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On the Cohomology of Arithmetic Subgroups of Unipotent Algebraic Groups

An Algebraic Proof of Van Est's Theorem

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Introduction

Let G be an algebraic group defined over $\mathbb{Q}$ and Γ be an arithmetic subgroup of G. When studying the properties of Γ, it is often advantageous to embed Γ into the Lie group of real points $G(\mathbb{R})$ of G. In this setting, Γ is a discrete subgroup of $G(\mathbb{R})$ and we can employ a variety of results from the general theory of discrete subgroups of Lie groups, as well as various geometric methods, to derive results on the structure of Γ. This work focuses on one of these results and tries to find a suitable adaption that does not require methods beyond the theory of algebraic groups and some homological algebra. More precisely, we are interested in the following theorem by van Est, which was published in [vE58]:

Theorem. *Let N be a nilpotent Lie group such that N modulo the maximal connected subgroup is finitely generated and Γ be a discrete cocompact subgroup of N. Then for any unipotent linear representation of N on a real finite dimensional vector space V the restriction $H^*_d(N, V) \to H^*(\Gamma, V)$ is an isomorphism.*

Proof. See [vE58][Theorem 8]. $\square$

The statement of this theorem itself requires some explanation. First of all, a unipotent linear representation is a smooth representation of N on a finite dimensional vector space V such that every $n \in N$ acts as a unipotent linear map on V. Moreover, the cohomology $H^*_d(N, V)$ is the cohomology based on "differentiable cochains" of N in V (see [HM62]), whereas $H^*(\Gamma, V)$ denotes the usual group cohomology of Γ with values in V.

One can apply this result to arithmetic subgroups of unipotent algebraic groups defined over $\mathbb{Q}$ as follows: Let V be a finite dimensional rational representation of a unipotent algebraic group U defined over $\mathbb{Q}$. This representation gives rise to a smooth unipotent representation of the Lie group of real points $U(\mathbb{R})$ on $V_{\mathbb{R}} = V \otimes \mathbb{R}$. Moreover, the group $U(\mathbb{R})$ is a nilpotent Lie group with finitely many connected components. Therefore, as soon as one observes that any arithmetic subgroup of U is cocompact, one obtains $H^*_d(U(\mathbb{R}), V_{\mathbb{R}}) \cong H^*(\Gamma, V_{\mathbb{R}})$. This implies that for any rational representation V of U, the cohomology of Γ with values in $V_{\mathbb{R}}$ is already completely determined by U. In particular, given any other arithmetic subgroup Γ' of U, the cohomology groups $H^*(\Gamma, V_{\mathbb{R}})$ and $H^*(\Gamma', V_{\mathbb{R}})$ are isomorphic.

In view of the preceding considerations, the theorem of van Est establishes, loosely speaking, the independence of the cohomology groups of Γ from the choice of Γ as an arithmetic subgroup of U, insofar as representations of Γ obtained from those of U are concerned. The question is if we can somehow eliminate the use of differential cohomology and replace it by something more closely related to the structure of U as an algebraic group. A first hint that this question can be answered in the affirmative is contained in the article *Cohomology of Lie Groups* by Hochschild and Mostow [HM62]. This paper takes a closer look at the different kinds of cohomology groups available for Lie groups. If one applies the results presented therein - in particular theorem 5.1, 11.1 and 12.1 - to a finite dimensional rational representation V of a unipotent group U, one obtains an isomorphism identifying the differentiable cohomology groups $H^*(U(\mathbb{R}), V)$ with the so called rational cohomology groups $H_r^*(U, V)$. The latter were defined and studied by Hochschild in [Hoc61] and depend only on the structure of U as an algebraic group. This suggests that one can prove van Est's theorem by regarding rational instead of differentiable cohomology, which is the fundamental idea this work is based on.

In passing, we shall reestablish another well known result of Hochschild, presented in [Hoc61], which asserts the existence of a similar isomorphism connecting rational cohomology of unipotent algebraic groups to the corresponding Lie algebra cohomology:

Theorem. *Let G be a unipotent algebraic group over a field F of characteristic 0 with Lie algebra $\mathfrak{g}$ and let A be a rational G-module. Then there is an isomorphism of the rational cohomology group $H(G, A)$ onto the Lie algebra cohomology group $H(\mathfrak{g}, A)$.*

Proof. See [Hoc61][Theorem 5.1] □

Interestingly, both results are consequences of strikingly similar observations and can be deduced using almost the same proof, once a suitable environment for stating them is available.

The first chapter therefore introduces group schemes as the technical framework of choice to formulate all later results. The connection with the ideas described above is made in Lemma 1.1.12, which reveals that the notion of a representation presented for group schemes yields precisely the rational representations considered in [Hoc61]. We then focus on unipotent groups and their representations. It turns out that both unipotent groups as well as their finite dimensional representations admit special filtrations. More precisely, Theorem 1.2.5 asserts that any unipotent algebraic group U over a field of characteristic 0 admits a closed normal subgroup V such that $U/V \cong G_a$, where G_a denotes the additive group and thus has a filtration

$$U = U_n \supset U_{n-1} \supset \ldots \supset U_0 = \{e\}$$

by closed subgroups U_i, each normal in its predecessor, such that $U_{i+1}/U_i \cong G_a$. Similarly, each finite dimensional U-module M admits a filtration

$$M = M_n \supset M_{n-1} \supset \ldots \supset M_0 = K_{tr}$$

with each M_i being a U-submodule of M such that $M_{i+1}/M_i = K_{tr}$ (see Proposition 1.2.7). The simple nature of these quotients suggests that unipotent groups are well suited for proofs involving induction on the dimension of U, an idea we will exploit later on. The first chapter concludes with a brief summary of constructions and results concerning the Lie algebra of an algebraic group.

In order to define the rational cohomology groups mentioned above, we need injective resolutions of modules over an algebraic group G. These can be obtained by considering so called induced representations, which are obtained from representations of a closed subgroup H of G and will be introduced in the second chapter of this work. The construction involved in this process is similar to the one used for Lie groups or finite groups (although one has to deal with slightly more technical difficulties). It also shares the main property of induced representations, a result commonly referred to as *Frobenius reciprocity*, which states that the functor mapping an H-module M to the induced G-module $\operatorname{ind}_H^G M$ is right adjoint to the restriction functor res_H^G, which assigns every representation N of G its restriction to H. In simpler terms, one has a canonical isomorphism

$$\operatorname{Hom}_G(N, \operatorname{ind}_H^G M) \to \operatorname{Hom}_H(\operatorname{res}_H^G N, M).$$

This result can be leveraged to show that for any representation N of G, the G-module $N \otimes K[G]$ is injective. Since N can easily be embedded into $N \otimes K[G]$, we obtain the existence of injective resolutions as well as a way to construct them in a simple manner.

With the foundations of the first two chapters, the third and last chapter can freely focus on our primary objects of interest: Arithmetic subgroups and the related cohomology groups. After a brief introduction of the notion of cohomology groups for algebraic as well as abstract groups and Lie algebras, we introduce spectral sequences as a technical tool necessary to obtain relations between the cohomology of a group or a Lie algebra with respect to a normal subgroup or an ideal, respectively. This is of particular interest to us, as it allows us to make use of the filtration of unipotent groups constructed in the first chapter. The focus then shifts to arithmetic subgroups. It is shown in Proposition 3.3.7 that any arithmetic subgroup Γ of a unipotent algebraic group U over the rationals admits a filtration similar to the one proven in Theorem 1.2.5, where the quotient of any two consecutive subgroups is isomorphic to the additive group of integers $\mathbb{Z}$. We subsequently consider the behaviour of the cohomology groups when applied to a direct limit of modules, such as the locally finite representations obtained from algebraic groups. The upshot is that arithmetic subgroups of unipotent algebraic groups permit the exchange of cohomology with direct limits.

All these facts combined enable us to prove the following results:

Theorem (van Est)**.** *Let U be a unipotent algebraic group over the field of rational numbers $\mathbb{Q}$ and let Γ be an arithmetic subgroup of U. Then for any U-module M, the module $M \otimes K[G]$ is acyclic for $H(\Gamma, -)$. In particular, $H(U, M) \cong H(\Gamma, M)$.*

Proof. This is shown in Theorem 3.4.2. □

Theorem (Hochschild)**.** *Let U be a unipotent algebraic group over a field K of characteristic 0 and let $\mathfrak{u}$ denote the Lie algebra of U. Then for any U-module M, the module $M \otimes K[G]$ is acyclic for $H(\mathfrak{u}, -)$. In particular, $H(U, M) \cong H(\mathfrak{u}, M)$.*

Proof. This is proven in Theorem 3.4.4. □

Both results rely on the same reduction steps. The first step uses the filtration of U and the induced filtrations on Γ and $\mathfrak{u}$ together with the associated spectral sequences to apply induction on the dimension of U, thus reducing to the case $U = G_a$. The second step then uses the local finiteness of the U-module M combined with the compatibility of $H(\Gamma, -)$ and $H(\mathfrak{u}, -)$ with direct limits to simplify to the case of a finite dimensional U-module. Finally, we can use the filtration of M to further lower the dimension of M, thus reducing to the case of the trivial module K_{tr}. In both cases, one is therefore left with computing either $H^1(\Gamma, K[t])$ (with $\Gamma \cong \mathbb{Z}$) or $H^1(\mathfrak{g}_a, K[t])$, which is done using the interpretation of these spaces in terms of derivations.

The idea for this work as well as the basic outline of the proof was provided by Prof. Schwermer, whom I would like to thank for his patience and invaluable support during the course of this work.

Chapter 1

Unipotent Algebraic Groups

This chapter is meant as a brief introduction to the different topics required to understand the basic structure of unipotent groups. We will first consider group schemes as a technical framework to deal with algebraic groups over various ground fields.

Through this chapter, K will denote an arbitrary field (until otherwise indicated). The term "K-algebra" will always refer to a commutative algebra over K, i.e. a commutative ring A together with a morphism of rings $K \to A$, called the structure morphism.

1.1 Group Schemes and Representations

Let R be a ring and denote the category of all commutative R-algebras by Alg_R. A set-valued functor (over R) is defined to be a covariant functor $Alg_R \to Sets$, where $Sets$ denotes the category of sets. If A is any R-algebra, there is an associated set-valued functor h_A, given by assigning each R-algebra B the set $\mathrm{Hom}_R(A, B)$ and each homomorphism of R-algebras $\varphi\colon B \to B'$ the map $h_A(\varphi)\colon \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, B')$ which maps $\alpha\colon A \to B$ to $\varphi \circ \alpha$.

A set-valued functor X is then called an *affine scheme* if it is naturally isomorphic to a functor h_A for some algebra A. By Yoneda's Lemma (cf. [Wat79][Section 1.3] or [HS97][Chapter II, Proposition 4.1]), the algebra A is uniquely determined (up to isomorphisms) and is said to represent X. If X is an affine scheme, we will often denote its representing R-algebra by $R[X]$ and call $R[X]$ the affine coordinate ring of X.

Given affine schemes X and Y, a morphism of affine schemes $X \to Y$ is simply a natural transformation of set-valued functors. Any morphism of R-algebras $\varphi^*\colon R[Y] \to R[X]$ determines a morphism $\varphi\colon X \to Y$ by mapping $\alpha \in X(B) = \mathrm{Hom}_R(R[X], B)$ to $\alpha \circ \varphi^*$ for any R-algebra B. Another application of Yoneda's Lemma shows that any morphism of affine schemes arises this way. More precisely, any morphism $\varphi\colon X \to Y$ is uniquely determined by a (unique) morphism of R-algebras $\varphi^*\colon R[Y] \to R[X]$, which will be called the *dual morphism* of φ.

These observations are the foundation of a dictionary to translate functorial constructions and properties to algebraic properties and constructions involving the R-algebra structure of the corresponding coordinate rings. For example, the direct product of two affine schemes X and Y is defined by $(X \times Y)(A) := X(A) \times Y(A)$. Translating the universal property of the direct product to its dual version in the category of R-algebras yields that $X \times Y$ is again an affine scheme, whose coordinate ring is given by the tensor product $R[X] \otimes R[Y]$.

Definition 1.1.1. *A* group scheme *over a ring R is a quadruple (G, μ, ν, ϵ) consisting of an affine scheme G and morphisms $\mu\colon G \times G \to G$, $\nu\colon G \to G$ and $\epsilon\colon h_R \to G$ satisfying the usual group identities. The morphisms μ, ν and ϵ are referred to as the multiplication, inversion and unit of G, respectively.*

We will often omit explicitly mentioning the morphisms μ, ν and ϵ and simply call G a group scheme. Note that h_R itself is a group scheme, often called the *trivial group scheme*, since h_R maps any R-algebra to the trivial group $\{e_A\}$, where e_A is the structure morphism of A as an R-algebra. We will therefore often denote h_R by $\{e\}$, to put additional emphasis on its role as the trivial group. This also implies that for any R-algebra A, the set $G(A)$ has a canonical group structure with multiplication $\mu(A)$, inversion $\nu(A)$ and identity $\epsilon(A)$ (note that $\epsilon(A)$ maps $\{e_A\}$ onto the identity element of $G(A)$, this is just the usual neutral element of $G(A)$ in the classical sense).

We will now consider the algebraic counterpart of this definition. For this, we fix a group scheme G over R. By our above observations, the morphisms μ, ν and ϵ correspond to R-algebra morphisms $\mu^*\colon A \to A \otimes A$, $\nu^*\colon A \to A$ and $\epsilon^*\colon A \to R$, which satisfy the dual identities of the group laws. An algebra A together with such morphisms is called a *Hopf algebra*, the morphisms μ^*, ν^* and ϵ^* are called the *comultiplication*, *coinverse* and *counit* (or augmentation) morphism respectively. From this point of view, group schemes over R correspond to Hopf algebras over R. This observation permits us to interpret the group structure on $G(A)$ in a slightly different way: Given f and g in $G(A)$ we define their composition to be $fg = \Delta^* \circ (f \otimes g) \circ \mu^*$, where $\Delta^*\colon B \otimes B \to B$ is the *diagonal map* mapping $x \otimes y$ to xy. An explicit calculation using the identities of the Hopf algebra structure on A shows that this composition is indeed associative, has a neutral element given by composing ϵ^* with the canonical structure map $R \to B$ and permits inverse elements, where the inverse of f is given by $f \circ \nu^*$. It is easy to see that the so defined group structure on $G(A)$ actually coincides with the one previously defined. The reader interested in the details of the computation may find them in [Yan77][Proposition 1.2].

In particular, any group scheme over R actually defines a group-valued functor (i.e. a functor $Alg_R \to Groups$) and it can be shown that any representable group-valued functor arises this way (see [Yan77][Proposition 1.5]). This observation will play an important role later on, when we define modules over group schemes.

We can also use the Hopf algebra structure to extend the base ring of a group scheme as follows. Given a group scheme G over a ring R and morphism of rings $\alpha\colon R \to S$, the tensor product $S[G] = R[G] \otimes_R S$ is again a Hopf algebra over S, where we regard $S[G]$ as an S-algebra via the canonical morphism $s \mapsto 1 \otimes s$. We will denote the associated group scheme by G_S and call G_S the *base extension* of G to S. From the functorial point of view, the morphism α induces a functor $R_\alpha\colon Alg_S \to Alg_R$, where we regard any S-algebra A as an algebra over R by defining $rx = \alpha(r)x$ for $r \in R$ and $x \in A$. This technique is often called restriction of scalars. The functor $G_S\colon Alg_S \to Groups$ can now be identified with the composition of G with R_α, which is tantamount to saying that any morphism of S-algebras $x\colon S[G] = R[G] \otimes S \to A$ is already uniquely determined by its restriction to $R[G]$ and that conversely, any morphism of R-algebras $R[G] \to A$ gives rise to a unique S-equivariant morphism $R[G] \otimes S \to A$. To put it simply, the base extension G_S of G to S is the unique group scheme over S satisfying $G_S(A) = G(A)$ for any S-algebra A.

If G and H are both R-group schemes and $\varphi\colon G \to H$ is a morphism of affine schemes, we will say that φ is a morphism of R-group schemes if it is compatible with the group structure on both G and H, i.e. if $\varphi \circ \mu_G = \mu_H \circ (\varphi \times \varphi)$. This is equivalent to saying that φ is actually a natural transformation of group-valued functors, since any morphism of abstract groups preserving multiplication necessarily preserves the neutral element and inversion. On the level of Hopf algebras, we may express this by saying that the dual morphism φ^* is a morphism of Hopf algebras, i.e. preserves μ^* in the sense that $\mu^* \circ \varphi^* = (\varphi^* \otimes \varphi^*) \circ \mu^*$. Note that just as φ necessarily preserves ν and ϵ if it preserves μ, the dual map φ^* preserves ν^* and ϵ^* once it is compatible with the comultiplication.

Having defined our objects of interest, we would like to have a good notion of a "subobject". Let X be an affine scheme with affine coordinate ring $R[X]$. We say that an affine scheme Y is a *closed subscheme* of X if its coordinate ring is given by a quotient $R[Y] = R[X]/I$ for some ideal I of $R[X]$. In this case, we shall say that I defines Y in X or that I is the defining ideal of Y in X. Note that the canonical projection $R[X] \to R[X]/I$ induces a canonical inclusion of Y into X.

For a group scheme G, one may now ask under which circumstances a closed subscheme H is again a group scheme with respect to a group structure induced by G, i.e. such that the canonical inclusion is a morphism of group schemes. As it turns out (see [Wat79][Section 2]), this is the case if and only if the ideal I defining H in G is a *Hopf ideal* of $R[G]$ i.e. if and only if I satisfies

$$\begin{aligned}
&\mu^*(I) \subseteq I \otimes R[G] + R[G] \otimes I,\\
&\nu^*(I) \subseteq I,\\
&\epsilon^*(I) = 0.
\end{aligned}$$

In this case, H is called a *closed subgroup scheme* (or simply a closed subgroup) of G.

Example 1.1.2. The most important example is the general linear group Gl_n over R, given by its affine coordinate ring $A = R[t_{ij}, t]/I$, with $i, j \in \{1, \ldots, n\}$, where I is the ideal generated by the polynomial $\det(t_{ij})t - 1$, together with the following morphisms as comultiplication, coinverse and augmentation

$$\mu^* : t_{ij} \mapsto \sum_k t_{ik} \otimes t_{kj},$$

$$\nu^* : t_{ij} \mapsto (-1)^{i+j} \det(t_{rs})_{r\neq i, s\neq j} t^{-1},$$

$$\epsilon^* : t_{ij} \mapsto \delta_{ij}.$$

In the special case of $n = 1$, we call Gl_1 the multiplicative group and denote it by G_m.
An example of a closed subgroup scheme of Gl_n is the group U_n of upper triangular matrices with 1 on the diagonal. It is given by the Hopf ideal I generated by the polynomials t_{ij} for $j < i$ and $t_{ii} - 1$ for all $i \in \{1, \ldots, n\}$. Hence the affine coordinate ring of U_n is given by the polynomial ring $R[t_{ij}]$ with $1 \leq i < j \leq n$ together with the induced Hopf algebra structure. Explicitly, we get

$$\mu^* : t_{ij} \mapsto \sum_k t_{ik} \otimes t_{kj},$$

$$\nu^* : t_{ij} \mapsto (-1)^{i+j} \det(t_{rs})_{r\neq i, s\neq j},$$

$$\epsilon^* : t_{ij} \mapsto 0.$$

We will also call U_n the standard unipotent group over R.
Another basic example is the additive group scheme G_a, given by the affine coordinate ring $R[t]$, together with the Hopf algebra structure

$$\mu^* : t \mapsto t \otimes 1 + 1 \otimes t,$$

$$\nu^* : t \mapsto -t,$$

$$\epsilon^* : t \mapsto 0.$$

Note that G_a is in fact isomorphic to U_2.

In all of the above examples, the affine coordinate rings had the special property that they were finitely generated R-algebras. While this is not necessarily true for all group schemes, in all the cases we are interested in, it will certainly be satisfied. In general, an affine scheme X whose coordinate ring is finitely generated as an R-algebra is said to be of *finite type*.

Definition 1.1.3. *Let K be a field. An* algebraic group *over K is a group scheme G whose coordinate ring is finitely generated as a K-algebra.*

For the reader familiar with algebraic groups in the classical setting, there is one subtle (but important) difference in the above definition compared to the classical case: We did not require algebraic groups to be reduced, which means that their coordinate rings may contain nilpotent elements (that is, so to speak, they no longer have varieties as their underlying surfaces, as varieties, in the classical sense, were assumed to be reduced). It turns out that this is indeed a more general definition, hence we will call an affine scheme *reduced* if its affine coordinate ring is reduced. Similarly, we will say that an affine scheme is *integral* if its coordinate ring is an integral domain and *connected* if the coordinate ring modulo the nilradical is an integral domain. These notions, while defined via algebraic conditions, have geometric origins, which may be found in Hartshorne's book [Har06], or many other books on algebraic geometry. The most important fact for the reader to keep in mind is the following theorem.

Theorem 1.1.4. *If K is a field and G is an algebraic group over K, then G, as an affine scheme, is connected if and only if it is integral.*

Proof. The proof, which relies on so-called separable subalgebras of the coordinate ring $K[G]$, can be found in [Wat79][Section 6.6]. □

In the theory of abstract groups, given a homomorphism of groups $G \to H$, the inverse image of any subgroup of H is a subgroup of G. The same is true for closed subgroups of group schemes, as we shall see in a moment.

In general, let X and X' be affine schemes over R and let Y' be a closed subscheme of X'. Given a morphism φ: $X \to X'$ of affine schemes, we define the inverse image of Y' under f to be the functor $\varphi^{-1}(Y')$ given by mapping any R-algebra A to $\varphi(A)^{-1}(Y'(A))$. This functor is actually a closed subscheme of X: Let I' be the ideal defining Y' in X' and consider the ideal I of $R[X]$ generated by $\varphi^*(I')$. We claim that $\varphi^{-1}(Y')$ is actually isomorphic to the affine scheme

Y given by the coordinate ring $R[G]/I$. To prove our claim, let $x \in Y(A)$. Then we may regard x as a homomorphism $R[G] \to A$ vanishing on I. Moreover, $\varphi(x) = x \circ \varphi^*$ implies that $f(x)$ vanishes on I', hence it is an element of $Y'(A)$ and therefore $x \in \varphi^{-1}(Y')(A)$. If on the other hand $x \in \varphi^{-1}(Y')(A)$ then $f(x) \in Y'(A)$ implies that I' is in the kernel of $f(x) = x \circ f^*$, which forces $f^*(I') \subset \ker(x)$. This implies $I \subset \ker(x)$ since $f^*(I')$ generates I and hence $x \in Y(A)$.

If we apply these considerations to the case of a morphism of group schemes φ: $G \to H$, we can even show slightly more: The inverse image of any closed subgroup H' of H is a closed subgroup of G. Indeed, let G' denote the inverse image of H' and let I and I' be the defining ideals of G' and H' respectively. In order to show that I is a Hopf ideal of $R[G]$, it is sufficient to check the necessary conditions on a generating set, hence on $\varphi^*(I')$. But since φ^* is a morphism of Hopf algebras we get

$$\begin{aligned}\mu_G^*(\varphi^*(I')) &= (\varphi^* \otimes \varphi^*)(\mu_H^*(I') \\ &\subseteq (\varphi^* \otimes \varphi^*)(I' \otimes R[H] + R[H] \otimes I') \\ &\subseteq I \otimes R[G] + R[G] \otimes I\end{aligned}$$

The other conditions of a Hopf ideal follow in a similar way. A particular consequence of this observation is the existence of a kernel for any homomorphism of group schemes:

Definition 1.1.5. *Let φ: $G \to H$ be a morphism of group schemes over R. We define the kernel of φ to be the closed subgroup* $\ker(\varphi)$ *of G defined by the inverse image of the trivial subgroup of H.*

Having defined closed subgroups, morphisms and kernels, we turn to another construction of group theory whose equivalent has not yet appeared in our discussion of group schemes, namely quotients of group schemes by closed subgroups. Up to now we have quietly avoided this subject as it is surprisingly difficult to deal with, especially in the general context we are working in. For this reason, our discussion will be quite brief and restricted to normal subgroups of algebraic groups over a field K.

Suppose N is a closed subgroup of a group scheme G over K. We will say that N is *normal* if for each K-algebra A, $N(A)$ is normal in $G(A)$. Note that the kernel of a morphism of group schemes is always a normal, closed subgroup.

Definition 1.1.6. *A morphism φ: $G \to H$ of group schemes is said to be a* quotient morphism *if the dual morphism φ^* is injective.*

Given a normal subgroup N of a group scheme G, a *quotient* of G by N is a quotient map π: $G \to H$ with $N = \ker(\pi)$. Most of the time, we will simply call H the quotient of G by N without explicitly mentioning the morphism π.

Theorem 1.1.7. *Let N be a normal subgroup of G. Then a quotient of G by N is unique up to isomorphism. If G is algebraic, then such a quotient always exists.*

Proof. This result can be found in [Wat79][Section 15.4 and 16.3]. □

We now have most of the basic concepts of group schemes at hand, except for one basic technique used to study groups: Representations. In order to define them, we need to keep in mind that the setup we are working in is based on functors. Thus we need to define functorial equivalents of R-modules and their "general linear groups". To do so, we first consider a more general situation. Suppose that M is an R-module and φ: $R \to S$ is a morphism of rings (commutative, as always in this section). Then we can form the tensor product $M \otimes S$ of M and S over R. This tensor product has a canonical structure as an S-module given by $s \cdot (m \otimes t) = m \otimes st$ and is usually called the extension of scalars of M from R to S. In particular, for any R-algebra A, $M \otimes A$ is an A-module in a canonical way (we simply extend the scalars via the structure morphism $R \to A$) and we may consider the group of A-linear automorphisms $\mathrm{Aut}_A(M \otimes A)$. Moreover, for any morphism of R-algebras $A \to B$ we naturally obtain morphisms of groups $M \otimes A \to M \otimes B$ and $\mathrm{Aut}_A(M \otimes A) \to \mathrm{Aut}_B(M \otimes B)$. These observations now motivate the following definition.

Definition 1.1.8. *Let M be an R-module. The* associated functor M_a *of M is given by $M_a(A) = M \otimes A$ and $M_a(\varphi) = \mathrm{id}_M \otimes \varphi$ for any R-algebra A and any morphism of R-algebras φ. The* general linear group *of M is given by* $\mathrm{Gl}(M)(A) = \mathrm{Aut}_A(M \otimes A)$.

Note that $\mathrm{Gl}(M)$ is, in general, not a group scheme. However, if M is free of finite rank n, then any choice of a basis of M yields an isomorphism $\mathrm{Gl}(M) \to \mathrm{Gl}_n$ of group-valued functors.

We can now define representations in just the same way as for abstract groups.

Definition 1.1.9. *A* representation *of a group scheme G on an R-module M is a morphism of group-valued functors ρ: $G \to \mathrm{Gl}(M)$.*

An R-module M together with such a morphism ρ is also called a *G-module*. This way, a representation of G on M is nothing more than an action of G on M_a by linear maps, i.e. a natural transformation ρ: $G \times M_a \to M_a$ satisfying the axioms of an action such that each $G(A)$ acts on $M_a(A)$ by A-linear maps (cf. [Wat79][Section 3.1]). By abuse of notation, we shall denote both the morphism $G \to \mathrm{Gl}(M)$ as well as the corresponding action $G \times M_a \to M_a$ by ρ.

As an interesting consequence of the latter description, we obtain the following lemma.

Lemma 1.1.10. *Let M be an R-module. Then there is a bijection between morphisms ρ: $G \to \mathrm{Gl}(M)$ and R-linear maps ρ^*: $M \to M \otimes K[G]$ satisfying $(\rho^* \otimes \mathrm{id}) \circ \rho^* = (\mathrm{id} \otimes \mu^*) \circ \rho^*$ and $(\mathrm{id} \otimes \epsilon^*) \circ \rho^* = \mathrm{id})$.*

Proof. See [Wat79][Section 3.2]. □

A linear map ρ^*: $M \to M \otimes K[G]$ satisfying the requirements of the lemma is called a *comodule map* and M together with such a map is called a $R[G]$-comodule. In accordance with our notational conventions, we shall denote comodule maps with raised stars, while denoting the associated representation without them.

Example 1.1.11. Any R-module M may be regarded as a trivial G-module via the trivial morphism $G \to Gl_n$ mapping $G(A)$ to $e \in Gl(A)$. We usually denote M with the trivial G action by M_{tr}. The associated comodule map is given by $m \mapsto m \otimes 1$.

Suppose G acts on an affine scheme X from the right, i.e. we have a morphism ρ: $G \times X \to X$ satisfying the properties of a right action, then the dual morphism ρ^*: $R[X] \to R[X] \otimes R[G]$ satisfies the properties of a comodule map. Hence any such action induces a representation of G on the affine coordinate ring of X. We call this structure the translation of functions on X induced by the action ρ. The name stems from the following interpretation, which we shall use later on: We may consider an element $f \in R[X]$ as a "rational function" on X. More precisely, for any $x \in X(A)$ and any R-algebra A we define $f(x) := x(f) \in A$ (in this sense, we may regard f as a natural transformation from X to the forgetful functor $Alg_R \to Sets$). Then acting with $g \in G(A)$ on f yields an element of $R[G] \otimes A$ and thus an element of the base extension $A[G]$. Hence gf is a "rational function" on G_A and given an A-algebra A' as well as $x \in X(A')$ we get

$$\begin{aligned}(gf)(x) &= x(gf)\\ &= (x \otimes g)(\rho^*(f))\\ &= f(xg),\end{aligned}$$

which justifies the idea of this action as right translation of functions in some sense.
A lot of important representations can be obtained this way: Consider the right action of G on itself by right multiplication $\mu\colon G \times G \to G$. The comodule structure on $R[G]$ obtained this way is called the *right regular representation*, and will be denoted by ρ_r^*. Note that $\rho_r^* = \mu^*$. We could also consider the action of G on itself by left multiplication. However, to turn it into a right action, we need to twist the factors and invert the element we act with, hence compose μ with $\mathrm{tw} \circ (\nu \otimes \mathrm{id})$. The representation of G on $R[G]$ obtained this way is called the *left regular representation*, which will be denoted by ρ_l.
Note that the inversion $\nu^*\colon R[G] \to R[G]$ defines an isomorphism between the left and the regular representation of G on its coordinate ring.

Similar to dual morphisms, we may now start to define concepts and properties functorially and translate them into more algebraic properties by using the comodule map. In particular, since we are dealing with group-valued functors, we can adapt most of the basic definitions from the representation theory of groups to group schemes. However, in the general case of a base ring R, one needs to pay close attention to the algebraic properties of a G-module M over R, such as flatness.

Given two G-modules M and N, an R-linear map $f\colon M \to N$ is said to be a morphism of G-modules or G-equivariant if $f(A)\colon M(A) \to N(A)$ commutes with the action of $G(A)$. If we denote the comodule maps of M and N by ρ_M^* and ρ_N^* respectively, then we can express this condition as

$$\rho_N^* \circ f = f \otimes \mathrm{id} \circ \rho_M^*$$

A *G-submodule* of a G-module M is an R-submodule N of M, together with a G-module structure such that the canonical inclusion $N \to M$ is a morphism of G-modules. In this case, an interpretation in terms of the comodule map is not immediately possible. However, if G is a *flat* group scheme (i.e. if the affine coordinate ring $R[G]$ is flat over R, which means that short exact sequences are preserved under the tensor product with $R[G]$), then $N \otimes R[G]$ is a $G(R[G])$-stable submodule of $M \otimes R[G]$ and thus $\rho^*(N) \subseteq N \otimes R[G]$. Conversely, any R-submodule N of M satisfying this condition is naturally a G-submodule of N with respect to the comodule map given by the restriction of ρ^* to N. This implies that in case of a flat group scheme, the G-module structure on N is uniquely determined by the one on M (cf. [Jan87][Section 2.9]).

To avoid technical difficulties, we shall now restrict ourselves to the case of group schemes over a field K, in which case any group scheme G is flat. Let G be a group scheme over K and M be a G-module. We define the module of *fixed points* of M to be the K-submodule M^G of M consisting of all m satisfying $g \cdot (m \otimes 1) = m \otimes 1$ for all $g \in G(A)$ and all K-algebras A. Taking

$g = \mathrm{id}_{K[G]} \in G(K[G])$ one sees that

$$M^G = \{m \in M \mid \rho^*(m) = m \otimes 1\}$$

As a consequence of this description, we can identify M^G with the group of G-homomorphism from K_{tr} to M: Any $m \in M^G$ determines a G-equivariant morphism $K_{tr} \to M$ by mapping x to xm and conversely, any such morphism is uniquely determined by the image m of the unit 1, which is then (due to G-equivariance) necessarily a fixed point. This implies that the fixed point functor which maps any G-module M to its fixed points M^G is naturally equivalent to $\mathrm{Hom}_G(K_{tr}, -)$ and hence left exact (this is true in general for flat group schemes).

When working over a field, representations of group schemes have another important property: They are *locally finite*, i.e. every element is contained in a finite dimensional invariant subspace.

Lemma 1.1.12. *Let G be a group scheme over K and M be a G-module. Then every finite dimensional subspace is contained in a finite dimensional G-submodule of M.*

Proof. Since any sum of G-submodules of M is again a G-submodule, it suffices to show that any element of M is contained in a finite dimensional G-stable subspace. Let $f_{i_{i \in I}}$ be a basis of $K[G]$ over K and $\rho^*: M \to M \otimes K[G]$ be the comodule map defining the G-module structure on M. Then for $m \in M$ we have

$$\rho^*(m) = \sum m_i \otimes f_i$$

with almost all $m_i = 0$. Furthermore, we may write $\mu^*(f_i) = \sum \lambda_{ijk} f_j \otimes f_k$ with $\lambda_{ijk} \in K$ and again almost all $\lambda_{ijk} = 0$. Then

$$\begin{aligned} \rho^*(m_i) \otimes f_i &= (\rho^* \otimes \mathrm{id})(\rho^*(m)) \\ &= (\mathrm{id} \otimes \mu^*)(\rho^*(m)) \\ &= \sum \lambda_{ijk} m_i \otimes f_j \otimes f_k. \end{aligned}$$

Comparing the coefficients of f_k yields

$$\rho(m_k) = \sum \lambda_{ijk} m_i \otimes f_j,$$

hence for $N = Km + \sum K m_i$ we get $\dim_K(N) < \infty$ and $\rho^*(N) \subseteq N \otimes K[G]$, which shows that N is a finite dimensional G-submodule of M containing m. □

This will be one of the key techniques in the last chapter, as it allows to reduce certain problems to finite dimensional modules, which are, to a certain extent, easier to deal with. Note that a similar statement holds for flat group schemes over arbitrary rings, yet the proof is different as the subspace is not necessarily described in such a nice way (see [Jan87][Section 2.13]).

Before beginning to consider unipotent groups, we need an additional notion from algebraic geometry. Suppose X is an affine scheme over a ring R and let $A = K[X]$ be its coordinate ring. A closed subscheme Y of X is said to be irreducible if the ideal I defining Y in X is a prime ideal. The maximal irreducible subschemes of X (i.e. closed subschemes such that the defining ideal I is a minimal prime ideal with respect to inclusion) are called the irreducible components of X. The *dimension* of X is then defined to be the supremum over all positive integers n such that there exists a strictly ascending sequence $p_1 \subsetneq \ldots \subsetneq p_n$ of prime ideals of A. It will be denoted by $\dim(A)$.

In general, dimension theory over an arbitrary base ring R can be very difficult, however one can compute the dimension of the polynomial ring $R[t_1, \ldots, t_n]$, which is in fact equal to $\dim(R) + n$. If $R = K$ is a is a field, then the affine space $\mathbb{A}^n_K$, whose underlying coordinate ring is precisely the polynomial ring $K[t_1, \ldots, t_n]$, therefore has dimension n (see [Liu02][Section 2.5, Corollary 5.17]). Since we may embed any algebraic scheme in some affine space, the dimension of any such scheme will also be finite. The most important result for us is the following proposition.

Proposition 1.1.13. *Suppose X is an integral algebraic scheme and $f \in K[X]$ be a non-nilpotent element. Then every irreducible component of $V(f) = K[X]/(f)$ has dimension* $\dim(X) - 1$

Proof. The proof can be found in [Liu02][Section 2.5, Corollary 5.26]. □

1.2 Unipotent Groups and Their Modules

Let G be an algebraic group over a field K. We will say that G is *unipotent* if G is isomorphic to a closed subgroup of U_n for some $n \in \mathbb{N}$. One can show that G is unipotent if and only if every simple G-module is isomorphic to K_{tr}, or, equivalently, if every G-module has nontrivial fixed points (see Waterhouse [Wat79][Section 8.3]).

Our particular interest is the structure of these groups if the given ground field is the field of rational numbers $\mathbb{Q}$. In what follows, we shall therefore assume that the field K is of characteristic 0. It turns out that working over such a field has a lot of advantages, as certain unpleasant phenomena such as nonreduced algebraic groups only arise over fields of positive characteristic. A particularly important consequence of the restriction of the characteristic of K is that all unipotent groups are in fact connected:

Theorem 1.2.1. *Over a field of characteristic* 0*, any unipotent group is connected.*

Proof. See Waterhouse [Wat79][Section 8.5]. □

We have already seen some examples of unipotent groups in the previous section. The "simplest" (nontrivial) such group is the additive group scheme G_a, which is isomorphic to U_2. It turns out to be worthwhile to take a closer look at G_a before moving on to arbitrary unipotent groups.

In general, given an arbitrary group scheme over K and a morphism $\varphi\colon G \to G_a$ of K-group schemes, the dual morphism $\varphi^*\colon K[t] \to K[G]$ is determined by an element $x = \varphi^*(t)$ of $K[G]$ satisfying $\mu_G^*(x) = x \otimes 1 + 1 \otimes x$. The elements of $K[G]$ satisfying this identity are said to be primitive and the set of all such elements is denoted by $P(K[G])$. It is then easy to check that any primitive element of $K[G]$ determines a morphism of Hopf algebras $K[t] \to K[G]$ and that the map $\varphi \mapsto \varphi^*(t)$ defines a bijection from the set of morphisms $\mathrm{Hom}(G, G_a)$ onto the set of primitive elements of $K[G]$. We can use this observation to classify the endomorphisms of G_a (as in [Wat79][8.4]).

Theorem 1.2.2. *If the characteristic of K is equal to* 0*, then* $\mathrm{Hom}(G_a, G_a) \cong K$*.*

Proof. By our above considerations, it is sufficient to classify the primitive elements of $K[t]$. Given $a \in K$, the element at is certainly primitive, thus we get an injective map $K \to P(K[t])$. It is therefore enough to show that any primitive element of $K[t]$ is actually of the form at for some $a \in K$. Suppose $f \in K[t]$ is primitive and let d be the degree of $f = \sum a_i t^i$. If we identify $K[t] \otimes K[t]$ with $K[x, y]$ then f has to satisfy

$$f(x) + f(y) = f(x + y)$$

and hence $\sum a_i(x^i + y^i) = \sum a_i(x + y)^i$. This immediately implies $a_0 = 0$ and by comparing the coefficients of the mixed terms resulting from expanding $a^i(x + y)^i$ starting from $i = 2$, we get $a_i = 0$ for $2 \leq i \leq d$. □

We are now able to relate the endomorphisms of G_a as a group scheme to its closed subgroups:

Proposition 1.2.3. *Every closed subgroup of G_a arises as the kernel of a morphism of group schemes $G_a \to G_a$.*

Proof. Let H be a closed subgroup of G_a and $I = (f)$, $f \in K[t]$, be the ideal defining H. Then $\epsilon^*(f) = 0$, i.e. $f(0) = 0$ and $\mu^*(f) \subseteq f \otimes K[t] + K[t] \otimes f$. For notational convenience we identify $K[t] \otimes K[t]$ with $K[x, y]$. Then the ideal $f \otimes K[t] + K[t] \otimes f$ is exactly the ideal generated by $f(x)$ and $f(y)$ and we claim that f is a primitive element of $K[t]$ i.e. $f(x + y) = f(x) + f(y)$. Since $f(x+y)$ is by assumption in the ideal generated by $f(x)$ and $f(y)$, there exist polynomials $a(x, y), b(x, y) \in K[x, y]$ such that

$$f(x + y) - f(x) - f(y) = a(x, y)f(x) + b(x, y)f(y).$$

Note that the left hand side has a degree in both x and y less than the degree of f. We then write $b(x, y) = b_1(x, y) + b_2(x, y)$ where $\deg_x b_1(x, y) < \deg f$ and $\deg_x b_2(x, y) \geq \deg f$. Suppose $a(x, y) \neq 0$ then $\deg_x a(x, y)f(x) \geq \deg f$, so since the left hand side has an x-degree smaller then $\deg f$ we must have $a(x, y)f(x) + b_2(x, y)f(y) = 0$, hence

$$f(x + y) - f(x) - f(y) = b_1(x, y)f(y).$$

By comparing y-degrees, this implies $b_1(x, y) = 0$, thus finally $f(x+y) = f(x)+f(y)$ as claimed. Since f is primitive, we get an endomorphism of Hopf algebras φ^*: $K[t] \to K[t]$ by mapping t to f. The dual morphism φ: $G_a \to G_a$ is a morphism of group schemes, whose kernel is clearly equal to H. □

Corollary 1.2.4. *If K is a field of characteristic 0, then G_a has no nontrivial closed subgroups.*

Proof. This is an immediate consequence of our above observations: Any closed subgroup H of G_a is defined by a primitive element $f \in K[t]$. Applying the theorem, we must have $f = \lambda t$ for some $\lambda \in K$. If $\lambda = 0$ then $H = G_a$, otherwise we may assume $\lambda = 1$. In this case, H is the trivial subgroup of G_a, consisting only of the neutral point. □

Keeping the assumption that the characteristic of K is 0, we now consider the standard unipotent group U_n over K. Let Λ be the set of all pairs (i, j) with $1 \leq i < j \leq n$, then the coordinate ring of U_n is given by $K[U_n] = K[t_{ij} \mid (i, j) \in \Lambda]$. By defining $(i, j) < (k, l)$ if $j < l$

or $j = l$ and $i > k$ we obtain an order on Λ, which will be used to construct a sequence of closed subgroups of U_n (in fact, we will prove [Hum75][Section 17.6, Exercise 7]). For this purpose, let I_{lk} be the ideal generated by all t_{ij} with $(i,j) < (l,k)$. These ideals are all Hopf ideals of $K[U_n]$ and each is normal in the preceding one. This can be checked on the matrix groups $U_n(A)$, since each subgroup deletes exactly one entry of its predecessor. More precisely, if we denote the scheme associated to $K[U_n]/I_{kl}$ by U_{kl} then $U_{kl}(A)$ consists of all upper triangular $n \times n$-matrices (a_{ij}) with $a_{ii} = 1$ and $a_{ij} = 0$ if $(i,j) < (k,l)$. Moreover, since all U_{kl} are again unipotent, they are connected, and since each step adds only one variable to the defining ideal, we get a sequence of closed subgroups U_{kl} of U_n, each normal and of codimension one in its predecessor. We now want to determine the quotient of two consecutive subgroups in this chain. Let $(l,k) \in \Lambda$ and (i,j) be the preceding index. Then $t_{ij} \notin I_{ij}$ but $t_{ij} \in I_{kl}$. Consider the morphism of group schemes $\varphi_{ij}\colon U_{ij} \to G_a$ defined by the dual $\varphi_{ij}^*\colon K[t] \to K[U_{ij}]$ with $\varphi_{ij}^*(t) = t_{ij}$. This morphism is injective and thus a quotient map. The kernel of φ_{ij} is given by the ideal of $K[U_{ij}] = K[U_n]/I_{ij}$ generated by t_{ij}, which is precisely the ideal defining the subgroup U_{kl} in U_{ij}. We therefore obtain a sequence of closed subgroups

$$U_n = U_{1,2} \supsetneq U_{2,1} \supsetneq \ldots \supsetneq U_{n-1,n} \cong G_a,$$

each normal in its predecessor, where each consecutive pair has quotient G_a.

The following theorem generalises this result to arbitrary unipotent groups.

Theorem 1.2.5. *Let U be a unipotent algebraic group of dimension n over a field K of characteristic 0. Then there exist a descending sequence of closed subgroups*

$$U = U_n \supsetneq U_{n-1} \supseteq \ldots \supseteq U_0 = \{e\}$$

such that each U_i is a normal subgroup of U_{i+1} of codimension 1 satisfying $U_{i+1}/U_i \cong G_a$.

Proof. We apply induction on the dimension n of U. If $n = 0$ then $K[U]$ is a finite dimensional integral domain, hence Artinian and must therefore be a field (this follows from [AM69][Theorem 8.5]). The morphism $\epsilon^*\colon K[U] \to K$ now forces $K[U] = K$, thus U is the trivial group scheme. Let $n > 0$ and assume the assertion to be valid for all unipotent groups of smaller dimension. After identifying U with a closed subgroup of U_n, we may assume U to be a closed subgroup of U_n to begin with. Then U is defined by some ideal I of $K[U_n]$. We now choose $(i,j) \in \Lambda$ maximal with respect to $I \supset I_{ij}$ and let (k,l) be the next larger index with respect to our chosen ordering. By construction $I \not\supset I_{kl}$, so in particular $t_{ij} \notin I$. We then consider the restriction of φ_{ij} to U. The dual map of this restriction is given by the composition of φ_{ij}^* with the canonical projection $K[U_{ij}] \to K[U] = K[U_{ij}]/I$. Then the kernel of this morphism of Hopf algebras $K[t] \to K[U]$ is a Hopf ideal of $K[t]$. But since G_a has no proper closed subgroups, this ideal is either zero or equal to (t). The latter case would imply that $t_{ij} = \varphi_{ij}^*(t) \in I$, hence $I_{jk} = I_{ij} + t_{ij} \subset I$ which contradicts our assumption on the maximality of (i,j). This implies that $\varphi_{ij}^*\colon K[t] \to K[U]$ is injective, hence φ_{ij} is a quotient map whose kernel V is given by the ideal of $K[U]$ generated by t_{ij}. In particular, V is a closed normal subgroup of codimension 1 in U with $U/V \cong G_a$. The assertion now follows by applying the induction hypothesis to V. □

Note that our proof of this result only works in characteristic 0, even though the statement itself is valid over any perfect field. But if we take a closer look at the proof, we can actually derive a slightly stronger result. Let U again be a closed subgroup of some U_n defined by the ideal I of $K[U_n]$ and choose $(i,j) \in \Lambda$ as in the proof above. Then the closed normal subgroup V is obtained as the kernel of $\varphi_{ij}\colon U \to G_a$. In particular, V is the zero set of t_{ij} in $K[U]$. We may express the fact that $U/V \cong G_a$ as having an exact sequence

$$0 \to V \to U \to G_a \to 0$$

of K-group schemes. On the other hand, if we consider $\sigma^*\colon K[U] \to K[t]$ given by the restriction of the K-algebra homomorphism $K[U_n] = K[t_{ij}] \to K[t]$ with $t_{ij} \mapsto t$ and $t_{kl} \mapsto 0$ for $(k,l) \neq (i,j)$, then it is easy to check that $\sigma\colon G_a \to U$ is a morphism of group schemes. In particular, σ splits φ_{ij}, i.e. the composition $\varphi_{ij} \circ \sigma$ is the identity on G_a. As a consequence, U is actually the *semidirect product* of V with G_a. This implies that U is isomorphic to the direct product $V \times G_a$ as an affine scheme, since this is valid as sets for any K-algebra A. If we consider the right action of V on $U = V \rtimes G_a$ via $((v',g),v) \mapsto (v^{-1},e)(v',g)$ (which is nothing but the left regular action of U on itself, restricted to an action of V) and the left regular action of V on $V \times G_a$ (which is defined in just the same way), then the isomorphism of affine schemes $V \times G_a \to V \rtimes G_a$ is compatible with this V-action. This implies that the coordinate ring $K[U]$ is isomorphic to $K[t] \otimes K[V]$ as a V-module, where we regard $K[t]$ as a trivial V-module. As a consequence, we can easily compute the fixed points of $K[U]$ under the left regular action of V:

Lemma 1.2.6. *Let M and N be G-modules. If the action on M is trivial then $(M \otimes N)^G = M \otimes N^G$.*

Proof. We first choose a basis $(m_i)_{i\in I}$ for M and $(n_j)_{j\in J}$ for N. Then the set of all $m_i \otimes n_j$ is a basis of $M \otimes N$, hence given $x \in (M \otimes N)^G$ we may write $x = \sum_{ij} \lambda_{ij} m_i \otimes n_j$ for some unique $\lambda_{ij} \in K$ almost all 0. For $g \in G(A)$ we thus get

$$\begin{aligned} x &= gx \\ &= g \sum_{ij} \lambda_{ij} m_i \otimes n_j \otimes 1 \\ &= \sum_{ij} \lambda_{ij} m_i \otimes g n_j \otimes 1. \end{aligned}$$

Hence $0 = \lambda_{ij}(n_j - g_n j)$, which implies that either $\lambda_{ij} = 0$ or $n_j \in N^G$. In particular, $x \in M \otimes N^G$ and hence $(M \otimes N)^G \subset M \otimes N^G$. Since the other inclusion is obvious, the assertion follows. □

Applying this lemma to the situation above now yields $K[U]^V \cong (K[t] \otimes K[V])^V \cong K[t]$ as V-modules.

Finally, we can show that a filtration similar to Theorem 1.2.5 exists for finite dimensional modules over unipotent groups. This is the content of the following proposition.

Proposition 1.2.7. *Let K be an arbitrary field, U be a unipotent algebraic group over K and M be a nontrivial U-module. If $n = \dim_K(M)$ is finite then there exists an ascending sequence*

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$$

of U-submodules of M such that $M_{i+1}/M_i \cong K_{tr}$.

Proof. We apply induction on n. If $n = 1$ there is nothing to show, so we may take $n \geq 2$ and assume the assertion for all positive integers smaller than n. Since (by definition) any nontrivial module over a unipotent group has nonzero fixed points, the module of fixed points M^U contains a nonzero element m_1. Let $M_1 = Km_1$ be the subspace of M generated by m_1. Then M_1 is a U-submodule of M isomorphic to K_{tr} and the quotient $M' = M/M_1$ has a canonical U-module structure. Moreover $\dim(M') = n - 1$, thus by induction hypothesis there exists a filtration of M' by U-submodules M'_i which satisfies the requirements of the lemma. Defining M_i to be the preimage of M'_i with respect to the canonical projection $M \to M' = M/M_1$ then yields the required sequence of U-submodules of M. □

1.3 The Lie Algebra of an Algebraic Group

Lie Algebras in General Let K be a field. A *Lie algebra* over K is a finite dimensional K-vector space $\mathfrak{g}$ together with a K-bilinear map $[.,.]:V \times V \to V$, called the Lie bracket, satisfying

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$
2. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{g}$ (Jacobi Identity)

In particular, any Lie algebra is anticommutative, i.e. satisfies $[x, y] = -[x, y]$. A morphism of Lie algebras over K is simply a K-linear map compatible with the respective Lie brackets.

Given a Lie algebra $\mathfrak{g}$, a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a Lie subalgebra of $\mathfrak{g}$ if it is closed under the Lie bracket, i.e. if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If in addition $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, then $\mathfrak{h}$ is called an ideal of $\mathfrak{g}$. In this case, the quotient $\mathfrak{g}/\mathfrak{h}$ has a canonical structure as a Lie algebra, given by $[\pi(x), \pi(y)] = \pi([x, y])$ where, $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denotes the canonical projection. Moreover, with respect to this structure, the projection π is a morphism of Lie algebras.

Example 1.3.1. Any associative algebra A over K gives rise to a Lie algebra in a canonical way by defining the Lie bracket to be the commutator bracket, i.e. $[x, y] = xy - yx$. To distinguish between A as an associative algebra and A with the canonical Lie algebra structure, we will denote the latter by $(A, [.,.])$.

Moreover, any K-space V may be regarded as a Lie algebra by defining $[x, y] = 0$. This is called the trivial Lie bracket.

A *representation* of a Lie algebra $\mathfrak{g}$ on a K-space V is defined to be a morphism of Lie algebras $\mathfrak{g} \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is regarded as a Lie algebra via the commutator bracket. Given such a morphism, we will also call V a $\mathfrak{g}$-module and say that $\mathfrak{g}$ acts on V. A morphism $V \to W$ of $\mathfrak{g}$-modules is a linear map compatible with the respective $\mathfrak{g}$-actions. Given a $\mathfrak{g}$-module V, a *fixed point* of $\mathfrak{g}$ in V is a vector $v \in V$ such that $Xv = 0$ for all $X \in \mathfrak{g}$. The set of all such points is a subspace of V, called the fixed point module and denoted by $V^{\mathfrak{g}}$. If $V = V^{\mathfrak{g}}$ then V is called a trivial $\mathfrak{g}$-module. Note that any K-space V may be regarded as a trivial $\mathfrak{g}$-module and that we may, as in the case of group schemes, identify the fixed point module of a $\mathfrak{g}$-module V with the space of homomorphisms $K_{tr} \to V$. Other notions, such as the tensor product of modules or the direct limit of modules, are defined in the obvious way.

One of the most important concepts in the representation theory of Lie algebras is that of a universal enveloping algebra (see [HS97][Section VIII.1]):

Definition 1.3.2. *Let $\mathfrak{g}$ be a Lie algebra over K. The universal enveloping $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the quotient $\mathcal{T}(\mathfrak{g})/\mathcal{I}$ where $\mathcal{T}(\mathfrak{g})$ is the tensor algebra on $\mathfrak{g}$ and $\mathcal{I}$ is the ideal of $\mathcal{T}(\mathfrak{g})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$.*

The universal enveloping algebra is an associative algebra over K and satisfies the following universal property: For any associative algebra A over K and any morphism of Lie algebras φ: $\mathfrak{g} \to (A, [.,.])$ there exists a unique morphism of associative algebras $\hat{\varphi}$: $\mathcal{U}(\mathfrak{g}) \to A$ such that $\varphi = \hat{\varphi} \circ i$, where i: $\mathfrak{g} \to (\mathcal{U}(\mathfrak{g}), [.,.])$ is the canonical morphism of Lie algebras given by $x \mapsto \bar{x} \in \mathcal{T}(\mathfrak{g})/\mathcal{I}$. In particular, representations of $\mathfrak{g}$ on a K-space V correspond bijectively to $\mathcal{U}(\mathfrak{g})$-module structures on V. Moreover, we have a canonical isomorphism of groups $\mathrm{Hom}_{\mathfrak{g}}(V, W) \to \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(V, U)$. In this sense, the category of $\mathfrak{g}$-modules is precisely the category of $\mathcal{U}(\mathfrak{g})$-modules.

Example 1.3.3. Let V be a K-space of dimension n regarded as a trivial Lie algebra. Then $\mathcal{I} = (0)$, so $\mathcal{U}(V) = \mathcal{T}(V) \cong K[t_1, \dots, t_n]$.

The Algebra of Distributions of a Group Scheme If G is a group scheme over K, then the augmentation morphism ϵ^*: $K[G] \to K$ is a K-algebra homomorphism defining the neutral element e of $G(K)$. The kernel $I_e = \ker(\epsilon^*)$ of ϵ^* is a maximal ideal of $K[G]$, called the *augmentation ideal.* We say that a linear map δ: $K[G] \to K$ is a *distribution* on G (with support in e) if there exists an $n \in \mathbb{N}$ such that $\delta(I_e^{n+1}) = 0$. The smallest such n is called the degree of δ. Note that the family of all distributions on G forms a K-subspace of the dual space $K[G]^*$ of $K[G]$ and we will denote this space by $\mathrm{Dist}(G)$. For any $n \in \mathbb{N}$, the distribution of degree less than or equal to n form a subspace of $\mathrm{Dist}(G)$, which will be denoted by $\mathrm{Dist}_n(G)$. Note that the space $\mathrm{Dist}(G)$ is the direct union of all $\mathrm{Dist}_n(G)$.

Given $\delta \in \mathrm{Dist}(G)$, we call $\delta(1)$ the *constant term* of δ. The evaluation map $\mathrm{Dist}(G) \to K$ which assigns to each distribution its constant term is a linear map, the kernel of which is given by all distributions with vanishing constant term. This yields a decomposition of $\mathrm{Dist}(G)$ into a direct sum $K \oplus \mathrm{Dist}^+(G)$, where $\mathrm{Dist}^+(G)$ is the kernel of our evaluation map. This decomposition is compatible with the filtration of $\mathrm{Dist}(G)$ given by the subspaces $\mathrm{Dist}_n(G)$ in the sense that $\mathrm{Dist}_n(G) = K \oplus \mathrm{Dist}_n^+(G)$ with $\mathrm{Dist}_n^+(G)$ being the space of all distributions of degree less than or equal to n with vanishing constant term.

We can now make use of the coalgebra structure on $K[G]$ to endow the K-space $\mathrm{Dist}(G)$ with the structure of an associative (but not necessarily commutative) algebra over K as follows. Given two distributions δ and γ of G, we define

$$\delta \cdot \gamma = \Delta^* \circ (\delta \otimes \gamma) \circ \mu^*$$

where Δ: $K \otimes K \to K$ is the canonical isomorphism given by mapping $x \otimes y$ to xy. It is easy to check that $\mathrm{Dist}(G)$ is an algebra with respect to this multiplication. Note that the unit of $\mathrm{Dist}(G)$ is precisely ϵ^*.

Definition 1.3.4. *Let G be a group scheme. The algebra* $\mathrm{Dist}(G)$ *is called the* algebra of distributions *on G.*

The space $\mathrm{Dist}_1^+(G)$ has a special meaning. By definition, it is the space of all linear maps $K[G] \to K$ such that I^2 is mapped to 0. Moreover, we have an exact sequence of K-spaces

$$0 \to I_e \to K[G] \to K \to 0,$$

which yields an isomorphism $K[G] \cong K \oplus I$ of vector spaces over K. The K-part of this decomposition is spanned by 1, therefore any distribution with vanishing constant term can be interpreted as a linear map $I \to K$. If $\delta \in \mathrm{Dist}_1^+(G)$, then δ vanishes on I^2. In particular, we may identify δ with an element of the dual space $(I/I^2)^*$. As any element of the latter space can be extended to a distribution $K[G] \to K$ with vanishing constant term by using the direct sum decomposition above, we obtain an isomorphism from Dist_1^+ to $(I/I^2)^*$. The latter space is precisely the *tangent space* of G at e in the classical sense (Mumford's "Red Book" contains an exposition of the different equivalent formulations of the tangent space, see [Mum99][Section 3.4, Theorem 3]).

We can use the same observation to obtain a decomposition of the tensor product $K[G] \otimes K[G] \cong (K \otimes K) \oplus (K \otimes I) \oplus (I \otimes K) \oplus (I \otimes I)$. Given $f \in I$ we can therefore find $\lambda_i \in K$ and $f_j \in I$ such that

$$\mu^*(f) = \lambda_1(1 \otimes 1) + \lambda_2 \otimes f_2 + f_3 \otimes \lambda_3 + f_4 \otimes f_5.$$

Then $(\epsilon^* \otimes \mathrm{id})(\mu^*(f)) = f$ implies

$$f = \lambda_1 + \lambda_3 f_3,$$

which forces $\lambda_1 = 0$ as $f - \lambda_3 f_3 \in I$ and thus $\lambda_3 = 1$, i.e. $f = f_3$. Similarly, $(\mathrm{id} \otimes \epsilon^*)(\mu^*(f)) = f$ implies $\lambda_2 = 1$ and $f = f_2$, hence

$$\mu^*(f) \in f \otimes 1 + 1 \otimes f + I \otimes I.$$

We can use this formula together with the fact that μ^* is a morphism of K-algebras to obtain

$$\mu^*(I^k) \in \sum_{i=0}^{k} I^i \otimes I^{k-i}.$$

In particular, the product of $\delta \in \mathrm{Dist}_n(G)$ and $\gamma \in \mathrm{Dist}_m(G)$ lies in $\mathrm{Dist}_{n+m}(G)$. We can even refine this to get

$$\mu^*(f_1 \dots f_k) \in \prod_{i=1}^{k} (f_i \otimes 1 + 1 \otimes f_i) + \sum_{i=1}^{k} I^i \otimes I^{k+1-i}.$$

If we now apply $\delta \otimes \gamma$ to a product $\mu^*(f_1 \dots f_{n+m})$, the sum $\sum_{i=1}^{n+m} I^i \otimes I^{k+1-i}$ vanishes as either $i < n$ and thus $\gamma(I^{k+1-i}) = 0$ or $i \geq n$ whence $\delta(I^i) = 0$. A similar observation applied to the expansion of the product $\prod_{i=1}^{n+m}(f_i \otimes 1 + 1 \otimes f_i)$ yields that the only nonvanishing terms are those of the form $f_J \otimes f_{J^c}$ where $J \subset \{1, \dots, n+m\}$ ranges over all subsets consisting of n elements, J^c denotes its complement and the element f_J (resp. f_{J^c}) denotes the product of all all f_j with $j \in J$ (resp. $j \in J^c$). The same discussion applies to $\gamma \otimes \delta$, with the only difference being that the nonvanishing terms are of the form $f_{J^c} \otimes f_J$. If we now consider the commutator $[\delta, \gamma] = \delta \otimes \gamma - \gamma \otimes \delta$, then

$$[\delta, \gamma](f_1 \dots f_{n+m}) = \sum_J \delta(f_J)\gamma(f_{J^c}) - \sum_J \gamma(f_{J^c})\delta(f_J) = 0$$

and hence $[\delta, \gamma] \in \mathrm{Dist}_{n+m-1}$. This implies that the tangent space $\mathrm{Dist}_1^+(G)$ of G at e is a Lie algebra with respect to the commutator bracket.

Definition 1.3.5. *Let G be a group scheme over K. The K-space* $\mathrm{Dist}_1^+(G)$ *together with the commutator bracket is called the* Lie algebra *of G and will be denoted by* $\mathrm{Lie}(G)$ *or* $\mathfrak{g}$.

Example 1.3.6. Consider the additive group G_a over a field K of characteristic 0. The coordinate ring of G_a is given by the polynomial ring $K[t]$ in one indeterminate. The augmentation ϵ^* simply maps t to 0 and the comultiplication μ^* maps t to $t \otimes 1 + 1 \otimes t$. Then the space $\mathrm{Dist}(G)$ consists of all linear maps $K[G] \to K$ with $\delta(t^k) = 0$ for all but finitely many k. Hence the maps δ_n: $K[t] \to K$ with $\delta_n(t^n) = 1$ and $\delta_n(t^k) = 0$ for all $k \neq n$ constitute a basis of $\mathrm{Dist}(G)$. Moreover,

$$\begin{aligned}(\delta_n\delta_m)(t^k) &= \delta_n \otimes \delta_m(\mu^*(t)^k) \\ &= \delta_n \otimes \delta_m((t \otimes 1 + 1 \otimes t)^k) \\ &= \sum_{i=0}^{k} \binom{k}{i} \delta_n(t^i) \otimes \delta_m(t^{k-i})\end{aligned}$$

shows that $(\delta_n\delta_m)(t^k) = 0$ if $k \neq n+m$ and $(\delta_n\delta_m)(t^{n+m}) = \binom{n+m}{n}$. This implies $\delta_n\delta_m = \binom{n+m}{n}\delta_{n+m}$. In particular, the algebra $\mathrm{Dist}(G)$ is commutative and the map $\mathrm{Dist}(G) \to K[t]$ given by $\delta_n \mapsto \frac{t^n}{n!}$ is an isomorphism of K-algebras (recall that we assumed $\mathrm{char}(K) = 0$ so dividing by $n!$ is possible). The Lie algebra $\mathfrak{g}_a = \mathrm{Dist}_1^+(G_a)$ is given by the dual space of $(t)/(t^2)$, which is one dimensional and isomorphic to K.

In the above example, the algebra $\mathrm{Dist}(G_a)$ coincides with the universal enveloping algebra of $\mathfrak{g}_a = \mathrm{Lie}(G_a)$. This is wrong in general (see [Jan87][7.10]), however we have the following result:

Proposition 1.3.7. *If K is a field of characteristic* 0 *and G is an algebraic group over K with Lie algebra $\mathfrak{g}$, then there is an isomorphism $\mathcal{U}(\mathfrak{g}) \to \mathrm{Dist}(G)$.*

Proof. See [DG80][II, §6, Section 1, "Cartier's Theorem"]. □

We can use this result to obtain representations of the Lie algebra $\mathfrak{g}$ from representations of the algebraic group G. If M is a G-module with comodule map ρ^*: $M \to M \otimes K[G]$ and δ is a distribution of G, we define

$$\delta \cdot m = \mathrm{id} \otimes \delta(\rho^*(m)).$$

It is easy to check that this yields a $\mathrm{Dist}(G)$-module structure on M. Since any morphism f: $M \to N$ of G-modules is also a morphism of $\mathrm{Dist}(G)$-modules with respect to this induced structure, we get an exact functor from the category of G-modules to the category of $\mathrm{Dist}(G)$-modules. As $\mathrm{Dist}(G) \cong \mathcal{U}(\mathfrak{g})$, the latter category is actually the same as the category of $\mathfrak{g}$-modules, thus we have constructed a functor from the category of G-modules to the category of $\mathfrak{g}$-modules.

Assume from now on that the characteristic of the base field K is equal to 0 and let G be an algebraic group over K with Lie algebra $\mathfrak{g}$. We want to study properties of a G-module M which are preserved when passing to the Lie algebra $\mathfrak{g}$. First of all, the local finiteness of M as a G-module immediately carries over to M as a $\mathfrak{g}$-module. Moreover, any G-submodule of M is certainly a $\mathfrak{g}$-submodule of M and as far as algebraic groups over a general field K are concerned, the converse is true as well provided that G is connected.

Lemma 1.3.8. *If G is a connected algebraic group over a field K of characteristic* 0 *and M is a G-module, then a subspace N of M is a G-submodule of M if and only if it is a $\mathfrak{g}$-submodule.*

Proof. The proof of this result can be found in Jantzen [Jan87][Lemma 7.15]. In fact, the result proven there is slightly more general. This result is only a special case if one takes into account that algebraic groups in characteristic 0 are always reduced and hence integral if and only if they are connected. □

If we assume our groups to be connected, then we can even show that our functor from G-modules to $\mathfrak{g}$-modules is actually *fully faithful*:

Lemma 1.3.9. *With the assumptions as in the previous lemma one has*

$$\mathrm{Hom}_G(M, N) \cong \mathrm{Hom}_{\mathfrak{g}}(M, N)$$

for any two G-modules M, N.

Proof. See Jantzen [Jan87][Lemma 7.16]. □

This lemma has an interesting consequence regarding fixed points. Suppose M is a G-module with comodule map ρ^* and $m \in M$ is a fixed point of the G-action on M. Then $\rho^*(m) = m \otimes 1$ and the action of any $X \in \mathfrak{g}$ on m is given by $Xm = (\mathrm{id} \otimes X)(\rho^*(m)) = X(1)m$. As $X \in \mathfrak{g} = \mathrm{Dist}_1^+(G)$ is a distribution with vanishing constant term, we have $X(1) = 0$ and hence $Xm = 0$. This implies that m is a fixed point of M under the associated $\mathfrak{g}$-action, hence passing to the $\mathfrak{g}$-module structure preserves fixed points, i.e. $M^G \subseteq M^{\mathfrak{g}}$. In particular, passing from G-modules to $\mathfrak{g}$-modules maps trivial modules to trivial modules. An immediate consequence of this is that the fixed point sets are actually equal, since

$$M^G \cong \mathrm{Hom}_G(K_{tr}, M) \cong \mathrm{Hom}_{\mathfrak{g}}(K_{tr}, M) \cong M^{\mathfrak{g}}.$$

Summing up, we have the following corollary.

Corollary 1.3.10. *Suppose K is a field of characteristic* 0 *and M is a module of a connected algebraic group G over K with Lie algebra $\mathfrak{g}$. Then $M^G = M^{\mathfrak{g}}$.*

We now turn to a slightly different topic. Suppose $\varphi\colon G \to H$ is a morphism of group schemes over K. Then its dual morphism $\varphi^*\colon K[H] \to K[G]$ preserves the augmentation ϵ^*, i.e. $\epsilon_H^* = \epsilon_G^* \circ \varphi^*$. This implies that $(\varphi^*)^{-1}(I_G) = I_H$ where I_G and I_H are the augmentation ideals of $K[G]$ and $K[H]$ respectively. In particular, $\varphi^*(I_H^{n+1}) \subset I_G^{n+1}$, hence φ induces a linear map $K[H]/I_H^{n+1} \to K[G]/I_G^{n+1}$. If we transpose this map we get a linear map $\mathrm{Dist}_n(G) \to \mathrm{Dist}_n(H)$ which in turn yields a linear map $\mathrm{Dist}(G) \to \mathrm{Dist}(H)$. This map will be denoted by $d\varphi$ and will be called the *differential* of φ. Note that $d\varphi$ maps $\mathrm{Dist}_n^+(G)$ to $\mathrm{Dist}_n^+(H)$, hence gives rise to a linear map $\mathfrak{g} \to \mathfrak{h}$, which, by abuse of notation, will also be denoted by $d\varphi$.

If $\varphi\colon G \to H$ has an injective dual morphism φ^*, then $(\varphi^*)^{-1}(I_G) = I_H$ implies that the resulting map $K[H]/I_H \to K[G]/I_G$ is injective, thus the dual morphism $d\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is onto. Using this observation, we can prove the following lemma.

Lemma 1.3.11. *Suppose N is a normal subgroup of an algebraic K group G. Then the Lie algebra $\mathfrak{n} = \mathrm{Lie}(N)$ is an ideal of $\mathfrak{g} = \mathrm{Lie}(G)$. If the characteristic of K is* 0 *and $H = G/N$ is* *the quotient of G by N then $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{h}$, where the isomorphism is induced by the differential of the projection $\pi\colon G \to H$.*

Proof. We will only sketch a proof. In general, over a field of characteristic 0, the dimension of any algebraic group is equal to the dimension of its Lie algebra (this follows from the fact that algebraic groups in characteristic 0 are smooth, see Waterhouse [Wat79][Section 11.6 and 12.2]). Moreover, one can show that in the situation of the lemma $\dim(G) = \dim(N) + \dim(H)$ (one reduces to the case of an algebraically closed K and uses [Bor91][Chapter I, Corollary 1.4]), hence in particular $\dim \mathfrak{g} = \dim \mathfrak{n} + \dim \mathfrak{h}$.

As π is a quotient map, its dual morphism π^* is injective and hence $d\pi\colon \mathfrak{g} \to \mathfrak{h}$ is onto. The assertion now follows if we prove that $\mathfrak{n} \subset \ker(d\pi)$, as the dimensions of these two spaces are equal. If one uses the interpretation of the Lie algebra in terms of derivations (see Waterhouse [Wat79][Section 12.2]), the differential $d\pi$ maps $X \in \mathfrak{g} = \mathrm{Der}_K(K[G], K)$ to $X \circ \pi^*$. As N is defined by the ideal I of $K[G]$ generated by $\pi^*(I_H)$, a derivation $K[N] \to K$ is nothing but a derivation $K[G] \to K$ vanishing on I. But if X is any such derivation, $d\pi(X)(I_H) = (X \circ \pi^*)(I_H) \subset X(I) = 0$, hence $d\pi(X)$ vanishes on I_H as well as on $K \cdot 1$ (since it is a K-derivation), and thus on all of $K[N] \cong K \oplus I_H$, which implies $X \in \ker(d\pi)$, as claimed. Moreover, as the kernel of a morphism of Lie algebras is always an ideal, so is $\mathfrak{n}$. $\square$

Chapter 2

Induced Representations

This chapter is essentially a brief summary of [Jan87], tailored to fit the needs of this work. It aims to provide the basic results required for a more in-depth study of representations of algebraic groups and introduces the key results necessary for the techniques deployed in the final chapter.

2.1 Restriction and Induction

We start with a slightly more general situation. Suppose $\varphi\colon H \to G$ is a morphism of group schemes over a field K and let M be a G-module with associated structure morphism $\rho\colon G \to \mathrm{Gl}(M)$. Then the morphism of group valued functors $\rho \circ \varphi\colon H \to \mathrm{Gl}(M)$ gives rise to an H-module structure on M. This is called the restricted H-module structure of M. Note that any morphism of G-modules $M \to N$ is also a morphism with respect to the restricted H-module structure, so we obtain an exact functor from the category of G-modules to the category of H-modules, called the restriction functor induced by φ. In terms of the comodule structure of M, this can be expressed by saying that the comodule map of M as an H-module is given by composing the comodule map $\rho^*\colon M \to M \otimes K[G]$ with $\mathrm{id} \otimes \varphi^*\colon M \otimes K[G] \to M \otimes K[H]$.

In the special case of a closed subgroup scheme H of G, we call the restriction along the canonical inclusion $i\colon H \to G$ the *restriction functor* and denote it by res_H^G. For notational convenience, we shall often simply write M for $\mathrm{res}_H^G M$, as long as no confusion is likely to arise from doing so.

Lemma 2.1.1. *Let H and H' be closed subgroup schemes of a K-group scheme G and suppose H' normalises H. Then for any G-module M, M^H is an H' submodule of M.*

Proof. Since H' normalises H, we may regard $K[H]$ as an H' module via the conjugation action. Then an explicit computation shows that the comodule map $\rho^*\colon M \to M \otimes K[H]$ as well as the map $m \mapsto \rho^*(m) - m \otimes 1$ are morphisms of H'-modules. The kernel of the latter map is just M^H, hence the assertion clearly follows. $\square$

As we have seen, it is quite easy to obtain modules of a subgroup from those of the whole group and it seems to be natural to ask whether one can also obtain modules of the whole group from those of a (closed) subgroup. As it turn out, this question can be answered affirmatively, however the necessary construction is slightly more involved.

Suppose H is a closed subgroup of a K-group scheme G and let M be an H-module. By regarding $K[G]$ as an H-module via the restriction of the right regular representation of G on $K[G]$ to H, we get an H-module structure on $M \otimes K[G]$. Similarly, $M \otimes K[G]$ has a G-module structure given by regarding M as a trivial G-module and $K[G]$ as a G-module via the left regular representation. These two structures are compatible in the sense that they give rise to a $(G \times H)$-module structure on $M \otimes K[G]$.

By identifying G and H with subgroups of $G \times H$ and applying Lemma 2.1.1 above, the fixed point module $(M \otimes K[G])^H$ is a G-submodule of $M \otimes K[G]$, which will be called the *induced module* of M and denoted by $\mathrm{ind}_H^G(M)$. It is easy to see that this defines a functor $\mathrm{ind}_H^G : \{H\text{-modules}\} \to \{G\text{-modules}\}$, which will be called the *induction functor*.

At first, this construction seems rather arbitrary, however there is a different interpretation of $\mathrm{ind}_H^G M$ in terms of natural transformations, detailed below, which also sheds some light on the similarity to the corresponding construction for Lie groups.

First of all, for any K-algebra A, we have a natural isomorphism of A-modules

$$M \otimes K[G] \otimes A \cong (M \otimes A) \otimes_A (K[G] \otimes A).$$

On the right hand side, the A-algebra $K[G] \otimes A$ is precisely the affine coordinate ring of the base extension G_A of G to A and the tensor product of $(M \otimes A)$ with $(K[G] \otimes A)$ over A is, after applying Yoneda's Lemma, isomorphic to $\mathrm{Nat}(G_A, (M \otimes A)_a)$ as an A-module. Since the left hand side is equal to $(M \otimes K[G])_a(A)$ we thus get an isomorphism of A-modules

$$(M \otimes K[G])_a(A) \cong \mathrm{Nat}(G_A, (M \otimes A)_a).$$

This shows that $(M \otimes K[G])_a$ is naturally isomorphic (as a K-group functor) to

$$A \mapsto \mathrm{Nat}(G_A, (M \otimes A)_a),$$

which is precisely the associated functor of the K-module $\mathrm{Nat}(G, M_a)$. We can endow this functor with a $(G \times H)$-module structure as follows. For $(g, h) \in (G \times H)(A)$ and $f \in \mathrm{Nat}(G_A, (M \otimes A)_a)$ we define for all $x \in G(A')$ and all A-algebras A'

$$((g, h) \cdot f)(x) = hf(g^{-1}xh).$$

Using these observations, we can identify the induced module $\mathrm{ind}_H^G(M)$ of an H-module M with a submodule of the natural transformations $G \to M_a$.

Lemma 2.1.2. *Let G, H and M be as above. Then the canonical isomorphism $M \otimes K[G] \to \mathrm{Nat}(G, M_a)$ is a morphism of $G \times H$-modules. Under this isomorphism, $\mathrm{ind}_H^G(M)$ corresponds to the submodule of $\mathrm{Nat}(G, M_a)$ consisting of all $f : G \to M_a$ satisfying*

$$f(gh) = h^{-1}f(g)$$

for all $g \in G(A)$, $h \in H(A)$ and all K-algebras A.

Proof. Given a K-algebra A and an element x of $(M \otimes K[G])_a(A)$, we denote the associated natural transformation $G_A \to (M \otimes A)_a$ by f_x. To prove the first assertion, we need to show that for any $(g,h) \in (G \times H)(A)$, any A-algebra A' and any $y \in G_A(A')$

$$f_{(g,h)x}(y) = ((g,h)f_x)(y).$$

As we already know that $(M \otimes K[G])_a \cong \mathrm{Nat}(G, M_a)_a$ as K-group functors, there is no loss of generality in assuming x to be of the form $m \otimes f \otimes 1$ for some $m \in M$ and some $f \in K[G]$. Then

$$\begin{aligned} f_{(g,h)x}(y) &= (\mathrm{id}_{M \otimes A} \otimes y)((g,h)x) \\ &= (\mathrm{id}_{M \otimes A} \otimes y)(hm \otimes 1 \otimes (g,h)f) \\ &= hm \otimes 1 \otimes ((g,h)f)(y) \\ &= hm \otimes 1 \otimes f(g^{-1}yh) \\ &= ((g,h)f_x)(y) \end{aligned}$$

which proves that we have an isomorphism of $(G \times H)$-modules. In particular, the H invariant elements of $M \otimes K[G]$ coincide with those of $\mathrm{Nat}(G, M_a)$. But $f \in \mathrm{Nat}(G, M_a)$ is a fixed point of the action of H if and only if for all K-algebras A, all $h \in H(A)$ and all $g \in G(A)$ the equation $((e,h) \cdot f)(g) = f(g)$ is satisfied. As $((e,h) \cdot f)(g) = h \cdot f(gh)$, we obtain

$$(M \otimes K[G])^H \cong \{f \in \mathrm{Nat}(G, M_a) \mid f(gh) = h^{-1} f(g)\}$$

as claimed. □

There is an important observation one should make at this point: The computation we made in the proof above to show that $M \otimes K[G] \cong \mathrm{Nat}(G, M_a)$ as $(G \times H)$-modules is actually valid in a more general context. Given any G-module M, we can identify the tensor product $M \otimes K[G]$ as a G-module with $\mathrm{Nat}(G, M_a)$, without regarding M as a trivial G-module. However, we need to choose a different G-module structure on $\mathrm{Nat}(G, M_a)$. More precisely, given $x = m \otimes f \in M \otimes K[G]$ with associated natural transformation f_x and elements g, $g' \in G(A)$ we have

$$\begin{aligned} f_{gx}(g') &= (\mathrm{id} \otimes g')(gx) \\ &= (\mathrm{id} \otimes g')(gm \otimes gf) \\ &= gm \otimes (gf)(g') \\ &= g((gf_x)(g')). \end{aligned}$$

So if we define a G-module structure on $\mathrm{Nat}(G, M_a)$ by $(gf_x)(g') = (g \otimes 1)((gf_x)(g'))$, we again obtain an isomorphism of G-modules $M \otimes K[G] \to \mathrm{Nat}(G, M_a)$.

Lemma 2.1.3. *Let M be a G-module. Then there is an isomorphism of G-modules*

$$M \otimes K[G] \to M_{tr} \otimes K[G].$$

Proof. We use the identification of both modules with $\mathrm{Nat}(G, M_a)$. Consider the linear endomorphism ϕ: $\mathrm{Nat}(G, M_a) \to \mathrm{Nat}(G, M_a)$ given by $\phi(f)(g) = gf(g)$. Then ϕ is an isomorphism of K-space with inverse $\tilde{\phi}$ defined by $\tilde{\phi}(f)(g) = g^{-1} f(g)$. Moreover, ϕ is actually G-equivariant since

$$\begin{aligned}(\phi(gf))(g') &= g'f(g^{-1}g') \\ &= g(g^{-1}g')f(g^{-1}g') \\ &= g\phi(f)(g^{-1}g') \\ &= g\phi(gf)(g').\end{aligned}$$

A similar computation shows that $\tilde{\phi}$ is G-equivariant as well, hence the assertion of the lemma follows. □

2.2 Frobenius Reciprocity

We will now use the interpretation of ind_H^G in terms of natural transformations to prove fundamental results on induced modules. The most important of these is the following proposition, which is commonly referred to as *Frobenius reciprocity.*

Proposition 2.2.1. *Let H be a closed subgroup scheme of a K-group scheme G. Then the induction functor ind_H^G is left exact and right adjoint to res_H^G, i.e. for all G-modules N and all H-modules M there is a (natural) isomorphism*

$$\mathrm{Hom}_G(N, \mathrm{ind}_H^G M) \to \mathrm{Hom}_H(\mathrm{res}_H^G N, M).$$

Proof. Let M be an H-module and consider the K-linear map $\epsilon_M\colon M \otimes K[G] \to M$ given by $\epsilon_M = id_M \otimes \epsilon^*$. If we identify $M \otimes K[G]$ with $\mathrm{Nat}(G, M_a)$ then, $\epsilon^*\colon \mathrm{Nat}(G, M_a) \to M$ is given by $\epsilon_M(f) = f(e)$. In particular, given a K-algebra A, $h \in H(A)$ and $f \in \mathrm{ind}_H^G M$ we have

$$\epsilon_M(hf) = (hf)(e) = f(h^{-1}) = hf(e) = h\epsilon_M(f)$$

This shows that $\epsilon_M\colon \mathrm{ind}_H^G M \to M$ is actually a morphism of H-modules.

As morphism of G-modules $\varphi\colon N \to \mathrm{ind}_H^G M$ is also a morphism of H-modules with respect to the respective restricted actions, composing φ with ϵ_M yields a morphism of H-modules $N \to M$. In particular, we obtain a map $\mathrm{Hom}_G(N, \mathrm{ind}_H^G M) \to \mathrm{Hom}_H(N, M)$ by mapping φ to $\epsilon_M \circ \varphi$. To show that map is an isomorphism, we construct an inverse as follows. For an H-module homomorphism $\psi\colon N \to M$, we define a map $\tilde{\psi}\colon N \to \mathrm{Nat}(G, M_a)$ by mapping $n \in N$ to the natural transformation $\tilde{\psi}(n)\colon G \to M_a$ with $\tilde{\psi}(n)(x) = (\psi \otimes \mathrm{id})(x^{-1}(n \otimes 1))$ for all $x \in G(A)$ and all K-algebras A. Then

$$\begin{aligned}\tilde{\psi}(n)(gh) &= (\psi \otimes \mathrm{id})((gh)^{-1}(n \otimes 1)) \\ &= h^{-1}(\psi \otimes \mathrm{id})(g^{-1}(n \otimes 1)) \\ &= h^{-1}\tilde{\psi}(n)(g)\end{aligned}$$

shows that $\tilde{\psi}\colon N \to \mathrm{Nat}(G_a, M_a)^H \cong \mathrm{ind}_H^G M$. It is now straightforward to check that $\varphi \mapsto \epsilon_M \circ \varphi$ and $\psi \mapsto \bar{\psi}$ are inverse homomorphisms. □

In the language of category theory, the above proposition simply states that the induction functor ind_H^G is *left adjoint* to res_H^G. One can use this fact to immediately deduce several properties of ind_H^G, the most important for us being the preservation of *injective modules*: A G-module I is said to be *injective*, if for any monomorphism $\varphi\colon A \to B$ of G-modules and any morphism $\psi\colon A \to I$, there is a unique map $\hat{\psi}\colon B \to I$ such that the diagram

$$\begin{tikzcd} A \arrow[r, hook, "\varphi"] \arrow[d, "\psi"'] & B \arrow[dl, dotted, "\hat{\psi}"] \\ I & \end{tikzcd}$$

commutes. It is easy to see that this is in fact equivalent to the exactness of the functor $\operatorname{Hom}_G(-, I)$ (cf. [HS97][Section I, Theorem 8.4]).

Corollary 2.2.2. *For any closed subgroup H of G, the functor ind_H^G preserves injective modules.*

Proof. By Frobenius reciprocity, we have a natural isomorphism

$$\epsilon\colon \operatorname{Hom}_G(N, \operatorname{ind}_H^G M) \to \operatorname{Hom}_H(\operatorname{res}_H^G N, M)$$

Now naturality asserts that for any morphism of G-modules $M \to M'$ and any morphism of H-modules $N' \to N$, the diagram

$$\begin{tikzcd} \operatorname{Hom}_G(N, \operatorname{ind}_H^G M) \arrow[r] \arrow[d, "\epsilon"'] & \operatorname{Hom}_G(N', \operatorname{ind}_H^G M') \arrow[d, "\epsilon"] \\ \operatorname{Hom}_H(\operatorname{res}_H^G N, M) \arrow[r] & \operatorname{Hom}_H(\operatorname{res}_H^G N', M') \end{tikzcd}$$

commutes, i.e. $\epsilon(\operatorname{ind}_H^G \beta \circ \varphi \circ \alpha) = \beta \circ \epsilon(\varphi) \circ \operatorname{res}_H^G \alpha$ of all $\varphi \in \operatorname{Hom}_G(N, \operatorname{ind}_H^G M)$. Similarly, by considering the same diagram for ϵ^{-1}, one obtains the identity $\epsilon^{-1}(\beta \circ \psi \circ \operatorname{res}_H^G \alpha) = \operatorname{ind}_H^G \beta \circ \epsilon^{-1}(\psi) \circ \alpha$ for all $\psi \in \operatorname{Hom}_H(\operatorname{res}_H^G N, M)$.
Suppose I is an injective H-module and consider the following diagram of G-modules.

$$\begin{tikzcd} & \operatorname{ind}_H^G I \\ N & N' \arrow[l, hook', "\alpha"] \arrow[u, "\varphi"'] \end{tikzcd}$$

By applying ϵ to φ we obtain

$$\begin{tikzcd} & I \\ \operatorname{res}_H^G N \arrow[ur, dotted] & \operatorname{res}_H^G N' \arrow[l, hook', "\operatorname{res}_H^G \alpha"] \arrow[u, "\epsilon(\varphi)"'] \end{tikzcd}$$

Therefore, because of the injectivity of I, there exists a unique morphism $\hat{\varphi} \in \operatorname{Hom}_G(N, \operatorname{ind}_H^G I)$ such that

$$\epsilon(\varphi) = \epsilon(\hat{\varphi}) \circ \operatorname{res}_H^G \alpha.$$

Applying ϵ^{-1} to this equation and using the identity obtained above (with $\beta = \mathrm{id}$), we get $\varphi = \hat{\varphi} \circ \alpha$, which proves that $\mathrm{ind}_H^G I$ is an injective G-module. $\square$

Remark: This proof works in just the same way for adjoint functors in general, provided that the left adjoint preserves monomorphisms, as is shown in [HS97][Proposition 10.2].

We can apply this observation to the special case $H = 1$. An H-module is then nothing other than a K-vector space and morphisms of H-modules are just the usual linear maps. Moreover, the induced module $\mathrm{ind}_1^G M$ is given by the tensor product $M \otimes K[G]$ with M regarded as a trivial G-module. In particular, since any K-space is certainly injective and induction preserves injective modules, $M \otimes K[G]$ is an injective G-module for any trivial G-module M. This assertion is actually valid without the assumption of triviality on M, since we know by Lemma 2.1.3 that $M \otimes K[G] \cong M_{tr} \otimes K[G]$.

Corollary 2.2.3. *For any G-module M, the module $M \otimes K[G]$ is injective.*

We can now show that any G-module M can be embedded into an injective G-module.

Lemma 2.2.4. *Let M be a G-module. Then there exists an injective morphism of G-modules $M \to M \otimes K[G]$.*

Proof. Consider the comodule map $\rho^*\colon M \to M \otimes K[G]$. Then for $g \in G(A)$ and $m \in M$ with $\rho^*(m) = \sum m_i \otimes f_i$

$$\begin{aligned} \rho^* \otimes \mathrm{id}_A(g(m \otimes 1)) &= \rho^* \otimes \mathrm{id}_A((\mathrm{id} \otimes g)(\rho^*(m)) \\ &= (\mathrm{id} \otimes g)((\rho^* \otimes \mathrm{id})(\rho^*(m))) \\ &= (\mathrm{id} \otimes g)((\mathrm{id} \otimes \mu^*)(\rho^*(m))) \\ &= \sum m_i \otimes (\mathrm{id} \otimes g)(\mu^*(f_i)). \end{aligned}$$

Since $(\mathrm{id} \otimes g)(\mu^*(f_i))$ is nothing else but gf_i with respect to the right regular representation, we get $\rho^*(g(m)) = g\rho^*(m)$ where we regard $M \otimes K[G]$ as a G-module with respect to the trivial G-module structure on M and the right regular representation on $K[G]$, hence ρ^* can be regarded as a morphism of G-modules $M \to M_{tr} \otimes K[G]$. As the right hand side of this equation is isomorphic to $M \otimes K[G]$ and ρ^* is injective (as a consequence of $(\mathrm{id} \otimes \epsilon^*)(\rho^*(m)) = m$), the assertion of the lemma follows. $\square$

In particular, this lemma shows that we can choose very special types of injective modules to embed a given module in. This fact will become important later on, when we want to compare different cohomology groups associated with a G-module M.

Chapter 3

Cohomology of Arithmetic Subgroups of Unipotent Groups

3.1 Cohomology in General

The previous chapter established the existence of injective presentations for modules of group schemes. The reader familiar with homological algebra will immediately realise the important implication behind this fact, as the existence of such presentations is an essential requirement for the construction of *right derived functors*. We will use this quite general technique to introduce the concept of cohomology for the various types of representations we're going to deal with.

Suppose $\mathfrak{A}$ is an abelian category with enough injectives (i.e. any object of A admits a monomorphism into an injective object) and F is a functor from $\mathfrak{A}$ to the category of abelian groups. The i-th right derived functor R^iF of F is defined as follows. For an object A of $\mathfrak{A}$, one chooses an injective resolution

$$0 \to A \to I^0 \to I^1 \to \ldots \to I^i \to \ldots,$$

and applies F to the resolution I^n. This yields a chain complex

$$F(I^0) \to F(I^1) \to \ldots \to F(I^i) \to \ldots$$

and $R^iF(A)$ is defined to be the i-th homology of this chain complex, i.e.

$$R^iF(A) = \ker(Fd^i)/\operatorname{im}(Fd^{i-1}),$$

where $d^i\colon I^i \to I^{i+1}$ and $d^{-1} = 0$. One can show that this construction does not depend on the injective resolution chosen for A (see [Wei94][Lemma 2.4.1 and Section 2.5.1]). Moreover, if F is left exact, then R^0F is always naturally isomorphic to F itself.

Example 3.1.1. Given an arbitrary ring R, the category of R-modules is an abelian category with enough injectives (see [HS97][Section I, Proposition 8.3]). In addition, for any R-module M, the functor $\operatorname{Hom}(M, -)$ is left exact (cf. [HS97][Section I, Theorem 2.1]). The right derived functors of $\operatorname{Hom}(M, -)$ are called the Ext-functors and will be denoted by $\operatorname{Ext}_R(M, -)$.

Using this general construction, we can easily introduce the notion of cohomology in all the settings we are interested in. Note that all of these cases fit into the general picture of deriving a fixed point functor.

Suppose G is an abstract group. A *G-module* is an abelian group M together with a group homomorphism ρ: $G \to \mathrm{Aut}(M)$. Any such automorphism clearly defines an action of G on M and we will usually denote $\rho(g)(m)$ by gm for convenience. Note that any abelian group can be regarded as a trivial G-module via the trivial homomorphism $G \to \mathrm{Aut}(M)$ which maps any $g \in G$ to the identity on M. One usually denotes M with this trivial G-module structure by M_{tr}.

Given G-modules M and N, a morphism of G-modules (or a G-equivariant morphism) is simply a homomorphism of groups f: $M \to N$ such that $f(gm) = gf(m)$ for all $g \in G$ and all $m \in M$. We will usually denote the group structure on G multiplicatively, while using an additive notion for the group structure on M.

One can almost effortlessly verify that the category of all G-modules, together with the G-equivariant morphisms defined above, constitutes an abelian category. However, the definition of cohomology groups usually requires injective resolutions, so we need to show that the category of all G-modules actually has enough injectives. Consider the *group ring* $\mathbb{Z}[G]$, given by the direct product $\sum_{g\in G} \mathbb{Z}$ together with the multiplication

$$(\sum_{g\in G} \lambda_g g) \cdot (\sum_{h\in G} \lambda'_h h) = \sum_{g,h\in G} \lambda_g \lambda'_h gh.$$

It is then easy to see that a G-module is actually nothing but a $\mathbb{Z}[G]$-module and that the G-equivariant homomorphisms are just the homomorphisms of $\mathbb{Z}[G]$-modules (see [HS97][Section VI, Proposition 1.1] as well as the discussion below the proof). In particular, the category of G-modules is naturally isomorphic to the category of modules over $\mathbb{Z}[G]$ and thus has enough injectives.

Given a G-module M, a fixed point of G in M is an element m of M such that $gm = m$ for all $g \in G$. Any such element determines a G-equivariant homomorphism $\mathbb{Z}_{tr} \to M$ by mapping 1 to m and conversely, any homomorphism $\mathbb{Z}_{tr} \to M$ determines a fixed point of M given by the image of 1. We can therefore identify the subgroup of all fixed points of G in M with the group $\mathrm{Hom}_G(\mathbb{Z}_{tr}, M)$.

Definition 3.1.2. *Let G be an abstract group. For a G-module M, the* i-th cohomology group *of M is defined as*

$$H^i(G, M) = R^i \mathrm{Hom}_G(\mathbb{Z}_{tr}, -)(M).$$

Note that $\mathrm{Hom}_G(\mathbb{Z}, -)$ is a left exact functor and thus $H^0(G, M) = M^G$ where M^G denotes the subgroup of fixed points of M.

It is often useful to have more concrete interpretations for certain cohomology groups. A particularly useful interpretation is available for H^1 and identifies $H^1(G, M)$ as a quotient of the group of derivations of M. In this setting, a *derivation* from G to M is a map d: $G \to M$ such that $d(gh) = d(g) + gd(h)$. The set of all derivations from G to M has a natural structure as an abelian group and will be denoted by $\mathrm{Der}(G, M)$. Note that any element $m \in M$ defines a derivation d_m via $d_m(g) = gm - m$. Derivations of this form will be called *inner derivations.* They form a subgroup of $\mathrm{Der}(G, M)$, denoted by $\mathrm{Ider}(G, M)$.

Proposition 3.1.3. *Given a group G and a G-module M, there is a natural isomorphism*

$$H^1(G, M) \to \operatorname{Der}(G, M)/\operatorname{Ider}(G, M)$$

Proof. See [HS97][Section IV, Corollary 5.2]. □

For Lie algebras, we can simply adapt the steps above. If $\mathfrak{g}$ is a Lie-algebra over some field K, we know that the category of $\mathfrak{g}$-modules is precisely the category of $\mathcal{U}(\mathfrak{g})$-modules, hence is again abelian with enough injectives. Moreover, we know that the fixed point module of a $\mathfrak{g}$-module M can be identified with the K-space of all $\mathfrak{g}$-homomorphisms $K_{tr} \to M$. This motivates the following definition.

Definition 3.1.4. *For a Lie-algebra $\mathfrak{g}$ over some field K, the i-th cohomology group of a $\mathfrak{g}$-module M is defined as*

$$H^i(\mathfrak{g}, M) = R^i \operatorname{Hom}_{\mathfrak{g}}(K_{tr}, M)$$

Similar to group cohomology, we can again interpret $H^1(\mathfrak{g}, M)$ in terms of derivations. In this case, a *derivation* from $\mathfrak{g}$ to M is a K-linear map $d\colon \mathfrak{g} \to M$ such that $d([x, y]) = xd(y) - yd(x)$. The set of all such maps is a K-vector space in a natural way and will be denoted by $\operatorname{Der}(\mathfrak{g}, M)$. Moreover, for any $m \in M$, we can define a derivation d_m by $d_m(x) = xm$. Derivations of this form are again called *inner derivations*. The subspace of $\operatorname{Der}(\mathfrak{g}, M)$ formed by all inner derivations is denoted by $\operatorname{Ider}(\mathfrak{g}, M)$.

Proposition 3.1.5. *If $\mathfrak{g}$ is a Lie algebra over a field K and M is a $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, M) \cong \operatorname{Der}(\mathfrak{g}, M)/\operatorname{Ider}(\mathfrak{g}, M)$.*

Proof. See [HS97][Section VII, Proposition 2.2]. □

Having defined these notions, we briefly step back to consider a more general situation. When we introduced the notion of cohomology for groups and Lie algebras above, we basically dealt with a category of modules over some ring R (in one case $R = \mathbb{Z}[G]$ was given by the group ring, in the other case $R = \mathcal{U}(\mathfrak{g})$ was given by the universal enveloping algebra) and we derived the fixed point functor by using its interpretation as a homomorphism functor of the form $\operatorname{Hom}_R(M, -)$. In particular, these cohomology functors are special instances of the Ext-functors $\operatorname{Ext}^i_R(M, -)$ introduced above. As it turns out, there is actually another way of computing the group $\operatorname{Ext}^i_R(M, N)$ by using a *projective resolution* of M instead of an injective resolution of N. By a projective resolution of M, we mean an exact sequence

$$\ldots \to P^i \to \ldots \to P^1 \to P^0 \to M$$

with each P^i projective over R. One can then compute $\operatorname{Ext}^i_R(M, N)$ by applying $\operatorname{Hom}(-, N)$ to this resolution and taking the i-th homology of the resulting chain complex (cf. [HS97][Section IV, Proposition 8.1]). One usually expresses this fact by saying that Ext is *balanced*. In particular, we could have defined the cohomology functors $H^i(G, -)$ and $H^i(\mathfrak{g}, -)$ above by using projective resolutions of $\mathbb{Z}_{tr}$ and K_{tr} respectively, which is slightly more useful as one can use one fixed resolution regardless of the module one considers.

We now want to address the question under which conditions $H^i(G, -)$ and $H^i(\mathfrak{g}, -)$ vanish for sufficiently large i. We can actually answer this question in a more general setting by using our above interpretations. If we consider the functors $\operatorname{Ext}^i_R(M,-)$ for a module M over some ring R, we can construct the groups $\operatorname{Ext}^i_R(M,N)$ from a projective resolution P^i of M. If this resolution is actually *finite*, i.e. $P^n = 0$ for some n (and hence also for all larger indices), then clearly $\operatorname{Ext}^i_R(M,N) = 0$ for all $i > n$. The minimal length of a finite resolution is called the *projective dimension* of M (over R), which we will denote by $\operatorname{pd}_R(M)$.

Definition 3.1.6. *Let G be an abstract group. The* cohomological dimension *of G, denoted by* $\operatorname{cd} G$, *is defined as the projective dimension of $\mathbb{Z}_{tr}$ as a G-module (and hence as a $\mathbb{Z}[G]$-module). Similarly, for a Lie algebra $\mathfrak{g}$, we define the cohomological dimension* $\operatorname{cd}\mathfrak{g}$ *as the projective dimension of K_{tr} as a $\mathfrak{g}$-module.*

This suggests that if we want to find conditions for our cohomology groups to vanish, we only need to find a way to determine if certain modules permit a finite projective resultion.

Example 3.1.7. Let $\Gamma = \mathbb{Z}$ be the additive group of integers. We claim that the cohomological dimension of Γ is equal to 1. To prove our claim, we consider the the group ring $\mathbb{Z}[\Gamma]$, which is isomorphic to $\mathbb{Z}[t,t^{-1}] = \mathbb{Z}[x,y]/(xy-1)$. The trivial Γ-module structure on $\mathbb{Z}$ corresponds to the module structure induced by the ring homomorphism $\pi\colon \mathbb{Z}[t,t^{-1}] \to \mathbb{Z}$ mapping t to 1. In particular, we get an exact sequence

$$0 \to I\Gamma \to \mathbb{Z}[\Gamma] \to \mathbb{Z} \to 0,$$

where $IG = \ker\pi$ is called the *augmentation ideal* of Γ. Thus, in order to show $\operatorname{cd}\Gamma = 1$ it is sufficient to show that $I\Gamma$ is projective. As the group Γ is a cyclic group generated by 1, the augmentation ideal, as a $\mathbb{Z}[\Gamma]$-module is generated by $t-1$ (see [HS97][Section VI, Lemma 1.2]) and therefore a principal ideal. Since $\mathbb{Z}[\Gamma]$ is an integral domain, mapping f to $f\cdot(t-1)$ yields an isomorphism $\mathbb{Z}[\Gamma] \to I\Gamma$ of $\mathbb{Z}[\Gamma]$-modules, hence the augmentation ideal is free and thus projective.

Example 3.1.8. Similarly, the one dimensional trivial Lie algebra $\mathfrak{g} = K$ has cohomological dimension 1. This follows directly from the fact that $\mathcal{U}(\mathfrak{g}) \cong K[t]$ is a principal ideal domain, since K as a trivial $\mathfrak{g}$-module corresponds to the $\mathcal{U}(\mathfrak{g})$-module structure on K induced by the map $\pi\colon \mathcal{U}(\mathfrak{g}) \cong K[t] \to K$ mapping t to 0 and the corresponding exact sequence

$$0 \to \ker\pi \to K[t] \to K \to 0$$

is a projective resolution for the same reason as in the above example.

With group schemes, the definition of cohomology follows the same basic idea. If G is a group scheme over a field K, then Lemma 2.2.4 shows that the category of G-modules has enough injectives. In particular, we can again derive the fixed point functor, just as we did above.

Definition 3.1.9. *Let G be a group scheme over a field K and M be a G-module. Then the i-th cohomology group of M is defined as*

$$H^i(G,M) = (R^i \operatorname{Hom}_G(K_{tr},-))(M).$$

Having introduced all these notions, the reader familiar with homological algebra (or any incarnation of "cohomology") may recall some of the fundamental properties of these cohomology functors, such as long exact sequences on the cohomology induced by short exact sequences on the respective modules. It turns out that all this fits into a more general picture, whose outline we shall coarsely sketch below. The reader interested in details may find them in Grothendieck's Tohoku paper [Gro57][Chapter 2], on which all of the following exposition is based. Another brief discussion of all the basic definitions and properties can be found in Hartshorne [Har06][III.1].

We first note that all of the cohomology functors considered so far are *additive*, i.e. they map a direct sum of modules to the direct sum of the respective cohomology groups. This definition makes sense for covariant functors between arbitrary abelian categories C and C'. We may then consider the cohomology functors H^i as a family of covariant additive functors $H = (H^i)$.

Definition 3.1.10. *Let C and C' be abelian categories. A δ-*functor *from C to C' is a collection $T = (T^i)_{i\geq 0}$ of covariant additive functors $T^i\colon C \to C'$ together with a morphism $\delta^i\colon T^i(A'') \to T^{i+1}(A')$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$ (called the "connecting" or "boundary" morphism) such that the following properties are satisfied:*

1. *Given a second short exact sequence $0 \to B' \to B \to B'' \to 0$ and any homomorphism from the first exact sequence to the second, the connecting morphisms are compatible in the sense that the diagram*
$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T^i(B'') \arrow[r, "\delta"] & T^{i+1}(B') \end{tikzcd}$$
commutes.

2. *The exact sequence $0 \to A' \to A \to A'' \to 0$ gives rise to a long exact sequence*
$$0 \to T^0(A') \to T^0(A) \to T \ldots \to T^0(A'') \to T^1(A') \ldots$$
$$\ldots \to T^i(A') \to T^i(A) \to T^i(A'') \to T^{i+1}(A') \to \ldots$$

In particular, all of our previously defined cohomology functors form δ-functors from the respective category of modules to the category of abelian groups. Given two δ-functors T and T', the notion of a *morphism* of δ-functors is quite natural. It is simply a family of natural transformations $T^i \to T'^i$ compatible with the boundary morphism for each short exact sequence, i.e. given an exact sequence
$$0 \to A' \to A \to A'' \to 0$$
one has a commutating diagram
$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T'^i(B'') \arrow[r, "\delta"] & T'^{i+1}(B') \end{tikzcd}$$
for all i.

Definition 3.1.11. *A δ-functor $T\colon C \to C'$ is said to be* universal *if for any other δ-functor*

T': $C \to C'$ and any natural transformation η: $T^0 \to T'^0$ there exists a unique morphism of δ-functors (η^i): $T \to T'$ with $\eta^0 = \eta$.

Of course, such a property is often rather unpleasant to check, but there is a criterion for universality which is sufficient most of the time.

Definition 3.1.12. *Let T: $C \to C'$ be an additive functor. The T is said to be* effaceable *if for any object A of C, there is a monomorphism i: $A \to I$, such that $T(i) = 0$.*

The cohomology functors H^i defined above certainly satisfy this property for $i > 0$. More precisely, given an injective module I, $H^i(I) = 0$ and we can embed any module into an injective module. As it turns out, this is a sufficient condition for the universality of $H = (H^i)$.

Lemma 3.1.13. *Let T: $C \to C'$ is a δ-functor. If T^i is effaceable for all $i > 0$ then T is universal.*

Proof. See [Gro57][Proposition 2.2.1] □

3.2 Spectral Sequences for Groups and Lie Algebras

Spectral sequences arise in a number of vastly different settings, two of which will occur in the sequel. However, since our application of this technical apparatus is of a rather simple nature, we shall refrain from an overly technical approach to this subject and focus on the properties and results important to us. The reader interested in the details can find them in various book on homological algebra, such as [Rot09] and [Wei94].

Let R be a ring. A graded R-module is a family of R-modules $(M^n)_{n\in\mathbb{Z}}$. If $d \in \mathbb{Z}$ is an integer, a morphism of graded R-modules φ:$(M^n) \to (N^n)$ of degree d is a family of morphisms $(\varphi^n)_{n\in\mathbb{Z}}$ where each φ^n: $M^n \to N^{n+d}$ is a morphism of R-modules. One often denotes the degree of φ by $\deg(\varphi)$.

Similarly, one defines *bigraded* R-modules $(A^{pq})_{p,q\in\mathbb{Z}}$ and their morphisms, where one replaces the degree with the *bidegree*, i.e. a morphism φ:$(A^{pq}) \to (B^{pq})$ of bidegree (a, b) is a family of morphism φ^{pq}: $A^{pq} \to B^{p+a,q+b}$.

Note that both graded and bigraded modules together with their respective morphisms form an abelian category, denoted by $(\mathrm{Mod}_R)_{\mathbb{Z}}$ and $(\mathrm{Mod}_R)_{\mathbb{Z}\times\mathbb{Z}}$ respectively.

A *differential bigraded module* over R is then defined to be a pair (A, d) consisting of a bigraded R-module $A = (A^{pq})$ and a morphism of bigraded modules d: $A \to A$ with $d^2 = 0$. This is nothing but a straightforward generalisation of the definition of a chain complex to the category of bigraded modules, just without any restriction on the degree of d. In particular, given a differential bigraded module (A, d) one defines its *homology* to be the bigraded module $H(A, d)$ with

$$H(A, d)^{pq} = \ker(d^{pq})/\operatorname{im}(d^{p-a,q-b}),$$

where (a, b) is the bidegree of d. Similar to the case of chain complexes, d is often called the differential of A.

Definition 3.2.1. *A* spectral sequence *over R is defined to be a family of differential bigraded modules (E_r, d_r) for $r \geq 1$, such that the bidegree of d_r is $(r, 1 - r)$ and $H(E_r, d_r) = E_{r+1}$.*

For our applications, the most important aspect of a spectral sequence is its *limit term*. Let (E_r, d_r) be a spectral sequence. Then $E_2 = H(E_1, d_1){=}Z_2/B_2$ where $Z_2 = \operatorname{im} d_1$ are the cycles and $B_2 = \ker d_1$ are the boundaries. We can now let Z_3 be the preimage of $\operatorname{im} d_2$ and B_3 be the preimage of $\ker d_2$ under the projection $Z_2 \to Z_2/B_2 \cong E_2$. In particular, we obtain $E_3 \cong Z_3/B_3$ and we can continue this way to construct

$$B_2 \subseteq \ldots \subseteq B_r \subseteq Z_r \subseteq \ldots \subseteq Z_2 \subseteq E_1$$

with $Z_r/B_r \cong E_r$. We then define the *limit term* E_∞ by letting $E_\infty \cong Z_\infty/B_\infty$ where $B_\infty = \bigcup B_r$ and $Z_\infty = \bigcap Z_r$.

In order to make meaningful use of these limit terms, we need to identify them with other objects of interest:

Definition 3.2.2. *Let $M = (M^n)$ be a graded module. A* filtration *of M is a family $(F^pM)_{p\in\mathbb{Z}}$ of submodules F^pM of M such that $F^pM \supseteq F^{p+1}M$. Such a filtration is said to be* bounded *if for each n there exist integers s and t such that $F^sM = M$ and $F^tM = \{0\}$.*

Definition 3.2.3. *A spectral sequence (E_r, d_r) is said to* converge *to a graded module M if there exists a bounded filtration (F^pM) of M such that $E_\infty^{pq} \cong F^pM^{p+q}/F^{p-1}M^{p+q}$. One denotes this by $E_2^{pq} \Rightarrow_p M^n$ and calls p the* filtration degree*.*

If we employ the common convention of denoting $n = p + q$, then the above reads $E_\infty^{pq} \cong F^pM^n/F^{p-1}M^n$, which is slightly more visually appealing.

Usually, spectral sequences reside in certain quadrants. Most of the time one encounters either first or third quadrant spectral sequences, depending on whether one is interested in homology or cohomology.

Definition 3.2.4. *A* first quadrant *(respectively* third quadrant*) spectral sequence is a spectral sequence (E_r, d_r) such that $E_r^{pq} = 0$ whenever $p < 0$ or $q < 0$ (respectively $p > 0$ or $q > 0$).*

Remark: Based on the choices made in this work, a spectral sequence in the sense of our definition is sometimes called a *cohomological spectral sequence*. Because of this, our spectral sequences will usually reside in the first quadrant when considering cohomology (which is a good thing, as this is the only thing we are interested in) instead of the seemingly more widespread third quadrant versions (which usually have raised indices and a slightly different degree on the derivations d_r). However, up to a minor change in the notation, all this is just a matter of personal taste.

Having all these notions at hand, we consider two cases where spectral sequences arise quite naturally. Both can be obtained as special instances of what is called the Grothendieck spectral sequence (see [Rot09][Section 10.6]). Since the latter result requires a more technical approach than the one chosen here, we shall refrain from even sketching a proof and simply state the special instances which will be required later.

Theorem 3.2.5 (Lyndon-Hochschild-Serre Spectral Sequence for Groups)**.** *Let N be a normal subgroup of a group G. Then for each G-module M there is a first quadrant spectral sequence with*

$$H^p(G/N, H^q(N, M)) \Rightarrow_p H^{p+q}(G, M).$$

Proof. See [Rot09][Theorem 10.52]. □

Theorem 3.2.6 (Hochschild-Serre Spectral Sequence for Lie algebras)**.** *Let $\mathfrak{n}$ be an ideal of a Lie algebra $\mathfrak{g}$. Then for each $\mathfrak{g}$-module M there is a first quadrant spectral sequence with*

$$H^p(\mathfrak{g}/\mathfrak{n}, H^q(\mathfrak{n}, M)) \Rightarrow_p H^{p+q}(\mathfrak{g}, M).$$

Proof. See [Wei94][7.5.2]. □

In general, given a spectral sequence (E_r, d_r), the more E_2-terms vanish, the simpler the spectral sequence is. In the particular case where $E_2^{pq} = 0$ for $q \neq 0$, we say that the spectral sequence *collapses* on the p-axis. Similarly, (E_r, d_r) is said to collapse on the q-axis if $E_2^{pq} = 0$ for $p \neq 0$.

Proposition 3.2.7. *If (E_r, d_r) collapses on the p-axis, then $E_\infty = E_2$. Moreover, if (E_r, d_r) is a first quadrant spectral sequence converging to a graded module M with respect to a filtration F^pM satisfying $F^{n+1}M^n = 0$ and $F^0M^n = M$ for all n, then $M^n = E_2^{n,0}$.*

Proof. For the first assertion, we consider the boundary operator d_2: $E_2 \to E_2$. By assumption, d_2 has bidegree $(2, -1)$ and $E_3 = H(E_2, d_2)$. In particular, for $q \neq 0$, $\ker d_2^{pq} \subseteq E_2^{pq} = 0$, hence $E_3^{pq} = \ker d_2^{pq} / \operatorname{im} d_2^{p-2,q+1} = 0$. For $q = 0$, $E_2^{p+2,-1} = E_2^{p-2,1} = 0$, so $\ker d_2^{pq} = E^{pq}$ and $\operatorname{im} d_2^{p-2,q+1} = 0$, whence $E_3^{pq} = E_2^{pq}$. In particular, $E_3 = E_2$ and by proceeding inductively, we obtain $E_r = E_2$ for all $r \geq 2$, thus clearly $E_\infty = E_2$.
Suppose (E_r, d_r) is a first quadrant spectral sequence collapsing in the p-axis. If (E_r, d_r) converges to some M with respect to a filtration as in the proposition, then for fixed n and $p \leq n$ we have

$$0 = E_2^{pq} = E_\infty^{pq} = F^pM^n / F^{p+1}M^n$$

and hence $F^pM^n = F^{p+1}M^n$. In particular, $M^n = F^0M^n = \ldots = F^nM^n$ and as moreover $F^{n+1}M^n = 0$ we get $E_2^{n0} = F^nM^n / F^{n+1}M^n = F^nM^n$ as claimed. □

Remark: The assumption on the filtration is automatically satisfied for first quadrant spectral sequences, see [Wei94][Example 5.2.6].

3.3 Arithmetic Subgroups

Up to this point we have primarily dealt with the fundamental results necessary to deal with the technical environment chosen for this work. We have yet to introduce the key objects of interest: Arithmetic subgroups of algebraic groups over $\mathbb{Q}$. This, along with a basic study of their properties, is done below. Note that the discussion is once again kept rather brief, as the study of these subgroups is, in general, quite involved. A much more extensive exposition of the theory of arithmetic groups can be found in [Bor69] as well as [PR94], on which the following material is based.

Recall that two subgroups H_1, H_2 of an abstract group G are said to be commensurable if their intersection $H_1 \cap H_2$ has finite index in both H_1 and H_2.

Definition 3.3.1. *Let G be an algebraic group defined over the field of rational numbers $\mathbb{Q}$. A subgroup Γ of $G(\mathbb{Q})$ is called an* arithmetic subgroup *of G if there exists a closed embedding ρ: $G \to \mathrm{Gl}_n$ such that $\rho(\Gamma)$ is commensurable with $\rho(G(\mathbb{Q})) \cap \mathrm{Gl}_n(\mathbb{Z})$.*

Commensurability ensures that the notion of an arithmetic subgroup is independent of the closed embedding of G: Given any other closed embedding $\rho' : G \to \mathrm{Gl}_m$ then $\rho'(\mathbb{Q})(\Gamma)$ is commensurable with $\rho'(G(\mathbb{Q})) \cap \mathrm{Gl}_m(\mathbb{Z})$ (this follows from [PR94][Chapter 4, Proposition 4.2]). More generally one has the following important proposition:

Proposition 3.3.2. *Let $\varphi : G \to H$ be a morphism of algebraic $\mathbb{Q}$-groups, surjective on the $\bar{\mathbb{Q}}$-points. If Γ is an arithmetic subgroup of G then $\varphi(\mathbb{Q})(\Gamma)$ is an arithmetic subgroup of H.*

Proof. See [PR94][Theorem 4.1]. □

The above definition includes another technical detail: An arithmetic subgroup was required to be contained in the rational points of the algebraic group G. This restriction is sometimes omitted to obtain a larger class of arithmetic groups (cf. [PR94][p. 173], especially the discussion following Corollary 2), however this condition is necessary in order to obtain suitable representations of Γ from those of G.

Lemma 3.3.3. *Let Γ be an arithmetic subgroup of an algebraic $\mathbb{Q}$-group G. Then any G-module M has a natural Γ-module structure.*

Proof. Let ρ: $G \times M_a \to M_a$ denote the action of G on M. Then $\rho(\mathbb{Q})$ is an action of the abstract group $G(\mathbb{Q})$ on $M_a(\mathbb{Q}) \cong M$ and since $\Gamma \subseteq G(\mathbb{Q})$ we may restrict this action to obtain a Γ-module structure on M. □

In particular, we obtain an exact functor from the category of G-modules to the category of Γ-modules. As we are interested in cohomology, it seems natural to ask how the fixed points of a G-module M relate to the fixed points of the induced Γ-action. First of all, any G-invariant element of M is Γ-invariant. This follows easily from the way the action of Γ on M is defined. More precisely, as Γ is a subgroup of the $\mathbb{Q}$-points of G and $G(\mathbb{Q})$ acts trivially on M^G, so does Γ. In general, the inclusion $M^G \subseteq M^\Gamma$ can be proper, however there is a coarse criterion for equality provided by the following lemma.

Lemma 3.3.4. *Let G be an algebraic group over a field K of characteristic 0 and $\Gamma \subseteq G(K)$ be a subgroup of $G(K)$ which is dense in G i.e. which is not contained in the K-points of any proper closed subscheme of G. Then $M^G = M^\Gamma$.*

Proof. Since $\Gamma \subset G(K) = \mathrm{Hom}(K[G], K)$, the kernel of any $\gamma \in \Gamma$ is a maximal ideal of $K[G]$. Translating the definition of density into an algebraic statement, we see that Γ is dense if and only if

$$\bigcap_{\gamma \in \Gamma} \ker \gamma = (0).$$

Now let m_i, $i \in I$, be a basis of M as a K-vector space such that m_j, $j \in J \subseteq I$, is a basis for M^Γ and write $\rho(m_k) = \sum_{j \in J} m_j \otimes f_{jk} + \sum_{i \notin J} m_i \otimes f_{ik}$. Then for $k \in J$ we have

$$m_k = \gamma \cdot m_k = (\mathrm{id}_M \otimes \gamma) \circ \rho(m_k)$$

for all $\gamma \in \Gamma$, so $\gamma(f_{jk}) = \delta_{jk}$ and $\gamma(f_{ik}) = 0$. By density we therefore get $\rho(m_k) = m_k \otimes 1$. □

We now specialise to arithmetic subgroups of unipotent groups. Let U be a unipotent algebraic group over $\mathbb{Q}$ and Γ be an arithmetic subgroup of U. Recall from Theorem 1.2.5 that U has a filtration by closed subgroups U_i satisfying $U_i/U_{i+1} \cong G_a$. We aim to prove a similar fact for arithmetic subgroups of U. To do so, we consider Γ as a subgroup of the group of real points $U(\mathbb{R})$. The latter has the canonical structure of a Lie group with finitely many connected components (see [PR94][Chapter 3, Theorem 3.6]) and, by convention, we will always consider the group of real points as a Lie group. The group Γ embeds into $U(\mathbb{R})$ via the canonical inclusion $U(\mathbb{Q}) \to U(\mathbb{R})$ and is in fact a discrete (and hence closed) subgroup of $U(\mathbb{R})$. If we start with the "simplest" (in the sense of our filtration) unipotent group we have, the additive group G_a, we thus see that any arithmetic subgroup of G_a is a discrete subgroup of the additive group of the real numbers $\mathbb{R}$. In this case, we have the following well known result.

Lemma 3.3.5. *Any discrete subgroup Γ of a finite dimensional vector space V over the real numbers $\mathbb{R}$ is isomorphic to $\mathbb{Z}^s$ for some $s \leq \dim(V)$.*

Proof. By replacing V with the $\mathbb{R}$-space spanned by Γ, we may assume that Γ generates V. Let $v_1, \ldots, v_n$ be a basis of V consisting of elements of Γ and consider the subgroup Γ_0 of Γ generated by $v_1 1, \ldots, v_n$. Then each element of Γ/Γ_0 has a unique representative $v = \sum \lambda_i v_i \in \Gamma$ with $0 \leq \lambda_i < 1$. As Γ is discrete, the set of such elements must be finite, hence Γ/Γ_0 is a finite group. In particular, since both Γ_0 and Γ/Γ_0 are finitely generated, Γ is finitely generated as well. This, together with the fact that Γ is torsion free implies that Γ is a free abelian group (see [Bea99][Theorem 2.7.6]). Moreover $\operatorname{rank}(\Gamma) = \operatorname{rank}(\Gamma_0) + \operatorname{rank}(\Gamma/\Gamma_0) = \dim(V)$, hence the assertion follows. □

Corollary 3.3.6. *Any arithmetic subgroup of the additive group G_a over $\mathbb{Q}$ is isomorphic to the additive group of integers $\mathbb{Z}$.*

From this result, we can directly derive a suitable adaption of the filtration of unipotent groups to arithmetic subgroups thereof.

Proposition 3.3.7. *Let U be a unipotent algebraic group over $\mathbb{Q}$ and Γ be an arithmetic subgroup of U. If V is a normal subgroup of U with $U/V \cong G_a$, then $\Gamma' = \Gamma \cap V(\mathbb{Q})$ is a normal subgroup of Γ with $\Gamma/\Gamma' \cong \mathbb{Z}$.*

Proof. It is clear that Γ' is a normal subgroup of Γ. Moreover, Γ/Γ' is an arithmetic subgroup of G_a, hence isomorphic to $\mathbb{Z}$ by the previous corollary. □

Since the subgroup Γ' is an arithmetic subgroup of V, hence we may inductively repeat this process to obtain a filtration

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_n = \{e\}$$

with Γ_{i+1} normal in Γ_i and $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$ for $0 \leq i < n$. Moreover, using the same fact, we can show the following important lemma.

Lemma 3.3.8. *Let Γ be an arithmetic subgroup of a unipotent algebraic group U over $\mathbb{Q}$. Then Γ is dense in U.*

Proof. We apply induction on the dimension n of U. If $n = 1$ then $U \cong G_a$. As $\Gamma \cong \mathbb{Z}$, it is clearly dense in U. If $n > 1$ then we may find a closed subgroup V of U with $U \cong V \rtimes G_a$ (cf. Lemma 1.2.5). Moreover, $\Gamma' = \Gamma \cap V(\mathbb{Q})$ is an arithmetic subgroup of V and hence by assumption dense in V and likewise $\Gamma'' = \Gamma/\Gamma' \subset G_a(\mathbb{Q})$ is dense in G_a. Now $U \cong V \otimes G_a$ as an affine scheme, and the closure of Γ contains both V and G_a, which implies that $\bar{\Gamma} = U$. □

As a consequence, the fixed points of any U-module under the corresponding Γ-action coincide with the fixed points under U-action.

Let us fix a unipotent group U over $\mathbb{Q}$ and suppose Γ is an arithmetic subgroup of U. As we have seen in Lemma 1.1.12, any U-module M is locally finite and hence a direct limit $\varinjlim M_j$ over a family $\{M_j\}_{j\in J}$ of finite dimensional U-modules. Since direct limits are preserved under tensor products, we see that $M \cong \varinjlim M_j$ as Γ-modules. If we now were to compute the cohomology of M as a Γ-module, we would end up computing $H^i(\Gamma, \varinjlim M_j)$. The natural question which comes to mind is whether we can "exchange" cohomology and direct limits i.e. if

$$H^i(\Gamma, \varinjlim M_j) \cong \varinjlim H^i(\Gamma, M_j).$$

This property, in general, depends on the type of resolution one can chose for the Γ-module $\mathbb{Z}_{tr}$. More precisely. we say that a projective resolution (P_n) of a module M over some ring R is of *finite type* (resp. *free*) if each P_n is finitely generated (resp. free). If (P_n) is of finite type and has finite length, i.e. if $P_i = 0$ for all i sufficiently large, then (P_n) is called a finite resolution.

Definition 3.3.9. *Let Γ be an abstract group. We say that Γ is of type* (FL) *if the trivial Γ-module $\mathbb{Z}$ has a finite free resolution.*

We already know that a finite length resolution guarantees that the cohomological dimension is finite. However, the existence of a resolution by finitely generated free modules yields an even stronger consequence.

Proposition 3.3.10. *If Γ is of type (FL) then $H^i(\Gamma, -)$ commutes with direct limits.*

Proof. The proof can be found in [Bro75][Corollary 1]. □

An example of a group of type (FL) is the additive group of integers $\mathbb{Z}$. This follows from an earlier example, where we showed that the exact sequence

$$0 \to I\Gamma \to \mathbb{Z}\Gamma \to \mathbb{Z} \to 0$$

with the morphism $\mathbb{Z}\Gamma \to \mathbb{Z}$ being the augmentation map, is a finite free resolution. This fact alone would be sufficient for our cause, however there is an interesting general result for discrete subgroups of unipotent groups, relating density to the (FL) property.

If we deal with discrete subgroups of real Lie groups with finitely many connected components (such as in our case), there is a useful criterion to check if the property (FL) is satisfied.

Theorem 3.3.11. *Suppose Γ is a discrete, torsion free subgroup of a real Lie group G with finitely many connected components. If G/Γ is compact, then Γ is of type (FL).*

Proof. This result can be found in *Cohomologie des groupes discrets* by J.P. Serre [Ser71][Proposition 18]. □

Hence our aim is to apply this criterion to an arithmetic subgroup Γ of our unipotent group U. We first show that Γ is torsion free. To do so, let $\gamma \neq e \in \Gamma$ be an element of finite order. As we have seen, there exists a filtration of Γ by a descending chain of subgroups Γ_i, each normal in its predecessor, such that the quotient of any two subsequent subgroups is equal to $\mathbb{Z}$. We now chose i to be maximal with respect to $\gamma \in \Gamma_i$ and let $\pi\colon \Gamma_i \to \Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$ be the canonical projection. Then $\pi(\gamma)$ is a torsion element of $\mathbb{Z}$, hence $\pi(\gamma) = 0$ which forces $\gamma \in \Gamma_{i+1}$, a contradiction to our choice of i. Thus, we only need to show that $U(\mathbb{R})/\Gamma$ is compact. This is provided by the following lemma, the proof of which can be found in an article by Raghunathan [Rag67][Lemma 1].

Proposition 3.3.12. *Let U be a unipotent algebraic group over the field of real numbers $\mathbb{R}$ and $\Gamma \subset U(\mathbb{R})$ be a discrete subgroup. Then the following conditions are equivalent:*

1. *The quotient $U(\mathbb{R})/\Gamma$ is compact.*
2. *The group Γ is dense in U.*

Summing up, we obtain the following corollary:

Corollary 3.3.13. *Let Γ be an arithmetic subgroup of a unipotent algebraic group over $\mathbb{Q}$. Then $H^i(\Gamma, -)$ commutes with direct limits for all i.*

3.4 The Cohomology Isomorphisms

Having all the necessary techniques at hand, we can now focus on the main results we want to derive. Suppose first that U is a unipotent algebraic group over the field of rational numbers $\mathbb{Q}$ with Lie algebra $\mathfrak{u} = \mathrm{Lie}(U)$ and let Γ be an arbitrary arithmetic subgroup of U. We want to show that the cohomology functors $H(U, -)$ and $H(\Gamma, -)$ coincide when applied to U-modules. In more technical terms, by composing the canonical functor $\{U\text{-modules}\} \to \{\Gamma\text{-modules}\}$ constructed in 3.3.3 with $H(\Gamma, -)$, we can regard both functors as δ-functors from the category of U-modules to the category of abelian groups and we claim these functors are naturally isomorphic. But before we prove this result, we require a rather inconspicuous lemma.

Lemma 3.4.1. *Let K be a field and $x \in K^\times$. The K-linear map $T\colon K[t] \to K[t]$, defined by $T(f) = f(t-x) - f$ is onto.*

Proof. Since T_x is K-linear, it is sufficient to show that for each $n \in \mathbb{N}$ there always exists $f \in K[t]$ such that $t^n = f(t-x) - f$. Suppose $f = \sum_{i=0}^{k} a_i t^i$, then

$$
\begin{aligned}
T_x(f) &= \sum_{i=0}^{k} a_i (t-x)^i - a_i t^i \\
&= \sum_{i=0}^{k} a_i \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} t^j - a_i t^i \\
&= \sum_{j=0}^{k-1} \Big(\sum_{i=j+1}^{k} \binom{i}{j} (-x)^{i-j} a_i \Big) t^j
\end{aligned}
$$

Thus, choosing $k = n + 1$ and considering

$$t^n = \sum_{j=0}^{n} \Big(\sum_{i=j+1}^{n+1} \binom{i}{j} (-x)^{i-j} a_i \Big) t^j$$

we obtain a system of equations which yields $a_{n+1} = 1$ and by recursively solving

$$\sum_{i=j+1}^{n+1} \binom{i}{j} (-x)^{i-j} a_i = 0$$

starting with $j = n$, we obtain the other coefficients and hence the necessary polynomial. □

We now consider the two δ-functors $H(U, -)$ and $H(\Gamma, -)$. In both cases, applying H^0 to a U-module M yields precisely the group of fixed points with respect to the action of either U or Γ on M. Thus, as a consequence of Lemma 3.3.8 and Lemma 3.3.4, we obtain that the functors $H^0(U, -)$ and $H^0(\Gamma, -)$ are naturally isomorphic. On the other hand, we know that $H(U, -)$ is a universal δ-functor. If we were able to prove that $H(\Gamma, -)$ is universal as well, we would be done since any two universal δ- functors coinciding on H^0 are isomorphic (this is an immediate consequence of the definition). By Lemma 3.1.13, it would be sufficient to show that $H(\Gamma, -)$ is effaceable as a functor from the category of U-modules to the category of groups. This is exactly what we will show.

Theorem 3.4.2. *Let U be a unipotent algebraic group over the field of rational numbers $\mathbb{Q}$ and let Γ be an arithmetic subgroup of U. Then given any U-module M, the module $M \otimes \mathbb{Q}[U]$ is acyclic for $H(\Gamma, -)$, i.e. $H^i(\Gamma, M \otimes \mathbb{Q}[U]) = 0$ for all $i \geq 1$.*

Proof. Since $M \otimes \mathbb{Q}[U] \cong M_{tr} \otimes \mathbb{Q}[U]$ as U-modules, there is no loss of generality in assuming $M = M_{tr}$. Let $n = \dim U$ and assume the assertion to be true for all unipotent groups of smaller dimension. By Theorem 1.2.5 there is a normal closed subgroup V of U with $U/V \cong G_a$. Moreover, as we have seen in Proposition 3.3.7 (and the discussion following its proof), $\Gamma' = \Gamma \cap V(\mathbb{Q})$ is an arithmetic subgroup of V, normal in Γ and $\Gamma'' = \Gamma/\Gamma' \cong \mathbb{Z}$. In particular, the exact sequence of groups

$$0 \to \Gamma' \to \Gamma \to \Gamma'' \to 0$$

yields a spectral sequence

$$H^p(\Gamma'', H^q(\Gamma', M \otimes \mathbb{Q}[U])) \Rightarrow_p H^{p+q}(\Gamma, M \otimes \mathbb{Q}[U]) \ .$$

Since U is the semidirect product of V and G_a, we get $\mathbb{Q}[U] \cong \mathbb{Q}[G_a] \otimes \mathbb{Q}[V]$ as V-modules (see the discussion following Theorem 1.2.5) and thus obtain an isomorphism of V-modules $M \otimes \mathbb{Q}[U] \to (M \otimes \mathbb{Q}[t]) \otimes \mathbb{Q}[V]$ with $\mathbb{Q}[t] = \mathbb{Q}[G_a]$ regarded as a trivial V-module. By assumption $(M \otimes \mathbb{Q}[t]) \otimes \mathbb{Q}[V]$ is acyclic for $H(\Gamma', -)$, hence $H^q(\Gamma', M \otimes \mathbb{Q}[U]) = 0$ for all $q \geq 1$. This implies that the spectral sequence considered above collapses on the p-axis and therefore

$$H^p(\Gamma'', H^0(\Gamma', M \otimes \mathbb{Q}[U])) \cong H^p(\Gamma, M \otimes \mathbb{Q}[U])$$

by Proposition 3.2.7. As M and $\mathbb{Q}[t]$ are trivial Γ'-modules, we can apply Lemma 1.2.6 to obtain

$$\begin{aligned} H^0(\Gamma', M \otimes \mathbb{Q}[U]) &\cong ((M \otimes \mathbb{Q}[t]) \otimes \mathbb{Q}[V])^{\Gamma'} \\ &= ((M \otimes \mathbb{Q}[t]) \otimes \mathbb{Q}[V])^{V} \\ &\cong (M \otimes \mathbb{Q}[t]) \otimes (\mathbb{Q}[V]^{V}) \\ &\cong M \otimes \mathbb{Q}[t]. \end{aligned}$$

Hence we end up with

$$H^p(\Gamma'', M \otimes \mathbb{Q}[t]) \cong H^p(\Gamma, M \otimes \mathbb{Q}[U]),$$

which shows that it is sufficient to prove the case $n = 0$ i.e. $U = G_a$.
Assuming $U = G_a$, we recall that M is locally finite and thus $M \cong \varinjlim M_i$ for some family of finite dimensional G_a-submodules of M. This implies that $M \otimes \mathbb{Q}[t] \cong \varinjlim M_i \otimes \mathbb{Q}[t]$, since direct limits commute with tensor products (see [Rot09][Chapter 5, Theorem 5.51]). Moreover, by Corollary 3.3.13, Γ is of type (FL), so we can exchange direct limits with cohomology to obtain

$$H^j(\Gamma, M \otimes \mathbb{Q}[t]) \cong \varinjlim H^j(\Gamma, M_i \otimes \mathbb{Q}[t]).$$

It thus suffices to show that $H^i(\Gamma, M \otimes \mathbb{Q}[t]) = 0$ holds for finite dimensional M and all $i \geq 1$. In this case, applying Lemma 1.2.7 yields a G_a-submodule M' of M such that $M/M' \cong K$, where $K = K_{tr}$ is regarded as a trivial G_a-module. This in turn gives rise to a long exact sequence on the cohomology groups

$$0 \to H^0(\Gamma, M' \otimes \mathbb{Q}[t]) \to H^0(\Gamma, M \otimes \mathbb{Q}[t]) \to H^0(\Gamma, \mathbb{Q}[t]) \to H^1(\Gamma, M' \otimes \mathbb{Q}[t]) \to \dots$$

$$\dots \to H^i(\Gamma, M' \otimes \mathbb{Q}[t]) \to H^i(\Gamma, M \otimes \mathbb{Q}[t]) \to H^i(\Gamma, \mathbb{Q}[t]) \to H^{i+1}(\Gamma, M'\mathbb{Q}[t]) \to \dots.$$

By Corollary 3.3.6 Γ is isomorphic to the additive group of integers $\mathbb{Z}$ and is therefore cyclic and has cohomological dimension 1. In particular, the long exact sequence above actually terminates at the first occurrence of H^2. Moreover, if we assume the assertion to be true for all G_a-modules of dimension less than M, we get $H^1(\Gamma, M' \otimes \mathbb{Q}[t]) = 0 = H^1(\Gamma, \mathbb{Q}[t])$ and thus $H^1(\Gamma, M \otimes \mathbb{Q}[t]) = 0$ by the exactness of

$$H^1(\Gamma, M' \otimes \mathbb{Q}[t]) \to H^1(\Gamma, M \otimes \mathbb{Q}[t]) \to H^1(\Gamma, \mathbb{Q}[t]) \to 0.$$

We have therefore reduced to the case of $\dim M = 1$, i.e. $M = \mathbb{Q}_{tr}$ and need to show that $H^1(\Gamma, \mathbb{Q}[t]) = 0$ (all larger cohomology groups vanish because of $\operatorname{cd} \Gamma = 1$). To do so, we use the interpretation of H^1 in terms of derivations, i.e. $H^1(\Gamma, \mathbb{Q}[t]) = \operatorname{Der}(\Gamma, \mathbb{Q}[t]) / \operatorname{Ider}(\Gamma, \mathbb{Q}[t])$. Therefore, it suffices to show that any derivation $d\colon \Gamma \to \mathbb{Q}[t]$ is inner. Since Γ is cyclic, it is generated by a single element γ. As a consequence, any other element of Γ can be written as either γ^n or γ^{-n} for some unique positive integer n. Using the fact that d is a derivation we get $d(\gamma^n) = d(\gamma) + \gamma d(\gamma^{n-1})$ and thus inductively

$$d(\gamma^n) = \sum_{i=0}^{n-1} \gamma^i d(\gamma).$$

Moreover $0 = d(\gamma\gamma^{-1}) = d(\gamma)+\gamma d(\gamma^{-1})$ yields $d(\gamma^{-1}) = -\gamma^{-1}d(\gamma)$. In particular, d is completely determined by $d(\gamma)$. Suppose now we could write $d(\gamma) = f - \gamma f$ with $f \in \mathbb{Q}[t]$. Then

$$\begin{aligned} d(\gamma^n) &= \sum_{i=0}^{n-1} \gamma^i d(\gamma) \\ &= \sum_{i=0}^{n-1} \gamma^i (f - \gamma f) \\ &= f - \gamma^n f \end{aligned}$$

and $d(\gamma^{-1}) = -\gamma^{-1}d(\gamma) = f - \gamma^{-1}f$. Thus d is an inner derivation if (and only if) we can solve the equation $d(\gamma) = f - \gamma f$. But this follows immediately from Lemma 3.4.1 since $\gamma f = f(t-\gamma)$. Summing up, we have $\mathrm{Der}(\Gamma, \mathbb{Q}[t]) = \mathrm{Ider}(\Gamma, \mathbb{Q}[t])$ and thus $H^1(\Gamma, \mathbb{Q}[t]) = 0$, which completes the proof. $\square$

As an immediate consequence, we obtain the following corollary, which is precisely the generalisation of van Est's Theorem we aimed for.

Corollary 3.4.3. *If Γ is an arithmetic subgroup of a unipotent algebraic group U over the field of rational numbers $\mathbb{Q}$, then for any U-module M*

$$H^i(U, M) = H^i(\Gamma, M).$$

We now turn to $H(\mathfrak{u}, -)$, which we want to identify with $H(U, -)$. The idea is largely the same as above, since we have all the properties we need to apply the same reduction steps, thus aside from slight differences, we can mimic the proof almost step by step for a general field K of characteristic 0.

Theorem 3.4.4. *Let U be a unipotent algebraic group over a field K of characteristic 0 and let M be a U-module. Then the module $M \otimes K[U]$ is acyclic for $H(\mathfrak{u}, -)$.*

Proof. We may again assume M to be a trivial U-module. Let V be a normal subgroup of U with $U/V \cong G_a$ as constructed in Theorem 1.2.5 and denote the Lie algebra of V by $\mathfrak{v}$. Then we have an exact sequence of Lie algebras

$$0 \to \mathfrak{v} \to \mathfrak{u} \to \mathfrak{g}_a \to 0$$

which gives rise to a spectral sequence

$$H^p(\mathfrak{g}_a, H^q(\mathfrak{v}, M \otimes K[U])) \Rightarrow_p H^{p+q}(\mathfrak{u}, M \otimes K[U]).$$

As before, we have $K[U] \cong K[t] \otimes K[V]$ as V-modules, so we can use Corollary 1.3.10 and Lemma 1.2.6 to obtain

$$\begin{aligned} H^0(\mathfrak{v}, M \otimes K[U]) &= (M \otimes K[U])^{\mathfrak{v}} \\ &= (M \otimes K[t] \otimes K[V])^V \\ &= M \otimes K[t]. \end{aligned}$$

Thus, if we assume the assertion of the theorem to be valid for all unipotent groups of dimension less than $\dim(U)$, we obtain $H^q(\mathfrak{v}, M \otimes K[U]) = 0$ for $q \geq 1$. In particular, the spectral sequence considered above collapses and yields

$$H^p(\mathfrak{g}_a, M \otimes K[t]) \cong H^p(\mathfrak{u}, M \otimes K[U]),$$

hence it is sufficient to consider the case $U = G_a$. Since G_a is abelian, its Lie algebra $\mathfrak{g}_a$ is simply the trivial Lie algebra K and its universal enveloping algebra is isomorphic to the polynomial ring $K[t]$. Therefore, $\mathfrak{g}_a$ has cohomological dimension one. The trivial $\mathfrak{g}_a$-module K corresponds to the $K[t]$-module structure on K obtained by mapping t to 0 and the augmentation ideal is precisely the principal ideal generated by t. This shows that K_{tr} has a finite free resolution of length 1 over $\mathfrak{g}_a$. Applying these observations to the cohomology groups $H^p(\mathfrak{g}_a, M \otimes K[U])$ we get

$$H^p(\mathfrak{g}_a, M \otimes K[U]) = 0 \text{ for } p > 1$$

and we can, just as in the proof of Theorem 3.4.2, reduce to the case of $\dim M < \infty$ by writing $M = \varinjlim M_j$ for a family $\{M_j\}$ of finite dimensional G_a-submodules of M and using

$$H^1(\mathfrak{g}_a, M \otimes K[U]) \cong \varinjlim H^1(\mathfrak{g}_a, M_j \otimes K[U]).$$

Assuming M to be a finite dimensional G_a-module, we can find a G_a-submodule M' of M with $M/M' \cong K_{tr}$. This yields an exact sequence

$$0 \to H^0(\mathfrak{g}_a, M' \otimes K[t]) \to H^0(\mathfrak{g}_a, M \otimes K[t]) \to H^0(\mathfrak{g}_a, K[t]) \to$$

$$H^1(\mathfrak{g}_a, M' \otimes K[t]) \to H^1(\mathfrak{g}_a, M \otimes K[t]) \to H^1(\mathfrak{g}_a, K[t]) \to 0$$

(recall that $H^2(\mathfrak{g}_a, -) = 0$), so if we assume the assertion to be true for all G_a-modules of dimension less than $\dim(M)$, we get $H^1(\mathfrak{g}_a, M \otimes K[t]) = 0$ as well. This shows that it is sufficient to establish the theorem for $\dim(M) = 1$ i.e. $M = K_{tr}$. Since we already know that $H^q(\mathfrak{g}_a, K[t]) = 0$, we only have to compute $H^1(\mathfrak{g}_a, K[t])$. To do so, we need to give an explicit description of the $\mathfrak{g}_a$-module structure on $K[t]$. Recall that the action of G_a on $K[t]$ is given by the right regular representation, i.e. by the comodule map μ: $K[t] \to K[t] \otimes K[t] \cong K[t, s]$ with $\mu(t) = t + s$. Furthermore, $\mathfrak{g}_a = \mathrm{Dist}_1^+(G_a) \subset K[G]^*$ consists of all K-linear maps γ: $K[s] \to K$ with $\gamma((s^2)) = 0$ and $\gamma(1) = 0$. In particular, any such map is uniquely determined by $\gamma(s) \in K$. By definition, the action of γ on f in $K[t]$ is now given by $\gamma \cdot f = (id \otimes \gamma)(\rho(f))$. In case $f = t^n$, we therefore get

$$\begin{aligned}
\gamma \cdot f &= (id \otimes \gamma)((t+s)^n) \\
&= \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \gamma(s^i) \\
&= n t^{n-1} \gamma(s).
\end{aligned}$$

We therefore get $\gamma \cdot t^n = a \frac{d}{dt} t^n$, where $a = \gamma(s) \in K$ and $\frac{d}{dt}$ is the standard differential operator on $K[t]$. By linearity of the action of γ on $K[t]$ this extends to $\gamma \cdot f = a \frac{d}{dt} f$ for any $f \in K[t]$, so $\mathfrak{g}_a$ acts on $K[t]$ by scalar multiples of the differential $\frac{d}{dt}$.

We now identify $H^1(\mathfrak{g}_a, K[t])$ with the quotient of $\mathrm{Der}(\mathfrak{g}_a, K[t])$ by $\mathrm{Ider}(\mathfrak{g}_a, K[t])$. Suppose d: $\mathfrak{g}_a \to K[t]$ is a derivation. Since $\mathfrak{g}_a$ is generated by 1 (or any nonzero element for that matter) as a vector space over K and d is K-linear, we get $d(x) = xd(1)$ for any $x \in \mathfrak{g}_a = K$. By definition, d is an inner derivation if and only if $d(x) = x \cdot f$ for some $f \in K[t]$. By using the linearity of the action of $\mathfrak{g}_a$ on $K[t]$ we can rewrite $x \cdot f = x(1 \cdot f)$. This shows that d is an inner derivation if $d(1) = 1 \cdot f$ i.e. if $d(1) = \frac{d}{dt} f$ for some $f \in K[t]$. But this equation clearly has a solution (the "antiderivative" of $d(1)$ exists since K has characteristic 0), hence any derivation d is an inner derivation and $H^1(\mathfrak{g}_a, K[t]) = 0$. $\square$

In particular, the equality $M^U = M^{\mathfrak{u}}$ extends to an isomorphism from $H^*(U, M)$ to $H^*(\mathfrak{u}, M)$. We have therefore reestablished Hochschild's Theorem:

Corollary 3.4.5. *Let U be a unipotent algebraic group over a field of characteristic 0 and let $\mathfrak{u}$ denote the Lie algebra of U. Then for any U-module M, there is a natural isomorphism*

$$H^i(U, M) \cong H^i(\mathfrak{u}, M).$$

Summary

The thesis at hand deals with the cohomology of arithmetic subgroups of unipotent algebraic groups over the field of rational numbers $\mathbb{Q}$. It is motivated by a result of van Est, which states that for a discrete cocompact subgroup Γ of a nilpotent real Lie group N and a finite dimensional unipotent linear representation V of N, there is an isomorphism from the differentiable cohomology group $H^*_d(N, V)$ into the cohomology group $H^*(\Gamma, V)$. The principal aim of this work is to establish an algebraic counterpart of this result, connecting the rational cohomology groups $H^*(U, V)$, associated to a rational representation V of a unipotent algebraic group U over $\mathbb{Q}$, to the cohomology groups $H^*(\Gamma, V)$ of an arithmetic subgroup Γ of U via a natural isomorphism. Aside from being different in the approach, the latter statement is not confined to finite dimensional representations, as a rational representation may well be of infinite dimension. Moreover, the new isomorphism considers the cohomology of Γ with values in the $\mathbb{Q}$-vector space V, thus does not require an extension of scalars to the field of real numbers $\mathbb{R}$.

In passing, we shall also consider a result of G. Hochschild, which relates the rational cohomology groups $H^*(U, V)$ to the Lie algebra cohomology groups $H^*(\mathfrak{u}, V)$ (see [Hoc61]). As it turns out, both results can be proven using quite similar techniques.

The exposition itself is split into three chapters. After a brief introduction to group schemes and their representations, the first chapter focuses on properties of unipotent groups over a field of characteristic 0. It is shown in Theorem 1.2.5, that any unipotent group U permits a closed subgroup V, such that the quotient U/V is isomorphic to G_a. Similarly, any finite dimensional representation M of U permits a U-submodule M' such that M/M' is a trivial 1-dimensional U-module (see Proposition 1.2.7). The last part of the first chapter contains a brief summary of the most important constructions and results regarding Lie algebras of algebraic groups.

Chapter 2 introduces the concept of induced representations for an algebraic group G, the notion of which is quite similar to the one used for finite groups or Lie groups. The most important property of these representations is called Frobenius reciprocity, which is established in Proposition 2.2.1. These results are applied to construct particularly simple injective resolutions for a given G-module M. More precisely, we shall establish that for any G-module M, the tensor product $M \otimes K[G]$ of M with the coordinate ring $K[G]$ of G is an injective module, in which M can easily be embedded.

The third and final chapter then deals with arithmetic groups and their cohomology groups. The first half contains a brief introduction to the concept of cohomology and spectral sequences. The upshot is that cohomology functors are all effaceable functors and thus special instances of universal δ-functors. Moreover, the spectral sequences for abstract groups and Lie Algebras provide a powerful tool for relating the cohomology groups of a group or Lie Algebra to those of

a normal subgroup or ideal, respectively.

In the second half of chapter 3 the focus shifts back to unipotent groups. After defining arithmetic subgroups of algebraic groups over $\mathbb{Q}$, we concentrate on the structure of those subgroups in case of a unipotent algebraic group U. We then leverage the construction of the filtration in the first chapter to show that any arithmetic subgroup Γ of U has normal subgroup Γ' of Γ such that $\Gamma/\Gamma' \cong \mathbb{Z}$. Moreover, Γ is a dense subgroup of U, cocompact in the Lie group of real points $U(\mathbb{R})$ and thus of type (FL). This permits us to exchange direct limits and cohomology.

Finally, we combine all these observations, to prove the effaceability of $H(\Gamma, -)$ and $H(\mathfrak{u}, -)$ as δ-functors from the category of U-modules to the category of abelian groups, which in turn yields the results of van Est and Hochschild, respectively.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Kohomologie arithmetischer Untergruppen von unipotenten algebraischen Gruppen über dem Körper der rationalen Zahlen $\mathbb{Q}$. Besonderes Augenmerk gilt dabei einem Resultat von van Est, das einen Isomorphismus zwischen den differenzierbaren Kohomologiegruppen $H_d^*(N,V)$ einer reellen nilpotenten Lie Gruppe N mit Werten in einer endlichdimensionalen unipotenten Darstellung V und der gewöhnlichen Gruppenkohomologie $H^*(\Gamma,V)$ einer diskreten, kokompakten Untergruppe Γ von N herstellt. Das erklärte Ziel der der Arbeit ist es, ein algebraisches Gegenstück zu diesem Resultat im Fall einer unipotenten algebraischen Gruppe U über $\mathbb{Q}$ und einer arithmetischen Untergruppe Γ von U zu finden, also die rationalen Kohomologiegruppen $H^*(U,V)$ von U mit Werten in einer rationalen Darstellung V durch einen natürlichen Isomorphismus mit den gewöhnlichen Kohomologiegruppen $H^*(\Gamma,V)$ zu identifizieren. Dies erweitert das Resultat von von Est nicht nur auf unendlichdimensionale Darstellungen, sondern bietet den zusätzlichen Vorteil, dass die erwähnten Kohomologiegruppen von Γ sich nur auf den $\mathbb{Q}$-Vektorraum V beziehen und damit keine skalare Erweiterung auf den Körper der reellen Zahlen mehr notwendig ist.

Die diesem Resultat zugrunde liegenden Ideen ähneln denen in [Hoc61]. In diesem Artikel wird die Existenz eines Isomorphismus zwischen den rationalen Kohomologiegruppen $H^*(U,V)$ und den zugehörigen Lie Algebra Kohomologiegruppen $H^*(\mathfrak{u},V)$ der Lie Algebra $\mathfrak{u}$ von U bewiesen. In der Tat eignen sich die in dieser Arbeit präsentierten Techniken, um einen alternativen Beweis für diese Aussage geben zu können.

Die Arbeit selbst gliedert sich in drei Kapitel. Das erste Kapitel beinhaltet eine kurze Einführung in die Theorie der Gruppenschemata und der zugehörigen Darstellungen. Besondere Beachtung gilt dabei der Struktur von unipotenten Gruppen über Körpern der Charakteristik 0 und ihren endlichdimensionalen Darstellungen. Beide erlauben besonders einfache Filtrierungen, die in Theorem 1.2.5 und Proposition 1.2.7 konstruiert werden. Das Kapitel schließt mit einer kurzen Zusammenfassung der wichtigsten Resultate und Konstruktionen für die Lie Algebren von algebraischen Gruppen.

Kapitel 2 beschreibt das Konzept von induzierten Darstellungen für algebraische Gruppen. Die Begriffe und Konstruktionen ähneln dabei denen für Lie Gruppen und endliche Gruppen. Insbesondere gilt für induzierte Darstellungen eine üblicherweise als Frobenius Reziprozität bezeichnete Eigenschaft, die in Proposition 2.2.1 bewiesen wird. Dies erlaubt die Konstruktion injektiver Auflösungen für Darstellungen algebraischer Gruppen. So ist für jeden Modul M einer algebraischen Gruppe G die Darstellung von G auf dem Tensorprodukt $M \otimes K[G]$, wobei $K[G]$ den Koordinatenring von G bezeichnet, injektiv. Da man M leicht in $M \otimes K[G]$ einbetten kann, ergibt sich damit eine besonders einfache Methode um injektive Auflösungen zu konstruieren.

Das dritte und letzte Kapitel befasst sich mit arithmetischen Untergruppen und ihrer Kohomologie. Die erste Hälfte beinhaltet dazu eine kurze Einführung in die grundlegenden Konzepte der Kohomologietheorie und Spektralsequenzen. Besonderes Augenmerk gilt hier der Interpretation von Kohomologiefunktoren als spezielle universelle δ-Funktoren.

Danach widmet sich die Arbeit den arithmetischen Untergruppen unipotenter algebraischer Gruppen über $\mathbb{Q}$. Hier kann die Filtrierung für unipotente Gruppen aus dem ersten Kapitel eingesetzt werden, um zu zeigen, dass jede arithmetische Untergruppe Γ einer unipotenten Gruppe U eine normale Untergruppe Γ' besitzt, die $\Gamma/\Gamma' \cong \mathbb{Z}$ erfüllt. Aus dieser Tatsache ergibt sich, dass Γ dicht in U ist. Des Weiteren ist Γ vom Typ (FL) und erlaubt damit das vertauschen der Kohomologiefunktoren $H^*(\Gamma, -)$ mit direkten Limiten.

Abschließend kann durch Kombination dieser Beobachtungen bewiesen werden, dass $H(\Gamma, -)$ und $H(\mathfrak{u}, -)$ als δ-Funktoren von der Kategorie der U-Moduln in die Kategorie der abelschen Gruppen auslöschbar sind. Dies beweist damit sowohl die oben beschriebene Version des Satzes von van Est als auch das erwähnte Theorem von Hochschild.

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