DISSERTATION

Titel der Dissertation

"Expectations in Monetary and Fiscal Policy"

Verfasser

Diplom-Volkswirt Emanuel Maximilian Gasteiger

angestrebter akademischer Grad

Doctor of Philosophy (PhD)

Wien, im Februar 2011

Studienkennzahl lt. Studienblatt:	A 094 140
Dissertationsgebiet lt. Studienblatt:	Volkswirtschaftslehre (Economics)
Betreuer:	Univ.-Prof. Dipl.-Ing. Dr. Gerhard Sorger

To my loved ones

Contents

List of Figures

List of Tables

Acknowledgements

Accomplishing a PhD in Vienna is a formidable task and there are indeed numerous people that have a share in this achievement.

First and foremost I want to thank my supervisor Gerhard Sorger for providing inspiration, thoughtful guidance and patience for my simple mind as well as being the perfect teacher. Moreover, I would like to thank my second supervisor and teacher Robert Kunst for valuable support and comments on many details.

Among the many other people I want to mention at least a few. I owe a great debt to my friend and fellow student Torsten Lisson for the great years of our undergraduate and graduate studies. He teached me a lot of useful lessons on economics and gave numerous important remarks on earlier versions of this dissertation - now its my turn.

Likewise, I want to acknowledge the efforts of my co-authors and friends Shoujian Zhang and Matteo Fragetta. It is a great pleasure to tackle economic questions jointly with them.

A frustrating exercise was to understand what other authors are doing due to the lack of transparency that appears to be present in many papers. Reproducing results in order to assess if they are convincing and/or correct consumed most of the time it took to write down these pieces of research. One fellow student put it that way: "there are too many papers circulating, where one should write a paper which explains what is done in the other paper". Therefore I want to remark my appreciation to Seppo Honkapohja and Bennett McCallum who supported me by making their results highly transparent and to be willing to discuss details in the course of insightful correspondences. In addition, I would like to thank Chryssi Giannitsarou, Pierpaolo Benigno, Fabrizio Germano and Yasuo Hirose for responding to my requests.

Besides, the efforts of more persons have been extremely beneficial: Lutz

Weinke with his support and by teaching me some important lessons about macroeconomics. My diploma supervisor and teacher Joachim Winter for his encouragement to stick to economics. Paul Pichler for crucial comments to the numerics of the last chapter. Ulrich Woitek for giving me a primer to proper coding and answering my requests. My friends and fellow students Klaus Prettner and Andreas Brunhart amongst others for sharing their time with me during the course of my studies and beyond. Karl Schlag and the participants of the Graduate Seminar for the challenge and many meaningful annotations.

While looking back, I hope to that I can also benefit from all these peoples' experience and criticism in the future. Note, that none of these people bears responsibility for the remaining weaknesses and errors as they are solely my fault.

In addition, my life has been enriched by numerous friends, fellows and colleagues that I met on my route so far. I would like to thank them for their hospitality, great memories and the fun we had together. They give me the pleasant feeling that they care about me - so do I.

Above all, I wish to express my indebtedness to my family. Among them especially my godfather and uncle Robert who is firm as a rock. He guided my family and me through all the difficult times. Moreover, I am grateful for my band of three brothers Christoph, Georg, Philipp and their loved ones. What can I say? You make me feel home at any place! Most importantly, I want to thank my mother and my - left too early - father. I am not sure if all this was part of your master plan, but I hope it fits in. At least it is the result of your efforts, love, trust and patience. You can choose everything in life except your family - how lucky I am!

Finally, it is the time to express my gratitude to Monika for the time being and the time yet to come - any plans?

Foreword

My research interests are *Monetary Economics*, *Fiscal Policy*, *Learning Theory* and *Macroeconometrics*. During my PhD studies I got fascinated by the role of (non-)rational expectations in monetary and fiscal policy. How do individuals make economic forecasts, choices and coordinate on equilibria? Therefore, the research papers that form this dissertation examine analytically and numerically the dynamic implications of simple and optimal monetary policy, monetary and fiscal policy interactions and anticipated fiscal policy, both in homogeneous and heterogeneous expectations models.

With regard to *Monetary Economics*, I am especially interested in the central bank's policy problem. Usually, a central bank has a mandate to ensure price stability and in many countries also to stabilize output. The central bank's main policy instrument is typically the nominal interest rate. It uses this instrument to react to aggregate shocks that hit the economy. The aim is to safeguard the economy against arbitrary large fluctuations that may be driven solely by expectations or against deflationary and inflationary spirals.

Recently, a popular economic framework to study such issues has been the *New Keynesian* model. In this model prices are sticky and individuals care about the future evolution of the economy. In consequence, individuals form expectations about the future. Frequently, the central bank's role is to set its nominal interest rate according to a simple rule or to a so-called reaction function derived from optimal policy. These rules and reaction functions are usually linear in inflation and output gap. A central finding of this literature is that in most cases the central bank is able to secure price stability and non-volatile output evolution when shocks hit the economy. The result is usually true, when the central bank reacts more than one-for-one to inflation (i.e. the so-called *Taylor-principle*) and modestly to deviations in the output gap.

Most of the results in this literature are derived under the assumption of rational expectations. That is when individuals act as *homo oeconomicus* while they form expectations about the future evolution of prices and other variables. Although I personally consider this assumption to be a useful benchmark case, ever since the beginning of my studies I have been critical about this concept, especially to what extent it reflects actual economic behaviour of individuals. In particular, it requires individuals to have perfect knowledge of all systematic characteristics of an economy. One approach that has relaxed the rational expectations assumption by a less restrictive one is the *Learning Theory*. In this strand of the literature it is assumed that individuals form their expectations adaptively. This means, that individuals are thought to behave like econometricians, when they forecast the future development of prices and other variables. Up to now, in most of the studies in *Monetary Economics*, the concept of learning served as a robustness-check. By robustness-check I mean that authors usually ask whether a unique stationary equilibrium, which the central bank is able to implement under rational expectations, remains stable if agents are non-rational but learn. If so, the equilibrium is denoted *expectational stable* and learning agents can coordinate on this equilibrium.

From my perspective, this might somehow be a comparison of two extreme types of behaviour, whereas the intermediate cases are also a relevant subject to study, as empirical evidence suggests. In consequence, one may ask, whether the central bank is able to implement a unique stationary equilibrium, if individuals with different types of expectations populate the economy. Based on recent advancements in modeling *heterogeneous expectations* in the New Keynesian model, I have taken up this question in two out of three papers of my PhD thesis. These papers combine analytical derivations with numerical methods. In particular, I examine an economy in which individuals with rational and adaptive expectations coexist.

In my job market paper *"Heterogeneous Expectations and the Merit of Monetary Policy Inertia"*, which forms the first chapter, the central bank is assumed to conduct policy by simple linear interest rate rules or reaction functions based on optimal policy. My results suggest that simple rules have (dis-)advantages similar to the standard New Keynesian model, but optimal policy is quite hazardous given the central bank ignores heterogeneous expectations and interest rate stabilization. Therefore, I provide an argument in favour of the inertial responses of central banks to aggregate shocks.

The second chapter is the related paper *"Monetary and Fiscal Policy Interaction in a World with Heterogeneous Expectations"* in which I extend the New Keynesian model with heterogeneous expectations and examine interactions between simple monetary and fiscal policy. The related literature has focused on how fiscal policies can limit the central bank's ability to ensure price stability. A central finding in homogeneous expectations New Keynesian models is that the combination of so-called active monetary and passive fiscal policy or passive monetary and active fiscal policy yield determinacy and expectational stability. With regard to determinacy, I find that such clear-cut insights do not prevail in a model of heterogeneous expectations, when the central bank conducts policy according to a simple rule. I also observe that the danger to trigger explosive paths of the price level increases, although determinacy remains feasible.

The third paper of my PhD thesis *"Anticipation, Learning and Welfare: the Case of Distortionary Taxation"* is a collaboration with my fellow student *Shoujian Zhang*. It emerged from our discussion of my interest in the learning literature. In particular, we became aware of the low number of contributions that treat anticipated fiscal policy issues under the assumption of learning agents. After reading through the existing literature we listed possible issues for further research in this area and started our joint project. Most of the work was done in face-to-face work. Sometimes one of us conducted an analytical derivation alone and the other one double-checked or vice-versa. The same is true for the scripts we coded in order to present numerical results. Thus, Shoujian Zhang and me equally contributed to this project. Our paper deals with a *Fiscal Policy* problem, which is very practical, but nevertheless expectations play an important role. In particular, fiscal policy changes are usually accompanied by implementation and/or legislation lags. Therefore, the date when a fiscal policy change is announced and the date when it becomes effective do not coincide. This opens the door to anticipation effects in the dynamic responses of individuals to a policy change. By anticipation effects we mean that agents might change their economic behaviour in response to the policy change before it becomes effective. These anticipation effects have already been studied under rational expectations, but until recently have been neglected in the learning literature. We are currently aware of only two other studies in this new field. Both of them employ a basic Ramsey growth model. One major insight is that the assumption of rational expectations is not necessary for the classic Ricardian equivalence result to hold. Another important finding is that under learning the dynamic effects of anticipated lump-sum tax changes remain smooth, but are strikingly different compared to their perfect foresight counterparts. We examine the latter issue in a more elaborate version of the Ramsey model with distortionary taxation and elastic labour supply. We detect oscillatory dynamic responses to anticipated tax changes. We also compare the welfare consequences of anticipated tax changes under perfect foresight and learning and find that the magnitude of consequences of reforms is much lower under learning.

In sum, with this dissertation I attempt to provide some new insights into the role of expectations in monetary and fiscal policy and to contribute to improvements in these fields of political decision-making. Likewise, I hope that this dissertation forms the first step in establishing my place in the economic literature and qualifies me to become a good university teacher.

Chapter 1

Heterogeneous Expectations and the Merit of Monetary Policy Inertia

1.1 Motivation

Nowadays, central banks in the industrialized economies usually have a mandate to ensure price stability and in most countries to stabilize economic output. Their preferred policy instrument in many cases is the nominal interest rate. In the theoretical monetary literature it is often recommended that monetary policy should be rule-based, i.e. that the central bank sets its policy instrument according to some specific code of conduct. Therefore, monetary policy rules still appear to be a popular subject to study. Advocates of rule-based monetary policy such as Clarida et al. (1999), Woodford (2003) and Galí (2008) among others provide theoretical justification for the use of rules in the conduct of monetary policy. The core argument is that such rules may provide a nominal anchor for the economy. This means that the central bank can control nominal variables such as inflation in a way that is beneficial for individual welfare. Controlling nominal variables is important as the common transversality conditions in macroeconomic models solely rule out explosions of real variables but not of nominal variables. This issue has been reemphasized by Cochrane (2007).[1]

[1]Note that the main point of Cochrane (2007) is a serious criticism of the theories that make the case for rule-based monetary policies in general. He has initialized

In the common rules discussed in monetary economics, the policy instrument of the central bank is usually a linear function of (expected) inflation and (expected) output gap. The policy coefficients, which premultiply these two variables of interest, express the magnitude of response to deviations in the two variables from a certain policy target. Such rules offer additional advantages. They allow the central bank to apparently relate its mandate to its policy instrument, which increases policy transparency. In addition, monetary policy can become more coherent, credible, accountable and easier to communicate.

Numerous monetary policy rules have been proposed and their dynamic properties have been assessed. In recent years, most of these studies embed the rules into a New Keynesian (NK) model, where it is usually assumed that agents have rational expectations (RE). Then authors ask, whether or not a specific rule can yield local determinacy, i.e. there exists a unique stationary rational expectations equilibrium (REE).[2] Some authors also conduct a robustness-check and assume that agents may not be fully rational but learn. In particular, it is assumed that agents act as econometricians and forecast the future development of prices and other endogenous variables. It is then asked, whether or not a unique stationary equilibrium, which the central bank is able to implement under RE, remains stable if agents learn. If so, the equilibrium is denoted expectational stable and learning agents can coordinate on this equilibrium.

A classical and widely-cited analysis of monetary policy rules is Bullard and Mitra (2002), who examine monetary policy rules with regard to determinacy and E-stability.[3] They apply two widely-used methodologies to assess the dynamic properties of monetary policy rules. First, they provide analytical conditions under which a certain rule yields determinacy. Second, they numerically illustrate so-called regions of local determinacy, local indeterminacy and local explosiveness.[4] In addition, studies also label regions of E-stability. The different regions

a vivid debate on the benefit of conducting monetary policy by the help of rules in forward-looking economies that has been joined by McCallum (2009b). This debate is still in progress and is not the focus of this study.

[2]Determinacy most importantly rules out undesirable evolutions of endogenous variables such as large fluctuations, see for example Woodford (1999, p.69).

[3]When an equilibrium is denoted expectational stable it is also often denoted learnable or it is said to have the property of E-stability. These concepts are all closely related. See Evans and Honkapohja (2001) for a rigorous discussion of this strand of the literature.

[4]We consider a situation to be locally determinate, when there is a unique bounded

are usually plotted in a plane where the axes measure the monetary policy coefficients of the specific rule. From our point of view, the former method is only sensible for low dimensional economic systems, whereas the latter method may in any case be a useful method of assessment.

Bullard and Mitra (2002) find that some monetary policy rules that condition interest rates on current values of endogenous variables (output gap and inflation) are relatively good tools to enforce determinacy. In addition, such rules appear to yield E-stability for a large fraction of the considered monetary policy parameter space on top. Most important, they find that a rule featuring contemporaneous expectations instead of current values yields the same results. A rule with contemporaneous expectations requires the central bank to have less information about contemporaneous economic conditions and therefore this rule is highly operational.[5] In consequence, Bullard and Mitra (2002) conclude that such rules are the most desirable ones. In addition, Bullard and Mitra (2002) investigate the dynamic properties of rules which depend on lagged values of endogenous variables or expected future values of the same. They find that the dynamic properties of these rules are less desirable in the sense that there is much more danger to render the economy in a situation of local indeterminacy or local explosiveness.

The results of Bullard and Mitra (2002) suggest that responding more than one-for-one to inflation, i.e. sticking to the Taylor-principle[6], and responding modestly to output gap deviations is a rather good policy independent of the particular rule.

Other noteworthy studies in the tradition of Bullard and Mitra (2002) are Bullard and Mitra (2007), Preston (2005) and Duffy and Xiao (2009). Bullard and

solution, to be locally indeterminate when there are multiple bounded solutions and to be locally explosive, when there exists no locally bounded solution.

[5]Expectations in a monetary policy rule can be thought of as the central bank's forecast of a variable. It is obviously easier to use a forecast of a contemporaneous aggregate variable than to correctly observe it, as mentioned by Bullard and Mitra (2002, p.1112) and emphasized by McCallum (1999a).

[6]Taylor (1993) suggests such a simple interest rate rule and assumes an inflation coefficient of 1.5, i.e. if inflation deviates from its target level, then the central bank should react with the nominal interest rate more than one-for-one, in this case one-and-a-half-for-one. In Taylor (1999) he denotes this suggestion from 1993 (with regard to the functional form) a "normative recommendation". In Taylor (1999) he explicitly advocates an inflation coefficient larger than one in such a policy rule. This policy stance towards inflation is denoted the "Taylor-principle" in the literature.

Mitra (2007) study similar monetary policy rules as Bullard and Mitra (2002), but the rules have the additional feature of policy inertia.[7] Bullard and Mitra (2007) demonstrate that the additional feature of policy inertia can make determinacy even more likely and in turn reduce the threats of local indeterminacy or local explosiveness. In addition, Preston (2005) analyzes rules in a situation, where the entire forecast horizon of agents is explicitly considered. Preston (2005) confirms most of the results of Bullard and Mitra (2002). Duffy and Xiao (2009) examine similar rules as Bullard and Mitra (2002) in two versions of a NK economy with capital accumulation with regard to determinacy and E-stability. One version of the model is without and the other with firm-specific capital. They find that some, though not all conclusions of Bullard and Mitra (2002) and Bullard and Mitra (2007) carry over to a NK economy with capital accumulation.

A potential shortcoming of these analyses is the fact that all assume homogeneity of agents in the economy, despite the fact that heterogeneity is a universal feature in reality. Heterogeneity can have an impact on the dynamics an economy and affect the dynamic properties of monetary policy rules if structural parameters capture it. We aim to focus on heterogeneity of expectations in the economy. Agents form either RE or adaptive expectations. In particular, we focus on heterogeneous expectations in a NK model as elaborated in Branch and McGough (2009). We examine the consequences for local stability when the central bank conducts monetary policy by several simple rules or rules derived from optimal policy. Thus, the analysis herein may be viewed as a kind of robustness-check for the performance of monetary policy rules with regard to local stability when expectations are heterogeneous. We follow the numerical method by Bullard and Mitra (2002) mentioned above.

It may be of interest that we are not the first to conduct that kind of analysis. Branch and McGough (2009, p.11ff.) did so before. They find that the presence of agents with purely adaptive expectations next to fully rational agents turns policies, which used to yield indeterminacy in the case of RE, into policies that yield determinacy ("Result 3"). Furthermore, the opposite is true if the non-rational agents have extrapolative expectations ("Result 4").[8] These results emerge in a

[7]Policy inertia denotes the modern central banks' practice to alter their policy instrument with remarkable inertia in response to economic shocks.

[8]Be aware that in our context non-rational expectations are always adaptive in the sense that agents use past observations of an endogenous variable to forecast its future

situation in which the central bank sticks to a forward-looking monetary policy rule. In consequence, they conclude that purely adaptive expectations may have a stabilizing effect, whereas extrapolative expectations may have a destabilizing effect.[9] Please be aware that Branch and McGough (2009, p.10) themselves claim that they considered other rules: "... we also checked for robustness when monetary policy adopts rules that depend on lagged and contemporaneous data. The qualitative results presented below are robust to the particular form of the policy rule". Unfortunately, no further reference is made to those alternative rules therein.

Overall, we think that a more detailed study of alternative simple rules and, in addition of optimal rules in an economy with heterogeneous expectations is necessary and interesting, especially when one slightly increases the level of heterogeneity compared to Branch and McGough (2009, p.11ff.). This can serve as a more detailed robustness-check for the rules. Moreover, it allows us to shed new light on the question, how important it is, that the central bank is aware of the expectational heterogeneity when it makes its interest rate decisions.

We start our analysis by studying the rules considered in Bullard and Mitra (2002) without or with policy inertia (as in Bullard and Mitra (2007)). Our results confirm their results for some monetary policy rules, but not for all. We detect new regions of local explosiveness. In consequence, purely adaptive expectations do not yield larger regions of determinacy in general, whereas extrapolative expectations yield larger regions of indeterminacy in general. With regard to other types of simple monetary policy rules we find that contemporaneous expectations in the policy rule remains the most desirable policy specification. There are three reasons for that. First, it does not require to measure current period aggregate variables and therefore is operational. This is a well-known argument. Second, given that the central bank sticks to the Taylor-principle and moderately feeds back to contemporaneous expectations about the output gap, such a rule renders the economy determinate for the whole parameter space under consider-

value. We distinguish "purely adaptive" and "extrapolative" expectations to make clear that the weight on the past observations is smaller than one in the former case and larger than one in the latter case.

[9] We suggest to stick to a different wording with regard to stability. More precisely, we suggest to stick to the mathematical perspective, where local explosiveness means instability, local determinacy means stability and local indeterminacy means too much stability and opens the door to extrinsic uncertainty.

ation. Finally, this result holds, no matter if the central bank is actually aware of the heterogeneity of expectations in the economy or not. We also find that our conclusions obtained without policy inertia remain valid in the presence of policy inertia for most rules. It is also noteworthy that policy inertia increases the regions of determinacy remarkably. This confirms the results of Bullard and Mitra (2007). Thus, policy inertia remains a highly desirable ingredient of a simple monetary policy rule even in the case of expectational heterogeneity. This is one aspect of the merit of policy inertia.

In the second building block of our analysis, we assume that the central bank's model of the economy is incorrectly based on the assumption of homogeneous rational agents. The central bank minimizes a given quadratic loss function that punishes inflation and output gap deviations. We let the central bank solve its RE model for the optimal paths under commitment. Subsequently, we examine the implementation of this policy by either a so-called fundamentals-based or expectations-based reaction function[10], whereas the true model of the economy still features heterogeneous expectations. This part of the analysis therefore focuses on a case in which the central banks assumption about expectation formation process of agents and the actual expectation formation process of agents do not coincide. This may therefore be regarded as a kind of a robustness-check for the central bank's model of the economy. This assumption might appear naive, but may have important consequences for economic modelling. It is desirable to find an implementation for the optimal policy stance that renders the economy with heterogeneous expectations determinate although the central bank is not aware of this fact. In this case, central bankers may be able to elaborate other aspects of policy analysis in a RE version of the model and be sure to guarantee price stability given its possible structural and parameter uncertainty about adaptive expectations. The advantage of the RE version is that it is usually much easier to handle and to analyze.[11]

We find that optimal monetary policy conducted in that particular fashion is not a guarantee for determinacy in general, when the actual economy exhibits

[10] For an excellent discussion of these issues see Evans and Honkapohja (2006) and Evans and Honkapohja (2010). Their main interest is E-stability.

[11] Note that from our perspective a central bank is not necessarily required to have a model of the economy when it implements a simple rule. In contrast, for the conduct of optimal policy it usually needs to have a model of the economy including an assumption about the nature of private sector expectations.

heterogeneous expectations. It is fair to say that the central bank is quite lucky when its optimal policy yields determinacy once expectational heterogeneity is in place. Therefore, we consider conventional optimal policies implemented by reaction functions to be hazardous.

Finally, we augment the central bank's quadratic objective by a term that makes interest rate stabilization (i.e. policy inertia) a desirable target for the central bank. Similar to the case before, the central bank solves its problem under RE and is considered to implement its optimal policy via an implicit instrument rule[12] into the actual economy with heterogeneous expectations. We find that in the presence of expectational heterogeneity the implicit instrument rule appears to be a desirable way of conducting optimal policy as the outcomes are determinate for the whole parameter space considered. This is another aspect of the merit of policy inertia.

The remainder of the paper is organized as follows. In Section 1.2 we briefly describe the economic model that is the subject of our study. We also explain how we numerically analyze the dynamic properties of rules and make some comments on our calibration. Section 1.3 contains the basic analysis of the dynamic properties of four simple monetary policy rules without and with policy inertia in a NK model with heterogeneous expectations. Section 1.4 studies the dynamic properties of optimal policy implemented either by a fundamentals-based or an expectations-based reaction function. It also contains the analysis for an implicit instrument rule. Finally, Section 1.5 concludes and points out directions for further research.

1.2 The Set-Up of the Analysis

The set-up of our analysis contains the economic environment, the methodology of numerical analysis and the calibration of the economy.

[12] See Woodford (2003) and Giannoni and Woodford (2005) for detailed analyses under fully RE.

1.2.1 The Economic Environment

We assume a heterogeneous expectations reduced form NK economy as derived by Branch and McGough (2009). Aggregate demand evolves according to

$$x_t = \widehat{E}_t\{x_{t+1}\} - \sigma^{-1}\left(i_t - \widehat{E}_t\{\pi_{t+1}\}\right) \tag{1.1}$$

and aggregate supply evolves according to

$$\pi_t = \beta\widehat{E}_t\{\pi_{t+1}\} + \lambda x_t. \tag{1.2}$$

The variable x_t denotes period t aggregate output gap, i_t is the nominal interest rate controlled by the central bank and π_t is the rate of inflation. The parameter σ denotes the coefficient of relative risk aversion, which in this setting equals the inverse of the inter-temporal elasticity of substitution of private consumption. The parameter β is the common discount factor and λ is a combination of structural parameters. $\widehat{E}_t\{z_{t+1}\}$ is the heterogeneous expectations operator for any aggregate variable z_{t+1} as specified in Branch and McGough (2009, p.3).[13] Following the latter, we assume that the heterogeneous expectations operator for any aggregate variable z_t is given by

$$\widehat{E}_t\{z_{t+1}\} = \alpha E_t^1\{z_{t+1}\} + (1-\alpha)E_t^2\{z_{t+1}\}.$$

Here $\alpha \in [0,1]$ is the share of agents that are rational and $E_t^1\{z_{t+1}\} = E_t\{z_{t+1}\}$ is the RE operator. The fraction $(1-\alpha)$ is not fully rational in the sense that they form expectations by the forecasting model $E_t^2\{z_{t+1}\} = \theta E_t^2\{z_t\} = \theta^2 z_{t-1}$, where the parameter θ governs the nature of the forecast that can either be purely adaptive ($\theta < 1$) or extrapolative ($\theta > 1$). As a consequence, the aggregate

[13] Please note that Branch and McGough (2009) make use of an "axiomatic approach" and impose some assumptions that may appear restrictive to other scholars, but are a necessity to achieve the aggregate equations (1.1) and (1.2). Briefly, the assumptions that may be regarded as critical are the specification of higher order beliefs and the assumption that wealth dynamics do not matter for the evolution of aggregate variables. For a detailed discussion of these issues we refer the reader to Branch and McGough (2009).

expectations for endogenous variables are given by

$$\widehat{E}_t\{x_{t+1}\} \;=\; \alpha E_t\{x_{t+1}\} + (1-\alpha)\theta^2 x_{t-1}, \tag{1.3}$$

$$\widehat{E}_t\{\pi_{t+1}\} \;=\; \alpha E_t\{\pi_{t+1}\} + (1-\alpha)\theta^2 \pi_{t-1}. \tag{1.4}$$

In what follows, we will close the model in each subsection with a different monetary policy rule and inspect its dynamic properties in the resulting system.

From (1.3) and (1.4) it should become clear that past values of aggregate endogenous variables can affect the aggregate demand and supply when RE and adaptive expectations coexist. In consequence, monetary policy rules that perform well in pure RE models may not necessarily do so under heterogeneous expectations.

1.2.2 The Numerical Approach to the Analysis

Given a reduced form model as sketched out above and a policy rule, we will usually end up with a second-order stochastic difference system of the form

$$\mathbf{y}_t \;=\; \mathbf{A}\; E_t\{\mathbf{y}_{t+1}\} + \mathbf{C}\; \mathbf{y}_{t-1}, \tag{1.5}$$

where $\mathbf{y}_t$ is a $m\times 1$ vector of endogenous variables and matrices $\mathbf{A}$ and $\mathbf{C}$ are $m\times m$ matrices. In order to assess the dynamic properties of such a system, one may choose a solution procedure that, as a by-product, yields the eigenvalues of the system matrix. Exactly these eigenvalues characterize the system dynamics. We may either apply the solution method outlined in Blanchard and Kahn (1980) or the more general and robust purely numerical method proposed by Klein (2000). A practical advantage of the latter method is that it allows matrices $\mathbf{A}$ and $\mathbf{C}$ to be singular. Therefore, we follow this approach in the analyses below. Our particular guide is McCallum (2009a, p.13ff.). We consider solutions to a system (1.5) of the type

$$\mathbf{y}_t = \boldsymbol{\Lambda}\mathbf{y}_{t-1}, \tag{1.6}$$

where $\mathbf{\Lambda}$ is a $m \times m$ matrix. One can also think of (1.6) as the Perceived Law of Motion (PLM). In Period $t+1$ (1.6) is given by

$$\begin{aligned} E_t\{\mathbf{y}_{t+1}\} &= \mathbf{\Lambda}\mathbf{y}_t \\ &= \mathbf{\Lambda}^2\mathbf{y}_{t-1}. \end{aligned} \tag{1.7}$$

If we plug (1.7) into the original model (1.5) we get the Actual Law of motion (ALM) in the economy

$$\begin{aligned} \mathbf{y}_t &= \mathbf{A}[\mathbf{\Lambda}^2\mathbf{y}_{t-1}] + \mathbf{C}\mathbf{y}_{t-1} \\ &= [\mathbf{A}\mathbf{\Lambda}^2 + \mathbf{C}]\mathbf{y}_{t-1}. \end{aligned} \tag{1.8}$$

In a REE, the PLM has to coincide with the ALM, which is

$$\mathbf{\Lambda} \overset{!}{=} [\mathbf{A}\mathbf{\Lambda}^2 + \mathbf{C}]. \tag{1.9}$$

We can augment condition (1.9) by the matrix identity $\mathbf{\Lambda} = \mathbf{\Lambda}$ and write the two of them as

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}^2 \\ \mathbf{\Lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{I} \end{bmatrix}, \tag{1.10}$$

or more compact as

$$\bar{\mathbf{A}} \begin{bmatrix} \mathbf{\Lambda}^2 \\ \mathbf{\Lambda} \end{bmatrix} = \bar{\mathbf{C}} \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{I} \end{bmatrix}. \tag{1.11}$$

Matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$ are of dimension $2m \times 2m$. Now, we look for the so-called generalized eigenvalues (GEVs) of $\bar{\mathbf{C}}$ with respect to $\bar{\mathbf{A}}$ or equivalently for the GEVs of the matrix pencil $[\bar{\mathbf{C}} - \lambda\bar{\mathbf{A}}]$. According to the Schur generalized decomposition theorem there exist some unitary $2m \times 2m$ matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that we can decompose matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{C}}$ into the upper triangular $2m \times 2m$ matrices $\mathbf{T}$ and $\mathbf{S}$ respectively, which is $\mathbf{Q}\bar{\mathbf{C}}\mathbf{Z} = \mathbf{T}$ and $\mathbf{Q}\bar{\mathbf{A}}\mathbf{Z} = \mathbf{S}$ respectively. The GEVs of the matrix pencil $[\bar{\mathbf{C}} - \lambda\bar{\mathbf{A}}]$ are defined as the ratio of the elements of the main diagonal of $\mathbf{T}$ to the main diagonal of $\mathbf{S}$, i.e. $\lambda_i = t_{ii}/s_{ii}$. We can calculate these GEVs for any combination of the monetary policy parameters. We can count the number of GEVs, whose moduli is inside or outside the unit circle for any combination of the monetary policy coefficients. We use this information to visualize regions of local determinacy, local indeterminacy or local explosiveness in the policy space as in Bullard and Mitra (2002). In particular, at any point the policy space, where the number of GEVs whose moduli lie outside the unit circle matches the number of free variables, there is local *determinacy*. Next, when the number of GEVs whose moduli lie outside the unit circle is lower than the number of free variables we have local indeterminacy of some order. The order measures the difference between the number of free variables and the number of GEVs whose moduli lie outside the unit circle. Consequently, when the difference is one, we label that *Order 1 Indeterminacy*. This denotes a situation with a system exhibiting a one dimensional continuum of stationary equilibria. When the difference is two, we label that *Order 2 Indeterminacy*. This denotes a situation with a system exhibiting a two dimensional continuum of equilibria and so on. The idea behind is to indicate "the number of independent sunspots required to specify the solution", see Evans and McGough (2005, p.1816). Finally, when the number of GEVs whose moduli lie outside the unit circle exceeds the number of free variables there is local *explosiveness*.[14]

1.2.3 The Calibration of the Economy

In order to carry out our numerical analysis, we need to choose a calibration of the structural parameters of the model. We calibrate our model according to Table 1.1 below. If we compare these choices to Evans and Honkapohja (2006, p.22),

Parameter	**Value**	**Source**
α	$\in \{1.00, 0.60\}$	-
β	0.99	-
λ	0.024	Bullard and Mitra (2002, p.1114)
σ	0.157	Bullard and Mitra (2002, p.1114)
θ	$\in \{0.90, 1.10\}$	Branch and McGough (2009, p.11ff.)
φ_π	$\in [0.00, 2.00]$	Branch and McGough (2009, p.11ff.)
φ_x	$\in [0.00, 2.00]$	Branch and McGough (2009, p.11ff.)
φ_i	$\in \{0.00, 0.65\}$	Bullard and Mitra (2007, p.1188)

Table 1.1: Calibration of the economy.

[14]In our analysis we ignore the special case, where one or more moduli of the GEVs may lie on the unit circle.

Bullard and Mitra (2002, p.1114) and Bullard and Mitra (2007, p.1182) we find that these studies provide results for the same choices of β, λ and σ. Moreover, all three studies cover the parameter-space with regard to the monetary policy coefficients of the simple rules φ_π and φ_x in Section 1.3 below. Thus, there is a high degree of comparability of our results with the ones of popular studies. Note that the choice of the monetary policy parameter φ_i is based on empirical evidence.[15] Recall that our analysis considers expectational heterogeneity. In particular, next to the case of only rational agents ($\alpha = 1.00$), we also study the coexistence of rational and non-rational agents ($\alpha \neq 1$), which in turn puts the parameter θ into action. The latter parameter characterizes the type of non-rational expectations. Compared to Branch and McGough (2009, p.11ff.) we allow for a higher degree of heterogeneity as we choose $\alpha \in \{1.00, 0.60\}$ in our analysis. We do so, as there is evidence for heterogeneous expectations among agents in micro data that corresponds to $\alpha = 0.60$, see Branch (2004).

1.3 Dynamic Properties of the Model with Simple Monetary Policy Rules

Herein, we carry out a numerical investigation of the dynamic consequences of simple monetary policy rules without and with policy inertia. These are linear rules that condition the central bank's instrument rate on the rate of inflation and the output gap which shall reflect the central bank's mandate. We also consider policy inertia to capture the tendency of central banks to gradually alter their policy instrument.

1.3.1 Monetary Policy Rule with Contemporaneous Data

Assume, as in Bullard and Mitra (2002, sec. 3.1.) that the central bank feeds back to contemporaneous data on inflation and the output gap.[16] Such a rule

[15] We highlight any additional parameter and the related numerical choice that is introduced in the text as the analysis proceeds.

[16] Be aware that each simple rule considered herein may have some advantages and shortcomings with regard to measurement issues etc. that are not related to the dynamic properties. For a discussion of these issues, we refer the interested reader to Bullard and Mitra (2002) or McCallum (1999a).

may be of the functional form

$$i_t \;=\; \varphi_\pi \pi_t + \varphi_x x_t + \varphi_i i_{t-1}. \tag{1.12}$$

For the moment, we ignore policy inertia, i.e. we set $\varphi_i = 0.00$. We can plug this version of (1.12) into (1.1), combine the latter with (1.2) and get a system as (1.5) with the vector $\mathbf{y}_t = [x_t, \pi_t]'$ and system matrices

$$\mathbf{A} \;=\; \frac{\alpha}{(\sigma + \varphi_x + \lambda\varphi_\pi)} \begin{bmatrix} \sigma & 1 - \beta\varphi_\pi \\ \sigma\lambda & \lambda + \beta(\sigma + \varphi_x) \end{bmatrix} \tag{1.13}$$

and

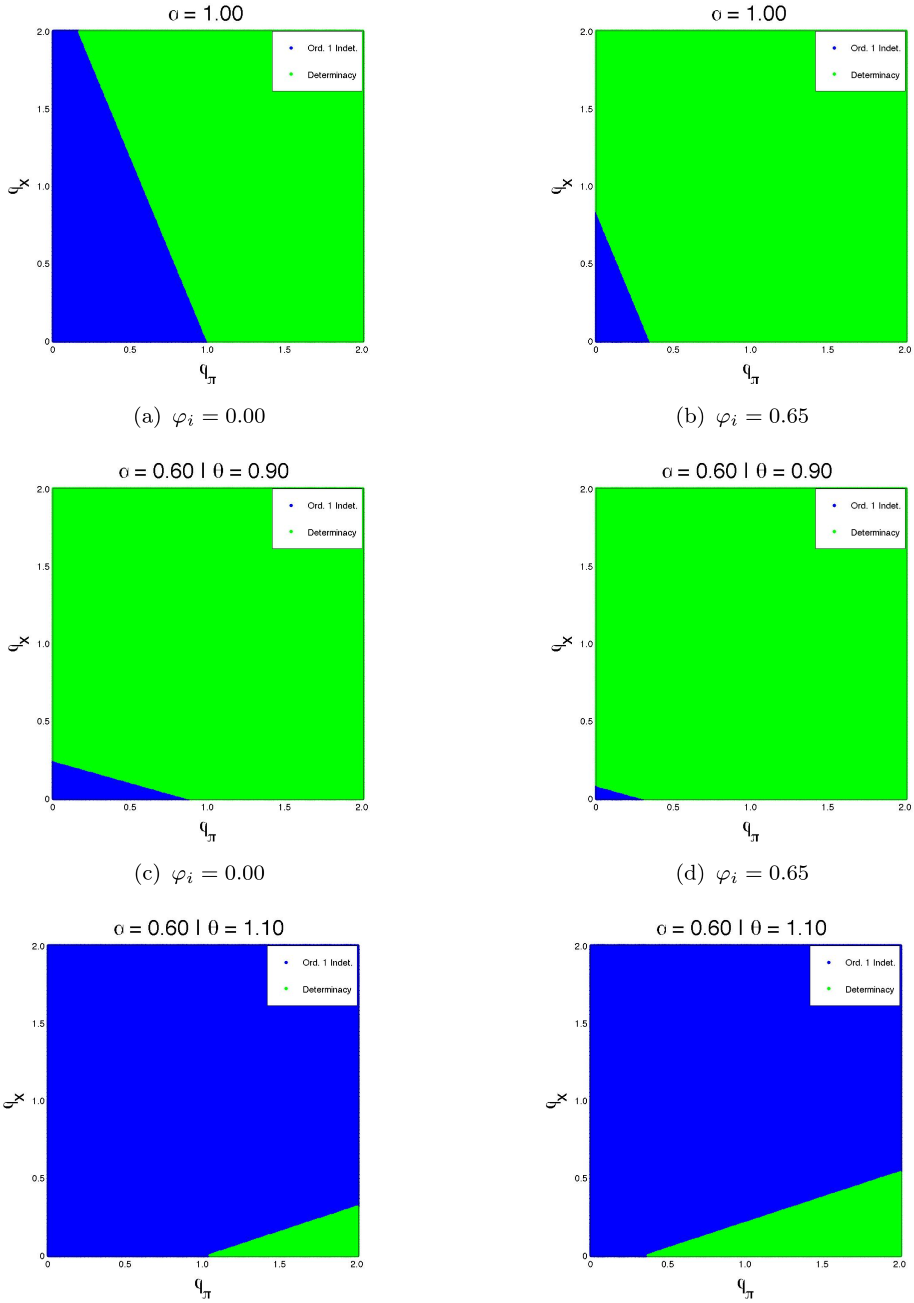
$$\mathbf{C} \;=\; \frac{(1-\alpha)\theta^2}{(\sigma + \varphi_x + \lambda\varphi_\pi)} \begin{bmatrix} \sigma & 1 - \beta\varphi_\pi \\ \sigma\lambda & \lambda + \beta(\sigma + \varphi_x) \end{bmatrix}. \tag{1.14}$$

Please be aware that with RE only ($\alpha = 1.00$) the matrix $\mathbf{C}$ is equal to zero and we are exactly in the case considered by Bullard and Mitra (2002, p.1115). In consequence, all the analytical proofs therein hold, both, with respect to determinacy and E-stability.

Now, we compare the case of homogeneous RE ($\alpha = 1.00$) to the case of heterogeneous expectations ($\alpha = 0.60$), where non-rational expectations are either purely adaptive ($\theta = 0.90$) or extrapolative ($\theta = 1.10$). Consider the numerical illustration in Figure 1.1 at the end of this subsection. Please note that in all figures below that plot regions the color-code is as follows: *red* regions label *Order 2 Indeterminacy*, *blue* regions label *Order 1 Indeterminacy*, *green* regions label *Determinacy* and *yellow* regions label *Local Explosiveness*. The horizontal axis measures the policy coefficient φ_π and the vertical axis measures the policy coefficient φ_x.

Realize that Panel 1.1(a) is nothing but an extract of Bullard and Mitra (2002, Fig.1, p.1117) and restates their numerical result with regard to determinacy. We observe that a large share of the policy space yields determinacy and the Taylor-principle yields determinacy throughout the parameter space.[17]

[17] Please note that we discuss our results in the light of the Taylor-principle as it appears to be a quite robust phenomenon that sticking to this principle yields determinacy. But be aware that this principle is not an exact and general condition (see Bullard and Mitra (2002)).



(a) $\varphi_i = 0.00$ (b) $\varphi_i = 0.65$

(c) $\varphi_i = 0.00$ (d) $\varphi_i = 0.65$

(e) $\varphi_i = 0.00$ (f) $\varphi_i = 0.65$

Figure 1.1: Regions of (in-)determinacy and explosiveness for the rule with feedback on contemporaneous data. The right column contains the results for this rule with policy inertia.

Furthermore, inspection of the differences between Panels 1.1(c) and 1.1(e) indicates two results. In case of contemporaneous data in the policy rule, where next to RE, purely adaptive expectations ($\theta = 0.90$) exist, the Taylor-principle still yields determinacy in the whole parameter space, whereas this is not true in the case of extrapolative expectations ($\theta = 1.10$). Next, the region of determinacy increases relatively to the region of indeterminacy for the case of purely adaptive expectations, whereas the reverse is true for the case of extrapolative expectations. Put differently, policies that used to lead to indeterminacy under homogeneous RE yield determinacy in the presence of purely adaptive expectations and the opposite is true in the presence of extrapolative expectations. This has been observed by Branch and McGough (2009, p.11) for a forward-looking monetary policy rule (as we will discuss in Section 1.3.3) and we can confirm that observation herein for a policy rule with contemporaneous data.

Now, consider the case with policy inertia, i.e. $\varphi_i = 0.65$. We can combine this version of (1.12) and (1.1) with (1.2) and get a system as (1.5) with the vector $\mathbf{y}_t = [x_t, \pi_t, i_t]'$ and matrices

$$\mathbf{A} = \frac{\alpha}{(\sigma + \varphi_x + \lambda\varphi_\pi)} \begin{bmatrix} \sigma & 1-\beta\varphi_\pi & 0 \\ \sigma\lambda & \lambda + \beta(\sigma + \varphi_x) & 0 \\ \sigma(\varphi_x + \varphi_\pi\lambda) & \varphi_x + \varphi_\pi(\lambda + \beta\sigma) & 0 \end{bmatrix} \tag{1.15}$$

and

$$\mathbf{C} = \frac{1}{(\sigma + \varphi_x + \lambda\varphi_\pi)} \times \begin{bmatrix} (1-\alpha)\theta^2\sigma & (1-\alpha)\theta^2(1-\beta\varphi_\pi) & -\varphi_i \\ (1-\alpha)\theta^2\sigma\lambda & (1-\alpha)\theta^2(\lambda + \beta(\sigma+\varphi_x)) & -\lambda\varphi_i \\ (1-\alpha)\theta^2\sigma(\varphi_x + \varphi_\pi\lambda) & (1-\alpha)\theta^2(\varphi_x + \varphi_\pi(\lambda+\beta\sigma)) & \sigma\varphi_i \end{bmatrix}. \tag{1.16}$$

When $\alpha = 1.00$ we are in the case of homogeneous RE. Numerical results are presented in the right column of Figure 1.1. First, compare Panel 1.1(b) to Panel 1.1(a), the case without policy inertia. One can observe that in an economy with homogeneous RE the set of policies $\{\varphi_\pi, \varphi_x\}$ that yield determinacy increases.[18] This is a result that was also observed by Bullard and Mitra (2007), but for policy

[18]Sensitivity analyses with parameter φ_i suggest that the larger the policy inertia, the larger the regions of determinacy throughout most of the cases in this study.

rules that we will to study in Sections 1.3.2 and 1.3.3 below. A comparison of Panel 1.1(d) to Panel 1.1(c) as well as Panel 1.1(f) to Panel 1.1(e) reveal that this pattern of observation is robust to heterogeneous expectations. This holds independent of the nature of the expectations of non-rational agents. Moreover, the Taylor-principle appears to be an appropriate policy recommendation in the case of homogeneous RE as well as in the case where the non-rational agents have purely adaptive expectations. Unfortunately this is not generally true, when non-rational agents have extrapolative expectations.

1.3.2 Monetary Policy Rule with Lagged Data

Next we assume, as in Bullard and Mitra (2002, sec. 3.2.) that the central bank feeds back to lagged data on inflation and the output gap, i.e.

$$i_t \;=\; \varphi_\pi \pi_{t-1} + \varphi_x x_{t-1} + \varphi_i i_{t-1}. \tag{1.17}$$

Notice that for the beginning we ignore policy inertia and set $\varphi_i = 0.00$. We combine this version of (1.17) with (1.1) and (1.2) in order to get a system as (1.5) with the vector $\mathbf{y}_t = [x_t, \pi_t]'$ and matrices[19]

$$\mathbf{A} \;=\; \alpha \begin{bmatrix} 1 & \sigma^{-1} \\ \lambda & \lambda\sigma^{-1} + \beta \end{bmatrix} \tag{1.18}$$

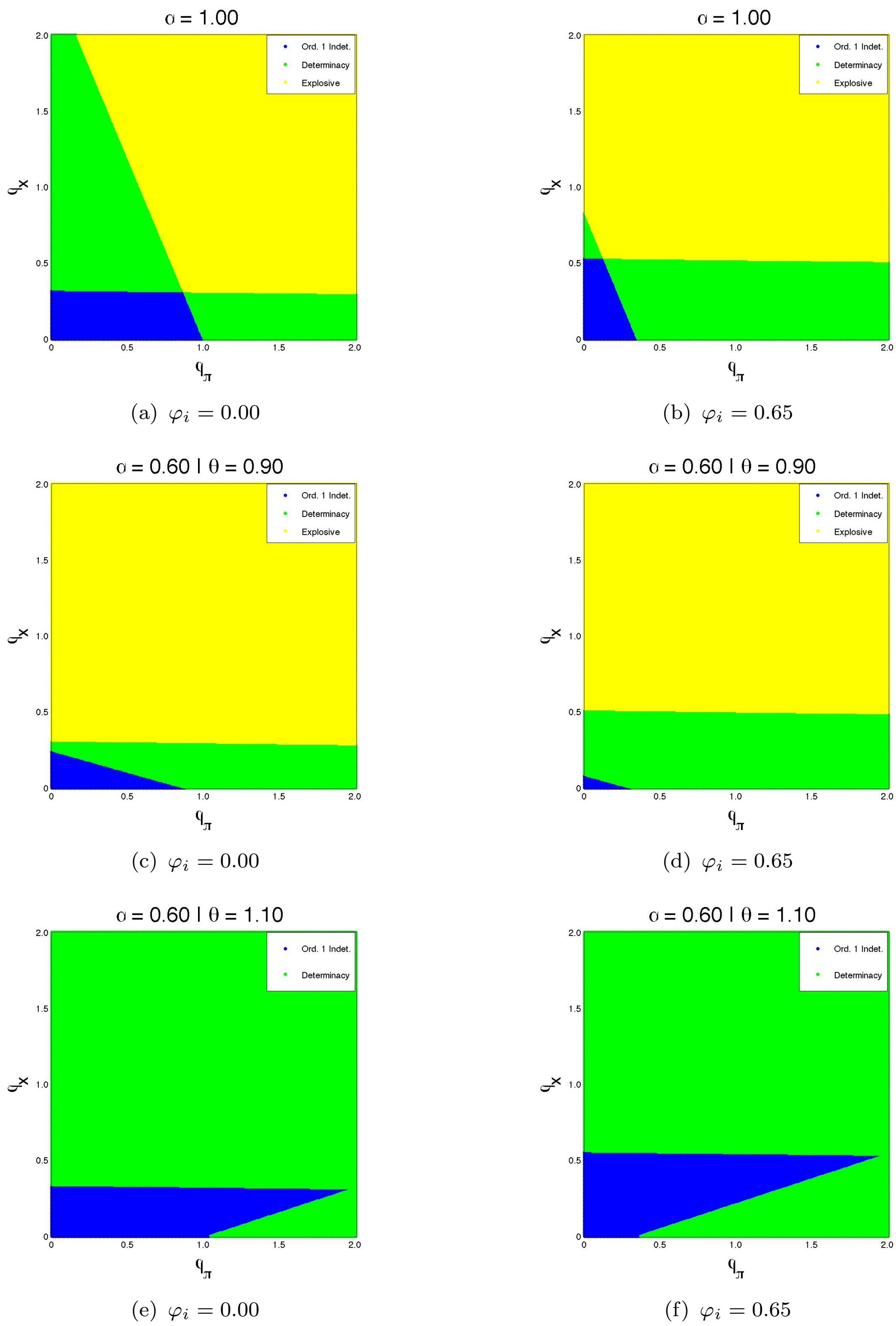
and

$$\mathbf{C} \;=\; \begin{bmatrix} (1-\alpha)\theta^2 - \varphi_x \sigma^{-1} & \sigma^{-1}[(1-\alpha)\theta^2 - \varphi_\pi] \\ \lambda[(1-\alpha)\theta^2 - \varphi_x\sigma^{-1}] & (1-\alpha)\theta^2\beta + \lambda\sigma^{-1}[(1-\alpha)\theta^2 - \varphi_\pi] \end{bmatrix}. \tag{1.19}$$

In the case when expectations are completely rational ($\alpha = 1.00$) matrix $\mathbf{C}$ is zero and we are exactly in the case of Bullard and Mitra (2002, p.1118). Thus, in this case all the analytical proofs therein hold, both, with respect to determinacy and E-stability.

When we turn to the numerical results in Figure 1.2, inspection of Panel 1.2(a)

[19]Note that Bullard and Mitra (2002, sec. 3.2.) forward (1.17) by one period and then combine it with (1.1) and (1.2) in order to get a system as (1.5) with $\mathbf{y}_t = [x_t, \pi_t, i_t]'$ for the derivation of the set of sufficient conditions and the related formal proof. Our analysis is purely numerical, and for the sake of simplicity, we eliminate as much variables as we can. The numerical results appear to be equivalent.



(a) $\varphi_i = 0.00$ (b) $\varphi_i = 0.65$

(c) $\varphi_i = 0.00$ (d) $\varphi_i = 0.65$

(e) $\varphi_i = 0.00$ (f) $\varphi_i = 0.65$

Figure 1.2: Regions of (in-)determinacy and explosiveness for the rule with feedback on lagged data. The right column contains the results for this rule with policy inertia.

makes clear that it is just an extract of Bullard and Mitra (2002, Fig.2, p.1120). We find regions of determinacy, indeterminacy and local explosiveness. In addition, the Taylor-principle only yields determinacy in case of modest feedback to output gap deviations.

Next, we observe that when non-rational agents are present and have purely adaptive expectations ($\theta = 0.90$), both the regions of determinacy and indeterminacy of order one become smaller and the region of explosiveness increases. Note that if the non-rational agents have extrapolative expectations ($\theta = 1.10$), then the reverse is true. The regions of determinacy and indeterminacy of order one increase but local explosiveness is no longer present. If we regard local explosiveness as a serious threat, then one cannot conclude that the presence of purely adaptive expectations is favourable to stability and the presence of extrapolative expectations is not. This is at odds with the numerical results in Branch and McGough (2009, p.11ff.).

Finally, there are two additional observations. First, sticking to the Taylor-principle is not a good policy in general, as it cannot rule out regions of indeterminacy or local explosiveness. Second, a policy that exclusively feeds back to the output gap ($\varphi_x \neq 0, \varphi_\pi = 0$) has the potential to yield determinacy, which is an unusual observation.

Now, we assume that the central bank favours policy inertia, which is similar to the rule studied in Bullard and Mitra (2007, p.1183ff.). We set $\varphi_i = 0.65$. This version of rule (1.17) together with equations (1.1) and (1.2) can be written as a system (1.5) with a vector $\mathbf{y}_t = [x_t, \pi_t, i_t]'$ and matrices[20]

$$\mathbf{A} = \frac{1}{(\varphi_x + \varphi_\pi \lambda - \varphi_i \sigma)} \begin{bmatrix} -\alpha\varphi_i\sigma & -\alpha(\varphi_\pi\beta + \varphi_i) & 1 \\ -\alpha\varphi_i\sigma\lambda & -\alpha[\varphi_i(\sigma\beta + \lambda) - \varphi_x\beta] & \lambda \\ \alpha\sigma(\varphi_x + \varphi_\pi\lambda) & \alpha[\varphi_x + \varphi_\pi(\sigma\beta + \lambda)] & -\sigma \end{bmatrix} \tag{1.20}$$

and

$$\mathbf{C} = \frac{(1-\alpha)\theta^2}{(\varphi_x + \varphi_\pi \lambda - \varphi_i \sigma)} \begin{bmatrix} -\varphi_i\sigma & -(\varphi_\pi\beta + \varphi_i) & 0 \\ -\varphi_i\sigma\lambda & -[\varphi_i(\sigma\beta + \lambda) - \varphi_x\beta] & 0 \\ \sigma(\varphi_x + \varphi_\pi\lambda) & [\varphi_x + \varphi_\pi(\sigma\beta + \lambda)] & 0 \end{bmatrix}. \tag{1.21}$$

[20] As in Bullard and Mitra (2007, p.1183ff.) we forward the rule by one period, before we build the system.

Again, it is an easy task to verify that for the case of homogeneous RE ($\alpha = 1.00$), we are exactly in the case of Bullard and Mitra (2007, p.1183ff.). Then their results with respect to determinacy and E-stability hold.

We present our numerical results in Figure 1.2 below. From comparison of Panel 1.2(b) to 1.2(a) it is hard to tell if the region of determinacy really increases in the case of policy inertia in an economy with homogeneous RE.[21] Furthermore, comparisons of Panel 1.2(d) to Panel 1.2(c) as well as Panel 1.2(f) to Panel 1.2(e) indicate that policy inertia does not improve the dynamic properties with regard to determinacy in general. This is only true for the case of purely adaptive expectations. In addition, with policy inertia the Taylor-principle is no suitable policy recommendation for a lagged data rule in general. Sticking to that principle cannot rule out indeterminacy or local explosiveness universally.

1.3.3 Forward-Looking Monetary Policy Rule

This section basically recapitulates the numerical analysis of Branch and McGough (2009, p.11ff.). We do so for completeness on the one hand and on the other hand because our calibration is slightly different, i.e. $\alpha \in \{1.00, 0.60\}$. We choose the latter in order to highlight the fact that heterogeneous expectations might cause local explosiveness in this specific setting. This is an observation possibly overlooked by Branch and McGough (2009, p.11ff.). Thus, similar as in Bullard and Mitra (2002, sec. 3.3.) or Branch and McGough (2009, p.11ff.) we assume that central bank feeds back to RE on period $t+1$ inflation and the output gap, i.e.

$$i_t = \varphi_\pi E_t\{\pi_{t+1}\} + \varphi_x E_t\{x_{t+1}\} + \varphi_i i_{t-1}. \tag{1.22}$$

One could also think of the expectations in the rule (1.22) as the central bank's forecast of the aggregate variables based on its period t information set. For the time being, we assume that there is no policy inertia, i.e. $\varphi_i = 0$. For the analysis, we plug (1.22) into (1.1), combine the latter with (1.2) and get a system

[21] Note that Bullard and Mitra (2007, p.1183ff.) attribute a beneficial role to policy inertia as the region that yields both determinate and E-stable outcomes increase with policy inertia.

as (1.5) with a vector $\mathbf{y}_t = [x_t, \pi_t]'$ and matrices

$$\mathbf{A} = \begin{bmatrix} \alpha - \sigma^{-1}\varphi_x & \sigma^{-1}(\alpha - \varphi_\pi) \\ \lambda(\alpha - \sigma^{-1}\varphi_x) & \alpha\beta + \lambda\sigma^{-1}(\alpha - \varphi_\pi) \end{bmatrix} \tag{1.23}$$

and

$$\mathbf{C} = (1-\alpha)\theta^2 \begin{bmatrix} 1 & \sigma^{-1} \\ \lambda & (\beta + \lambda\sigma^{-1}) \end{bmatrix}. \tag{1.24}$$

Note that for the case of RE only ($\alpha = 1.00$) the matrix $\mathbf{C}$ is equal to zero. In this case all the analytical proofs with respect to determinacy and E-stability in Bullard and Mitra (2002, p.1121) hold.

Next, consider the illustration of numerical results in Figure 1.3 below. Panel 1.3(a) is an exact reproduction of north-west panel in Branch and McGough (2009, Fig.1, p.12) which is an extract of Bullard and Mitra (2002, Fig.3, p.1123), but in the latter study, there is no distinction between indeterminacy of different orders and for that reason labels in Panel 1.1(a) are different compared to the latter.[22]

In Panel 1.1(a) we observe regions of indeterminacy of order 1 and order 2 next to regions of determinacy. In addition, it is obvious that the Taylor-principle does not hold in general, but only for modest feedback to output gap deviations.

Next, Panels 1.3(c) and 1.3(e) make clear that in presence of heterogeneous agents, regions of explosiveness may arise. Interestingly, these regions seem to originate and expand from an area around ($\varphi_\pi \approx 1, \varphi_x = 0$) with decreasing α, the fraction of non-rational agents. As a consequence, sticking too close to the Taylor-principle might turn out to be a rather dangerous policy in an economy with heterogeneous expectations. As a matter of fact, such a policy could trigger explosive paths of the price level under the rule (1.22) without policy inertia.

Our findings for this particular rule make clear that the results in Branch and McGough (2009, p.11ff.) are heavily dependent on the fraction of non-rational

[22] If one compares the two figures Branch and McGough (2009, Fig.1, p.12) and Bullard and Mitra (2002, Fig.3, p.1123), one realizes that regions of indeterminacy of order one, are found to be E-stable and regions of indeterminacy of order two, are found to be E-unstable by Bullard and Mitra (2002, p.1121ff.). From our perspective, it would be interesting to examine, whether or not there is a link between the concepts of E-stability and indeterminacy of some order.

agents. For our choice of expectational heterogeneity ($\alpha = 0.60$) explosive regions emerge for both the case of purely adaptive expectations and the case of extrapolative expectations. Therefore, one cannot claim that the former type of adaptive expectations may improve the dynamic properties with regard to determinacy in general, whereas for the latter type the opposite is true.

Finally, note from Panel 1.3(c) that in the presence of purely adaptive expectations policies that solely feed back to output gap deviations ($\varphi_x \neq 0, \varphi_\pi = 0$) again have the potential to yield determinacy. This is a rather unusual observation.

Let us get back to rule (1.22) and assume that central bank attaches importance to policy inertia as in Bullard and Mitra (2007, p.1184ff.). Then the system to analyze (1.5) has matrices

$$\mathbf{A} = \begin{bmatrix} \alpha - \sigma^{-1}\varphi_x & \sigma^{-1}(\alpha - \varphi_\pi) & 0 \\ \lambda(\alpha - \sigma^{-1}\varphi_x) & \alpha\beta + \lambda\sigma^{-1}(\alpha - \varphi_\pi) & 0 \\ \varphi_x & \varphi_\pi & 0 \end{bmatrix} \tag{1.25}$$

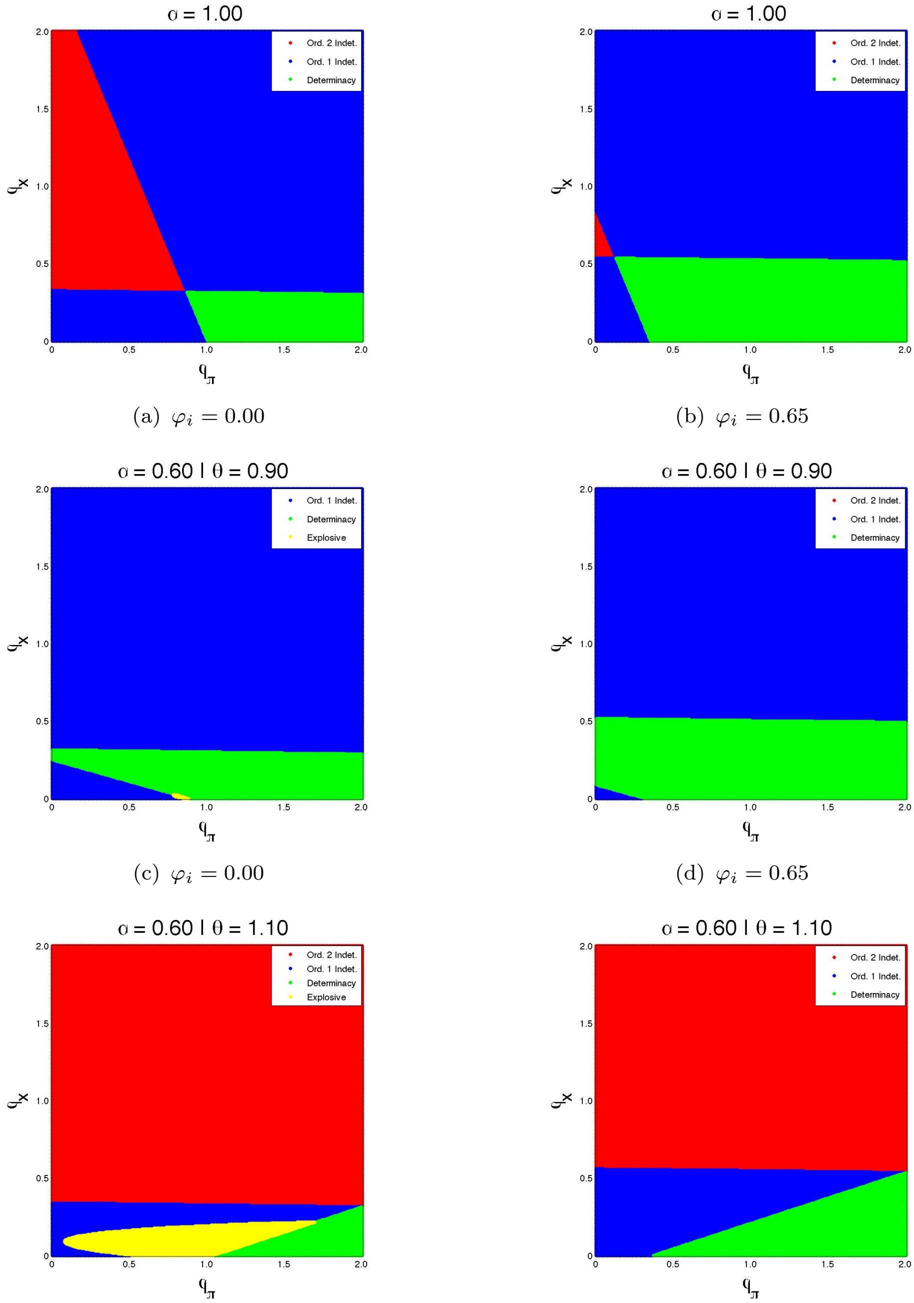
and

$$\mathbf{C} = \begin{bmatrix} (1-\alpha)\theta^2 & (1-\alpha)\theta^2\sigma^{-1} & -\varphi_i\sigma^{-1} \\ (1-\alpha)\theta^2\lambda & (1-\alpha)\theta^2(\beta + \lambda\sigma^{-1}) & -\varphi_i\sigma^{-1}\lambda \\ 0 & 0 & \varphi_i \end{bmatrix} \tag{1.26}$$

corresponding to a vector $\mathbf{y}_t = [x_t, \pi_t, i_t]'$. If there are only fully rational agents ($\alpha = 1.00$), we are exactly in the case of Bullard and Mitra (2007, p.1184ff.). Hence their results with respect to determinacy and E-stability hold. The numerical results are illustrated in the right column of Figure 1.3.

By comparing Panel 1.3(b) to Panel 1.3(a) we find that in the case of homogeneous RE the region of determinacy increases. This pattern remains stable for the case of heterogeneous expectations, independent of the nature of expectations of non-rational agents as Panels 1.3(d) and 1.3(f) reveal. Most notably, policy inertia eliminates regions of local explosiveness in the case of heterogeneous expectations. Moreover, the Taylor-principle does not hold in general as in the case without policy inertia.

A priori, it is not clear, why the central bank should feedback to RE of aggregate variables. It may simply do so, because it assumes a pure RE model of



(a) $\varphi_i = 0.00$

(b) $\varphi_i = 0.65$

(c) $\varphi_i = 0.00$

(d) $\varphi_i = 0.65$

(e) $\varphi_i = 0.00$

(f) $\varphi_i = 0.65$

Figure 1.3: Regions of (in-)determinacy and explosiveness for the rule with feedback on expectations of period $t+1$ values. The right column contains the results for this rule with policy inertia.

the economy. Alternatively, as Branch and McGough (2009, p.9) propose that one could assume that the central bank is aware of the exact nature of heterogeneous expectations and conditions its instrument on these expectations, which is

$$i_t \;=\; \varphi_\pi \widehat{E}_t\{\pi_{t+1}\} + \varphi_x \widehat{E}_t\{x_{t+1}\} + \varphi_i i_{t-1}. \tag{1.27}$$

From our perspective, this appears to be a strong assumption in practice. We presume that tracking the exact shares (α) of agents with their different types (γ) of expectations demands a non-negligible effort from central banks. This may come at large information costs. Nevertheless, it is of interest, whether or not the potential benefit of such a rule could justify the costs. As before, we start with rule (1.27) without considering policy inertia ($\varphi_i = 0$). This leads to a system with a vector $\mathbf{y}_t = [x_t, \pi_t]'$ and matrices

$$\mathbf{A} \;=\; \alpha \begin{bmatrix} 1-\sigma^{-1}\varphi_x & \sigma^{-1}(1-\varphi_\pi) \\ \lambda(1-\sigma^{-1}\varphi_x) & \beta+\lambda\sigma^{-1}(1-\varphi_\pi) \end{bmatrix} \tag{1.28}$$

and

$$\mathbf{C} \;=\; (1-\alpha)\theta^2 \begin{bmatrix} 1-\sigma^{-1}\varphi_x & \sigma^{-1}(1-\varphi_\pi) \\ \lambda(1-\sigma^{-1}\varphi_x) & \beta+\lambda\sigma^{-1}(1-\varphi_\pi) \end{bmatrix}. \tag{1.29}$$

Obviously we end up in the case of Bullard and Mitra (2002, p.1121) if we set $\alpha = 1.00$. In this case, all the analytical proofs with respect to determinacy and E-stability therein hold. Our numerical results are outlined in the left column of Figure 1.4 below.

Panel 1.4(a) does coincide with Panel 1.3(a) by construction. But how do things change once expectational heterogeneity is in place? We observe that the locally explosive regions in Panels 1.3(c) and 1.3(e) are not longer present in Panels 1.4(c) and Panel 1.4(e). Thus, it is evident that when the central bank makes use of a monetary policy rule featuring feedback on heterogeneous expectations, it may at least be able to rule out explosive paths of nominal variables. With regard to indeterminacy the results for rules (1.22) and (1.27) appear to be observationally equivalent in the absence of policy inertia.

Now, we may again ask how policy inertia in rule (1.27) affects the dynamics.

Then, the system (1.5) with vector $\mathbf{y}_t = [x_t, \pi_t, i_t]'$ has matrices

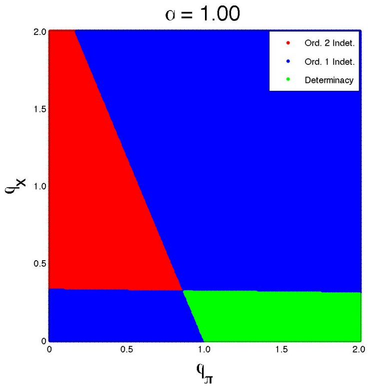
$$\mathbf{A} \;=\; \alpha \begin{bmatrix} 1-\sigma^{-1}\varphi_x & \sigma^{-1}(1-\varphi_\pi) & 0 \\ \lambda(1-\sigma^{-1}\varphi_x) & \beta+\lambda\sigma^{-1}(1-\varphi_\pi) & 0 \\ \varphi_x & \varphi_\pi & 0 \end{bmatrix} \tag{1.30}$$

and

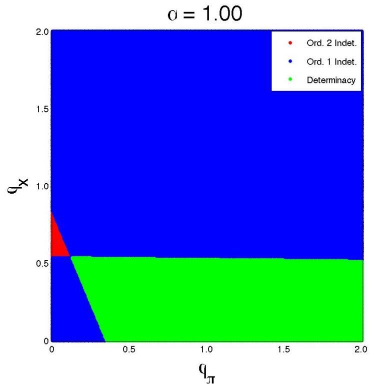
$$\mathbf{C} = \begin{bmatrix} (1-\alpha)\theta^2(1-\sigma^{-1}\varphi_x) & (1-\alpha)\theta^2\sigma^{-1}(1-\varphi_\pi) & -\varphi_i\sigma^{-1} \\ (1-\alpha)\theta^2[\lambda(1-\sigma^{-1}\varphi_x)] & (1-\alpha)\theta^2[\beta+\lambda\sigma^{-1}(1-\varphi_\pi)] & -\varphi_i\sigma^{-1}\lambda \\ (1-\alpha)\theta^2\varphi_x & (1-\alpha)\theta^2\varphi_\pi & \varphi_i \end{bmatrix}. \tag{1.31}$$

Results are displayed in the right column of Figure 1.4. Panels 1.4(b), 1.4(d) and 1.4(f) reveal that at least qualitatively the results do no change compared to the situation, where the central bank is not aware of expectational heterogeneity.

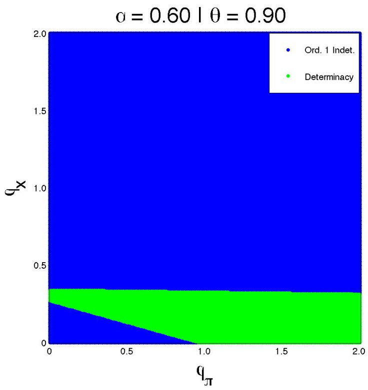
The observations in this section suggest that if a forward-looking rule is in place there are two ways of ruling out local explosiveness. One way is to track the exact nature of expectations as is done by rule (1.27). The second way is to simply add policy inertia to rule (1.22). The latter option is less costly with regard to information and may therefore be preferred by central banks that implement a forward-looking instrument rule. This is de facto another merit of policy inertia.



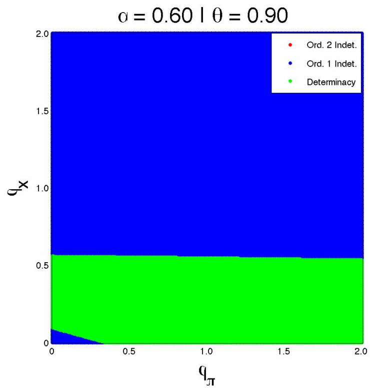
(a) $\varphi_i = 0.00$



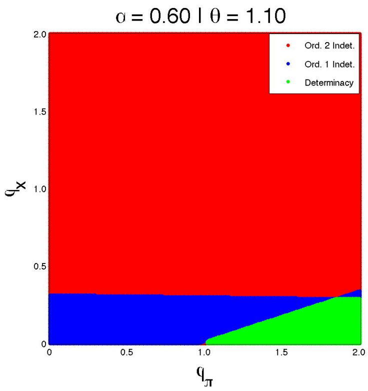
(b) $\varphi_i = 0.65$



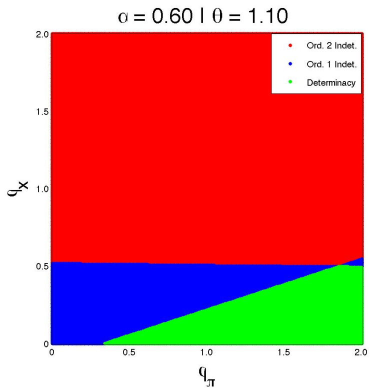
(c) $\varphi_i = 0.00$



(d) $\varphi_i = 0.65$



(e) $\varphi_i = 0.00$



(f) $\varphi_i = 0.65$

Figure 1.4: Regions of (in-)determinacy and explosiveness for the rule with feedback on heterogeneous expectations of period $t + 1$ values. The right column contains the results for this rule with policy inertia.

1.3.4 Monetary Policy Rule with Contemporaneous Expectations

The final simple rule we are going to consider is the one in which the central bank feeds back to contemporaneous expectations on inflation and the output gap as in Bullard and Mitra (2002, sec. 3.4.), i.e.

$$i_t \;=\; \varphi_\pi E_t\{\pi_t\} + \varphi_x E_t\{x_t\} + \varphi_i i_{t-1}. \tag{1.32}$$

One can motivate such a rule by the fact that real time data of aggregate variables usually are not available for central bankers or only with high imprecision and it may be far more realistic to assume that the policy makers feed back to their RE forecast of period t variables, rather than actual contemporaneous data. In such a situation, the information set of the central bank contains observations up to period $t-1$. In order to ensure symmetry in information sets, we follow Bullard and Mitra (2002, sec. 3.4.) and assume that policy makers as well as agents in the economy form expectations with an information set as of period $t-1$.[23] Otherwise private sector agents would observe more data then the central bank. Thus, our economy now evolves according to

$$x_t \;=\; \widehat{E}_{t-1}\{x_{t+1}\} - \sigma^{-1}\left(i_t - \widehat{E}_{t-1}\{\pi_{t+1}\}\right) \tag{1.33}$$

and

$$\pi_t \;=\; \beta\widehat{E}_{t-1}\{\pi_{t+1}\} + \lambda x_t. \tag{1.34}$$

The average expectations of aggregate variables are now given by

$$\widehat{E}_{t-1}\{x_{t+1}\} \;=\; \alpha E_{t-1}\{x_{t+1}\} + (1-\alpha)\theta^2 x_{t-1} \tag{1.35}$$

$$\widehat{E}_{t-1}\{\pi_{t+1}\} \;=\; \alpha E_{t-1}\{\pi_{t+1}\} + (1-\alpha)\theta^2 \pi_{t-1} \tag{1.36}$$

[23] From our understanding the assumptions in Branch and McGough (2009, sec. 2.1.) are general enough to allow for a change in the timing of expectations.

instead of (1.3) and (1.4). Finally, (1.32) is transformed to

$$i_t \;=\; \varphi_\pi E_{t-1}\{\pi_t\} + \varphi_x E_{t-1}\{x_t\} + \varphi_i i_{t-1}. \tag{1.37}$$

We can rewrite the resulting system (1.33)-(1.37) as

$$\mathbf{A}_0\ \mathbf{s}_t \;=\; \mathbf{A}_1\ E_{t-1}\{\mathbf{s}_t\} + \mathbf{A}_2\ E_{t-1}\{\mathbf{s}_{t+1}\} + \mathbf{A}_3\ \mathbf{s}_{t-1}, \tag{1.38}$$

where $\mathbf{s}_t = [x_t, \pi_t]'$ is a $p \times 1$ vector and matrices are given by

$$\mathbf{A}_0 \;=\; \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}, \tag{1.39}$$

$$\mathbf{A}_1 \;=\; \begin{bmatrix} -\varphi_x\sigma^{-1} & -\varphi_\pi\sigma^{-1} \\ 0 & 0 \end{bmatrix}, \tag{1.40}$$

$$\mathbf{A}_2 \;=\; \begin{bmatrix} \alpha & \sigma^{-1}\alpha \\ 0 & \beta\alpha \end{bmatrix} \tag{1.41}$$

and

$$\mathbf{A}_3 \;=\; \begin{bmatrix} (1-\alpha)\theta^2 & \sigma^{-1}(1-\alpha)\theta^2 \\ 0 & \beta(1-\alpha)\theta^2 \end{bmatrix}. \tag{1.42}$$

In order to bring this system into our standard form (1.5), we follow Binder and Pesaran (1999, p.140ff.) as (1.38) matches their general multivariate structural RE model

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \mathbf{M}_{ij}\ E(\mathbf{s}_{t+j-i}|\mathbf{\Omega}_{t-i}) = \mathbf{0}. \tag{1.43}$$

The matrices $\mathbf{M}_{ij}$ are of dimension $p \times p$ and the vectors $\mathbf{s}_{t+j-i}$ are of dimension $p \times 1$. $\mathbf{\Omega}_{t-i}$ is the non-decreasing information set. In our specific case it is convenient to consider two lags $n_1 = 2$ and two leads $n_2 = 2$, thus

$$\begin{aligned}\mathbf{0} &= \mathbf{M}_{00}\ \mathbf{s}_t + \mathbf{M}_{01}\ E_t\{\mathbf{s}_{t+1}\} + \mathbf{M}_{02}\ E_t\{\mathbf{s}_{t+2}\} + \mathbf{M}_{10}\ \mathbf{s}_{t-1} + \mathbf{M}_{20}\ \mathbf{s}_{t-2} \\ &\quad + \mathbf{M}_{11}\ E_{t-1}\{\mathbf{s}_t\} + \mathbf{M}_{21}\ E_{t-2}\{\mathbf{s}_{t-1}\} \\ &\quad + \mathbf{M}_{12}\ E_{t-1}\{\mathbf{s}_{t+1}\} + \mathbf{M}_{22}\ E_{t-2}\{\mathbf{s}_t\}. \end{aligned} \tag{1.44}$$

Note that $\mathbf{M}_{00} = -\mathbf{A}_0$, $\mathbf{M}_{10} = \mathbf{A}_3$, $\mathbf{M}_{11} = \mathbf{A}_1$, $\mathbf{M}_{12} = \mathbf{A}_2$ and $\mathbf{0}_2 = \mathbf{M}_{01} = \mathbf{M}_{02} = \mathbf{M}_{20} = \mathbf{M}_{21} = \mathbf{M}_{22}$. Next, we can recast the latter expression as

$$\begin{aligned}\mathbf{0} &= \begin{bmatrix} \mathbf{M}_{00} & \mathbf{M}_{01} & \mathbf{M}_{02} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{s}_t \\ E_t\mathbf{s}_{t+1} \\ E_t\mathbf{s}_{t+2} \end{bmatrix} + \begin{bmatrix} \mathbf{M}_{10} & \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s}_{t-1} \\ E_{t-1}\mathbf{s}_t \\ E_{t-1}\mathbf{s}_{t+1} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{M}_{20} & \mathbf{M}_{21} & \mathbf{M}_{22} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s}_{t-2} \\ E_{t-2}\mathbf{s}_{t-1} \\ E_{t-2}\mathbf{s}_t \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} E_t\mathbf{s}_{t+1} \\ E_t\mathbf{s}_{t+2} \\ E_t\mathbf{s}_{t+3} \end{bmatrix}\end{aligned}$$

or with $\mathbf{z}_t = [\mathbf{s}'_t, E_t\mathbf{s}'_{t+1}, E_t\mathbf{s}'_{t+2}]'$ more compact as

$$\mathbf{0} = \mathbf{\Gamma}_0\ \mathbf{z}_t + \mathbf{\Gamma}_1\ \mathbf{z}_{t-1} + \mathbf{\Gamma}_2\ \mathbf{z}_{t-2} + \mathbf{\Gamma}_{-1}\ E_t\mathbf{z}_{t+1}. \tag{1.45}$$

Now, we can rewrite equation (1.45) as

$$\mathbf{0} = \begin{bmatrix} \mathbf{\Gamma}_0 & \mathbf{\Gamma}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{z}_{t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_2 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{t-1} \\ \mathbf{z}_{t-2} \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} E_t\mathbf{z}_{t+1} \\ \mathbf{z}_t \end{bmatrix},$$

or by defining $\mathbf{y}_t = [\mathbf{z}'_t, \mathbf{z}'_{t-1}]'$ more compactly as a second-order stochastic difference system, which in general can be written as

$$\begin{aligned}\mathbf{\Lambda}_0\ \mathbf{y}_t &= -\mathbf{\Lambda}_{-1}\ E_t\{\mathbf{y}_{t+1}\} - \mathbf{\Lambda}_1\ \mathbf{y}_{t-1} \\ \mathbf{y}_t &= -\mathbf{\Lambda}_0^{-1}\mathbf{\Lambda}_{-1}\ E_t\{\mathbf{y}_{t+1}\} - \mathbf{\Lambda}_0^{-1}\mathbf{\Lambda}_1\ \mathbf{y}_{t-1} \\ \mathbf{y}_t &= \mathbf{A}\ E_t\{\mathbf{y}_{t+1}\} + \mathbf{C}\ \mathbf{y}_{t-1}.\end{aligned} \tag{1.46}$$

This is the same as our standard form (1.5).[24] The numerical results appear to be observationally similar to the left column of Figure 1.1 above for the rule with

[24] $\mathbf{\Lambda}_0$ is non-singular and invertible as matrices $\mathbf{\Gamma}_0$ and $\mathbf{A}_0$ are non-singular. We omit matrices $\mathbf{A}$ and $\mathbf{C}$ as they are both of dimension 12×12 in this case.

contemporaneous actual data (1.12).[25]

This is good news for the central bank. The interest rate rule depending on contemporaneous expectations (1.32) does only require data up to period $t-1$, as mentioned above. Therefore, it is easier to implement compared to the contemporaneous data rule (1.12) and still yields similar results. Consequently, rule (1.32) is preferable to rule (1.12) even in an economy of heterogeneous expectations and not only in an economy of homogeneous RE as argued by Bullard and Mitra (2002, p.1108).

Next, we would like to consider the effect of policy inertia in rule (1.32), i.e. $\varphi_i = 0.65$. Similar steps as detailed above yield a system

$$\mathbf{A}_0\ \mathbf{s}_t \;=\; \mathbf{A}_1\ E_{t-1}\{\mathbf{s}_t\} + \mathbf{A}_2\ E_{t-1}\{\mathbf{s}_{t+1}\} + \mathbf{A}_3\ \mathbf{s}_{t-1}, \tag{1.47}$$

where $\mathbf{s}_t = [x_t, \pi_t, i_t]'$ and matrices are given by

$$\mathbf{A}_0 \;=\; \begin{bmatrix} 1 & 0 & \sigma^{-1} \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.48}$$

$$\mathbf{A}_1 \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varphi_x & \varphi_\pi & 0 \end{bmatrix}, \tag{1.49}$$

$$\mathbf{A}_2 \;=\; \begin{bmatrix} \alpha & \sigma^{-1}\alpha & 0 \\ 0 & \beta\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.50}$$

[25] For the analysis we may also replace expected values by their actual counterparts in (1.38) as is done by Bullard and Mitra (2002, p.1123ff.). We understand the latter approach as a kind of shortcut. Then it is easy to verify that the matrices for the case of contemporaneous data rule and contemporaneous expectations rule coincide and that for $\alpha = 1$ we are in the same case as in Bullard and Mitra (2002, p.1123ff.). Then all the analytical proofs with respect to determinacy and E-stability therein hold. We choose to analyze the system in a rigorous way as we are not aware of the argument behind “shortcut” of Bullard and Mitra (2002, p.1123ff.).

and

$$\mathbf{A}_3 \;=\; \begin{bmatrix} (1-\alpha)\theta^2 & \sigma^{-1}(1-\alpha)\theta^2 & 0 \\ 0 & \beta(1-\alpha)\theta^2 & 0 \\ 0 & 0 & \varphi_i \end{bmatrix}. \tag{1.51}$$

Once more we make use of (1.43) and the subsequent steps outlined above to bring the system (1.47) into our standard form (1.5).[26] We find that the numerical results are the same as in the right column of Figure 1.4 for the contemporaneous data rule. Nevertheless, once more we would like to emphasize that the contemporaneous expectations rule (1.32) is preferable compared to the contemporaneous data rule (1.12) as it is operational.

Next, one could again assume that the central bank is aware of the heterogeneous expectations as in Section 1.3.3 above. Then the central bank sets the nominal interest rate not according to (1.32) but according to

$$i_t \;=\; \varphi_\pi \widehat{E}_{t-1}\{\pi_t\} + \varphi_x \widehat{E}_{t-1}\{x_t\} + \varphi_i i_{t-1}. \tag{1.52}$$

Also note that, given the assumptions in Branch and McGough (2009, p.3), we have

$$\widehat{E}_{t-1}\{x_t\} \;=\; \alpha E_{t-1}\{x_t\} + (1-\alpha)\theta x_{t-1}, \tag{1.53}$$

$$\widehat{E}_{t-1}\{\pi_t\} \;=\; \alpha E_{t-1}\{\pi_t\} + (1-\alpha)\theta \pi_{t-1}. \tag{1.54}$$

For the moment, we omit policy inertia, i.e. $\varphi_i = 0$. We can rewrite the system (1.33)-(1.36) and (1.52)-(1.54) as

$$\mathbf{A}_0\ \mathbf{s}_t \;=\; \mathbf{A}_1\ E_{t-1}\{\mathbf{s}_t\} + \mathbf{A}_2\ E_{t-1}\{\mathbf{s}_{t+1}\} + \mathbf{A}_3\ \mathbf{s}_{t-1}, \tag{1.55}$$

where the vector of variables is $\mathbf{s}_t = [x_t, \pi_t]'$ and the system matrices are given

[26] Again we omit matrices $\mathbf{A}$ and $\mathbf{C}$ as they are both of dimension 18×18 in this case.

by

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}, \tag{1.56}$$

$$\mathbf{A}_1 = \begin{bmatrix} -\varphi_x\sigma^{-1}\alpha & -\varphi_\pi\sigma^{-1}\alpha \\ 0 & 0 \end{bmatrix}, \tag{1.57}$$

$$\mathbf{A}_2 = \begin{bmatrix} \alpha & \sigma^{-1}\alpha \\ 0 & \beta\alpha \end{bmatrix} \tag{1.58}$$

and

$$\mathbf{A}_3 = \begin{bmatrix} (1-\alpha)\theta(\theta-\varphi_x\sigma^{-1}) & \sigma^{-1}(1-\alpha)\theta(\theta-\varphi_x) \\ 0 & \beta(1-\alpha)\theta^2 \end{bmatrix}. \tag{1.59}$$

Again we use the general form (1.43) and the subsequent steps outlined above to bring the system into our standard form (1.5).[27] The numerical results are illustrated in the left column of Figure 1.5. It appears that the numerical results look similar to the ones for the contemporaneous data rule in Section 1.3.1 above. Therefore, they are also observationally similar to the results for the contemporaneous expectations rule (1.37). This makes clear that it does not make a difference whether or not the central bank is aware of expectational heterogeneity in case of the contemporaneous expectations rule. This is true at least for the parameter space considered herein.

Finally, we study the impact of policy inertia in rule (1.52) on the dynamics, i.e. $\varphi_i = 0.65$. With assumptions (1.53)-(1.54) we can derive a system similar to (1.47) with matrices

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 & \sigma^{-1} \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.60}$$

[27] Again we omit matrices $\mathbf{A}$ and $\mathbf{C}$ as they are both of dimension 12×12.

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varphi_x \alpha & \varphi_\pi \alpha & 0 \end{bmatrix}, \tag{1.61}$$

$$\mathbf{A}_2 = \begin{bmatrix} \alpha & \sigma^{-1}\alpha & 0 \\ 0 & \beta\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.62}$$

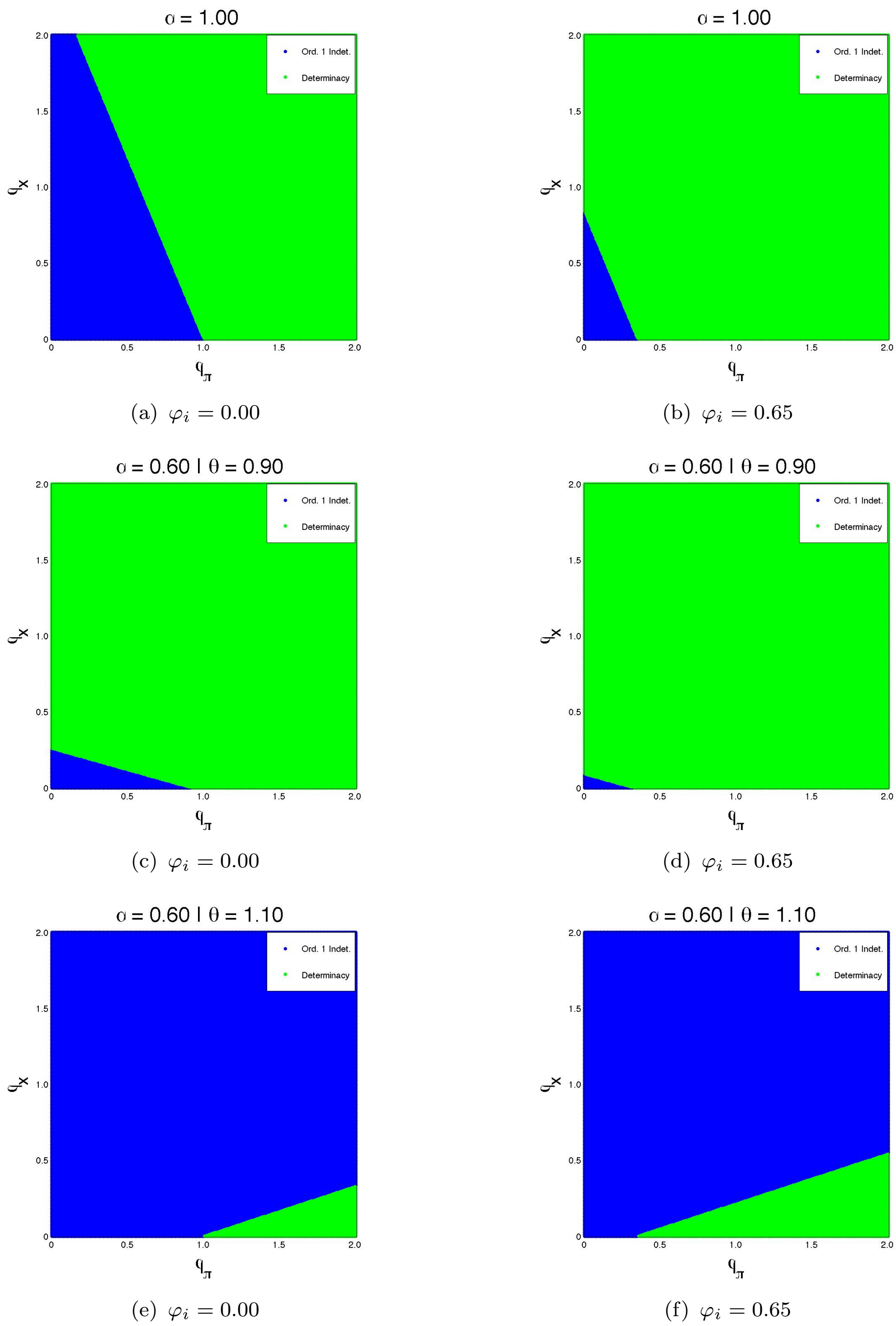
and

$$\mathbf{A}_3 = \begin{bmatrix} (1-\alpha)\theta^2 & \sigma^{-1}(1-\alpha)\theta^2 & 0 \\ 0 & \beta(1-\alpha)\theta^2 & 0 \\ \varphi_x(1-\alpha)\theta & \varphi_\pi(1-\alpha)\theta & \varphi_i \end{bmatrix}. \tag{1.63}$$

Again we can bring this version of (1.47) into our standard form (1.5).[28] The numerical results are illustrated in the right column of Figure 1.5. It appears that the numerical results look similar to the ones obtained for the contemporaneous data rule (1.12) above. Therefore, they are also similar to the results for the rule (1.32) with policy inertia. Thus, also for the rule that depends on contemporaneous expectations, it does not make a qualitative difference if the central bank tracks heterogeneous expectations or not. Furthermore, these results again indicate that in an economy with expectational heterogeneity the central bank can instead choose a rule that is easier to implement, i.e. the rule that depends on contemporaneous expectations. It will not encounter a disadvantage with regard to determinacy compared to the rule that depends on contemporaneous data.

In this section we have observed that the simple contemporaneous expectations rule is more desirable than other simple rules in an economy with heterogeneous expectations. This is due to the fact that this policy prescription rules out explosiveness and does not require to track individuals' expectations. Furthermore, the Taylor-principle holds under this rule for a large share of the parameter space. If there is a moderate feedback to the output gap, it can hold in general. We have also noticed that rules that depend on forecasts can be improved by tracking the nature of expectations and applying this information to

[28] Once more we omit matrices **A** and **C** as they are both of dimension 18×18 in this case.



(a) $\varphi_i = 0.00$ (b) $\varphi_i = 0.65$

(c) $\varphi_i = 0.00$ (d) $\varphi_i = 0.65$

(e) $\varphi_i = 0.00$ (f) $\varphi_i = 0.65$

Figure 1.5: Regions of (in-)determinacy and explosiveness for the rule with feedback on heterogeneous expectations of period t values. The right column contains the results for this rule with policy inertia.

the forecast. Most importantly, policy inertia can improve the properties of all rules and is much easier to implement compared to expectations tracking. This is an important aspect of the merit of policy inertia.

1.4 Dynamic Properties of the Model Under Conventional Optimal Policy

So far we have only considered simple monetary policy rules. These rules can implement some REE. Now, we will discuss several policies that aim to implement the optimal REE. By optimal we mean that central banks are committed to the economic well-being of the individuals that populate the economy. Thus, they should maximize the utility of individuals. Therefore, optimal policies assessed in this section are based on the assumption that the central bank tries to minimize welfare losses caused by large volatility in variables that matter for the utility of individuals.

1.4.1 Dynamic Properties of the Model with Reaction Functions

Let us consider the case in which the central bank incorrectly assumes the standard NK model with fully rational agents. Thus, the central bank is not aware of the expectational heterogeneity and the actual aggregate demand and supply relations (1.1) and (1.2) respectively, but assumes that the aggregate demand is given by

$$x_t = E_t\{x_{t+1}\} - \sigma^{-1}\left(i_t - E_t\{\pi_{t+1}\}\right) \tag{1.64}$$

and aggregate supply evolves according to

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \lambda x_t. \tag{1.65}$$

Furthermore, it aims to minimize a welfare loss function[29]

$$E_t\left\{\sum_{s=0}^{\infty}\beta^s\frac{1}{2}(\pi_{t+s}^2+\omega_x x_{t+s}^2)\right\} \tag{1.66}$$

subject to (1.65), similar as in Evans and Honkapohja (2006, p.18). Intuitively, the central bank is now concerned about stabilizing inflation and output gap over the whole time horizon. The parameter ω_x in (1.66) is the relative weight that the central bank assigns to stabilizing the output gap relative to inflation. We assume that the central bank is free to choose this parameter. The subsequent analysis will therefore focus on this parameter. Next, the Lagrangian of the central bank's problem is

$$\mathcal{L} \;=\; E_t\left\{\textstyle\sum_{s=0}^{\infty}\beta^s\frac{1}{2}(\pi_{t+s}^2+\omega_x x_{t+s}^2)+\kappa_{t+s}\left[\pi_{t+s}-\beta E_t\{\pi_{t+s+1}\}-\lambda x_{t+s}\right]\right\}.$$

The related first-order conditions are:

$$\begin{aligned}
(a)\quad \tfrac{\partial\mathcal{L}}{\partial\pi_{t+s}} &: \quad E_t\left\{\beta^s\{\pi_{t+s}+\kappa_{t+s}\}+\beta^{s-1}\{\kappa_{t+s-1}[-\beta]\}\right\} \overset{!}{=} 0\\
(b)\quad \tfrac{\partial\mathcal{L}}{\partial x_{t+s}} &: \quad E_t\left\{\beta^s\{\omega_x x_{t+s}+\kappa_{t+s}[-\lambda]\}\right\} \overset{!}{=} 0.
\end{aligned}$$

The central bank deduces from (a) and (b) that the *specific targeting rule* should be

$$\pi_{t+s} \;=\; -\frac{\omega_x}{\lambda}(x_{t+s}-x_{t+s-1}) \tag{1.67}$$

for $s \geq 0$, given that the central bank employs a commitment to the specific targeting rule from a timeless perspective.[30] Thus, we can neglect the s in the subscript. In case the central bank wants to know the optimal REE, it now needs to solve its model for this REE. As discussed in Evans and Honkapohja (2006,

[29] Such a welfare loss function can be derived by a second-order approximation of the utility function of the individuals, see Galí (2008, p.95ff.) or Woodford (2003, p.379). One can therefore regard this loss-function as micro-founded. Notice that as in Evans and Honkapohja (2006, p.18) we consider the case where the output gap target is zero to omit the problem of *inflation bias*.

[30] This requires that the central bank respects "the optimality conditions from the full inter-temporal optimization under commitment, except for the current decision-making period", see Evans and Honkapohja (2006, p.16).

p.19ff.), it can apply the *method of undetermined coefficients*. A conjecture for the equilibrium paths of inflation and output gap under commitment may be of the form

$$x_t = b_x x_{t-1} \tag{1.68}$$

$$\pi_t = b_\pi x_{t-1}, \tag{1.69}$$

which could be written in the form of (1.6). Step-by-step, we follow the exposition in McCallum and Nelson (2004, p.46) for the sake of clarity. It follows that a forecast of inflation based on (1.69) would be

$$\begin{aligned} E_t\{\pi_{t+1}\} &= b_\pi x_t \\ &= b_\pi b_x x_{t-1}, \end{aligned} \tag{1.70}$$

which corresponds to (1.7).[31] One can then plug this conjecture into (1.65) as well as into (1.67) in order to derive *undetermined coefficient conditions*. Equation (1.65) becomes

$$\begin{aligned} x_t &= \lambda^{-1}\left[\pi_t - \beta E_t\{\pi_{t+1}\}\right] \\ &= \lambda^{-1}\left[b_\pi x_{t-1} - \beta\left(b_\pi b_x x_{t-1}\right)\right] \\ &= \lambda^{-1}\left[b_\pi - \beta b_\pi b_x\right] x_{t-1}. \end{aligned} \tag{1.71}$$

Similarly (1.67) becomes

$$\begin{aligned} \pi_t &= -\frac{\omega_x}{\lambda} x_t + \frac{\omega_x}{\lambda} x_{t-1} \\ &= \frac{\omega_x}{\lambda} x_{t-1} - \frac{\omega_x}{\lambda} b_x x_{t-1} \\ &= \frac{\omega_x}{\lambda}\left(1 - b_x\right) x_{t-1}. \end{aligned} \tag{1.72}$$

[31] One could simply augment (1.70) by a forecast for the output gap $E_t\{x_{t+1}\} = b_x^2 x_{t-1}$ and write the two forecasts in matrix form similar to (1.7).

Inspection of (1.71) and (1.72) makes clear that the *undetermined coefficient conditions*

$$b_x \stackrel{!}{=} \lambda^{-1}\left[b_\pi - \beta b_\pi b_x\right] \tag{1.73}$$

$$b_\pi \stackrel{!}{=} \frac{\omega_x}{\lambda}\left(1 - b_x\right) \tag{1.74}$$

must hold. From (1.73) and (1.74) one can find a condition that is satisfied by b_x, that is

$$\begin{aligned} 0 &= -\beta b_\pi b_x + b_\pi - \lambda b_x \\ 0 &= -\beta \frac{\omega_x}{\lambda}\left(1 - b_x\right) b_x + \frac{\omega_x}{\lambda}\left(1 - b_x\right) - \lambda b_x \\ 0 &= \beta b_x^2 - \left[1 + \beta + \frac{\lambda^2}{\omega_x}\right] b_x + 1 \\ 0 &= \beta b_x^2 - \gamma b_x + 1, \end{aligned} \tag{1.75}$$

where $\gamma \equiv \left[1 + \beta + \frac{\lambda^2}{\omega_x}\right]$. Following the arguments in McCallum (1999b, p.626ff.), the relevant root, both in terms of stability as well as the *minimal state variable* criterion[32], is given by

$$\bar{b}_x \equiv \frac{\gamma - \sqrt{\gamma^2 - 4\beta}}{2\beta}, \tag{1.76}$$

which satisfies $0 < \bar{b}_x < 1$. Given $\bar{b}_x$ one directly gets

$$\bar{b}_\pi = \frac{\omega_x}{\lambda}\left(1 - \bar{b}_x\right) \tag{1.77}$$

from (1.74). Now, from (1.68) and (1.69) the central bank gets

$$x_t = \bar{b}_x x_{t-1} \tag{1.78}$$

$$\pi_t = \bar{b}_\pi x_{t-1} \tag{1.79}$$

[32]Intuitively speaking this is the solution with the minimal set of predetermined endogenous and exogenous variables that pins down free endogenous variables.

and can consequently compute expectations

$$E_t\{x_{t+1}\} = \bar{b}_x^2 x_{t-1} \quad (1.80)$$

$$E_t\{\pi_{t+1}\} = \bar{b}_\pi \bar{b}_x x_{t-1}. \quad (1.81)$$

Given the specific targeting rule under RE and the optimal REE, the central bank must now take a stand on how it may try to implement this REE. Usually implementation works via a so-called reaction function and we will discuss two alternatives in what follows.

1.4.1.1 Fundamentals-Based Reaction Function

For the first implementation alternative, the central bank may plug (1.78)-(1.81) into (1.64) and solve this equation for the nominal interest rate. This leads to

$$\begin{aligned} i_t &= \bar{b}_x[\sigma(\bar{b}_x - 1) + \bar{b}_\pi]x_{t-1}, \\ i_t &= \psi_x x_{t-1}, \end{aligned} \quad (1.82)$$

where $\psi_x \equiv \bar{b}_x[\sigma(\bar{b}_x - 1) + \bar{b}_\pi]$. This reaction function (1.82) corresponds to the one in Evans and Honkapohja (2006, p.21) and is often called "fundamentals-based" as it explicitly depends on the RE solution coefficients $\bar{b}_x$ and $\bar{b}_\pi$.

What happens if the central bank sticks to this reaction function, but the actual economy evolves not according to (1.64) and (1.65), as the central bank assumes, but according to (1.1) and (1.2)? In order to answer this question, we plug (1.82) into (1.1), combine the latter with (1.2) and get a system as (1.5) with a vector $\mathbf{y}_t = [x_t, \pi_t]'$ and matrices

$$\mathbf{A} = \alpha \begin{bmatrix} 1 & \sigma^{-1} \\ \lambda & \lambda\sigma^{-1} + \beta \end{bmatrix} \quad (1.83)$$

and

$$\mathbf{C} = \begin{bmatrix} (1-\alpha)\theta^2 - \psi_x\sigma^{-1} & (1-\alpha)\theta^2\sigma^{-1} \\ \lambda[(1-\alpha)\theta^2 - \psi_x\sigma^{-1}] & [(1-\alpha)\theta^2 - \psi_x\sigma^{-1}]\lambda\sigma^{-1} + (1-\alpha)\theta^2\beta \end{bmatrix}. \quad (1.84)$$

Recall that ψ_x depends on the relative weight for output gap stabilization ω_x. This is the free policy preference parameter and therefore we plot the resulting four GEVs[33] for $\omega_x \in (0, 2]$.[34] However, it is important that the lines in the figures below represent borders. This means that we sort the GEVs for each value of ω_x in descending order. Thus, a line does not always necessarily represent the same GEVs for all values of ω_x. Instead, for each value of ω_x, we can assess how many GEVs are smaller or larger than one in modulus by counting the number of lines that are below or above unity for any value of ω_x. We choose to do so as the figures become much easier accessible. Note that we have two free variables in this system. Thus determinacy is obtained if exactly two GEVs lie outside the unit circle for a value of ω_x. In Figure 1.6 below, this is equivalent to two lines above and two lines below unity for a value of ω_x.

In Panel 1.6(a) one of the GEVs is always larger than one in modulus for all values of ω_x considered here and a second one is always equal to zero. A third one becomes larger than one in modulus for $\omega_x \geq 0.16$. Thus, left to this value, there is indeterminacy of order 1 and on the right to this value there is determinacy in the parameter-space. It is important to mention that in a situation of indeterminacy no matter which non-explosive equilibrium is to be reached, it will not satisfy the optimality condition (1.67), thus equilibria for $\omega_x < 0.16$ are suboptimal.[35]

Inspection of Panel 1.6(b) makes clear that in the presence of purely adaptive expectations there is once more one GEV larger than one in modulus for all values of ω_x considered here. This holds for a second one for $\omega_x \geq 0.10$ and for a third one for $\omega_x \geq 0.15$. Thus we can only find determinacy within the interval $[0.10, 0.15)$. Left from that interval there is indeterminacy of order 1 and right of that interval there is local explosiveness.

Finally, in Panel 1.6(c) we observe again that there is always one GEV larger than one in modulus for all values of ω_x. This is also true for a second one for $\omega_x \geq 0.17$, which yields determinacy for this and higher values. For smaller values of ω_x there is indeterminacy of order 1.

A sensitivity analysis in Appendix 1.6.1 reveals that in case of purely adaptive expectations of type $\gamma = 2$ agents these observations hold for $\alpha \in \{0.50, ..., 0.90\}$, whereas for $\alpha \in \{0.10, ..., 0.40\}$ the system is locally explosive for almost all values of ω_x. In case of extrapolative expectations the observations are true for $\alpha \in \{0.60, ..., 0.90\}$. For $\alpha \in \{0.10, ..., 0.50\}$ the system is locally explosive for all

[33]Note that matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$ are of dimension 4×4 in this case.

[34]We cannot include $\omega_x = 0$ into the analysis as in this case $\bar{b}_x$ is not defined.

[35]See Evans and Honkapohja (2006, p.22) for the details.

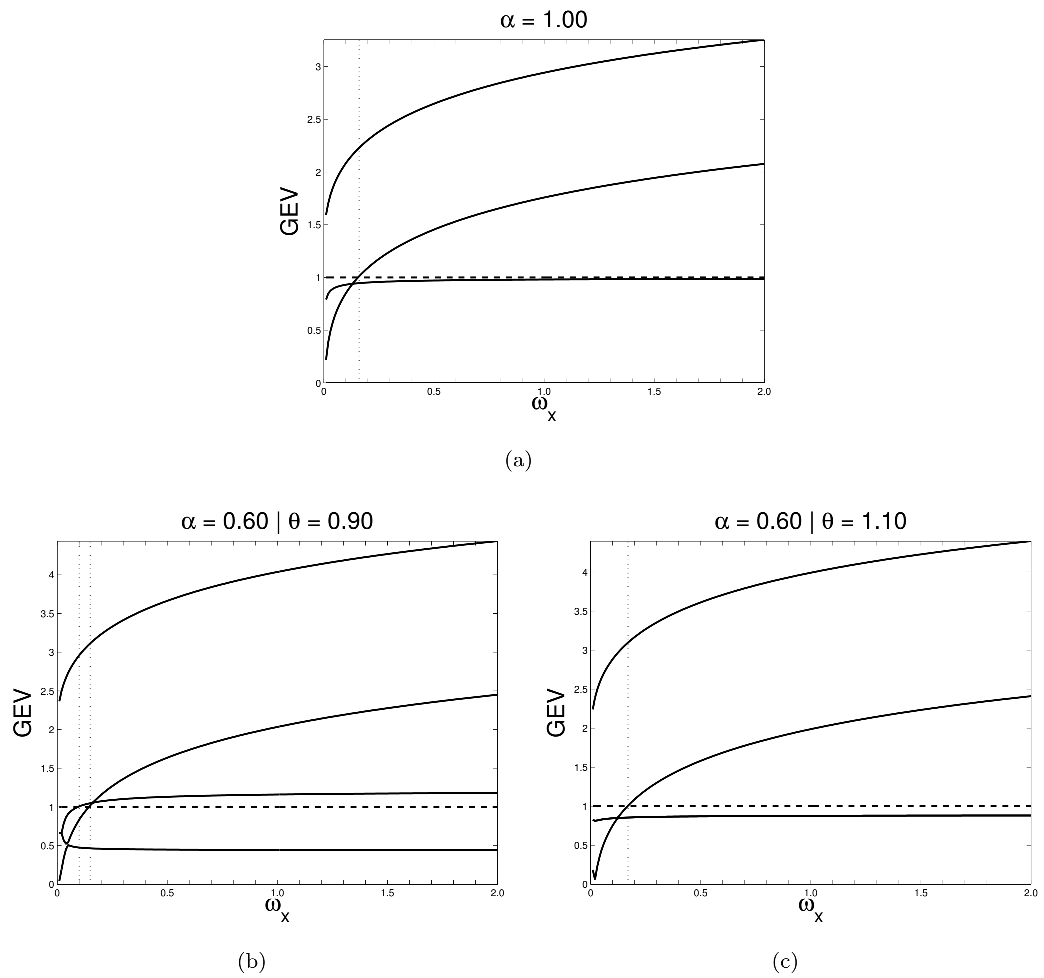


Figure 1.6: The generalized eigenvalues (GEVs) for various types of expectational settings for the fundamentals-based reaction function.

values of ω_x.

Overall, our results suggest that a fundamentals-based reaction function is a risky way of implementing optimal monetary policy in case the central bank bases its decision on a pure RE version of the NK model in world with heterogeneous expectations. For example, consider a situation in which the central bank chooses $\omega_x \geq 0.17$. If the central bank is lucky, non-rational agents have extrapolative expectations and the economy is determinate. But if the central bank is less lucky and non-rational agents have purely adaptive expectations, then the central bank triggers locally explosive behaviour of nominal variables.

1.4.1.2 Expectations-Based Reaction Function

The central bank may also choose to implement the optimal monetary policy by an alternative reaction function. To do so, it can plug (1.67) with its assumption about aggregate supply (1.65) in order to eliminate current period inflation π_t. Next, it solves the resulting equation for the output gap x_t, which leads to

$$x_t = \frac{\lambda}{\omega_x + \lambda^2}\left[\frac{\omega_x}{\lambda}x_{t-1} - \beta E_t\{\pi_{t+1}\}\right]. \tag{1.85}$$

Finally, the central bank can substitute (1.85) into its assumption about aggregate demand (1.64) and solve this expression for the nominal interest rate i_t. This yields

$$\begin{aligned} i_t &= -\frac{\omega_x\sigma}{\omega_x + \lambda^2}x_{t-1} + \left[1 + \frac{\lambda\sigma}{\omega_x + \lambda^2}\beta\right] E_t\{\pi_{t+1}\} + \sigma E_t\{x_{t+1}\} \\ &= \delta_L x_{t-1} + \delta_\pi E_t\{\pi_{t+1}\} + \delta_x E_t\{x_{t+1}\}, \end{aligned} \tag{1.86}$$

where $\delta_L \equiv -\frac{\omega_x\sigma}{\omega_x+\lambda^2}$, $\delta_\pi \equiv \left[1 + \frac{\lambda\sigma}{\omega_x+\lambda^2}\beta\right]$ and $\delta_x \equiv \sigma$. Note that (1.86) corresponds to the reaction function in Evans and Honkapohja (2006, p.26) and is often called “expectations-based”. The name originates from the fact that this reaction function conditions on private sector expectations (in our case at least from the perspective of the central bank). Note also that (1.86) does not explicitly depend on the optimal RE solution.

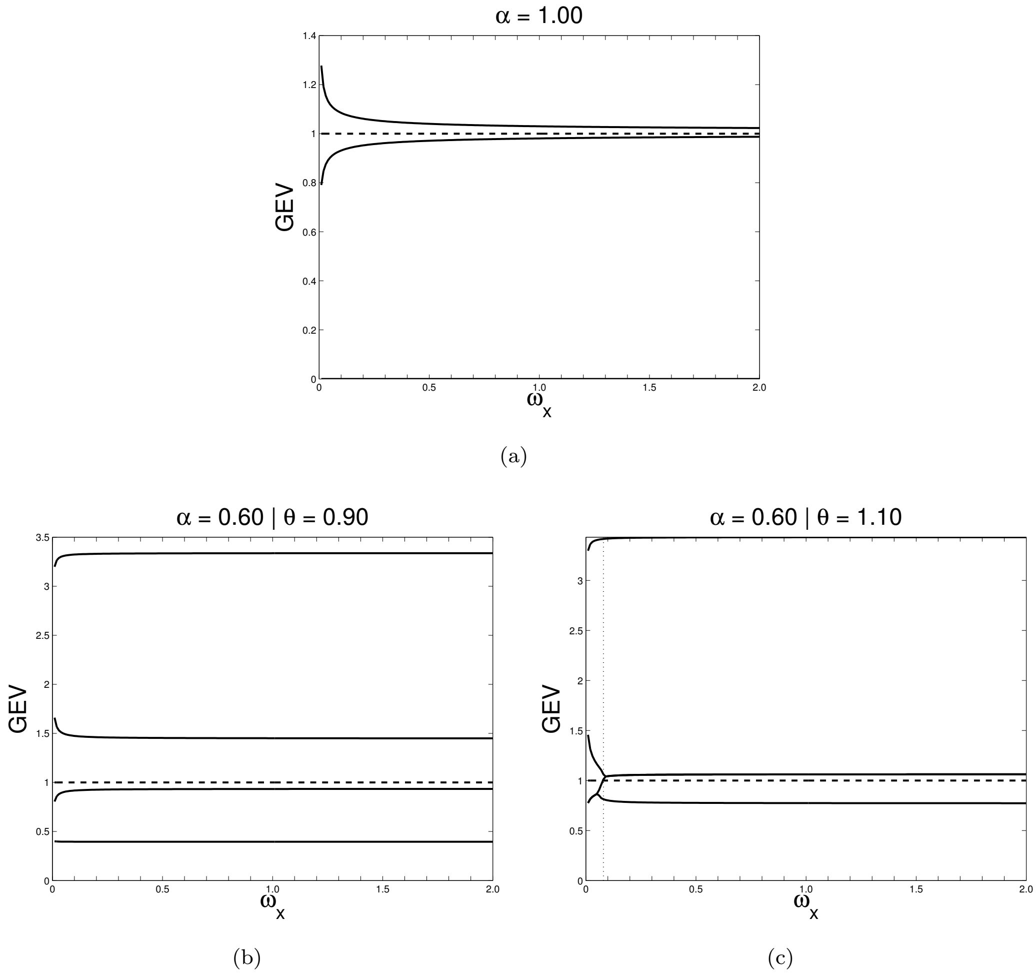
Once more, we are interested in the case in which the central bank sticks to its reaction function, but the actual economy evolves according to (1.1) and (1.2) instead of (1.64) and (1.65). Thus, we plug (1.86) into the actual aggregate demand curve (1.1), combine the latter with the actual aggregate supply curve (1.2) and get a system as (1.5) with a vector $\mathbf{y}_t = [x_t, \pi_t]'$ and matrices

$$\mathbf{A} = \begin{bmatrix} \alpha - \delta_x\sigma^{-1} & \sigma^{-1}(\alpha - \delta_\pi) \\ \lambda(\alpha - \delta_x\sigma^{-1}) & \lambda\sigma^{-1}(\alpha - \delta_\pi) + \alpha\beta \end{bmatrix} \tag{1.87}$$

and

$$\mathbf{C} = \begin{bmatrix} (1-\alpha)\theta^2 - \delta_L\sigma^{-1} & (1-\alpha)\theta^2\sigma^{-1} \\ \lambda[(1-\alpha)\theta^2 - \delta_L\sigma^{-1}] & \lambda\sigma^{-1}(1-\alpha)\theta^2 + (1-\alpha)\theta^2\beta \end{bmatrix}. \tag{1.88}$$

As in the subsection before, we now plot the resulting four GEVs for $\omega_x \in (0, 2]$.



(a) (b) (c)

Figure 1.7: The generalized eigenvalues (GEVs) for various types of expectational settings for the expectations-based reaction function.

Again there are two free variables in the system. Thus, if there are two GEVs outside the unit circle, determinacy follows. We observe that in Panel 1.7(a) one of the GEVs is always infinite. Thus, for all values of ω_x considered herein, there are always two GEVs larger than one in modulus and two GEVs smaller than one in modulus. In consequence, determinacy prevails in the complete parameter-space. This is not a surprise as there are only agents with RE in this case ($\alpha = 1.00$) and it is well known that optimal monetary policy under commitment in the pure RE version of the NK model yields determinacy when implemented in this way. For a formal proof see for example Evans and Honkapohja (2006, p.35ff.) or Woodford (2003, ch.7).

In Panel 1.7(b) we observe a qualitatively similar pattern, although there is no infinite GEV within the parameter-space. Thus, it appears that in the presence of non-rational agents with purely adaptive expectations, the central bank can still render the economy determinate independent of its choice of $\omega_x \in (0, 2]$ although it is not aware of the heterogeneity of expectations.

In Panel 1.7(c) the picture changes. In the presence of non-rational agents with extrapolative expectations, two GEVs are always larger than one in modulus in the parameter space. A third one becomes larger than one in modulus for $\omega_x \geq 0.08$. Thus, for this and larger values of ω_x the central bank triggers locally explosive behaviour of nominal variables, due to the fact that it is not aware of the expectational heterogeneity. Put differently, we now observe qualitatively the result opposite from the fundamentals-based reaction function. For values $\omega_x \geq 0.08$ the central bank is lucky if non-rational agents have purely adaptive expectations, but less lucky when non-rational agents have extrapolative expectations.

The sensitivity analysis in Appendix 1.6.1 makes clear that our observation for purely adaptive expectations is true within $\alpha \in \{0.10, ..., 0.90\}$. For extrapolative expectations, the observations are true for $\alpha \in \{0.40, ..., 0.90\}$, but for $\alpha \in \{0.10, ..., 0.30\}$ the system is locally explosive.

Overall, our results for the two reaction functions above suggest that in a NK economy with heterogeneous expectations it is quite hazardous for a central bank to incorrectly base its optimal policy on a RE version of the NK model.

1.4.2 Dynamic Properties of the Model with an Implicit Instrument Rule

So far we dealt with simple interest rate rules with and without policy inertia and two variants of implementing optimal policy. In the latter case the central bank is concerned about variations in inflation and the output gap based on its (incorrectly assumed) NK model with RE. Under expectational heterogeneity, one lesson learned is that conducting optimal policy as outlined above is rather risky. Another lesson learned is that policy inertia in simple rules may be a good tool in order to reduce regions of indeterminacy or local explosiveness. Consequently, one may ask whether or not policy inertia has similar effects in case of optimal policy? For a pure RE version of the NK model, Woodford (2003, p.582ff.) as well as Giannoni and Woodford (2005, p.106ff.) present related analyses that extend the loss function (1.66) by an interest stabilization objective, i.e.

$$E_t\left\{\sum_{s=0}^{\infty}\beta^s\frac{1}{2}(\pi_{t+s}^2+\omega_x x_{t+s}^2+\omega_i i_{t+s}^2)\right\}. \tag{1.89}$$

In this case $\omega_x, \omega_i > 0$ are the relative weights that the central bank may attach to output and interest rate stabilization.[36] Note that this objective can be justified by a micro-foundation. See Woodford (2003, p.420ff.) for the technical derivation and Woodford (2003, p.582ff.) as well as Giannoni and Woodford (2005, p.106ff.) for details of two theoretical arguments in favour of such an objective. Briefly these are non-negligible transaction frictions as discussed in Friedman (1969) or punishment of potential violations of the zero lower bound on the policy instrument.

As before, we assume that the central bank minimizes (1.89) subject to (1.64) and (1.65). This means that the central bank again incorrectly assumes that it operates in an economy of homogeneous RE. The Lagrangian of this problem is given by

$$\begin{aligned}\mathcal{L} \;=\; & E_t\{\textstyle\sum_{s=0}^{\infty}\beta^s\frac{1}{2}(\pi_{t+s}^2+\omega_x x_{t+s}^2+\omega_i i_{t+s}^2)\\ & +\kappa_{1|t+s}\left[x_{t+s}-E_t\{x_{t+s+1}\}+\sigma^{-1}i_{t+s}-\sigma^{-1}E_t\{\pi_{t+s+1}\}\right]\\ & +\kappa_{2|t+s}\left[\pi_{t+s}-\beta E_t\{\pi_{t+s+1}\}-\lambda x_{t+s}\right]\}.\end{aligned}$$

The related first-order conditions are given by

$$(a)\ \frac{\partial\mathcal{L}}{\partial\pi_{t+s}}: E_t\left\{\beta^s\{\pi_{t+s}+\kappa_{2|t+s}\}+\beta^{s-1}\{\kappa_{1|t+s-1}[-\sigma^{-1}]+\kappa_{2|t+s-1}[-\beta]\}\right\} \overset{!}{=} 0$$

$$(b)\ \frac{\partial\mathcal{L}}{\partial x_{t+s}}: E_t\left\{\beta^s\{\omega_x x_{t+s}+\kappa_{1|t+s}+\kappa_{2|t+s}[-\lambda]+\beta^{s-1}\{-\kappa_{1|t+s-1}\}\right\} \overset{!}{=} 0.$$

$$(c)\ \frac{\partial\mathcal{L}}{\partial i_{t+s}}: E_t\left\{\beta^s\{\omega_i i_{t+s}+\kappa_{1|t+s}\sigma^{-1}\}\right\} \overset{!}{=} 0,$$

[36] Compared to Woodford (2003, p.582ff.) and Giannoni and Woodford (2005, p.106ff.) we consider the case where the optimal constant interest rate target is zero. The output gap target is also zero once more.

for each date $s \geq 0$ and initial conditions $\kappa_{1|-1} = \kappa_{2|-1} = 0$, given that the central bank employs a commitment to its optimality conditions from a timeless perspective. Thus, we can again ignore the s in the subscript and equivalently write (a), (b) and (c) as

$$\pi_t - \beta^{-1}\sigma^{-1}\kappa_{1|t-1} + \kappa_{2|t} - \kappa_{2|t-1} = 0 \tag{1.90}$$

$$\kappa_{2|t} = \frac{\omega_x}{\lambda}x_t + \frac{1}{\lambda}\kappa_{1|t} - \frac{1}{\beta\lambda}\kappa_{1|t-1} \tag{1.91}$$

$$\kappa_{1|t-1} = -\sigma\omega_i i_t. \tag{1.92}$$

It is easy to eliminate both Lagrange multipliers. First, we plug (1.92) into (1.91) both for period t and $t-1$. Second, we can use the resulting version of (1.91) as well as (1.92) to express (1.90) free of Lagrange multipliers as

$$i_t = \underbrace{\frac{\omega_x}{\sigma\omega_i}}_{\equiv\gamma_1}\triangle x_t + \underbrace{\frac{\lambda}{\sigma\omega_i}}_{\equiv\gamma_2}\pi_t + \underbrace{(1 + \frac{\lambda}{\beta\sigma} + \beta^{-1})}_{\equiv\gamma_3} i_{t-1} - \underbrace{\beta^{-1}}_{\equiv\gamma_4} i_{t-2}. \tag{1.93}$$

Woodford (2003, p.582ff.) denotes (1.93) as the *implicit instrument rule*. Furthermore, he proves that commitment to this rule yields a determinate REE that is optimal from a timeless perspective as long as (1.64) and (1.65) are true.

In the case of expectational heterogeneity (1.64) and (1.65) do not hold, but instead the economy evolves according to (1.1) and (1.2). Nevertheless, we assume that the central bank commits to (1.93) in all periods. We recognize that we now have a difference equation with one lead and two lags. Therefore, we recast our system once more in the way outlined by Binder and Pesaran (1999, p.140ff.) for

$n_1 = 2$ and $n_2 = 1$

$$\begin{aligned}
\mathbf{0} \;=\; & \begin{bmatrix} -1 & 0 & -\sigma^{-1} \\ \lambda & -1 & 0 \\ \gamma_1 & \gamma_2 & -1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ i_t \end{bmatrix} + \begin{bmatrix} \alpha & \sigma^{-1}\alpha & 0 \\ 0 & \beta\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ E_t i_{t+1} \end{bmatrix} \\
& + \begin{bmatrix} (1-\alpha)\theta^1 & -\sigma^{-1}(1-\alpha)\theta^1 & 0 \\ 0 & \beta(1-\alpha)\theta^1 & 0 \\ -\gamma_1 & 0 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma_4 \end{bmatrix} \begin{bmatrix} x_{t-2} \\ \pi_{t-2} \\ i_{t-2} \end{bmatrix} + \mathbf{0} \begin{bmatrix} E_{t-1} x_t \\ E_{t-1} \pi_t \\ E_{t-1} i_t \end{bmatrix} + \mathbf{0} \begin{bmatrix} E_{t-2} x_{t-1} \\ E_{t-2} \pi_{t-1} \\ E_{t-2} i_{t-1} \end{bmatrix}
\end{aligned}$$

or with $\mathbf{s}_t = [x_t, \pi_t, i_t]'$ more compact as

$$\begin{aligned}
\mathbf{0} \;=\; & \mathbf{M}_{00}\ \mathbf{s}_t + \mathbf{M}_{01}\ E_t \mathbf{s}_{t+1} + \mathbf{M}_{10}\ \mathbf{s}_{t-1} + \mathbf{M}_{20}\ \mathbf{s}_{t-2} \\
& + \mathbf{M}_{11}\ E_{t-1}\mathbf{s}_t + \mathbf{M}_{21}\ E_{t-2}\mathbf{s}_{t-1}. \qquad (1.94)
\end{aligned}$$

We can rewrite the latter as

$$\begin{aligned}
\mathbf{0} \;=\; & \begin{bmatrix} \mathbf{M}_{00} & \mathbf{M}_{01} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{s}_t \\ E_t \mathbf{s}_{t+1} \end{bmatrix} + \begin{bmatrix} \mathbf{M}_{10} & \mathbf{M}_{11} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s}_{t-1} \\ E_{t-1} \mathbf{s}_t \end{bmatrix} \\
& + \begin{bmatrix} \mathbf{M}_{20} & \mathbf{M}_{21} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s}_{t-2} \\ E_{t-2} \mathbf{s}_{t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} E_t \mathbf{s}_{t+1} \\ E_t \mathbf{s}_{t+2} \end{bmatrix}
\end{aligned}$$

or with $\mathbf{z}_t = [\mathbf{s}_t', E_t \mathbf{s}_{t+1}']'$ more compact as

$$\mathbf{0} \;=\; \boldsymbol{\Gamma}_0\ \mathbf{z}_t + \boldsymbol{\Gamma}_1\ \mathbf{z}_{t-1} + \boldsymbol{\Gamma}_2\ \mathbf{z}_{t-2} + \boldsymbol{\Gamma}_{-1}\ E_t \mathbf{z}_{t+1}. \qquad (1.95)$$

Again, we can rewrite this equation as

$$\mathbf{0} \;=\; \begin{bmatrix} \boldsymbol{\Gamma}_0 & \boldsymbol{\Gamma}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{z}_{t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \boldsymbol{\Gamma}_2 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{t-1} \\ \mathbf{z}_{t-2} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Gamma}_{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} E_t \mathbf{z}_{t+1} \\ \mathbf{z}_t \end{bmatrix},$$

or by defining $\mathbf{y}_t = [\mathbf{z}_t', \mathbf{z}_{t-1}']'$ more compactly as

$$\boldsymbol{\Lambda}_0\ \mathbf{y}_t \;=\; -\boldsymbol{\Lambda}_{-1}\ E_t\{\mathbf{y}_{t+1}\} - \boldsymbol{\Lambda}_1\ \mathbf{y}_{t-1}.$$

Obviously, $\mathbf{\Lambda}_0$ is non-singular and invertible (as $\mathbf{\Gamma}_0$ and $\mathbf{M}_{00}$ are non-singular too) and we can finally arrive at the familiar system (1.5)

$$\mathbf{y}_t = \mathbf{A}\, E_t\{\mathbf{y}_{t+1}\} + \mathbf{C}\, \mathbf{y}_{t-1}, \tag{1.96}$$

with $\mathbf{A} = -\mathbf{\Lambda}_0^{-1}\mathbf{\Lambda}_{-1}$ and $\mathbf{C} = -\mathbf{\Lambda}_0^{-1}\mathbf{\Lambda}_1$.[37] As we have two relative weights ω_x and ω_i that the central bank may choose freely, we return to plot regions in a plane. For each combination of the relative weights we exhibit the number of GEVs in or outside the unit circle. Panels 1.8(a)-1.8(c) indicate that the implicit instrument rule yields determinate outcomes for all cases within the parameter space. This finding makes such an implicit instrument rule a highly desirable way of implementing optimal monetary policy in the case of expectational heterogeneity. Put differently, if the NK model with heterogeneous expectations herein is the true model of the economy, then the central bank may ignore non-rational expectations as long as its objective function contains a term for instrument stabilization. This demonstrates the merit of policy inertia in the context of optimal monetary policy.

[37] We omit details of matrices $\mathbf{A}$ and $\mathbf{C}$ due to space limitations, as in our case they are both of dimension 12×12 in this case.

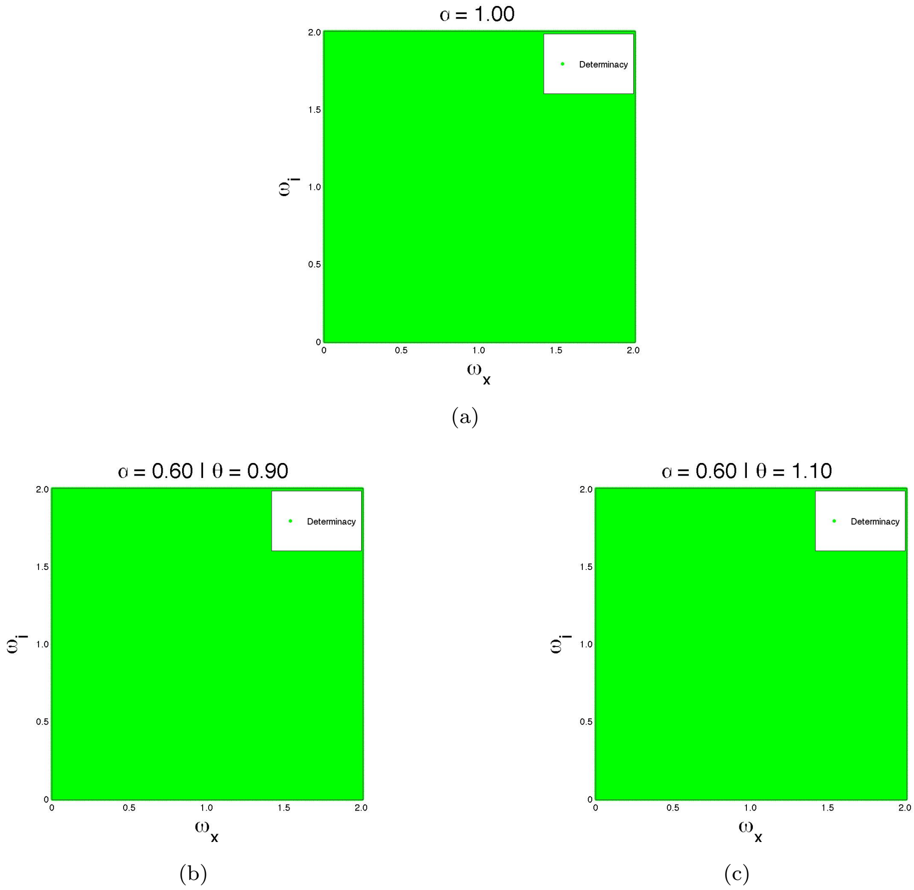


Figure 1.8: The generalized eigenvalues (GEVs) for various types of expectational settings for the implicit instrument rule.

1.5 Conclusion

In our analysis of simple monetary policy rules, we find that a rule that feeds back to contemporaneous data of inflation and output gap yields qualitatively the same results as the ones presented in Branch and McGough (2009). There are both regions of indeterminacy and determinacy in the parameter space for the case of homogeneous rational agents. In the presence of heterogeneous expectations, if non-rational agents have purely adaptive expectations, the region of determinacy increases relative to the region of indeterminacy, if non-rational agents have extrapolative expectations the opposite is true. The Taylor-principle does not hold in general.

Next, the interest rule that depends on lagged data does not yield qualitatively similar results as presented in Branch and McGough (2009). The reason is, that we observe regions of local explosiveness next to regions of (in-)determinacy which is not a desirable feature of a policy rule. Therefore, we can only partly confirm Branch and McGough (2009)'s results for those two rules, i.e. that the presence of purely adaptive expectations improves the situation with regard to determinacy, whereas the presence of extrapolative expectations worsens the situation.

Moreover, the forward-looking interest rate rule that feeds back on purely RE, does also exhibit regions of local explosiveness. Interestingly these regions occur in the area, in which the central bank would fight inflation expectations moderately by more than one-for-one, i.e. sticking to the Taylor-principle. Once the central bank is aware of the nature of expectations in the economy and feeds back to heterogeneous expectations, it is able to rule out local explosiveness and results are qualitatively the same as in Branch and McGough (2009). But one should be aware that this type of rule imposes large informational requirements for the central bank. The central bank must perfectly track the nature of each type of expectations as well as the exact fraction of agents of each type within the economy, which appears to be quite unrealistic. In addition, both variants of the forward-looking policy rule enable the central bank to bring about determinacy by solely giving feedback to expectations about the output gap when non-rational agents have purely adaptive expectations. This is a rather unusual observation.

Next, simple monetary policy rules that depend on contemporaneous expectations, no matter whether the central bank considers purely RE or is aware of expectational heterogeneity, yield similar results as rules that depend on contemporaneous data. Most importantly, both of these rules do not depend on measurement of contemporaneous data. This makes them easier to implement. Furthermore, as both rules yield similar results, the central bank is not necessarily required to have an idea about the nature and fractions of different types of heterogeneous expectations.

Once a realistic degree of policy inertia is present in the simple rules, our conclusions with regard to the Taylor-principle do not change. In addition, we observe for almost all simple rules that with an increasing level of policy inertia the regions of determinacy appear to increase relative to regions of indeterminacy and local explosiveness. This holds no matter whether expectations in the economy are homogeneous RE or heterogeneous. Remarkable is the case of the forward-looking monetary policy rule. We observe that the presence of policy inertia rules out locally explosive paths of nominal variables in that case. Overall, we conclude that this finding of Bullard and Mitra (2007) remains robust also in the case of heterogeneous expectations. Policy inertia is a merit of simple monetary policy rules.

In the subsequent analysis we examine optimal monetary policy in a setting in which the central bank is not aware of heterogeneous expectations. It implements optimal monetary policy based on the assumption of homogeneous RE via a fundamentals-based reaction function or an expectations-based reaction function. We find that both reaction functions are a rather hazardous way of conducting optimal policy in presence of expectational heterogeneity.

Finally, we examine optimal monetary policy when the central bank's objective enforces policy inertia and is implemented via an implicit instrument rule. We find that this implicit instrument rule renders the economy determinate throughout the parameter space considered and that this finding is robust to heterogeneous expectations. This is another important aspect of the merit of policy inertia.

Policy recommendations in the light of our results are as follows. A central bank that prefers a simple rule may conduct monetary policy by a rule that depends on contemporaneous expectations with policy inertia and stick to the Taylor-principle in the sense that it feeds back to contemporaneous inflation expectations more than one-for-one and in addition it may moderately feed back to contemporaneous expectations about the output gap. In case, the central bank wants to implement optimal policy it should consider policy inertia in its objective, as the implicit instrument rule always yields determinacy within the parameter space in an economy with heterogeneous expectations.

Naturally the question arises, how these results for optimal monetary policy change, once the central bank is aware of the expectational heterogeneity? From our point of view, this is an interesting subject for future research, but we want to remind the reader that tracking heterogeneous expectations could be costly. Thus the implicit instrument rule considered may still be the superior policy choice for central banks. Another path of future inquiry may be to study a NK model with heterogeneous expectations that allows for three different types of expectations at the same time. Such results would help to evaluate the robustness of our findings. In future research we could also replace one of the types of expectations considered herein by a type of expectations not considered herein as has been emphasized by Branch and McGough (2009, p.14). Another direction of future research could aim to study simple and optimal monetary policy in larger scale versions of the NK model with heterogeneous expectations. Possible examples are models with capital accumulation or monetary and fiscal policy interactions.

1.6 Appendices

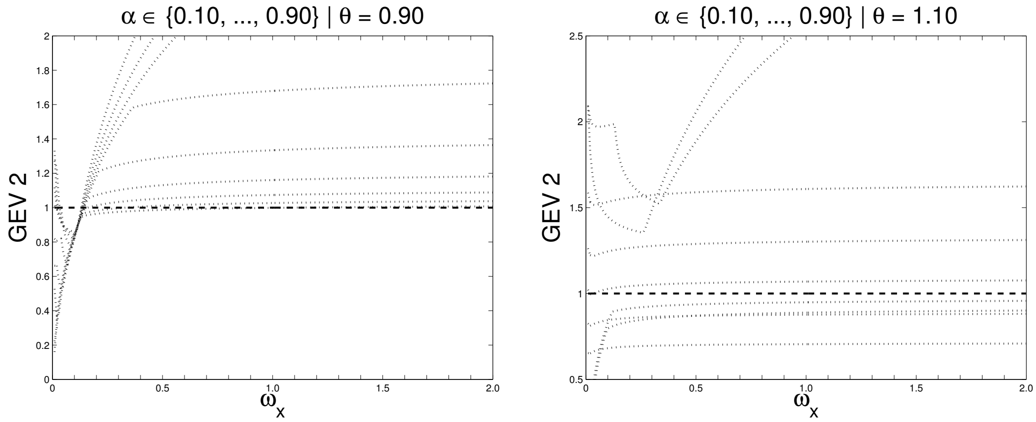
1.6.1 Sensitivity Analysis for the Reaction Functions

A relevant issue is the robustness of our findings with regard to the expectational set-up. The key factor that determines this set-up is the fraction of non-rational agents $(1-\alpha)$. Note that both in the case of the fundamentals-based and expectations-based reaction function, there are two free variables in the system. Therefore, determinacy requires that there are two GEVs outside the unit circle and two GEVs inside the unit circle. We label the GEVs according to the size of their modulus in descending order GEV4, GEV3, GEV2 and GEV1.

In order to assess the robustness of results, we vary $\alpha \in [0, 1]$ and plot the GEV of interest, which is GEV2. Note that GEV4 is always outside the unit circle and GEV3 is usually outside the unit circle for $\alpha \in [0, 1]$, except we explicitly mention the opposite. Furthermore, GEV1 is usually inside the unit circle for

$\alpha \in [0, 1]$. If GEV2 remains inside the unit circle for $\alpha \in [0, 1]$, then determinacy occurs.

Figure 1.9 illustrates the outcome of the sensitivity analysis for the fundamentals-based reaction function of Subsection 1.4.1.1.



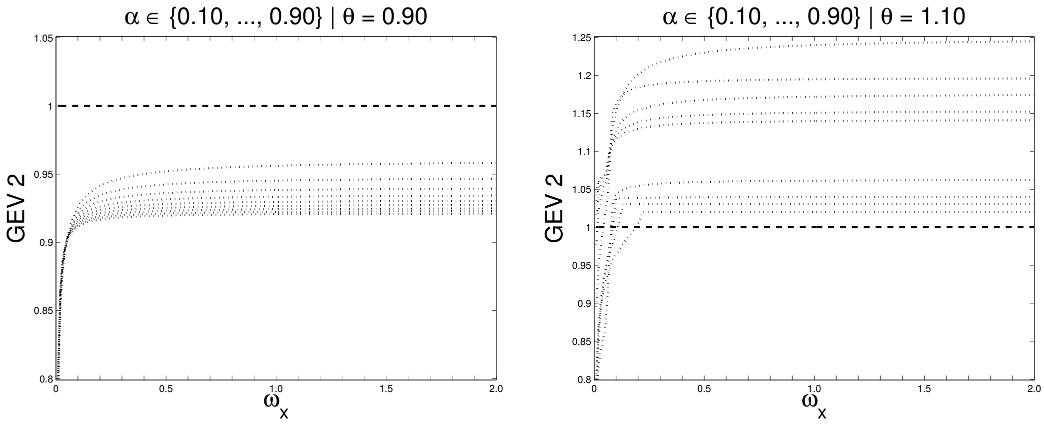
(a) (b)

Figure 1.9: The second generalized eigenvalue (GEV2) in the sensitivity analysis for the fundamentals-based reaction function.

The case of purely adaptive expectations is outlined in Panel 1.9(a). We find a pattern comparable to the one described Subsection 1.4.1.1 above for $\alpha \in \{0.50, ..., 0.90\}$. For small values of ω_x the economy is indeterminate of order one as both GEV3 (not displayed) and GEV2 are inside the unit circle. As ω_x increases, first GEV3 becomes larger than one in modulus and shortly thereafter GEV2 becomes larger than one in modulus. Thus, determinacy prevails only for a relatively small range of ω_x. For $\alpha \in \{0.10, ..., 0.40\}$ the economy is even more unstable. The upper four lines indicate the GEV2 for these values of α. Only where these U-shaped lines are below unity, determinacy arises.

Next, the case of extrapolative expectations is outlined in Panel 1.9(b). We observe that in the range $\alpha \in \{0.60, ..., 0.90\}$ the pattern is as reported above. GEV2 remains inside the unit circle for the values of ω_x considered herein, as the lower four lines indicate. Thus, as mentioned above, once GEV3 becomes larger than one in modulus, determinacy occurs. This is not true for $\alpha \in \{0.10, ..., 0.50\}$ as the upper five lines indicate. For these values of α, there is local explosiveness because GEV3 is also always outside the unit circle.

In Figure 1.10, we can observe the outcome of the sensitivity analysis for the expectations-based reaction function of Subsection 1.4.1.2.



(a) (b)

Figure 1.10: The second generalized eigenvalue (GEV2) in the sensitivity analysis for the expectations-based reaction function.

For the case of purely adaptive expectations we find that there are always two GEVs outside the unit circle and one GEV inside the unit circle for $\alpha \in \{0.10, ..., 0.90\}$. The crucial question is, whether the fourth GEV that we label GEV2 remains inside the unit circle for $\alpha \in \{0.10, ..., 0.90\}$. Panel 1.10(a) reveals that this is indeed the case as all lines remain below unity. Thus, the expectations-based reaction function ensures determinacy under purely adaptive expectations for all values of α considered herein.

Finally, Panel 1.10(b) illustrates the GEV of interest for $\alpha \in \{0.10, ..., 0.90\}$ given extrapolative expectations. We find that for $\alpha \in \{0.40, ..., 0.90\}$ GEV2 is inside the unit circle for small values of ω_x, which results in determinacy, but becomes locally explosive for larger values of ω_x. In addition, for $\alpha \in \{0.10, ..., 0.30\}$ the GEV2 is outside the unit circle for all values of ω_x considered herein, which yields local explosiveness.

Chapter 2

Monetary and Fiscal Policy Interaction in a World with Heterogeneous Expectations

2.1 Motivation

The interaction of fiscal and monetary policies and its consequences for the dynamics of an economy have been subject of ongoing research for many years. The main motivation is that in standard monetary models fiscal policy is often neglected. It is assumed that the fiscal authority in the economy ensures government solvency at any time. This behaviour of the fiscal authority may not always be true and therefore it may become the central bank's task to enforce government solvency. It is usually argued that in this case, fiscal policy may have the potential to limit the central bank's ability to control inflation. In particular reducing the public debt burden and fighting inflation may represent a trade-off for the interest rate policy of a central bank. A classic example is Leeper (1991) who considers interactions of fiscal and monetary policies in a Neo-Classical economy with homogeneous rational agents and its consequences for the equilibrium path of the price level. He models policy interactions between authorities as responses of the policy instrument of one authority to a shock in the instrument of another authority. He considers shocks to be additive components of the authorities' policy rules. His findings may be summarized in Table 2.1.

What Leeper (1991) denotes *active* monetary policy, is when the central bank

	Fiscal Policy	
Monetary Policy	**passive**	**active**
passive	(iii) indeterminacy	(ii) determinacy
active	(i) determinacy	(iv) explosive paths

Table 2.1: Types of fiscal and monetary policies and resulting outcomes for the price level in Leeper (1991).

fights inflation by sticking to the Taylor principle.[1] *Passive* monetary policy means weak responses to inflation. *Passive* fiscal policy means that the government ensures a balanced budget in each period, whereas under *active* fiscal policy this may not necessarily be the case.[2]

As one can see from Table 2.1, different combinations of policy stances lead to alternative outcomes for the price level. The outcome is *determinate*, when there exists a locally unique stationary solution. Whether or not, or under which conditions there is determinacy is an important issue in monetary economics and has generated an immense body of literature. As it appears from our reading, the conventional wisdom is that only policies that render the economy determinate, are desirable. These policies rule out arbitrary large fluctuations in response to aggregate shocks. Be also aware that *indeterminacy* describes a situation in which there are multiple stationary solutions. Intuitively one can think of this as a situation in which there is too much stability in the economy. A drawback of such a situation is that some solutions potentially depend on extrinsic uncertainty (i.e. sunspots). *Sunspots* are known to lead to large fluctuations. If there exists no local stationary solution this results in local divergence from a steady-state. We denote such a situation *explosive*. Note that local divergence can result in deflation or hyperinflation. It is important to mention that such explosive paths of nominal variables are not prohibited by the common assumption of transversality conditions as they only limit the paths of real variables. This point was recently highlighted by Cochrane (2007).

[1]Taylor (1993) suggests a simple interest rate rule and assumes an inflation coefficient of 1.5, i.e. if inflation deviates from its target level, then the central bank should react with the nominal interest rate more than one-for-one, in this case one-and-a-half-for-one. In Taylor (1999) he denotes this suggestion from 1993 (with regard to the functional form) a “normative recommendation”. In Taylor (1999) he explicitly advocates an inflation coefficient larger than one in such a policy rule. This policy stance towards inflation is denoted the “Taylor-principle” in the literature.

[2]Nowadays, an active fiscal policy is also often denoted a non-Ricardian fiscal policy.

Notice that both cases (i) and (ii) in Table 2.1 yield determinacy. The former case is also often denoted as the *monetarist view* and the latter case is often denoted the *fiscalist view*.[3] One should mention, that the two cases differ most importantly in the issue whether or not fiscal variables have an impact on inflation. An important implication of the results of Leeper (1991) is that in case of active fiscal policy it is dangerous for the central bank to conduct active monetary policy as this could trigger explosive paths of the price-level.

Evans and Honkapohja (2007) emphasize the central role of expectations in economics in a straightforward extension of the analysis of Leeper (1991). They follow the approach sketched out by Leeper (1991) and compare the dynamic properties of an economy around a steady-state under the assumption of fully rational agents to an economy where agents are engaged in one-step ahead forecasts of the structural parameters of the economy. This behaviour is usually labeled as adaptive learning.[4] They find that the rational expectations equilibrium (REE) in both cases (i) and (ii) of Leeper (1991) (compare Table 2.1) is expectational stable and therefore robust to a modest deviation from the rational expectations (RE) assumption of Leeper (1991).

Be aware that so far monetary and fiscal policy interactions have only been analyzed when out of equilibrium behaviour of agents depends either on RE or on adaptive expectations. From our point of view, it may be a natural extension to abandon the idea of comparing two different assumptions on homogeneous agents' expectations formation process and instead focus on a world, in which there are heterogeneous agents. In particular, one can assume that agents' heterogeneity stems from different expectations formation processes. We make this conjecture as there is convincing micro-data evidence for heterogeneous expectations among agents. See for example Branch (2004, p.607ff.) who, depending on the specification, estimates shares roughly varying around 7%, 40% and 50% for naive, adaptive and VAR-forecast expectations respectively from the Michigan

[3] Leeper (2009, p.9ff.) consolidates the diverse nomenclatures.

[4] One can find the foundations of the learning theory in Evans and Honkapohja (2001). Evans and Honkapohja (2007) consider what Preston (2005, p.96) labels the "Euler Equation Approach", to properly distinguish it from the "Long-Horizon Approach" therein. Be aware that a recent equivalence result by Bullard and Eusepi (2008, p.8ff) makes the point that explicitly considering long-horizons may be dispensable for a quite general class of models as long as there exists only a single discount factor in the economy.

Survey of Professional Forecasters.

In the recent years the Neo-Classical framework has been to large extent replaced by the New Keynesian (NK) framework. Thus, we consider heterogeneous expectations in a NK framework.[5] The challenge of introducing heterogeneous expectations into the NK framework has recently been tackled by Branch and McGough (2009). The model features the coexistence of agents with RE and simple adaptive expectations. In their "axiomatic approach" fiscal policy has only a passive role. This is an assumption that the majority of authors impose as Leeper (2009) criticises. Nevertheless, Branch and McGough (2009) present some very important insights with regard to the dynamics of the economy. In presence of a quite modest fraction of agents with adaptive expectations the dynamic implications of sticking to the Taylor principle can change dramatically. In particular, the implications depend on the exact nature of adaptive expectations. Remarkably, the Taylor principle does not yield determinacy in general.

We add the additional layer of fiscal and monetary policy interaction to the NK framework with heterogeneous expectations. In contrast to Branch and McGough (2009), we model a decentralized market economy instead of a yeoman farmer model. Our main goal is to ask how heterogeneous expectations affect the interaction of fiscal and monetary policies with regard to determinacy. Can we still distinguish the cases (i) and (ii) of Leeper (1991) in a clear-cut manner? Can we relate particular policy combinations (active/passive monetary policy and/or active/passive fiscal policy) to particular outcomes? Can we still find combinations of policy stances that lead to determinacy in general?

We find that our model does not lead to clear-cut analytical results as in Leeper (1991), therefore we follow a numerical approach. Similar as Branch and McGough (2009) we impose that monetary policy is conducted by a simple linear rule that depends on expected inflation and expected output gap. We therefore can illustrate regions of (in-)determinacy or explosiveness in a plane, in which the axes measure the monetary policy feedback coefficients of the rule given active or passive fiscal policy.

For passive fiscal policy, we are able to confirm the results of Branch and McGough (2009) with regard to the impact of the presence of heterogeneous ex-

[5]Note that Evans et al. (2008) provide a global analysis of monetary and fiscal policy interaction in a NK framework with homogeneous agents that are assumed to be adaptive learners where they explicitly take into account the zero lower bound.

pectations on determinacy and indeterminacy. Once heterogeneous expectations are in place, regions of determinacy increase relative to regions of indeterminacy if non-RE are purely adaptive. The opposite is true if non-RE are extrapolative.[6]

Surprisingly, we detect locally explosive regions, independent of the fiscal policy stance. As a consequence one might label some monetary policies as dangerous. In order to be concrete, a monetary policy that feeds back to inflation expectations with around one-for-one, and features a low or no feedback to expected output gap deviations appears to be explosive. This is an important finding in the light that data from 1979 to 1993 suggest that since 1979 some countries have been conducting comparable policies with a forward-looking inflation targeting rule. In particular, Clarida et al. (1998, p.1045ff.) found that feedback on expected inflation was 1.31/1.79 for Germany and the US respectively and feedback on expected output gap was 0.25/0.07. Moreover, we find that active fiscal policy tends to increase the size of locally explosive regions as well as regions of indeterminacy. Nevertheless, monetary policies that render the economy determinate remain feasible.

Another important finding is that in our set-up a passive monetary / active fiscal policy regime economy is unlikely to yield determinacy even under RE.

In Section 2.2, we outline our model, discuss derivations and refer to the assumptions used to achieve them. In addition, we clarify what we denote as active/passive monetary policies and active/passive fiscal policies. Section 2.3 characterizes the appropriate equilibrium conditions. We also derive the natural and efficient levels of output and account for distortions caused by the fiscal policy framework. These conditions and our assumptions about expectations form the fully specified system we are going to analyze in Section 2.4. Therein, in Section 2.4.3, we visualize regions of (in-)determinacy in the parameter-space for monetary policy. Section 2.5 concludes.

[6]Be aware that in this context non-RE are always adaptive. Agents use past observations of an endogenous variable to forecast its future value. We distinguish "purely adaptive" and "extrapolative" expectations to make clear that the coefficient on the past observations is smaller than one in the former case and larger than one in the latter case.

2.2 The Economic Environment

The model in this study is a mash-up of a standard decentralized market economy with a continuum of households and a continuum of firms, the model in Branch and McGough (2009) and ingredients of economies considered in the literature on *sticky information*.[7,8] We will in turn outline households that consist of a worker and a consumer. The ultimate goal of both members of the household is to maximize the household's lifetime utility by their decisions. Moreover, there is an actuary fair insurance agency that serves households to ensure themselves against income risk. Next, firms consist of a hiring and a sales department that maximize the firm's profit. Finally, there will be a fiscal authority and a central bank. Figure 2.1 captures the main points of this model. The following subsections give the details.

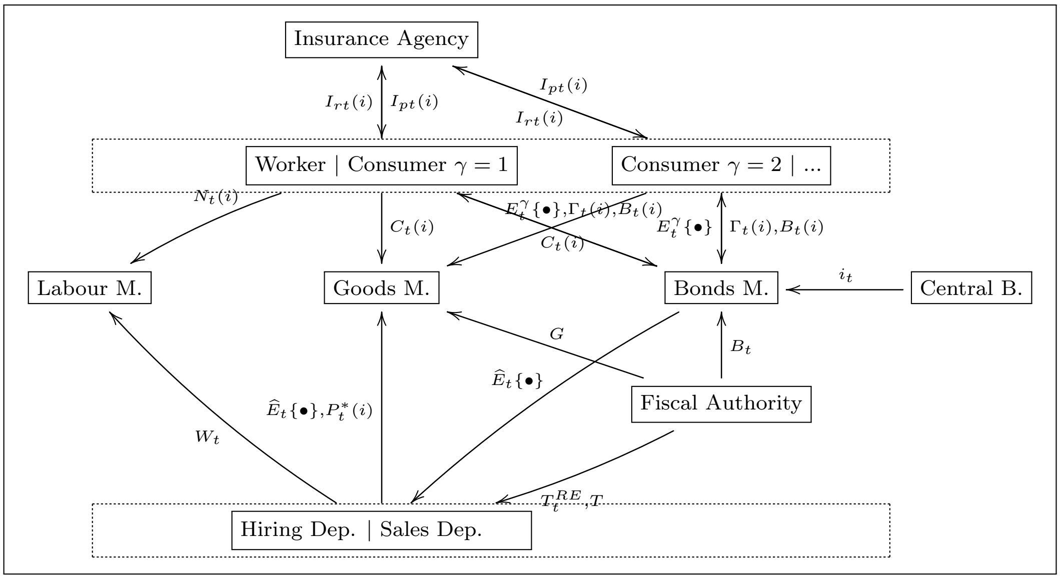


Figure 2.1: Model overview outlining the interactions of the different sectors. A dashed box indicates a continuum.

[7]In this way we are able to ensure that household income is dependent on household effort. This route was suggested by Branch and McGough (2009, p.6) but not pursued therein.

[8]Sticky information models go back to Mankiw and Reis (2002) and base the derivation of the New Keynesian Phillips Curve on the assumption of inattentive agents instead of sticky prices. Some remarks regarding the insurance mechanism we are going to use are provided in Mankiw and Reis (2007).

2.2.1 The Household Sector

We assume a continuum of infinitely-lived households, where each household $i \in [0,1]$ can be of one of the two types $\gamma \in \{1,2\}$. Please be aware that all households are completely identical except for the way they form expectations about the future. For this particular reason we introduce the two different types γ.

A single household consists of two decision makers, the worker and the consumer. The household's worker offers one differentiated type of labour $N_t(i)$ at a perfectly competitive labour market in period t and earns labour income given his intra-temporal decision.[9] The household's consumer is responsible for the inter-temporal decisions, where he forms expectation given household's type γ. In particular, he decides how much to consume of a consumption aggregate at given prices $C_t(i)$ and how much to invest in bonds $B_t(i)$. Furthermore, the household's consumer holds a portfolio of the continuum of firms (described below) and therefore earns a share of the firms' profit stream $\Gamma_t(i)$. The household members jointly maximize their life-time utility given by

$$E_t^{\gamma}\left\{\sum_{t=0}^{\infty}\beta^t\ U(C_t(i),N_t(i))\right\},$$

where $C_t(i)$ is the quantity of real consumption of agent i of a consumption aggregate defined below and $N_t(i)$ represents individual labour supply. The parameter β is the common discount rate. A fraction χ of the households are of type $\gamma=1$ and form period t expectations about unobserved and future values of variables using the expectations operator E_t^1. The remaining $(1-\chi)$ households are of type $\gamma=2$ and form period t expectations about unobserved and future values of variables using the expectations operator E_t^2. Thus, we impose a heterogeneous expectations operator $\hat{E}_t = \chi E_t^1 + (1-\chi)E_t^2$ as in Branch and McGough (2009, p.3). E_t^{γ} is usually denoted a subjective expectations operator. In our study this

[9] Note that in a world with homogeneous agents it is equivalent to assume a representative household that offers all varieties of labour. In the case of homogeneous agents the assumption of complete financial markets is sufficient to ensure that each individual household earns average income, although its individual labour income may vary due to the Calvo (1983)-pricing of the firm that hires this particular variety to produce a differentiated good. Woodford (2003, p.144ff.) discusses risk sharing issues in models with homogeneous agents in context of a representative agent, a continuum of agents as well as a yeoman farmer setting. For a more general treatment of this kind of risk sharing see Cochrane (2005, p.54ff.)

operator is assumed to fulfill exactly the same assumptions as detailed in Branch and McGough (2009, p.3).[10] With regard to household preferences we assume additive separable instantaneous utility

$$U(C_t(i), N_t(i)) = \frac{C_t(i)^{1-\sigma_C}}{(1-\sigma_C)} - \frac{N_t(i)^{1+\varphi}}{(1+\varphi)}.$$

The parameter σ_C is the coefficient of relative risk aversion, which in this setting equals the inverse of the inter-temporal elasticity of substitution of private consumption and the parameter φ is the degree of convexity of labour disutility. $C_t(i)$ is a Dixit and Stiglitz (1977) consumption aggregate consisting of a continuum of differentiated goods on the interval $j \in [0, 1]$ defined as

$$C_t(i) \equiv \left(\int_0^1 C_t(i,j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}},$$

where $\epsilon > 1$ is the price elasticity of demand[11], i.e. the elasticity of substitution between the differentiated goods. $C_t(i,j)$ denotes the quantity of real consumption of good j by household i. The corresponding price index P_t is defined as

$$P_t \equiv \left(\int_0^1 P_t(j)^{(1-\epsilon)} dj \right)^{\frac{1}{1-\epsilon}}. \tag{2.1}$$

$P_t(j)$ is the price charged for good j in period t. Demand for a good j of a household i is derived by assuming that for any given level of consumption expenditures the household i maximizes his consumption basket. These choices result in household i's demand equations for a good j

$$C_t(i,j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} C_t(i),\ \forall j \in [0, 1]. \tag{2.2}$$

Please be also aware that the consumption goods are non-durable, i.e. households buy them and consume them within period t. These are the only types of goods

[10] We listed the assumptions in Appendix 2.6.1 and will refer to them in the derivations below. Note that one can also find a critical discussion about the assumptions in Branch and McGough (2009, p.7ff.).

[11] Assuming $\epsilon > 1$ ensures that we look at an economy with monopolistic competition among firms, where each firm has a constant desired markup, see Section 2.2.2 below for the details.

in the economy. Thus, the government purchases the same types of goods (see Section 2.2.3 below on government spending).

We also want to remind the reader that we will ignore some features of more elaborate models in order to keep the analysis as simple as possible. Among these features are investment and capital accumulation.[12] In addition, we consider a single, closed economy and ignore any kind of open economy issues like international trade, exchange rates or purchasing power parities.[13]

Note that each household i faces a nominal flow budget constraint

$$\begin{aligned} P_tC_t(i) + E_t^{\gamma}\{Q_{t,t+1}\}B_t(i) + M_t(i) + I_{pt}(i) + P_tT \quad \leq \quad & W_tN_t(i) + B_{t-1}(i) \\ & + M_{t-1}(i) + I_{rt}(i) + \Gamma_t. \end{aligned} \tag{2.3}$$

The left hand side lists how household i distributes his means in period t. The list includes expenses on real consumption at aggregate price level P_t, zero-bond holdings $B_t(i)$ at price $Q_{t,t+1}$ (the nominal stochastic discount factor[14]), money holdings $M_t(i)$ and payments to an actuary fair insurance agency $I_{pt}(i)$.[15] T denotes a fixed lump-sum tax. The means originate from several types of nominal income of household i in period t. First, there is the household's share of firm-sector nominal after-tax profit[16] $\Gamma_t(i) = \Gamma_t$. This has to be the same for all households as we assume that each household owns an equal share of the portfolio of all firms in the economy. Furthermore, there is the nominal labour income $W_tN_t(i)$, where W_t is the aggregate nominal period t wage faced by each household. Next, we have the nominal interest income from bonds $B_{t-1}(i)$, money holdings at the beginning of period t, $M_{t-1}(i)$ and receipts $I_{rt}(i)$ from the actuary fair insurance agency.

[12] See Woodford (2003, ch.5, sec.3), the related corrections in Woodford (2005) or Sveen and Weinke (2005).

[13] See Obstfeld and Rogoff (1995), Clarida et al. (2002) or Galí (2008, ch.7).

[14] See for example Cochrane (2005, p.10) for this concept in general.

[15] It is assumed that at the outset of time the household's consumer signs a contract with the actuary fair insurance agency. The contract guarantees the household the type-dependent average income. As a result, all households of one type will face the same initial as well as subsequent inter-temporal budget constraints in each point in time and in each state of the world, given initial financial assets,.

[16] For the moment we simply accept that there is a revenue tax levied on each firm. This tax is common to all households and is at rate $T_t^{RE} \in [0, 1]$. In Section 2.2.3 below we will detail the nature of the fiscal policy framework.

Be aware that we look at the cashless limit, i.e. $M_{t-1}(i) = M_t(i) = M(i)$.[17] We also impose that agents pay their profits to the actuary fair insurance agency ($I_{pt}(i) = \Gamma_t$). From the agency, they receive an individual payment that ensures them the average nominal income of their type γ given their individual labour income $I_{rt}(i) = P_t\Omega_t^\gamma - W_t N_t(i)$. Thus, Ω_t^γ is the average type-dependent real income net of taxes.

We will now detail consequences of the insurance mechanism for a household. First, we discuss the average type-dependent income net of taxes. It can be calculated by dividing the sum of aggregate profit shares and aggregate labour income of households of a type γ by the number of households of this type. Formally this is

$$\Omega_t^1 = \frac{1}{\chi P_t}\left[\int_0^\chi \Gamma_t di + \int_0^\chi W_t N_t(i) di\right] = \frac{1}{\chi P_t}\left[\chi\Gamma_t + \int_0^\chi W_t N_t(i) di\right]$$

for type $\gamma = 1$ and similarly

$$\begin{aligned}\Omega_t^2 &= \frac{1}{(1-\chi)P_t}\left[\int_\chi^1 \Gamma_t di + \int_\chi^1 W_t N_t(i) di\right] \\ &= \frac{1}{(1-\chi)P_t}\left[(1-\chi)\Gamma_t + \int_\chi^1 W_t N_t(i) di\right],\end{aligned}$$

for type $\gamma = 2$. Thus, aggregate after-tax income in the economy can be written as the sum of aggregate after-tax profits and the aggregate wage bill

$$\begin{aligned}P_t\Omega_t &= P_t(\chi\Omega_t^1 + (1-\chi)\Omega_t^2) \\ &= \Gamma_t + W_t\left[\int_0^\chi N_t(i) di + \int_\chi^1 N_t(i) di\right] \\ &= (1 - T_t^{RE})P_t Y_t - W_t N_t + W_t N_t \\ &= (1 - T_t^{RE})P_t Y_t. \qquad (2.4)\end{aligned}$$

As a direct consequence on the aggregate level in real terms we have $\Omega_t = (1 -$

[17] Considering the cashless limit is common practice when monetary policy is modeled by interest rate rules (as is detailed in Section 2.2.4 below). Woodford (2003, p.31ff.) provides the reasoning for the cashless limit. Be also aware that in the context of analyzing optimal monetary policy, the cashless limit case may lead to distinctive outcomes compared to an analysis which gives a role to money, see Carlstrom and Fuerst (2004, p.328ff.).

$T_t^{RE})Y_t$, in steady-state we have $\Omega = (1 - T^{RE})Y$ and in log-deviations

$$\widehat{\omega}_t = \widehat{y}_t - \frac{T^{RE}}{(1 - T^{RE})}\widehat{\tau}_t^{RE} \tag{2.5}$$

holds.[18] In general, we define a log-deviation of a variable Z_t (in levels) of its steady-state level Z as $\log(Z_t) - \log(Z) \equiv z_t - z \equiv \widehat{z}_t$.

Now, we want to emphasize that the household's worker in each period t will make his labour-supply decision on the base of the flow budget constraint

$$P_t C_t(i) + E_t^\gamma\{Q_{t,t+1}\}B_t(i) + P_t T = W_t N_t(i) + B_{t-1}(i) + P_t\Omega_t^\gamma(i). \tag{2.6}$$

Next, turn attention to the household's consumer and his insurance contract. Each individual consumer, given his type is paid

$$P_t\Omega_t^1(i) = \Gamma_t + \frac{1}{\chi}W_t\int_0^\chi N_t(i)di - W_t N_t(i) = \Gamma_t + W_t\left[\frac{1}{\chi}\int_0^\chi N_t(i)di - N_t(i)\right]$$

or

$$\begin{aligned} P_t\Omega_t^2(i) &= \Gamma_t + \frac{1}{(1-\chi)}W_t\int_\chi^1 N_t(i)di - W_t N_t(i) \\ &= \Gamma_t + W_t\left[\frac{1}{(1-\chi)}\int_\chi^1 N_t(i)di - N_t(i)\right]. \end{aligned}$$

In each of the two identities above, the first term is the individual share of aggregate after-tax profits and the second term is the difference between average type dependent labour income and individual labour income. The latter term is the core of the risk sharing mechanism. It becomes clear that, with the risk sharing mechanism in place, for the household's consumer, we can equivalently write (2.6) as

$$P_t C_t(i) + E_t^\gamma\{Q_{t,t+1}\}B_t(i) + P_t T = B_{t-1}(i) + P_t\Omega_t^\gamma,$$

[18] The derivation of the latter identity is outlined in Appendix 2.6.2.1. Please be aware that we follow DeJong and Dave (2007, p.13ff.) in all log-linear approximations.

which in real terms is

$$C_t(i) + E_t^\gamma\{Q_{t,t+1}\}\frac{B_t(i)}{P_t} + T \leq \frac{B_{t-1}(i)}{P_{t-1}}\Pi_{t-1,t}^{-1} + \Omega_t^\gamma, \tag{2.7}$$

where $\Pi_{t-1,t} \equiv P_t/P_{t-1}$ is the gross inflation rate from period $t-1$ to period t. Furthermore, we assume that each household faces the *subjective solvency condition*

$$\lim_{T\mapsto\infty} E_t^\gamma\{Q_{t,T+1}\}\frac{B_T(i)}{P_T} \geq 0$$

for all t. Loosely speaking, households need to have zero or positive real bond holdings at Judgement Day. The initial period endowment of households with bonds is assumed to be zero, i.e. $B_0(i) = 0$. Given the insurance mechanism above, we can set up the Lagrangian for a household i, which is given by

$$\begin{aligned}\mathcal{L} \;=\; & E_t^\gamma \textstyle\sum_{t=0}^\infty \beta^t\{\frac{(C_t(i))^{1-\sigma_C}}{(1-\sigma_C)} - \frac{(N_t(i))^{1+\varphi}}{(1+\varphi)} \\ & -\lambda_t\left[P_tC_t(i) + E_t^\gamma\{Q_{t,t+1}\}B_t(i) + P_tT - W_tN_t(i) - B_{t-1}(i) - P_t\Omega_t^\gamma(i)\right]\}.\end{aligned}$$

The related first-order conditions are

$$\begin{aligned}(a) \quad \tfrac{\partial\mathcal{L}}{\partial C_t(i)} &: & \beta^t\{(C_t(i))^{-\sigma_C} - \lambda_tP_t\} &\overset{!}{=} 0\\ (b) \quad \tfrac{\partial\mathcal{L}}{\partial B_t(i)} &: & E_t^\gamma\{-\beta^t\lambda_tQ_{t,t+1} + \beta^{t+1}\lambda_{t+1}\} &\overset{!}{=} 0\\ (c) \quad \tfrac{\partial\mathcal{L}}{\partial N_t(i)} &: & -\beta^t\{(N_t(i))^\varphi - \lambda_tW_t\} &\overset{!}{=} 0,\end{aligned}$$

whereby the household's consumer chooses (a) and (b) and the worker chooses (c). Conditions (a) and (b) yield the (general) household Euler equation

$$E_t^\gamma\{Q_{t,t+k}\} = \beta^k E_t^\gamma\left\{\left(\frac{C_{t+k}(i)}{C_t(i)}\right)^{-\sigma_C}\frac{P_t}{P_{t+k}}\right\}. \tag{2.8}$$

Taking logs and subtracting c on both sides yields

$$\widehat{c}_t(i) = E_t^\gamma\{\widehat{c}_{t+1}(i)\} - \frac{1}{\sigma_C}\left[i_t - E_t^\gamma\{\pi_{t+1}\} - \rho\right], \tag{2.9}$$

for $k = 1$, where $i_t \equiv -\log(E_t^\gamma\{Q_{t,t+1}\})$ denotes the short-term nominal interest rate and $\rho \equiv -\log(\beta)$ denotes the discount rate. This equation expresses the *inter-temporal* decision of each household i between consumption today and tomorrow depending on the real rate of interest, given his type of expectations.

Condition (a) and (c) relate the *intra-temporal* consumption-leisure trade-off to the real wage

$$\frac{W_t}{P_t} = N_t^{\varphi}(i) C_t^{\sigma_C}(i).$$

We recognize that this decision is the same for all households independent of their type and therefore, in log-linear terms, we get

$$w_t - p_t = \varphi n_t + \sigma_C c_t. \tag{2.10}$$

Next we get back to the household's flow budget constraint (2.7). In a zero-inflation steady-state, in which $E_t^{\gamma}\{Q_{t,t+k}\}) = \beta^k$ and $\Pi_{t,t+k} = \Pi = 1$ hold, a log-linear approximation around the steady-state[19] is

$$\begin{aligned} \widehat{c}_t(i) &= \frac{B/P}{[C + \beta(B/P)\sigma_C]}\left[\widehat{b}_{t-1}(i) - \pi_t\right] + \frac{\beta(B/P)\sigma_C}{[C + \beta(B/P)\sigma_C]} E_t^{\gamma}\{\widehat{c}_{t+1}(i)\} \\ & \quad - \frac{\beta(B/P)}{[C + \beta(B/P)\sigma_C]}\left[\widehat{b}_t(i) - E_t^{\gamma}\{\pi_{t+1}\}\right] + \frac{C + T - (B/P)(1-\beta)}{C + \beta(B/P)\sigma_C}\,\widehat{\omega}_t^{\gamma} \\ &\equiv \widehat{\mathcal{W}}_t^{\gamma}. \end{aligned} \tag{2.11}$$

Loosely speaking a household can only consume (here in terms of deviations) what he owns in each period. In the household's individual perception this is inflation-adjusted real bond income plus deviations in expected next period consumption minus inflation adjusted real bond purchases plus the average real income of his type. One can think of $\widehat{\mathcal{W}}_t^{\gamma}$ as wealth deviations from steady-state of a household i of type γ. We can plug (2.11) into the Euler equation (2.9) to get

$$\widehat{\mathcal{W}}_t^{\gamma} = E_t^{\gamma}\{\widehat{\mathcal{W}}_{t+1}^{\gamma}\} - \frac{1}{\sigma_C}\left[i_t - E_t^{\gamma}\{\pi_{t+1}\} - \rho\right] \tag{2.12}$$

[19]See Appendix 2.6.2.2 for the derivation.

and iterate this equation forward to obtain

$$\widehat{\mathcal{W}}_t^{\gamma} = \underbrace{\lim_{k \mapsto \infty} E_t^{\gamma}\{\widehat{\mathcal{W}}_{t+k+1}^{\gamma}\}}_{\equiv \widehat{\mathcal{W}}_{\infty}^{\gamma}} - \frac{1}{\sigma_C} E_t^{\gamma} \left\{ \sum_{k=0}^{\infty} [i_{t+k} - \pi_{t+k+1} - \rho] \right\}$$

or

$$\widehat{\mathcal{W}}_t^{\gamma} = \widehat{\mathcal{W}}_{\infty}^{\gamma} - \frac{1}{\sigma_C} E_t^{\gamma} \left\{ \sum_{k=0}^{\infty} [i_{t+k} - \pi_{t+k+1} - \rho] \right\}, \tag{2.13}$$

where we make use of assumptions *A1, A3, A4 and A5.*[20]

2.2.2 The Firm Sector

There exists a continuum of firms $i \in [0,1]$, where each firm produces one differentiated good $j \in [0,1]$. Every individual firm consist of two decision makers, the hiring department and the sales department. Before turning to an individual firm i's production technology and pricing problem, we will outline the evolution of the aggregate price level in the economy.

Aggregate Price Level Dynamics

We assume that firms set prices according to mechanism introduced by Calvo (1983). In each period t a fraction of firms $(1-\theta)$ will, at random, receive a signal to reset prices and will consequently choose the optimal price P_t^*. The fraction θ, at random has to stick to its individual price already in place at the beginning of period t. On the aggregate this is P_{t-1}. If we recall the definition of the aggregate price level (2.1), it becomes clear that the aggregate price level

[20] Appendix 2.6.2.3 discusses the iteration procedure and the application of the assumptions in greater detail.

evolves according to

$$\begin{aligned} P_t &= \left[\theta P_{t-1}^{(1-\epsilon)} + (1-\theta)(P_t^*)^{(1-\epsilon)}\right]^{\frac{1}{(1-\epsilon)}} \\ \text{or} \quad 1 &= \left[\theta(\tfrac{P_{t-1}}{P_t})^{(1-\epsilon)} + (1-\theta)(\tfrac{P_t^*}{P_t})^{(1-\epsilon)}\right]^{\frac{1}{(1-\epsilon)}} \\ \text{or} \quad 1 &= \left[\theta(\Pi_{t-1,t})^{-(1-\epsilon)} + (1-\theta)(\tfrac{P_t^*}{P_t})^{(1-\epsilon)}\right]^{\frac{1}{(1-\epsilon)}} \\ \text{or} \quad 1 &= \theta(\Pi_{t-1,t})^{-(1-\epsilon)} + (1-\theta)(\tfrac{P_t^*}{P_t})^{(1-\epsilon)}. \end{aligned}$$

A log-linear approximation[21] around a *zero inflation steady-state* is given by

$$\widehat{p}_t^* - \widehat{p}_t = \frac{\theta}{(1-\theta)}\pi_t. \tag{2.14}$$

Thus, in our economy inflation arises from the firms' optimal price deviations from the aggregate price level.

Technology

As mentioned before there is a continuum of monopolistically competitive firms where each firm $i \in [0,1]$ produces a differentiated good $j \in [0,1]$. Each firm has access to the identical technology

$$Y_t(i,j) = Y_t(i) = A_t\ N_t(i)^{1-\alpha}, \tag{2.15}$$

where $Y_t(i,j) = Y_t(i)$ is period t output of the variety j by firm i. Thus, indices i and j coincide. The parameter $\alpha \in [0,1]$ is the output elasticity of labour and the variable A_t is the multi-factor productivity evolving exogenous over time according to

$$a_t = \rho_a\ a_{t-1} + \epsilon_t^a. \tag{2.16}$$

This law of motion is stated in terms of the logarithm, i.e. $a_t \equiv \log(A_t)$. We assume that $\rho_a \in [0,1)$ and ϵ_t^a is a white-noise process with mean zero and

[21] See Appendix 2.6.2.4 for the derivation.

constant variance σ_a^2. Next, firm i's real marginal costs by definition are

$$MC_t(i) = \frac{W_t}{P_t \; MPN_t(i)}. \tag{2.17}$$

Here $MPN_t(i)$ denotes the marginal product of labour which is given by

$$MPN_t(i) \equiv \frac{\partial Y_t(i)}{\partial N_t(i)} = (1-\alpha) A_t N_t(i)^{-\alpha}. \tag{2.18}$$

Therefore we can rewrite (2.17) as

$$\begin{aligned} MC_t(i) &= \tfrac{W_t \; N_t(i)^{\alpha}}{P_t \; (1-\alpha) \; A_t} \\ \text{or} \qquad & \\ &= \tfrac{W_t \; N_t(i)}{P_t \; (1-\alpha) \; Y_t(i)}. \end{aligned} \tag{2.19}$$

Optimal Price Setting

Recall that each household owns an equal share of the portfolio of the continuum of firms. This is crucial with regard to the optimal price setting behaviour of a single firm. Exactly on these grounds we assume that every firm will act in proportion to the owners expectations when it forms expectations about the future. Thus, we directly apply the heterogeneous expectations operator $\widehat{E}_t$ to the firm's problem.[22] In our case, it is sufficient to restrict attention to a firm i that revises its price optimally in period t and takes the whole time-span k into account, in which the price is valid. Overall, the firm faces the problem of choosing a price for its good $P_t^*(i)$, the level of employment $N_{t+k|t}(i)$ and the quantity of its output $Y_{t+k|t}(i)$ given its constraints and a revenue tax T_{t+k}^{RE}. Formally, that is

$$\max_{N_{t+k|t}(i), Y_{t+k|t}(i), P_t^*(i)} \sum_{k=0}^{\infty} \theta^k \widehat{E}_t \left\{ Q_{t,t+k} \left[(1 - T_{t+k}^{RE}) \; P_t^*(i) \; Y_{t+k|t}(i) - W_{t+k} N_{t+k|t}(i) \right] \right\}$$

[22] This is clearly a deviation from Branch and McGough (2009, p.6ff.), who derive the type-dependent optimal pricing decision in a yeoman-farmer setting. From our understanding, the results must be equivalent.

s.t.

$$(i) \quad Y_{t+k|t}(i) \quad = \quad \left(\frac{P_t^*(i)}{P_{t+k}}\right)^{-\epsilon} Y_{t+k}$$

$$(ii) \quad Y_{t+k|t}(i) \quad = \quad A_{t+k} N_{t+k|t}(i)^{(1-\alpha)}.$$

Condition (i) is the market demand for the good produced by firm i, i.e. the household demand schedule (2.2) together with a similar schedule for constant government purchases G given aggregate price level (2.1) (see details for the latter in Section 2.2.3 below). The private and public demand schedule add up to the total demand schedule (see Section 2.3.1 below). Condition (ii) is the production technology (2.15).[23] Thus, the Lagrangian of a firm is given by

$$\begin{aligned} \mathcal{L} \quad = \quad & \textstyle\sum_{k=0}^{\infty} \theta^k \widehat{E}_t \{ Q_{t,t+k} [(1 - T_{t+k}^{RE}) \; P_t^*(i) \; Y_{t+k|t}(i) - W_{t+k} N_{t+k|t}(i)) \\ & -\lambda_{t+k}(i)(Y_{t+k|t}(i) - \left(\frac{P_t^*(i)}{P_{t+k}}\right)^{-\epsilon} Y_{t+k})) \\ & -\xi_{t+k}(i)(Y_{t+k|t}(i) - A_{t+k} N_{t+k|t}(i)^{(1-\alpha)})]\}. \end{aligned}$$

The first-order condition with respect to $N_{t+k|t}(i)$ is given by

$$\frac{\partial \mathcal{L}}{\partial N_{t+k|t}(i)} \quad : \quad -W_{t+k} + \xi_{t+k}(i) A_{t+k} (1-\alpha) N_{t+k|t}(i)^{-\alpha} \quad \stackrel{!}{=} \quad 0$$

$$\underbrace{\frac{W_{t+k} \; N_{t+k|t}(i)}{Y_{t+k|t}(i) \; (1-\alpha)}}_{MC_{t+k|t}^n(i)} \quad = \quad \xi_{t+k}(i),$$

where $MC_{t+k|t}^n(i)$ are the period $t+k$ nominal marginal costs of a firm that last reset its price in period t.[24] The first-order condition with respect to $Y_{t+k|t}(i)$ is

[23] Please be aware that a revenue tax for firms is equivalent to a distortionary tax on labour income as both types affect the marginal after-tax profit of a firm in exactly the same way in this economy.

[24] The decreasing returns to scale technology causes asymmetry among individual marginal cost. This technology therefore introduces some endogenous price stickiness which usually brings the behaviour of inflation closer to empirical evidence in this type of models. In order to understand this, note, that a firm always will serve the market demand for the specific good it offers. Now, suppose that a firm that receives the exogenous signal to change the price and has to fix this price over the expected lifetime of the price, expects the average marginal cost to increase over this life-span.

given by

$$\frac{\partial \mathcal{L}}{\partial Y_{t+k|t}(i)} \quad : \quad (1 - T_{t+k}^{RE})P_t^*(i) - \lambda_{t+k}(i) - \xi_{t+k}(i) \quad \stackrel{!}{=} \quad 0$$

$$\underbrace{(1 - T_{t+k}^{RE})P_t^*(i) - \xi_{t+k}(i)}_{\text{marginal after-tax profit}} \quad = \quad \lambda_{t+k}(i).$$

The first-order condition with respect to $P_t^*(i)$ is given by

$$\frac{\partial \mathcal{L}}{\partial P_t^*(i)} \quad : \quad \widehat{E}_t\{\textstyle\sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \times$$

$$\left[(1 - T_{t+k}^{RE})Y_{t+k|t}(i) - \lambda_{t+k}(i)(-(-\epsilon)\left(\frac{P_t^*(i)}{P_{t+k}}\right)^{-\epsilon-1} \frac{1}{P_{t+k}} Y_{t+k|t})\right]\} \quad \stackrel{!}{=} \quad 0.$$

We now plug in constraint (i), multiply by $P_t^*(i)$ and rearrange terms to get

$$\widehat{E}_t \left\{\textstyle\sum_{k=0}^{\infty} \theta^k Q_{t,t+k} Y_{t+k|t}(i) \left[(1 - T_{t+k}^{RE})P_t^*(i) - \lambda_{t+k}(i)\epsilon\right]\right\} \quad = \quad 0.$$

If it does not change the price, the market demand for its specific good will increase and this will decrease its individual marginal product of labour and increase its individual marginal cost leading to suboptimal profits for a monopolistically competitive firm.

If it does change the price to exactly match the expected increase in average marginal cost, this will decrease the market demand for its specific good and this will increase its individual marginal product of labour and decrease its individual marginal cost leading again to suboptimal profits for monopolistically competitive firm.

Consequently, in order to maintain a situation of optimal profits, the monopolistically competitive firm that is aware of the impact of its price-setting behaviour on the market demand for the specific good it offers and expects average marginal cost to increase, will set a price below the price that will exactly match the expected increase in average marginal cost. The reason is that the increased price will, via the reduced demand for the specific good offered by the firm, also decrease the individual marginal cost to some extent. In this way the firm is able to maintain optimal profits.

Thus, this channel does limit the size of price changes. In consequence, a larger fraction of firms that is allowed to reset its price may forgo a change of its price at all and exactly this causes a higher degree of price stickiness via endogenous firm behaviour as the fraction of firms that changes prices, via the law of large numbers, determines the frequency of price changes.

Using the other two first-order conditions from above, we get

$$\begin{aligned}
&\widehat{E}_t\{\textstyle\sum_{k=0}^{\infty}\theta^k Q_{t,t+k}Y_{t+k|t}(i)\times \\
&\Big[(1-T^{RE}_{t+k})P^*_t(i)-\epsilon(1-T^{RE}_{t+k})P^*_t(i)+\epsilon MC^n_{t+k|t}(i)\Big]\} &=& \quad 0 \\
&\widehat{E}_t\left\{\textstyle\sum_{k=0}^{\infty}\theta^k Q_{t,t+k}Y_{t+k|t}(i)\left[(1-T^{RE}_{t+k})P^*_t(i)-\frac{\epsilon}{\epsilon-1}MC^n_{t+k|t}(i)\right]\right\} &=& \quad 0 \\
&\widehat{E}_t\left\{\textstyle\sum_{k=0}^{\infty}\theta^k Q_{t,t+k}Y_{t+k|t}(i)\left[(1-T^{RE}_{t+k})P^*_t(i)-\mathcal{M}MC^n_{t+k|t}(i)\right]\right\} &=& \quad 0,
\end{aligned}$$

where $\mathcal{M}\equiv\frac{\epsilon}{\epsilon-1}$. We can see that for $\theta=0$, i.e. in the absence of nominal price rigidity, $(1-T^{RE}_t)P^*_t(i)=\mathcal{M}MC^n_{t|t}(i)$ holds. Thus, $\mathcal{M}$ can be interpreted as the desired constant after-tax mark-up. In addition, we can divide the equation above by the aggregate price level in the economy P_t, which yields

$$\widehat{E}_t\left\{\textstyle\sum_{k=0}^{\infty}\theta^k Q_{t,t+k}Y_{t+k|t}(i)\left[(1-T^{RE}_{t+k})\frac{P^*_t(i)}{P_t}-\mathcal{M}\ \frac{P_{t+k}}{P_{t+k}}\frac{1}{P_t}\ MC^n_{t+k|t}(i)\right]\right\} = 0$$

$$\widehat{E}_t\left\{\textstyle\sum_{k=0}^{\infty}\theta^k Q_{t,t+k}Y_{t+k|t}(i)\left[(1-T^{RE}_{t+k})\frac{P^*_t(i)}{P_t}-\mathcal{M}\ \Pi_{t,t+k}\ MC_{t+k|t}(i)\right]\right\} = 0,$$

where $MC_{t+k|t}(i)\equiv\frac{MC^n_{t+k|t}(i)}{P_{t+k}}$ are real marginal costs and $\Pi_{t,t+k}\equiv\frac{P_{t+k}}{P_t}$ is gross inflation. Furthermore, we can use the household Euler equation (2.8) in order to get

$$\begin{aligned}
&\widehat{E}_t\{\sum_{k=0}^{\infty}(\beta\theta)^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma_C}\Pi^{-1}_{t,t+k}\ Y_{t+k|t}(i)\times \\
&\left[(1-T^{RE}_{t+k})\frac{P^*_t(i)}{P_t}-\mathcal{M}\ \Pi_{t,t+k}\ MC_{t+k|t}(i)\right]\} \quad = \quad 0.
\end{aligned} \tag{2.20}$$

A log-linearized version of (2.20) is[25]

$$\widehat{p^*_t}(i)-\widehat{p_t}=(1-\beta\theta)\widehat{E}_t\left\{\sum_{k=0}^{\infty}(\beta\theta)^k\left[\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}^{RE}_{t+k}+\left(\widehat{\pi}_{t|t+k}+\widehat{mc}_{t+k|t}(i)\right)\right]\right\}. \tag{2.21}$$

[25]See Appendix 2.6.2.5 for the derivation.

2.2.3 The Fiscal Authority's Policy

The fiscal authority is in charge of financing constant government expenditures G and considers the nominal flow government budget constraint

$$B_{t-1} + M_{t-1} + P_t G \leq \widehat{E}_t\{Q_{t,t+1}\}B_t + M_t + P_t T_t^{RE} Y_t + P_t T,$$

stating that the government in each period can refinance its debt outstanding (left-hand side) by issuing new bonds, money, levying a revenue tax at rate $T^{RE} \in [0, 1]$ on nominal output or a lump-sum tax T. Considering the cashless limit case yields

$$\frac{B_{t-1}}{P_{t-1}}\,\Pi_{t-1,t}^{-1} + G \leq \widehat{E}_t\{Q_{t,t+1}\}\frac{B_t}{P_t} + T_t^{RE} Y_t + T \tag{2.22}$$

in real terms. In addition, we assume that the government must obey the transversality condition

$$\lim_{T\mapsto\infty} \widehat{E}_t\{Q_{t,T+1}\}B_T \leq 0$$

for all t and initial value B_0. Thus, the government must ensure that at the end of all times its debt outstanding equals either zero or is negative. Next, basically following the approach of Leeper (1991) or more recently Evans and Honkapohja (2007, p.669), we assume that the government sets the revenue tax rate T_t^{RE} to back a fraction $\varrho_1 \in [0, 1]$ of the real debt outstanding by tax revenue. Formally the tax rule is

$$T + T_t^{RE} Y_t = \left[\varrho_0 + \varrho_1 \frac{B_{t-1}}{P_{t-1}}\,\Pi_{t-1,t}^{-1}\right]\Psi_t. \tag{2.23}$$

The parameter ϱ_0 is a constant and Ψ_t is a tax-revenue shock. A log-linear approximation is[26]

$$\widehat{\tau}_t^{RE} = \frac{B/P}{T^{RE}Y}\varrho_1[\widehat{b}_{t-1} - \pi_t] - \widehat{y}_t + \frac{(T + T^{RE}Y)}{T^{RE}Y}\widehat{\Psi}_t, \tag{2.24}$$

where we assume that $\widehat{\Psi}_t$ follows an AR(1) process according to

$$\widehat{\Psi}_t = \rho_\Psi \widehat{\Psi}_{t-1} + \epsilon_t^\Psi$$

[26]See Appendix 2.6.2.6 for the derivation.

with the coefficient $\rho_\Psi \in [0,1)$. ϵ_t^Ψ is a white-noise process with mean zero and constant variance σ_Ψ^2.

Thus, the revenue tax rate deviations increase with inflation adjusted real debt outstanding and decrease with output.[27] We want to emphasize the latter fact, as it makes clear that we do not exactly follow Leeper (1991) or Evans and Honkapohja (2007, p.669). Our revenue tax rate is responsive to the business-cycle. Higher employment or an increase in multi-factor productivity both increase output and this in turn increases the tax base. Given the level of real-debt outstanding, this tax base increase reduces the tax rate deviations. Furthermore, a log-linear approximation to the flow government budget constraint (2.22) is given by[28]

$$\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\} = \sigma_C \widehat{E}_t\{\widehat{c}_{t+1}\} - \sigma_C \widehat{c}_t + \frac{1}{\beta}(\widehat{b}_{t-1} - \pi_t) - \frac{T^{RE}Y}{\beta(B/P)}(\widehat{\tau}_t^{RE} + \widehat{y}_t). \quad (2.25)$$

We may plug (2.24) into (2.25) to get

$$\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\} = \sigma_C \widehat{E}_t\{\widehat{c}_{t+1}\} - \sigma_C \widehat{c}_t + \frac{(1-\varrho_1)}{\beta}(\widehat{b}_{t-1} - \pi_t) - \frac{(T + T^{RE}Y)}{\beta(B/P)}\widehat{\Psi}_t,$$

which indicates that except for a passive fiscal policy ($\varrho_1 = 1$) current real debt outstanding increases in last periods inflation adjusted real debt outstanding as well as the aggregate expected consumption growth. Moreover, a positive surprise in the tax revenue decreases current real debt outstanding to some extent. Given our fiscal policy setting, defining *active* and *passive* fiscal policy in the spirit of Leeper (1991) and Evans and Honkapohja (2007) means that fiscal policy is *passive* if $\varrho_1 = 1$, i.e. all debt outstanding is repaid each period and *active* if $\varrho_1 \in [0,1)$.[29]

[27] This appears to be a very volatile tax rate setting mechanism that may not be in line with empirical evidence.

[28] See Appendix 2.6.2.7 for the derivation.

[29] Note that our system has more dimensions and complexity than the systems of Leeper (1991) and Evans and Honkapohja (2007) by construction. Therefore we could not manage to link the characterization of *active* and *passive* fiscal policy to the eigenvalues of the system matrix. The same is true for the characterization of *active* and *passive* monetary policy in section 2.2.4 below.

2.2.4 The Central Bank's Policy

The central bank's policy instrument is the nominal interest rate i_t. The central bank is assumed to implement its policy by the use of a forward-looking version of a Taylor-type rule

$$i_t = \phi_{\widehat{x}}\ E_t\{\widehat{x}_{t+1}\} + \phi_\pi\ E_t\{\pi_{t+1}\} + v_t. \tag{2.26}$$

The parameters $\phi_{\widehat{x}} \geq 0$, $\phi_\pi \geq 0$ are the monetary policy response coefficients with regard to expected output gap deviations and expected inflation deviations respectively.[30] Furthermore, v_t denotes a monetary policy shock that follows an AR(1) process

$$v_t = \rho_v v_{t-1} + \epsilon_t^v$$

with the coefficient $\rho_v \in [0, 1)$. ϵ_t^v is a white-noise process with mean zero and constant variance σ_v^2.

This is a deviation from the assumption about monetary policy by Leeper (1991) or Evans and Honkapohja (2007) who both assume that the central bank follows a Taylor-type rule that feeds back solely on current inflation and ignores output. We believe that our assumption on the conduct of monetary policy is closer to actual practice. Nowadays central banks are often concerned with price stability and the business cycle. Furthermore, this specific functional form mimics the basic features of some reaction functions. These functions implement optimal monetary policy based on a specific target criterion, as is demonstrated by Woodford (2003, p.293ff.) or Svensson and Woodford (2005, p.71ff.) for example. Another argument in favour of the choice of this particular interest rule is to keep results more comparable to the analysis of Branch and McGough (2009), who assume the same functional form.

Please be aware that if a central bank sticks to the Taylor principle, this corresponds to $\phi_\pi > 1$ for the simple rule (2.26). At least under RE it is a quite robust finding in the literature that sticking to this principle is sufficient to render an economy determinate. But the principle is only the exact formal sufficient condition for determinacy if $\phi_{\widehat{x}} = 0$. This was demonstrated in Bullard and Mitra (2002). In our setting, as in Leeper (1991) as well as in Evans and

[30] Note that $\widehat{x}_t$ measures the welfare relevant output gap deviations in period t. We account for these deviations in Section 2.3.7 below.

Honkapohja (2007), sticking to this principle is equivalent to an *active* monetary policy, while $\phi_\pi \in [0, 1]$ is denoted as a *passive* monetary policy.

2.3 Components of Equilibrium

We will now examine under which conditions the goods market, bonds market and labour market clear. Furthermore, we derive the natural levels and the efficient levels of output and account for the tax distortions in the equilibrium conditions.

2.3.1 Goods Market Clearing

Each firm j satisfies total demand for its good, that is

$$Y_t(j) = C_t(j) + G(j),$$

where $C_t(j)$ is total private sector demand for good j and $G(j)$ is total public sector demand for good j. By definition[31] we can rewrite the total private sector demand for good j as the aggregate of individual household demands for good j, that is

$$Y_t(j) = \int_0^1 C_t(i, j)di + G(j)$$

and making use of (2.2) as well as the counterpart for public demand $G(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} G$ yields

$$Y_t(j) = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} C_t(i)di + \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} G,$$

or

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} \left(\int_0^1 C_t(i)di + G\right).$$

It follows that

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} (C_t + G) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t. \tag{2.27}$$

[31] Please keep in mind that aggregating a variable $Z_t(i)$ over all agents $i \in [0, 1]$ requires to take the integral $\int_0^1 Z_t(i)di = Z_t$.

The latter identity holds, as the aggregate output of the economy is defined as

$$Y_t \equiv \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}.$$

Applying this definition to (2.27), yields the aggregate goods market clearing condition

$$Y_t = C_t + G. \tag{2.28}$$

By log-linearizing this condition around a steady-state[32], we obtain

$$\widehat{y}_t = \frac{C}{Y}\widehat{c}_t \tag{2.29}$$

in terms of steady-state deviations. Notice that government spending G plays no role as we assumed it to be constant.

2.3.2 Bonds Market Clearing

First, for the bond market to clear, the total amount of bonds issued by the government must either be held by agents of type $\gamma = 1$ or of type $\gamma = 2$, that is

$$B_t = \chi B_t^1 + (1-\chi)B_t^2. \tag{2.30}$$

A log-linear approximation to (2.30) in real terms using assumption *A2* is[33]

$$\widehat{b}_t = \chi \widehat{b}_t^1 + (1-\chi)\widehat{b}_t^2. \tag{2.31}$$

Given (2.31) the overall period t average real wealth in the economy

$$\widehat{\mathcal{W}}_t = \chi \widehat{\mathcal{W}}_t^1 + (1-\chi)\widehat{\mathcal{W}}_t^1$$

[32] See Appendix 2.6.2.8 for the derivation.
[33] See Appendix 2.6.2.9 for the derivation.

can be computed by

$$
\begin{aligned}
\widehat{\mathcal{W}}_t \;=\;& \chi[\frac{(B/P)}{[C+\beta(B/P)\sigma_C]}\left[\widehat{b}^1_{t-1}-\pi_t\right]+\frac{\beta(B/P)\sigma_C}{[C+\beta(B/P)\sigma_C]}E^1_t\{\widehat{c}_{t+1}\} \\
&-\frac{\beta(B/P)}{[C+\beta(B/P)\sigma_C]}\left[\widehat{b}^1_t-E^1_t\{\pi_{t+1}\}\right]+\frac{C+T-(B/P)(1-\beta)}{C+\beta(B/P)\sigma_C}\,\widehat{\omega}^1_t] \\
&+(1-\chi)[\frac{(B/P)}{[C+\beta(B/P)\sigma_C]}\left[\widehat{b}^2_{t-1}-\pi_t\right]+\frac{\beta(B/P)\sigma_C}{[C+\beta(B/P)\sigma_C]}E^2_t\{\widehat{c}_{t+1}\} \\
&-\frac{\beta(B/P)}{[C+\beta(B/P)\sigma_C]}\left[\widehat{b}^2_t-E^2_t\{\pi_{t+1}\}\right]+\frac{C+T-(B/P)(1-\beta)}{C+\beta(B/P)\sigma_C}\,\widehat{\omega}^2_t] \\
=\;& \frac{(B/P)}{[C+\beta(B/P)\sigma_C]}\left[\widehat{b}_{t-1}-\pi_t\right]+\frac{\beta(B/P)\sigma_C}{[C+\beta(B/P)\sigma_C]}\widehat{E}_t\{\widehat{c}_{t+1}\} \\
&-\frac{\beta(B/P)}{[C+\beta(B/P)\sigma_C]}\left[\widehat{b}_t-\widehat{E}_t\{\pi_{t+1}\}\right] \\
&+\frac{C+T-(B/P)(1-\beta)}{C+\beta(B/P)\sigma_C}\,\widehat{\omega}_t, \qquad (2.32)
\end{aligned}
$$

where we make use of (2.11), assume that within-type decisions are symmetric and each household $i\in[0,\chi]$ is of type $\gamma=1$ and each household $i\in(\chi,1]$ is of type $\gamma=2$. Furthermore, from taking into account (2.31), (2.32), (2.11) as well as (2.29) (for each type γ) and (2.5), we get

$$
\begin{aligned}
\widehat{y}_t \;=\;& \underbrace{\left[\frac{C+\beta(B/P)\sigma_C}{C+T-(B/P)(1-\beta)}\right]}_{\equiv\delta_1}\left[\chi\widehat{\mathcal{W}}^1_t+(1-\chi)\widehat{\mathcal{W}}^2_t\right] \\
&-\underbrace{\left[\frac{(B/P)}{C+T-(B/P)(1-\beta)}\right]}_{\equiv\delta_2}\left[\widehat{b}_{t-1}-\pi_t\right] \\
&+\underbrace{\left[\frac{\beta(B/P)\sigma_C}{(B/P)(1-\beta)-C-T}\frac{Y}{C}\right]}_{\equiv\delta_3}\widehat{E}_t\{\widehat{y}_{t+1}\} \\
&+\underbrace{\left[\frac{\beta(B/P)}{C+T-(B/P)(1-\beta)}\right]}_{\equiv\delta_4}\left[\widehat{b}_t-\widehat{E}_t\{\pi_{t+1}\}\right]+\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}^{RE}_t,
\end{aligned}
$$

or more compact

$$\begin{aligned}
\widehat{y}_t &= \delta_1\left[\chi\widehat{\mathcal{W}}_t^1+(1-\chi)\widehat{\mathcal{W}}_t^2\right]-\delta_2\left[\widehat{b}_{t-1}-\pi_t\right]+\delta_3\widehat{E}_t\{\widehat{y}_{t+1}\}\\
&\quad+\delta_4\left[\widehat{b}_t-\widehat{E}_t\{\pi_{t+1}\}\right]+\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE}\\
&= \delta_1[\chi\left(\widehat{\mathcal{W}}_\infty^1-\frac{1}{\sigma_C}E_t^1\left\{\sum_{k=0}^{\infty}[i_{t+k}-\pi_{t+k+1}-\rho]\right\}\right)\\
&\quad+(1-\chi)\left(\widehat{\mathcal{W}}_\infty^2-\frac{1}{\sigma_C}E_t^2\left\{\sum_{k=0}^{\infty}[i_{t+k}-\pi_{t+k+1}-\rho]\right\}\right)]\\
&\quad-\delta_2\left[\widehat{b}_{t-1}-\pi_t\right]+\delta_3\widehat{E}_t\{\widehat{y}_{t+1}\}+\delta_4\left[\widehat{b}_t-\widehat{E}_t\{\pi_{t+1}\}\right]+\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE}\\
&= \delta_1\left[\chi\widehat{\mathcal{W}}_\infty^1+(1-\chi)\widehat{\mathcal{W}}_\infty^2-\frac{1}{\sigma_C}\widehat{E}_t\left\{\sum_{k=0}^{\infty}[i_{t+k}-\pi_{t+k+1}-\rho]\right\}\right]\\
&\quad-\delta_2\left[\widehat{b}_{t-1}-\pi_t\right]+\delta_3\widehat{E}_t\{\widehat{y}_{t+1}\}+\delta_4\left[\widehat{b}_t-\widehat{E}_t\{\pi_{t+1}\}\right]+\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE},
\end{aligned} \tag{2.33}$$

where we make use of (2.13). One period ahead, (2.33) reads

$$\begin{aligned}
\widehat{E}_t\{\widehat{y}_{t+1}\} &= \delta_1\left[\widehat{E}_t\left\{\chi\widehat{\mathcal{W}}_\infty^1+(1-\chi)\widehat{\mathcal{W}}_\infty^2\right\}-\frac{1}{\sigma_C}\widehat{E}_t\left\{\sum_{k=1}^{\infty}[i_{t+k}-\pi_{t+k+1}-\rho]\right\}\right]\\
&\quad-\delta_2\left[\widehat{b}_t-\widehat{E}_t\{\pi_{t+1}\}\right]+\delta_3\widehat{E}_t\{\widehat{y}_{t+2}\}+\delta_4\left[\widehat{E}_t\{\widehat{b}_{t+1}\}-\widehat{E}_t\{\pi_{t+2}\}\right]\\
&\quad+\frac{T^{RE}}{(1-T^{RE})}\widehat{E}_t\{\widehat{\tau}_{t+1}^{RE}\}\\
&\quad-\underbrace{\text{term of higher order beliefs}}_{=0,\text{ due to A6}}.
\end{aligned}$$

We can subtract the two equations above to finally get a *Dynamic IS Curve*

$$\begin{aligned}
\widehat{y}_t &= \widehat{E}_t\left\{\widehat{y}_{t+1}\right\} - \delta_1\frac{1}{\sigma_C}\left[i_t - \widehat{E}_t\{\pi_{t+1}\} - \rho\right] \\
&\quad -\delta_2\left[\widehat{b}_{t-1} - \pi_t\right] + \delta_3\left[\widehat{E}_t\{\widehat{y}_{t+1}\} - \widehat{E}_t\{\widehat{y}_{t+2}\}\right] + (\delta_4 - \delta_2)\left[\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\}\right] \\
&\quad -\delta_4\left[\widehat{E}_t\{\widehat{b}_{t+1}\} - \widehat{E}_t\{\pi_{t+2}\}\right] + \frac{T^{RE}}{(1-T^{RE})}(\widehat{\tau}_t^{RE} - \widehat{E}_t\left\{\widehat{\tau}_{t+1}^{RE}\right\}) \\
&\quad \underbrace{+\delta_1\left[\chi\widehat{\mathcal{W}}_\infty^1 + (1-\chi)\widehat{\mathcal{W}}_\infty^2 - \widehat{E}_t\left\{\left(\chi\widehat{\mathcal{W}}_\infty^1 + (1-\chi)\widehat{\mathcal{W}}_\infty^2\right)\right\}\right]}_{=0,\ \text{due to A7}},
\end{aligned}$$

or

$$\begin{aligned}
\widehat{y}_t &= \widehat{E}_t\left\{\widehat{y}_{t+1}\right\} - \delta_1\frac{1}{\sigma_C}\left[i_t - \widehat{E}_t\{\pi_{t+1}\} - \rho\right] \\
&\quad -\delta_2\left[\widehat{b}_{t-1} - \pi_t\right] + \delta_3\left[\widehat{E}_t\{\widehat{y}_{t+1}\} - \widehat{E}_t\{\widehat{y}_{t+2}\}\right] + (\delta_4 - \delta_2)\left[\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\}\right] \\
&\quad -\delta_4\left[\widehat{E}_t\{\widehat{b}_{t+1}\} - \widehat{E}_t\{\pi_{t+2}\}\right] + \frac{T^{RE}}{(1-T^{RE})}(\widehat{\tau}_t^{RE} - \widehat{E}_t\left\{\widehat{\tau}_{t+1}^{RE}\right\}). \qquad (2.34)
\end{aligned}$$

We realize that this version of the Dynamic IS curve features not only the (expected) output and real rate dynamics as is standard in NK literature, but also the (expected) real debt dynamics and the expected change in revenue tax rate deviations.

2.3.3 Labour Market Clearing

Aggregate labour supply is given by

$$N_t = \int_0^1 N_t(i)di.$$

Combined with firm i's production technology (2.15) and the economy-wide demand (2.27) we get:

$$N_t = \int_0^1 \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} di = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} di.$$

Up to a first-order log-linear approximation, we can write aggregate output as[34]

$$y_t = a_t + (1-\alpha)n_t. \tag{2.35}$$

2.3.4 Marginal Cost

From (2.17) and by imposing symmetry among firms, we see that economy's average real marginal cost in logs is

$$\begin{aligned} mc_t &= (w_t - p_t) - mpn_t \\ &\overset{(2.37)}{=} (w_t - p_t) - (a_t - \alpha n_t - \log(1-\alpha)) \\ &\overset{(2.35)}{=} (w_t - p_t) - \frac{1}{(1-\alpha)}a_t + \frac{\alpha}{(1-\alpha)}y_t + \log(1-\alpha), \end{aligned} \tag{2.36}$$

where the log-linearized form of the marginal product of labour (2.18) is

$$mpn_t = a_t - \alpha n_t - \log(1-\alpha). \tag{2.37}$$

Similarly, period $t+k$ marginal cost for a firm, that last reset its price in period t, is

$$\begin{aligned} mc_{t+k|t}(i) &= (w_{t+k} - p_{t+k}) - mpn_{t+k|t}(i) \\ &\overset{(2.37)}{=} (w_{t+k} - p_{t+k}) - (a_{t+k} - \alpha n_{t+k|t}(i) - \log(1-\alpha)) \\ &\overset{(2.35)}{=} (w_{t+k} - p_{t+k}) - (a_{t+k} - \alpha \tfrac{1}{(1-\alpha)}[y_{t+k|t}(i) - a_{t+k}] - \log(1-\alpha)) \\ &= (w_{t+k} - p_{t+k}) - \tfrac{1}{(1-\alpha)}a_{t+k} + \log(1-\alpha) + \tfrac{\alpha}{(1-\alpha)}y_{t+k|t}(i) \end{aligned}$$

[34] See Galí (2008, p.46) and the related proof therein.

$$
\begin{aligned}
&= \underbrace{(w_{t+k} - p_{t+k}) - \frac{1}{(1-\alpha)} a_{t+k} + \frac{\alpha}{(1-\alpha)} y_{t+k} + \log(1-\alpha)}_{=mc_{t+k}} \\
&\quad + \tfrac{\alpha}{(1-\alpha)} [y_{t+k|t}(i) - y_{t+k}] \\
&\overset{(2.27)}{=} mc_{t+k} - \tfrac{\alpha\epsilon}{(1-\alpha)} [p_t^*(i) - p_{t+k}],
\end{aligned}
$$

where we make use of the aggregate output (2.35) as well as the log-linearized demand schedule (2.27) and the marginal product of labour (2.37). In consequence, we can express the marginal cost in log-deviations from the steady-state as

$$
\begin{aligned}
\widehat{mc}_{t+k|t}(i) &= mc_{t+k|t}(i) - mc \\
&= mc_{t+k} - \frac{\epsilon\alpha}{(1-\alpha)} [\widehat{p}_t^*(i) - \widehat{p}_{t+k}] - mc \\
&= \widehat{mc}_{t+k} - \frac{\epsilon\alpha}{(1-\alpha)} [\widehat{p}_t^*(i) - \widehat{p}_{t+k}]. \qquad (2.38)
\end{aligned}
$$

This illustrates formally the asymmetry among individual marginal costs that may result in endogenous price stickiness as explained above. Next, if we apply (2.38) to (2.21), we get

$$
\begin{aligned}
\widehat{p}_t^*(i) - \widehat{p}_t &= (1-\beta\theta)\widehat{E}_t\{\sum_{k=0}^{\infty} (\beta\theta)^k \times \\
&\quad \left[\Theta\left(\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_{t+k}^{RE} + \widehat{mc}_{t+k}\right) + \frac{\beta\theta}{(1-\beta\theta)}\pi_{t+k+1}\right]\}, \quad (2.39)
\end{aligned}
$$

where $\Theta \equiv \frac{(1-\alpha)}{1+\alpha(\epsilon-1)} \leq 1$. We can forward this equation one period, apply *A1, A3, A4 and A5* to arrive at the difference equation[35]

$$
\begin{aligned}
\widehat{p}_t^*(i) - \widehat{p}_t &= (1-\beta\theta)\Theta\left[\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE} + \widehat{mc}_t\right] + \beta\theta\widehat{E}_t\left\{\pi_{t+1}\right\} \\
&\quad + \beta\theta\widehat{E}_t\left\{\widehat{p}_{t+1}^*(i) - \widehat{p}_{t+1}\right\}. \qquad (2.40)
\end{aligned}
$$

[35] In Appendix 2.6.2.10 we outline the steps and the application of the assumptions.

Now, as decision making is equal among all firms, we get

$$\widehat{p}_t^* - \widehat{p}_t = (1-\beta\theta)\Theta\left[\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE} + \widehat{mc}_t\right] + \beta\theta\widehat{E}_t\left\{\pi_{t+1}\right\} + \beta\theta\widehat{E}_t\left\{\widehat{p}_{t+1}^* - \widehat{p}_{t+1}\right\}.$$

Using the Calvo (1983) assumption about aggregate price setting (2.14) yields

$$\frac{\theta}{(1-\theta)}\pi_t = (1-\beta\theta)\Theta\left[\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE} + \widehat{mc}_t\right] + \beta\theta\widehat{E}_t\left\{\pi_{t+1}\right\} + \beta\theta\frac{\theta}{(1-\theta)}\widehat{E}_t\left\{\pi_{t+1}\right\}.$$

Finally, we rearrange terms, define $\lambda \equiv \frac{(1-\beta\theta)(1-\theta)}{\theta}\Theta$ and get the *inflation equation*

$$\pi_t = \lambda\left[\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE} + \widehat{mc}_t\right] + \beta\widehat{E}_t\left\{\pi_{t+1}\right\}. \tag{2.41}$$

We can iterate this equation forward and see that inflation can be expressed as a weighted average of the actual and the expected future steady-state deviations of the tax-rate as well as the marginal cost, which are determinants of the firms pricing decisions. Formally this is

$$\pi_t = \lambda\ \widehat{E}_t\left\{\sum_{k=0}^{\infty}\beta^k\left[\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_{t+k}^{RE} + \widehat{mc}_{t+k}\right]\right\}.$$

Be aware that an innovation in the fiscal authorities policy rule can affect aggregate inflation through this channel and may therefore affect monetary policy.

Next, we impose goods market clearing (2.29), labour market clearing (2.35), as well as an efficient consumption labour choice (2.10) to the average real marginal cost (2.36) and get

$$mc_t \quad = \quad \left[\frac{(\varphi+\alpha)}{(1-\alpha)} + \sigma_C\frac{Y}{C}\right]y_t - \sigma_C\frac{G}{C}g - \left[\frac{(1+\varphi)}{(1-\alpha)}\right]a_t + \log(1-\alpha).$$

Consequently for the steady-state follows that[36]

$$mc \quad = \quad \left[\frac{(\varphi+\alpha)}{(1-\alpha)} + \sigma_C\frac{Y}{C}\right]y_t^n - \sigma_C\frac{G}{C}g - \left[\frac{(1+\varphi)}{(1-\alpha)}\right]a_t + \log(1-\alpha)$$

[36] We denote the natural level of a variable in logs z_t by z_t^n. The natural level is the result of a flexible price equilibrium.

is true. Furthermore, we have

$$\widehat{mc}_t = mc_t - mc = \left[\frac{(\varphi+\alpha)}{(1-\alpha)} + \sigma_C \frac{Y}{C}\right] \underbrace{(y_t - y_t^n)}_{\equiv \widetilde{y}_t}, \tag{2.42}$$

where $\widetilde{y}_t$ is the difference between actual and natural level of output. Combining (2.42) with (2.41) yields the *New Keynesian Phillips Curve*

$$\begin{aligned} \pi_t &= \beta \widehat{E}_t \{\pi_{t+1}\} + \lambda \frac{T^{RE}}{(1-T^{RE})} \widehat{\tau}_t^{RE} + \underbrace{\lambda \left(\frac{(\varphi+\alpha)}{(1-\alpha)} + \sigma_C \frac{Y}{C}\right)}_{\equiv \kappa} \widetilde{y}_t \\ &= \beta \widehat{E}_t \{\pi_{t+1}\} + \lambda \frac{T^{RE}}{(1-T^{RE})} \widehat{\tau}_t^{RE} + \kappa \widetilde{y}_t. \end{aligned} \tag{2.43}$$

Compared to the functional form of the standard New Keynesian Phillips Curve there is the additional term with the revenue tax rate deviations.

2.3.5 The Flex-Price Equilibrium

Under flexible prices each monopolistically competitive firm i facing given prices and wages will maximize its profits

$$\max_{P_t^*(i), Y_t(i), N_t(i)} \; (1 - T_t^{RE}) \; P_t^*(i) \; Y_t(i) - W_t \; N_t(i)$$

subject to technology (2.15) as well as market demand (2.27). The Lagrangian of this problem is

$$\begin{aligned} \mathcal{L} &= (1 - T_t^{RE}) \; P_t^*(i) \; Y_t(i) - W_t \; N_t(i) \\ &\quad - \lambda_1 \left[Y_t(i) - A_t N_t(i)^{1-\alpha}\right] - \lambda_2 \left[Y_t(i) - \left(\frac{P_t^*(i)}{P_t}\right)^{-\epsilon} Y_t\right]. \end{aligned}$$

Combining first-order conditions indicates the optimal firm behaviour. A firm should increase employment until the real wage equals the constant markup over after-tax marginal product of labour

$$\frac{W_t}{P_t^*(i)} = \frac{(1-T_t^{RE})}{\mathcal{M}}(1-\alpha)A_t N_t^{-\alpha}(i)$$
$$\frac{W_t}{P_t} = \frac{(1-T_t^{RE})}{\mathcal{M}}(1-\alpha)A_t N_t^{-\alpha}. \tag{2.44}$$

The latter identity makes use of the assumption of symmetry among firm behaviour. A log-linear approximation of (2.44) yields[37]

$$\widehat{w}_t - \widehat{p}_t = a_t - \alpha\widehat{n}_t - \frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE}. \tag{2.45}$$

Goods market clear (2.29) and households supply labour according to (2.10), therefore

$$\varphi\widehat{n}_t + \sigma_C\frac{Y}{C}\widehat{y}_t = a_t - \alpha\widehat{n}_t - \frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE}$$

should hold. Labour market clearing condition (2.35) and rearranging terms yields

$$\widehat{y}_t^n = \overbrace{\left[\frac{(1+\varphi)C}{C(\varphi+\alpha)+\sigma_C(1-\alpha)Y}\right]}^{\equiv\psi_{ya}^n} a_t - \overbrace{\left[\frac{T^{RE}}{(1-T^{RE})}\frac{(1-\alpha)C}{C(\varphi+\alpha)+\sigma_C(1-\alpha)Y}\right]}^{\equiv\psi_{yT^{RE}}^n}\widehat{\tau}_t^{RE}$$

or

$$\widehat{y}_t^n = \psi_{ya}^n a_t - \psi_{yT^{RE}}^n\ \widehat{\tau}_t^{RE} \tag{2.46}$$

as the *natural level of output* in terms of steady-state deviations.

2.3.6 Efficient Levels

Under flexible prices, perfect competition and without taxes each firm i facing given prices and wages will maximize its profits

$$\max_{Y_t(i),N_t(i)}\ P_t\ Y_t(i) - W_t\ N_t(i)$$

[37] See Appendix 2.6.2.11 for the derivation.

subject to technology (2.15). Optimal firm behaviour now requires increasing employment until the real wage equals the marginal product of labour

$$\frac{W_t}{P_t} = (1-\alpha)A_t N_t^{-\alpha}, \tag{2.47}$$

where we again made use of the assumption of symmetry among firms. Taking logs yields

$$w_t - p_t = a_t - \alpha n_t + \log(1-\alpha). \tag{2.48}$$

Goods market clear (2.29), and households supply labour according to (2.10), therefore

$$\varphi n_t + \sigma_C \left[\frac{Y}{C} y_t - \frac{G}{C} g\right] = a_t - \alpha n_t + \log(1-\alpha)$$

should hold. Labour market clearing (2.35) and rearranging terms yields[38]

$$\begin{aligned} y_t^e &= \overbrace{\left[\frac{C(1+\varphi)}{C(\varphi+\alpha)+\sigma_C(1-\alpha)Y}\right]}^{\equiv \psi_{ya}^e = \psi_{ya}^n} a_t + \overbrace{\left[\frac{G(1-\alpha)}{C(\varphi+\alpha)+\sigma_C(1-\alpha)Y}\right]}^{\equiv \psi_{yg}^e} g \\ &\quad + \underbrace{\left[\frac{C(1-\alpha)}{C(\varphi+\alpha)+\sigma_C(1-\alpha)Y}\right] \log(1-\alpha)}_{\equiv \vartheta_y^e} \end{aligned}$$

or

$$y_t^e = \psi_{ya}^e a_t + \psi_{yg}^e g + \vartheta_y^e \tag{2.49}$$

as the *efficient level of output*. In terms of steady-state deviations, this is given by

$$\widehat{y}_t^e = \psi_{ya}^e a_t.$$

2.3.7 Accounting for Distortions

Comparing (2.44) to (2.47) makes clear that the flex price equilibrium is distorted by the revenue tax rate as well as the monopolistic competition and, in order to write the equilibrium conditions in terms of the welfare relevant output gap, we need to account for these distortions.[39] In order to do so, we have to rewrite

[38] We denote the efficient level of a variable in logs z_t by z_t^e.

[39] This is necessary as we do assume the more realistic case, in which the authorities cannot perfectly offset the distortions by lump-sump transfers.

the Dynamic IS Curve (2.34), the New Keynesian Philips Curve (2.43), the tax rule (2.24) and the government budget constraint (2.25) in terms of the welfare relevant output gap deviation $\widehat{x}_t \equiv x_t - x = (y_t - y_t^e) - (y - y^e)$. So it follows that $\widehat{y}_t = \widehat{x}_t + \widehat{y}_t^e$ as well as $\widetilde{y}_t = \widehat{x}_t + (\widehat{y}_t^e - \widehat{y}_t^n)$. The Dynamic IS curve (2.34) becomes

$$\begin{aligned}(\widehat{x}_t + \widehat{y}_t^e) &= (1+\delta_3)\widehat{E}_t\left\{(\widehat{x}_{t+1} + \widehat{y}_{t+1}^e)\right\} - \delta_3\widehat{E}_t\{(\widehat{x}_{t+2} + \widehat{y}_{t+2}^e)\} \\ &-\delta_1\frac{1}{\sigma_C}\left[i_t - \widehat{E}_t\{\pi_{t+1}\} - \rho\right] - \delta_2\left[\widehat{b}_{t-1} - \pi_t\right] \\ &+(\delta_4-\delta_2)\left[\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\}\right] - \delta_4\left[\widehat{E}_t\{\widehat{b}_{t+1}\} - \widehat{E}_t\{\pi_{t+2}\}\right] \\ &+\frac{T^{RE}}{(1-T^{RE})}(\widehat{\tau}_t^{RE} - \widehat{E}_t\left\{\widehat{\tau}_{t+1}^{RE}\right\}),\end{aligned}$$

$$\begin{aligned}\widehat{x}_t &= (1+\delta_3)\widehat{E}_t\left\{\widehat{x}_{t+1}\right\} + \delta_1\frac{1}{\sigma_C}\underbrace{\left[\rho + \frac{\sigma_C}{\delta_1}\triangle\widehat{E}_t\{\widehat{y}_{t+1}^e\} - \delta_3\frac{\sigma_C}{\delta_1}\triangle\widehat{E}_t\{\widehat{y}_{t+2}^e\}\right]}_{\equiv r_t^e} \\ &-\delta_1\frac{1}{\sigma_C}i_t + \delta_1\frac{1}{\sigma_C}\widehat{E}_t\{\pi_{t+1}\} - \delta_3\widehat{E}_t\{\widehat{x}_{t+2}\} - \delta_2\left[\widehat{b}_{t-1} - \pi_t\right] \\ &+(\delta_4-\delta_2)\left[\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\}\right] - \delta_4\left[\widehat{E}_t\{\widehat{b}_{t+1}\} - \widehat{E}_t\{\pi_{t+2}\}\right] \\ &+\frac{T^{RE}}{(1-T^{RE})}(\widehat{\tau}_t^{RE} - \widehat{E}_t\left\{\widehat{\tau}_{t+1}^{RE}\right\}),\end{aligned}$$

or

$$\begin{aligned}\widehat{x}_t &= (1+\delta_3)\widehat{E}_t\left\{\widehat{x}_{t+1}\right\} - \delta_3\widehat{E}_t\{\widehat{x}_{t+2}\} - \delta_1\frac{1}{\sigma_C}\left[i_t - \widehat{E}_t\{\pi_{t+1}\} - r_t^e\right] \\ &-\delta_2\left[\widehat{b}_{t-1} - \pi_t\right] + (\delta_4-\delta_2)\left[\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\}\right] - \delta_4\left[\widehat{E}_t\{\widehat{b}_{t+1}\} - \widehat{E}_t\{\pi_{t+2}\}\right] \\ &+\frac{T^{RE}}{(1-T^{RE})}(\widehat{\tau}_t^{RE} - \widehat{E}_t\left\{\widehat{\tau}_{t+1}^{RE}\right\}). \qquad (2.50)\end{aligned}$$

Here r_t^e is the *efficient real rate of interest*, i.e. the rate of interest that prevails in absence of any nominal rigidity as well as distortions. Note that, given the efficient level of output (2.49), the process for multi-factor productivity (2.16) and the expectations specified in Section 2.4.1 below, we can express this interest rate in terms of structural parameters and exogenous forces only

$$r_t^e = \rho + \left[\frac{\sigma_C}{\delta_1}\left(\chi\rho_a + (1-\chi)\iota - 1\right) - \delta_3 \frac{\sigma_C}{\delta_1}\left(\chi\rho_a^2(1-\chi)\iota^2 - \chi\rho_a - (1-\chi)\iota\right)\right]\psi_{ya}^e a_t.$$

Next, the New Keynesian Philips Curve (2.43) becomes

$$\begin{aligned}
\pi_t &= \beta\widehat{E}_t\{\pi_{t+1}\} + \lambda\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE} + \kappa[\widehat{x}_t + (\widehat{y}_t^e - \widehat{y}_t^n)] \\
&= \beta\widehat{E}_t\{\pi_{t+1}\} + \lambda\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE} + \kappa\widehat{x}_t + \underbrace{\kappa(\widehat{y}_t^e - \widehat{y}_t^n)}_{\equiv u_t} \\
&= \beta\widehat{E}_t\{\pi_{t+1}\} + \lambda\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE} + \kappa\widehat{x}_t + u_t. \qquad (2.51)
\end{aligned}$$

u_t is usually interpreted in the literature as a cost-push shock. We assume that it follows an AR(1) process

$$u_t = \rho_u u_{t-1} + \epsilon_t^u,$$

where $\rho_u \in [0,1)$ and ϵ_t^u is a white-noise process with mean zero and constant variance σ_u^2. Next, the tax rule (2.24) becomes

$$\begin{aligned}
\widehat{\tau}_t^{RE} &= \frac{B/P}{T^{RE}Y}\varrho_1(\widehat{b}_{t-1} - \pi_t) - [\widehat{x}_t + \widehat{y}_t^e] + \frac{(T + T^{RE}Y)}{T^{RE}Y}\widehat{\Psi}_t \\
&= \frac{B/P}{T^{RE}Y}\varrho_1(\widehat{b}_{t-1} - \pi_t) - \widehat{x}_t - \widehat{y}_t^e + \frac{(T + T^{RE}Y)}{T^{RE}Y}\widehat{\Psi}_t. \qquad (2.52)
\end{aligned}$$

Finally, we make use of (2.29) and the government budget constraint (2.25) becomes

$$\begin{aligned}
\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\} &= \sigma_C \widehat{E}_t\{\frac{Y}{C}\widehat{y}_{t+1}\} - \sigma_C \frac{Y}{C}\widehat{y}_t + \frac{1}{\beta}(\widehat{b}_{t-1} - \pi_t) - \frac{T^{RE}Y}{\beta\frac{B}{P}}(\widehat{\tau}_t^{RE} + \widehat{y}_t) \\
&= \sigma_C \frac{Y}{C}\widehat{E}_t\{\widehat{x}_{t+1} + \widehat{y}^e_{t+1}\} - \sigma_C \frac{Y}{C}(\widehat{x}_t + \widehat{y}^e_t) + \frac{1}{\beta}(\widehat{b}_{t-1} - \pi_t) \\
&\quad - \frac{T^{RE}Y}{\beta\frac{B}{P}}(\widehat{\tau}_t^{RE} + \widehat{x}_t + \widehat{y}^e_t) \\
&= \sigma_C \frac{Y}{C}\widehat{E}_t\{\widehat{x}_{t+1}\} - \sigma_C \frac{Y}{C}\widehat{x}_t + \frac{1}{\beta}(\widehat{b}_{t-1} - \pi_t) - \frac{T^{RE}Y}{\beta\frac{B}{P}}(\widehat{\tau}_t^{RE} + \widehat{x}_t) \\
&\quad + \sigma_C \frac{Y}{C}\widehat{E}_t\{\widehat{y}^e_{t+1}\} - (\sigma_C \frac{Y}{C} + \frac{T^{RE}Y}{\beta\frac{B}{P}})\widehat{y}^e_t. \qquad (2.53)
\end{aligned}$$

2.4 The Dynamics of the Model

In this section we detail the expectational setting. In addition, we write down the fully specified system we are going to analyze and the methodology that is used for this purpose. Finally, we characterize the dynamic properties of the economy.

2.4.1 The Nature of the Heterogeneous Expectations

Exactly following Branch and McGough (2009, ch.4.2), we assume that agents of type $\gamma = 1$ have one-step ahead perfect foresight with regard to endogenous variables. Agents of type $\gamma = 2$ have adaptive expectations on the same variables. Therefore, for any variable z_t that is not observable in period t, agents of type $\gamma = 2$ compute $E^2_t\{z_t\} = \iota z_{t-1}$ or $E^2_t\{z_{t+1}\} = \iota E^2_t\{z_t\} = \iota^2 z_{t-1}$, where $\iota > 0$. From $A6$ it follows that on the aggregate for any variable z_t that is not observable for agents in period t we get

$$\widehat{E}_t\{z_{t+1}\} = \chi E_t\{z_{t+1}\} + (1-\chi)\iota^2 z_{t-1}.$$

If z_t is observable in period t, we get

$$\widehat{E}_t\{z_{t+1}\} = \chi E_t\{z_{t+1}\} + (1-\chi)\iota z_t.$$

Furthermore, notice that a rate of change $\triangle z_t$ is in general given by

$$\widehat{E}_t\{\triangle z_{t+1}\} = \widehat{E}_t\{z_{t+1}\} - \widehat{E}_t\{z_t\},$$

thus for any variable that is observable at the beginning of period t, we get

$$\widehat{E}_t\{\triangle z_{t+1}\} = \chi E_t\{z_{t+1}\} + [(1-\chi)\iota - 1]\, z_t,$$

and for any variable that is not observable at the beginning of period t, we get

$$\widehat{E}_t\{\triangle z_{t+1}\} = \chi E_t\{\triangle z_{t+1}\} + (1-\chi)\iota(\iota - 1) z_{t-1}.$$

Be aware that in what follows, we will stick to the learning literature's widely used assumption that for agents of type $\gamma = 2$ current period's exogenous shocks are observable, whereas aggregate levels of endogenous variables are not. Nevertheless, all agents are aware of the individual level they choose (see Evans and Honkapohja (2001, p.200) or Branch and McGough (2009, p.4)). The idea behind is, that we rule out situations, in which agents' expectations may affect current values of aggregate endogenous variables and vice versa. This could potentially introduce some strategic behaviour. This may be viewed as a crucial shortcoming of the learning approach in general as has been emphasized by Bullard (1991, p.57), who states "making this assumption is unsatisfactory, however, because it means that individuals ignore relevant and potentially useful information when forming their forecasts".

Next we need to specify the coefficient ι. With regard to our subsequent analysis, as emphasized by Branch and McGough (2009) we distinguish three cases. $0 < \iota < 1$ is what usually denotes adaptive expectations in a conventional sense. Therefore, we call this purely adaptive. $\iota = 1$ is often referred to naive expectations. $\iota > 1$ is what Branch and McGough (2009) denote extrapolative expectations. Below, we will parallel the analysis of Branch and McGough (2009) and consider the cases $\iota \in \{0.90, 1.10\}$, where we will refer to the first case as *purely adaptive* expectations and to the second case as *extrapolative* expectations.

2.4.2 The Fully Specified System

The complete system at hand consists of the conditions (2.50), (2.51), (2.26), (2.52) and (2.53). We have a non-policy block

$$
\begin{aligned}
\widehat{x}_t &= (1+\delta_3)\widehat{E}_t\left\{\widehat{x}_{t+1}\right\} - \delta_3\widehat{E}_t\{\widehat{x}_{t+2}\} - \delta_1\frac{1}{\sigma_C}\left[i_t - \widehat{E}_t\{\pi_{t+1}\} - r^e_t\right] \\
&\quad -\delta_2\left[\widehat{b}_{t-1} - \pi_t\right] + (\delta_4 - \delta_2)\left[\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\}\right] \\
&\quad -\delta_4\left[\widehat{E}_t\{\widehat{b}_{t+1}\} - \widehat{E}_t\{\pi_{t+2}\}\right] + \frac{T^{RE}}{(1-T^{RE})}(\widehat{\tau}^{RE}_t - \widehat{E}_t\left\{\widehat{\tau}^{RE}_{t+1}\right\}) \\
\pi_t &= \beta\widehat{E}_t\left\{\pi_{t+1}\right\} + \lambda\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}^{RE}_t + \kappa\widehat{x}_t + u_t.
\end{aligned}
$$

Furthermore there is monetary policy

$$
i_t = \phi_{\widehat{x}}\ E_t\{\widehat{x}_{t+1}\} + \phi_\pi\ E_t\{\pi_{t+1}\} + v_t,
$$

fiscal policy

$$
\widehat{\tau}^{RE}_t = \frac{B/P}{T^{RE}Y}\varrho_1(\widehat{b}_{t-1} - \pi_t) - \widehat{x}_t - \widehat{y}^e_t + \frac{(T + T^{RE}Y)}{T^{RE}Y}\widehat{\Psi}_t
$$

$$
\begin{aligned}
\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\} &= \sigma_C\frac{Y}{C}\widehat{E}_t\{\widehat{x}_{t+1}\} - \sigma_C\frac{Y}{C}\widehat{x}_t + \frac{1}{\beta}(\widehat{b}_{t-1} - \pi_t) - \frac{T^{RE}Y}{\beta\frac{B}{P}}(\widehat{\tau}^{RE}_t + \widehat{x}_t) \\
&\quad +\sigma_C\frac{Y}{C}\widehat{E}_t\{\widehat{y}^e_{t+1}\} - (\sigma_C\frac{Y}{C} + \frac{T^{RE}Y}{\beta\frac{B}{P}})\widehat{y}^e_t
\end{aligned}
$$

and expectations

$$
\begin{aligned}
\widehat{E}_t\{\widehat{x}_{t+1}\} &= \chi E_t\{\widehat{x}_{t+1}\} + (1-\chi)\iota^2\widehat{x}_{t-1} \\
\widehat{E}_t\{\widehat{x}_{t+2}\} &= \chi E_t\{\widehat{x}_{t+2}\} + (1-\chi)\iota^3\widehat{x}_{t-1} \\
\widehat{E}_t\{\pi_{t+1}\} &= \chi E_t\{\pi_{t+1}\} + (1-\chi)\iota^2\pi_{t-1}, \\
\widehat{E}_t\{\pi_{t+2}\} &= \chi E_t\{\pi_{t+2}\} + (1-\chi)\iota^3\pi_{t-1} \\
\widehat{E}_t\{\widehat{b}_{t+1}\} &= \chi E_t\{\widehat{b}_{t+1}\} + (1-\chi)\iota^2\widehat{b}_{t-1} \\
\widehat{E}_t\{\widehat{\tau}^{RE}_{t+1}\} &= \chi E_t\{\widehat{\tau}^{RE}_{t+1}\} + (1-\chi)\iota^2\widehat{\tau}^{RE}_{t-1} \\
\widehat{E}_t\{\widehat{y}^{e}_{t+1}\} &= \chi E_t\{\widehat{y}^{e}_{t+1}\} + (1-\chi)\iota\widehat{y}^{e}_{t}
\end{aligned}
$$

as well as the processes for natural and efficient levels. We can boil this system down to four equations

$$\begin{pmatrix} 1 & \left(\sigma_C \frac{Y}{C} + \frac{T^{RE}Y}{\beta B/P}\right) & \frac{1}{\beta} & \frac{T^{RE}Y}{\beta B/P} \\ -(\delta_4 - \delta_2) & 1 & -\delta_2 & \frac{T^{RE}}{(1-T^{RE})} \\ 0 & -\kappa & 1 & -\lambda\frac{T^{RE}}{(1-T^{RE})} \\ 0 & 1 & \frac{B/P}{T^{RE}Y}\varrho_1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{b}_t \\ \widehat{x}_t \\ \pi_t \\ \widehat{\tau}_t^{RE} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \sigma_C \frac{Y}{C}\chi & \chi & 0 \\ -\delta_4\chi & \left[(1+\delta_3)\chi - \delta_1 \frac{\phi_{\widehat{x}}}{\sigma_C}\right] & \left[\frac{\delta_1}{\sigma_C}(\chi - \phi_\pi) - (\delta_4 - \delta_2)\chi\right] & -\frac{T^{RE}}{(1-T^{RE})}\chi \\ 0 & 0 & \beta\chi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_t\{\widehat{b}_{t+1}\} \\ E_t\{\widehat{x}_{t+1}\} \\ E_t\{\pi_{t+1}\} \\ E_t\{\widehat{\tau}_{t+1}^{RE}\} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\delta_3\chi & \delta_4\chi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_t\{\widehat{b}_{t+2}\} \\ E_t\{\widehat{x}_{t+2}\} \\ E_t\{\pi_{t+2}\} \\ E_t\{\widehat{\tau}_{t+2}^{RE}\} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{1}{\beta} & \sigma_C \frac{Y}{C}(1-\chi)\iota^2 & (1-\chi)\iota^2 & 0 \\ -\left[\delta_2 + \delta_4(1-\chi)\iota^2\right] & \left[(1+\delta_3)(1-\chi)\iota^2 - \delta_3(1-\chi)\iota^3\right] & \left[\left(\frac{\delta_1}{\sigma_C} - (\delta_4 - \delta_2)\right)(1-\chi)\iota^2 + \delta_4(1-\chi)\iota^3\right] & -\frac{T^{RE}}{(1-T^{RE})}(1-\chi)\iota^2 \\ 0 & 0 & \beta(1-\chi)\iota^2 & 0 \\ \frac{B/P}{T^{RE}Y}\varrho_1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{b}_{t-1} \\ \widehat{x}_{t-1} \\ \pi_{t-1} \\ \widehat{\tau}_{t-1}^{RE} \end{pmatrix}$$

$$+ \begin{pmatrix} \left[\sigma_C \frac{Y}{C}(\chi\rho_a + (1-\chi)\iota) - \left(\sigma_C \frac{Y}{C} + \frac{T^{RE}Y}{\beta B/P}\right)\right] & 0 & 0 & 0 & 0 \\ 0 & -\delta_1 \frac{1}{\sigma_C} & \delta_1 \frac{1}{\sigma_C} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & \frac{(T+T^{RE}Y)}{T^{RE}Y} \end{pmatrix} \begin{pmatrix} \widehat{y}_t^e \\ v_t \\ r_t^e \\ u_t \\ \widehat{\Psi}_t \end{pmatrix}.$$

For the sake of convenience, we may rewrite the system as in Binder and Pesaran (1999, p.140ff). They demonstrate how one can recast any multivariate RE model with any finite number of leads and lags as a second-order system. With $\mathbf{s}_t = [\widehat{b}_t, \widehat{x}_t, \pi_t, \widehat{\tau}_t^{RE}]'$ of dimension $e = 4$ and $\mathbf{U}_t = [\widehat{y}_t^e, v_t, r_t^e, u_t, \widehat{\Psi}_t]'$ of dimension $K = 5$ we write the system in general as

$$\begin{aligned}\mathbf{M}_U\ \mathbf{U}_t \;&=\; \mathbf{M}_{00}\ \mathbf{s}_t + \mathbf{M}_{01}\ E_t\mathbf{s}_{t+1} + \mathbf{M}_{02}\ E_t\mathbf{s}_{t+2} + \mathbf{M}_{10}\ \mathbf{s}_{t-1} + \mathbf{M}_{20}\ \mathbf{s}_{t-2}\\ &\quad + \mathbf{M}_{11}\ E_{t-1}\mathbf{s}_t + \mathbf{M}_{21}\ E_{t-2}\mathbf{s}_{t-1}\\ &\quad + \mathbf{M}_{12}\ E_{t-1}\mathbf{s}_{t+1} + \mathbf{M}_{22}\ E_{t-2}\mathbf{s}_t, \end{aligned} \tag{2.54}$$

where in our case matrices $\mathbf{M}_{20} = \mathbf{M}_{11} = \mathbf{M}_{12} = \mathbf{M}_{21} = \mathbf{M}_{22} = \mathbf{0}$. We can rewrite the latter as

$$\begin{aligned}\begin{bmatrix} \mathbf{M_U} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{U}_t \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \;&=\; \begin{bmatrix} \mathbf{M}_{00} & \mathbf{M}_{01} & \mathbf{M}_{02} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{s}_t \\ E_t\mathbf{s}_{t+1} \\ E_t\mathbf{s}_{t+2} \end{bmatrix}\\ &\quad + \begin{bmatrix} \mathbf{M}_{10} & \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{s}_{t-1} \\ E_{t-1}\mathbf{s}_t \\ E_{t-1}\mathbf{s}_{t+1} \end{bmatrix}\\ &\quad + \begin{bmatrix} \mathbf{M}_{20} & \mathbf{M}_{21} & \mathbf{M}_{22} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{s}_{t-2} \\ E_{t-2}\mathbf{s}_{t-1} \\ E_{t-2}\mathbf{s}_t \end{bmatrix}\\ &\quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} E_t\mathbf{s}_{t+1} \\ E_t\mathbf{s}_{t+2} \\ E_t\mathbf{s}_{t+3} \end{bmatrix}\end{aligned}$$

or with $\mathbf{z}_t = [\mathbf{s}_t', E_t\mathbf{s}_{t+1}', E_t\mathbf{s}_{t+2}']'$ more compact as

$$\mathbf{\Gamma_U}\ \nu_t \;=\; \mathbf{\Gamma}_0\ \mathbf{z}_t + \mathbf{\Gamma}_1\ \mathbf{z}_{t-1} + \mathbf{\Gamma}_2\ \mathbf{z}_{t-2} + \mathbf{\Gamma}_{-1}\ E_t\mathbf{z}_{t+1}. \tag{2.55}$$

Again, we can rewrite this equation as

$$\begin{bmatrix} \mathbf{\Gamma_U} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \nu_t \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_0 & \mathbf{\Gamma}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{z}_t \\ \mathbf{z}_{t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_2 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{t-1} \\ \mathbf{z}_{t-2} \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} E_t\mathbf{z}_{t+1} \\ \mathbf{z}_t \end{bmatrix},$$

or by defining $\mathbf{y}_t = [\mathbf{z}_t', \mathbf{z}_{t-1}']'$ and $\mathbf{Z}_t = [\nu_t', 0]'$ more compactly as a second-order stochastic difference system which in general can be written as

$$\breve{\mathbf{A}}_0\mathbf{y}_t = -\breve{\mathbf{A}}_1 E_t\{\mathbf{y}_{t+1}\} - \breve{\mathbf{A}}_2\mathbf{y}_{t-1} + \breve{\mathbf{A}}_3\mathbf{Z}_t, \tag{2.56}$$

where $\mathbf{y}_t$ is a $e \times 1$ vector of endogenous variables and $\mathbf{Z}_t$ is a $K \times 1$ vector of exogenous variables. Furthermore, the matrices $\breve{\mathbf{A}}_0$, $\breve{\mathbf{A}}_1$, $\breve{\mathbf{A}}_2$ are of dimension $e \times e$ and the matrix $\breve{\mathbf{A}}_3$ is of dimension $e \times K$. $\mathbf{Z}_t$ is assumed to follow a dynamically stable process

$$\mathbf{Z}_t = \mathbf{R}\mathbf{Z}_{t-1} + \epsilon_t, \tag{2.57}$$

where $\mathbf{R}$ is a $K \times K$ matrix, ϵ_t is the $K \times 1$ vector of innovations and $E_{t-1}\{\epsilon_t\} = 0$. If $\breve{\mathbf{A}}_1$ were non-singular, i.e. invertible, then we could simply follow the analysis of Branch and McGough (2009, p.11ff.). But we face a singular matrix $\breve{\mathbf{A}}_1$ so we cannot bring our system into the standard form of Blanchard and Kahn (1980, p.1305). A more general approach to analyze the dynamics of such a system numerically has, for example, been proposed by Klein (2000), who makes use of a generalized complex Schur decomposition. We follow this approach as outlined in the recent paper by McCallum (2009a, p.13ff.).[40] Starting from (2.56) we can write the system as

$$\mathbf{y}_t = \underbrace{-\breve{\mathbf{A}}_0^{-1}\breve{\mathbf{A}}_1}_{=\mathbf{A}} E_t\{\mathbf{y}_{t+1}\} \underbrace{-\breve{\mathbf{A}}_0^{-1}\breve{\mathbf{A}}_2}_{=\mathbf{C}} \mathbf{y}_{t-1} + \underbrace{\breve{\mathbf{A}}_0^{-1}\breve{\mathbf{A}}_3}_{=\mathbf{D}} \mathbf{Z}_t$$

or

$$\mathbf{y}_t = \mathbf{A}E_t\{\mathbf{y}_{t+1}\} + \mathbf{C}\mathbf{y}_{t-1} + \mathbf{D}\mathbf{Z}_t, \tag{2.58}$$

[40] Other useful references for solution procedures may be Sims (2002), Lubik and Schorfheide (2003) and Uhlig (2006).

where $\mathbf{A}$ and $\mathbf{C}$ are $e \times e$ matrices and $\mathbf{D}$ is a $e \times K$ matrix. This system corresponds to the class of linear models that is discussed by McCallum (2009a, p.13ff.).[41] Following this approach, we consider solutions to (2.57) and (2.58) of the type

$$\mathbf{y}_t = \mathbf{\Lambda}\mathbf{y}_{t-1} + \mathbf{\Upsilon}\mathbf{Z}_t, \tag{2.59}$$

where $\mathbf{\Lambda}$ is a $e \times e$ matrix and $\mathbf{\Upsilon}$ is a $e \times K$ matrix. One can also think of (2.59) as the Perceived Law of Motion (PLM). In Period $t+1$ (2.59) is

$$\begin{aligned} E_t\{\mathbf{y}_{t+1}\} &= \mathbf{\Lambda}\mathbf{y}_t + \mathbf{\Upsilon}E_t\{\mathbf{Z}_{t+1}\} \\ &= \mathbf{\Lambda}(\mathbf{\Lambda}\mathbf{y}_{t-1} + \mathbf{\Upsilon}\mathbf{Z}_t) + \mathbf{\Upsilon}(\mathbf{R}\mathbf{Z}_t) \\ &= (\mathbf{\Lambda}^2\mathbf{y}_{t-1}) + (\mathbf{\Lambda}\mathbf{\Upsilon} + \mathbf{\Upsilon}\mathbf{R})\mathbf{Z}_t. \end{aligned} \tag{2.60}$$

If we plug (2.60) into the original model (2.58) we get the Actual Law of motion (ALM) in the economy

$$\begin{aligned} \mathbf{y}_t &= \mathbf{A}[(\mathbf{\Lambda}^2\mathbf{y}_{t-1}) + (\mathbf{\Lambda}\mathbf{\Upsilon} + \mathbf{\Upsilon}\mathbf{R})\mathbf{Z}_t] + \mathbf{C}\mathbf{y}_{t-1} + \mathbf{D}\mathbf{Z}_t \\ &= [\mathbf{A}\mathbf{\Lambda}^2 + \mathbf{C}]\mathbf{y}_{t-1} + [\mathbf{A}\mathbf{\Lambda}\mathbf{\Upsilon} + \mathbf{A}\mathbf{\Upsilon}\mathbf{R} + \mathbf{D}]\mathbf{Z}_t. \end{aligned} \tag{2.61}$$

In a REE, the PLM has to coincide to the ALM, that is

$$\mathbf{\Lambda} \overset{!}{=} [\mathbf{A}\mathbf{\Lambda}^2 + \mathbf{C}] \tag{2.62}$$

and

$$\mathbf{\Upsilon} \overset{!}{=} [\mathbf{A}\mathbf{\Lambda}\mathbf{\Upsilon} + \mathbf{A}\mathbf{\Upsilon}\mathbf{R} + \mathbf{D}]. \tag{2.63}$$

In order to analyze the dynamics we focus on (2.62).[42] We can augment condition (2.62) by the matrix identity $\mathbf{\Lambda} = \mathbf{\Lambda}$ and write the two conditions as

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}^2 \\ \mathbf{\Lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{I} \end{bmatrix}, \tag{2.64}$$

[41]Note that our matrices $\mathbf{A}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{R}$ correspond to their counterparts in McCallum (2009a, p.13ff.).

[42]Inspecting (2.62)-(2.63) makes clear that given $\mathbf{\Lambda}$ we can pin down a unique solution for $\mathbf{\Upsilon}$. But clearly there exist multiple solutions as (2.62) is a matrix quadratic.

or more compact as

$$\bar{\mathbf{A}}\begin{bmatrix}\mathbf{\Lambda}^2\\ \mathbf{\Lambda}\end{bmatrix}=\bar{\mathbf{C}}\begin{bmatrix}\mathbf{\Lambda}\\ \mathbf{I}\end{bmatrix}, \tag{2.65}$$

where $\bar{\mathbf{A}}$, $\bar{\mathbf{C}}$ are $(e+e)\times(e+e)$ matrices. Now we look for the generalized eigenvalues (GEVs) of $\bar{\mathbf{C}}$ with respect to $\bar{\mathbf{A}}$ or equivalently for the GEVs of the matrix pencil $[\bar{\mathbf{C}}-\lambda\bar{\mathbf{A}}]$. According to the Schur generalized (complex) decomposition theorem there exist unitary $(e+e)\times(e+e)$ matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that we can decompose matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{C}}$ into the upper triangular $(e+e)\times(e+e)$ matrices $\mathbf{S}$ and $\mathbf{T}$ respectively. The GEVs of the matrix pencil $[\bar{\mathbf{C}}-\lambda\bar{\mathbf{A}}]$ are defined as the ratio of the elements of the main diagonal of $\mathbf{T}$ to the main diagonal of $\mathbf{S}$, i.e. $\lambda_i = t_{ii}/s_{ii}$. Given these GEVs, we can now proceed as in Branch and McGough (2009, p.11ff.) and visualize regions of determinacy and indeterminacy depending on policy parameters.[43]

The decomposition itself is quite an easy numerical task, but in order to characterize regions of (in-) determinacy we have to be conceptually clear about the number of GEVs and under which conditions there prevails determinacy or indeterminacy of some order.[44]

In order to be clear, notice that we may deal with N variables predetermined and M variables determined within period t. Thus the $(e+e)\times(e+e)$ or equivalently $(N+M)\times(N+M)$ matrix pencil $[\bar{\mathbf{C}}-\lambda\bar{\mathbf{A}}]$ has $(N+M)$ GEVs. In general, such a system is considered determinate if exactly M of the $(N+M)$ GEVs are outside the unit circle. Consequently, if $M-1$ GEVs are outside the

[43] Note that comparing eigenvalues of a system matrix à la Blanchard and Kahn (1980) and GEVs à la Klein (2000) for the same economy may lead to slightly different results, if one does not account properly for complex eigenvalues. By accounting we mean that one needs to ensure that one calculates the moduli of possibly complex numbers. In the software Mathematica for example, simple use of the command *SchurDecomposition*[] is not sufficient. We became aware of this issue while reproducing the results of Branch and McGough (2009, p.11ff.), see Appendix 2.6.3 for an illustration. Note that using Matlab one can reproduce the same results with the function *qz.m*, the difference is that, whereas in Mathematica a QR iteration is employed, in Matlab a QZ factorization is used. In consequence the matrices we label Q and Z differ but T and S are the same.

[44] Evans and McGough (2005, p.1816) denote a situation with a system exhibiting a one dimensional continuum of equilibria *Order 1 Indeterminacy* and a situation with a system exhibiting a two dimensional continuum of equilibria *Order 2 Indeterminacy*. The idea behind is to indicate “the number of independent sunspots required to specify the solution”.

unit circle, we consider it Order 1 Indeterminacy. If $M-2$ GEVs are outside the unit circle, we consider it Order 2 Indeterminacy and so on. Generally speaking we talk about

$$\binom{(N+M)}{M} = \frac{(N+M)!}{M!\ ((N+M)-M)!} = \frac{(N+M)!}{M!\ N!} \tag{2.66}$$

combinations of GEVs, where N of the $(N+M)$ GEVs are outside the unit circle and determinacy prevails. Furthermore there are

$$\sum_{m=0}^{M-1} \binom{(N+M)}{m} = \sum_{m=0}^{M-1} \frac{(N+M)!}{m!\ ((N+M)-m)!} \tag{2.67}$$

combinations of indeterminacy, where $m = 0, 1, ...$ corresponds to Order $M, M-1, ...$ Indeterminacy respectively.

2.4.3 Regions of (In-)determinacy

Now we have to assign values to our free parameters. One can find an overview in Table 2.2 below.

Param.	**Description**	**Value(s)**	**Reference**
$\phi_{\widehat{x}}$	Monetary policy response coefficient with regard to output gap deviations	$\in [0, 2]$	
ϕ_{π}	Monetary policy response coefficient with regard to inflation deviations	$\in [0, 2]$	
ϱ_1	Tax revenue backing parameter for real debt outstanding	$\in \{1.00, 0.50\}$	
χ	Fraction of households of type $\gamma = 1$	$\in \{1.00, 0.90, 0.80, 0.60\}$	Branch and McGough (2009, p.11) Branch (2004, p.607ff.)
ι	Weight on past data in expectations formation of type $\gamma = 2$ agents	$\in \{0.90, 1.10\}$	Branch and McGough (2009, p.11ff.)
$1/\sigma_C$	Inter-temporal elasticity of substitution of private consumption	(1/0.157)	Branch and McGough (2009, p.11)
β	Discount factor	0.99	Galí (2008, p.52)
α	Output elasticity of labour	(1/3)	Galí (2008, p.52)
$1/\varphi$	Frisch elasticity of labour supply	1.00	Galí (2008, p.52)
ϵ	Elasticity of substitution between differentiated goods	6.00	Galí (2008, p.52)
θ	Calvo (1983) parameter	(2/3)	Galí (2008, p.52)
Y	Steady-state level of output	1.00	
T^{RE}	Steady-state level of the tax rate	0.20	Benigno and Woodford (2004, p.295)
B	Steady-state level of nominal debt	2.40	Benigno and Woodford (2004, p.295)
C	Steady-state level of consumption	0.70	
P	Steady-state level of the price level	1.00	

Table 2.2: Calibration of structural parameters.

Note that these values lead to an annual public debt to GDP ratio of 60%. Moreover, we have $\rho \equiv -\log(\beta) \approx 0.0101$, $\Theta \equiv \frac{(1-\alpha)}{1+\alpha(\epsilon-1)} = 0.25$, $\lambda \equiv \frac{(1-\beta\theta)(1-\theta)}{\theta}\Theta = 0.0425$ and $\kappa \equiv \lambda\left(\frac{(\varphi+\alpha)}{(1-\alpha)} + \sigma_C \frac{Y}{C}\right) \approx 0.095$.

First, we want to make a note on so-called infinite GEVs. In our case, these infinite GEVs may occur if the diagonal of matrix $\mathbf{S}$ has elements with absolute value of zero, i.e. $s_{ii} = 0$ for some i. As emphasized by Klein (2000, p.1410), one can treat such infinite GEVs in the same way as finite unstable eigenvalues, that is eigenvalues outside the unit circle.

According to Klein (2000, p.1410), given the fact that our matrix $\bar{\mathbf{A}}$ is singular, we have to deal with the infinite GEVs. The crucial question is, how many of them are present in our setting. We are not aware of a straightforward procedure to determine the number of infinite GEVs. Our conjecture is

$$\max\{Dimensions[\bar{\mathbf{A}}]\} - MatrixRank[\bar{\mathbf{A}}].$$

We base this conjecture on the fact that it works for the whole parameter space, given our calibration in Table 2.2. A proof of that conjecture is beyond the scope of this paper.

In the illustration of numerical results that follow in this subsection, we adopt the following legend: yellow regions are regions of explosiveness, red regions are regions of Order 2 Indeterminacy, blue regions are regions of Order 1 Indeterminacy and green regions are regions of determinacy. Note also that in each figure the vertical axis measures monetary policy feedback on the output gap via the coefficient $\phi_{\widehat{x}}$ and the horizontal axis measures feedback on inflation via the coefficient ϕ_{π}.

2.4.3.1 Purely Adaptive Expectations and Passive Fiscal Policy

Figure 2.2 shows the results for the case of purely adaptive expectations ($\iota = 0.90$) combined with passive fiscal policy ($\varrho_1 = 1.00$). Please keep in mind that we denote such a fiscal policy as *passive*. Panel 2.2(a) displays the outcome for the economy when only fully rational agents ($\chi = 1.00$) are present. From Panel 2.2(b) over Panel 2.2(c) to Panel 2.2(d), χ takes values of 0.90, 0.80 and 0.60 respectively, i.e. the fraction of type $\gamma = 2$ households increases to up to 40%.

An obvious qualitative feature of our results, that coincides with "Result 3"

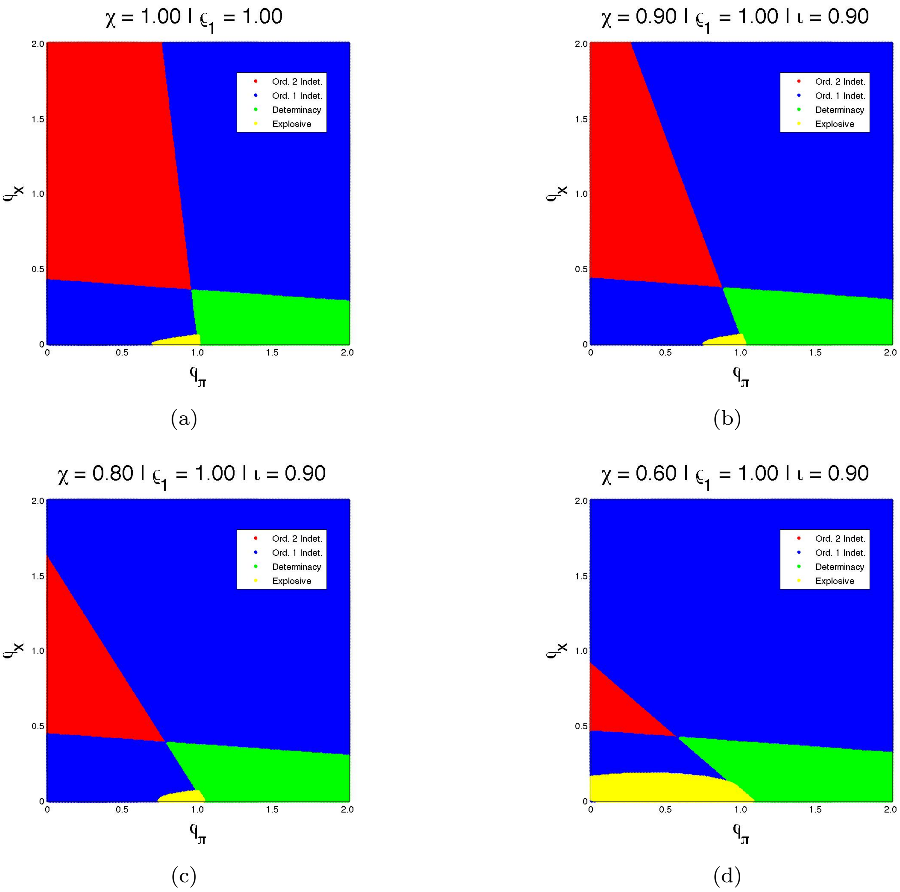


Figure 2.2: Regions of (in-)determinacy and explosiveness given purely adaptive expectations and passive fiscal policy.

in Branch and McGough (2009, p.11), is that an increase of the fraction of type $\gamma = 2$ households increases the region of determinacy and Order 1 Indeterminacy relative to Order 2 Indeterminacy. Moreover, we observe locally explosive regions in our model. This local divergence from steady-state is associated with policies that give slightly less than one-for-one feedback to inflation and none or minor feedback to the output gap.

Taking Panels 2.2(b)-2.2(d) into account, reveals our most notable result. The small locally explosive region grows with the increase in the fraction of $\gamma = 2$ agents. These agents with purely adaptive expectations have a destabilizing effect in the mathematical sense, that has not been observed by Branch and McGough (2009, p.10ff.) in their model. In addition, note that the locally explosive area expands around ($\phi_\pi \approx 1.0$). Thus, in this set-up it appears to be a wise monetary policy to give feedback on inflation that is significantly larger than one.

Furthermore, we observe that sticking to the Taylor principle ($\phi_\pi > 1.0$), i.e. *active* monetary policy is no guarantee to render the economy determinate. This is another qualitative result, we share with Branch and McGough (2009, p.12), but in our case, this insight must be viewed in the light of the given *passive* fiscal policy ($\varrho_1 = 1.00$). Thus, in our setting, we cannot confirm the result of Leeper (1991) and Evans and Honkapohja (2007) that combinations of *active* monetary policy and *passive* fiscal policy lead to determinacy in general (compare case (i) in Table 2.1).

2.4.3.2 Extrapolative Expectations and Passive Fiscal Policy

Figure 2.3 displays the outcome for the case of extrapolative expectations ($\iota = 1.10$) under *passive* fiscal policy.

Please note that Panel 2.3(a) by construction exactly matches Panel 2.2(a). But once we put heterogeneity of expectations into action, the picture changes, compared to the case of purely adaptive expectations.

Inspection of Panels 2.3(b) and 2.3(c) gives the impression that, with an increasing fraction of extrapolative households, policies $\{\phi_\pi, \phi_{\hat{x}}\}$, that used to lead to Order 1 Indeterminacy now may lead to Order 2 Indeterminacy. This effect of extrapolative expectations has been documented by Branch and McGough (2009, p.13) in their "Result 4". Furthermore, policies that used to lead to determinate outcomes may lead to indeterminacy of some order or locally explosive outcomes. Thus, in our economy it is not a necessary condition for local explosiveness to have the presence of both *active* monetary policy and *active* fiscal policy as in Leeper (1991) and Evans and Honkapohja (2007). Compared to Branch and McGough (2009, p.13) this is a new level of destabilization, as it is destabilizing in a mathematical sense.

Moreover, again we observe that sticking to the Taylor principle and therefore following combinations of *active* monetary policy and *passive* fiscal policy can lead to Order 1 Indeterminacy, in contrast to Leeper (1991) and Evans and Honkapohja (2007) (compare case (i) in Table 2.1).

Inspection of Panel 2.3(d) illustrates our finding that around $\alpha \approx 0.60$ the picture changes dramatically. The parameter space turns out to become a minefield, where we can no longer distinguish closed regions of any type. Our observations

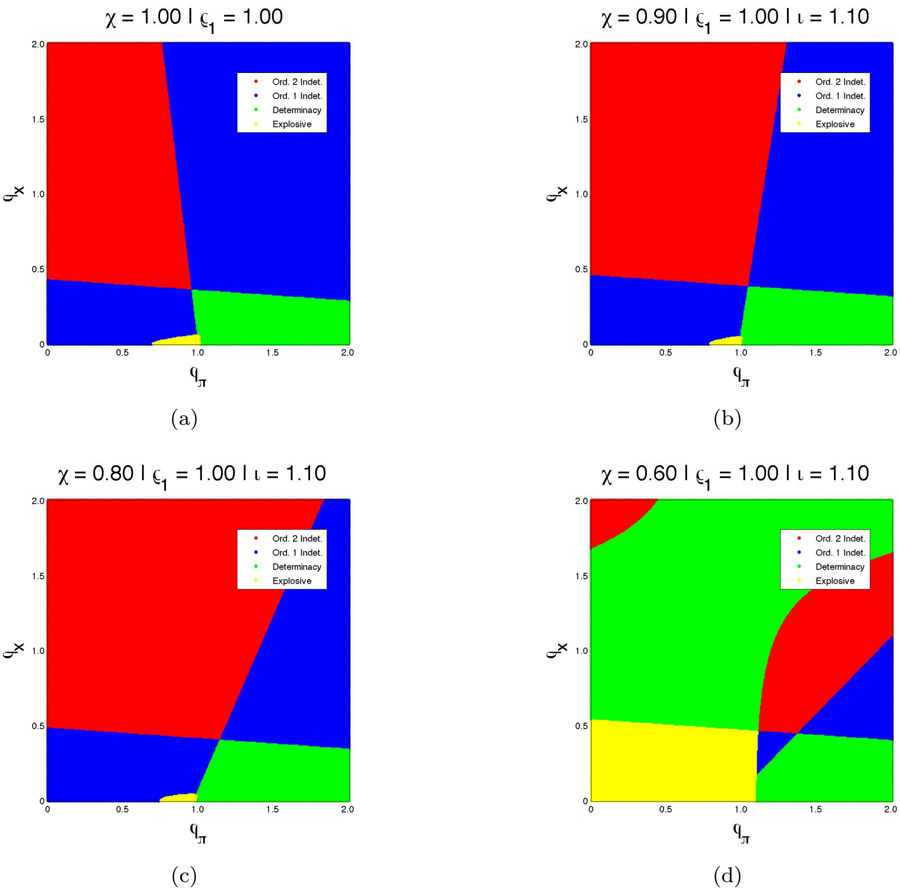


Figure 2.3: Regions of (in-)determinacy and explosiveness given extrapolative expectations and passive fiscal policy.

from Panels 2.3(a) to 2.3(c) do no longer hold. In particular, sensitivity analyses reveal that for $\alpha \lesssim 0.60$ an increasing fraction of agents with extrapolative expectations appears to reduce regions of indeterminacy of some order but increase locally explosive regions, which is a destabilizing effect.

2.4.3.3 Purely Adaptive Expectations and Active Fiscal Policy

No we draw attention to *active* fiscal policy ($\varrho_1 = 0.50$). In Figure 2.4 below we find results for the case of purely adaptive expectations ($\iota = 0.90$).

First of all, let us suspend non-rational households and focus on an economy in which only households of type $\gamma = 1$ are present. We compare the dynamics of our economy under passive fiscal policy as in Panel 2.2(a) with active fiscal policy as exhibited in Panel 2.4(a).

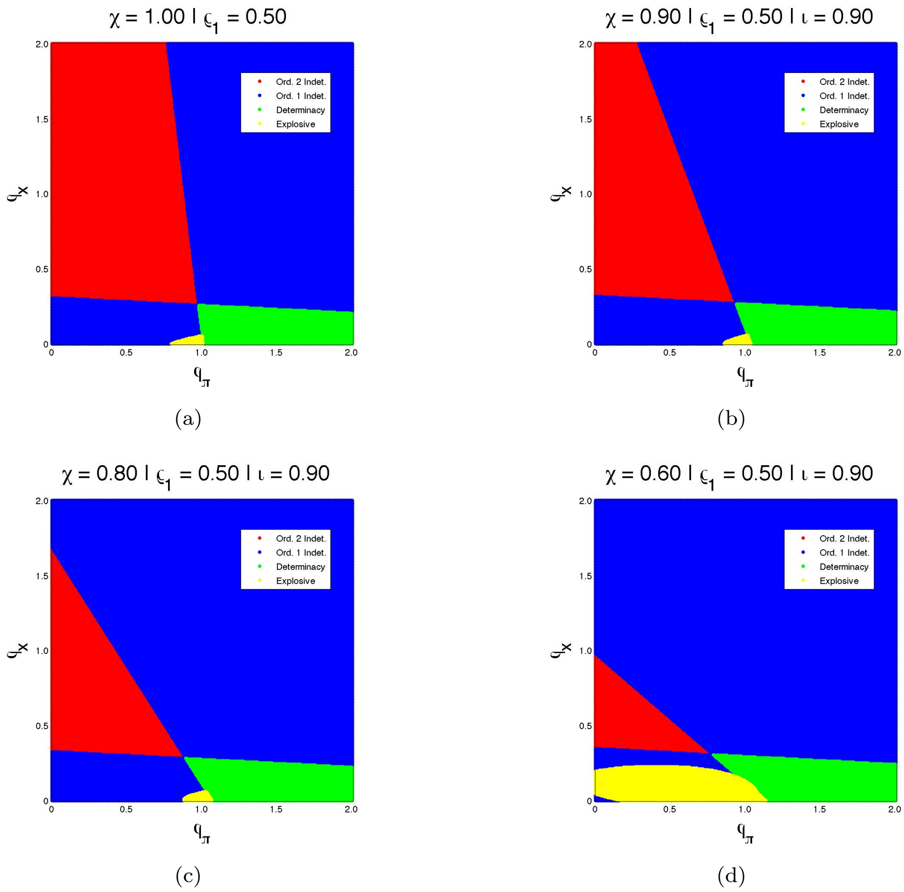


Figure 2.4: Regions of (in-)determinacy and explosiveness given purely adaptive expectations and active fiscal policy.

Obviously the consequence of an *active* fiscal policy are not that dramatic. We can only observe a relatively small increase in the set of policies $\{\phi_\pi,\ \phi_{\widehat{x}}\}$, that lead to indeterminacy of some order and a decrease in the set of policies $\{\phi_\pi,\ \phi_{\widehat{x}}\}$ that render the economy determinate. Thus, in our economy *active* fiscal policies have a minor impact apart from any interaction with heterogeneous expectations. Moreover, we once more observe the locally explosive region similar to the situation of passive fiscal policy.

Most notably, a *passive* monetary / *active* fiscal policy regime in our economy is unlikely to yield determinacy. This is not in line with the results of Leeper (1991) and Evans and Honkapohja (2007) (compare case (ii) in Table 2.1) in their model. In addition, we may also observe that the Taylor principle once more is not a guarantee for determinacy.

Now, what happens, once heterogeneity in expectations among households comes into effect? A first observation in Panels 2.4(b) and Panel 2.4(c) is that with an increasing fraction of purely adaptive agents, the region of Order 2 Indeterminacy is decreasing, while the region of Order 1 Indeterminacy and determinacy is increasing. But, also the region of local explosiveness is increasing in the fraction of purely adaptive agents. Thus, once more we find that the presence of purely adaptive agents does not lead to more stability in a mathematical sense.

With regard to active fiscal policy we observe that both regions of indeterminacy and local explosiveness are larger relative to the region of determinacy compared to the case of passive fiscal policy. Thus, active fiscal policy has a destabilizing influence on the economy.

2.4.3.4 Extrapolative Expectations and Active Fiscal Policy

Finally, we turn to the case of extrapolative expectations in an economy with *active* fiscal policy. Figure 2.5 provides the results.

Note that Panel 2.5(a) by construction has to match Panel 2.4(a).

Once heterogeneity is introduced, the pattern of Figure 2.3 is closely resembled. On the one hand, with an increasing fraction of extrapolative households, policies $\{\phi_\pi, \phi_{\widehat{x}}\}$ that used to lead to Order 1 Indeterminacy, may then lead to Order 2 Indeterminacy and on the other hand, policies $\{\phi_\pi, \phi_{\widehat{x}}\}$ that used to lead to determinate outcomes may then lead to Order 1 Indeterminacy. Moreover, we once more observe a region of locally explosive outcomes that expands around $\phi_\pi \approx 1.00$.

In addition, when expectations of type $\gamma = 2$ households are extrapolative and fiscal policy is *active*, regions of indeterminacy of some order are larger relative to the region of determinacy compared to the case of *passive* fiscal policy.

In addition, the Taylor principle does not hold in general, as in the cases before.

Finally, from Panel 2.5(d) we can again observe that around $\alpha \approx 0.60$ the parameter space turns out to become a minefield. Once more, the observations from Panels 2.5(a) to 2.5(c) do no longer hold. If $\alpha \lesssim 0.60$ an increasing fraction of agents with extrapolative expectations again reduces regions of indeterminacy of some order but increases locally explosive regions. Thus, extrapolative expectations seem to have a destabilizing effect.

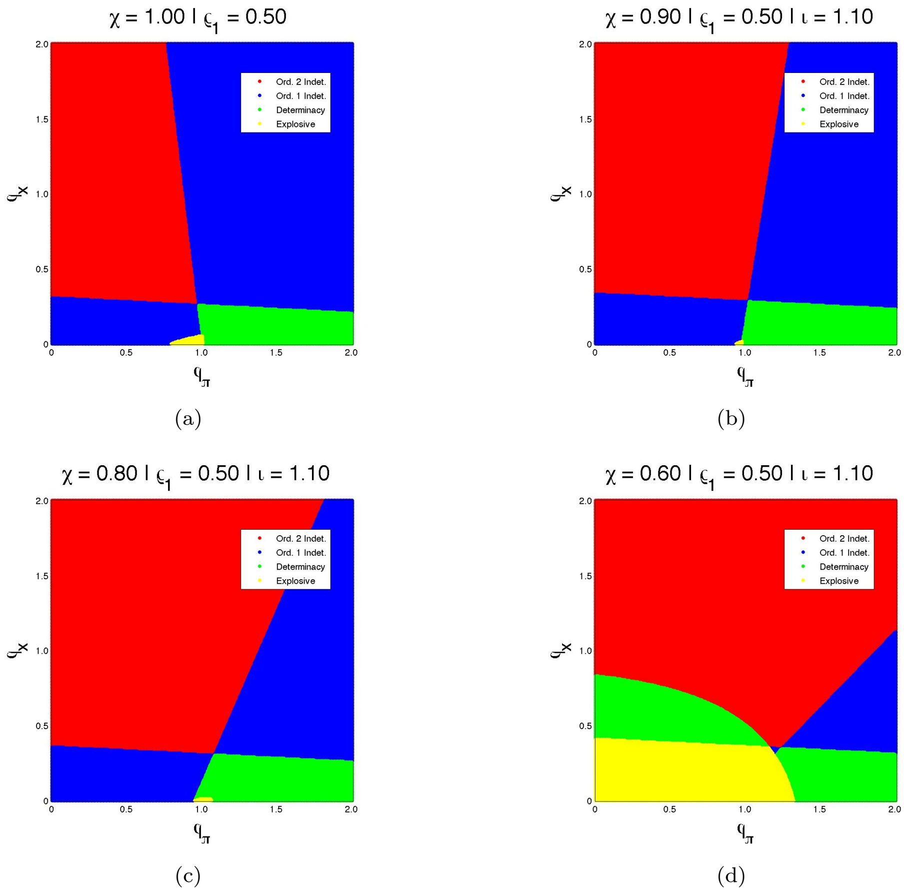


Figure 2.5: Regions of (in-)determinacy and explosiveness given extrapolative expectations and active fiscal policy.

2.5 Conclusion

Our study sheds light on the ability of a central bank to ensure price stability given interactions with fiscal policy as well as heterogeneous expectations of private sector agents.

Thereby, we contribute to the advancements in economic modelling by setting up a New Keynesian model with heterogeneous expectations and monetary and fiscal policy interaction. Moreover, in contrast to the previous literature, we derive aggregate equilibrium conditions from the assumption of a decentralized market.

The stability analysis reveals that the Taylor principle breaks down for policies that respond exclusively to expected inflation by one or slightly more than one-for-one as we detect a region of local explosiveness in this area of the parameter space. The size of the locally explosive region increases with an increasing fraction of agents with non-RE expectations.

Remarkably, when agents with RE and purely adaptive expectations coexist, the latter type of expectations does not have a stabilizing effect in general. Moreover, when we replace purely adaptive expectations with extrapolative expectations, the latter expectations do not destabilize the economy in general.

Another important finding is that even under RE a *passive* monetary / *active* fiscal policy regime is unlikely to yield determinacy which contradicts with the fiscalist view.

Finally, it turns out that active fiscal policy reduces the set of policies that yield price stability, but price stability remains feasible. Future research may explore the dynamic responses of key variables to structural shocks under monetary policy that yields price stability. A special emphasis should be given on the model's ability to qualitatively mimic stylized facts we observe in the data.

2.6 Appendices

2.6.1 Assumptions for the Subjective Expectations Operator

Note that the assumptions A1 to A7 are word by word the same as in Branch and McGough (2009, p.3). We restate them for the convenience of the interested reader.

A1. Expectations operators fix observables.

A2. If x is a variable forecasted by agents and has steady state $\bar{x}$ then $E^1\{\bar{x}\} = E^2\{\bar{x}\} = \bar{x}$.

A3. If x, y, $x+y$ and χx are variables forecasted by agents then $E_t^\gamma\{(x+y)\} = E_t^\gamma\{x\} + E_t^\gamma\{y\}$ and $E_t^\gamma\{\chi x\} = \chi E_t^\gamma\{x\}$.

A4. If for all $k \geq 0$, x_{t+k} and $\sum_{k=0}^{\infty} \beta^{t+k} x_{t+k}$ are forecasted by agents, then

$$E_t^\gamma \left\{ \sum_{k=0}^{\infty} \beta^{t+k} x_{t+k} \right\} = \sum_{k=0}^{\infty} \beta^{t+k} E_t^\gamma\{x_{t+k}\}.$$

A5. E_t^γ satisfies the law of iterated expectations: If x is a variable forecasted by agents at time t and time $t+k$ then $E_t^\gamma \circ E_{t+k}^\gamma\{x\} = E_t^\gamma\{x\}$.

A6. If x is a variable forecasted by agents at time t and time $t+k$ then $E_t^\gamma E_{t+k}^{\gamma'}\{x_{t+k}\} = E_t^\gamma\{x_{t+k}\}, \gamma \neq \gamma'$.

A7. All agents have common expectations on expected differences in limiting wealth.

2.6.2 Model Derivations

2.6.2.1 Logarithmic Approximation of Aggregate Real After-Tax Income

Here we follow DeJong and Dave (2007, p.13ff.). Consider an identity $\mathcal{F}_1(Z_t) = \mathcal{F}_2(Z_t)$. The operator $\mathcal{F}_\bullet(Z_t)$ denotes a function of a vector of variables Z_t. The logarithmic approximation to such functions in general terms is

$$\log[\mathcal{F}_1(e^{\log(Z_t)})] \approx \log[\mathcal{F}_1(Z)] + \frac{\partial \log[\mathcal{F}_1(Z)]}{\partial \log[Z_t]}[\log(Z_t) - \log(Z)].$$

for the left-hand side and the same holds for the right-hand side $\mathcal{F}_2(Z_t)$. Now lets turn to our result (2.4)

$$\underbrace{\Omega_t}_{\mathcal{F}_1(Z_t)} = \underbrace{(1 - T_t^{RE})Y_t}_{\mathcal{F}_2(Z_t)}$$

$$\begin{aligned} \log[\Omega_t] &= \log[(1 - T_t^{RE})Y_t] \\ \log[e^{\log(\Omega_t)}] &= \log[e^{\log((1-T_t^{RE})Y_t)}], \end{aligned}$$

and please note that in steady-state

$$\underbrace{\Omega}_{\mathcal{F}_1(Z)} = \underbrace{(1 - T^{RE})Y}_{\mathcal{F}_2(Z)}.$$

In our case the left-hand side is approximated by

$$\log[e^{\log(\Omega_t)}] \simeq \log[\Omega] + \frac{1}{\Omega}e^{\log(\Omega)}\ 1\ [\log(\Omega_t) - \log(\Omega)]$$

and the right-hand side is

$$\log[e^{\log((1-T_t^{RE})Y_t)}] \simeq \underbrace{\log[(1-T^{RE})Y]}_{=\log[\Omega]} + \frac{(1-T^{RE})\,Y}{(1-T^{RE})Y}[\log(Y_t)-\log(Y)] + \frac{-Y\;T^{RE}}{(1-T^{RE})Y}[\log(T_t^{RE})-\log(T^{RE})].$$

Equating the two approximations and recalling the definition $\log(Z_t)-\log(Z) = z_t - z = \widehat{z}_t$ yields (2.5)

$$\widehat{\omega}_t = \widehat{y}_t - \frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_t^{RE}. \tag{2.68}$$

2.6.2.2 Logarithmic Approximation of the Household Budget Constraint

Here again we follow DeJong and Dave (2007, p.13ff.). The household budget constraint (2.7) is

$$\underbrace{C_t(i)}_{\mathcal{F}_1(Z_t)} = \underbrace{\frac{B_{t-1}(i)}{P_{t-1}}\Pi_{t-1,t}^{-1} - E_t^{\gamma}\{Q_{t,t+1}\}\frac{B_t(i)}{P_t} + \Omega_t^{\gamma} - T}_{\mathcal{F}_2(Z_t,Z_{t-1})}. \tag{2.69}$$

Note that in steady-state

$$C = \frac{B}{P}\Pi^{-1} - \beta\frac{B}{P} + \Omega^{\gamma} - T, \tag{2.70}$$

where we make use of assumption *A2* as $E_t^{\gamma}\{Q_{t,t+1}\}$ in steady-state (as one can see from the household Euler equation (2.8)) becomes β. One can take natural logs and write (2.69) as

$$\log[\mathcal{F}_1(e^{\log(Z_t)})] = \log[\mathcal{F}_2(e^{\log(Z_t,Z_{t-1})})].$$

This is

$$\log[e^{\log(C_t(i))}] = \log[e^{\log(\frac{B_{t-1}(i)}{P_{t-1}}\Pi_{t-1,t}^{-1})} - e^{\log(\beta E_t^{\gamma}\left\{\left(\frac{C_{t+1}(i)}{C_t(i)}\right)^{-\sigma_C}\frac{P_t}{P_{t+1}}\right\}\frac{B_t(i)}{P_t})} + e^{\log(\Omega_t^{\gamma})} - e^{\log(T)}]. \tag{2.71}$$

The approximation to the left-hand side is

$$\log[C] + 1\ [\log(C_t(i)) - \log(C)].$$

The same holds for the right-hand side. Thus we get

$$\begin{aligned}
\log[\frac{B}{P}\Pi^{-1} - \beta\frac{B}{P} + \Omega^\gamma] \quad &+ \quad \frac{B/P}{C}\ [\log(\frac{B_{t-1}(i)}{P_{t-1}}\Pi^{-1}_{t-1,t}) - \log(\frac{B}{P}\Pi^{-1})] \\
&- \quad \beta\frac{B/P}{C}\ [\log(E^\gamma_t\{\left(\frac{C_{t+1}(i)}{C_t(i)}\right)^{-\sigma_C}\frac{P_t}{P_{t+1}}\}\frac{B_t(i)}{P_t}) \\
&- \quad \log(\left(\frac{C}{C}\right)^{-\sigma_C}\frac{P}{P}\frac{B}{P})] \\
&+ \quad \frac{\Omega^\gamma}{C}\ [\log(\Omega^\gamma_t) - \log(\Omega^\gamma)] - \frac{T}{C}\ [\log(T) - \log(T)].
\end{aligned}$$

We can combine left-hand side and right-hand side approximations and apply the definition $\log(Z_t) - \log(Z) = z_t - z = \widehat{z}_t$ as well as use (2.70) to eliminate Ω^γ which yields[45]

$$\begin{aligned}
\widehat{c}_t(i) \quad = \quad & \frac{B/P}{C}[\widehat{b}_{t-1}(i) - \pi_t] - \beta\frac{B/P}{C}[-\sigma_C E^\gamma_t\{\widehat{c}_{t+1}(i)\} + \sigma_C\widehat{c}_t(i) \\
& +\widehat{b}_t(i) - E^\gamma_t\{\pi_{t+1}\}] + \left[\frac{(C+T)}{C} - \frac{B/P}{C}(1-\beta)\right]\widehat{\omega}^\gamma_t.
\end{aligned} \tag{2.72}$$

We can rearrange terms and finally get (2.11)

$$\begin{aligned}
\widehat{c}_t(i) \quad = \quad & \frac{B/P}{[C+\beta(B/P)\sigma_C]}\left[\widehat{b}_{t-1}(i) - \pi_t\right] + \frac{\beta(B/P)\sigma_C}{[C+\beta(B/P)\sigma_C]}E^\gamma_t\{\widehat{c}_{t+1}(i)\} \\
& -\frac{\beta(B/P)}{[C+\beta(B/P)\sigma_C]}\left[\widehat{b}_t(i) - E^\gamma_t\{\pi_{t+1}\}\right] \\
& +\frac{C+T-(B/P)(1-\beta)}{C+\beta(B/P)\sigma_C}\ \widehat{\omega}^\gamma_t.
\end{aligned} \tag{2.73}$$

[45]It is valid to split $-\sigma_C E^\gamma_t\{\triangle\widehat{c}_{t+1}(i)\}$ into $-\sigma_C E^\gamma_t\{\widehat{c}_{t+1}(i)\} + \sigma_C\widehat{c}_t(i)$ as every agent knows his own period t choice and there is no need to forecast that. Otherwise assumption *A1* would not be satisfied.

2.6.2.3 The Forward Iteration of Household Wealth

We derived

$$\widehat{\mathcal{W}}_t^\gamma = E_t^\gamma\{\widehat{\mathcal{W}}_{t+1}^\gamma\} - \frac{1}{\sigma_C}\left[i_t - E_t^\gamma\{\pi_{t+1}\} - \rho\right], \tag{2.74}$$

which originally is

$$E_t^\gamma\{\widehat{\mathcal{W}}_t^\gamma\} = E_t^\gamma\{\widehat{\mathcal{W}}_{t+1}^\gamma - \frac{1}{\sigma_C}\left[i_t - \pi_{t+1} - \rho\right]\}, \tag{2.75}$$

but could be rewritten as (2.74) due to assumptions *A1* and *A3*. One period ahead (2.75) is

$$E_{t+1}^\gamma\{\widehat{\mathcal{W}}_{t+1}^\gamma\} = E_{t+1}^\gamma\{\widehat{\mathcal{W}}_{t+2}^\gamma - \frac{1}{\sigma_C}\left[i_{t+1} - \pi_{t+2} - \rho\right]\},$$

or

$$\widehat{\mathcal{W}}_{t+1}^\gamma = E_{t+1}^\gamma\{\widehat{\mathcal{W}}_{t+2}^\gamma\} - \frac{1}{\sigma_C}\left[i_{t+1} - E_{t+1}^\gamma\{\pi_{t+2}\} - \rho\right], \tag{2.76}$$

again applying assumptions *A1* and *A3*. We can plug this into (2.74) and get

$$\begin{aligned}\widehat{\mathcal{W}}_t^\gamma &= E_t^\gamma\{E_{t+1}^\gamma\{\widehat{\mathcal{W}}_{t+2}^\gamma\} - \frac{1}{\sigma_C}\left[i_{t+1} - E_{t+1}^\gamma\{\pi_{t+2}\} - \rho\right]\} \\ &\quad -\frac{1}{\sigma_C}\left[i_t - E_t^\gamma\{\pi_{t+1}\} - \rho\right]\end{aligned}$$

or

$$\begin{aligned}\widehat{\mathcal{W}}_t^\gamma &= E_t^\gamma\{E_{t+1}^\gamma\{\widehat{\mathcal{W}}_{t+2}^\gamma\}\} - \frac{1}{\sigma_C}\left[E_t^\gamma\{i_{t+1}\} - E_t^\gamma\{E_{t+1}^\gamma\{\pi_{t+2}\}\} - \rho\right] \\ &\quad -\frac{1}{\sigma_C}\left[i_t - E_t^\gamma\{\pi_{t+1}\} - \rho\right],\end{aligned}$$

where we used assumptions *A1* and *A3* again. Applying assumption *A5* allows us to write

$$\begin{aligned}\widehat{\mathcal{W}}_t^\gamma &= E_t^\gamma\{\widehat{\mathcal{W}}_{t+2}^\gamma\} - \frac{1}{\sigma_C}\left[E_t^\gamma\{i_{t+1}\} - E_t^\gamma\{\pi_{t+2}\} - \rho\right] \\ &\quad -\frac{1}{\sigma_C}\left[i_t - E_t^\gamma\{\pi_{t+1}\} - \rho\right].\end{aligned} \tag{2.77}$$

Iteration of the steps that lead to (2.77) yield

$$\widehat{\mathcal{W}}_t^\gamma = \underbrace{\lim_{k\mapsto\infty} E_t^\gamma\{\widehat{\mathcal{W}}_{t+k+1}^\gamma\}}_{\equiv\widehat{\mathcal{W}}_\infty^\gamma} - \frac{1}{\sigma_C}\sum_{k=0}^{\infty} E_t^\gamma\{[i_{t+k} - \pi_{t+k+1} - \rho]\},$$

which we can write as (2.13)

$$\widehat{\mathcal{W}}_t^\gamma = \widehat{\mathcal{W}}_\infty^\gamma - \frac{1}{\sigma_C} E_t^\gamma \left\{\sum_{k=0}^{\infty} [i_{t+k} - \pi_{t+k+1} - \rho]\right\}$$

due to assumption *A4* and the definition of $\widehat{\mathcal{W}}_\infty^\gamma$.

2.6.2.4 Logarithmic Approximation of Aggregate Price Level Dynamics

We start from

$$1 \quad = \quad \theta(\Pi_{t-1,t})^{-(1-\epsilon)} + (1-\theta)(\frac{P_t^*}{P_t})^{(1-\epsilon)}, \tag{2.78}$$

which might be written as

$$\log[1] \quad = \quad \log\left[e^{\log(\theta(\Pi_{t-1,t})^{-(1-\epsilon)})} + e^{\log((1-\theta)(\frac{P_t^*}{P_t})^{(1-\epsilon)})}\right]. \tag{2.79}$$

The left-hand side of (2.79) is exactly 0 and the right-hand side's approximation is

$$\begin{aligned}
\simeq \qquad & \log\left[e^{\log(\theta(\Pi)^{-(1-\epsilon)})} + e^{\log((1-\theta)(\frac{P^*}{P})^{(1-\epsilon)})}\right] \\
& +\theta[\log(\theta(\Pi_{t-1,t})^{-(1-\epsilon)}) - \log(\theta(\Pi)^{-(1-\epsilon)})] \\
& +(1-\theta)[\log((1-\theta)(\frac{P_t^*}{P_t})^{(1-\epsilon)}) - \log((1-\theta)(\frac{P^*}{P})^{(1-\epsilon)})].
\end{aligned}$$

Consequently we achieve (2.14)

$$\widehat{p}_t^* - \widehat{p}_t = \frac{\theta}{(1-\theta)}\pi_t. \tag{2.80}$$

2.6.2.5 Logarithmic Approximation of the Firm's FOC

We start from (2.20)

$$0 = \widehat{E}_t \Bigg\{ \underbrace{\sum_{k=0}^{\infty} (\beta\theta)^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma_C} \Pi_{t,t+k}^{-1}\; Y_{t+k|t}(i) \times \left[(1 - T_{t+k}^{RE}) \frac{P_t^*(i)}{P_t} - \mathcal{M}\; \Pi_{t,t+k}\; MC_{t+k|t}(i)\right] \Bigg\}}_{\mathcal{F}_1(Z_{t+k}, Z_t)}. \tag{2.81}$$

In steady-state, we can write this as

$$0 = \underbrace{(1-\beta\theta)\left[(1 - T^{RE})\frac{P^*(i)}{P} - \mathcal{M}\; \Pi\; MC\right]}_{\mathcal{F}_1(Z)} \tag{2.82}$$

We now equate the left-hand side of (2.81) to a log-linear approximation of its right-hand side

$$0 = \underbrace{\log\left[\widehat{E}_t \left\{\sum_{k=0}^{\infty} (\beta\theta)^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma_C} \Pi_{t,t+k}^{-1}\; Y_{t+k|t}(i) \times \left[e^{\log(\frac{P_t^*(i)}{P_t})} - e^{\log(T_{t+k}^{RE}\frac{P_t^*(i)}{P_t})} - \mathcal{M} e^{\log(\Pi_{t,t+k} MC_{t+k|t}(i))}\right]\right\}\right]}_{\mathcal{F}_1(e^{\log(Z_{t+k})}, e^{\log(Z_t)})}.$$

The right-hand side can be approximated by

$$\log[\mathcal{F}_1(Z)]$$
$$+\frac{1}{\mathcal{F}_1(Z)}\times$$
$$\left(\widehat{E}_t\left\{\sum_{k=0}^{\infty}(\beta\theta)^k(1-\beta\theta)e^{\log(\frac{P^*(i)}{P})}[\log\left(\frac{P_t^*(i)}{P_t}\right)-\log\left(\frac{P^*(i)}{P}\right)]\right\}\right)$$
$$-\frac{1}{\mathcal{F}_1(Z)}\times$$
$$\left(\widehat{E}_t\left\{\sum_{k=0}^{\infty}(\beta\theta)^k(1-\beta\theta)e^{\log(T^{RE}\frac{P^*(i)}{P})}[\log\left(T_{t+k}^{RE}\frac{P_t^*(i)}{P_t}\right)-\log\left(T^{RE}\frac{P^*(i)}{P}\right)]\right\}\right)$$
$$-\frac{1}{\mathcal{F}_1(Z)}\times$$
$$\left(\widehat{E}_t\left\{\sum_{k=0}^{\infty}(\beta\theta)^k\mathcal{M}(1-\beta\theta)e^{\log(\Pi MC)}[\log\left(\Pi_{t,t+k}MC_{t+k|t}(i)\right)-\log\left(\Pi MC\right)]\right\}\right).$$

Combining left-hand side and approximation of right-hand side, using the fact that $P^*(i) = P$ as well as applying the definition $\log(Z_t) - \log(Z) = z_t - z = \widehat{z}_t$ yields

$$\begin{aligned} 0 &= (1-\beta\theta)\widehat{E}_t\{\sum_{k=0}^{\infty}(\beta\theta)^k \times \\ &\quad \left[(1-T^{RE})[\widehat{p}_t^*(i)-\widehat{p}_t]-T^{RE}\widehat{\tau}_{t+k}^{RE}-\mathcal{M}\Pi MC[\widehat{\pi}_{t,t+k}+\widehat{mc}_{t+k|t}(i)]\right]\}. \end{aligned}$$

Inspecting (2.82) indicates $\mathcal{M}\Pi MC = (1 - T^{RE})$ so we can divide by $(1 - T^{RE})$ and bring $\widehat{p}_t^*(i)$ on the left-hand side and therefore a log-linearized version is (2.21)

$$\begin{aligned} [\widehat{p}_t^*(i)-\widehat{p}_t] &= (1-\beta\theta)\times \\ &\quad \widehat{E}_t\left\{\sum_{k=0}^{\infty}(\beta\theta)^k\left[\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_{t+k}^{RE}+\left[\widehat{\pi}_{t|t+k}+\widehat{mc}_{t+k|t}(i)\right]\right]\right\}, \end{aligned} \quad (2.83)$$

where we make use of assumption *A1*.

2.6.2.6 Logarithmic Approximation of the Tax Rule

The tax rule (2.23)

$$\begin{aligned}\underbrace{T + T_t^{RE} Y_t}_{\mathcal{F}_1(Z_t)} &= \underbrace{\left[\varrho_0 + \varrho_1 \frac{B_{t-1}}{P_{t-1}}\ \Pi_{t-1,t}^{-1}\right] \Psi_t}_{\mathcal{F}_2(Z_t, Z_{t-1})} \\ \log[T + T_t^{RE} Y_t] &= \log[\left[\varrho_0 + \varrho_1 \frac{B_{t-1}}{P_{t-1}}\ \Pi_{t-1,t}^{-1}\right] \Psi_t] \\ \log[e^{\log(T)} + e^{\log(T_t^{RE} Y_t)}] &= \log[e^{\log(\varrho_0 \Psi_t)} + e^{\log(\varrho_1 \frac{B_{t-1}}{P_{t-1}}\ \Pi_{t-1,t}^{-1} \Psi_t)}],\end{aligned}$$

and please note that in steady-state

$$\underbrace{T + T^{RE} Y}_{\mathcal{F}_1(Z)} = \underbrace{\left[\varrho_0 + \varrho_1 \frac{B}{P}\ \Pi^{-1}\right] \Psi}_{\mathcal{F}_2(Z)}$$

In our case the left-hand side is approximated by

$$\begin{aligned}\log[e^{\log(T)} + e^{\log(T_t^{RE} Y_t)}] \simeq\ & \log[T + T^{RE} Y] \\ & + \frac{e^{\log(T^{RE} Y)}}{(T + T^{RE} Y)} [\log(T_t^{RE} Y_t) - \log(T^{RE} Y)] \\ & + \frac{e^{\log(T)}}{(T + T^{RE} Y)} [\log(T) - \log(T)]\end{aligned}$$

and the right-hand side yields

$$\begin{aligned}\log[e^{\log(\varrho_0 \Psi_t)} + e^{\log(\varrho_1 \frac{B_{t-1}}{P_{t-1}}\ \Pi_{t-1,t}^{-1} \Psi_t)}] \simeq\ & \underbrace{\log[\mathcal{F}_2(Z)]}_{=\log[T + T^{RE} Y]} \\ & + \frac{1}{\mathcal{F}_2(Z)}\ e^{\log(\varrho_0 \Psi)} [\log(\varrho_0 \Psi_t) - \log(\varrho_0 \Psi)] \\ + \frac{1}{\mathcal{F}_2(Z)}\ e^{\log(\varrho_1 \frac{B}{P}\ \Pi^{-1} \Psi)} [\log(\varrho_1 & \frac{B_{t-1}}{P_{t-1}}\ \Pi_{t-1,t}^{-1} \Psi_t) - \log(\varrho_1 \frac{B}{P}\ \Pi^{-1} \Psi)]\end{aligned}$$

Equating the two approximations and recalling the definition $\log(Z_t) - \log(Z) = z_t - z = \widehat{z}_t$ yields (2.24)

$$\widehat{\tau}_t^{RE} = \frac{B/P}{T^{RE} Y} \varrho_1 [\widehat{b}_{t-1} - \pi_t] - \widehat{y}_t + \frac{(T + T^{RE} Y)}{T^{RE} Y} \widehat{\Psi}_t. \tag{2.84}$$

2.6.2.7 Logarithmic Approximation of the Government Budget Constraint

The government budget constraint in real terms is given by (2.22)

$$\underbrace{G}_{\mathcal{F}_1(Z_t)} = \underbrace{\widehat{E}_t\{Q_{t,t+1}\}\frac{B_t}{P_t} + T_t^{RE}Y_t + T - \frac{B_{t-1}}{P_{t-1}}\,\Pi_{t-1,t}^{-1}}_{\mathcal{F}_2(Z_t,Z_{t-1})} \tag{2.85}$$

Note that in steady-state

$$\underbrace{G}_{\mathcal{F}_1(Z)} = \underbrace{\beta\frac{B}{P} + T^{RE}Y + T - \frac{B}{P}\,\Pi^{-1}}_{\mathcal{F}_2(Z)}$$

where we make use of assumption *A2* as $\widehat{E}_t\{Q_{t,t+1}\}$ in steady-state (as one can see from the household Euler equation (2.8)) becomes β. One can take natural logs and write (2.85) as

$$\begin{aligned}\log[e^{\log(G)}] \;&=\; \log[e^{\log(\beta\widehat{E}_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma_C}\frac{P_t}{P_{t+1}}\right\}\frac{B_t}{P_t})} + e^{\log(T_t^{RE}Y_t)} + e^{\log(T)}\\ &\quad -e^{\log\left(\frac{B_{t-1}}{P_{t-1}}\,\Pi_{t-1,t}^{-1}\right)}]\end{aligned}$$

The approximation to the left-hand side is

$$\log[G] + \frac{1}{G}e^{\log(G)}1\;[\log(G) - \log(G)].$$

The same holds for the right-hand side. Thus we get

$$\begin{aligned}\log[\mathcal{F}_2(Z)] + \frac{1}{\mathcal{F}_2(Z)}e^{\log(\beta\frac{B}{P})}1\;[\log(\beta\widehat{E}_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma_C}\frac{P_t}{P_{t+1}}\right\}\frac{B_t}{P_t})\\ -\log(\beta\left(\frac{C}{C}\right)^{-\sigma_C}\frac{P}{P}\frac{B}{P})]\\ +\frac{1}{\mathcal{F}_2(Z)}e^{\log(T^{RE}Y)}1\;[\log(T_t^{RE}Y_t) - \log(T^{RE}Y)]\\ -\frac{1}{\mathcal{F}_2(Z)}e^{\log\left(\frac{B}{P}\,\Pi^{-1}\right)}1\;[\log\left(\frac{B_{t-1}}{P_{t-1}}\,\Pi_{t-1,t}^{-1}\right) - \log\left(\frac{B}{P}\,\Pi^{-1}\right)].\end{aligned}$$

We can combine left-hand side and right-hand side approximations and apply the definition $\log(Z_t) - \log(Z) = z_t - z = \widehat{z}_t$ and finally get to equation (2.25), which is[46]

$$\widehat{b}_t - \widehat{E}_t\{\pi_{t+1}\} = \sigma_C \widehat{E}_t\{\widehat{c}_{t+1}\} - \sigma_C \widehat{c}_t + \frac{1}{\beta}(\widehat{b}_{t-1} - \pi_t) - \frac{T^{RE}Y}{\beta \frac{B}{P}}(\widehat{\tau}_t^{RE} + \widehat{y}_t). \quad (2.86)$$

2.6.2.8 Logarithmic Approximation of the Goods Market Clearing Condition

The goods market clearing condition (2.28)

$$Y_t = C_t + G$$

might be written as

$$\log[e^{\log(Y_t)}] \quad = \quad \log[e^{\log(C_t)} + e^{\log(G)}],$$

The left-hand side of (2.87) is approximated by

$$\log[e^{\log(Y_t)}] \quad \simeq \quad \log[Y] + \frac{1}{Y} e^{\log(Y)} \ 1 \ [\log(Y_t) - \log(Y)]$$

and similarly the right-hand side's approximation is

$$\begin{aligned} \log[e^{\log(C_t)} + e^{\log(G)}] \quad \simeq \quad & \log[C + G] \\ & + \frac{1}{C+G} e^{\log(C)} \ 1 \ [\log(C_t) - \log(C)] \\ & + \frac{1}{C+G} e^{\log(G)} \ 1 \ [\log(G) - \log(G)]. \end{aligned}$$

Consequently we achieve (2.29)

$$\widehat{y}_t = \frac{C}{Y} \widehat{c}_t. \quad (2.87)$$

[46]Splitting $\sigma_C \widehat{E}_t\{\triangle \widehat{c}_{t+1}\}$ into $\sigma_C \widehat{E}_t\{\widehat{c}_{t+1}\} - \sigma_C \widehat{c}_t$ is valid as we assume that the government observes period t aggregate consumption. Otherwise assumption *A1* is not satisfied.

2.6.2.9 Logarithmic Approximation of the Bonds Market Clearing Condition

The bonds market clearing condition (2.28) in real terms is

$$\frac{B_t}{P_t} = \chi\frac{B_t^1}{P_t} + (1-\chi)\frac{B_t^2}{P_t}$$

and might be written as

$$\log[e^{\log(\frac{B_t}{P_t})}] \;=\; \log[\chi e^{\log(\frac{B_t^1}{P_t})} + (1-\chi)e^{\log(\frac{B_t^2}{P_t})}].$$

The left-hand side of (2.88) is approximated by

$$\log[e^{\log(\frac{B_t}{P_t})}] \;\simeq\; \log[\frac{B}{P}] + \frac{1}{\frac{B}{P}}e^{\log(\frac{B}{P})}\ 1\ [\log(\frac{B_t}{P_t}) - \log(\frac{B}{P})]$$

and similarly the right-hand side's approximation is

$$\begin{aligned}\log[\chi e^{\log(\frac{B_t^1}{P_t})} + (1-\chi)e^{\log(\frac{B_t^2}{P_t})}] \;\simeq\; & \log[\frac{B}{P}] \\ & +\frac{1}{\frac{B}{P}}\chi e^{\log(\frac{B^1}{P})}\ 1\ [\log(\frac{B_t^1}{P_t}) - \log(\frac{B^1}{P})] \\ & +\frac{1}{\frac{B}{P}}(1-\chi)e^{\log(\frac{B^2}{P})}\ 1\ [\log(\frac{B_t^2}{P_t}) - \log(\frac{B^2}{P})].\end{aligned}$$

Recognize assumption *A2* which implies that beliefs of agents, no matter what type, coincide in steady-state. Consequently all agents will hold the same amount of bonds, that is

$$B = \chi B^1 + (1-\chi)B^1 = \chi B^2 + (1-\chi)B^2 = B^1 = B^2.$$

By making use of this fact we achieve (2.31)

$$\widehat{b}_t = \chi\widehat{b}_t^1 + (1-\chi)\widehat{b}_t^2. \tag{2.88}$$

2.6.2.10 The Forward Iteration of the Firm's FOC

Starting with (2.39)

$$\begin{aligned}\widehat{p}_t^*(i)-\widehat{p}_t &= (1-\beta\theta)\widehat{E}_t\{\sum_{k=0}^{\infty}(\beta\theta)^k \times \\ &\left[\Theta\left(\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_{t+k}^{RE}+\widehat{mc}_{t+k}\right)+\frac{\beta\theta}{(1-\beta\theta)}\pi_{t+k+1}\right]\},\end{aligned}$$

where $\Theta \equiv \frac{(1-\alpha)}{1+\alpha(\epsilon-1)} \leq 1$, we know that by assumption *A1* this comes from

$$\begin{aligned}\widehat{E}_t\left\{\widehat{p}_t^*(i)-\widehat{p}_t\right\} &= (1-\beta\theta)\widehat{E}_t\{\sum_{k=0}^{\infty}(\beta\theta)^k \times \\ &\left[\Theta\left(\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_{t+k}^{RE}+\widehat{mc}_{t+k}\right)+\frac{\beta\theta}{(1-\beta\theta)}\pi_{t+k+1}\right]\}.\end{aligned} \tag{2.89}$$

One period ahead, that is $t+1$, this becomes

$$\begin{aligned}\widehat{E}_{t+1}\left\{\widehat{p}_{t+1}^*(i)-\widehat{p}_{t+1}\right\} &= (1-\beta\theta)\widehat{E}_{t+1}\{\sum_{k=0}^{\infty}(\beta\theta)^k \times \\ &[\Theta\left(\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_{t+k+1}^{RE}+\widehat{mc}_{t+k+1}\right)+ \\ &\frac{\beta\theta}{(1-\beta\theta)}\pi_{t+k+2}]\}\end{aligned}$$

or

$$\beta\theta\widehat{E}_{t+1}\left\{\widehat{p}_{t+1}^*(i)-\widehat{p}_{t+1}\right\} = (1-\beta\theta)\times$$

$$\widehat{E}_{t+1}\left\{\sum_{k=0}^{\infty}(\beta\theta)^{k+1}\left[\Theta\left(\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}_{t+k+1}^{RE}+\widehat{mc}_{t+k+1}\right)+\frac{\beta\theta}{(1-\beta\theta)}\pi_{t+k+2}\right]\right\},$$

where we multiplied both sides with $(\beta\theta)$ in the latter expression. We can evaluate this expression with time t expectations $\widehat{E}_t\{\bullet\}$ and apply assumption *A5* to get

$$\beta\theta\widehat{E}_t\left\{\widehat{p}^*_{t+1}(i)-\widehat{p}_{t+1}\right\}=(1-\beta\theta)\times$$

$$\widehat{E}_t\left\{\sum_{k=0}^{\infty}(\beta\theta)^{k+1}\left[\Theta\left(\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}^{RE}_{t+k+1}+\widehat{mc}_{t+k+1}\right)+\frac{\beta\theta}{(1-\beta\theta)}\pi_{t+k+2}\right]\right\}. \tag{2.90}$$

Now we can subtract (2.90) from (2.89), where we make use of assumption *A4*, which leaves us with

$$\begin{aligned}\widehat{p}^*_t(i)-\widehat{p}_t &= (1-\beta\theta)\widehat{E}_t\left\{\left[\Theta\left(\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}^{RE}_t+\widehat{mc}_t\right)+\frac{\beta\theta}{(1-\beta\theta)}\pi_{t+1}\right]\right\}\\ &\quad+\beta\theta\widehat{E}_t\left\{\widehat{p}^*_{t+1}(i)-\widehat{p}_{t+1}\right\}.\end{aligned}$$

When we apply assumptions *A1* and *A3* we finally arrive at the difference equation (2.40)

$$\begin{aligned}\widehat{p}^*_t(i)-\widehat{p}_t &= (1-\beta\theta)\Theta\left[\frac{T^{RE}}{(1-T^{RE})}\widehat{\tau}^{RE}_t+\widehat{mc}_t\right]+\beta\theta\widehat{E}_t\left\{\pi_{t+1}\right\}\\ &\quad+\beta\theta\widehat{E}_t\left\{\widehat{p}^*_{t+1}(i)-\widehat{p}_{t+1}\right\}.\end{aligned}$$

2.6.2.11 Logarithmic Approximation of the Optimality Condition Under Flexible Prices

The optimality condition under flexible prices is (2.44)

$$\frac{W_t}{P_t}=\frac{(1-T^{RE}_t)}{\mathcal{M}}(1-\alpha)A_tN_t^{-\alpha}.$$

We can equivalently write this as

$$\log[e^{\log(\frac{W_t}{P_t})}] = \log[\frac{(1-\alpha)}{\mathcal{M}}(e^{\log(A_tN_t^{-\alpha})}-e^{\log(T^{RE}_tA_tN_t^{-\alpha})})], \tag{2.91}$$

The left-hand side of (2.91) is approximated by

$$\log[e^{\log(\frac{W_t}{P_t})}] \simeq \log[\frac{W}{P}] + \frac{1}{\frac{W}{P}} e^{\log(\frac{W}{P})} \ 1 \ [\log(\frac{W_t}{P_t}) - \log(\frac{W}{P})]$$

and similarly the right-hand side's approximation is

$$\begin{aligned}
\log[\frac{(1-\alpha)}{\mathcal{M}}(e^{\log(A_t N_t^{-\alpha})} - e^{\log(T_t^{RE} A_t N_t^{-\alpha})})] &\simeq \log[(1-T^{RE})\frac{(1-\alpha)}{\mathcal{M}} A N^{-\alpha}] \\
+\frac{1}{[(1-T^{RE})\frac{(1-\alpha)}{\mathcal{M}} A N^{-\alpha}]} & \frac{(1-\alpha) e^{\log(AN^{-\alpha})}}{\mathcal{M}} [\log(A_t N_t^{-\alpha}) - \log(AN^{-\alpha})] \\
&- \frac{1}{[(1-T^{RE})\frac{(1-\alpha)}{\mathcal{M}} A N^{-\alpha}]} \frac{(1-\alpha) e^{\log(T^{RE} AN^{-\alpha})}}{\mathcal{M}} \times \\
&\left[\log(T_t^{RE} A_t N_t^{-\alpha}) - \log(T^{RE} A N^{-\alpha})\right].
\end{aligned}$$

Combining the two approximations yields (2.45)

$$\widehat{w}_t - \widehat{p}_t = a_t - \alpha \widehat{n}_t - \frac{T^{RE}}{(1-T^{RE})} \widehat{\tau}_t^{RE}. \tag{2.92}$$

2.6.3 A Comparison of Two Solution Methods

In this section we compare the eigenvalues of the system matrix $\mathbf{M}$ in Branch and McGough (2009, p.11ff.) derived according to the methodology of Blanchard and Kahn (1980) to GEVs according to the methodology of Klein (2000) for the same system matrix $\mathbf{M}$. We twice demonstrate the methodology of Klein (2000), first, we do not account for complex eigenvalues, whereas for the second time we do account for complex eigenvalues. $\mathbf{M}$ is given by

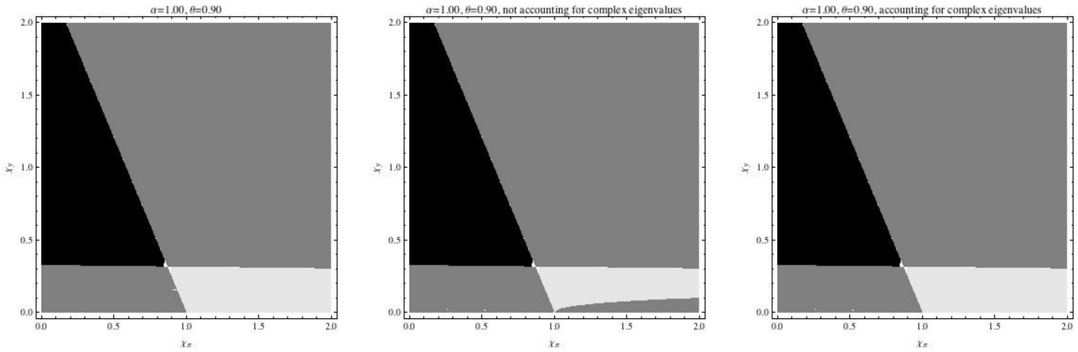
$$\mathbf{M} = \begin{bmatrix}
\frac{\alpha\beta}{\alpha^2\beta - \frac{\alpha\beta\chi_y}{\sigma}} + \frac{\lambda(\alpha-\chi_\pi)}{\sigma\left(\alpha^2\beta - \frac{\alpha\beta\chi_y}{\sigma}\right)} & -\frac{\alpha-\chi_\pi}{\sigma\left(\alpha^2\beta - \frac{\alpha\beta\chi_y}{\sigma}\right)} & -\frac{(1-\alpha)\alpha\beta\theta^2}{\alpha^2\beta - \frac{\alpha\beta\chi_\mathrm{y}}{\sigma}} & \frac{(1-\alpha)\beta\theta^2(\alpha-\chi_\pi)}{\sigma\left(\alpha^2\beta - \frac{\alpha\beta\chi_y}{\sigma}\right)} - \frac{(1-\alpha)\alpha\beta\theta^2}{\sigma\left(\alpha^2\beta - \frac{\alpha\beta\chi_y}{\sigma}\right)} \\
-\frac{\lambda\left(\alpha - \frac{\chi_y}{\sigma}\right)}{\alpha^2\beta - \frac{\alpha\beta\chi_y}{\sigma}} & \frac{\alpha - \frac{\chi_y}{\sigma}}{\alpha^2\beta - \frac{\alpha\beta\chi_y}{\sigma}} & 0 & \frac{(1-\alpha)\beta\theta^2\left(\frac{\chi_y}{\sigma} - \alpha\right)}{\alpha^2\beta - \frac{\alpha\beta\chi_y}{\sigma}} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},$$

where all symbols in this appendix are the same as in Branch and McGough (2009, p.11ff.) and the calibration in what follows is also identical with their study as listed in Table 2.3 below.

Parameter	Value	Parameter	Value
θ	0.900	α	1.000
σ	0.157	λ	0.024
β	0.9900	ϕ	6.3694

Table 2.3: Calibration of structural parameters for the appendix.

In order to be precise, we only reproduce the north-west Panel of Branch and McGough (2009, p.12)'s Figure 1 for both methodologies. The result is displayed Figure 2.6 below.



(a) (In-)determinacy regions resulting from the methodology of Blanchard and Kahn (1980).

(b) (In-)determinacy regions resulting from the methodology of Klein (2000) without accounting for complex eigenvalues.

(c) (In-)determinacy regions resulting from the methodology of Klein (2000), accounting for complex eigenvalues.

Figure 2.6: Reproduction of the results of the north-west Panel of Branch and McGough (2009, p.12)'s Figure 1.

First of all, 2.6(a) reproduces the north-west Panel of Branch and McGough (2009, p.12)'s Figure 1. Second the regions of Order 2 Indeterminacy of Panels 2.6(a) and 2.6(b) coincide (black region) . Third and more important, the regions of Determinacy (light grey region) of Panels 2.6(a) and 2.6(b) do not coincide. The same is true for the regions of Order 1 Indeterminacy (dark grey region). Once we account for complex eigenvalues, that is Panel 2.6(c), the picture changes. We can again exactly reproduce the north-west Panel of Branch and McGough (2009, p.12)'s Figure 1, as our Panels 2.6(a) and 2.6(c) coincide.

Please be aware that in an economy with fully rational agents only ($\alpha = 1.00$), the matrix **M** becomes

$$\mathbf{M} = \begin{bmatrix} \frac{\beta}{\beta - \frac{\beta \chi \mathrm{y}}{\sigma}} + \frac{\lambda(1-\chi \mathrm{pi})}{\sigma\left(\beta - \frac{\beta \chi \mathrm{y}}{\sigma}\right)} & -\frac{1-\chi \mathrm{pi}}{\sigma\left(\beta - \frac{\beta \chi \mathrm{y}}{\sigma}\right)} & 0 & 0 \\ -\frac{\lambda\left(1-\frac{\chi \mathrm{y}}{\sigma}\right)}{\beta - \frac{\beta \chi \mathrm{y}}{\sigma}} & \frac{1-\frac{\chi \mathrm{y}}{\sigma}}{\beta - \frac{\beta \chi \mathrm{y}}{\sigma}} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

This is equivalent to

$$\mathbf{M} = \begin{bmatrix} \mathbf{B}^{-1} & & 0 & 0 \\ & & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where matrix **B** in this case is the same as in Section 3.3 of Bullard and Mitra (2002, p.1121ff.). In addition the calibrations of Bullard and Mitra (2002) and Branch and McGough (2009) coincide. In consequence the NW Panel in Figure 1 of Branch and McGough (2009, p.12) is an extract of Figure 3 of Bullard and Mitra (2002, p.1123). One can observe, that the region of Order 2 Indeterminacy in Branch and McGough (2009, p.12) is E-unstable in Bullard and Mitra (2002, p.1123) and the regions of Order 1 Indeterminacy and determinacy in Branch and McGough (2009, p.12) are E-stable in Bullard and Mitra (2002, p.1123). Logically the question arises, how the concepts of indeterminacy of some order and E-stability are related?

Chapter 3

Anticipation, Learning and Welfare: the Case of Distortionary Taxation

3.1 Motivation

Nowadays, fiscal policy is usually accompanied by legislation and implementation lags. These lags create a non-negligible span of time between the announcement and effective date of a fiscal policy change. This gives individuals in the economy the opportunity to anticipate the tax changes. The economic literature denotes this aspect of fiscal policy either anticipated fiscal policy or fiscal foresight. From our reading, those two terms are equivalents and will be used as such.[1]

When agents anticipate, their resulting actions may to some extent depend on the way they form expectations about the future. The standard assumption of expectations in economics is perfect-foresight / rational expectations (RE). This assumption might be questioned. One prominent deviation of RE that imposes weaker requirements on the agent's information set when making his decisions, is the learning literature (see Evans and Honkapohja (2001) for the

[1]Recently Leeper (2009, p.11ff.) has listed empirical evidence for fiscal foresight and reemphasized the relevance of expectations for sound fiscal policy. Furthermore, Leeper et al. (2009) is another good example of empirical evidence of fiscal foresight. Therein they also demonstrate the challenges for econometricians that aim to quantify the impact of fiscal policy actions and at the same time account adequately for fiscal foresight.

foundations of this approach). The main idea is that agents form expectations about future values of variables they cannot observe by engaging in a kind of statistical inference when making their economic choices.

Although the learning approach has gained significant popularity in some areas of macroeconomics, anticipated fiscal policy has, until recently, been neglected. A pioneering contribution to studying the consequences of anticipated fiscal policy when agents learn factor prices, has been made by Evans et al. (2009). They demonstrate the adaptive constant gain learning approach in several deterministic economic environments, taking changes in lump-sum taxation as an example. The choice of a constant gain therein is motivated by the fact that fiscal policy moves may state structural change. First Evans et al. (2009, p.932ff.) consider permanent, temporary and repeated tax changes in an endowment economy with a balanced-budget policy. The core message of their results is that under learning, anticipated fiscal policy changes have instant effects on key variables as in the perfect foresight case, but the transition paths are remarkably different from the latter. This result, at least with regard to the volatility of key variables' time paths may not come as a surprise. It is well known that constant gain learning causes excess volatility compared to the case of RE (see Evans and Honkapohja (2001, p.49) for an illustration). Thereafter, Evans et al. (2009, p.941ff.) turn attention to the scenario of debt financing of anticipated fiscal policy changes and find that, given agents understand the structure of government financing, the so-called "near Ricardian equivalence" holds under learning. Finally, Evans et al. (2009, p.943ff.) introduce the adaptive learning approach to the basic Ramsey model. For an anticipated balanced-budget permanent tax change they once more confirm that under learning the time paths of key variables are strikingly different from their perfect foresight counterparts.

In subsequent work, Evans et al. (2010) focus on Ricardian equivalence in the basic Ramsey model with anticipated fiscal policy under learning. Most important, Evans et al. (2010, p.8ff.) formally proof that the assumption of RE is not necessary for the classic Ricardian equivalence result. Furthermore, Evans et al. (2010, p.10ff.) provide new departures from the Ricardian equivalence proposition. First, if government expenditures are endogenous, i.e. depend on a fiscal rule, then Ricardian equivalence holds only under RE but fails under learning. Second, Ricardian equivalence breaks down, if the expected interest rates depend on changes in the level of public debt.

Building on the contribution of Evans et al. (2009), we aim to generalize their analysis of anticipated fiscal policy under learning into an economy featuring distortionary taxes and elastic labour supply. More specifically, we derive the dynamic paths of key variables for permanent changes in distortionary taxes in a deterministic version of the prominent Ramsey model. In particular we consider permanent changes in distortionary labour income, capital income and consumption tax in turn. In addition, we examine more sophisticated fiscal policy reforms, in the presence of several tax instruments. There are fundamental differences between lump-sum taxation and distortionary taxation: a labour income tax under inelastic labour supply does not affect household margins and therefore causes no distortion, but under elastic labour supply the labour income tax affects the intra-temporal choice between consumption and leisure of the household and may cause an intra-temporal distortion. Next, a capital income tax has the potential to cause up to two types of distortion. First, the capital income tax in any case affects the inter-temporal household Euler equation. In case of elastic labour supply, the capital income tax also affects the intra-temporal choice between consumption and leisure of the household due to its distortion of the consumption choice. Finally, a consumption tax may also cause an inter-temporal distortion by affecting the household Euler equation, but there is an important difference compared to capital income taxation. The consumption tax affects the price of consumption in both periods considered in the household Euler equation whereas the capital income tax always affects only the price of next period's consumption in the household Euler equation. Loosely speaking, a consumption tax can distort consumption and investment decision via the household's Euler equation, only when it is changed, i.e. time-varying, whereas a capital income tax always causes distortions in the Ramsey economy. Thus, we may expect that the dynamics of the economy for a capital income tax reform may be fundamentally different from the economic dynamics for a consumption tax reform.[2]

Furthermore, the assumption of elastic labour supply implies that endogenous variables such as factor prices as well as employment and consumption are not predetermined as in Evans et al. (2009, p.943ff.) or in Evans et al. (2010), but

[2]Note that a consumption tax may also be a desirable subject of study, as it has special stability properties. See Giannitsarou (2007) for the details.

determined simultaneously in each period.

Next to the analytical derivations, we also calibrate our model and calculate welfare consequences for several policy experiments under perfect foresight as well as under learning. For this purpose, we make use of the welfare measure proposed by Lucas (1990) and also applied by Cooley and Hansen (1992) (for discrete time), which takes into account the whole transition path between the initial and new steady-states associated with initial and changed tax rate. Thus, putting it differently, we ask, to what extent the excess volatility caused by constant gain learning affects the well-being of households compared to the perfect foresight case. Using such a measure of welfare consequences, may even allow comparison of results for learning dynamics to previous studies such as Lucas (1990), Cooley and Hansen (1992) or Garcia-Milà et al. (2010). All these studies evaluate and rank various distortionary tax reforms according to their welfare consequences under perfect foresight.

Our main results are as follows. When we assume that agents use adaptive learning rules to forecast factor prices, our model predicts oscillatory dynamic responses to anticipated permanent tax changes. Unfortunately we cannot isolate an exclusive source of the oscillatory dynamics. Sensitivity analyses suggest that there are at least two sources. In addition, policy experiments indicate that these volatile responses may have a major impact on the welfare consequences of tax reforms. In particular we consider experiments that improve welfare but do so to a much lower extent under learning compared to perfect foresight.

Note that our approach links the learning literature to that part of the public finance literature that is concerned with the welfare consequences of different types of taxation. See Chamley (1981) for an example of a comparative statics analysis or Judd (1987) for differences in unanticipated and anticipated changes in factor taxes. In addition, there have been studies in stochastic set-ups, like Cooley and Hansen (1992). With regard to the implementation of anticipated optimal fiscal policy an example is Domeij and Klein (2005) or its extension for public goods and capital by Trabandt (2007). Moreover, Garcia-Milà et al. (2010) have recently conducted research on welfare consequences of fiscal policy experiments in the spirit of Cooley and Hansen (1992) in a heterogeneous agents model.

The remainder of the paper is organized as follows. In Section 3.2 we outline the economic model, derive optimality conditions and detail our approach of learning. Section 3.3 compares the dynamics with and without elastic labour supply for the case of lump-sum tax changes. This section also provides sensitivity analysis for some structural parameters. In Section 3.4 we consider changes in distortionary taxation and compare the resulting dynamics to the case of lump-sum taxation. The last part of this section contains the welfare analysis of selected policy experiments. Section 3.5 concludes and points out directions for further research.

3.2 The Model

Our economy is a version of the Ramsey economy outlined in detail in Ljungqvist and Sargent (2000, p.305ff.). The capital stock k_t evolves according to the economy-wide resource constraint

$$k_{t+1} \quad = \quad F(k_t, n_t) - c_t - g_t + (1-\delta)k_t, \tag{3.1}$$

where $F(k_t, n_t)$ is the economy's production function (equalling output) showing that the firm sector uses capital k_t and labour n_t as inputs to produce the single good of the economy (see Section 3.2.2 for the details). Output can either be consumed by households (c_t) or the government (g_t) or added to the capital stock. Capital is assumed to depreciate at a constant rate δ.

3.2.1 Households

With regard to the household sector, we assume a continuum of households, where we normalize the size of the economy to unity and each household faces the problem

$$\max_{c_t, n_t} E_t^* \left\{ \sum_{t=0}^{\infty} \beta^t \left[\log(c_t) + \eta \log(\bar{L} - n_t) \right] \right\} \tag{3.2}$$

s.t.

$$\begin{aligned} k_{t+1} + \frac{b_{t+1}}{R_t} + (1+\tau_t^c)c_t \quad &= \quad (1-\tau_t^l)w_t n_t + (1-\tau_t^k)r_t k_t + (1-\delta)k_t \\ &\quad\; +b_t - \tau_t + \pi_t, \end{aligned} \tag{3.3}$$

where all variables are in per capita terms. Thus, the variable k_{t+1} denotes the level of capital in period $t+1$ and b_{t+1} is the level of government debt holdings chosen in period t. Furthermore, r_t is the rental rate of capital and R_t is the gross real interest rate in period t. The level of consumption chosen in period t is indicated by c_t. Next, $\tau_t^\bullet$ denotes a distortionary tax either on consumption, labour income or capital income[3]. The real wage in period t is given by w_t and $l_t = \bar{L} - n_t$ denotes leisure. In consequence, n_t is labour supply of the household. τ_t is a per capita lump-sum tax and $\pi_t = 0$ is the profit under perfect competition. Furthermore, the parameter $\eta \geq 0$ measures the elasticity of labour supply.

$E_t^*\{\bullet\}$ denotes subjective period t expectations for future values of variables. Households apply this operator, if they do not have perfect foresight. This assumption is commonly used in the learning literature. Furthermore, note that we abstract from aggregate uncertainty, i.e. we conduct our analysis in a deterministic economy. Thus, if households do not have perfect foresight, their expectations are so-called point expectations, i.e. agents base their economic choices on the mean of their expectations, see Evans and Honkapohja (2001, p.61). In Section 3.2.4 below we outline our concept of learning. An important aspect of this concept is that forecasts of single variables are independent of each other. In consequence, we can assume that for any two variables X and Y it is true that $E_t^*\{XY\} = E_t^*\{X\}E_t^*\{Y\}$ holds.

Now, we detail the household's decisions. Each household solves the Lagrangian

$$
\begin{aligned}
\mathcal{L} \;=\; & E_t^* \sum_{t=0}^{\infty} \beta^t \{\log(c_t) + \eta \log(\bar{L} - n_t) \\
& -\lambda_t[k_{t+1} + \frac{b_{t+1}}{R_t} + (1+\tau_t^c)c_t - (1-\tau_t^l)w_t n_t - (1-\tau_t^k)r_t k_t - (1-\delta)k_t \\
& -b_t + \tau_t]\}
\end{aligned}
$$

[3]We use the symbol $\bullet$ as a placeholder throughout our analysis.

with first-order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} \quad : \quad \beta^t \left\{ c_t^{-1} - \lambda_t (1+\tau_t^c) \right\} \stackrel{!}{=} 0 \tag{3.4}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} \quad : \quad \beta^t \{-\lambda_t\} + \beta^{t+1} E_t^* \left\{ \lambda_{t+1} \left[(1-\delta) + (1-\tau_{t+1}^k) r_{t+1} \right] \right\} \stackrel{!}{=} 0 \tag{3.5}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} \quad : \quad \beta^t \left\{ -\lambda_t R_t^{-1} \right\} + \beta^{t+1} E_t^* \{\lambda_{t+1}\} \stackrel{!}{=} 0 \tag{3.6}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} \quad : \quad \beta^t \left\{ -\eta (\bar{L} - n_t)^{-1} - \lambda_t [-(1-\tau_t^l) w_t] \right\} \stackrel{!}{=} 0. \tag{3.7}$$

From (3.4) and (3.6) we get the household Euler condition

$$c_t^{-1} \quad = \quad \beta R_t E_t^* \left\{ c_{t+1}^{-1} \frac{(1+\tau_t^c)}{(1+\tau_{t+1}^c)} \right\}, \tag{3.8}$$

(3.5) and (3.6) yield the no-arbitrage condition for capital and bonds

$$R_t \quad = \quad \left[(1-\delta) + (1 - E_t^* \left\{ \tau_{t+1}^k \right\}) E_t^* \{ r_{t+1} \} \right], \tag{3.9}$$

and from (3.4) and (3.7) we get the consumption leisure trade-off

$$n_t \quad = \quad \bar{L} - \frac{\eta (1+\tau_t^c) c_t}{(1-\tau_t^l) w_t}. \tag{3.10}$$

3.2.2 Firms

In our economy, there is a unit continuum of firms who compete perfectly. Each firm in each period t rents capital at given price r_t and labour at given price w_t and produces the numeraire good with constant returns to scale production function

$$\begin{aligned} y_t &= F(k_t, n_t) \\ y_t &= A k_t^{\alpha} n_t^{(1-\alpha)}, \end{aligned} \tag{3.11}$$

where $\alpha \in (0,1)$. The optimal firm behaviour requires that

$$r_t \overset{!}{=} \frac{\partial y_t}{\partial k_t} = A\alpha k_t^{\alpha-1} n_t^{1-\alpha}, \tag{3.12}$$

as well as

$$w_t \overset{!}{=} \frac{\partial y_t}{\partial n_t} = A(1-\alpha) k_t^{\alpha} n_t^{-\alpha}, \tag{3.13}$$

i.e. each production factor earns its marginal product. Finally, we have the per capita national income identity

$$\begin{aligned} y_t &= r_t k_t + w_t n_t, \\ \pi_t &= y_t - r_t k_t - w_t n_t = 0, \end{aligned} \tag{3.14}$$

which means zero profits, as one can expect from perfect competition.

3.2.3 Government

The government finances its expenses on goods and debt repayment by tax revenues and the issuance of new bonds in each period t

$$g_t + b_t = \tau_t^c c_t + \tau_t^l w_t n_t + \tau_t^k r_t k_t + \tau_t + \frac{b_{t+1}}{R_t}.$$

For the remainder, we will assume that the government operates a balanced-budget rule in each period t, thus tax revenues will fully cover expenses such that bonds are in zero net supply as a direct consequence. Thus the government sets $g_t, \tau_t^c, \tau_t^l, \tau_t^k$ and τ_t constrained by

$$g_t = \tau_t^c c_t + \tau_t^l w_t n_t + \tau_t^k r_t k_t + \tau_t \tag{3.15}$$

in each period t.

3.2.4 Learning

Now, we aim to detail our concept of learning that was elaborated in Evans et al. (2009, p.943ff.). For completeness we restate the crucial assumptions on learning.

Under learning, households are supposed to know the entire history of endogenous variables. They observe the current period value of exogenous variables and they know the state variables. Furthermore, they know the structure of the economy with regard to the fiscal policy sector. Agents understand the implications of the announced policy change for the government budget constraint. They are also convinced that the intertemporal government budget constraint will always hold (see Evans et al. (2009, p.944)). Agents then forecast factor prices such as interest rates and wages $r^e_{t+j}(t), w^e_{t+j}(t), j \geq 1$, by making use of constant-gain steady-state adaptive learning rules[4]

$$r^e_{t+j}(t) = r^e(t) \quad \text{and} \quad w^e_{t+j}(t) = w^e(t), \tag{3.16}$$

where

$$\begin{aligned} r^e(t) &= r^e(t-1) + \gamma(r_{t-1} - r^e(t-1)) \\ w^e(t) &= w^e(t-1) + \gamma(w_{t-1} - w^e(t-1)), \end{aligned} \tag{3.17}$$

where $0 < \gamma \leq 1$ is the gain parameter.[5] Our choice of this specific learning rule is motivated by two well known arguments in the learning literature. First, as Evans and Honkapohja (2001, p.332) outline, choosing a constant gain learning rule is the appropriate choice for agents, when they are aware of structural change, as in such a learning rule agents discount past data exponentially. Note that rules (3.17) are equivalent to $r^e(t) = \gamma \sum_{i=0}^{\infty}(1-\gamma)^i r_{t-i-1}$ and $w^e(t) = \gamma \sum_{i=0}^{\infty}(1-\gamma)^i w_{t-i-1}$. Second, the timing of the learning rule, i.e. that agents' update in period t uses data up to period $t-1$, is chosen in order to avoid simultaneity between $r^e(t)$ and r_t as well as $w^e(t)$ and w_t (see for example Evans and Honkapohja (2001, p.51)). Think of simultaneity in this context as a

[4]Here we apply the same short-hand notation as Evans et al. (2009). Thus for any variable say z, its period t expected future value in period $t+j$ derived by a learning rule may either be denoted $E^*_t\{z_{t+j}\}$ or equivalently $z^e_{t+j}(t)$. An additional notation we introduce is $z^p_{t+j}(t)$ which denotes the agent's planned choice of the variable z in period $t+j$ based on expected values formed via the learning rule in period t.

[5]The gain parameter measures the responsiveness of the forecast to new observations, see Evans and Honkapohja (2001, p.18). Be aware that in our model the gain parameter is exogenous. See Branch and Evans (2007) for a recent example where agents can choose the gain parameter.

situation in which agents' expectations affect current values of aggregate endogenous variables and vice versa, which may potentially introduce some strategic behaviour.

Such a learning rule yields a sequence of so-called *temporary equilibria*, which consist of sequences of (planned) time paths for all endogenous variables. These sequences satisfy the learning rule above, the expectation history, household and firm optimality conditions, the government budget constraint and the economy-wide resource constraint given the exogenous variables as well as the current stock of capital in each period. These plans are revisited and potentially altered in each period after expectations have been updated.

3.3 Base Case: Lump-Sum Tax

Before pursuing our core issue, i.e. the case of distortionary taxation, we would like to illustrate the applied methodology for the case of lump-sum taxation for two reasons: first, we want to illustrate the consequences of the introduction of elastic labour supply compared to the case of inelastic labour supply as assumed in Evans et al. (2009, p.943ff.) and its effect on the dynamic paths of the key variables such as consumption and capital, given their calibration (see Table 3.1 below); second, below in Subsection 3.3.2, we aim to present a sensitivity analysis for the very basic version of the model under examination.

Let us now derive the dynamic paths under learning for an anticipated lump-sum tax change. Consequently we assume all other types of taxation away, i.e. $\tau_t^c = \tau_t^l = \tau_t^k = 0$. The Euler equation (3.8) is standard

$$c_t^{-1} \quad = \quad \beta(c_{t+1}^p(t))^{-1}\left[(1-\delta) + r_{t+1}^e(t)\right]$$

and forward substitution of this yields

$$c_{t+j}^p(t) \quad = \quad \beta^j D_{t,t+j}^e(t) c_t, \tag{3.18}$$

where we define $D_{t,t+j}^e(t) \equiv \Pi_{i=1}^j[(1-\delta) + r_{t+i}^e(t)]$. One can think of this term as "expectations of the interest rate factor $D_{t,t+j}$ at time t" (see Evans et al. (2009, p.933)). Next, we notice that the consumption leisure trade-off in this case is

$$n_t = \bar{L} - \frac{\eta c_t}{w_t}. \tag{3.19}$$

Given the adequate transversality condition for capital

$$\lim_{T \mapsto \infty} \left(D^e_{t,t+T}(t)\right)^{-1} k^p_{t+T+1}(t) = 0, \tag{3.20}$$

the inter-temporal budget constraint of the consumer is

$$\begin{aligned} c_t + \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} c^p_{t+j}(t) &= [(1-\delta)+r_t]k_t + w_t n_t - \tau_t \\ &\quad + \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \left[w^e_{t+j}(t) n^p_{t+j}(t) - \tau^e_{t+j}(t)\right], \end{aligned}$$

which by the virtue of (3.18) and (3.19) yields

$$\begin{aligned} c_t \frac{(1+\eta)}{(1-\beta)} &= [(1-\delta)+r_t]k_t + w_t \bar{L} - \tau_t \\ &\quad + \underbrace{\sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} w^e_{t+j}(t) \bar{L}}_{\equiv SW_1} - \underbrace{\sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \tau^e_{t+j}(t)}_{\equiv ST_1}. \end{aligned} \tag{3.21}$$

Equations (3.12) and (3.13) hold for firms. Finally, government faces the constraint

$$g_t = \tau_t \tag{3.22}$$

in each period t and the economy-wide resource constraint is given by (3.1).

We now need to think about the policy experiment we will study. We are looking at a scenario of a credible permanent change in taxes announced at the outset of period $t = 1$ and effective from period $t = T_p$ onwards. In particular a tax change from τ_0 to τ_1 at some point in time T_p. The dynamics under perfect foresight are standard.[6] Under learning we can directly follow Evans et al. (2009,

[6]Ljungqvist and Sargent (2000, p.305ff.) illustrate the analytical derivations and numerical simulation alternatives for the perfect foresight case. We will simply make use of the DYNARE toolbox throughout all calculations to compute dynamics under

p.943ff.). The crucial step is to calculate the infinite sums on the right-hand side of (3.21), i.e. SW_1 and ST_1. Directly following the appendix in Evans et al. (2009, p.951ff.) we calculate

$$SW_1 = \frac{w^e(t)\bar{L}}{r^e(t)-\delta}. \tag{3.23}$$

With regard to ST_1, we have[7]

$$ST_1 = \frac{\tau_0}{r^e(t)-\delta} + (\tau_1-\tau_0)\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}} \tag{3.24}$$

for $1 \leq t < T_p$ and

$$ST_1 = \frac{\tau_1}{r^e(t)-\delta}. \tag{3.25}$$

for $t \geq T_p$. From (3.21) follows that we have

$$\begin{aligned} c_t = {} & \frac{(1-\beta)}{(1+\eta)}\{[(1-\delta)+r_t]k_t + w_t\bar{L} - \tau_0 + \frac{w^e(t)\bar{L}}{r^e(t)-\delta} \\ & -\frac{\tau_0}{r^e(t)-\delta} - (\tau_1-\tau_0)\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}}\} \end{aligned} \tag{3.26}$$

for $1 \leq t < T_p$ and

$$c_t = \frac{(1-\beta)}{(1+\eta)}\left[[(1-\delta)+r_t]k_t + w_t\bar{L} - \tau_1 + \frac{w^e(t)\bar{L}}{r^e(t)-\delta} - \frac{\tau_1}{r^e(t)-\delta}\right] \tag{3.27}$$

for $t \geq T_p$. Given a calibration, we can then compute the dynamics of consumption and other endogenous variables.

3.3.1 Inelastic Labour Supply vs. Elastic Labour Supply

We believe that it is of importance to use a model that features elastic labour supply in order to calculate welfare implications of fiscal policy reforms adequately. Completely inelastic labour supply is a quite unrealistic assumption itself and at least some moderately elastic labour supply should be considered. Moreover, it

perfect foresight. Note that this toolbox employs linearization methods.

[7] See Appendix 3.6.1.2 for details on derivations.

implies that agents' choices of current period endogenous variables are in fact predetermined as is pointed out in Evans et al. (2009, p.944). In order to illustrate differences in the dynamics of endogenous variables based on the assumption of inelastic and elastic labour supply, we return to the simulation exercise of Evans et al. (2009, p.943ff.). Note that $\tau_t^c = \tau_t^l = \tau_t^k = \delta = 0$ and $\eta = 0$ imply that $n_t = \bar{L}$ (i.e. inelastic labour supply) for all t (see equation (3.19)). Therefore, we are exactly in the same scenario as in Evans et al. (2009, p.943ff.). Although we do not fully agree with the calibration of Evans et al. (2009), we will stick to their calibration in this subsection to keep our results comparable. We will indicate, when we deviate from their calibration. The basic reason for this disagreement is the combination of parameters $\beta = 0.95$ and $T_p = 20$. These parameter choices imply that a government, which in reality is usually in charge of a legislation period of four to six years, may announce a tax policy change that will be effective in 20 years' time. From our perception of political execution and our confidence in fiscal policy makers' ability to commit, this appears to be unrealistic in most cases. Nevertheless, we would like to mention, that in all the subsequent numerical illustrations of our analytical derivations, we experienced severe difficulties in finding calibrations that could yield convergence for the dynamics under learning. Our experience is, that it is quite a difficult task, to perfectly calibrate the model to empirically estimated structural parameters and achieve convergence, at least with the numerical methods, we have at our disposal.

For the moment, we calibrate the model according to Table 3.1 below. The

Parameter	**Value**	**Parameter**	**Value**
A	1.00	δ	0.00
α	0.33	T_p	20
β	0.95	γ	0.10

Table 3.1: Parameters similar as in Evans et al. (2009, p.945)

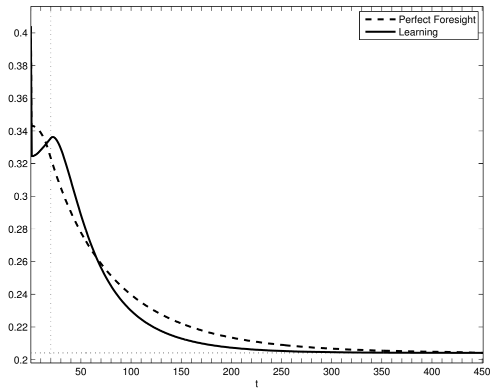
policy experiment considered in Evans et al. (2009, p.943ff.) is a permanent increase in government purchases from $g_0 = \tau_0 = 0.9$ to $g_1 = \tau_1 = 1.1$ that is announced credibly in period $t = 1$ and will be effective from period $T_p = 20$ onwards. It is assumed that the economy is in steady-state in period $t = 0$. Simulations in Evans et al. (2009, p.943ff.) for consumption and capital are recalculated (with $\eta = 0, \bar{L} = 0.5182$) and displayed in Figures 3.1(a) and 3.1(b) below. Furthermore, Figures 3.1(c) and 3.1(d) exhibit the dynamics for elastic labour supply with $\eta = 2.00$ and $\bar{L} = 1.00$, values that match $n_0 = 0.5182$ and $g_0 = 0.9$ in this set-up.[8]

Two distinct features emerge from Figure 3.1. First, when we compare the dynamic paths of consumption (as well as capital) under perfect foresight and learning, they are different from each other no matter with or without elastic labour supply. Therefore, it may be quite important to consider learning when evaluating fiscal policies as learning is a more realistic assumption of human behaviour from our point of view.[9] Second, obviously the learning paths in Figures 3.1(a) and 3.1(b) for inelastic labour supply are strikingly different to the ones under elastic labour supply in Figures 3.1(c) and 3.1(d). In particular, elastic labour supply yields much more volatility in the time paths of consumption and capital (as well as other variables in the model) compared to the inelastic labour supply case. In fact, the variables oscillate around their steady-state until they converge to it. This implies, that the tax reforms may have different welfare implications in an economy with elastic labour supply, when one compares the case of perfect foresight against the case of learning.

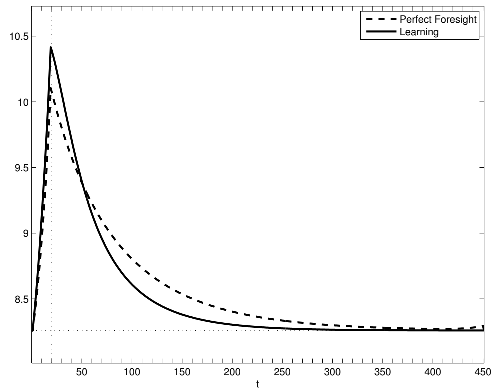
From our point of view, possible reasons for the significant differences in the dynamics under learning between elastic and inelastic labour supply could be as follows. Consider agents' behaviour under perfect foresight. Agents fix their current and future choices once and for all. They do not form expectations about current and future factor prices. Second, agents without perfect foresight forecast current period factor prices in each period. Therefore, they make an update of their expectations of factor prices. Thereby agents also make an expectational error. Based on their updated expectations of factor prices they revise current and planned future choices of variables in each period. In addition, actual factor prices in that period are determined based on the agents updated expectations of factor prices. Be aware that the first and the second point above are true for inelastic labour supply as well as elastic labour supply. So the learning itself cannot explain the differences in the dynamics. Furthermore, note that with inelastic labour supply, factor prices are predetermined, whereas with elastic labour supply factor prices are free variables. Moreover, with elastic labour supply, households

[8] Note that $n_0 = 0.5182$ corresponds to 12.44 hours per day. This appears to be quite unrealistic, but we choose those numbers in order to achieve both comparable magnitudes in Figure 3.1 below as well as convergence under learning.

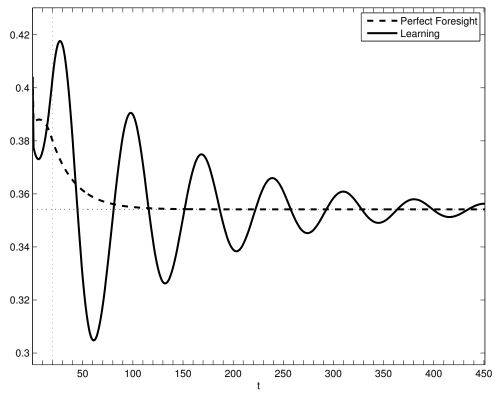
[9] This is the core result of Evans et al. (2009).



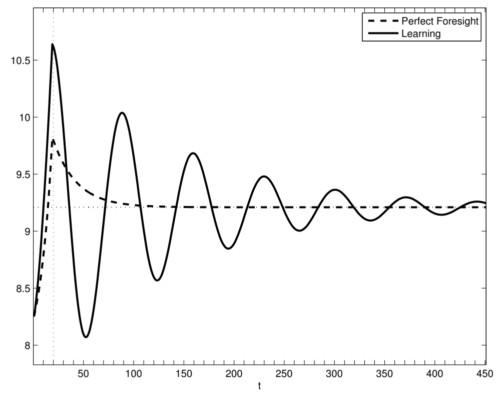
(a) Consumption



(b) Capital



(c) Consumption with elastic n_t



(d) Capital with elastic n_t

Figure 3.1: Consumption (a) and capital (b) dynamics under learning (solid curve) and perfect foresight (dashed curve) with inelastic labour supply as in Evans et al. (2009, p.943ff.) as well as consumption (c) and capital (d) dynamics under learning (solid curve) and perfect foresight (dashed curve) with elastic labour supply. The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

can react to structural changes by substitution of consumption for leisure or vice versa in order to sustain a certain level of utility. For agents with perfect foresight nothing really changes when factor prices are no longer predetermined. Now, as labour supply is elastic, they choose a plan for leisure in addition to their plan for consumption, but they still do that in a once and for all manner. Transition paths should be smooth as before. But for agents that use adaptive learning, it might make a difference. In particular, we suspect that the expectational error could be larger in the case in which factor prices are no longer predetermined. This could lead to more volatility in the expectations of factor prices which translates into higher volatility of actual factor prices as well as consumption and leisure choices. We suggest that the correction of the expectational error in each period could explain the oscillations.

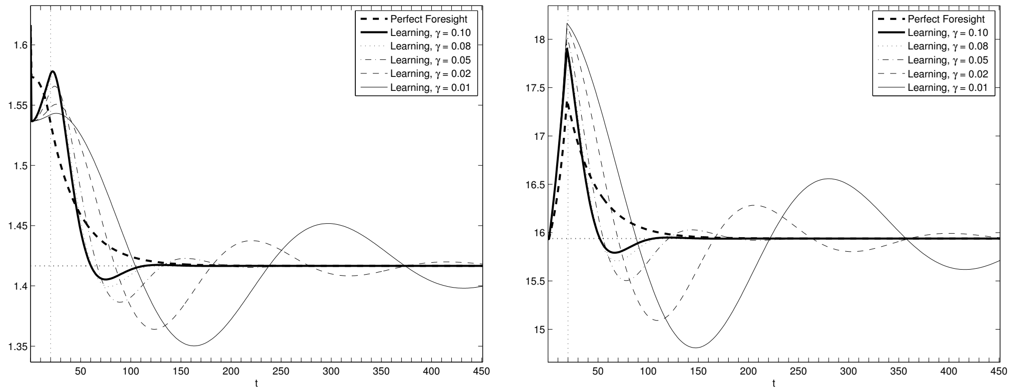
3.3.2 Sensitivity Analysis

Compared to the previous literature on welfare evaluation of tax reforms, our learning approach introduces two additional structural parameters. One is γ, the gain parameter and a second one is T_p, the period, in which the pre-announced tax change becomes effective. Therefore, we are interested in how these two parameters affect the dynamic properties of the model.

3.3.2.1 Sensitivity Analysis for the Gain Parameter

No matter what calibration, one always has to choose a gain parameter γ in the adaptive learning literature. In this subsection we would therefore like to illustrate the consequences of different choices of the gain parameter. The sole empirical estimate we are aware of is provided by Milani (2007, p.2074) for quarterly frequency and is $\gamma = 0.0183$. This number indicates that agents use approximately $1/\gamma \approx 55$ quarters of data. But a reason to be cautious to use the estimate of Milani (2007, p.2074) is that it is based on a data set containing output, inflation and the nominal interest rate, whereas in our setting agents forecast the rental rate of capital and the real wage. Next, Milani (2007, p.2074) mentions that for constant gain learning a range of $\gamma \in [0.01, 0.03]$ is commonly used. Evans and Honkapohja (2009, p.154) note a range of $\gamma \in [0.01, 0.06]$ as known estimates.

Below we will present sensitivity of the dynamics under learning for $\gamma \in \{0.01, 0.02, 0.05, 0.08, 0.10\}$. We do so for the original numerical analysis of Evans et al. (2009, p.943ff.) ($\bar{L} = 1.00$, $\eta = 0.00$), as in this case, there is inelastic labour supply and we can focus solely on the possible fluctuations introduced by varying the gain parameter γ. Note that the two thick lines in Figures 3.2(a) and 3.2(b) exactly replicate the Figures 8 and 9 in Evans et al. (2009, p.943ff.).



(a) Consumption

(b) Capital

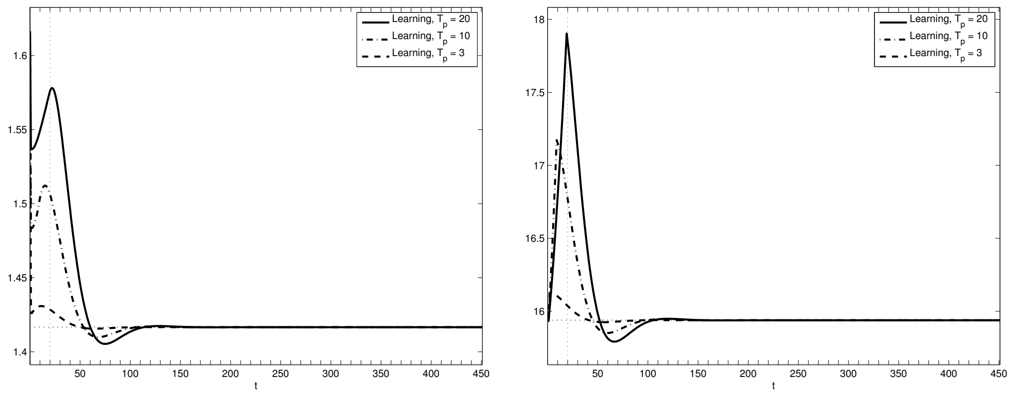
Figure 3.2: Consumption (a) and capital (b) dynamics under learning and perfect foresight with inelastic labour supply as in Evans et al. (2009, p.943ff.) for alternating values of γ. The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

In Figure 3.2(a) we observe that the smaller the gain γ, the smaller the increase in consumption until the period of the tax change T_p (after the initial drop). Furthermore, as we recognize from Figure 3.2(b), the smaller the gain γ, the larger the increase in capital accumulation until the period of the tax change T_p. However, in both Figure 3.2(a) and 3.2(b), we observe that with decreasing γ the dynamics fluctuate around the steady-state with increasing amplitude and it takes an increasing number of periods to converge to the steady-state. These observations are partly at odds with what Evans and Honkapohja (2001, p.332) report: "a larger gain is better at tracking changes but at the cost of a larger variance". In our case it holds, that, the smaller the gain, the larger the volatility.

Summing up, we find that for the parameter range considered in this sensitivity analysis, the choice of the gain parameter γ is not crucial for the shape of the dynamic response.

3.3.2.2 Sensitivity Analysis for the Implementation Date

Another issue that may be of interest is the implementation date T_p. As mentioned above a tax policy change that is going to be effective in 20 years time appears to be unrealistic from our point of view. Therefore, we examine sensitivity of dynamics under learning for various implementation dates, in particular $T_p \in \{3, 10, 20\}$. Figures 3.3(a) and 3.3(b) below display the results.



(a) Consumption

(b) Capital

Figure 3.3: Consumption (a) and capital (b) dynamics under learning with inelastic labour supply as in Evans et al. (2009, p.943ff.) for alternating values of T_p. The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period $T_p = 20$.

In Figure 3.3(a) we observe that the shorter the distance between the announcement date and implementation date of the tax change, the higher the initial drop in consumption and the lower the increase in consumption until the implementation date thereafter. Focusing on capital, in Figure 3.3(b) we observe that with decreasing distance between the announcement date and implementation date of the tax change, the level that capital reaches until the implementation date, is also lower. Finally, for implementation in three years time, i.e. $T_p = 3$, learning dynamics are not significantly different from $T_p \in \{10, 20\}$, but lower in scale. Overall, we observe that the shorter the distance between announcement date and implementation date of the tax change, the earlier the learning dynamics approach the steady-state, but, at least for the parameter range considered herein, the nature of dynamics is not seriously affected.

Thus, we learn that in the subsequent numerical analysis, next to the elasticity of labour supply η (and the commonly known candidate parameters β and δ), the choice of the gain parameter γ as well as the implementation date T_p may also be crucial in achieving convergence on the one hand and determining the magnitude of volatility of the dynamics on the other hand. But these choices may not affect the general nature of the dynamics. Furthermore, our experience with β and δ suggests that they strongly affect the scale of results, next to their impact on convergence.

In order to summarize, there are three important insights from the analysis above. First, there are at least qualitative differences between the case of inelastic labour supply ($\eta = 0$) and elastic labour supply ($\eta > 0$). Therefore, if one regards the latter assumption as more realistic, a model that allows for elastic labour supply is a more appropriate framework to study anticipated fiscal policy under learning. Second, our sensitivity analysis suggests that the choice of the gain parameters γ and the implementation date T_p does not affect the nature of transition paths so we consider ourselves free to choose any of the values considered in the sensitivity analysis.[10] Finally and most notably, we observed at least a qualitative difference in the dynamics under learning compared to the dynamics under perfect foresight. The former appear to be much more volatile than the latter. This stylized fact, from our point of view, justifies the quantification and comparison of welfare cost of anticipated fiscal policy reforms under learning and under perfect foresight. In order to be able to mimic, at least to some extent, a realistic fiscal policy reform, we will introduce distortionary taxes. Before we look at complex fiscal policy reforms, we qualitatively inspect isolated changes in distortionary taxes and the resulting dynamics for each type of tax. Thereafter, we analyze more sophisticated fiscal policy reforms with regard to their welfare costs in a realistic calibration.

3.4 The Case of Distortionary Taxation

After the base case of lump-sum taxation, we now study the case of distortionary taxes. In the remainder, we will assume elastic labour supply. We first characterize the dynamics for a permanent change in a single distortionary tax. This follows closely from Evans et al. (2009, p.943ff.) similar to the last section. Next,

[10] In particular, in the subsequent analysis, we will choose $\gamma = 0.08$ and $T_p = 8$, which will correspond to 8 quarters.

we simulate the dynamic paths of the economy for a change in each type of distortionary tax in turn, given there are no other tax instruments. We inspect the associated dynamics for each distortionary tax with regard to qualitative differences compared to the lump-sum tax and the other distortionary taxes. Thereafter, in Section 3.4.4 below, we derive the dynamic paths of the economy in presence of all types of taxes considered in this economy. Moreover, we evaluate some specific tax reforms with regard to welfare, given our suggested calibration.

3.4.1 Labour Income Tax

Let us now assume that $\tau_t^c = \tau_t^k = \tau_t = 0$ for all t and $\tau_t^l \in [0,1]$. The Euler equation (3.8) is standard and forward substitution again yields (3.18). Next we notice that the consumption leisure trade-off in this case is

$$n_t = \bar{L} - \frac{\eta c_t}{(1-\tau_t^l)w_t}. \tag{3.28}$$

Given the adequate transversality condition for capital (3.20), the inter-temporal budget constraint of the consumer is

$$\begin{aligned} c_t + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^e(t)} c_{t+j}^p(t) &= [(1-\delta)+r_t]k_t + (1-\tau_t^l)w_t n_t \\ &+ \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^e(t)} \left[(1-\tau_{t+j}^{l,e}(t)) w_{t+j}^e(t) n_{t+j}^p(t) \right]. \end{aligned}$$

Given (3.18) and (3.28) we can rewrite the latter as

$$\begin{aligned} c_t \frac{(1+\eta)}{(1-\beta)} &= [(1-\delta)+r_t]k_t + (1-\tau_t^l)w_t \bar{L} \\ &+ \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^e(t)} \left[(1-\tau_{t+j}^{l,e}(t)) w_{t+j}^e(t) \bar{L} \right]. \end{aligned} \tag{3.29}$$

For firms, nothing changes compared to the base case in Section 3.3. Finally, the government now faces the constraint

$$g_t = \tau_t^l w_t n_t \tag{3.30}$$

in each period t and the economy-wide resource constraint is given by (3.1).

We now once more consider the scenario of a credible permanent change in the tax rate announced in period $t = 1$ and effective from period $t = T_p$ onwards. In particular, the labour income tax is changed from τ_0^l to τ_1^l at some point in time T_p. The dynamics under perfect foresight are again standard. Under learning we can directly follow Evans et al. (2009, p.943ff.). The crucial step is to calculate the infinite sums on the right-hand side of (3.29). That is

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{e}(t)} \left[(1-\tau_{t+j}^{l,e}(t)) w_{t+j}^{e}(t)\bar{L}\right] &= \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{e}(t)} w_{t+j}^{e}(t)\bar{L} \\ &\quad - \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{e}(t)} \tau_{t+j}^{l,e}(t) w_{t+j}^{e}(t)\bar{L} \\ &= SW_1 - ST_2. \end{aligned}$$

Given (3.16) and (3.17), we get the term $SW_1 = \frac{w^e(t)\bar{L}}{r^e(t)-\delta}$ as before in Section 3.3. With regard to ST_2, for $1 \le t < T_p$ we calculate[11]

$$ST_2 = w^e(t)\bar{L}\left[\frac{\tau_0^l}{r^e(t)-\delta} + \left(\tau_1^l - \tau_0^l\right)\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}}\right] \tag{3.31}$$

and for $t \ge T_p$ we calculate

$$ST_2 = \frac{\tau_1^l\, w^e(t)\bar{L}}{r^e(t)-\delta}. \tag{3.32}$$

Given (3.29), it follows that we have

$$\begin{aligned} c_t &= \frac{(1-\beta)}{(1+\eta)}[[(1-\delta)+r_t]k_t + (1-\tau_0^l)w_t\bar{L} + (1-\tau_0^l)\frac{w^e(t)\bar{L}}{r^e(t)-\delta} \\ &\quad - w^e(t)\bar{L}(\tau_1^l - \tau_0^l)\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}}] \end{aligned} \tag{3.33}$$

for $1 \le t < T_p$ and

$$c_t = \frac{(1-\beta)}{(1+\eta)}\left[[(1-\delta)+r_t]k_t + (1-\tau_1^l)w_t\bar{L} + (1-\tau_1^l)\frac{w^e(t)\bar{L}}{r^e(t)-\delta}\right]. \tag{3.34}$$

[11]See Appendix 3.6.1.3 for details on derivations.

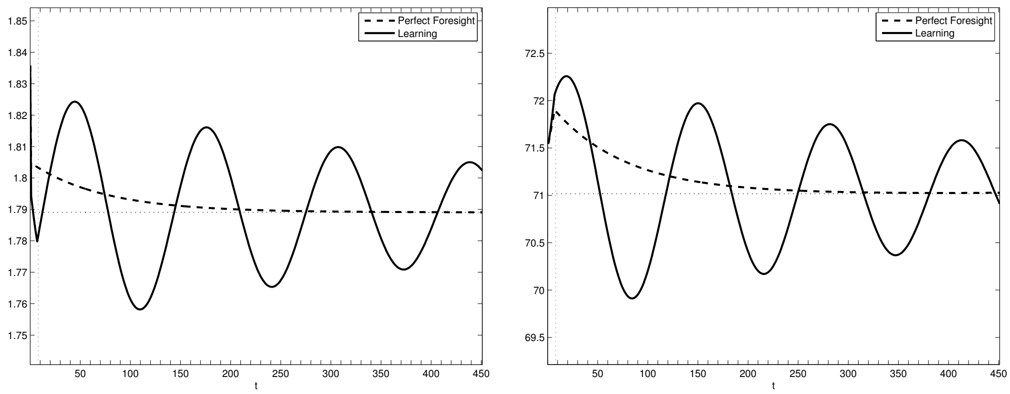
for $t \geq T_p$. Given a calibration, we are then able to compute the dynamics of consumption and other endogenous variables.

Now let us return to the numerical example. Here we calibrate the model according to Table 3.2 below.

Parameter	**Value**	**Parameter**	**Value**
A	1.00	δ	0.00
α	0.33	T_p	8
β	0.99	γ	0.08
η	1.00	$\bar{L}$	1.00

Table 3.2: Calibration for the case with labour income tax only.

We choose the initial labour income tax rate to be $\tau_0^l = 0.23$ as in Cooley and Hansen (1992, p.305) and assume a credible pre-announced permanent increase by 10% to $\tau_1^l = 0.2530$. These parameter choices yield initial steady-state employment of $n_0 = 0.3774$, which corresponds to 9.06 hours per day. Simulations for the first 450 periods are displayed in Figures 3.4(a) to 3.4(b) below.



(a) Consumption (b) Capital

Figure 3.4: Consumption (a) and capital dynamics under learning (solid curve) and perfect foresight (dashed curve). The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

We find that the only qualitative difference in the dynamics compared to the case of lump-sum taxation with elastic labour supply, is the remarkably slower convergence. We conjecture that this is due to the different calibration of key parameters such as β, η and γ.

3.4.2 Capital Income Tax

Let us now assume that $\tau_t^c = \tau_t^l = \tau_t = 0$ for all t and $\tau_t^k \in [0,1]$. The Euler equation (3.8) now changes to

$$c_t^{-1} = \beta(c_{t+1}^p(t))^{-1}[(1-\delta) + (1-\tau_{t+1}^{k,e}(t))r_{t+1}^e(t)]$$

and forward substitution of this equation yields

$$c_{t+j}^p(t) = \beta^j(D_{t,t+j}^{k,e}(t))c_t, \tag{3.35}$$

where we define $D_{t,t+j}^{k,e}(t) \equiv \Pi_{i=1}^j[(1-\delta) + (1-\tau_{t+i}^{k,e}(t))r_{t+i}^e(t)]$. Furthermore, notice that the consumption leisure trade-off is again given by (3.19). Given the adequate transversality condition for capital

$$\lim_{T\mapsto\infty}\left(D_{t,t+T}^{k,e}(t)\right)^{-1}k_{t+T+1}^p(t) = 0, \tag{3.36}$$

the inter-temporal budget constraint of the consumer is given by

$$\begin{aligned} c_t + \sum_{j=1}^{\infty}\frac{1}{D_{t,t+j}^{k,e}(t)}c_{t+j}^p(t) &= [(1-\delta) + (1-\tau_t^k(t))r_t]k_t + w_t n_t \\ &+ \sum_{j=1}^{\infty}\frac{1}{D_{t,t+j}^{k,e}(t)}[w_{t+j}^e(t)\bar{L} - \eta c_{t+j}^p(t)]. \end{aligned}$$

By the virtue of (3.35) as well as (3.19) we can rewrite the latter as

$$\frac{(1+\eta)}{(1-\beta)}c_t = [(1-\delta) + (1-\tau_t^k(t))r_t]k_t + w_t\bar{L} + \sum_{j=1}^{\infty}\frac{1}{D_{t,t+j}^{k,e}(t)}w_{t+j}^e(t)\bar{L}. \tag{3.37}$$

For firms again nothing changes compared to the base case in Section 3.3. Finally, the government now faces the constraint

$$g_t = \tau_t^k r_t k_t \tag{3.38}$$

in each period t. The economy-wide resource constraint is again given by (3.1).

We now consider the scenario of a permanent change in the capital income tax rate. The rate is changed from τ_0^k to τ_1^k at some point in time T_p. The dynamics under perfect foresight are again standard. Under learning we follow the approach of Evans et al. (2009, p.943ff.). The infinite sum on the right-hand side of (3.37) is

$$SW_2 \quad = \quad \sum_{j=1}^{\infty} \frac{1}{D^{k,e}_{t,t+j}(t)} w^e_{t+j}(t)\bar{L}. \tag{3.39}$$

Given (3.16) and (3.17), for $1 \leq t < T_p$ we calculate[12]

$$SW_2 \quad = \quad \frac{w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L} \times$$
$$\left[\frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\right] \tag{3.40}$$

and for $t \geq T_p$ we calculate

$$SW_2 \quad = \quad \frac{w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}. \tag{3.41}$$

From (3.37) follows that we have

$$c_t \quad = \quad \frac{(1-\beta)}{(1+\eta)}\{[(1-\delta)+(1-\tau_0^k)r_t]k_t + w_t\bar{L} + \frac{w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L} \times$$
$$\left[\frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\right]\} \tag{3.42}$$

for $1 \leq t < T_p$ and

$$c_t \quad = \quad \frac{(1-\beta)}{(1+\eta)}\left[[(1-\delta)+(1-\tau_1^k)r_t]k_t + w_t\bar{L} + \frac{w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}\right] \tag{3.43}$$

for $t \geq T_p$. Given a calibration, we can compute the dynamics of consumption and other endogenous variables.

Now let's return to the numerical example. Here we calibrate the model

[12]See Appendix 3.6.1.4 for details on derivations.

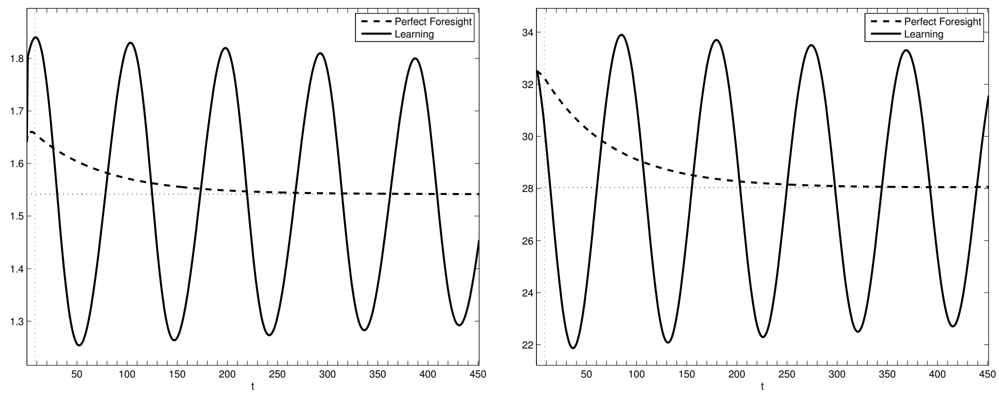
according to Table 3.3 below.

Parameter	**Value**	**Parameter**	**Value**
A	1.00	δ	0.00
α	0.33	T_p	8
β	0.99	γ	0.08
η	0.85	$\bar{L}$	1.00

Table 3.3: Calibration for the case with capital income tax only.

We choose the initial capital income tax rate to be $\tau_0^k = 0.5000$ as in Cooley and Hansen (1992, p.305) and assume a credible pre-announced permanent increase by 10% to $\tau_1^k = 0.5500$. These parameter choices yield initial steady-state employment of $n_0 = 0.4848$, which corresponds to 11.6 hours per day.

Simulation results are displayed in Figures 3.5(a) to 3.5(b) below. The only



(a) Consumption (b) Capital

Figure 3.5: Consumption (a) and capital (b) dynamics under learning (solid curve) and perfect foresight (dashed curve). The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

qualitative difference we can find in the dynamics, is the larger size of fluctuations and higher frequency of them. We can also observe that under learning time paths need more periods to converge to the steady-state compared to the case of lump-sum tax or labour income tax.

3.4.3 Consumption Tax

Let us now assume that $\tau_t^l = \tau_t^k = \tau_t = 0$ for all t and $\tau_t^c \in [0,1]$. The Euler equation (3.8) now changes to

$$c_t^{-1} \;=\; \beta(c_{t+1}^p(t))^{-1}\frac{(1+\tau_t^c)}{(1+\tau_{t+1}^{c,e}(t))}[(1-\delta)+r_{t+1}^e(t)]$$

and forward substitution of this expression yields

$$c_{t+j}^p(t) \;=\; \beta^j D_{t,t+j}^{c,e}(t)c_t, \tag{3.44}$$

where we define $D_{t,t+j}^{c,e}(t) \equiv \left[\frac{(1+\tau_t^c)}{(1+\tau_{t+j}^{c,e}(t))}\right] D_{t,t+j}^e(t)$. Next we notice that the consumption leisure trade-off in this case is

$$n_t \;=\; \bar{L} - \frac{\eta(1+\tau_t^c)c_t}{w_t}. \tag{3.45}$$

Given the adequate transversality condition for capital (3.18), the inter-temporal budget constraint of the consumer is given by

$$\begin{aligned}(1+\tau_t^c)c_t + \sum_{j=1}^{\infty}\frac{1}{D_{t,t+j}^e(t)}(1+\tau_{t+j}^{c,e}(t))c_{t+j}^p(t) \;&=\; [(1-\delta)+r_t]k_t + w_t n_t \\ &\quad + \sum_{j=1}^{\infty}\frac{1}{D_{t,t+j}^e(t)}w_{t+j}^e(t)n_{t+j}^p(t).\end{aligned}$$

which by the virtue of (3.44) as well as (3.45) yields

$$\frac{(1+\eta)}{(1-\beta)}(1+\tau_t^c)c_t \;=\; [(1-\delta)+r_t]k_t + w_t\bar{L} + SW_1. \tag{3.46}$$

For firms nothing changes compared to the base case in Section 3.3. Finally government now faces the constraint

$$g_t \;=\; \tau_t^c c_t \tag{3.47}$$

in each period t. The economy-wide resource constraint again is given by (3.1).

We now consider the scenario of a permanent change in the consumption tax rate from τ_0^c to τ_1^c at some point in time T_p. The dynamics under perfect foresight are again standard. Under learning we follow again the methodology of Evans et al. (2009, p.943ff.). The infinite sum on the right-hand side of (3.46) is equal to $SW_1 = \frac{w^e(t)\bar{L}}{r^e(t)-\delta}$ as in Section 3.3. Obviously from (3.46) follows that we have

$$c_t = \frac{(1-\beta)}{(1+\eta)(1+\tau_0^c)}\left[[(1-\delta)+r_t]k_t + w_t\bar{L} + \frac{w^e(t)\bar{L}}{r^e(t)-\delta}\right] \tag{3.48}$$

for $1 \leq t < T_p$ and

$$c_t = \frac{(1-\beta)}{(1+\eta)(1+\tau_1^c)}\left[[(1-\delta)+r_t]k_t + w_t\bar{L} + \frac{w^e(t)\bar{L}}{r^e(t)-\delta}\right] \tag{3.49}$$

for $t \geq T_p$. Given a calibration, we have everything at hands to compute the dynamics of consumption and the other endogenous variables.

Note that inspection of (3.48) and (3.49) makes clear that in the case of a consumption tax the dynamics of consumption are independent of the implementation date T_p. At least, this is true in our economy.[13] This fact may have a major impact on the dynamics.

In order to illustrate the dynamics numerically we calibrate the model according to Table 3.4 below.

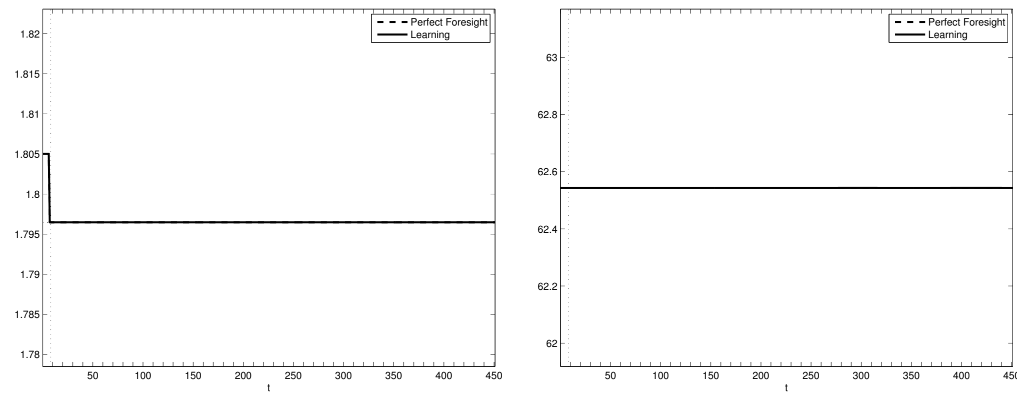
Parameter	**Value**	**Parameter**	**Value**
A	1.00	δ	0.00
α	0.33	T_p	8
β	0.99	γ	0.08
η	1.25	$\bar{L}$	1.00

Table 3.4: Calibration for the case with consumption tax only.

Initial consumption tax rate is $\tau_0^c = 0.0500$ as in Giannitsarou (2007, p.1424) and assume a credible pre-announced permanent increase by 10% to $\tau_1^c = 0.0550$. These parameter choices yield initial steady-state employment of $n_0 = 0.3299$, which approximately corresponds to 8.0 hours per day. Simulation results are displayed in Figures 3.6(a) to 3.6(b) below.

We observe that the dynamics of the consumption tax reform coincide for perfect foresight and learning. It appears, that in both cases, the consumption tax only matters in the period when it is changed, as suspected in our motivation above. We presume that this result depends on our utility specification with

[13]Compare (3.42) for the case of the capital income tax or for the labour income tax.



(a) Consumption (b) Capital

Figure 3.6: Consumption (a) and capital (b) dynamics under learning (solid curve) and perfect foresight (dashed curve). The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

regard to consumption, that is log-utility. Ljungqvist and Sargent (2000, p.318ff.) present results for a power utility function under perfect foresight. The different perfect foresight dynamics therein suggest that the specification of utility may be the source of our result.

Nevertheless it would be an interesting subject of study to check, whether one can formally prove that the dynamics in response to a change in the consumption tax under learning and perfect foresight are similar in general, but we leave that to future research.

3.4.4 Policy Experiments

Let us now assume that $\tau_t^c, \tau_t^l, \tau_t^k \in [0,1]$ and $\tau_t \neq 0$ for all t. The Euler equation (3.8) now changes to

$$c_t^{-1} = \beta(c_{t+1}^p(t))^{-1}\left[\frac{(1+\tau_t^c)}{(1+\tau_{t+1}^{c,e}(t))}\right]\left[(1-\delta)+(1-\tau_{t+1}^{k,e}(t))r_{t+1}^e(t)\right]$$

and forward substitution of this expression yields

$$c_{t+j}^p(t) = \beta^j D_{t,t+j}^{k,e}(t)\left[\frac{(1+\tau_t^c)}{(1+\tau_{t+j}^{c,e}(t))}\right]c_t. \tag{3.50}$$

Furthermore, notice that the consumption leisure trade-off is now given by (3.10). Given the adequate transversality condition for capital (3.36), the inter-temporal budget constraint of the consumer is

$$\begin{aligned}(1+\tau_t^c)c_t + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)}(1+\tau_{t+j}^{c,e}(t))c_{t+j}^p(t) &= [(1-\delta)+(1-\tau_t^k)r_t]k_t \\ &\quad +(1-\tau_t^l)w_t n_t - \tau_t \\ &\quad + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)}[(1-\tau_{t+j}^{l,e}(t))w_{t+j}^e(t)n_{t+j}^p(t) - \tau_{t+j}^e(t)],\end{aligned}$$

which by the virtue of (3.50) as well as (3.10) yields

$$\begin{aligned}\frac{(1+\eta)}{(1-\beta)}(1+\tau_t^c)c_t &= [(1-\delta)+(1-\tau_t^k)r_t]k_t + (1-\tau_t^l)w_t\bar{L} - \tau_t \\ &\quad + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)}[(1-\tau_{t+j}^{l,e}(t))w_{t+j}^e(t)\bar{L} - \tau_{t+j}^e(t)] \\ &= [(1-\delta)+(1-\tau_t^k)r_t]k_t + (1-\tau_t^l)w_t\bar{L} - \tau_t \\ &\quad + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)}[w_{t+j}^e(t)\bar{L} - \tau_{t+j}^{l,e}(t)w_{t+j}^e(t)\bar{L} - \tau_{t+j}^e(t)] \\ &= [(1-\delta)+(1-\tau_t^k)r_t]k_t + (1-\tau_t^l)w_t\bar{L} - \tau_t \\ &\quad + SW_2 - ST_3 - ST_4. \qquad (3.51)\end{aligned}$$

For firms nothing changes compared to the base case in Section 3.3. Finally government now faces the constraint (3.15) in each period t. The economy-wide resource constraint is again given by (3.1).

We now consider the scenario of a permanent (simultaneous) change in (some of the) taxes at some point in time T_p. The dynamics under perfect foresight are again standard. Under learning we again follow the approach Evans et al. (2009, p.943ff.). The infinite sum SW_2 on the right-hand side of (3.51) is already known to be

$$\begin{aligned} SW_2 &= \frac{w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L} \times \\ & \left[\frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\right] \end{aligned} \quad (3.52)$$

for $1 \le t < T_p$ and

$$SW_2 = \frac{w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]} \quad (3.53)$$

for $t \ge T_p$ from Section 3.4.2. ST_3 on the right-hand side of (3.51) is

$$ST_3 = \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} \tau_{t+j}^{l,e}(t) w_{t+j}^e(t)\bar{L}. \quad (3.54)$$

Given (3.16) and (3.17), for $1 \le t < T_p$ we calculate[14]

$$\begin{aligned} ST_3 &= \frac{\tau_0^l \; w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L} \times \\ & \left[\frac{\tau_1^l \; [(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{\tau_0^l \; [(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\right] \end{aligned} \quad (3.55)$$

and for $t \ge T_p$ we calculate

$$ST_3 = \frac{\tau_1^l \; w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}. \quad (3.56)$$

Finally, ST_4 on the right-hand side of (3.51) is

$$ST_4 = \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} \tau_{t+j}^e(t). \quad (3.57)$$

[14]See appendices 3.6.1.5 and 3.6.1.6 for the details on derivations of ST_3 and ST_4.

Given (3.16) and (3.17), for $1 \leq t < T_p$ we calculate

$$
\begin{aligned}
ST_4 &= \frac{\tau_0}{[(1-\tau_0^k)r^e(t)-\delta]} + [\ \frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}}\ \tau_1 \\
&\quad -\frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\ \tau_0\ ] \qquad (3.58)
\end{aligned}
$$

and for $t \geq T_p$ we calculate

$$
ST_4 = \frac{\tau_1}{[(1-\tau_1^k)r^e(t)-\delta]}. \qquad (3.59)
$$

Given (3.51) we can then compute the dynamics responses for consumption and the other endogenous variables as before. Now, we will conduct several policy experiments numerically and compute welfare measures following the approach of Cooley and Hansen (1992, p.301ff.).[15] Intuitively speaking, we compute the increase in consumption that an individual would require to be as well off as under the equilibrium allocation without taxes. We express that number in percentage of output. First, we will do so for our initial choice of tax levels (see line 1 in Table 3.6 below). Thereafter, we carry out policy reforms, where we change taxes in a certain way and each time recalculate welfare measure both for learning and perfect foresight. As a result we can then compare the welfare implications for a tax change under perfect foresight against the case under learning. Note that we use the measure of Cooley and Hansen (1992, p.301ff.) for the transition paths. We do so because their measure for static comparison would lead to the same number for perfect foresight and learning, as in both cases the initial and new steady-states are identical.

An additional parameter needs to be chosen. That is the evaluation horizon T. Cooley and Hansen (1992, p.301ff.) choose a horizon $T \geq 2000$ and give no further detail on the motivation of that choice. Garcia-Milà et al. (2010) use $T = 200$ and give no motivation either. We will choose the latter in our welfare evaluations. For the series of experiments in Table 3.6 below, our calibration of the model is according to Table 3.5 below.

We choose the initial tax rates to be $\tau_0 = 0.0000$, $\tau_0^l = 0.2300$, $\tau_0^k = 0.5000$ and $\tau_0^c = 0.0500$. These non-zero tax rates lead to distortions. The first row

[15] We detail the computation in Appendix 3.6.2.

Parameter	**Value**	**Parameter**	**Value**
A	1.00	δ	0.00
α	0.33	T_p	8
β	0.99	γ	0.08
η	0.99	$\bar{L}$	1.00

Table 3.5: Model calibration for policy experiments 1. – 4.

in Table 3.6 reveals the welfare loss between the steady-state of the economy without taxes and the steady-state of the economy with our initially chosen tax rates amounts to 73.72%. This number tells us the change in consumption (in percentage of output) which is required so that households in the economy with initial tax levels are as well off as in the case with zero taxes is 73.72%. Be aware that Table 3.6 also indicates that without taxes our calibration yields a first best steady-state employment of $n_{FB} = 0.4024$, which implies 9.66 hours. With the initial taxes in place, the steady-state employment is $n_0 = 0.4326$, which implies 10.38 hours.

Now we assume a credible pre-announced permanent tax reform that favours capital accumulation, i.e. we lower the capital income tax to a level of $\tau_1^k = 0.2500$. As suggested by Judd (1987), Lucas (1990) and Cooley and Hansen (1992) this is expected to reduce the welfare costs of distortionary taxation. In each experiment reported lines 2 to 4 in Table 3.6 below, one of the other tax instruments, $\tau_\bullet$, $\tau_\bullet^l$ or $\tau_\bullet^c$ will be raised to a level that ensures that the periodic tax revenue in the new steady-state is the same as in the initial steady-state.[16] The second row of Table 3.6 indicates that compensating the cut in the capital income tax to τ_1^k by an increase in the labour income tax to τ_1^l leads to a welfare improvement under perfect foresight as well as under learning as both welfare measures decrease. But the numbers also reveal that the magnitude of the improvement differs. Whereas under learning the welfare measure goes down from

[16]Note, that as long as the dynamics under learning and perfect foresight differ, one is not able to equalize present values of tax revenues under learning and perfect foresight to the present value of tax revenues in the initial steady-state by manipulating tax rates in the same way. This approach was used in the analysis of Cooley and Hansen (1992) for perfect foresight only, but is not feasible in our case. In addition, we believe that keeping present values constant is not the kind of fiscal policy change that governments conduct in reality. Moreover, we believe that our comparison of welfare costs under learning to welfare costs under perfect foresight is valid even without equalizing present values of the tax revenue.

73.72% to 72.12%, under perfect foresight it decreases much more to 64.47%.[17] We can also observe that the new steady-state employment n_1 is lower than the initial steady-state employment n_0.

The pattern just described is also true, if we compensate the cut in $\tau_\bullet^k$ by an increase in $\tau_\bullet^c$ or $\tau_\bullet$ as the third and fourth row in Table 3.6 indicate. It is noteworthy that using the the lump-sum tax to compensate for the cut in the capital income tax yields the largest welfare improvement and keeps steady-state employment at the highest level independent of the assumption about expectations.

Thus, experiments 2 to 4 indicate that the resulting welfare improvements of an anticipated tax reform might be much smaller in magnitude under learning compared to its improvements under perfect foresight.

[17]We would like to emphasize that we set the rate of depreciation to $\delta = 0$ in order to achieve convergence for the dynamics under learning. That might be the reason, why the scale of $\mathcal{W}$ both under learning and perfect foresight is approximately twice the scale as the results in Cooley and Hansen (1992).

no.	$\tau_\bullet$		$\tau^l_\bullet$		$\tau^k_\bullet$		$\tau^c_\bullet$		$\mathcal{W}$		n_{FB}	$n_\bullet$	
	0	1	0	1	0	1	0	1	P	L		0	1
1.	0.0		0.2300		0.5000		0.0500		73.72%		0.4024	0.4326	
2.	0.0		0.2300	0.2931	0.5000	0.2500	0.0500		64.47%	72.12%	0.4024	0.4326	0.3976
3.	0.0		0.2300		0.5000	0.2500	0.0500	0.1027	63.71%	71.28%	0.4024	0.4326	0.4045
4.	0.0	0.0600	0.2300		0.5000	0.2500	0.0500		63.49%	70.23%	0.4024	0.4326	0.4139

0: Value before the tax change
P: Value under perfect foresight
1: Value after the tax change
L: Value under learning

Table 3.6: Simulation results of various policy experiments.

3.5 Conclusion

We demonstrate that under the assumption of elastic labour supply the responses to anticipated permanent lump-sum tax changes when agents learn are remarkably different compared to their counterparts under perfect foresight. The dynamics under learning appear to oscillate around the steady-state to which they converge slowly. Thus, there is more volatility under learning.

However sensitivity analyses show that even under inelastic labour supply these oscillations may be present for some choices of the gain parameter. We also find that a smaller gain parameter leads to higher volatility in our framework. This result is at odds with conventional wisdom about the link between the gain parameter and the dynamic responses in the learning literature. Overall, we detect two sources that may lead to oscillatory dynamics under learning given an anticipated permanent lump-sum tax change. These are the assumption of elastic labour supply and the choice of the gain parameter in the learning rule.

In the subsequent analysis we derive the dynamics for several distortionary taxes and illustrate that the dynamics for labour income as well as capital income tax rate changes are quite similar to changes of the lump-sum tax given elastic labour supply. Again we observe oscillating time paths. In case of a consumption tax, there is no oscillatory behaviour for the dynamics under learning, at least when agents have a log-utility function.

Moreover, policy experiments in the presence of multiple tax instruments indicate that the magnitude of welfare improvements due to the tax reform considered herein appears to be substantially lower under the assumption of learning compared to the case of perfect foresight. The reason may be the oscillatory behaviour of the dynamics under learning.

Form our point of view these results raise two major issues. First, oscillatory dynamic responses to exogenous shocks are rarely found in actual economic data. This fact questions the suitability of the model herein for policy analysis. Second, given that this model would be suitable for policy analysis, our results indicate that permanent tax changes may lead to lower welfare improvements under learning compared to perfect foresight.

We believe that future research in this area needs to come up with convincing empirical evidence on whether or how agents learn about fiscal policy. In addition, we also need to clarify from actual economic data, how the dynamic responses to anticipated permanent tax changes look like. Are they smooth or oscillatory?

With regard to theoretical considerations, it would also be desirable to derive a version of the model that allows for changing different tax rates at different points in time and therefore allows for public debt accumulation. But this task is beyond the focus of this paper and we aim to pursue that idea in subsequent research.

Furthermore, we think that perfect foresight and the implied once and for all choices of agents on the one hand and learning which implies periodic revision of current and future choices of agents on the other hand are extreme cases. One could also imagine agents that use adaptive learning, but infrequently and with differing interval length update their expectations and revise their current and future choices. Alternatively, agents randomly receive a signal to update their expectations.

In addition, more sophisticated computational methods may allow to calibrate the rate of depreciation different from zero or more realistic values of the elasticity of labour supply and still ensure convergence for the dynamics under learning on the other side. This could facilitate numerical results that are directly comparable to the existing literature in public finance.

3.6 Appendices

3.6.1 Model Derivations

3.6.1.1 Timing

We believe that the understanding of the timing is crucial to follow the derivations. For time periods indexed by t, discounting periods indexed by j, and an implementation date T_p announced in $t = 1$ and $T \equiv T_p - t$ denoting the number of periods until T_p we got the following picture:

$$\begin{aligned} t &= 1, 2, 3, 4, 5, 6, ... \\ j &= 0, 1, 2, 3, 4, 5, ... \\ T \equiv T_p - t &= 4, 3, 2, 1, 0, -1, ..., \end{aligned}$$

thus for the infinite sum over index j

$$\sum_{j=1}^{T-1}\{\bullet\} + \sum_{j=T}^{\infty}\{\bullet\} \tag{3.60}$$

from period $t = 1$ perspective, given exemplary $T_p = 5$ on the line $1 \leq t \leq T_p - 1$, until $j = 3 = T - 1$ we have the old tax rate. Furthermore, on the line $t \geq T_p$ from $j = 4 = T$ onwards we have the new tax rate. Equivalently for the infinite sum

$$\sum_{j=0}^{T-2}\{\bullet\} + \sum_{j=T-1}^{\infty}\{\bullet\} \tag{3.61}$$

from period $t = 1$ perspective, given exemplary $T_p = 5$ on the line $1 \leq t \leq T_p - 1$, until $j = 2 = T - 2$ we have the old tax rate. Furthermore, on the line $t \geq T_p$ from $j = 3 = T - 1$ onwards we have the new tax. This allows us later on to replace T with $T_p - t$ for $1 \leq t \leq T_p - 1$ and $T - 1$ with 0 for $t \geq T_p$.

3.6.1.2 Derivation of ST_1

Here we want to illustrate the methodology we apply in all derivations under learning for the example of ST_1. Starting from

$$ST_1 = \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \tau^e_{t+j}(t)$$

we split this infinite sum into

$$ST_1 = \left[\sum_{j=1}^{T-1} \frac{1}{D^e_{t,t+j}(t)} \tau_0 + \sum_{j=T}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \tau_1 \right].$$

Next we go back to the definition of $D^e_{t,t+j}(t)$. Given the learning rules (3.16) and (3.17) we get

$$D^e_{t,t+j}(t) = \Pi_{i=1}^{j} \left[(1-\delta) + r^e(t) \right] = \left[(1-\delta) + r^e(t) \right]^j . \tag{3.62}$$

Consequently we get

$$ST_1 = \left[\sum_{j=1}^{T-1}\left([(1-\delta)+r^e(t)]^{-1}\right)^j\tau_0+\sum_{j=T}^{\infty}\left([(1-\delta)+r^e(t)]^{-1}\right)^j\tau_1\right],$$

or

$$\begin{aligned}ST_1 = {} & [(1-\delta)+r^e(t)]^{-1}\times \\ & \left[\sum_{j=0}^{T-2}\left([(1-\delta)+r^e(t)]^{-1}\right)^j\tau_0+\sum_{j=T-1}^{\infty}\left([(1-\delta)+r^e(t)]^{-1}\right)^j\tau_1\right].\end{aligned}$$

Given the property of a finite geometric series $\sum_{j=m}^{n} f^j = \frac{f^{n+1}-f^m}{f-1}$ for some constant f, we get

$$\begin{aligned}ST_1 = {} & [(1-\delta)+r^e(t)]^{-1}\times \\ & \left[\left(\frac{[(1-\delta)+r^e(t)]^{1-T}-1}{[(1-\delta)+r^e(t)]^{-1}-1}\right)\tau_0+\left(\frac{-[(1-\delta)+r^e(t)]^{1-T}}{[(1-\delta)+r^e(t)]^{-1}-1}\right)\tau_1\right],\end{aligned}$$

which can be rewritten as

$$ST_1 = \frac{\tau_0}{r^e(t)-\delta}+\frac{(\tau_1-\tau_0)}{[(1-\delta)+r^e(t)]}\,\frac{[(1-\delta)+r^e(t)]^{1-T}}{1-[(1-\delta)+r^e(t)]^{-1}}. \tag{3.63}$$

Now, considering the timing outlined in Appendix 3.6.1.1 above, for $1 \le t \le T_p-1$ we plug in $T_p - t$ for T and get (3.24)

$$ST_1 = \frac{\tau_0}{r^e(t)-\delta}+(\tau_1-\tau_0)\,\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}}, \tag{3.64}$$

and for $t \ge T_p$ we have $T-1=0$, thus we get (3.25)

$$ST_1 = \frac{\tau_1}{r^e(t)-\delta}. \tag{3.65}$$

3.6.1.3 Derivation of ST_2

Starting from

$$ST_2 = \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \tau^{l,e}_{t+j}(t) w^e_{t+j}(t) \bar{L}$$

and given the learning rules (3.16) and (3.17) as well as (3.62) from above and $\tau^{l,e}_{t+j}$ being either τ^l_0 or τ^l_1, we may split the infinite sum above into

$$ST_2 = w^e(t)\bar{L} \left[\sum_{j=1}^{T-1} \left([(1-\delta) + r^e(t)]^j \right)^{-1} \tau^l_0 + \sum_{j=T}^{\infty} \left([(1-\delta) + r^e(t)]^j \right)^{-1} \tau^l_1 \right]$$

or

$$\begin{aligned} ST_2 &= \frac{\tau^l_0 \; w^e(t)\bar{L}}{[(1-\delta) + r^e(t)]} \sum_{j=0}^{T-2} \left([(1-\delta) + r^e(t)]^{-1} \right)^j \\ &\quad + \frac{\tau^l_1 \; w^e(t)\bar{L}}{[(1-\delta) + r^e(t)]} \sum_{j=T-1}^{\infty} \left([(1-\delta) + r^e(t)]^{-1} \right)^j . \end{aligned}$$

Now, as above, the properties of the geometric series allow us to rewrite this as

$$\begin{aligned} ST_2 &= \frac{\tau^l_0 \; w^e(t)\bar{L}}{[(1-\delta) + r^e(t)]} \left(\frac{[(1-\delta) + r^e(t)]^{1-T} - 1}{[(1-\delta) + r^e(t)]^{-1} - 1} \right) \\ &\quad + \frac{\tau^l_1 \; w^e(t)\bar{L}}{[(1-\delta) + r^e(t)]} \left(\frac{-[(1-\delta) + r^e(t)]^{1-T}}{[(1-\delta) + r^e(t)]^{-1} - 1} \right). \end{aligned}$$

For the timing outlined in Appendix 3.6.1.1 above, for $1 \leq t \leq T_p - 1$ we plug in $T_p - t$ for T and get (3.31)

$$ST_2 = w^e(t)\bar{L} \left[\frac{\tau^l_0}{r^e(t) - \delta} + \left(\tau^l_1 - \tau^l_0 \right) \frac{[(1-\delta) + r^e(t)]^{t-T_p}}{1 - [(1-\delta) + r^e(t)]^{-1}} \right] \tag{3.66}$$

and for $t \geq T_p$ we have $T - 1 = 0$, thus we get (3.32)

$$ST_2 = \frac{\tau^l_1 \; w^e(t)\bar{L}}{r^e(t) - \delta}. \tag{3.67}$$

3.6.1.4 Derivation of SW_2

We start from (3.39)

$$SW_2 = \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} w_{t+j}^{e}(t)\bar{L}.$$

Next, we recall the definition of $D_{t,t+j}^{k,e}(t)$. Given the learning rules (3.16) and (3.17) we get

$$D_{t,t+j}^{k,e}(t) = \Pi_{i=1}^{j}\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right] = \left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^j \quad (3.68)$$

for $\tau_{t+j}^{k,e}(t) = \tau_0^k$ and

$$D_{t,t+j}^{k,e}(t) = \Pi_{i=1}^{j}\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right] = \left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^j \quad (3.69)$$

for $\tau_{t+j}^{k,e}(t) = \tau_1^k$. Thereafter, we split this infinite sum into

$$\begin{aligned} SW_2 &= \bar{L}\left[\sum_{j=1}^{T-1} \frac{1}{D_{t,t+j}^{k,e}(t)} w^e(t) + \sum_{j=T}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} w^e(t)\right] \\ &= \bar{L}[\sum_{j=1}^{T-1}(\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^j)^{-1}w^e(t) + \\ &\quad \sum_{j=T}^{\infty}(\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^j)^{-1}w^e(t)\ ], \end{aligned}$$

or

$$\begin{aligned} SW_2 = &\frac{w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]} \sum_{j=0}^{T-2}\left(\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1}\right)^j \\ &+\frac{w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]} \sum_{j=T-1}^{\infty}\left(\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1}\right)^j. \end{aligned}$$

As in Section 3.6.1.2 above, we exploit the properties of geometric series and derive

$$\begin{aligned} SW_2 &= \frac{w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]}\left(\frac{1-\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{1-T}}{1-\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1}}\right) \\ &+\frac{w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]}\left(\frac{\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{1-T}}{1-\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1}}\right). \end{aligned}$$

Now we get back to the timing outlined in Appendix 3.6.1.1 above, for $1 \leq t \leq T_p - 1$ we plug in $T_p - t$ for T and get (3.40)

$$\begin{aligned} SW_2 &= \frac{w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L}\times \\ &\left[\frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\right] \end{aligned} \tag{3.70}$$

and for $t \geq T_p$ we have $T - 1 = 0$, thus we get (3.41)

$$SW_2 = \frac{w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}. \tag{3.71}$$

3.6.1.5 Derivation of ST_3

Starting from (3.54)

$$ST_3 = \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} \tau_{t+j}^{l,e}(t) w_{t+j}^e(t)\bar{L}$$

for (3.68) and (3.69) and $\tau^{l,e}_{t+j}(t)$ is either given by τ^l_0 or τ^l_1, we may once more split the infinite sum into

$$\begin{aligned} ST_3 &= w^e(t)\bar{L}\times \\ &\quad [\sum_{j=1}^{T-1}\left(\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^j\right)^{-1}\tau^l_0 \\ &\quad +\sum_{j=T}^{\infty}\left(\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^j\right)^{-1}\tau^l_1\ ], \end{aligned}$$

or

$$\begin{aligned} ST_3 &= \frac{\tau^l_0\ w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]}\ \sum_{j=0}^{T-2}\left(\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^{-1}\right)^j \\ &\quad +\frac{\tau^l_1\ w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]}\ \sum_{j=T-1}^{\infty}\left(\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^{-1}\right)^j. \end{aligned}$$

Now, the properties of the geometric series allow us to rewrite this as

$$\begin{aligned} ST_3 &= \frac{\tau^l_0\ w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]}\ \left(\frac{\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^{1-T}-1}{\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^{-1}-1}\right) \\ &\quad +\frac{\tau^l_1\ w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]}\ \left(\frac{-\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^{1-T}}{\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^{-1}-1}\right). \end{aligned}$$

For the timing outlined in Appendix 3.6.1.1 above, for $1\le t\le T_p-1$ we plug in T_p-t for T and get (3.55)

$$\begin{aligned} ST_3 &= \frac{\tau^l_0\ w^e(t)\bar{L}}{[(1-\tau^k_0)r^e(t)-\delta]}+w^e(t)\bar{L}\times \\ &\quad \left[\frac{\tau^l_1\ [(1-\delta)+(1-\tau^k_1)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau^k_1)r^e(t)]^{-1}}-\frac{\tau^l_0\ [(1-\delta)+(1-\tau^k_0)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau^k_0)r^e(t)]^{-1}}\right] \end{aligned} \tag{3.72}$$

and for $t \geq T_p$ we have $T-1=0$, thus we get (3.56)

$$ST_3 \;=\; \frac{\tau_1^l \; w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}. \tag{3.73}$$

3.6.1.6 Derivation of ST_4

Starting from (3.57)

$$ST_4 \;=\; \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} \tau_{t+j}^e(t)$$

given (3.68) and (3.69) are true and $\tau_{t+j}^e(t)$ is either τ_0 or τ_1, we again split the infinite sum into

$$\begin{aligned} ST_4 \;=\; & [\sum_{j=1}^{T-1} \left(\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^j \right)^{-1} \tau_0 \\ & + \sum_{j=T}^{\infty} \left(\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^j \right)^{-1} \tau_1 \;], \end{aligned}$$

or

$$\begin{aligned} ST_4 \;=\; & \left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1} \left[\sum_{j=0}^{T-2} \left(\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1} \right)^j \tau_0 \right] \\ & + \left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1} \left[\sum_{j=T-1}^{\infty} \left(\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1} \right)^j \tau_1 \right]. \end{aligned}$$

Given the properties of geometric series we can rewrite the latter as

$$\begin{aligned} ST_4 \;=\; & \left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1} \left(\frac{\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{1-T}-1}{\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1}-1} \tau_0 \right) \\ & + \left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1} \left(\frac{-\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{1-T}}{\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1}-1} \tau_1 \right). \end{aligned}$$

Now given the timing outlined in Appendix 3.6.1.1 above, for $1 \leq t \leq T_p - 1$ we plug in $T_p - t$ for T and get (3.58)

$$\begin{aligned} ST_4 &= \frac{\tau_0}{[(1-\tau_0^k)r^e(t) - \delta]} \\ &\quad +[\ \frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}}\ \tau_1 \\ &\quad -\frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\ \tau_0\ ] \end{aligned} \tag{3.74}$$

and for $t \geq T_p$ we have $T - 1 = 0$, thus we get (3.59)

$$ST_4 = \frac{\tau_1}{[(1-\tau_1^k)r^e(t) - \delta]}. \tag{3.75}$$

3.6.2 Computing Welfare Changes

3.6.2.1 Comparative Statics

We follow the approach of Cooley and Hansen (1992, p.301ff.) based on Lucas (1990). Their measure of welfare change for a given policy change is derived by solving

$$U_0 = \log[c_1(1+x^\bullet)] + \eta \log[1-n_1] \tag{3.76}$$

for x in our case.[18] U_0 is the utility a household obtains in the steady-state without any tax and c_1 and n_1 are the values of consumption and employment at the new steady-state after the tax change either under perfect foresight or learning. It follows that

$$x^\bullet = \frac{\exp(U_0)}{c_1(1-n_1)^\eta} - 1. \tag{3.77}$$

[18] $x^\bullet$ is either x under perfect foresight or x^* under learning.

Thus, in general, we need to solve for x for the perfect foresight dynamics and another x^* for the dynamics under learning.[19] Given $x^\bullet$ we can calculate

$$\overline{\mathcal{W}} = \frac{\triangle C}{y_1} = \frac{x^\bullet c_1}{y_1}, \tag{3.78}$$

where $\triangle C$ is the restoration value of consumption, which in our case may be interpreted as the total change in consumption required to restore a household to the level of utility obtained under the allocation associated with zero taxes. y_1 is the level of output at the new steady-state.

3.6.2.2 Transition Measure

Again we follow the approach of Cooley and Hansen (1992, p.301ff.) based on Lucas (1990). Their measure of welfare change accounting for transition given a policy change is derived by solving

$$\sum_{t=1}^{T} \beta^t \left\{ \log[c_t(1+x^\bullet)] + \eta \log[1-n_t] - U_0 \right\} \stackrel{!}{=} 0 \tag{3.79}$$

for x under perfect foresight and x^* under learning. T is the terminal period, c_t is period t consumption either under perfect foresight or learning and y_t is period t output either under perfect foresight or learning.

$$x^\bullet = \left[\frac{\exp\left(U_0\left[\beta^1 + ... + \beta^T\right]\right)}{\left(c_1^{\beta^1} ... \, c_T^{\beta^T}\right) \times \left[(1-n_1)^{\eta\beta^1} ... \, (1-n_T)^{\eta\beta^T}\right]} \right]^{\frac{1}{[\beta^1+...+\beta^T]}} - 1.$$

$$x^\bullet = \left[\frac{\exp\left(U_0 \sum_{t=1}^{T} \beta^t\right)}{\Pi_{t=1}^{T} c_t^{\beta^t} \times \Pi_{t=1}^{T} (1-n_t)^{\eta\beta^t}} \right]^{\frac{1}{\sum_{t=1}^{T}\beta^t}} - 1. \tag{3.80}$$

Given $x^\bullet$ we can calculate

$$\mathcal{W}^\bullet = \frac{\sum_{t=1}^{T} \beta^t \{x c_t\}}{\sum_{t=1}^{T} \beta^t \{y_t\}}, \tag{3.81}$$

[19] Of course we are aware that this must yield the same $x = x^*$ both under perfect-foresight and under learning, but this number may be useful to compare different policy experiments.

which will be reported as $\mathcal{W}$ for the perfect foresight dynamics and as $\mathcal{W}^*$ for the dynamics under learning.

Bibliography

Benigno, P. and Woodford, M. (2004). Optimal Monetary and Fiscal Policy: A Linear-Quadratic Approach. In Gertler, M. and Rogoff, K. S., editors, *NBER Macroeconomics Annual 2003*, volume 18, pages 271–364. National Bureau of Economic Research, MIT Press.

Binder, M. and Pesaran, M. H. (1999). Multivariate Rational Expectations Models and Macroeconometric Modeling: A Review and Some New Results. In Pesaran, M. H. and Wickens, M. R., editors, *Handbook of Applied Econometrics. Volume I: Macroeconomics*, volume 1 of *Blackwell Handbooks in Economics*, pages 139–187, Oxford, UK. Blackwell Publishing.

Blanchard, O. J. and Kahn, C. M. (1980). The Solution of Linear Difference Models under Rational Expectations. *Econometrica*, 48(5):1305–1311.

Branch, W. A. (2004). The Theory of Rationally Heterogeneous Expectations: Evidence from Survey Data on Inflation Expectations. *The Economic Journal*, 114(497):592–621.

Branch, W. A. and Evans, G. W. (2007). Model Uncertainty and Endogenous Volatility. *Review of Economic Dynamics*, 10(2):207–237.

Branch, W. A. and McGough, B. (2009). A New Keynesian Model with Heterogeneous Expectations. *Journal of Economic Dynamics and Control*, Forthcoming.

Bullard, J. B. (1991). Learning, Rational Expectations and Policy: A Summary of Recent Research. *Federal Reserve Bank of St. Louis Review*, 73(1):50–60.

Bullard, J. B. and Eusepi, S. (2008). When Does Determinacy Imply Expectational Stability? *Federal Reserve Bank of St. Louis Working Paper Series*, 007B.

Bullard, J. B. and Mitra, K. (2002). Learning about Monetary Policy Rules. *Journal of Monetary Economics*, 49(6):1105–1129.

Bullard, J. B. and Mitra, K. (2007). Determinacy, Learnability, and Monetary Policy Inertia. *Journal of Money, Credit and Banking*, 39(5):1177–1212.

Calvo, G. A. (1983). Staggered Prices in a Utility-Maximizing Framework. *Journal of Monetary Economics*, 12(3):383–398.

Carlstrom, C. T. and Fuerst, T. S. (2004). Thinking about Monetary Policy without Money. *International Finance*, 7(2):325–347.

Chamley, C. (1981). The Welfare Cost of Capital Income Taxation in a Growing Economy. *Journal of Political Economy*, 89(3):468–496.

Clarida, R. H., Galí, J., and Gertler, M. (1998). Monetary Policy Rules in Practice Some International Evidence. *European Economic Review*, 42(6):1033–1067.

Clarida, R. H., Galí, J., and Gertler, M. (1999). The Science of Monetary Policy: A New Keynesian Perspective. *Journal of Economic Literature*, 37(4):1661–1707.

Clarida, R. H., Galí, J., and Gertler, M. (2002). A Simple Framework for International Monetary Policy Analysis. *Journal of Monetary Economics*, 49(5):879–904.

Cochrane, J. H. (2005). *Asset Pricing*. Princeton University Press, second edition.

Cochrane, J. H. (2007). Inflation Determination With Taylor Rules: A Critical Review. *NBER Working Paper*, 13409.

Cooley, T. F. and Hansen, G. D. (1992). Tax Distortions in a Neoclassical Monetary Economy. *Journal of Economic Theory*, 58(2):290–316.

DeJong, D. N. and Dave, C. (2007). *Structural Macroeconometrics*. Princeton University Press.

Dixit, A. K. and Stiglitz, J. E. (1977). Monopolistic Competition and Optimum Product Diversity. *The American Economic Review*, 67(3):297–308.

Domeij, D. and Klein, P. (2005). Preannounced Optimal Tax Reform. *Macroeconomic Dynamics*, 9(2):150–169.

Duffy, J. and Xiao, W. (2009). Investment and Monetary Policy: Learning and Determinacy of Equilibrium. *Working Paper*.

Evans, G. W., Guse, E., and Honkapohja, S. (2008). Liquidity Traps, Learning and Stagnation. *European Economic Review*, 52(8):1438–1463.

Evans, G. W. and Honkapohja, S. (2001). *Learning and Expectations in Macroeconomics*. Frontiers of Economic Research. Princeton University Press, Princeton, NJ.

Evans, G. W. and Honkapohja, S. (2006). Monetary Policy, Expectations and Commitment. *Scandinavian Journal of Economics*, 108(1):15–38.

Evans, G. W. and Honkapohja, S. (2007). Policy Interaction, Learning, and the Fiscal Theory of Prices. *Macroeconomic Dynamics*, 11(5):665–690.

Evans, G. W. and Honkapohja, S. (2009). Robust Learning Stability with Operational Monetary Policy Rules. In Schmidt-Hebbel, K. and Walsh, C. E., editors, *Monetary Policy Under Uncertainty and Learning*, volume XIII of *Series on Central Banking, Analysis, and Economic Policies*, pages 145–170, Santiago, Chile. Central Bank of Chile, Central Bank of Chile.

Evans, G. W. and Honkapohja, S. (2010). Corrigendum: Monetary Policy, Expectations and Commitment. *Scandinavian Journal of Economics*, 112(3):640–641.

Evans, G. W., Honkapohja, S., and Mitra, K. (2009). Anticipated Fiscal Policy and Adaptive Learning. *Journal of Monetary Economics*, 56(7):930–953.

Evans, G. W., Honkapohja, S., and Mitra, K. (2010). Does Ricardian Equivalence Hold When Expectations are not Rational? *Centre For Dynamic Macroeconomic Analysis Working Paper Series*, 10/08.

Evans, G. W. and McGough, B. (2005). Monetary Policy, Indeterminacy and Learning. *Journal of Economic Dynamics and Control*, 29(11):1809–1840.

Friedman, M. (1969). The Optimum Quantity of Money. In *The Optimum Quantity of Money and Other Essays*, pages 1–50, Chicago. Aldine.

Galí, J. (2008). *Monetary Policy, Inflation, and the Business Cycle*. Princeton University Press.

Garcia-Milà, T., Marcet, A., and Ventura, E. (2010). Supply Side Interventions and Redistribution. *The Economic Journal*, 120(543):105–130.

Giannitsarou, C. (2007). Balanced Budget Rules and Aggregate Instability: The Role of Consumption Taxes. *The Economic Journal*, 117(523):1423–1435.

Giannoni, M. P. and Woodford, M. (2005). Optimal Inflation-Targeting Rules. In Bernanke, B. S. and Woodford, M., editors, *The Inflation-Targeting Debate*, volume 32 of *Studies in Business Cycles*, pages 93–162, Chicago and London. National Bureau of Economic Research, The University of Chicago Press.

Judd, K. L. (1987). The Welfare Cost of Factor Taxation in a Perfect-Foresight Model. *Journal of Political Economy*, 95(4):675–709.

Klein, P. (2000). Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model. *Journal of Economic Dynamics and Control*, 24(10):1405–1423.

Leeper, E. M. (1991). Equilibria under Active and Passive Monetary and Fiscal Policies. *Journal of Monetary Economics*, 27(1):129–147.

Leeper, E. M. (2009). Anchoring Fiscal Expectations. *NBER Working Paper*, 15269.

Leeper, E. M., Walker, T. B., and Yang, S.-C. S. (2009). Fiscal Foresight and Information Flows. *NBER Working Paper*, 14630.

Ljungqvist, L. and Sargent, T. J. (2000). *Recursive Macroeconomic Theory*. MIT Press, Cambridge, MA, 2nd edition.

Lubik, T. and Schorfheide, F. (2003). Computing Sunspot Equilibria in Linear Rational Expectations Models. *Journal of Economic Dynamics and Control*, 28(2):273–285.

Lucas, Jr., R. E. (1990). Supply-Side Economics: An Analytical Review. *Oxford Economic Papers*, 42(2):293–316.

Mankiw, N. G. and Reis, R. (2002). Sticky Information Versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve. *The Quarterly Journal of Economics*, 117(4):1295–1328.

Mankiw, N. G. and Reis, R. (2007). Sticky Information in General Equilibrium. *Journal of the European Economic Association*, 5(2-3):603–613.

McCallum, B. T. (1999a). Issues in the Design of Monetary Policy Rules. In Taylor, J. B. and Woodford, M., editors, *Handbook of Macroeconomics*, volume 1A of *Handbooks in Economics*, pages 1483–1530. North-Holland, Amsterdam.

McCallum, B. T. (1999b). Role of the Minimal State Variable Criterion in Rational Expectations Models. *International Tax and Public Finance*, 6(4):621–639.

McCallum, B. T. (2009a). Causality, Structure, and the Uniqueness of Rational Expectations Equilibria. *NBER Working Paper*, 15234.

McCallum, B. T. (2009b). Inflation Determination with Taylor Rules: Is New Keynesian Analysis Critically Flawed? *Journal of Monetary Economics*, 56(8):1101–1108.

McCallum, B. T. and Nelson, E. (2004). Timeless Perspective vs. Discretionary Monetary Policy in Forward-Looking Models. *Federal Reserve Bank of St. Louis Review*, 86(2):43–56.

Milani, F. (2007). Expectations, Learning and Macroeconomic Persistence. *Journal of Monetary Economics*, 54(7):2065–2082.

Obstfeld, M. and Rogoff, K. S. (1995). Exchange Rate Dynamics Redux. *Journal of Political Economy*, 103(3):624–660.

Preston, B. (2005). Learning about Monetary Policy Rules when Long-Horizon Expectations Matter. *International Journal of Central Banking*, 1(2):81–126.

Sims, C. A. (2002). Solving Linear Rational Expectations Models. *Computational Economics*, 20(1-2):1–20.

Sveen, T. and Weinke, L. (2005). New Perspectives on Capital, Sticky Prices, and the Taylor Principle. *Journal of Economic Theory*, 123(1):21–39.

Svensson, L. E. and Woodford, M. (2005). Implementing Optimal Policy through Inflation-Forecast Targeting. In Bernanke, B. S. and Woodford, M., editors, *The Inflation-Targeting Debate*, volume 32 of *Studies in Business Cycles*, pages 19–83, Chicago. National Bureau of Economic Research, The University of Chicago Press.

Taylor, J. B. (1993). Discretion versus Policy Rules in Practice. *Carnegie-Rochester Conference Series on Public Policy*, 39:195–214.

Taylor, J. B. (1999). A Historical Analysis of Monetary Policy Rules. In Taylor, J. B., editor, *Monetary Policy Rules*, volume 31 of *Studies in Business Cycles*, Chicago. National Bureau of Economic Research, The University of Chicago Press.

Trabandt, M. (2007). Optimal Pre-Announced Tax Reform Revisited. *EUI Economics Working Papers*, 52.

Uhlig, H. (2006). Approximate Solutions to Dynamic Models - Linear Methods. *SFB 649 Discussion Papers*, 030.

Woodford, M. (1999). Optimal Monetary Policy Inertia. *NBER Working Paper*, 7261.

Woodford, M. (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

Woodford, M. (2005). Firm-Specific Capital and the New Keynesian Phillips Curve. *International Journal of Central Banking*, 1(2).

Appendix A

Abstract

English Version

This dissertation is comprised of three papers concerned with monetary and fiscal policy issues. Each paper forms one chapter of the dissertation.
The first chapter is titled "Heterogeneous Expectations and the Merit of Monetary Policy Inertia." We study a central bank's ability to ensure price stability by conventional policies in the recently developed New Keynesian model with heterogeneous expectations. Agents have either rational or adaptive expectations. We find that linear rules that mechanically react to inflation and output exhibit similar (dis-)advantages as in the standard New Keynesian model. It is optimal policy, which appears to be hazardous given the central bank ignores heterogeneous expectations. Policy inertia in the central bank's objective can resolve this threat. Therefore, we provide favorable arguments for nowadays central banks' practice to adjust interest rates with notable inertia in response to shocks.
The second chapter is titled "Monetary and Fiscal Policy Interaction in a World with Heterogeneous Expectations." We assess the central bank's capability to keep prices stable in scenarios of monetary and fiscal policy interaction. We do so in the New Keynesian model with heterogeneous expectations, where agents with rational and adaptive expectations coexist. No traditional clear-cut distinctions between policy regimes that do and do not yield to price stability are possible. Moreover, for most policy regimes explosive paths of the price level become feasible. Consequently conventional monetary policy recommendations can become risky. Nevertheless, price stability remains attainable.

The third and final chapter is titled "Anticipation, Learning and Welfare: the Case of Distortionary Taxation." This is a collaboration with my fellow student *Shoujian Zhang*. We study the impact of anticipated fiscal policy changes in the Ramsey economy when agents form expectations using adaptive learning. We extend the existing framework by distortionary taxes as well as elastic labour supply, which makes agents' decisions non-predetermined but more realistic. We detect that the dynamic responses to anticipated tax changes under learning have oscillatory behavior. Moreover, we demonstrate that this behavior can have important implications for the welfare consequences of fiscal reforms.

German Version

Die vorliegende kumulative Dissertation umfasst drei Essays, welche sich mit Fragestellungen der Geld- und Fiskalpolitik beschäftigen. Jeder Essay stellt ein Kapitel der Dissertation dar.
Das erste Kapitel trägt den Titel "Heterogeneous Expectations and the Merit of Monetary Policy Inertia." Es untersucht die Fähigkeit einer Zentralbank, mittels populärer Zinsregeln Preisstabilität in einer Ökonomie sicherzustellen. Es handelt sich um eine Neu-Keynesianische Ökonomie mit heterogenen Erwartungen. Die Individuen formen ihre Erwartungen entweder rational oder adaptiv. Es stellt sich heraus das simple Zinsregeln die gleichen Stärken und Schwächen haben, wie in einer Ökonomie, in welcher nur rationale Agenten präsent sind. Dagegen kann Zinspolitik, welche unter rationalen Erwartungen optimal ist, unter heterogenen Erwartungen schwerwiegende Konsequenzen im Hinblick auf die Preisstabilität haben. Wendet eine Zentralbank jedoch ein geeignetes Wohlfahrtskriterium an, so stellen heterogene Erwartungen keine Bedrohung für Preisstabilität dar. Die Ergebnisse liefern ein weiteres Argument dafür, dass Zentralbanken heutzutage den Leitzins in Reaktion auf ökonomische Turbulenzen nur graduell ändern.
Das zweite Kapitel trägt den Titel "Monetary and Fiscal Policy Interaction in a World with Heterogeneous Expectations." Ein weiteres mal ist der Untersuchungsgegenstand die Fähigkeit einer Zentralbank, mittels Zinspolitik Preisstabilität in einer Ökonomie mit heterogenen Erwartungen sicherzustellen. Wiederum formen Individuen ihre Erwartungen entweder rational oder adaptiv. Dabei wird insbesondere auf die Herausforderungen durch so genannte aktive Fiskalpolitik

für die Zentralbank eingegangen. Es zeigt sich, dass im Gegensatz zum Fall homogen rationaler Erwartungen keine Politikregime analytisch identifiziert werden können, welche zu Preisstabilität führen. Numerische Simulationen zeigen, dass in den meisten Politikszenarien die Gefahr lokaler Divergenz besteht, falls konventionelle Geldpolitik zum Einsatz kommt. Trotzdem kann Preisstabilität durch geeignete Geldpolitik erreicht werden.
Das dritte Kapitel trägt den Titel "Anticipation, Learning and Welfare: the Case of Distortionary Taxation." Es handelt sich hierbei um eine Zusammenarbeit mit Shoujian Zhang. Im Fokus der Untersuchung stehen antizipierte Steuerreformen. Die Individuen in der Ökonomie haben nicht-rationale Erwartungen. Sie verhalten sich wie empirische Wirtschaftsforscher. Die Individuen nutzen Beobachtungen aus der Vergangenheit um Erwartungen über die Zukunft mittels statistischer Verfahren zu bilden. Besonderes Augenmerk richtet die Untersuchung auf die Änderung verzerrender Steuern. Es stellt sich heraus, dass eine Steuerreform unter nicht-rationalen Erwartungen der Individuen zu oszillierenden Anpassungspfaden ökonomischer Variablen wie Konsum oder Kapital führt. Numerische Simulationen zeigen, dass die durch Steuerreformen verursachten Oszillationen wichtige Implikationen für die Wohlfahrt einer Volkswirtschaft, gemessen am Nutzen der Individuen, haben.

Appendix B

Curriculum Vitae

Emanuel Maximilian Gasteiger

Email ::: emanuel.gasteiger@gmail.com

Personal Information

Born in Freising, Bavaria, Germany in January 1981.

Graduate Studies

- PhD Studies in Economics, University of Vienna, 2006 to present.
  - *Dissertation:* Expectations in Monetary and Fiscal Policy.
  - *Supervisors:* Gerhard Sorger, Robert Kunst.
  - *External Examiner:* Seppo Honkapohja, Member of the Board, Bank of Finland.
  - *Expected Completion Date:* April 2011.
- Summer Schools
  - DYNARE Summer School 2009, CEPREMAP and Banque de France, Paris, France.
  - Summer School in Applied Macroeconometrics 2009, University of Salento, Lecce, Italy.

Undergraduate Studies

- Diploma in Economics, Ludwig-Maximilians-University Munich, 2001 to 2006.
  - *Diploma Thesis:* Pension Reform and Private Retirement Saving.
  - *Supervisor:* Joachim Winter.

Research

Research Fields

- Monetary Economics, Fiscal Policy, Learning Theory, Macroeconometrics.

Working Papers

- Heterogeneous Expectations and the Merit of Monetary Policy Inertia. (Job Market Paper)
- Monetary and Fiscal Policy Interaction in a World with Heterogeneous Expectations.
- Anticipation, Learning and Welfare: the Case of Distortionary Taxation. (with Shoujian Zhang)

Work in Progress

- Indeterminacy and Welfare.
- Optimal Inflation Targeting and Fiscal Policy Under Imperfect Knowledge.
- Fiscal Multipliers Considering a Large Dataset: A FAVAR Approach. (with Matteo Fragetta)
- Anticipated Fiscal Policy and Adaptive Learning in an Overlapping Generations Model. (with Shoujian Zhang)

Presentations

- Heterogeneous Expectations and the Merit of Monetary Policy Inertia. (Job Market Paper)

  - 6th PhD Presentation Meeting of the Royal Economic Society, City University London, January 15th-16th, 2011.
  - Annual Conference of the Scottish Economic Society, Perth, April 4th-6th, 2011.

- Anticipation, Learning and Welfare: the Case of Distortionary Taxation.
  - 4th RGS Doctoral Conference in Economics, TU Dortmund University, February 21st-23rd, 2011.

References

Univ.-Prof. Dipl.-Ing. Dr. Gerhard Sorger (Principal Supervisor)
Department of Economics
University of Vienna
Hohenstaufengasse 9
A-1010 Vienna
Office: +43 (0) 1 4277 37443
Email: gerhard.sorger@univie.ac.at
Homepage: homepage.univie.ac.at/gerhard.sorger

Univ.-Prof. Dipl.-Ing. Dr. Robert Kunst (Secondary Supervisor)
Department of Economics
University of Vienna
Brünner Straße 72
A-1210 Vienna
Office: +43 (0) 1 4277 37479
Email: robert.kunst@univie.ac.at
Homepage: homepage.univie.ac.at/robert.kunst

Prof. Lutz Weinke, PhD
Institute of Economic Policy
Humboldt-Universität zu Berlin
Spandauer Straße 1
D-10099 Berlin
Office: +49 (0) 30 2093 5927
Email: lutz.weinke@wiwi.hu-berlin.de
Homepage: lehre.wiwi.hu-berlin.de/Professuren/vwl/wipo

Prof. Dr. Joachim Winter
Department of Economics
Ludwig-Maximilians-University Munich
Ludwigstraße 28 RG
D-80539 München
Office: +49 (0) 89 2180 2459
Email: winter@lmu.de
Homepage: empwi.vwl.lmu.de

Non-Academic Work Experience

- INSET Research & Advisory GmbH, Vienna, Austria, Senior Consultant, 2007 to 2008.
- Beck et al. Services GmbH, Munich, Germany, Consultant, 2004 to 2006.
- O_2(UK) Ltd., London, United Kingdom, Intern, 2004.
- O_2(Germany) GmbH & Co. OHG, Munich, Germany, Student Trainee in Business Consulting, 2001 to 2004.
- O_2(Germany) GmbH & Co. OHG, Munich, Germany, Intern, 2001.
- BBBank eG, Karlsruhe and Munich, Germany, Intern, 2000.

Miscellaneous

- Languages: German (native), English (fluent), Arabic (working knowledge), French (working knowledge), Swedish (basic).