Diplomarbeit

Titel der Diplomarbeit

Semileptonic processes
in Chiral Perturbation Theory

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Foreword

Semileptonic processes have played a crucial role in our understanding of flavour physics. In this thesis we consider the $K_{\ell 3}$ decays ($\ell = e, \mu$)

$$\begin{aligned} K^+(p_K) &\to \pi^0(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu), \\ K^0(p_K) &\to \pi^-(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu) \end{aligned}$$

(and their charge conjugate modes). These decays provide the theoretically cleanest and most precise measurement of the Cabbibo-Kobayashi-Maskawa matrix element $|V_{us}|$ [1], which is one of the main input parameters in the standard model of particle physics, formed by the Glashow-Weinberg-Salam theory of electroweak interactions [2] and Quantum Chromodynamics (QCD) [3], the quantum field theory of strong interactions. Therefore it is important to have a deep theoretical understanding of these processes.

The (fully inclusive) $K_{\ell 3}$ decay rate is given by [1]

$$\Gamma(K_{\ell 3(\gamma)}) = \frac{G_F^2 M_K^5 C_K^2}{192\pi^3} S_{\mathrm{EW}} |V_{us}|^2 \left| f_+^{K^0\pi^-}(0) \right|^2 I_K^\ell (1 + \delta_K^\ell + \delta_{\mathrm{SU}(2)}),$$

where G_F is the Fermi constant as determined from muon decays, $S_{EW} = 1.0232(3)$ [4] is the short-distance electroweak correction, C_K is a Clebsch-Gordan coefficient (1 for K^0 and $1/\sqrt{2}$ for $K^\pm$ decays), δ_K^ℓ represents the channel-dependent long-distance EM corrections, $\delta_{\mathrm{SU}(2)}$ the correction for isospin breaking, $f_+^{K_0\pi^-}(0)$ is the $K_{\ell 3}^0$ vector form factor at zero momentum transfer, and I_K^ℓ is a phase-space integral that is sensitive to the momentum dependence of the form factors. The latter describe the hadronic matrix elements

$$\langle \pi(p_\pi) | \, \bar{u}\gamma_\mu s \, | K(p_K) \rangle = (p_\pi + p_K)_\mu f_+^{K\pi}(t) + (p_\pi - p_K)_\mu f_-^{K\pi}(t),$$

where $t = (p_K - p_\pi)^2 = (p_\ell + p_\nu)^2$. In the experiment, the values of the vector form factor $f_+^{K\pi}(t)$ and the scalar form factor

$$f_0^{K\pi}(t) = f_+^{K\pi}(t) + \frac{t}{M_K^2 - M_\pi^2} f_-^{K\pi}(t)$$

are measured. These form factors are (usually) parameterized by the vector slope (λ'_+) and curvature (λ''_+) parameters and the scalar slope parameter λ_0, respectively [1]:

$$f_+^{K\pi}(t) = f_+^{K\pi}(0) \left[1 + \lambda'_+ \frac{t}{M_{\pi^+}^2} + \frac{1}{2}\lambda''_+ \left(\frac{t}{M_{\pi^+}^2} \right)^2 \right], \quad f_0^{K\pi}(t) = f_+^{K\pi}(0) \left(1 + \lambda_0 \frac{t}{M_{\pi^+}^2} \right).$$

On the other hand, this form factors can be calculated in theory to provide a comparison with the experimental outcomes.

Although recent high statistics data from ISTRA+ [5], KTeV [6], NA48 [7] and KLOE [8] are available now, our picture of the scalar form factor has not become much clearer. While the values of λ'_+ and λ''_+ of the different experiments are consistent with each other, the actual value of λ_0 remains unclear. Especially if both of the values of λ_0 from ISTRA+ and NA48 were true, this would signalize an enormous isospin violation in the $K_{\ell 3}$ decays. Therefore it is important to know if such a huge isospin violation can be understood within the standard model.

For a comparison with the experimental outcomes, we need to know the theoretical prediction for the behaviour of the scalar form factors of $K^0_{\ell 3}$ and $K^+_{\ell 3}$ as precisely as possible. In this thesis we wish to address the following questions:

- Which of the values of the slope parameter λ_0 found by the different experimental groups are compatible with the standard model of particle physics?
- Which magnitude of isospin violation can be expected for the scalar form factors?

The natural tool of this analysis is Chiral Perturbation Theory (χPT) [9, 10], the effective theory of the standard model at low energies. The Lagrangian of this theory contains all operators invariant under transformations of the chiral symmetry group $G = SU(3)_L \times SU(3)_R$, which is an infinite number of terms, but makes sense as an expansion in powers of the momentum. QCD becomes non-perturbative in the low-energy regime (due to confinement). In χPT, on the other hand, the relevant degrees of freedom are no longer quarks and gluons, but the pseudoscalar mesons. The octet of the lightest pseudoscalar mesons plays a special role as the pseudo-Goldstone bosons (GBs) of spontaneously broken (approximate) chiral symmetry. χPT exploits this feature and describes the strong interaction by an exchange of these pseudo-GBs. Due to Goldstone's theorem [11], the interaction among them vanishes at zero momentum – one can apply perturbation theory at low energies ($p \ll 1$ GeV).

The drawback of such an effective theory is that one gets an increasing number of new low-energy constants (LECs) with each order in the momentum expansion [9, 10]. These free parameters must be fixed with experimental input, additional model-dependent assumptions or lattice calculations.

The outline of this thesis is as follows. In Part I we give a short introduction to χPT. Part II is dedicated to the $K_{\ell 3}$ decays and especially the slope parameters of the scalar form factors. This part follows our work [12]. In section 2 we summarize the basic facts about and present the kinematics of $K_{\ell 3}$ decays and take a closer look on the current experimental situation. We describe the determination of

$$\frac{F_K}{F_\pi f_+^{K^0\pi^-}(0)},$$

which is one of our main input parameters. F_K and F_π denote the kaon and the pion decay constant, respectively. In section 3 we review the next-to-leading order (NLO) results for the vector and the scalar form factors, including pure QCD isospin violation ($m_d \neq m_u$) as well as isospin violation due to electromagnetic effects. After updating the parameter $\varepsilon^{(2)}$, which determines the size of isospin breaking, we turn to the numerical determination of the size of isospin violation in order to obtain numerical results for the slope parameter of the scalar form factor with a separate determination of the contributions of both sources of isospin violation. Finally, we analyze the Callan-Treiman relations [13] at NLO, again including isospin violating effects.

In section 4 we consider effects arising at next-to-next-to-leading order (NNLO). We estimate the order p^6 low-energy couplings C_{12}^r and C_{34}^r using $1/N_c$ expansion and truncating the hadronic spectrum to the lowest lying resonances [14]. With these results and the two-loop calculations of Bijnens and Talavera [15] we calculate the scalar slope and curvature parameters in the isospin limit. We give an update of the vector form factor at zero momentum transfer, $f_+^{K\pi}(0)$. We compare our results for the scalar slope λ_0 and curvature c_0 with the values recently obtained by dispersive methods [16–21]. We continue with extending the results obtained at the order $(m_d - m_u)p^4$ [22] on the $K_{\ell 3}$ scalar form factors by an estimate of the associated local contributions relevant for the splitting $\lambda_0^{K^0\pi^+} - \lambda_0^{K^+\pi^0}$. Finally, we analyze the size of the scalar form factor in the isospin limit at the Callan-Treiman point and discuss the possible size of corrections to the Callan-Treiman relation induced by isospin violation at this chiral order.

Contents

Part I

Introduction

1 Chiral Perturbation Theory

1.1 QCD in the chiral limit

In the last decades, the standard model of particle physics had amazing success in describing almost all observed phenomena in high-energy pyhsics. QCD, the quantum field theory of strong interactions, has two fundamental properties: Asymptotic freedom [23] and confinement. Due to the latter, QCD becomes non-perturbative at low energies - the usual perturbative techniques of calculating decay widths and cross sections are no longer applicable.

Fortunately, in the late seventies, Steven Weinberg came up with the concept of effective field theories. He formulated his idea as a conjecture [24]:

> "...if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analycity, perturbative unitarity, cluster decomposition and the assumed symmetry principles."

The basic idea of an effective theory is not to attempt to construct a so called "Theory of Everything", but rather to look for specific classes of phenomena where only a certain subset of degrees of freedom is relevant. Based on Weinberg's idea, Gasser and Leutwyler worked out the effective field theory for the standard model at low energies, chiral perturbation theory (χPT) [9, 10]. While Weinberg's statement was just a conjecture, it has been shown [26] that with an appropriately chosen Lagrangian, χPT is mathematically equivalent to the low-energy limit of the standard model of particle physics. This effective Lagrangian must contain all terms allowed by the symmetry of the fundamental theory for the given set of fields [24], in χPT these are the light mesons (ie. π^0,$\pi^\pm$,$K^0/\bar{K}^0$,$K^\pm$ and η). Although the number of these operators is infinite, they can be ordered in powers of momenta and one can isolate the more relevant terms from the less important ones. The drawback of an effective theory is that one gets more and more low-energy constants (LECs) at each order in this expansion.

In the following, we describe the construction of χPT from the QCD Lagrangian in the chiral limit[1]. The hierarchy of the quark masses suggests to separate them in a group of light quarks (u, d, s) and a group of heavy quarks (c, b, t). The hierarchy of the quark masses is shown in Figure 1. The masses of the heavy quarks and the light quarks are separated by more than an order of magnitude, therefore the mass terms of the light quarks in the QCD Lagrangian $\mathcal{L}_{QCD}$ [3] can be seen as a small perturbation,

$$\mathcal{L}_{QCD} \; = \; \mathcal{L}^0_{QCD} - \bar{q}\mathcal{M}_q q, \tag{1.1}$$

where $\mathcal{L}^0_{QCD}$ is the QCD Lagrangian in the chiral limit $(m_u = m_d = m_s = 0)$,

$$\begin{aligned} \mathcal{L}^0_{QCD} &= \bar{q}\left(\partial_\mu + ig_s\frac{\lambda_a}{2}G^a_\mu\right)q + \mathcal{L}_{\text{heavy quarks}} + \mathcal{L}_{\text{gluons}} \\ &= \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R + \mathcal{L}_{\text{heavy quarks}} + \mathcal{L}_{\text{gluons}}, \end{aligned} \tag{1.2}$$

with the light quark fields

$$q = (u, d, s)^T, \tag{1.3}$$

their left- and right-handed projections

$$q_{L,R} = \frac{1}{2}(1 \pm \gamma_5)q, \tag{1.4}$$

the quark mass matrix

$$\mathcal{M}_q = \text{diag}(m_u, m_d, m_s) \tag{1.5}$$

and the covariant derivative acting in colour space

$$D_\mu = \partial_\mu + ig_s\frac{\lambda_a}{2}G^a_\mu, \tag{1.6}$$

with the Gell-Mann matrices λ_a $(a = 1, \ldots, 8)$. The QCD Lagrangian in the chiral limit (1.2) is invariant under transformations of the chiral group

$$G = SU(3)_L \times SU(3)_R. \tag{1.7}$$

[1]The discussion in sections 1.1, 1.2, 1.3 and 1.5 follows the lines of the introductory paper [25].

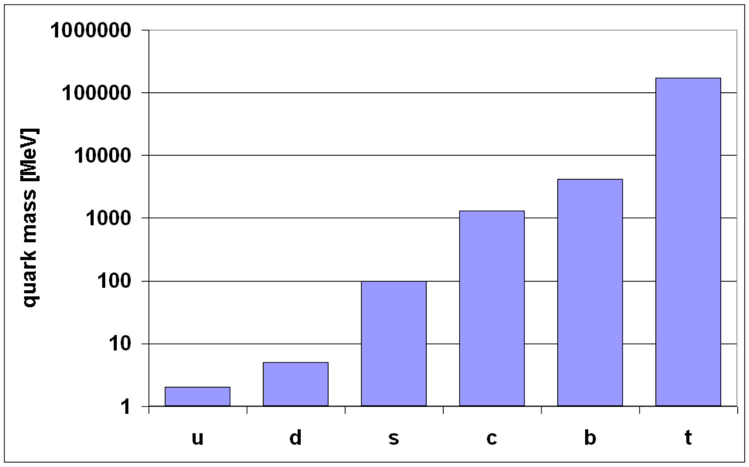

Figure 1: Hierarchy of the quark masses. The numerical values entering the diagram were taken from Amsler et al. [1]. For the light quarks (u, d, s) the values of the quark masses correspond to the scale $\mu = 2$ GeV.

The Noether currents associated with the chiral group G are

$$J^{a,\mu}_{R,L} = \bar{q}_{R,L}\gamma^\mu \frac{\lambda_a}{2} q_{R,L} \quad (a = 1, \ldots, 8), \tag{1.8}$$

where γ^μ denote the Dirac matrices, the corresponding Noether charges are

$$Q^a_{R,L} = \int d^3x \; J^{a,0}_{R,L}. \tag{1.9}$$

It is a well known fact that chiral symmetry is spontaneously broken down to $H = SU(3)_V$ through the non-vanishing vacuum expectation value

$$\langle 0 \left| \bar{q}q \right| 0 \rangle \neq 0, \tag{1.10}$$

the quark condensate. A few arguments for this can be found in [27].

According to Goldstone's theorem [11], as a consequence of a spontaneously broken (continuous) symmetry a set of massless particles enters a theory. Denoting the number of generators of the groups G and H by n_G and n_H, respectively, in the case of χPT this mechanism gives rise to $n = n_G - n_H = 8$ Goldstone bosons which transform as an octet under the subgroup H and can be identified with the lowest-lying pseudoscalar mesons π, K, η.

The Goldstone fields ϕ^a $(a = 1, \ldots, 8)$ parameterize the chiral coset space $SU(3)_L \times SU(3)_R / SU(3)_V$. G acts non-linearly on the ϕ^a, but in the case of chiral symmetry, the

Goldstone fields can be collected in a unitary matrix field $U(\phi)$ transforming as

$$U(\phi) \stackrel{G}{\rightarrow} g_R U(\phi) g_L^{-1}, \quad g_{R,L} \in SU(3)_{R,L}, \tag{1.11}$$

under chiral rotations. The group-theoretical foundations for a nonlinear realization of chiral symmetry were developed in [28–30]. There are different possible representations of $U(\phi)$ corresponding to different coordinates of the chiral coset space. In the original work Gasser and Leutwyler used the exponential parametrization [10]

$$U(\phi) = \exp\left(\frac{i\phi}{F_0}\right), \quad \phi = \sum_a \lambda_a \phi^a. \tag{1.12}$$

At this stage, F_0 is just an arbitrary constant (with dimension of energy), its physical meaning will become clear later. In this work we use a more general representation with the coset variables $u_{L,R}(\phi)$ transforming as [29, 30]

$$\begin{aligned} u_L(\phi) &\stackrel{G}{\rightarrow} g_L u_L(\phi) h(g,\phi)^{-1}, \\ u_R(\phi) &\stackrel{G}{\rightarrow} h(g,\phi) u_R(\phi) g_R^{-1}, \end{aligned} \tag{1.13}$$

where $h(g, \phi)$ is the nonlinear realization of G, and the parametrization

$$u_R(\phi) = u_L(\phi)^\dagger = u(\phi) = \exp\left(\frac{i\Phi}{\sqrt{2}F_0}\right), \tag{1.14}$$

where

$$\Phi = \sum_{a=1}^{8} \frac{\lambda_a \phi^a}{\sqrt{2}}. \tag{1.15}$$

The most general Lagrangian density one can construct containing all possible terms compatible with assumed symmetry principles will then describe the dynamics of these eight degrees of freedom resulting from the spontaneous symmetry breaking of the QCD Lagrangian density.

Of course, in reality there is no chiral symmetry in nature: Due to the non-vanishing quark masses $m_u, m_d, m_s \neq 0$, the chiral limit is only an approximate symmetry. As a consequence, the octet of Goldstone particles acquires mass [9, 10] (see section 1.4). The chiral expansion is not only an expansion in the momenta, but a simultaneous expansion in the momenta and the masses of the light quarks.

1.2 External fields and explicit symmetry breaking

To include terms that break the chiral symmetry explicitly, we follow Gasser and Leutwyler [9, 10] in extending the chiral invariant QCD Lagrangian (1.2) by coupling the quarks to external hermitian matrix fields – vectors v_μ, axial-vectors a_μ, scalars s and pseudoscalars p:

$$\mathcal{L} = \mathcal{L}^0_{QCD} + \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \bar{q}(s - ip\gamma_5)q, \tag{1.16}$$

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu. \tag{1.17}$$

The Lagrangian (1.16) exhibits a local $SU(3)_R \times SU(3)_L$ symmetry with the transformation properties [10]

$$\begin{aligned}
q &\stackrel{G}{\to} g_R\frac{1}{2}(1+\gamma_5)q + g_L\frac{1}{2}(1-\gamma_5)q, \\
r_\mu &\stackrel{G}{\to} g_R r_\mu g_R^\dagger + i g_R \partial_\mu g_R^{-1}, \\
l_\mu &\stackrel{G}{\to} g_L l_\mu g_L^\dagger + i g_L \partial_\mu g_L^{-1}, \\
s + ip &\stackrel{G}{\to} g_R(s+ip)g_L^{-1}, \\
g_{L,R} &\in SU(3)_{L,R}.
\end{aligned} \tag{1.18}$$

The effective Lagrangian of QCD including external fields must of course contain all terms with external fields allowed by the chiral symmetry, especially the lowest order term

$$\mathcal{L}_m = \frac{1}{2}F_0^2 B_0 \left\langle u_R^\dagger(s+ip)u_L + u_L^\dagger(s+ip)^\dagger u_R \right\rangle, \tag{1.19}$$

which provides a very convenient way of including explicit chiral symmetry breaking through the quark masses and therefore non-vanishing meson masses by setting

$$v_\mu = a_\mu = p = 0 \tag{1.20}$$

and

$$s = \mathcal{M}_q = \mathrm{diag}(m_u, m_d, m_s) \tag{1.21}$$

after constructing the most general Lagrangian invariant under chiral transformations including external fields.

1.3 The Chiral Lagrangian

The effective chiral lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \tag{1.22}$$

contains all terms allowed by the gauge group of the underlying theory, organized in powers of momenta and the masses of the light quarks. In the chiral limit, this Lagrangian is invariant under $SU(3)_L \times SU(3)_R$. It contains eight pseudoscalar degrees of freedom transforming as an octet under the subgroup $H = SU(3)_V$. The explicit form of $\mathcal{L}_2$ and $\mathcal{L}_4$ is given below. The Lagrangian $\mathcal{L}_6$ already has 94 independent terms, each coming with its own low-energy constant. A full listing can be found in [31].

In the chiral power counting scheme of χPT, the building blocks are counted as [25]:

$$\begin{aligned} u_{L,R}: &\quad \mathcal{O}(p^0), \\ \partial_\mu, v_\mu, a_\mu: &\quad \mathcal{O}(p), \\ s, p: &\quad \mathcal{O}(p^2). \end{aligned} \tag{1.23}$$

To lowest order in the chiral expansion, the effective Lagrangian in the chiral limit is given by [9, 10]

$$\mathcal{L}_2^{(0)} = \frac{F_0^2}{4} \langle u_\mu u^\mu \rangle \,, \tag{1.24}$$

where $\langle \ldots \rangle$ denotes the trace in three-dimensional flavour space and

$$U(\phi) := u_R(\phi) u_L(\phi)^\dagger = u(\phi)^2. \tag{1.25}$$

The vielbein field u_μ is the covariant derivative of the scalar field,

$$u_\mu = i \left[u_R^\dagger (\partial_\mu - i r_\mu) u_R - u_L^\dagger (\partial_\mu - i l_\mu) u_L \right] \tag{1.26}$$

and therefore also of $\mathcal{O}(p)$ in the chiral power counting scheme (1.23).

This Lagrangian exhibits an important feature of the Goldstone theorem: The Goldstone bosons (contained in the matrix field u_μ) have derivative couplings only – the interaction among them vanishes at zero momentum. Expanding the exponentials u_L, u_R in the first term of (1.24) and switching off the external sources results in

$$\mathcal{L}_2^{(0)} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a + \mathcal{L}_{int}. \tag{1.27}$$

Since there are no other terms containing only two fields ($\mathcal{L}_{int}$ starts with interaction terms containing at least four Goldstone bosons) the eight fields ϕ_a describe eight *massless* particles[2].

The pseudoscalar masses are introduced through explicit chiral symmetry breaking in χ_+ by substituting the external fields by the quark mass matrix,

$$\chi = 2B_0(s + ip) \rightarrow 2B_0 \mathcal{M}_q. \tag{1.28}$$

[2]At this stage, this is only a tree-level argument. We will see in section 3.1 that the Goldstone bosons remain massless in the chiral limit even when loop corrections have been included.

To lowest order in the chiral expansion, the effective Lagrangian is then given by [9, 10]

$$\mathcal{L}_2 = \frac{F_0^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \,, \tag{1.29}$$

where

$$\chi_+ = u_R^\dagger \chi u_L + u_L^\dagger \chi^\dagger u_R, \quad \chi = 2B_0(s + ip). \tag{1.30}$$

The pseudoscalar decay constants F_a are defined by

$$\langle 0| A_\mu^a(0) |\phi^a(p)\rangle = i\sqrt{2} p_\mu F_a, \tag{1.31}$$

where A_μ^a is the axial-current

$$A_\mu^a = \bar{q}\gamma^5\gamma^\mu \frac{\lambda_a}{2} q. \tag{1.32}$$

The effective Lagrangian of order p^2 contains two low-energy constants (LECs): F_0 is the pion decay constant in the chiral limit and in absence of electroweak interactions,

$$F_\pi \;=\; F_0 \left(1 + \mathcal{O}(m_q)\right) \;=\; 92.2 \pm 0.2 \text{ MeV}, \tag{1.33}$$

where the numerical value was taken from [32], and B_0 is related to the quark condensate in the chiral limit [10],

$$\langle 0| \bar{q}^i q^j |0\rangle \;=\; -F_0^2 B_0 \delta^{ij} \left(1 + \mathcal{O}(\mathcal{M}_q)\right). \tag{1.34}$$

For example, the $\bar{u}u$ component of the scalar quark condensate in the chiral limit, $\langle 0| \bar{u}u |0\rangle_0$, is given by

$$\begin{aligned} &\langle 0| \bar{u}u |0\rangle_0 = \\ &\frac{i}{2}\left[\sqrt{\frac{2}{3}}\frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_3(x)} + \frac{1}{\sqrt{3}}\frac{\delta}{\delta s_8(x)}\right] \exp(iZ[v,a,s,p])\Big|_{v=a=s=p=0}, \end{aligned} \tag{1.35}$$

where $Z[v, a, s, p]$ is the generating functional [10].

1.4 Masses of the light mesons I

The mass terms of the pseudoscalars are contained in

$$\begin{aligned} \mathcal{L}_m &= \frac{1}{2} F_0^2 B_0 \left\langle u_R^\dagger \mathcal{M}_q u_L + u_L^\dagger \mathcal{M}_q^\dagger u_R \right\rangle \\ &= \frac{1}{2} F_0^2 B_0 \left\langle \mathcal{M}_q U^\dagger + \mathcal{M}_q^\dagger U \right\rangle, \end{aligned} \tag{1.36}$$

with

$$\mathcal{M}_q = \mathrm{diag}(m_u, m_d, m_s). \tag{1.37}$$

Since $\mathcal{M}_q^\dagger = \mathcal{M}_q$, $\mathcal{L}_m$ contains only terms even in ϕ. The expansion in powers of the pseudoscalar fields ϕ yields the following expression for the quadratic terms:

$$\mathcal{L}_m = -\frac{1}{2} B_0 \cdot \langle \lambda_a \lambda_b \mathcal{M}_q \rangle \phi^a \phi^b + \dots \tag{1.38}$$

We get the result

$$\begin{aligned} \frac{1}{4}\left\langle \phi^2 \chi \right\rangle \quad = \quad & - B_0(m_u + m_d)\pi^+\pi^- - B_0(m_u + m_s)K^+K^- \\ & - B_0(m_d + m_s)K^0\bar{K}^0 - \frac{1}{\sqrt{3}} B_0(m_u - m_d)\pi^0\eta \\ & - B_0 \frac{m_u + m_d}{2}\pi^0\pi^0 - B_0 \frac{m_u + m_d + 4m_s}{6}\eta^2. \end{aligned} \tag{1.39}$$

From this expression we see that we have mixing in the neutral π^0/η-sector. However, in the isospin limit ($m_d = m_u$) the mixing vanishes and the mass eigenvalues are given by [10]

$$\begin{aligned} M^2_{\pi^\pm} &= M^2_{\pi^0} = B_0\left(m_u + m_d\right), \\ M^2_{K^\pm} &= B_0\left(m_u + m_s\right), \\ M^2_{\overset{(-)}{K}{}^0} &= B_0\left(m_d + m_s\right), \\ M^2_\eta &= \frac{B_0}{3}\left(m_u + m_d + 4m_s\right). \end{aligned} \tag{1.40}$$

Up to terms of $\mathcal{O}(\mathcal{M}_q^2)$ the pseudoscalar octet obeys the Gell-Mann-Okubo formula [33],

$$4M_K^2 = 3M_\eta^2 + M_\pi^2 + \mathcal{O}\left(\mathcal{M}_q^2\right). \tag{1.41}$$

The explicit expression of the meson field matrix in terms of the real fields ϕ_i and of the mass eigenstates in the isospin limit reads

$$\begin{aligned} \phi &= \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix} \\ &= \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \end{aligned} \tag{1.42}$$

Until now we neglected isospin breaking effects. For $m_u \neq m_d$ the states π^0 and η undergo mixing. The eigenstates described by the fields $\pi^0(x)$ and $\eta(x)$ – the diagonal elements of the ϕ matrix – are given by

$$\begin{aligned} & \lambda_3\phi_3(x) + \lambda_8\phi_8(x) \\ = & \left[\lambda_3 \cos\varepsilon^{(2)} + \lambda_8 \sin\varepsilon^{(2)}\right]\pi^0(x) + \left[-\lambda_3 \sin\varepsilon^{(2)} + \lambda_8 \cos\varepsilon^{(2)}\right]\eta(x). \end{aligned} \tag{1.43}$$

The π^0/η-mixing angle at $\mathcal{O}(p^2)$, $\varepsilon^{(2)}$, is determined by

$$\tan 2\varepsilon^{(2)} = \frac{\sqrt{3}}{2}\frac{m_d - m_u}{m_s - \widehat{m}}, \tag{1.44}$$

the symbol $\widehat{m}$ stands for the mean value of the light quark masses,

$$\widehat{m} = \frac{1}{2}\left(m_u + m_d\right). \tag{1.45}$$

Expanded in powers of $m_d - m_u$ this reads

$$\varepsilon^{(2)} = \frac{\sqrt{3}}{4}\frac{m_d - m_u}{m_s - \widehat{m}} + \mathcal{O}\left([m_d - m_u]^2\right). \tag{1.46}$$

Due to the π^0/η-mixing the mass of the neutral pion is pushed down slightly by

$$M_{\pi^0}^2 = M_{\pi^+}^2 - \frac{1}{4}\left(\frac{m_d - m_u}{m_s - \widehat{m}}\right)^2\left(M_K^2 - M_\pi^2\right). \tag{1.47}$$

While the pion mass difference is of order $(m_d - m_u)^2$, the kaon mass difference is not protected from isospin breaking, but is proportional to the first power of $m_d - m_u$.

1.5 The effective Lagrangian of order p^4 and loops

At order p^4, the most general Lagrangian is given by [10]

$$\begin{aligned} \mathcal{L}_4 = & L_1 \langle u_\mu u^\mu\rangle^2 + L_2 \langle u_\mu u^\nu\rangle \langle u^\mu u_\nu\rangle + L_3 \langle u_\mu u^\mu u_\nu u^\nu\rangle \\ & + L_4 \langle u_\mu u^\mu\rangle \langle\chi_+\rangle + L_5 \langle u_\mu u^\mu \chi_+\rangle + L_6 \langle\chi_+\rangle^2 + L_7 \langle\chi_-\rangle^2 \\ & + \frac{1}{4}(2L_8 + L_{12})\left\langle\chi_+^2\right\rangle + \frac{1}{4}(2L_8 - L_{12})\left\langle\chi_-^2\right\rangle - iL_9 \langle f_+^{\nu\mu} u_\mu u_\nu\rangle \\ & + \frac{1}{4}(L_{10} + 2L_{11})\langle f_{+\mu\nu} f_+^{\mu\nu}\rangle - \frac{1}{4}(L_{10} - 2L_{11})\langle f_{-\mu\nu} f_-^{\mu\nu}\rangle, \end{aligned} \tag{1.48}$$

where

$$\begin{aligned} \chi_- &= u_R^\dagger \chi u_L - u_L^\dagger \chi^\dagger u_R, \\ f_\pm^{\mu\nu} &= u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u, \\ F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i\left[l^\mu, l^\nu\right], \\ F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i\left[r^\mu, r^\nu\right]. \end{aligned} \tag{1.49}$$

While the terms with L_{11} and L_{12} in (1.48) are contact terms, i.e. they contain only external fields and are therefore of no physical relevance, the LECs $L_1, \ldots, L_{10}$ are not restricted by chiral symmetry. They are parameters containing information on the dynamics of the underlying fundamental theory, QCD. Although the number of arbitrary constants in $\mathcal{L}_4$ seems quite big, only a few of them contribute to a given observable. Their numerical values are extracted from experimental input, estimated with additional model dependent assumptions or obtained from lattice calculations. Numerical values of the LECs can be found in Table 1.

When calculating one-loop diagrams arising from vertices of $\mathcal{L}_2$, one encounters divergences which cannot be absorbed by a renormalization of the $\mathcal{O}(p^2)$ LECs F_0 and B_0 (as it would be the case in a renormalizable theory)[3]. According to Weinberg's power counting rules [24], the counterterms that cancel these infinities are of order p^4. Since dimensional regularization preserves the symmetries and the Lagrangian $\mathcal{L}_4$ already contains all allowed operators of this order, these divergences can be absorbed in a renormalization of the coupling constants L_i.

The twelve low-energy coupling constants $L_1, ..., L_{12}$ arising in (1.48) are divergent (except L_3 and L_7). They absorb the divergences of the one-loop graphs via the renormalization [10]

$$L_i = L_i^r(\mu) + \Gamma_i \Lambda(\mu), \tag{1.50}$$

$$\Lambda(\mu) = \frac{\mu^{D-4}}{(4\pi)^2}\left(\frac{1}{D-4} - \frac{1}{2}\left[\ln(4\pi) + \Gamma'(1) + 1\right]\right), \tag{1.51}$$

where $D = 4 - 2\varepsilon$ is the dimension of space-time, in the dimensional regularization scheme. The coefficients Γ_i are shown in Table 1. This is the crucial point about χPT (and effective field theories in general): The low-energy behavior of the observables is governed by the tree-contributions, the loop diagrams represent contributions of higher order in the chiral power counting scheme, i.e. in the momenta [9].

The scale dependence of the (measurable) renormalized LECs $L_i^r(\mu)$ follows directly from (1.50):

$$\begin{aligned} L_i^r(\mu_2) &= L_i^r(\mu_1) + \lim_{D\to 4} \frac{\Gamma_i}{(4\pi)^2} \frac{\mu_1^{D-4} - \mu_2^{D-4}}{D-4} \\ &= L_i^r(\mu_1) + \frac{\Gamma_i}{(4\pi)^2} \ln \frac{\mu_1}{\mu_2}. \end{aligned} \tag{1.52}$$

[3]In this thesis, we use dimensional regularization, since it preserves the symmetries of the Lagrangian.

i	$\mathcal{O}(p^4)$	$\mathcal{O}(p^6)$	Γ_i
1	0.7 ± 0.3	0.43 ± 0.12	$3/32$
2	1.3 ± 0.7	0.73 ± 0.12	$3/16$
3	-4.4 ± 2.5	-2.35 ± 0.37	0
4	-0.3 ± 0.5	$\equiv 0$	$1/8$
5	1.4 ± 0.5	0.97 ± 0.11	$3/8$
6	-0.2 ± 0.3	$\equiv 0$	$11/144$
7	-0.4 ± 0.2	-0.31 ± 0.14	0
8	0.9 ± 0.3	0.60 ± 0.18	$5/48$
9	6.9 ± 0.7		$1/4$
10	-5.5 ± 0.7		$-1/4$
11			$-1/8$
12			$5/24$

Table 1: Phenomenological values for the LECs $L_i^r(M_\rho)$ in units of 10^{-3}. The first column shows the original values of [10], the second column displays the values taken from fit 10 of [34], which we use for our calculations. The coefficients Γ_i in the third column are taken from [10].

This scale dependence is of course canceled by that of the loop amplitude in any measurable quantity. A short remark on higher orders: In the same sense as the counterterms that cancel the divergences of the one-loop diagrams arising from $\mathcal{L}_2$ are of order p^4 and have the structure of $\mathcal{L}_4$ [9, 10], the two-loop diagrams are of order p^6 and so on. The loop diagrams therefore do not modify the leading low energy behavior, but contribute to higher orders in the chiral expansion scheme.

1.6 The electroweak interaction in χPT

Apart from introducing mass terms for the pseudoscalars, the external field technology provides another important feature: It allows the systematic inclusion of the electroweak interaction in the framework of χPT.

Electroweak processes where photons A_μ and leptons ℓ, ν_ℓ ($\ell = e, \mu$) are present only as external legs can be treated within the framework of χPT by simply adding appropriate terms to the usual external vector and axial-vector sources v_μ, a_μ [35],

$$\begin{aligned} l_\mu &= v_\mu - a_\mu - eQ_L^{\mathrm{em}} A_\mu + \sum_\ell \left(\bar{\ell}\gamma_\mu \nu_{\ell L} Q_L^W + \bar{\nu_{\ell L}}\gamma_\mu \ell Q_L^{W\dagger} \right), \\ r_\mu &= v_\mu + a_\mu - eQ_L^{\mathrm{em}} a_\mu, \end{aligned} \tag{1.53}$$

with the electromagnetic coupling $e = \sqrt{4\pi\alpha}$, the quark charge matrix

$$Q_L^{\mathrm{em}} = Q_R^{\mathrm{em}} = Q^{\mathrm{em}} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \tag{1.54}$$

and

$$Q_L^W = -2\sqrt{2}G_F \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.55}$$

where G_F is the Fermi coupling constant and V_{ud}, V_{us} are Cabbibo-Kobayashi-Maskawa matrix elements.

If we want to calculate diagrams with virtual photons, we have to include the photon field as an additional dynamical degree of freedom by adding a kinetic term for the photon,

$$\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1.56}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength tensor, to the Lagrangian of the theory. With the substitution $v_\mu \to v_\mu - eQA_\mu$, χPT automatically generates all diagrams with virtual (and real) photons. However, loop diagrams with virtual photons will in general be divergent and therefore require appropriate counterterms.

The relevant chiral Lagrangian for virtual photons is, in addition to the replacements (1.53), given by the most general chiral invariant Lagrangian that is bilinear in the spurion fields $\mathcal{Q}_L(x), \mathcal{Q}_R(x)$ with the transformation properties [36]

$$\mathcal{Q}_{L,R} \xrightarrow{G} h(\phi) \mathcal{Q}_{L,R} h(\phi)^\dagger. \tag{1.57}$$

At leading order e^2p^0, the electromagnetic effective Lagrangian contains a single term [37],

$$\mathcal{L}_{e^2p^0} = F_0^4 e^2 Z \left\langle \mathcal{Q}_L \mathcal{Q}_R \right\rangle, \tag{1.58}$$

with a real and dimensionless coupling constant Z. After constructing the chiral invariant Lagrangian at order e^2p^0 one can express $\mathcal{Q}_L$ and $\mathcal{Q}_L$ through the new spurion fields Q_L

and Q_R transforming as [37]

$$Q_{L,R}(x) \stackrel{G}{\rightarrow} g_{L,R} Q_{L,R}(x) g_{L,R}^{-1}, \tag{1.59}$$

$$\mathcal{Q}_L^{\rm em} = u Q_L^{\rm em} u^\dagger, \quad \mathcal{Q}_R^{\rm em} = u^\dagger Q_R^{\rm em} u, \tag{1.60}$$

which can be identified with the quark charge matrix

$$Q_L^{\rm em} = Q_R^{\rm em} = Q^{\rm em}. \tag{1.61}$$

At next-to-leading order e^2p^2, one finds the following list of local counterterms [38]:

$$\begin{aligned}
\mathcal{L}_{e^2p^2} \quad = \quad & F_0^2 e^2 \Big(\frac{1}{2} K_1 \left\langle \mathcal{Q}_L^2 + \mathcal{Q}_R^2 \right\rangle \langle u_\mu u^\mu \rangle + K_2 \langle \mathcal{Q}_L \mathcal{Q}_R \rangle \langle u_\mu u^\mu \rangle \\
& - K_3 \left[\langle \mathcal{Q}_L u_\mu \rangle \langle \mathcal{Q}_L u^\mu \rangle + \langle \mathcal{Q}_R u_\mu \rangle \langle \mathcal{Q}_R u^\mu \rangle \right] \\
& + K_4 \langle \mathcal{Q}_L u_\mu \rangle \langle \mathcal{Q}_R u^\mu \rangle + K_5 \left\langle \left(\mathcal{Q}_L^2 + \mathcal{Q}_R^2 \right) u_\mu u^\mu \right\rangle \\
& + K_6 \langle \left(\mathcal{Q}_L \mathcal{Q}_R + \mathcal{Q}_R \mathcal{Q}_L \right) u_\mu u^\mu \rangle + \frac{1}{2} K_7 \left\langle \mathcal{Q}_L^2 + \mathcal{Q}_R^2 \right\rangle \langle \chi_+ \rangle \\
& + K_8 \langle \mathcal{Q}_L \mathcal{Q}_R \rangle \langle \chi_+ \rangle + K_9 \left\langle \left(\mathcal{Q}_L^2 + \mathcal{Q}_R^2 \right) \chi_+ \right\rangle \\
& + K_{10} \langle \left(\mathcal{Q}_L \mathcal{Q}_R + \mathcal{Q}_R \mathcal{Q}_L \right) \chi_+ \rangle - K_{11} \langle \left(\mathcal{Q}_L \mathcal{Q}_R - \mathcal{Q}_R \mathcal{Q}_L \right) \chi_- \rangle \\
& - i K_{12} \left\langle \left(\hat{\nabla}_\mu \mathcal{Q}_L \mathcal{Q}_L - \mathcal{Q}_L \hat{\nabla}_\mu \mathcal{Q}_L - \hat{\nabla}_\mu \mathcal{Q}_R \mathcal{Q}_R + \mathcal{Q}_R \hat{\nabla}_\mu \mathcal{Q}_R \right) u^\mu \right\rangle \\
& + K_{13} \left\langle \hat{\nabla}_\mu \mathcal{Q}_L \hat{\nabla}^\mu \mathcal{Q}_R \right\rangle + K_{14} \left\langle \hat{\nabla}_\mu \mathcal{Q}_L \hat{\nabla}^\mu \mathcal{Q}_L + \hat{\nabla}_\mu \mathcal{Q}_R \hat{\nabla}^\mu \mathcal{Q}_R \right\rangle \Big),
\end{aligned} \tag{1.62}$$

where

$$\begin{aligned}
\hat{\nabla}_\mu \mathcal{Q}_L \quad &= \quad \nabla_\mu \mathcal{Q}_L + \frac{i}{2} \left[u_\mu, \mathcal{Q}_L \right] \quad = \quad u D_\mu Q_L u^\dagger, \\
\hat{\nabla}_\mu \mathcal{Q}_R \quad &= \quad \nabla_\mu \mathcal{Q}_R - \frac{i}{2} \left[u_\mu, \mathcal{Q}_R \right] \quad = \quad u^\dagger D_\mu Q_R u.
\end{aligned} \tag{1.63}$$

The low-energy couplings $K_1, ..., K_{14}$ arising here are divergent (except K_7, K_{13} and K_{14}). The divergences of the one-loop graphs with a virtual photon or one vertex from $\mathcal{L}_{e^2p^0}$ are absorbed by an appropriate renormalization of the coupling constants in (1.62), in the dimensional regularization scheme this reads [38]:

$$K_i = K_i^r(\mu) + \Sigma_i \Lambda(\mu), \tag{1.64}$$

with $\Lambda(\mu)$ defined in (1.51). The coefficients Σ_i can be found in [38].

The renormalized electromagnetic low-energy constants $K_i^r(\mu)$ are measurable quantities, numerical results [39] are given in Table 2. The constants Σ_i govern the scale dependence of the $K_i^r(\mu)$,

$$K_i^r(\mu_2) = K_i^r(\mu_1) + \frac{\Sigma_i}{(4\pi)^2} \ln\left(\frac{\mu_1}{\mu_2}\right). \tag{1.65}$$

$10^3 K_1^r$	$10^3 K_2^r$	$10^3 K_3^r$	$10^3 K_4^r$	$10^3 K_5^r$	$10^3 K_6^r$
-2.71	0.69	2.71	1.38	11.59	2.77

Table 2: Numerical results obtained for $K_i^r(\mu)$ with $\mu = 0.77$ GeV taken from [39].

In any physical amplitude, the scale dependence always cancels between the loop and the counterterm contributions containing the renormalized coupling constants.

Finally, for the correct treatment of semileptonic processes, also virtual leptons and appropriate counterterms have to be taken into account. This framework was worked out in [35].

1.7 Masses of the light mesons II

With the framework described in section 1.6 we are in a position to calculate the contribution of the electromagnetic interaction to the meson masses. The masses of the charged mesons receive corrections from the effective Lagrangian $\mathcal{L}_{e^2p^0}$ (1.58) [37],

$$\begin{aligned} M^2_{\pi^\pm} &= B\,(m_u + m_d) + 2e^2 Z F_0^2, \\ M^2_{K^\pm} &= B\,(m_u + m_s) + 2e^2 Z F_0^2, \end{aligned} \tag{1.66}$$

while the (squared) masses of the neutral mesons $M^2_{\pi_0}$, $M^2_{K_0}$ and M^2_η stay unchanged. For later convenience we give the (lowest-order) expressions of the pseudoscalar masses in dependence of the isospin violating parameters $\varepsilon^{(2)}$ and e,

$$\begin{aligned} M^2_{\pi^\pm} &= 2B_0\widehat{m} + 2e^2 Z {F_0}^2, \\ M^2_{\pi^0} &= 2B_0\widehat{m}, \\ M^2_{K^\pm} &= B_0\left[(m_s + \widehat{m}) - \frac{2\varepsilon^{(2)}}{\sqrt{3}}(m_s - \widehat{m})\right] + 2e^2 Z {F_0}^2, \\ M^2_{\overset{(-)}{K}{}^0} &= B_0\left[(m_s + \widehat{m}) + \frac{2\varepsilon^{(2)}}{\sqrt{3}}(m_s - \widehat{m})\right], \\ M^2_\eta &= \frac{4}{3}B_0\left(m_s + \frac{\widehat{m}}{2}\right). \end{aligned} \tag{1.67}$$

The effective Lagrangian (1.58) does not contribute to the π^0/η-mixing angle. At leading order, the masses of the charged mesons receive the same contribution from the electromagnetic interaction (1.66). This is Dashen's theorem [40],

$$(\Delta_{K^0K^+} - \Delta_{\pi^+\pi^0})_{\text{EM}} = \mathcal{O}(e^2p^2). \tag{1.68}$$

The mass difference of the pions is dominated by (1.58) because the contributions of π^0/η-mixing are of order $(m_u - m_d)^2$. Neglecting this tiny quantity $(M^2_{\pi^+} - M^2_{\pi^0})_{QCD}$, the mass difference of the pions implies $Z \cong 0.8$.

For later convenience we note that with (1.67) one can easily express the pseudoscalar masses in the isospin limit through the physical ones,

$$\begin{aligned} M_\pi^2 &= M_{\pi^0}^2 = 2B_0\widehat{m}, \\ M_K^2 &= \frac{1}{2}\left(M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+} + M^2_{\pi^0}\right) = B_0\left(m_s + \widehat{m}\right). \end{aligned} \tag{1.69}$$

Within χPT, one cannot calculate the quark masses:

> "The quark masses depend on the QCD renormalization scale. Since the effective Lagrangians cannot depend on this scale, the quark masses always appear multiplied by quantities that transform contragrediently under changes of the renormalization scale. The chiral Lagrangian (1.22) contains the quark masses via the scalar field χ defined in (1.28). As long as one does not use direct or indirect information on B_0, one can only extract ratios of quark masses." [25]

The lowest-order mass formulas (1.69) together with Dashen's theorem (1.68) lead to the Weinberg rations [41]

$$\begin{aligned} \frac{m_u}{m_d} &= \frac{-M^2_{K^0} + M^2_{K^+} - M^2_{\pi^+} + 2M^2_{\pi^0}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}}, \\ \frac{m_s}{m_d} &= \frac{M^2_{K^0} + M^2_{K^+} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}}. \end{aligned} \tag{1.70}$$

With the numerical values for the meson masses given in [1], these formulas yield the quark-mass ratios

$$\frac{m_u}{m_d} = 0.56, \quad \frac{m_s}{m_d} = 20.2. \tag{1.71}$$

Part II

The $K_{\ell 3}$ scalar form factors in the standard model

2 Basics

Before we turn to the analysis of the $K_{\ell 3}$ form factors, we will briefly review the main features of $K_{\ell 3}$ decays, including the kinematics of the process and the experimental situation, which we will both need for the determination of the quantity $F_K/F_\pi f_+^{K^0\pi^-}(0)$, which is one of the basic input parameters in the subsequent analysis.

2.1 Structure of the invariant amplitude

The coupling of the W^+ vector boson to the fermions is the standard model coupling, the coupling of the pseudoscalar mesons to the W^+ is effectively taken into account (1.53).

The invariant amplitude of the $K_{\ell 3}$ decays ($\ell = e, \mu$)

$$\begin{aligned} K^+(p_K) &\rightarrow \pi^0(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu), &(2.1)\\ K^0(p_K) &\rightarrow \pi^-(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu) &(2.2)\end{aligned}$$

(and their charge conjugate modes) reads

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{us}^* \ell^\mu C_K \left[f_+^{K\pi}(t)(p_K + p_\pi)_\mu + f_-^{K\pi}(t)(p_K - p_\pi)_\mu \right], \tag{2.3}$$

where

$$\ell^\mu = \bar{u}(p_\nu)\gamma^\mu(1-\gamma_5)v(p_\ell) \tag{2.4}$$

denotes the weak leptonic current,

$$t = (p_K - p_\pi)^2 = (p_\ell + p_\nu)^2 \tag{2.5}$$

is the squared momentum transfer to the leptons and C_K is a Clebsch-Gordan coefficient,

$$C_K = \begin{cases} 1 & \text{for } K^0_{e3} \\ 1/\sqrt{2} & \text{for } K^+_{e3} \end{cases} . \tag{2.6}$$

The hadronic matrix element of the $K_{\ell 3}$ decays has the general form

$$\left\langle \pi^-(p_\pi) \right| \bar{u}\gamma_\mu s \left| K^0(p_K) \right\rangle = (p_\pi + p_K)_\mu f_+(t) + (p_\pi - p_K)_\mu f_-(t). \tag{2.7}$$

The currents entering in this formula are defined on the quark level. The connection to the effective theory is established by identifying these currents with the Noether currents of the chiral symmetry,

$$\bar{u}\gamma_\mu s = V_{\mu,4} - iV_{\mu,5}, \tag{2.8}$$

where $V_a^\mu = J_L^{\mu,a} + J_R^{\mu,a}$ $(a = 1, \ldots, 8)$ denotes the vector current in the effective theory.

Every diagram contributing to $K_{\ell 3}$ decay contains one vertex where the external W-boson couples to the mesons. The Feynman rules of the corresponding vertices result from the terms in $\mathcal{L}$ that are linear in the gauge fields. Thus, the left- and right-handed mesonic currents that couple to the external pseudo-scalar mesons are given by [42]

$$J^L_{\mu,a} = \left.\frac{\delta \mathcal{L}}{\delta l^{\mu,a}}\right|_{r_\mu = l_\mu = 0}, \quad J^R_{\mu,a} = \left.\frac{\delta \mathcal{L}}{\delta r^{\mu,a}}\right|_{r_\mu = l_\mu = 0}. \tag{2.9}$$

The $K_{\ell 3}$ decay rate is given by the frequently used formula [1]

$$\Gamma(K_{\ell 3(\gamma)}) = \frac{G_F^2 M_K^5 C_K^2}{192\pi^3} S_{\mathrm{EW}} |V_{us}|^2 \left| f_+^{K^0\pi^-}(0) \right|^2 I_K^\ell (1 + \delta_K^\ell + \delta_{\mathrm{SU}(2)}). \tag{2.10}$$

This formula contains both short-distance (S_{EW}) and long-distance (δ_K^ℓ) radiative corrections. The phase space integral I_K^ℓ is given by

$$I_K^\ell = \int_{\mathcal{D}_3} dy dz \rho(y, z), \tag{2.11}$$

where the integral extends on the physical domain $\mathcal{D}_3$ defining the three-body Dalitz plot (see [43] for the explicit definition). The spin-averaged decay distribution $\rho(y, z)$ depends on two (independent) kinematical variables. We follow the choice in [43],

$$y = \frac{2p_\pi \cdot p_K}{M_K^2} = \frac{2E_\pi}{M_K}, \quad z = \frac{2p_K \cdot p_\ell}{M_K^2} = \frac{2E_\ell}{M_K}, \tag{2.12}$$

where E_π (E_ℓ) is the pion (charged lepton) energy in the kaon rest frame. With this set of variables the distribution $\rho(y, z)$ reads [43]

$$\rho(y, z) = A_1(y, z) \left| f_+^{K\pi}(t) \right|^2 + A_2(y, z) f_+^{K\pi}(t) f_-^{K\pi}(t) + A_3(y, z) \left| f_-^{K\pi}(t) \right|^2, \tag{2.13}$$

where the kinematical densities are given by [44]

$$\begin{aligned}
A_1(y,z) &= 4(z+y-1)(1-y)+r_\ell(4y+3z-3)-4r_\pi+r_\ell(r_\pi-r_\ell), \\
A_2(y,z) &= 2r_\ell(3-2y-z+r_\ell-r_\pi), \\
A_3(y,z) &= r_\ell(1+r_\pi-z-r_\ell),
\end{aligned} \tag{2.14}$$

with the squared ratios $r_\ell = (m_\ell/M_K)^2$ and $r_\pi = (M_\pi/M_K)^2$. The form factor $f_+(t)$ is accessible in K_{e3} and $K_{\mu3}$ decays, while the form factor $f_-(t)$ is only accessible in $K_{\mu3}$ decays, because it is suppressed by the quantity m_ℓ^2/M_K^2, see (2.14).

The physical domain $\mathcal{D}$ is defined by [44]

$$\begin{aligned}
2\sqrt{r_\ell} &\leq y \leq 1+r_\ell-r_\pi, \\
a(y)-b(y) &\leq z \leq a(y)+b(y),
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
a(y) &= \frac{(2-y)\,(1+r_\ell+r_\pi-y)}{2(1+r_\ell-y)}, \\
b(y) &= \frac{\sqrt{y^2-4r_\ell}\,(1+r_\ell-r_\pi-y)}{2(1+r_\ell-y)},
\end{aligned} \tag{2.16}$$

or, equivalently, [44]

$$\begin{aligned}
2\sqrt{r_\pi} &\leq z \leq 1+r_\pi-r_\ell, \\
c(z)-d(z) &\leq y \leq c(z)+d(z),
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
c(z) &= \frac{(2-z)\,(1+r_\pi+r_\ell-z)}{2(1+r_\pi-z)}, \\
d(z) &= \frac{\sqrt{z^2-4r_\pi}\,(1+r_\pi-r_\ell-z)}{2(1+r_\pi-z)}.
\end{aligned} \tag{2.18}$$

The vector form factor $f_+^{K\pi}$ describes the P-wave projection of the crossed channel matrix element $\langle 0|\, V_\mu^{4-i5}(0)\,|K\pi\rangle$, while the scalar form factor

$$f_0^{K\pi}(t) = f_+^{K\pi}(t) + \frac{t}{\Delta_{K\pi}} f_-^{K\pi}(t) \tag{2.19}$$

describes the S-wave projection. It directly follows that

$$f_0^{K\pi}(0) = f_+^{K\pi}(0). \tag{2.20}$$

2.2 Experimental situation

One possibility for the parametrization of the form factors for the fit of the measured distribution of the $K_{\ell 3}$ decays is a Taylor expansion. Older measurments usually used the linear parametrization of the form factors [1]

$$f_{+,0}^{K\pi}(t) = f_{+}^{K\pi}(0)\left(1 + \lambda_{+,0}\frac{t}{M_{\pi^+}^2}\right) \tag{2.21}$$

for the fit. With the newer high-statistics measurements also the quadratic term in the expansion of the vector form factor [1]

$$f_{+}^{K\pi}(t) = f_{+}^{K\pi}(0)\left[1 + \lambda'_{+}\frac{t}{M_{\pi^+}^2} + \frac{1}{2}\lambda''_{+}\left(\frac{t}{M_{\pi^+}^2}\right)^2\right] \tag{2.22}$$

became accessible. The parameters describing higher order terms of the form factor expansion are in principle free to be determined from data. In practice, this additional freedom greatly complicates the use of such parameterizations. As noted in [45], if a quadratic parametrization is used for both the vector and scalar terms, fits to experimental data will provide no sensitivity to λ''_0 because of the strong parameter correlations, especially between λ'_0 and λ''_0 . For this reason, existing power-series fits use a parametrization in λ'_+, λ''_+ and λ_0.

Alternatively, also a pole fit,

$$f_{+}^{K\pi}(t) = f_{+}^{K\pi}(0)\frac{M_V^2}{M_V^2 - t}, \tag{2.23}$$

$$f_{0}^{K\pi}(t) = f_{+}^{K\pi}(0)\frac{M_S^2}{M_S^2 - t}, \tag{2.24}$$

has been employed. We will see that this parametrization assume additional physical constraints – to reduce the number of independent parameters – which are not fulfilled in the standard model.

Recently, a dispersive representation of the scalar form factor based on a twice subtracted dispersion relation was proposed [19–21]. We will return to this topic in section 4.4.

Recent high-statistics measurements of the $K_{\ell 3}$ form factor parameters λ'_+, λ''_+, λ_0 are available from ISTRA+ [5], KTeV [6], NA48 [7] and KLOE [8]. In particular for the scalar slope, the NA48 results are difficult to accommodate with these of the other experiments

ISTRA+ ($K^+_{\mu 3}$)	KTeV ($K_{L\mu 3}$)	KTeV ($K_{L\mu 3}+K_{Le3}$)
17.1 ± 2.2	12.8 ± 1.8	13.7 ± 1.3
NA48 ($K_{L\mu 3}$)	KLOE ($K_{L\mu 3}$)	KLOE ($K_{L\mu 3}+K_{Le3}$)
9.5 ± 1.4	9.1 ± 6.5	15.4 ± 2.2

Table 3: Experimental results for $\lambda_0^{K\pi} \times 10^3$

(The results are displayed in Table 3, where the ISTRA+ result has been rescaled by $M^2_{\pi^+}/M^2_{\pi^0}$). The actual value of that slope parameter is still unclear. We want to analyze the current situation from a phenomenological point of view - the two main questions we want to concentrate on in the following are

- which of the measured values of λ_0 are compatible with the standard model of particle physics, and
- which size of isospin violation is predicted by theory?

The natural framework of such analysis is χPT [Part 1], the low-energy effective theory of the standard model.

2.3 The determination of $F_K/F_\pi f_+^{K^0\pi^-}(0)$

From the theoretical point of view, the scalar $K_{\ell 3}$ form factor has a remarkable property: the low-energy theorem of Callan and Treiman [13] predicts the size of $f_0^{K\pi}(t)$ at the (unphysical) momentum transfer $t = \Delta_{K\pi}$ to be

$$f_0^{K\pi}(\Delta_{K\pi}) = \frac{F_K}{F_\pi} + \Delta_{CT}, \qquad \Delta_{CT} = \mathcal{O}(m_u, m_d). \tag{2.25}$$

In the isospin limit ($m_u = m_d,\ e = 0$) and at first non-leading order, Δ_{CT} was calculated already some time ago [46]:

$$\Delta_{CT} = -3.5 \times 10^{-3}. \tag{2.26}$$

Assuming for a moment a strict linear behavior of the scalar form factor in the range between $t = 0$ and the Callan-Treiman point $t = \Delta_{K\pi}$, the slope parameter would be given by [12]

$$\lambda_0 \simeq \frac{M_{\pi^+}^2}{\Delta_{K\pi}} \left(\frac{F_K}{F_\pi f_+^{K\pi}(0)} - 1 \right) \tag{2.27}$$

as a consequence of (2.25). The ratio $F_K/F_\pi f_+^{K\pi}(0)$ appearing in (2.27) can be determined with remarkable precision from the experimental input, independent of $K_{\mu 3}$ data.

Before we turn to the results for the vector and the scalar form factors, we demonstrate the determination of the quantity

$$\frac{F_K}{F_\pi f_+^{K^0\pi^-}(0)}, \tag{2.28}$$

which will be one of our main input parameters in our subsequent analysis. We want to point out that the decay constants used here always refer to the respective charged pseudoscalars ($F_\pi \equiv F_{\pi^+}$, $F_K \equiv F_{K^+}$). In the case of the pion, the distiction between charged and neutral decay constant amounts to a tiny effect of order $(m_d - m_u)^2$, whereas F_{K^+} differs from F_{K^0} by terms of order $m_d - m_u$ [10].

Including electromagnetic corrections [4, 35], the ratio of the (fully inclusive) $K_{\ell 2(\gamma)}$ and $\pi_{\ell 2(\gamma)}$ widths can be written as

$$\begin{aligned} \frac{\Gamma(K_{\ell 2(\gamma)})}{\Gamma(\pi_{\ell 2(\gamma)})} &= \frac{|V_{us}|^2 F_K^2 M_{K^\pm}(1 - z_{K\ell})^2}{|V_{ud}|^2 F_\pi^2 M_{\pi^\pm}(1 - z_{\pi\ell})^2} \\ &\times \left\{ 1 + \frac{\alpha}{4\pi} \left[H(z_{K\ell}) - H(z_{\pi\ell}) + (3 - Z) \ln \frac{M_K^2}{M_\pi^2} + \dots \right] \right\}, \end{aligned} \tag{2.29}$$

where $z_{P\ell} = m_\ell^2/M_P^2$. The kinematical function

$$\begin{aligned} H(z) &= \frac{23}{2} - \frac{3}{1-z} + 11 \ln z - \frac{2 \ln z}{1-z} - \frac{3 \ln z}{(1-z)^2} - 8 \ln(1-z) \\ &\quad - \frac{4(1+z)}{1-z} \ln z \ln(1-z) + \frac{8(1+z)}{1-z} \int_0^{1-z} dt\, \frac{\ln(1-t)}{t} \end{aligned} \tag{2.30}$$

ist taken from [35]. The chiral coupling [35] $Z \simeq 0.8$ arises from the electromagnetic mass difference of the pion,

$$M_{\pi^\pm}^2 - M_{\pi^0}^2 = 2e^2 Z F_0^2, \tag{2.31}$$

where F_0 denotes the pion decay constant in the chiral limit. The dots in (2.29) refer to contributions arising at $\mathcal{O}(e^2 p^4)$. Inserting the measured widths [1]

$$\Gamma(K_{\mu 2(\gamma)}) = 0.5122(15) \times 10^8\,\mathrm{s}^{-1}, \tag{2.32}$$

$$\Gamma(\pi_{\mu 2(\gamma)}) = 0.38408(7) \times 10^8\,\mathrm{s}^{-1}, \tag{2.33}$$

we find[4] [12]

$$\frac{|V_{us}|F_K}{|V_{ud}|F_\pi} = 0.27567(40)(2)(29) = 0.27567(50). \tag{2.34}$$

The first two separated errors correspond to the experimental uncertainties of the $K_{\mu 2(\gamma)}$ and $\pi_{\mu 2(\gamma)}$ width, respectively. The third one is an estimate[5] of the unknown electromagnetic contributions of $\mathcal{O}(e^2p^4)$. Using (2.34), the quantity (2.28) we are interested in, can be written as

$$\frac{F_K}{F_\pi f_+^{K^0\pi^-}(0)} = 0.27567(50) \times \frac{|V_{ud}|}{|V_{us}|f_+^{K^0\pi^-}(0)}. \tag{2.35}$$

For the determination of the product $|V_{us}|f_+^{K^0\pi^-}(0)$, we employ the master formula (2.10). For the short-distance enhancement factor S_{EW} we use the value $S_{EW}(M_\rho, M_Z) = 1.0232$ given in [4] including leading logarithmic and QCD corrections.

In order to avoid any bias from K_{e3}^+ (which would require additional theoretical input for the determination of $\delta_{\mathrm{SU}(2)}$) or $K_{\mu 3}$ data (involving also information about λ_0, the quantity we actually want to determine), we are exclusively using input from K_{Le3}^0 decays [6, 47] as given in [1]:

$$\Gamma(K^0_{Le3(\gamma)}) = 0.0792(4) \times 10^8\,\mathrm{s}^{-1}, \tag{2.36}$$

$$\lambda'_+ = 0.0249(13),\ \ \lambda''_+ = 0.0016(5),\ \ \rho_{\lambda',\lambda''} \simeq -0.95. \tag{2.37}$$

Taking into account the recently determined values [32] of the electromagnetic low energy couplings X_i [35], we obtain [12]

$$\delta^e_{K^0} = 0.0114(30) \tag{2.38}$$

as an update of the electromagnetic corrections presented in [48]. Putting everything together, we find [12]

$$|V_{us}|f_+^{K^0\pi^-}(0) = 0.21616(68). \tag{2.39}$$

With [49]

$$|V_{ud}| = 0.97418(26), \tag{2.40}$$

extracted from superallowed nuclear Fermi transitions, we finally obtain [12]

$$\frac{F_K}{F_\pi f_+^{K^0\pi^-}(0)} = 1.2424(23)(39)(3) = 1.2424(45), \tag{2.41}$$

where the first error comes from (2.34), the second one from (2.39) and the third one from (2.40)[6]. Note that the small difference between our number and the one obtained in [18] within a similar approach is due to the slightly different input parameters.

[4]With the new value [50] $\Gamma(K_{\mu 2(\gamma)}) = 0.5133(13) \times 10^8\,\mathrm{s}^{-1}$ we find $|V_{us}|F_K/|V_{ud}|F_\pi = 0.27597(45)$.

[5]See also [51] for a recent calculation of $\mathcal{O}(e^2p^4)$ contributions to the ratio $R^{\pi,K}_{e/\mu}$.

[6]Our update of (2.34) together with the recent value $|V_{ud}| = 0.97425(22)$ [52] yields the slightly different result $F_K/F_\pi f_+^{K\pi}(0) = 1.2438(20)(39)(3) = 1.2438(44)$.

Figure 2: Tree diagrams with vertices of $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$.

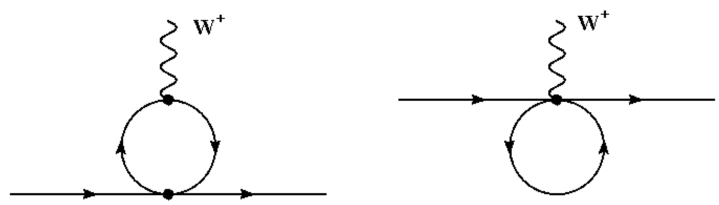

Figure 3: One-loop diagrams with vertices of $\mathcal{O}(p^2)$.

3 Analysis at NLO

The NLO amplitude of the $K_{\ell 3}$ decays consists of four types of Feynman diagrams:

(a) the tree diagram with wave function renormalization,

(b) the loop graph with a weak current and a purely mesonic vertex,

(c) the loop graph with a $W^+\phi^4$ vertex and

(d) a counterterm diagram from $\mathcal{L}_4$.

3.1 Mass and wave function renormalization

To do loop calculations, one has to renormalize the two-point function first. The order p^4 results for the wave function and the mass renormalization are well known [10]:

$$\begin{aligned}
\delta Z_\pi &= -\frac{1}{3F_0^2}\Big[A_0(M_K^2)+A_0(M_\pi^2) \\
&\quad +24L_4(2M_K^2+M_\pi^2)+24L_5M_\pi^2\Big]\,, \\
\delta Z_K &= -\frac{1}{4F_0^2}\Big[A_0(M_\eta^2)+2A_0(M_K^2)+A_0(M_\pi^2)
\end{aligned}$$

$$
\begin{aligned}
& +32L_4(2M_K^2+M_\pi^2)+32L_5M_K^2\Big], \\
\delta M_\pi^2 &= \frac{1}{6F^2}\Big[M_\pi^2 A_0(M_\eta^2)-3M_\pi^2A_0(M_\pi^2)-48L_4M_\pi^2(2M_K^2+M_\pi^2) \\
&\qquad -48L_5M_\pi^4+96L_6M_\pi^2(2M_K^2+M_\pi^2)+96L_8M_\pi^4\Big], \\
\delta M_K^2 &= \frac{1}{12F^2}\Big[-4M_K^2A_0(M_\eta^2)-96L_4M_K^2(2M_K^2+M_\pi^2) \\
&\qquad -96L_5M_K^4+192L_6M_K^2(2M_K^2+M_\pi^2)+192L_8M_K^4\Big]. \qquad (3.1)
\end{aligned}
$$

The function $A_0(m^2)$ is the standard tadpole integral

$$
A_0(m^2)=\mu^{4-D}\int\frac{d^Dk}{i(2\pi)^D}\,\frac{1}{k^2-m^2}, \qquad (3.2)
$$

where $D=4-2\varepsilon$ is the dimension of space-time.

One easily checks that the expressions of the masses are finite. The bare (infinite) coefficients L_i cancel the infinities resulting from the divergent loop integrals. As we had expected from QCD in the chiral limit, the masses of the Goldstone bosons vanish at $\mathcal{O}(p^4)$, if the quark masses are sent to zero.

Each external meson propagator in the tree diagram must be multiplied with a factor

$$
\sqrt{Z}=1+\frac{\delta Z}{2}. \qquad (3.3)
$$

3.2 The loop function $\bar{J}(t)$

In this section we define the function appearing in the loop integrals used in the text. We consider a loop with two propagators with different masses, M_P and M_Q. In the calculation of the $K_{\ell 3}$ form factors to order $p^4, (m_d-m_u)p^2, e^2p^2$ all needed functions can be given in terms of the subtracted scalar integral $\bar{J}(t)=J(t)-J(0)$. We define the loop function $J(t)$ by [10]

$$
J(t):=-i\int d^Dz\; e^{ipz}\Delta_P(z)\Delta_Q(z), \qquad (3.4)
$$

where $\Delta_P(z)$ is the Feynman propagator for a scalar field of mass M_P in D dimensions. In dimensional regularization ($D=4-2\varepsilon$), the loop function $J(t)$ reads

$$
J(t)=\pi^{D/2}(2\pi)^{-D}\Gamma(2-D/2)\int_0^1 dx\; g(x;t)^{(D-4)/2}, \qquad (3.5)
$$

with

$$g(x,t) = M_P^2(1-x) + M_Q^2 x - tx(1-x). \tag{3.6}$$

The quantity $\bar{J}(t)$ defined by

$$\bar{J}(t) := J(t) - J(0) \tag{3.7}$$

remains finite as $D \to 4$. Explicitly, the loop functions $\bar{J}_{PQ}(t)$ is given by [10]

$$\begin{aligned}
\bar{J}_{PQ}(t) &= -\frac{1}{16\pi^2}\int_0^1 \ln\frac{g(x,t)}{g(x,0)}dx \\
&= \frac{1}{32\pi^2}\Bigg[2 + \frac{\Delta_{PQ}}{t}\ln\frac{M_Q^2}{M_P^2} - \frac{\Sigma_{PQ}}{\Delta_{PQ}}\ln\frac{M_Q^2}{M_P^2} \\
&\qquad - \frac{\lambda^{1/2}(t,M_P^2,M_Q^2)}{t}\ln\left(\frac{[t+\lambda^{1/2}(t,M_P^2,M_Q^2)]^2 - \Delta_{PQ}^2}{[t-\lambda^{1/2}(t,M_P^2,M_Q^2)]^2 - \Delta_{PQ}^2}\right)\Bigg],
\end{aligned} \tag{3.8}$$

with

$$\Delta_{PQ} = M_P^2 - M_Q^2, \quad \Sigma_{PQ} = M_P^2 + M_Q^2 \tag{3.9}$$

and λ being the Källén-function

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz). \tag{3.10}$$

3.3 The $K_{\ell 3}$ form factors at NLO in the isospin limit

The f_+ form factor in the isospin limit was already calculated more than twenty years ago [10, 46]. The expressions for the individual diagrams are given by [42]

$$\begin{aligned}
\Delta^{(0)} f_+ &= 1 - \frac{2}{F_0^2}\left\{4L_4(M_\pi^2 + 2M_K^2) + 2L_5(M_\pi^2 + M_K^2)\right\} \\
&\quad - \frac{1}{24F_0^2}\left\{3A_0(M_\eta^2) + 10A_0(M_K^2) + 11A_0(M_\pi^2)\right\}, \\
\Delta^{(1a)} f_+ &= -\frac{3}{2F_0^2}\left\{B_{21}(q^2, M_\eta^2, M_K^2) + B_{21}(q^2, M_K^2, M_\pi^2)\right\}, \\
\Delta^{(1b)} f_+ &= \frac{1}{6F_0^2}\left\{3A_0(M_\eta^2) + 7A_0(M_K^2) + 5A_0(M_\pi^2)\right\}, \\
\Delta^{(1c)} f_+ &= \frac{2}{F_0^2}\left\{4L_4(M_\pi^2 + 2M_K^2) + 2L_5(M_\pi^2 + M_K^2) + q^2 L_9\right\},
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
B_{21}(q^2, m_1^2, m_2^2) &= \frac{1}{4q^2(1-D)}\Big\{(m_2^2 - m_1^2 - q^2)A_0(m_1^2) + (m_1^2 - m_2^2 - q^2)A_0(m_2^2) \\
&\qquad + \lambda(q^2, m_1^2, m_2^2)B_0(q^2, m_1^2, m_2^2)\Big\}
\end{aligned} \tag{3.12}$$

and

$$B_0(q^2, m_1^2, m_2^2) = \mu^{4-D} \int \frac{d^D k}{i(2\pi)^D} \, \frac{1}{[(k+q)^2 - m_1^2][k^2 - m_2^2]}. \tag{3.13}$$

In the final result the scale dependence of the low energy constant $L_9^r(\mu)$ is canceled by the chiral logs $A_P(\mu)$ and one gets [46]

$$\begin{aligned} f_+^{K\pi}(t) &= 1 + \frac{3}{2F_\pi^2} \left[h_{K\pi}(t,\mu) + h_{K\eta}(t,\mu) \right] + \frac{2}{F_\pi^2} t L_9^r(\mu) \\ &= 1 + \frac{3}{2} \left[H_{K\pi}(t) + H_{K\eta}(t) \right], \end{aligned} \tag{3.14}$$

with

$$\begin{aligned} h_{PQ}(t,\mu) &= \frac{1}{12t} \lambda(t, M_P^2, M_Q^2) \, \bar{J}_{PQ}(t) + \frac{1}{18(4\pi)^2} (t - 3\Sigma_{PQ}) \\ &\quad - \frac{1}{12} \left\{ \frac{2\Sigma_{PQ} - t}{\Delta_{PQ}} [A_P(\mu) - A_Q(\mu)] - 2[A_P(\mu) + A_Q(\mu)] \right\}, \end{aligned} \tag{3.15}$$

where

$$A_P(\mu) = -\frac{M_P^2}{(4\pi)^2} \ln \frac{M_P^2}{\mu^2} \tag{3.16}$$

and

$$H_{PQ}(t) = \frac{1}{F_0^2} \left[h_{PQ}^r(t,\mu) + \frac{2}{3} t L_9^r(\mu) \right]. \tag{3.17}$$

The analogous expression for the f_- form factor is given by

$$\begin{aligned} f_-^{K\pi} &= \frac{4\Delta_{KP}}{F_0^2} \left[L_5^r(\mu) - \frac{3}{256\pi^2} \ln \frac{M_\pi^2}{\mu^2} \right] \\ &\quad - \frac{1}{128\pi^2 F_0^2} \left[2M_K^2 \ln \frac{M_K^2}{M_\pi^2} + 3M_\eta^2 \ln \frac{M_\eta^2}{M_\pi^2} - 3M_\pi^2 \ln \frac{M_\eta^2}{M_\pi^2} \right] \\ &\quad + \frac{(5t^2 - 2t\Sigma_{K\pi} - 3\Delta_{K\pi}^2) \, K_{K\pi}(t)}{4F_0^2 t} \\ &\quad + \frac{(-3t^2 + 2t\Sigma_{K\pi} - \Delta_{K\pi}^2) \, K_{K\eta}(t)}{4F_0^2 t} \\ &\quad - \frac{3\Delta_{K\pi}}{2t} \left[H_{K\pi}(t) + H_{K\eta}(t) \right]. \end{aligned} \tag{3.18}$$

In the isospin conserving case the low-energy representation of the scalar form factor,

$$f_0^{K\pi}(t) := f_+^{K\pi}(t) + \frac{t}{\Delta_{K\pi}} f_-^{K\pi}(t), \tag{3.19}$$

is given by [46]

$$f_0(t) = 1 + \frac{1}{8F_0^2} \left(5t - 2\Sigma_{K\pi} - 3\frac{\Delta_{K\pi}^2}{t} \right) \bar{J}_{K\pi}(t)$$

$$+\frac{1}{24F_0^2}\left(3t-2\Sigma_{K\pi}-\frac{\Delta_{K\pi}^2}{t}\right)\bar{J}_{K\eta}(t)$$
$$+\frac{t}{\Delta_{K\pi}}\left(\frac{F_K}{F_\pi}-1\right), \tag{3.20}$$

where the dependence of the low energy constant L_5^r was expressed through the ratio [10]

$$\frac{F_K}{F_\pi}=1+\frac{1}{4}(5\mu_\pi-2\mu_K-3\mu_\eta)+\frac{4}{F_0^2}(M_K^2-M_\pi^2)L_5^r(\mu). \tag{3.21}$$

3.4 The f_+ form factors at order $p^4, (m_d-m_u)p^2, e^2p^2$

Now we include isospin breaking effects arising from strong and electromagnetic interaction. It is convenient to use the notation introduced in [43],

$$\begin{aligned} f_\pm^{K^+\pi^0} &= \widetilde{f}_\pm^{K^+\pi^0}+\widehat{f}_\pm^{K^+\pi^0}, \\ F_\pm^{K^0\pi^-} &= \widetilde{f}_\pm^{K^0\pi^-}+\widehat{f}_\pm^{K^0\pi^-}, \end{aligned} \tag{3.22}$$

where the first one represents the pure QCD contributions (in principle at any order in the chiral expansion) plus the electromagnetic contributions up to order e^2p^2 generated by the non-derivative Lagrangian

$$\mathcal{L}_{e^2p^0}=e^2F_0^4Z\langle\mathcal{Q}_L^{\rm EM}\mathcal{Q}_R^{\rm EM}\rangle. \tag{3.23}$$

Diagrammatically, they arise form purely mesonic graphs. In the definition of $\widetilde{f}_\pm^{K^+\pi^0}$ we have included also the electromagnetic counterterms relevant to π^0/η-mixing. The second term in (3.22) represents the local effects of virtual photon exchange of order e^2p^2. Using this convention, we have to perform the replacement

$$f_\pm^{K\pi}\rightarrow\widetilde{f}_\pm^{K\pi} \tag{3.24}$$

in the master formula (2.10).

The contributions of order $(m_d-m_u)p^2$ were already calculated in the Eighties by Gasser and Leutwyler [46], while these arising from the Lagrangian $\mathcal{L}_{e^2p^2}$ were calculated for the first time in [43]. The results are

$$\begin{aligned} \widetilde{f}_+^{K^+\pi^0}(t) &= 1+\sqrt{3}\Big(\varepsilon^{(2)}+\varepsilon_{\rm S}^{(4)}+\varepsilon_{\rm EM}^{(4)}\Big) \\ &\quad +\frac{1}{2}H_{K^+\pi^0}(t)+\frac{3}{2}H_{K^+\eta}(t)+H_{K^0\pi^-}(t) \\ &\quad +\sqrt{3}\varepsilon^{(2)}\Big[\frac{5}{2}H_{K\pi}(t)+\frac{1}{2}H_{K\eta}(t)\Big] \end{aligned} \tag{3.25}$$

for the K^+ decays and

$$\begin{aligned}\widetilde{f}_+^{K^0\pi^-}(t) &= 1 + \frac{1}{2}H_{K^+\pi^0}(t) + \frac{3}{2}H_{K^+\eta}(t) + H_{K^0\pi^-}(t) \\ &\quad + \sqrt{3}\varepsilon^{(2)}[H_{K\pi}(t) - H_{K\eta}(t)]\end{aligned} \tag{3.26}$$

for the K^0 decays.

The expression for $\widetilde{f}_+^{K^+\pi^0}(t)$ is more complicated because of π^0/η-mixing: The quantity $\varepsilon_{\rm S}^{(4)}$ is the strong contribution to the π^0/η-mixing angle arising at first nonleading order [46] and $\varepsilon_{\rm EM}^{(4)}$ is the corresponding term generated at $\mathcal{O}(e^2p^2)$ [36]. They are given by [43]

$$\begin{aligned}\varepsilon_{\rm S}^{(4)} &= -\frac{2\,\varepsilon^{(2)}}{3(4\pi F_0)^2(M_\eta^2 - M_\pi^2)} \\ &\times \Big\{(4\pi)^2\,64\,[3L_7 + L_8^r(\mu)]\,(M_K^2 - M_\pi^2)^2 \\ &\quad - M_\eta^2(M_K^2 - M_\pi^2)\log\frac{M_\eta^2}{\mu^2} + M_\pi^2(M_K^2 - 3M_\pi^2)\log\frac{M_\pi^2}{\mu^2} \\ &\quad -2M_K^2(M_K^2 - 2M_\pi^2)\log\frac{M_K^2}{\mu^2} - 2M_K^2(M_K^2 - M_\pi^2)\Big\}\end{aligned} \tag{3.27}$$

and [43]

$$\begin{aligned}\varepsilon_{\rm EM}^{(4)} &= \frac{2\sqrt{3}\,\alpha\,M_K^2}{108\,\pi\,(M_\eta^2 - M_\pi^2)} \\ &\times \Big\{2(4\pi)^2\Big[-6K_3^r(\mu) + 3K_4^r(\mu) + 2K_5^r(\mu) + 2K_6^r(\mu)\Big] \\ &\quad - 9Z\Big(\log\frac{M_K^2}{\mu^2} + 1\Big)\Big\}.\end{aligned} \tag{3.28}$$

For completeness we note that (3.25) and (3.26) imply the relation

$$\widetilde{f}_+^{K^+\pi^0}(0) = \widetilde{f}_+^{K^0\pi^-}(0)\Big[1 + \sqrt{3}\Big(\varepsilon^{(2)} + \varepsilon_{\rm S}^{(4)} + \varepsilon_{\rm EM}^{(4)}\Big)\Big], \tag{3.29}$$

which defines

$$\delta_{\rm SU(2)} = \begin{cases} 0 & \text{for } K_{\ell 3}^0 \\ 2\sqrt{3}\Big(\varepsilon^{(2)} + \varepsilon_{\rm S}^{(4)} + \varepsilon_{\rm EM}^{(4)}\Big) & \text{for } K_{\ell 3}^+ \end{cases} \tag{3.30}$$

to the order $(m_d - m_u)p^2$, e^2p^2.

3.5 The $f_-^{K\pi}$ form factors at order $p^4, (m_d - m_u)p^2, e^2p^2$

The analogous expressions for the f_- form factors are given by [43]

$$
\begin{aligned}
\tilde{f}_-^{K^+\pi^0}(t) &= \frac{4\Delta_{K\pi}}{F_0^2}\left(1+\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\left[L_5^r(\mu) - \frac{3}{256\pi^2}\ln\frac{M_{K^\pm}^2}{\mu^2}\right] \\
&- \frac{1}{128\pi^2F_0^2}\Big[(3+\sqrt{3}\varepsilon^{(2)})M_\eta^2\ln\frac{M_\eta^2}{M_{K^\pm}^2} + 2(3-\sqrt{3}\varepsilon^{(2)})M_{K^0}^2\ln\frac{M_{K^0}^2}{M_{K^\pm}^2} \\
&\qquad - 2(3-\sqrt{3}\varepsilon^{(2)})M_{\pi^\pm}^2\ln\frac{M_{\pi^\pm}^2}{M_{K^\pm}^2} + (1+3\sqrt{3}\varepsilon^{(2)})M_{\pi^0}^2\ln\frac{M_{\pi^0}^2}{M_{K^\pm}^2}\Big] \\
&+ \sum_{PQ}\left\{\left[a_{PQ}(t) + \frac{\Delta_{PQ}}{2t}b_{PQ}\right]K_{PQ}(t) + b_{PQ}\frac{F_0^2}{t}H_{PQ}(t)\right\}
\end{aligned}
\tag{3.31}
$$

and [43]

$$
\begin{aligned}
\tilde{f}_-^{K^0\pi^-}(t) &= \frac{4\Delta_{K\pi}}{F_0^2}\left(1+\frac{2\varepsilon^{(2)}}{\sqrt{3}}\right)\left[L_5^r(\mu) - \frac{3}{256\pi^2}\ln\frac{M_{\pi^\pm}^2}{\mu^2}\right] \\
&- \frac{1}{128\pi^2F_0^2}\Big[2M_{K^0}^2\ln\frac{M_{K^0}^2}{M_{\pi^\pm}^2} + (3+2\sqrt{3}\varepsilon^{(2)})M_\eta^2\ln\frac{M_\eta^2}{M_{\pi^\pm}^2} \\
&\qquad - (3+2\sqrt{3}\varepsilon^{(2)})M_{\pi^0}^2\ln\frac{M_{\pi^0}^2}{M_{\pi^\pm}^2}\Big] \\
&+ \sum_{PQ}\left\{\left[c_{PQ}(t) + \frac{\Delta_{PQ}}{2t}d_{PQ}\right]K_{PQ}(t) + d_{PQ}\frac{F_0^2}{t}H_{PQ}(t)\right\},
\end{aligned}
\tag{3.32}
$$

where the sum runs over all meson pairs in the loop diagrams ($K^+\pi^0, K^0\pi^+, K^+\eta$). The loop function $K_{PQ}(t)$ is defined by [10]

$$
K_\mu(p^2) := \frac{i}{2}\int d^Dz\, e^{-ipz}\left(\partial_\mu\Delta_P\Delta_Q - \Delta_P\partial_\mu\Delta_Q\right), \tag{3.33}
$$

$K_\mu(p^2) = p_\mu K(p^2)$, it remains finite as $D \to 4$ and reads

$$
K_{PQ}(t) = \frac{\Delta_{PQ}}{2t}\bar{J}_{PQ}(t). \tag{3.34}
$$

The coefficients $a_{PQ}(t), b_{PQ}, c_{PQ}(t)$ and d_{PQ} are given in [43] and are displayed in Appendix A.

Analogously to the isospin conserving case, we can trade in the the low-energy constant $L_5^r(\mu)$ for the ratio F_K/F_π. At one loop, the pion decay constant F_π, defined by

$$
\left\langle 0|A_\mu^3|\pi^0(p)\right\rangle = ip_\mu F_\pi, \quad A_\mu^a = \bar{q}\gamma_\mu\gamma_5\frac{1}{2}\lambda^a q, \tag{3.35}
$$

is given by the following (scale invariant) expression [10]

$$
\begin{aligned}
F_\pi &= F_0\Bigg\{1 + \frac{4}{{F_0}^2}\left[L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_\pi^2\right] \\
&\qquad - \frac{1}{2(4\pi)^2{F_0}^2}\left[2M_\pi^2 \ln\frac{M_\pi^2}{\mu^2} + M_K^2 \ln\frac{M_K^2}{\mu^2}\right]\Bigg\}.
\end{aligned}
\tag{3.36}
$$

Together with the decay constant of the charged kaons [35],

$$
\begin{aligned}
F_{K^\pm} &= F_0\Bigg\{1 + \frac{4}{{F_0}^2}\left[L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_K^2\right] \\
&- \frac{1}{8(4\pi)^2{F_0}^2}\left[3M_\pi^2 \ln\frac{M_\pi^2}{\mu^2} + 6M_K^2 \ln\frac{M_K^2}{\mu^2} + 3M_\eta^2 \ln\frac{M_\eta^2}{\mu^2}\right] \\
&- \frac{8\sqrt{3}\,\varepsilon^{(2)}}{3F_0^2}L_5^r(\mu)(M_K^2 - M_\pi^2) \\
&- \frac{\sqrt{3}\,\varepsilon^{(2)}}{4(4\pi)^2F_0^2}\left[M_\pi^2 \ln\frac{M_\pi^2}{\mu^2} - M_\eta^2 \ln\frac{M_\eta^2}{\mu^2} - \frac{2}{3}(M_K^2 - M_\pi^2)\left(\ln\frac{M_K^2}{\mu^2} + 1\right)\right]\Bigg\},
\end{aligned}
\tag{3.37}
$$

we can express the low-energy constant $L_5^r(\mu)$ in terms of the ratio F_K/F_π [35],

$$
\begin{aligned}
\frac{F_K}{F_\pi} &= 1 + \frac{4\Delta_{K\pi}}{F_0^2}L_5^r(\mu)\left(1 - \frac{2\varepsilon^{(2)}}{\sqrt{3}}\right) \\
&- \frac{1}{8(4\pi)^2{F_0}^2}\left[3M_\eta^2 \ln\frac{M_\eta^2}{\mu^2} + 2M_K^2 \ln\frac{M_K^2}{\mu^2} - 5M_\pi^2 \ln\frac{M_\pi^2}{\mu^2}\right] \\
&+ \frac{\sqrt{3}\varepsilon^{(2)}}{4(4\pi)^2F_0^2}\left[M_\eta^2 \ln\frac{M_\eta^2}{\mu^2} - M_\pi^2 \ln\frac{M_\pi^2}{\mu^2} + \frac{2}{3}\Delta_{K\pi}\left(\ln\frac{M_K^2}{\mu^2} + 1\right)\right].
\end{aligned}
\tag{3.38}
$$

Performing this replacement in our form factors we arrive at [12]

$$
\begin{aligned}
\widetilde{f}_-^{K^+\pi^0}(t) &= \left(\frac{F_K}{F_\pi} - 1\right)(1 + \sqrt{3}\varepsilon^{(2)}) \\
&- \frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\left(\Delta_{K\pi} - M_\pi^2 \ln\frac{M_K^2}{M_\pi^2}\right) + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\left(5 - 3\ln\frac{M_K^2}{M_\pi^2}\right) \\
&+ \sum_{PQ}\left\{\left[a_{PQ}(t) + \frac{\Delta_{PQ}}{2t}b_{PQ}\right]K_{PQ}(t) + b_{PQ}\,\frac{F_0^2}{t}H_{PQ}(t)\right\}
\end{aligned}
\tag{3.39}
$$

and [12]

$$
\begin{aligned}
\widetilde{f}_-^{K^0\pi^-}(t) &= \left(\frac{F_K}{F_\pi} - 1\right)\left(1 + \frac{4\varepsilon^{(2)}}{\sqrt{3}}\right) \\
&- \frac{\varepsilon^{(2)}}{\sqrt{3}(4\pi F_0)^2}\left(\Delta_{K\pi} - M_\pi^2 \ln\frac{M_K^2}{M_\pi^2}\right) + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2} \\
&+ \sum_{PQ}\left\{\left[c_{PQ}(t) + \frac{\Delta_{PQ}}{2t}d_{PQ}\right]K_{PQ}(t) + d_{PQ}\,\frac{F_0^2}{t}H_{PQ}(t)\right\}.
\end{aligned}
\tag{3.40}
$$

3.6 Scalar form factors at order $p^4, (m_d - m_u)p^2, e^2p^2$

In the scalar form factor

$$\widetilde{f}_0^{K^+\pi^0}(t) = \widetilde{f}_+^{K^+\pi^0}(t) + \frac{t}{\Delta_{K^+\pi^0}} \widetilde{f}_-^{K^+\pi^0}(t). \tag{3.41}$$

the loop functions $H_{PQ}(t)$ cancel because of the relation

$$\begin{aligned}
&\frac{1}{2} H_{K^+\pi^0}(t) + \frac{3}{2} H_{K^+\eta}(t) + H_{K^0\pi^-}(t) \\
&+\sqrt{3}\,\varepsilon^{(2)} \left[\frac{5}{2} H_{K\pi}(t) + \frac{1}{2} H_{K\eta}(t)\right] \\
&+ \sum_{PQ} b_{PQ} F_0^2 H_{PQ}(t)/\Delta_{K^+\pi^0} = 0.
\end{aligned} \tag{3.42}$$

Using (3.25) and (3.39) one obtains [12]

$$\begin{aligned}
\widetilde{f}_0^{K^+\pi^0}(t) \;=\; & \widetilde{f}_0^{K^+\pi^0}(0) + \frac{t}{\Delta_{K^+\pi^0}} \Bigg\{ \left(\frac{F_K}{F_\pi} - 1\right)(1 + \sqrt{3}\varepsilon^{(2)}) \\
& - \frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2} \left(\Delta_{K\pi} - M_\pi^2 \ln \frac{M_K^2}{M_\pi^2}\right) + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2} \left(5 - 3 \ln \frac{M_K^2}{M_\pi^2}\right) \\
& + \sum_{PQ} \left[\frac{1}{2} a_{PQ}(0) \Delta_{PQ} \bar{J}'_{PQ}(0) + \frac{1}{8} b_{PQ} \Delta_{PQ}^2 \bar{J}''_{PQ}(0)\right]\Bigg\} \\
& + \frac{1}{\Delta_{K^+\pi^0}} \sum_{PQ} \Bigg\{ \frac{1}{2} a'_{PQ}(0) \Delta_{PQ} t \bar{J}_{PQ}(t) \\
& + \frac{1}{2} a_{PQ}(0) \Delta_{PQ} [\bar{J}_{PQ}(t) - t \bar{J}'_{PQ}(0)] \\
& + \frac{1}{4} b_{PQ} \Delta_{PQ}^2 \frac{\bar{J}_{PQ}(t) - t \bar{J}'_{PQ}(0) - t^2 \bar{J}''_{PQ}(0)/2}{t} \Bigg\}.
\end{aligned} \tag{3.43}$$

From the terms linear in t one can directly read off the expression for the slope parameter.

Analogously the scalar form factor of the $K_{\ell 3}^0$ decay is given by

$$\widetilde{f}_0^{K^0\pi^-}(t) = \widetilde{f}_+^{K^0\pi^-}(t) + \frac{t}{\Delta_{K^0\pi^-}} \widetilde{f}_-^{K^0\pi^-}(t). \tag{3.44}$$

Again the loop functions $H_{PQ}(t)$ cancel because of the relation

$$\begin{aligned}
&\frac{1}{2} H_{K^+\pi^0}(t) + \frac{3}{2} H_{K^+\eta}(t) + H_{K^0\pi^-}(t) \\
&+ \sqrt{3}\,\varepsilon^{(2)} [H_{K\pi}(t) - H_{K\eta}(t)] \\
&+ \sum_{PQ} d_{PQ} F_0^2 H_{PQ}(t)/\Delta_{K^0\pi^-} = 0.
\end{aligned} \tag{3.45}$$

Inserting (3.26) and (3.40) one obtains [12]

$$
\begin{aligned}
\widetilde{f}_0^{K^0\pi^-}(t) &= \widetilde{f}_0^{K^0\pi^-}(0) \\
&+ \frac{t}{\Delta_{K^0\pi^-}}\Bigg\{\Big(\frac{F_K}{F_\pi}-1\Big)\Big(1+\frac{4\varepsilon^{(2)}}{\sqrt{3}}\Big) \\
&\qquad - \frac{\varepsilon^{(2)}}{\sqrt{3}(4\pi F_0)^2}\left(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right)+\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2} \\
&\qquad + \sum_{PQ}\left[\frac{1}{2}c_{PQ}(0)\Delta_{PQ}\bar{J}'_{PQ}(0)+\frac{1}{8}d_{PQ}\Delta_{PQ}^2\bar{J}''_{PQ}(0)\right]\Bigg\} \\
&+ \frac{1}{\Delta_{K^0\pi^-}}\sum_{PQ}\Bigg\{\frac{1}{2}c'_{PQ}(0)\Delta_{PQ}t\bar{J}_{PQ}(t) \\
&\qquad + \frac{1}{2}c_{PQ}(0)\Delta_{PQ}[\bar{J}_{PQ}(t)-t\bar{J}'_{PQ}(0)] \\
&\qquad + \frac{1}{4}d_{PQ}\Delta_{PQ}^2\frac{\bar{J}_{PQ}(t)-t\bar{J}'_{PQ}(0)-t^2\bar{J}''_{PQ}(0)/2}{t}\Bigg\}. \qquad (3.46)
\end{aligned}
$$

The values of the derivatives of the loop function $\bar{J}(0)$ at $s=0$ are easily obtained from the integral representation (3.5) [10],

$$
\begin{aligned}
\bar{J}'_{PQ}(0) &= \frac{1}{32\pi^2}\left(\frac{\Sigma}{\Delta^2}+2\frac{M_P^2M_Q^2}{\Delta^3}\ln\frac{M_Q^2}{M_P^2}\right), \\
\bar{J}''_{PQ}(0) &= \frac{1}{32\pi^2}\left(\frac{2}{3\Delta^4}(3\Sigma^2-2\Delta^2)+4\frac{M_P^2M_Q^2}{\Delta^5}\Sigma\ln\frac{M_Q^2}{M_P^2}\right). \qquad (3.47)
\end{aligned}
$$

3.7 Slope parameters

In the following section we turn to the slope parameters of the scalar form factors, the quantities we are actually interested in. For the slope parameter of the $K_{\ell 3}^+$ decays,

$$
\lambda_0^{K^+\pi^0} := \frac{M_{\pi^+}^2}{\widetilde{f}_+^{K^+\pi^0}(0)}\left.\frac{d\widetilde{f}_0^{K^+\pi^0}(t)}{dt}\right|_{t=0}, \qquad (3.48)
$$

we use (3.29) and obtain the result [12]

$$
\begin{aligned}
\lambda_0^{K^+\pi^0} &= \frac{M_{\pi^+}^2}{\Delta_{K^+\pi^0}}\Bigg\{\frac{F_K}{F_\pi\widetilde{f}_+^{K^0\pi^-}(0)}-\frac{1}{\widetilde{f}_+^{K^0\pi^-}(0)} \\
&\qquad - \frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\Big(\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\Big) \qquad (3.49) \\
&\qquad + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\Big(5-3\ln\frac{M_K^2}{M_\pi^2}\Big)+(1-\sqrt{3}\varepsilon^{(2)})
\end{aligned}
$$

$$\times \sum_{PQ}\Big[\frac{1}{2}a_{PQ}(0)\Delta_{PQ}\bar{J}'_{PQ}(0)+\frac{1}{8}b_{PQ}\Delta^2_{PQ}\bar{J}''_{PQ}(0)\Big]\bigg\}$$

and for the slope parameter of the $K^0_{\ell 3}$ decays,

$$\lambda_0^{K^0\pi^-} := \frac{M^2_{\pi^+}}{\widetilde{f}_+^{K^0\pi^-}(0)}\,\frac{d\widetilde{f}_0^{K^0\pi^-}(t)}{dt}\bigg|_{t=0}, \tag{3.50}$$

we obtain the result [12]

$$\begin{aligned}\lambda_0^{K^0\pi^-} \quad = \quad & \frac{M^2_{\pi^+}}{\Delta_{K^0\pi^-}}\bigg\{\Big(\frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)}-\frac{1}{\widetilde{f}_+^{K^0\pi^-}(0)}\Big)\Big(1+\frac{4\varepsilon^{(2)}}{\sqrt{3}}\Big) \\ & -\frac{\varepsilon^{(2)}/\sqrt{3}}{(4\pi F_0)^2}\Big[\Delta_{K\pi}-M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\Big]+\frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2} \\ & +\sum_{PQ}\Big[\frac{1}{2}c_{PQ}(0)\Delta_{PQ}\bar{J}'_{PQ}(0)+\frac{1}{8}d_{PQ}\Delta^2_{PQ}\bar{J}''_{PQ}(0)\Big]\bigg\}.\end{aligned} \tag{3.51}$$

3.8 Size of isospin breaking

The size of strong isospin violation is determined by the π^0/η-mixing angle $\varepsilon^{(2)}$ defined in (1.46) or, equivalently, by the ratio of quark mass differences

$$R := \frac{m_s-\widehat{m}}{m_d-m_u}. \tag{3.52}$$

Up to corrections of order m_q^2, the double ratio

$$Q^2 := \frac{m_s^2-\widehat{m}^2}{m_d^2-m_u^2} = R\frac{m_s/\widehat{m}+1}{2} \tag{3.53}$$

is given by meson masses and a purely electromagnetic contribution [10]:

$$Q^2 = \frac{\Delta_{K\pi}M_K^2[1+\mathcal{O}(m_q^2)]}{M_\pi^2[\Delta_{K^0K^+}+\Delta_{\pi^+\pi^0}-(\Delta_{K^0K^+}+\Delta_{\pi^+\pi^0})_{\rm EM}]}. \tag{3.54}$$

As a consequence of Dashen's theorem [40], the electromagnetic term vanishes at lowest order e^2p^0. It can be expressed through chiral logarithms and a certain combination of electromagnetic couplings of $\mathcal{L}_{e^2p^0}$ and $\mathcal{L}_{e^2p^2}$ [36, 38]:

$$\begin{aligned}(\Delta_{K^0K^+}+\Delta_{\pi^+\pi^0})_{\rm EM} \quad = \quad & e^2M_K^2\bigg[\frac{1}{4\pi^2}\Big(3\ln\frac{M_K^2}{\mu^2}-4+2\ln\frac{M_K^2}{\mu^2}\Big) \\ & +\frac{4}{3}(K_5+K_6)^r(\mu)-8(K_{10}+K_{11})^r(\mu) \\ & +16ZL_5^r(\mu)\bigg]+\mathcal{O}(e^2M_\pi^2).\end{aligned} \tag{3.55}$$

The numerical values of the electromagnetic coupling constants appearing in this expression have been determined by several authors [39, 53, 54]. Here we are using the most recent result by Ananthanarayan and Moussallam [39]. They obtain a rather large deviation from Dashen's limit,

$$(\Delta_{K^0K^+} + \Delta_{\pi^+\pi^0})_{\mathrm{EM}} = -1.5\,\Delta_{\pi^+\pi^0}, \tag{3.56}$$

which corresponds to [12]

$$Q = 20.7 \pm 1.2, \tag{3.57}$$

where we have added a rather generous error to account for higher order corrections.

For the determination of

$$R = \frac{2Q^2}{m_s/\widehat{m} + 1} \tag{3.58}$$

we also need information about the quark mass ratio $m_s/\widehat{m}$ as our second input parameter. Employing different methods [55], typical values around $m_s/\widehat{m} \sim 24$ have been obtained in the literature. We want to corroborate this size of the quark mass ratio by a numerical update of the determination of $m_s/\widehat{m}$ with a method proposed by Leutwyler [56] using the decay widths of $\eta \to \gamma\gamma$ and $\eta' \to \gamma\gamma$. Defining the parameters c_η and $c_{\eta'}$ by [56]

$$\Gamma(P \to \gamma\gamma) = \frac{\alpha^2 M_P^3}{64\pi^3 F_\pi^2} c_P^2, \tag{3.59}$$

the experimental values for the decay widths given in [1] correspond to $c_\eta = 0.991 \pm 0.025$ and $c_{\eta'} = 1.245 \pm 0.022$. The quark mass ratio can be obtained from the system of equations [56] (see also [57, 58])

$$F_\eta^8 c_\eta + F_{\eta'}^8 c_{\eta'} = \frac{F_\pi}{\sqrt{3}}, \tag{3.60}$$

$$(F_\eta^8)^2 + (F_{\eta'}^8)^2 = \frac{4F_K^2 - F_\pi^2}{3}, \tag{3.61}$$

$$(F_\eta^8)^2 M_\eta^2 + (F_{\eta'}^8)^2 M_{\eta'}^2 = \frac{8F_K^2 M_K^2 m_s/\widehat{m}}{3(m_s/\widehat{m}+1)} - \frac{F_\pi^2 M_\pi^2 (2m_s/\widehat{m} - 1)}{3}. \tag{3.62}$$

Eq.(3.61) can be written in the form [56]

$$F_\eta^8 = F_8 \cos\vartheta_8, \quad F_{\eta'}^8 = F_8 \sin\vartheta_8 \tag{3.63}$$

with

$$(F_8)^2 = \frac{4F_K^2 - F_\pi^2}{3}. \tag{3.64}$$

Using (3.60), the observed values of c_η and $c_{\eta'}$ require $\vartheta_8 = -22.0°$. Inserting this in (3.62) yields the quark mass ratio [12]

$$\frac{m_s}{\widehat{m}} = 24.7 \pm 1.0 \pm 0.3 \pm 0.1 = 24.7 \pm 1.1, \tag{3.65}$$

where the errors refer to the uncertainties of $\Gamma(\eta \to \gamma\gamma)$, $\Gamma(\eta' \to \gamma\gamma)$ and F_K/F_π. This value is perfectly consistent with $m_s/\widehat{m} = 24.4 \pm 1.5$ obtained in [55] based on different arguments.

Combining (3.57) and (3.65), the relation (3.58) finally gives [12]

$$R = 33.5 \pm 4.0 \pm 1.5 = 33.5 \pm 4.3. \tag{3.66}$$

A value for R of this size has been suggested in [34]. Note however that a recent analysis of $\eta \to 3\pi$ at the two-loop level [59] favours the values $R = 42.2$ and $Q = 23.2$. A review of recent lattice results gives the values $R = 37.2 \pm 4.1$ and $Q = 23.1 \pm 1.5$. The result (3.66) corresponds to [12]

$$\varepsilon^{(2)} = (1.29 \pm 0.17) \times 10^{-2} \tag{3.67}$$

and will be used in our subsequent numerical analysis. We also note that (3.67) leads to the numerical value [12]

$$\delta_{\mathrm{SU}(2)} = 0.058(8) \tag{3.68}$$

for the parameter (3.30) in $K_{\ell 3}$ decays.

3.9 Numerics at order $p^4, (m_d - m_u)p^2, e^2p^2$

For our subsequent numerical evaluations we use the PDG08 values [1] for $M_{\pi^\pm}$, M_{π^0}, $M_{K^\pm}$ and M_{K^0}. Since we have used the Gell-Mann-Okubo formula [33]

$$3\Delta_{\eta K} = \Delta_{K\pi} \tag{3.69}$$

in our previous calculations of the form factors and slope parameters, it is the only unambiguous choice at the considered chiral order to use it also to obtain a numerical value of M_η.

Plugging all our numerical input parameters in (3.49) and (3.51), we arrive at the following results [12]

$$\lambda_0^{K^0\pi^-} = (\underbrace{16.64}_{m_u=m_d} + \underbrace{0.17}_{m_u\neq m_d} + \underbrace{0.14}_{\mathrm{EM}}) \times 10^{-3}$$

$$= \quad 16.95(40)(5) \times 10^{-3}, \tag{3.70}$$

$$\lambda_0^{K^+\pi^0} = (\underbrace{16.64}_{m_u=m_d} - \underbrace{0.12}_{m_u\neq m_d} - \underbrace{0.08}_{\mathrm{EM}}) \times 10^{-3}$$

$$= \quad 16.44(39)(4) \times 10^{-3}, \tag{3.71}$$

where the contributions of strong isospin violation and of the electromagnetic interaction are given separately. The latter two pieces turn out to be of the same size. In the total results, the first error refers to (2.41) and the second one to (3.67). Both sources of isospin violation generate only tiny shifts with respect to the result in the isospin limit, with a splitting of the two slope parameters given by [12]

$$\Delta\lambda_0 := \lambda_0^{K^0\pi^-} - \lambda_0^{K^+\pi^0} = (5.1 \pm 0.9) \times 10^{-4}. \tag{3.72}$$

3.10 Callan-Treiman relations at the NLO

For the investigation of the Callan-Treiman relations in the presence of isospin breaking effects, it is convenient to consider the ratios

$$\frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^+\pi^0}(0)}, \qquad \frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)}. \tag{3.73}$$

In the case of $K_{\ell 3}^+$ decays, we find [12]

$$\begin{aligned}
\frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^+\pi^0}(0)} \quad = \quad & \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} - \frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\left(\Delta_{K\pi} - M_\pi^2 \ln \frac{M_K^2}{M_\pi^2}\right) \\
& + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\left(5 - 3\ln\frac{M_K^2}{M_\pi^2}\right) \\
& + (1+\sqrt{3}\varepsilon^{(2)}) \sum_{PQ}\left[a_{PQ}(\Delta_{K^+\pi^0}) + \frac{\Delta_{PQ} b_{PQ}}{2\Delta_{K^+\pi^0}}\right] K_{PQ}(\Delta_{K^+\pi^0}).
\end{aligned} \tag{3.74}$$

A further evaluation of the coefficients $a_{PQ}(\Delta_{K^+\pi^0})$, b_{PQ} and of $K_{PQ}(\Delta_{K^+\pi^0})$ leads to the alternative form [12]

$$\begin{aligned}
\frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^+\pi^0}(0)} \quad = \quad & \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} - \frac{\sqrt{3}\varepsilon^{(2)}}{(4\pi F_0)^2}\left(\Delta_{K\pi} - M_\pi^2 \ln \frac{M_K^2}{M_\pi^2}\right) \\
& + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2}\left(5 - 3\ln\frac{M_K^2}{M_\pi^2}\right) \\
& + \frac{M_\pi^2}{2F_0^2}\left(1 + \frac{12\varepsilon^{(2)}}{\sqrt{3}} - \frac{4\varepsilon^{(2)} M_K^2}{\sqrt{3} M_\pi^2}\right) \bar{J}_{K^+\pi^0}(\Delta_{K^+\pi^0})
\end{aligned}$$

$$
\begin{aligned}
&- \frac{M_\pi^2}{F_0^2}\left(1+\frac{2\varepsilon^{(2)}}{\sqrt{3}}+\frac{4\varepsilon^{(2)}M_K^2}{\sqrt{3}M_\pi^2}-\frac{2\Delta_{\pi^\pm\pi^0}}{\Delta_{K\pi}}\right)\bar{J}_{K^0\pi^-}(\Delta_{K^+\pi^0}) \\
&- \frac{M_\pi^2}{6F_0^2}\left(1+\frac{8\varepsilon^{(2)}}{\sqrt{3}}+\frac{4\varepsilon^{(2)}M_K^2}{\sqrt{3}M_\pi^2}-\frac{4\Delta_{\pi^\pm\pi^0}}{\Delta_{K\pi}}\right)\bar{J}_{K^+\eta}(\Delta_{K^+\pi^0}).
\end{aligned}
\tag{3.75}
$$

The analogous formula in the case of $K^0_{\ell 3}$ decays is given by [12]

$$
\begin{aligned}
\frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)} \quad = \quad & \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} + \frac{4\varepsilon^{(2)}}{\sqrt{3}}\left(\frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} - \frac{1}{\widetilde{f}_+^{K^0\pi^-}(0)}\right) \\
& - \frac{\varepsilon^{(2)}}{\sqrt{3}(4\pi F_0)^2}\left(\Delta_{K\pi} - M_\pi^2 \ln\frac{M_K^2}{M_\pi^2}\right) + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2} \\
& + \sum_{PQ}\left[c_{PQ}(\Delta_{K^0\pi^-}) + \frac{\Delta_{PQ}d_{PQ}}{2\Delta_{K^0\pi^-}}\right]K_{PQ}(\Delta_{K^0\pi^-}),
\end{aligned}
\tag{3.76}
$$

after inserting $c_{PQ}(\Delta_{K^0\pi^-})$, b_{PQ} and $K_{PQ}(\Delta_{K^0\pi^-})$ we arrive at [12]

$$
\begin{aligned}
\frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)} \quad = \quad & \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} + \frac{4\varepsilon^{(2)}}{\sqrt{3}}\left(\frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} - \frac{1}{\widetilde{f}_+^{K^0\pi^-}(0)}\right) \\
& - \frac{\varepsilon^{(2)}}{\sqrt{3}(4\pi F_0)^2}\left(\Delta_{K\pi} - M_\pi^2 \ln\frac{M_K^2}{M_\pi^2}\right) + \frac{\Delta_{\pi^\pm\pi^0}}{4(4\pi F_0)^2} \\
& - \frac{M_\pi^2}{2F_0^2}\left(1 - \frac{2\varepsilon^{(2)}}{\sqrt{3}} + \frac{2\Delta_{\pi^\pm\pi^0}M_K^2}{\Delta_{K\pi}M_\pi^2}\right)\bar{J}_{K^+\pi^0}(\Delta_{K^0\pi^-}) \\
& - \frac{M_\pi^2}{6F_0^2}\left(1 + \frac{6\varepsilon^{(2)}}{\sqrt{3}} - \frac{2\Delta_{\pi^\pm\pi^0}}{\Delta_{K\pi}}\right)\bar{J}_{K^+\eta}(\Delta_{K^0\pi^-}).
\end{aligned}
\tag{3.77}
$$

We note that in the isospin limit ($\varepsilon^{(2)} = \Delta_{\pi^+\pi^0} = 0$), (3.75) as well as (3.77) reduce to the well known result [46]

$$
f_0^{K\pi}(\Delta_{K\pi}) = \frac{F_K}{F_\pi} - \frac{M_\pi^2}{6F_0^2}\Big[3\bar{J}_{K\pi}(\Delta_{K\pi}) + \bar{J}_{K\eta}(\Delta_{K\pi})\Big]. \tag{3.78}
$$

The quantity

$$
\Delta_{\mathrm{CT}}^{K^+\pi^0} = \widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0}) - \frac{F_K}{F_\pi} = \sqrt{3}\,\varepsilon^{(2)} + \ldots \tag{3.79}
$$

receives a large (but trivial) contribution already at the tree level, making it less convenient for the discussion of deviations from the Callan-Treiman limit in the presence of isospin violation. In contrast, the quantities

$$
\delta_{\mathrm{CT}}^{K^+\pi^0} := \frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^+\pi^0}(0)} - \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} \tag{3.80}
$$

and

$$
\delta_{\mathrm{CT}}^{K^0\pi^-} := \frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)} - \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} \tag{3.81}
$$

vanish at lowest order and will be used in the following to measure the size of corrections to the Callan-Treiman relation in the cases of the charged kaon decays and the neutral kaon decays, respectively.

Finally, after inserting (2.41), at order $p^4, (m_u - m_d)p^2, e^2p^2$ we find the numerical results [12]

$$\delta_{CT}^{K^0\pi^-} = \frac{\widetilde{f}_0^{K^0\pi^-}(\Delta_{K^0\pi^-})}{\widetilde{f}_0^{K^0\pi^-}(0)} - \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} = 1.7(1)(7) \times 10^{-3}, \tag{3.82}$$

$$\delta_{CT}^{K^+\pi^0} = \frac{\widetilde{f}_0^{K^+\pi^0}(\Delta_{K^+\pi^0})}{\widetilde{f}_0^{K^0\pi^-}(0)} - \frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} = -10.4(0)(7) \times 10^{-3}, \tag{3.83}$$

where in both cases the first error originates from (2.41) and the second one from (3.67). Switching off the electromagnetic contributions in (3.82) and (3.83), we obtain [12]

$$\delta_{\mathrm{CT}}^{K^0\pi^-}\Big|_{e=0} = 1.9 \times 10^{-3}, \quad \delta_{\mathrm{CT}}^{K^+\pi^0}\Big|_{e=0} = -9.9 \times 10^{-3}, \tag{3.84}$$

the result in the isospin limit is given by [12]

$$\frac{f_0^{K\pi}(\Delta_{K\pi})}{f_0^{K\pi}(0)} - \frac{F_K}{F_\pi f_+^{K\pi}(0)} = -3.6 \times 10^{-3}. \tag{3.85}$$

One learns from this results that at NLO the Callan-Treiman theorem holds with excellent precision even if isospin breaking contributions are taken into account.

4 Analysis at NNLO

4.1 The scalar form factor in the isospin limit

At NNLO, the result for the slope parameter in the isospin limit is given by [15]

$$\begin{aligned} f_0^{K\pi}(t) + \frac{t}{\Delta_{K\pi}}\left(1 - \frac{F_K}{F_\pi}\right) &= 1 + \bar{\Delta}(t) + \Delta(0) \\ &- \frac{8\Delta_{K\pi}^2}{F_\pi^4}[C_{12}^r(M_\rho) + C_{34}^r(M_\rho)] \\ &+ \frac{8t\Delta_{K\pi}}{F_\pi^4}[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)] \\ &+ \frac{16tM_\pi^2}{F_\pi^4}[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)] \\ &- \frac{8t^2}{F_\pi^4}C_{12}^r(M_\rho). \end{aligned} \tag{4.1}$$

The loop functions $\bar{\Delta}(t)$ and $\Delta(0)$ were calculated numerically in [15]:

$$\begin{aligned}
\bar{\Delta}(t) &= -0.25763t/\mathrm{GeV}^2 + 0.833045(t/\mathrm{GeV}^2)^2 + 1.25252(t/\mathrm{GeV}^2)^3[K^0_{e3}], \\
\Delta(0) &= -0.0080 \pm 0.0057[loops] \pm 0.0028[L^r_i].
\end{aligned} \tag{4.2}$$

Following the strategy proposed in [14], we pull out the tree-level pieces $\sim L^r_i \times L^r_j$ from $\bar{\Delta}(t)$ and $\Delta(0)$ by defining[7]

$$D(0) = \Delta(0) - \frac{8\Delta^2_{K\pi}}{F^4_\pi} L^r_5(M_\rho)^2, \tag{4.3}$$

$$\bar{D}(t) = \bar{\Delta}(t) + \frac{8t\Delta_{K\pi}}{F^4_\pi} L^r_5(M_\rho)^2. \tag{4.4}$$

Expressing (4.1) through the functions $\bar{D}(t)$ and $D(0)$, we obtain [12]

$$\begin{aligned}
f^{K\pi}_0(t) &= f^{K\pi}_+(0) + \frac{t}{\Delta_{K\pi}}\Big(\frac{F_K}{F_\pi} - 1\Big) \\
&\quad + \frac{8t\Delta_{K\pi}}{F^4_\pi}[2C^r_{12}(M_\rho) + C^r_{34}(M_\rho) - L^r_5(M_\rho)^2] \\
&\quad + \frac{16tM^2_\pi}{F^4_\pi}[2C^r_{12}(M_\rho) + C^r_{34}(M_\rho)] \\
&\quad - \frac{8t^2}{F^4_\pi} C^r_{12}(M_\rho) + \bar{D}(t),
\end{aligned} \tag{4.5}$$

where

$$f^{K\pi}_+(0) = 1 + D(0) - \frac{8\Delta^2_{K\pi}}{F^4_\pi}\left[C^r_{12}(M_\rho) + C^r_{34}(M_\rho) - L^r_5(M_\rho)^2\right]. \tag{4.6}$$

The expression for the normalized scalar form factor takes the form [12]

$$\begin{aligned}
\frac{f^{K\pi}_0(t)}{f^{K\pi}_+(0)} &= 1 + \frac{t}{\Delta_{K\pi}}\Big(\frac{F_K}{F_\pi f^{K\pi}_+(0)} - \frac{1}{1+D(0)}\Big) \\
&\quad + \frac{8t(\Delta_{K\pi} - t)}{F^4_\pi} C^r_{12}(M_\rho) \\
&\quad + \frac{16tM^2_\pi}{F^4_\pi}[2C^r_{12}(M_\rho) + C^r_{34}(M_\rho)] \\
&\quad + \frac{\bar{D}(t)}{1+D(0)},
\end{aligned} \tag{4.7}$$

allowing the following conclusion: Apart from the very small contribution

$$\frac{16tM^2_\pi}{F^4_\pi}[2C^r_{12}(M_\rho) + C^r_{34}(M_\rho)] = \Delta^{\mathrm{tree},\,p^6}_{\mathrm{CT}}\frac{t}{\Delta_{K\pi}}, \tag{4.8}$$

[7]Note that terms $\sim L^r_4 \times L^r_5, L^r_5 \times L^r_6, L^r_5 \times L^r_8$, etc. cancel in the combination of terms entering in (4.1).

which is suppressed by a factor M_π^2/M_K^2, the slope as well as the curvature of (4.7) depend only on the counterterm $C_{12}^r(M_\rho)$ if the loop functions $\bar{D}(t)$, $D(0)$ are known and the quantity $F_K/F_\pi f_+^{K\pi}(0)$ is used as input parameter.

Taking $\Delta(0)$ and $\bar{\Delta}(t)(K^0_{\ell 3})$ from [15] and $L_5^r(M_\rho)$ (fit 10) from [34], (4.3) and (4.4) assume the numerical values

$$\begin{aligned} D(0) &= -0.0134 \pm 0.0005, \\ \bar{D}(t) &= -0.23407t/\mathrm{GeV}^2 + 0.833045(t/\mathrm{GeV}^2)^2 + 1.25252(t/\mathrm{GeV}^2)^3. \end{aligned} \tag{4.9}$$

4.2 Renormalization group equations

The relevant p^6 counterterms have been determined by using the $1/N_C$ expansion and truncating the hadronic spectrum to the lowest lying resonances [14]. In this framework, the leading term in the large-N_C expansion of the relevant couplings can be expressed in terms of the scalar and pseudoscalar octet masses (M_S and M_P) and the pion decay constant [14]:

$$\begin{aligned} L_5^{\mathcal{SP}} &= \frac{F_\pi^2}{4M_S^2}, \quad C_{12}^{\mathcal{SP}} = -\frac{F_\pi^4}{8M_S^4}, \\ C_{34}^{\mathcal{SP}} &= \frac{3F_\pi^4}{16M_S^4} + \frac{F_\pi^4}{16M_S^4}\left(1 - \frac{M_S^2}{M_P^2}\right)^2. \end{aligned} \tag{4.10}$$

One assumes that the expressions given above determine the corresponding renormalized coupling constants at some typical hadronic matching scale μ:

$$C_i^r(\mu) = C_i^{\mathcal{SP}}. \tag{4.11}$$

The renormalization of the order p^6 LECs gives [60]

$$C_i^r(M_\rho) = C_i^r(\mu) + \delta C_i(\mu, M_\rho), \tag{4.12}$$

where

$$\delta C_i(\mu, M_\rho) = \frac{1}{(4\pi)^2}\left\{\frac{\Gamma_i^{(2)}}{(4\pi)^2}\left(\ln\frac{\mu}{M_\rho}\right)^2 - [2\Gamma_i^{(1)} + \Gamma_i^{(L)}(M_\rho)]\ln\frac{\mu}{M_\rho}\right\} \tag{4.13}$$

is determined by the renormalization group equations

$$\begin{aligned} \mu\frac{\partial C_i^r(\mu)}{\partial\mu} &= \frac{1}{(4\pi)^2}[2\Gamma_i^{(1)} + \Gamma_i^{(L)}(\mu)], \\ \mu\frac{\partial\Gamma_i^{(L)}(\mu)}{\partial\mu} &= -\frac{\Gamma_i^{(2)}}{8\pi^2}. \end{aligned} \tag{4.14}$$

With this formula we can obtain the value of the coupling constant at our standard reference scale M_ρ. For our analysis we need coefficients [60]

$$\begin{aligned}\Gamma_{12}^{(2)} &= \frac{19}{64}, \quad \Gamma_{12}^{(1)} = -\frac{13}{768(4\pi)^2}, \\ \Gamma_{12}^{(L)} &= \frac{2}{3}L_1^r + \frac{4}{3}L_2^r + \frac{8}{9}L_3^r + \frac{3}{4}L_5^r\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\Gamma_{34}^{(2)} &= -\frac{13}{32}, \quad \Gamma_{34}^{(1)} = -\frac{31}{2304(4\pi)^2}, \\ \Gamma_{34}^{(L)} &= -L_1^r - \frac{3}{2}L_2^r - \frac{11}{12}L_3^r + L_4^r - \frac{3}{2}L_5^r.\end{aligned} \tag{4.16}$$

The analysis of [61] (scenario A) suggests the value $M_S = 1.48$ GeV for the lightest scalar nonet that survives the large-N_c limit. With this choice of the mass parameter one gets [12]

$$L_5^{\mathcal{SP}} = 0.97 \times 10^{-3}, \tag{4.17}$$

which agrees exactly with the mean value of $L_5^r(M_\rho)$ obtained in fit 10 of [34]. For the pseudoscalar mass parameter, spectroscopy and chiral symmetry [1,61] suggest the value $M_P = 1.3$ GeV. With this input we obtain the results [12]

$$C_{12}^r(M_\rho) = (-1.9^{+2.0}_{-0.4}) \times 10^{-6} \tag{4.18}$$

and

$$C_{34}^r(M_\rho) = (2.9^{+1.3}_{-5.0}) \times 10^{-6}. \tag{4.19}$$

The errors were estimated by evaluating (4.13) using (4.15) and (4.16), respectively. The numerical values of the couplings $L_i^r(M_\rho)$ (together with their errors) were taken from fit 10 of [34] and are shown in Table 4. Varying the matching scale μ between M_η and 1 GeV provides us with an estimate of the intrinsic uncertainty due to subleading contributions in $1/N_c$. Note that the asymmetric errors in (4.18) and (4.19) originate from the quadratic term in (4.13) as a consequence of the two-loop renormalization group equation. This is shown in Figure 4.

4.3 Slope parameter at order p^6

Expanding the scalar form factor as

$$\frac{f_0^{K\pi}(t)}{f_+^{K\pi}(0)} = 1 + \lambda_0^{K\pi}\frac{t}{M_{\pi^+}^2} + \frac{1}{2}c_0^{K\pi}\left(\frac{t}{M_{\pi^+}^2}\right)^2 + \ldots, \tag{4.20}$$

$10^3 L_1^r$	$10^3 L_2^r$	$10^3 L_3^r$	$10^3 L_4^r$
0.43 ± 0.12	0.73 ± 0.12	-2.35 ± 0.37	$\equiv 0$
$10^3 L_5^r$	$10^3 L_6^r$	$10^3 L_7^r$	$10^3 L_8^r$
0.97 ± 0.11	$\equiv 0$	-0.31 ± 0.14	0.60 ± 0.18

Table 4: Results for $L_i^r(\mu)$ at the scale $\mu = 0.77$ GeV taken from fit 10 of [34].

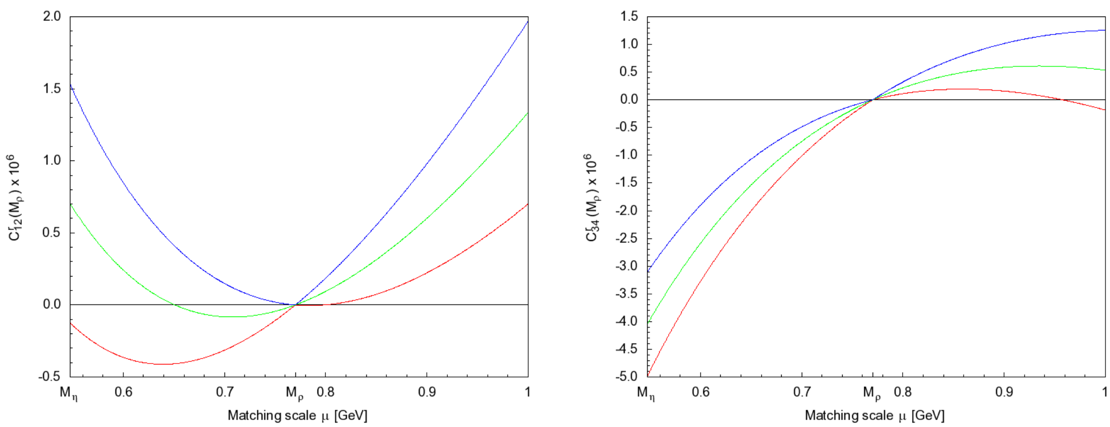

Figure 4: The uncertainties of the order $\mathcal{O}(p^6)$ LECs $C_{12}^r(M_\rho)$ and $C_{34}^r(M_\rho)$ in dependence of the matching scale μ.

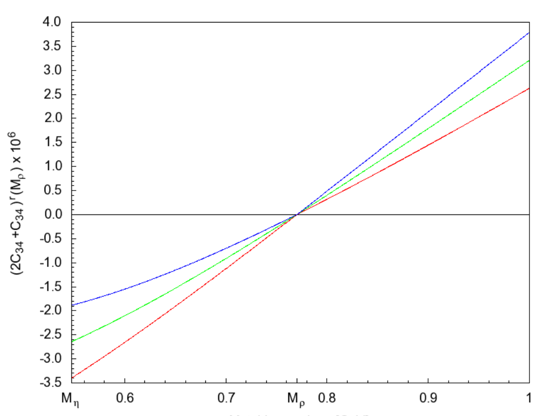

Figure 5: The uncertainty of the combination $(2C_{12} + C_{34})^r(M_\rho)$ entering in $\Delta_{\rm CT}^{{\rm tree},\, p^6}$ in dependence of the matching scale μ.

(4.7) implies [12]

$$\begin{aligned}
\lambda_0^{K\pi} &= M_{\pi^+}^2\Big\{\frac{1}{\Delta_{K\pi}}\Big(\frac{F_K}{F_\pi f_+^{K\pi}(0)} - \frac{1}{1+D(0)}\Big) \\
&\quad + \frac{8\Delta_{K\pi}}{F_\pi^4}C_{12}^r(M_\rho) \\
&\quad + \frac{16M_\pi^2}{F_\pi^4}[2C_{12}^r(M_\rho) + C_{34}^r(M_\rho)] \\
&\quad + \frac{\bar{D}'(0)}{1+D(0)}\Big\}
\end{aligned} \tag{4.21}$$

for the slope parameter.

Using the two-loop results $D(0)$ (4.3) and $\bar{D}(t)$ (4.4) and estimating the relevant combination of low-energy couplings using the renormalization group equations in the way described above, we find [12]

$$\lambda_0^{K\pi} = (13.9^{+1.3}_{-0.4} \pm 0.4) \times 10^{-3}. \tag{4.22}$$

The first error is related to the uncertainties in the determination of the C_i and the second one to those in $F_K/F_\pi f_+(0)$ and $D(0)$.

The expression for the curvature reads [12]

$$c_0^{K\pi} = M_{\pi^+}^4\Big\{-\frac{16}{F_\pi^4}C_{12}^r(M_\rho) + \frac{\bar{D}''(0)}{1+D(0)}\Big\}, \tag{4.23}$$

which leads to the numerical result [12]

$$c_0^{K\pi} = (8.0^{+0.3}_{-1.7}) \times 10^{-4}, \tag{4.24}$$

once $\bar{D}(t)$ together with $C_{12}^r(M_\rho)$ have been inserted. Note that the naive pole parametrization (2.24) would predict

$$c_0^{K\pi}|_{\text{pole fit}} = 2(\lambda_0^{K\pi})^2 \simeq 4 \times 10^{-4}, \tag{4.25}$$

where the numerical value was obtained by inserting $\lambda_0^{K\pi}$. This discrepancy is due to the fact, that the pole parametrization assumes a relation between the slope and the curvature parameters which is not fulfilled in the standard model. Therefore the pole fit should be avoided when analyzing the experimental data.

Using our estimates for the order p^6 coupling constants we are also able to calculate $f_+^{K\pi}(0)$. The relevant combination [12]

$$C_{12}^r(M_\rho) + C_{34}^r(M_\rho) - L_5^r(M_\rho)^2 = (0.1^{+1.1}_{-1.2}) \times 10^{-6} \tag{4.26}$$

corresponds to the result [12]

$$f_+^{K\pi}(0) = 0.986 \pm 0.007_{1/N_c} \pm 0.002_{M_S,M_P}. \tag{4.27}$$

Apart from varying the matching scale, we have also added a second error to account for the uncertainty in the choice of the resonance masses, as our central value for $f_{p^6}^{\text{tree}}$ given by

$$f_{p^6}^{\text{tree}} = -\frac{\Delta_{K\pi}^2}{2M_S^4}\left(1 - \frac{M_S^2}{M_P^2}\right)^2 \tag{4.28}$$

depends strongly on the (relative) size of the mass parameters. The number given in (4.27) is to be compared with the still currently used Leutwyler-Roos value $f_+^{K\pi}(0) = 0.961(8)$ [62]. An average of various lattice calculations is given by $f_+^{K\pi}(0) = 0.956(8)$ [63].

4.4 Dispersive analysis

In the following section we want to check our numerical two-loop χPT results for the slope parameter (4.22) and the curvature (4.24) by comparing them with independent approaches using a dispersive representation of the scalar form factor [16–21].

These parameterizations are based on the observation that the vector and scalar form factors are analytic functions in the complex t-plane, except for a cut along the positive real axis for $t > t_{\text{lim}} = (M_K + M_\pi)^2$, where they develop discontinuities. One can therefore write [16]

$$f_{+,0}(t) = \frac{1}{\pi}\int_{t_{\text{lim}}}^{\infty} ds' \frac{\text{Im} f_{+,0}(s)}{(s - t - i\epsilon)} + \text{subtractions}, \tag{4.29}$$

where the imaginary part, $\text{Im} f_{+,0}(s')$, can be determined from data on $K\pi$ scattering, and the ultraviolet component of the integral is absorbed into the (polynomial) subtraction terms.

In addition to the analyticity constraints, the scalar form factor must satisfy an additional theoretical constraint dictated by chiral symmetry. The Callan-Treiman (CT) theorem [13] implies that the scalar form factor at $t = \Delta_{K\pi} \equiv M_K^2 - M_\pi^2$ is determined in terms of f_K/f_π and $f_+(0)$ up to $\mathcal{O}(m_u, m_d)$ corrections. The quantity Δ_{CT} can be evaluated in χPT, see (2.26), (3.82) and (3.83).

Motivated by the existence of the CT theorem, a particularly appealing dispersive parametrization for the scalar form factor has been proposed [19]. Two subtractions are performed, one at $t = 0$, where by definition $\bar{f}_0(0) = 1$, and the other at the CT point, $t = \Delta_{K\pi}$. With this parametrization, only on free parameter, C, has to be determined from data.

The analysis of Bernard et al. [19–21] based on a twice subtracted dispersion relation gives the expression

$$
\begin{aligned}
f(t) \;:=\; \frac{f_0^{K\pi}(t)}{f_+^{K\pi}(0)} &= \exp\left[\frac{t}{\Delta_{K\pi}}(\ln C - G(t))\right], \\
G(t) &= \frac{\Delta_{K\pi}(\Delta_{K\pi} - t)}{\pi} \int_{t_{K\pi}}^{\infty} \frac{ds}{s} \frac{\phi(s)}{(s - \Delta_{K\pi})(s - t - i\epsilon)},
\end{aligned} \tag{4.30}
$$

for the normalized form factor. The quantity $t_{K\pi}$ denotes the threshold of $K\pi$ scattering and $\phi(t)$ is the phase of $f(t)$,

$$
f(t) = |f(t)| \exp(i\phi(t)). \tag{4.31}
$$

The main advantage of the dispersive relation is that it introduces the value of the form factor at the Callan-Treiman point $\Delta_{K\pi} = M_K^2 - M_\pi^2$, a quantity $C = f(\Delta_{K\pi})$ which is not affected by chiral corrections beyond $SU(2) \times SU(2)$. Thus these are of $\mathcal{O}(m_u, m_d)$ while the slopes get larger corrections of $\mathcal{O}(m_s)$. Expanding (4.30) in the momentum transfer t leads to the expression

$$
\lambda_0^{K\pi} = \frac{M_{\pi^+}^2}{\Delta_{K\pi}}(\ln C - G(0)). \tag{4.32}
$$

for the slope parameter. Evaluating (4.30) at the Callan-Treiman point $t = \Delta_{K\pi}$, one finds the relation

$$
C = \frac{F_K}{F_\pi f_+^{K\pi}(0)} + \frac{\Delta_{\mathrm{CT}}}{f_+^{K\pi}(0)}. \tag{4.33}
$$

Using (2.41) and the estimate ± 0.01 for the uncertainty due to $\Delta_{\mathrm{CT}}/f_+^{K\pi}(0)$, the parameter C assumes the value [12]

$$
C = 1.2424 \pm 0.0045 \pm 0.01, \tag{4.34}
$$

or, equivalently,

$$
\ln C = 0.2170 \pm 0.0036 \pm 0.0080. \tag{4.35}
$$

Together with [19]

$$
G(0) = 0.0398 \pm 0.0036 \pm 0.002, \tag{4.36}
$$

the dispersive analysis gives the numerical value [12]

$$
\lambda_0^{K\pi} = (15.1 \pm 0.8) \times 10^{-3} \tag{4.37}
$$

for the slope parameter, which is consistent with our result based on resonance saturation.

The expression for the curvature reads [19]

$$
\begin{aligned}
c_0^{K\pi} &= (\lambda_0^{K\pi})^2 - \frac{2M_{\pi^+}^4 G'(0)}{\Delta_{K\pi}} \\
&= (\lambda_0^{K\pi})^2 + (4.16 \pm 0.50) \times 10^{-4}.
\end{aligned} \tag{4.38}
$$

Inserting their value of $\lambda_0^{K\pi}$, the curvature is given by

$$
c_0^{K\pi} = (6.4 \pm 0.6) \times 10^{-4}, \tag{4.39}
$$

which is again consistent with the result of our analysis.

The dispersive approach of Jamin, Oller and Pich [16–18] using a method based on a coupled-channel solution of the dispersive relation for the form factor which includes also the $K\eta'$ channel gives the result [18]

$$
\left.\frac{d}{dt}\frac{f_0^{K\pi}(t)}{f_0^{K\pi}(0)}\right|_{t=0} = 0.773(21)\,\mathrm{GeV}^{-2}, \tag{4.40}
$$

$$
\left.\frac{d^2}{dt^2}\frac{f_0^{K\pi}(t)}{f_0^{K\pi}(0)}\right|_{t=0} = 1.599(52)\,\mathrm{GeV}^{-4}, \tag{4.41}
$$

which corresponds to the values

$$
\lambda_0^{K\pi} = (14.7 \pm 0.4) \times 10^{-3}, \tag{4.42}
$$

and

$$
c_0^{K\pi} = (6.07 \pm 0.20) \times 10^{-4}. \tag{4.43}
$$

This results are in good agreement with those obtained by Bernard et al. and also with our results (4.22) and (4.24) obtained in χPT.

4.5 Contributions of order $(m_d - m_u)p^4$

Recently, isospin breaking in the $K_{\ell 3}$ form factors has also been studied at the two-loop level [22]. The results for the scalar form factor of $K^0 \to \pi^- \ell^+ \nu_\ell$ with $C_i^r = 0$ turn out to be essentially the same as those in the isospin limit. From Fig. 13 of [22] one extracts [12]

$$
\Delta\lambda_0|_{C_i^r = e = 0} \simeq 5 \times 10^{-4}. \tag{4.44}
$$

The remaining contributions to the form factors containing the order p^6 LECs C_i^r were given in [22] and are shown in Appendix B. In the splitting of the two slope parameters this results simplify to [12]

$$\Delta\lambda_0\big|_{C_i^r} \;=\; \frac{32\varepsilon^{(2)}\Delta_{K\pi}M_{\pi^+}^2}{\sqrt{3}F_\pi^4} \times (2C_{12}+6C_{17}+6C_{18}+3C_{34}+3C_{35})^r(M_\rho). \tag{4.45}$$

Using the resonance estimates of the LECs appearing in (4.45) given in [64], we find [12]

$$(2C_{12}+6C_{17}+6C_{18}+3C_{34}+3C_{35})^{\mathcal{SP}} \;=\; \frac{F_\pi^4}{4M_S^4}\Big(1-\frac{3M_S^2}{2M_P^2}-\frac{M_S^2}{M_{\eta'}^2}+6\lambda_2^{\mathcal{SS}}\Big). \tag{4.46}$$

With our standard values for the resonance masses M_S, M_P and our usual determination of the uncertainty of the large N_C estimate, we find [12]

$$(2C_{12}+6C_{17}+6C_{18}+3C_{34}+3C_{35})^r(M_\rho) = (\,-1.25+2.26\lambda_2^{\mathcal{SS}}\pm 0.7_{1/N_C})\times 10^{-5}. \tag{4.47}$$

Varying the unknown $\lambda_2^{\mathcal{SS}}$ in the interval

$$-1 \lesssim \lambda_2^{\mathcal{SS}} \lesssim 1 \tag{4.48}$$

and combining the two-loop results given in [22] with an estimate of a further combination of low-energy couplings, we expect the total value for the difference of the two slope parameters to lie within the rather small range [12]

$$0 \lesssim \Delta\lambda_0 \lesssim 10^{-3}. \tag{4.49}$$

4.6 Callan-Treiman relations at NNLO

The combination of counterterms entering in (4.8) is given by [12]

$$2C_{12}^r(M_\rho)+C_{34}^r(M_\rho) = (\,-0.9^{+3.8}_{-3.4})\times 10^{-6} \tag{4.50}$$

which translates into [12]

$$\Delta_{\rm CT}^{\text{tree},\,p^6} = (\,-0.8^{+3.5}_{-3.1})\times 10^{-3}. \tag{4.51}$$

Combined with the two-loop result given in [22], the total p^6 result (in the isospin-limit) reads [12]

$$\Delta_{\rm CT} = (\,-7.0^{+3.5}_{-3.1})\times 10^{-3}. \tag{4.52}$$

The two-loop contributions to the correction terms of the Callan-Treiman relation in the presence of isospin violation were also given in [22]. Translated in terms of the quantities defined in (3.81) and (3.80), they find

$$\delta_{\rm CT}^{K^0\pi^-}\Big|_{C_i^r=e=0} = -5.6\times 10^{-3} \tag{4.53}$$

and

$$\delta_{\rm CT}^{K^+\pi^0}\Big|_{C_i^r=e=0} = -13.3\times 10^{-3}, \tag{4.54}$$

respectively. These results should be supplemented by the associated local contributions arising at this order [22], which are, however, also plagued by partly undetermined low-energy couplings. We demonstrate this only for the purely isospin violating combination [12]

$$\begin{aligned}\left(\delta_{\rm CT}^{K^0\pi^-} - \delta_{\rm CT}^{K^+\pi^0}\right)\Big|_{C_i^r} &= \frac{32\varepsilon^{(2)}M_K^4}{\sqrt{3}F_\pi^4}(2C_{12}+2C_{14}\\ &\quad +2C_{15}+6C_{17}+6C_{18}+4C_{34}+3C_{35})^r(M_\rho),\end{aligned} \tag{4.55}$$

where terms $\sim \varepsilon^{(2)}M_\pi^2$ have been discarded. In addition to the undetermined parameter λ_2^{SS} already encountered in (4.46), the resonance estimate for the p^6 low-energy coupling C_{14} is still incomplete [64], preventing a reliable numerical determination of (4.55) (and even more for the individual terms) for the time being.

Nevertheless, based on the numbers (3.82) and (3.83) found at NLO, the partial NNLO results shown in (4.53) and (4.54), our estimate of the isospin symmetric local p^6 contribution (4.51) and a rough order-of-magnitude estimate of not yet determined local terms of the order $(m_d-m_u)p^4$ (a typical term is shown in (4.55)), we expect numerically small corrections to the Callan-Treiman relation also in the presence of isospin violation with [12]

$$|\delta_{\rm CT}^{K^0\pi^-}|,\ |\delta_{\rm CT}^{K^+\pi^0}| \lesssim 10^{-2}. \tag{4.56}$$

5 Summary and conclusions

In this thesis we have discussed the theoretical predictions for the scalar form factors of $K_{\ell 3}$ decays within the standard model. The principal theoretical tool for this analysis is chiral perturbation theory (χPT), the effective field theory of the standard model at low energies. We have given a short introduction to χPT.

We have given an introduction to $K_{\ell 3}$ decays, including a description of the kinematics and a summary of the experimental situation. The leading non-vanishing contribution to the scalar slope arises at order p^4 in the chiral expansion. The theoretical expression for the scalar form factor was worked out already more than twenty years ago [46] in the limit of isospin conservation. In this case, the slope parameter is uniquely determined by the pseudoscalar masses, the pion decay constant and the ratio [12]

$$\frac{F_K}{F_\pi \widetilde{f}_+^{K^0\pi^-}(0)} = 1.2424(45). \tag{5.1}$$

The remarkably precise numerical value given here can be obtained by combining the latest experimental data on $K_{\mu 2(\gamma)}$, $\pi_{\mu 2(\gamma)}$, K^0_{Le3} and V_{ud} with the corresponding theoretical expressions. Using this input, one finds [12]

$$\lambda_0^{K\pi}|_{p^4} = (16.64 \pm 0.39) \times 10^{-3}. \tag{5.2}$$

The isospin violating contributions of order $(m_d - m_u)p^2$ and e^2p^2 to the $K_{\ell 3}$ form factors were considered for the first time in [43]. The effects of strong isospin breaking are proportional to the mixing angle [12]

$$\varepsilon^{(2)} = \frac{\sqrt{3}}{4}\,\frac{m_d - m_u}{m_s - \widehat{m}} = (1.29 \pm 0.17) \times 10^{-2}. \tag{5.3}$$

The numerical value shown here was obtained by using the corrections to Dashen's limit given in [39]. The electromagnetic contributions of order e^2p^2 entering in the slope parameters $\lambda_0^{K^0\pi^-}$ and $\lambda_0^{K^+\pi^0}$ can be expressed through the electromagnetic pieces of the pseudoscalar masses as well as the coupling Z associated to the chiral Lagrangian of order e^2p^0, which can also be related to the pion mass difference (to the considered order). Both sources of isospin violation generate only a tiny shift of the two slope parameters (compared to the isospin symmetric limit) with a splitting $\Delta\lambda_0 = \lambda_0^{K^0\pi^-} - \lambda_0^{K^+\pi^0}$ given by [12]

$$\Delta\lambda_0|_{(m_d-m_u)p^2, e^2p^2} = (5.1 \pm 0.9) \times 10^{-4} \tag{5.4}$$

at this chiral order.

The corrections arising at order p^6 (in the isospin limit) turn out to be quite sizeable. Combining the two-loop results of χPT [15] with an updated estimate of the necessary p^6 low-energy couplings, the numerical value of the slope parameter in the isospin symmetric limit is given by [12]

$$\lambda_0^{K\pi} = (13.9^{+1.3}_{-0.4} \pm 0.4) \times 10^{-3}. \tag{5.5}$$

The main uncertainty in this result comes from a certain combination of p^6 low energy couplings which has been determined by an updated analysis based on [14, 61].

Using the dispersive representation proposed in [19] with (2.41), we find [12]

$$\lambda_0^{K\pi} = (15.1 \pm 0.8) \times 10^{-3}, \tag{5.6}$$

being in good agreement with the value (5.5) obtained in χPT and also with other results [17, 18] using dispersion techniques.

The inclusion of isospin violating contributions of order $(m_d - m_u)p^4$ does not change this picture substantially. We expect an additional uncertainty for the values of the slope parameters of at most $\pm 10^{-3}$, mainly due to not yet fully determined low-energy couplings. Combining the two-loop results given in [22] with an estimate of a further combination of low-energy couplings, the difference of the two slope parameters should be confined to the rather small range [12]

$$0 \lesssim \Delta\lambda_0 \lesssim 10^{-3}. \tag{5.7}$$

In other words, if a difference of the size of the two slope parameters is detected at all, $\lambda_0^{K^0\pi^-}$ should be slightly larger than $\lambda_0^{K^+\pi^0}$.

At the Callan-Treiman point $t = \Delta_{K\pi}$, the size of the scalar form factor is predicted as [13]

$$f_0^{K\pi}(\Delta_{K\pi}) = \frac{F_K}{F_\pi} + \Delta_{\mathrm{CT}}, \tag{5.8}$$

where Δ_{CT} is of the order m_u, m_d, e. At order p^4 (in the isospin limit) the correction term $\Delta_{\mathrm{CT}} = -3.5 \times 10^{-3}$ was calculated in [46]. If isospin violation is included, it is advantageous to consider the quantities defined in (3.81) and (3.80). At the order $p^4, (m_d - m_u)p^2, e^2p^2$, we find [12]

$$\delta_{\mathrm{CT}}^{K^0\pi^-}\big|_{p^4,(m_d-m_u)p^2,e^2p^2} = (1.7 \pm 0.7) \times 10^{-3} \tag{5.9}$$

and

$$\delta_{\mathrm{CT}}^{K^+\pi^0}\big|_{p^4,(m_d-m_u)p^2,e^2p^2} = (-10.4 \pm 0.7) \times 10^{-3}. \tag{5.10}$$

In spite of the large corrections to the correction term itself, the Callan-Treiman relation still holds with excellent precision also if isospin violating contributions are taken into account.

Corrections to Δ_{CT} arising at NNLO are also (potentially) large. At the same time, the uncertainty of the theoretical result is increased by the presence of p^6 low-energy couplings. Combining the two-loop result given in [22] with our estimate for $2C_{12}^r + C_{34}^r$, we find (in the isospin limit) [12]

$$\Delta_{\mathrm{CT}} = (- 7.0^{+3.5}_{-3.1}) \times 10^{-3}. \tag{5.11}$$

The loop contributions of order $(m_d - m_u)p^4$ were considered in [22]. The associated counterterm contributions depend on partly undetermined low-energy couplings. In spite of these theoretical uncertainties, we expect only small corrections to the Callan-Treiman relation with [12]

$$|\delta_{\mathrm{CT}}^{K^0\pi^-}|, \, |\delta_{\mathrm{CT}}^{K^+\pi^0}| \lesssim 10^{-2}. \tag{5.12}$$

The experimental results for the scalar slope parameter found by ISTRA+, KTeV and KLOE are in agreement with the predictions of the standard model. On the other hand, the value found by NA48 can hardly be reconciled with our theoretical results. Furthermore, an isospin violation in $\Delta\lambda_0$ as it would be suggested by the simultaneous validity of the results of ISTRA+ and NA48 is definitely ruled out within the standard model.

The naive pole parametrization of the scalar form factor should be avoided. It contains an implicit assumption of a relation between slope and curvature which is not fulfilled in the standard model.

At the present theoretical and experimental level of precision, the correct treatment of electromagnetic corrections in $K_{\mu 3}$ decays is mandatory for the extraction of form factor parameters from experimental data. The appropriate procedure was described in [43], a more detailed presentation of the numerics is given in [65].

A Coefficients

In this section we list the coefficients $a_{PQ}(t)$, b_{PQ}, $c_{PQ}(t)$, and d_{PQ} given in [43].

$$\begin{aligned}
a_{K^+\pi^0}(t) &= \frac{2M_K^2+2M_\pi^2-t}{4F_0^2}+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-2M_K^2+22M_\pi^2-9t}{4F_0^2}+4\pi\alpha Z,\\
a_{K^0\pi^-}(t) &= \frac{-2M_K^2-2M_\pi^2+3t}{2F_0^2}+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-2M_K^2+6M_\pi^2-3t}{2F_0^2}-16\pi\alpha Z,\\
a_{K^+\eta}(t) &= \frac{2M_K^2+2M_\pi^2-3t}{4F_0^2}+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{6M_K^2-2M_\pi^2-3t}{4F_0^2}+12\pi\alpha Z.
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
b_{K^+\pi^0} &= -\frac{\Delta_{K\pi}}{2F_0^2}-\left(\frac{7\varepsilon^{(2)}}{2\sqrt{3}}\right)\frac{\Delta_{K\pi}}{F_0^2}-4\pi\alpha Z,\\
b_{K^0\pi^-} &= -\frac{\Delta_{K\pi}}{F_0^2}-\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{\Delta_{K\pi}}{F_0^2}-8\pi\alpha Z,\\
b_{K^+\eta} &= -\frac{3\Delta_{K\pi}}{2F_0^2}+\left(\frac{\sqrt{3}\varepsilon^{(2)}}{2}\right)\frac{\Delta_{K\pi}}{F_0^2}-12\pi\alpha Z.
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
c_{K^+\pi^0}(t) &= -\frac{2M_K^2+2M_\pi^2-3t}{4F_0^2}+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{-4M_K^2+3t}{2F_0^2}-8\pi\alpha Z,\\
c_{K^0\pi^-}(t) &= \frac{t}{2F_0^2},\\
c_{K^+\eta}(t) &= \frac{2M_K^2+2M_\pi^2-3t}{4F_0^2}+\left(\frac{\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{4M_K^2-3t}{2F_0^2}.
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
d_{K^+\pi^0} &= -\frac{\Delta_{K\pi}}{2F_0^2}-\left(\frac{4\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{\Delta_{K\pi}}{F_0^2}+4\pi\alpha Z,\\
d_{K^0\pi^-} &= -\frac{\Delta_{K\pi}}{F_0^2}-\left(\frac{2\varepsilon^{(2)}}{\sqrt{3}}\right)\frac{\Delta_{K\pi}}{F_0^2}+8\pi\alpha Z,\\
d_{K^+\eta} &= -\frac{3\Delta_{K\pi}}{2F_0^2}+12\pi\alpha Z.
\end{aligned} \tag{A.4}$$

B The order p^6 LECs dependent part

In this appendix we give the part of the $K_{\ell 3}$ form factors dependent on the order p^6 LECs C_i^r entering in (4.45) taken from [22].

$$
\begin{aligned}
f_\pm^{K^+\pi^0}(t)\Big|_{C_i^r} &= \frac{1}{F_\pi^4}\left(f_\pm^A(t) + \frac{\sin\varepsilon^{(2)}}{\sqrt{3}} f_\pm^B(t) + \frac{\sin\varepsilon^{(2)}}{\sqrt{3}\left(m_{\pi^0}^2 - m_\eta^2\right)} f_\pm^E(t)\right), \\
f_\pm^{K^0\pi^-}(t)\Big|_{C_i^r} &= \frac{1}{F_\pi^4}\left(f_\pm^A(t) - \frac{\sin\varepsilon^{(2)}}{\sqrt{3}} f_\pm^D(t)\right).
\end{aligned} \tag{B.1}
$$

The C_i^r dependence is now given by

$$
\begin{aligned}
f_+^A(t) &= +t^2\,(-4\,C_{88}^r + 4\,C_{90}^r) + M_\sigma^2\,t\,(-4\,C_{12}^r - 16\,C_{13}^r - 4\,C_{63}^r - 4\,C_{64}^r - 2\,C_{90}^r) \\
&\quad + M_\pi^2\,t\,(-12\,C_{12}^r - 32\,C_{13}^r - 4\,C_{63}^r - 8\,C_{64}^r - 4\,C_{65}^r - 6\,C_{90}^r) + M_\sigma^4\,(-2\,C_{12}^r - 2\,C_{34}^r) \\
&\quad + M_\pi^2\,M_\sigma^2\,(4\,C_{12}^r + 4\,C_{34}^r) + M_\pi^4\,(-2\,C_{12}^r - 2\,C_{34}^r), \\
f_+^B(t) &= +t^2\,(-12\,C_{88}^r + 12\,C_{90}^r) + M_\sigma^2\,t\,(-4\,C_{12}^r - 48\,C_{13}^r - 4\,C_{63}^r - 12\,C_{64}^r - 2\,C_{90}^r) \\
&\quad + M_\pi^2\,t\,(-44\,C_{12}^r - 96\,C_{13}^r - 20\,C_{63}^r - 24\,C_{64}^r - 12\,C_{65}^r - 22\,C_{90}^r) \\
&\quad + M_\sigma^4\,(2\,C_{12}^r + 16\,C_{14}^r + 16\,C_{17}^r + 48\,C_{18}^r - 14\,C_{34}^r - 24\,C_{35}^r) \\
&\quad + M_\pi^2\,M_\sigma^2\,(-4\,C_{12}^r - 32\,C_{14}^r - 32\,C_{17}^r - 96\,C_{18}^r + 28\,C_{34}^r + 48\,C_{35}^r) \\
&\quad + M_\pi^4\,(2\,C_{12}^r + 16\,C_{14}^r + 16\,C_{17}^r + 48\,C_{18}^r - 14\,C_{34}^r - 24\,C_{35}^r), \\
f_+^E(t) &= +M_\sigma^6\,(96\,C_{19}^r + 64\,C_{20}^r + 64\,C_{31}^r + 64\,C_{32}^r + 128\,C_{33}^r) \\
&\quad + M_\pi^2\,M_\sigma^4\,(-32\,C_{14}^r - 32\,C_{17}^r - 96\,C_{18}^r) + M_\pi^4\,M_\sigma^2\,(64\,C_{14}^r + 64\,C_{17}^r + 192\,C_{18}^r \\
&\quad - 288\,C_{19}^r - 192\,C_{20}^r - 192\,C_{31}^r - 192\,C_{32}^r - 384\,C_{33}^r) \\
&\quad + M_\pi^6\,(-32\,C_{14}^r - 32\,C_{17}^r - 96\,C_{18}^r + 192\,C_{19}^r + 128\,C_{20}^r + 128\,C_{31}^r + 128\,C_{32}^r \\
&\quad + 256\,C_{33}^r), \\
f_+^D(t) &= +M_\sigma^2\,t\,(8\,C_{12}^r - 8\,C_{63}^r + 8\,C_{65}^r + 4\,C_{90}^r) + M_\pi^2\,t\,(-8\,C_{12}^r + 8\,C_{63}^r - 8\,C_{65}^r - 4\,C_{90}^r) \\
&\quad + M_\sigma^4\,(8\,C_{12}^r + 8\,C_{34}^r) + M_\pi^2\,M_\sigma^2\,(-16\,C_{12}^r - 16\,C_{34}^r) + M_\pi^4\,(8\,C_{12}^r + 8\,C_{34}^r), \\
f_-^A(t) &= +M_\sigma^2\,t\,(-4\,C_{12}^r + 2\,C_{88}^r - 2\,C_{90}^r) + M_\pi^2\,t\,(4\,C_{12}^r - 2\,C_{88}^r + 2\,C_{90}^r) \\
&\quad + M_\sigma^4\,(6\,C_{12}^r + 8\,C_{13}^r + 4\,C_{14}^r + 4\,C_{15}^r + 2\,C_{34}^r + 2\,C_{63}^r + 2\,C_{64}^r + C_{90}^r) \\
&\quad + M_\pi^2\,M_\sigma^2\,(12\,C_{12}^r + 8\,C_{13}^r + 4\,C_{15}^r + 8\,C_{17}^r + 4\,C_{34}^r + 2\,C_{64}^r + 2\,C_{65}^r + 2\,C_{90}^r) \\
&\quad + M_\pi^4\,(-18\,C_{12}^r - 16\,C_{13}^r - 4\,C_{14}^r - 8\,C_{15}^r - 8\,C_{17}^r - 6\,C_{34}^r - 2\,C_{63}^r - 4\,C_{64}^r \\
&\quad - 2\,C_{65}^r - 3\,C_{90}^r), \\
f_-^B(t) &= +M_\sigma^2\,t\,(-4\,C_{12}^r + 2\,C_{88}^r - 2\,C_{90}^r) + M_\pi^2\,t\,(4\,C_{12}^r - 2\,C_{88}^r + 2\,C_{90}^r) \\
&\quad + M_\sigma^4\,(-6\,C_{12}^r + 8\,C_{13}^r - 4\,C_{14}^r + 4\,C_{15}^r - 32\,C_{17}^r - 48\,C_{18}^r - 18\,C_{34}^r - 24\,C_{35}^r - 2\,C_{63}^r
\end{aligned}
$$

$$
\begin{aligned}
& +2\,C_{64}^r - C_{90}^r) \\
& +M_\pi^2\,M_\sigma^2\,(36\,C_{12}^r + 8\,C_{13}^r + 16\,C_{14}^r + 4\,C_{15}^r + 72\,C_{17}^r + 96\,C_{18}^r + 44\,C_{34}^r + 48\,C_{35}^r \\
& +8\,C_{63}^r + 2\,C_{64}^r + 2\,C_{65}^r + 6\,C_{90}^r) \\
& +M_\pi^4\,(-30\,C_{12}^r - 16\,C_{13}^r - 12\,C_{14}^r - 8\,C_{15}^r - 40\,C_{17}^r - 48\,C_{18}^r - 26\,C_{34}^r - 24\,C_{35}^r \\
& -6\,C_{63}^r - 4\,C_{64}^r - 2\,C_{65}^r - 5\,C_{90}^r)\,, \\
f_-^E(t) &= 0, \\
f_-^D(t) &= +M_\sigma^2\,t\,(8\,C_{12}^r - 4\,C_{88}^r + 4\,C_{90}^r) + M_\pi^2\,t\,(-8\,C_{12}^r + 4\,C_{88}^r - 4\,C_{90}^r) \\
& +M_\sigma^4\,(-24\,C_{12}^r - 16\,C_{13}^r - 8\,C_{15}^r - 16\,C_{17}^r - 8\,C_{34}^r - 4\,C_{64}^r - 4\,C_{65}^r - 4\,C_{90}^r) \\
& +M_\pi^2\,M_\sigma^2\,(-16\,C_{13}^r - 16\,C_{14}^r - 8\,C_{15}^r + 16\,C_{17}^r - 8\,C_{63}^r - 4\,C_{64}^r + 4\,C_{65}^r) \\
& +M_\pi^4\,(24\,C_{12}^r + 32\,C_{13}^r + 16\,C_{14}^r + 16\,C_{15}^r + 8\,C_{34}^r + 8\,C_{63}^r + 8\,C_{64}^r + 4\,C_{90}^r), \qquad \text{(B.2)}
\end{aligned}
$$

where

$$
M_\sigma^2 = M_{K^+}^2 + M_{K^0}^2 - M_\pi^2. \qquad \text{(B.3)}
$$

The pion mass is used generically since M_{π^+} and M_{π^0} are the same to the considered order.

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Zusammenfassung

In dieser Diplomarbeit werden die semileptonischen Kaon-Zerfälle ($\ell = e, \mu$)

$$\begin{aligned} K^+(p_K) &\rightarrow \pi^0(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu), \\ K^0(p_K) &\rightarrow \pi^-(p_\pi)\ell^+(p_\ell)\nu_\ell(p_\nu) \end{aligned}$$

(und ihre ladungskonjugierten Moden) und insbesondere isospinverletzende Effekte durch die starke und die elektromagnetische Wechselwirkung untersucht. Die Untersuchung dieser Prozesse erfolgt mit Hilfe der Chiralen Störungstheorie (χPT), einer effektiven Feldtheorie des Standardmodells der Teilchenphysik bei niedrigen Energien ($E \ll 1$GeV). Diese Zerfälle, insbesondere die K_{e3}-Mode, sind die wichtigste Quelle zur Bestimmung des Kobayashi-Maskawa-Matrixelements $|V_{us}|$ im Standardmodell. Daher ist es von großer Bedeutung, diese Prozesse so gut wie möglich zu verstehen.

Die Zerfallsbreite dieser Prozesse wird durch die vektoriellen und skalaren Formfaktoren $f_+^{K^+\pi^0}(t)$ und $f_0^{K^+\pi^0}(t)$ bzw. $f_+^{K^0\pi^-}(t)$ und $f_0^{K^0\pi^-}(t)$ bestimmt. Im Experiment wird der sklalare Formfaktor üblicherweise durch die Steigung $\lambda_0^{K\pi}$ parametrisiert,

$$f_0^{K\pi}(t) = f_+^{K\pi}(0)\left(1 + \lambda_0^{K\pi}\frac{t}{M_{\pi^+}^2}\right).$$

Die aktuellen Experimente ISTRA+, KTeV, KLOE und NA48 liefern allerdings Werte für $\lambda_0^{K\pi}$, die nur schwer miteinander vereinbar sind. Das Ziel dieser Arbeit ist es, die Ergebnisse dieser Experimente mit den Vorhersagen des Standardmodells zu vergleichen.

Der Aufbau der Diplomarbeit ist folgender: In Abschnitt 1 wird eine kurze Einführung in die χPT gegeben. In Abschnitt 2 werden die experimentelle Situation der $K_{\ell3}$-Zerfälle geschildert und ihre Kinematik beschrieben. In Abschnitt 3 werden zunächst die Ergebnisse für die vektoriellen und skalaren Formfaktoren und deren Steigungen in Einschleifennäherung wiedergegeben. Die Steigung des skalaren Formfaktors wird insbesondere durch die Größe $F_K/F_\pi f_+^{K^0\pi^-}(0)$ und den isospinverletzenden Parameter $\varepsilon^{(2)}$ bestimmt, die Bestimmung dieser Größen wird ausführlich beschrieben. Mit diesen Ergebnissen werden $\lambda_0^{K^+\pi^0}$ und $\lambda_0^{K^0\pi^-}$ numerisch ausgewertet. Am Ende dieses Abschnitts werden die Callan-Treiman-Relationen in Einschleifennäherung (inklusive Isospinverletzung) ausgearbeitet.

Die Ergebnisse in Zweischleifennäherung werden in Abschnitt 4 präsentiert und mit einem unabhängigen Ansatz, der auf Dispersionsrelationen basiert, verglichen. Weiters werden die isospinverletzenden Beiträge der Ordnung $(m_d - m_u)p^4$ in $\lambda_0^{K\pi}$ und in den Callan-Treiman-Relationen studiert.

Zusammenfassend kann gesagt werden, dass die Ergebnisse von ISTRA+, KTeV und KLOE in Übereinstimmung mit den Vorhersagen des Standardmodells sind. Die Resultate von NA48 sind hingegen nur schwer mit den Ergebnissen dieser Diplomarbeit vereinbar. Insbesondere kann eine Isospinverletzung, wie sie der Fall wäre, wenn man die Ergebnisse von ISTRA+ und NA48 als richtig annimmt, im Standardmodell ausgeschlossen werden.

Curriculum vitae

Ich wurde am 6. Februar 1982 in Mistelbach (Niederösterreich) als Sohn von Franz und Barbara Kastner geboren.

1988-1992 Besuch der Volksschule Staatz-Kautendorf.

1992-1996 Besuch der Hauptschule Laa/Thaya.

1996-2001 Besuch der Bundeshandelsakademie in Laa/Thaya.

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