DIPLOMARBEIT

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Affine crystallographic groups

verfasst von

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Abstract

Abstract

This diploma thesis is dedicated to the study of *affine crystallographic groups*, i.e. *properly discontinuous cocompact* subgroups of $\mathrm{Aff}(\mathbb{R}^n)$.

We start off with a brief overview of the historical roots of such groups by giving a heuristic introduction to *crystallographic groups*, see section 0.1. The structure of these groups is very well understood owing to the famous *Bieberbach theorems*, see 1.2.

Crystallographic groups are chosen as a starting point because of two reasons. First, they are easier to handle than affine crystallographic groups. However, there are still many similarities between them and their affine counterparts, see for example sections 1.3 and 2.3, and thus they serve as a fitting introduction to the topic at hand. Second, examinations of affine crystallographic groups are in large part inspired by the classical case of crystallographic groups, see chapter 3.

Chapter 4 is devoted to a famous conjecture by *Auslander*, namely the question whether every affine crystallographic group is *virtually solvable*. We will not only present said conjecture, 4.1.3, and variations thereof, we will also cite another conjecture, namely one by *Milnor*, 4.1.5, which asks whether properly discontinuous affine groups are virtually solvable. For the latter, we will present a counterexample due to *Margulis*, see section 4.2. Auslander's conjecture on the other hand is still an open matter. Until today there is no known counterexample and proofs have been given for special cases only. In sections 4.3 through 4.7 we will present some of these proofs. If Auslander's conjecture were proven to be true in general, we would gain profound insight into the structure of affine crystallographic groups. In particular, we would have nice generalizations of the Bieberbach theorems for the affine setting, see theorems 3.3.6, 3.4.6 and 3.5.3.

The last chapter of this diploma thesis serves to outline possible future developments in the field of affine crystallographic groups. In section 5.1 we cite two problems by *Abels*, *Margulis*, and *Soifer* which could be central for proving Auslander's conjecture in dimension 7. Section 5.2 is concerned with a very natural generalization of affine crystallographic groups, namely so called *NIL-affine crystallographic groups*. A part of the motivation for examining this setting stems from Milnor's '*converse Auslander conjecture*', 4.1.4, which does not hold in the affine case, yet holds in the NIL-affine one, 5.2.4.

Zusammenfassung

Im Zentrum dieser Diplomarbeit stehen *affin-kristallographische Gruppen*, i.e. *eigentlich diskontinuierliche kokompakte* Untergruppen von $\mathrm{Aff}(\mathbb{R}^n)$.

Wir geben zunächst einen Überlick über die historischen Ursprünge von diesen Gruppen, indem wir kurz *kristallographische Gruppen* anschaulich präsentieren, siehe Abschnitt

0.1. Die Struktur ebendieser Gruppen ist dank der *Bieberbach Theoreme* sehr gut verstanden, siehe dazu 1.2.
Das Studium der affin-kristallographischen Gruppen wird aus zwei Gründen mit einer Auseinandersetzung mit kristallographischen Gruppen begonnen. Erstens gibt es einige Parallelen zwischen diesen beiden Klassen von Gruppen, vergleiche beispielsweise 1.3 und 2.3, wobei die Theorie der kristallographischen Gruppen deutlich einfacher ist als die ihrer affinen Pendants. Zweitens ist ein großer Teil der Theorie affin-kristallographischer Gruppen von dem klassischen Fall der kristallographischen Gruppen inspiriert, siehe dazu Kapitel 3.
In Kapitel 4 beschäftigen wir uns mit *Auslanders* berühmter Vermutung, also mit der Frage, ob jede affin-kristallographische Gruppe *virtuell auflösbar* ist, siehe 4.1.3. Wir werden nicht nur einige Variationen dieser Vermutung präsentieren, sondern auch noch eine zweite, nämlich *Milnors* Vermutung. Diese fragt, ob jede eigentlich diskontinuierliche affine Gruppe virtuell auflösbar ist, 4.1.5. *Margulis* zeigte, dass dies nicht der Fall ist, siehe Abschnitt 4.2. Ob Auslanders Vermutung wahr ist oder nicht, ist bis heute offen - weder ist ein Gegenbeispiel bekannt, noch wurde ein Beweis gefunden. Lediglich Spezialfälle konnten bewiesen werden, einige davon werden wir in den Abschnitten 4.3 bis 4.7 präsentieren. Ein Beweis für Auslanders Vermutung wäre ein großer Schritt in Richtung eines umfassenden Verständnisses der Struktur von affin-kristallographischen Gruppen. So wären beispielsweise schöne Verallgemeinerungen der Bieberbach-Theoreme möglich, siehe 3.3.6, 3.4.6 und 3.5.3.
Das letzte Kapitel dieser Diplomarbeit bietet einen Ausblick auf mögliche zukünftige Entwicklungen im Bereich affin-kristallographischer Gruppen. Zunächst werden zwei Probleme präsentiert, die zentral für einen Beweis von Auslanders Vermutung in Dimension 7 sein könnten, siehe 5.1. In Abschnitt 5.2 werden so genannte *NIL-affin-kristallographische Gruppen* vorgestellt, welche eine natürliche Verallgemeinerung von affin-kristallographischen Gruppen sind. Ein Teil der Motivation für die Auseinandersetzung mit ebendiesen folgt aus Milnors '*umgekehrter Auslander Vermutung*', 4.1.4, welche im affinen Fall falsch ist, im NIL-affinen jedoch stimmt, 5.2.4.

Danksagung

Diese Diplomarbeit hätte nicht ohne die Hilfe bestimmter Personen geschrieben werden können. Bei ebendiesen will ich mich nun bedanken.

Zunächst möchte ich meinem Betreuer Dietrich Burde für seine intensive Betreuung sowie vielfältigen Ratschläge und Anregungen danken. Im Großen wie im Kleinen stand er mir mit Geduld und Tipps zur Seite.

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Für die letzten drei auf meiner Dankesliste - Agnes, Franz, Susanne - habe ich eigentlich eine wunderbare Danksagung vorbereitet. Leider reicht dafür der Platz auf dieser Seite nicht aus und eine verkürzte Version würde dem notwendigen Ausmaß der Danksagung nicht gerecht. Insofern halte ich es kurz und sage: Danke.

Contents

0 Preface

0.1 Origin & generalizations of crystallographic groups

Crystallographic groups arose during the task of classifying *space fillings*. Heuristically speaking, a space filling is just some (real) affine space (of finite dimension) endowed with a periodic pattern that fills the whole space. For example in dimension 2 we may have something like this:

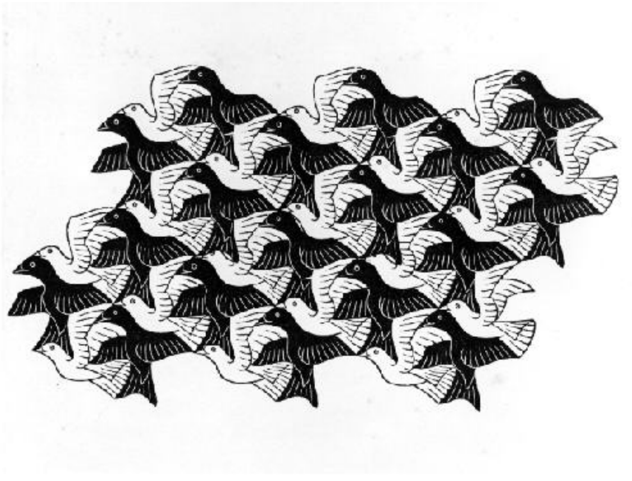

Figure 0.1: M.C. Escher, Regular Division of the Plane with Birds, 1949.

Speaking a little bit more mathematically, we can describe this as follows: we start with a small (compact and connected) section of the space, a so called *fundamental domain*, and translate it until we have covered the whole space. For example, to achieve a space filling such as in figure 0.1, we could choose any white bird and an adjacent black one.

Obviously, we can also let the translations operate on the space itself and see that they don't change the way the space looks, i.e. they respect the pattern - those translations are *symmetries*.

Of course, depending on the chosen pattern, one may also find other symmetries: rotations and/or reflections. The set of such symmetries forms a group. In dimension 2 such a group of symmetries is called a *wallpaper group*, in dimension 3 we call such groups *space groups*.

In general dimension, one calls such groups of symmetries *crystallographic groups*.

This notion is derived from the following: if we 'forget' that a crystal is a finite object, we can describe it as a set of atoms which are arranged in a periodic pattern that fills the 3-dimensional affine space.

The beauty of the correspondence of space fillings and their accompanying symmetries is that by classifying all possible groups which can act as such groups of symmetries, one also classifies all the possible space fillings. On a side note: crystals were also classified in this way. For every crystallographic group a crystal was found, and there is not one crystal that does not have a crystallographic group as its group of symmetries.

Although plane fillings by regular polygons were already studied in 1618 by Johannes Kepler, it was not before the end of the 19$^{\text{th}}$ century that space fillings received rigid mathematical inquiry. Independently, Fedorov, Schönflies and Barlow classified the 3-dimensional crystallographic groups, and thus also all possible periodical fillings of the 3-dimensional affine space.

Further investigation of crystallographic groups was then fueled by one of Hilbert's famous problems. One part of his 18$^{\text{th}}$ problem was the question whether or not there are only finitely many different crystallographic groups in any given dimension. Not only was Bieberbach able to positively answer this question, he also provided other theorems which give a very good insight into crystallographic groups. Due to him, we know the 'structure' of such groups. Furthermore, one of his theorems yields a suitable equivalence relation for crystallographic groups.

There are many ways to generalize crystallographic groups.

For example, instead of filling some affine space, i.e. a space with zero curvature, we could do the same with a space of positive or negative curvature. The corresponding groups of symmetry are then called *non-Euclidean*-crystallographic groups. Figure 2 shows such a filling for the 2-dimensional hyperbolic space in the Poincaré disk model.

Another possible generalization is the following: by definition we have that if Γ is a crystallographic group for the n-dimensional affine space, then we can find n linearly independent translations in Γ. If one requires Γ to have at most $n-1$ linearly independent translations, one arrives at so called *quasicrystallographic* groups which correspond to aperiodic space fillings and quasicrystals.

The generalization we want to pursue is as follows: Obviously, crystallographic groups are not only groups of symmetry, they are even isometries. We will drop this assumption and instead analyze so called *affine crystallographic* groups. To put this into mathematical terms, rather than examining certain subgroups of the group of all isometries $\mathrm{Isom}(\mathbb{E}) \cong \mathbb{R}^n \rtimes O(\mathbb{R}^n)$, we will be concerned with subgroups of $\mathrm{Aff}(\mathbb{E}) \cong \mathbb{R}^n \rtimes GL(\mathbb{R}^n)$, with $\mathbb{E}$ denoting the real affine space of dimension n.

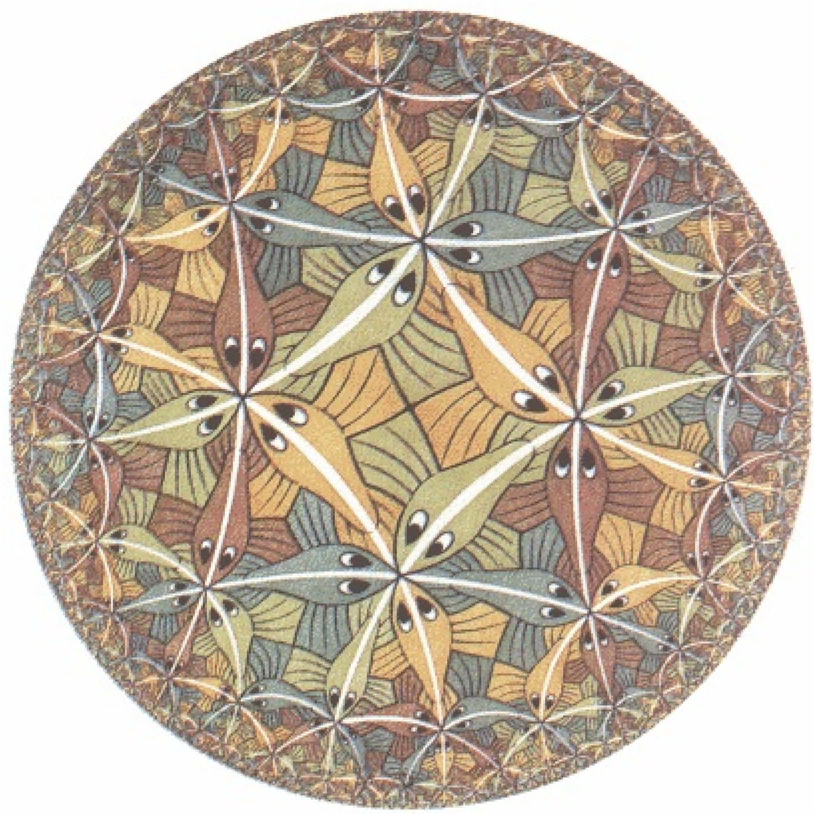

Figure 0.2: M.C. Escher, Circle Limit III, 1959.

0.2 Preliminaries

This chapter serves two purposes. First, it settles basic notations. Second, it is a reminder of a variety of different definitions that will be used thereafter without further reference.

0.2.1 Notations

We start with

Sets

$\mathbb{N} \ldots$ the set of positive integers including 0
$\mathbb{Z} \ldots$ the set of integers
$\mathbb{R} \ldots$ the set of real numbers
$\mathbb{N}^\times, \mathbb{Z}^\times, \mathbb{R}^\times \ldots$ the respective set without 0
$A \times B \ldots$ direct product of A and B
$A \oplus B \ldots$ direct sum A and B of sets with an algebraic structure
$B \subseteq A \ldots$ B is subset of A
$B \subset A \ldots$ B is a proper subset of A

Groups

$G \cong H \ldots$ G and H are isomorphic as abstract groups
$H \leq G \ldots$ H is a subgroup of G
$H < G \ldots$ H is a proper subgroup of G
$H \trianglelefteq G \ldots$ H is a normal subgroup of G
$N \rtimes Q \ldots$ semidirect product of N by Q, i.e. $N \trianglelefteq N \rtimes Q$
$Aut(X) \ldots$ automorphism group of X
$\mathrm{Aff}(X) \ldots$ group of affine motions of X
$\mathrm{Isom}(X) \ldots$ group of isometries of X
$\mathrm{rank}_{\mathbb{R}}(G) \ldots$ real rank of G
$\overline{G} \ldots$ the topological closure of (a topological group) G
$G^0 \ldots$ identity component of G
$G/H \ldots$ the cosets of H in G for $H \leq G$
$G \backslash X \ldots$ the orbit space for a group action of some group G on some space X
$<S> \ldots$ the span of a $S \subseteq G$, i.e. the set of all finite products of elements in S

0.2.2 Definitions

This section is dedicated to basic definitions. First we give an overview of central definitions concerning groups. After that, we have a brief summary of the basics of group actions, exact sequences and the cohomological dimension of groups. The last portion then adresses some basics concerning manifolds.

Groups Let G be a group.

Given a subgroup $H \leq G$, the quantity of cosets $|G/H|$ is called the *index* of H in G and will be denoted by $[G : H]$. Recall that for $H_2 \leq H_1 \leq G$ we have multiplicity of the index, i.e. $[G : H_2] = [G : H_1][H_1 : H_2]$.

If S is a subset, $S \subseteq G$, then the normal subgroup $C_G(S) := \{g \in G \mid gs = sg \; \forall s \in S\}$ is called the *centralizer* of S in G and by the subgroup $N_G(S) := \{g \in G \mid gS = Sg\}$ we denote the *nomalizer* of S in G.

Given $g \in G$ with $g \neq e$. If there is an $n \in \mathbb{N}$, such that $g^n = e$, we say that g is a *torsion element* and accordingly, that G is a *torsion group* if every element of G is a torsion element. On the other hand, if there is no such g, then we call G *torsion free.*

We say that a group G is *linear*, if there is an embedding $G \hookrightarrow GL(\mathbb{R}^n)$.

If a group G is endowed with a topology, such that both the multiplication $(g, h) \mapsto gh$ and the inversion $g \mapsto g^{-1}$ are continuous with respect to the topology in question, G is called a *topological* group. If the topology in question is the discrete one, G is said to be a *discrete group.* And if G is a topological group, any subgroup $H \leq G$ is called a *discrete subgroup*, if it is discrete with respect to the induced topology.

A group, which is endowed with a smooth manifold structure in such a way, that the multiplication as well as the inversion are smooth maps, is called a *Lie group.* For a detailed study of the theory of Lie groups and Lie algebras see for example [Kn02] and for the theory of discrete subgroups of Lie groups see [OV00].

If G is a group and at the same time an algebraic variety with the property that the multiplication and inversion are regular functions, G is said to be an *algebraic* group. If $H \leq G$ is closed with respect to the Zariski topology, then we say that H is an *algebraic subgroup.*

Group actions Given a triple $(G, \mathcal{O}, \varphi)$ consisting of a group G, some set $\mathcal{O}$ and a group homomorphism $\varphi\colon G \to Aut(\mathcal{O})$, we call the map

$$\begin{aligned} G \times \mathcal{O} &\to \mathcal{O} \\ (g, x) &\mapsto \varphi(g)(x) \end{aligned}$$

a *group action* of G on $\mathcal{O}$. If the homomorphism φ is unambiguous, for example if $G \leq Aut(\mathcal{O})$ and $\varphi = id$, we will omit it and just write gx rather than $\varphi(g)(x)$. Furthermore, if a group G is acting on some $\mathcal{O}$, we will write $G \circlearrowleft \mathcal{O}$.

Given any subgroup G of the automorphism group of some X, we call the action given by $(g, x) \mapsto id(g)(x)$ the natural action from G on X. For example, if $G \leq GL(\mathbb{R}^n)$, the natural action $G \circlearrowleft \mathbb{R}^n$ is given by matrix multiplication.

A group action $G \circlearrowleft \mathcal{O}$ is called *free*, if for every $g \in G$ with $gx = x$ for any $x \in \mathcal{O}$ we have that $g = e$. Equivalently, an action is free, if $gx = hx$, $g, h \in G$, $x \in \mathcal{O}$, implies $g = h$.

If S is a subset of $\mathcal{O}$, we call $Stab_G(S) := \{g \in G \mid gS \subseteq S\}$ the *stabilizer* of S in G. If S is a singleton, i.e. $S = \{x\}$, we will write $G_x := Stab_G(x)$.

If there is a subset S of $\mathcal{O}$ with $GS := \{gs \mid g \in G, s \in S\} \subseteq S$, we say that G acts *invariantly* on S. Of course, in this situation there is also an induced action $G \circlearrowleft S$.

For a group action of G on $\mathcal{O}$ and a fixed $x \in \mathcal{O}$, the set $Gx := \{gx \mid g \in G\}$ is called the *orbit* of x under G. Trivially, $\bigcup_{x \in \mathcal{O}, g \in G} gx = \mathcal{O}$. Furthermore, such orbits, or rather the action of G on $\mathcal{O}$, define an equivalence relation $\sim_G$ via $x \sim_G y$, $x, y \in \mathcal{O}$, iff there is a $g \in G$ such that $gx = y$. Of course, this yields a partition into equivalence classes $[x] := \{y \in \mathcal{O} \mid x \sim_G y\} = Gx$. The set of all these classes $G \backslash \mathcal{O} := \mathcal{O}/\sim_G := \bigcup_{x \in \mathcal{O}} [x]$ is called the *orbit space* or *quotient space* of $\mathcal{O}$ modulo G.

Exact sequences of groups A sequence of groups G_i with homomorphisms φ_i

$$\cdots \longrightarrow G_{i-1} \xrightarrow{\varphi_i} G_i \xrightarrow{\varphi_{i+1}} G_{i+1} \longrightarrow \cdots$$

is called *exact*, if for every j we have $\operatorname{im}(\varphi_i) = \ker(\varphi_{i+1})$.

Thus, given a *short* exact sequence

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1,$$

we immediately know that ι is injective, $\ker(\iota) = 1$, and hence N may be viewed as a subgroup of G. Furthermore, N is even a normal subgroup, because $N = \operatorname{im}(\iota) = \ker(\pi)$. Also, π is surjective, $\operatorname{im}(\pi) = \ker(Q \to 1) = Q$. In this situation we will say that G is an extension of N by Q.

Such a short exact sequence *splits*, i.e. there is a homomorphism $\sigma \colon Q \to G$ with $\pi \circ \sigma = id$, iff G is the semidirect product of N by Q, written $G = N \rtimes Q$. It should be noted that in general there is no *splitting homomorphism* σ, although there is always a set theoretical map that is a right inverse for π.

Cohomological dimension of a group Let G be a group and R a ring with unit and let RG be the group ring. Furthermore, $H^k(G, M)$ shall denote the k-th cohomology group of G. The smallest $n \in \mathbb{N}$ such that $H^k(G, M) = 0 \ \forall k > n$, with $k \in \mathbb{N}$, for any RG-module M, is called the *cohomological dimension* of G with *coefficients* in R. If $R = \mathbb{Z}$, we just say that G has cohomological dimension n and drop the references on the coefficients.

Equivalently, the cohomological dimension of G with coefficients in R equals n, if there is a projective resolution of length n of the trivial RG-module R, and this resolution is minimal with respect to the length.

More on the cohomology of groups can be found in [Br82].

Manifolds We say that M is a *Riemannian* manifold, if for every tangent space T_pM, there is an inner product g_p, such that for all vector fields X and Y the map $p \mapsto g_p(X(p), Y(p))$ is smooth

. A Riemannian manifold M is called a *complete*, if M satisfies one of the following equivalent assertions:

(i) when viewed as a metric space, M is complete

(ii) M is geodesically complete

(iii) if K is a closed and bounded subset of M, then K is compact.

This equivalence is due to the so called Hopf-Rinow theorem.

If M is Riemannian and if furthermore the curvature tensor $R(u,v)w := \nabla_u \nabla_v(w) - \nabla_v \nabla_u(w) - \nabla_{[u,v]} w$ is identically 0, with u, v, w tangential vectors and ∇ the Levi-Civita connection, then M is said to be *flat*.

We say that M with atlas $\mathfrak{A}$ is *compact*, if there is a finite atlas $\mathfrak{B} \leq \mathfrak{A}$ for M.

1 Crystallographic groups & the Bieberbach theorems

1.1 Crystallographic groups

In the preface we have used a geometrical approach to introduce crystallographic groups. Besides looking at them as certain groups of symmetries, there are a couple of ways to describe crystallographic groups and we will present two of them. At first we will give an algebraic definition and later, subsection 1.3, we will show a topological viewpoint on such groups.

We start by defining crystallographic groups via group actions.

1.1.1 Definition (Properly discontinuous & crystallographic actions)**.** *Given some group Γ and a locally compact Hausdorff space X. An action $\Gamma \circlearrowleft X$ is called* proper *if for every compact $K \subseteq X$ we have that $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is compact. If this set is even finite, the action is said to be* properly discontinuous*.*

If we have a properly discontinuous group action $\Gamma \circlearrowleft X$ with the additional property that the orbit space $\Gamma\backslash X$ is compact with respect to the quotient topology, said action is called crystallographic*.*

An equivalent property to *cocompactness*, i.e. compactness of the quotient $\Gamma\backslash X$, is *quasitransitivity*, which means that there is a compact set $K \subset X$ such that $X = \bigcup_{\gamma\in\Gamma} \gamma K$.

Of course, if we are given a crystallographic action $\Gamma \circlearrowleft X$ and a compact K with $X = \bigcup_{\gamma\in\Gamma} \gamma K$, such that K is minimal, i.e. there is no compact $\tilde{K}$ with $\tilde{K} \subset K$ and $X = \bigcup_{\gamma\in\Gamma} \gamma\tilde{K}$, then the connected components of K are the fundamental domains of a space filling.

1.1.2 Remark (Properly discontinuous versus discrete)**.** While in general a group that admits a properly discontinuous action is discrete, the converse is not true, i.e. not every action of a discrete group is automatically properly discontinuous.

E.g. consider the group $\Lambda(\lambda) := \{\lambda^k \mathbb{I} \mid k \in \mathbb{Z}\}$ for a fixed $\lambda \in \mathbb{R}\backslash\{0, \pm 1\}$, with $\mathbb{I}$ being the identity element of $GL(\mathbb{R}^n)$. Of course, $\Lambda(\lambda)$ is a discrete group. The natural action $\Lambda(\lambda) \circlearrowleft \mathbb{R}^n$, however, is not properly discontinuous, because by the very definition of a properly discontinuous action, the stabilizer group $\Lambda(\lambda)_x := \{g \in \Lambda(\lambda) \mid g(x) = x\}$ has to be finite for every $x \in \mathbb{R}^n$. Setting $x_0 := (0, \ldots, 0)$ it is obvious that $\Lambda(\lambda)_{x_0}$ is not finite.

Furthermore, the same argument can be used to show that there is no discrete infinite subgroup of $GL(\mathbb{R}^n)$ whose natural action is properly discontinuous.

The setting of crystallographic groups is a very well known one: let $\mathbb{E}$ denote the real affine space of dimension n and let $\text{Isom}(\mathbb{E})$ be the group of Euclidean motions of $\mathbb{E}$. Crystallographic groups are certain subgroups of $\text{Isom}(\mathbb{E})$:

1.1.3 Definition (Properly discontinuous, crystallographic & Bieberbach groups)**.** *Assume that* $\Gamma \leq \text{Isom}(\mathbb{E})$*. If the natural action of* Γ *on* $\mathbb{E}$ *is properly discontinuous,* Γ *is called a* properly discontinuous *group. Analogously, we will call* Γ *a* crystallographic *group if the natural action* $\Gamma \circlearrowleft \mathbb{E}$ *is crystallographic.*

If Γ *is a torsion free crystallographic group, it is called a* Bieberbach *group.*

As stated before, not every discrete group acts properly discontinuously. However, we have that a subgroup $\Gamma \leq \text{Isom}(\mathbb{E})$ is discrete iff it is properly discontinuous. This follows directly from the fact that $\text{Isom}(\mathbb{E}) = \mathbb{R}^n \rtimes O(\mathbb{R}^n)$ acts properly on $\mathbb{E}$, because the action $\mathbb{R}^n \circlearrowleft \mathbb{E}$ is clearly proper, and so the same holds for the action of an extension of $\mathbb{R}^n$ by a compact group and $O(\mathbb{R}^n)$ is well known to be compact.

It is also possible to define crystallographic groups without employing group actions. One can show that the following definition is equivalent to the one given before:

1.1.4 Definition ((Uniform) lattices & crystallographic groups)**.** *Let* Γ *be a discrete subgroup of a locally compact topological group* G*.*

(i) Γ *is called a* lattice*, if the quotient space* G/Γ *has finite invariant Haar-measure.*

(ii) If G/Γ *is even compact, then* Γ *is said to be a* uniform *lattice.*

(iii) A unifrom lattice $\Gamma \leq \text{Isom}(\mathbb{E})$ *is called a* crystallographic group*.*

Crystallographic groups are very well understood, especially the three *Bieberbach theorems* offer a lot of insight into the structure of such groups.

1.2 The Bieberbach theorems

The three Bieberbach theorems were first proven in [Bi11] and [Bi12]. Before citing these three theorems, we give a definition in order to be able to state the first theorem.

1.2.1 Definition (Virtually)**.** *Given a group* G *and a property* P*, we say that* G *is* virtually P*, if we can find a subgroup* $H \leq G$ *of finite index that has property* P*.*

1.2.2 Theorem (Bieberbach's first theorem)**.** *If* Γ *is a crystallographic group, then* Γ *is virtually abelian.*

We can actually be more precise. The above theorem tells us that there is an abelian subgroup of finite index in Γ. However, we do know more about this subgroup. Namely, it is even a normal subgroup of Γ and can be written as $\Gamma \cap \mathbb{R}^n$. Furthermore, it is a uniform lattice in $\mathbb{R}^n$, and thus isomorphic to $\mathbb{Z}^n$.

Hence, if we are given a crystallographic group, we can always think of it as $\mathbb{Z}^n$ extended by a finite group. Note, however, that in general Γ is only an extension of $\mathbb{Z}^n$ by some finite group but not a semidirect product of $\mathbb{Z}^n$ by some finite group.

Also note that because Γ is a finite extension of $\mathbb{Z}^n$, we also immediately have that Γ is finitely generated.

Viewing Γ as a group of symmetries, we have that Γ is virtually a group of (discrete) translations.

While the theorem above tells us about the nature of a single crystallographic group, Bieberbach's second theorem reveals information about the possible relationship of two such groups:

1.2.3 Theorem (Bieberbach's second theorem)**.** *Let* $\mathrm{Aff}(\mathbb{E})$ *be the group of affine motions of* $\mathbb{E}$ *and let* Γ, $\tilde{\Gamma}$ *be crystallographic groups. Then we have:*

$$\Gamma \cong \tilde{\Gamma} \iff \Gamma \textit{and } \tilde{\Gamma} \textit{ are } \text{conjugate} \textit{ in } \mathrm{Aff}(\mathbb{E})$$

One part of *Hilbert's* $18^t h$ *problem* is the question whether one can find infinitely many different crystallographic groups in $\mathrm{Isom}(\mathbb{E})$ for a fixed dimension. The last of the three Bieberbach theorems answers this question:

1.2.4 Theorem (Bieberbach's third theorem)**.** *Up to dimension and up to isomorphisms, there are only* finitely many different *crystallographic groups.*

In addition to this, crystallographic groups have been completely classified for $\dim(\mathbb{E}) \leq 6$. For instance, there are, up to isomorphism, 17 'different', i.e. not isomorphic/conjugate, such groups if $\dim(\mathbb{E}) = 2$. If $\dim(\mathbb{E}) = 3$, we can find 219 'different' crystallographic groups in $\mathrm{Isom}(\mathbb{E})$.

The classification for $\dim(\mathbb{E}) = 3$ was given independently by Fedorov [Fe91], Schönflies [Sc91] and Barlow [Ba94]. For the case $\dim(\mathbb{E}) = 4$ see [BBNWZ78], $\dim(\mathbb{E}) = 5$ and $\dim(\mathbb{E}) = 6$ are treated in [PS00].

Over time, quite a few considerably different proofs for these theorems have been given. Bieberbach's proof for his first theorem uses a theorem by Minkowksi about simultaneous Diophantine approximation [Mi07]. In [Fr11] Frobenius used the commutativity of unitary matrices. A more modern approach that relies on results by Gromov on almost flat manifolds can be found in [Bu85]. While all these proofs are more geometrical in nature, there are also purely algebraic ways to prove the Bieberbach theorems.

For example, there is a cohomological approach to the Bieberbach theorems. In this regard the main theorem is the following.

Let Γ be an n-dimensional crystallographic group and $G := \Gamma/\mathbb{Z}^n$ and let $N(G, \mathbb{Z}^n)$ denote the normalizer of G in $Aut(\mathbb{Z}^n)$. Furthermore, $H^1(G, \mathbb{R}^n/\mathbb{Z}^n)$ shall be the first cohomology group of G with coefficients in $\mathbb{R}^n/\mathbb{Z}^n$. Then we have the following result.

1.2.5 Theorem. *There is a* bijection *between the conjugation classes (in* $\mathrm{Aff}(\mathbb{R}^n)$*) of crystallographic groups and the orbits* $N(G, \mathbb{Z}^n)\backslash H^1(G, \mathbb{R}^n/\mathbb{Z}^n)$*.*

Proof. A proof may be found in [Sch80]. □

It should be noted that there is an isomorphism: $H^1(G, \mathbb{Z}^n) \cong H^1(G, \mathbb{R}^n/\mathbb{Z}^n)$.
From the above theorem, Bieberbach's third follows immediately. The only thing that needs to be shown is that $H^1(G, \mathbb{R}^n/\mathbb{Z}^n)$ is finite, which is easy to show, see for example [Sch80] or [Br82].

As mentioned before, we present three viewpoints on crystallographic groups. So far we have described crystallographic groups as symmetries as well as by their algebraic properties - lattices of $\mathrm{Isom}(\mathbb{E})$ that act on $\mathbb{E}$ in a certain way. Now we will turn to a topological viewpoint:

1.3 Crystallographic groups and flat manifolds

Let us assume that Γ is a Bieberbach group, i.e. a subgroup of $\mathrm{Isom}(\mathbb{E})$, that, in addition to being crystallographic, is torsion free. Then we have that $\Gamma \circlearrowleft \mathbb{E}$ is *free*, if we let Γ act on $\mathbb{E}$ in the usual way.

It can be shown that the quotient space $M := \Gamma\backslash\mathbb{E}$ has the following properties: it is a complete, compact, flat Riemannian manifold M with fundamental group $\pi_1(M) \cong \Gamma$. Moreover, any complete, compact, flat Riemannian manifold can be constructed this way.

It should be noted, that being torsion free is indeed necessary for $\Gamma\backslash\mathbb{E}$ to be a manifold.

Before we move on, we want to state what is known about the quantity of torsion free crystallographic groups for small dimension. Let $\dim(\mathbb{E}) = n$, then:

(i) if $n = 2$, we can only find two crystallographic groups in $\mathrm{Isom}(\mathbb{E})$ that are torsion free: $\mathbb{Z}^2$ and $\mathbb{Z} \rtimes \mathbb{Z}$. The corresponding manifolds are the torus and the Kleinian bottle, respectively [Ku53].

(ii) if $n = 3, 4$ respectively, there are only 6, 75 respectively, torsion free crystallographic groups.

In the last section we presented the Bieberbach theorems in algebraic terms. Now that we have acquired a new viewpoint on crystallographic groups, we want to carry over the said theorems. In the light of the geometric interpretation, one can state Bieberbach's first and third theorem in the following way:

1.3.1 Theorem (Bieberbach's first theorem - geometric version)**.** *Assume M to be a complete, compact, flat Riemannian manifold of dimension n. It follows that M is covered by the n-torus.*

1.3.2 Theorem (Bieberbach's third theorem - geometric version)**.** *With respect to homeomorphism we can only find finitely many different n-dimensional manifolds, that are complete, compact, flat and Riemannian.*

Of course, viewing crystallographic groups as fundamental groups is not something entirely different as opposed to viewing them as symmetries of the affine space. After all, those symmetries are deck transformations of the cover $\mathbb{E} \to (\Gamma \backslash \mathbb{E})$. This also explains, why we have to assume Γ to be torsion free. The affine space $\mathbb{E}$ is path connected, and thus, using the unique lifting property of paths, we immediately have that the action $\Gamma \circlearrowleft \mathbb{E}$ is free and certainly any $\gamma \in \Gamma$ that acts freely on $\mathbb{E}$ cannot be of finite order. Actually, freely acting properly discontinuous Γ are torsion free and vice versa, see 2.3.2.

2 Affine crystallographic groups

2.0.3 Remark. If we speak about linear spaces, we assume them to be real and of finite dimension.

As stated above, there are a lot of ways one can try to generalize the theory of crystallographic groups, the direction we are interested in is the one of so called *affine crystallographic groups*. At first, they will be defined in algebraic terms and later the connection between them and certain manifolds will be given. However, before we give a definition, we start with a short reminder.

2.1 Affine spaces, affine maps & affine transformations

Affine crystallographic groups are certain symmetry groups of real, finite dimensional affine spaces. In order to understand these groups better, we will give a little overview of such affine spaces.

2.1.1 Definition (Simply transitive action and affine spaces)**.** *Given a group G acting on some set X. This action is called* simply transitive*, if we have that for any $x, y \in G$ there is exactly one $g \in G$ such that $gx = y$. If we have such an action $G \circlearrowleft X$, then X is said to be a* principal homogeneous space *for G.*

Assume $G = V$, with V a linear space, $\dim(V) = n$*. Then we call a principal homogeneous space $\mathbb{E}$ the* affine space *of dimension n.*

Let us examine simply transitive actions a little bit. They are a powerful tool because such actions $G \circlearrowleft X$ allow us to define a group structure on X in such a way that $G \cong X$ via the following: at first, we have to choose a basepoint $x_0 \in X$. For the pointed set (X, x_0) we can write $gx_0 = x$ for any $x \in X$ with an appropriate choice of $g \in G$. Thus,

$$\begin{aligned}(X, x_0) \times (X, x_0) &\to (X, x_0)\\ (x, y) = (gx_0, hx_0) &\mapsto (ghx_0).\end{aligned}$$

Clearly, this composition gives rise to a group structure on (X, x_0). x_0 is the neutral element and for $x = gx_0$ we have $x^{-1} = g^{-1}x_0$. Now look at

$$\begin{aligned}G &\to (X, x_0),\\ g &\mapsto \varphi(g) = gx_0.\end{aligned}$$

Obviously, φ is not only a homomorphism but, by the virtue of the simple transitivity of $G \circlearrowleft X$, even an isomorphism. However, although $G \cong (X, x_0)$, there is no canonical isomorphism between G and X, because one has to choose a basepoint in order to define a group structure.

In view of this, a more informal approach to affine spaces is to say that they are just vector spaces without addition or, maybe better, without origin.
If we are given an action $V \circlearrowleft \mathbb{E}$ we will write this action as

$$\begin{aligned} V \times \mathbb{E} &\to \mathbb{E}, \\ (v, x) &\mapsto x + v, \end{aligned}$$

while for a fixed $v \in V$ we will write

$$\begin{aligned} T_v \colon \mathbb{E} &\to \mathbb{E}, \\ T_v(x) &\mapsto x + v \end{aligned}$$

and call this mapping a *translation* by v. Accordingly, $T\mathbb{E} := V$ is called the *linear space of translations*. Of course, any affine space can be endowed with a smooth atlas, for example via an isomorphism like φ given above. So we can speak in differential geometric terms about $\mathbb{E}$: we have that $T\mathbb{E}$ is the *tangent space* of any $x \in \mathbb{E}$ and, of course, no tangent space comes without *tangent maps* Lf:

Suppose we are given a mapping $f : \mathbb{E} \to \mathbb{B}$ between two affine spaces. We will call f an *affine map*, if we can find an $x \in \mathbb{E}$ and a linear map A such that $f(x+v) = f(x) + A(v)$ for all v. If we have found such an x, the same holds for any other element of $\mathbb{E}$. Now the tangent map of an affine map is defined as its linear part, i.e. $Lf := A$. Of course, we have: f is an isomorphism of affine spaces iff Lf is an isomorphism of linear spaces.

The group of *affine transformations* of $\mathbb{E}$, i.e. automorphisms that respect the action $V \circlearrowleft \mathbb{E}$, will be denoted by $\mathrm{Aff}(\mathbb{E})$. It is easy to see that $\mathrm{Aff}(\mathbb{E}) \cong \mathrm{Aff}(V) = \{\psi(v) \mapsto Av + b \mid A \in GL(V),\ b \in V\}$, i.e. the group of affine transformations of V. Furthermore, we have that $T\mathbb{E}$ is a normal subgroup of $\mathrm{Aff}(\mathbb{E})$ and that the quotient $\mathrm{Aff}(\mathbb{E})/T\mathbb{E}$ is isomorphic to $GL(T\mathbb{E})$. This leads to a short exact sequence:

$$1 \longrightarrow T\mathbb{E} \longrightarrow \mathrm{Aff}(\mathbb{E}) \xrightarrow{L} GL(T\mathbb{E}) \longrightarrow 1,$$

It is also easy to check that this sequence splits, thus we have $\mathrm{Aff}(\mathbb{E}) \cong T\mathbb{E} \rtimes GL(T\mathbb{E})$ - actually, we even have for every $x \in \mathbb{E}$ a splitting homomorphism $\sigma_x : GL(T\mathbb{E}) \to \mathrm{Aff}(\mathbb{E})$ via $\sigma_x(A)(x + v) = x + Av$.

Taking all of the above in consideration, we see that affine spaces and vector spaces are, from a certain point of view, essentially the same thing. Furthermore, of course every linear space acts simply transitively on itself by addition. Thus, it is somewhat justified if we switch from affine to linear spaces, and vice versa, when ever we see fit.

To complete this section, a few words about *homogenization*. We described an affine space $\mathbb{E}$ as a principal homogeneous space for some linear space V, as well as a linear space without origin. Another point of view would be to characterize affine spaces of dimension n as *affine hyperplanes* in a linear space of dimension $n+1$. Starting with an affine space $\mathbb{E}$ and $T\mathbb{E} = V$ we can embed our affine space into $V \oplus \mathbb{R}$ via the following:

(i) Of course, the affine hyperplane $\mathbb{B} := \{(v, 1) \mid v \in V\} \subseteq V \oplus \mathbb{R}$ is an affine space with $T\mathbb{B} = V$.

(ii) On the other hand, let us start with an affine space $\mathbb{E}$ and fix an $x \in \mathbb{E}$. Then any $y \in \mathbb{E}$ can be written unambiguously as $y = T_v(x)$ for some $v \in V$. Now the map

$$\phi_x\colon \mathbb{E} \to \mathbb{B}$$
$$\phi_x(y) = \phi_x(T_v(y)) \mapsto (v, 1)$$

is an isomorphism of affine spaces.

This identification allows us to view $\mathrm{Aff}(\mathbb{E})$ in another light, namely as a subgroup of $GL(V \oplus \mathbb{R})$ via the obvious isomorphism

$$\mathrm{Aff}(\mathbb{E}) \cong \left\{ \begin{pmatrix} Lf & v \\ 0 & 1 \end{pmatrix} \mid f \in \mathrm{Aff}(\mathbb{E}), v \in V \right\} \leq GL((V \oplus \mathbb{R}),$$

or, in other words: every affine group is also a linear one.

Lastly, the composition of two affine maps $f(x) := Ax + v$ and $g(x) := Bx + w$ is given by $fg(x) = B(Ax + v) + w$, or in the homogenized version

$$fg(x) = \begin{pmatrix} AB & Bv + A \\ 0 & 1 \end{pmatrix},$$

and we immediately see that any group G of affine motions gives rise to a short exact sequence

$$1 \to L(G) \to G \to G/L(G) \to 1,$$

whereas the map $L\colon G \to GL(V)$ is defined to be the projection on the linear part, i.e. for of every $f \in G$ with $f(x) = Ax + v$ we have $L(f) = A$.

Be aware that in general such an exact sequence does not split.

As promised, we are now introducing *affine crystallographic* groups.

2.2 Affine crystallographic groups

Whereas crystallographic groups are certain subgroups of $\mathrm{Isom}(\mathbb{E})$, their affine counterparts lie in $\mathrm{Aff}(\mathbb{E})$:

2.2.1 Definition (Affine crystallographic groups). *Let $\Gamma \leq \mathrm{Aff}(\mathbb{E})$ act on the affine space $\mathbb{E}$ in the natural way. If this action is crystallographic, we call Γ an* affine crystallographic *group.*

In section 1.3 we showed the connection of crystallographic groups and differential geometry, i.e. that they are exactly the fundamental groups of compact, complete, flat, Riemannian manifolds, if they are torsion free. We will now establish such a connection for affine crystallographic groups:

2.3 Affine crystallographic groups and flat manifolds

2.3.1 Definition (Affine manifolds & affine lines). *Given a smooth manifold M and an atlas $\mathfrak{A}$. We call $\mathfrak{A}$ an* affine *atlas, if for all charts (U, φ), $(V, \psi) \subseteq \mathfrak{A}$ the transition maps*

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

are locally restrictions of affine maps. M is then said to be an affine *manifold.*
An injective map

$$\iota \colon I \to M, \text{ with } I \text{ an open interval in } \mathbb{R}$$

into such a manifold is called affine line segment*. If this map is even defined on all of $\mathbb{R}$, it is called* affine line*.*

If we now have a flat affine manifold M in which every affine line segment is the restriction of an affine line, we will say M is complete*.*

We now present a few well-known facts:

2.3.2 Facts. Let M be a flat affine manifold, $\dim(M) = n$. It follows,

(i) that the universal cover $\tilde{M}$ of M is a flat affine manifold, too. Furthermore,

(ii) M is complete iff $\tilde{M}$ is complete, and

(iii) if $\tilde{M}$ is indeed complete, we have: $\tilde{M} \cong \mathbb{R}^n$.

It can be shown, that these facts yield a variation of the Hopf-Killing theorem, i.e. they imply that the fundamental group $\Gamma := \pi_1(M)$ is actually a properly discontinuous subgroup of $\mathrm{Aff}(\mathbb{R}^n)$, and thus an affine crystallographic group, with the addition that all stabilizer groups Γ_x, $x \in \mathbb{R}^n$, are all trivial. On a group level this means the following: Γ is torsion free, because we have the following: let G be a subgroup of $\mathrm{Aff}(\mathbb{R}^n)$, that acts, in the usual way, properly discontinuous on $\mathbb{R}^n$, then G is torsion free iff G_x is the trivial group for every $x \in \mathbb{R}^n$:

(i) "$\Rightarrow$" Assume G is torsion free and acts properly discontinuously on $\mathbb{R}^n$ and suppose that $1 < |G_x|$ for some $x \in \mathbb{R}^n$ and fix a $g \in G_x$. Of course, $gx = x$ implies $g^k = x$ for every $k \in \mathbb{Z}$, thus $<g> \leq G_x$. Now $|G_x|$, and a fortiori $<g>$, has to be finite, otherwise it would be a contradiction to our assumption that $G \circlearrowleft \mathbb{R}^n$ is properly discontinuous. On the other hand, the span $<g>$ also cannot be a finite subgroup of G, otherwise the assumption on G to be torsion free would be contradicted. Thus, all stabilizer groups G_x are trivial, i.e. G acts freely on $\mathbb{R}^n$.

(ii) "$\Leftarrow$" It is easily shown that every finite group H in $\mathrm{Aff}(\mathbb{R}^n)$ acting on $\mathbb{R}^n$ in the standard way has a fixed point. But this immediately implies the claim: if a group G is acting on $\mathbb{R}^n$ freely, there is no subgroup H of finite order, which means that G is torsion free.

As stated above, the fundamental group of a compact, complete, affine manifold is torsion free and affine crystallographic. This also works the other way around: if $\Gamma \leq \mathrm{Aff}(\mathbb{E})$ is a torsion free, affine crystallographic group, one can show the quotient space $\Gamma\backslash\mathbb{E}$ is a complete flat affine manifold with $\pi_1(\Gamma\backslash\mathbb{E}) \cong \Gamma$. Of course, by the very definition of crystallographic, $\Gamma\backslash\mathbb{E}$ is also compact.

Later, right after corollary 3.4.5, we will show, that any compact, complete, affine manifold is covered by a complete affine solvmanifold.

It should also be noted, that when one is talking about compact, complete, affine manifolds, one may not simply drop the assumption on completeness, because in contrast to the Riemannian case, the Hopf-Rinow theorem does not hold for affine manifolds, or in other words, there are compact, affine manifolds that are not complete.

Now that we have defined affine crystallographic groups algebraically and given a geometric description of them, we ask ourselves which properties these groups possess. Of course, an obvious question is, whether we can prove Bieberbach-like theorems for these groups as well.

3 Bieberbach-like theorems for affine crystallographic groups?

This section is devoted to two tasks. At first we show the following:

(i) Affine crystallographic groups are not virtually abelian.

(ii) (Abstract) Isomorphic affine crystallographic groups are not conjugate in $\mathrm{Aff}(\mathbb{E})$.

(iii) There are infinitely many isomorphism classes of affine crystallographic groups.

To summarize: we cannot directly generalize Bieberbach's theorems for affine crystallographic groups. In fact, we can show that the first and the third Bieberbach theorem already fail in dimension 3, if we replace *crystallographic* with *affine crystallographic*. see 3.1. To show that a naive 'affine' version of the second Bieberbach theorem does not hold, we will assume $\mathbb{E}$ to be of dimension 6, section 3.2.

After we have shown via examples that the above claims hold, we will pursue the second task of this section: we will explore possible generalizations of Bieberbach's theorems for affine crystallographic groups.

While the generalizations of Bieberbach's first theorem, which will be presented in this section, are somewhat technical, we will examine a conjectured generalization in section 4, which is more straight-forward. This conjecture - Auslander's - states that every affine crystallographic group is virtually solvable. For a thorough definition of the term *solvable* see definition 4.1.1.

Back to our examples: At first we will construct a subgroup $\Gamma(1)$ of $\mathrm{Aff}(\mathbb{R}^3)$, that will be affine crystallographic on the one hand, but on the other hand has no free abelian subgroup of rank 3. To deduce the latter, we will show that for any set S, $|S| = 3$, that generates a free abelian subgroup, one can easily construct a generating set $\tilde{S}$, $|\tilde{S}| = 2$, that yields the same subgroup, hence every free abelian subgroup is of rank at most 2. As a last step, we will use all those generating sets to deduce that $\Gamma(1)$ cannot be virtually abelian and thus invalidating a direct generalization of Bieberbach's first theorem.

Next we will generalize the construction of $\Gamma(1)$ to get infinitely many affine crystallographic groups $\Gamma(i), i \in \mathbb{N}^{\times}$, which are pairwise not isomorphic. These constructions are due to Auslander [Au56].

In the last example, which is due to Grunewald and Segal, [GS94], we will show how to construct a familiy of affine crystallographic and virtually polycyclic groups $\{\Gamma_k \mid k \in \mathbb{N}\}$, which are all isomorphic to each other. Then we will deduce that they are conjugate iff $k = l$, $l \in \mathbb{N}$.

Thereafter, we will first give a generalization of Bieberbach's first theorem, which not only covers the affine situation but is more general. The second generalization is only concerned with the affine setting.

Then we will present a generalization of Bieberbach's second theorem.

The main theorem of the last subsection is a generalization of Bieberbach's third theorem.

3.1 Bieberbach's first and third theorem do not hold in the affine case

As promised, we will now construct an affine crystallographic group Γ that is not virtually abelian. As stated before, this example along with the necessary propositions and their respective proofs are due to Auslander [Au56].

3.1.1 Example (There are affine crystallographic groups that are not virtually abelian)**.** To construct such a group $\Gamma(1)$ we will think of its homogenized version, i.e. $\Gamma(1) < \mathrm{Aff}(\mathbb{R}^3) < GL(\mathbb{R}^4)$. We set $\Gamma(1) := < I, J >$, with

$$I := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } J := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore we set $K := JIJ^{-1}I^{-1}$. We immediately see that

$$K = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$KI = IK, \tag{3.1}$$
$$KJ = JK, \text{ and that} \tag{3.2}$$
$$J^n I^m = K^{nm} I^m J^n \;\forall n, m \in \mathbb{Z}. \tag{3.3}$$

Now the equalities (3.1) through (3.3) immediately yield that any fixed $\gamma \in \Gamma(1)$ may be written as

$$\gamma = K^n I^m J^l = \begin{pmatrix} 1 & l & 0 & n \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

with an appropriate choice of $n, m, l \in \mathbb{Z}$. We will now show that $\Gamma(1)$ is indeed an affine crystallographic group.

3.1.2 Proposition. *The orbit space $M(\Gamma(1)) := \Gamma(1)\backslash\mathbb{E}$ is a compact, complete, flat, affine manifold.*

Proof. According to [Th97] we have the following: let Γ be a group which acts freely and wandering on a connected (Hausdorff) manifold X. Then the quotient space $\Gamma\backslash X$ is a manifold, whereas by wandering action $\Gamma \circlearrowleft X$ we mean, that for every point in X we can find a neighborhood N such that only finite many γN intersect with N, $\gamma \in \Gamma$. We will now show that the natural action $\Gamma(1) \circlearrowleft \mathbb{E}$, with $\mathbb{E}$ being the real affine space of dimension 3, is a free action without accumulation points, and thus also wandering. By the virtue of equation (3.4), we may find $n, m, l \in \mathbb{Z}$, such that any $\gamma(x)$, $\gamma \in \Gamma, x \in \mathbb{E}$, can be written in the following way:

$$\gamma(x) = K^n I^m J^l(x) = \begin{pmatrix} x_1 + lx_2 + n \\ x_2 + m \\ x_3 + l \\ 1 \end{pmatrix} \tag{3.5}$$

If we can show that the Euclidean distance between the two points x and $\gamma(x)$ is always at least 1, as long as n, m and l are not equal to 0 at the same time, which is only the case if γ is the neutral element $e \in \Gamma(1)$, we immediately have that $\Gamma(1)$ acts freely and wandering on $\mathbb{E}$, which in turn yields that $M(\Gamma(1)$ is indeed a manifold.

For the distance we have

$$\begin{aligned} d(\gamma(x), x) &= \sqrt{(x_1 + lx_2 + n - x_1)^2 + (x_2 + m - x_2)^2 + (x_3 + l - x_3)^2 + (1-1)^2} \\ &= \sqrt{(lx_2 + n)^2 + m^2 + l^2}. \end{aligned}$$

Obviously, if at least one of $\{n, m, l\}$ is not equal to 0, we have $d(\gamma(x), x) \geq 1$.

Let us move on to the question of compactness of $M(\Gamma(1))$. Let F denote the unit cube of $\mathbb{E}$:

$$F := \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x \in \mathbb{E}, 0 \leq x_i \leq 1, 1 \leq i \leq 3 \right\}$$

From equation (3.5) we deduce that under the action of $\Gamma(1)$ any $x \in \mathbb{E}$ is equivalent to an element in F and furthermore that any two elements of F are certainly not equivalent under this action, because, as stated above, the distance between x and $\gamma(x)$ is always at least one $\forall \gamma \neq e$. Thus F is a fundamental domain and therefore $M(\Gamma(1))$ is compact.

We will now deduce that $M(\Gamma(1))$ admits an affine structure. Of course, the affine space $\mathbb{E}$ has a canonical flat linear connection ∇, which is invariant under affine transformations. It can be shown, that ∇ induces a flat affine connection on every $\Gamma\backslash\mathbb{E}$, if Γ is a discrete and torsion free subgroup of $\mathrm{Aff}(\mathbb{E})$. But from the very definition of $\Gamma(1)$ and equation (3.4) it follows that $\Gamma(1)$ is discrete as well as torsion free, which yields the existence of a flat affine connection for $M(\Gamma(1))$. Considering the fact that the existence of such a connection is equivalent to the existence of an affine atlas, we follow that $M(\Gamma(1))$ is an affine manifold. There is one last property, completeness of $M(\Gamma(1))$, to be checked, in order to finalize our proof.

Clearly, $\mathbb{E}$ is complete. It is also the universal cover for $M(\Gamma(1))$. Therefore $M(\Gamma(1))$ is complete as well because a manifold is complete iff its universal cover is complete. □

By using the considerations from section 1.3 in conjunction with the proposition, i.e. $\Gamma(1)$ is the fundamental group of the compact, complete, flat, affine manifold $M(\Gamma(1))$, we deduce that $\Gamma(1)$ is an affine crystallographic group. We will now show that it is not virtually abelian by proving the following proposition.

3.1.3 Proposition. $\Gamma(1)$ *has no free abelian subgroup of rank 3.*

Proof. Let $\gamma_1, \gamma_2, \gamma_3 \in \Gamma(1)$. Once again, with an appropriate choice of integers n_i, m_i, l_i, $1 \leq i \leq 3$, we have

$$\gamma_i = K^{n_i} I^{m_i} J^{l_i}.$$

This, together with the equations (3.1) - (3.3), yields

$$\begin{aligned} \gamma_i\gamma_j &= K^{n_i+n_j+l_i m_j} I^{m_i m_j} J^{l_i l_j}, \text{ and} \\ \gamma_j\gamma_i &= K^{n_i+n_j+l_j m_i} I^{m_i m_j} J^{l_i l_j}. \end{aligned} \tag{3.6}$$

We conclude that $\gamma_i\gamma_j = \gamma_j\gamma_i$ if and only if $l_i m_j = m_i l_j$. In order to prove the proposition, we will look at the five different cases, in which γ_1, γ_2 and γ_3 commute with each other. Then we will examine the rank of the free abelian subgroup $< \gamma_1, \gamma_2, \gamma_3 > \leq \Gamma(1)$.

Case 1: $m_1 = m_2 = m_3 = 0$.
In this case all γ_i are reduced to the form $\gamma_i = K^{n_i} J^{l_i}$. Recall that $KJ = JK$. Hence, we see that the span $< \gamma_1, \gamma_2, \gamma_3 >$ is a subgroup of the free abelian group $< K, J >$. Clearly a free abelian group of rank 2 does not have a free abelian subgroup of rank 3. It follows that $<\gamma_1, \gamma_2, \gamma_3>$ is of rank at most 2.

Case 2: $l_1 = l_2 = l_3 = 0$.
In this case we can make the exact same argument as we did in *Case 1*.

Case 3: $m_1 \neq 0, m_2 = m_3 = 0$.
Here we have to have $l_2 = l_3 = 0$ if we want γ_1 to commute with γ_2 as well as with γ_3. Looking at *Case 2*, the only thing to check is what happens if $l_1 \neq 0$. Assuming this is case, we have $\gamma_1 = K^{n_1} I^{m_1} J^{l_1}$, $\gamma_2 = K^{n_2}$ and $\gamma_3^{n_3}$. This time, we conclude that the span of the γ_i has to be equal to the span of K and $I^{m_1} J^{l_1}$. hence $<\gamma_1, \gamma_2, \gamma_3>$ is the free abelian group of rank 2.

Case 4: $m_1 \neq 0, m_2 \neq 0, m_3 = 0$.
We want γ_3 to commute with γ_1 and γ_2. For this to be the case l_3 has to be 0. This

implies that $\gamma_1 = K^{m_1} I^{n_1} J^{l_1}$, $\gamma_2 = K^{m_2} I^{n_2} J^{l_2}$ and $\gamma_3 = K^{m_3}$. In order for γ_1 and γ_2 to commute, we need that $l_1 m_2 = m_1 l_2$. This is true, if either $l_1 = l_2 = 0$ or if $l_1 \neq 0$ and $l_2 \neq 0$. The first possibility would lead to *Case 2*, so let us move on to the second one. If both l_1 and l_2 are not equal to 0 it follows that $\gamma_1^{l_2}\gamma_2^{-l_1} = K^a$ for some integer a. Furthermore, n_3 may not be equal to 0, otherwise γ_3 would be the neutral element and $<\gamma_1, \gamma_2, \gamma_3>$ would be the free abelian group of rank 2. Thus we assume $n_3 \neq 0$ and define d to be the greatest common divisor of n_3 and a. Thus we have $a = dr$ and $n_3 = ds$ for some $r, s \in \mathbb{Z}$. From $\gamma_3 = K^{n_3} = (K^d)^s$ we deduce $<\gamma_3> \leq <K^d>$, which yields $<\gamma_1, \gamma_2, \gamma_3> \leq <\gamma_1, \gamma_2, K^d>$. Let's have a closer look at $<\gamma_1, \gamma_2, K^d>$:
The equation

$$\gamma_1^{l_2}\gamma_2^{-l_1} = K^a = (K^d)^r$$

yields the following relation for our group $<\gamma_1, \gamma_2, K^d>$:

$$\gamma_1^{l_2}\gamma_2^{-l_1}(K^d)^{-r} = e.$$

Now, a free abelian group does not have any relations, besides the ones denoting commutativity of course. Therefore, the group $<\gamma_1, \gamma_2, K^d>$ cannot be of rank 3, and consequently its subgroup $<\gamma_1, \gamma_2, \gamma_3>$ cannot be of rank 3 either.

Case 5: $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0$ as well as $l_1 \neq 0, l_2 \neq 0, l_3 \neq 0$.
At first assume $m_1 = m_2 = m_3 = m$, $m \in \mathbb{Z}$. For $<\gamma_1, \gamma_2, \gamma_3>$ to be abelian we then also have to have $l_1 = l_2 = l_3 = l \in \mathbb{Z}$ and therefore

$$\begin{aligned}
\gamma_1 &= K^{n_1} I^m J^l \\
\gamma_2 &= K^{n_2} I^m J^l \\
\gamma_3 &= K^{n_3} I^m J^l.
\end{aligned}$$

From this, it immediately follows that $<\gamma_1, \gamma_2, \gamma_3> \leq <K, I^m J^l>$ with $<K, I^m J^l>$ obviously being free abelian with rank 2 and so its subgroup $<\gamma_1, \gamma_2, \gamma_3>$ cannot be free abelian of rank 3. Hence we drop the assumption $m_1 = m_2 = m_3$ and instead demand $m_1 \neq m_2$ as well as $n_1 \neq n_2$. This gives us

$$\begin{aligned}
m_1\frac{(l_2 - l_3)}{(l_2 - l_1)} + m_1\frac{(l_3 - l_1)}{(l_2 - l_1)} &= m_3 \\
l_1\frac{(m_2 - m_3)}{(m_2 - m_1)} + l_2\frac{(m_3 - m_1)}{(m_2 - m_1)} &= l_3.
\end{aligned}$$

With $m_i l_j = m_j l_i, 1 \leq i, j \leq 3$ in mind, so that $\gamma_1, \gamma_2, \gamma_3$ commute pairwise, we follow

$$\begin{aligned}
\frac{(l_2 - l_3)}{(l_2 - l_1)} &= \frac{(m_2 - m_3)}{(m_2 - m_1)} =: k_1, \text{ and} \\
\frac{(l_3 - l_1)}{(l_2 - l_1)} &= \frac{(m_3 - m_1)}{(m_2 - m_1)} =: k_2.
\end{aligned}$$

These two sets of equations yield

$$\begin{aligned} m_1k_1 + m_2k_2 &= m_3, \\ l_1k_1 + l_2k_2 &= l_3. \end{aligned}$$

Using this and formula (3.6) above, which tells us what $\gamma_i\gamma_j$ looks like, and an appropriate choice of $a \in \mathbb{Z}$, we get

$$\begin{aligned} \gamma_1^{k_1}\gamma_2^{k2} &= K^a I^{m_1k_1+m_2k_2} J^{l_1k_1+l_2k_2} \\ &= K^a I^{m_2} J^{l_2} \\ &= K^a(K^{-r}K^r)Im_2J^{l_2} \\ &= K^{a-r}\gamma_3. \end{aligned}$$

This yields $<\gamma_1,\gamma_2,\gamma_3> = <\gamma_1,\gamma_2,K^{a-r}>$. In order to prove that $<\gamma_1,\gamma_2,\gamma_3>$ is not free abelian of rank 3, it suffices to proof this for $<\gamma_1,\gamma_2,K^{a-r}>$, but this situation is exactly the same as in *Case 4*.

Furthermore, these were obviously all the cases in which $\gamma_,\gamma_2$ and γ_3 commute pairwise, and of course all the cases are without loss of generality. Hence we have finished our proof, as well as our example, i.e. we showed that $\Gamma(1)$ is an affine crystallographic group that is not virtually abelian of rank 3. □

Our next task is to use the generating sets for free abelian subgroups of $\Gamma(1)$ we constructed in the above proposition, to show that

3.1.4 Proposition. $\Gamma(1)$ *is not virtually abelian.*

Proof. We start by observing that no abelian subgroup $\tilde{\Gamma} \leq \Gamma$ is finite. That this is indeed the case, can be immediately deduced by looking at (3.4) or (3.6). Hence, any abelian subgroup of $\Gamma(1)$ is also free abelian and further, in view of proposition 3.1.3, any such subgroup is of rank either 1 or 2.

(i) If the rank of $\tilde{\Gamma}$ is 1, we trivially have that $|\Gamma(1)/\tilde{\Gamma}|$ is not finite by using that $\tilde{\Gamma} = <K^nI^mJ^l>$, for some $n, m, l \in \mathbb{Z}$, at least one of them being not equal to 0, and our multiplication formula (3.6).

(ii) Similarly, if the rank of $\tilde{\Gamma}$ is 2, we immediately know generating elements of $\tilde{\Gamma}$ by looking at the different cases we examined in proposition 3.1.3. Using this along with (3.6), we also have that $|\Gamma(1)/\tilde{\Gamma}|$ cannot be finite.

The main point in both arguments is, that we always have at least $\pi(<I>) \cong <I>$ or $\pi(<J>) \cong <J>$, with π being the natural projection from $\Gamma(1)$ onto $\Gamma(1)/\tilde{\Gamma}$. □

We will now construct infinitely many affine crystallographic subgroups $\Gamma(i) \leq \mathrm{Aff}(\mathbb{E})$, $i \in \mathbb{N}^+$, $\mathbb{E}$ being the real affine space of dimension 3, that are pairwise not isomorphic and thus we show that Bieberbach's third theorem does not hold if one substitutes *affine crystallographic* for *crystallographic*.

3.1.5 Example (There are infinitely many different conjugacy classes of affine crystallographic groups in $\mathrm{Aff}(\mathbb{E})$)**.** The groups we construct build upon the definition of $\Gamma(1)$. So, as in the example above, we think of $\mathrm{Aff}(\mathbb{R}^3)$ as a subgroup of $GL(\mathbb{R}^4)$ and define

$$I := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } J := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, we fix an $i \in \mathbb{N}^+$ and set

$$K(i) := \begin{pmatrix} 1 & 0 & 0 & \frac{1}{i} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By definition $K(1)$ is our K from above, i.e. $K(1) = K = JIJ^{-1}I^{-1}$. Now we define $\Gamma(i) := < I, J, K(i) >$. Every element $\gamma \in \Gamma(i)$ is of the form

$$\gamma = (K(i))^a (K(1))^n I^m J^l$$

for some integers a, n, m, l. Additionally we have $0 \leq a < i$. From the equation above we deduce that we can write the product of two elements γ_1 and γ_2 in $\Gamma(i)$ in this way:

$$\begin{aligned} \gamma_1\gamma_2 &= (K(i))^{a_1}(K(1))^{n_1} I^{m_1} J^{l_1} (K(i))^{a_2} (K(1))^{n_2} I^{m_2} J^{l_2} \\ &= (K(i))^{b} (K(1))^{n_1+n_2+l_1 m_2 + k} I^{m_1+m_2} J^{l_1 l_2}, \end{aligned}$$

whereas

$$b + ki = a_1 + a_2, \text{ with } 0 \leq b < i.$$

Of course, looking at the multiplication formula, we see that $K(1)$ commutes with I, J and $K(i)$. Actually, we have more: The commutator subgroup $[\Gamma(i), \Gamma(i)]$ is exactly the span of $K(1)$. Using this, together with the rule for multiplication, we deduce that $\Gamma(i)/[\Gamma(i), \Gamma(i)] \cong \mathbb{Z}_i \times \mathbb{Z} \times \mathbb{Z}$. This yields $\Gamma(i) \cong \Gamma(j)$ iff $i = j$. Hence, there are infinitely many isomorphism classes. Of course, this immediately yields that there are also infinitely many different conjugacy classes.

To show that $\Gamma(i)\backslash\mathbb{E}$ is a compact, complete, flat affine manifold, and thus that $\Gamma(i)$ is a (torsion free) affine crystallographic group, an analogous argument to the one given can be made to show that the special case for $\Gamma(1)$ holds. Therefore we have shown that there are infinitely many isomorphism classes of affine crystallographic groups in $\mathrm{Aff}(\mathbb{E})$.

3.2 Bieberbach's second theorem does not hold in the affine case

Now that we have shown that one cannot go the easy route when generalizing Bieberbach's first and third theorem, we will present that a direct generalization of the second one also fails.

The construction, which yields the desired example as well as necessary propositions are due to Grunewald and Segal, [GS94].

3.2.1 Example (There are isomorphic affine crystallographic groups that are not conjugate in $\mathrm{Aff}(\mathbb{E})$)**.** To prove the claim, the procedure is as follows: At first we will construct a simply transitive group G and afterwards an affine crystallographic subgroup $\Gamma \leq G$. We will go on by defining a family of groups Γ_k, $k \in \mathbb{N}^\times$, such that $\Gamma = \Gamma_1 \cong \Gamma_k$ for any k. Then we will show that two such groups Γ_k and Γ_l are conjugate iff $k = l$, which clearly gives us a counterexample to Bieberbach's second theorem in the affine case.

We start by defining a couple of maps. The first one is defined as

$$\begin{aligned} \nu\colon \mathbb{R}^6 &\to \mathrm{Aff}(\mathbb{R}^6) \\ (r, y, \tilde{x}, \tilde{z}) &\mapsto \nu(r, y, \tilde{x}, \tilde{z}) = \begin{pmatrix} 1 & 0 & \tilde{x}^t & 0 & r + \frac{1}{2}||\tilde{x}|| \\ 0 & 1 & 0 & 0 & y \\ 0 & 0 & \mathbb{I}_2 & 0 & \tilde{x} \\ 0 & 0 & 0 & \mathbb{I}_2 & \tilde{z} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

with $r, y \in \mathbb{R}$, $\tilde{x}, \tilde{z} \in \mathbb{R}^2$ and $\mathbb{I}_2$ denoting the 2×2-identity matrix. Clearly, ν is a homomorphism, and thus $U := \nu(\mathbb{R}^6)$ is an abelian and unipotent group. Furthermore, ν is injective and we have $\mathbb{R}^6 \cong U$.

The second map we will make use of is defined as follows

$$\begin{aligned} E\colon \mathbb{R} &\to Gl(\mathbb{R}^2) \\ r &\mapsto E(r) = \begin{pmatrix} \cos 2\pi r & \sin 2\pi r \\ -\sin 2\pi r & \cos 2\pi r \end{pmatrix}. \end{aligned}$$

We also set

$$\begin{aligned} \tau\colon \mathbb{R}^6 &\to \mathrm{Aff}(\mathbb{R}^6) \\ (r, y, \tilde{x}, \tilde{z}) &\mapsto \tau((r, y, \tilde{x}, \tilde{z})) = diag(1, 1, (E(r), e^y, e^{-y}, 1), \end{aligned}$$

with $(r, y, \tilde{x}, \tilde{z})$ as before and $diag(\cdot)$ denoting the diagonal matrix with the respective entries. The map τ is a homomorphism and consequently so is the map $\theta := \tau \circ \nu^{-1}\colon U \to S =: \mathrm{im}(\tau)$. For the kernel of θ we have

$$\ker(\theta) = \nu(\ker(\tau)) = \nu(\mathbb{Z} \oplus 0 \oplus \mathbb{R}^4). \tag{3.7}$$

We fix $g_1, g_2 \in \mathbb{R}^6$, $g_i = (r_i, y_i, \tilde{x}_i, \tilde{z}_i)$, $1 \in \{1, 2\}$ and set $\tilde{z}_1 = (z_1, z_2)$. Then we have

$$\begin{aligned} \theta(\nu(h)^{-1}\nu(g)\nu(h)) &= diag(r_1, y_1, \tilde{x}_1 E(r_2), e^{-y_2} z_1, e^{y_2} - z_2), \text{ and} \\ [\theta(g), \theta(h)] &= diag(0, 0, \tilde{x}E(r_2) - \tilde{x}, e^{-y_2} z_1 - z_1, e^{y_2} z_2 - z_2). \end{aligned} \tag{3.8}$$

We see $[U, S] \leq \ker(\theta)$ and hence

$$G := \{g\theta(g) \mid g \in U\} = \{\nu(v)\tau(v) \mid v \in \mathbb{R}^6\} \leq \mathrm{Aff}(\mathbb{R}^6\}$$

is a simply transitive group. This can be deduced from [GS94], theorem 5.5. Furthermore, $\psi\colon \mathbb{R}^6 \to G$, $\psi(v) := \nu(v)\tau(v)$ is a surjective homeomorphism.

We go on by fixing an $L \leq \mathbb{R}^4$ with $L \cong \mathbb{Z}^4$ and choosing an approriate $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^\times$, such that

$$L^t \begin{pmatrix} E(\alpha) & 0 & 0 \\ 0 & 0 & e^{-\beta} \\ 0 & 0 & e^{\beta} \end{pmatrix} = L^t. \tag{3.9}$$

At the end of this section, we will show that such L, α and β exist indeed.

We set

$$\gamma := \psi(\alpha, \beta, 0, 0, 0, 0)$$

and deduce from equation (3.8) that for $A \leq \mathbb{R}^2$ we have

$$\gamma^{-1}\nu(A \oplus L)\gamma = \nu(A \oplus L).$$

This yields

$$\Gamma_A := \nu(A \oplus L) <\gamma> \leq \mathrm{Aff}(\mathbb{R}^6).$$

Let $(a, b) \in A \oplus L$. Using equation (3.7) we see

$$\nu(a, b) \in G \iff (ab) \in \mathbb{Z} \oplus 0 \iff \nu(a, b) = \psi(a, b).$$

Now, if $\mathbb{Z}\oplus 0 \leq A$, then $\Gamma_A \cap G = \Gamma_{\mathbb{Z}\oplus 0} =: \Gamma$. Furthermore, recalling that U is abelian, we have $\Gamma \lhd \Gamma_A$. Equation (3.8) implies that γ lies in the normalizer of $\nu(\mathbb{Z}\oplus 0 \oplus L)$. We also have $\Gamma_a/\Gamma \cong A/(\mathbb{Z} \oplus 0)$, because ν is an isomorphism. Thus, setting $A(k) := k^{-1}\mathbb{Z} \oplus 0$, $k \in \mathbb{N}^\times$, and $\Gamma_k := \Gamma_{A(k)}$, we see $\Gamma \lhd \Gamma_k$ as well as $|\Gamma_k : \Gamma| = k$ and $\Gamma_k \cap G = \Gamma$.

Of course, we also have $\Gamma_k \cong \Gamma_l$ for any $k, l \in \mathbb{N}^\times$, especially $\Gamma = \Gamma_1 \cong \Gamma_k\ \forall k$.

3.2.2 Proposition. *Γ is an affine crystallographic group, if α is irrational.*

Proof. Our group Γ can be written as

$$\Gamma = \{\psi(m + n\alpha, n\beta, a) \mid m, n \in \mathbb{Z}, a \in L\},$$

and we see that $\mathbb{Z}^6 \cong \psi^{-1}(\Gamma) \leq \psi^{-1}(G) = \mathbb{R}^6$. In other words, $\psi^{-1}(\Gamma)$ is discrete and cocompact in $\psi^{-1}(G)$. Thus, because ψ is a homeomorphism, we have that Γ is discrete and cocompact in G. Furthermore, one can show that Γ is Zariski dense in G iff $<E(\alpha)>$ is Zariski dense in $SO(\mathbb{R}^2)$ iff α is irrational.

Because G is simply transitive, we can employ [GS94], theorem D, which basically says that in such a situation, Γ is an affine crystallographic group. □

From now on we will assume α to be irrational, so that Γ is affine crystallographic. For L we require

$$(1, 0, z_1, z_2) \in L, z_1, z_2 \in \mathbb{R}, \text{ and} \tag{3.10}$$
$$(\tilde{x}, \tilde{z}) \in L \to ||\tilde{x}|| \text{ is an algebraic integer.} \tag{3.11}$$

That we may impose such obstructions on L and α will also be shown at the end of this section.

We are now ready to prove the main proposition of this section.

3.2.3 Proposition. *Γ_k and Γ_l are conjugate in* $\mathrm{Aff}(\mathbb{R}^6)$ *iff $k = l$.*

Proof. Now let $k, l \in \mathbb{N}^\times$. We assume there is a $g \in \mathrm{Aff}(\mathbb{R}^6)$, such that $g^{-1}\Gamma_k g = \Gamma_l$. Because $u(\Gamma_A) = \nu(A \oplus L)$ is the maximal normal unipotent subgroup of Γ_A, we have

$$\nu(A_k \oplus L)g = g\nu(A_l \oplus L).$$

The center of Γ_A is exactly $\nu(A \oplus 0)$, because by assumption α is irrational. Furthermore, we have

$$\nu(A_k \oplus)g = g\nu(A_l \oplus 0). \tag{3.12}$$

If g_{ij} is the (i,j)-entry of g, then we get from equation (3.12), that

$$\begin{aligned} k^{-1}\mathbb{Z} &= g_{11} l^{-1}\mathbb{Z}, \text{ and} \\ 0 &= g_{i1} l^{-1}\mathbb{Z} \;\forall\, 2 \le i \le 6. \end{aligned}$$

Consequently,

$$g_{11} = \pm\frac{l}{k}, \text{ and} \tag{3.13}$$

$$g_{21} = \cdots = g_{61} = 0. \tag{3.14}$$

We choose $a \in A_k$, $b \in A_l$ and $x, y \in L$, such that

$$\nu(a + x)g = g\nu(b + y).$$

Then we have the following set of linear equations:

$$x_1 g_{33} + x_2 g_{43} = g_{11} y_1, \tag{3.15}$$

$$x_1 g_{34} + x_2 g_{44} = g_{11} y_2, \text{ as well as} \tag{3.16}$$

$$x_1 g_{35} + x_2 g_{45} = 0, \text{ and} \tag{3.17}$$

$$x_1 g_{36} + x_2 g_{46} = 0. \tag{3.18}$$

By the power of equation (3.13), we have

$$x_1 = g_{33} y_1 + g_{34} y_2 + g_{35} y_3 + g_{36} y_4, \text{ and} \tag{3.19}$$

$$x_2 = g_{43} y_1 + g_{44} y_2 + g_{45} y_3 + g_{46} y_4. \tag{3.20}$$

Choosing x_1 and x_2 linearly independent, equations (3.17) and (3.18) imply

$$g_{35} = g_{45} = g_{36} = g_{46} = 0. \tag{3.21}$$

Recalling assumption (3.10), we set $(y_1, y_2) := (1, 0)$ and deduce from (3.19) through (3.21) that $(x_1, x_2) = (g_{33}, g_{43})$. Furthermore, equations (3.15) and (3.16) yield $||(x_1, x_2)|| = g_{11}$, which equals $\pm\frac{l}{k}$ according to (3.13). We also assumed (3.11), and hence, $\frac{l}{k}$ is an algebraic integer. Consequently, $k|l$ and, because of symmetry, $l|k$. But this means $k = l$. □

Though we have shown that our claim holds, there is one last thing to check, namely that such L, α and β exist indeed. To proof this, it suffices to give an algebraic number field $\mathbb{K}$ over $\mathbb{Q}$ of degree 4 that admits two real and two non-real complex embeddings

$$\sigma_1, \sigma_2 \colon \mathbb{K} \to \mathbb{R}, \text{ and}$$
$$\mu, \overline{\mu} \colon \mathbb{K} \to \mathbb{C},$$

and furthermore has a unit $\epsilon \neq \pm 1$ in the ring of integers $O_{\mathbb{K}}$ such that

$$\sigma_1(\epsilon) = \sigma_2(\epsilon) > 0, \text{ and}$$
$$||\mu(\epsilon)|| = 1.$$

To achieve this, we can for example define ϵ to be a root of $x^4 - 4x^3 + 4x^2 - 4x + 1$ and set $\mathbb{K} := \mathbb{Q}(\epsilon)$.

We can now define L via the map

$$\xi := \mu \times \sigma_1 \times \sigma_2 \colon O_{\mathbb{K}} \to \mathbb{C} \oplus \mathbb{R}^2.$$

If we set

$$\begin{aligned} L &:= \operatorname{im}(\xi), \\ \alpha &:= \frac{1}{2} arg(\mu(\epsilon)), \text{ and} \\ \beta &:= -log(\sigma_1(\epsilon)), \end{aligned}$$

one can check that $L \cong \mathbb{Z}^4$ and also that assumption (3.9) holds. The definitions above also satisfy $\beta \neq 0$ and the irrationality of α. And from $(1, 0, 1, 1) = \xi(1) \in L$ we deduce that we may also assume (3.10). Lastly, (3.11) is also not violated because, if $(\tilde{x}, \tilde{z}) = \xi(c)$, then we have $||\tilde{x}|| = \mu(c)\overline{\mu}(c)$.

3.3 Three generalizations of Bieberbach's first theorem for the affine case

To state the three generalizations, we need the Zariski topology. Thus, we start with a short recap thereof.

3.3.1 Definition (Zariski topology, affine varieties and algebraic (sub)groups)**.** *For a fixed set S of polynomials over $\mathbb{C}$ with n variables we set*

$$V(S) := \{x \in \mathbb{B} \mid f(x) = 0 \; \forall f \in S\},$$

and call $V(S)$ an algebraic *set. The* Zariski *topology on $\mathbb{C}^n$ is then defined via these algebraic sets, namely they are exactly the closed sets of this topology.*

Of course, the real linear space $\mathbb{R}^n$ embeds into $\mathbb{C}^n$, and we define the Zariski topology on $\mathbb{R}^n$ to be the restriction of the Zariski topology on $\mathbb{C}^n$ to $\mathbb{R}^n$.

If we are given a nonempty algebraic set V, that is not the union of two proper algebraic sets, we say that V is an affine variety*. Any affine variety V can naturally be equipped with the Zariski topology by defining a set $W \subseteq$ to be closed if W is an algebraic set.*

An affine variety that is also a group is called an algebraic *group. Lastly, if G is an algebraic group, then we call $H \leq G$ an* algebraic subgroup*, if H is closed with respect to the Zariski topology on G.*

The definition of the Zariski topology we just presented is the classical approach. A more modern one would be to define closed sets as follows: Let A be an abelian ring. Then the Zariski closed sets are defined to be the sets $V(I) := \{P \in Spec(A) \mid I \subseteq P\}$, with $I \trianglelefteq R$ being an ideal and $Spec(A) := \{P \trianglelefteq R \mid P \text{ is a proper prime ideal}\}$ being the spectrum of A.

One of the most prominent examples of algebraic groups is $\mathrm{Aff}(\mathbb{C}^n)$. Thus, it makes sense to talk about the Zariski closure of any (real) linear group.

We can now state a generalized version of Bieberbach's first theorem.

3.3.2 Theorem (Bieberbach's first theorem - generalized version)**.** *Given a Lie group G, a closed connected solvable subgroup $R \trianglelefteq G$ with natural projection $\pi\colon G \to G/R$. If H is a closed subgroup of G, and if the identity component H^0 is solvable, then the identity component of the Zariski closure of the projection of H, $\overline{\pi(H)}^0$, is also* solvable.

Proof. A proof may be found in [Au63] or [Ra72]. □

Of course, setting $G := \mathrm{Aff}(\mathbb{R}^n)$, $R := \mathbb{Z}^n$ and assuming H to be crystallographic, we immediately see that Bieberbach's theorem is a corollary.

Besides Bieberbach's first theorem, there is another useful corollary derived in [Ra72]. We will make use of both the theorem and the corollary later in section 4.

3.3.3 Corollary. *Given a Lie group G with radical R. If $\Gamma \leq G$ is discrete and $\pi(\Gamma)$ is Zariski dense in G/R, then $\pi(\Gamma)$ is also* discrete*.*

It should be noted that two other corollaries in [Ra72], 82.5 and 8.28, do not hold, as Witte showed according to Abels, [Ab01].

We go on by giving another generalization of the first of the Bieberbach theorems, namely for discrete subgroups of $\mathrm{Aff}(\mathbb{R}^n)$. In order to state the proposition we need a definition and a little more notation.

3.3.4 Definition (Unipotent element and unipotent group)**.** *Given a ring R with 1 being the identity element of the multiplication. An $r \in R$ is called* unipotent*, if $(r-1)$ is nilpotent, i.e. if there is a $k \in \mathbb{N}^\times$, such that $(r-1)^k = 0$.*

Accordingly, a linear group is called unipotent*, if all its elements are unipotent.*

Furthermore, set $\Gamma_{nd} := \Gamma \cap L^{-1}(\overline{L(\Gamma)}^0)$. Abels suggests, [Ab01], to think of Γ_{nd}, respectively $L(\Gamma)_{nd} := L(\Gamma) \cap \overline{L(\Gamma)}^0$ as the non-discrete part of Γ, $L(\Gamma)$ respectively.

3.3.5 Proposition (Bieberbach's first theorem - an affine version)**.** *Given a discrete subgroup $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$. In this case Γ_{nd} is a nilpotent, finitely generated group. If furthermore Γ is crystallographic, we have that Γ_{nd} as well as $L(\Gamma_{nd})$ are* unipotent.

Proof. This assertion was proven in [CD89]. □

To end this section, we present yet another generalization that holds for affine crystallographic groups, if we furthermore assume them to be virtually solvable.

To put the upcoming theorem into perspective we recall that Bieberbach's first theorem says, that any crystallographic group Γ contains a subgroup $\Delta \cong \mathbb{Z}^n$, that is given via $\Gamma \cap \mathbb{R}^n = \Delta$. Trivially, Δ is a subgroup of an abelian group that acts simply transitively, via translations, on $\mathbb{R}^n$. In view of this, we have the following generalization for virtually solvable affine crystallographic groups:

3.3.6 Theorem (Bieberbach's first theorem - another affine version)**.** *Given a virtually solvable affine crystallographic group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$, there is closed connected solvable Lie group $G \leq \mathrm{Aff}(\mathbb{R}^n)$, such that*

(i) $G \circlearrowleft \mathbb{R}^n$ is simply transitive,

(ii) $\Delta := \Gamma \cap G$ is a finite index subgroup of Γ and the quotient G/Δ is compact, and

(iii) for the Zariski closures of Δ and G we have $\overline{\Delta} = \overline{H}$.

Proof. See [FG83]. □

On a side note, it is also possible to achieve more general results, see theorems 3.4.1 and 3.4.3.

We can also put the above assertion into geometric terms: because G acts simply transitively, it is possible to pull back the affine structure on $\mathbb{E}$, i.e. the affine atlas, to G via any evaluation map from G to $\mathbb{E}$. Furthermore, if the pulled back affine structure on G is invariant under muliplication from the left, it induces an affine atlas on $\Gamma\backslash G$. If we require Γ to be torsion free, we then have that the complete affine manifold $\Gamma\backslash\mathbb{E}$ is affinely equivalent to the complete affine solvmanifold $\Gamma\backslash G$. In the light of this and the above corollary, we deduce that M is a compact complete affine manifold, such that if the fundamental group is virtually solvable, then M is covered by a complete affine solvmanifold.

3.4 A generalization of Bieberbach's second theorem for the affine case

While the generalizations we want to present for Bieberbach's second and third theorem are quite different from each other, they are both based on common definitions and results due to Fried and Goldman, [FG83]. These results also serve as a corner stone for the classification of affine crystallographic virtually solvable groups - once again we refer to 4.1.1 for a comprehensive definition of the term *solvable*.

3.4.1 Theorem. *Suppose $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ is a properly discontinuous group and virtually solvable. Then there is a $G \leq \mathrm{Aff}(\mathbb{R}^n)$ with $\Gamma \leq G$, such that*

(i) G decomposes into finitely many components $G_1, \dots, G_k$ with $G_i \cap \Gamma \neq \emptyset\ \forall 1 \leq i \leq k$,

(ii) the quotient G/Γ is compact,

(iii) for the Zariksi closures of Γ and G we have $\overline{\Gamma} = \overline{G}$, and

(iv) G_x is finite $\forall x \in \mathbb{E}$.

Furthermore, G is unique, if there is a $\tilde{\Gamma} \leq \Gamma$ with finite $[\Gamma : \tilde{\Gamma}]$, such that all $\gamma \in L(\tilde{\Gamma})$ have only real eigenvalues.

Proof. This theorem is proven in [FG83]. □

3.4.2 Definition (Crystallographic hull)**.** *Let Γ and G be as in theorem 3.4.1. Then G is called a* crystallographic hull *of Γ.*

Before we go on, we present a short example of crystallographic hulls, in order to see that in general they are indeed not unique. This example is taken from [FG83].

We think 'homogenizised' and set

$$\Gamma := \left\{ \begin{pmatrix} r^k\cos(k\theta) & -r^k\sin(k\theta) & 0 & 0 \\ r^k sin(k\theta) & r^k\cos(k\theta) & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\},$$

with $r \in \mathbb{R}^+$ and $\frac{\theta}{2\pi} \in \mathbb{R}$ being irrational. It can be checked that Γ is a properly discontinuous, yet not affine crystallographic, group, and that for any $m \in \mathbb{Z}$ the group

$$G_m := \left\{ \begin{pmatrix} r^t\cos t(\theta+2\pi m) & -r^t\sin t(\theta+2\pi m) & 0 & 0 \\ r^t sint(\theta+2\pi m) & r^t\cos t(\theta+2\pi m) & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

is a crystallographic hull for Γ.

Note that uniqueness of G is also not given if Γ is affine crystallographic. For example, let $\Gamma \cong \mathbb{Z}^3 \rtimes \mathbb{Z}$. We pick an $A \in SL(\mathbb{Z}^3)$, such that A has only one real eigenvalue and is of infinite order. If Γ acts on $\mathbb{R}^4$ via the following, it is easy to see that Γ is an affine crystallographic group:

We decompose $\mathbb{R}^4 = V \oplus \mathbb{R}$. The $\mathbb{Z}^3$-part of Γ shall act by three linearly independent translations on V and the part corresponding to $\mathbb{Z}$ shall act by A on V and by a (non-trivial) translation in the fourth direction, i.e. on $\{0\} \times \mathbb{R}$.

From the fact that there are infinitely many $B \in \mathfrak{sl}(\mathbb{R}^3)$, one can deduce that there are infinitely many crystallographic hulls for Γ.

A more general analogon for crystallographic hulls is motivated by the following theorem:

3.4.3 Theorem. *Suppose $H \le GL(\mathbb{R}^n)$ is virtually solvable. Then there is a $G \le GL(\mathbb{R}^n)$ with $H \le G$, such that*

(i) G decomposes into finitely many components $G_1, \dots, G_k$ with $G_i \cap H \neq \emptyset \; \forall 1 \le i \le k$,

(ii) there is a compact set $K \subset G$ with $G = KH$,

(iii) for the Zariksi closures of H and G we have $\overline{H} = \overline{G}$, and

(iv) $\dim G \le \operatorname{rank} H$.

Furthermore, G is unique, if there is a $\tilde{H} \le G$ with finite $[H : \tilde{H}]$, such that all $h \in L(\tilde{H})$ have only real eigenvalues.

Proof. This theorem is proven in [FG83]. □

3.4.4 Definition (Syndetic hull)**.** *Let H and G be as in theorem 3.4.3. Then G is called a* syndetic hull *of H.*

An immediate corollary to theorem 3.4.1 is the generalized version of Bieberbach's first theorem we presented at the end of the last subsection, theorem 3.3.6:

3.4.5 Corollary. *Suppose $\Gamma \le \operatorname{Aff}(\mathbb{R}^n)$ is an affine crystallographic group and virtually solvable. Then there is a $G \le \operatorname{Aff}(\mathbb{R}^n)$, such that $G \circlearrowleft \mathbb{R}^n$ is simply transitive, $\Gamma \cap G \le G$ is discrete, $G/(\Gamma \cap G)$ is compact and $[\Gamma : \Gamma \cap G]$ is finite.*

Back to the main concern of this section, i.e. generalizing Bieberbach's second theorem. We present an approach by Fried and Goldman, [FG83].

Given a linear space $\mathbb{R}^n$, and a diffeomorphism $f \colon \mathbb{R}^n \to \mathbb{R}^n$, with the property that both f and its inverse f^{-1} are polynomial mappings, we call f a *polynomial automorphism* of $\mathbb{R}^n$.

3.4.6 Theorem (Bieberbach's second theorem - an affine version)**.** *Suppose we are given two affine crystallographic and virtually solvable groups $\Gamma_1, \Gamma_2 \le \operatorname{Aff}(\mathbb{R}^n)$. If there is an isomorphism $\varphi \colon \Gamma_1 \to \Gamma_2$, then it is induced by a polynomial automorphism f of $\mathbb{R}^n$, i.e. there is an f such that following diagram commutes for every $\gamma \in \Gamma_1$:*

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^n \\
\gamma \big\downarrow & & \big\downarrow \varphi(\gamma) \\
\mathbb{R}^n & \xrightarrow[f]{} & \mathbb{R}^n
\end{array}$$

In other words: $\Gamma_1 \cong \Gamma_2$ *iff there is such an* f *with* $f\Gamma_1 f^{-1} = \Gamma_2$*.*

Proof. The proof, which is based on the properties of crystallographic hulls, can be found in [FG83]. $\square$

There are two more observations to be made.
Of course, this statement holds also for crystallographic groups, because they are conjugate by linear map, i.e. a constant polynomial function.
The second observation is the fact that any property of an affine crystallographic and virtually solvable group Γ that can be expressed via polynomials holds for any other affine crystallographic group in the isomorphism class of Γ. For example, if Γ is affine crystallographic and virtually solvable, then the de Rham cohomology of the manifold $\Gamma\backslash\mathbb{R}^n$ can be computed from the complex of differential forms, with coefficients being expressable as polynomial functions of the affine coordinates, on M - for a proof see [Go82].

3.5 A generalization of Bieberbach's third theorem for the affine case

Before we state the main theorem of this subsection, which is due to Grunewald and Segal, [GS94], we first like to recall certain properties of crystallographic groups. We also have to introduce some definitions.

We have seen that any crystallographic group $\Gamma \leq \mathrm{Isom}(\mathbb{R}^n)$ is an extension of $\mathbb{Z}^n$ by some finite group. In other words, there is a finite index subgroup $\mathbb{Z}^n \cong \Delta \leq \Gamma$. Combining Bieberbach's second and third theorem, we see that up to conjugation there are only finitely many different extensions of Δ, i.e. only finitely many isomorphism classes of crystallographic groups.
The main point of the theorem we are going to present shortly is that it provides a 'reasonable' way to separate virtually solvable affine crystallographic groups into classes, such that in each class there are only finitely many conjugacy classes. Thus, the said theorem provides a finiteness result for affine crystallographic groups.

Our first task at hand is finding a suitable replacement for $\mathbb{Z}^n$ in the affine situation.

3.5.1 Definition (Fitting subgroup)**.** *It can be shown that any virtually polycyclic group* Γ *has a unique maximal nilpotent normal subgroup. Such a subgroup is called a* fitting subgroup *for* Γ *and denoted by* $Fitt(\Gamma)$*.*

A definition of *polycyclic* can be found in 4.1.1. Also note that any (virtually) polycyclic group is (virtually) solvable, whereas the contrary does not hold. It does however for discrete subgroups of $\mathrm{Aff}(\mathbb{R}^n)$. Hence, any discrete virtually solvable affine group possesses a fitting subgroup and consequently the same is true for virtually solvable affine crystallographic groups.
We can now define the replacement for $\mathbb{Z}^n$:

3.5.2 Definition ((Geometrically) strict (normal) extension)**.** *Given two affine crystallographic and virtually solvable groups $\Delta \leq \Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$. If $|\Gamma : \Delta]$ is finite and $Fitt(\Gamma) \leq \Delta$, then we say that Γ is a* strict extension *of Δ.*

If furthermore $\Delta \trianglelefteq \Gamma$, then we call Γ a strict normal *extension of Δ.*

Given two polycyclic subgroups $\Delta \leq \Gamma \leq GL(\mathbb{R}^n)$ with $|\Gamma : \Delta|$ finite, we say that Γ is a geometrically strict *extension of Δ, if there is a syndetic hull G for Δ with $G \cap \Gamma = \Delta$.*

On a side note: while the definition of syndetic hull used in [GS94] differs from the one we presented, definition 3.4.4 which is due to [FG83], they coincide for affine crystallographic groups.

We need one more definition: if we have for an affine crystallographic group $\Gamma \leq G \leq \mathrm{Aff}(\mathbb{R}^n)$, we will say that Γ is of type G. For example, if Γ is of type $\mathrm{Isom}(\mathbb{R}^n)$, then this is just the classical case of crystallographic groups.

Accordingly, if Γ is additionally also torsion free, we say that the manifold $\Gamma \backslash \mathbb{R}^n$ is of type G.

We can now state the promised finiteness result.

3.5.3 Theorem. *[Bieberbach's third theorem - an algebraical affine version] Given a virtually solvable affine crystallographic group Δ of type G. Then we have the following:*

(i) there are only finitely many conjugacy classes in $N_G(\Delta)$ of strict normal extensions of Δ of type G, and

(ii) there are only finitely many conjugacy classes in G of strict extensions of Δ. Furthermore, the index of Δ in these strict extensions is bounded.

Proof. The proof, which relies on finiteness properties of so called arithmetic groups, may be found in [GS94]. In the cited paper, one may also find a thorough definition of arithmetic groups at the beginning of chapter 3. $\square$

The above theorem is not the only one concerned with finiteness of equivalence classes of extensions of virtually solvable affine crystallographic groups. We also have the following.

3.5.4 Theorem. *Assume Δ to be a solvable affine crystallographic group. It follows, that there are only finitely many isomorphism classes of geometrically strict extensions of Δ.*

Proof. This assertion is also shown to be true in [GS94]. $\square$

Based on theorem 3.5.3, Grunewald and Segal prove the following: To state this next result, once again, we need some notation. We say that a Lie group K is of type $\mathcal{NP}$, if its identity component K^0, with respect to its manifold structure, is nilpotent and K/K^0 is polycyclic

3.5.5 Theorem. *If Γ is a virtually solvable affine crystallographic group of type G, then there is a closed normal subgroup $K \leq N_G(\Delta)$, such that K is of tpye $\mathcal{NP}$ and $N_G(\Gamma)/K$ is isomorphic to an arithmetic group.*

The geometric interpretation of this result is as follows: suppose that Γ and $\tilde{\Gamma}$ are virtually solvable torsion free affine crystallographic groups. We set $M := \Gamma\backslash\mathbb{R}^n$ and $N := \tilde{\Gamma}\backslash\mathbb{R}^n$ and suppose that both manifold are of type G. A map $\alpha\colon M \to N$ is called an isometry, if α is a homeomorphism and if its lift $\tilde{\alpha}\colon \mathbb{R}^n \to \mathbb{R}^n$ is an automorphism. That this definition of isometry is exactly the usual one, is shown in [Wo67], lemma 2.5.6.

Let $Aut_G(M)$ be the group of all self-isometries of M. Then, by using the identity, one can show the following corollary to the above theorem.

3.5.6 Corollary. *Let M be as above. Then the group of self-isometries $Aut_G(M)$ is an extension of a Lie group of type $\mathcal{NP}$ by some arithmetic group.*

There is also a geometric version of theorem 3.5.3. Suppose that $\Delta \leq \Gamma$ are two torsion free affine crystallographic groups of type G. Then there is a covering map $p\colon \Delta\backslash\mathbb{R}^n \to \Gamma\backslash\mathbb{R}^n$. If Γ is a strict (normal) extension of Δ, p is called a strict (normal) covering.

Furthermore, given two such covers $p\colon M \to N$ and $q\colon M \to L$, we say that these covers are equivalent, if there is an isometry $\beta\colon N \to L$ and an $\alpha \in Aut_G(M)$, such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "p"'] & M \arrow[d, "q"] \\ N \arrow[r, "\beta"'] & L \end{tikzcd}$$

3.5.7 Theorem (Bieberbach's third theorem - a geometrical affine version)**.** *Given two complete compact affine flat manifolds M and N, both of type G. Then we have the following:*

(i) there are only finitely many equivalence classes of strict normal coverings $M \to N$, and

(ii) there are only finitely many isometry classes for N, if $M \to N$ is a strict covering. Furthermore, these coverings have bounded multiplicites.

In the last three subsections we have seen that there are powerful theorems - 3.3.6, 3.4.6 and 3.5.3 -, which generalize the three Bieberbach theorems in a very nice way. However, each of these theorems relies on the assumption that the groups in question are not only affine crystallographic but also virtually solvable. Thus, it is worth asking in what way these two terms are related to each other. The whole next section is devoted to this question.

4 Auslander's conjecture

4.1 Two conjectures - Auslander's & Milnor's

In this section we will examine two conjectures - Auslander's and Milnor's. To put them in perspective we will start with definitions:

4.1.1 Definition (Nilpotent, solvable and polycyclic groups and the radical of groups). *Let G be a group and set $[H, G] := (h^{-1}g^{-1}hg \mid h \in H, g \in G)$ for any subgroup $H \leq G$. $[G, G]$ is called the* commutator subgroup *of G. Now define $G_{(0)} := G, G_{(n-1)} := [G_{(n)}, G], \forall n \in \mathbb{N}^+$.*
Consider the following series, called the lower central series *of G:*

$$G = G_{(0)} \rhd G_{(1)} \rhd G_{(2)} \cdots$$

Assume this series terminates, i.e. there is an n such that $G_{(n)} = [G_{(n)}, G] = G_{(n+1)}$. Furthermore, assume that

$$G = G_{(0)} \rhd G_{(1)} \rhd G_{(2)} \cdots G_{(n-1)} \rhd G_{(n)} = \{e\}, \tag{4.1}$$

with $\{e\}$ being the trivial group. If this is the case, G is called nilpotent. *If we want to be more specific we say that G is n-*step-nilpotent *or G is of* nilpotency class *n, assuming that n is minimal with respect to property 4.1. We now define $G^{(0)} := G, G^{(n)} := [G^{(n-1)}, G^{(n-1)}], \forall n \in \mathbb{N}^+$. We call*

$$G = G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \cdots$$

the derived series *of G. Assume this series terminates, i.e. there is an n such that*

$$G = G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \cdots G^{(n)} = [G^{(n)}, G^{(n)}] = G^{(n+1)}.$$

$G^{(n)}$ is then said to be a perfect *group.*
If we furthermore have that $G^{(n)}$ is the trivial group, G is called solvable. *It follows that any solvable group can be constructed via extending an abelian group finitely many times.*
The maximal solvable subgroup $rad(G) \leq G$ is called the radical *of G.*
Given a solvable group G with the addition that every quotient $G^{(n)}/G^{(n-1)}$ is finitely generated and abelian, G is called polycylic.

Before we continue, a few words on the relationship of the just defined properties as well as on solvable groups:

If G is an abelian group, we have $[G, G] = \{e\}$, i.e. G is nilpotent. It is also easy to check that $G^{(n)} \leq G_{(n)}$, and thus we deduce that nilpotency implies solvability. It can also be shown that finitely generated nilpotent groups are polycyclic, whereas the condition of being finitely generated is indeed necessary.

On the other hand, polycyclic does not imply solvable does not imply nilpotent does not imply abelian.

Trivially, if G is solvable, so is every subgroup $H \leq G$ and the image of G under any group homomorphism is also solvable. From this, in conjunction with the fundamental theorem on group isomorphisms, it follows directly that if G is a group, N is a normal subgroup of G, then G is solvable iff N as well as G/N are solvable.

Now we turn back to affine crystallographic groups. In [Au64] Auslander stated that the radical $rad(\Gamma)$ of a finitely generated fundamental group Γ of a complete locally affine manifold has finite index in Γ.

As a second step in his paper, Auslander used this result to show that if Γ is the fundamental group of compact, complete, flat affine manifold M of dimension n, then M is finitely covered by a solvmanifold, or equivalently: one may find a subgroup $\tilde{\Gamma} \leq \Gamma$, $[\Gamma : \tilde{\Gamma}]$ finite, such that $\tilde{\Gamma}\backslash\mathbb{E}$, $\mathbb{E}$ being the real n-dimensional affine space, is homeomorphic to a solvmanifold. By using the technique concerning the transition from such manifolds via their fundamental groups to properly discontinuous subgroups of $\mathrm{Aff}(\mathbb{E})$, we can put the theorem into algebraic terms: affine crystallographic groups are virtually solvable.

Auslander then went on by using these results to associate certain Lie groups, so called algebraic hulls, to the fundamental groups in question in order to prove that two compact, complete, flat affine manifolds are homemorphic, if their respective fundamental groups are isomorphic.

However, as Auslander later discovered, there unfortunately is a gap in the proof given for his second theorem - he did not use the assumption of compactness. This led to the following conjecture:

4.1.2 Conjecture (Auslander - geometric version)**.** *Every compact, complete, flat, affine manifold is finitely covered by a solvmanifold.*

As before, we can state this conjecture algebraically:

4.1.3 Conjecture (Auslander - algebraic version)**.** *Every affine crystallographic group is virtually solvable.*

There is also a slight variation of the algebraic version of Auslander conjecture. Of course, a polycyclic group is always solvable. However, the converse is certainly not true, i.e. solvable groups in general are not polycyclic. On the other hand, every discrete solvable subgroup of $GL(\mathbb{R}^n)$ is indeed polycyclic, for a proof see [Mi77] or [Ra72]. Thus, one could ask as well, whether every affine crystallographic subgroup of $\mathrm{Aff}(\mathbb{E})$ is virtually polycyclic.

It should be noted that the 'converse' of Auslander's cojecture does not hold. In [Mi77] Milnor asked the following:

4.1.4 Problem (Milnor's converse Auslander conjecture)**.** *Can any torsion free virtually polycyclic group be realized as an affine crystallographic group? I.e., given a torsion free virtually polycyclic group* Γ*, is there always an embedding* $\iota\colon \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$*, such that the induced action of* Γ *on* $\mathbb{R}^n$ *is crystallographic?*

There are indeed such Γ that cannot be realized as affine crystallographic groups, see [BG95] or [Be95]. On the other hand, there is a least one setting in which a variation of the above problem holds, see theorem 5.2.3.

Now we go on to the second conjecture we like to examine - compared to Auslander's, it is a less strict one.

4.1.5 Conjecture (Milnor - algebraic version)**.** *Every properly discontinuous subgroup of* $\mathrm{Aff}(\mathbb{E})$ *is virtually solvable.*

Of course, we can also express the conjecture geometrically:

4.1.6 Conjecture (Milnor - geometric version)**.** *Every complete, flat, affine manifold is finitely covered by a solvmanifold.*

Milnor's conjecture stems from the following: in [Mi77] Milnor proved that a torsion free virtually polycyclic group Γ is always isomorphic to the fundamental group of some complete, affine, flat manifold M. In the light of this, he asked, whether these Γ are the only possible fundamental groups, which is exactly the same question the conjecture yields, considering the discussion above about the relationship of polycyclic and solvability and the following theorem. A proof can be found in [Se60] or [Ra72].

4.1.7 Theorem (Selberg's lemma)**.** *If* Γ *is a finitely generated subgroup of* $GL(\mathbb{C}^n)$*, then we can find a torsion free subgroup* $\tilde{\Gamma} \leq \Gamma$ *such that* $\tilde{\Gamma}$ *has finite index in* Γ*.*

Thus, when one is examining a virtual property P of some finitely generated group $\Gamma \leq GL(\mathbb{C}^n)$, one may, without loss of generality, always pass on to a subgroup $\tilde{\Gamma}$ that is torsion free, because, due to the multiplicity of the index ($[\Gamma : \Gamma_1][\Gamma_1 : \Gamma_2] = [\Gamma : \Gamma_2]$ for any two subgroups $\Gamma_2 \leq \Gamma_1 \leq \Gamma$), the torsion free subgroup $\tilde{\Gamma}$ virtually hands over all its (virtual) properties to Γ. Trivially, we have $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n) \leq \mathrm{Aff}(\mathbb{C}^n)$, and thus can apply Selberg's lemma to affine crystallographic groups.

Yet another way of looking at Auslander's conjecture, respectively Milnor's, can be achieved via Tits' famous theorem, a proof can be found in [Ti72].

4.1.8 Theorem (Tits alternative)**.** *Given a group* $\Gamma \leq \mathrm{Aff}(\mathbb{C}^n)$*. Then* Γ *is either virtually solvable or* Γ *has a free subgroup that is not abelian.*

In view of this theorem, we may transform the conjectures into the following: does a (cocompact) properly discontinuous group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ contain a free subgroup?

In section 3.1.1 we have seen that in general affine crystallographic groups are not (virtually) abelian. Yet, the groups $\Gamma(i)$ we constructed are nilpotent. Indeed, they are subgroups of the group of stricly upper triangular matrices, which are very well-known to be nilpotent and obviously subgroups of nilpotent groups are nilpotent. Thus, one could be inclined to propose a stronger version of Auslander's conjecture, respectively Milnor's, namely the question if all (cocompact) properly discontinuous subgroups of $\mathrm{Aff}(\mathbb{R}^n)$ are (virtually) nilpotent.

However, an easy example shows, that affine crystallographic groups, and thus properly discontinuous groups, are in general not (virtually) nilpotent.

4.1.9 Example (There are (cocompact) properly discontinuous groups, that are not (virtually) nilpotent)**.** Let $\Gamma \leq \mathrm{Aff}(\mathbb{E})$, $\mathbb{E}$ being the real affine space of dimension n, and let Γ be generated by $\{e_i, \gamma \mid 1 \leq i \leq n-1\}$, with e_i being the i-th standard vector and

$$\gamma(x) = Ax + e_n, \text{ whereas } A = \begin{pmatrix} & & & 0 \\ & B & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \text{ with } B \in GL(\mathbb{Z}^{n-1}).$$

Now Γ is an affine crystallographic group that is nilpotent iff B is unipotent.

And of course, if B is not unipotent, we immediately see that our group cannot be virtually nilpotent either.

Now that we looked at various variations of the two stated conjectures, equivalent and less strict ones, our next task is to present what is actually known about them.

So far, Auslander's conjecture could neither be proven nor disproven. On the one hand, to date no counterexample has been constructed. On the other hand, until today proofs can only be given for certain special cases. Auslander's conjecture holds, if

(i) $L(\Gamma) \leq O(\mathbb{R}^n)$, which is just the classic case of the Bieberbach theorems,

(ii) $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ for $n \leq 6$. A proof for the case $n \leq 3$ was first given by Fried and Goldman, [FG83], see section 4.3. The cases $4 \leq n \leq 6$ are proven by Abels, Margulis and Soifer in [AMS12],

(iii) $L(\Gamma) \leq O(\mathbb{R}^{n-1,1})$, due to Goldman and Kamishima, [GK84], see section 4.4,

(iv) $L(\Gamma) \leq O(\mathbb{R}^{n-2,2})$, due to Abels, Margulis and Soifer, [AMS05],see section 4.5,

(v) $L(\Gamma) \leq G$ whereas G is a reductive group and $\mathrm{rank}_{\mathbb{R}} = 1$, due to Grunevald and Margulis, [GM88], see 4.6,

(vi) if a special representation of $\overline{\Gamma}^0$ satifies certain properties, due to Kawabe, [Ka98]. These conditions are presented in 4.7, and

(vii) $L(\Gamma) \leq G$, whereas for the maximal semisimple subgroup S of G^0 we have that S is an almost direct product of simple Lie groups $S_1 \ldots, S_r$ and $\mathrm{rank}_{\mathbb{R}}(S_i) = 1$ $\forall 1 \leq i \leq r$, due to Tomanov, [To90].

Milnor's conjecture however has been settled. As it turns out, already in dimension 3 it is not true. Be aware that this also invalidates the 'proof' Auslander has given for the second step in his paper [Au64] discussed above.

The first counterexample to Milnor's conjecture is due to Margulis, [Ma87], and it has not only impacted said conjecture but also Auslander's. The techniques used by Margulis to construct a properly discontinuous group, which has a free non-abelian subgroup, are heavily used by Abels, Margulis and Soifer in the course of proving Auslander's conjecture in the special cases of $L(\Gamma) \leq O(\mathbb{R}^{n-2,2})$, section 4.5, and $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ for $4 \leq n \leq 6$.

We will now present Margulis' counterexample. Counterexamples constructed by a different approach can be found in [Bu96] for example.

4.2 Margulis' counterexample to Milnor's conjecture

4.2.1 Example (There are properly discontinuous groups that are not virtually solvable)**.** The main result in this example, which is, along with the needed propositions and their proofs, due to Margulis, [Ma87], is a theorem that provides us with a class of free groups that act properly discontinuously on $\mathbb{R}^3$, and hence anwsers Milnor's conjecture negatively.

In order to do so we first have to introduce some notation and concepts and then break the example into parts, i.e. into a couple of lemmata and theorems.

Let $x \in \mathbb{R}^3$. As always, let $||x||$ denote the Euclidean norm of x. If $\mathcal{A}, \mathcal{B}$ are two straight lines or planes in $\mathbb{R}^3$, we denote by $\angle(\mathcal{A}, \mathcal{B})$ the angle between $\mathcal{A}$ and $\mathcal{B} \mod \frac{\pi}{2}$. For the distance, with respect to the Euclidean metric, between points/sets and other points/sets we write $d(\cdot, \cdot)$. We furthermore define a bilinear form:

$$B(x, y) = x_1y_1 + x_2y_2 - x_3y_3 \; \forall x, y \in \mathbb{R}^3,$$

and set $B(x, A) = c = \min\{B(x, y) \mid y \in A\}$. Moreover, $W := \{x \mid (B(x, x) = 0\}$. By H we denote the identity component of the indefinite orthogonal group $O(2, 1)$, which clearly leaves the bilinear form B invariant. Set $G = \mathbb{R}^3 \rtimes H$, and thus we can write any $g \in G$ as $g = vh$ with appropriately chosen $v \in \mathbb{R}^3$ and $h \in H$. As before, let L be the natural projection:

$$\begin{aligned} L \colon G &\to H \\ x = vh &\mapsto L(vh) = h \end{aligned}$$

A $g \in G$ is called hyperbolic, if $L(g)$ has a real eigenvalue that is not equal to 1. If g is hyperbolic, $L(g)$ is diagonalizable and every eigenvalue is greater than 0. Actually, we can be a bit more precise: one eigenvalue is 1; for the second eigenvalue, called $\lambda(g)$, we have $\lambda(g) < 1$. Finally, the third one is $\lambda(g)^{-1}$ and thus greater 1. All of this is may be shown by using the fact that $H \cong PSL(\mathbb{R}^2)$ and that the given presentation of H is equivalent to its adjoint representation.

A basic observation:

$$\lambda(g^k) = \lambda(g)^{|k|},\ \forall k \in \mathbb{Z}\backslash\{0\}. \tag{4.2}$$

From now on, let and $x \in \mathbb{R}^3$ and $g \in G$, g fixed and hyperbolic. Then the eigenspaces are

$$\begin{aligned}
\mathcal{A}_g^0 &= \{x \mid \pi(g)x = x\},\\
\mathcal{A}_g^- &= \{x \mid \pi(g)x = \lambda(g)x\},\\
\mathcal{A}_g^+ &= \{x \mid \pi(g)x = \lambda(g)^{-1}x\}, \text{ and we set}\\
\mathcal{D}_g^- &:= \mathcal{A}_g^0 \oplus \mathcal{A}_g^-, \text{ as well as}\\
\mathcal{D}_g^+ &:= \mathcal{A}_g^0 \oplus \mathcal{A}_g^+.
\end{aligned}$$

Trivially, we have

$$\mathbb{R}^3 = \mathcal{D}_g^- \oplus \mathcal{A}_g^+ = \mathcal{D}_g^+ \oplus \mathcal{A}_g^-.$$

Using the fact, that $L(g)$ preserves our bilinear form B, we see that

$$\begin{aligned}
&\mathcal{A}_g^-, \mathcal{A}_g^+ \subset W,\\
&\mathcal{D}_g^- = \{x \mid B(x, \mathcal{A}_g^-) = 0\}, \text{ and}\\
&\mathcal{D}_g^+ = \{x \mid B(x, \mathcal{A}_g^+) = 0\}.
\end{aligned}$$

Now consider a straight line $L \subset W$. It follows that $\{x \mid (B(x, L) = 0\}$ is a plane that is tangential to W and hence the planes $\mathcal{D}_g^-$ and $\mathcal{D}_g^+$ are tangential to W. Clearly, $\mathcal{A}_g^0 = \mathcal{D}_g^- \cap \mathcal{D}_g^-$ and therefore $B(x, x) > 0$, if $x \in \mathcal{A}_g^0\backslash\{0\}$.

If we choose an orientation for $\mathbb{R}^3$, we also have a unique positively oriented basis associated to g, namely $(x^0(g), x^-(g), x^+(g))$. Appropriately choosing a decomposition of $W\backslash\{0\} = W^- \cup W^+$, we have for this basis

$$\begin{gathered}
x^0(g) \in \mathcal{A}_g^0,\ x^-(g) \in \mathcal{A}_g^- \cap W^-,\ x^+(g) \in \mathcal{A}_g^+ \cap W^+, \text{ and}\\
B(x^0(g), x^0(g)) = ||x^-(g)|| = ||x^+(g)|| = 1.
\end{gathered}$$

Furthermore

$$x^0(g^n) = x^0(g),\ x^-(g^n) = x^-(g),\ x^+(g^n) = x^+(g)\ \forall k \in \mathbb{Z}^+, \text{ and} \tag{4.3}$$

$$x^0(g^n) = -x^0(g),\ x^-(g^n) = x^-(g),\ x^+(g^n) = x^+(g)\ \forall k \in \mathbb{Z}^-. \tag{4.4}$$

Because $L(g)$ is diagonalizable and the dimension of $\mathcal{A}_g^0$ is 1, there exists exactly one line $\mathcal{C}_g$, that is parallel to $\mathcal{A}_g^0$ as well as invariant under the action of g. To be more precise, the restricted action $g \circlearrowleft C_g$ is just a translation by some $t_g \in \mathcal{A}_g^0$. For C_g and t_g we have the equalities

$$C_{g^k} = C_g,\ t_{g^k} = kt_g\ \forall k \in \mathbb{Z}\backslash\{0\}. \tag{4.5}$$

Set

$$\alpha(g) := B(t_g, x^0(g)).$$

From $B(x^0(g), \mathcal{A}_g^-) = B(x^0(g), \mathcal{A}_g^+) = 0$ we deduce

$$\alpha(g) = B(gx - x, x^0(g)). \tag{4.6}$$

This equation, together with (4.3) through (4.5), yields

$$\alpha(g^k) = |k|\alpha(g) \; \forall k \in \mathbb{Z}\backslash\{0\}. \tag{4.7}$$

From this we deduce that g and g^{-1} have the same sign. We go on by defining $\mathcal{E}_g^-$ to be the plane that contains $\mathcal{C}_g$ and is parallel to $\mathcal{D}_g^-$. Analogously, let $\mathcal{E}_g^+$ be the plane containing $\mathcal{C}_g$ that is parallel to $\mathcal{D}_g^+$. It follows

$$\mathcal{E}_{g^k}^- = \mathcal{E}_g^-, \; \mathcal{E}_{g^k}^+ = \mathcal{E}_g^+ \; \forall k \in \mathbb{Z}^+, \text{ and} \tag{4.8}$$

$$\mathcal{E}_{g^k}^- = \mathcal{E}_g^+, \; \mathcal{E}_{g^k}^+ = \mathcal{E}_g^- \; \forall k \in \mathbb{Z}^-. \tag{4.9}$$

Further, we denote by $\mathcal{B}_g^-(x)$, $\mathcal{B}_g^+(x)$ respectively, the line that goes parallel to $\mathcal{A}_g^-$, $\mathcal{A}_g^+$ respectively, through the point x.

Assume that $g \in G$ and, in addition to g being hyperbolic, we have that the distance $d(x^-(g), x^+(g)) > \varepsilon > 0$ for some $\varepsilon \in \mathbb{R}$, then g is called ε-hyperbolic. Given another hyperbolic element $h \in G$, we will say that g and h are ε-transversal, or just transversal, if the distance between $x^-(g)$ or $x^+(g)$ and $x^-(h)$ or $x^+(h)$ is greater than $\varepsilon > 0$. Equivalently, g and h are transversal, if $\{x^-(g), x^+(g)\} \cap \{x^-(h), x^+(h)\} = \emptyset$.

Now consider two functions f_1 and f_2 from an arbitrary but fixed subset $S \subset G^k = \{(g_1, \ldots, g_k \mid g_i \in G\}, 1 \leq i \leq k \in \mathbb{N}^+$, to $\mathbb{R}^+$. Assume we can find a $p(\varepsilon)$, such that $f_1(g_1, \ldots, g_k) \leq p(\varepsilon) f_2(g_1, \ldots, g_k)$ for all $(g_1, \ldots, g_k) \in S$, given that all the g_i are ε-hyperbolic as well as pairwise ε-transversal. If this is the case, f_1 is said to be of an order not exceeding f_2, written $f_1 \preccurlyeq f_2$. If we have both $f_1 \preccurlyeq f_2$ as well as $f_2 \preccurlyeq f_1$, we call them equivalent and denote this by $f_1 \sim f_2$.

For some hyperbolic $g, h \in G$ we define

$$\begin{aligned} \tau(g) &:= \angle(\mathcal{D}_g^-, \mathcal{D}_g^+), \text{ and} \\ \theta(g,h) &:= \min\{\angle(X,Y) \mid X \in \{\mathcal{D}_g^-, \mathcal{D}_g^+\}, \; Y \in \{\mathcal{A}_h^-, \mathcal{A}_h^+\}\}. \end{aligned}$$

Since $\mathcal{D}_g^-$, $\mathcal{D}_g^+$ respectively, and W have $x^-(g)$, $x^-(g)$ respecively, as a common point we get

$$\tau(g) \sim 1 \text{ and } \theta(g,h) \sim 1. \tag{4.10}$$

This, along with $\mathcal{A}_g^0 = \mathcal{D}_g^- \cap \mathcal{D}_g^+$, implies

$$d(x^0(g), W) \sim 1 \text{ and } ||x^0(g)|| \sim 1, \tag{4.11}$$

which in turn gives us

$$|\alpha(g)| \sim ||t_g||. \tag{4.12}$$

Now that we have settled our notations and definitions, we start with the actual proof. As said before, it is broken down into a few parts.

4.2.2 Lemma. *Let $g, h \in G = \mathbb{R}^3 \rtimes H$, with H the identity component of $O(\mathbb{R}^{2,1})$, be ε-hyperbolic and ε-transversal. Then one can find an $a(\varepsilon) > 1$ with $\lambda(g), \lambda(h) < a(\varepsilon)^{-1}$, such that*

(i) $<g, h>$ is a free group,

(ii) gh is hyperbolic,

(iii) $a(\varepsilon)^{-1}\lambda(g)\lambda(h) < \lambda(gh) < a(\varepsilon)\lambda(g)\lambda(h)$,

(iv) $d(x^+(gh), x^+(g)) < a(\varepsilon)\lambda(g)$,

(v) $d(x^-(gh), x^-(h)) < a(\varepsilon)\lambda(h)$,

(vi) $\angle(\mathcal{E}^+_{gh}, \mathcal{E}^+_g) < a(\varepsilon)\lambda(g)$,

(vii) $\angle(\mathcal{E}^-_{gh}, \mathcal{E}^-(h)) < a(\varepsilon)\lambda(h)$,

(viii) $d(x^0(gh), \mathcal{B}^+_g(x^0(g)) < a(\varepsilon)\lambda(g)$,

(ix) $d(x^0(gh), \mathcal{B}^-_h(x^0(h)) < a(\varepsilon)\lambda(h)$.

Proof. Some further needed notations:

$$\begin{aligned}
\overline{g}(x) &:= \frac{g(x)}{||x||}, \overline{h}(x) := \frac{h(x)}{||x||},\\
U(x,\varepsilon) &:= \{y \in \mathbb{R}^3 \mid d(x,y) \le \varepsilon),\\
D(g,h) &:= (\{x^-(g), x^+(g)\}, \mathcal{D}^-_h \cup \mathcal{D}^+_h), \text{ and}\\
V(g,h) &:= \left(x^-(g), \frac{B(g,h)}{3}\right) \cup \left(x^+(g), \frac{B(g,h)}{3}\right).
\end{aligned}$$

The equalities in (4.10) imply $D(g,h) \sim 1$. Using this, one can show that indeed an $a(\varepsilon) > 1$ exists for any ε and any $g, h \in G$, g, h ε-hyperbolic and ε-transversal. The other requirement, $\lambda(g), \lambda(h) < a(\varepsilon)^{-1}$, also follows.
Now, we fix a $p \in W^+, ||p|| = 1$. We have

$$d(x^+(h), \overline{h}^k(\{p\} \cup V(g,h))) < \min\left(a(\varepsilon)\lambda(h), \frac{D(h,g)}{3}\right) \ \forall k \in \mathbb{Z}^+ \tag{4.13}$$

$$d(x^-(h), \overline{h}^k(\{p\} \cup V(g,h))) < \min\left(a(\varepsilon)\lambda(h), \frac{D(h,g)}{3}\right) \ \forall k \in \mathbb{Z}^-, \text{ and} \tag{4.14}$$

$$a(\varepsilon)^{\frac{-1}{2}}\lambda(h)||x|| < ||h(x)|| < a(\varepsilon)^{\frac{1}{2}}\lambda(h)||x|| \ \forall x \in V(g,h). \tag{4.15}$$

Certainly, the inequalities above still hold if we switch g and h. From (4.13) and (4.14) we deduce that for all $k \in \mathbb{Z}\backslash\{0\}$

$$\overline{h}^k(\{p\} \cup V(g,h)) \subset V(h,g), \text{ and} \tag{4.16}$$

$$\overline{g}^k(\{p\} \cup V(h,g)) \subset V(g,h). \tag{4.17}$$

Because of this and $p \notin V(g,h) \cup V(h,g)$, $||p|| = 1$, we deduce that we can apply Proposition 1.1 of [Ti72] and hence *(i)* holds.

Using inequality (4.13) and the inclusions (4.16) and (4.17), we see that

$$\overline{g}\overline{h}(V(g,h)) \subset V(g,h), \text{ and}$$
$$d(x^+(g), \overline{g}\overline{h}(V(g,h))) < a(\varepsilon)\lambda(g)$$

Restricting $\overline{g}\overline{h}$ to $V(g,h)$, we see that, by the virtue of (4.15), (4.16), (4.17) and Brouwer's fixed point theorem, there is an eigenvector $x \in V(g,h), ||x|| = 1$, for $\pi(gh)$, that lies in the open ball around x with radius $a(\varepsilon)$. The associated eigenvalue $\lambda(gh)$ has the property that $a(\varepsilon)^{-1}\lambda(g)\lambda(h) < \lambda(gh) < a(\varepsilon)\lambda(g)\lambda(h)$. This gives us *(ii)* through *(iv)*.

Of course, we get *(v)* from *(iv)*, by simply substituting g, respectively h, with h^{-1}, g^{-1} respectively.

That *(vi)* - *(ix)* hold, can be followed from (4.11), *(iv)* and *(v)*, along with using the following: we have that $\mathcal{B}_g^+(x^0)) = \{x \in \mathcal{D}_g^+ \mid B(x,x) = 1\}$ and $\mathcal{B}_h^-(x^0)) = \{x \in \mathcal{D}_h^- \mid B(x,x) = 1\}$, as well as that the two bases $(x, y, x^+(g))$ and $(x, x^-(h), y)$ have positive orientation if $y \in W^+ \backslash \{x^+(g)\}$, $y \in W^+ \backslash \{x^-(h)\}$ respectively. $\square$

4.2.3 Lemma. *Let $g, h \in G = \mathbb{R}^3 \rtimes H$, with H the identity component of $O(\mathbb{R}^{2,1})$, be ε-hyperbolic and ε-transversal, and fix an $v \in \mathbb{R}^3$. Then one can find an $c(\varepsilon) > 1$ such that*

$$\begin{aligned} d(v, \mathcal{E}_{gh}^+) &\leq d(v, \mathcal{E}_g^+ + c(\varepsilon)\lambda(g)(|\alpha(g)| \\ &\quad + d(v, \mathcal{C}_g) + d(v, \mathcal{C}_h)) + c(\varepsilon)\lambda(gh)|\alpha(h)| \end{aligned} \tag{4.18}$$

$$\begin{aligned} d(v, \mathcal{C}_{gh}) &< c(\varepsilon)(\lambda(g) + \lambda(h)(d(v, \mathcal{C}_g) + c(\varepsilon)(d(v, \mathcal{C}_h) \\ &\quad + d(v, \mathcal{E}_g^+ + \lambda(g)|\alpha(g)| + \lambda(h)|\alpha(h)|), \end{aligned} \tag{4.19}$$

$$|\alpha(gh) - \alpha(g) - \alpha(h)| \leq c(\varepsilon)(d(\mathcal{C}_g, \mathcal{C}_h) + \lambda(g)|\alpha(g)| + \lambda(h)|\alpha(h)|). \tag{4.20}$$

Proof. We choose an $x \in \mathcal{C}_g$ with the property $d(v,x) = d(v, \mathcal{C}_g)$ and let $y := \mathcal{B}_g^-(g^{-1}x) \cap \mathcal{E}^-(h^{-1})$ and $z := \mathcal{B}_{h^{-1}}^- \cap \mathcal{C}_h$. From (4.10) together with $t_g = x - g^{-1}x$ follows

$$d(g^{-1}x, y) \preccurlyeq d(x, \mathcal{C}_h) + ||t_g||, \text{ and} \tag{4.21}$$

$$d(y, z) \preccurlyeq d(x, \mathcal{C}_h) + ||t_g||. \tag{4.22}$$

Using (4.21) together with the fact $y \in B_g^-(g^{-1}x)$, we get

$$d(x, gy) \preccurlyeq \lambda(g)(d(x, \mathcal{C}_h) + ||t_g||). \tag{4.23}$$

Combining (4.21) with (4.23) and once again $t_g = x - g^{-1}x$, we deduce

$$d(gy, y) \leq d(x, gy) + d(x, g^{-1}x) + d(g^{-1}x, y) \preccurlyeq d(x, \mathcal{C}_g) + ||t_g||. \tag{4.24}$$

Analogously, we have the following by using $t_h = z - h^{-1}z$ along with (4.22):

$$d(y, h^{-1}) \preccurlyeq d(x, \mathcal{C}_h) + ||t_g|| + ||t_h||. \tag{4.25}$$

Now we combine (4.24) and (4.25) and get

$$d(h^{-1}y, gy) \preccurlyeq d(x, \mathcal{C}_h) + ||t_g|| + ||t_h||.$$

But we also have $h^{-1}y = (gh)^{-1}gy$. With this in mind we see

$$d(gy, \mathcal{E}_{gh}^+) \preccurlyeq \lambda(gh)(||t_g|| + ||t_h|| + d(x, \mathcal{C}_h)), \tag{4.26}$$

if we further use the following inequality, which is implied by the fact that $d(g^{-1}x, \mathcal{E}_g^+) = \lambda(g)^{-1}d(x, \mathcal{E}_g^+)$,

$$d(x, \mathcal{E}_g^+) \leq \frac{\lambda(g)}{1-\lambda(g)} d(x, g^{-1}x).$$

Now recall that we have chosen our x such that $x \in \mathcal{E}_g^+$ and $d(v, x) = d(v, \mathcal{C}_g)$. From (4.23) we deduce

$$\begin{aligned} d(gy, \mathcal{E}_g^+) &\preccurlyeq \lambda(g)(||t_g|| + d(x, \mathcal{C}_h), \text{ and} \\ d(v, gy) &\preccurlyeq d(v, \mathcal{C}_g) + \lambda(g)(||t_g|| + d(x, \mathcal{C}_h)). \end{aligned}$$

We also have

$$d(x, \mathcal{C}_h) \leq d(v, x) + d(v, \mathcal{C}_h) = d(v, \mathcal{C}_g) + d(v, \mathcal{C}_h),$$

and, using lemma 4.2.2,

$$\angle(\mathcal{E}_g^+, \mathcal{E}_{gh}^+) \preccurlyeq \lambda(g) \preccurlyeq \lambda(gh).$$

Taking this into account, we conclude, in conjunction with (4.26), that

$$\begin{aligned} |d(v, \mathcal{E}_{gh}^+) - d(v, \mathcal{E}_g^+)| &\preccurlyeq d(gy, \mathcal{E}_{gh}^+) + d(gy, \mathcal{E}_g^+) + \angle(\mathcal{E}_g^+, \mathcal{E}_{gh}^+) d(v, gy) \preccurlyeq \\ &\preccurlyeq \lambda(g)(||t_g|| + d(v, \mathcal{C}_g) + d(v, \mathcal{C}_h)) + \lambda(gh)||t_g||. \end{aligned} \tag{4.27}$$

The above inequality, together with (4.12), yields the inequality (4.18).

In order to proof (4.19), we need gh to be $\frac{\varepsilon}{2}$-hyperbolic. Thus, from now on we assume $\lambda(g), \lambda(h) < a(\varepsilon)^{-1}\frac{\varepsilon}{4}$. That we can do so without loss of generality is ensured by 4.2.2. Thus, gh is $\frac{\varepsilon}{2}$-hyperbolic. We also have $\mathcal{C}_{gh} = \mathcal{E}_{gh}^+ \cap \mathcal{E}_{gh}^-$ and hence $d(v, \mathcal{C}_{gh}) \preccurlyeq d(v, \mathcal{E}_{gh}^+) + d(v, \mathcal{E}_{gh}^-)$. Along with (4.18), (4.27) and $\lambda(gh) \preccurlyeq \lambda(g), \lambda(h)$, which can once again be deduced from our lemma above, we arrive at

$$\begin{aligned} d(v, \mathcal{C}_g h) &\preccurlyeq d(v, \mathcal{E}_g^+) + d(v, \mathcal{E}_h^-) + \lambda(g)|\alpha(g)| + \lambda(h)|\alpha(h)| \\ &\quad + (\lambda(g) + \lambda(h))(d(v, \mathcal{C}_g) + d(v, \mathcal{C}_h)) \preccurlyeq \\ &\quad \preccurlyeq d(v, \mathcal{E}_g^+) + d(v, \mathcal{C}_h) + \lambda(g)|\alpha(g)| + \lambda(h)|\alpha(h)| \\ &\quad + (\lambda(g) + \lambda(h))d(v, \mathcal{C}_g), \end{aligned}$$

and thus have proven (4.19).

In order to prove the last part of this lemma, (4.20), we pick a $u \in \mathcal{C}_g$ with $d(u, \mathcal{C}_h) = d(\mathcal{C}_g, \mathcal{C}_h)$ and set $a := \mathcal{B}_g^-(u) \cap \mathcal{E}_h^+, b := \mathcal{B}_{h^{-1}}^-(a) \cap \mathcal{C}_h$. Using (4.10) yet another time, we see $d(u,a), d(u,w) \preccurlyeq d(\mathcal{C}_g,\mathcal{C}_h)$, which in turn yields $d(gu,ga) \preccurlyeq \lambda(g)d(\mathcal{C}_g,\mathcal{C}_h)$ as well as $d(h^{-1}a,h^{-1}b) \preccurlyeq \lambda(h)d(\mathcal{C}_g,\mathcal{C}_h)$. We also have $t_g = gu-u$ and $t_h = b-h^{-1}b$. Consequently, $d(ga-a,t_g), d(a-h^{-1}a,t_h) \preccurlyeq d(\mathcal{C}_g,\mathcal{C}_h)$ and thus

$$d(ga-h^{-1}a, t_g+t_h) \preccurlyeq d(\mathcal{C}_g,\mathcal{C}_h).$$

Now, we combine the above inequality with the following: (4.6), $ga = ghh^{-1}a$ and $||x^0(gh)||$, which follows from (4.10) and the assumption on gh to be $\frac{\varepsilon}{2}$-hyperbolic, we have

$$\begin{aligned} |\alpha(gh) - B(t_g+t_h, x^0(gh)| &= |B(ga-h^{-1}v, x^0(gh)) - B(t_g+t_h, x^0(gh))| \\ &\preccurlyeq d(\mathcal{C}_g,\mathcal{C}_h). \end{aligned} \tag{4.28}$$

From *(viii)* and *(ix)* it follows

$$\begin{aligned} d(x^0(gh), \mathcal{B}_g^+(x^0(g))) &\preccurlyeq \lambda(g), \\ d(x^0(gh), \mathcal{B}_g^-(x^0(h))) &\preccurlyeq \lambda(h). \end{aligned}$$

Together with the equalities $B(t_g, \mathcal{B}_g^+(x^0(g))) = \alpha(g)$ and $B(t_h, \mathcal{B}_g^-(x^0(h))) = \alpha(h)$, this yields

$$\begin{aligned} |B(t_g, x^0(gh)) - \alpha(g)| &\preccurlyeq \alpha(g)||t_g||, \text{ and} \\ |B(t_h, x^0(gh)) - \alpha(h)| &\preccurlyeq \alpha(h)||t_h||. \end{aligned}$$

Now, applying (4.28), we arrive at

$$|\alpha(gh) - \alpha(g) - \alpha(h)| \preccurlyeq d(\mathcal{C}_g,\mathcal{C}_h) + \lambda(g)||t_g|| + \lambda(h)||t_h||.$$

This inequality, in the light of (4.12), yields (4.20) and we have finished the proof. □

Before we go on, we want to talk a little about (4.20): assume that the distance between the lines $\mathcal{C}_g$ and $\mathcal{C}_h$ as well as the eigenvalues $\lambda(g)$ and $\lambda(h)$ are 'sufficiently' small, in such a way that $\alpha(gh) - \alpha(g) - \alpha(h)$ is 'almost' equal 0, or in turn that we 'almost' have $\alpha(gh) = \alpha(g) + \alpha(h)$. Thus, for g and h which have the same sign with respect to α, considering (4.12), we would get

$$||t_g|| + ||t_h|| \sim |\alpha(g) + \alpha(h) = |\alpha(g) + \alpha(h)| = |\alpha(gh)| \sim ||t_{gh}||.$$

Also note the following: If one would substitute g, h respectively, with gh, h^{-1}, one might conclude that not only is $\alpha(gh)$ 'almost' the same as $\alpha(g) + \alpha(h)$, but also that $\alpha(gh)$ 'almost' equals $\alpha(g) - \alpha(h)$. But this is not possible. One cannot use (4.20) for gh and h^{-1} because lemma 4.2.2 *(iv)* tells us that gh and h^{-1} are not ε-transversal if $\lambda(h)$ is 'sufficiently' small.

4.2.4 Proposition. *Given hyperbolic and transversal $h_1, h_2 \in G = \mathbb{R}^3 \rtimes H$, with H the identity component of $O(\mathbb{R}^{2,1})$, we set $\Delta := <h_1, h_2>$. If we have either* *(i)* $t_{h_i} = 0$ *for any* $i \in \{1,2\}$*, or*

(ii) h_1 *and* h_2 *do not have the same sign,*

then the action $\Delta \circlearrowleft \mathbb{R}^3$ *is not properly discontinuous.*

Proof. We want our group Δ to be a free group and we can assume so without loss of generality by remembering equation (4.2). We may always find large enough k_1 and k_2 in $\mathbb{N}^0$ such that $h_1^{k_1}$ and h_2^{k2} satisfy the condition of lemma 4.2.2 and then *(i)* of said lemma tells us that $<h_1^{k_1}, h_2^{k_2}>$ is a free group. Hence, by possibly replacing h_1 and h_2 with $h_1^{k_1}$ and $h_2^{k_2}$ we may assume Δ to be a free group.

Assuming that either *(i)* or *(ii)* holds, we can find a sequence $\{m_i, n_i\}_{i \in \mathbb{N} \setminus \{0\}}$ such that $\sup_i \{|m_i \alpha(h_1) + n_i \alpha(h_2)|\} < \infty$. We set $h(i) := h_1^{m_i} h_2^{n_i}$ and pick a $v \in \mathbb{R}^3$. By using (4.2), (4.5), (4.7), (4.8), (4.9) as well as (4.19) and (4.20) we see that $L(v) := \sup_i \{d(v, \mathcal{C}_{h(1)})\} < \infty$ and also $M := \sup_i \{|\alpha(h(i))|\} < \infty$. Let g be any hyperbolic element of G and consider the following sets:

$$
\begin{aligned}
&U(v, d(v, \mathcal{C}_g) + |\alpha(g)|), \text{ and} \\
&gU(v, d(v, \mathcal{C}_g)).
\end{aligned}
$$

Both sets contain gw, $w \in \mathcal{C}_g$ with $d(v, w) = d(v, \mathcal{C}_g)$, and therefore their intersection is not empty. Now because g was arbitrary, we have for any i that

$$U(v, d(v, L(v) + M) \cap h(i)U(v, L(v)) \neq \emptyset.$$

This clearly yields that $\Delta \circlearrowleft \mathbb{R}^3$ is not properly discontinuous. □

There is one more lemma that we need to prove in order to get the final results of this section.

4.2.5 Lemma. *Given* $h_1, h_2 \in G = \mathbb{R}^3 \rtimes H$*, with* H *the identity component of* $O(2,1)$*, such that* h_1 *and* h_2 *are* 2ε*-hyperbolic and* 2ε*-transversal, with* $0 < \varepsilon < \frac{1}{10}$*, and assume that they act freely on* $\mathbb{R}^3$ *and that they have the same sign with respect to* α*. We set*

$$
\begin{aligned}
&L := \min(|\alpha(h_1)|, |\alpha(h_2)|), \text{ and} \\
&\delta := d(\mathcal{C}_{h_1}, \mathcal{C}_{h_2}),
\end{aligned}
$$

and define $a(\varepsilon), c(\varepsilon) > 1$ *as in lemmata 4.2.2 and 4.2.3 and let* h *be a word of length* j *composed by* h_1 *and* h_2*. If we have*

$$\lambda(h_i)^{-1} > 20(c(\varepsilon)^2 + a(\varepsilon))\varepsilon^{-1}, i \in \{1,2\}, \text{ and} \tag{4.29}$$

$$L \leq 60c(\varepsilon)^2\delta, \tag{4.30}$$

then there exists a $g \in G$ *with* gh *being* ε*-hyperbolic and the additional property*

$$|\alpha(gh)| \geq \frac{jL}{2}. \tag{4.31}$$

Proof. Because of $\varepsilon < \frac{1}{10}$ there are $x_1, x_2 \in W^+$ with $||x_1|| = ||x_2|| = 1$ and $d(x_1, x_2) < 2\varepsilon$. Moreover, we can choose x_1 and x_2 such that the distances $d(x_i, x^-(h_j))$ and $d(x_i, x^+(h_j))$, $i, j \in \{1, 2\}$, are always greater than 2ε.

Let $x_0 \in \mathbb{R}^3\backslash\{0\}$ such that $B(x_0, x_1) = B(x_0, x_2) = 0$. Our three vectors x_0, x_1, x_2 will then be a basis for $\mathbb{R}^3$. Now let $\tilde{g} \in H$ be the linear transformation defined by $\tilde{g}x_0 = x_0$, $\tilde{g}x_1 = 2x_1$, $\tilde{g}x_2 = \frac{1}{2}x_2$. From $x^+(\tilde{g}) = x_1$ and $x^-(\tilde{g}) = x_2$ we deduce that $\tilde{g}$ is 2ε-hyperbolic and 2ε-transversal with respect to h_1 and h_2. We fix an $v \in \mathcal{C}_{h_1}$ with the property $d(v, \mathcal{C}_{h_2}) = \delta$. Multiplying $\tilde{g}$ by an appropriately chosen translation yields an $g \in G$ that is 2ε-hyperbolic and 2ε-transversal with respect to h_1 and h_2 and furthermore has the property $\alpha(g) = 0$. Moreover, we have $v \in \mathcal{C}_g$. By possibly substituting g with g^n, $n \in \mathbb{N}$, we may assume

$$\lambda(g) < a(\varepsilon)^{-1}\frac{\varepsilon}{4}, c(\varepsilon)^2\varepsilon. \tag{4.32}$$

Let $\{i_l\}_{l\in\mathbb{N}} \subset \{1, 2\}$ and $\{k_l\}_{l\in\mathbb{N}} \subset \mathbb{Z}\backslash\{0\}$ be sequences with $i_l \neq i_{l+1}$. We now set

$$g_0 := g,\ g_l := g_{l-1}h_{i_l}^{k_l}, \text{ and}$$
$$K_0 := 1,\ K_l = K - l - 1 + |k_l| = 1 + \sum_{m=2}^{l} |k_m|.$$

By the virtue of lemma 4.2.2, (4.2), (4.3), (4.4), (4.29), (4.32) and induction on l, we see that the following inequalities hold for all l:

$$d(x^-(g_{l+1}), \{x^+(h_{i_{l+1}}), x^-(h_{i_{l+1}})\}) < \frac{\varepsilon}{4}, \tag{4.33}$$
$$d(x^+(g_{l+1}), x^+(g_l)) < \varepsilon 2K_l, \tag{4.34}$$
$$\varepsilon(1 + 2^{1-K_l}) < d(x + (g_l)\{x^-(h_1), x^+(h_1), x^-(h_2), x^+(h_2)\}) \tag{4.35}$$
$$\lambda(g_{l+1}) < 2^{-K_{l+1}}\lambda(g), \tag{4.36}$$

and furthermore we have that g_l and $h_{i_{l+1}}$ are hyperbolic and ε-transversal and we will denote this statement by *(A)*.
We set

$$a_l := d(v, \mathcal{E}_{g_l}^+),\ b_l := d(g, \mathcal{C}_{g_l}),\ t_l := \alpha(g_l),$$
$$f_l := \alpha(h_{i_l}),\ s_l := \lambda(h_{i_l}),\ T := \max(|\alpha(h_1)|, |\alpha(h_2)|).$$

Combining *(A)* and (4.18) and with the help of (4.5), (4.7), (4.32) and (4.33) through (4.36), we see

$$\begin{aligned} a_{l+1} &\leq a_l + c(\varepsilon)\lambda(g_l)(|t_l| + b_l + \delta) + c(\varepsilon)\lambda(g_{l+1})|k_{l+1}||f_{l+1} \leq \\ &\leq a_l + 2^{-K_l}(b_l + |t_l| + \delta) + 2^{-K_{l+1}}|k_{l+1}|T \leq \\ &\leq a_l + 2^{-K_l}(b_l + |t_l| + \delta + T). \end{aligned} \tag{4.37}$$

Using *(A)* and (4.19) together with (4.2), (4.5), (4.7), (4.29) and (4.33) through (4.36), we have

$$b_{l+1} \leq c(\varepsilon)\left(\lambda(g_l) + s_{l+1}^{|k_{l+1}|}\right) b_l + c(\varepsilon)\left(\delta + a_l + \lambda(g_l)|t_l| + s_{l+1}^{|k_{l+1}|}|k_{l+1}||f_{l+1}|\right) \leq$$
$$\leq \frac{b_l}{2} + c(\varepsilon)\delta + c(\varepsilon)a_l + c(\varepsilon)2^{-K_l}\lambda(g)|t_l| + \frac{T}{20c(\varepsilon)}. \tag{4.38}$$

Now we use *(A)* and (4.20) together with (4.2), (4.5), (4.7), (4.29), (4.33) through (4.36) and $d(\mathcal{C}_{g_l}, \mathcal{C}_{h_{i_l}}) \leq b_l + \delta$ in order to deduce

$$|t_{l+1} - t_l - |k_{l+1}|f_{l+1}| \leq c(\varepsilon)(b_l + \delta + \lambda(g_l)|t_l| + s_{l+1}^{|k_{l+1}|}|k_{l+1}||f_{l+1}|) \leq \tag{4.39}$$
$$c(\varepsilon)b_l + c(\varepsilon)\delta + c(\varepsilon)2^{-K_l}\lambda(g)|t_l| + \frac{T}{20c(\varepsilon)}.$$

By the virtue of this inequality along with (4.32) we have

$$|t_{l+1}| \leq (1 + 2^{-K_l})|t_l| + c(\varepsilon)b_l + c(\varepsilon)\delta + 2|k_{l+1}|T. \tag{4.40}$$

We set

$$\kappa_l := \max\{100c(\varepsilon)^2 a_l, 10c(\epsilon)b_l, |t_l|K_l^{-1}, 100c(\epsilon)^2\delta, 100c(\epsilon)T\}, \text{ and}$$
$$Q := \prod_{i=1}^{\infty}(1 + 200c(\epsilon)^2 iK_l 2^{-i}).$$

For κ_l we have, by the power of (4.32), (4.37), (4.38) and (4.40)

$$\kappa_{l+1} \leq \kappa_l(1 + 200c(\epsilon)^2 K_l 2^{-K_l}), \tag{4.41}$$

and hence for any l it holds that

$$\kappa_l \leq \kappa_0 Q. \tag{4.42}$$

Inequalities (4.33) through (4.37) and (4.42) give us

$$a_{l+1} \leq a_l + 2c(\epsilon)\kappa_0 Q K_l 2^{-K_l}\lambda(g). \tag{4.43}$$

Furthermore, recall $v \in \mathcal{C}_g$ and $\alpha(g) = 0$, we have

$$a_0 = b_0 = t_0 = 0. \tag{4.44}$$

This result, together with (4.44) and $L \leq T$, yields

$$\kappa_0 = 100c(\epsilon)T. \tag{4.45}$$

If we combine (4.43) through (4.45) and set $R := 2c(\epsilon)Q\sum_{i=1}^{\infty} i2^{-i}$ we arrive at

$$a_l \leq a_0 + R\kappa_0\lambda(g) = 100c(\epsilon)R\lambda(g)T. \tag{4.46}$$

Once again, by substituting g with an $g^k, k \in \mathbb{N}^+$ if necessary, we have

$$100c(\epsilon)^2(R+Q)\lambda(g) < \frac{1}{30c(\epsilon)}. \tag{4.47}$$

The following inequalities can be derived from putting together (4.30), (4.38), (4.42), (4.45) - (4.47) as well as $L \leq T$, $|t_l| \leq \kappa_l K_l$ and $K_l 2^{-K_l} < 1$.

$$\begin{aligned} b_{l+1} &\leq \frac{b_l}{2} + \frac{L}{50c(\epsilon)} + 100c(\epsilon)^2 R\lambda(g)T + 100c(\epsilon)^2 Q K_l 2^{-K_l}\lambda(g)T + \frac{t}{20c(\epsilon)} \leq \\ &\leq \frac{b_l}{2} + \frac{T}{60c(\epsilon)} + 100c(\epsilon)^2(R+Q)\lambda(g)T + \frac{T}{20c(\epsilon)} \leq \\ &\leq \frac{b_L}{2} + \frac{T}{60c(\epsilon)} + \frac{T}{30c(\epsilon)} + \frac{T}{20c(\epsilon)} = \frac{b_l}{2} + \frac{T}{10c(\epsilon)}. \end{aligned}$$

As stated above we have $b_0 = 0$, so, in the light of the above inequality, via induction on l, we see

$$b_l \leq \frac{T}{5c(\epsilon)}. \tag{4.48}$$

By using (4.42) and (4.45), we get from $|t_l| \leq \kappa_l K_l$ to $|t_l| \leq 100c(\epsilon)TQK_l$, which gives us, considering (4.47) and $2^{-K_l}K_l < 1$ the following:

$$b_l \leq \frac{T}{5c(\epsilon)}. \tag{4.49}$$

Now, we use (4.39), via (4.30), (4.32), (4.49) and $L \leq T$, to see

$$\begin{aligned} t_{l+1} &\geq t_l + |k_{l+1}|f_{l+1} - c(\epsilon)b_l - c(\epsilon)\delta - c(\epsilon)2^{-K_l}\lambda(g)|t_l| - \frac{T}{20c(\epsilon)} \geq \\ &\geq |t_l| + |k_{l+1}|f_{l+1} - \frac{T}{5} - \frac{T}{60c(\epsilon)} - \frac{T}{30} - \frac{T}{20c(\epsilon)} \\ &> t_l + |k_{l+1}|f_{l+1} - \frac{T}{3}. \end{aligned} \tag{4.50}$$

By assumption h_1 and h_2 have the same sign. Without loss of generality we can choose the orientation for $\mathbb{R}^3$ such that $\alpha(h_1), \alpha(h_2) > 0$. From this we infer $f_l > 0$. We also defined that $i_{l+1} \neq i_{l+2}$. Hence, $f_{l+1} + f_{l+2} = T + L$. This equation and (4.50) imply

$$\begin{aligned} t_{l+2} &\geq t_2 + |k_{l+1}|f_{l+1} + |k_{l+2}|f_{l+2} - \frac{2T}{3} = \\ &= t_l + f_{l+1} + f_{l+2} + ((|k_{l+1}| - 1)f_{l+1} + (|k_{l+2}| - 1)f_{l+2}) - \frac{2T}{3} \geq \\ &\geq t_l + T + L - \frac{2T}{3} + (|k_{l+1}| + |k_{l+2}| - 2)L > t_l + (|k_{l+1}| + |k_{l+2}| - 1)L \geq \\ &\geq t_l + \frac{1}{2}(|k_{l+1}| + |k_{l+2}|)L, \end{aligned} \tag{4.51}$$

whereas the last inquality follows from the fact that $1 < |k_{l+1}, |k_{l+2}|$. Also, be aware that (4.51) is a strict inequality.

To finish the proof we use $b_0 = t_0 = 0$ and $1 < |k_1|$ in conjunction with (4.29), (4.30) and (4.39) to see

$$t_1 \geq |k_1|f_1 - c(\epsilon)\delta - c(\epsilon)s_1^{|k_1|}|k_1|f_1 \geq \frac{1}{2}|k_1|L.$$

This inequality, together with (4.51), shows using $t_0 = 0$ and induction on l, yields

$$\alpha(g) = t_l \geq \frac{K_l}{2},$$

which, considering *(A)*, proves the lemma. □

Now we are able to prove the main theorem of this section:

4.2.6 Theorem. *Given $\varepsilon > 0$ and $h_1, h_2 \in G = \mathbb{R}^3 \rtimes H$, with H the identity component of $O(\mathbb{R}^{2,1})$, with h_1 and h_2 satisfying the assumptions of lemma 4.2.5. Then the group $\Gamma := < h_1, h_2 >$ is free and the action $\Gamma \circlearrowleft \mathbb{R}^3$ is properly discontinuous.*

Proof. Define $g \in G$ as in lemma 4.2.5: let $\tilde{g} \in G$ be defined by the equations $\tilde{g}x_0 = x_0$, $\tilde{g}x_1 = 2x_1$, $\tilde{g}x_2 = \frac{1}{2}x_2$, whereas we set x_1, x_2 such that $||x_1|| = ||x_2|| = 0, d(x_1, x_2) < 2\varepsilon$, and furthermore $d(x_1, x^-(h_j)), d(x_i, x^+(h_j)) > 2\varepsilon$ for $i, j \in \{1, 2\}$. We picked x_0 such that $B(x_0, x_i) = 0$. Finally, it may be necessary, but possible without loss of generality, to substitute $\tilde{g}$ with g^k for a large enough $k \in \mathbb{N}^+$.

Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^3$. Lemma 4.2.5 tells us then, in view of (4.6) and (4.11), that $\{\gamma \in \Gamma \mid g\gamma\mathcal{K} \cap g\mathcal{K} \neq \emptyset\} = \{\gamma \in \Gamma \mid \gamma\mathcal{K} \cap \mathcal{K} \neq \emptyset\} < \infty$. This finiteness together with lemma 4.2.2 *(i)* yields the theorem. □

In proposition 4.2.4 we have seen that for hyperbolic and transversal $h_1, h_2 \in G$ acting freely on $\mathbb{R}^3$, $< h_1, h_2 >$ is not a properly discontinuous subgroup, if h_1 and h_2 have different sign with respect to α. The following corollary tells us what happens if we assume the opposite:

4.2.7 Corollary. *Given hyperbolic and transversal $h_1, h_2 \in G = \mathbb{R}^3 \rtimes H$, that have the same sign with respect to α and act freely on $\mathbb{R}^3$, one can find a $k \in \mathbb{N}$, such that for all $m, n > k$ with $m, n \in \mathbb{N}$ the group $< h_1^m, h_2^n >$ is free and $< h_1^m, h_2^n > \circlearrowleft \mathbb{R}^3$ is properly discontinuous.*

Proof. This corollary follows from theorem 4.2.6 in conjunction with (4.2), (4.5) and (4.7). □

In order to apply theorem 4.2.6 to hyperbolic and transversal h_1 and h_2 one has to check quite a few things. The next theorem we present is a special case of theorem 4.2.6, which imposes fewer assumptions on h_1 and h_2.

4.2.8 Theorem. *Given $h_1, h_2 \in G = \mathbb{R}^3 \rtimes H$, with H the identity component of $O(2,1)$. Let h_1 and h_2 be 2ε-hyperbolic and 2ε-transversal and suppose further that h_1 and h_2 have the same sign with respect to α and that they act freely on $\mathbb{R}$.*
If we have that

(i) $\mathcal{C}_{h_1} \cap \mathcal{C}_{h_2} \neq \emptyset$, *and*

(ii) $\lambda(h_i)^{-1} > 20(c(\varepsilon)^2 + a(\varepsilon))\varepsilon^{-1}$ *for* $i \in \{1,2\}$, *then*

the group $<h_1, h_2>$ *is free and* $<h_1, h_2> \circlearrowleft \mathbb{R}^3$ *is properly discontinuous.*

Now, by using proposition 4.2.4 and theorem 4.2.8, along with lemma 4.2.2 *(i)* and a little more notation, we arrive at the final statement of this section. Said statement not only wraps things up nicely, but also tells us how one may find free groups that act properly discontinuously on $\mathbb{R}^3$, once hyperbolic h_1 and h_2 are found.

If we are given hyperbolic $h, g \in G$, we will call them *similar* if we have $\mathcal{C}_g = \mathcal{C}_h$ as well as $\lambda(g) = \lambda(h)$.

4.2.9 Corollary. *One can find hyperbolic $h_1, h_2 \in G$, with $\mathcal{C}_{h_1} \cap \mathcal{C}_{h_2} \neq \emptyset$, such that: if we have $\tilde{h}_1, \tilde{h}_2 \in G$ with $\tilde{h}_i$ being similar to h_i for $i \in \{1,2\}$, then we have*

(i) $<\tilde{h}_1, \tilde{h}_2>$ *is a free group, and*

(ii) the action $<\tilde{h}_1, \tilde{h}_2> \circlearrowleft \mathbb{R}^3$ *is properly discontinuous iff* $\tilde{h}_1$ *and* $\tilde{h}_2$ *have the same sign with respect to* α *and act freely on* $\mathbb{R}^3$.

Now that we have shown that Milnor's conjecture does not hold, we would like to give some positive results, i.e. present some cases in which Auslander's conjecture is known to be true.

4.3 Auslander's conjecture holds in dimension 2 & dimension 3

This section is dedicated to affine crystallographic groups of 'small' dimension, i.e. 2 and 3. For these dimensions, it is not only possible to show that Auslander's conjecture holds, one can even proof stronger theorems, i.e. that these affine crystallographic groups are even solvable. All of these proofs are due to Fried and Goldman, [FG83].

At first we will examine the case of dimension 2 and later go on to dimension 3. For dimension 2 we will actually prove that properly discontinuous groups are virtually solvable. Similarly, we will prove that a properly discontinuous group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^3)$ is virtually solvable, if either $\Gamma\backslash\mathbb{R}^3$ is compact, and thus Γ crystallographic, or if the projection onto the linear part $L(\Gamma)$ has no subgroup that preserves a Lorentzian inner product.

In either case, by the power of Selberg's lemma, theorem 4.1.7, we may assume that an affine crystallographic group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$, $n \in \{2,3\}$, is torsion free and therefore no $\gamma \neq e, \gamma \in \Gamma$, has a fixed point. Because of this, we can, and will, make use of the following lemma.

4.3.1 Lemma (Eigenvalue-1-criterion)**.** *Given a $\gamma \in \mathrm{Aff}(\mathbb{R}^n)$ that acts freely on $\mathbb{R}^n$. Then we have that 1 is an eigenvalue of $L(\gamma)$.*

Proof. As stated before, the action of any $\gamma \in \mathrm{Aff}(\mathbb{R}^n)$ can be written as $\gamma(v) = Av + b$, with $L(\gamma) = A \in GL(\mathbb{R}^n), b \in \mathbb{R}^n$. Using this, we deduce the following:

$$\gamma \text{ has a fixed point } \iff \exists v \in \mathbb{R}^n : Av + b = v \iff b \in Im(A - \mathbb{I}).$$

Hence, if γ does not have any fixed point, it follows that $b \notin Im(A - \mathbb{I})$, or equivalently, that $(A - \mathbb{I})\mathbb{R}^n < \mathbb{R}^n$, which yields that $(A - \mathbb{I})$ is not invertible. It is a standard result of linear algebra, that if T is a linear map and $(T - \lambda\mathbb{I})$ is not invertible, then λ is an eigenvalue for T. Thus, we conclude that for any γ acting freely, $L(\gamma) = A$ has 1 as an eigenvalue. □

To proceed, we need two important observations about the Zariski closure. First, $\mathrm{Aff}(\mathbb{R}^n)$ is an algebraic subgroup of $GL(\mathbb{R}^{n+1})$, thus for any $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ we have $\overline{\Gamma} \leq \mathrm{Aff}(\mathbb{R}^n)$. This yields the second important fact, which is due to Konstant and Sullivan, [KS75], namely the following corollary to our lemma about eigenvalues of freely acting affine groups, .

4.3.2 Corollary. *Given a $\Gamma \in \mathrm{Aff}(\mathbb{R}^n)$ that acts freely on $\mathbb{R}^n$. Then we have that 1 is an eigenvalue of $L(g)$ for every $g \in \overline{\Gamma}$.*

Proof. This follows immediately from the fact that the condition on having 1 as an eigenvalue is based on the polynomial equation $\det(L(g) - \mathbb{I}) = 0$. □

The second important fact about the Zariski closure is the following: the identity component, with respect to the Zariski closure, $\overline{G}^0$ of $\overline{G}$ is always a normal subgroup of finite index in $\overline{G}$. Thus, when dealing with virtual properties of an affine crystallographic group Γ, we may always assume, that, in addition to Γ being torsion free and it acting freely, $\Gamma \leq \overline{\Gamma}^0$, because $\Gamma \cap \overline{\Gamma}^0$ has finite index in Γ.

There is one more thing we need, before we can start to prove that Auslander's conjecture is true in dimensions 2 and 3.

4.3.3 Definition (Levi decomposition and Levi factor)**.** *In the light of the Levi-Malcev theorem, every Lie algebra $\mathfrak{g}$ can be seen as a semi-direct product $\mathfrak{g} = \mathfrak{rad}(\mathfrak{g}) \rtimes \mathfrak{s}$, whereas $\mathfrak{rad}(\mathfrak{g})$ denotes the radical of $\mathfrak{g}$, i.e. its (unique) maximal solvable ideal, and $\mathfrak{s}$ is an arbitrary but fixed maximal semisimple subalgebra of $\mathfrak{g}$. For such an $\mathfrak{s}$ we always have $\mathfrak{g}/\mathfrak{rad}(\mathfrak{g}) \cong \mathfrak{s}$.*

This semidirect product is called a Levi decomposition *of $\mathfrak{g}$ and the maximal semisimple subalgebra is called a* Levi factor *of $\mathfrak{g}$. Note that while in general $\mathfrak{s}$ is not unique, every two maximal semisimple subalgebras are not only isomorphic but also conjugate.*

Now let G be a Lie group with Lie algebra $\mathfrak{g} = \mathfrak{rad}(\mathfrak{g}) \rtimes \mathfrak{s}$. The maximal semisimple connected subgroup S of G corresponding to the Levi factor $\mathfrak{s}$ of $\mathfrak{g}$ will be called a Levi factor of G. If G is connected, it can be decomposed as $G = rad(G)S$.

If G is a subgroup of $\mathrm{Aff}(\mathbb{R}^n)$, it can be shown that its Levi factor S can be conjugated to lie in $SL(\mathbb{R}^n)$.

The Levi factors of the Zariski closure of properly discontinuous/crystallographic groups are the corner stones for the proofs of Auslander's conjecture in dimension 2 and 3. They are Lie groups, and thus by the classification of Lie algebras and Lie groups, we know a great deal about them, especially because they are of small dimension.

Basically, the approach to the proofs we present is as follows: we use the fact, that the Levi factor S of $\overline{G}^0$ is isomorphic to a subgroup $\tilde{S}$ of $SL(\mathbb{R}^n)$. This allows us to split the proof into disjoint cases.

4.3.4 Theorem. *A properly discontinuous subgroup Γ of $\mathrm{Aff}(\mathbb{R}^2)$ is virtually solvable.*

Proof. Every such Γ has a torsion free subgroup Γ_1 of finite index. Additionally, we assume $\Gamma_1 \leq \overline{\Gamma_1}^0$. According to corollary 4.3.2, every $g \in \overline{\Gamma_1}^0$ has 1 as an eigenvalue. Furthermore, because $\overline{\Gamma_1}^0$ is a connected Lie group, we have $\overline{\Gamma_1}^0 = rad(\overline{\Gamma_1})^0 S$, with S being the Levi factor of $\overline{\Gamma_1}^0$.

As stated above, because $\overline{\Gamma_1}^0$ is an affine group, S can be conjugated into $SL(\mathbb{R}^2)$. However, by virtue of the classification of simple Lie algebras, we know that $SL(\mathbb{R}^2)$ has no proper semisimple subgroups, and thus we either have that S is the trivial group or that $S \cong SL(\mathbb{R}^2)$. But $SL(\mathbb{R}^2)$ certainly has elements that do not have 1 as an eigenvalue, and hence, assuming the latter case would yield a contradiction. Therefore, the first case is the only possible one, but we immediately have that $\overline{\Gamma_1}^0 = rad(\overline{\Gamma_1})^0$, if S is trivial. By definition, $\overline{\Gamma_1}^0$ is solvable, and a fortiori so is Γ_1. This proves that Γ is virtually solvable. $\square$

Before we go on to examine affine crystallographic subgroups of $\mathrm{Aff}(\mathbb{R}^3)$, a few more words on the 2-dimensional case.

We start by defining three subgroups of $\mathrm{Aff}(\mathbb{R}^2)$:

$$H := \left\{ \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\},$$
$$T := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} \mid s, t \in \mathbb{R} \right\},$$
$$P := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} \mid s, t \in \mathbb{R} \right\}.$$

In this definition the letters H, T and P correspond to properties of their respective groups: *hyperbolic*, *translation* and *parabolic*.

Using these groups, we get a nice characterization of affine subgroups of $\mathrm{Aff}(\mathbb{R}^2)$. A proof for the following assertion can be found in [Ku53].

4.3.5 Proposition. *The natural actions of H, T and P on $\mathbb{R}^2$ are proper, therefore if Γ is a discrete subgroup of any of these groups, the natural action $\Gamma \circlearrowleft \mathbb{R}^2$ is properly discontinuous.*

On the other hand, in every properly discontinuous group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^2)$ there is a *subgroup $\tilde{\Gamma} \leq \Gamma$ of finite index, that is isomorphic to a discrete subgroup of either H, T or P. In other words, every Γ is virtually (isomorphic to) a subgroup of one of these three groups.*

It is worth noting that, aside from the trivial case $\Gamma = \{e\}$, there is only one possibility for Γ to virtually be a subgroup of two of these groups at the same time, namely if Γ is a group of translations, then it lies virtually in $T \cap P$.

The following corollary follows directly from the classification above.

4.3.6 Corollary. *Given $\Gamma \leq \mathrm{Aff}(\mathbb{R}^2)$, Γ properly discontinuous. Then we have*

(i) *Γ is virtually abelian, and*

(ii) *Γ is crystallographic iff it is virtually isomorphic to $\mathbb{Z}^2$.*
In geometric terms we have, assuming that Γ is torsion free, that

(iii) *$\Gamma\backslash\mathbb{R}^2$ is diffeomorphic to either the cylinder $S^1 \times \mathbb{R}^2$ or the torus $S^1 \times S^1$.*

The second assertion of this corollary can also be stated in the following way, which also holds for higher dimensions: a properly discontinuous group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ is crystallographic iff its cohomological dimension is n. This assertion will be used shortly, namely during the course of proving Auslander's conjecture in dimension 3, to which we will proceed now.

4.3.7 Theorem. *A properly discontinuous group Γ of $\mathrm{Aff}(\mathbb{R}^3)$ is virtually solvable, if either*

(i) *there is no subgroup of $L(\Gamma)$ of finite index, that preserves a Lorentzian inner product, or*

(ii) *$\Gamma\backslash\mathbb{R}^3$ is compact, i.e. Γ is affine crystallographic.*

Proof. The first assertion follows from propositions 4.3.9 and 4.3.13. Proposition 4.3.14 then completes the proof. □

We can narrow down the situation at hand, i.e. we present what we can assume without loss of generality analogously to the 2-dimensional case:

From now on we are only concerned with finite-index subgroups Γ_1 of properly discontinuous groups Γ, such that Γ_1 is torsion free and acts freely on $\mathbb{R}^3$. Furthermore, we may assume such a group Γ_1 to lie in the identity component of its algebraic hull, i.e. $\Gamma_1 \leq \overline{\Gamma_1}^0$. Recall that we have that every element in $\overline{\Gamma_1}$ has 1 as an eigenvalue and so the same applies to the Levi factor S of $\overline{\Gamma_1}^0$. Moreover, S is isomorphic to a connected subgroup of $SL(\mathbb{R}^3)$, which has to be proper because of the eigenvalue-criterion and lastly we have $\overline{\Gamma_1}^0 = rad(\overline{\Gamma_1}^0)S$. This, together with $S \cong L(S)$, immediately yields that the Levi factor of $L(\overline{\Gamma_1}^0) = L$ has to be isomorphic to S.

The fact $S \cong L(S)$ follows directly from $S \cong \tilde{S} \leq SL(\mathbb{R}^n)$, because we immediately see that every element of S acts without a translation.

Before we start the proof, we want to put *(i)* and *(ii)* into perspective. To do so we need the following lemma:

4.3.8 Lemma. *$SL(\mathbb{R}^3)$ has only three proper semisimple connected subgroups, namely:*

(i) $SO(\mathbb{R}^2) \times \{1\} := \begin{pmatrix} SO(\mathbb{R}^2) & 0 \\ 0 & 1 \end{pmatrix}$,

(ii) $SO(\mathbb{R}^3)$, *and*

(iii) $SO(\mathbb{R}^{2,1})^0$, *i.e. the identity component of the group that preserves the bilinear form* $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $B(x,y) \mapsto x_1y_1 + x_2y_2 - x_3y_3$.

Proof. This assertions follows from the classification of Lie algebras as well as from the classification of Lie groups. □

In our situation, this lemma means that the Levi factor S of $L(\overline{\Gamma_1}^0)$ has to be conjugate to one of these three groups, and thus assertion *(i)* of theorem 4.3.7 is just a combination of the first two cases of the lemma. Indeed, to prove that Γ_1 is virtually solvable, if S is conjugate to either $SO(\mathbb{R}^2) \times \{1\}$ or $SO(\mathbb{R}^3)$, we do not have to require cocompactness of Γ. Propositions 4.3.9 and 4.3.13 will be concerned with these cases.

Only for the last case, proposition 4.3.14, we certainly need the quotient $\Gamma\backslash\mathbb{R}^3$ to be compact. Indeed, this assumption cannot be dropped: Looking at Margulis' counterexample, we see the following: in corollary 4.2.7 a properly discontinuous group is constructed, that is not cocompact but a free group on two elements and thus not virtually solvable. Said group is a subgroup of $\mathbb{R}^3 \rtimes O(\mathbb{R}^{2,1})^0$ and hence we see that the Levi factor of its algebraic hull is conjugate to $SO(\mathbb{R}^{2,1})^0$.

Now that we have settled the situation at hand, let us start with our proof.

4.3.9 Proposition. *Let $\Gamma_1 \leq \Gamma$ be defined as above. If we have for the Levi factor S of $\overline{\Gamma_1}^0$ that $S \cong SO(\mathbb{R}^2) \times \{1\}$, then Γ is virtually solvable.*

Proof. It can be shown that any connected group G with Levi factor S may be conjugated in such a way, that it is a subgroup of one of the following groups.

$$\text{Either } G \leq G_1 = \begin{pmatrix} \multicolumn{2}{c}{GL(2,\mathbb{R})} & \begin{matrix} a \\ b \end{matrix} \\ 0 & 0 & c \end{pmatrix}, \text{ or}$$

$$G \leq G_2 = \begin{pmatrix} \multicolumn{2}{c}{GL(2,\mathbb{R})} & \begin{matrix} 0 \\ 0 \end{matrix} \\ a & b & c \end{pmatrix},$$

with $a, b, c \in \mathbb{R}$, not fixed. Moreover, the conjugation can be chosen, such that, under the projection to $GL(\mathbb{R}^2)$, G is mapped onto at least $SL(\mathbb{R}^2)$.

Now, as above, $S \cong L(S)$ and hence $L(\overline{\Gamma_1}^0) \cong L(rad(\overline{\Gamma_1})^0)S$. From this we see that S is also the maximal semisimple subgroup of $L(\overline{\Gamma_1}^0)$. If we set $G = L\overline{\Gamma_1}^0)$, we have that the last slot on the main diagonal has to be 1 in either of the two cases. Indeed,

$SO(\mathbb{R}^2) \leq L(\overline{\Gamma_1}^0)$, but every $\gamma \in L(\overline{\Gamma_1}^0)$ has to have 1 as an eigenvalue, and hence,

$$\text{either } L(\overline{\Gamma_1}^0) \leq \tilde{G_1} = \begin{pmatrix} GL(2,\mathbb{R}) & \begin{matrix} a \\ b \end{matrix} \\ 0 \quad 0 & 1 \end{pmatrix}, \text{ or}$$

$$L\overline{\Gamma_1}^0) \leq \tilde{G_2} = \begin{pmatrix} GL(2,\mathbb{R}) & \begin{matrix} 0 \\ 0 \end{matrix} \\ a \quad b & 1 \end{pmatrix},$$

At first we examine what happens if $L(\overline{\Gamma_1}^0) \leq \tilde{G_1}$. If we look at the linear functional $x_3 : (x, y, z) \mapsto z$, we see that it is invariant under $L(\Gamma_1)$. Hence, we have that the parallel form dx_3 on $\mathbb{E}$ is invariant under the action of Γ_1. Now we use the linear functional x_3 to construct a homomorphism:

$$\begin{aligned} &\varphi\colon \Gamma_1 \to \mathbb{R} \\ &\varphi(\gamma) \mapsto x_3(\gamma(v))) - x_3(v), \text{ with } v \in \mathbb{R}^3,\ v \text{ arbitrary but fixed.} \end{aligned}$$

Clearly, due to the invariance mentioned above, $\ker(\varphi) = \{\gamma \in \Gamma \mid \gamma((x_1, y_1, z)) = (x_2, y_2, z)\}$. In other words, every subspace $(x, y, \overline{z})$, with $x, y \in \mathbb{R}$, x, y not fixed, and $\overline{z} \in \mathbb{R}$, $\overline{z}$ fixed, is invariant under the action of $\ker(\varphi)$. Of course, $\ker(\varphi) \leq \Gamma_1$ acts properly discontinuous, and considering that it acts invariantly on a 2-dimensional space, we may view $\ker(\varphi)$ as a subgroup of $\mathrm{Aff}(\mathbb{R}^2)$, therefore it is virtually solvable by the power of theorem 4.3. This yields that Γ_1 itself is virtually solvable, because it is an extension of $\ker(\varphi)$ by an abelian group, namely $\mathbb{R}$, and extending a (virtually) solvable group by a solvable group results in a (virtually) solvable group.

Now, on to the case $L(\overline{\Gamma_1}^0) \leq \tilde{G_2}$ for which we need the two following lemmata:

4.3.10 Lemma. *Let $G \leq \mathrm{Aff}(\mathbb{R}^n)$. Then we have:*

$$G \textit{ is solvable} \iff L(G) \cap SL(\mathbb{R}^n) \textit{ is solvable.}$$

Proof. "⇒" G is solvable, and thus $L(G)$ is, too. Of course, every subgroup of $L(G)$ is also solvable.
"⇐" Recall that $GL(\mathbb{R}^n) = SL(\mathbb{R}^n) \rtimes \mathbb{R}^\times$, and hence we have a short exact sequence:

$$1 \to SL(\mathbb{R})^n \to GL(\mathbb{R}^n) \overset{\det}{\to} \mathbb{R}^\times \to 1$$

Similarly, for $L(G) \leq GL(\mathbb{R}^n)$, we have

$$1 \to L(G) \cap SL(\mathbb{R})^n \to L(G) \overset{\det}{\to} \det(L(G)) \to 1$$

Certainly, $\det(L(G)) \leq \mathbb{R}^\times$ is abelian and thus solvable. By assumption, the section $L(G) \cap SL(\mathbb{R})^n$ is also solvable and we deduce that the same is true for $L(G)$, because it is the extension of a solvable group by a solvable group.

Since L is an homomorphism, there is an exact sequence

$$1 \to T \to G \xrightarrow{L} L(G) \to 1.$$

Looking at

$$1 \to \mathbb{R}^n \to \mathrm{Aff}(\mathbb{R}^n) \xrightarrow{L} GL(\mathbb{R}^n) \to 1.$$

we see that T is abelian and thus solvable. Using the exact same argument as above we see that G has to be solvable. $\square$

4.3.11 Lemma. *Let $G \leq GL(2,\mathbb{R})$ and suppose G is not solvable. Then there are hyperbolic g_1, $g_2 \in G \cap SL(2,\mathbb{R})$ such that they do not have a common eigenspace.*

Proof. Without loss of generality, we may assume $G \leq SL(\mathbb{R}^2)$. Because G is not solvable, the representation of G on $\mathbb{R}^2$ has to be irreducible. This fact allows us to use the Burnside theorem as done in [CoGu74], which implies the existence of some hyperbolic $g_1 \in G$. From the irreducibility of the representation we deduce that there is an $h \in G$ which is either elliptic, i.e. h has complex conjugate eigenvalues, or its eigenspaces are different from the eigenspaces of g_1. Now, if h is hyperbolic, we set $g_2 := h$. If it is not, there is an $n \in \mathbb{N}$ such that $g_2 := g_1^n h$ is hyperbolic and that g_1 and g_2 have no shared eigenspace. $\square$

In situation of $L(\overline{\Gamma_1}^0) \leq \tilde{G}_2$ we have that every vector $(0,0,z)$ is invariant under the action of $L(\Gamma_1)$ and thus the linear functional x_3, defined as above, is also invariant under $L(\Gamma_1)$. Consequently, the vector field $\frac{d}{dx_3}$ is invariant under the action of Γ_1. We set $M := \Gamma\backslash\mathbb{R}^3$ and ζ shall be the parallel vector field on M, that is induced by $\frac{d}{dx_3}$, and by $\{\zeta_t\}_{t\in\mathbb{R}}$ we denote its flow. The vector field ζ is completely integrable on M, because $\frac{d}{dx_3}$ is completely integrable on $\mathbb{R}^3$.

From now on, we will assume that Γ_1 is not solvable. Our goal is to show that this leads to a contradiction.

According to the first of the two lemmata 4.3.10 and 4.3.11, we have that the section $L(\Gamma_1) \cap SL(\mathbb{R}^2)$ is not solvable. Recalling that $L(\Gamma_1) \leq G_2$, we see that we can map $L(\Gamma_1) \cap SL(\mathbb{R}^2)$ homomorphically into $GL(\mathbb{R}^2)$, such that the image is not solvable. Using the second lemma, we know that in this image there are hyperbolic $g_1, g_2 \in GL(\mathbb{R}^2)$ that share no eigenspace. Now, fix γ_i such that the $L(\gamma_i)$ is in the preimage of g_i, for $i \in \{1,2\}$. $L(\gamma_i)$ has to look as follows:

$$L(\gamma_i) = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{g_i}} & 0 \\ & & 0 \\ a & b & 1 \end{pmatrix},$$

with $a, b \in \mathbb{R}$. From the hyperbolicity of g_i we follow that there is a unique line l_i in $\mathbb{R}^3$ that is invariant under the action of γ_i as well as parallel to $\frac{\partial}{\partial x_3}$. This yields that the image σ_i of l_i in M under the natural projection is a closed orbit of ζ. The Poincaré map around this orbit is $g_i^{\pm 1}$. Furthermore this orbit is hyperbolic in the sense of a smooth dynamic system (for more details see [Sm67]).

We set

$$W_i := \{m \in M \mid \xi_t(m) \to \sigma_i \text{ for } t \to +\infty\}.$$

These stable manifolds W_i are covered by planes P_i in $\mathbb{R}^3$ parallel to the $(v_i, \frac{\partial}{\partial x_3})$-plane, if v_i is an eigenvector of $L(\gamma_i)$ with eigenvalue $\neq 1$ - this can be seen by passing on to the covering space $\mathbb{R}^3/\{\gamma_i^n \mid n \in \mathbb{Z}\}$. Because the eigenvalue associated to v_i is not equal 1, v_i has to correspond to the g_i-part of $L(\gamma_i)$. As stated above, g_1 and g_2 do not share an eigenspace, so the eigenvectors v_1 and v_2 are not parallel and we conclude that $m := P_1 \cap P_2$ is a line parallel to $\frac{\partial}{\partial x_3}$.

Now we fix a complete Riemannian metric on M and lift it to $\mathbb{R}^3$. Let $0 < \varepsilon \in \mathbb{R}$. By N_i we denote the ε-tubular neighborhood of l_i in $\mathbb{R}^3$. Because N_i is invariant under γ_i, we see that the positive rays of m lie in $N_1 \cap N_2$, see figure 4.1.

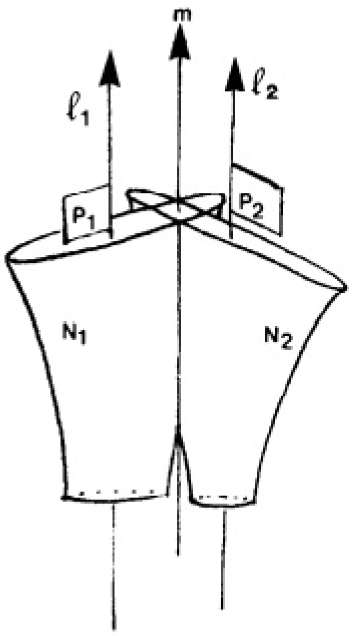

Figure 4.1: Fried and Goldman, Stable manifolds of an impossible parallel flow, [FG83].

We conclude that the following inequalities for distances: $d(\sigma_1, \sigma_2) \leq d(l_1, l_2) \leq 2\varepsilon$. But ε was chosen arbitrarily, thus, from the compactness of the σ_i, we deduce that σ_1 and σ_2 are identical. Consequently, the two lifts l_1 and l_2 of σ_1 also have a distance of 0 and therefore $l_1 = l_2$. This yields that the properly discontinuous action $<\gamma_1, \gamma_2> \circlearrowleft l_1$ consists of pure translations and hence our assumption on no shared eigenspaces is contradicted.

Hence, we have proven that Γ_1 is solvable if the Levi factor S of the algebraic hull of Γ_1 is isomorphic to $SO(\mathbb{R}^2) \times \{1\}$ and consequently Γ is virtually solvable. $\square$

For the cases $S \cong SO(\mathbb{R}^3)$ and $S \cong SO(\mathbb{R}^{2,1})^0$ we need the following well-known proposition on connected Lie groups.

4.3.12 Proposition. *Let G be a connected Lie group G and let K be a maximal compact subgroup. It follows that G decomposes into a topological direct product $G = K \times \mathbb{R}^k$, for some $k \in \mathbb{N}$.*

It is worth noting that in general K is not unique, yet all maximal compact subgroups of G are conjugate in G. Also, a maximal compact subgroup is not always a maximal subgroup. Further, while G can be written as a topolocigal direct product, there is not necessarily a group-isomorphism between G and the direct product of the groups K and $\mathbb{R}^k$.

4.3.13 Proposition. *Let $\Gamma_1 \leq \Gamma$ be defined as above. If we have for the Levi factor S of $\overline{\Gamma_1}^0$ that $S \cong SO(\mathbb{R}^3)$, then Γ is virtually solvable.*

Proof. By assumption S contains only elements without a translational part, thus we have $S \cong L(S)$, which implies that the Levi factor of $L(\overline{\Gamma_1}^0) = L(rad(\overline{\Gamma_1}^0))L(S)$ is isomorphic to S, because homomorphisms respect solvability, and hence, $L(S)$ is a maximal semisimple subgroup.

Of course, $L(\overline{\Gamma_1}^0)$ is a connected Lie group and S a compact subgroup, because L is continuous and thus preserves connectedness and compactness. In the light of proposition 4.3.12, it is now easy to see that $GL(\mathbb{R}^3)$ contains only two connected subgroups whose Levi factor is isomorphic to S, namely S itself and $S \times \{\lambda \mathbb{I} \mid \lambda \in \mathbb{R}^+\}$ and hence $L(\overline{\Gamma_1}^0)$ has to be isomorphic to either one of these groups. However, once again, using the fact that every $\gamma \in \overline{\Gamma_1}^0$ has to have 1 as an eigenvalue, we deduce that $L(\overline{\Gamma_1}^0) \cong SO(\mathbb{R}^3)$, but this means that $L(\overline{\Gamma_1}) \leq SO(\mathbb{R}^3)$. Consequently, we have $\Gamma_1 \leq \text{Isom}(\mathbb{R})^3$ and hence, by applying the first Bieberbach theorem, we see that Γ_1, and thus Γ, is virtually abelian. $\square$

Clearly, in the view of lemma 4.3.8, by combining propositions 4.3.9 and 4.3.13 we see that 4.3.7 *(i)* indeed holds. And thus, by showing that the next proposition is true, we will complete the proof of 4.3.7.

4.3.14 Proposition. *Let $\Gamma_1 \leq \Gamma$ be defined as above with the addition that Γ is crystallographic. If we have for the Levi factor S of $\overline{\Gamma_1}^0$ that $S \cong SO(\mathbb{R}^{2,1})^0$, then Γ is virtually solvable.*

In [So96] this proposition is shown to even hold for semigroups. This proof is also different in nature to the approach we will present - Soifer's proof is more geometric and relies on the dynamics of affine maps.

Proof. Analogously to the proof for proposition 4.3.13, we observe that besides $SO(2,1)^0$ there is only one connected subgroup of $GL(\mathbb{R}^3)$: the product $SO(2,1)^0 \times \mathbb{R}^+$, but we can rule out the latter, once again, by applying the lemma about eigenvalues. This leads to $L(\Gamma_1) \leq L(\overline{\Gamma_1}^0) \cong SO(\mathbb{R}^{2,1})$.

We are now interested in restriction of L to Γ_1. There are two possibilities: either $L(\Gamma_1)$ is Zariski dense in $SO(\mathbb{R}^{2,1})$, or it is not.

We start by assuming the first possibility and claim that in this situation

(i) $L(\Gamma_1)$ is discrete in $SO(\mathbb{R}^{2,1})$, and

(ii) $L|_{\Gamma_!} \colon \Gamma_1 \to SO(\mathbb{R}^{2,1})$ is injective.

We will then show that *(i)* in conjunction with *(ii)* contradicts crystallographicness.

The first assertion follows directly from corollary 3.3.3.

In order to prove the second assertion we will now, for the first time in this proof for our theorem 4.3.7, require that Γ, and thus Γ_1, is not only properly discontinuous but crystallographic, i.e. we assume that $\Gamma\backslash\mathbb{R}^3$ is compact.

Now we will prove that $L|_{\Gamma_1}$ is injective. Suppose the opposite, i.e. that $\ker(L|_{\Gamma_1})$ is not trivial. This implies that $\ker(L|_{\Gamma_1})$ is a lattice in the group of translations T, because the representation of $SO(\mathbb{R}^{2,1})$ on $\mathbb{R}^3$ is irreducible, and so the same is true for $L(\Gamma_1)$, from which we deduce that $\ker(L|_{\Gamma_1})$ is an $L(\Gamma_1)$-module and that $\ker(L|_{\Gamma_1})$ is discrete in T. Furthermore, the compact space $\Gamma_1\backslash\mathbb{R}^3$ is a covering space for $\ker(L|_{\Gamma_1})\backslash\mathbb{R}^3$, which is also compact, because Γ_1 is torsion free. This leads to the fact that $\ker(L|_{\Gamma_1})$ is of finite index in Γ_1. However, this means that $L(\Gamma_1) \cong \tilde{\Gamma}/\ker(L|_{\Gamma_1})$ is finite, which contradicts that Γ_1 is Zariski dense in $SO(\mathbb{R}^{2,1})$ and assertion *(ii)* follows.

Let X be the space, such that $Sym(X) \cong SO(\mathbb{R}^{2,1})$. For example, the complex upper half plane is a model of X and we deduce that X is an aspherical manifold of dimension 2. Now, because every discrete subgroup of $SO(\mathbb{R}^{2,1})$ acts properly discontinuously on X, the same holds for $L(\Gamma_1)$ and therefore $L(\Gamma_1)$ has virtual cohomological dimension of at most 2. However, the action $\Gamma_1 \circlearrowleft \mathbb{R}^3$ is crystallographic and therefore $\Gamma_1\backslash\mathbb{R}^3$ is a closed aspherical manifold of dimension 3. We conclude that the virtual cohomological dimension of Γ_1 has to be 3 and arrive at a contradiction.

To complete the proof, we have to examine what happens if $L(\Gamma_1)$ is not Zariski dense in $SO(\mathbb{R}^{2,1})$. As seen above, at least one of the following statements has to be true:

(i) $L(\Gamma_1)$ is not discrete in $SO(\mathbb{R}^{2,1})$, or

(ii) $L|_{\Gamma_1} \colon \tilde{\Gamma} \to SO(\mathbb{R}^{2,1})$ is not injective.

We start with *(i)*: if $L(\Gamma_1)$ is not discrete in $SO(\mathbb{R}^{2,1})$, then $\overline{L(\Gamma_1)}^0$ is not the trivial group and furthermore it is normalized by $L(\Gamma_1)$. By virtue of the generalized first theorem of Bieberbach, 3.3.2, $\overline{L(G)}^0$ is solvable, if $G \leq \mathrm{Aff}(\mathbb{R}^3)$ is discrete. Consequently, $L(\Gamma_1)$ normalizes a nontrivial solvable subgroup of $SO(\mathbb{R}^{2,1})$ and therefore must be solvable.

Now we suppose that $L|_{\Gamma_1}$ is not injective: we start with the observation that any subgroup of $SO(\mathbb{R}^{2,1})$ which leaves invariant a linear space of dimension either 1 or 2 has to be solvable. Of course, Γ_1 and thus $L(\Gamma_1)$ acts invariantly on

$$\ker(L|_{\Gamma_1}) = \{\gamma \in L(\Gamma_1) \mid \gamma \text{ is a translation}\}.$$

Hence, if $L(\Gamma_1)$ is not solvable, $\ker(L|_{\Gamma_1})$ has to be of dimension 3 and we can find three linear independent translations in Γ_1. Because $\ker(L|_{\Gamma_1})\backslash\mathbb{R}^3$ is compact, we have that Γ_1, and thus Γ, is virtually solvable. $\square$

In the course of the proofs for propositions 4.3.9, 4.3.13 and 4.3.14, we have seen that actually there is only one situation in which Γ is not virtually solvable, namely if the linear projection $L(\Gamma_1)$ is Zariski dense in $SO(\mathbb{R}^{2,1})$ - in all the other cases and subcases thereof we do not need Γ to be crystallographic.

More on the Zariski closure of properly discontinuous groups can be found in [AMS97] and [AMS02]. The main results are:

4.3.15 Theorem. *One can find a properly discontinuous group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^{2n+1})$, with $n \in \mathbb{N}$, such that $L(\Gamma)$ is Zariski dense in $SO(\mathbb{R}^{n+1,n})$, if n is odd.*

4.3.16 Theorem. *One cannot find a properly discontinuous group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^{2n+1})$, with $n \in \mathbb{N}$, such that $L(\Gamma)$ is Zariski dense in $SO(\mathbb{R}^{n+1,n})$, if n is even.*

4.3.17 Theorem. *One cannot find a properly discontinuous group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^{p+q})$, with $p, q \in \mathbb{N}$, such that $L(\Gamma)$ is Zariski dense in $O(\mathbb{R}^{p,q})$, if $|p - q| \neq 1$.*

The proof of the first of these theorems is in part based on Margulis' construction of a free properly discontinuous group, see section 4.2.

4.4 Auslander's conjecture holds if $L(\Gamma) \leq O(\mathbb{R}^{n-1,1})$

In this section we will examine affine crystallographic Γ, whose linear part $L(\Gamma)$ is a subgroup of the orthogonal group $O(\mathbb{R}^{n-1}, \mathbb{R}^1)$, that preserves the Lorentzian bilinear form $B(x, y) = x_1y_1 + \ldots + x_{n-1}y_{n-1} - x_ny_n$. We will derive that such Γ are virtually solvable. The proof we present is due to Goldman and Kamishima, [GK84].

4.4.1 Theorem. *A crystallographic subgroup Γ of $\mathrm{Aff}(\mathbb{R}^n)$ is virtually solvable, if $L(\Gamma) \leq O(\mathbb{R}^{n-1,1})$.*

Proof. The proof for this theorem is done in three parts. At first, we will prove a special case of the theorem, namely we will assume that the linear projection of $\Gamma\backslash\mathbb{R}^n$ is discrete - proposition 4.4.9. Then, we will drop this assumption and face two disjoint cases: either $\overline{L(\Gamma)}^0$ is compact (propostion 4.4.11) or it is not (proposition 4.4.12).

To prove that the theorem holds if the first case is true, we will show that Γ, respectively a 'nice' subgroup of it, $\tilde{\Gamma}$, projects to a discrete group $L(\tilde{\Gamma})$, and thus we can employ proposition 4.4.9.

For the second case, it suffices to assume Γ to be discrete instead of requiring it to be crystallographic.

Throughout this section we will use the theory of connected subgroups of $O(\mathbb{R}^k)$ - a detailed examination thereof can be found in [ChGr74] or [Gr82]. □

This theorem is not the only thing known about such crystallographic groups. A complete classification of such Γ up to abstract commensurability was given in [GM88]. Before we are to prove the just presented theorem, we state the said classification. To write it down we need definitions and notation.

4.4.2 Definition (abstract commensurable groups)**.** *Given two groups G_1 and G_2 with finite index subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$. If $H_1 \cong H_2$, we say that G_1 and G_2 are* abstractly commensurable*.*

We start with a positive definite quadratic form $q_m(x) := m_1x_1^2 + \cdots + m_kx_k^2$, with $m := (m_1, \ldots, m_k) \in (2\mathbb{N})^k$, and let

$$L(\sqrt{m}) := \{x_1\sqrt{m_1}, \ldots, x_k\sqrt{m_k} \mid x_1, \ldots; x_k \in \mathbb{Z}\}.$$

We define, for an appropriate choice of d_i, $1 \leq i \leq 3$,

$$\Gamma_1(n+1, k, m) :=$$
$$\left\{ \begin{pmatrix} 1 & 0 & 0 & -x & -\frac{1}{2}x^tx & r \\ 0 & \mathbb{I}_{d_1} & 0 & 0 & 0 & z \\ 0 & 0 & \mathbb{I}_{d_2} & 0 & 0 & x \\ 0 & 0 & 0 & \mathbb{I}_{d_3} & x & y \\ 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid r, s \in \mathbb{Z}, x, y \in L(\sqrt{m}), z \in \mathbb{Z}^{n-1-2k} \right\},$$

and if $k < \frac{n-1}{2}$, we set e_1 to be the first standard vector in $\mathbb{Z}^{n-1-2k}$ and define

$$\Gamma_2(n+1, k, m) :=$$
$$\left\{ \begin{pmatrix} 1 & -2se_1^t & 0 & -x & -\frac{1}{2}(4s^2 + x^tx) & r \\ 0 & \mathbb{I}_{d_1} & 0 & 0 & 2se_1 & z \\ 0 & 0 & \mathbb{I}_{d_2} & 0 & 0 & x \\ 0 & 0 & 0 & \mathbb{I}_{d_3} & x & y \\ 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| r, s \in \mathbb{Z}, x, y \in L(\sqrt{m}), z \in \mathbb{Z}^{n-1-2k} \right\}.$$

Given two groups $\Gamma_i(n+1, k, m)$ and $\Gamma_j(\tilde{n}+1, \tilde{k}, \tilde{m})$, they are abstractly isomorphic iff $i = j$, $n = \tilde{n}$, $k = \tilde{k}$ and if the quadratic forms q_m and $q_{\tilde{m}}$ are integrally equivalent up to their respective sign. Two such groups are abstractly commensurable iff one replaces the equivalence of q_m and $q_{\tilde{m}}$ with the following property: there is a $\alpha \in \mathbb{Q}^\times$ such that q_m and $\alpha q_{\tilde{m}}$ are equivalent over $\mathbb{Q}$.

These two classes of groups yield the following classification of virtually nilpotent Γ.

4.4.3 Theorem. *Given an affine crystallographic group Γ with $L(\Gamma) \leq O(\mathbb{R}^{n-1,1})$. If Γ is virtually nilpotent, then it is abstractly commensurable to either a group of type $\Gamma_1(n+1, k, m)$ or to a group of type $\Gamma_2(n+1, k, m)$.*

For those Γ, which are not virtually nilpotent, we need the following.

Let $A \in GL(\mathbb{Z}^n)$ be of Lorentz-type, i.e. A shall be diagonalizable. We also require for the eigenvalues to be $1, \lambda, \lambda^{-1}, a_1, \ldots, a_{n-3}$, with $\lambda \in \mathbb{R}^\times$ and $|a_i| = 1$, for all $1 \leq i \leq n-3$.

We now define $\Gamma(n+1, A) := \mathbb{Z}^n \rtimes_A \mathbb{Z}$, with the action of $\mathbb{Z}$ on $\mathbb{Z}^n$ is given by $1_\mathbb{Z} z := Az$

for $z \in \mathbb{Z}^n$.

For two maps A and B, both being of Lorentz-type, we have that $\Gamma(n+1, A) \cong \Gamma(n+1, B)$ iff A is conjugate to either B or B^{-1} in $GL(\mathbb{Z}^n)$. And they are commensurable iff there is an r and an s in $\mathbb{Z}^\times$, such that A^r is conjugate to B^{rs} in $GL(\mathbb{Q})$.

4.4.4 Theorem. *Given an affine crystallographic group Γ with $L(\Gamma) \leq O(\mathbb{R}^{n-1,1})$. If Γ is not virtually nilpotent, then it is isomorphic to a group $\Gamma(n+1, A)$ with $A \in GL(\mathbb{Z}^n)$ being of Lorentz-type.*

Furthermore, any such $\Gamma(n+1, A)$ can be embedded into $\mathrm{Aff}(\mathbb{R}^{n-1,1})$, such that $\Gamma(n+1, A) \circlearrowleft \mathbb{R}^n$ is a properly discontinuous action.

We now start to collect some information needed in order to prove theorem 4.4.1. Let $\mathrm{Isom}(\mathbb{R}^{n-1,1})$ be the group of isometries $\mathbb{R}^n \to \mathbb{R}^n$ that preserves the Lorentzian inner product $B(x, y) = x_1y_1 + \ldots + x_{n-1}y_{n-1} - x_ny_n$. Certainly, we can decompose $\mathrm{Isom}(\mathbb{R}^{n-1,1}) = T \rtimes O(\mathbb{R}^{n-1,1})$, with T being the group of pure translations. The identity component of $\mathrm{Isom}(\mathbb{R}^{n-1,1})$ can be written as $\mathrm{Isom}(\mathbb{R}^{n-1,1})^0 = T \rtimes SO(\mathbb{R}^{n-1,1})^0$. And as always, let L denote the natural projection $L\colon \mathrm{Isom}(\mathbb{R}^{n-1,1}) \to O(\mathbb{R}^{n-1,1})$.

The following theorem plays a prominent role in our proof. Before stating it, we give a definition:

4.4.5 Definition (amenable Lie group)**.** *A connected Lie group G is called amenable, if there is a solvable $S \trianglelefteq G$ and a compact $K \leq G$, such that $G = S \rtimes K$.*

4.4.6 Theorem. *Let Δ be a discrete subgroup of a connected Lie group G. If G is amenable, then Δ is virtually polycyclic.*

Proof. See [Mi77], 2.2. □

In view of this theorem, we can prove the following:
an affine crystallographic $\Gamma \leq \mathrm{Isom}(\mathbb{R}^{n-1,1})$ is virtually solvable, if its linear part $L(\Gamma)$ is a discrete subgroup of an amenable subgroup G of $O(\mathbb{R}^{n-1,1})$, such that G has only finitely many components.

We fix $k, n \in \mathbb{N}\backslash\{0\}$, such that $k < n$ and set $V := \mathbb{R}^k < \mathbb{R}^n$. By G_V we denote the stabilizer of V. Let B the quadratic form that corresponds to $O(\mathbb{R}^{n-1,1})$ and set $\tilde{B}(x) := B(x, x)$. The restriction of $\tilde{B}$ to V will be called $\tilde{B}_V$. This restricted form may be positive or negative definite, indefinite or degenerate. Now we take a look at these different situations and then use the results to prove an upcoming lemma.

4.4.7 Facts. If $\tilde{B}_V$ is

(i) positive definite, one may find a $g \in O(\mathbb{R}^{n-1,1})$ with

$$gV = \mathbb{R}^k \times \{0\} < \mathbb{R}^n.$$

Furthermore, conjugating G_V by g yields the subgroup

$$O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1}) < O(\mathbb{R}^{n-1,1}).$$

(ii) indefinite or negative definite, in the case of $k = 1$, V is of the form

$$\{0\} \times \mathbb{R}^k < \mathbb{R}^n$$

and the stabilizer G_V is conjugate to

$$(\mathbb{R}^{n-k}) \times O(\mathbb{R}^{k-1,1}) < O(\mathbb{R}^{n-1,1}).$$

(iii) is degenerate, then we immediately see $\ker(\tilde{B}_V) = 1$. In this situation

$$G_V \cong \mathbb{R}^{n-2} \rtimes (\mathbb{R}^\times \times SO(\mathbb{R}^{n-2}))$$

and G_V clearly is an amenable Lie group. With the help of theorem 4.4.6 we deduce that $\Gamma \leq \mathrm{Isom}(\mathbb{R}^{n-1,1})$ is virtually solvable, if $L(\Gamma)$ normalizes a linear subspace on which the restriction of a Lorentzian inner product is degenerate.

4.4.8 Lemma. *Given a subgroup G of $O(\mathbb{R}^{n-1,1})$ with normalizer $N(G)$. Suppose G to be nontrivial, closed, connected and amenable. Then we have:*

(i) G is noncompact $\Rightarrow N(G)$ is amenable, and

(ii) G is compact $\Rightarrow N(G)$ is either compact, or there is an $k \in \mathbb{N}$, $1 \leq k \leq n-1$, and a $g \in O(\mathbb{R}^{n-1,1})$ with

$$g^{-1}N(G)g < O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1}) < O(\mathbb{R}^{n-1,1}), \text{ and}$$
$$g^{-1}Gg < O(\mathbb{R}^k) \times \{1\} < O(\mathbb{R}^{n-1,1}).$$

Proof. The proof for this lemma is based on properties of the set $\mathcal{C} := \{x \in \mathbb{R}^n \mid 0 < x_n, \tilde{B}(x) < 0$, i.e. the positive half of a light cone. By $\overline{\mathcal{C}}$ we denote its closure, and $\delta\mathcal{C}$ shall be its boundary, i.e. $\delta\mathcal{C} = \tilde{B}^{-1}(0) \cap \{x \in \mathbb{R}^n \mid 0 \leq x_n\}$.

By assumption G is amenable, and thus there is at least one $x \in \overline{\mathcal{C}}$, such that x is not only an eigenvector for any $g \in G$, but also $gx \in \overline{\mathcal{C}}$ - for a proof see [Gr82]. We will denote the set of all such vectors by Λ and set $\Sigma := <\Lambda>$. Because Λ is invariant under the action of $N(G)$, so is Σ.

From now on let G be noncompact. We start by showing $\Lambda \subset \delta\mathcal{C}$ by contradiction. Assuming the opposite, one may find an $x \in \mathcal{C}$ such that $gx \in \mathcal{C}$ for every $g \in G$. Actually, because the restriction of $\tilde{B}$ on $\mathcal{C}$ is nondegenerate, we have $gx = x$ for every $g \in G$. From this, together with the fact that G respects $\tilde{B}$, we conclude there is a quadratic form $\hat{B}$, also respected by G, such that $\tilde{B} \equiv \hat{B}$ on $x^\perp$ while restricting both forms to x yields $-\tilde{B} \equiv \hat{B}$. This shows that $\hat{B}$ is positive definite. But a real linear Lie group G preserves a positive definite bilinear form iff G is compact, which contradicts our assumption on G and therefore $\Lambda \subset \delta\mathcal{C}$.

We are now ready to prove assertion *(i)* and at this point we split the proof into two disjoint cases, namely $\Sigma \subset \delta\mathcal{C}$ and $\Sigma \not\subset \delta\mathcal{C}$.

In the first case, $\Sigma \subset \delta\mathcal{C}$, Σ has to be 1-dimensional. It follows that its stabilizer

$O(\mathbb{R}^{n-1,1})$ is isomorphic to $\mathbb{R}^{n-2} \rtimes (\mathbb{R}^{\times} \times O(\mathbb{R}^{n-2}))$ and thus it is amenable. Once again, because closed subgroups of amenable groups are amenable, we deduce that $N(G)$ is amenable.

In order to prove the second case of *(i)* we need to show that Σ is a proper subspace of $\mathbb{R}^n$. Suppose the opposite: $\Sigma = \mathbb{R}^n$. This implies that there are linearly independent $v_1, \ldots, v_n \in \overline{\mathcal{C}}$ and every $v_i, 1 \leq i \leq n$, would be an eigenvector for every $g \in G$. Let us look at the orthogonal complement O_j of $<v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n>$. In any such O_j we can find a line l_j that does not lie in $\mathcal{C} \backslash \{0\}$, such that there is a vector $w_j \subset l_j$ with $\tilde{B}(w_j) = 1$. Now the set of all w_j, $1 \leq j \leq n$, is linearly independent and all w_j are eigenvectors for all $g \in G$ with eigenvalue 1, because G respects $\tilde{B}$. But trivially, if G is a group with $gw_j = w_j$ for all $g \in G$, G is the trivial group and we arrive at a contradiction. Thus, Σ is a proper subspace.

Now we assume $\Sigma \not\subset \delta\mathcal{C}$. At first we observe that the intersection of Σ and $\mathcal{C}$ is nonempty and hence the restricted form $\tilde{B}_{|\Sigma}$ is indefinite. This yields that there is a $g \in O(\mathbb{R}^{n-1,1})$ such that $g\Sigma = \{0\} \times \mathbb{R}^{n-k}$, with k being the dimension of Σ. Furthermore we have $g\Sigma^{\perp} = \mathbb{R}^k \times \{0\}$. Because both Σ and $\Sigma^{\perp}$ are invariant subspaces, we have $g^{-1}N(G)g < O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1})$. As shown above, Σ is a proper subspace, thus $0 < k < n$. This allows us now to use induction on n:

In the case of $n = 2$, $\mathrm{Isom}(\mathbb{R}^{1,1})$ is an amenable group and thus we have that the closed subgroup $N(G)$ is also amenable. So let's suppose *(i)* has been shown for all $2 \leq m < n$, with $m \in \mathbb{N}$. By assumption G is noncompact and therefore its projection into $O(\mathbb{R}^{n-k-1,1})$ has to be a noncompact amenable subgroup. Now $g^{-1}N(G)g < O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1})$ together with our induction hypothesis implies that the projection of $N(G)$ into $O(\mathbb{R}^{n-k-1,1})$ yields a closed amenable subgroup. The projection of $N(G)$ into $O(\mathbb{R}^k)$ is a closed, and thus compact, group. Consequently, $N(G)$ is an extension of a closed amenable group by a compact group and thus amenable. Hence, we have shown that *(i)* holds.

On to the second assertion of the lemma. Because G is compact, so is Gv for any $v \in \mathcal{C}$. The barycenter of this orbit lies in $\mathcal{C}$ and for the vector v corresponding to it we have $Gv = v$. This implies that $\Lambda \cap \mathcal{C} \neq \emptyset$ and $\Sigma = <\Lambda \cap \mathcal{C}>$. But the action $G \circlearrowleft (\Lambda \cap \mathcal{C})$ is trivial and therefore $G_{|\Sigma}$ is the trivial group.

If $\dim \Sigma \geq 2$, then, analogous to the proof for *(i)*, we know there is a $g \in O(\mathbb{R}^{n-1,1})$, such that $g\Sigma = \{0\} \times \mathbb{R}^{n-k}$, $g\Sigma^{\perp} = \mathbb{R}^k \times \{0\}$ and hence $g^{-1}N(G)g < O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1})$. But G is trivial on Σ and thus $g^{-1}Gg < O(\mathbb{R}^k) \times \{e\}$.

If $\dim \Sigma = 1$, we have that G fixes a unique vector in $\mathcal{C}$, then the same applies to $N(G)$ and $N(G)$ has to be compact. This shows that *(ii)* holds and we have completed the proof. $\square$

We are now going to prove the following special case of theorem 4.4.1.

4.4.9 Proposition. *A crystallographic subgroup Γ of $\mathrm{Aff}(\mathbb{R}^n)$ is virtually solvable, if $L(\Gamma) \leq O(\mathbb{R}^{n-1,1})$ is discrete.*

Proof. Exactly as in section 4.3, we may without loss of generality replace Γ with a torsion free subgroup $\tilde{\Gamma}$, such that $\tilde{\Gamma}$ has finite index in Γ and $\tilde{\Gamma} \leq \mathrm{Isom}(\mathbb{R}^{n-1,1})^0$.
To prove the proposition we will use induction on the dimension n. If $n = 1$, then there are exactly two possibilites. Either $\Gamma \cong \mathbb{Z}$ or $\Gamma \cong \mathbb{Z} \rtimes \mathbb{Z}_2$. In both cases, Γ evidently is virtually solvable. From now on, let $1 < n$ and assume that the proposition holds for every $m < n$ with $m \in \mathbb{N}$.

Also as before, for the projection L onto the linear part, we have that the kernel is the set of all translations, i.e. $\ker(L_{|\tilde{\Gamma}})$ is the discrete subgroup $\tilde{\Gamma} \cap T$. Our first task is to show that this is indeed a proper group.

Assume $\ker(L_{|\tilde{\Gamma}})$ to be the trivial group. Equivalently, $\tilde{\Gamma} \cong L(\tilde{\Gamma})$. If this is the case, then the space of double cosets $L(\tilde{\Gamma})\backslash(SO(\mathbb{R}^{n-1,1})/O(\mathbb{R}^{n-1})$ is a hyperbolic complete Riemannian manifold of dimension n-1 with fundamental group $\tilde{\Gamma}$. Thus, the cohomological dimension of $\tilde{\Gamma}$ cannot exceed $n-1$. However, $\tilde{\Gamma}$ is also the fundamental group of the compact aspherical manifold $\tilde{\Gamma}\backslash\mathbb{R}^n$, which has dimension n, and therefore the cohomological dimension of $\tilde{\Gamma}$ is n and we arrive at a contradiction. This yields, that $\ker(L_{|\tilde{\Gamma}})$ is indeed a proper group and the linear space $V := <\ker(L_{|\tilde{\Gamma}})>$ has dimension $k > 0$. Of course, V is invariant under the action of $L(\tilde{\Gamma})$.

We denote by $B_{|V}$ the restriction of the Lorentzian bilinear form to V. There are three possibilities:

(i) $B_{|V}$ is degenerate,

(ii) $B_{|V}$ is positive definite, i.e. has signature $(k, 0)$, and

(iii) $B_{|V}$ is indefinite, i.e. has signature $(k-1, 1)$.

If $B_{|V}$ is degenerate, and because V is invariant under $L(\tilde{\Gamma})$, we deduce from 4.4.7, that $\tilde{\Gamma}$, and thus Γ, is virtually solvable.

In order to prove the other two cases, we first have to establish a few things. Of course, $\mathbb{R}^n$ can be decomposed into $V \times V^\perp$. By $O(V, V^\perp) \leq O(\mathbb{R}^{n-1,1})$ we denote the symmetry group of $(V \times V^\perp, B_{|V} \times B_{|V^\perp})$. This group naturally splits into $O(V) \times O(V^\perp)$, i.e. the symmetry groups of V and $V^\perp$ together with the respective restrictions of B.

Clearly, we have the following isomorphism:

$$\mathrm{Isom}(V \times V^\perp) \cong T \rtimes (\mathbb{R}^\times \times O(V \times V^\perp)) \cong \mathrm{Isom}(V) \times \mathrm{Isom}(V^\perp)$$

and

$$\mathrm{Isom}(V) \cong T_{|V} \rtimes (\mathbb{R}^\times \times O(V))$$

as well as

$$\mathrm{Isom}(V^\perp) \cong T_{|V^\perp} \rtimes (\mathbb{R}^\times \times O(V^\perp)),$$

with $T_{|V}$, respectively $T_{|V^\perp}$, denoting the translations restricted to V, respectively $V^\perp$. Lastly, by $\tilde{L}$ we denote the projection from $\mathrm{Isom}(V \times V^\perp)$ onto $\mathrm{Isom}(V^\perp)$.

Using Selberg's lemma one more time, $\tilde{\Gamma}$ may be chosen in such a way that $\tilde{\Gamma}$ yields a torsion free group if its mapped from $\mathrm{Isom}(V \times V^{\perp})$ into $O(V)$.

Let $\hat{M} := (\tilde{\Gamma} \cap T)\backslash\mathbb{R}^n$. Of course, $\hat{M}$ is a covering space for $M := \tilde{\Gamma}\backslash\mathbb{R}^n$. If we let V act on $\mathbb{R}^n$ by translations, we have an induced action $\mathbb{T}^k \circlearrowleft \hat{M}$, with $\mathbb{T}^k = (\tilde{\Gamma} \cap T)\backslash V$ being the k-torus - recall $0 < \dim V = k$.

To prove the proposition in cases *(ii)* and *(iii)*, we need the following lemma.

4.4.10 Lemma. *The action $\tilde{L}(\tilde{\Gamma}) \circlearrowleft V^{\perp}$ is properly discontinuous.*

Proof. We start by showing that the action $V \rtimes \tilde{\Gamma} \circlearrowleft \mathbb{R}^n$ is proper, whereas the semidirect product is the one induced from the natural action of $\tilde{\Gamma}$ on $V^{\perp}$. We fix a compact subset $F \subset V$, with the property that it meets every $(\tilde{\Gamma} \cap T)$-coset. Furthermore, let $K_1, K_2 \subset \mathbb{R}^n$ be compact subsets. It follows that

$$\{(v\gamma) \in V \rtimes \tilde{\Gamma} \mid (v\gamma)K_1 \cap K_2 \neq \emptyset\}$$

is a closed subset of $F(\{\gamma \in \tilde{\Gamma} \mid \gamma K_1 \cap K_2 \neq \emptyset\})$. Hence, $V \rtimes \tilde{\Gamma} \circlearrowleft \mathbb{R}^n$ is a proper action.

Of course, V is mapped under $\tilde{L}$ to the trivial group and $\tilde{\Gamma}$ is a discrete subgroup of $\mathrm{Isom}(V \times V^{\perp}$, therefore $\tilde{L}(\tilde{\Gamma})$ is a discrete subgroup of $\mathrm{Isom}(V^{\perp})$ and we deduce that $\tilde{L}(\tilde{\Gamma}) \circlearrowleft V^{\perp}$ is indeed properly discontinuous. □

To complete the proof of proposition 4.4.9 we need one more thing, namely that $\ker(\tilde{L}) = \ker(\mathrm{L}_{|\tilde{\Gamma}}) = \tilde{\Gamma} \cap T$. If $\tilde{L}(\gamma)$, $\gamma \in \tilde{\Gamma}$, is the identity element, then the action of γ on $V^{\perp}$ is trivial. Thus, if we look at the induced action of γ on the covering space $\hat{M}$, we conclude that γ acts invariantly on every $\mathbb{T}^k$-fiber of $\hat{M} \to V^{\perp}$. Furthermore, γ acts properly discontinuously on $\hat{M}$ and thus it acts properly discontinuously on every $\mathbb{T}^k$ as well and we see that $L(\gamma)$ coincides with $L(\gamma)_{|V}$, its restriction to V. Furthermore, $L(\gamma)_{|V}$ has to have finite order, but $\tilde{\Gamma}$ was chosen to be a torsion free group, and so its image must be under any homomorphism, too. Thus, there is only one choice for $L(\gamma) = L(\gamma)_{|V}$ to avoid a contradiction, namely $L(\gamma) = 1$, i.e. γ is a translation and we have proven the claim.

This immediately translates into the existence of an exact sequence

$$\mathbb{Z}^k \to \tilde{\Gamma} \to \tilde{L}(\tilde{\Gamma}),$$

and we immediately see that $\tilde{\Gamma}$ is virtually solvable iff $\tilde{\Gamma}$ is virtually solvable.

We are finally ready to finalize the proof. Assume that $B_{|V}$ is positive definite. It follows that the orthogonal complement $V^{\perp}$ has a flat Lorentz-structure that is invariant under $\tilde{L}(\tilde{\Gamma})$ and the quotient space $\tilde{L}(\tilde{\Gamma})\backslash V$ is compact. This, together with lemma 4.4.10 yields that $\tilde{L}(\tilde{\Gamma})$ is a crystallographic group and because $\dim V^{\perp} < n$ we can make use of the induction hypothesis from the beginning of this proof. We get that $\tilde{L}(\tilde{\Gamma})$ is virtually solvable and by virtue of the discussion above the same applies to $\tilde{\Gamma}$. Thus we have proven that *(ii)* holds.

The last thing to check is what happens if B_V is indefinite. But in this situation $\tilde{L}(\tilde{\Gamma})$ is a subgroup of Euclidean isometries, i.e. $\tilde{L}(\tilde{\Gamma}) < \mathrm{Isom}(\mathbb{R}^{n-k-1})$ and hence we can apply Bieberbach's first theorem to conclude that $\tilde{L}(\tilde{\Gamma})$ is virtually abelian. A fortiori, $\tilde{\Gamma}$ is virtually solvable. □

Now that we have seen that a crystallographic Γ is virtually solvable, if $L(\Gamma) < O(\mathbb{R}^{n-1,1})$ is discrete, we are left to examine what happens if we drop this assumption.

As stated above, we will now look into two disjoint cases, based on whether $\overline{L(\Gamma)}^0$ is compact (proposition 4.4.11) or not (proposition 4.4.12).

4.4.11 Proposition. *If Γ is a crystallographic subgroup of $\mathrm{Isom}(\mathbb{R}^{n-1,1})$, such that $\overline{L(\Gamma)}^0$ is compact, then Γ is virtually solvable.*

Proof. Once again, applying theorem 3.3.2 yields that $G := \overline{L(\Gamma)}^0$ is solvable. Actually, it is even abelian, because G is compact and connected. Furthermore, using assertion *(i)* of lemma 4.4.8, we have that the normalizer $N(G)$ is either compact or conjugate to a subgroup of $O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1})$. In the latter case, we also have $G \cong O(\mathbb{R}^k)$.

If $N(G)$ is compact, then $\Gamma \leq T \rtimes N(G)$, which clearly is an amenable group. From 4.4.6 we conclude that Γ is virtually solvable.

To complete this proof, we will assume $G \leq O(\mathbb{R}^k) \times \{1\}$ as well as

$$L(\tilde{\Gamma}) \leq N(G) \leq O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1}),$$

with $\tilde{\Gamma}$ being a torsion free subgroup of Γ, such that $[\Gamma : \tilde{\Gamma}]$ is finite.

There is a composition of natural projections:

$$\tilde{\Gamma} \to L(\tilde{\Gamma}) < O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1}) \to O(\mathbb{R}^{n-k-1,1}).$$

We call this composition $\pi \colon \tilde{\Gamma} \to O(\mathbb{R}^{n-k-1,1})$ and set $\hat{\Gamma} := \ker(\pi)$. Immediately we see that $\hat{\Gamma}$ is a group of Euclidean isometries. Since the action $\tilde{\Gamma} \circlearrowleft \mathbb{R}^n$ is free, the same holds for the action of $\hat{\Gamma}$ on $\mathbb{R}^n$ and hence $\hat{\Gamma}\backslash\mathbb{R}^n$ is a complete flat Riemannian manifold. Due to a result by Wolf, [Wo67], the submanifold $\hat{\Gamma}\backslash\mathbb{B}$ is a compact, totally geodesic deformation retract, with $\mathbb{B}$ being an affine subspace of $\mathbb{R}^n$ on which $\hat{\Gamma}$ acts invariantly. Any two such subspaces $\mathbb{B}_1$ and $\mathbb{B}_2$ are parallel and the union $\mathbb{E} := \bigcup_{i \in I} \mathbb{B}_i$, I being an appropriate index set, is also an affine subspace of $\mathbb{R}^n$. Of course, $\hat{\Gamma} \trianglelefteq \tilde{\Gamma}$ and thus $\tilde{\Gamma}(\mathbb{B}_i) = \mathbb{B}_j$ for some $j \in I$. Consequently, $\mathbb{E}$ is a $\tilde{\Gamma}$-invariant subspace. However, in this situation we have $\mathbb{E} = \mathbb{R}^n$, according to [FGH81].

We fix an $\mathbb{B} \subset \mathbb{R}^n$, such that $\hat{\Gamma} \circlearrowleft \mathbb{B}$ is crystallographic and Euclidean. This yields that $\hat{\Gamma} \circlearrowleft (v + \mathbb{B})$, $v \in \mathbb{R}^n$, is also crystallographic and Euclidean and thus if $\hat{\gamma} \in L(\hat{\Gamma})$, then all eigenvalues of $\hat{\gamma}$ are roots of unity. However, $\tilde{\Gamma}$ was chosen to be torsion free, thus there is only one value that may occur as an eigenvalue for any element γ in Γ, namely 1, because otherwise there would be an $0 < m \in \mathbb{N}$, such that $\gamma^m = \mathbb{I}$. In other words, $\hat{\Gamma}$ is a group of translations.

As a last step of this proof, we show that $L(\tilde{\Gamma})$ is discrete. From $G < O(\mathbb{R}^k) \times \{1\}$ we deduce that $\pi(\tilde{\Gamma})$ is a discrete subgroup of $O(\mathbb{R}^{n-k-1,1})$. But this means that the image of $L(\tilde{\Gamma}) < O(\mathbb{R}^k) \times O(\mathbb{R}^{n-k-1,1})$ is discrete. From the fact that $L(\tilde{\Gamma}) = 1$ if $\pi(\tilde{\Gamma}) = 1$ we conclude that $L(\tilde{\Gamma})$ is discrete. Proposition 4.4.9 completes the proof. □

There is only one more case to check in order to prove that 4.4.1 holds.

4.4.12 Proposition. *If Γ is a discrete subgroup of* $\mathrm{Isom}(\mathbb{R}^{n-1,1})$*, such that* $\overline{L(\Gamma)}^0$ *is noncompact, then Γ is virtually solvable.*

Proof. From theorem 3.3.2 we conclude that $G := \overline{L(\Gamma)}^0 < O(\mathbb{R}^{n-1,1})$ is a connected, closed and solvable group, which by assumption is noncompact. We also have that $L(\Gamma)$ normalizes G. By assertion *(i)* of 4.4.8 $L(\Gamma)$ is a subgroup of the amenable group $N(G) < O(\mathbb{R}^{n-1,1})$. This yields that Γ is a subgroup of the amenable group $T \rtimes N(G) <$ $\mathrm{Isom}(\mathbb{R}^{n-1,1})$. Because we assumed Γ to be discrete, we can employ theorem 4.4.6 to conclude that Γ is virtually solvable. □

Now that the main theorem of this section is shown to be true, we are ready to move on to our next task.

4.5 Auslander's conjecture holds if $L(\Gamma) \leq O(\mathbb{R}^{n-2,2})$

In this section we are concerned with affine crystallographic groups Γ, whose linear part is a subgroup of $O(\mathbb{R}^{n-2,2})$, i.e. the orthogonal group that preserves a nondegenerate bilinear form B of signature $(n-2,2)$. We will prove that for such Γ Auslander's conjecture holds, however we will give the proof, which is due to Abels, Margulis and Soifer, [AMS05], not in full detail.

One might think that methods similar to those used to prove the case of $L(\Gamma) <$ $O(\mathbb{R}^{n-1,1})$ will be applied in order to show that the claim holds, however the techniques of this section will be vastly different in nature - they built upon Margulis' counterexample to Milnor's conjecture, see section 4.2

A different approach can be found in [KW96], where the assertion is proven under the assumption that in the unipotent radical of the Zariski closure of Γ there is a g with $L(g) \neq \mathbb{I}$.

4.5.1 Theorem. *A crystallographic subgroup Γ of* $\mathrm{Aff}(\mathbb{R}^n)$ *is virtually solvable, if* $L(\Gamma) \leq O(\mathbb{R}^{n-2,2})$*.*

Proof. To prove this theorem it suffices to show that the Zariski closure $\overline{L(\Gamma)}$ of $L(\Gamma)$ is virtually solvable, because in this case Γ is the extension of an abelian group - $\ker(L)$ is just the group of translations - by a virtually solvable group $L(\Gamma)$ and thus Γ itself is virtually solvable. For the Zariski closure we can assume connectedness, because as in section 4.3, there is a finite-index subgroup in Γ, such that its linear part maps to $\overline{L(\Gamma)}^0$.

Our claim will be shown by contradiction. Hence, we assume Γ and $\overline{L(\Gamma)}$ to not be virtually solvable and choose n in such a way that n is minimal with respect to all dimensions in which our claim fails.

Before we start this section properly, we give a rough sketch of the proof. A more detailed one is then to be found on page 80. Every $\gamma \in \Gamma$ can be decomposed into two components corresponding to the action of γ on either W or $W^\perp$, whereas $W \oplus W^\perp = \mathbb{R}^n$ is a certain decomposition, that will be introduced shortly.

Proposition 4.5.11 tells us, that there is some sort of 'coming-back effect' for certain elements of Γ. Furthermore, corollary 4.5.12 states that the word length of such an element can be controlled. There are exponentially many of these elements such that the W-component comes back closely to where it started. Comparing the exponential growth of the quantity of such elements with the polynomial growth of the volume of some compact $K \subset \mathbb{R}^n$ yields that one can find infinitely many elements in Γ, such that the $W^\perp$ component also comes back closely to where it started, which ultimately leads to a contradiction to the assumption of Γ being properly discontinuous. $\square$

We start with a definition.

4.5.2 Definition (Standard subgroup)**.** *Given a vector space V and a quadratic form B with $O(B)$ denoting the orthogonal group $Sym(V, B)$. Let $H < O(B)$ be a connected, simple group. If there exists an orthogonal decomposition of $V = W \oplus W^\perp$ with respect to B, such that*

(i) W and $W^\perp$ are invariant under the action of H,

(ii) $\pi\colon H \to O(B_{|W})$ is an epimorphism, and

(iii) $h_{W^\perp}$ is the trivial action for every $h \in H$,

then we call H a standard subgroup *of $O(B)$.*

We are now ready to formulate the first lemma of this section.

4.5.3 Lemma. *Let H be a connected, simple subgroup of $O(\mathbb{R}^{n-2,2})$ with* $\operatorname{rank}_{\mathbb{R}}(H) = 2$*. It follows that H is a standard subgroup.*

A thorough definition of the *real rank* $\operatorname{rank}_{\mathbb{R}}$ of some Lie group can be found in definition 4.6.3.

Proof. According to [OV00], the Lie algebra $\mathfrak{g}$ of $O(\mathbb{R}^{n-2,2})$ can be written as $\mathfrak{g} = \{A \in M_n(\mathbb{R}) \mid AJ = JA^t\}$, with $M_n(\mathbb{R})$ denoting the real $n \times n$-matrices and

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \mathbb{I}_{n-4} & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

In $O(\mathbb{R}^{n-2,2})$ we can find a maximal $\mathbb{R}$-split torus $\mathbb{T} = (\mathbb{R}^\times)^d$, for an appropriate $d \in \mathbb{N}$, with Lie algebra $\mathfrak{t}$. Any $t \in \mathfrak{t}$ can be written as

$$t = \begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_2 & \cdots & \cdots & \\ \vdots & \vdots & 0_{n-4} & \vdots & \vdots \\ 0 & 0 & \cdots & -\varepsilon_2 & 0 \\ 0 & 0 & \cdots & 0 & -\varepsilon_1 \end{pmatrix},$$

for an appropriate choice of $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. By $0_{n-4} \in M_{n-4}$ we denote the matrix in which every slot is equal to 0.

We see that the positive roots are $\alpha = \varepsilon_1, \beta = \varepsilon_2, \alpha+\beta = \varepsilon_1+\varepsilon_2$ and $\alpha - \beta = \varepsilon_1 - \varepsilon_2$ and for the dimension of the respective root spaces we have: $\dim V_\alpha = \dim V_\beta = n-2$ and $\dim V_{\alpha+\beta} = \dim V_{\alpha-\beta} = 1$.

If we define G_0 to be the smallest connected, simple subgroup of $O(\mathbb{R}^{n-2,2})$, such that $\mathbb{T} \leq G_0$, then we have $G_0 \leq H$, because by assumption H is connected and simple and furthermore $\operatorname{rank}_{\mathbb{R}} = 2$ and thus we may suppose without loss of generality that $\mathbb{T} \leq H$. Of course, the Lie algebra $\mathfrak{g}_0$ of G_0 contains $\mathfrak{t}$ and thus also the root spaces $V_{\alpha+\beta}, V_{-(\alpha+\beta)}, V_{\alpha-\beta}$ and $V_{-(\alpha-\beta)}$.

We set $U^+ := V_\alpha \oplus V_\beta$ and $U^- := V_{-\alpha} \oplus V_{-\beta}$. The Lie algebra of H will be denoted by $\mathfrak{h}$. Then we have that the intersection $\mathfrak{h} \cap U^+$ is a proper subspace and invariant under the action of T, which yields that $\mathfrak{h} \cap V_\alpha \neq \{0\} \neq \mathfrak{h} \cap V_\beta$. Furthermore we can find a w_1 in the Weyl group of G_0 with $w_1 V_\alpha w_1^{-1} = V_\beta$, which immediately gives us $w_1(\mathfrak{h} \cap V_\alpha) w_1^{-1} = \mathfrak{h} \cap V_\beta$.

The centralizer of $\mathbb{T}$ in $O(\mathbb{R}^{n-2,2})$ acts transitively on U^+ and U^-. Furthermore, there is a w_2 in the Weyl group of G_0 with $w_2 U^+ w_2^{-1} = U^-$ and hence there is a standard subgroup $\tilde{H}$ with Lie algebra $\tilde{\mathfrak{h}}$, such that $\tilde{\mathfrak{h}} \cap U^+ = \mathfrak{h} \cap U^+$ and $\tilde{\mathfrak{h}} \cap U^- = \mathfrak{h} \cap U^-$. We deduce that every unipotent element of H and every unipotent element of $\tilde{H}$ lies in the intersection $H \cap \tilde{H}$. But since the unipotent elements are a generating set of H as well as of $\tilde{H}$ we have $H = \tilde{H}$ and H indeed is a standard subgroup. □

In [AMS05] the authors claim that this proof can be generalized in such a way, that if $H \leq O(\mathbb{R}^{n-k,k})$, H connected and simple with $\operatorname{rank}_{\mathbb{R}} = k$, then H is also a standard subgroup.

Before we can make use of the just proven lemma, we need some definitions.

4.5.4 Definition (Unipotent radical and reductive algebraic groups)**.** *Let G be an algebraic group with radical $rad(G)$. The subgroup $U \leq rad(G)$ that consists of all the unipotent elements is called* unipotent radical *of G. If the unipotent radical is trivial, G is said to be a* reductive group.

4.5.5 Lemma. *Given a crystallographic Γ that is not virtually solvable with $L(\Gamma) \leq O(\mathbb{R}^{n-2,2})$. It follows that the Zariski closure G of $L(\Gamma)$ is a reductive group.*

Proof. We denote by S the semisimple part of G. Our crystallographic group Γ is virtually solvable, if for every connected, simple subgroup $\tilde{S}$ of S we have that $\operatorname{rank}_{\mathbb{R}}(\tilde{S}) \leq 1$, see [To90]. But by assumption Γ is not virtually solvable and thus there is such an $\tilde{S} \leq S$ with $\operatorname{rank}_{\mathbb{R}}(S) = 2$.

By the virtue of lemma 4.5.3, $\mathbb{R}^n$ decomposes into $W \oplus W^\perp$, such that these two spaces are invariant under the action of $\tilde{S}$ and furthermore W and $W^\perp$ are orthogonal with respect to our bilinear form B. We also have $\tilde{S}_{|W} = O(B_{|W})$ as well as the fact that $B_{|W^\perp}$ is positive definite. We define V to be the subspace $\{v \in V \mid tv = v \; \forall t \in U\} \leq \mathbb{R}^n$.Now there are two possibilities for this G-invariant space: either $V \leq W^\perp$ or $V = \mathbb{R}^n$. However, it can be checked that the first of these two possibilities is not possible. But if

$V = \mathbb{R}^n$, then U has to be the trivial group because every element in U fixes every vector in $\mathbb{R}^n$. Thus, G is reductive. □

To proceed we need some notation. Given a bilinear form B on some linear space $\mathbb{R}^n$ with signature (p, q), such that $p + q = n$ and $q \leq p$. Not surprisingly we set $SO(B) := \{t \in O(B) \mid \det(t) = 1\}$. Let $g \in SO(B)$ be semisimple, i.e. if V is g-invariant subspace of $\mathbb{R}^n$, then there is a $W \leq \mathbb{R}^n$, such that W is also g-invariant and W is a complete for V in $\mathbb{R}^n$.

If $g \in SO(B)$ is semisimple, we can decompose $\mathbb{R}^n = \mathcal{A}_g^+ \oplus \mathcal{A}_g^- \oplus \mathcal{A}_g^0$ such that we have the following correspondence: $\mathcal{A}_g^+$ is the subspace on which all eigenvalues of the restriction of g are greater than 1, $\mathcal{A}_g^-$ is the subspace on which all eigenvalues of the restriction of g are less than 1 and $\mathcal{A}_g^0$ is the subspace on which all eigenvalues of the restriction of g are equal to 1.

Such a semisimpile g is called hyperbolic if $\dim \mathcal{A}_g^0 = p - q$ and of course the definition given in Margulis' counterexample, section 4.2, is just a special case of the one just presented. Accordingly, we set $\mathcal{D}_g^- := \mathcal{A}_g^0 \oplus \mathcal{A}_g^-$, as well as $\mathcal{D}_g^+ := \mathcal{A}_g^0 \oplus \mathcal{A}_g^+$. Also recall $\mathcal{D}_g^- \cap \mathcal{D}_g^+ = \mathcal{A}_g^0$.

Furthermore, we set

$$\begin{aligned} s^+(g) &:= \max\{|\lambda_g| \mid \lambda_g \text{ is an eigenvalue of } g \text{ and } \lambda_g < 1\}, \\ s^-(g) &=: s^+(g^{-1}), \text{ and} \\ s(g) &:= \max\{s^+(g), s^-(g)\}. \end{aligned}$$

The metric d on $\mathbb{R}^n$, derived from the standard inner product $||\cdot||$, induces a metric $\hat{d}$ on the projective space $\mathbb{PR}^n$. Similarly to the situation in Margulis' counterexample, we say that a hyperbolic g is ε-hyperbolic if $\hat{d}(\mathcal{A}_g^+, \mathcal{A}_g^-) > \varepsilon > 0$ for some $\varepsilon \in \mathbb{R}^+$ and if g and h are both hyperbolic and additionally $\hat{d}(\mathcal{A}_g^+, \mathcal{A}_h^-), \hat{d}(\mathcal{A}_h^+, \mathcal{A}_g^-) > \varepsilon > 0$, then we call them ε-transversal.

For two ε-hyperbolic and ε-transversal elements of H_b we have an isometry $\rho\colon \mathcal{A}_h^0 \to \mathcal{A}_g^0$ via the following: We set $\mathcal{A}_{g,h}^0 := \mathcal{D}_g^- \cap \mathcal{D}_h^+$ and

$$\rho\colon \mathcal{A}_h^0 \to \mathcal{A}_{g,h}^0 \to \mathcal{A}_g^0.$$

Thus ρ is the composition of two maps, both being chosen to be the respective natural projection. For the first one we have that it is parallel to $\mathcal{A}_h^+$, whereas the second one is parallel to $\mathcal{A}_g^-$. If we are given ε-hyperbolic and pairwise ε-transversal $g_0, g_1, \ldots, g_n \in SO(B)$, then we have for every $(g_i, g_{i-1}), 0 < i \leq n$, an isometry ρ_i from $\mathcal{A}_{g_i}^0$ to $\mathcal{A}_{g_{i-1}}^0$. The composition of these isometries will be denoted by $\pi_i := \rho_0 \cdots \rho_i$. Let $o_g := g_{|\mathcal{A}_g^0}$ and $\tilde{o}_{g_i} := \pi_i o_{g_i} \pi_i^{-1}$, i.e. we have a composition

$$\tilde{o}_{g_i} \colon \mathcal{A}_{g_0}^0 \xrightarrow{\pi_i^{-1}} \mathcal{A}_{g_i}^0 \xrightarrow{o_{g_i}} \mathcal{A}_{g_i}^0 \xrightarrow{\pi_i} \mathcal{A}_{g_0}^0,$$

which is an orthogonal transformation on $\mathcal{A}_{g_0}^0$. We also immediately see that $\tilde{o}_{g_i}^{l_i} = \tilde{o}_{g_i^{l_i}}$ for every $l_i \in \mathbb{N}$. If $l = (l_0, l_1, \ldots, l_n) \in \mathbb{N}^n$, then we set $g_l := g_0^{l_0} g_1^{l_1} \cdots g_n^{l_n}$ and $o_l :=$

$\tilde{o}_{g_0}^{l_0}\tilde{o}_{g_1}^{l_1}\cdots\tilde{o}_{g_n}^{l_n}$.

The following lemma was first proven in [PR03].

4.5.6 Lemma. *Given a Zariski dense group $\Gamma \leq SO(B)$ and $\tilde{\varepsilon}$-hyperbolic and $\tilde{\varepsilon}$-transversal g and h in Γ. Then there is an $\varepsilon \in \mathbb{R}^+$ and $g_i \in \Gamma$, $1 \leq i < n$, such that*

- *(i) $g = g_0, g_1, \ldots, g_{n_1}, g_n = h$ are ε-hyperbolic as well as pairwise ε-transversal, and*
- *(ii) $\{o_l\}_{l\in\mathbb{N}^n}$ is Zariski dense in the connected component of $O(B_{g_0})$.*

Proof. Assertion *(i)* follows from the fact that Γ is a Zariski dense subgroup of $O(\mathbb{R}^n)$ and results from [AMS02].

To show that the second assertion also holds, it suffices to prove that there are ε-hyperbolic and pairwise ε-transversal $g_1, \ldots, g_n$ such that there is a Zariski open $U \subset \overline{\{o_l\}}_{l\in\mathbb{N}^n}$. Among all $(n-1)$-tupels of elements of Γ provided by *(i)* we choose the g_i, $1 \leq i < n$ in such a way that $\overline{\{o_l\}}_{l\in\mathbb{N}^n}$ has maximal dimension. Now, $\overline{\{o_l\}}_{l\in\mathbb{N}^n}$ is a constructible set. This implies that there are subsets $K_i, U_i \subset O(B_{g_0})$, $1 \leq i \leq m$, K_i Zariski closed, U_i Zariski open, such that $\overline{\{o_l\}} = \bigcup_{i=i}^m (K_i \cap U_i)$. According to [PR03] the set

$$S_1 := \{\gamma \in \Gamma \mid \gamma \text{ is hyperbolic and } \mathbb{R}\text{-irreducible}\}$$

is nonempty. And because we also have that the set

$$S_2 := \{\gamma \in \Gamma \mid \gamma \text{ and } g_{n-1} \text{ are transversal}\}$$

is nonempty, the same of course holds for the intersection $S := S_1 \cap S_2 \neq \emptyset$.

We fix a $\gamma \in S$ and denote by $\rho\colon \mathcal{A}_\gamma^0 \to \mathcal{A}_{g_{n-1}}^0$ the isometry we constructed right before this lemma. Furthermore we set $\pi_{n+1} = \pi_n \rho$ and

$$T_\gamma := \{t \in SO(B) \mid t \text{ is regular}, \mathcal{A}_\gamma^+ = \mathcal{A}_t^+, \mathcal{A}_\gamma^- = \mathcal{A}_t^-\}.$$

Of course, for any $t \in T_\gamma$ we also have $\mathcal{A}_\gamma^0 = \mathcal{A}_t^0$. Using that γ is $\mathbb{R}$-irreducible, it can be shown that

$$\tilde{T} := \{o_t o_\gamma^k o_{t^{-1}} \mid k \in \mathbb{N}, t \in T_\gamma\}$$

is Zariski dense in $O(B_\gamma)$. Putting $\gamma(t) := t\gamma t^{-1}$ the above yields that

$$T := \{t \in SO(B) \mid \overline{\{\tilde{o}(\gamma(t))^k\}}_{k\in\mathbb{N}} \subset K := \bigcup_{i=1}^m K_i\}$$

is open as well as nonempty. We now fix a $t \in T \cap S$ and a $\gamma \in S$ and set $g_n := \gamma(t)$. Adding g_n to the set of g_i, $1 \leq i \leq n-1$, would increase the dimension of $\{o_l\}_{l\in\mathbb{N}^n}$, which cannot be because $\{o_l\}_{l\in\mathbb{N}^n}$ was chosen to be of maximal dimension, which yields that there has to be an i such that K_i equals $O(B_{g_0})$. Hence, there are Zariski open subsets in $\{o_l\}_{l\in\mathbb{N}^n}$ and we have finished the proof. ∎

We go on by giving some more notation.

$$\mathfrak{D}(B) := \{(W,g) \mid W \leq \mathbb{R}^n \text{ maximal among all } B\text{-anisotropic spaces}, g \in O(B_{|W}\}.$$

By anisotropic we mean that there are no null vectors, i.e. there is no $0 \neq w \in W$ with $B(w,w) = 0$.

Furthermore let $g \in O(B)$ be hyperbolic. Then we set $X_g := (\mathcal{A}_g^0, o_g) \in \mathfrak{D}(B)$. We will also use $\mathfrak{D}(\Gamma) := \{X_g \mid g \in \Gamma, g \text{ is hyperbolic}\}$ and $\mathfrak{D}^\varepsilon(\Gamma) := \{X_g \mid g \in \Gamma, g \text{ is } \varepsilon\text{-hyperbolic}\}$.

4.5.7 Lemma. *Given a group Γ, that is Zariski dense in $O(B)$. Then*

(i) $W = \mathcal{A}_\Gamma^0$ with $\gamma \in \Gamma$, γ hyperbolic $\Rightarrow (W,\tilde{g}) \in \overline{\mathfrak{D}(\Gamma)}$ if $\tilde{g} \in O(B_{|W})$, and

(ii) $(W,g) \in \overline{\mathfrak{D}^\varepsilon(\Gamma)} \Rightarrow (W,\tilde{g}) \in \overline{\mathfrak{D}^\varepsilon(\Gamma)}$ if $\tilde{g} \in O(B_{|W})$.

Let $X_i := (W_i, g_i)$ for some $i \in \mathbb{N}$. Given a sequence $\{X_i\}_{i\in\mathbb{N}}$, we say it converges to $X \in \mathfrak{D}(B)$ and write $\{X_i\}_{i\in\mathbb{N}} \rightrightarrows X$, if

(i) $\hat{d}(W_i, W) \to 0$ for $i \to \infty$, and

(ii) for every $\varepsilon \in \mathbb{R}^+$ and every $(a,b) \in (W, W_i)$ with $|a| = |b| = 1$ there is an $m \in \mathbb{N}$, such that for all $i > m$ we have

$$||a-b|| - \varepsilon \leq ||ga - g_i b|| \leq ||a-b|| + \varepsilon.$$

4.5.8 Remark (From linear hyperbolic and transversal maps to affine hyperbolic and transversal maps)**.** In contrast to section 4.2, where we started right away with hyperbolic and transversal affine maps, so far we have only been talking about linear maps. Of course, we are more interested in affine ones. The transition from the linear to the affine setting is done in the obvious way.

By A_B we denote the subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ whose elements are mapped under the projection L onto their linear part into $SO(B)$. If for such a $g \in A_B$ the linear part $L(g)$ is hyperbolic, we say that g itself is hyperbolic. Accordingly, we will use the denotation $\mathcal{A}_g^-, \mathcal{A}_g^+$ and $\mathcal{A}_g^0$ instead of $\mathcal{A}_{L(g)}^-, \mathcal{A}_{L(g)}^+$ and $\mathcal{A}_{L(g)}^0$. Subsequently, we will also write $\mathcal{D}_g^-, \mathcal{D}_g^+$ and $\mathcal{D}_g^0$ rather than $\mathcal{D}_{L(g)}^-, \mathcal{D}_{L(g)}^+$ and $\mathcal{D}_{L(g)}^0$

Also as in Margulis' counterexample, if $g \in A_B$ is hyperbolic and there is a unique line that is invariant under the action of g, we will denote the said line by $\mathcal{C}_g$. Of course, if g is not only hyperbolic but lies in a group Γ with $\Gamma \circlearrowleft \mathbb{R}^n$ being a properly discontinuous action, then there will be such a $\mathcal{C}_g$. Also, if g is an element of such a Γ, then $g \circlearrowleft \mathcal{C}_g$ is a parallel translation given by some vector $t_g \in \mathcal{A}_g^0$. Evidently, $B_g(t_g, t_g) > 0$. If $v_g := \frac{t_g}{B_g(t_g,t_g)}$, then $B_g(v_g, v_g) = 1$ and thus for any $x \in \mathbb{R}^n$ we have $B(gx - x, v_g) = t_g$.

Furthermore $\mathcal{E}_g^+$, respectively $\mathcal{E}_g^-$, shall denote the plane that contains $\mathcal{C}_g$ and is parallel to $\mathcal{D}_g^+$, respectively $\mathcal{D}_g^-$. The intersection $\mathcal{E}_g^+ \cap \mathcal{E}_g^-$ will by denoted by $\mathcal{F}_g$. Obviously, we have $\mathcal{C}_g \subseteq \mathcal{F}_g$. Lastly, we call π_g the natural projection $\pi_g \colon \mathbb{R}^n \to \mathcal{F}_g$, which is parallel to $\mathcal{A}_g^+ \oplus \mathcal{A}_g^-$.

Using these notations, the following lemma, which is a generalization from results derived in the course of Margulis' counterexample, can be achieved.

4.5.9 Lemma. *Given 2ε-hyperbolic elements $g, h_1, \ldots, h_m \in A_B$, which are pairwise 2ε-transversal. We set $H := < h_1, \ldots, h_m > \leq A_B$ and define $c_{gh} = d(v, \mathcal{F}_{gh})$ for a fixed $v \in \mathbb{R}^n$. It follows that there is a $c \in \mathbb{R}$ such that for all $h \in H$*

(i) gh is ε-hyperbolic, and

(ii) $c_{gh} \leq c$.

Let $\mathfrak{D}_a(B) := \{(X, v) \mid X = (W, g) \in \mathfrak{D}(B), v \in W, B(v, v) = 1\}$. The motivation for adding the index a to this set is the following: from the said set we can deduce information not only about $L(g)$ but also about the *a*ffine map g.

Given a sequence $\{(X_i, v_i)\}_{i \in \mathbb{N}}, (X_i, v_i) \in \mathfrak{D}_a(B)$, and a pair $(X, v) \in \mathfrak{D}_a(B)$ with $X_n \rightrightarrows X$ and $v_n \to v$, then way say that $\{(X_i, v_i)\}_{i \in \mathbb{N}}$ converges to (X, v). We also set

$$\mathfrak{D}_a^{\varepsilon}(\Gamma) := \{(X, v) \in \mathfrak{D}_a(B) \mid X \in \mathfrak{D}^{\varepsilon}(\Gamma), v \in W, B(V, v) = 1\}.$$

We are now ready to state two key results for the proof of the main theorem of this section.

4.5.10 Proposition. *Given a group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ and a nondegenerate bilinear form B of signature $(p, q), p \geq q$. If*

(i) either $2 \leq p - q$ or $q \leq 2$,

(ii) $L(\Gamma)$ is a Zariski-dense subgroup of $O(B)$, and

(iii) every hyperbolic $\gamma \in \Gamma$ acts freely,

then there are $X_1, \ldots, X_m, Y_1, \ldots, Y_m \in \mathfrak{D}(B)$ with

(i) $X_i, Y_j \in \overline{\mathfrak{D}_a^{\varepsilon}(\Gamma)}$, $i, j \in \{1, \ldots, m\}$, such that

(ii) $X_i = (W, g_i, v_i)$ and $Y_j = (W, g_j, -v_j)$, and

(iii) the set $\{v_1, \ldots, v_m\}$ is a basis for W.

4.5.11 Proposition. *If a group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ and a bilinear form B are both defined as in proposition 4.5.10, one can find an $\varepsilon \in \mathbb{R}^+$, such that there are ε-hyperbolic and ε-transversal $\gamma_1, \ldots, \gamma_s \in \Gamma$, a compact $K \subset \mathbb{R}^n$ and constants*

$$\begin{aligned} &c_1(\gamma_1, \ldots, \gamma_s), c_2(\gamma_1, \ldots, \gamma_s), \\ &1 < a(\gamma_1, \ldots, \gamma_s), \text{ and} \\ &b(\gamma_1, \ldots, \gamma_s) < 1, \end{aligned}$$

such that

(i) $\tilde{\Gamma} :=< \gamma_1, \dots, \gamma_s >$ *is a free group and the* γ_i*,* $1 \leq i \leq s$*,* $i \in \mathbb{N}$*, form a free generating set for* $\tilde{\Gamma}$*,*

(ii) if $\gamma \in \Gamma$ *is* ε*-hyperbolic and* ε*-transversal with respect to every* γ_i*,* $1 \leq i \leq s$*, and if* $a(\gamma_1, \dots, \gamma_s) \leq a(\gamma, \gamma_1, \dots, \gamma_s)$*, then* $\hat{\Gamma} :=< \gamma, \gamma_1, \dots, \gamma_s >$ *is a free group and* $\gamma, \gamma_1, \dots, \gamma_s$ *are free generators for* $\hat{\Gamma}$*, and*

(iii) if for such a γ *we have* $c_1(\gamma_1, \dots, \gamma_s) < d^B_\gamma(K)$ *as well as* $a(\gamma_1, \dots, \gamma_s) \leq a(\gamma, \gamma_1, \dots, \gamma_s)$*, then there are* $k, l \in \mathbb{N}$*,* $1 \leq k \leq s$*,* $l \leq d^B_\gamma(K) c_2$*, such that* $\tilde{\gamma} := \gamma_k^l \gamma$ *yields* $d^B_{\tilde{\gamma}}(K) \leq b d^B_\gamma(K)$*.*

Propositions 4.5.10 and 4.5.11, along with Lemma 4.5, yield the following corollary.

4.5.12 Corollary. *Let the same notation and assumptions apply as in proposition 4.5.11 and let furthermore* w *denote the word metric on* $\tilde{\Gamma} =< \gamma_1, \dots, \gamma_2 >$*. If* $\gamma \in \Gamma$ *is an element provided by proposition 4.5.11 then there is a* $\hat{\gamma} \in \tilde{\Gamma}$ *and a* $c \in \mathbb{R}$*, with* $w(\hat{\gamma}) \leq (d^B_\gamma(K) c)^2$ *and* $d^B_{\hat{\gamma}\gamma}(K) \leq c_1$*.*

We are now ready to sketch the proof for theorem 4.5.1.

proof of theorem 4.5.1. As stated above, we will prove this proposition by contradiction. Thus, let $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ be a properly discontinuous, yet not virtually solvable group. This yields that $G := \overline{L(\Gamma)}$ is semisimple and that we can decompose $\mathbb{R}^n = W \oplus W^\perp$ such that both subspaces are invariant under the action of G and W is even irreducible with respect to G. Furthermore, $B_{|W}$ is a bilinear form of signature $(n-2, 2)$ whereas $B_{|W^\perp}$ is positive definite. Of course, the natural projections π_w respectively $\pi_{W^\perp}$ from $\mathbb{R}^n$ onto W respectively $W^\perp$ induce homomorphisms from Γ into $\mathrm{Aff}(W)$ and $\mathrm{Aff}(W^\perp)$.

It can be shown that $\pi_W(\Gamma)$ is Zariski dense in $O(B_W)$. Thus, there are $\gamma_1, \dots, \gamma_s \in \Gamma$ with $\pi_W(\gamma_1), \dots, \pi_W(\gamma_s)$ satisfying the assumptions of proposition 4.5.11. Then, using results from [AMS95] or [AMS97], it follows that there are $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \Gamma$ such that $\pi(\tilde{\gamma}_i), \pi(\gamma_j)$, $1 \leq i \leq 2, 1 \leq j \leq s$, are $\tilde{\varepsilon}$-hyperbolic and pairwise $\tilde{\varepsilon}$- transversal.

One can show then that there is an $k \in \mathbb{N}$, such that for any $\gamma \in < \tilde{\gamma}_1^k, \tilde{\gamma}_2^k >$ the following holds:

(i) $\pi_W(\gamma)$ is $\frac{\tilde{\varepsilon}}{2}$-hyperbolic,

(ii) $\pi_W(\gamma), \pi_W(\gamma_1), \dots, \pi_W(\gamma_s)$ are pairwise $\frac{\tilde{\varepsilon}}{2}$-transversal, and

(iii) $a(\pi_W(\gamma_1), \dots, \pi_W(\gamma_s)) \leq a(\pi_W(\gamma), \pi_W(\gamma_1), \dots, \pi_W(\gamma_s))$.

Without loss of generality one can assume $k = 1$. For ease of notation we also set $\varepsilon := \frac{\tilde{\varepsilon}}{2}$. Furthermore, let $\Gamma^* :=< \tilde{\gamma}_1, \tilde{\gamma}_2, \gamma_1, \dots, \gamma_s >$ with word metric w. Then we have for compact $K_W \subset W$ and $K_{W^\perp} \subset W^\perp$

(i) $d^{B_W}_{\pi_W(\gamma)}(K_W) < w(\gamma)$, and

(ii) $d^{B_{W^\perp}}_{\pi_{W^\perp}(\gamma)}(K_{W^\perp}) < w(\gamma)$.

Set $D_l := \{\gamma \in <\tilde{\gamma}_1, \tilde{\gamma}_2> \mid w(\gamma) \leq l\}$ for $l \in \mathbb{N}$. It can be shown that $|D_l| \geq 3^l - 1$. By corollary 4.5.12, we have that if $\gamma \in D_l$, then there is a $\hat{\gamma} \in \tilde{\Gamma}$, such that $w(\hat{\gamma}) \leq (d^B_\gamma(K)c)^2$ and $d^B_{\hat{\gamma}\gamma}(K) \leq c_1$.

Thus, choosing an appropriate c_3, we have for

$$T_l := \{\gamma \in \Gamma^* \mid w(\gamma) \leq c_3 l^2, d^{B_W}_{\pi_W(\gamma)}\},$$

that $|T_l| \geq 3^l - 1$.

Let $v \in W$ and $\gamma \in T_L$. Then *(i)* and *(ii)* show, that γv is in the $c_3 l^2$-ball around w. This ball has volume smaller than $l^{2 \dim W^\perp}$. However, $|T_l|$ grows exponentially for $l \to \infty$. We deduce that for every $\delta \in \mathbb{R}^+$ we have $|P_l| \to \infty$ for $l \to \infty$, whereas

$$P_l := \{(\hat{\gamma}, \hat{\gamma}_2) \in T_l \times T_l \mid B_{W^\perp}(\pi_{W^\perp}(\hat{\gamma}_1)p - \pi_{W^\perp}(\hat{\gamma}_2)p) \leq \delta\}.$$

If $w \in \mathbb{R}^n$ is properly chosen, then $d(\gamma(w), w)$ is bounded if $\gamma = \hat{\gamma}_1\hat{\gamma}_2^{-1}$. This shows that $\Gamma \circlearrowleft \mathbb{R}^n$ cannot be properly discontinuous, we arrive at a contradiction and have thus finished the proof. □

4.6 Auslander's conjecture holds if $L(\Gamma) \leq G$, G reductive, $\mathrm{rank}_{\mathbb{R}}(G) = 1$

In this section, we are going to prove that

4.6.1 Theorem. *A crystallographic subgroup Γ of $\mathrm{Aff}(\mathbb{R}^n)$ is virtually solvable, if $L(\Gamma) \leq G$, with G having only finitely many connected components as well as being reductive and of real rank 1.*

Proof. As in the sections before, the proof, which is due to Grunewald and Margulis, [GM88], will be done in a couple of steps. As always, we begin by collecting some definitions and results.

Then, the actual proof will be done by using induction on the dimension of $\mathbb{R}^n$. Also, there will be a case-by-case study. At first we assume that Γ contains no subgroup of translations.

After having settled the situation above, we will look into the opposite one, i.e. the case of Γ having a subgroup of translations. We will examine what happens if $L(\Gamma)$ is either discrete or not. In both these subcases, by using cohomological arguments exactly as in the cases $\Gamma \leq \mathrm{Aff}(\mathbb{R}^4)$ and $L(\Gamma) \leq O(\mathbb{R}^{n-1,1})$, we will show that either of these possibilities leads to a contradiction, i.e. there has to be a subgroup of translations in Γ.

The proof can be found on page 84. □

As a consequence we have

4.6.2 Corollary. *Let Γ be as in theorem 4.6.1. Then there is a series of groups*

$$1 \trianglelefteq \Gamma_0 \trianglelefteq \Gamma_1 \trianglelefteq \Gamma_2 \trianglelefteq \Gamma_3 \trianglelefteq \Gamma,$$

such that

(i) Γ_0 *is abelian,*

(ii) Γ_1/Γ_0 *is 2-step nilpotent,*

(iii) Γ_2/Γ_1 *is abelian,*

(iv) Γ_3/Γ_2 *is abelian, and*

(v) Γ/Γ_3 *is finite.*

Proof. Right after we finish the proof for the main theorem, we will show that this assertion holds, see page 85. □

To start, we give a detailed definition of the real rank of a connected Lie group; a definition of the term reductive can be found in 4.5.4.

4.6.3 Definition (Cartan decomposition and real rank of a connected Lie group)**.** *Let* G *be a semisimple real Lie group and denote its associated Lie algebra by* $\mathfrak{g}$*. Furthermore let* $B(\cdot,\cdot)$ *be the Killing form of* $\mathfrak{g}$*, i.e. for* $X, Y \in \mathfrak{g}$ *we have* $B(X,Y) = trace(ad(X)ad(Y))$ *with* $ad(X)(\cdot) = [X, \cdot\,]$*.*

An automorphism θ *of* $\mathfrak{g}$ *is called* Cartan involution *if* $\theta^2 = id$ *and* $B_\theta(X,Y) = -B(X,\theta Y)$*. Such an automorphism exists for any Lie algebra and any two such automorphisms are equal modulo the inner automorphisms of* $\mathfrak{g}$*. Because* $\theta^2 = id$*,* θ *has only two distinct eigenvalues, namely, 1, with eigenspace* $\mathfrak{k}$*, and -1, with eigenspace* $\mathfrak{p}$*. We call* $(\mathfrak{k},\mathfrak{p})$ *a* Cartan pair *and the decomposition* $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ *a* Cartan decomposition *of* $\mathfrak{g}$*.*

The real rank *of* G*, denoted by* $\mathrm{rank}_{\mathbb{R}}(G)$*, is then defined as the dimension of a maximal abelian subalgebra of* $\mathfrak{p}$*.*

On a sidenote: the Cartan decomposition can be viewed as the generalization of the so called polar-*decomposition of a single matrix.*

We will now give a series of results needed for the proof of the main theorem of this section.

We start by defining two homomorphisms. Let $V \leq \mathbb{R}^n$. We have the natural projections

$$r_V\colon Stab_{GL(\mathbb{R}^n)}(V) \to GL(\mathbb{R}^n/V), \text{ and}$$
$$\rho_V\colon L^{-1}(Stab_{GL(\mathbb{R}^n)}(V)) \to \mathrm{Aff}(\mathbb{R}^n/V),$$

whereas for the latter homomorphism we have

$$\rho_v\left(\begin{pmatrix} A & b \\ 0\cdots 0 & 1\end{pmatrix}\right)\begin{pmatrix} w+V \\ 1\end{pmatrix} = \begin{pmatrix} Aw+b+V \\ 1\end{pmatrix},$$

and the kernel of ρ_V looks as follows:

$$\ker(\rho_V) = Stab_{\mathrm{Aff}(\mathbb{R}^n)}(V) \cap L^{-1}(\ker_r(V)).$$

4.6.4 Lemma. *Given* $V \leq \mathbb{R}^n$ *and a crystallographic group* $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$*. If* *(i)* $L(\Gamma) \leq Stab_{GL(\mathbb{R}^n)}(V)$, *and*

(ii) the action $\ker(\rho_V) \cap \Gamma \circlearrowleft \mathbb{R}^n$ *is properly discontinuous,*

then $\rho_V(\Gamma) \leq \mathrm{Aff}(\mathbb{R}^n/V)$ *is also crystallographic.*

Proof. Cocompactness of $\rho_V(\Gamma)$ is obvious, thus we are left to check whether the action is properly discontinuous.

We fix a compact subset $K \subset \mathbb{R}^n/V$ and let $\tilde{K} \subset \mathbb{R}^n$, such that $\tilde{K}$ is also compact and maps onto K under the natural projection $\mathbb{R}^n \to \mathbb{R}^n/V$. Furthermore, we require

$$\bigcup_{\gamma \in (\ker(\rho_V) \cap \Gamma)} \gamma(\tilde{K} \cap V) = V.$$

It follows, that for every coset $\theta \in \rho_v(\Gamma)$ with $K \cap \theta K \neq \emptyset$ there is a $\gamma \in \theta$ such that $\gamma\tilde{K} \cap \tilde{K} \neq \emptyset$. Consequently, $\rho_V(\Gamma)$ is a properly discontinuous group and we finished the proof. ◻

To state the next proposition of this section, we will first give a definition.

4.6.5 Definition (Borel subgroup and parabolic subgroup)**.** *Given an algebraic group G and an algebraic subgroup $B \leq G$. If B is a maximal Zariski closed, connected and solvable group, we call B a* Borel *group.*

Subgroups P with $B < P < G$ are called parabolic *groups.*

4.6.6 Proposition. *Given a linear algebraic group G and a Zariski closed subgroup $H < G$. If G is reductive, and if furthermore the unipotent radical of H is not trivial, then the normalizer $N_G(H)$ lies in a parabolic subgroup of G.*

Proof. For a proof see [Mo56] and [Pl69]. ◻

As an immediate corollary we have:

4.6.7 Corollary. *Let $H < G$ be defined as in proposition 4.6.6. If* $\mathrm{rank}_{\mathbb{R}}(G) \leq 1$*, then either*

(i) the 1-component of H is a reductive group with real rank at most 1, or

(ii) $N_G(H) \leq S \rtimes K$, with S solvable and K compact.

Proof. This follows from the fact, that any parabolic $P < G$ can be written as a semidirect product of a solvable group by a compact one. ◻

The proof for the next result can be found in [Ra72].

4.6.8 Lemma. *Given a connected Lie group $H = S \rtimes K$ with S solvable and K compact. If $\Gamma \leq H$ is discrete, then Γ is virtually polycyclic.*

The following proposition is the special case of a result by Auslander, [Au63]. See also theorem 3.3.2.

4.6.9 Proposition. *If $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ is discrete, then $\overline{L(\Gamma)}^0$ is solvable.*

Once again, in order to proceed, we need more definitions.

4.6.10 Definition (Almost direct product of groups)**.** *Given an algebraic group G with algebraic subgroups $G_1, \ldots, G_n$, $n \in \mathbb{N}$. If*

$$\begin{aligned} G_1 \times \cdots G_n &\to G \\ (g_1, \ldots, g_n) &\mapsto g_1 \cdots g_n \end{aligned}$$

is an endomorphism with finite kernel, then we say that G is an almost direct product of the G_i, $1 \leq i \leq n$.

4.6.11 Definition (Isogeny of Lie groups)**.** *Given two Lie groups G and H. A map $\phi\colon G \to H$ is called* isogeny *if ϕ is a Lie group homomorphism as well as a covering map.*

Two groups G, H are then called isogenous *if there is an isogeny either from G onto H or from H onto G.*

4.6.12 Definition (Symmetric space attached to G)**.** *Given a linear semisimple Lie group G, such that G has only finitely many components and $\mathrm{rank}_{\mathbb{R}} = 1$. Let K be a maximal compact subgroup of G. We set $X_G = G^0/K$ and call X_G the* symmetric space attached *to G.*

4.6.13 Proposition. *Given a real semisimple Lie group G, such that G has only finitely many components and $\mathrm{rank}_{\mathbb{R}} = 1$. Choose n such that $\rho\colon G \to GL(\mathbb{R}^n)$ is a faithful representation. If G is not isogenous to an almost direct product of some compact $\tilde{K}$ and $O(\mathbb{R}^{2,1})$, then we have $\dim X_G < \dim \mathbb{R}^n$.*

Proof. Our group in question can be viewed as an almost direct product of a compact group K and an almost simple group H of real rank 1. Of course, ρ induces also a faithful representation of H and thus there is a nontrivial representation of $\mathfrak{h}_{\mathbb{C}}$, i.e. of the complexification of the Lie algebra $\mathfrak{h}$ associated to H. Such an $\mathfrak{h}_{\mathbb{C}}$ is always simple, unless H is isogenous to $O(\mathbb{R}^{3,1})$. If this is indeed the case, then $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

Using [Ti67], one can determine which irreducible representations of $\mathfrak{h}_{\mathbb{C}}$ are real. Then checking the lower bound for the minimal dimension of an irreducible representation of $\mathfrak{h}_{\mathbb{C}}$ finishes the proof.

This lower bound can be deduced from Weyl's dimension formula, see [Ma49]. □

We have now collected enough data to prove theorem 4.6.1.

proof for theorem 4.6.1. Of course, the theorem holds for dimension up to 3. Thus, because we use induction on the dimension of $\mathbb{R}^n$, we may assume $n \geq 3$.

If $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ has a proper subgroup of pure translation, the claim follows from our results 4.6.4, 4.6.7 and 4.6.8 in conjunction with the induction hypothesis.

Thus, from now on we assume that the subgroup of pure translations is trivial. There are two disjoint cases, either $L(\Gamma)$ is a discrete subgroup of G or it is not.

Case 1: $L(\tilde{\Gamma})$ is discrete.
As before, we are not concerned with Γ directly, but instead pass on to a finite-index subgroup Γ that is torsion free. Furthermore, without loss of generality, we may assume that $\tilde{\Gamma}$ preserves orientation.

We start by observing that $\tilde{\Gamma}\backslash\mathbb{R}^n$ is a compact, orientable n-dimensional manifold. Using Poincaré-duality, we have the following isomorphism:

$$H^n(\tilde{\Gamma}, \mathbb{R}) \cong H^n(\pi_1(\tilde{\Gamma}\backslash\mathbb{R}^n), \mathbb{R}) \cong H^n(\tilde{\Gamma}\backslash\mathbb{R}^n, \mathbb{R}^n) \cong \mathbb{R}.$$

However, because $\tilde{\Gamma}$ is discrete and torsion free, the action $\tilde{\Gamma} \circlearrowleft X_G$ is properly discontinuous. Furthermore, from $\pi_1(\tilde{\Gamma}\backslash X_G) \cong \tilde{\Gamma}$ and $\dim(\tilde{\Gamma}\backslash X_G) < n$, proposition 4.6.6, we deduce that the cohomological dimension of $\tilde{\Gamma}$ is also less than n and we arrive at a contradiction.

Also note that, because we assumed $n \geq 3$, the exceptional case mentioned in proposition 4.6.6 cannot arise.

Case 2: $L(\tilde{\Gamma})$ is not discrete.
In this situation $H := \overline{L(\tilde{\Gamma})}^0$ is a nontrivial, connected and solvable group, according to propostion 4.6.9.

There are two possibilities for $\overline{H}$, either the unipotent radical is trivial or not. If it is nontrivial, then corollary 4.6.7 and lemma 4.6.8 yield the desired result.

Thus, we assume that $\overline{H}$ contains no unipotent elements at all, otherwise they lie in the radical. In this case $\overline{H}$ is a torus. It follows that the centralizer $C_G(\overline{H})$ is of finite index in $N_G(\overline{H})$. We choose a subgroup $\Gamma_0 \leq \tilde{\Gamma}$, such that $[\tilde{\Gamma} : \Gamma_0] < \infty$ and $\Gamma_0 \leq N_G(H)$. Furthermore, let $\Gamma_1 := L^{-1}(H) \cap \Gamma_0 \leq \tilde{\Gamma}$.

Because Γ_0 centralizes $L(\Gamma_1)$, we have for the commutator $[\Gamma_1, \Gamma_0] \leq T_{\tilde{\Gamma}} := \{\gamma \in \tilde{\Gamma} \mid \gamma \text{ is a translation}\}$. Thus, $\Gamma_1 \leq Z(\Gamma_0)$.

We fix a $\gamma \in \Gamma_1$, $\gamma \neq e$, and set $V_\gamma := \{v \in \mathbb{R}^n \mid (\mathbb{I} - L(\gamma))v = 0\}$. Because γ commutes with every element of Γ_0, V_γ is left invariant under $L(\Gamma_0)$.

The map $L(\gamma)$ is semisimple and therefore $(1 - L(\gamma))$ is invertible on $\mathbb{R}^n/V_\gamma$. Consequently, there is a unique coset $w + V_\gamma$, such that the affine space $(w + V_\gamma, 1)$ is left invariant under γ. It is possible to conjugate Γ_0 in such a way that $w = 0$. This yields that Γ_0 acts invariantly on V_γ and that $\Gamma_0 \circlearrowleft V_\gamma$ is properly discontinuous.

Without loss of generality, we may assume Γ_0 to be torsion free. We deduce that $\Gamma_0\backslash V_\gamma$ and $\Gamma_0\backslash\mathbb{R}^n$ are both compact manifolds. However, they are of different dimension. Once again employing the cohomological dimension of Γ_0, we arrive at a contradiction. □

Now that we have proven our theorem, as promised we are going to show that corollary 4.6.2 holds.

proof for corollary 4.6.2. It is easy to show that the assertion holds if G is not algebraic, thus we assume it is.

Because Γ is virtually solvable, there is torsion free solvable subgroup $\tilde{\Gamma} \leq \Gamma$, such that $\tilde{\Gamma}$ has finite index. This yields that $\tilde{\Gamma}$ is the extension of the abelian group $\ker(L_{|\tilde{\Gamma}})$ by the group $L(\tilde{\Gamma})$.

The Zariski closure $\overline{L(\tilde{\Gamma})}$ is either a torus or it has unipotent elements. In the first case, it is an abelian group.

If there are unipotent elements in $\overline{L(\tilde{\Gamma})}$, then, by the power of proposition 4.6.6, there is a parabolic subgroup $P < G$, such that $L(\tilde{\Gamma}) \leq P$. Such a parabolic P is an almost semidirect product by some compact K and a group $S = U \rtimes \mathbb{R}$, with U being unipotent of class less than 2. Using that $L(\tilde{\Gamma}) \leq K/(K \cap S)$ is abelian, finishes the proof. □

4.7 Auslander's conjecture holds if Kawabe's conditions are met

In order to state the main theorem of [Ka98], we need a little notation and a proposition.

Given an n-dimensional crystallographic group Γ. From now, let G be the Zariski closure of $L(\Gamma)$, i.e. $G := \overline{L(\Gamma)}$. For G we then have $G^0 = SR$, with S being a maximal semisimple subgroup of G^0 and R being the solvable radical of G^0. Furthermore, P shall be a maximal reductive subgroup of G^0 such that there is a Levi factor for G with $S \leq P$.

By using representation theory of algebraic groups, one can show that the following proposition holds:

4.7.1 Proposition. *There is a basis of $\mathbb{R}^n$, such that G^0 can be written with respect to the said basis as*

$$\left\{ \begin{bmatrix} \rho_1 & * & \cdots & * \\ 0 & \rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \rho_r \end{bmatrix} \right\},$$

such that ρ_i is an irreducible presentation of P.

Furthermore, if we have that the restriction $\rho_{i|\tilde{S}}$ is reducible we have the decomposition

$$\rho_{i|\tilde{S}} = \sigma_{i1} \oplus \cdots \oplus \sigma_{it_i}(S),$$

with the σ_{ij} being irreducible representations of $\tilde{S}$ and $1 \leq j \leq t_i$.

Let $H \in \{SL(\mathbb{R}^n), Sp(\mathbb{R}^{2m}), SO(\mathbb{R}^{2m})\}$ and ν shall be the standard representation of H. Then we say that the representation $\tau \colon S \to GL(\mathbb{R}^m)$ is of type S_m, if

(i) it is irreducible, and if

(ii) $\tau = \nu \circ \tilde{\tau}$ with $\tilde{\tau}$ being an homomorphism from S into H.

We can now state the main theorem of [Ka98].

4.7.2 Theorem (Kawabe's conditions)**.** *A crystallographic subgroup* Γ *of* $\mathrm{Aff}(\mathbb{R}^n)$ *is virtually solvable, if the following holds for the identity component of* $G = \overline{L(\Gamma)}$ *represented as in proposition 4.7.1:*

(i) every $\sigma_{ij}(S)$ *is of type* S_{m_j}*, and*

(ii) every trivial representation is either at top $\sigma_{1j}(S)$*, i.e.* $1 \leq j \leq t_1$*, or at bottom* σ_{rj}*, i.e.* $s \leq j \leq t_r$*.*

Proof. This theorem is proven in [Ka98]. The crucial part of the proof is the construction of a certain virtually torsion free $\tilde{\Gamma} \leq \mathrm{Aff}(\mathbb{R}^n)$ which admits a properly discontinuous action $\tilde{\Gamma} \circlearrowleft \mathbb{R}^n$. $\square$

5 Perspectives

As the title suggests, in the last chapter we would like to offer perspectives on future developments of the study of affine crystallographic groups.

In the first subsection we will present two problems and one conjecture, all of which are due to Abels, Margulis and Soifer, [AMS12]. Solving these three problems could be a big step towards answering Auslander's conjecture in dimension 7.

For reasons explained in said subsection, one might have to come up with techniques vastly different from those that were used in the proofs we presented for special cases of Auslander's conjecture.

At this point it should be noted that approaches to Auslander's conjecture already exist, which are entirely different from those we have seen used in section 4. For example, Friedland uses invariant probability measures to examine a so called weak version of the conjecture, see [Fr95].

The second subsection is devoted to so called *NIL-affine crystallographic groups*, which turn out to be a very natural generalization of (affine) crystallographic groups. Furthermore, in this setting a question posed by Milnor, which was answered negatively in the affine case, can be confirmed. Also, one can state a generalized version of Auslander's conjecture and prove it in special cases.

5.1 Auslander's conjecture in dimension 7

In [AMS12], Abels, Margulis and Soifer proved that Auslander's conjecture holds if $4 \leq \dim(\mathbb{E}) \leq 6$ by mainly using dynamical arguments, similar to the techniques employed in Margulis' counterexample, 4.2 as well as to the ones presented in the proof of the special case $L(\Gamma) \leq O(\mathbb{R}^{n-2,2})$, see section 4.5.

One of the crucial parts in the course of proving that Auslander's conjecture holds in dimension 6 is the possibility of changing the sign of hyperbolic elements - said sign is similar to the one defined in order to construct Margulis' counterexample, see equation (4.7).

However, according to Abels, Margulis and Soifer, for a hyperbolic $g \in SO(\mathbb{R}^{4,3})$, there is no easy way to change its sign. This fact motivated the authors to pose the following question in [AMS12]:

5.1.1 Problem (AMS-Problem 1)**.** *Is there a crystallographic* $\Gamma \leq \mathrm{Aff}(\mathbb{R}^7)$ *with* $L(\Gamma)$ *being a Zariski dense subgroup of* $SO(\mathbb{R}^{4,3})$*?*

In the same paper, the three authors also pose a second problem, which they believe to be also of utter importance for a possible proof of Auslander' conjecture in dimension 7.

If G is the simplest representation of a simple Lie group of type $G_2 \cong Aut(\mathbb{O})$, i.e. the automorphism group of the octonions, it can be shown that $G < O(\mathbb{R}^{4,3})$.

5.1.2 Problem (AMS-Problem 2)**.** *Is there a crystallographic $\Gamma \leq \mathrm{Aff}(\mathbb{R}^7)$ with $L(\Gamma)$ being a Zariski dense subgroup of G?*

Abels, Margulis and Soifer claim, [AMS12], that a negative answer to the problem above results in a positive one to the following conjecture:

5.1.3 Conjecture (Abels, Margulis and Soifer)**.** *Given a connected Lie group, such that no simple non-abelian connected subgroup of G has real rank greater 2. It follows that every crystallographic group $\Gamma \leq G$ is virtually solvable.*

Also note that another argument widely used in proofs for special cases of Auslander's conjecture cannot be used in dimension 7.

In dimension 3, 4.3, in the cases of $L(\Gamma) \leq O(\mathbb{R}^{n-1,1})$, section 4.4, and $L(\Gamma) \leq H$, H reductive, $\mathrm{rank}_{\mathbb{R}}(H) = 1$, as well as in [AMS12] an argument concerning the cohomological dimension of the crystallographic group $\Gamma \leq \mathrm{Aff}(\mathbb{R}^n)$ was used, namely that said dimension equals $n = \dim(\mathbb{R}^n)$.

Unfortunately, this argument cannot be employed in dimension 7, because a Γ defined as in one of the above problems has virtual cohomological dimension 7, yet the corresponding symmetric spaces are of dimension at least 8.

5.2 NIL-affine crystallographic groups

A part of the motivation for studying NIL-affine crystallographic groups is what we called the 'converse Auslander conjecture', 4.1.4, which was stated by Milnor, [Mi77].

5.2.1 Problem. *Can any torsion free virtually polycyclic group be realized as an affine crystallographic group?*

As stated right after problem 4.1.4, the answer to this problem is known - there are indeed such groups which cannot be realized as affine crystallographic groups.

However, there is a setting in which the problem above can be answered affirmatively. Furthermore, one can carry over many concepts from affine geometry to this new setting, which then can be used to approach a generalized version of Auslander's conjecture.

Recall that an affine crystallographic group is a subgroup of the group of affine motions $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(\mathbb{R}^n)$. In this section, we want to present subgroups of $\mathrm{Aff}(N) = N \rtimes Aut(N)$, with N being a simply connected nilpotent Lie group, which is diffeomorphic to some $\mathbb{R}^n$. The action $\mathrm{Aff}(N) \circlearrowleft N$ is given by

$$\begin{aligned} \mathrm{Aff}(N) \times N = (N \rtimes Aut(N)) \times N &\to N \\ (n, \alpha)(m) &\mapsto n\alpha(m) \end{aligned}$$

Before we move on, a few words on the justification of the notation $\mathrm{Aff}(N)$: similar to the case of $\mathrm{Aff}(\mathbb{R}^n)$, $\mathrm{Aff}(N)$ is exactly the group of diffeomorphism of N which preserve any left invariant affine connection on N. A proof for this assertion can be found in [KT68].

5.2.2 Definition (NIL-affine crystallographic group)**.** *Given a subgroup $\Gamma \leq \mathrm{Aff}(N)$. If the action $\Gamma \circlearrowleft N$ is crystallographic, we call Γ a* NIL-affine crystallographic *group.*

For such groups, a variation of the problem mentioned above holds.

5.2.3 Theorem. *If Γ is a torsion free virtually polycyclic group, then it can be realized as a NIL-affine crystallographic group, i.e. there is an embedding*

$$\iota \colon \Gamma \to \mathrm{Aff}(N)$$

for some simply connected nilpotent Lie group N, such that $\iota(\Gamma) \circlearrowleft N$ is crystallographic.

Proof. The proof can be found in [De03]. $\square$

Consequently, an obvious question is the following:

5.2.4 Conjecture (Auslander's conjecture - generalized version)**.** *Given a simply connected nilpotent Lie group N and a crystallographic group $\Gamma \leq \mathrm{Aff}(N)$. It follows that Γ is virtually polycyclic.*

A proof for this conjecture in low dimension was given by Burde, Dekimpe and Deschamps in [BDD05].

5.2.5 Theorem. *A crystallographic subgroup Γ of $\mathrm{Aff}(N)$ is virtually solvable, if $\dim(N) \leq 5$.*

The proof for the case $\dim(n) \leq 3$ can be deduced from the following proposition in conjunction with the fact that any nilpotent Lie algebra of dimension at most 3 is of nilpotency class 2, which implies the same for any nilpotent Lie group of dimension at most 3. The final step then is to use the main results from section 4.3, namely that Auslander's conjecture holds in dimensions 3 and 4.

5.2.6 Proposition. *Given a simply connected Lie group N of nilpotency class 2. If $\Gamma \leq \mathrm{Aff}(N)$ is NIL-affine crystallographic, then there is an affine crystallographic action $\Gamma \circlearrowleft \mathbb{R}^n$.*

A proof of this assertion is given in [BDD05].

The proofs of theorem 5.2.5 for dimensions 4 and 5 are even more involved. They rely on the translation of ideas, for example (pseudo)hyperbolic elements and transversality of two such elements, developed in [AMS02] for the affine setting to the NIL-affine one.

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