10/2010 | Student assistant at the University of Natural Resources and Life Sciences, Vienna |
| 10/2009 – 10/2010 | Performance Scholarship (StudFG) offered by the bm:bwk |