universität wien

DIPLOMARBEIT

Titel der Diplomarbeit

Functional Analytic Methods and Applications to Quasilinear Boundary Value Problems

Verfasser

Ronald Quirchmayr

angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag. rer. nat.)

Wien, Dezember 2012

Sudienkennzahl lt. Studienblatt:	A 405
Studienrichtung lt. Studienblatt:	Mathematik
Betreuer:	ao. Univ.-Prof. Dr. Günther Hörmann

Contents

Notational Conventions iii

Preface v
Introduction v
Acknowledgments vi

Chapter 1. Preliminary Material 1
Orientation 1
1.1. Functional analytic principles 1
1.2. Theory of integration 8
1.3. Sobolev spaces 19

Chapter 2. Functional Analytic Methods 25
Orientation 25
2.1. Weak compactness in reflexive Banach spaces 26
2.2. Main theorem for extremal points 29
2.3. Convex functionals 32
2.4. Differentiable functionals 34
2.5. Potential operators 40
2.6. Examples 42

Chapter 3. Quasilinear Boundary Value Problems 45
Orientation 45
3.1. A first closer inspection of problem (3.1) 45
3.2. Weak formulation of (3.1) 46
3.3. Specification of the data in (3.1) 49
3.4. Variational analysis of problem (3.1) 49

Chapter 4. Weak Lower Semicontinuity of Integral Functionals 63
Orientation 63
4.1. Preliminairy remarks 64
4.2. Convexity in the main part: a sufficient condition 64
4.3. Convexity in the main part: a necessary condition 70

References 75

Index 77

Curriculum Vitae 79

Notational Conventions

$\mathbb{N}$	$\{1, 2, 3, ...\}$
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{\pm\infty\}$
Ω	an open subset in $\mathbb{R}^d$, often with further restrictions
$C(\Omega)$	the set of continuous functions $u\colon \Omega \to \mathbb{R}$
$C^k(\Omega)$	the set of k-times continuously differentiable functions $u\colon \Omega \to \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$
$\mathcal{D}(\Omega)$	the set of compactly supported functions in $C^\infty(\Omega)$
$\mathcal{N}_\varphi$	Nemytskii operator for the function φ, see Section 1.2.3
Vol_d	d-dimensional Lebesgue measure
$\mathrm{B}_\varepsilon(u_0)$	ball of radius $\varepsilon > 0$ centered at u_0
p	an element of $[1, \infty]$
p'	the Hölder conjugate of p, see Theorem 1.37 in Section 1.2.2
p^*	the Sobolev conjugate of p, see Section 1.3
$p^\#$	the trace conjugate of p, see Theorem 1.67 in Section 1.3
”.”	scalar product in $\mathbb{R}^m$, $m \geq 2$

Preface

This diploma thesis deals with functional analytic methods in the calculus of variations (Chapter 2) and an application to the existence theory for solutions to boundary value problems for stationary quasilinear second-order partial differential equations (Chapter 3). In Chapter 4 we discuss weak sequential lower semicontinuity - a key ingredient in the calculus of variations - of integral functionals. Chapter 1 provides tools from the theory of normed spaces, integration theory and the theory of Sobolev spaces.

Introduction

The modern calculus of variations is intimately connected with Hilbert's justification of the *Dirichlet principle*. Let us for this reason give a short summery of the developments in the second half of the 19th century concerning this principle.

Consider *Dirichlet's problem* for a function $u\colon \Omega \to \mathbb{R}$:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

Here, $\Omega \subseteq \mathbb{R}^d$ is open and bounded with C^1-boundary $\partial\Omega$, and $f \in C(\Omega)$ is a given function. Problem (0.1) is closely related to the problem of minimizing the functional

$$F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - f u \, dx$$

defined on the set of admissible functions

$$\mathcal{A} := \{u \in C^2(\overline{\Omega})\colon u = 0 \text{ on } \partial\Omega\}.$$

The modern general treatment of Dirichlet's problem (0.1) starts with the study of its equivalent *variational formulation* and goes back to Gauss (1839), Lord Kelvin and Dirichlet [**2**]. This equivalence can be stated as following:

Theorem. *A function* $u_0 \in C^2(\overline{\Omega})$ *solves* (0.1) *if and only if* u_0 *solves the* variational problem

$$F(u_0) = \min_{u \in \mathcal{A}} F(u).$$

The proof (which is not very difficult) can be found e.g. in [**7**], section 2.2.5 or [**21**], section 18.3.

Hence, in order to solve (0.1), the integral functional F has to be minimized, and one made the mistake of confusing the concepts of a greatest lower bound and a minimum value [**11**]. The existence of a minimum u_0 seemed to be be obvious and was therefore taken for granted. Riemann based his theory of complex functions on this incorrect procedure, which he called *Dirichlet's principle* [**11**], [**21**].

In studying Riemann's work, Weierstraß found the Dirichlet principle unsatisfactory. In 1870 he constructed a counterexample which showed that the existence of solutions to variational problems is no trivial matter [**21**]. His essential result is as follows:

(i) Each variational problem which is bounded below possesses an infimum and hence a minimizing sequence exists.
(ii) However, those minimal sequences do not necessarily converge to a solution of the variational problem.

Hence a compactness argument is needed in order to guarantee that the minimizing sequence actually converges to a solution. By recognizing this fact, Hilbert was able to rigorously justify Dirichlet's principle in a *direct*[1] way in 1900 under additional restrictions [**11**], [**21**]. We will give a functional analytic justification of the Dirichlet principle in Example 2.43 within the concept of *generalized solutions*. By applying the regularity theory for linear elliptic partial differential equations (see e.g. [**7**], Chapter 6.3) one can then prove, that a generalized solution to (0.1) is also a classical solution, provided that Ω and f satisfy further requirements. For a sufficiently smooth boundary $\partial\Omega$ and $f \in C^1(\overline{\Omega})$, problem (0.1) possesses a unique classical solution $u_0 \in C^2(\overline{\Omega})$. However, (0.1) need not have a classical solution if f is only continuous [**21**].

Acknowledgments

First of all I want to thank my parents for their support. Moreover I would like to thank my advisor Günther Hörmann. Not only for supervising this thesis, but also for his excellent lectures on calculus and vector analysis, which provided a solid basis for my mathematical development.

[1]The *"indirect"* way is to solve (0.1) for a large class of domains and sufficiently smooth f to deduce the existence of a minimizer for F on those domains.

CHAPTER 1

Preliminary Material

Orientation

This chapter provides basic results from linear functional analysis, in particular the theory of normed spaces, integration theory and the theory of Sobolev spaces. These results will be needed in the remaining chapters.

1.1. Functional analytic principles

If not specified differently, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ denote real normed spaces throughout this section. We point out that there exists a corresponding complex version to each of the presented results. However we will only need the theorems for the real case later on. Most theorems can be found in any textbook on (linear) functional analysis, e.g. [**15**] or [**19**].

1.1.1. Mappings between normed spaces

Definition 1.1. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X and let $u \in X$. We say that $(u_n)_{n\in\mathbb{N}}$ *converges* to u, and write $u = \lim_{n\to\infty} u_n$ or $u_n \to u$, if

$$\lim_{n\to\infty} \|u_n - u\| = 0.$$

This type of convergence is sometimes called *norm convergence*, *strong convergence* or *original convergence*.

Definition 1.2 (Continuity modes). Let $A\colon X \to Y$ be an arbitrary, not necessarily linear operator. Let u be an arbitrary element in X.

(i) A is *continuous* if $A(u_n) \to A(u)$ for any sequence $(u_n)_{n\in\mathbb{N}}$ in X converging to u.
(ii) A is *weakly sequentially continuous* if $A(u_n) \rightharpoonup A(u)$ for any sequence $(u_n)_{n\in\mathbb{N}}$ in X converging weakly[1] to u.
(iii) A is *compact* if A is continuous and maps bounded sets into relatively compact sets.

[1]Weak convergence is defined in Definition 1.7.

(iv) A is *bounded* if A maps bounded sets into bounded sets.
(v) A is *locally bounded* if u has a neighborhood $V(u) \subseteq X$ such that $A(V(u))$ is bounded.
(vi) A is *Lipschitz continuous* on a subset $B \subseteq X$ if there exists $L > 0$ such that

$$\|A(v) - A(w)\| \leq L\,\|v - w\|$$

for all $u, v \in B$.
(vii) A is *locally Lipschitz continuous* if u has a neighborhood $V(u) \subseteq X$ on which A is Lipschitz continuous.

If $Y = \mathbb{R}$, we call A a *functional*.

Notation 1.3. If $T\colon X \to Y$ is *linear*, we write Tu rather then $T(u)$ to mean the evaluation of T at a point $u \in X$.

The following basic result on linear operators between normed spaces can be found in any textbook on functional analysis:

Proposition 1.4. *If $T\colon X \to Y$ is linear, the following are equivalent:*
(i) *T is bounded.*
(ii) *There is some $C > 0$ such that $\|Tu\|_Y \leq C\|u\|_X$ for all $u \in X$.*
(iii) *T is continuous at* 0.
(iv) *T is continuous.*
(v) *T is Lipschitz continuous.*

Proposition 1.5. *Consider the vector space*

$$\mathcal{L}(X, Y) := \{T\colon X \to Y \mid T \text{ is linear and } \|T\|_{\mathcal{L}(X,Y)} < \infty\}$$

where

$$\|T\|_{\mathcal{L}(X,Y)} := \inf\{C > 0\colon \|Tu\|_Y \leq C\|u\|_X \text{ for all } u \in X\} = \sup_{\|u\|_X \leq 1} \|Tu\|_Y.$$

Then the following assertions hold.
(i) $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X,Y)})$*, the space of* bounded linear operators *from X into Y is a normed space.*
(ii) *If Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.*

Notation 1.6. We denote by

$$X' := (\mathcal{L}(X, \mathbb{R}), \|\cdot\|_{\mathcal{L}(X,\mathbb{R})})$$

the space of *bounded linear functionals* on X and call X' the *topological dual*, or just *dual*, of X; X is called the *predual* of X'. Note that X' is a Banach space by Proposition 1.5(ii).

We denote the action of a functional $f \in X'$ on $u \in X$ by $\langle f, u\rangle := f(u)$.

Definition 1.7. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X and let $u \in X$. We say that $(u_n)_{n\in\mathbb{N}}$ *converges weakly* to u, and write $u = \text{w-lim}_{n\to\infty} u_n$ or just $u_n \rightharpoonup u$, if

$$\lim_{n\to\infty} \langle f, u_n\rangle = \langle f, u\rangle$$

for all $f \in X'$.

Definition 1.8. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in X' and let f in X'. We say that $(f_n)_{n\in\mathbb{N}}$ *weak* converges* to f, and write $f = \text{w*-lim}_{n\to\infty} f_n$ or just $f_n \overset{*}{\rightharpoonup} f$, if

$$\lim_{n\to\infty} \langle f_n, u\rangle = \langle f, u\rangle$$

for all $u \in X$.

Remark 1.9. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X and let $u \in X$. If $(u_n)_{n\in\mathbb{N}}$ converges to u, then $(u_n)_{n\in\mathbb{N}}$ converges weakly to u, because of the estimate

$$|\langle f, u_n - u\rangle| \le \|f\|_{X'}\|u_n - u\|, \quad f \in X'.$$

The converse statement is generally not true; see Theorem 1 in [**19**], Chapter 5.1 for a counterexample.

On the dual space X' of some normed space X one can consider both weak and weak* convergence. In this situation, a weakly convergent sequence is always weak* convergent; c.f. Corollary 1.16.

It is a well-known fact that in a finite dimensional real normed space X, $u_n \rightharpoonup u$ implies $u_n \to u$.

Theorem 1.10 (Uniform Boundedness Principle). *Let $\{T_\iota\}_{\iota\in I}$ be a family of bounded linear operators defined on a Banach space X into a normed space Y. If the collection $\{\|T_\iota u\|_Y\}_{\iota\in I}$ is bounded for each $u \in X$, then $\{\|T_\iota\|_{\mathcal{L}(X,Y)}\}_{\iota\in I}$ is bounded as well.*

Proof. See Corollary 1 in [**19**], Chapter 2.1. □

Theorem 1.10 is also known as Banach-Steinhaus' theorem. As an immediate consequence we deduce the following

Theorem 1.11. *Let X be a Banach space, let Y be a normed space and let $A_n : X \to Y$ be a sequence of bounded linear operators with pointwise limit $A : X \to Y$. Then A is linear and bounded, and the sequence $(\|A_n\|_{\mathcal{L}(X,Y)})_{n\in\mathbb{N}}$ is bounded.*

Proof. For $u \in X$ we have that $Au = \lim_{n\to\infty} A_n u$, thus A is obviously linear. Since $(A_n)_{n\in\mathbb{N}}$ converges pointwise in Y, the sequence of norms $\{\|A_n u\|_Y\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ for every $u \in X$. Due to Theorem 1.10, $\{\|A_n\|_{\mathcal{L}(X,Y)}\}_{n\in\mathbb{N}}$ is bounded, say by $C > 0$, i.e. $\|A_n u\|_Y \le C\|u\|$ for all $n \in \mathbb{N}$ and for all $u \in X$. Letting $n \to \infty$ yields that $\|Au\|_Y \le C\|u\|$ for all $u \in X$, thus A is bounded. □

Theorem 1.12. *A sequence of bounded linear operators A_n between two Banach spaces X and Y converges pointwise to a bounded linear operator $A\colon X \to Y$ if and only if the following conditions are satisfied:*

(i) *the sequence of norms $(\|A_n\|_{\mathcal{L}(X,Y)})_{n\in\mathbb{N}}$ is bounded, and*
(ii) *the sequence $(A_n v)_{n\in\mathbb{N}}$ converges (in Y) for all elements v of a dense subset $E \subseteq X$.*

Proof. First of all, (i) is a necessary condition by Theorem 1.11, whereas the necessity of (ii) is trivial.

Conversely, let (i) and (ii) be satisfied and let u be an arbitrary element in X. Choose $\varepsilon > 0$. We set $C := \sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{L}(X,Y)}$ and pick $v \in E$ such that $\|u - v\| < \varepsilon/(3C)$. The sequence $(A_n v)_{n\in\mathbb{N}}$ converges by (ii), hence there exists some $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\|A_n v - A_m v\|_Y < \varepsilon/3$. Thus,

$$\begin{aligned}\|A_n u - A_m u\| &\leq \|A_n u - A_n v\|_Y + \|A_n v - A_m v\|_Y + \|A_m v - A_m u\|_Y \\ &< C\frac{\varepsilon}{3C} + \frac{\varepsilon}{3} + C\frac{\varepsilon}{3C}\end{aligned}$$

for all $n, m \geq N$. This shows that $(A_n u)_{N\in\mathbb{N}}$ is a Cauchy sequence in Y and hence converges to some $Au \in Y$, since Y is complete. Thus $(A_n)_{N\in\mathbb{N}}$ converges pointwise to the mapping $A\colon X \to Y$, $u \mapsto \lim_{n\to\infty} A_n u$. Due to Theorem 1.11, A is linear and bounded. □

1.1.2. The Hahn-Banach theorems

Theorem 1.13 (Hahn-Banach's Theorem). *Let M be a subspace of a real vector space X and let $\varphi\colon X \to \mathbb{R}$ be a functional which satisfies*

$$\varphi(u + v) \leq \varphi(u) + \varphi(v) \quad \textit{and} \quad \varphi(tu) = t\varphi(u)$$

for all $u, v \in X$ and for all $t \geq 0$.

If $f\colon M \to \mathbb{R}$ is linear and satisfies $f(w) \leq \varphi(w)$ for all $w \in M$, then there exists a linear functional $\tilde{f}\colon X \to \mathbb{R}$ such that

$$\tilde{f}(w) = f(w) \quad \textit{for all } w \in M$$

and

$$-\varphi(-u) \leq \tilde{f}(u) \leq \varphi(u) \quad \textit{for all } u \in X.$$

Proof. See Theorem 3.2 in [**15**]. □

Theorem 1.14. *Let M be a subspace of X and let $f \in M'$. Then there exists an extension $\tilde{f} \in X'$ of f preserving the norm, i.e. $\|\tilde{f}\|_{X'} = \|f\|_{M'}$.*

Proof. For $u \in X$ set $\varphi(u) := \|f\|_{M'}\|u\|$ and note that φ satisfies the assumptions of Theorem 1.13. Thus there exists an extension $\tilde{f}\colon X \to \mathbb{R}$, $\tilde{f}\big|_M = f$ with $|\tilde{f}(u)| \leq \|\tilde{f}\|_{M'}\|u\|$ for all $u \in X$. This shows that $\tilde{f} \in X'$ and that $\|\tilde{f}\|_{X'} \leq \|f\|_{M'}$. On the other hand, $\|f\|_{M'} \leq \|\tilde{f}\|_{X'}$ since $\tilde{f}$ extends f. □

Theorem 1.15. *For every $u_0 \in X$, there exists a functional $f \in X'$ such that*

$$f(u_0) = \|u_0\| \quad \text{and} \quad |f(u)| \leq \|u\| \quad \text{for all } u \in X.$$

In particular, $X' \neq \{0\}$, if $X \neq \{0\}$.

Proof. If $u_0 = 0$, set $f := 0$. For $u_0 \neq 0$, consider the one-dimensional subspace $M \subseteq X$ generated by x_0. We define the linear functional f_0 on M by $f_0(\alpha u_0) := \alpha\|u_0\|$. Then $f_0 \in M'$, $\|f_0\|_{M'} = 1$ and Theorem 1.14 yields the claim. □

We denote by $X'' := (X')'$ the *bidual* of a normed space X. The corollary below follows directly from Theorem 1.15.

Corollary 1.16. *The mapping $\iota \colon X \to X''$ given by*

$$\iota(u)(f) := f(u) \quad u \in X,\ f \in X'$$

defines a norm preserving embedding. It is called the canonical embedding *of X into X''.*

Definition 1.17. We call X *reflexive*, if the canonical embedding $\iota \colon X \to X''$ is bijective.

Note, that ι can not be surjective unless X is a Banach space.

Definition 1.18. A subset M of X is *convex* if for all $u, v \in M$ and for all $t \in [0, 1]$,

$$(1 - t)u + tv \in M.$$

In other words, for any two points $u, v \in M$, the straight line $\{(1 - t)u + tv \colon 0 \leq t \leq 1\}$ connecting u and v is contained in M.

Equivalently, M is convex, if

$$\sum_{i=1}^{n} \alpha_i u_i \in M$$

whenever $u_1, ..., u_n \in M$ and $\alpha_1, ..., \alpha_n \in [0, 1]$ such that $\sum_i^n \alpha_i = 1$ for arbitrary $n \in \mathbb{N}$.

Definition 1.19. Let $A \subseteq X$ be some arbitrary subset. The *convex hull* $\mathrm{co}(A)$ of A is set of all finite convex combinations of points in A:

$$\mathrm{co}(A) := \left\{ \sum_{i=1}^{n} \alpha_i a_i \,\middle|\, n \in \mathbb{N}, a_i \in A, \alpha_i \in [0, 1] \text{ such that } \sum_{i=1}^{n} \alpha_i = 1 \right\};$$

it is the smallest convex subset of X which contains A.

Equivalently, $\mathrm{co}(A)$ can be defined as the intersection of all convex supersets of A.

Theorem 1.20 (Separation theorem). *Let $A, B \subseteq X$ be disjoint, nonempty, convex sets such that A is compact and B is closed. Then there exist $f \in X'$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that*

$$f(u) < \gamma_1 < \gamma_2 < f(v)$$

for all $u \in A$ and for all $v \in B$. In particular, X' separates points on X.

Proof. See Theorem 3.4 in [**15**]. □

Remark 1.21. Note, that Theorem 1.20 guarantees that the weak limit of a weakly convergent sequence is unique. Alternatively this can already be seen from Theorem 1.15. The same holds true for weak* limits in X'. This can be seen by identifying $u \in X$ with $\iota u \in X''$ and noting that the set $\{\iota u \colon u \in X\} \subseteq X''$ still separates points in X'. Indeed if $f_1, f_2 \in X'$ and $\langle \iota u, f_1\rangle = \langle \iota u, f_2\rangle$ for all $u \in X$ then $\langle f_1, u\rangle = \langle f_2, u\rangle$ for all $u \in X$ and hence $f_1 = f_2$.

A direct consequence of the separation theorem is the following

Theorem 1.22. *Suppose M is a subspace of X, and $u_0 \in X$. If $u_0 \notin \overline{M}$, then there exists a functional $f \in X'$ such that $f(u_0) = 1$ but $f(u) = 0$ for all $u \in M$.*

The separation theorem has a very important consequence, concerning *closures* of convex sets:

Theorem 1.23. *Let A be a convex subset of X. The following are equivalent:*

(i) *A is closed.*
(ii) *A is weakly sequentially closed.*

Proof. (ii)⇒(i). This holds true for any subset $A \subseteq X$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in A which converges to $u \in X$. Then $u_n \rightharpoonup u$ by Remark 1.9, hence $u \in A$ by (ii).

(i)⇒(ii). Let A be a closed convex subset of X and let $u_0 \in X$ lie in the weak sequential closure of A. Then there is a sequence $(u_n)_{n\in\mathbb{N}}$ in A, which converges weakly to u_0. We claim that $u_0 \in A$. Let us suppose the contrary. Then Theorem 1.20 grants us some $f \in X'$ and $\gamma \in \mathbb{R}$ such that

$$f(u_0) < \gamma < f(u_n) \quad \text{for all } n \in \mathbb{N}.$$

Thus $u_0 \neq \text{w-lim}_{n\to\infty} u_n$. This contradiction implies $u_0 \in A$. □

We state another useful corollary of the separation theorem, which in certain situations allows one to deduce strong convergence from weak convergence:

Theorem 1.24 (Mazur's Theorem). *Assume that $(u_n)_{n\in\mathbb{N}}$ is a sequence in X converging weakly to $u \in X$:*

$$u_n \rightharpoonup u.$$

Then there exists a sequence $(v_k)_{k\in\mathbb{N}}$ in the convex hull $\mathrm{co}(\{u_n : n \in \mathbb{N}\})$ *which converges strongly to u:*

$$v_k \to u.$$

Proof. By assumption, u is in the weak sequential closure of $\mathrm{co}(\{u_n : n \in \mathbb{N}\})$. Since the convex hull of any set is convex by definition, u is in the original closure of $\mathrm{co}(\{u_n : n \in \mathbb{N}\})$ due to Theorem 1.23. Hence there exists a sequence $(v_k)_{k\in\mathbb{N}}$ lying entirely in $\mathrm{co}(\{u_n : n \in \mathbb{N}\})$ such that $v_k \to u$. □

1.1.3. Convergence principles in Banach spaces

Theorem 1.25. *Let $(u_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in X. Then $(u_n)_{n\in\mathbb{N}}$ is bounded, i.e. there exists a constant $C > 0$ such that $\|u_n\| \leq C$ for all $n \in \mathbb{N}$.*

Proof. We define the functional $F_n \in X''$ by $F_n(f) := \langle \iota u_n, f\rangle = \langle f, u_n\rangle$, with ι denoting the canonical embedding $X \to X''$. Since $(u_n)_{n\in\mathbb{N}}$ converges weakly, $\{|F_n(f)|\}_{n\in\mathbb{N}}$ is bounded for every $f \in X'$. Hence, Theorem 1.10 implies[2] that $\{\|F_n\|_{X''}\}_{n\in\mathbb{N}}$ is bounded. This finishes the proof, because $\{\|F_n\|_{X''}\}_{n\in\mathbb{N}} = \{\|u_n\|\}_{n\in\mathbb{N}}$ due to Corollary 1.16. □

Theorem 1.25 has the following dual version. Its proof is obtained in a similar way, with the difference, that one works on X instead of X'. For this reason, X must be complete in order to apply the uniform boundedness principle.

Theorem 1.26. *Let X be a Banach space with dual X' and let $(u'_n)_{n\in\mathbb{N}}$ be a weak* convergent sequence in X'. Then $(u'_n)_{n\in\mathbb{N}}$ is bounded, i.e. there exists a constant $C > 0$ such that $\|u'_n\|_{X'} \leq C$ for all $n \in \mathbb{N}$.*

Theorem 1.27. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence the Banach space X, let $u \in X$. Moreover, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in X' and let $f \in X'$.*

(i) *If every subsequence of $(u_n)_{n\in\mathbb{N}}$ has a subsequence which converges to u, then*

$$u_n \to u.$$

(ii) *If every subsequence of $(u_n)_{n\in\mathbb{N}}$ has a subsequence which converges weakly to u, then*

$$u_n \rightharpoonup u.$$

[2] Recall that X' is Banach space, c.f. Proposition 1.5(ii).

(iii) *If* $f_n \to f$ *and* $u_n \rightharpoonup u$, *then*

$$\lim_{n\to\infty} \langle f_n, u_n\rangle = \langle f, u\rangle\,. \tag{1.1}$$

(iv) *If* $f_n \overset{*}{\rightharpoonup} f$ *and* $u_n \to u$, *then*

$$\lim_{n\to\infty} \langle f_n, u_n\rangle = \langle f, u\rangle\,.$$

Proof. (i). If $(u_n)_{n\in\mathbb{N}}$ does not converge to u, then there exists an $\varepsilon_0 > 0$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$\|u_{n_k} - u\| \geq \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Hence, there is no subsequence of $(u_{n_k})_{k\in\mathbb{N}}$, which converges to u.

(ii). If $(u_n)_{n\in\mathbb{N}}$ does not converge weakly to u, then there exists a functional $f_0 \in X'$ such that $\langle f_0, u_n\rangle$ does not converge to $\langle f_0, u\rangle$. Thus there exists an $\varepsilon_0 > 0$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$|\,\langle f_0, u_{n_k}\rangle - \langle f_0, u\rangle\,| \geq \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Hence, there is no subsequence of $(u_{n_k})_{k\in\mathbb{N}}$, which converges weakly to u.

(iii). Let $f_n \to f$ and $u_n \rightharpoonup u$. Due to Theorem 1.25, $(u_n)_{n\in\mathbb{N}}$ is bounded, say by $C > 0$. Hence

$$\begin{aligned}|\,\langle f_n, u_n\rangle - \langle f, u\rangle\,| &= |\,\langle f_n - f, u_n\rangle + \langle f, u_n\rangle - \langle f, u\rangle\,| \\ &\leq C\,\|f_n - f\|_{X'} + |\,\langle f, u_n\rangle - \langle f, u\rangle\,| \xrightarrow{n\to\infty} 0.\end{aligned}$$

(iv). Let $f_n \overset{*}{\rightharpoonup} f$ and $u_n \to u$. By Theorem 1.26, $(f_n)_{n\in\mathbb{N}}$ is bounded in X', say by $C > 0$. Hence

$$\begin{aligned}|\,\langle f_n, u_n\rangle - \langle f, u\rangle\,| &= |\,\langle f_n, u_n - u\rangle + \langle f_n, u\rangle - \langle f, u\rangle\,| \\ &\leq C\,\|u_n - u\| + |\,\langle f_n, u\rangle - \langle f, u\rangle\,| \xrightarrow{n\to\infty} 0.\end{aligned}$$

□

1.2. Theory of integration

The purpose of Section 1.2.1 and Section 1.2.2 is mainly to fix notation and provide the most important results about Lebesgue integrable functions and the Lebesgue space $L^p(\Omega;\mathbb{R}^m)$. We refer to the literature for the proofs, e.g. [**6**], [**8**], [**14**].

Section 1.2.3 addresses Carathéodory mappings and superposition operators, so-called Nemytsii mappings, between Lebesgue spaces.

1.2.1. Integrable functions

When speaking about a *measurable* function $f\colon \Omega \to \overline{\mathbb{R}}$ on a *Lebesgue measurable* set $\Omega \subseteq \mathbb{R}^d$, we will in particular mean a $\mathcal{L}_{\mathbb{R}^d}$-$\mathcal{B}_{\overline{\mathbb{R}}}$-measurable function, where $\mathcal{L}_{\mathbb{R}^d}$ is the *Lebesgue σ-algebra* on $\mathbb{R}^d$ and $\mathcal{B}_{\overline{\mathbb{R}}}$ is the *Borel σ-algebra* on $\overline{\mathbb{R}} := [\infty, \infty]$. Similarly a measurable function $f\colon \Omega \to \mathbb{R}^m$ is understood to be a $\mathcal{L}_{\mathbb{R}^d}$-$\mathcal{B}_{\mathbb{R}^m}$-measurable function, where $\mathcal{B}_{\mathbb{R}^m}$ is the Borel σ-algebra on $\mathbb{R}^m$. In the applications in Chapter 3 and Chapter 4, Ω is always an open subset in $\mathbb{R}^d$ and hence Lebesgue measurable. We will sometimes denote the d-dimensional *Lebesgue measure* λ_d by Vol_d. The symbol $\int$ denotes the *Lebesgue integral*. A measurable function $u\colon \Omega \to \overline{\mathbb{R}}$ is called *integrable* if

$$\int_\Omega |u(x)|\, d\lambda_d(x) < \infty.$$

In the sequel we write $\int_\Omega f(x)\, dx$ or $\int_\Omega f\, dx$ rather than $\int_\Omega f(x)\, d\lambda_d(x)$.

We say that two functions $f, g\colon \Omega \to \overline{\mathbb{R}}$ agree *almost everywhere (a.e.)*, if they agree in every $x \in \Omega \setminus N$ where $N \subseteq \Omega$ is a set with measure $\lambda_d(N) = 0$, a so-called *null set*. Note, that the relation "agreeing a.e." defines an equivalence relation $\sim$ on the set of all measurable functions $\Omega \to \overline{\mathbb{R}}$.

In the following we state three remarkable convergence results of Lebesgue's integration theory:

Theorem 1.28 (Monotone Convergence Theorem). *Let f, $f_n\colon \Omega \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$ be measurable functions, $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ such that*

$$f = \lim_{n\to\infty} f_n \text{ a.e.} \quad \text{and} \quad \int_\Omega f_1^-\, dx < \infty.$$

Then f is measurable and

$$\lim_{n\to\infty} \int_\Omega f_n\, dx = \int_\Omega f\, dx \quad (\in [0, \infty]).$$

Theorem 1.29 (Fatou's Lemma). *Let g, $f_n\colon \Omega \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$ be measurable functions.*

(i) *If $\int_\Omega g^-\, dx < \infty$ and $f_n \geq g$ a.e. for all $n \in \mathbb{N}$, then*

$$\int_\Omega \liminf_{n\to\infty} f_n\, dx \leq \liminf_{n\to\infty} \int_\Omega f_n\, dx.$$

(ii) *If $\int_\Omega g^+\, dx < \infty$ and $f_n \leq g$ a.e. for all $n \in \mathbb{N}$, then*

$$\leq \limsup_{n\to\infty} \int_\Omega f_n\, dx \leq \int_\Omega \limsup_{n\to\infty} f_n\, dx.$$

(iii) *If g is integrable and $|f_n| \le g$ a.e. for all $n \in \mathbb{N}$, then*

$$\int_\Omega \liminf_{n\to\infty} f_n\,dx \le \liminf_{n\to\infty} \int_\Omega f_n\,dx \\ \le \limsup_{n\to\infty} \int_\Omega f_n\,dx \le \int_\Omega \limsup_{n\to\infty} f_n\,dx.$$

Theorem 1.30 (Dominated Convergence Theorem). *Let $f, f_n \colon \Omega \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$ be measurable functions with $\lim_{n\to\infty} f_n = f$ a.e. If there exists an integrable function $g \colon \Omega \to \overline{\mathbb{R}}$ such that $|f_n| \le g$ a.e. and for all $n \in \mathbb{N}$, then f is integrable and*

$$\lim_{n\to\infty} \int_\Omega f_n\,dx = \int_\Omega f\,dx.$$

The following assertion concerning the change of the order of integration and differentiation is a consequence of the dominated convergence theorem:

Theorem 1.31. *Let $I \subseteq \mathbb{R}$ be some open interval and $r_0 \in I$. Assume that $\varphi \colon \Omega \times I \to \mathbb{R}$ satisfies the following properties:*

(i) *$\varphi(\cdot, r) \colon \Omega \to \mathbb{R}$ is integrable for all $r \in I$.*
(ii) *There is some $\delta > 0$ such that the partial derivative $\frac{\partial\varphi}{\partial r}(x, r)$ exists for all $r \in U := (r_0 - \delta, r_0 + \delta) \cap I$, for a.e. $x \in \Omega$.*
(iii) *There exists an integrable function $g \colon \Omega \to \mathbb{R}$ such that*

$$\left|\frac{\partial\varphi}{\partial r}(x, r)\right| \le g(x)$$

for all $r \in U$ and for a.e. $x \in \Omega$.

Then the function $\Phi \colon I \to \mathbb{R}$, $r \mapsto \int_\Omega \varphi(x, r)\,dx$ is differentiable at r_0, $\frac{\partial\varphi}{\partial r}(\cdot, r_0)$ is integrable, and

$$\frac{\mathrm{d}}{\mathrm{d}r}\Phi(r_0) = \int_\Omega \frac{\partial\varphi}{\partial r}(x, r_0)\,dx.$$

Proof. First we note that Φ actually defines a function $I \to \mathbb{R}$ by (i). Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in U such that $r_n \neq r_0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = r_0$. Due to property (ii),

$$\frac{\varphi(\cdot, r_n) - \varphi(\cdot, r_0)}{r_n - r_0} \xrightarrow{n\to\infty} \frac{\partial\varphi}{\partial r}(\cdot, r_0)$$

a.e. in Ω. Moreover, for every $n \in \mathbb{N}$ and every fixed $x \in \Omega$ such that $\frac{\partial\varphi}{\partial r}(x, r)$ ($r \in U$) exists, the mean value theorem provides us with some $\xi(n, x) \in U$ such that $\frac{\varphi(x, r_n) - \varphi(x, r_0)}{r_n - r_0} = \frac{\partial\varphi}{\partial r}(x, \xi(n, x))$. Thus (ii) and (iii) imply that

$$\left|\frac{\varphi(x, r_n) - \varphi(x, r_0)}{r_n - r_0}\right| = \left|\frac{\partial\varphi}{\partial r}(x, \xi(n, x))\right| \le g(x),$$

for a.e. $x \in \Omega$. Lebesgue's dominated convergence theorem, Theorem 1.30, yields that $\frac{\partial\varphi}{\partial r}(\cdot, r_0)$ is integrable and furthermore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\Phi(r_0) &= \lim_{n\to\infty}\frac{\Phi(r_n)-\Phi(r_0)}{r_n-r_0} = \lim_{n\to\infty}\int_\Omega \frac{\varphi(x,r_n)-\varphi(x,r_0)}{r_n-r_0}\,dx \\ &= \int_\Omega \lim_{n\to\infty}\frac{\varphi(x,r_n)-\varphi(x,r_0)}{r_n-r_0}\,dx = \int_\Omega \frac{\partial\varphi}{\partial r}(x,r_0)\,dx.\end{aligned}$$

□

Next we present Lebesgue's version of the *fundamental theorem of calculus* and *Fubini's theorem*:

Definition 1.32. A function $F\colon [a,b] \to \mathbb{R}$ is called *absolutely continuous*, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $n \in \mathbb{N}$ and all $a \le \alpha_1 < \beta_1 \le \dots \le \alpha_n < \beta_n \le b$ with $\sum_{i=1}^n(\beta_i - \alpha_i) < \delta$,

$$\sum_{i=1}^n |F(\beta_i) - F(\alpha_i)| < \varepsilon.$$

Theorem 1.33 (Fundamental Theorem of Calculus, Part 1). *Let the function $f\colon [a,b] \to \mathbb{R}$ be integrable. Then $F\colon [a,b] \to \mathbb{R}$ given by*

$$F(x) := \int_a^x f(t)\,dt$$

is absolutely continuous and $F' = f$ a.e. on $[a,b]$.

Theorem 1.34 (Fundamental Theorem of Calculus, Part 2). *Let the function $F\colon [a,b] \to \mathbb{R}$ be absolutely continuous and set $F'(x) := 0$ in every point $x \in [a,b]$ where F is not differentiable. Then F' is integrable on $[a,b]$ and*

$$F(x) - F(a) = \int_a^x F'(t)\,dt \quad (a \le x \le b).$$

Theorem 1.35 (Fubini's Theorem). *Let $\Omega_1 \subseteq \mathbb{R}^{d_1}$ and $\Omega_2 \subseteq \mathbb{R}^{d_2}$ be two measurable sets, then $\Omega_1 \times \Omega_2$ is a measurable subset of $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \cong \mathbb{R}^{d_1+d_2}$. Suppose that $f\colon \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}$ is a measurable function, such that one of the following integrals*

$$\int_{\Omega_1\times\Omega_2} |f|\,d(x_1,x_2), \quad \int_{\Omega_1}\int_{\Omega_2} |f|\,dx_2\,dx_1, \quad \text{or} \quad \int_{\Omega_2}\int_{\Omega_1} |f|\,dx_1\,dx_2$$

is finite. Then f is integrable and

$$\int_{\Omega_1\times\Omega_2} f\,d(x_1,x_2) = \int_{\Omega_1}\int_{\Omega_2} f\,dx_2\,dx_1 = \int_{\Omega_2}\int_{\Omega_1} f\,dx_1\,dx_2.$$

1.2.2. The space L^p

In this section, we define the space $L^p(\Omega;\mathbb{R}^m)$ and discuss its most important properties.

For a measurable function $u\colon \Omega \to \mathbb{R}^m$ and $1 \le p \le \infty$ we define

$$\|u\|_{L^p(\Omega;\mathbb{R}^m)} := \begin{cases} \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{for } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\Omega} |u(x)| & \text{for } p = \infty, \end{cases}$$

where ”ess sup” stands for the *essential supremum*.

Lemma 1.36. *Let $k \in \mathbb{N}$ and let $\alpha_1, ..., \alpha_k \in [0,1]$ such that $\alpha_1 + ... + \alpha_k = 1$. For $a_1, ..., a_k \in [0,\infty]$ one has*

$$a_1^{\alpha_1} ... a_k^{\alpha_k} \le \alpha_1 a_1 + ... + \alpha_k a_k.$$

Lemma 1.36 is also known as the *inequality for the weighted arithmetic mean and the weighted geometric mean*. It is the key to the following two fundamental inequalities:

Theorem 1.37 (Hölder's Inequality). *Let $p \in [1,\infty]$ and let p' denote its* Hölder conjugate*, that is $\frac{1}{p} + \frac{1}{p'} = 1$ with the convention $\frac{1}{\infty} := 0$. Then any two measurable functions $u, v\colon \Omega \to \mathbb{R}^m$ satisfy*

$$\|uv\|_{L^1(\Omega;\mathbb{R}^m)} \le \|u\|_{L^p(\Omega;\mathbb{R}^m)} \|v\|_{L^{p'}(\Omega;\mathbb{R}^m)}.$$

Theorem 1.38 (Minkowski's Inequality). *Let $p \in [1,\infty]$. Then any two measurable functions $u, v\colon \Omega \to \mathbb{R}^m$ satisfy*

$$\|u+v\|_{L^p(\Omega;\mathbb{R}^m)} \le \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \|v\|_{L^p(\Omega;\mathbb{R}^m)}.$$

Lemma 1.39. *Let $u\colon \Omega \to \mathbb{R}^m$ be a measurable function and let $p \in [1,\infty]$. Then*

$$\|u\|_{L^p(\Omega;\mathbb{R}^m)} = 0 \iff u = 0 \quad \text{a.e. in } \Omega.$$

Before we define the space $L^p(\Omega;\mathbb{R}^m)$, let us consider the space

$$\mathcal{L}^p(\Omega;\mathbb{R}^m) := \left\{u\colon \Omega \to \mathbb{R}^m : u \text{ is measurable and } \|u\|_{L^p(\Omega;\mathbb{R}^m)} < \infty\right\},$$

equipped with the semi-norm $\|\cdot\|_{L^p(\Omega;\mathbb{R}^m)}$.

Next we consider the equivalence relation

$$u \sim v \Leftrightarrow u - v \in \mathcal{N}(\Omega;\mathbb{R}^m) := \left\{w\colon \Omega \to \mathbb{R}^m : w \text{ measurable, } w = 0 \text{ a.e.}\right\}$$

on the set of measurable functions $\Omega \to \mathbb{R}^m$. Note that $\mathcal{N}(\Omega;\mathbb{R}^m) \subseteq \mathcal{L}^p(\Omega;\mathbb{R}^m)$ is a closed subspace.

Definition 1.40. The *Lebesgue space* $L^p(\Omega;\mathbb{R}^m)$, $1 \le p \le \infty$ is the quotient space

$$\begin{aligned} L^p(\Omega;\mathbb{R}^m) &:= \mathcal{L}^p(\Omega;\mathbb{R}^m)/\mathcal{N}(\Omega;\mathbb{R}^m) \\ &= \Big\{[u] := u + \mathcal{N}(\Omega;\mathbb{R}^m) \colon u \in \mathcal{L}^p(\Omega;\mathbb{R}^m)\Big\}, \end{aligned}$$

equipped with $\|\cdot\|_{L^p(\Omega;\mathbb{R}^m)}$. We set $L^p(\Omega) := L^p(\Omega;\mathbb{R})$

Remark 1.41. Although the space $L^p(\Omega;\mathbb{R}^m)$ actually consists of classes of functions, we will nonetheless often speak e.g. of a "function" $u \in L^p(\Omega;\mathbb{R}^m)$. Sometimes we will implicitly mean a certain representative $u \in [u]$ rather then its corresponding class $[u]$.

$L^p(\Omega;\mathbb{R}^m)$ is a normed space, which is actually complete, and moreover reflexive if $p \in (1,\infty)$:

Theorem 1.42. *$L^p(\Omega;\mathbb{R}^m)$ is a Banach space for every $p \in [1,\infty]$.*

Theorem 1.43. *For $1 < p < \infty$, $L^p(\Omega;\mathbb{R}^m)$ is a separable reflexive Banach space and*

$$(L^p(\Omega;\mathbb{R}^m))' \cong L^{p'}(\Omega;\mathbb{R}^m).$$

In particular, $L^2(\Omega;\mathbb{R}^m)$ is a separable Hilbert space with the inner product

$$(u,v)_{L^2(\Omega;\mathbb{R}^m)} = \int_\Omega u \cdot v \, dx.$$

Remark 1.44. If $\Omega \subseteq \mathbb{R}^d$ has finite measure, i.e. $\lambda_d(\Omega) < \infty$, one has the following chain of inclusions (see Theorem 2.14 in [**1**]):

$$L^\infty(\Omega;\mathbb{R}^m) \subseteq ... \subseteq L^p(\Omega;\mathbb{R}^m) \subseteq ... \subseteq L^q(\Omega;\mathbb{R}^m) \subseteq ... \subseteq L^1(\Omega;\mathbb{R}^m) \quad (1.2)$$

for all $1 \le q \le p \le \infty$. For a general Ω, (1.2) is wrong.

Theorem 1.45. *Let $\Omega \subseteq \mathbb{R}$ be open, and let $1 \le p < \infty$. Then $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. See Proposition 2.2.3 in [**2**]. □

Definition 1.46. Let $\Omega \subseteq \mathbb{R}^d$ be measurable. A measurable function $u\colon \Omega \to \mathbb{R}$ is called *simple function*, if u takes only a finite number of values in $\mathbb{R}$. We denote by $\mathcal{T}(\Omega)$ the set of all simple functions on Ω.

Theorem 1.47. *Let $\Omega \subseteq \mathbb{R}$ be open, and let $1 \le p < \infty$. Then $\mathcal{T}(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. See Satz 2.28 in [**6**], Kapitel VI. □

Let $1 \le p < \infty$. A sequence $(u_n)_{n\in\mathbb{N}}$ in $L^p(\Omega)$ with $u_n \to u$ in $L^p(\Omega)$

does generally neither converge pointwise to u, nor pointwise almost everywhere. One can easily construct a sequence in $L^p(0,1)$ that converges to zero in $L^p(0,1)$ yet does not converge pointwise at any $x \in (0,1)$, see [**6**] Beispiel 2.8 in Kapitel VI. Given a convergent sequence in L^p, one can at least extract a subsequence which converges almost everywhere to the limit function, as the next very useful lemma asserts.

Lemma 1.48. *Let $1 \leq p \leq \infty$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega)$ converging to $u \in L^p(\Omega)$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function $v \in L^p(\Omega)$ such that*

$$\lim_{k\to\infty} u_{n_k}(x) = u(x) \quad \textit{for a.e. } x \in \Omega$$

and

$$|u_{n_k}(x)| \leq v(x) \quad \textit{for a.e. } x \in \Omega, \quad \textit{for every } k \in \mathbb{N}.$$

1.2.3. Nemytskii mappings

Let $\Omega \subseteq \mathbb{R}^d$ be an open subset and let $j, m_0, m_1, ..., m_j \in \mathbb{N}$.

Definition 1.49. We say that a mapping $\varphi\colon \Omega \times \mathbb{R}^{m_1} \times ... \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_0}$ is a *Carathéodory mapping*, if $\varphi(\cdot, r_1, ..., r_j)\colon \Omega \to \mathbb{R}^{m_0}$ is measurable for all $(r_1, ..., r_j) \in \mathbb{R}^{m_1} \times ... \times \mathbb{R}^{m_j}$, and $\varphi(x, \cdot)\colon \mathbb{R}^{m_1} \times ... \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_0}$ is continuous a.e. on Ω.

Definition 1.50. Consider the functions $\varphi\colon \Omega \times \mathbb{R}^{m_1} \times ... \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_0}$ and $u_i\colon \Omega \to \mathbb{R}^{m_i}$, $i = 1, ..., j$. The *Nemytskii operator* $\mathcal{N}_\varphi$ maps the functions $(u_1, ..., u_j)$ to a function $\mathcal{N}_\varphi(u_1, ..., u_j)\colon \Omega \to \mathbb{R}^{m_0}$ defined by

$$\Big(\mathcal{N}_\varphi(u_1, ..., u_j)\Big)(x) := \varphi(x, u_1(x), ..., u_j(x)).$$

Lemma 1.51. *The following assertions hold:*

(i) *If $u\colon \Omega \to \overline{\mathbb{R}}$ is the pointwise limit of a sequence of measurable functions $u_n\colon \Omega \to \overline{\mathbb{R}}$, then u is measurable.*

(ii) *Every measurable function $u\colon \Omega \to \overline{\mathbb{R}}$ is the pointwise limit of a sequence of simple functions $u_n \in \mathcal{T}(\Omega)$. Every simple function $u \in \mathcal{T}(\Omega)$ can be written as*[3]

$$u = \sum_{i=1}^{m} \alpha_i \chi_{A_i},$$

where $m \in \mathbb{N}$, $\alpha_1, ..., \alpha_m \in \mathbb{R}$, and $A_i \subseteq \Omega$ are measurable for $i = 1, ..., m$ such that $\bigcup_{i=1}^m A_i = \Omega$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. Here, χ_A denotes the characteristic function of the set A, i.e. $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ elsewhere.

[3] We say that the simple function u is given in *standard form*.

Proof. See Satz 4.3, Definition 4.12 and Korollar 4.14 in [**6**], Kapitel III. ◻

Lemma 1.52. *Let $\alpha, a_1, ..., a_j \in [0, \infty)$. Then*

$$\left(\sum_{i=1}^{j} a_i\right)^{\alpha} \leq C \sum_{i=1}^{j} a_i^{\alpha}$$

for some $C > 0$ depending only on j and α.

Theorem 1.53. *Let $\varphi\colon \Omega \times \mathbb{R}^{m_1} \times ... \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_0}$ be a Carathéodory mapping and let the functions $u_i\colon \Omega \to \mathbb{R}^{m_i}$, $i = 1, ..., j$ be measurable. Then $\mathcal{N}_\varphi(u_1, ..., u_j)\colon \Omega \to \mathbb{R}^{m_0}$ is measurable. Moreover, $\mathcal{N}_\varphi$ is a bounded continuous mapping $L^{p_1}(\Omega; \mathbb{R}^{m_1}) \times ... \times L^{p_j}(\Omega; \mathbb{R}^{m_j}) \to L^{p_0}(\Omega; \mathbb{R}^{m_0})$, $1 \leq p_i < \infty$, $i = 0, 1, ..., j$, with*

$$\|\mathcal{N}_\varphi(u)\|_{L^{p_0}(\Omega)} \leq \|\gamma\|_{L^{p_0}(\Omega)} + c \sum_{i=1}^{j} \|u_i\|_{L^{p_i}(\Omega;\mathbb{R}^{m_i})}^{p_i/p_0} \tag{1.3}$$

for some $c > 0$, provided that φ satisfies the growth condition

$$|\varphi(x, r_1, ..., r_j)| \leq \gamma(x) + C \sum_{i=1}^{j} |r_i|^{p_i/p_0} \tag{1.4}$$

for all $(r_1, ...r_j) \in \mathbb{R}^{m_1} \times ... \times \mathbb{R}^{m_j}$, for some $\gamma \in L^{p_0}(\Omega)$ and a constant $C > 0$. If some $p_i = \infty$, $i = 1, ..., j$, the same holds if the respective term $|\cdot|^{p_1/p_0}$ is replaced by any continuous function.

Proof. (i) Measurability. Let us for the sake of notational simplicity prove the assertion for the special case that $\varphi\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathédory mapping and $u\colon \Omega \to \mathbb{R}$ is measurable. Thus we have to show that $\mathcal{N}_\varphi(u)\colon \Omega \to \mathbb{R}$ is measurable.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of simple functions $\Omega \to \mathbb{R}$ which converges pointwise to u. That is, each u_n can be chosen of the form

$$u_n = \sum_{i=1}^{m} \alpha_i \chi_{A_i},$$

for some $m \in \mathbb{N}$, where $\alpha_i \in \mathbb{R}$, $A_i \subseteq \Omega$ are measurable such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{m} A_i = \Omega$. For $x \in \Omega$ we have that

$$\mathcal{N}_\varphi(u_n)(x) = \varphi\left(x, \sum_{i=1}^{m} \alpha_i \chi_{A_i}(x)\right) = \sum_{i=1}^{m} \varphi(x, \alpha_i)\chi_{A_i}(x). \tag{1.5}$$

Since φ is measurable in the first argument, the mappings $A_k \to \mathbb{R}$, $x \mapsto \varphi(x, \alpha_k)$, $k = 1, ..., m$ can be represented by a countable sum of simple functions defined on A_k. Thus (1.5) implies that each $\mathcal{N}_\varphi(u_n)\colon \Omega \to \mathbb{R}$ is a countable sum of simple functions on Ω and hence measurable. Since φ is continuous in the second argument for a.e. $x \in \Omega$,

$$\mathcal{N}_\varphi(u)(x) = \lim_{n\to\infty} \mathcal{N}_\varphi(u_n)(x)$$

for a.e. $x \in \Omega$. Thus $\mathcal{N}_\varphi(u)$ is (a.e.) the limit of a sequence of measurable functions and therefore measurable.

Let us assume that $1 \le p_i < \infty$ for all $i = 0, 1, ..., j$. We do not treat the case $p_i = \infty$, $i = 1, ..., j$, since this special case runs quite the same way and the required estimates are easier.

(ii) Boundedness. Condition (1.4) yields that

$$|\varphi(x, u(x))| \le \gamma(x) + C \sum_{i=1}^{j} |u_i(x)|^{p_i/p_0}$$

for all $x \in \Omega$. Thus,

$$|\varphi(x, u(x))|^{p_0} \le \gamma(x)^{p_0} + \tilde{c} \sum_{i=1}^{j} |u_i(x)|^{p_i}$$

by Lemma 1.52. Integration over Ω gives us

$$\|\mathcal{N}_\varphi(u)\|^{p_0}_{L^{p_0}(\Omega)} \le \|\gamma\|^{p_0}_{L^{p_0}(\Omega)} + \tilde{c} \sum_{i=1}^{j} \|u_i\|^{p_i}_{L^{p_i}(\Omega;\mathbb{R}^{m_i})},$$

and Lemma 1.52 implies (1.3).

(iii) Continuity. We set $j := 1$, $u := u_1$ and $p := p_1$. Let $u \in \Omega$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega)$ with

$$u_n \xrightarrow{n\to\infty} u$$

in $L^p(\Omega)$. By Lemma 1.48 there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function $v \in L^p(\Omega)$ such that

$$\lim_{k\to\infty} u_{n_k}(x) = u(x) \quad \text{and} \quad |u_{n_k}(x)| \le v(x)$$

for a.e. $x \in \Omega$. Therefore, by (1.4) and Lemma 1.52,

$$\begin{aligned} \|\mathcal{N}_\varphi(u_{n_k}) - \mathcal{N}_\varphi(u)\|^{p_0}_{p_0} &= \int_\Omega |\varphi(x, u_{n_k}(x)) - \varphi(x, u(x))|^{p_0}\, dx \\ &\le \int_\Omega (|\varphi(x, u_{n_k}(x))| + |\varphi(x, u(x))|)^{p_0}\, dx \\ &\le \tilde{C} \int_\Omega |\varphi(x, u_{n_k}(x))|^{p_0} + |\varphi(x, u(x))|^{p_0}\, dx \\ &\le C \int_\Omega |\gamma(x)|^{p_0} + |v(x)|^p + |u(x)|^p\, dx < \infty \end{aligned} \tag{1.6}$$

for some $\gamma \in L^{p_0}(\Omega)$ and $C > 0$. Since φ is a Carathéodory function, we have that

$$\varphi(x, u_{n_k}(x)) \xrightarrow{k\to\infty} \varphi(x, u(x)) \tag{1.7}$$

for a.e. $x \in \Omega$. Thus, the Dominated Convergence Theorem 1.30 together with (1.6) and (1.7) yield that

$$\mathcal{N}_\varphi(u_{n_k}) \xrightarrow{k\to\infty} \mathcal{N}_\varphi(u) \quad \text{in } L^{p_0}(\Omega).$$

Due to Proposition 1.27, the entire sequence converges:

$$\mathcal{N}_\varphi(u_n) \xrightarrow{n\to\infty} \mathcal{N}_\varphi(u) \quad \text{in } L^{p_0}(\Omega).$$

□

Remark 1.54. Nemytskii operators are in general not weakly sequentially continuous as we shall see in Example 1.56.

Example 1.55 (Weak convergence and wild oscillation). Let $\Omega := (a, b)$, where $a, b \in \mathbb{R}$ with $a < b$. For $n \in \mathbb{N}$ we define $u_n \colon (a, b) \to \mathbb{R}$ by $u_n(x) := \sin(nx)$. Each u_n oscillates between -1 and 1 with period $T_n = 2\pi/n$, thus $T_n \to 0$ as n goes to ∞.

We show that $u_n \rightharpoonup 0$ in $L^p(\Omega)$ for each $1 \le p < \infty$, that is

$$\int_a^b u_n(x)\, v(x)\, dx \xrightarrow{n\to\infty} 0$$

for every $v \in L^{p'}(\Omega)$. Actually we show a slightly stronger[4] result: for any $z \in L^1(\Omega)$,

$$\int_a^b \sin(nx)\, z(x)\, dx \xrightarrow{n\to\infty} 0. \tag{1.8}$$

Let us fist show that (1.8) holds for any $z \in \mathcal{D}(\Omega)$ and then apply a density argument to prove the general case.

So let z be an arbitrary element in $\mathcal{D}(\Omega)$. Applying integration by parts yields

$$\int_a^b \sin(nx)\, z(x)\, dx = \int_a^b \frac{\cos(nx)}{n}\, z'(x)\, dx,$$

thus

$$\left|\int_a^b \sin(nx)\, z(x)\, dx\right| \le \frac{1}{n} \int_a^b |z'(x)|\, dx \xrightarrow{n\to\infty} 0.$$

Now take some arbitrary $z \in L^1(\Omega)$ and let $\varepsilon > 0$. By the density of $\mathcal{D}(\Omega)$ in $L^1(\Omega)$, Theorem 1.45, we can find $z_\varepsilon \in \mathcal{D}(\Omega)$ such that $\|z - z_\varepsilon\|_{L^1(\Omega)} < \varepsilon$. Then

$$\begin{aligned}\left|\int_a^b \sin(nx)\, z(x)\, dx\right| &\le \left|\int_a^b \sin(nx)\, z_\varepsilon(x)\, dx\right| + \int_a^b |z(x) - z_\varepsilon(x)|\, dx \\ &\le \left|\int_a^b \sin(nx)\, z_\varepsilon(x)\, dx\right| + \varepsilon,\end{aligned}$$

[4]Recall that $L^{p'}(\Omega) \subseteq L^1(\Omega)$, see Remark 1.44.

and since $z_\varepsilon \in \mathcal{D}(\Omega)$,

$$\limsup_{n\to\infty} \left| \int_a^b \sin(nx)\, z(x)\, dx \right| \le \varepsilon.$$

Thus we proved that (1.8) holds for any $z \in L^1(\Omega)$, hence $u_n \rightharpoonup 0$ in $L^p(\Omega)$ for $1 \le p < \infty$.

Finally we emphasize that $(u_n)_{n\in\mathbb{N}}$ is not convergent in $L^p(\Omega)$; the only possible candidate for a limit is 0, see Remark 1.9, but $u_n \nrightarrow 0$. If $p = 2$, for example, we find that

$$\begin{aligned} \|u_n\|_{L^2(\Omega)} &= \int_a^b \big(\sin(nx)\big)^2\, dx = \int_a^b \frac{1}{2}(1 - \cos(2nx))\, dx \\ &= \frac{b-a}{2} - \frac{1}{4n}\big(\sin(2nb) - \sin(2na)\big) \xrightarrow{n\to\infty} \frac{b-a}{2} \neq 0. \end{aligned}$$

Example 1.56. Let $\Omega := (0, \pi)$ and let us consider the map $\varphi\colon \mathbb{R} \to \mathbb{R}$, $r \mapsto r^2$. We claim that the Nemytskii operator $\mathcal{N}_\varphi$ is not weakly sequentially continuous as a mapping $L^p(\Omega) \to L^q(\Omega)$ for any $1 \le p, q < \infty$. To see this, take the sequence $(u_n)_{n\in\mathbb{N}}$ in $L^p(\Omega)$ given by $u_n(x) := \sqrt{\frac{2}{\pi}}\sin(nx)$. From Example 1.55 we get that $u_n \rightharpoonup 0$ in any $L^p(\Omega)$, $1 \le p < \infty$. Similarly one shows that the sequence $(w_n)_{n\in\mathbb{N}}$, $w_n(x) := \cos(nx)$ converges weakly to zero in every $L^p(\Omega)$, $1 \le p < \infty$. So from $\sin^2(nx) = \frac{1}{2}(1 - \cos(2nx))$ it follows that $(u_n)^2 \rightharpoonup \frac{1}{\pi} \neq 0$ in $L^q(\Omega)$ for every $1 \le p < \infty$.

Note that by Theorem 1.53, $\mathcal{N}_\varphi$ is continuous as an operator $L^p(\Omega) \to L^q(\Omega)$ whenever p, q satisfy $1 \le p < \infty$ and $1 \le q \le p/2$.

The following theorem is a generalization of Lusin's[5] theorem, which roughly states that measurable functions are continuous except for a set with arbitrary small but positive measure.

Theorem 1.57 (Scorza-Dragoni's Theorem). *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, let $S \subseteq \mathbb{R}^m$ be compact and let $\varphi\colon \Omega\times\mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function. Then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \Omega$ such that the following properties hold:*

(i) $\mathrm{Vol}_d(\Omega \setminus K_\varepsilon) \le \varepsilon$.
(ii) *The restriction $\varphi|_{K_\varepsilon\times S} : K_\varepsilon \times S \to \mathbb{R}$ is continuous.*

Proof. See Theorem 3.8 in [**5**]. □

[5] See [**6**].

1.3. Sobolev spaces

Sobolev spaces provide a suitable foundation for the modern theory of partial differential equations. We present some important results and refer to [**1**] for a detailed exposition.

Let $\Omega \subseteq \mathbb{R}^d$ be open, and let $\nabla u := \left(\frac{\partial}{\partial x_1}u, ..., \frac{\partial}{\partial x_d}u\right)$ denote the *gradient* of a distribution[6] $u \in \mathcal{D}'(\Omega)$[7], where $\frac{\partial}{\partial x_i}$, $i = 1, ..., d$ is taken in the distributional sense. For $1 \leq p \leq \infty$ we define the *Sobolev space*

$$W^{1,p}(\Omega) := \left\{u \in L^p(\Omega) \colon \nabla u \in L^p(\Omega; \mathbb{R}^d)\right\},$$

equipped with the norm

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega)} &:= \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^d)}\right)^{\frac{1}{p}} \quad \text{if} \quad 1 \leq p < \infty, \\ \|u\|_{W^{1,\infty}(\Omega)} &:= \max\left(\|u\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega;\mathbb{R}^d)}\right). \end{aligned}$$

We denote by $W^{1,p}_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ with respect to $\|u\|_{W^{1,p}(\Omega)}$. For $p = 2$ we use the notation $H^1(\Omega) := W^{1,2}(\Omega)$ and $H^1_0(\Omega) := W^{1,2}_0(\Omega)$. Note that the norm $\|u\|_{H^1(\Omega)}$ stems from the scalar product

$$(u, v)_{H^1(\Omega)} := \int_\Omega uv\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx$$

Theorem 1.58. *$W^{1,p}(\Omega)$ is a Banach space. Furthermore, $W^{1,p}(\Omega)$ is separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$. In particular, $H^1(\Omega)$ is a separable Hilbert space.*

Proof. See Theorems 3.3 and 3.6 in [**1**]. □

The next result holds true in dimension $d = 1$ only, see [**7**] end of section 5.2.2.

Proposition 1.59. *Let $I \subseteq \mathbb{R}$ be an open interval. Then $u \in W^{1,p}(I)$ if and only if u equals a.e. an absolutely continuous function $\tilde{u}\colon I \to \mathbb{R}$ whose classical derivative (which exists a.e. in I) belongs to $L^p(I)$.*

From now on, let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, i.e. open and connected with C^1-boundary $\Gamma := \partial\Omega$. More generally, all theorems in this section remain true if Ω is a bounded Lipschitz domain, i.e. Ω has a Lipschitz boundary[8] Γ (c.f. the discussion of geometric properties of domains at the

[6] Consult e.g. Chapter 6 in [**15**] for an introduction to the theory of distributions.

[7] The space of real-valued distributions.

[8] That is, that each point $x \in \Gamma$ has a neighborhood $U_x \subseteq \mathbb{R}^d$ such that $U_x \cap \Gamma$ is the graph of a Lipschitz continuous function.

beginning of Chapter 4 in [**1**]). Additionally, unless differently specified, we assume that $1 \leq p < \infty$.

Theorem 1.60 (Approximation Theorem). *Let $\mathbb{W}^{1,p}(\Omega)$ denote the closure of $C^\infty(\overline{\Omega})$ in $L^p(\Omega)$ with respect to $\|u\|_{W^{1,p}(\Omega)}$. Then*

$$W^{1,p}(\Omega) = \mathbb{W}^{1,p}(\Omega).$$

Proof. See Theorem 3.22 in [**1**]. □

Remark 1.61. The statement of Theorem 1.60 is wrong if $p = \infty$: put $\Omega := (-1, 1)$ and $u(x) := |x|$ for $x \in \Omega$. Then $u'(x) = x/|x|$ for a.e. $x \in \Omega$, so $u \in W^{1,\infty}(\Omega)$. But there is no sequence of smooth functions $\varphi_n \in C^\infty(\Omega)$ such that $\|\varphi_n' - u'\|_{L^\infty(\Omega)} \to 0$, because the class $[u'] \in L^\infty(\Omega)$ has no continuous representative.

We define the *Sobolev conjugate* p^* of p by

$$p^* := \begin{cases} \frac{dp}{d-p} & \text{for } p < d, \\ \text{an arbitrary element of } [1, \infty) & \text{for } p = d, \\ \infty & \text{for } p > d. \end{cases}$$

Theorem 1.62 (Sobolev's Embedding Theorem). *Let $1 \leq p \leq \infty$. Then*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

for all $q \in [1, p^*]$. More precisely, any element $u \in W^{1,p}(\Omega)$ belongs to $L^q(\Omega)$, and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$.

Proof. See Theorem 4.12 in [**1**]. □

Theorem 1.63 (The Rellich-Kondrachov Theorem). *The embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq p \leq \infty$, is compact for all $q \in [1, p^*)$. We use the shorthand notation:*

$$W^{1,p}(\Omega) \Subset L^q(\Omega).$$

Proof. See Theorem 6.3 in [**1**]. □

Remark 1.64. We will apply Theorem 1.63 in the following way. Suppose a given sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$. Then its image under the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q < p^*$ is relatively compact in $L^q(\Omega)$ (c.f. Definition 1.2), i.e. there exists $u \in L^q(\Omega)$, and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ in $L^q(\Omega)$. In the particular case, that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, we find that $u_n \to u$ in $L^q(\Omega)$, $1 \leq q < p^*$; we applied Theorem 1.25 to obtain boundedness of $(u_n)_{n\in\mathbb{N}}$ and Theorem 1.27 to get convergence of the entire sequence.

Theorem 1.65 (Trace Theorem). *There exists a unique bounded linear operator*

$$T\colon W^{1,p}(\Omega) \to L^p(\Gamma)$$

such that

$$Tu = u|_\Gamma$$

for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. *More precisely,* T *satisfies*

$$\|Tu\|_{L^p(\Gamma)} \le N\|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$, *where* $N > 0$ *depends only on* p *and* Ω.

T *is called* trace operator, *and* $u|_\Gamma := Tu$ *is called the* trace *of* u.

Proof. See Theorem 1 in [**7**], Part II, Chapter 5.5. □

Theorem 1.66 (Trivial Traces). *It holds that*

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega)\colon\ u|_\Gamma = 0\right\}.$$

Proof. See Theorem 2 in [**7**], Part II, Chapter 5.5. □

Theorem 1.67 (Trace Embedding Theorem). *The trace operator* T *remains continuous as an operator*

$$T\colon W^{1,p}(\Omega) \to L^q(\Gamma)$$

for all $q \in [1, p^\#]$, *where*

$$p^\# := \begin{cases} \frac{dp-p}{d-p} & \text{for } p < d,\\ \text{an arbitrary element of } [1,\infty) & \text{for } p = d,\\ \infty & \text{for } p > d. \end{cases}$$

Moreover, $T\colon W^{1,p}(\Omega) \to L^q(\Gamma)$ *is compact for all* $q \in [1, p^\#)$.

Proof. This is a special case of Theorem 1.1 in [**3**]. □

Theorem 1.68 (Green's Formula). *Let* $\nu = (\nu_1, ..., \nu_d)\colon \Gamma \to \mathbb{R}^d$ *denote the unit outward field on* Γ. *For any* $u \in W^{1,p}(\Omega)$ *and* $w \in W^{1,p'}(\Omega;\mathbb{R}^d)$,

$$\int_\Omega \Big(u\ \mathrm{div}(w) + w\cdot\nabla u\Big)\,dx = \int_\Gamma u\,(w\cdot\nu)\,dS. \tag{1.9}$$

Proof. See Lemma 14.4 in [**17**]. □

A special case of Theorem 1.68 is the following integration by parts formula for trace-zero functions:

Theorem 1.69. *Let $u, v \in W^{1,p}(\Omega)$ with either u or v having trivial trace. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i} v \, dx = - \int_\Omega u \frac{\partial v}{\partial x_i} \, dx$$

for $i = 1, ..., d$.

Theorem 1.70 (Poincaré's Inequality). *There exists a constant $N > 0$ depending on p, d and Ω such that for all $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \leq N \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}. \tag{1.10}$$

Theorem 1.70 is a special case of the following

Theorem 1.71. *Let X be a closed subspace of $W^{1,p}(\Omega)$ such that the only constant function belonging to X is the function which is identically zero. Then there exists a constant $N > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq N \|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)} \tag{1.11}$$

for all $u \in X$.

Proof. Let us assume the contrary. That is, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in X such that for all $n \in \mathbb{N}$

$$\|v_n\|_{L^p(\Omega)} > n \|\nabla v_n\|_{L^p(\Omega;\mathbb{R}^d)}.$$

We set $u_n := v_n / \|v_n\|_{L^p(\Omega)}$, then $(u_n)_{n\in\mathbb{N}}$ is a sequence in X such that for every $n \in \mathbb{N}$

$$\|u_n\|_{L^p(\Omega)} = 1 \quad \text{and} \quad \|\nabla u_n\|_{L^p(\Omega;\mathbb{R}^d)} < \frac{1}{n}.$$

Thus, $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$ and Theorem 1.63 tells us, that $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L^p(\Omega)$. Hence we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ with

$$u_{n_k} \xrightarrow{k\to\infty} u \quad \text{in } L^p(\Omega)$$

for some $u \in L^p(\Omega)$. Since

$$\nabla u_{n_k} \xrightarrow{k\to\infty} 0 \quad \text{in } L^p(\Omega;\mathbb{R}^d),$$

we conclude that $Du = 0$, thus $u = c =$ constant and

$$u_{n_k} \xrightarrow{k\to\infty} c \quad \text{in } W^{1,p}(\Omega).$$

Since X is closed, we have that $c \in X$, hence $c = 0$ due to our assumptions. But this contradicts the fact that $1 = \|u_{n_k}\|_{L^p(\Omega)} \leq \|u_{n_k}\|_{W^{1,p}(\Omega)}$ for all $k \in \mathbb{N}$. □

Theorem 1.72. *Let $1 \leq q \leq p^*$. Then there exists a constant $N > 0$ depending on p, q, d and Ω such that for all $u \in W^{1,p}(\Omega)$*

$$\|u\|_{W^{1,p}(\Omega)} \leq N \Big(\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)} + \|u\|_{L^q(\Omega)} \Big). \tag{1.12}$$

Proof. The proof is very similar to the proof of Theorem 1.71. Let $q \in [1, p^*]$. Then any $u \in W^{1,p}(\Omega)$ belongs to $L^q(\Omega)$ by Theorem 1.62. Let us assume that there is no constant N such that (1.12) holds for every $u \in W^{1,p}(\Omega)$. Then there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ with

$$\|v_n\|_{W^{1,p}(\Omega)} > n\Big(\|\nabla v_n\|_{L^p(\Omega;\mathbb{R}^d)} + \|v_n\|_{L^q(\Omega)}\Big).$$

We set $u_n := v_n/\|v_n\|_{W^{1,p}(\Omega)}$, then

$$\|u_n\|_{W^{1,p}(\Omega)} = 1 \quad \text{and} \quad \|\nabla u_n\|_{L^p(\Omega;\mathbb{R}^d)} + \|u_n\|_{L^q(\Omega)} < \frac{1}{n} \tag{1.13}$$

for all $n \in \mathbb{N}$. Hence, $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\Omega)$, and by Theorem 1.63 there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging in $L^p(\Omega)$ to some $u \in L^p(\Omega)$. It follows from (1.13), that the gradient of u vanishes. Therefore u is constant almost everywhere in Ω. On the other hand, (1.13) implies that $\|u\|_{L^q(\Omega)} = 0$, hence $u = 0$. This is a contradiction, because $\|u_{n_k}\|_{W^{1,p}(\Omega)} = 1$ for all $k \in \mathbb{N}$. □

CHAPTER 2

Functional Analytic Methods

Orientation

This chapter is largely based on the Chapters 38-42 in [**23**]. Let us outline the basic ideas. One of the main results of this chapter is Theorem 2.37. It yields existence of a solution to the abstract operator equation

$$A(u) = f \tag{2.1}$$

where A denotes an operator $X \to X'$, and f stands for an arbitrary element in X'. The requirements for Theorem 2.37 are, roughly speaking, the following:

(i) A is the derivative of a functional $F: X' \to \mathbb{R}$,
(ii) X is a reflexive Banach space.
(iii) F satisfies a certain continuity property,
(iv) F tends faster then linear to ∞ for $\|u\|_X \to \infty$.

In this connection, equation (2.1) is sometimes called the abstract *Euler equation* to the *Euler-Lagrange functional* F.

In Section 2.4 we will make precise, what "differentiable" means in this context. And we will see, that, as in classical calculus, any local minimum u_0 of a differentiable functional F is a critical point, i.e. $F'(u_0) = 0$; see Proposition 2.24. Hence any global minimum of $G := F - f$ solves (2.1).

We point out, that not every operator A is a *potential operator*, i.e. the derivative of a functional F. In Section 2.5, we characterize potential operators and thereby give a complete answer the question whether property (i) is satisfied. In a real Hilbert space H for example, a bounded linear operator $B: H \to H \cong H'$ is a potential operator if and only if B is symmetric; see Example 2.42 in 2.6.

Theorem 2.37 is based on Theorem 2.13 in Section 2.2. It asserts the existence of a global minimum of a (not necessarily differentiable) functional F under the requirements (ii)-(iv). Of course it would suffice to identify a local minimum or a local maximum or a saddle point of the above functional G in order to solve (2.1). However we restrict ourselves to global minimization (and global maximization by considering $-G$). The proof or Theorem 2.13 relies on the *direct method in the calculus of variations*. This method goes back to Hilbert, who used it in giving a proof to Dirichlet's principle under certain restrictions in 1900. The importance of Hilbert's method is not only the fact, that he succeeded in giving a rigorous proof of Dirichlet's principle, but perhaps more in the fact that he operated with the admissible functions themselves as mathematical objects [**11**]. The direct method can be sketched as follows. One chooses a minimizing sequence, which always exists by the definition of the infimum. Property (iv) now guarantees that this sequence is bounded. Since X is assumed to be reflexive, this sequence possesses a weakly convergent subsequence; c.f. Theorem 2.4. The required continuity property yields that the limit function attains the infimum.

Theorem 2.13 can be considered as a generalization of Weierstraß' classical extreme value theorem for continuous functions on a compact interval $[a, b]$.

Throughout this chapter let $X = (X, \|\cdot\|)$ denote a real Banach space.

2.1. Weak compactness in reflexive Banach spaces

The closed unit ball in an infinite-dimensional Banach space X is not compact:

Theorem 2.1 (Riesz). *The closed unit ball* $\overline{\mathrm{B}_1(0)} = \{u \in X\colon \|u\| \leq 1\}$ *is compact if and only if X is finite-dimensional.*

If X is reflexive and $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in X, we can at least extract a subsequence which converges weakly to some element $u \in X$. This is the assertion of Theorem 2.4. For the proof, we need the following two lemmas.

Lemma 2.2. *The following assertions hold true.*

(i) *If X is reflexive, so is every closed subspace.*
(ii) *X is reflexive if and only if X' is.*

Proof. (i). Let V be a closed subspace of X, and let j denote the inclusion mapping $V \hookrightarrow X$. We define $j''\colon V'' \to X''$ via

$$\langle j''F, f\rangle := \langle F, f|_V\rangle, \qquad F \in V'',\ f \in X'.$$

The linear mapping j'' is isometric, since by Hahn-Banach's theorem, Theorem 1.14,

$$\|F\|_{V''} = \sup_{\substack{\|f\|_{V'}\leq 1\\ f\in V'}} |\langle F, f\rangle| = \sup_{\substack{\|f\|_{X'}\leq 1\\ f\in X'}} |\langle F, f|_V\rangle| = \|j''F\|_{X''}$$

for all $F \in V''$. This shows, that j'' is injective with inverse mapping $(j'')^{-1}\colon j''(V'') \to V''$.

Next, let ι_X and ι_V denote the canonical embeddings $X \hookrightarrow X''$ and $V \hookrightarrow V''$ respectively according to Theorem 1.16. For any $v \in V$ and $f \in X'$,

$$\langle j''(\iota_V v), f\rangle = \langle \iota_V v, f|_V\rangle = \langle f|_V, v\rangle = \langle f, jv\rangle = \langle \iota_X(jv), f\rangle.$$

So far we have proved that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\iota_V"] \arrow[d, "j"'] & V'' \arrow[d, "j''"] \\ X \arrow[r, "\iota_X"'] & X'' \end{tikzcd}$$

After the preceding preparations it is not hard to see that ι_V is onto. Let $F \in V''$ be arbitrary. Since ι_X is bijective, there exists a unique $u \in X$ such that $j''F = \iota_X u$. Now for any $f \in X'$ which vanishes on V it holds that

$$0 = \langle F, f|_V\rangle = \langle j''F, f\rangle = \langle \iota_X u, f\rangle = \langle f, u\rangle.$$

Therefore Theorem 1.22 implies that $u \in V$. This shows that $j''(V'') = \iota_X(j(V))$, hence $\iota_V = (j'')^{-1} \circ \iota_X \circ j$ is bijective.

(ii). Let $\iota_X, \iota_{X'}$ denote the canonical embeddings $X \to X''$ and $X' \to (X'')' =: X'''$ respectively.

First, suppose that X is reflexive and consider the two mappings

$$\begin{aligned} j_1 \colon X' &\to X''' & \qquad j_2 \colon X''' &\to X' \\ x' &\mapsto x' \circ \iota_X^{-1} & x''' &\mapsto x''' \circ \iota_X \end{aligned}$$

which are inverse to each other. For $x' \in X'$ and $x'' \in X''$,

$$j_1(x')(x'') = \langle x', \iota_X^{-1}x''\rangle = \langle \iota_X(\iota_X^{-1}x''), x'\rangle = \langle x'', x'\rangle = \iota_{X'}(x')(x''),$$

thus $j_1 = \iota_{X'}$ and therefore $j_2 = \iota_{X'}^{-1}$. This shows that X' is reflexive.

On the other hand, if X' is reflexive, so is X''. Therefore the closed subspace $\iota_X(X) \cong X$ of X'' is reflexive by (i). $\square$

Lemma 2.3. *Let X be a separable, reflexive Banach space. Then its dual X' is a separable, reflexive Banach space.*

Proof. By Lemma 2.2, X' is a reflexive Banach space. We prove, that X' is separable. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of unit vectors such that its linear hull is dense in X, i.e.

$$X = \overline{\operatorname{span}(\{u_n : n \in \mathbb{N}\})}.$$

This is possible, since X is separable. Hahn-Banach's theorem, Theorem 1.15, implies that for every $n \in \mathbb{N}$ there is some $f_n \in X'$ with $\|f_n\|_{X'} = 1$, such that $\langle f_n, u_n\rangle = 1$. Let E' denote the closure of the linear hull of $(f_n)_{n\in\mathbb{N}}$:

$$E' := \overline{\operatorname{span}(\{f_n : n \in \mathbb{N}\})}.$$

Then E' is a closed subspace of X'. We claim that $E' = X'$.

Assume the contrary, i.e. there is some $f \in X' \setminus E'$. Since $E' \subseteq X'$ is a closed subspace, Hahn-Banach's theorem, Theorem 1.22, gives us some $F \in X''$ such that $\langle F, f \rangle = 1$ and $\langle F, g \rangle = 0$ for any $g \in E'$. Set $F_0 := F/\|F\|_{X''}$. By reflexivity of X, there is a unique $u_0 \in X$ such that $F_0 = \iota u_0$, where ι denotes the canonical embedding $X \hookrightarrow X''$. Note that $\|u_0\|_X = 1$. Now, for every $n \in \mathbb{N}$ we have that

$$\|u_0 - u_n\|_X = \|\iota(u_0 - u_n)\|_{X''} \geq |\langle \iota(u_0 - u_n), f_n \rangle| = |\langle F_0, f_n \rangle - \langle f_n, u_n \rangle| = 1,$$

which contradicts our assumption that $(u_n)_{n\in\mathbb{N}}$ is dense in the unit sphere $\{x \in X : \|x\|_X = 1\}$. Hence $E' = X'$, therefore X is separable. □

Theorem 2.4. *Each bounded sequence in a reflexive Banach space X has a weakly convergent subsequence.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in X, set $C := \sup_{n\in\mathbb{N}} \|u_n\|$. Let $E \subseteq X$ denote the subspace generated by $(u_n)_{n\in\mathbb{N}}$, i.e.

$$E := \overline{\operatorname{span}(\{u_n : n \in \mathbb{N}\})}.$$

Thus E is a separable Banach space, which is reflexive by Lemma 2.2. Due to Lemma 2.3, E' is separable and reflexive as well. Let $\{f_n\}_{n\in\mathbb{N}}$ be a dense subset in E'. We find that each sequence

$$(\langle f_n, u_i \rangle)_{i\in\mathbb{N}}, \quad n \in \mathbb{N}$$

is bounded in $\mathbb{R}$, and hence possesses a convergent subsequence. By a simple diagonal argument we can extract one subsequence $(u_{i_k})_{k\in\mathbb{N}}$, such that for every $n \in \mathbb{N}$,

$$\lim_{k\to\infty} \langle f_n, u_{i_k} \rangle \in \mathbb{R}$$

exists. For $k \in \mathbb{N}$ let us define the functionals $G_k : E' \to \mathbb{R}$ by

$$G_k(f) := \langle f, u_{i_k} \rangle .$$

Each G_k is an element of E'', the sequence $(\|G_k\|_{E''})_{k\in\mathbb{N}}$ is bounded by C and the sequence $(G_k(f_n))_{k\in\mathbb{N}}$ converges for every f_n as $k \to \infty$. Thus Theorem 1.11 tells us, that the limit functional

$$G : E' \to \mathbb{R}, \quad G(f) := \lim_{k\to\infty} \langle f, u_{i_k} \rangle$$

exists and lies in E''.

Since E is reflexive, the functional G has the unique representation $G = \iota u$, $u \in E$, where ι is the canonical embedding $X \hookrightarrow X''$. This shows

$$\lim_{k\to\infty} \langle f, u_{ik} \rangle = G(f) = \langle \iota(u), f \rangle = \langle f, u \rangle , \quad f \in E',$$

thus we proved weak convergence of the sequence $(u_{ik})_{k\in\mathbb{N}}$ in E.

Finally, note that for any $f \in X'$,

$$\lim_{k\to\infty} \langle f, u_{i_k} \rangle = \lim_{k\to\infty} \langle f|_E , u_{i_k} \rangle = \langle f|_E , u \rangle = \langle f, u \rangle ,$$

since $u_{i_k} \in E$ for all $k \in \mathbb{N}$ and $u \in E$. Hence $(u_{i_k})_{k\in\mathbb{N}}$ converges weakly in X. □

2.2. Main theorem for extremal points

Definition 2.5. Let $u \in X$ and let $F\colon X \to \mathbb{R}$ be a functional.

(i) F is called *lower semicontinuous* in u if

$$F(u) \le \liminf_{n\to\infty} F(u_n) \tag{2.2}$$

for any sequence $(u_n)_{n\in\mathbb{N}}$ in X which converges to u.

(ii) F is called *weakly sequentially lower semicontinuous* in u if (2.2) holds for any sequence $(u_n)_{n\in\mathbb{N}}$ in X which converges weakly to u.

(iii) If X is the dual of some normed space Y, i.e. $X = Y'$, then F is called *weak* sequentially lower semicontinuous* in u if (2.2) holds for any sequence $(u_n)_{n\in\mathbb{N}}$ in X which weak* converges to u.

If F is (weakly sequentially/weak* sequentially) lower semicontinuous in every $u \in X$, then F is called *(weakly sequentially/weak* sequentially) lower semicontinuous*.

Remark 2.6. Let $F\colon X \to \mathbb{R}$ be a functional.

(i) F is lower semicontinuous in $u \in X$ if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \forall v \in \mathrm{B}_\delta(u)\colon F(v) > F(u) - \varepsilon.$$

(ii) F is continuous if and only if both F and $-F$ are lower semicontinuous.

(iii) F is usually called *upper semicontinuous*, if $-F$ is lower semicontinuous.

(iv) F is called *weakly (weak*) sequentially continuous* if both F and $-F$ are weakly (weak*) sequentially lower semicontinuous, i.e.

$$u_n \rightharpoonup u \quad \text{implies} \quad \lim_{n\to\infty} F(u_n) = F(u),$$

and accordingly

$$u_n \overset{*}{\rightharpoonup} u \quad \text{implies} \quad \lim_{n\to\infty} F(u_n) = F(u).$$

Example 2.7. Let $\Omega \subseteq \mathbb{R}^d$ be open and consider the Banach space $X := (L^1(\Omega), \|\cdot\|_{L^1(\Omega)})$ consisting of all integrable functions $u\colon \Omega \to \mathbb{R}$. The L^1-norm $\|\cdot\|_{L^1(\Omega)} =: F\colon L^1(\Omega) \to \mathbb{R}$ is weakly sequentially lower semicontinuous.

Let us first note that F is continuous, hence in particular lower semicontinuous, because the norm of a normed space is always continuous.

In order to demonstrate that F is weakly sequentially lower semicontinuous, take some arbitrary $u \in X$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence which converges weakly to u:

$$u_n \rightharpoonup u.$$

Let

$$L := \liminf_{n\to\infty} F(u_n)$$

and observe that $L < \infty$ due to Theorem 1.25 and the continuity of F. Restricting our attention, if necessary, to a subsequence, we can without loss of generality assume that $L = \lim_{n\to\infty} F(u_n)$. Therefore we can find for every $\varepsilon > 0$ some $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$,

$$F(u_n) \leq L + \varepsilon.$$

Next, fix $\varepsilon > 0$. By Mazur's theorem, Theorem 1.24, there exists a sequence $(v_k)_{k\in\mathbb{N}}$ lying in the convex hull $\operatorname{co}(\{u_n : n \geq n_\varepsilon\})$ and converging strongly to u, that is

$$v_k \to u \quad \text{in } X.$$

Every v_k can be written as $v_k = \sum_{i=n_\varepsilon}^{N_k} \alpha_k^i u_i$, where $N_k \geq n_\varepsilon$ and $\alpha_k^i \in [0, 1]$ such that $\sum_{i=n_\varepsilon}^{N_k} \alpha_k^i = 1$. Exploiting the sublinearity of F, we deduce that

$$F(v_k) \leq \sum_{i=n_\varepsilon}^{N_k} \alpha_k^i F(u_i) < L + \varepsilon$$

for any $k \in \mathbb{N}$. Due to lower semicontinuity of F we conclude that

$$F(u) \leq \liminf_{k\to\infty} F(v_k) \leq L + \varepsilon = \liminf_{n\to\infty} F(u_n) + \varepsilon.$$

Thus F is weakly sequentially lower semicontinuous, since ε was arbitrary.

Note that $\|\cdot\|_{L^1(\Omega)}$ is *not* weakly sequentially continuous, see Example 1.55.

Remark 2.8. The procedure in the above example to deduce weak sequential lower semiconinuity from lower semicontinuity does not only apply to sublinear functionals like norms, but generally to *convex* functionals. In this case one may apply Jensen's inequality, Lemma 2.15, in the last step.

This important feature of convex functionals is discussed in Theorem 2.19. Its proof does not rely on Mazur's theorem, but on Theorem 1.23. However, both ways are ultimately based on the separation theorem, Theorem 1.20.

The superadditivity of the limit inferior yields that (weak sequential) lower semicontinuity is preserved under finite summation:

Proposition 2.9. *Let $F, G\colon X \to \mathbb{R}$ be two (weakly sequentially) lower semicontinuous functionals. Then $F+G$ is (weakly sequentially) lower semicontinuous.*

Proof. We prove the weak case only. Let $u \in X$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X which converges weakly to u. Then

$$\begin{aligned}\liminf_{n\to\infty} (F+G)(u_n) &= \liminf_{n\to\infty} (F(u_n) + G(u_n)) \\ &\geq \liminf_{n\to\infty} F(u_n) + \liminf_{n\to\infty} G(u_n) \\ &\geq F(u) + G(u).\end{aligned}$$

□

Definition 2.10. Let $F\colon X \to \mathbb{R}$ be a functional and let $A\colon X \to X'$ be an operator.

(i) F is called *coercive*, if

$$\lim_{\|u\|\to\infty} \frac{F(u)}{\|u\|} = \infty.$$

(ii) F is called *weakly coercive*, if

$$\lim_{\|u\|\to\infty} F(u) = \infty.$$

(iii) A is called *coercive* if

$$\lim_{\|u\|\to\infty} \frac{\langle A(u), u\rangle}{\|u\|} = \infty.$$

Definition 2.11. For a functional $F\colon X \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we define the *sublevel set* (or *lower λ-level set*) $\mathrm{lev}_\lambda(F)$ of F by

$$\mathrm{lev}_\lambda(F) := \{u \in X\colon F(u) \le \lambda\} \subseteq X.$$

Proposition 2.12. *Let $F\colon X \to \mathbb{R}$ be a functional. The following are equivalent:*

(i) *F is weakly coercive.*

(ii) *$\mathrm{lev}_\lambda(F)$ is bounded in X for any $\lambda \in \mathbb{R}$.*

Proof. (i)⇒(ii). If for some $\lambda_0 \in \mathbb{R}$ the sublevel set $M_{\lambda_0}(F)$ is not bounded, then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in X such that $\|u_n\| \to \infty$ as $n \to \infty$ and $F(u_n) \le \lambda_0$ for all $n \in \mathbb{N}$. Hence F is not weakly coercive.

(ii)⇒(i). If F is not weakly coercive, there exists an unbounded sequence $(u_n)_{n\in\mathbb{N}}$ in X and some $\lambda_0 \in \mathbb{R}$ such that $F(u_n) \le \lambda_0$ for all $n \in \mathbb{N}$. □

Theorem 2.13. *Let $F\colon X \to \mathbb{R}$ be a weakly sequentially lower semicontinuous and weakly coercive functional on the real, reflexive Banach space X. Then there exists an element $u_0 \in X$ with*

$$F(u_0) = \min_{u\in X} F(u) = \alpha \in \mathbb{R}.$$

Proof. Take a sequence $(u_n)_{n\in\mathbb{N}}$ in X such that

$$\lim_{n\to\infty} F(u_n) = \inf_{u\in X} F(u) =: \alpha$$

By weak coercivity, (u_n) is bounded, c.f. Proposition 2.12. Since X is assumed to be reflexive, there exists a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ and an element $u_0 \in X$ such that $u_{n_j} \rightharpoonup u_0$ due to Theorem 2.4. As F is weakly lower semicontinuous,

$$F(u_0) \le \liminf_{j\to\infty} F(u_{n_j}) = \lim_{j\to\infty} F(u_{n_j}) = \alpha$$

This implies that $\alpha = F(u_0) \in \mathbb{R}$ is a minimum of F. □

2.3. Convex functionals

In this section we discuss a very important class of functionals: convex functionals. They possess two remarkable features:

(i) strictly convex functionals attain at most one minimum (uniqueness).
(ii) convex functionals are weakly sequentially lower semicontinuous, if they are lower semicontinuous (existence).

Definition 2.14. Let $M \subseteq X$ be a convex subset and let $F\colon M \to \mathbb{R}$ be a functional. F is said to be *convex* if

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v)$$

for all $u, v \in M$ and for all $t \in (0,1)$. F is *strictly convex* if this inequality is strict whenever $u \neq v$.

Inductively one obtains

Lemma 2.15 (Jensen's Inequality). *Let $M \subseteq X$ be a convex subset and let $F\colon M \to \mathbb{R}$ be a convex functional. For any finite number of points $u_1, ..., u_n \in X$ and $\alpha_1, ..., \alpha_n \in [0,1]$ such that $\sum_{i=1}^n \alpha_i = 1$,*

$$F\left(\sum_{i=1}^n \alpha_i u_i\right) \leq \sum_{i=1}^n \alpha_i F(u_i).$$

Theorem 2.16. *The functional $F\colon X \supseteq M \to \mathbb{R}$ has at most one minimum on M if the following conditions hold:*

(i) *$M \subseteq X$ is convex*
(ii) *F is strictly convex.*

Proof. Having two minima $u \neq v$, (ii) implies

$$F(\frac{1}{2}u + \frac{1}{2}v) < \frac{1}{2}F(u) + \frac{1}{2}F(v) = \min_{w \in M} F(w),$$

a contradiction showing $u = v$. □

Proposition 2.17. *Let $F\colon X \supseteq M \to \mathbb{R}$ be a convex functional. Then any local minimum of F is a global minimum.*

Proof. Let u_0 be a local minimum of F, i.e. there exists $\varepsilon > 0$ such that $F(u_0) \leq F(w)$ for all $w \in \mathrm{B}_\varepsilon(u_0) \cap M$. Let $u \in M \setminus \{u_0\}$. Then there is some $\lambda \in (0,1]$ such that $u_0 + \lambda(u - u_0) \in \mathrm{B}_\varepsilon(u_0) \cap M$. Since F is convex, we get

$$F(u_0) \leq F(u_0 + \lambda(u - u_0)) \leq \lambda F(u) + (1-\lambda)F(u_0).$$

This implies $F(u_0) \leq F(u)$, thus u_0 is a global minimum. □

Lemma 2.18. *Let $F\colon X \to \mathbb{R}$ be a functional.*

(1) *The following are equivalent:*
  (i) *F is lower semicontinuous.*
  (ii) $\mathrm{lev}_\lambda(F)$ *is closed for each* $\lambda \in \mathbb{R}$.
(2) *The following are equivalent:*
  (i) *F is weakly sequentially lower semicontinuous.*
  (ii) $\mathrm{lev}_\lambda(F)$ *is weakly sequentially closed for each* $\lambda \in \mathbb{R}$.

Proof. We only proof assertion (2). The proof of (1) is almost identical to the proof of (2).

(i)⇒(ii). For an arbitrary $\lambda \in \mathbb{R}$ let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{lev}_\lambda(F)$, which converges weakly to $u_0 \in X$. Since F is weakly sequentially lower semicontinuous in u_0, we have that

$$F(u_0) \leq \liminf_{n\to\infty} F(u_n) \leq \lambda.$$

Thus $u_0 \in \mathrm{lev}_\lambda(F)$.

(ii)⇒(i). If F is not weakly sequentially lower semicontinuous, then there exists some $u_0 \in X$ and a sequence $(u_n)_{n\in\mathbb{N}}$ in X such that

$$u_n \rightharpoonup u_0 \quad \text{and} \quad \liminf_{n\to\infty} F(u_n) < F(u_0).$$

Choose $\lambda \in \mathbb{R}$ with

$$\liminf_{n\to\infty} F(u_n) < \lambda < F(u_0).$$

Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which lies entirely in $\mathrm{lev}_\lambda(F)$ and

$$u_{n_k} \rightharpoonup u_0.$$

Hence $\mathrm{lev}_\lambda(F)$ is not weakly sequentially closed, because $u_0 \notin \mathrm{lev}_\lambda(F)$. □

Theorem 2.19. *Let X be a real Banach space and let $F\colon X \to \mathbb{R}$ be a convex functional. The following assertions are equivalent:*

(i) *F is lower semicontinuous.*
(ii) *F is weakly sequentially lower semicontinuous.*

Proof. Let $F\colon X \to \mathbb{R}$ be convex and observe that $\mathrm{lev}_\lambda(F)$ is convex for all $\lambda \in \mathbb{R}$: for $u, v \in \mathrm{lev}_\lambda(F)$ and $t \in [0, 1]$ we have that

$$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v) \leq t\lambda + (1-t)\lambda = \lambda,$$

hence $tu + (1-t)v \in \mathrm{lev}_\lambda(F)$. Therefore

$$\begin{aligned} \text{(i)} &\overset{2.18}{\iff} \mathrm{lev}_\lambda(F) \text{ is closed for all } \lambda \in \mathbb{R} \\ &\overset{1.23}{\iff} \mathrm{lev}_\lambda(F) \text{ is weakly sequentially closed for all } \lambda \in \mathbb{R} \\ &\overset{2.18}{\iff} \text{(ii)}. \end{aligned}$$

□

Remark 2.20. We will later[1] prove a result that is similar to Theorem 2.19. Under the additional assumption of differentiability of F we will then be able to give a constructive proof which does not make use of Hahn-Banach's separation theorem.

Example 2.21. The norm $\|\cdot\|_X$ is weakly sequentially lower semicontinuous. For a sequence $(u_n)_{n\in\mathbb{N}}$ in X with $u_n \rightharpoonup u$, we have that

$$\liminf_{n\to\infty} \|u_n\|_X \geq \|u\|_X.$$

Let us conclude this section with the following useful convexity criterion.

Proposition 2.22. *Let M be a convex subset of a real vector space X and let $F\colon M \to \mathbb{R}$ be a functional. For $u, v \in M$ we set $\varphi_{u,v}(t) := F(u + t(v - u))$, $\varphi_{u,v}\colon [0,1] \to \mathbb{R}$. Then:*

F is (strictly) convex $\iff$ $\varphi_{u,v}$ is (strictly) convex on $[0,1]$ for all $u, v \in M$

Proof. "$\Leftarrow$": Let $t \in (0,1)$. For all $u, v \in M$,

$$\begin{aligned} F((1-t)u + tv) &= F(u + t(v-u)) \\ &= \varphi_{u,v}(t) \\ &\leq (1-t)\varphi_{u,v}(0) + t\varphi_{u,v}(1) \\ &= (1-t)F(u) + tF(v) \end{aligned}$$

"$\Rightarrow$": Let $\lambda \in (0,1)$ and let $t, s \in [0,1]$, then for all $u, v \in M$,

$$\begin{aligned} \varphi_{u,v}((1-\lambda)t + \lambda s) &= F(u + ((1-\lambda)t + \lambda s)(v-u)) \\ &= F((1-\lambda)(u + t(v-u)) + \lambda(u + s(v-u))) \\ &\leq (1-\lambda)F(u + t(v-u)) + \lambda F(u + s(v-u)) \\ &= (1-\lambda)\varphi_{u,v}(t) + \lambda_{u,v}\varphi(s) \end{aligned}$$

□

2.4. Differentiable functionals

We discuss G-differentiable functionals. The main result of this section is Theorem 2.31, which states that a G-differentiable functional F is convex if and only if its G-derivative F' is monotone. Another highlight is Theorem 2.37 on the resolvability of the abstract operator equation $F'(u) = f$. It is the basis for the existence theory of solutions to the boundary value problem (3.1) in Chapter 3.

[1] C.f. Corollary 2.33 (ii).

Definition 2.23. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two real Banach spaces, let $A\colon X_1 \to X_2$ be a mapping and let $u, h \in X_1$. If there exists an element $\delta \in X_2$ such that

$$\lim_{t\to\infty} \left\| \frac{A(u+th) - A(u)}{t} - \delta \right\|_2 = 0,$$

then we say that A has the *directional derivative* $DA(u,h) := \delta$ at u in direction h. If the mapping $h \mapsto DA(u,h)$, $X_1 \to X_2$ is linear and continuous, we say that A has the *Gâteaux differential* - or shorter: *G-differential* - at u, denoted by $A'(u) \in \mathcal{L}(X_1, X_2)$. We call A *G-differentiable* if $A'(u)$ exists for every $u \in X_1$; then the *G-derivative* A' of A defines a mapping $X_1 \to \mathcal{L}(X_1, X_2)$.

In the special case $X_2 = \mathbb{R}$, a G-differentiable functional $F\colon X_1 \to \mathbb{R}$ has the G-derivative $F'\colon X_1 \to X_1{}'$ and $DF(u,h)$ is usually written as $\langle F'(u), h\rangle$.

Proposition 2.24. *Let $F\colon X \to \mathbb{R}$ be a G-differentiable functional. If F has a local minimum at $u_0 \in X$, then u_0 is a* critical point, *i.e.* $F'(u_0) = 0$.

Proof. For $h \in X$ we define the function $\varphi_h\colon \mathbb{R} \to \mathbb{R}$, $\varphi_h(t) := F(u_0 + th)$. By our assumptions every φ_h is differentiable on $\mathbb{R}$ and attains a local minimum at 0. Therefore

$$0 = \varphi_h{}'(0) = DF(u_0, h) \text{ for all } h \in X.$$

□

Definition 2.25. Let $A\colon X \to X'$ be an operator.

(i) A is called *monotone* if
$$\langle A(u) - A(v), u - v\rangle \geq 0$$
for all $u, v \in X$.

(ii) A is called *strictly monotone* if A is monotone and
$$\langle A(u) - A(v), u - v\rangle > 0$$
whenever $u \neq v$.

(iii) A is called *uniformly monotone* if there exists a continuous strictly monotonically increasing function $a\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $a(0) = 0$ such that $\lim_{\xi\to\infty} a(\xi) = \infty$ and
$$\langle A(u) - A(v), u - v\rangle \geq a(\|u - v\|)\,\|u - v\|$$
for all $u, v \in X$.

(iv) A is called *strongly monotone* if there exists a constant $c > 0$ such that
$$\langle A(u) - A(v), u - v\rangle \geq c\|u - v\|^2$$
for all $u, v \in X$.

Remark 2.26. Note that

(i) strongly monotone operators are uniformly monotone, and that

(ii) uniformly monotone operators are both strictly monotone and coercive.

We start with the observation, that monotone operators are locally bounded:

Proposition 2.27. *Let $A\colon X \to X'$ be monotone. Then A is locally bounded, that is*[2]

$$\forall u \in X\ \exists \varepsilon > 0\ \exists M > 0\ \forall v \in \mathrm{B}_\varepsilon(u)\colon \|A(u)\|_{X'} \leq M. \tag{2.3}$$

Proof. Let us suppose, that (2.3) does not hold. Then there exists some $u \in X$ and a sequence $(u_n)_{n\in\mathbb{N}}$ in X converging to u such that $\|A(u_n)\|_{X'} \to \infty$. We assume without loss of generality that $u = 0$ and put $c_n := 1 + \|A(u_n)\|_{X'}\,\|u_n\| \geq 1$ for $n \in \mathbb{N}$. Since A is monotone, we have for any $v \in X$ that

$$0 \leq \langle A(v) - A(u_n), v - u_n\rangle = \langle A(u_n), u_n\rangle + \langle A(v), v - u_n\rangle - \langle A(u_n), v\rangle\,.$$

Hence

$$\begin{aligned}\left\langle c_n^{-1}A(u_n), v\right\rangle &\leq \frac{\langle A(u_n), u_n\rangle + \langle A(v), v - u_n\rangle}{c_n}\\ &\leq \frac{\|A(u_n)\|_{X'}\,\|u_n\|}{c_n} + \frac{\|A(v)\|_{X'}(\|v\| + \|u_n\|)}{c_n}\\ &\leq 1 + \|A(v)\|_{X'}(\|v\| + \|u_n\|) \xrightarrow{n\to\infty} 1 + \|A(v)\|_{X'}\,\|v\|.\end{aligned}$$

Since the above estimate holds for $-v$ as well, we conclude that for all $v \in X$

$$\limsup_{n\to\infty}\left|\left\langle c_n^{-1}A(u_n), v\right\rangle\right| < \infty.$$

Therefore the uniform boundedness principle, Theorem 1.10, yields that the family

$$\left\{c_n^{-1}\|A(u_n)\|_{X'}\right\}_{n\in\mathbb{N}}$$

is uniformly bounded, say by $N \in \mathbb{R}^+$. Thus $\|A(u_n)\| \leq N(1 + \|A(u_n)\|_{X'})$ for all $n \in \mathbb{N}$. This implies

$$\|A(u_n)\|_{X'} \leq \frac{N}{1 - N\|u_n\|} \xrightarrow{n\to\infty} N,$$

which contradicts our assumption that $\|A(u_n)\|_{X'} \to \infty$. □

Corollary 2.28. *If $A\colon X \to X'$ is linear and monotone, then A is continuous.*

The next theorem establishes the connection between the convexity of a differentiable functional F and the monotonicity of its derivative F'. It is based on the following proposition for real functions:

Proposition 2.29. *Let $\varphi\colon [a, b] \to \mathbb{R}$ be differentiable*[3].

[2] C.f. Definition 1.2.

[3] The derivatives in the boundary points are understood to be the corresponding one-sided derivatives. We will recurrently make this abuse of notation in the sequel.

(i) *φ is (strictly) convex on $[a,b]$ if and only if φ' is (strictly) monotonically increasing on $[a,b]$.*
(ii) *If φ is convex on $[a,b]$ then φ' is continuous on (a,b).*

Proof. The proof of assertion (i) can be found in most textbooks on calculus, see e.g. Satz 49.7 in [**10**]. See Problem 42.3 in [**23**] for a proof of assertion (ii). □

Corollary 2.30. *Let $\varphi\colon [a,b] \to \mathbb{R}$ be differentiable and convex. Then φ is Lipschitz continuous on any compact interval $I \subseteq (a,b)$.*

Proof. Let $I \subseteq (a,b)$ be a compact interval. By Proposition 2.29 (ii), φ' is bounded on I. Therefore, having in mind the mean value theorem, one can choose $L = \sup_{t\in I} |\varphi'(t)|$ as Lipschitz constant. □

Theorem 2.31. *Let $F\colon X \to \mathbb{R}$ be a G-differentiable functional. Then*

(1) *The following are equivalent:*
  (i) *F is convex on X.*
  (ii) *F' is monotone on X.*
  (iii) *$F(v) - F(u) \geq \langle F'(u), v-u\rangle$ for all $u, v \in X$.*
(2) *The following are equivalent:*
  (i) *F is strictly convex on X.*
  (ii) *F' is strictly monotone on X.*
  (iii) *$F(v) - F(u) > \langle F'(u), v-u\rangle$ for all $u, v \in X$ such that $u \neq v$.*

Proof. (1) As in Proposition 2.22 we define $\varphi(t) := \varphi_{u,v}(t) = F(u + t(v - u))$ for $u, v \in X$ and $t \in [0,1]$, then $\varphi'(t) = \langle F'(u + t(v-u)), v-u\rangle$ for all $t \in [0,1]$, and:

$$
\begin{aligned}
\text{(i)} &\overset{2.22}{\Longleftrightarrow} \varphi \text{ is convex on } [0,1] \text{ for all } u, v \in X \\
&\overset{2.29}{\Longleftrightarrow} \varphi' \text{ is monotonically increasing on } [0,1] \text{ for all } u, v \in X \\
&\overset{(\triangle)}{\Longleftrightarrow} \text{(ii)}.
\end{aligned}
$$

We have to verify $(\triangle)$:
"$\Rightarrow$":

$$
\begin{aligned}
0 \leq \varphi'(1) - \varphi'(0) &= \langle F'(v), v-u\rangle - \langle F'(v), v-u\rangle \\
&= \langle F'(v) - F'(u), v-u\rangle\,.
\end{aligned}
$$

"$\Leftarrow$": Let $0 \leq s < t \leq 1$, then

$$
\begin{aligned}
\varphi'(t) - \varphi'(s) &= (t-s)^{-1} \langle F'(u + t(v-u)) - F'(u + s(v-u)), (t-s)(v-u)\rangle \\
&\geq 0.
\end{aligned}
$$

It remains to prove the equivalence (ii) $\Longleftrightarrow$ (iii):

$$\begin{aligned}
\text{(ii)} &\Rightarrow \varphi' \text{ is monotonically increasing on } [0,1] \text{ for all } u, v \in X \\
&\overset{(\triangle)}{\Rightarrow} \varphi(1) - \varphi(0) = \varphi'(\xi) \geq \varphi'(0) \text{ for some } \xi \in (0,1) \text{ for all } u, v \in X \\
&\Rightarrow F(v) - F(u) \geq \langle F'(u), v - u \rangle \text{ for all } u, v \in X \text{ [this is (iii)]} \\
&\Rightarrow F(u) - F(v) \geq \langle F'(v), u - v \rangle \text{ for all } u, v \in X \\
&\Rightarrow 0 \geq \langle F'(u) - F'(v), v - u \rangle \text{ for all } u, v \in X \\
&\Rightarrow \text{(ii)}.
\end{aligned}$$

The proof of (2) is almost identical to the proof of (1). □

Remark 2.32. Let us consider the mapping $a\colon \mathbb{R}^d \to \mathbb{R}^d \cong (\mathbb{R}^d)'$. According to Definition 2.25, a is monotone if and only if

$$\big(a(x) - a(y)\big) \cdot (x - y) \geq 0$$

for all $x, y \in \mathbb{R}^d$, where "$\cdot$" denotes the standard inner product on $\mathbb{R}^d$. Let furthermore $f\colon \mathbb{R}^d \to \mathbb{R}$ be a differentiable function with gradient $\nabla f = a$. Then Theorem 2.31 applies, and gives

$$\begin{aligned}
a \text{ is monotone } &\Longleftrightarrow f \text{ is convex} \\
&\Longleftrightarrow f(y) - f(x) \leq a(x) \cdot (y - x) \quad \forall x, y \in \mathbb{R}^d.
\end{aligned}$$

Likewise, the equivalences with regard to the respective strict cases remain true in this finite dimensional setting.

Theorem 2.31 has two important consequences.

Corollary 2.33. *If $F\colon X \to \mathbb{R}$ is a convex G-differentiable functional, then the following two assertions hold:*

(i) *F has a minimum at u if and only if $F'(u) = 0$*

(ii) *F is weakly lower semicontinuous.*

Proof. Assertion (i) is a direct consequence of Theorem 2.31 (1) and Proposition 2.17.
(ii). Inequality (iii) in Theorem 2.31 (1) and weak continuity of $\langle \cdot, \cdot \rangle$ yield:

$$F(u) \leq \liminf_{v \rightharpoonup u} (F(v) + \langle F'(u), u - v \rangle) = \liminf_{v \rightharpoonup u} F(v)$$

□

Corollary 2.34. *Let $F := F_1 + F_2\colon X \to \mathbb{R}$ be a functional with $F_1\colon X \to \mathbb{R}$ being convex and G-differentiable, and with $F_2\colon X \to \mathbb{R}$ being weakly sequentially lower semicontinuous and G-differentiable. Then F is G-differentiable and weakly lower semicontinuous.*

Proof. F_1 is weakly sequentially lower semicontinuous by Corollary 2.33. Thus F is weakly sequentially lower semicontinuous due to Proposition 2.9; the G-differentiability of F is obvious. □

Proposition 2.36 deals with further relations between F and F'. But first, we need to prove the following

Lemma 2.35. *If $\varphi\colon [a,b] \to \mathbb{R}$ is Lipschitz continuous, then φ is absolutely continuous.*

Proof. Let L denote the Lipschitz constant of φ, i.e.

$$|\varphi(x) - \varphi(y)| \leq L\,|x - y|$$

for all $x, y \in [a,b]$. Let $\varepsilon > 0$ and set $\delta := \varepsilon/L$. Then the criterion for the absolute continuity of φ, see Definition 1.32, is satisfied. □

Proposition 2.36. *Let $F\colon X \to \mathbb{R}$ be G-differentiable.*

(i) *If F' is monotone, then F is locally Lipschitz continuous.*
(ii) *If F' is uniformly monotone, then F is coercive.*

Proof. (i). By Proposition 2.27, F' is locally bounded. Thus F, having a locally bounded G-derivative, is locally Lipschitz.

(ii). For $u \in X$ we define the map $\varphi = \varphi_u\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto F(tu)$. By assumption φ is differentiable and the proof of Theorem 2.31 shows that φ is convex. Corollary 2.30 implies that φ is Lipschitz continuous on $[0,1]$, thus φ is absolutely continuous on $[0,1]$ by Lemma 2.35. Applying Lebesgue's fundamental theorem of calculus, Theorem 1.34, yields:

$$F(u) - F(0) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\,dt = \int_0^1 \langle F'(tu), u\rangle\;dt.$$

Now choose $\varepsilon \in (0,1)$. Supposing without loss of generality that $F(0) = 0$ we get:

$$\begin{aligned}
F(u) &= \int_0^1 \langle F'(tu), u\rangle\;dt = \int_0^1 t^{-1}\,\langle F'(tu) - F'(0), tu - 0\rangle\;dt + \langle F'(0), u\rangle \\
&\geq \int_\varepsilon^1 t^{-1}\,\langle F'(tu) - F'(0), tu - 0\rangle\;dt + \langle F'(0), u\rangle \\
&\geq \int_\varepsilon^1 t^{-1} a(\|tu\|)\,\|tu\|\,dt + \langle F'(0), u\rangle \\
&\geq \int_\varepsilon^1 t^{-1} a(\|\varepsilon u\|)\,\|\varepsilon u\|\,dt + \langle F'(0), u\rangle \\
&\geq -\varepsilon\log(\varepsilon)\;a(\varepsilon\,\|u\|)\,\|u\| - \|F'(0)\|_{X'}\;\|u\| \\
&= \|u\|\left(\varepsilon\log\left(\varepsilon^{-1}\right)\;a(\varepsilon\,\|u\|) - \|F'(0)\|_{X'}\right),
\end{aligned}$$

where $a\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function that is strictly monotonically increasing such that $\lim_{\xi\to\infty} a(\xi) = \infty$ according to Definition 2.25. Thus,

$$\lim_{\|u\|\to\infty} \frac{F(u)}{\|u\|} = \infty.$$

□

Theorem 2.37. *Let X be a real, reflexive Banach space and let $F\colon X \to \mathbb{R}$ be a G-differentiable, weakly sequentially lower semicontinuous functional. We set $A := F'$. Then the following assertions hold:*

(i) *If F is coercive, then $A(u) = f$ has a solution for any $f \in X'$.*
(ii) *If F is strictly convex, then $A(u) = f$ has at most one solution.*

Proof. Let $f \in X'$. We define $G\colon X \to \mathbb{R}$, $u \mapsto F(u) - \langle f, u\rangle$.
(i). The functional G is weakly sequentially lower semicontinuous and G-differentiable with G-differential $G'(u) = A(u) - f$. Observe that G is weakly coercive. Therefore G attains a minimum $u_0 \in X$ by Theorem 2.13. Due to Proposition 2.24 we get $G'(u_0) = 0$, thus u_0 solves the equation $A(u) = f$.

(ii). G is strictly convex. Corollary 2.33 tells us, that any solution to $A(u) = f$ minimizes G, thus due to Theorem 2.16 the equation $A(u) = f$ has at most one solution. □

2.5. Potential operators

Definition 2.38. An operator $A\colon X \to X'$ is called *hemicontinuous* if the mapping $t \mapsto \langle A(u + tv), w\rangle$ is continuous as a mapping $[0, 1] \to \mathbb{R}$ for all $u, v, w \in X$, i.e. A is directionally weakly continuous.

Definition 2.39. Let $A\colon X \to X'$ be an operator.

(i) A is called a *potential operator*, if there exists a G-differentiable functional $F\colon X \to \mathbb{R}$ such that $A = F'$. In this case, we call F a *potential* of A.
(ii) If A is hemicontinuous, then we define $F_A\colon X \to \mathbb{R}$ by

$$F_A(u) := \int_0^1 \langle A(tu), u\rangle \, dt \tag{2.4}$$

and call F_A a *pseudo potential* of A.

Next, we consider the following two equations:

$$F_A(u) - F_A(v) = \int_0^1 \langle A(v + t(u - v)), u - v\rangle \, dt \quad \text{for all } u, v \in X \tag{2.5}$$

and

$$\langle A'(u)(v), w\rangle = \langle A'(u)(w), v\rangle \quad \text{for all } u, v, w \in X \tag{2.6}$$

together with the condition

$$\begin{aligned} &(t, s) \mapsto \langle A'(w + tu + sv)(x), y\rangle \text{ is continuous on } [0, 1] \times [0, 1] \\ &\text{for all } u, v, w, x, y \in X. \end{aligned} \tag{2.7}$$

Proposition 2.40. *Let $A\colon X \to X'$ be a hemicontinuous operator. Then the following assertions hold:*

(i) Integral criterion. *A is a potential operator if and only if* (2.5) *holds. Then the pseudo potential F_A is a potential, and an arbitrary potential of A differs from F_A only by a constant.*

(ii) Derivative criterion. *If A' exists on X as a G-differential satisfying* (2.7)*, then A is a potential operator if and only if* (2.6) *holds.*

Proof. (i), "⇒"-part. Let $F\colon X \to \mathbb{R}$ be a G-differentiable functional such that $F' = A$, let $u, v \in X$. For $\varphi(t) := F(v + t(u - v))$, $t \in \mathbb{R}$ we have that $\varphi'(t) := \langle A(v + t(u - v)), u - v\rangle$, and therefore

$$F(u) - F(v) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\,dt = \int_0^1 \langle A(v + t(u - v)), u - v\rangle\;dt. \tag{2.8}$$

Note that (2.8) holds true for all $u, v \in X$, thus $F = F(0) + F_A$ and F_A satisfies (2.5).

(i), "⇐"-part. Let F_A satisfy (2.5), let $u, v \in X$. We calculate

$$\begin{aligned}
\langle {F_A}'(u), v\rangle &= \lim_{s\to 0} s^{-1}(F_A(u + sv) - F_A(u)) \\
&\overset{(2.5)}{=} \lim_{s\to 0} s^{-1} \int_0^1 \langle A(u + t(u + sv - u)), u + sv - u\rangle\;dt \\
&= \int_0^1 \lim_{s\to 0} \langle A(u + tsv), v\rangle\;dt \\
&= \langle A(u), v\rangle\,,
\end{aligned}$$

where we interchanged passage to the limit with integration, thanks to the continuity of the integrand. This shows that F_A is G-differentiable with G-derivative ${F_A}' = A$, because $u, v \in X$ have been chosen arbitrarily.

(ii), "⇒"-part. Let F be a potential for A, let $u, v, w \in X$. We define a function $W\colon [0, 1] \times [0, 1] \to \mathbb{R}$ by $W(t, s) := F(w + tu + sv)$. Then:

$$\frac{\partial}{\partial t \partial s} W(t, s) = \langle A'(w + tu + sv)(v), u\rangle \tag{2.9a}$$

$$\frac{\partial}{\partial s \partial t} W(t, s) = \langle A'(w + tu + sv)(u), v\rangle\,. \tag{2.9b}$$

Now (2.7) and (2.9) imply that W is twice continuously differentiable on $[0, 1] \times [0, 1]$. Therefore it follows by Schwarz' theorem that the second derivatives of W are symmetric on $[0, 1] \times [0, 1]$. In particular we find that

$$\frac{\partial}{\partial t \partial s} W(0, 0) = \frac{\partial}{\partial s \partial t} W(0, 0),$$

but this is (2.6), since $u, v, w \in X$ have been chosen arbitrarily.

(ii) "⇐"-part. Let $u, v \in X$. We define two functions $U, V\colon [0, 1] \to \mathbb{R}$ by

$$U(t, s) := \langle A(tv + su), v\rangle$$
$$V(t, s) := \langle A(tv + su), u\rangle\,.$$

Then (2.6) means that

$$\frac{\partial}{\partial s}U(t,s) = \frac{\partial}{\partial t}V(t,s) \quad \text{for all } t, s \in [0,1]$$

and (2.7) implies that the vector field $(U, V)\colon [0,1] \times [0,1] \to \mathbb{R}^2$ is continuously differentiable. Therefore integration over any closed piecewise continuously differentiable path γ in $[0,1] \times [0,1]$ gives zero:

$$\int_\gamma U(t,s)\,dt + V(t,s)\,ds = 0.$$

In particular, we obtain (2.5) by choosing γ to be the triangle $\triangle = \triangle ABC$ with vertices $A := (0,0)$, $B := (0,1)$, $C := (1,0)$ and with clockwise traversal. Note once again, that all calculations and arguments we have made, are independent of the particular choice of $u, v \in X$. Thus A is a potential operator by (i). □

We want to close this section with a remark on non-potential problems. If a given hemicontinuous operator $A\colon X \to X'$ does not satisfy (2.5), Proposition 2.40 tells us, that there is no functional $F\colon X \to \mathbb{R}$ such that $F' = A$. Hence the direct method, Theorem 2.37, does not apply to the problem

$$A(u) = f, \quad f \in X'. \tag{2.10}$$

However one might find answers concerning resolvability of (2.10) within the *theory of monotone operators*. This theory can be considered as a generalization of the calculus of variations, since it applies to both potential and non-potential problems. The main theorem on monotone operators is due to Browder and Minty. It can be found e.g. in [**22**]. We state a simplified version of this theorem.

Theorem 2.41 (Browder (1963), Minty (1963)). *Let $A\colon X \to X'$ be a monotone, coercive and hemicontinuous operator on the real, separable and reflexive Banach space X. Then the equation*

$$A(u) = f$$

has a solution for every $f \in X'$.

2.6. Examples

Example 2.42. Let H be a real Hilbert space and let $A\colon H \to H \cong H'$ be a continuous linear operator. We have that A is G-differentiable and $A'(u) = A$ for all $u \in H$, hence A' clearly satisfies (2.7). By Proposition 2.40, A is a potential operator if and only if A is symmetric.

So let A be symmetric, then the potential $F_A \colon H \to \mathbb{R}$ is given by

$$F_A(u) = \int_0^1 \langle A(tu), u\rangle \, dt = \int_0^1 t \langle Au, u\rangle \, dt = \frac{1}{2} \langle Au, u\rangle . \tag{2.11}$$

Let us additionally assume that A is strongly monotone. Then A is coercive and strictly monotone, c.f. Remark 2.26, and F_A is coercive - either by Proposition 2.36 or directly from (2.11). Thus, by Theorem 2.31 and Corollary 2.33, F_A is strictly convex and weakly lower semicontinuous. Now Theorem 2.37 says that for any $b \in H'$ the equation $Au = b$ possesses a unique solution $u_0 \in H$.

Example 2.43. Let $\Omega \subseteq \mathbb{R}^d$, be a bounded Lipschitz domain and let $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i{}^2}$ denote the *Laplace operator* in $\mathbb{R}^d$. Consider the Dirichlet boundary value problem:

$$-\Delta u = f \quad \text{in}\, \Omega \tag{2.12a}$$

$$u = 0 \quad \text{on}\, \partial\Omega \tag{2.12b}$$

We want to show that in a certain sense, the boundary value problem (2.12) has a unique solution. Therefore we choose the Sobolev space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ as solution space and we interpret Δ as an operator acting between $H_0^1(\Omega)$ and $(H_0^1(\Omega))' = H^{-1}(\Omega)$:

$$\Delta \colon H_0^1(\Omega) \to H^{-1}(\Omega), \quad u \mapsto (\Delta u, \cdot)_{L^2(\Omega)}.$$

We set $\|\cdot\| := \|\cdot\|_{H^1(\Omega)}$. The operator Δ is linear, moreover Δ is bounded:

$$\begin{aligned}
\sup_{\|u\|\le 1} \|\Delta u\|_{H^{-1}(\Omega)} &= \sup_{\|u\|\le 1} \sup_{\|v\|\le 1} |(\Delta u, v)_{L^2(\Omega)}| \\
&\overset{1.69}{=} \sup_{\|u\|\le 1} \sup_{\|v\|\le 1} |(\nabla u, \nabla v)_{L^2(\Omega)}| \\
&\le \sup_{\|u\|\le 1} \sup_{\|v\|\le 1} |(u, v)_{H_0^1(\Omega)}| = 1.
\end{aligned}$$

The above estimation also shows, that the image of Δ is contained in $H^{-1}(\Omega)$.

Let $\tilde{f} \in L^2(\Omega)$. We set $f := (\tilde{f}, \cdot)_{L^2(\Omega)} \in H^{-1}(\Omega)$ and call $u_0 \in H_0^1(\Omega)$ a *weak solution*[4] to the boundary value problem (2.12) if

$$\int_\Omega \nabla u_0 \cdot \nabla v \, dx = \int_\Omega \tilde{f} v \, dx \tag{2.13}$$

holds for all $v \in H_0^1(\Omega)$.

Formula (2.13) is motivated by Theorem 1.69: assume that $u_0 \in H_0^1(\Omega)$ with $\nabla u_0 \in H^1(\Omega)$ satisfies (2.13) for every $v \in H_0^1(\Omega)$. Then for all $v \in H_0^1(\Omega)$,

$$\int_\Omega -\Delta u_0 v \, dx \overset{1.69}{=} \int_\Omega \nabla u_0 \cdot \nabla v \, dx.$$

That is $-\Delta u_0 = f$ holds true in $H^{-1}(\Omega)$

[4] See Section 3.2 for a detailed discussion of this solution concept.

We claim that (2.12) possesses a unique weak solution for every $f \in L^2(\Omega)$. According to Example 2.42 and by exploiting the fact that $-\Delta$ is symmetric, it is sufficient to verify that $-\Delta$ is strongly monotone in order to find the unique weak solution $u_0 \in H_0^1(\Omega)$ to (2.12). Thus we have to show that there exists a constant $c > 0$ such that

$$\int_\Omega (\nabla u)^2 \, dx \geq c \int_\Omega u^2 + (\nabla u)^2 \, dx = c\|u\|^2 \tag{2.14}$$

for all $u \in H_0^1(\Omega)$. It follows directly from Poincaré's inequality (1.10), that

$$\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq (N^2 + 1)\|\nabla u\|_{L^2(\Omega)}^2,$$

hence (2.14) holds true with $c := (N^2 + 1)^{-1}$.

CHAPTER 3

Quasilinear Boundary Value Problems

Orientation

This chapter is devoted to the study of the following mixed boundary value problem for a function $u\colon \Omega \to \mathbb{R}$. It is given in *divergence form*:

$$-\operatorname{div}(a(x,u,\nabla u)) + c(x,u,\nabla u) = g \qquad \text{in}\,\Omega \tag{3.1a}$$

$$u|_\Gamma = u_D \qquad \text{on}\,\Gamma_D \tag{3.1b}$$

$$\nu \cdot a(x,u,\nabla u) + b(x,u) = h \qquad \text{on}\,\Gamma_N. \tag{3.1c}$$

The subset $\Omega \subseteq \mathbb{R}^d$ is supposed to be a bounded domain with C^1-boundary $\Gamma := \partial\Omega$. More general, Ω can be a bounded Lipschitz domain. The vector field $\nu = (\nu_1, ..., \nu_d)$ denotes the unit outward normal to Γ. It is defined at any point $x \in \Gamma$, if Γ is a C^1-boundary. If Γ is a Lipschitz boundary, ν is defined a.e. on Γ. Furthermore, $\Gamma_D, \Gamma_N \subseteq \Gamma$ are disjoint open[1] subsets of Γ such that $\Gamma \setminus (\Gamma_D \cup \Gamma_N)$ has measure zero with respect to the $(d-1)$-dimensional Lebesgue measure. Note, that either Γ_D or Γ_N might be empty.

We are particularly interested in showing existence of weak solutions u to (3.1) by means of the abstract functional analytic theory presented in the previous chapter.

We follow the exposition in [**13**], Chapter 4.

3.1. A first closer inspection of problem (3.1)

The equation (3.1a) is a quasilinear second-order partial differential equation. In more detail it reads:

$$-\sum_{i=1}^{d} \frac{\partial}{\partial x_i} a_i(x,u(x),\nabla u(x)) + c(x,u(x),\nabla u(x)) = g(x) \tag{3.2}$$

where $x = (x_1, ..., x_d) \in \Omega$, and $a_i\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i = 1, ..., d$ and $c\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are given functions. Instead of (3.2) we will often use the abbreviated form (3.1a). For $x \in \Omega$ we will assume $a_i(x,\cdot,\cdot)$, $i = 1, ..., d$, and $c(x,\cdot,\cdot)$ to be smooth in the $W^{1,1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$ sense, and we require the

[1] In the relative topology.

following symmetry conditions:

$$\frac{\partial a_i(x,r,s)}{\partial s_j} = \frac{\partial a_j(x,r,s)}{\partial s_i} \quad \text{and} \quad \frac{\partial a_i(x,r,s)}{\partial r} = \frac{\partial c(x,r,s)}{\partial s_i} \tag{3.3}$$

for all $1 \le i, j \le d$ and for almost all $(x, r, s) \in \Omega \times \mathbb{R} \times \mathbb{R}^d$. That is, the Jacobian matrix of the mapping

$$(r, s) \mapsto (c(x, r, s), a(x, r, s)), \quad \mathbb{R}^{1+d} \to \mathbb{R}^{1+d} \tag{3.4}$$

is symmetric for almost all $x \in \Omega$.

Condition (3.1b) is a so-called *Dirichlet boundary condition* for the trace $u|_\Gamma$ of u on Γ_D with $u_D\colon \Gamma_D \to \mathbb{R}$ being a given function.

Relation (3.1c) is a condition for the boundary flux $\nu \cdot a$ of u on Γ_N, where $\nu = (\nu_1, ..., \nu_d)$ denotes the unit outward normal to Γ, and $h\colon \Gamma_N \to \mathbb{R}$ and $b\colon \Gamma_N \times \mathbb{R}$ are given functions. This condition is a so-called *Newton boundary condition*; if b is identically zero, it is called a *Neumann boundary condition*.

We call (3.1) a *Dirichlet boundary value problem* if $\Gamma_D = \Gamma$, or a *Newton/Neumann boundary value problem* if $\Gamma_N = \Gamma$, or else a *mixed boundary value problem*, that is if Γ_D and Γ_N both have nonzero $(d-1)$-dimensional volume.

3.2. Weak formulation of (3.1)

Next we will derive a *weak formulation* of the mixed boundary value problem (3.1). We emphasize that the calculations we are carrying out are not rigorously justified at the moment. In the next section we will set up further restrictions on the data appearing in (3.1) and thereby subsequently give a meaning to all calculations we have done.

First, we multiply equation (3.1a) by a so-called *test function* z with $z|_{\Gamma_D} = 0$ and integrate over Ω:

$$\int_\Omega (-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u))z\, dx = \int_\Omega gz\, dx.$$

Next, we apply Green's formula (1.9):

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla z + c(x, u, \nabla u)z\, dx - \int_\Gamma (\nu \cdot a(x, u, \nabla u))z\, dS = \int_\Omega gz\, dx;$$

and substitute the Newton boundary condition (3.1b):

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla z + c(x, u, \nabla u)z\, dx + \int_\Gamma (b(x, u) - h)z\, dS = \int_\Omega gz\, dx.$$

Finally, noting that the boundary integral vanishes on Γ_D since $z|_{\Gamma_D} = 0$, we arrive at

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla z + c(x,u,\nabla u) z\,dx + \int_{\Gamma_N} b(x,u) z\,dS = \int_\Omega g z\,dx + \int_{\Gamma_N} h z\,dS. \tag{3.5}$$

Formula (3.5) leads to the following

Definition 3.1. Let $1 < p < \infty$. We call $u \in W^{1,p}(\Omega)$ a *weak solution* to the mixed boundary value problem (3.1) if $u|_{\Gamma_D} = u_D$ and if (3.5) holds for any

$$z \in X := \left\{ z \in W^{1,p}(\Omega) \colon\ z|_{\Gamma_D} = 0 \right\}.$$

Remark 3.2. The reason why we have chosen the Sobolev space $W^{1,p}(\Omega)$ as appropriate solution space will be clarified in the next section, where we will restrict the data in (3.1) to functions of p-polynomial growth.

Note, that the cases $p = 1$ and $p = \infty$ are excluded. This is because we want to apply the existence theory from Chapter 2, which relies on compactness properties of reflexive Banach spaces.

The next proposition makes sure that our definition of a weak solution to (3.1) is a consistent generalization of the concept of *classical solutions*. We call a function $u \in C^2(\overline{\Omega})$ a classical solution for (3.1), if u satisfies (3.1) in every $x \in \overline{\Omega}$.

Proposition 3.3. *Consider the mixed boundary value problem* (3.1). *Then:*

(i) *Every classical solution is also a weak solution.*

(ii) *Let $a \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d)$, $c \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d)$ and $b \in C(\Gamma_N \times \mathbb{R})$. Furthermore let $g \in C(\overline{\Omega})$, $h \in C(\Gamma_N)$ and $u_D \in C(\Gamma_D)$. If $u \in W^{1,p}(\Omega) \cap C^2(\overline{\Omega})$ is a weak solution, then u is also a classical solution.*

Proof. Assertion (i) follows from the above derivation of (3.5).
(ii). By assumption, (3.5) is satisfied for every $z \in X$. Due to Green's formula (1.9),

$$\begin{aligned}\int_\Omega \Big(\operatorname{div}(a(x,u,\nabla u)) - c(x,u,\nabla u) + g \Big) z\,dx \\ + \int_{\Gamma_N} \Big(h - b(x,u) - \nu \cdot a(x,u,\nabla u) \Big) z\,dS = 0\end{aligned} \tag{3.6}$$

for every $z \in X$. First, we show that u solves (3.1a) in every $x \in \Omega$. For any $z \in W_0^{1,p}(\Omega) \subseteq X$, the boundary integral in (3.6) vanishes, hence the first integral vanishes too. We set

$$\alpha(x) := \operatorname{div}(a(x,u(x),\nabla u(x))) - c(x,u(x),\nabla u(x)) + g(x), \quad \alpha\colon \Omega \to \mathbb{R},$$

and claim that $\alpha = 0$. By our hypothesis, $\alpha \in C(\Omega)$. If $\alpha \neq 0$ there exists some $x \in \Omega$ such that $\alpha(x) \neq 0$, say $\alpha(x) > 0$. Then α is positive on $\mathrm{B}_\varepsilon(x)$ for some $\varepsilon > 0$. This contradicts the fact that the first integral in (3.6) is zero for all $z \in W_0^{1,p}(\Omega)$. Hence $\alpha = 0$ and therefore u satisfies (3.1a) in every $x \in \Omega$.

Now we know that the first integral in (3.6) vanishes for any $z \in X$. This implies that the second integral is zero for any $z \in X$. Observe that the set $\left\{z|_{\Gamma_N} : z \in X\right\}$ is dense[2] in $L^1(\Gamma_N)$. Therefore, the function

$$\beta(x) := h(x) - b(x,u(x)) - \nu(x)\cdot a(x,u(x),\nabla u(x)), \quad \beta\colon \Gamma_N \to \mathbb{R}$$

is zero almost everywhere on Γ_N. Since by assumption $\beta \in C(\Gamma_N)$, we have that β is identically zero on Γ_N. This shows that u satisfies (3.1c) in every point $x \in \Gamma_N$.

Finally note, that the trace $(u - u_D)|_{\Gamma_D}$ is zero almost everywhere on $L^p(\Gamma_D)$ by Definition 3.1. Thus (3.1b) is satisfied in every $x \in \Gamma_D$, because $u|_{\Gamma_D} \in C(\Gamma_D)$ by our assumption. □

Remark 3.4. We make some comments on the space X:

(i) X equipped with the Sobolev norm $\|\cdot\|_{W^{1,p}(\Omega)}$ is a closed subspace of the separable, reflexive Banach space $W^{1,p}(\Omega)$. Therefore X is also a separable and reflexive Banach space by Lemma 2.2. The closedness of $X \subseteq W^{1,p}(\Omega)$ is a direct consequence of the continuity of the trace operator; c.f. Theorem 1.65.

(ii) We have seen in the proof of Proposition 3.3 that the density of $\left\{z|_{\Gamma_N} : z \in X\right\}$ in $L^1(\Gamma_N)$ was important to obtain consistency with the classical problem. This density property is guaranteed, if Γ_D and Γ_N are sufficiently "regular". In particular we are on the save side with our assumption, that $\Gamma_D, \Gamma_N \subseteq \Gamma$ are open.

If we would presume Γ_D and Γ_N to be measurable only, which would be fine from a pure theoretical point of view, the following scenario could happen. One could imagine Γ_D to be a dense null set in Γ. Then any continuous function satisfying the Dirichlet condition $0 := u_D = u|_{\Gamma_D}$ is identically zero on Γ and the Newton boundary condition (3.1c) has no meaning, although $\mathrm{Vol}_{d-1}(\Gamma_N) = \mathrm{Vol}_{d-1}(\Gamma)$.

[2] See Remark 3.4.

3.3. Specification of the data in (3.1)

We will now set up further requirements on the functions a, b, c, g, h and u_D that appear in (3.1).

First, we have to make sure that both integrands on the left-hand side of (3.5) are measurable. To ensure this, we assume that

$$a_i, c\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \quad b\colon \Gamma \times \mathbb{R} \to \mathbb{R} \quad \text{are Carathéodory functions}$$

for $i = 1, ..., d$. Furthermore the integrands in (3.5) should be integrable for all $u, z \in W^{1,p}(\Omega)$, and additionally we will later need certain continuity properties[3] of the Nemytskii mappings

$$\begin{aligned}
&\mathcal{N}_a\colon W^{1,p}(\Omega) \times L^p(\Omega; \mathbb{R}^d) \to L^{p'}(\Omega; \mathbb{R}^d),\\
&u \mapsto \mathcal{N}_b(u|_\Gamma)\colon W^{1,p}(\Omega) \to L^{p^{\#'}}(\Gamma),\\
&\mathcal{N}_c\colon W^{1,p}(\Omega) \times L^p(\Omega; \mathbb{R}^d) \to L^{p^{*'}}(\Omega).
\end{aligned}$$

These considerations lead to the following growth conditions on the nonlinearities a, b and c:

$$|a(x, r, s)| \le \gamma(x) + C\,|r|^{(p^*-\epsilon)/p'} + C\,|s|^{p-1} \quad \text{for some } \gamma \in L^{p'}(\Omega), \tag{3.7a}$$

$$|b(x, r)| \le \gamma(x) + C\,|r|^{p^\#-\epsilon-1} \quad \text{for some } \gamma \in L^{p^{\#'}}(\Gamma), \tag{3.7b}$$

$$|c(x, r, s)| \le \gamma(x) + C\,|r|^{p^*-\epsilon-1} + C\,|s|^{p/p^{*'}} \quad \text{for some } \gamma \in L^{p^{*'}}(\Omega). \tag{3.7c}$$

Convention 3.5. For $p > d$, the terms $|r|^\infty$ occurring in (3.7) are to be understood such that $|a(x, \cdot, s)|$, $|b(x, \cdot)|$ and $|c(x, \cdot, s)|$ may be growing arbitrarily fast as $|r| \to \infty$.

Furthermore let us suppose:

$$g \in L^{p^{*'}}(\Omega) \quad \text{and} \quad h \in L^{p^{\#'}}(\Gamma).$$

This ensures that $gz \in L^1(\Omega)$ and $(hz)|_\Gamma \in L^1(\Gamma)$ for all $z \in W^{1,p}(\Omega)$.

Finally we have to qualify u_D occurring in the Dirichlet boundary condition (3.1b). The simplest way is to assume that u_D is the trace of some $w \in W^{1,p}(\Omega)$:

$$u_D := w|_\Gamma \quad \text{on } \Gamma_D. \tag{3.8}$$

3.4. Variational analysis of problem (3.1)

In order to apply the theory in Chapter 2 to the boundary value problem (3.1), we have to set up an abstract operator equation, that corresponds to (3.1) or rather to its weak formulation (3.5). Let us therefore consider the

[3] According to Theorem 1.53.

Banach space $(X, \|\cdot\|)$, where $\|\cdot\| := \|\cdot\|_{W^{1,p}(\Omega)}$. We define the operator $A\colon W^{1,p}(\Omega) \to X'$ and the functional $f \in X'$ by

$$\langle A(u), z\rangle := \text{ left-hand side of (3.5)} \tag{3.9}$$

$$\langle f, z\rangle := \text{ right-hand side of (3.5).} \tag{3.10}$$

Lemma 3.6. *We have that*

(i) $f \in X'$, *and*

(ii) $A(u) \in X'$ *for all* $u \in W^{1,p}(\Omega)$.

Proof. (i). The linear functional f on X satisfies

$$\begin{aligned}
\|f\|_{X'} \le \|f\|_{W^{1,p}(\Omega)} &= \sup_{\substack{\|v\|\le 1,\\ v\in W^{1,p}(\Omega)}} \left| \int_\Omega gv\,dx + \int_{\Gamma_N} hv\,dS \right| \\
&\le \sup_{\substack{\|v\|\le 1,\\ v\in W^{1,p}(\Omega)}} \left(\|g\|_{L^{p^{*\prime}}(\Omega)} \|v\|_{L^{p^*}(\Omega)} + \|h\|_{L^{p^{\#\prime}}(\Gamma_N)} \|v\|_{L^{p^\#}(\Gamma_N)} \right) \\
&\le N_1 \|g\|_{L^{p^{*\prime}}(\Omega)} + N_2 \|h\|_{L^{p^{\#\prime}}(\Gamma_N)};
\end{aligned}$$

where N_1 is the norm of the embedding operator $W^{1,p}(\Omega) \to L^{p^*}(\Omega)$, and N_2 denotes the norm of the trace operator $W^{1,p}(\Omega) \to L^{p^\#}(\Gamma_N)$. Hence $f \in X'$.

(ii). Clearly, $A(u)$ defines a linear functional on X for any $u \in W^{1,p}(\Omega)$. We prove that $A\left(\left\{u \in W^{1,p}(\Omega)\colon \|u\| \le \rho\right\}\right)$ is contained in X' for any $\rho > 0$. The growth conditions (3.7) allow us to make the following estimates:

$$\begin{aligned}
&\sup_{\|u\|\le\rho} \|A(u)\|_{V'} \\
&\quad\le \sup_{\|u\|\le\rho} \|A(u)\|_{(W^{1,p}(\Omega))'} \\
&\quad= \sup_{\|u\|\le\rho} \sup_{\substack{\|v\|\le 1,\\ v\in W^{1,p}(\Omega)}} \left| \int_\Omega a(x,u,\nabla u)\cdot\nabla v + c(x,u,\nabla u) v\,dx + \int_{\Gamma_N} b(x,u) v\,dS \right| \\
&\quad\le \sup_{\|u\|\le\rho} \sup_{\substack{\|v\|\le 1,\\ v\in W^{1,p}(\Omega)}} \Bigg(\|a(x,u,\nabla u)\|_{L^{p'}(\Omega;\mathbb{R}^d)} \underbrace{\|\nabla v\|_{L^p(\Omega;\mathbb{R}^d)}}_{\le 1} \\
&\qquad\qquad + \|c(x,u,\nabla u)\|_{L^{p^{*\prime}}(\Omega)} \|v\|_{L^{p^*}(\Omega)} + \|b(x,u)\|_{L^{p^{\#\prime}}(\Gamma_N)} \|v\|_{L^{p^\#}(\Gamma_N)} \Bigg) \\
&\quad\le \sup_{\|u\|\le\rho} \|a(x,u,\nabla u)\|_{L^{p'}(\Omega;\mathbb{R}^d)} + N_1 \|c(x,u,\nabla u)\|_{L^{p^{*\prime}}(\Omega)} + N_2 \|b(x,u)\|_{L^{p^{\#\prime}}(\Gamma_N)} \\
&\quad< \infty.
\end{aligned}$$

As in the proof of assertion (i), N_1 is the norm of the embedding operator $W^{1,p}(\Omega) \to L^{p^*}(\Omega)$, and N_2 denotes the norm of the trace operator $W^{1,p}(\Omega) \to L^{p^\#}(\Gamma_N)$. This proves, that $A(u) \in X'$ for every $u \in W^{1,p}(\Omega)$. □

Remark 3.7. The above proof yields a stronger statement: $A(u)$ is uniformly bounded in X' for u ranging over a bounded set in $W^{1,p}(\Omega)$.

Next, we need to find a potential F of A. According to (2.4) we make the ansatz

$$\begin{aligned}
F(u) &= \int_0^1 \langle A(tu), u\rangle \, dt \\
&= \int_0^1 \left(\int_\Omega a(x, tu, \nabla tu) \cdot \nabla u + c(x, tu, \nabla tu)\, u \, dx + \int_{\Gamma_N} b(x, tu)\, u \, dS \right) dt \\
&= \int_\Omega \left(\int_0^1 a(x, tu, t\nabla u) \cdot \nabla u + c(x, tu, t\nabla u)\, u \, dt \right) dx \\
&\qquad\qquad + \int_{\Gamma_N} \left(\int_0^1 b(x, tu)\, u \, dt \right) dS,
\end{aligned} \tag{3.11}$$

where we employed Fubini's theorem, Theorem 1.35, in the last equality in (3.11). Note that $\int_0^1 |\langle A(tu), u\rangle| \, dt$ is finite for every $u \in W^{1,p}(\Omega)$ due to Remark 3.7.

This ansatz leads to the following definition of $F\colon W^{1,p}(\Omega) \to \mathbb{R}$:

$$F(u) := \int_\Omega \varphi(x, u, \nabla u)\, dx + \int_{\Gamma_N} \psi(x, u)\, dS, \quad \text{where} \tag{3.12a}$$

$$\varphi(x, r, s) := \int_0^1 s \cdot a(x, tr, ts) + r\, c(x, tr, ts)\, dt, \quad \text{and} \tag{3.12b}$$

$$\psi(x, r) := \int_0^1 r\, b(x, tr)\, dt. \tag{3.12c}$$

The strategy to find weak solutions to the mixed boundary value problem (3.1) reads as follows. We show that F is G-differentiable with derivative $F' = A$. This is done in Lemma 3.9. Next we verify that F is weakly sequentially lower semicontinuous, see Lemma 3.10. In Lemma 3.12, we will set up further restrictions on the nonlinearities a, b and c to guarantee that F is coercive. Finally we can apply the direct method of the calculus of variations, Theorem 2.37, to the functional F to find weak solutions for problem (3.1).

First we show that F is continuous.

Lemma 3.8. *F is continuous. This remains true with $\epsilon = 0$ in (3.7).*

Proof. We have that

- $u \mapsto \nabla u\colon W^{1,p}(\Omega) \to L^p(\Omega; \mathbb{R}^d)$ is continuous,
- the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is continuous, and
- the trace operator $W^{1,p}(\Omega) \to L^{p^\#}(\Gamma_N)$, $u \mapsto u|_{\Gamma_N}$ is continuous.

Thus it remains to prove that the Nemytskii mappings

$$\mathcal{N}_\varphi\colon L^{p^*}(\Omega) \times L^p(\Omega; \mathbb{R}^d) \to L^1(\Omega) \quad \text{and} \quad \mathcal{N}_\psi\colon L^{p^\#}(\Gamma_N) \to L^1(\Gamma_N)$$

are continuous in order deduce continuity of F.

We begin with $\mathcal{N}_\varphi$. In view of Theorem 1.53 our task is to find a function $\gamma \in L^1(\Omega)$ and a constant $C > 0$ such that

$$|\varphi(x, r, s)| \leq \gamma(x) + C|r|^{p^*} + C|s|^p. \tag{3.13}$$

The growth conditions (3.7a,c) on a and c weakened by putting $\epsilon := 0$ imply

$$\begin{aligned}
|\varphi(x, r, s)| &\leq \int_0^1 |s \cdot a(x, tr, ts)|\, dt + \int_0^1 |r\, c(x, tr, ts)|\, dt \\
&\leq \int_0^1 \Big(|s| \left(\gamma_a(x) + C_a|tr|^{p^*/p'} + C_a|ts|^{p-1}\right) \\
&\qquad\qquad + |r| \left(\gamma_c(x) + C_c|tr|^{p^*-1} + C_c|ts|^{p/p^{*\prime}}\right)\Big)\, dt \\
&= |s|\, \gamma_a(x) + |s|\, \frac{C_a|r|^{p^*/p'}}{p^*/p' + 1} + \frac{C_a|s|^p}{p} \\
&\qquad + |r|\, \gamma_c(x) + |r|\, \frac{C_c|s|^{p/p^{*\prime}}}{p/p^{*\prime} + 1} + \frac{C_c|r|^{p^*}}{p^*} \\
&\leq 2\frac{|s|^p}{p} + \frac{\gamma_a(x)^{p'}}{p'} + \frac{C_a{}^{p'}|r|^{p^*}}{p'(p^*/p' + 1)^{p'}} + \frac{C_a|s|^p}{p} \\
&\qquad + 2\frac{|r|^{p^*}}{p^*} + \frac{\gamma_c(x)^{p^{*\prime}}}{p^{*\prime}} + \frac{C_c{}^{p^{*\prime}}|s|^p}{p^{*\prime}(p/p^{*\prime} + 1)^{p^{*\prime}}} + \frac{C_c|r|^{p^*}}{p^*} \\
&\leq \frac{\gamma_a(x)^{p'}}{p'} + \frac{\gamma_c(x)^{p^{*\prime}}}{p^{*\prime}} + \frac{C_a{}^{p'}|r|^{p^*}}{p'(p^* + 1)} + \frac{C_c|r|^{p^*}}{p^*} + 2\frac{|r|^{p^*}}{p^*} \\
&\qquad + \frac{C_c{}^{p^{*\prime}}|s|^p}{p^{*\prime}(p + 1)} + 2\frac{|s|^p}{p} + \frac{C_a|s|^p}{p} \\
&=: \underbrace{\gamma(x)}_{\in L^1(\Omega)} + C|r|^{p^*} + \tilde{C}|s|^p,
\end{aligned}$$

where we employed Lemma 1.36 in the third estimate and the real version of Bernoulli's inequality in the last estimate. This verifies (3.13).

Finally the growth condition (3.7b) on b yields:

$$\begin{aligned}
|\psi(x, r)| &\leq \int_0^1 |r\, b(x, tr)|\, dt \leq \int_0^1 |r|\, (\gamma_b(x) + C_b|tr|^{p^\#-1})\, dt = |r|\, \gamma_b(x) + \frac{C_b|r|^{p^\#}}{p^\#} \\
&\leq \frac{\gamma_b(x)^{p^{\#\prime}}}{p^{\#\prime}} + \frac{|r|^{p^\#}(C_b + 1)}{p^\#} =: \underbrace{\gamma(x)}_{\in L^1(\Gamma_N)} + C|r|^{p^\#};
\end{aligned} \tag{3.14}$$

here we applied Lemma 1.36 once again. This shows that $\mathcal{N}_\psi$ is continuous. □

We will utilize the continuity of the Nemytskii operators $\mathcal{N}_\varphi$ and $\mathcal{N}_\psi$ to proof Lemma 3.10.

Lemma 3.9. *F is G-differentiable and $F' = A$. This remains true, if $\epsilon = 0$ in* (3.7).

Proof. We organize the proof in three steps.

Step 1. Let $\nabla_s := (\frac{\partial}{\partial s_1}, ..., \frac{\partial}{\partial s_d})$. We prove the identities

$$\nabla_s\varphi(x, r, s) = a(x, r, s), \quad \frac{\partial\varphi}{\partial r}(x, r, s) = c(x, r, s), \quad \frac{\partial\psi}{\partial r}(x, r) = b(x, r). \tag{3.15}$$

Let us start with $\nabla_s\varphi$. First, we claim that for a.e. fixed $x \in \Omega$, every fixed $r \in \mathbb{R}$ and every $i = 1, ..., d$,

$$\frac{\partial}{\partial s_i}\int_0^1 s \cdot a(x, tr, ts)\, dt = \int_0^1 \frac{\partial}{\partial s_i}\Big(s \cdot a(x, tr, ts)\Big)\, dt \tag{3.16}$$

and

$$\frac{\partial}{\partial s_i}\int_0^1 r\, c(x, tr, ts)\, dt = \int_0^1 \frac{\partial}{\partial s_i}\Big(r\, c(x, tr, ts)\Big)\, dt. \tag{3.17}$$

for a.e. $s \in \mathbb{R}^d$. We emphasize, that the partial derivatives appearing on the right-hand sides in (3.16) and (3.17) have to be understood in the distributional sense, because the functions $a(x, \cdot, \cdot)$ and $c(x, \cdot, \cdot)$ are only continuous. Hence we can not apply Theorem 1.31 directly in order to justify the change of differentiation and integration. On the other hand we realize that both sides in (3.16) and (3.17) make sense (a.e.) and that they are finite. In order to prove the respective equalities, we make use of a density argument. We will only demonstrate the validity of (3.16), since the proof of (3.17) runs the same way. Consider the map $\tilde{\phi}\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $(t, s) \mapsto s \cdot a(x, tr, ts)$. By assumption[4], $\tilde{\phi} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$, hence the restriction $\phi := \tilde{\phi}\big|_{(0,1)\times \mathrm{B}_\varepsilon(s_0)} \in W^{1,1}((0, 1) \times \mathrm{B}_\varepsilon(s_0))$ for any $\varepsilon > 0$ and for any $s_0 \in \mathbb{R}^d$. Theorem 1.60 (combined with Lemma 1.48) enables us to approximate[5] ϕ by a sequence of smooth functions $\phi_n \in C^\infty([0, 1] \times \overline{\mathrm{B}_\varepsilon(s_0)})$ such that for a.e. $s \in \mathrm{B}_\varepsilon(s_0)$

$$\frac{\partial}{\partial s_i}\int_0^1 \phi_n(t, s)\, dt \xrightarrow{n\to\infty} \frac{\partial}{\partial s_i}\int_0^1 s \cdot a(x, tr, ts)\, dt$$

and

$$\int_0^1 \frac{\partial\phi_n}{\partial s_i}(t, s)\, dt \xrightarrow{n\to\infty} \int_0^1 \frac{\partial}{\partial s_i}\Big(s \cdot a(x, tr, ts)\Big)\, dt.$$

Every ϕ_n satisfies the assumptions of Theorem 1.31:

(i) $\phi_n(\cdot, s) \in L^1((0, 1))$ for all $s \in \mathrm{B}_\varepsilon(s_0)$,

---

[4]C.f. Section 3.1.

[5]In $W^{1,1}((0, 1) \times \mathrm{B}_\varepsilon(s_0))$.

(ii) $\frac{\partial \phi_n}{\partial s_i}(t, s)$ exists for all $s \in \mathrm{B}_\varepsilon(s_0)$ and for all $t \in [0, 1]$,

(iii) there exists[6] an integrable function $g \in L^1((0, 1))$ such that

$$\left|\frac{\partial \phi_n}{\partial s_i}(t, s)\right| \le g(t)$$

for all $s \in \mathrm{B}_\varepsilon(s_0)$ and for all $t \in [0, 1]$.

Thus the functions $\Phi_n \colon \mathrm{B}_\varepsilon(s_0) \to \mathbb{R},\ s \mapsto \int_0^1 \phi_n(t, s)\, dt$ are differentiable at s_0, $\frac{\partial \phi_n}{\partial s_i}(\cdot, s_0)$ are integrable, and

$$\frac{\partial}{\partial s_i}\Phi_n(s_0) = \frac{\partial}{\partial s_i}\int_0^1 \phi_n(t, s_0)\, dt = \int_0^1 \frac{\partial \phi_n}{\partial s_i}(t, s_0)\, dt$$

for every $n \in \mathbb{N}$. Letting $n \to \infty$ and recalling that $s_0 \in \mathbb{R}^d$ was arbitrary implies that (3.16) holds a.e. in $\mathbb{R}^d$.

Analogously for a.e. fixed $x \in \Omega$ and for every fixed $s \in \mathbb{R}^d$,

$$\frac{\partial}{\partial r}\int_0^1 s \cdot a(x, tr, ts)\, dt = \int_0^1 \frac{\partial}{\partial r}\Big(s \cdot a(x, tr, ts)\Big)\, dt \tag{3.18}$$

and

$$\frac{\partial}{\partial r}\int_0^1 r\, c(x, tr, ts)\, dt = \int_0^1 \frac{\partial}{\partial r}\Big(r\, c(x, tr, ts)\Big)\, dt \tag{3.19}$$

hold a.e. in $\mathbb{R}$.

Now, thanks to (3.16) and (3.17),

$$\begin{aligned}
\frac{\partial \varphi(x, r, s)}{\partial s_i} &= \frac{\partial}{\partial s_i}\int_0^1 s \cdot a(x, tr, ts) + r\, c(x, tr, ts)\, dt \\
&= \int_0^1 \frac{\partial}{\partial s_i}\Big(s \cdot a(x, tr, ts) + r\, c(x, tr, ts)\Big)\, dt \\
&= \int_0^1 a_i(x, tr, ts) + t\left(\sum_{j=1}^d s_j \frac{\partial a_j}{\partial s_i}(x, tr, ts) + r\frac{\partial c}{\partial s_i}(x, tr, ts)\right) dt \\
&\overset{(3.3)}{=} \int_0^1 a_i(x, tr, ts) + t\left(\sum_{j=1}^d s_j \frac{\partial a_i}{\partial s_j}(x, tr, ts) + r\frac{\partial a_i}{\partial r}(x, tr, ts)\right) dt \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big(t\, a_i(x, tr, ts)\Big)\, dt = t\, a_i(x, tr, ts)\Big|_{t=0}^{t=1} = a_i(x, r, s),
\end{aligned} \tag{3.20}$$

which holds for a.e. $x \in \Omega$, for a.e. $r \in \mathbb{R}$ and for a.e. $s \in \mathbb{R}^d$. In the last line in (3.20) we applied the fundamental theorem of calculus, Theorem 1.34, in combination with Proposition 1.59.

[6] W.l.o.g., c.f. Lemma 1.48.

Similarly (3.18) and (3.19) imply that

$$
\begin{aligned}
\frac{\partial \varphi(x,r,s)}{\partial r} &= \int_0^1 \frac{\partial}{\partial r}\Big(s\cdot a(x,tr,ts) + r\,c(x,tr,ts)\Big)\,dt \\
&= \int_0^1 c(x,tr,ts) + t\left(r\frac{\partial c}{\partial r}(x,tr,ts) + \sum_{i=1}^d s_i \frac{\partial a_i}{\partial r}(x,tr,ts)\right)dt \\
&\overset{(3.3)}{=} \int_0^1 c(x,tr,ts) + t\left(r\frac{\partial c}{\partial r}(x,tr,ts) + \sum_{i=1}^d s_i \frac{\partial c}{\partial s_i}(x,tr,ts)\right)dt \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big(t\,c(x,tr,ts)\Big)\,dt = c(x,r,s);
\end{aligned}
\tag{3.21}
$$

again for a.e. $x \in \Omega$, a.e. $r \in \mathbb{R}$ and a.e. $s \in \mathbb{R}^d$.

The fact, that

$$
\frac{\partial \psi(x,r)}{\partial r} = b(x,r) \tag{3.22}
$$

a.e. in Ω follows directly from the fundamental theorem of calculus, Theorem 1.33, via the following change of variables:

$$
\int_0^r b(x,\tau)\,d\tau = \int_0^1 r\,b(x,tr)\,dt = \psi(x,r).
$$

Note moreover, that by (3.20), (3.21) and (3.22), the partial derivatives

$$
\frac{\partial \varphi(x,r,s)}{\partial s_i}, \quad \frac{\partial \varphi(x,r,s)}{\partial r} \quad \text{and} \quad \frac{\partial \psi(x,r)}{\partial r}
$$

can be interpreted as classical partial derivatives for a.e. $x \in \Omega$, because a, b and c are Carathéodory functions.

Step 2. Next, we show that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\int_\Omega \varphi(x,u+\varepsilon v,\nabla u+\varepsilon\nabla v)\,dx + \int_{\Gamma_N}\psi(x,u+\varepsilon v)\,dS\right)\Bigg|_{\varepsilon=0} \\
&\quad = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varphi(x,u+\varepsilon v,\nabla u+\varepsilon\nabla v)\Big|_{\varepsilon=0}\,dx + \int_{\Gamma_N}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\psi(x,u+\varepsilon v)\Big|_{\varepsilon=0}\,dS
\end{aligned}
\tag{3.23}
$$

holds for every $u \in W^{1,p}(\Omega)$ and every $v \in C^\infty(\overline{\Omega})$.

To do so, we want to apply Theorem 1.31 once more. We find that the right-hand side of (3.23) equals

$$\begin{aligned}
&\int_\Omega \underbrace{\sum_{i=1}^d \frac{\partial\varphi(x,u,\nabla u)}{\partial s_i}\frac{\partial v}{\partial x_i}}_{\nabla_s\varphi(x,u,\nabla u)\cdot\nabla v} + \frac{\partial\varphi(x,u,\nabla u)}{\partial r} v\,dx + \int_{\Gamma_N} \frac{\partial\psi(x,u)}{\partial r} v\,dS \\
&\qquad = \int_\Omega a(x,u,\nabla u)\cdot\nabla v + c(x,u,\nabla u)v\,dx + \int_{\Gamma_N} b(x,u)v\,dS.
\end{aligned} \tag{3.24}$$

We employed the results of step 1 in (3.24). Thus for arbitrary $\varepsilon_0 > 0$:

(i) the mappings $\Omega \to \mathbb{R}$, $(x,\varepsilon) \mapsto \varphi(x, u+\varepsilon v, \nabla u+\varepsilon\nabla v)$ and $\Gamma_N \to \mathbb{R}$, $(x,\varepsilon) \mapsto \psi(x, u+\varepsilon v)$ are integrable[7] for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

(ii) the derivatives $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varphi(x, u+\varepsilon v, \nabla u+\varepsilon\nabla v)$ and $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\psi(x, u+\varepsilon v)$ exist in the classical sense for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and for a.e. $x \in \Omega$,

(iii) the families

$$\Big\{\nabla_s\varphi(\cdot, u+\varepsilon v, \nabla u+\varepsilon\nabla v)\cdot\nabla v\Big\}_{0\le|\varepsilon|\le\varepsilon_0}$$

$$\Big\{\frac{\partial\varphi}{\partial r}(\cdot, u+\varepsilon v, \nabla u+\varepsilon\nabla v)v\Big\}_{0\le|\varepsilon|\le\varepsilon_0}$$

$$\Big\{\frac{\partial\psi}{\partial r}(\cdot, u+\varepsilon v)v\Big\}_{0\le|\varepsilon|\le\varepsilon_0}$$

posses a common integrable majorant.

It remains to verify property (iii). By exploiting (3.7a) with $\epsilon = 0$ we find that

$$\begin{aligned}
&|\nabla_s\varphi(x,u+\varepsilon v, \nabla u+\varepsilon\nabla v)\cdot\nabla v| \le \Big(\gamma(x) + C|u+\varepsilon v|^{p^*/p'} + C|\nabla u+\varepsilon\nabla v|^{p-1}\Big)|\nabla v| \\
&\le \Big(\gamma(x) + C_p(|u|^{p^*/p'} + \varepsilon_0^{p^*/p'}|v|^{p^*/p'}) + C_p(|\nabla u|^{p-1} + \varepsilon_0^{p-1}|\nabla v|^{p-1})\Big)|\nabla v| \in L^1(\Omega),
\end{aligned}$$

which holds for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and a suitable constant $C_p > 0$, depending only on p and C from (3.7a). Analogously, employing (3.7b,c) we get

$$\left|\frac{\partial\psi}{\partial r}(x, u+\varepsilon v)v\right| \le \Big(\gamma(x) + C_p(|u|^{p^\#-1} + \varepsilon_0^{p^\#-1}|v|^{p^\#-1})\Big)|v| \in L^1(\Gamma_N),$$

and

$$\begin{aligned}
\left|\frac{\partial\varphi}{\partial r}(x, u+\varepsilon v, \nabla u+\varepsilon\nabla v)v\right| &\le \Big(\gamma(x) + C_p(|u|^{p^*-1} + \varepsilon_0^{p^*-1}|v|^{p^*-1}) \\
&\quad + C_p(|\nabla u|^{p/p^{*\prime}} + \varepsilon_0^{p/p^{*\prime}}|\nabla v|^{p/p^{*\prime}})\Big)|v| \in L^1(\Omega).
\end{aligned}$$

Theorem 1.31 yields that (3.23) holds whenever $u \in W^{1,p}(\Omega)$ and $v \in C^\infty(\overline{\Omega})$.

Step 3. Finally we show that F is G-differentiable with derivative $F' = A$.

[7] C.f. Remark 3.7.

Therefore let u and v be arbitrary elements in $W^{1,p}(\Omega)$ and observe that (3.23) still holds true by the density of $C^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega)$ (Theorem 1.60). Therefore,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{F(u+\varepsilon v)-F(u)}{\varepsilon}\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}F(u+\varepsilon v)\Big|_{\varepsilon=0}\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega \varphi(x,u+\varepsilon v,\nabla u+\varepsilon\nabla v)\,dx+\int_{\Gamma_N}\psi(x,u+\varepsilon v)\,dS\Big|_{\varepsilon=0}\\
&\quad\overset{(3.23)}{=} \int_\Omega \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varphi(x,u+\varepsilon v,\nabla u+\varepsilon\nabla v)\Big|_{\varepsilon=0}\,dx+\int_{\Gamma_N}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\psi(x,u+\varepsilon v)\Big|_{\varepsilon=0}\,dS\\
&\quad= \int_\Omega \sum_{i=1}^d \frac{\partial\varphi(x,u,\nabla u)}{\partial s_i}\frac{\partial v}{\partial x_i}+\frac{\partial\varphi(x,u,\nabla u)}{\partial r}v\,dx+\int_{\Gamma_N}\frac{\partial\psi(x,u)}{\partial r}v\,dS\\
&\quad= \int_\Omega a(x,u,\nabla u)\cdot\nabla v+c(x,u,\nabla u)v\,dx+\int_{\Gamma_N}b(x,u)v\,dS\\
&\quad= \langle A(u),v\rangle.
\end{aligned}$$

This shows that $DF(u,v)$ exists for all $u,v\in W^{1,p}(\Omega)$ and equals $\langle A(u),v\rangle$. Hence, F is G-differentiable by Lemma 3.6 and A is a potential for F. □

Lemma 3.10. *F is weakly sequentially lower semicontinuous if one of the following two additional requirements are satisfied:*

(i) *the growth conditions* (3.7) *hold for $\epsilon>0$ and $a(x,r,\cdot)\colon\mathbb{R}^d\to\mathbb{R}^d$ is monotone for almost all $x\in\Omega$ and for all $r\in\mathbb{R}$.*
(ii) *the growth conditions* (3.7) *hold with $\epsilon=0$ and the mappings* (3.4) *and*
$b(x,\cdot)\colon\mathbb{R}\to\mathbb{R}$ are monotone for almost all $x\in\Omega$.

Proof. (i): By (3.15), the monotonicity of $a(x,r,\cdot)$ is just the monotonicity of $\nabla_s\varphi(x,r,\cdot)\colon\mathbb{R}^d\to\mathbb{R}^d$. Therefore $\varphi(x,r,\cdot)\colon\mathbb{R}^d\to\mathbb{R}$, the potential of $\nabla_s\varphi(x,r,\cdot)$, is convex for all $r\in\mathbb{R}$ and for almost all $x\in\Omega$; c.f. Remark 2.32. Since we assume that $\epsilon>0$, φ satisfies the growth condition (3.13) with $p^*-\epsilon$ instead of p^*. Let $u\in W^{1,p}(\Omega)$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W^{1,p}(\Omega)$ with $u_k\rightharpoonup u$ for $k\to\infty$. By the compact embedding $W^{1,p}(\Omega)\Subset L^{p^*-\epsilon}(\Omega)$ (see Theorem 1.63) we have that $u_k\to u$ in $L^{p^*-\epsilon}(\Omega)$. Hence the continuity of the Nemytskii mapping $\mathcal{N}_\varphi\colon L^{p^*-\epsilon}(\Omega)\times L^p(\Omega;\mathbb{R}^d)\to L^1(\Omega)$ implies: $\mathcal{N}_\varphi(u_k,\nabla u)\to\mathcal{N}_\varphi(u,\nabla u)$ in $L^1(\Omega)$ for $k\to\infty$. Analogously the continuity of the Nemytskii mapping $\mathcal{N}_a\colon L^{p^*-\epsilon}(\Omega)\times L^p(\Omega)\to L^{p'}(\Omega)$, which is a direct consequence of the growth condition (3.7a) and Theorem 1.53, yields that $\mathcal{N}_a(u_k,\nabla u)\to\mathcal{N}_a(u,\nabla u)$ in $L^{p'}(\Omega)$ as $k\to\infty$. Due to the convexity of $\varphi(x, r, \cdot)$ and basic properties the integral we have that:

$$\begin{aligned}\liminf_{k\to\infty} \int_\Omega \varphi(x, u_k, \nabla u_k)\, dx &\geq \lim_{k\to\infty} \int_\Omega \varphi(x, u_k, \nabla u)\, dx \\ &\quad + \lim_{k\to\infty} \int_\Omega \nabla_s \varphi(x, u_k, \nabla u) \cdot (\nabla u_k - \nabla u)\, dx \\ &= \int_\Omega \varphi(x, u, \nabla u)\, dx,\end{aligned} \tag{3.25}$$

because the second limit on the right-hand side in (3.25) is zero due to the continuity of $\mathcal{N}_a = \mathcal{N}_{\nabla_s \varphi}$ and Proposition 1.27.(iii). This proves that the functional $F_1 \colon W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$F_1(u) := \int_\Omega \varphi(x, u, \nabla u)\, dx$$

is weakly sequentially lower semicontinuous. Furthermore, $F_2 \colon W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$F_2(u) := \int_{\Gamma_N} \psi(x, u)\, dS$$

is weakly continuous: the trace operator $W^{1,p}(\Omega) \to L^{p^\# - \epsilon}(\Gamma)$ is compact (see Theorem 1.67) and the Nemytskii mapping $\mathcal{N}_\psi \colon L^{p^\# - \epsilon}(\Gamma_N) \to L^1(\Gamma_N)$ is continuous by (3.14) with $p^\# - \epsilon$ replaced by $p^\#$. Thus $F_2(u_k) \to F_2(u)$ as $k \to \infty$. Due to Proposition 2.9, we finally get that $F = F_1 + F_2$ is weakly sequentially lower semicontinuous.

(ii): In the proof of Lemma 3.9 we have seen, that $\varphi(x, \cdot, \cdot) \colon \mathbb{R}^{1+d} \to \mathbb{R}$ is a potential for $(c, a)(x, \cdot, \cdot) \colon \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$, and that $\psi(x, \cdot) \colon \mathbb{R} \to \mathbb{R}$ is a potential for $b(x, \cdot) \colon \mathbb{R} \to \mathbb{R}$; in both cases for almost all $x \in \Omega$. From the monotonicity of $(c, a)(x, \cdot, \cdot)$ and $b(x, \cdot)$ it follows that both $\varphi(x, \cdot, \cdot)$ and $\psi(x, \cdot)$ are convex functionals for almost all $x \in \Omega$. Basic properties of the integral now yield that F is convex on $W^{1,p}(\Omega)$. Since F is differentiable too, Corollary 2.33 tells us, that F is weakly sequentially lower semicontinuous. Alternatively one can apply Theorem 2.19 combined with Lemma 3.8 to deduce sequential lower semicontinuity of F. □

Corollary 3.11. *Let the mappings* (3.4) *and* $b(x, \cdot) \colon \mathbb{R} \to \mathbb{R}$ *be strictly monotone for almost all* $x \in \Omega$. *Then* F *is strictly convex and weakly sequentially lower semicontinuous.*

Lemma 3.12. *Let us assume that there exist* $\varepsilon_0, \varepsilon_1 > 0$, $p \geq q > 1$ *and* $k_0 \in L^{p'}(\Omega)$, $k_1 \in L^{p^{*\prime}}(\Omega)$ *and* $k_2 \in L^{p^{\#\prime}}(\Gamma)$ *such that*

$$\begin{aligned}&a(x, r, s) \cdot s + c(x, r, s) r \geq \varepsilon_1 |s|^p + \varepsilon_2 |r|^q - k_0(x)|s| - k_1(x)|r| \\ &b(x, r) r \geq -k_2(x)|r|.\end{aligned}$$

Then

(i) *F is coercive on $W^{1,p}(\Omega)$.*

(ii) *F is coercive on X*[8] *even if $\varepsilon_2 = 0$.*

Proof. (i). We have

$$\begin{aligned}\varphi(x,r,s) &= \int_0^1 s\cdot a(x,tr,ts) + r\,c(x,tr,ts)\,dt\\ &= \int_0^1 \frac{ts\cdot a(x,tr,ts) + tr\,c(x,tr,ts)}{t}\,dt\\ &\geq \int_0^1 \frac{\varepsilon_1|ts|^p + \varepsilon_2|tr|^q - k_0|ts| - k_1|tr|}{t}\,dt\\ &= \frac{\varepsilon_1}{p}|s|^p + \frac{\varepsilon_2}{q}|r|^q - k_0|s| - k_1|r|,\end{aligned}$$

and

$$\psi(x,r) = \int_0^1 r\,b(x,tr)\,dt = \int_0^1 \frac{tr\,b(x,tr)}{t}\,dt \geq -\int_0^1 \frac{k_2|tr|}{t}\,dt = -k_2|r|.$$

Hence for $u \in W^{1,p}(\Omega)$ we have that

$$\begin{aligned}F(u) &\geq \int_\Omega \frac{\varepsilon_1}{p}|\nabla u|^p + \frac{\varepsilon_2}{q}|u|^q - k_0|\nabla u| - k_1|u|\,dx - \int_{\Gamma_N} k_2|u|\,dS\\ &\geq \underbrace{\frac{\varepsilon_1}{p}\|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^d)} + \frac{\varepsilon_2}{q}\|u\|^q_{L^q(\Omega)}}_{(\triangle)}\\ &\quad - \Big(\|k_0\|_{L^{p'}(\Omega)}\,\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)}\\ &\qquad + \|k_1\|_{L^{p^{*\prime}}(\Omega)}\,\|u\|_{L^{p^*}(\Omega)} + \|k_2\|_{L^{p^{\#\prime}}(\Gamma_N}\,\|u\|_{L^{p^\#}(\Gamma_N)}\Big).\end{aligned} \tag{3.26}$$

We make a few further estimates on $(\triangle)$:

$$\begin{aligned}(\triangle) &\geq \tilde{\varepsilon}\Big(\|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^d)} + \|u\|^q_{L^q(\Omega)}\Big) \geq \tilde{\varepsilon}\Big(\|\nabla u\|^q_{L^p(\Omega;\mathbb{R}^d)} + \|u\|^q_{L^q(\Omega)} - C_1\Big)\\ &\overset{[1.52]}{\geq} \tilde{\varepsilon}\Big(C_2\big(\underbrace{\|\nabla u\|_{L^p(\Omega;\mathbb{R}^d)} + \|u\|_{L^q(\Omega)}}_{\geq N^{-1}\|u\|_{W^{1,p}(\Omega)}}\big)^q - C_1\Big) \geq \varepsilon\,\|u\|^q_{W^{1,p}(\Omega)} - C,\end{aligned}$$

where we employed the constants $\tilde{\varepsilon} \leq \min(\frac{\varepsilon_1}{p}, \frac{\varepsilon_2}{q})$, $C_1 = C_1(p,q) \in (0,1)$, $C_2 = C_2(q) > 0$ and N from inequality (1.12).

By inserting the above estimation of $(\triangle)$ in (3.26) we arrive at

$$\begin{aligned}F(u) &\geq \varepsilon\,\|u\|^q_{W^{1,p}(\Omega)} - C\\ &\quad - \Big(\|k_0\|_{L^{p'}(\Omega)} + N_1\,\|k_1\|_{L^{p^{*\prime}}(\Omega)} + N_2\,\|k_2\|_{L^{p^{\#\prime}}(\Gamma_N}\Big)\|u\|_{W^{1,p}(\Omega)},\end{aligned} \tag{3.27}$$

[8]It is implicitly assumed that $\mathrm{Vol}_{d-1}(\Gamma_D) > 0$, because otherwise $X = W^{1,p}(\Omega)$ and (i) applies.

where N_1 stands for the norm of the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and N_2 denotes the norm of the trace operator $W^{1,p}(\Omega) \to L^{p^\#}(\Gamma)$. Dividing (3.27) by $\|u\|_{W^{1,p}(\Omega)}$ yields that F is coercive on $W^{1,p}(\Omega)$, since we assumed that $q > 1$.

(ii). Let finally $u \in X$ and let $\varepsilon_2 = 0$. Theorem 1.71 yields that

$$\|u\|^p_{W^{1,p}(\Omega)} \leq C\|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^d)}, \tag{3.28}$$

where $C = N^p + 1$ with N from (1.11). It follows that

$$\begin{aligned}
F(u) &\geq \int_\Omega \frac{\varepsilon_1}{p}|\nabla u|^p - k_0|\nabla u| - k_1|u|\,dx - \int_{\Gamma_N} k_2|u|\,dS \\
&\geq \frac{\varepsilon_1}{p}\|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^d)} \\
&\qquad - \left(\|k_0\|_{L^{p'}(\Omega)} + N_1\,\|k_1\|_{L^{p^{*\prime}}(\Omega)} + N_2\,\|k_2\|_{L^{p^{\#\prime}}(\Gamma_N)}\right)\|u\|_{W^{1,p}(\Omega)} \\
&\geq \varepsilon\,\|u\|^p_{W^{1,p}(\Omega)} - \left(\|k_0\|_{L^{p'}(\Omega)} + N_1\,\|k_1\|_{L^{p^{*\prime}}(\Omega)} + N_2\,\|k_2\|_{L^{p^{\#\prime}}(\Gamma_N)}\right)\|u\|_{W^{1,p}(\Omega)}
\end{aligned}$$

for all $u \in X$, where $\varepsilon = \varepsilon_1(Cp)^{-1}$ with C from (3.28), and the constants N_1, N_2 taken from the first part of the proof. Thus F is coercive on X, because $p > 1$. □

Now we are almost ready to prove the existence result Theorem 3.15. Let us assume for the moment that the given Dirichlet boundary condition is trivial, i.e. $u_D = 0$ in (3.1b). Then every function $u \in X$ automatically satisfies (3.1b). Since $F' = A$ by Lemma 3.9, every minimizer

$$u_0 = \min_{u \in X} F(u) - \langle f, u\rangle$$

solves the equation $A(u) = f$ (see Proposition 2.24) and hence is a weak solution for (3.1).

If the Dirichlet boundary condition is nonzero, say $u_D = w|_\Gamma \neq 0$ on Γ_D, $w \in W^{1,p}(\Omega)$, we arrange a *boundary shift*. This is done as follows. For w satisfying (3.8), we define the operator $A_0\colon X \to X'$ by

$$A_0(u) := A(u + w).$$

Then $\langle A_0(u), z\rangle$ equals the left-hand side of (3.5) with a_0, b_0, c_0 given by:

$$\begin{aligned}
a_0(x, r, s) &:= a(x, r + w(x), s + \nabla w(x)) \\
b_0(x, r) &:= b(x, r + w(x)) \\
c_0(x, r, s) &:= c(x, r + w(x), s + \nabla w(x)).
\end{aligned}$$

For $u_D = 0$ we can choose $w = 0$ in (3.8), then $A_0 = A|_X$. If $\mathrm{Vol}_{d-1}(\Gamma_D) = 0$ we simply get $A_0 = A$.

Remark 3.13. Due to Lemma 3.6, A_0 is well-defined as an operator $X \to X'$.

Proposition 3.14. *An element $u_0 \in X$ solves the abstract equation $A_0(u) = f$ if and only if $u := u_0 + w \in W^{1,p}(\Omega)$ is a weak solution to the boundary value problem* (3.1)*.*

Proof. The assertion follows directly from the respective definitions via

$$A_0(u_0) = A_0(u - w) = A(u - w + w) = A(u).$$

□

Theorem 3.15. *Additionally to all assumptions set up in Sections 1-3, suppose that the requirements of Lemma 3.12 hold.*

(i) *If in addition at least one of the two conditions of Lemma 3.10 is satisfied, then the mixed boundary value problem* (3.1) *has a weak solution.*
(ii) *If in addition the assumptions of Corollary 3.11 hold, then the mixed boundary value problem* (3.1) *has a unique weak solution.*

Proof. (i). We have to find an element $u_0 \in W^{1,p}(\Omega)$ with $u_0|_{\Gamma_D} = u_D$ such that $A(u_0) = f$ with A and f defined by (3.9) and (3.10) respectively. Consider the functional $F_0 \colon X \to \mathbb{R}$, $F_0(u) := F(u + w)$ with F given by (3.12), and choose $w \in W^{1,p}(\Omega)$ such that $w|_{\Gamma_D} = u_D$. Due to Lemma 3.9 F_0 is G-differentiable on X with G-derivative $A_0 := F_0{}'$. The operator $A_0 \colon X \to X'$ satisfies $A_0(u) = F_0{}'(u) = F'(u+w) = A(u+w)$ with $A := F'$. By Lemma 3.10 and Lemma 3.12, F is weakly sequentially lower semicontinuous and coercive. Clearly, both properties are inherited by F_0. Since X is a reflexive Banach space (see Remark 3.4), we may apply Theorem 2.37 (i), which provides us with a solution $\tilde{u}_0 \in X$ for $A_0(u) = f$. Proposition 3.14 tells us that

$$u_0 := \tilde{u}_0 + w \in X + w := \left\{u \in W^{1,p}(\Omega)\colon\ u|_{\Gamma_D} = u_D\right\}$$

solves (3.1).

(ii). The existence of a solution $u_0 \in X + w$ for problem (3.1) follows by (i), and due to Corollary 3.11 and Theorem 2.37 (ii) there is no other solution. □

CHAPTER 4

Weak Lower Semicontinuity of Integral Functionals

Orientation

In Chapter 2 we have seen, that the weak sequential lower semicontinuity of a functional $F\colon X \to \mathbb{R}$ on a Banach space X is a key property to apply the direct method of the calculus of variations in order to find a minimum $u_0 = \min_{u \in X} F(u)$; see Theorem 2.13 and Theorem 2.37.

This chapter is devoted to the study of weak sequential lower semicontinuity of integral functionals having the form

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx, \tag{4.1}$$

and acting on the Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$. We can easily deduce that F is weakly sequentially lower semicontinuous, if the following conditions are satisfied:

(i) the integrand f is a Carathéodory function,
(ii) the Nemytskii operator $\mathcal{N}_f\colon L^{p^*}(\Omega) \times L^p(\Omega; \mathbb{R}^d) \to L^1(\Omega)$ is continuous,
(iii) the mapping $(r, s) \mapsto f(x, r, s)$, $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is convex for a.e. $x \in \Omega$.

Because in this case, F is continuous and convex and therefore weakly sequentially lower semicontinuous by Theorem 2.19. Let us emphasize that those three conditions are *not* necessary for F being weakly sequentially lower semicontinuous. We have already discovered in Chapter 3, that *"convexity in the main part"*, that is convexity of $s \mapsto f(x, r, s)$ for a.e. $x \in \Omega$ and for all $r \in \mathbb{R}$, can be sufficient in particular situations, where the lack of convexity of f in the variable r is compensated by compactness properties; see Lemma 3.10(i).

We will study the functional F in (4.1) in a more general framework and find out, roughly speaking, that the convexity of the integrand f in the main part $s = \nabla u(x)$ is sufficient *and* necessary that F is weakly sequentially lower semicontinuous; see Corollary 4.8 and Theorem 4.12.

This chapter is widely based on [**5**], only the proof of Theorem 4.11 is taken from [**2**].

4.1. Preliminairy remarks

In Chapter 3 we were able to prove weak lower semicontinuity of F (see Lemma 3.10) without employing a particular representation of the duals of the Sobolev spaces $W^{1,p}(\Omega)$, $1 < p < \infty$. The result was obtained by means of abstract functional analytic considerations. In Section 4.2 we will deduce $W^{1,p}$-weak sequential lower semicontinuity from weak lower semicontinuity of suitable Lebesgue spaces. The theorems in Section 4.3 are formulated in terms of weak* convergence in the Sobolev space $W^{1,\infty}(\Omega)$. Since we do not discuss the duality theory for Sobolev spaces, we give the following alternative definitions of weak and weak* convergence of sequences in Sobolev spaces.

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^d$ be open and let $1 \leq p < \infty$. We say that a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ converges weakly to $u \in W^{1,p}(\Omega)$ and write $u_n \rightharpoonup u$, if $u_n \rightharpoonup u$ in $L^p(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega;\mathbb{R}^d)$.

Definition 4.2. Let $\Omega \subseteq \mathbb{R}^d$ be open. We say that a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,\infty}(\Omega)$ weak* converges to $u \in W^{1,\infty}(\Omega)$ and write $u_n \overset{*}{\rightharpoonup} u$, if $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ and $\nabla u_n \overset{*}{\rightharpoonup} \nabla u$ in $L^\infty(\Omega;\mathbb{R}^d)$.

Remark 4.3. Definition 4.1 is compatible with Definition 1.7 due to Theorem 3.8 in [**1**]. If we identify the predual of $W^{1,\infty}(\Omega)$ with the space $W^{-1,1}(\Omega)$, we obtain compatibility of Definitions 4.2 and 1.8.

Remark 4.4. Assume that $\Omega \subseteq \mathbb{R}^d$ is a bounded open set. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,\infty}(\Omega)$ and let $u \in W^{1,\infty}(\Omega)$. We easily find that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(\Omega) \quad \Rightarrow \quad u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega)$$

for all $1 \leq p < \infty$. Hence a weakly sequentially lower semicontinuous functional $F\colon W^{1,p}(\Omega) \to \mathbb{R}$ is automatically weak* sequentially lower semicontinuous on $W^{1,\infty}(\Omega)$, if Ω is bounded.

4.2. Convexity in the main part: a sufficient condition

We begin with a particular case, which will be used in the proof of the general result, Theorem 4.7.

Theorem 4.5. *Let Ω be an open subset of $\mathbb{R}^d$ and let $1 \leq q < \infty$. Assume $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ to be a Carathéodory function satisfying*

$$f(x,s) \geq a(x) \cdot s + b(x) \tag{4.2}$$

for a.e. $x \in \Omega$ *and for all* $s \in \mathbb{R}^N$ *for some* $a \in L^{q'}(\Omega; \mathbb{R}^N)$ *and a function* $b \in L^1(\Omega)$. *Let* $F\colon L^q(\Omega; \mathbb{R}^N) \to \mathbb{R}$ *be given by*

$$F(u) := \int_\Omega f(x, u(x))\, dx.$$

If $s \mapsto f(x, s)$ *is convex for a.e.* $x \in \Omega$*, then* F *is weakly sequentially lower semicontinuous on* $L^q(\Omega; \mathbb{R}^N)$*.*

Proof. Let $u \in L^q(\Omega; \mathbb{R}^N)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^q(\Omega; \mathbb{R}^N)$ such that

$$u_n \rightharpoonup u.$$

We have to show that

$$F(u) \le \liminf_{n\to\infty} F(u_n).$$

We divide the proof in three steps.

Step 1. We demonstrate, that we can without loss of generality assume that $f \ge 0$. Therefore let

$$h(x, s) := a(x) \cdot s + b(x).$$

Then

$$\lim_{n\to\infty} \int_\Omega h(x, u_n(x))\, dx = \int_\Omega h(x, u(x))\, dx,$$

by the definition of weak convergence. Hence weak sequential lower semicontinuity of F is equivalent to the weak sequential lower semicontinuity of G given by

$$G(u) := \int_\Omega g(x, u(x))\, dx,$$

where $g(x, s) := f(x, s) - h(x, s)$ which is nonnegative by (4.2).

Step 2. We show that F is lower semicontinuous. Let $w \in L^q(\Omega; \mathbb{R}^N)$ and let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $L^q(\Omega; \mathbb{R}^N)$ converging strongly to w:

$$w_n \to w.$$

Now let us take some arbitrary subsequence from $(w_n)_{n\in\mathbb{N}}$, which we do not relabel. From this subsequence, we can extract a subsequence, still labeled $(w_n)_{n\in\mathbb{N}}$ such that by Lemma 1.48

$$w_n \to w \quad \text{a.e. in } \Omega;$$

this is due to Lemma 1.48. Since $f \ge 0$, we can apply Fatou's lemma, Theorem 1.29 to find that

$$\liminf_{n\to\infty} \int_\Omega f(x, w_n(x))\, dx \ge \int_\Omega \liminf_{n\to\infty} f(x, w_n(x))\, dx. \tag{4.3}$$

The Carathéodory condition on f implies

$$\int_\Omega \liminf_{n\to\infty} f(x, w_n(x))\,dx = \int_\Omega \lim_{n\to\infty} f(x, w_n(x))\,dx = \int_\Omega f(x, w(x))\,dx, \tag{4.4}$$

so (4.3) and (4.4) yield lower semicontinuity of F.

Step 3. We have to pass from lower semicontinuity to weak sequential lower semicontinuity. Since by assumption $s \mapsto f(x, s)$ is convex for a.e. $x \in \Omega$, F is convex on $L^q(\Omega); \mathbb{R}^N)$: let $u_1, u_2 \in L^q(\Omega); \mathbb{R}^N)$ and let $t \in [0, 1]$, then

$$\begin{aligned} F(tu_1 + (1-t)u_2) &= \int_\Omega f(x, tu_1(x) + (t-1)u_2(x))\,dx \\ &\le \int_\Omega tf(x, u_1(x)) + (t-1)f(x, u_2(x))\,dx \\ &= tF(u_1) + (1-t)F(u_2). \end{aligned}$$

Thus F is weakly sequentially lower semicontinuous by Theorem 2.19. □

The following corollary is a direct consequence of Theorem 4.5. Note that it holds generally for vector-valued functions $u \in W^{1,p}(\Omega; \mathbb{R}^m) \cong (W^{1,p}(\Omega))^m$.

Corollary 4.6. *Let $1 \le p < \infty$, let $\Omega \subseteq \mathbb{R}^d$ be open and let $f\colon \Omega \times \mathbb{R}^{md} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$f(x, s) \ge a(x) \cdot s + b(x)$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}^{md}$ for some $a \in L^{p'}(\Omega; \mathbb{R}^{md})$ and a function $b \in L^1(\Omega)$. Let $F\colon W^{1,p}(\Omega; \mathbb{R}^m) \to \mathbb{R}$ be defined as

$$F(u) := \int_\Omega f(x, \nabla u(x))\,dx.$$

If $s \mapsto f(x, s)$ is convex for a.e. $x \in \Omega$, then F is weakly sequentially lower semicontinuous.

Theorem 4.7. *Let Ω be an open subset of $\mathbb{R}^d$ and $1 \le p, q < \infty$. Let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function satisfying*

$$f(x, r, s) \ge a(x) \cdot s + b(x) + c\,|u|^p$$

for a.e. $x \in \Omega$, for every $(r, s) \in \mathbb{R}^m \times \mathbb{R}^N$, for a function $a \in L^{q'}(\Omega; \mathbb{R}^N)$ and some $b \in L^1(\Omega)$, $c \in \mathbb{R}$. Let $F\colon L^p(\Omega; \mathbb{R}^m) \times L^q(\Omega; \mathbb{R}^N) \to \mathbb{R}$ be a functional given by

$$F(u, v) := \int_\Omega f(x, u(x), v(x))\,dx$$

Furthermore let $u \in L^p(\Omega; \mathbb{R}^m)$ and $v \in L^q(\Omega; \mathbb{R}^N)$ be arbitrary points and let some sequence $(u_n)_{n\in\mathbb{N}}$ in $L^p(\Omega; \mathbb{R}^m)$ converge strongly to u and some $(v_n)_{n\in\mathbb{N}}$ in $L^q(\Omega; \mathbb{R}^N)$ converge weakly to v:

$$u_n \to u \quad \text{and} \quad v_n \rightharpoonup v.$$

If $s \mapsto f(x, r, s)$ is convex for a.e. $x \in \Omega$ and for all $r \in \mathbb{R}^m$, then

$$F(u, v) \leq \liminf_{n\to\infty} F(u_n, v_n).$$

In other words, F is sequentially lower semicontinuous on the product space $L^p(\Omega; \mathbb{R}^m) \times L^q(\Omega; \mathbb{R}^N)$ endowed with the original topology on $L^p(\Omega; \mathbb{R}^m)$ and the weak topology on $L^q(\Omega; \mathbb{R}^N)$.

Proof. Let $u_n \to u$ in $L^p(\Omega; \mathbb{R}^m)$ and $v_n \rightharpoonup v$ in $L^q(\Omega; \mathbb{R}^N)$.

Step 1. Without loss of generality we may assume that $f \geq 0$. Otherwise we can work with the functional

$$G(u, v) := \int_\Omega g(x, u, v)\, dx$$

instead, where $g(x, r, s) := f(x, r, s) - h(x, r, s) \geq 0$ with

$$h(x, r, s) := a(x) \cdot s + b(x) + c\, |u|^p.$$

F satisfies the asserted properties if and only if G satisfies them, because

$$\lim_{n\to\infty} \int_\Omega h(x, u_n, v_n)\, dx = \int_\Omega h(x, u, v)\, dx.$$

Step 2. Set

$$L := \liminf_{n\to\infty} F(u_n, v_n).$$

Then $L \geq 0$ since $f \geq 0$. The assertion of the theorem is trivial for the case $L = \infty$, hence we assume that $L < \infty$. By considering appropriate subsequences if necessary, we may furthermore assume that

$$L = \lim_{n\to\infty} F(u_n, v_n).$$

We show that we can without loss of generality assume that Ω is bounded. The emphasize the dependence on the domain let us write

$$F(u, v, \Omega) := \int_\Omega f(x, u, v)\, dx.$$

Summarizing the above considerations, we have that

$$L = \lim_{n\to\infty} F(u_n, v_n, \Omega) < \infty.$$

Suppose that the assertion of the theorem holds true for any bounded open $\Omega_\mu \subseteq \Omega$, i.e.

$$F(u, v, \Omega_\mu) \leq \liminf_{n\to\infty} F(u_n, v_n, \Omega_\mu).$$

Since $f \geq 0$, for every $n \in \mathbb{N}$,

$$F(u_n, v_n, \Omega_\mu) \leq F(u_n, v_n, \Omega),$$

thus

$$F(u, v, \Omega_\mu) \leq L.$$

We can then obtain the result by choosing a sequence of bounded open sets $\Omega_\mu \subseteq \Omega$, $\mu \in \mathbb{N}$ such that $\Omega_\mu \nearrow \Omega$ and applying Lebesgue's monotone convergence, Theorem 1.28.

Step 3. We assume from now on that Ω is bounded, $f \geq 0$ and

$$\lim_{n\to\infty} F(u_n, v_n) = L < \infty.$$

We will show, that for arbitrary $\varepsilon > 0$ there exists a measurable set $\Omega_\varepsilon \subseteq \Omega$ and a subsequence $(n_j)_{j\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that for all n_j,

$$\begin{cases} \mathrm{Vol}_d(\Omega \setminus \Omega_\varepsilon) < \varepsilon, \quad \text{and} \\ \int_{\Omega_\varepsilon} |f(x, u, v_{n_j}) - f(x, u_{n_j}, v_{n_j})|\, dx < \varepsilon \,\mathrm{Vol}_d(\Omega). \end{cases} \tag{4.5}$$

In the following, we construct Ω_ε having the property (4.5). Note that both $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are bounded in $L^p(\Omega;\mathbb{R}^m)$ and $L^q(\Omega;\mathbb{R}^N)$ respectively; apply Theorem 1.25 in order to obtain boundedness of $(v_n)_{n\in\mathbb{N}}$. Hence for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$, such that for every $n \in \mathbb{N}$,

$$\mathrm{Vol}_d(K^1_{\varepsilon,n}) < \frac{\varepsilon}{6} \quad \text{and} \quad \mathrm{Vol}_d(K^2_{\varepsilon,n}) < \frac{\varepsilon}{6},$$

where

$$\begin{aligned} K^1_{\varepsilon,n} &:= \{x \in \Omega \colon |u(x)| \geq M_\varepsilon > 0 \quad \text{or} \quad |u_n(x)| \geq M_\varepsilon > 0\} \\ K^2_{\varepsilon,n} &:= \{x \in \Omega \colon |v_n(x)| \geq M_\varepsilon > 0\}. \end{aligned}$$

Hence for

$$\Omega^1_{\varepsilon,n} := \Omega \setminus (K^1_{\varepsilon,n} \cup K^2_{\varepsilon,n}),$$

we obtain

$$\mathrm{Vol}_d(\Omega \setminus \Omega^1_{\varepsilon,n} < \frac{\varepsilon}{3}. \tag{4.6}$$

By assumption, f is a Carathéodory function. Therefore, Scorza-Dragoni's theorem, Theorem 1.57, grants us a compact set $\Omega^2_{\varepsilon,n} \subseteq \Omega^1_{\varepsilon,n}$ such that

$$\begin{cases} \mathrm{Vol}_d(\Omega^1_{\varepsilon,n} \setminus \Omega^2_{\varepsilon,n}) < \frac{\varepsilon}{3}, \quad \text{and} \\ f|_{\Omega^2_{\varepsilon,n}\times S_\varepsilon} \colon \Omega^2_{\varepsilon,n} \times S_\varepsilon \to \mathbb{R} \text{ is continuous,} \end{cases} \tag{4.7}$$

where

$$S_\varepsilon := \{(r,s) \in \mathbb{R}^k \times \mathbb{R}^m \colon |r| \leq M_\varepsilon \text{ and } |s| \leq M_\varepsilon\} \subseteq \mathbb{R}^m \times \mathbb{R}^N$$

is compact. Hence there exists $\delta(\varepsilon) > 0$ such that for every $x \in \Omega^2_{\varepsilon,n}$ and every $|r_1|, |r_2|, |s| < M_\varepsilon$,

$$|r_1 - r_2| < \delta(\varepsilon) \;\Rightarrow\; |f(x, r_1, s) - f(x, r_2, s)| < \varepsilon \tag{4.8}$$

By using the fact that $u_n \to u$ in $L^p(\Omega;\mathbb{R}^m)$, we can find[1] $n_\varepsilon = n_{\varepsilon,\delta(\varepsilon)} \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$,

$$\mathrm{Vol}_d(\Omega \setminus \Omega^3_{\varepsilon,n}) < \frac{\varepsilon}{3}, \tag{4.9}$$

[1] Possibly by considering an appropriate subsequence of $(u_n)_{n\in\mathbb{N}}$; c.f. Lemma 1.48.

where
$$\Omega^3_{\varepsilon,n} := \{x \in \Omega \colon |u_n(x) - u(x)| < \delta(\varepsilon)\}.$$
Therefore, letting
$$\Omega_{\varepsilon,n} := \Omega^2_{\varepsilon,n} \cap \Omega^3_{\varepsilon,n},$$
we obtain from (4.6), (4.7), (4.8) and (4.9) that
$$\begin{cases} \mathrm{Vol}_d(\Omega \setminus \Omega_{\varepsilon,n}) < \varepsilon, \quad \text{and} \\ \int_{\Omega_{\varepsilon,n}} |f(x,u,v_n) - f(x,u_n,v_n)|\,dx < \varepsilon\,\mathrm{Vol}_d(\Omega) \end{cases} \tag{4.10}$$
for all $n \geq n_\varepsilon$.

Note that $\varepsilon > 0$ was fixed so far. Now let us set $\varepsilon_j := \varepsilon/2^j$ for $j \in \mathbb{N}$. Then (4.10) holds with ε and n_ε replaced by ε_j and n_{ε_j}. We then pick every $n_j \in \mathbb{N}$ with $n_j \geq n_{\varepsilon_j}$, $j \in \mathbb{N}$ and set
$$\Omega_\varepsilon := \bigcap_{j=1}^{\infty} \Omega_{\varepsilon_j, n_j},$$
and finally deduce (4.5).

Step 4. Let
$$\chi_{\Omega_\varepsilon}(x) := \begin{cases} 1 & \text{if } x \in \Omega_\varepsilon \\ 0 & \text{if } x \in \Omega \setminus \Omega_\varepsilon, \end{cases}$$
and set
$$f_{u,\varepsilon}(x,s) := \chi_{\Omega_\varepsilon}(x)\, f(x,u(x),s).$$
Then $f_{u,\varepsilon} \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and $s \mapsto f_{u,\varepsilon}(x,s)$ is by assumption convex for a.e. $x \in \Omega$. Applying Theorem 4.5 to
$$J(s) := \int_\Omega f_{u,\varepsilon}(x, v(x))\,dx,$$
yields
$$\begin{aligned} \liminf_{n_j \to \infty} J(v_{n_j}) &= \liminf_{n_j \to \infty} \int_\Omega \chi_{\Omega_\varepsilon}(x)\, f(x,u(x),v_{n_j}(x))\,dx \\ &\geq J(v) = \int_\Omega \chi_{\Omega_\varepsilon}(x)\, f(x,u(x),v(x))\,dx. \end{aligned} \tag{4.11}$$
By employing (4.5) we have for all n_j, that
$$\begin{aligned} \int_{\Omega_\varepsilon} f(x,u_{n_j},v_{n_j})\,dx \geq \int_{\Omega_\varepsilon} & f(x,u,v_{n_j})\,dx \\ & - \int_{\Omega_\varepsilon} |f(x,u,v_{n_j}) - f(x,u_{n_j},v_{n_j})|\,dx \\ = \int_{\Omega_\varepsilon} & f(x,u,v_{n_j})\,dx - \varepsilon\,\mathrm{Vol}_d(\Omega). \end{aligned}$$

Therefore, exploiting the fact that $f \geq 0$, we find that

$$\begin{aligned}\int_\Omega f(x, u_{n_j}, v_{n_j})\,dx &\geq \int_{\Omega_\varepsilon} f(x, u_{n_j}, v_{n_j})\,dx \\ &\geq \int_{\Omega_\varepsilon} f(x, u, v_{n_j})\,dx - \varepsilon\,\mathrm{Vol}_d(\Omega) \\ &= J(v_{n_j}) - \varepsilon\,\mathrm{Vol}_d(\Omega)\end{aligned}$$

for all n_j. Letting $n_j \to \infty$ and using (4.11) we obtain

$$\begin{aligned}L &= \liminf_{n_j\to\infty} \int_\Omega f(x, u_{n_j}, v_{n_j})\,dx \\ &\geq \int_\Omega \chi_{\Omega_\varepsilon}(x)\, f(x, u(x), v(x))\,dx - \varepsilon\,\mathrm{Vol}_d(\Omega).\end{aligned}$$

Finally let $\varepsilon \to 0$, then $\mathrm{Vol}_d(\Omega \setminus \Omega_\varepsilon) \to 0$ and Lebesgue's monotone convergence theorem ($f \geq 0$), applied to the integrand on the right-hand side of the above inequality, implies the assertion of the theorem. □

Corollary 4.8. *Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{md} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$f(x, r, s) \geq a(x) \cdot s + b(x) + c\,|u|^q$$

for a.e. $x \in \Omega$, for all $(r, s) \in \mathbb{R}^m \times \mathbb{R}^{md}$, for some $a \in L^{p'}(\Omega; \mathbb{R}^{md})$, $b \in L^1(\Omega)$, $c \in \mathbb{R}$ and $q \in [1, p^*)$. Let $F\colon W^{1,p}(\Omega; \mathbb{R}^m) \to \mathbb{R}$ be defined by

$$F(u) := \int_\Omega f(x, u, \nabla u)\,dx.$$

If $s \mapsto f(x, r, s)$ is convex for a.e. $x \in \Omega$ and for every $r \in \mathbb{R}^m$, then F is weakly sequentially lower semicontinuous.

Proof. Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,p}(\Omega; \mathbb{R}^m)$ such that

$$u_n \rightharpoonup u.$$

Then $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega; \mathbb{R}^{md})$, and by the Rellich-Kondrachov theorem, Theorem 1.63,

$$u_n \to u \quad \text{in } L^q(\Omega; \mathbb{R}^m).$$

An application of Theorem 4.7 with $q = p$ and $p = q$ yields the assertion of the corollary. □

4.3. Convexity in the main part: a necessary condition

We start with a lemma; it deals with a wild oscillating sequence which weak* converges to its average in L^∞.

Lemma 4.9. *Let $a, b \in \mathbb{R}^m$ and $t \in [0,1]$. Let $u\colon [0,1] \to \mathbb{R}^m$ be given by*

$$u(x) := \begin{cases} a & \text{if } x \in [0,t) \\ b & \text{if } x \in [t,1]. \end{cases}$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions $[0,1] \to \mathbb{R}^m$ defined by $u_n(x) := u(nx)$. Then

$$u_n \overset{*}{\rightharpoonup} u_0$$

in $L^\infty(0,1)$, where $u_0 = ta + (1-t)b$ is a constant function.

Proof. We have to show that

$$\lim_{n\to\infty} \int_0^1 u_n(x)\phi(x)\,dx = (ta + (1-t)b) \int_0^1 \phi(x)\,dx \tag{4.12}$$

holds for every $\phi \in L^1(0,1)$.

Equation (4.12) is evidently true for functions ϕ of the form

$$\phi(x) := c\chi_I(x) = \begin{cases} c & \text{if } x \in I \\ 0 & \text{if } x \notin I, \end{cases}$$

for any constant $c \in \mathbb{R}^m$ and subinterval $I \subseteq [0,1)$. Thus (4.12) holds true for every simple function on $[0,1)$ given in canonical form. Since the set of simple functions on $[0,1)$ is dense in $L^1(0,1)$, see Theorem 1.47, relation (4.12) holds for any $\phi \in L^1(0,1)$. □

Remark 4.10. The assertion of the above lemma remains true, if one replaces the interval $[0,1]$ by any bounded interval $J \subseteq \mathbb{R}$. An analogous construction yields a generalization of the lemma for cubes $C \subseteq \mathbb{R}^d$ parallel to the coordinate axes, if the functions u_n oscillate in a single coordinate direction. More generally an analogous statement of Lemma 4.9 holds true for open bounded $\Omega \subseteq \mathbb{R}^d$ and corresponding functions u_n oscillating in one (arbitrary) direction.

Let us first discuss a special case, where F has the form $F(u) := \int_\Omega f(\nabla u)\,dx$:

Theorem 4.11. *Let Ω be a bounded open subset of $\mathbb{R}^d$ and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Consider the functional $F\colon W^{1,\infty}(\Omega) \to \mathbb{R}$,*

$$F(u) := \int_\Omega f(\nabla u)\,dx.$$

If F is sequentially weak lower semicontinuous, then f is convex.*

Proof. We have to show that for arbitrary $z_1, z_2 \in \mathbb{R}^d$ and $t \in [0,1]$,

$$f(tz_1 + (1-t)z_2) \le tf(z_1) + (1-t)f(z_2). \tag{4.13}$$

So take $z_1, z_2 \in \mathbb{R}^d$, $t \in [0,1]$ and set $z := tz_1 + (1-t)z_2$. Let us denote the linear function $x \mapsto z \cdot x$, $\Omega \to \mathbb{R}$ by u_z. Next, we make a few definitions:

$$\begin{aligned}
z_0 &:= \frac{z_2 - z_1}{|z_2 - z_1|},\\
\Omega^1_{hj} &:= \left\{ x \in \Omega \colon \frac{j-1}{h} < z_0 \cdot x < \frac{j-1+t}{h} \right\}, \quad j \in \mathbb{Z}, h \in \mathbb{N},\\
\Omega^2_{hj} &:= \left\{ x \in \Omega \colon \frac{j-1+t}{h} < z_0 \cdot x < \frac{j}{h} \right\}, \quad j \in \mathbb{Z}, h \in \mathbb{N},\\
\Omega^1_h &:= \bigcup_{j \in \mathbb{Z}} \Omega^1_{hj},\\
\Omega^2_h &:= \bigcup_{j \in \mathbb{Z}} \Omega^2_{hj},\\
u_h(x) &:= \begin{cases} c^1_{hj} + z_1 \cdot x & \text{if } x \in \Omega^1_{hj} \\ c^2_{hj} + z_2 \cdot x & \text{if } x \in \Omega^2_{hj}, \end{cases}
\end{aligned}$$

where

$$c^1_{hj} := \frac{(j-1)(1-t)}{h}|z_2 - z_1| \quad \text{and} \quad c^2_{hj} := -\frac{jt}{h}|z_2 - z_1|.$$

Obviously for $h \to \infty$,

$$\frac{\mathrm{Vol}_d(\Omega^1_h)}{\mathrm{Vol}_d(\Omega)} \to t \quad \text{and} \quad \frac{\mathrm{Vol}_d(\Omega^2_h)}{\mathrm{Vol}_d(\Omega)} \to 1-t. \tag{4.14}$$

The heights c^1_{hj} and c^2_{hj} are defined in such a way, that the piecewise affine functions $u_h \colon \Omega \to \mathbb{R}$ are continuous.

We show that $u_h \to u_z$ uniformly on Ω for $h \to \infty$. Indeed, for every $x \in \overline{\Omega^1_{hj}}$,

$$\begin{aligned}
|u_h(x) - u_z(x)| = |c^1_{hj} + (z_1 - z) \cdot x| &= (1-t)\left| \frac{j-1}{h}|z_2 - z_1| + (z_1 - z_2) \cdot x \right|\\
&= (1-t)|z_2 - z_1| \left| \frac{j-1}{h} - z_0 \cdot x \right| \le \frac{(1-t)|z_2 - z_1| t}{h},
\end{aligned}$$

and for every $x \in \overline{\Omega^2_{hj}}$,

$$\begin{aligned}
|u_h(x) - u_z(x)| = |c^2_{hj} + (z_2 - z) \cdot x| &= t\left| -\frac{j}{h}|z_2 - z_1| + (z_2 - z_1) \cdot x \right|\\
&= t|z_2 - z_1| \left| -\frac{j}{h} + z_0 \cdot x \right| \le \frac{t(1-t)|z_2 - z_1|}{h}.
\end{aligned}$$

Moreover the distributional gradients ∇u_h are uniformly bounded on Ω; in fact they take just the two values z_1 and z_2 on all of Ω for all $h \in \mathbb{N}$. By a slight generalization of Lemma 4.9, c.f. Remark 4.10, it follows that $\nabla u_h \overset{*}{\rightharpoonup} \nabla u_z = z$ in $L^\infty(\Omega)$. This shows that the sequence $(u_h)_{h \in \mathbb{N}}$ weak* converges to u_z in $W^{1,\infty}(\Omega)$. Since F is sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega)$, we obtain

$$\begin{aligned} f(z)\mathrm{Vol}_d(\Omega) = F(u_z) &\leq \liminf_{h\to\infty} F(u_h) \\ &= \liminf_{h\to\infty} \Big(f(z_1)\mathrm{Vol}_d(\Omega_h^1) + f(z_2)\mathrm{Vol}_d(\Omega_h^2) \Big) \\ &\overset{(4.14)}{=} t f(z_1)\mathrm{Vol}_d(\Omega) + (1-t) f(z_2)\mathrm{Vol}_d(\Omega). \end{aligned}$$

Dividing the above inequality by $\mathrm{Vol}_d(\Omega) \in (0,\infty)$ yields (4.13) and finishes the proof. □

The proof of the general case is very technical and relies on sophisticated approximation methods. We only state the theorem and refer to [**5**] for the proof. Note also that the domain Ω is not assumed to be bounded.

Theorem 4.12. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{md} \to \mathbb{R}$ be a Carathéodory function satisfying for a.e. $x \in \Omega$ and for every $(r,s) \in \mathbb{R}^m \times \mathbb{R}^{md}$,*

$$|f(x,r,s)| \leq a(x) + b(r,s)$$

for some $a \in L^1(\Omega)$ and some $b \in C(\mathbb{R}^m \times \mathbb{R}^{md})$. Moreover let

$$F(u) = F(u,\Omega) := \int_\Omega f(x,u,\nabla u)\,dx$$

and assume that there exists some $u_0 \in W^{1,\infty}(\Omega;\mathbb{R}^m) \cong \big(W^{1,\infty}(\Omega;\mathbb{R})\big)^m$ such that

$$|F(u_0,\Omega)| < \infty.$$

If F is sequentially weak lower semicontinuous on $W^{1,\infty}(\Omega;\mathbb{R}^m)$ and if either $m = 1$ or $d = 1$, then $s \mapsto f(x,r,s)$ is convex for a.e. $x \in \Omega$ and for every $r \in \mathbb{R}^m$.*

If the functional F in the above theorem is weak* sequentially continuous, i.e. both F and $-F$ are weak* sequentially lower semicontinuous, then $s \mapsto f(x,r,s)$ is affine and, for bounded Lipschitz domain Ω, vice versa:

Corollary 4.13. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{md} \to \mathbb{R}$ be a Carathéodory function satisfying for a.e. $x \in \Omega$ and for every $(r,s) \in \mathbb{R}^m \times \mathbb{R}^{md}$,*

$$|f(x,r,s)| \leq a(x) + b(r,s) \tag{4.15}$$

for some $a \in L^1(\Omega)$ and some $b \in C(\mathbb{R}^m \times \mathbb{R}^{md})$. Moreover assume that either $m = 1$ or $d = 1$ and let

$$F(u) := \int_\Omega f(x,u,\nabla u)\,dx$$

Then F is sequentially weak continuous on $W^{1,\infty}(\Omega;\mathbb{R}^m)$ if and only if $s \mapsto f(x,r,s)$ is affin, i.e. there exist Carathéodory functions $g := \Omega \times \mathbb{R}^m \to \mathbb{R}^{md}$*

and $h\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ *such that*

$$f(x, r, s) = g(x, r) \cdot s + h(x, r).$$

Proof. In order to prove the necessity part it remains to show that g and h are Carathéodory functions. Setting $s := 0$ yields that h is Carathéodory, since $(x, r) \mapsto f(x, r, 0)$ is Carathéodory. But then, g has to be Carathéodory as well, otherwise f would not be a Carathéodory function.

For the sufficiency part, note that $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$ implies $u_n \to u$ in $L^\infty(\Omega)$, see Theorem 1.63. Exploiting the fact that g and h are Carathéodory, and applying Theorem 1.27(iv) yields the claim. Note that the Nemytskii operators $\mathcal{N}_g\colon L^\infty(\Omega) \to L^1(\Omega)$ and $\mathcal{N}_h\colon L^\infty(\Omega) \to L^1(\Omega)$ are continuous by (4.15), see Theorem 1.53. □

Remark 4.14. An analogous statement of Corollary 4.13 holds true in the $W^{1,p}$-case if one imposes further restrictions on g and h such that the corresponding Nemytskii operators $\mathcal{N}_g\colon W^{1,p}(\Omega) \to L^{p'}(\Omega)$ and $\mathcal{N}_g\colon W^{1,p}(\Omega) \to L^1(\Omega)$ are sequentially (weak-norm)-continuous.

Remark 4.15. The theorems in Section 4.2 hold generally true for vector-valued functions $u\colon \Omega \to \mathbb{R}^m$. Contrary, the functions u appearing in Theorem 4.11 and Theorem 4.12 (if $d \geq 2$) are scalar valued. If $d > 1$ and $m > 1$, there exist weakly sequentially lower semicontinuous functionals having integrands f with $s \mapsto f(x, r, s)$ not being convex. The systematic study of functionals on spaces of vector-valued functions leads to the notion of *quasiconvexity*; see [**5**].

References

[1] R. A. Adams and John J. F. Fournier, *Sobolev spaces*, 2nd ed., Academic Press, Amsterdam, 2003.

[2] Hedy Attouch, Giuseppe Buttazzo, and Gerard Michaille, *Variational Analysis in Sobolev and BV Spaces*: *Applications to PDEs and optimization*, Society for Industrial and Applied Mathematics, Philadelphia, 2006.

[3] Markus Biegert, *On Traces of Sobolev Functions on the Boundary of Extension Domains*, Proceedings of the American Mathematical Society **137** (2009), no. 12, 4169–4176, available at `http://www.ams.org/journals/proc/2009-137-12/S0002-9939-09-10045-X/S0002-9939-09-10045-X.pdf`.

[4] Philippe Blanchard and Erwin Brüning, *Direkte Methoden der Variationsrechnung*, Springer, Wien, 1982.

[5] Bernard Dacorogna, *Direct Methods in the Calculus of Variations*, 2nd ed., Springer, New York, 2008.

[6] Jürgen Elstrodt, *Maß- und Integrationstheorie*, 7th ed., Springer, Berlin, 2011.

[7] Lawrence C. Evans, *Partial differential equations*, American Mathematical Society, Providence, RI, 1998.

[8] Gerald B. Folland, *Real Analysis*: *Modern Techniques and their Applications*, 2nd ed., John Wiley & Sons, New York, 1999.

[9] Harro Heuser, *Funktionalanalysis*: *Theorie und Anwendungen*, 3rd ed., Teubner, Stuttgart, 1992.

[10] ———, *Lehrbuch der Analysis*: *Teil 1*, 15th ed., Teubner, Stuttgart, 2003.

[11] Antonie Frans Monna, *Dirichlet's principle*: *A mathematical comedy of errors and its influence on the development of analysis*, Oosthoek, Scheltema & Holkema, Utrecht, 1975.

[12] Michael Renardy and Robert C. Rogers, *An Introduction to Partial Differential Equations*, 2nd ed., Springer, New York, 2004.

[13] Tomáš Roubíček, *Nonlinear Partial Differential Equations with Applications*, Birkhäuser, Basel, 2005.

[14] Halsey Royden and Patrick Fitzpatrick, *Real Analysis*, 4th ed., Pearson, New Jersey, 2010.

[15] Walter Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[16] Ralph E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, Providence, RI, 1997.

[17] Luc Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Springer, Berlin, 2007.

[18] Gerald Teschl, *Topics in Real and Functional Analysis*, available at `http://www.mat.univie.ac.at/~gerald/ftp/book-fa/index.html`.

[19] Kôsaku Yosida, *Functional Analysis*, 6th ed., Springer, Berlin, 1980.

[20] Eberhard Zeidler, *Nonlinear Functional Analysis and its Applications. Part I*: *Fixed-point theorems*, Springer, New York, 1986.

[21] ———, *Nonlinear Functional Analysis and its Applications. Part II/A*: *Linear monotone operators*, Springer, New York, 1990.

[22] ———, *Nonlinear Functional Analysis and its Applications. Part II/B*: *Nonlinear monotone operators*, Springer, New York, 1990.
[23] ———, *Nonlinear Functional Analysis and its Applications. Part III*: *Variational Methods and Optimization*, Springer, New York, 1985.

Index

boundary shift, 60
boundary value problem
  Dirichlet, 46
  mixed, 46
  Neumann, 46
  Newton, 46

Carathéodory mapping, 14
classical solution, 47
coercive
  functional, 31
  operator, 31
convex
  functional, 32
  set, 5
convex hull, 5
critical point, 35

direct method, 25
Dirichlet's principle, vi
Dirichlet's problem, v
divergence form, 45
dominated convergence theorem, 10
dual space, 2

Euler equation, 25
Euler-Lagrange functional, 25

Fatou's lemma, 9
Fubini's theorem, 11
functional, 2
  linear bounded, 2
  lower semicontinuous, 29
fundamental theorem of calculus, 11

G-derivative, 35
G-differentiability, 35
Green's formula, 21

Hölder conjugate, 12
Hölder's inequality, 12
Hahn-Banach's theorem, 4

Jensen's inequality, 32

Laplace operator, 43
Lebesgue space, 13
Lipschitz domain, 19

Mazur's theorem, 6
Minkowski's inequality, 12
monotone
  function, 37, 38
  operator, 35
    strictly, 35
    strongly, 35
    uniformly, 35
monotone convergence theorem, 9

Nemytskii operator, 14

operator
  bounded, 2
  compact, 1
  continuous, 1
  hemicontinuous, 40
  linear bounded, 2
  Lipschitz continuous, 2
    locally, 2
  locally bounded, 2
  weakly sequentially continuous, 1

Poincaré's inequality, 22
potential, 40
  pseudo potential, 40
potential operator, 40
predual, 2

reflexive Banach space, 5
Rellich-Kondrachov theorem, 20

Scorza-Dragoni's theorem, 18
separation theorem, 6
simple function, 13
Sobolev
  conjugate, 20

embedding theorem, 20
space, 19
sublevel set, 31

trace embedding theorem, 21
trace theorem, 21

uniform boundedness principle, 3

weak convergence, 2
weak formulation, 46
weak solution, 43, 47
weak* convergence, 3

Curriculum Vitae

Personal data

Name:	Ronald Quirchmayr
Date of birth:	November 24th, 1987
Place of birth:	Tulln, Austria
Nationality:	Austria
Mother:	Gertrude Quirchmayr
Father:	Friedrich Quirchmayr
Sister:	Kerstin Quirchmayr

Education

2007-2012:	Diploma studies in mathematics, University of Vienna
1998-2006:	BG/BRG Krems, Piaristengasse 2
1994-1998:	VS Sitzenberg-Reidling