# MAGISTERARBEIT

Titel der Magisterarbeit

## Constructing Imaginary Points and Lines:
## von Staudt's Interpretation and Locher-Ernst's Method

verfasst von

## John Bouchier

angestrebter akademischer Grad

## Magister der Naturwissenschaften

Wien, 2013

# <u>Contents</u>

# Foreword

The original meaning of the word geometry was `earth measurement´. In the course of time the study of geometry developed from `school geometry´, where compass and straight edge were used to construct images of the pictorial forms we have in mind while retaining in the back of our heads the conviction that the results put to paper are only rough representations of the idealised, to the development of axiomatic systems related to the aforesaid.

This necessary development leaves far behind the `earth measurement´ with its carefully constructed geometrical figures and offers the interested person the opportunity to climb the ladder to the realm of pure reasoning almost without reference to anything imaginable. The undefined elements of point, line, passes through (intersects) and lies in (is incident with) etc. are remnants of the original `earth measurement´.

While there is much to be said for the modern development of remaining in the abstract without any reference to anything imaginable, my interest lies in both directions: the richness of the multitude of imaginable and symbolically constructible expressions of the abstract concept as well as the possibility of relating and integrating each of these back into their conceptual origin. I have therefore attempted to provide for this aspect by including a number of possible, symbolic images of these all inclusive unimaginable concepts.

During the course of my axiomatic development of this fascinating geometry, I therefore present constructions giving a wider perspective of what will later be developed out of the axiomatic development of the subject and which can be later incorporated into the end result.

My particular interest lies in von Staudt's interpretation of an elliptic involution as an expression of an imaginary point or line and the further development of this by Locher-Ernst enabling us to depict imaginary points and also imaginary lines (as opposed to the algebraic solution of an equation resulting in complex conjugated points and lines)[1] in a plane which can then easily be extended to include imaginary points, lines and planes in three dimensions. A consequence of this is that we can attain to a *geometry of movement* of points and lines in the plane.

I follow very closely H. M. S. Coxeter's development of the subject as presented in his book `The Real Projective Plane´ up to the section on conics as this is widely considered to be the best presentation of the matter. At this point though the thesis takes a turn and starts concentrating on a lead in to geometrical constructions of imaginary points and lines.

The exact wording of the definitions and theorems is almost exclusively my own and all geometrical constructions were done by me.

At this point I would like to express my deeply felt gratitude to <u>Prof. Dr. Gerhard Kowol</u> of the University of Vienna's mathematics department for giving me the opportunity to write this thesis.

---

[1] In the text the words complex and imaginary are used as synonyms.

# Introduction

**Basic Elements and Relations**

The *basic elements* in plane projective geometry can be taken to be the *point* and *line*, both lying in the same plane and both imaginable and roughly able to be represented on paper as a depiction of what we are referring to. They are widened to include the points and line at infinity and the joining lines and intersection points of these with any other line or point or line on the plane. The line at infinity is no longer imaginable but the projection of it onto our paper and the possibility of creating a depiction of resulting relations is. The infinite line and points on it are therefore nothing exceptional in projective geometry and are included in any definition or theorem.

The *basic relations* between lines and points are those of *incidence* (one or more points lying in a line or lines passing through a point) and *separation* (two or more pairs of points dividing a line into two or more parts or the plane into two or more areas) will be considered more closely when we look at sense of movement and continuity.

The goal of an axiomatic system is of course to have as few undefined elements and relations as possible (As Aristotle said, one shouldn't increase the number of principles unnecessarily) and the development of the topic is therefore based on incidence and separation, although metrical properties will occasionally be mentioned.

**The Principle of Duality**

A point can be imagined as either an entity in itself or alternatively as inherently constituted of a *pencil of lines* (Fig. 0.1 top left seen in two dimensions) or a *sheaf* of at least three lines or a *bundle* of at least three planes (Fig. 0.1 top left seen in three dimensions, top right) and thus encompassing the whole plane or respectively, space.

A line can be imagined as an entity in itself or alternatively as inherently constituted of a *range of points* (Fig. 0.1 middle left; one dimensional) or a *pencil of planes* (middle right).

A plane can be thought of as an entity in itself or alternatively as constituted of a *field of points* or lines (Fig. 0.1 left and right).

In three dimensions, the *principle of duality* states that we can interchange the words point with plane;



**Fig. 0.1**

(line remains line) *lie in* with *pass through* and *join* with *intersect* in any theorem and the theorem remains valid. We have to be careful that when we say that a plane passes through a line we *also* mean that the line lies in the plane. A few easily imaginable examples are the following:

| | |
|---|---|
| The join of two distinct points is a line. | The intersect of two distinct planes is a line |
| Three distinct points not lying in a line lie in a plane. | Three distinct planes not passing through a line pass through a point. |
| The join of a line and a point not lying in that line is a plane. | The intersect of a line and a plane not passing through that line is a point. |

The second statement right is an axiomatic requirement.

Pursued further, the principle of duality leads to Polar Euclidean geometry, where for example a cube with its 8 points, 12 lines and 6 planes and an octahedron with its 6 points (corners), 12 lines (edges) and 8 planes (surfaces) are *polar* to each other (see Fig. 0.2) and depict a *polarity*. We could imagine the octahedron to be `solid´ and inherently composed of points as entities in themselves and the cube to be inherently composed of planes as entities in themselves while setting up a relation whereby every point of the octahedron corresponds to a plane of the cube and a field of points of the planes of the octahedron correspond to the intersection of planes of the cube while
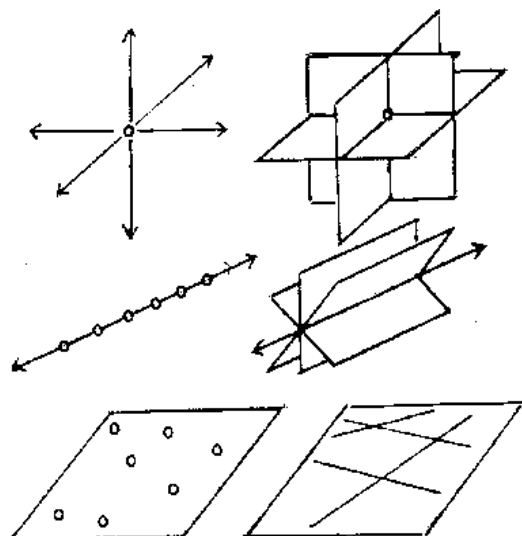
lines correspond to lines.

Applying our principle of duality, we can say that a point moving from *S* to *W* then corresponds to a plane turning in the line corresponding to *SW*, in line *DA*. Point *A*, where three lines of the cube intersect in a point, corresponds to plane *SWT* of the octahedron where the three corresponding lines lie in a plane and *D* to the plane *SWV* and *DA*, the join of the two points, corresponds to *SW*, the intersect of the corresponding planes.  Since the four lines *SW*, *WU*, *UX* and *XS* all lie in the same plane, their corresponding lines *DA*, *CB*, *GF* and *HE* must then all go through a point; the infinitely distant point( in both directions)!
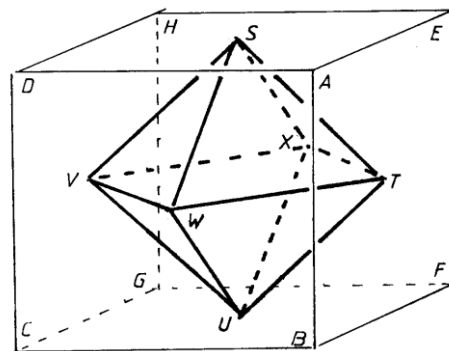


**Fig. 0.2**

We can call this the *infinite point* in which the parallel lines meet; provided we don't want to destroy the universality of the duality principle by insisting on <u>Euclid's 5<sup>th</sup> axiom</u>; that parallel lines don't intersect. Furthermore, the three planes *SWUX*, *SVUT* and *VWTX* all intersect in a point at the centre of the octahedron.  The corresponding points must then all lie in a plane containing the intersection points of the sets of lines corresponding to those lying in the aforementioned planes. We call this plane the *infinite plane*. It is of course completely flat and not a sphere.

**Notation: Joining Line and Intersection Point**

The joining line (join) of two points, *A* and *B*, is denoted by *AB* . The intersection point of line *AB* and line *CD*  is denoted by  (*AB*)·(*CD*).

The intersection point (intersect) of two lines, *a* and *b* is denoted by  *a·b*. The joining line of  *a·b* and *c·d*  is  denoted by  (*a·b*)(*c·d*).

Thus the two processes of joining and intersecting somewhat resemble the processes of addition and multiplication in algebra and are normally denoted by the same symbolism.

As customary, I use *upper case italic* letters for points*, lower case italic* for lines and Greek letters for planes.

**The General Case of Relating Two Triangles or Trilaterals to Each Other**



**Fig. 0.3a**



**Fig. 0.3b**

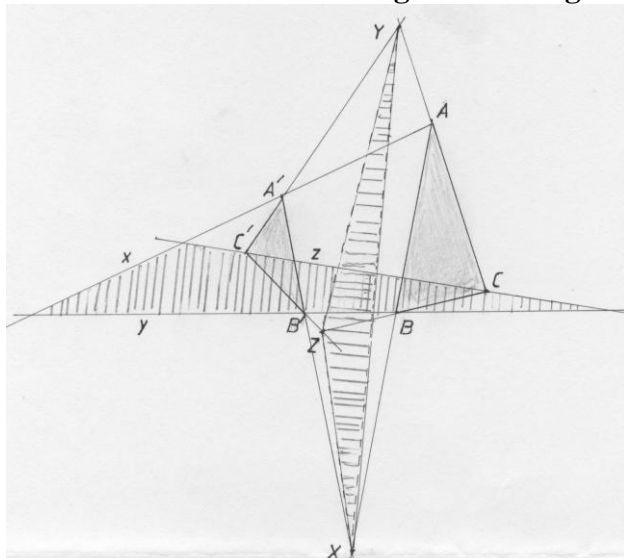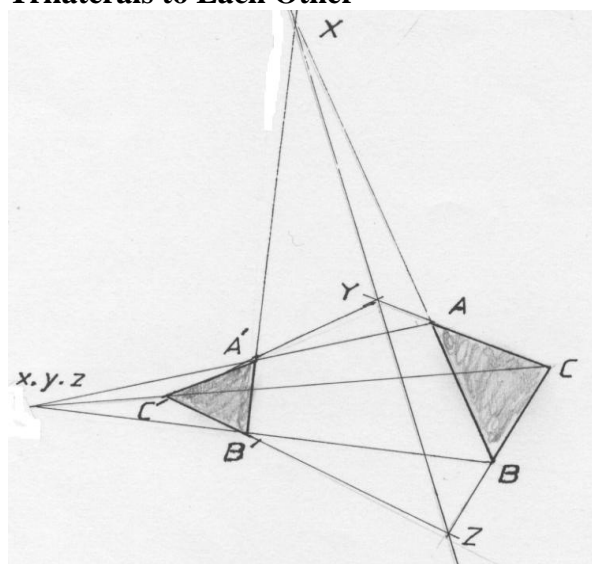If we imagine any two triangles in a plane and relate them to each other by assigning the vertices (points) and sides (lines) of the first triangle to those of the second triangle, we find that, with one exception, we always create a secondary trilateral Δ*xyz* and triangle Δ*XYZ* (see Fig. 0.3a). If however the secondary trilateral is concentrated to a point then the secondary triangle collapses to a

4

line and visa verse. In either case the other secondary form also always collapses. An example is shown in Fig. 0.3a and 0.3b, below. This is a beautiful example of a fundamental phenomenon describing what happens when we relate two simple planar forms to each other.

## Notation: Projectivity

$X \barwedge x$ is the notational form for the correspondence of a variable point $X$ of the *range of points* on the line $o$ to a variable line $x$ of a *pencil of lines* in $O$, where $A$, $B$, $C$ . . . are particular positions of $X$ relate to particular lines $a$, $b$, $c$ . . . of $O$, whereby $O$ does not lie in $o$.

We make a subtle distinction here: $X \barwedge x$ transforms the points on a line into the lines in a point, but $x \barwedge X$ , its inverse, transforms the lines (for example in $O$) into the points (for example in $o$). There can of course be a whole sequence of transformations, but the important aspect here is that a range is transformed onto a pencil and visa versa. We say that the points and lines are projectively related.

## Def. 0.3 Projectivity as a Product of Transformations

| A range of points in $o$ can be transformed into a pencil of lines in $O^n$ using the multiple *product* of transformations of the previously described relation. (see Fig. 0.4) | A pencil of lines in $O^n$ can be transformed into a range of points in $o$ using the multiple *product* of transformations of the previously described relation. |
|---|---|
| We write $X \barwedge x$ to describe the whole series of steps. | We write $x \barwedge X$ to describe the whole series of steps. |
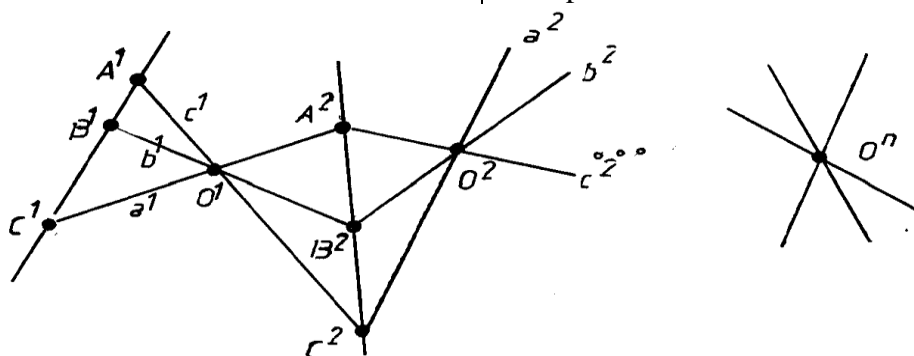


**Fig. 0.4**

For the dual statement, Fig. 0.4 just has to be read from right to left.

As one correspondence relation doesn't change anything but only sets up the correspondence between points and lines, we'll use the word projectivity for cases where the number of transformations is $\geq 3$. As previously stated, there are two basic initial actions that can be carried out: that of *taking a section* of a pencil by intersecting the pencil with a line not passing through the *carrier point* of the pencil and thereby creating a range of points (Fig. 0.4 starting with $O^n$ and reading right to left), and secondly, that of *making a radiation* of a range of points by joining the range of points with a point not lying in the *carrier line* of the range (Fig. 0.4 starting with $o$ and read left to right).

## Definition 0.4: Perspectivity

An important projectivity, with which we often have to work with, is *perspectivity*. This projectivity of $X \barwedge x \barwedge X'$ or alternatively $x \barwedge X \barwedge x'$ , that is; a product of exactly two perspectivities resulting in projecting a range onto a range or a pencil onto a pencil is central to all of the proofs that will follow.

## Notation: Perspectivity

We use the symbol $\overline{\barwedge}$ to denote the direct transformation of a range onto a range or pencil onto a pencil.

**Definition 0.5: Perspectivity Centre and Axis, Section, Radiation**

We call two ranges *perspective from a point O, the perspectivity centre,* when they are sections of the *same* pencil of lines in *O;* thus of *two* correspondence relations. The point *A* in $o^1$ transforms into the line *a* in *O* and further into *A´* in $o^2$.

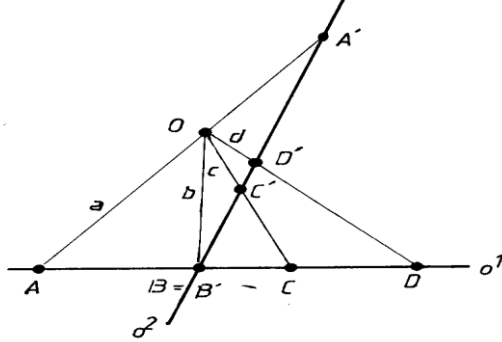We write *ABCD* $\overset{O}{\overline{\wedge}}$ *A´B´C´D´* (see Fig 0.5a).

We call two pencils *perspective from a line o, the perspectivity axis,* when they are radiations of the *same* range of points in *o;* thus of *two* correspondence relations The line *a* in $O^1$ transforms into the point *A* in $o^1$ and further into *a´* in $O^2$.

We write *abcd* $\overset{o}{\overline{\wedge}}$ *a´b´c´d´* (see Fig 0.5b).
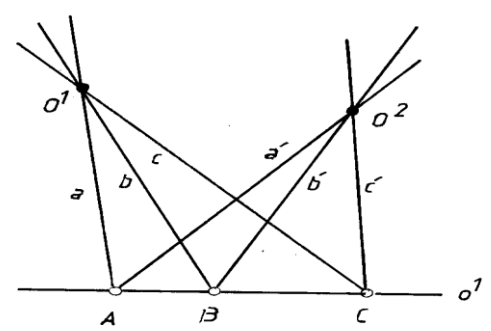
**Fig. 0.5a**

**Fig. 0.5**

Said differently, pairs of related points lie in a line passing through a common point of the lines and pairs of related lines pass through a point lying in a common line of the points.

**The Theorem of Desargues'and its Dual[2]**

Because of its importance for the whole of the following, we include a proof of Desargues' theorem here. It will appear in its two dimensional form again in the axioms. It turns out to be easier to prove the dual first and then, using the results, prove Desargues' theorem.

**Fig. 0.6**

In either case there are two configurations to be considered here: the three-dimensional and the two-dimensional configuration. I prove consecutively the *three*-dimensional and two-dimensional cases of the dual before proving Desargues' theorem itself.

[2] See Coxeter [5], p. 19.

*The dual to Desargues theorem.*
*If two triangles are perspective from a line then they are perspective from a point.*
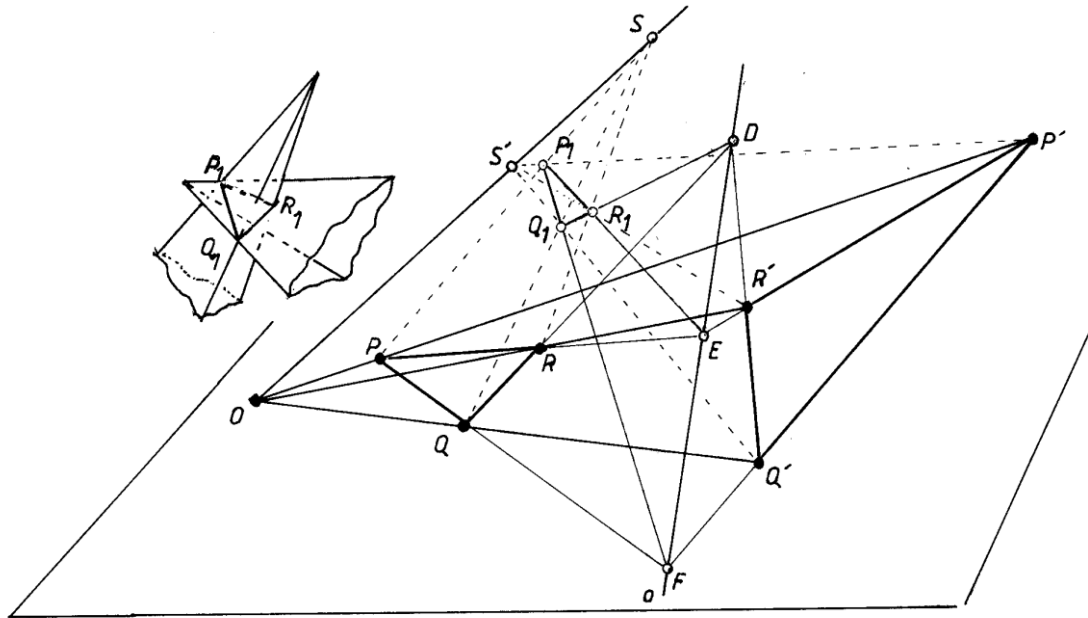
Proof: Triangles in different planes.
Let the two triangles, $\Delta PQR$ and $\Delta P_1Q_1R_1$ *lie in different planes*. (see fig 0.6)
Given that $F = PQ \cdot P_1Q_1$, $E = PR \cdot P_1R_1$ and $D = QR \cdot Q_1R_1$ lie in a line *o* which is the intersection line of the two planes in which the two triangles lie, we have to show that $PP_1$, $QQ_1$ and $RR_1$ all go through the same point. As $P$, $Q$, $P_1$, $Q_1$ all lie in the same plane $FPP_1$ (the two lines $FP$ and $FP_1$ define a plane) then we can state that $PP_1$ and $QQ_1$ intersect in a point, for example $S$ (as any two lines meet in a point). Similarly, $QQ_1$ and $RR_1$ meet in some point $T$ and $RR_1$ and $PP_1$ in a point $U$. If the planes ($\Delta PQS$, $\Delta QRT$ and $\Delta RPU$) are distinct, then the three intersection points of the three lines $PP_1$, $QQ_1$ and $RR_1$ must be concurrent as otherwise $R_1$ for example, would not lie in both $EP_1$ and $Q_1D$ nor $Q_1$ in $FP_1$ and $RD_1$ nor $P_1$ in $ER_1$ and $Q_1F$. The perspectivity from line *o* (see def. 0.5 perspectivity centre and axis) forces S, T and U to be concurrent. Thus the two triangles $PQR$ and $P_1Q_1R_1$ are perspective from point $S$.

Proof: Triangles in the same plane
Let $\Delta PQR$ and $\Delta P'Q'R'$ *lie in the same plane*. (see fig 0.6)
Three non-concurrent lines in another plane (here plane $P_1FD$) through *o* are constructed through $D$, $E$ and $F$ respectively so as to form $\Delta P_1Q_1R_1$ with $P_1Q_1$ through $F$, $P_1R_1$ through $E$ and $Q_1R_1$ through $D$. Applying the above, $\Delta P_1Q_1R_1$ lies in the same triangular pyramid as $\Delta PQR$ and including $S$ and therefore $PP_1$, $QQ_1$ and $RR_1$ all pass through one point, $S$. Similarly, the three lines $P'P_1$, $Q'Q_1$ and $R'R_1$ all pass through another point $S'$ ($S'$ projects $P_1Q_1R_1$ onto $P'Q'R'$ and is unique as stated in the two different planes part of the proof). $S$ and $S'$ are not concurrent, otherwise $P_1$ for example, would lie in PP' and therefore in the plane of $\Delta PQR$ and also of $\Delta P'Q'R'$. As $P_1$ lies in both $PS$ and $P'S'$ then $SS'$ intersects $PP'$ in $O$ in plane $SOP$ ($P'$, $O$ and $S'$define a plane). The same approach to $Q_1$ and $R_1$ requires $SS'$ to also intersect $QQ'$ in the same point $O$, the same applying to $RR'$ and $PP'$. As line $SS'$ does not lie in the plane of $PP'$, $QQ'$ and $RR'$, but intersects all three point pairs individually, then all three must meet in a point through which $SS'$ passes – point $O$.

We are now in a position to prove the converse; Desargues well-known theorem.

*Desargues' Theorem:*
*If two triangles are perspective from a point, they are perspective from a line.*

Proof: let the two triangles $\Delta PQR$ and $\Delta P'Q'R'$ (either coplanar or non-coplanar) be perspective from a point *O*.
Corresponding sides intersect in say $D$, $E$ and $F$. (As $QQ'$ intersects $RR'$, then $PQ$ intersects $Q'R'$). We have to prove that $D$, $E$ and $F$ are collinear. When considering the two triangles $\Delta QQ'F$ and $\Delta RR'E$, we see that, as their corresponding sides intersect in three collinear points $O$, $P$ and $P'$, these triangles are perspective from a line and therefore, as proved above, from a point – the point $QR \cdot Q'R' = D$. Hence the three points $D$, $E$ and $F$ lie in a line.

**Four points in the plane**
If we decided to look at the properties of the square and how the sides and vertices relate to each other, then we'd find that opposite sides and angles are equal, diagonals bisect each other, there are for axes of symmetry etc. etc. This is hardly surprising as we would already have imparted these special features to the figure by making it a square. Suppose though, that we impart nothing to our figure other than stating that it has 4 points $P$, $Q$, $R$ and S (see Fig. 0.7) no three of which are collinear and all lying in the same plane. We might well expect nothing of significance to arise out of investigation of the relationships between lines and points.

There is an astonishing ordering of lines and points in the plane that arises out of the implications of this simple action. The intersection of the 6 joining lines with a line $AB$, defined by the intersection points of $A = RQ \cdot PS$ and $B = QS \cdot RP$ immediately gives us two more points; $D = (RS) \cdot (AB)$ and $C = (QP) \cdot (AB)$. If we then consider the relation between the internal ratio of the distances $AD:DB$ with respect to $A$ and $B$ and the external ratio $AC:CB$ (again with respect to $A$ and $B$, we find that this is always equal to exactly negative 1 (negative as $CB$ is in the other direction).
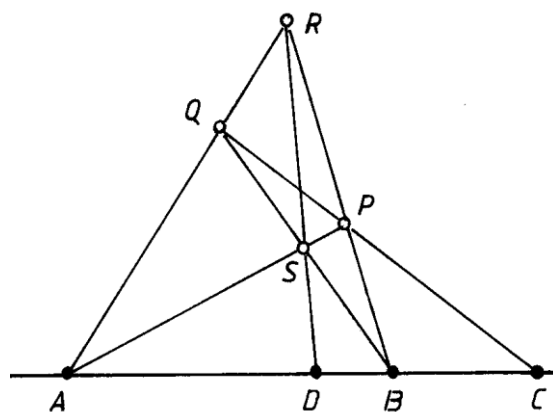


**Fig. 0.6**

This ordering (except for the ordering and directionalising of the segments) obviously wasn't put there by us artificially but instead must be deeply embedded in the qualities of space itself. Of course one could always say that our constructions are an algebraic structure and that this is therefore not surprising. But this only transfers the question of how it comes about, that what we have thought out (our algebraic structure), expresses itself in this harmonious and beautiful way. May the astonishment and feeling that we have touched on something more deep never leave us. The four randomly chosen points or lines immediately imply their six joining lines or points and the accompanying diagonal triangle/lateral (see chapter 1) and mark the start of our investigation into what can be developed by taking sections of pencils or radiations of ranges and considering their interrelationships.

**The Imaginary**

If we intersect an ellipse, for example $4x^2 + 9y^2 = 36$ with a line, for example $y = {}^5/_6\, x - 5$ we arrive at the result $x_1 = 3.66 + 2.25i$ and $x_2 = 3.66 - 2.25i$.

By treating the real and imaginary parts as coordinates of a vector and set these results into the equation of the line, we obtain the results: $y_1 = -1.95 + 1.87i$ and $y_2 = -1.95 - 1.87i$.

If we construct points $P(3.66 - 2.25, -1.95 - 1.87)$ and $Q(3.66 + 2.25, -1.95 + 1.87)$ and then construct or calculate the polars of each of these points,



**Fig. 0.7**

the polar $p$ of $P$ being the join of the two intersection points of the two tangents from $P$ to the conic and $q$, the polar of $Q$ likewise, we find that the polar of $P$ passes through $Q$ and the polar of $Q$ passes through $P$.

(see Fig. 0.7) As we find the same is true for any line not intersecting the ellipse, we can confidently conclude that the poles and polars related in this way have something to do with complex numbers. The aim of the following is to give the background into the geometry behind von Staudt's ingenious interpretation of these results and thereby attain to a useful method of representing complex points and lines in an enlightening, imaginable and constructible manner.

This thesis has been written in order to elucidate exactly the aforesaid.
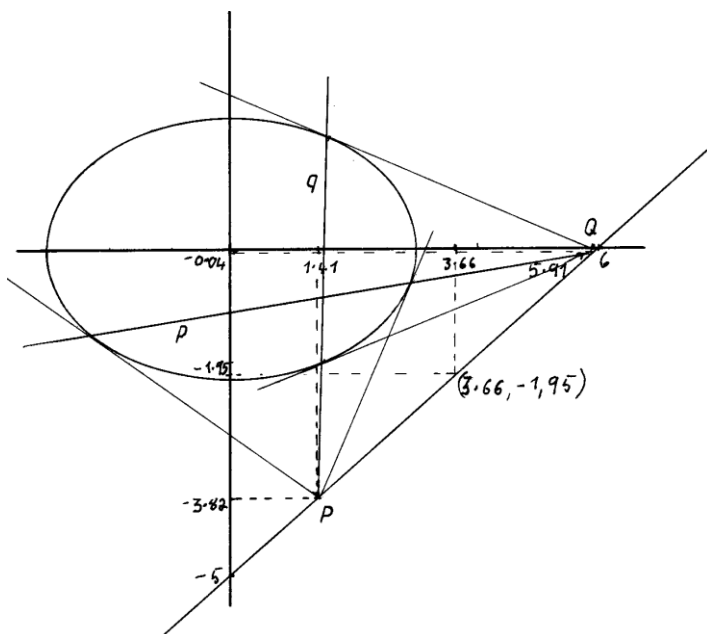
8

# Chapter 1          Incidence

## The Principle of Duality in Two Dimensions

The following development will be that of the concept of the projective plane: that is in two dimensions. Every definition remains valid and every theorem remains valid when we interchange the words:

<div align="center">

*point and line,*
*join and intersection*

</div>

From the aforesaid, we can see that we have chosen the duality principle to be generally valid and therefore reject for example Euclid's 5th axiom. This follows directly from the dual of axiom 1. In the following I use Poncelet's system of parallel columns to emphasize the Duality principle which will be extensively demonstrated in two dimensions in the first part of this thesis. First though, we need to introduce our axioms for plane geometry.

## Axioms of Incidence[3]

**1.1a**      **Any two points are incident with at least one line.**
**1.1b**      **Any two lines are incident with one point.**
**1.2**      **Two distinct points cannot both be incident with two distinct lines at the same time.**
**1.3**      **There are at least two points and two lines such that each of the points is incident with just one of the lines.**
**1.4**      **There are at least two points and two lines with the points not incident with the lines such that the join of the points is incident with the intersection of the lines.**
**1.5**      **If four points $O, P, Q, R$ with their six distinct joins, and four lines $o, p, q, r$ with their six distinct intersections, are situated so that the five joins $OP, OQ, OR, PR, RQ$ are incident with the respective intersections $q{\cdot}r, r{\cdot}p, p{\cdot}q, q{\cdot}o, o{\cdot}p$, then the sixth join $PQ$ is incident with the sixth intersection $o{\cdot}r$.**

Checking axioms 1.1 to 1.4 against Fig. 1.1a and b, we see that the constructions are a geometrical representation of the first four axioms. Axiom 1.5 is of course just a restatement of Desargue's theorem in two dimensions. This leads directly to the following useful definition: that of the complete quadrangle and complete quadrilateral. These will prove central to an understanding of all that follows.

### Def. 1.1  Complete Quadrangle, Quadrilateral

| | |
|---|---|
| A complete plane quadrangle *PQRS* is made up of four points (the vertices) of which no three are collinear, and six joining lines *QR*, *PS, RP, QS, PQ* and *RS*. | A complete plane quadrilateral *pqrs* is made up of four lines of which no three are concurrent., and six intersection points *q·r, p·s, r·p, q·s, p·q* and *r·s*. |



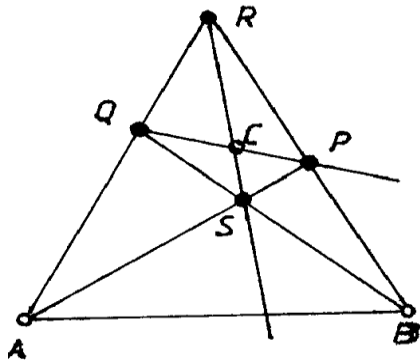<div align="center">
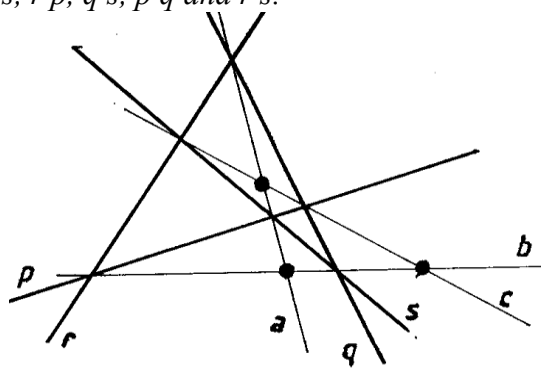
**Fig. 1.1a**                  **Fig. 1.1b**

</div>

---

[3] Axioms 1.1 to 1.4 from Karl Menger. For axiom 1.5. see Veblen and Young [11], p. 53.

The intersection points of opposite ´ joining lines namely:

$$A = QR·PS$$
$$B = RP·QS$$
$$C = PQ·RS$$

are called *diagonal points* and are the vertices of the *diagonal triangle* (see Fig. 1.1a).

The joining lines of opposite intersection points namely:

$$a = (q·r)(p·s)$$
$$b = (r·p)(q·s)$$
$$c = (p·q)(r·s)$$

are called *diagonal lines* and are the sides of the *diagonal trilateral* (see Fig. 1.1b).

We will now have to look more closely at the definition of a complete quadrangle and quadrilateral with its accompanying diagonal triangle and use this figure so that we can define the projectively invariant harmonic ratio in terms of incidence and not as is normally the case, in terms of distances. This will prove to be fundamental to our whole development of the matter.

### Def. 1.2 Harmonic Conjugates

Four collinear points *A, B, C, D* are said to form a *harmonic set* if there is a quadrangle of which two opposite sides pass through *A* and the remaining two through *B* while the remaining sides pass through *C* and *D* respectively. We say that *C* and *D* are *harmonic conjugates* of each other with respect to *A* and *B* and write H(*AB,CD*).
 (see Fig. 1.2a)

Four concurrent lines *a*, *b*, *c*, *d* are said to form a *harmonic set* if there is a quadrilateral of which two opposite vertices lie in *a* and the remaining two in *b* while the remaining vertices lie in *c* and *d* respectively. We say that *c* and *d* are *harmonic conjugates* of each other with respect to *a* and *b* and write H(*ab,cd*).
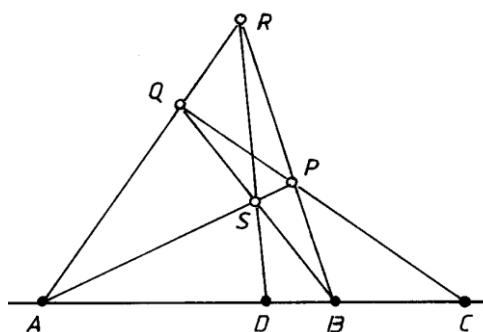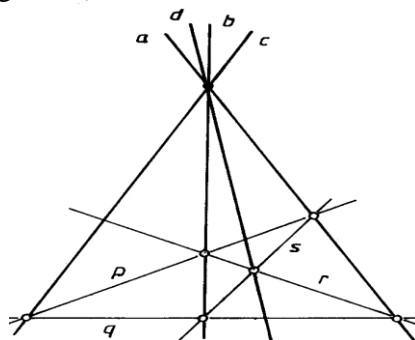(see Fig. 1.2b)



**Fig. 1.2a**



**Fig. 1.2b**

To construct *D* given *A,B,C* we draw any triangle *PQR* whose sides *QR*, *RP*, *PQ*, go through *A*, *B*, *C,* respectively. This determines a quadrangle *PQRS*, where

$$S = AP.BQ,$$

as in the following figure. We thus obtain

$$D = RS. AB$$

To construct *d* given *a* , *b* , *c* we draw any trilateral *p q r* whose points *q.r, r.p, p.q,* lie in *a, b, c,* respectively. This determines a quadrilateral *pqrs*, where

$$s = (a.p)(b.q),$$

as in the following figure. We thus obtain

$$d = (r.s)(a.b)$$

These definitions will prove fundamental to what follows. We have thus defined the harmonic ratio, not in metrical terms but in terms of incidence! This will greatly simplify the whole development of the subject. The harmonic ratio, that of the internal to the external ratio (or in the case of the pencil, that of the sines of the respective angles) of course holds in all cases.

We can set up a correspondence between a range of points on a line and a pencil of lines in a point: a pencil of lines is intersected by a range of points with corresponding elements incident. We initially exclude the case of a pencil of lines *O* lying in a range of points *o,* as all points in *o* would then be transformed into the point *O*. This will later be significant when we look at a parabolic involution where exactly this happens.
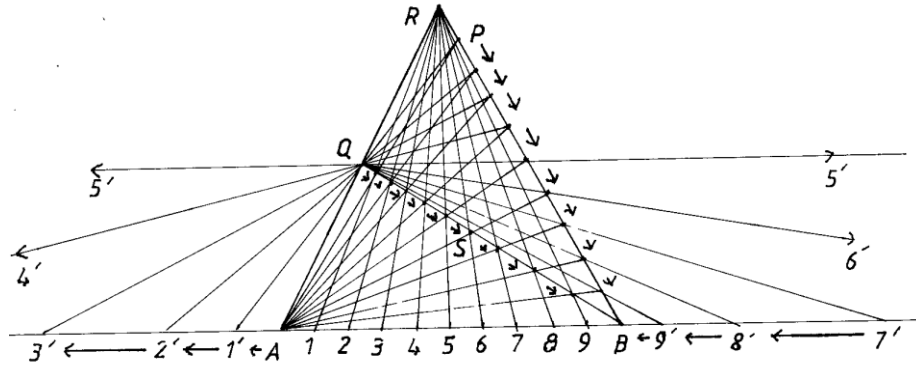
**Fig. 1.3a**

It's important to get a feeling for the movement of the pairs of conjugates, 1, 1´ and 2, 2´ etc. It's one of the two fundamental movements involved in an understanding of the geometrical representation of complex points and lines presented in the last chapter (see Fig.1.3a and b). As a point moves steadily away to 1, 2, 3 etc. its conjugate starts off slowly moving to the left and, rapidly accelerating, flashes through the infinite point and comes back from the right, decelerating (as shown by the arrows) and crawls in to meet its conjugate in point $B$. The same relative movement is shown in Fig.1.3b for lines turning in a point.
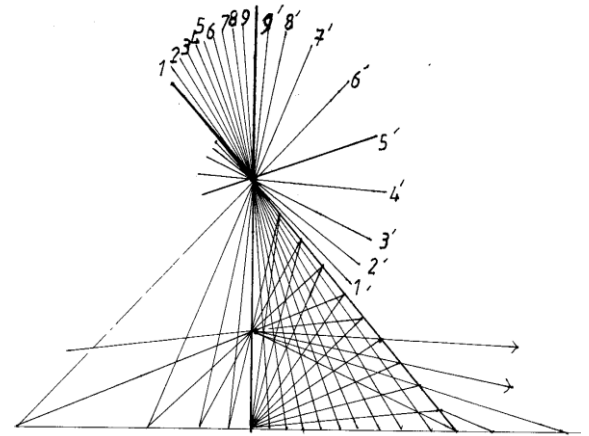


**Fig. 1.3b**

**Theorem 1.1 Complete Quadrangle/Quadrilateral Construction given a Diagonal Triangle[4]**

| | |
|---|---|
| *If $\Delta ABC$ is the diagonal triangle of a quadrangle PQRS, then the three points* $A_1 = BC{\cdot}QR$ , $B_1 = CA{\cdot}RP$, $C_1 = AB{\cdot}PQ$ *are collinear.* | *If $\Delta abc$ is the diagonal trilateral of a quadrilateral pqrs, then the three lines* $a_1 = (b{\cdot}c)(q{\cdot}r)$, $b_1 = (c{\cdot}a)(r{\cdot}p)$, $c_1 = (a{\cdot}b)(p{\cdot}q)$ *are concurrent.* |
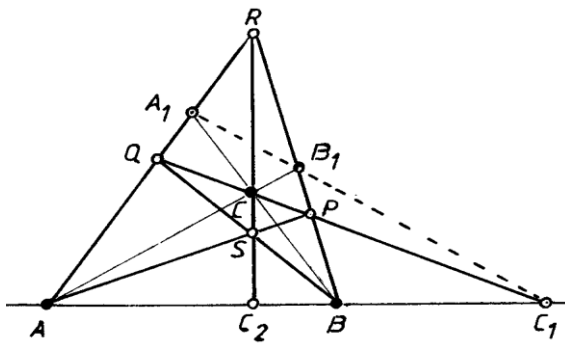


**Fig. 1.4a**



**Fig. 1.4b**

Proof:
If we apply Desargues' theorem to $\Delta ABC$ and $\Delta PQR$ (see Fig. 1.4a) then the three joining lines of corresponding points, $AP$, $BQ$ and $CR$ pass through the perspectivity centre S and therefore the three intersection points $A_1, B_1$ and $C_1$ of corresponding lines lie in the perspectivity axis which is a line.

Proof:
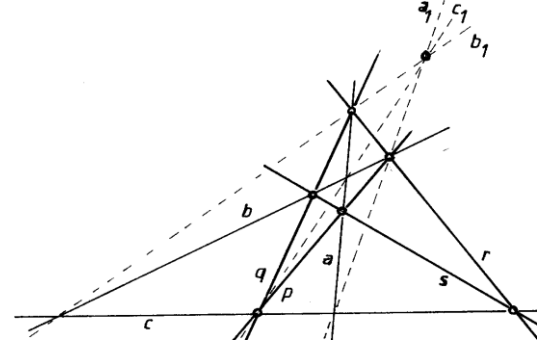If we apply the dual of Desargues' theorem to $\Delta abc$ and $\Delta pqr$ (see Fig. 1.4b) then the three intersection points of corresponding lines, $a{\cdot}p$, $b{\cdot}q$ and $c{\cdot}r$ lie in the Perspectivity axis $s$ and therefore the three joining lines $a_1, b_1$ and $c_1$ of corresponding points go through the perspectivity centre which is a point.

---

[4] See Coxeter [6], p. 19.

**Corollary**

If we had only been given the diagonal triangle and one vertex of the quadrangle, we could construct the remaining three vertices using incidences. Given for example $\Delta ABC$ and $P$, we can construct $B_1=CA{\cdot}BP$, $C_1=AB{\cdot}CP$, $A_1=BC{\cdot}B_1C_1$, $R=BP{\cdot}AA_1$, $Q=CP{\cdot}AA_1$ and $S=AP{\cdot}BQ$. The dual also applies of course.

By using different quadrilaterals, we can easily see that:

the sides of the diagonal triangle intersect the sides of the quadrilateral in harmonic conjugates. Using $PSBC$ we have on side $QR$; $H(QR,AA_1)$. Using $AQCB$ we have on side $PS$; $H(PS,EF)$. Using $ACQS$ we have on side $RP$; $H(RP,BB_1)$ etc. The complete quadrangle and quadrilateral are apparently astonishingly harmonious figures.

From the above definition of harmonic conjugates, it remains to be shown that the fourth point or line is unique; that is, independent of the construction. For this we need the theorem of Desargues where two triangles were `projected down´ from three to two dimensions.

## Theorem 1.2 Independence of Construction of Harmonic Conjugates[5]

*The harmonic conjugate of $C$ with respect to A and B is independent of our choice of triangle PQR used to construct D.*

Proof:

Let's assume that another such triangle $P´Q´R´$, giving us, together with $A$ and $B$ a Quadrangle $P´Q´R´S´$ (see Fig. 1.5). We have to show that $RS$ and $R´S´$ both determine the same point $O$. We can consider three pairs of triangles:

a.) $PQR$ and $P´Q´R´$: their corresponding sides meet in $A$, $B$, $C$ and hence, according to Desargues' Dual, $RR´$ must pass through the Perspectivity centre $O=PP´QQ´$



**Fig. 1.5**

b.) $PQS$ and $P´Q´S´$: by applying the same theorem we see that $SS´$ also passes through $O$

c.) $RSP$ and $R´S´P´$: applying Desargues' dual once again, leads to $PP´$ passing through $O$ as their sides meet in $A$, $B$ and $D$. Applying Desargues theorem directly, as the joining lines of corresponding points in $c$.) are perspective from $O$, the intersection points of corresponding sides meet in the collinear points; that is $RS$ and $R´S´$ meet in $D$ on $A$.
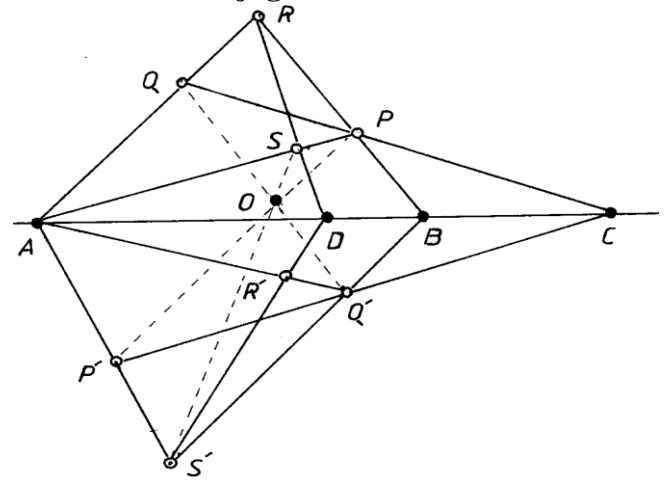
The following two important theorems demonstrate the usage of the perspectivity symbol when using multiple perspectivities in a projectivity. They're also important for grouping together eight of the 24 permutations of four different items.

## Theorem 1.3: Interchanging Pairs[6]

*Using a sequence of three perspectivities we can interchange pairs of points among any four collinear points.*

Proof:

If we want to interchange $A$ with $A´$ and $B$ with $B´$ (see Fig. 1.6a) we can always draw any triangle $UTR$ whose sides $RU$, $RT$ and $UT$ pass through $A$, $B$ and $B´$.

*Using a sequence of three perspectivities we can interchange pairs of lines among any four concurrent lines.*

Proof:

If we want to interchange $a$ with $a´$ and $b$ with $b´$ (see Fig. 1.6b) we can always draw any trilateral $utr$ whose points $r{\cdot}u$, $r{\cdot}t$ and $u{\cdot}t$ lie in $a$, $b$ and $b´$.

---

[5] See Coxeter [6], p. 19.
[6] See Coxeter [6], p. 22.

This determines two points: $S = A'R \cdot UT$ and $V = AS \cdot RB$. Applying the product of perspectivities.

$$AA'BB' \overset{R}{\barwedge} USTB' \overset{A}{\barwedge} RVTB \overset{S}{\barwedge} A'AB'B$$
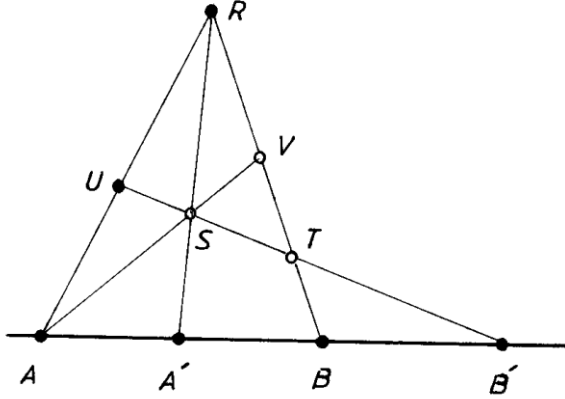
we obtain the desired result.

This determines two lines: $s = (a' \cdot r)(u \cdot t)$ and $v = (a \cdot s)(r \cdot b)$. Applying the product of perspectivities.

$$aa'bb' \overset{r}{\barwedge} ustb' \overset{a}{\barwedge} rvtb \overset{s}{\barwedge} a'ab'b$$
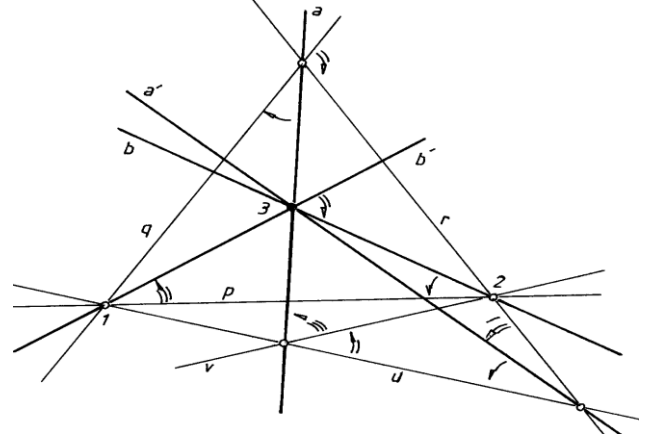
we obtain the desired result.



**Fig. 1.6a**



**Fig. 1.6b**

The following is a precursor to the fundamental theorem of projective geometry and is confined to any two related pairs of three points.

**Theorem 1.4: Three Perspectivities, Three Pairs of Elements on two Ranges or in two Pencils[7]**

*Using a sequence of not more than three perspectivities, we can relate any three distinct collinear points to any other three distinct collinear points.*

*Using a sequence of not more than three perspectivities, we can relate any three distinct concurrent lines to any other three distinct concurrent lines.*

Proof:
The points lie on distinct lines:
we construct a line $A'B$. (see Fig. 1.7a) This gives us an intermediary Perspectivity axis $q$. The points $(A'A) \cdot (C'C)$ and $(B'B) \cdot (CC')$ give us the perspectivity centres $R$ and $S$. Applying the product of perspectivities:

$$ABC \overset{R}{\barwedge} A'BC_0 \overset{S}{\barwedge} A'B'C'.$$

Proof:
The lines go through distinct points:
we construct a point $a' \cdot b$. (see Fig. 1.7b) This gives us an intermediary Perspectivity centre $Q$. The lines $(a \cdot a)(c' \cdot c)$ and $(b' \cdot b)(c \cdot c')$ give us the Perspectivity axes $r$ and $s$. Applying the product of perspectivities:

$$abc \overset{r}{\barwedge} a'bc_0 \overset{s}{\barwedge} a'b'c'.$$
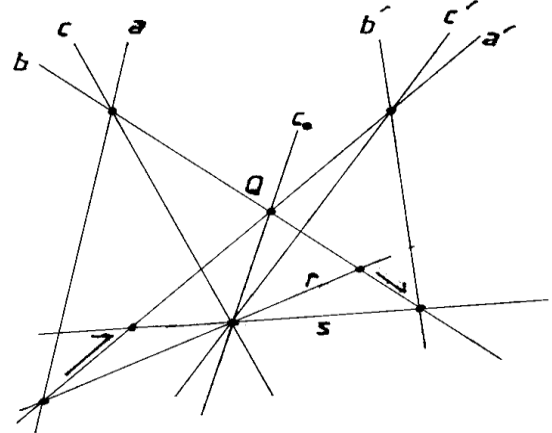


**Fig. 1.7a**



**Fig. 1.7b**

---

[7] See von Staudt [12] p. 59. The dual is original.

13

The points lie on the same line or one of the points is an intersection point of the two ranges: we just have to make a radiation of $A$, $B$, $C$ onto another line (by joining $A$, $B$, $C$ to an arbitrary point and taking a section of the lines) and then use the same construction as described above. If the intersection point is a common point of the two ranges, the two pairs of three points are of course perspective to each other.

The lines go through the same point or one of the lines is a joining line of the two pencils: we just have to take a section of $a$, $b$, $c$ in another point (by intersecting a, b, c in an arbitrary line and making a radiation of the points) and then use the same construction as described above. If the joining line is a common line of the two pencils, the two pairs of three lines are of course perspective to each other.

**Theorem 1.5:  A Harmonic Section of Lines Corresponds to a Harmonic Radiation of Points[8]**
*Any section of a harmonic set of lines in a point is a harmonic set of points in a line, and any radiation of a harmonic set of points in a line is a harmonic set of lines in a point.*
Proof:
Let P be a vertex of triangle *PQR* of our four point construction to determine *A, B, C, D* as in Fig 1.8.
The six points of the *quadrilateral ASBRQD* has two opposite vertices on $AS = a$, two opposite vertices on $BR = b$ , one vertex $Q$ on $c$, and one vertex $D$ on $d$.
Hence H($ab,cd$). From this we can conclude that perspectivities preserve the harmonic relation. If $ABCD \overline{\barwedge} A´B´C´D´$ and H($AB,CD$) then H($A´B´,C´D´$).
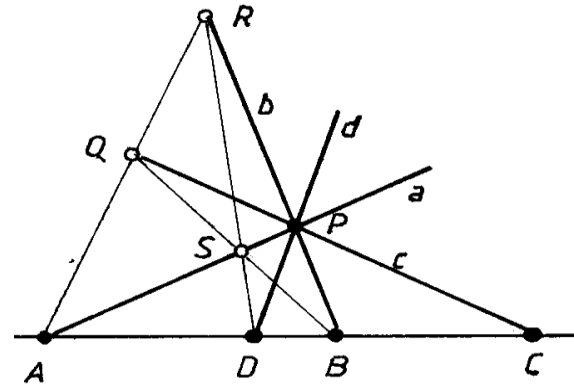

**Fig. 1.8**

Considering this together with our definition of harmonic conjugates and the fact that we can not only interchange A with B and C with D but also A with D and B with C (see theorem 1.3) we can conclude that the eight relations
H($AB,CD$);  H($BA,CD$), H($AB,DC$), H($BA,DC$), H($CD,AB$), H($CD,BA$), H($DC,AB$), and H($DC,BA$) are all equivalent.
We defined the harmonic relation on the basis of incidences and defined two pairs of points separating each other. The first pair separating the second is equivalent to the second pair separating the first. This, combined with the present non-metrical and non-directional state of the investigation, leads to the stated result.

After having introduced the axioms of order we will then be in a position of being able to define the two senses of movement along a line, which is necessary for creating a basis for von Staudt's interpretation of an elliptic involution as an expression of complex conjugates.

---

[8] See Coxeter [6], p. 23.

# Chapter 2     Separation, Axioms of Order, the Sense of Movement, Continuity

With the axioms of incidence and Desargues' theorem we've ascertained that intersection points are projected onto intersection points and joining lines onto joining lines. What we haven't yet considered is what happens to the ordering of the points within a line or the lines within a point From the consequences of axiom1.4 we have seen that a line is a straight but "closed" entity, with the point at both "ends" being the same point. We therefore cannot say that a point separates the line into two segments, as we cannot say that a line separates a plane into two areas, (A line is therefore a one-sided element!).

Although two points separate the projective line into two segments, we cannot say of three points that one lies between the other two, just as we cannot say of three lines that one lies between the other two because there are two possible positions for `between´, as the order of arrangement of points in a line or lines in a point is cyclic. We can say though, that given four points in a line or lines in a point, two *separate* the other two.

(The concept of *between* belongs to affine geometry where the fourth point is the point at infinity.) This cyclic arrangement then gives us a watertight correspondence between a range of points and a pencil of lines. The beautiful symmetry of this geometry starts to emerge!

**Definition 2.1: Notation Separation**

In Fig 2.1, *A* and *B separate C* and *D* as do *a* and *b*, *c* and *d*. We write *AB//CD,* respectively *ab//cd*. Put differently, it's not possible to move *C* along the line to *D* without passing through either *A* or *B*.

The following axioms of order are not quite enough to characterise the projective plane. A more complete characterisation will follow when we come to the concept of ordered correspondences and continuity.
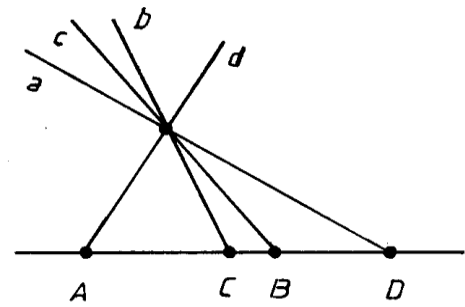


**Fig. 2.1**

## Axioms of Order [9]

**2.1  There exists a line containing four distinct points.**
**2.2  If *AB//CD*, then  *A,  B,  C,  D*  are four distinct collinear points.**
**2.3  If *AB//CD*   then *AB//DC* .**
**2.4  If  *A, B, C, D*  are four distinct collinear points then at least one of the relations :**
***BC//AD* ,  *CA//BD* ,  *AB//CD*  must hold.**
**2.5   If  *AB//CD* and *AC//BE*   then  *AB//DE*.**
**2.6.   If  *AB//CD* and *ABCD* $\overline{\wedge}$ *A´B´C´D´*  then   *A´B´//C´D´* .**

The first five axioms are one dimensional and the first four are illustrated in fig. 2.2, but axiom 2.6 is two dimensional. With the fifth axiom though, we have solved a problem. We've already shown that H(*AB,CD*) = H(*AB,DC*) and we therefore have to bring the following in line with this. The position of a fifth point E can lie either to the "right" of *A* (see fig. 2.2 left 1.) or to the "left" of *A* (see fig. 2.2 left 2.). In both cases it lies between *A* and *C* but axiom 2.3 doesn't distinguish between configurations 1. and 2. We therefore need 2.5 to clear this matter.

In considering the axioms together, we see that the concept of separation is again characterised by incidence. Either two pairs of points separate each other or they don't. The order of the points within the pair with respect to separation is irrelevant as, using theorem 1.3 and interchanging. Thus *AB//CD* is the same as *CD//AB*.

---

[9] See Enriques [7],  pp. 71-75.

By using axiom 2.6 together with Theorem 1.3 for interchanging pairs of points we can see that the following relations are equivalent:

*AB//CD*; *BA//CD*; *AB//DC*; *BA//DC*; *CD//AB*; *CD//BA*; *DC//AB*; *DC//BA*   (see Fig. 2.2, below left). The other two groups of equivalent possibilities for four points separating each other can easily be obtained by applying the aforesaid to the points as shown in Fig. 2.2 middle and right below.
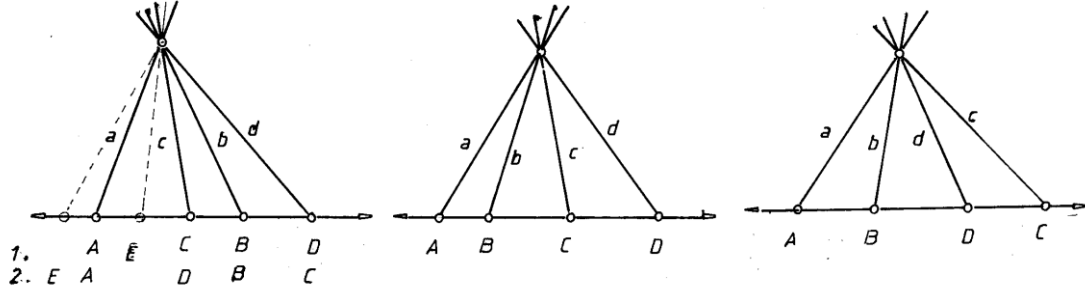


**Fig. 2.2**

From the aforesaid and Fig.2.2 above, we can see that the following theorem must hold.

**Theorem 2.1:  The Three Separation Relations for Four Elements**
*The three relations AB//CD*, *AC//BD*, *AD//BC*, *are mutually exclusive.*

If however, we have five elements then the following also holds

**Theorem 2.2:  The Three Separation Relations for Five Elements[10]**
*The three separation relations AB//DE*, *AC//DE*, *BC//DE*, *cannot all hold simultaneously*.
Proof: (by contradiction) (see Fig. 2.2 above)
If all three were valid and assuming that *AD//BC* (the third case in theorem 2.1),
then by axiom 2.5 *AD//BC* and *AB//DE* imply *AB//CE*.
On the other hand *AD//BC* and *AC//DE* imply *AC//BE*.
But the conclusions *AB//CE* and *AC//BE* are incompatible as *E* would lie *AB/C* as well as in *AC/B*.
As *A*, *B*, and *C* can be considered symmetrically, the proof is complete.


From the fact that we can relate three distinct points of a range or lines of a point (Theorem 1.4 ) to each other or any other three distinct points or lines using at most three perspectivities and given axioms 2.1, 2.4 and 2.6 we can deduce:

**Theorem 2.3:  Given Four Points AB//CD**
If *A, B, C* are three distinct collinear points, their line contains a point *D* such that *AB//CD*.
Axiom 2.1 says that there are four points; Axiom 2.4 gives us the possible positioning of *D* and axiom 2.6 requires *A* and *B* to separate *C* and *D*.

**Definition 2.2:  Segment, Interval, Interior, Between**
Given three different collinear points  *A* , *B* , *C*   we introduce the symbol  *AB/C*  to signify the *segment  AB*  on a line *l* but *neither* including the point  *C nor A* nor *B*: these being the points for which *AB* clearly separates *C from for example D*.
An *interval* is written  *"AB"/C*. The quotation marks indicate that the interval includes its end points.
If *X* and *Y* are included in the interval  *"AB"/C* then the interval  *"XY"/C* is said to be *interior* to  *"AB"/C*  including the case where *X* or *Y* coincides with *A* or *B*.
Despite the aforesaid, we introduce here the concept of `between´.
A point *D* in  *"AB"/C* lies *between X* and *Y* if it belongs to  *XY/C* , that is, *XY//CD*. Thus the concept of between is valid for the interval but not for the whole line.

---

[10] From Coxeter [5],  p. 27.

From the previous examples it's obvious that the conjugate of *C* with respect to *A* and *B* lies in *AB/C* but we still have to prove this.

**Theorem 2.4:  The Harmonic Conjugate of *C* always lies in Segment *AB/C*[11]**

*If A, B, C are all distinct,* H(AB,CD) *implies AB//CD*.

Proof:

According to theorem  2.3  we can choose a point *M* such that *AS//PM* (see Fig. 2.3a). Using our quadrangle *PQRS*, we let *QM* intersect *AB* in *Y* and *RS* in *O*. Further, *PO* intersects *AB* in *X* and *AR* in *N*. If *Y* coincides with *D*, we immediately get *AB//CD* as *AMSP* and *ADBC* are perspective from *Q*. (Axiom 2.6). If not, then we carry out the following series of transformations: firstly;

$$ASPM \ \overset{Q}{\overline{\overline{\wedge}}} \ ABCY$$ trapping *D* between *Y* and *B,* then

$$ASPM \ \overset{O}{\overline{\wedge}} \ ARNQ \ \overset{P}{\overline{\wedge}} \ ABXC$$
trapping *D* between *A* and *X,* and finally

$$ASPM \ \overset{O}{\overline{\overline{\wedge}}} \ ADXY$$
trapping D between *X* and *Y*. Hence, according to 2.6, we get *AB//XC, AB//CY*, and *AD//XY*. We can see that both *X* and *Y* are in the segment *AB/C*, and *D* is securely trapped between *A* and *B*. Fig. 2.3b shows us that the theorem is valid for any line intersecting the four sides of the quadrilateral *PQRS*. As a corollary we can see that  if *A,B,  C* are all distinct points, then H(*AB,CD*) implies that *D* ≠ *C*. Put differently, the diagonal points of a quadrangle are not collinear. We assumed this when defining Harmonic conjugates; if they were, then *S* and *C* would be concurrent, as would *Q* with *A* and *P* with *B*, thus leaving us at this point with no separation and no possibility of projecting anything anywhere. Considering the quadrangle *APBQ* we can conclude that the diagonal points of a quadrilateral are not collinear!



**Fig. 2.3a** (above) **b**(below)

## Sense of Movement

With the concept of points on a projective line being arranged cyclically, like the lines in a point, we can now introduce the concept of sense of movement. The incidence axioms, with the resulting theorems and the definitions haven't yet distinguished between the two possible directions of movement of a point on a line, although the axioms of order have already hinted at, and given us the basis for this distinction.

We stated before that two points decompose their line into two segments. This might at first appear to be obvious but is in fact quite difficult to prove.
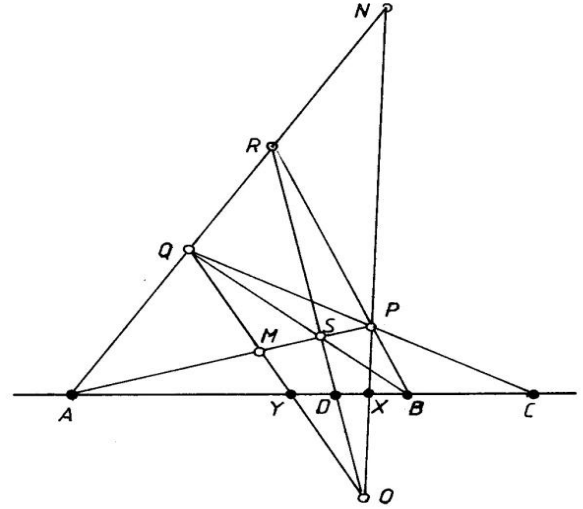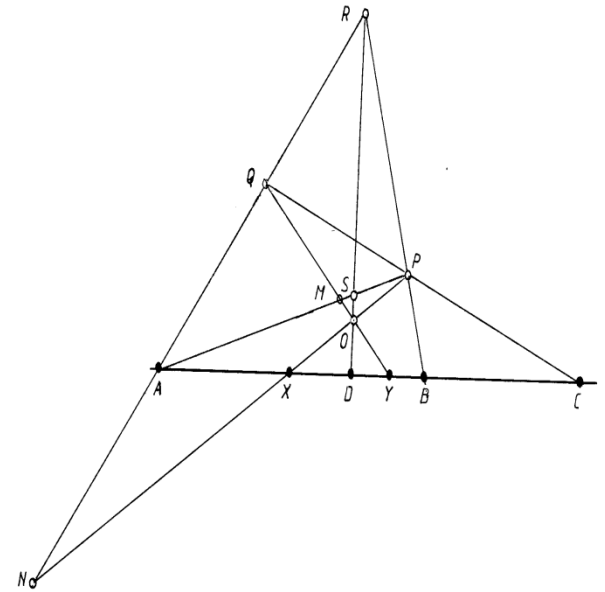
---

[11] See Veblen and Young [11],  p. 46.

**Theorem 2.5: Two Points Divide the Line into Two Segments.**[12]
*If AB//CD, the two points A and B divide their line into exactly two segments: AB/C and AB/D.*
Proof:

A point *X* cannot lie in both *AB/C* and *AB/D* (see Fig. 2.2, left). If it did, we'd then obtain *AB//XC, AB//CD*, and *AB//XD* which, according to theorem 2.2 (Three Separation Relations) is a contradiction. We still have to prove that any point *X* not lying on *A* or *B* has to lie in one of the segments. If we assume that *X* does not lie in *AB/C*, then it has to lie in *AB/D*. This is of course true when *X* coincides with *C*. According to 2.4 (replacing *D* with *X*.) the only other possibility is either *AC//BX* or *AX//BC* (see Fig. 2.2 and theorem 2.1 for the three mutually exclusive possibilities). Assuming *AC//BX* and *AB//CD*. This implies that *AB//DX* and thus *X* lies in *AB/D*. Assuming *AX //BC* and *AB//CD*. This implies that *AB//DX* and thus *X* lies in *AB/D*. In either case the line is separated into two segments.

**Definition 2.3: Supplementary Segments**
Given four points, *A, B, C* and *D* whereby *AB//CD* the two segments and with their corresponding intervals *"AB"/C* and *"AB"/D* are said to be *supplementary*.

**Theorem 2.6: n Points and n Segments**
*Given n points, the line can be divided into n segments.*
Proof:[13]
This is not actually so easy to prove. For a detailed proof see footnote 12

**Definition 2.4: Sense of Movement**
Given three collinear points *A,B,C* in a line and three further points *D, E, F* on the same line, we can state whether the *sense of movement* of *A, B, C* is the same or not the same as that of *D, E, F*.
We write either: S(*ABC*) = S(*DEF*) or S(*ABC*) ≠ S(*DEF*) which is the same as S(*ABC*) = − S(*DEF*).
Furthermore S(*ABC*) = S(*BCA*) =S(*CAB*) ≠ S(*CBA*) as the order on a line or in a point is cyclic.
Thus, when we give a particular sequence of points on a line to follow, for example *ABC* the direction is uniquely defined and is the same as for example *BCA* but different from the sense of movement of *CBA* or *ACB*.

**Theorem 2.7: AB//CD gives us the Two Senses of Movement**
*The relation AB//CD is equivalent to S(ABC) ≠ S(ABD)*
This is obvious. In Fig. 2.2 left S(*ABC*) is from `right´ to `left´ whereas S(*ABD*) is from `left´ to `right´. If *AB//CD* then two intervals, *"AB"/C* and *"AB"/D* are created: the sense of movement from one interval into the next is opposite to the sense of movement of remaining in the same interval.

**Definition 2.5: Ordered Correspondence**
A correspondence, as we have already seen in the case of a perspectivity, preserves the relation of separation and therefore of sense of movement within a cyclic ordering in any relation. As this property is shared by other kinds of correspondences, we define more closely its meaning to, on the one hand, the preservation of separation and extend it on the other to including segments (and intervals) separating segments (and intervals).
An *ordered correspondence* exists whenever the separation relation is preserved when applying perspectivity and therefore also a product of perspectivities. Thus, AB//CD automatically applies A´B´//C´D´.

---

[12] See Robinson [10], p. 120.
[13] See Kowol [8], Folgerung 3.27, pp. 106, 107.

We can now move on to considering the correspondence between two ranges on the same line.

### Definition 2.6: Invariant Point.

Any point $X$ which coincides with its corresponding point is called an *invariant* or double point. If the ranges are on two different lines, this is their intersection point, or with respect to pencils, their joining line.

Two ranges on the same line (or pencils in the same point) may have no, one or two double points, as we shall see.

### Definition 2.7: Direct and Indirect Correspondences.

When two ranges are on one line, an ordered correspondence is either *direct* or *opposite* according to whether

$$S(ABC) = S(A´B´C´) \qquad \text{or} \qquad S(ABC) \neq S(A´B´C´)$$

In particular the identity, namely for a variable point $X$, whereby $X = X´$, is direct.

### Theorem 2.8: Harmonic Conjugates: Opposite Correspondence[14]

*The correspondence between the points of a range and their harmonic conjugates with respect to two fixed points M and N is an opposite correspondence with fixed points M and N.*
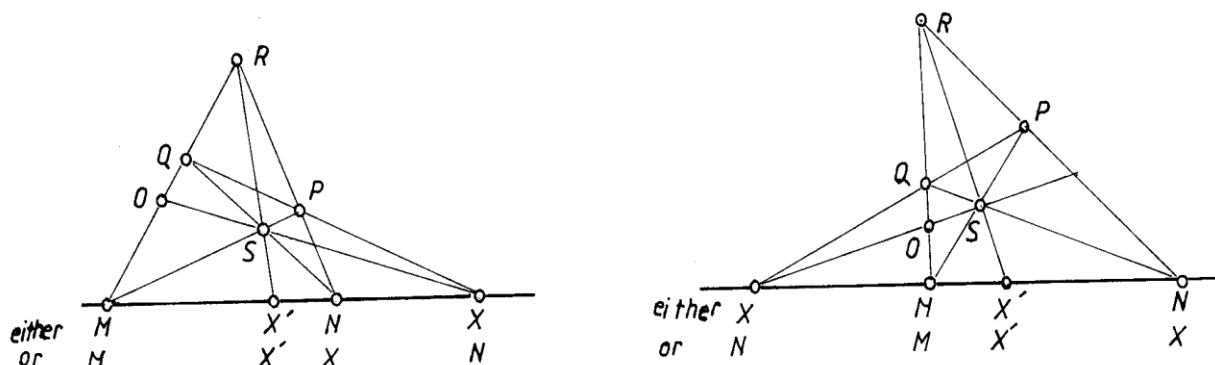


**Fig. 2.4**

Proof:

As a perspectivity preserves order then so does a sequence or product of perspectivities.

If we construct a quadrangle $PQRS$ above a line $o$ with M and N being two points of the diagonal triangle (see Fig. 2.4 left; the '*either*' configurations) with $Q$ and $S$ collinear with $N$, $P$ and $S$ collinear with $M$ i.e. $P = NR \cdot QX$, $S = MP \cdot NQ$ and $X´ = (MN) \cdot (RS)$ and then project range $MNX$ sequentially using the following;

$$MNX \; \overset{Q}{\barwedge} \; RNP \; \overset{M}{\barwedge} \; QNS \; \overset{R}{\barwedge} \; MNX´.$$

We see that the correspondence mapping $X$ onto $X´$ is ordered and has $M$ and N as invariant points. Further, it's opposite as $S(MNX) \neq S(MNX´)$.

Looked at another way, we could hold $M$, $N$, $R$ and $Q$ in position and move $X$ towards $N$. $P$ and $X´$ then also move towards $N$ on their respective lines. $S$ of course moves down $RX´$, also approaching $N$ and moving $X$ towards $N$ moves $X´$ towards $N$.

In proving the above, we used three perspectivities. By including a further point $O = MQ \cdot SX$, this can be reduced to two (see Fig. 2.4 right, '*either*' configuration again). Taking $O$ and $R$ as perspectivity centres we get

$$MNX \; \overset{O}{\barwedge} \; QNS \; \overset{R}{\barwedge} \; MNX´$$

Line $QN$ is then used as a perspectivity axis with the points on the line being radiated up onto $QN$ by $O$ and down onto the line again from $R$.

---

[14] See Coxeter [6], pp. 32, 33.

**Theorem 2.9: Separation of Pairs of Harmonic Conjugates[15]**

*Two pairs of harmonic conjugates with respect to just two invariant points cannot separate each other.*

Proof:

Suppose that $M$ and $N$ formed a harmonic set with both $X_1$ and $X_1´$ as well as $X_2$ and $X_2´$.

$X_1$, $X_1´$ and $X_2´$ are then three positions of $X$ in the correspondence of theorem 2.8 above.
The respective positions of $X´$ are then $X_1$, $X_1´$ and $X_2´$.
As $X_1X_1´//X_2X_2´$ is equivalent to $S(X_1X_1´X_2) \neq S(X_1X_1´X_2´)$ the correspondence is opposite and we can conclude that $S(X_1X_1´X_2) \neq S(X_1´X_1X_2´)$. Thus the harmonic conjugates don't separate each other. This was obvious from the construction in Fig. 2.5 though.

**Theorem 2.10: Harmonic Conjugates: Direct Correspondence[16]**

*Given H(MX,NX´) with M and N invariant, the correspondence between harmonic conjugates is a direct correspondence.*

Proof:

By interchanging $N$ and $X$ we've now got a new correspondence with $X$ moving to $X´$ and H(MX,NX´). If we take fixed points $O, Q$ and R ;not on line $MN$ (see Fig. 23, *'or'* configuration).with H(MQ,OR) we get $QX$ meeting $RX´$ in $S$ in line $ON$. Using perspectivity centres $Q$ and $R$ we obtain:

$$MNX \stackrel{Q}{\overline{\wedge}} ONS \stackrel{R}{\overline{\wedge}} MNX´.$$

Here S($MNX$) = S($MNX´$), otherwise we'd have $MN//XX´$ but H($MX,NX´$) implies $MX//NX´$.
We see: the correspondence is direct.
Alternatively, we could think of $M$, $Q$ and $R$ as invariant points and move $N$, while watching what happens to $X$ and $X'$: if we then move $X$ a little, we see that $X´$ moves in the same direction as $X$.

## Axiom of Continuity

**C1:** *If an ordered correspondence relates an interval "AB"/C to an interior interval "A´B´"/C, the latter contains an invariant point M such that there is no invariant point between A and M in "AB"/C.*

If the correspondence is opposite and we move $X$ from $A$ to $B$ while demanding that $X´$ moves contemporaneously from $B´$ to $A´$, then $M$ is the first point where they meet.
If the correspondence is direct and we move $X$ from $A$ to $B$ while demanding that $X´$ moves contemporaneously from $A´$ to $B´$ then $M$ is the first point in "$AB$"/C in which $X$ catches up with $X´$. $X$ might overtake $X´$ or accompany it for a while before moving ahead, but there will always be a first invariant point $M$.
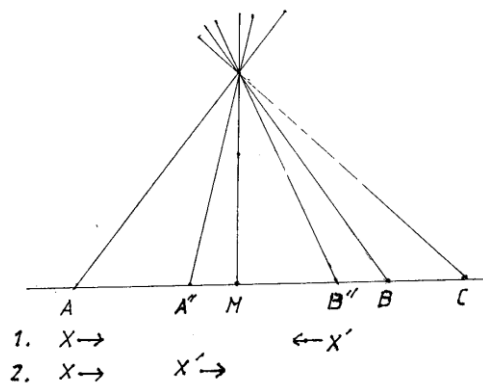
**Fig. 2.5**

**Theorem 2.11: Invariant Points**

*Every opposite correspondence has exactly two invariant points.*

Proof:

As the identity is direct, an opposite correspondence gives us a point $A$ which is not invariant. Assuming the correspondence relates $A$ to $A´$ and this further to $A´´$ and choosing a point C such that $AA´//A´´C$ then the opposite correspondence relates "$AA´´´$"/$C$ to the interior interval "$A´A´´´$"/$C$.

---

[15] See Coxeter [6], p. 33.
[16] See Coxeter [6], p.35.

Hence there is only one invariant point *M* in the exterior interval. But there is also an invariant point *N* in the supplementary segment *AA´/M*, because the inverse correspondence relates "*A´A*"/*M* to the interior interval "*AA´´´*"/*M*. As we saw before, H(*MN,AB*) and H(*MN,CD*) exclude *AB//CD*.

**Theorem 2.12: The Harmonic Conjugate of Non-Separating Pairs [17]**
*If AB and CD are two pairs of points that are collinear but don't separate each other, then there exist two points M and N such that H(AB,MN) and H(CD,MN).*
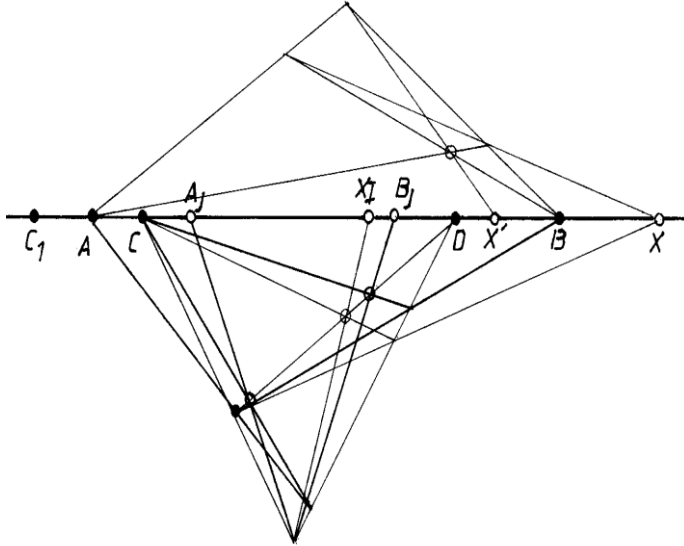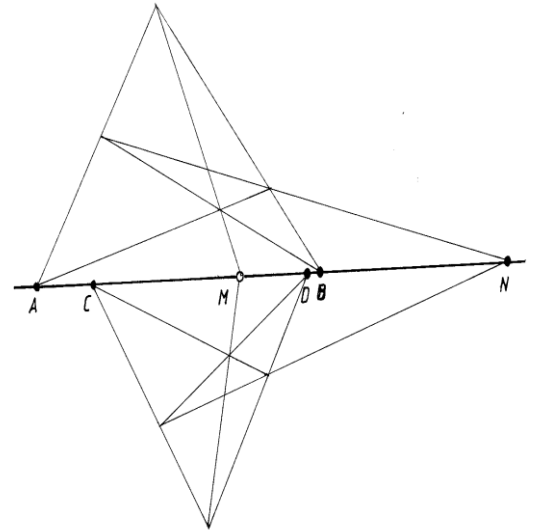


| **Fig. 2.6a** | **Fig. 2.6b** |

Proof:

Any point *X* has a harmonic conjugate $X_I$ with respect to *A* and *B* (see Fig. 2.6a) and a harmonic conjugate $X_J$ with respect to *C* and *D*.

If *X* moves from *A* to *B* in the interval "*AB*"/*C*, then $X_I$ moves along the supplementary interval "*AB*"/$C_I$ which includes *D* and *C* as *AB* and *CD* don't separate each other.

Meanwhile $X_J$ moves from $A_J$ to $B_J$ through part of the same interval.

Let's consider the combined correspondence $X_I \rightarrow X_J$. This relates the interval "*AB*"/$C_I$ to the interior interval ``$A_JB_J$´´/*C*. According to the Continuity axiom, the latter interval contains a point *M*. We could also name *M*, $N_I$ or $N_J$ as it's the harmonic conjugate of some point *N* with respect to the pairs *AB* and *CD*.

We now move on to a lead up to the Fundamental Theorem of Projective Geometry: that three pairs of corresponding points (either on the same line or on distinct lines) are all that is needed to define completely projectivity. As stated above, projectivity is an ordered correspondence preserving the harmonic relation. But we'll have to prove this though.

**Theorem 2.13: Preservation of the Harmonic Relation**
*Every projectivity is an ordered correspondence preserving the harmonic relation.*
Proof:

Suppose it were possible that *ABCD* $\overline{\wedge}$ *A´B´C´D´* and AB//CD but not *A´B´//C´C´*. According to theorem 2.11 above (see Fig. 2.6b), there are two points *M´* and *N´* with which the Harmonic relations H(*A´B´;M´N´*) and H(*C´D´;M´N´*) can be constructed. The two points *M´* and *N´* correspond to *M* and *N* giving us H(*AB,CD*) and H(*CD,MN*). According to theorem 1.13 though we then have H(*MN,AB*) and H(*MN,CD*). As we assumed that *AB//CD* theorem 2.9 would then be contradicted.

**Theorem 2.14 : Identity**

*A projectivity having more than two invariant points is the identity*

Proof:

I prove that the converse is not true.

If we assume that there are three invariant points  *A* , *B*  and  *C*  and a movable point  *P*    then *ABCP* ⊼ *A´B´C´P´*  with  *P* = *P´*.  If  *P*  lies on  *AB*/*C*  and  *P´* in *PB*/*C*  then *PB*/*C*  is related to *P´B*/*C*.  Thus there is a first invariant point with no invariant points between *P* and *M*. Similarly the inverse projectivity relating  *AP´*/*C*   to  *AP*/*C*  contains a last invariant point  *N*  with no invariant points between  *N*  and  *P´*. As *NP´*/*C* and *PM* /*C* overlap there is no invariant point in *NM* /*C*.  If  *D* is the harmonic conjugate of  *C*   with respect to *M* and *N,* then *MNCD* ⊼ *MNCD´* which means that we have  H(*MN* ; *CD*) and  H(*MN* ; *CD´*). Thus *D* = *D´* and *D* is in the forbidden segment *MN*/*C*. Thus there are either not three invariant points, or all points are invariant.

# Chapter 3

## The Fundamental Theorem, The Theorems of Pappus and Pascal, Classification of Projectivities, Involutions

We have seen that a cyclic order of points or lines is projected onto a cyclic order of points or lines and a harmonic set of points or lines is projected onto a harmonic set of points or lines.

### Theorem 3.1: The Fundamental Theorem of Projective Geometry[18]

*A projectivity is completely determined when three of one range and the corresponding three points of another range are given.*

Proof:



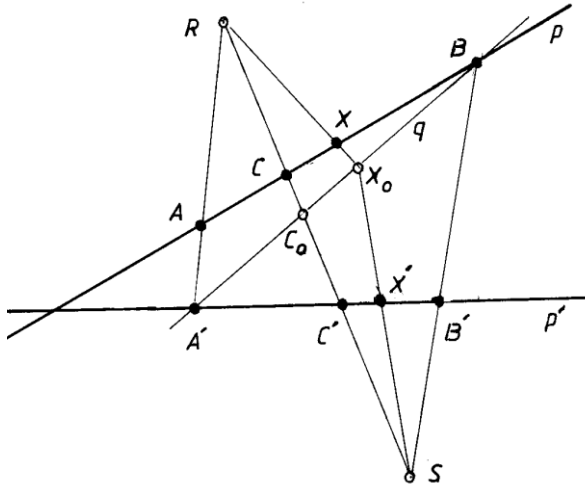**Fig. 3.1a**                                  **Fig. 3.1b**

We use the results and construction of theorem 1.4. If the two ranges are on one line we use an arbitrary perspectivity to project *A, B, C, X* onto $A_1$, $B_1$, $C_1$, $X_1$ on line $l_1$ and then use the following to relate the result to *A´, B´, C´, X´.´*. If the two pencils are in distinct points we only need two perspectivities. We let $R$, $S$, $C_0$ be the points where $A_1A´$, $B_1B´$ and $B_1A´$ intersect $CC´$ respectively (see Fig. 3.1a). Any point $X$ on line $l_1$ determines $X´$ on line $l$.

Byby applying the two perspectivities

$$ABCX \ \overset{R}{\overline{\wedge}}\ A´BC_0X_0 \ \overset{S}{\overline{\wedge}}\ A´B´C´X´$$ we obtain the desired result.

We use the results and construction of theorem 1.4. If the two pencils are in one point we use an arbitrary perspectivity to project *a, b, c, x* onto $a_1$, $b_1$, $c_1$, $x_1$ in point $P_1$ and then use the following to relate the result to *a´, b´, c´, x´*. If the two pencils are in distinct points we only need two perspectivities. We let $r$, $s$, $c_o$ be the lines in which $a·a´$, $b·b´$ and $b·a´$ join $c·c´$ (see Fig. 3.1b). Any line $x$ in point $P$ determines $x´$ in point $P´$.

By applying the two perspectivities

$$abcx \ \overset{r}{\overline{\wedge}}\ a´bc_0x_0 \ \overset{s}{\overline{\wedge}}\ a´b´c´x´$$ we obtain the desired result.

To prove that a different construction gives the same $X´$ for a given $X$ we could assume that one construction results in $ABCX \ \overline{\wedge}\ A´B´C´X´$ while another results in $ABCX \overline{\wedge} A´B´C´X_1´$. By relating the results of the two constructions we obtain $A´B´C´X´ \overline{\wedge} A´B´C´X_1´$ which has three invariant points and therefore according to theorem 2.14, $X_1´$ must coincide with $X´$.

As a corollary we can state that: *any projectivity can be reduced to a product of three perspectivities and if the two ranges are on different lines then two perspectivities.*
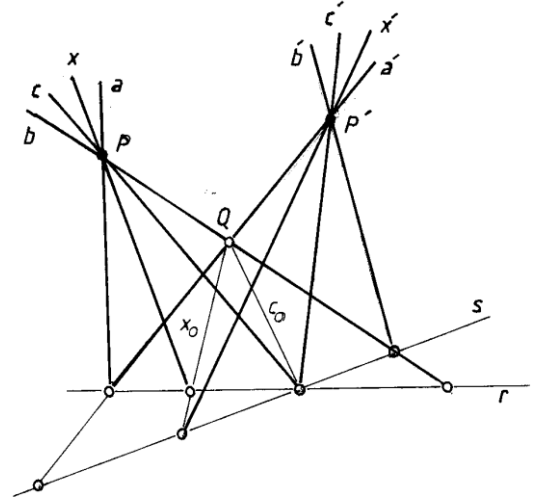
---

[18] See von Staudt [12], p. 52.

This is a powerful and astonishing result! What it means is that we can project a whole range of points or pencil of lines around the page as often as we want and, from what's been developed above, we require only a maximum of three perspectivities to relate the end result to the range or pencil we started with. We can also be sure that the axiom of continuity holds, that order is maintained and that the harmonic relation is unaffected.

## Classification and methods of ascertaining Projectivities

*For the rest of this chapter, projectivity will denote a projectivity between two ranges on the same line. Put another way, we'll be looking at the way points on a line move when subjected to a projectivity.*

As we have seen, a projectivity may be either direct (same sense of movement) or opposite (reversed sense of movement). The identity is direct.
Apart from the identity there are three types of projectivity:
an *elliptic* projectivity having no invariant points,
a *parabolic* projectivity, having one invariant point, and
a *hyperbolic* projectivity, having exactly two invariant points.
The *identity* of course, contains infinitely many invariant points, and as we have seen, hyperbolic projectivities can be either opposite or direct, while the remaining projectivities are all direct.

### Periodic Projectivities

A periodic projectivity is where we project a point $A$ onto $A^{(1)}$ and this further onto $A^{(2)}$ etc. to $A^{(n)}$ and $A^{(n)}$ back onto $A^{(1)}$. The smallest number $n$ for which this is the case is called the period. The identity is of period one, the correspondence between harmonic conjugates with respect to the double points is of period two and an elliptic projectivity where $ABC$ is projected onto $BCA$ is of period three.
We find that $A^{(1)}$ does not in general project back onto $A$ but onto another point $A^{(2)}$ on the line. We then have to do with product of projectivities. Doing this has the advantage that we can make ourselves a clearer picture of the `movement´ induced by the projectivity.
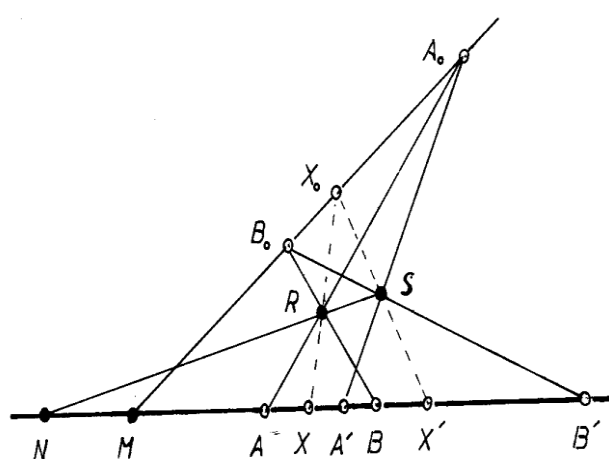
### <u>Hyperbolic Projectivities[19]</u>

### Theorem 3.2: Two Perspectivities Required for a Hyperbolic/Parabolic Projectivity
*All parabolic and hyperbolic projectivities can be constructed as the product of two perspectivities.*
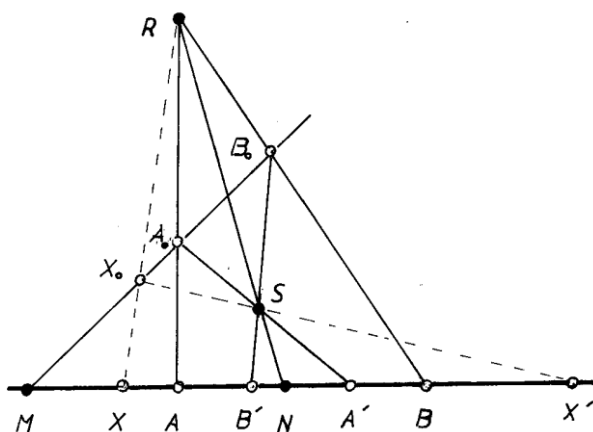


Hyperbolic - opposite
**Fig. 3.2a.**

Hyperbolic - direct
**Fig. 3.2b.**

---

[19] See Coxeter [6], p. 48.

Proof:

According to the fundamental theorem, a projectivity containing an invariant point $M$ can be described by $MAB \barwedge MA'B'$ as three pairs of corresponding points define the projectivity. Constructing any line through $M$ and taking any two convenient points $A_o$ and $B_o$ on this line, we can construct $R = AA_o \cdot BB_o$ and $S = A_oA' \cdot B_oB'$. (see Fig. 3.2a ) Using the two perspectivities $R$ and $S$ we can determine any $X'$ for a given $X$. Thus, using R and S as perspectivity centres, we obtain:

$$MABX \;\overset{R}{\barwedge}\; MA_oB_oX_o \;\overset{S}{\barwedge}\; MA'B'X'$$

with $M$ being the invariant point. Any other invariant point $N$ must then lie on $RS$ as it must be projected onto itself. A hyperbolic projectivity is then determined when the two invariant points and a third point are given so that $MNA \barwedge MNA'$. The resulting sense of movement can then be either opposite (Fig. 3.2a.) if $MN//AA'$ or direct otherwise (Fig. 3.2b).

In the case of a direct movement we can construct the resulting projectivity using an adaptation of the above. As above we choose any two points $R$ and $S$ collinear with $N$ and, locating $A_o = AR \cdot A'S$, we use the line $MA_o$ as before. For any other point $X$ on our line we can use the two perspectivities $R$ and S to obtain:

$$MNAX \;\overset{R}{\barwedge}\; MN_o A_oX_o \;\overset{S}{\barwedge}\; MNA'X'.$$

Product of Direct and Opposite Hyperbolic Projectivities

The following is included to give an impression of the complete result of the product of infinitely many hyperbolic projectivities. Using the above and projecting *1* onto *2* and *2* onto *3* etc. on the *carrier line of the projectivity,* we gain an impression of a movement. The point moves slowly away from *M*, accelerating until it reaches the 'middle' position between *M* and *N* (in either interval. We could also have continued our repeated projection backwards from point *1* (left) towards M. In the interval 'outside' of *MN*, the 'middle' is the infinite point.) and then decelerating as it approaches *N*. Of course, if we had taken a different starting point, we would have obtained a different sequence of following points but the gesture of the movement would still remain the same: that of accelerating to a maximum before decelerating and that of *M* and *N* being stationary points. (There are also infinitely many points between *M* and *1* on either side of M and infinitely many points between point *8* and *N*, again on either side of *N*.)
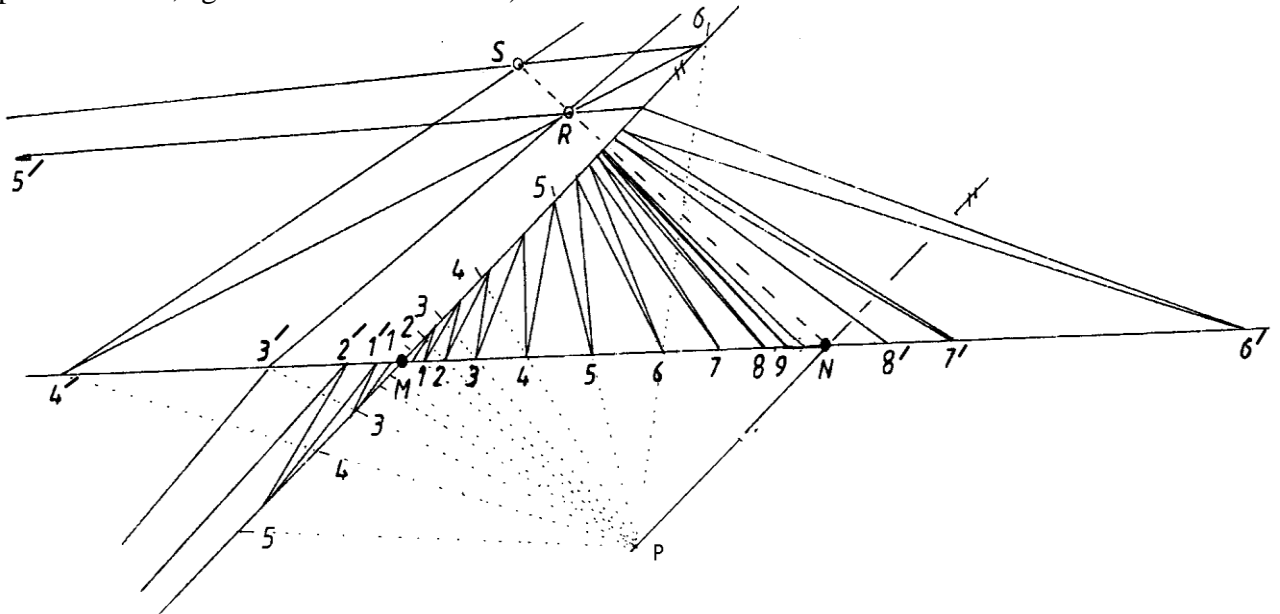


**Fig. 3.3 Hyperbolic direct**

(If we had based our investigation on metric considerations instead of on incidence and separation, we would have considered the *cross ratio* of any two consecutive points, the double ratio of the distances (*M1*):(*12*)*divided by* (*MN*):(*N2*) we would have found that it would remain the same for

25

all such sets of four points and only vary according to where we had placed points *1* and *2* (or *S* and *R*) to start with; which is astonishing. Furthermore, we would have found that if we project the thus obtained points from any point *P* not on the line *MN* with *N* being projected to the infinitely distant point, the size of the distances between respective points would provide us with a perfect geometric sequence having a constant factor for the increase between consecutive segments (see Fig. 3.3, the projection onto the slanting line through *M*).

Applying the same to an opposite hyperbolic projectivity, we see that the same applies to the points with respect to their crowding around the double points but with the difference that successive points lie on alternate sides on the double points (see Fig 3.4). The result of projecting the points on the carrier line onto any line through *M*, with *N* being projected onto the infinite point, again gives us a geometric sequence of respective distances but with a negative factor!
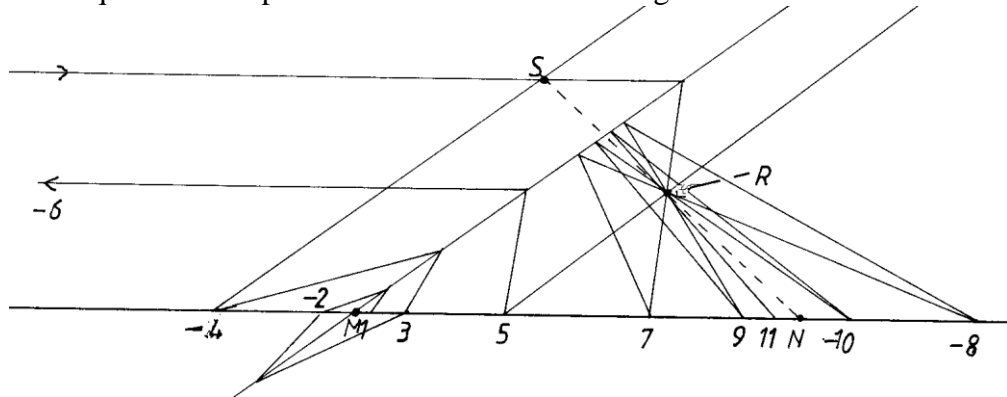


**Fig. 3.4  Hyperbolic opposite**
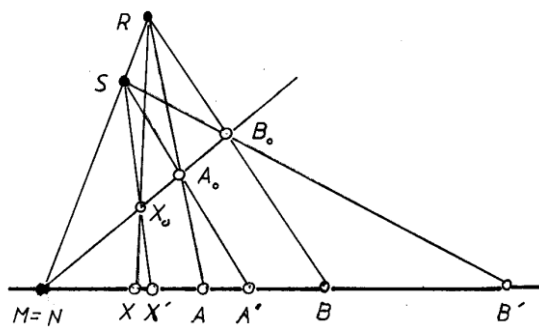
**Parabolic Projectivities**



**Fig. 3.5  Parabolic**

If *M* coincides with *N* then the projectivity has only one invariant point and is parabolic. We can use the method above to construct further points by drawing any line through *M* = *N* and radiating *A* and *B* up onto it using *R*. (see Fig. 3.5) If *R*, *S* and *M* lie on a line then there is clearly only one invariant point: *M*. Further points can then be constructed by radiating up onto the line $B_0 A_0$ M using R and back down onto the line using *S*.
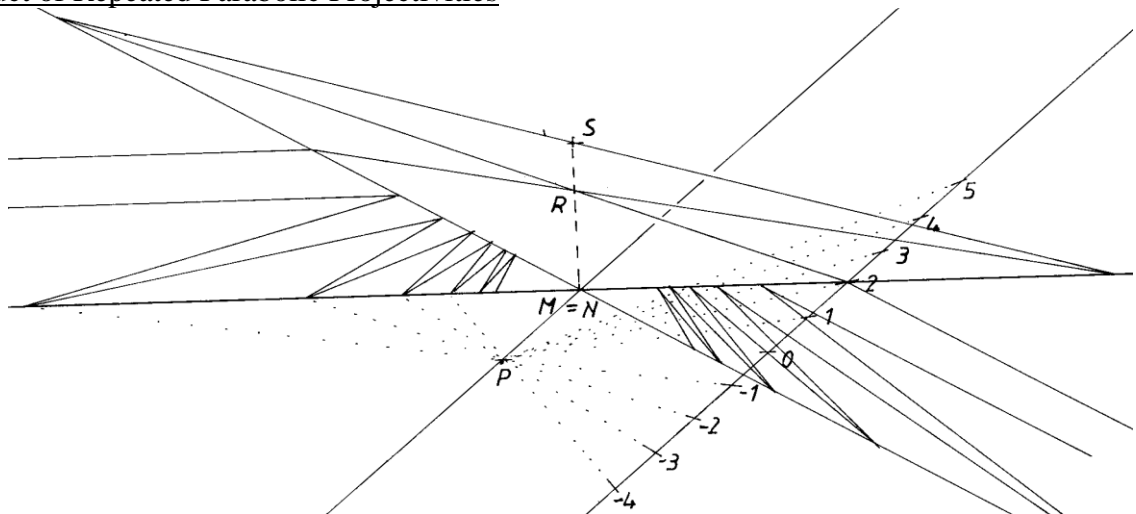
Product of Repeated Parabolic Projectivities



**Fig. 3.6 Repeated Parabolic**

26

Again, by radiating the result of the periodic projectivity with period ∞ from a point $P$ onto any line, whereby $N$ is projected onto the infinite point, we obtain a perfect arithmetic sequence of lengths of consecutive segments (see Fig. 3.6).

**Elliptic Projectivities**
An example a projectivity where we can be sure that it is elliptic is given by

$$ABC \overset{O}{\overline{\wedge}} A'B'C' \overset{R}{\overline{\wedge}} CB'C_0 \overset{S}{\overline{\wedge}} BCA$$

Where $A$, $B$ and $C$ are collinear. As each of these points is not invariant and the three segments $BC/A$, $CA/B$ and $AB/C$ are related to each other, there is no invariant point anywhere.
The construction of an elliptic projectivity requires three perspectivities.

Product of Repeated Elliptic Projectivities

By repeatedly projecting points in the above projectivity we don't arrive at helpful picture of the overall movement of the projectivity. The points tend to crowd in towards the three given points but the required number of repetitions of the projectivity (the author tried it using a 0.13 mm pen and 20 repetitions) leads to inaccuracies. The exercise is useful though as it becomes clear that the result of the projected points moves along the line-carrier of the projectivity in one direction and repeatedly crosses the infinitely distant point. An overview of what's happening can more easily be gained by considering the set of points on the carrier-line as a section of a set of lines in a point with the set of lines in a point being a radiation from a point of the range.
Considering Fig. 3.3 which depicts the projective movement along a line of a product of direct projectivities and using the section-radiation concept, we could consider constructing a point not on the carrier-line (for example the point $P$) consisting of a pencil of lines and intersecting the carrier-line in their corresponding points of the range; $1, 2, 3$ . . . and so on (see Fig. 3.5). This then gives us a depiction of how a line, starting from position $1,$ slowly turns stepwise away from a parallel line in $P$, generating a set of points in which crowd around the middle of the picture and are more widely spaced on either side.

Any three lines of the pencil together with their corresponding lines in the same pencil will then define the projectivity. As in the above projectivities, we can then project line 1 onto line 2 and this onto line 3 and so on. We get a depiction of a section of a line rotating according to some rule. The simplest rule would then be to let a line in the pencil turn stepwise in regular angles, say 10°. This gives us the following figure (see Fig. 3.7).
The picture is essentially the opposite of the direct hyperbolic projectivity in that the stepwise movement of the points of the range (section) decelerates towards `the middle´ and then accelerates and similar to the picture of a repeated parabolic projectivity in that the points crowd in `the middle´. In this particular case we have a periodic projectivity with period of 18. If however, 180 divided by the angle we had chosen were irrational, we would still arrive at a picture of the points and lines crowding around `the middle´ but the projectivity would not be periodic.
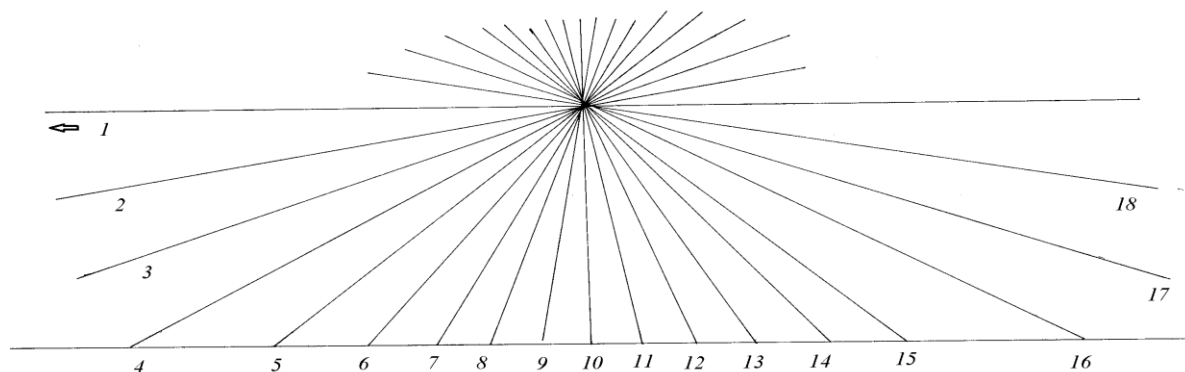


**Fig. 3.7**

## An Equivalent Case: Conic Sections

To the present point we have been dealing with projecting ranges onto ranges on the same or different lines and pencils onto pencils through the same or different points. In the following, we look at what happens to the intersection points of two different pencils or the joining lines of two different ranges which are *not* perspective to each other. In this case we are not dealing with a transformation of the whole plane as presented in the section on collineations or correlations in chapter 4 but instead the result of relating two pencils projectively to each other.

We can project lines $c$, $d$, $e$ onto points $X_3$, $X_2$, $X_1$ on $g$ and, using a pencil in $N$, project these further onto points $X'_3$, $X'_2$, $X'_1$ on another line $f$. Relating these directly to the lines $c'$, $d'$, $e'$, in another pencil $B$ gives us automatically the intersection points $e \cdot e'$, $d \cdot d'$, and $c \cdot c'$. If we then continue this process of finding the intersection points of all lines in $A$ with those in $B$, (which are now projectively related) we find that they definitely do not lie on a straight line but instead on a closed curve: a curve of second order. Depending on where we have placed the five elements $A$, $B$, $N$ and $f$ and $g$, we arrive at something looking like an ellipse, hyperbola or a parabola. For the rest of this chapter we will define a conic, following the great mathematician Jacob Steiner, to be the result of this construction.



**Fig. 3.8**

### Definition: Steiner's Conic Definition
*The conic is the locus of the points of intersection of corresponding lines of two projective, but not perspective pencils.*
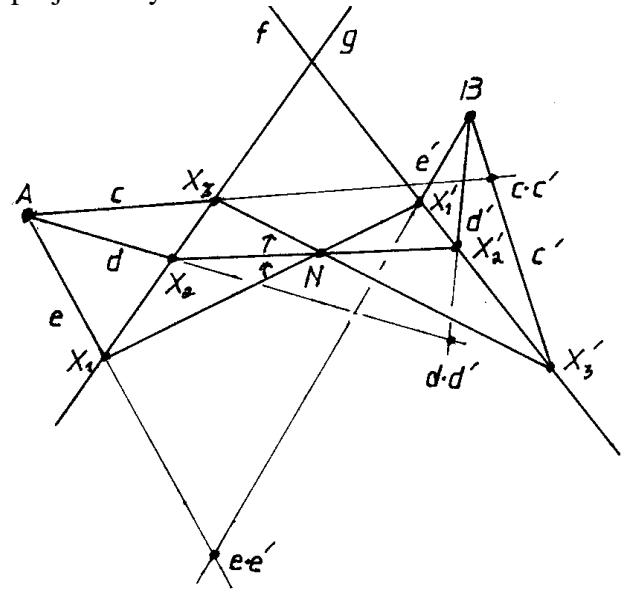
Before we continue with this development in chapter 5 though, we include two important theorems which will enrich the whole subject.

### Theorem 3.3: Crossing lines: The Pappos and Pascal Theorems[20]

| | |
|---|---|
| *If alternate vertices of a hexagon AB´CA´BC´ lie on two different lines then the intersection points of the three pairs of opposite sides are collinear. We call line NML the Pappus line or perspectivity axis.* | *If alternate sides of a hexalateral ab´ca´bc´ pass through two different points then the joining lines of the three pairs of opposite vertices are concurrent. We call the point n·m·l the Pascal point or perspectivity centre.* |
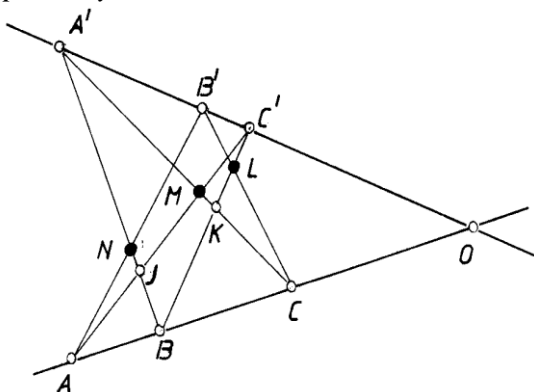


**Fig. 3.9a**
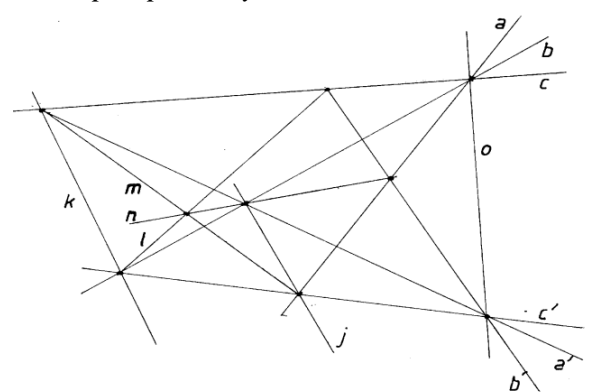


**Fig. 3.9b**

---

[20] See O'Hara and Ward [9], p. 53. The dual is original.

Proof:

If $AB'CA'BC'$ is a hexagon then we have to prove that: $L = BC''\cdot CB'$, $M = C\,A'\cdot AC'$ and $N = AB'\cdot BA'$ lie on a line (see Fig. 3.9a and Fig. 3.9b for its dual). If we take $J = AC'\cdot BA'$, $K = BC'\cdot CA'$ and $O = AB\cdot A'B'$ then we have

$$A'NJB \overset{A}{\overline{\wedge}} A'B'CO' \overset{C}{\overline{\wedge}} KLC'B$$

and thus $B$ is an invariant point of the projectivity $A'N J \overline{\wedge} KLC'$. As this has an invariant point then it is a perspectivity i.e. $A'NJ \overline{\overline{\wedge}} KLC'$ and thus $NL$ goes through $M$.

If we take this together with the following, a surprisingly refreshing overview emerges.

**Theorem 3.4: Pascal and Brianchon**

*If we inscribe the vertices of a hexangle in a conic then the intersection points of the three pairs of opposite sides lie in a line called* Pascal's Line*, and conversely.*

*If we circumscribe the sides of a hexalateral on a conic then the joining lines of the three pairs of opposite vertices go through a point called* Brianchon's Point*, and conversely.*

Proof :

If $A\,B'C\,A'B\,C'$ is the hexagon then we have to prove that $L = BC'.CB'$, $M = CA'.AC'$, and $N = AB.BA'$ lie in a line (see Fig 3.10 for Pascal and Fig. 3.10b for Brianchon).. If we use $J = AC'.BA'$ and $K = BC.CA'$ then we have a perspectivity series:
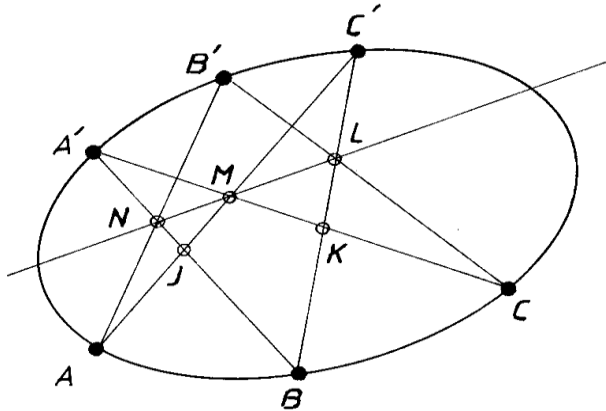


**Fig. 3.10a**   **Fig. 3.10b**
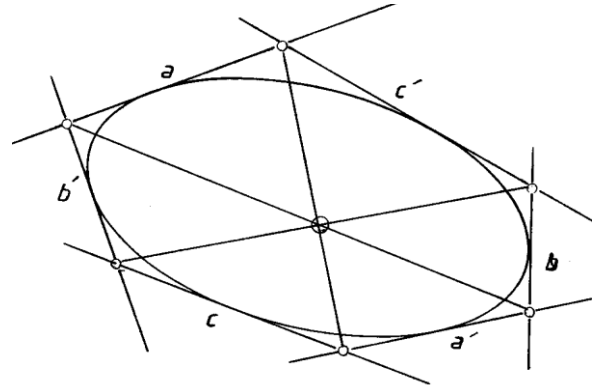
$$A'N\,J\,B \overset{A}{\overline{\wedge}} A'B'C'B \overset{C}{\overline{\wedge}} KLC'B$$

and B is the invariant point of the projectivity $A'NJ \overline{\wedge} KLC$
which then gives us the perspectivity $A'N\,J \overline{\overline{\wedge}} KLC'$ and thus $NL$ passes through $M$
Considering the equivalence of the Pappos-Pascal and Pascal-Brianchon theorems we could say that *we can apparently consider a pair of straight line ranges to be projectively equivalent to a conic section!*
What this also means is that if we are given five points which should lie on a conic section or on two straight lines, we can construct the join of any two pairs of points, choose any convenient line as a perspectivity axis and then, using the fifth point, construct a sixth point which then lies on the same conic! Together with Desargues theorem and the resulting harmonic relationship, this fact appears to be something like what Goethe would call an 'Urphänomen': the basis of our geometry.

A projectivity on a conic can be parabolic (having one invariant point), hyperbolic (either direct or opposite and having two invariant points) or elliptic, in which there are no real invariant points although as we shall later see, having two imaginary points.
The *axis* of a projectivity from a conic onto itself is the Pascal line determined as in theorem 3.4 above.

**Theorem 3.5:**

*Given three pairs of corresponding points on a conic, we can locate the point X´ corresponding to any point X.*

 Proof:

The Pascal line divides the conic into two arcs and using any pair of corresponding points, for example A and A´ we can project the points A, B, C, X from A´ onto the Pascal line and from A onto the other section of the conic (see Fig. 3.11).

$$ABCX \;\overset{A´}{\overline{\wedge}}\; GNMF \;\overset{A}{\overline{\wedge}}\; A´B´C´X´$$

The invariant points (if any) are the intersection points of the Pascal line with the conic. Furthermore we can see from the construction that the projectivity is determined when the axis and a pair of corresponding points are given. We could also have considered the Pascal line as the *polar p* of *pole P*, both with respect to the conic. As point X moves towards the perspectivity axis, F will move towards its intersection with the conic and the three points X, X´ and F will meet in the double point of the tangent.
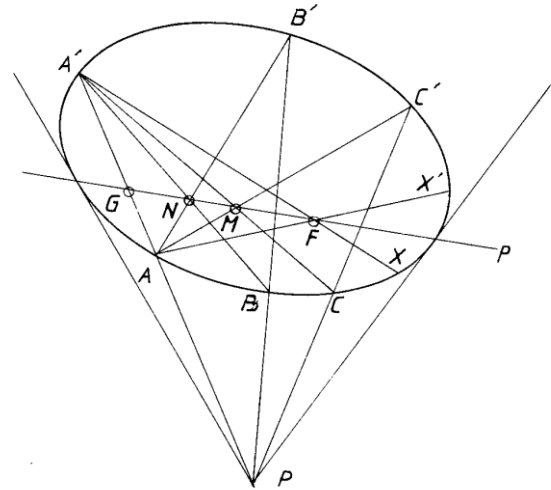


**Fig. 3.11**

We could also consider the special case of a perspectivity where the pairs of corresponding points are projections from a point *P* (see Fig 3.10). (In general AA´, BB´ and CC´ won't pass through P.) In that case we could just as well have projected A, B, C, X directly onto the other arc. In the above construction we obviously have a hyperbolic projectivity with its two invariant points (the tangent points).

Both cases suggest a useful construction for finding the invariant points of a projectivity on a line.

**Theorem 3.6: Steiners Construction for the Invariant Points of a Hyperbolic Projectivity**

1) $ABC \;\overline{\wedge}\; A´B´C´$ is a projectivity on a line *l*.

2) Select any point  G  on a conic constructed above the line and using G as a centre, radiate the points on *l* onto the conic. This gives us $A_1$ ,$A_1´$, $B_1$, $B_1´$.etc. (see Fig.3.12). In general the corresponding points won't pass through a perspectivity centre O).

3) Locate the perspectivity axis of the points on the conic using the crossing line theorem as applied to a conic. The points where this axis intersects the conic are the double points  $M_1$  and  $N_1$  on the conic.

5) Radiate these back down onto *l* from G.

Proof:



**Fig. 3.12**

$$MNABC \;\overset{P}{\overline{\wedge}}\; M_1N_1A_1B_1C_1 \;\overline{\wedge}\; M_1N_1A_1B_1C_1 \;\overset{P}{\overline{\wedge}}\; M´N´A´B´C´$$

Thus the projectivity on the line is perspectively related to the projectivity on the conic. If we use the same construction for the case of an elliptic projectivity, we find that the points $M_1$ and $N_1$ lie on a line outside the circle. As we shall see later, they are imaginary.
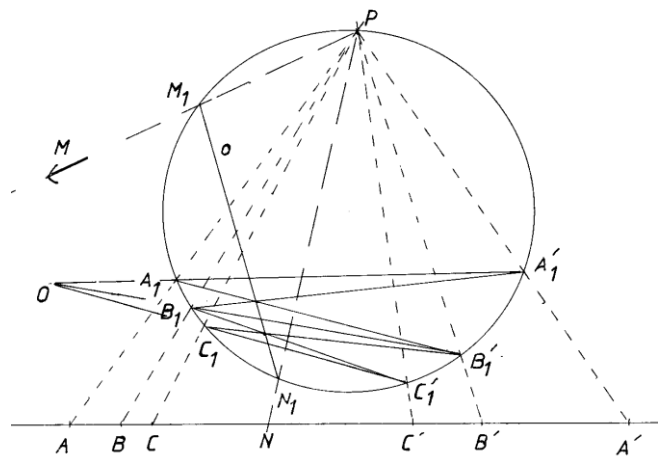
## Involutions in One Dimension
The following definition is that of von Staudt.


**Def:** *An involution is a periodic projectivity of period two*:

$$XX´ \barwedge X´X$$

## Theorem 3.7:  Interchanging Two Points Determines an Involution[21]
*A projectivity that interchanges two points is an involution.*
Proof:

If we are given $AA´ \barwedge A´A$ and, considering any other point $X$ , then according to the fundamental theorem the projectivity given by $AA´X \barwedge A´AX´$ is uniquely determined as we then have three pairs of corresponding points. As we can use a sequence of three perspectivities to interchange pairs of points, then $AA´XX´ \barwedge A´AX´X$ which is the same projectivity as the given one with $XX´$ being a pair of corresponding points. As we can see, an involution is determined by any two of its pairs.


### Notation
We denote the involution $AA´BB´ \barwedge A´AB´B$ by $(AA´)(BB´)$. If there is an invariant point, we can denote $AA´M \barwedge A´AM$ by $(AA´)(MM)$.


## Theorem  3.8:  An Invariant Point $\rightarrow$ Hyperbolic Involution
*If an involution has one invariant point it has another and the involution is just the correspondence between harmonic conjugates with respect to these two points.*
Proof:

If we have an involution  $(AA´)(MM´)$ and $N$ is the harmonic conjugate of $M$ with respect to $A$ and $A´$. then the involution, being a projectivity , preserves the harmonic relation.
$N$ is therefore a distinct second invariant point. If another point pair $XX´$ is used instead of $AA´$, we would still obtain the same harmonic conjugate $N$ as we would otherwise have three invariant points and the involution would then be a so-called degenerate parabolic involution.
The appropriate symbol for this would be $(MA)(MB)$. In other words: M is interchanged with A and M (the position) is interchanged with B with A $\neq$ B. All the points of the line are projected onto point M and M is projected onto all points of the line.


There are then two types of involution:
Opposite sense or hyperbolic or harmonic involutions and
same sense or elliptic involutions.  Two points to note!
1) If two pairs of conjugate points *don't* separate each other then the involution is hyperbolic.
2) If two pairs of conjugate points *do* separate each other then the involution is elliptic with no double points.

## Theorem  3.9: Point Number Requirements for Involutions
*A necessary and sufficient condition for three pairs of conjugate points to belong to an involution is* $ABCC´ \barwedge B´A´CC´.$
Proof: As a sequence of three projectivities can interchange conjugate pairs of points, and if CC´ is such a pair, we have:

$$ABCC´ \barwedge AB´C´C \barwedge B´A´CC´.$$

Conversely; $ABCC´ \barwedge B´A´CC´ \barwedge A´B´C´C$ implies that all three pairs belong to the involution (We can have $A =A´$ or $B = B´$ but *not* $C = C´$).


[21] See Coxeter  [6], p. 52.

## Theorem 3.10: Generating an Involution on a Complete Quadrangle

There is an eloquent construction for generating both hyperbolic and elliptic involutions *using only a straight edge*. (Later we will provide a method using straight edge and compass.) If we take a section of the six sides of a complete quadrangle using a line not going through any of the angles, the intersection points of opposite sides deliver us with an involution.
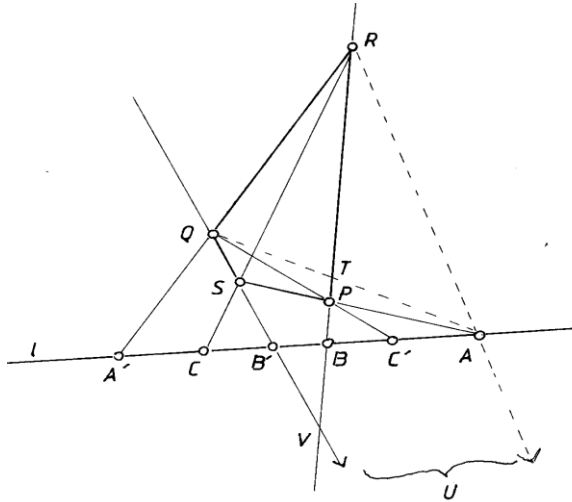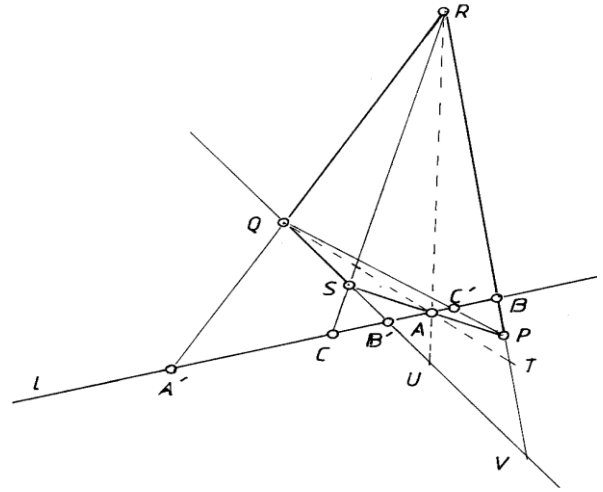


**Fig. 3.13a  Hyperbolic.**                    **Fig. 3.13b Elliptic.**

Proof:

$P, Q, R$ and $S$ are the angles of a complete quadrangle (see Fig. 1.13)

Line $l$ cuts opposite sides $SP$ in $A$, $RQ$ in $A'$; $RP$ in $B$, $QS$ in $B'$; $RS$ in $C$.

If we project the row of points $l$ up onto $QS$ from centre $R$ and then this row of points onto $PR$ from centre $A$ and finally this row of points from $Q$ back into $l$ we have :

$$AA'BB'C \overset{R}{\overline{\wedge}} UQVB'S \overset{A}{\overline{\wedge}} RTVBP \overset{Q}{\overline{\wedge}} A'AB'BC'.$$

As an involution is determined by any two of its pairs of conjugates ($AA'BB' \overline{\wedge} A'AB'B$) then $C$ is also projected onto its conjugate C´.

If the line $l$ goes through a diagonal point of the quadrilateral then this is a double point of the involution and it's therefore hyperbolic. If $l$ passes through two diagonal points then these are the two invariant points of a hyperbolic involution. It is also hyperbolic if the line lies in one of the three quadrilateral regions of the quadrangle $PQRS$ (That is: 1. Any line not passing through the 'area' enclosed by $PQRS$; 2. Any line passing 'between' the points $Q$ and $S$ on the one side and $P$ and $R$ on the other; 3. Any line passing 'between' the points $Q$ and $R$ on the one side and $P$ and $S$ on the other.). If the line passes through an angle of the quadrilateral then we have a *parabolic degenerate involution* where all points on the line correspond to this one point; through neither of these and also lies in one of the four trilateral regions of the quadrilateral (1. 'between' $S$ and $R$ on the one 'side' and $P$ on the other; 2. 'between' $S$ and $R$ on the one 'side' and $Q$ on the other; 3. 'between' $Q$ and $P$ on the one 'side' and $R$ on the other; 4. 'between' $Q$ and $P$ on the one 'side' and $S$ on the other) then we have an *elliptic involution*.

The construction delivers more though. Having set up the quadrilateral on the basis of the given two pairs of the involution on line *l*, we can then allow side *SP* to turn in point *A* and thereby obtain the conjugates of any other point *C* on line *l*.

We can now look at the equivalent case of involutions on a conic and develop the method for finding a harmonic representation of both hyperbolic and elliptic involutions.

## Involutions on a Conic

If a projectivity is given by $ABC \barwedge ABC$ we can consider the special case where $C = B'$ and $C' = B$. In this case we then have $ABB' \barwedge AB'B$. The axis of the projectivity on a conic is then given by $o = (AB' \cdot BA')(AC' \cdot CA') = (AB' \cdot BA')(AB \cdot A'B')$ see Fig. 3.14a and b.

There are then two cases to consider. Firstly, the hyperbolic involution where the axis $o$ passes through the conic and the pole of the axis $O$ with respect to the conic is an exterior point; in this case we have a hyperbolic involution of points on the conic with the invariant points being given by the tangent lines through $O$ and the projection of any point $X$ onto $X'$ being given by the line $OX$ and secondly, the elliptic involution where the axis $o$ does not pass through the conic
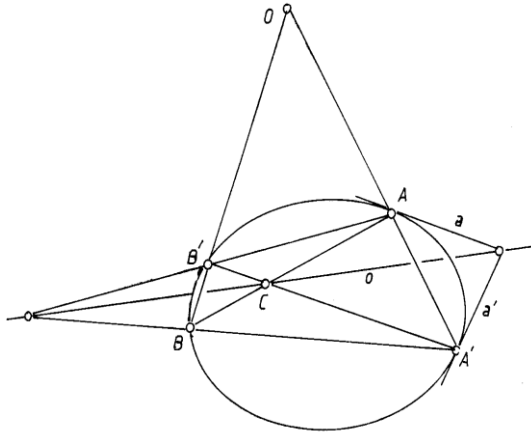


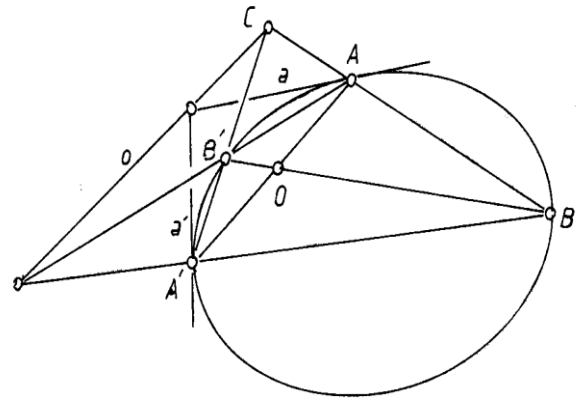**Fig. 3.14a**                                    **Fig. 3.14b**

and the pole of the axis $O$ is interior ; in that case we have an elliptic involution of points on the conic with the 'invariant points' being given by the 'tangent lines' through $O$ and the projection of any point $X$ onto $X'$ being given by the line $OX$. In the latter case, the 'invariant points' and 'tangent lines' will turn out to be imaginary and also imaginably representable by an elliptic involution. This will be developed in the course of the following.

We can project the involution from any point on the conic onto a line, thereby obtaining an expression of the involution on a line, or we could have started with an involution on a line and, using any convenient point on the conic, project the involution up onto the same.

Before we leave this section in which we regarded the conic more or less as a range of points (although second order) and move on to regarding the projective plane including all points and lines in it, there is one more important relationship which we have to take into account: that of the relationship between an involution of points on a conic and the same on a straight line. In particular, we have to consider the question of whether or not it is possible to give a harmonic representation of the points of a hyperbolic or an elliptic involution on a line. The answer is yes; with the following construction.

**Theorem 3.10: Harmonic Representation of a Hyperbolic Involution**

*If we are given two pairs of points of an involution on a line (AA′)(BB′) whereby the pairs of the involution don't separate each other then we can find a harmonic representation of the involution by locating the invariant points of the involution.*

As an involution is determined by any two pairs of its elements then the joining lines of pairs of corresponding points on a conic involution go through the involution centre $O$ which is the pole of the perspectivity axis $o$ of the involution. The axis $o$ is given by $(AB \cdot A'B')(AB' \cdot A'B)$.

To find the two points which divide two pairs of a hyperbolic involution on a line harmonically we only have to consider that the intersection of the perspectivity axis with the conic provides us with exactly these invariant points.

Projecting for example $A$, $A'$, $M_1$ ($=M_2$), and $N_1$ ($=N_2$) through the same point back onto the line gives us the required H($MN,AA'$). As a constructional check I included the complete quadrilateral to show that $A$ and $A'$ are harmonic conjugates with respect to $M$ and $N$.
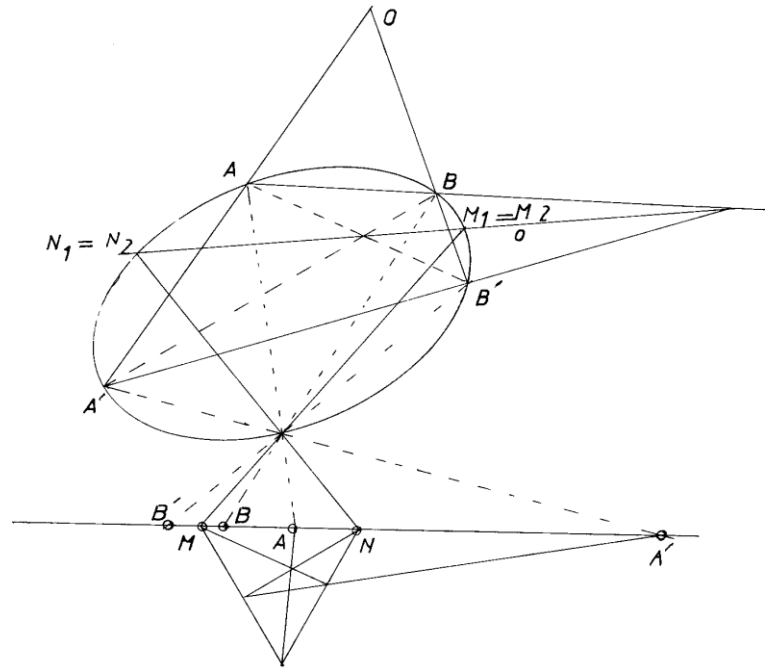


**Fig. 3.15**

**Theorem 3.11: Harmonic Representation of an Elliptic Involution**
We can also use Steiner's method to create a harmonic representation of an elliptic involution. Two pairs of corresponding elements in an elliptic involution do separate each other.

Our aim is to find another expression of the involution using two pairs ($CC'$)($DD'$) whereby we have H($CC',DD'$).
1) Project the involution $AA'.BB$ onto the conic. $P = (A_1 A_2)\cdot(B_1 B_2)$ is the perspectivity centre.
2) Select any points $C_1$ and $C_2$ whose joining line goes through and locate $P_0$ whereby $P_0$ and $P$ divide $C_1$ and $C_2$ harmonically on the line through $C_1$ and $C_2.$
3) Locate $D_1$ and $D_2$, the tangents to the conic through $P_0$ and project $C_1$ ,$C_2$ , $D_1$ , $D_2$ back onto line $l$.
H($CC';DD'$) is a harmonic representation of the elliptic involution.
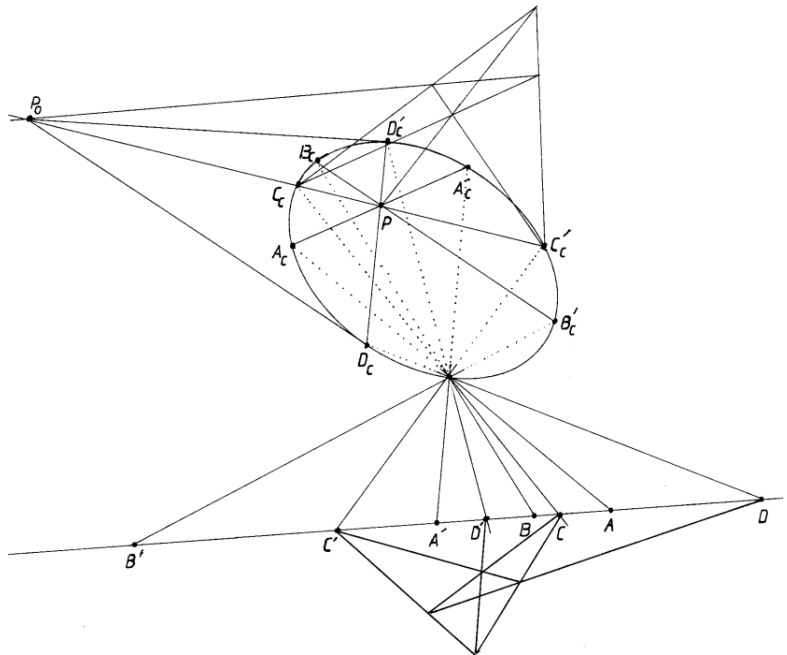


**Fig. 3.16**

**Projectivity on a Conic**
Briefly summarising we can say that if the centre $C$ is interior the involution on the conic is elliptic; if exterior then hyperbolic. Any two involutions will have just one and only one pair of conjugates in common.

If the involutions are elliptic involutions they will have a pair of real conjugates in common as in the diagram right (top left). The line through $C_1$ and $C_2$ gives us the pair of common conjugates.

If one involution is an elliptic involution and the other is a hyperbolic involution then the same is true. Again, the intersection of the line through $C_1$ and $C_2$ gives us the pair of common elements on the curve (diagram middle right).

Two hyperbolic involutions may have either a common pair of real elements or a pair of imaginary common conjugates in common (diagram bottom left and right).



**Fig. 3.17**

# Chapter 4                    Two dimensional Projectivities

## Collineations and Correlations

We now extend our investigation from the one-dimensional case, where we were interested in what happens to the individual ranges on one or more lines and the individual p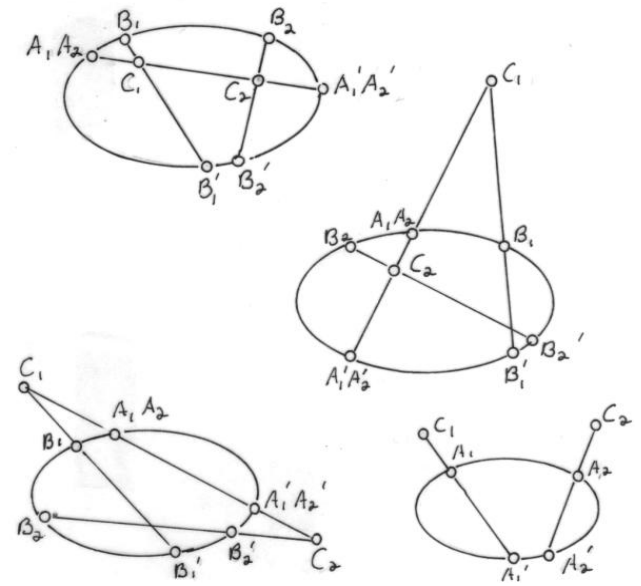encils through one or more points, to all points and lines lying in the projective plane. Previously we used lines and points outside the range or pencil to study the relationships between the points of one or more ranges, or alternatively lines of one or more pencils, these functioned as 'auxiliary' tools for our investigation. In plane projective geometry the tolls are included in the investigation. There are two cases to consider: that of relating points to points and lines to lines and that of relating points to lines and vice versa. The first is termed *collineation* (it could also be called co-punctuation) and the second is called a *correlation*. We consider first the collineations.

## Collineations

### Definition 4.1:  Collineation
*We call a point to point transformation from one line onto another preserving collinearity, together with a line to line transformation preserving concurrence, a collineation. Thus projectivity between corresponding ranges and pencils in the same plane is introduced.*
(If the domain plane $\alpha$ and the plane it is mapped onto $\alpha'$ are not coplanar then we need a quadrilateral or quadrangle in each to define the transformation.) If four lines of a quadrilateral or four points of a quadrangle are left invariant then three intersection points on any line of the quadrilateral or joining lines of the quadrangle are also invariant as are the all lines of the plane. The projectivity is then the identity. The equivalent of the fundamental theorem is then:

### Theorem 4.0: The General Collineation[22]
*A unique collineation is defined when two complete quadrilaterals or two complete quadrangles together with their corresponding sides or points are given.*
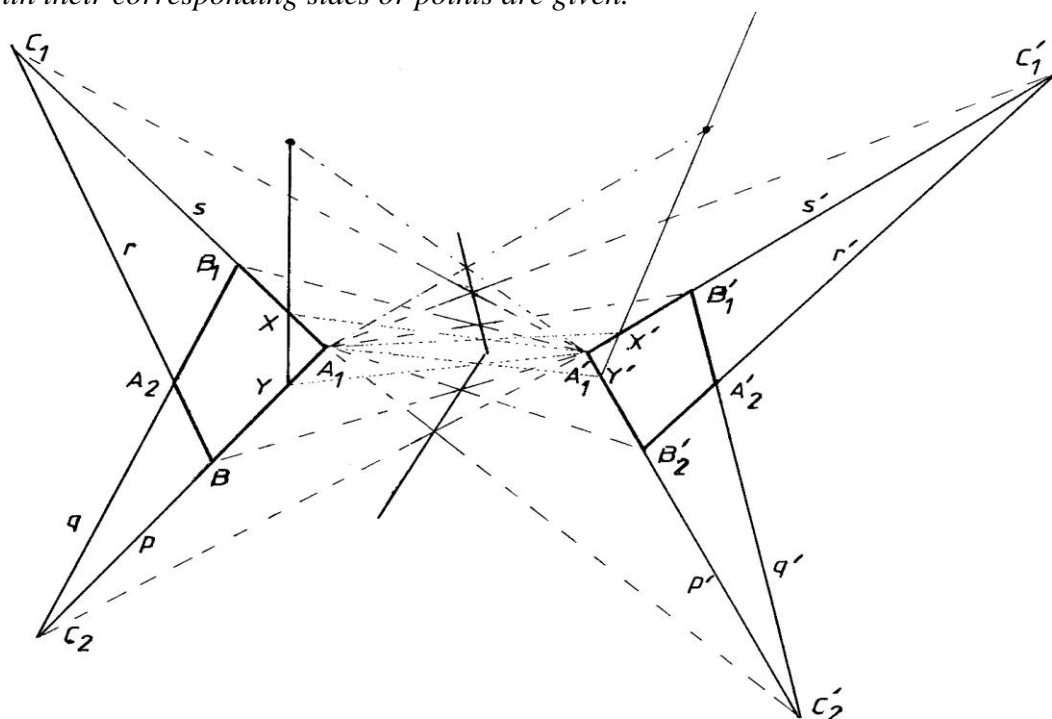


**Fig 4.0**

[22] See Coxeter  [5], p. 51.

Proof:

If $p, q, r, s$ and $p´, q´, r´, s´$ are the two given quadrilaterals, then an arbitrary line a must collinate uniquely onto a line $a´$ (see Fig. 4.0). Line a must intersect any two other lines, say $s$ in $X$ and $p$ in $Y$. The collineation giving us p $\overline{\wedge}$ p´ and s $\overline{\wedge}$ s´ determines $a´= X´Y´$ as it also collineates points $A_1B_1C_1X$ on s onto the points $A´_1B´_1C´_1X´$ on s´ and the points $A_1B_2C_2Y$ on p onto the points $A´_1B´_2C´_2Y´$ on p´. If the correspondence is a collineation then we have to show that incidences are preserved.

If we vary $a$ in a pencil so that $X \overline{\overline{\wedge}} Y$ then we get $X´ \overline{\wedge} X \overline{\overline{\wedge}} Y \overline{\wedge} Y´$.

As $X$ runs down $s$ to $A_1$, $Y$ runs up $p$ to the same point and $A_1$ is then an invariant point, as must $A_1´$ be. (If two ranges intersect in a common point then the two ranges are perspective and the common point is projected onto itself) Thus a´ must also vary in a pencil with concurrent lines being transformed onto concurrent lines. Added to that we have shown that collinear points are transformed onto collinear points and thus we have a collineation.

## Perspective Collineations

Here we are interested in perspectively transforming a plane onto itself (lines onto lines, points onto points) and we have a mapping which leaves every line through a point $O$ invariant and every point in a line $o$ invariant. If the invariant point lies in the invariant line then we call the transformation an *elation* and if not then a *homology*.

## Theorem 4.1 : Centre, Axis , two Points/lines determine a Perspective Collineation[23]

*A perspective collineation is determined when its centre and axis and one pair of corresponding points are given.*
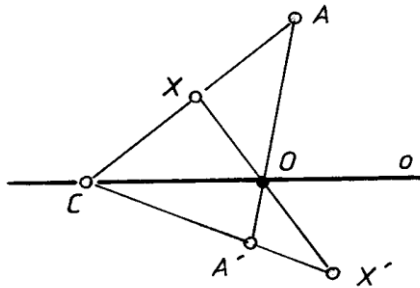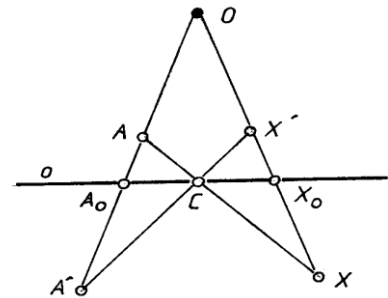
Proof:



| Fig. 4.1a | Fig. 4.1b |

If we put down a pair of points $A$ and $A´$, collinear with centre $O$ then any point $X$ not on $OA$ determines $C = AX \cdot o$ and $X´= OX \cdot CA´$ (see Fig. 4.1a,b). As all points in $o$ and all lines through $O$ are invariant The correlation must relate $X= OX \cdot CA$ to the point $X´$ as defined.

By including a further point $B$ we get two triangles, $ABX$ and $A´B´X´$ and the pair of Desargues' triangles is related by a homology or an elation. Thus an elation is determined when the axis and one pair of corresponding points are given and a perspective collineation induces a projectivity on any line through its centre. If it is an elation then a parabolic projectivity is induced and if a homology then a hyperbolic one. A pair of Desargues' triangles is then related by a perspective collineation. A harmonic homology is determined if H($OA;A_0 A´$) whereby $A_0$ is the intersection of $OA$ and $o$.

## Theorem 4.2: Homologies and Involutary Collineations.

*All pairs of corresponding points and lines of a homology are harmonically separated by the centre P and the axis p .*

Proof:

If the homology is given by $ABCD \rightarrow BADC$, then $A, B$ and $P, P_1$ separate each other harmonically ( complete quadrangle $DCRQ$ ).

---

[23] See Coxeter [5], p. 62.

If $E$ and $E'$ correspond to each other then $(AE).(BE')$ lies on the axis $p$ so that $E, E'$ and $P$, $P_2 = (EE').p$ are a projection of harmonically separated pairs from centre $(EA).(E'B')$ onto $EE'$. The latter are thus harmonically separated by $P$ and $P_2$, the projection of $P_1$.

We call the harmonic reflection *central symmetry* if the axis $p$ is the infinitely distant line, *skew axial symmetry* if the centre $P$ is the infinitely distant point and *axial symmetry* or mirror reflection if the direction to the centre $P$ is perpendicular to the axis and infinitely distant. The harmonic reflections are *involutory collineations*. The harmonic reflection in a line through $P$ or in point in $p$ generates a projectivity which is an involution.
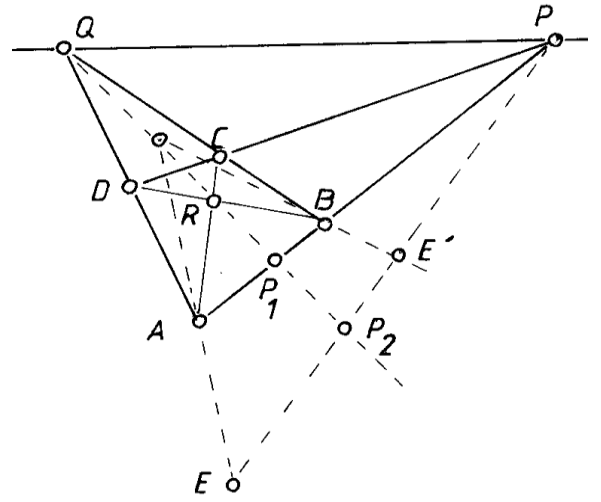


**Fig. 4.2**

Conversely an involutary collineation is a *harmonic homology*.

## Correlations

### Definition 4.2: Correlation
*We call a point to line and line to point transformation a correlation if they are ordered such that every point or respectively every line corresponds to one and only one line or respectively point, whereby the relationship of separate is retained.*

Thus points $X$ and $Y$ correspond to lines $x'$ and $y'$ respectively and line $XY$ transforms onto point $x'.y'$.

### Theorem 4.3: A Quadrangle and a Quadrilateral determine a Correlation[24]
*A quadrilateral defpqr and the corresponding quadrangle D'E'F'P'Q'R' are related by just one projective correlation.*

Proof:

An arbitrary point $A$, (see fig. 4.3) the intersection $x \cdot y$ should be projected onto exactly one line, the join $X'Y'$, with $x$ being part of the pencil $d \cdot e$ and $y$ of $d \cdot q$. The projectivities $def \barwedge D'E'F'$ by projecting $d, e, f$ first onto line $l$ and then, using another pencil and $m$ as the perspectivity axis further onto $D'E'F'$. The other perspectivity axis for $dqr \barwedge D'Q'R'$, is $n$ which is similarly constructed. These determine $a'$ where $a'$ is the join of $X'Y'$. Thus:

$$defx \barwedge D'E'F'X' \text{ and } dqry \barwedge D'Q'R'Y'.$$

In order to prove that $A \rightarrow a'$ we have to be sure that it preserves incidence in relating lines to their corresponding points.

We let $A$ move the so that $x \doublebarwedge y$. We then have for $a'$,

$$X' \barwedge x \doublebarwedge y \barwedge Y'.$$

As $d$ is an invariant line of the perspectivity $x \doublebarwedge y$, then $D'$ is an invariant point of the projectivity $X' \barwedge Y'$. Thus $a'$ turns in a pencil and collinear points correspond to concurrent lines. This relation goes in both directions: point to line and line to point and is therefore a correlation. The projectivity is of course unique and, using the principle of duality, we could use the projective correlation to relate a given quadrilateral to a given quadrangle.
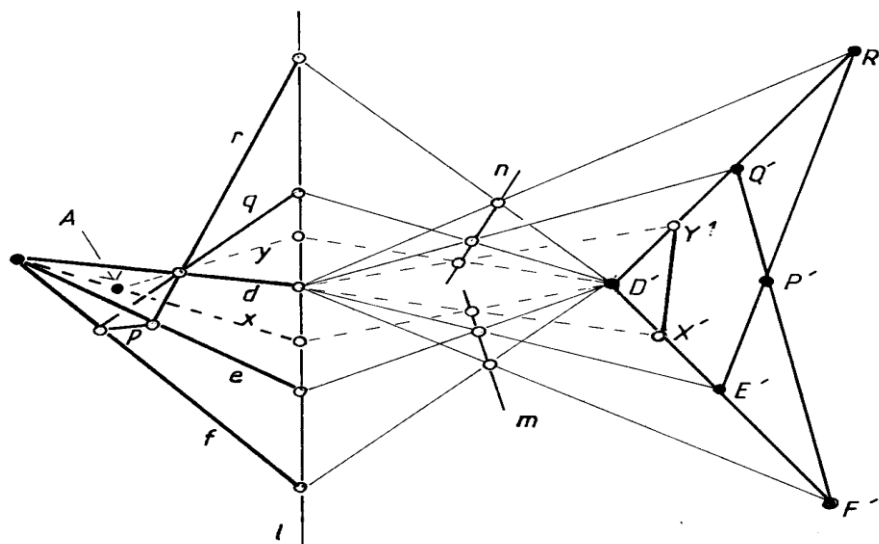
---

[24] See Coxeter [5], p. 58.

**Fig. 4.3**

## Involuting Correlations or Polar Systems

If the quadrilateral and quadrangle both lie in the same plane and if a point $X$ is related to a line $x'$ and this again to point $X''$ whereby $X'' = X$ then we have a correlation of period two and thus an involuting correlation. We can omit the $[\,']$ of $x'$ and say we are relating a point to a line and conversely.

### Definition 4.3: Pole, Polar, Polarity
*We call a point X and the line x to which it is correlated pole and polar and the relation a polarity. A polarity is then a projective correlation of period two.*

As a result of the properties of a correlation we can see that the polars of all points on a line $a$ form a projectively related pencil of lines through the pole $A$.

### Definition 4.4: (Self-) Conjugate Points, Conjugate Lines
*If A lies in b and a passes through B and we call A and B conjugate points; a and b conjugate lines. If A lies in a then A is a self-conjugate point and a a self-conjugate line.*

### Theorem 4.4: Self conjugate Points[25]
*The join of two self conjugate points cannot be a self-conjugate line.*
If the join $a$ of two self-conjugate points were a self-conjugate line, it would include its own pole $A$ together with at least one other self-conjugate point, for example $B$. The polars of $B$ including both $A$ and $B$ would then lie in $a$: because if point $B$ lies on line a then line $b$ goes through point $A$. The two distinct points would then both have the same polar, which is impossible as a polarity is a one to one correlation.


### Theorem 4.5: At Most Two Self-conjugate Points
*A line cannot contain more than two self-conjugate points*
Proof:
If $A$ and $B$ are two self-conjugate points on a line $s$, (see Fig. 4.4) and $P$ is a point on $AS$ or $a$ and its polar $p$ meets $b$ in $Q$ then $Q = b \cdot p$ is the pole of $BP = q$ which meets $p$ in $R$. Also $R = p \cdot q$ is the pole of $PQ = r$ which meets $s$ in $C$. But $C = r \cdot s$ is the pole of $RS = c$ which meets $s$ in $D$, the harmonic conjugate of $C$ w.r.t. $A$ and $B$. $C$ therefore cannot coincide with $A$ or $B$ as then

---

[25] See Enriques [7], pp. 184, 185. The same for theorem 4.5.

$P$ would coincide with $A$ or $S$. The points on $s$ are the invariant points of the projectivity $X \barwedge x \cdot s$ induced on $s$ by the polarity. Therefore the projectivity is not the identity and can't have more than two self conjugate points.

Therefore $C$ does not lie in $c$ and is not self-conjugate. On $s$ we then have two self conjugate points $A$ and $B$ and a non self conjugate point $C$.
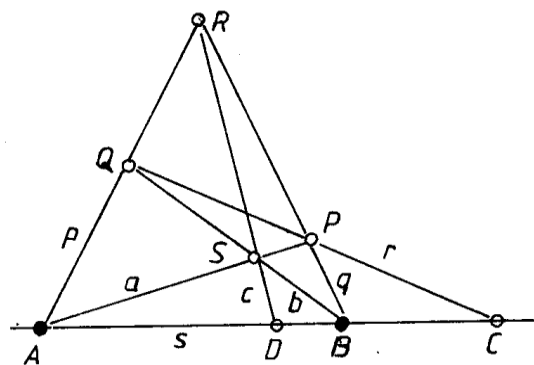


**Fig. 4.4**

## Theorem 4.6: A Polarity Induces an Involution[26]

*A polarity induces an involution of conjugate points on any non-self-conjugate line and an involution of conjugate lines through any non-self-conjugate point.*

Proof: Taking a non-self-conjugate line $c$ of the triangle $ABC$, (see Fig. 4.5) the projectivity, which is a polarity transforming $X = c \cdot y$ on $c$ into $x = CY$ in $C$ and $x = CY$ onto $y = CX$, transforms any non-self-conjugate point $B = a \cdot c$ into another point $A = b \cdot c$ whose polar is $BC$. The same projectivity therefore interchanges $A$ and $B$ as well as $X$ and $Y$ and must therefore be an involution there that interchanging two pairs of points defines an involution.
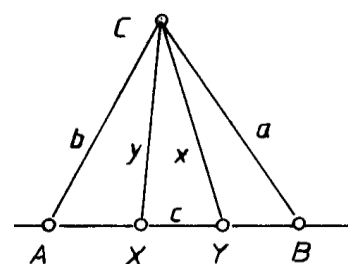


**Fig. 4.5**

## Theorem 4.7: [27]

*If the four lines of a quadrilateral are self-conjugate then at most one pair of opposite vertices are conjugate to each other.*

Proof:

If $pqrs$ is a quadrilateral of self-conjugate lines with $s$ through its own pole $S$ (see Fig. 4.6) and also through $A = p \cdot s$, $B = q \cdot s$, and $C = r \cdot s$ then its polars are of course $a = PS$, $b = QS$ and $c = RS$ (see Fig. 4.6). Taking $q \cdot r$ and $p \cdot s$ as conjugate points with $o$ as their joining line, $a$ then goes through $q \cdot r$ and is the conjugate of $o$ in the involution of lines through $q \cdot r$. As $q$ and $r$ are invariant lines of the involution, we obtain H($qr,oa$) and hence H($BC,AS$). If $r \cdot p$ were conjugate to $q \cdot s$, we
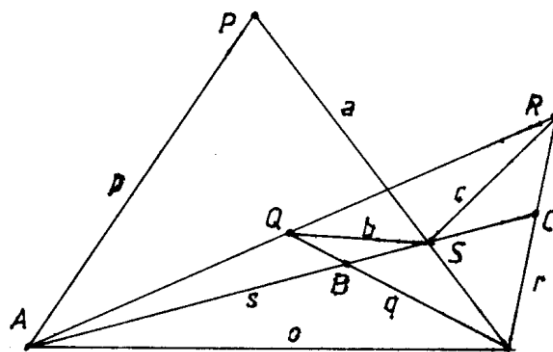


**Fig. 4.6**

would get H($CA,BS$) and if $p \cdot q$ were conjugate to $r \cdot s$ we would then obtain H($AB,CS$). But according to axiom 2.4 and theorem 2.1, the three separation relations are mutually exclusive. Thus, at most one pair of vertices are conjugates.

## Theorem 4.8: Hesse's theorem

*If two pairs of opposite vertices of a quadrilateral are pairs of conjugate points in a given polarity then the third pair of opposite vertices is also a pair of conjugate points.*

*If two pairs of opposite sides of a quadrangle are pairs of conjugate lines in a given polarity then the third pair of opposite sides is also a pair of conjugate lines.*

---

[26] See Coxeter [5], p. 62.
[27] See Coxeter [6], p. 69.

Proof:

If $QR$ is conjugate to $PS$ and $PR$ is conjugate to $QS$, we have to show that $PQ$ is conjugate to $RS$. If the six sides meet s, the polar of $S$ in (see Fig. 4.7) $A$, $A'$, $B$, $B'$, $C$, $C'$ then, assuming s is non-self-conjugate, as $PS$ goes through $S$ and is conjugate to $QR$, its pole is $s.QR = A$ and thus $PS = a$. Similarly $QS = b$ and the involution of conjugate points on s in $(AA')(BB')$ is established. As the three pairs of opposite sides meet s in pairs or an involution $RSC'$ is the polar of $C$ and conjugate to $PQC$.
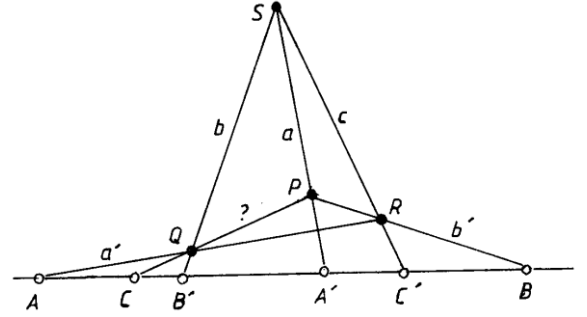


**Fig. 4.7**

## The Correlation of a Self-Polar Triangle

The polar triangle of a given triangle is formed by the polars of the vertices and the poles of the sides. If each vertex is the pole of the opposite side then the triangle is said to be *self-polar*. A correlation of a plane field onto itself is a *polar system* when there is a polar triangle in which vertices and opposite sides correspond to each other.

**Theorem 4.9: A Polarity is Determined by a Self-Polar triangle, Pole and Polar[28]**
*Any projective correlation that relates the three points of a self-polar triangle to its three opposite sides determines a polarity.*

Proof:
We consider the correlation where the quadrangle $ABCP$ corresponds to the quadrangle $abcp$, whereby $a$, $b$ and $c$ are the sides and polars of $A$, $B$ and $C$ respectively and $P$ is any point not lying on the triangle and $p$ any line cot going through any of the vertices. $P$ and $p$ determine the six points (see Fig. 4.8).

$$P_a = a \cdot AP, \quad P_b = b \cdot BP,$$
$$P_c = c \cdot CP, \quad A_p = a \cdot p,$$
$$B_p = b \cdot p, \quad C_p = c \cdot p$$
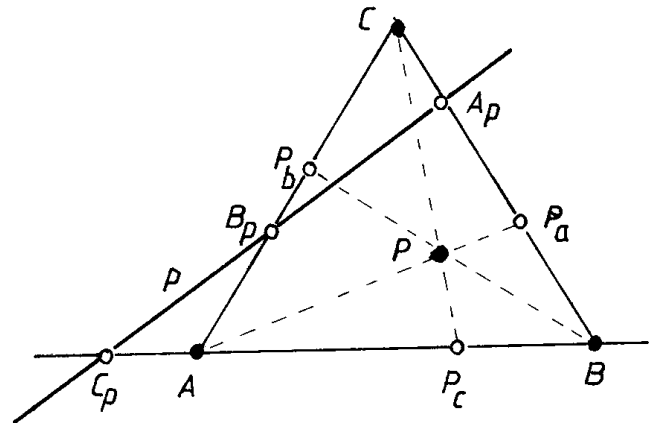


**Fig. 4.8**

The correlation transforming $A$, $B$ and $C$ onto $a$, $b$ and $c$ transforms $a = BC$ into $A = b \cdot c$, $AP$ into $C_p = a \cdot p$, and so on, and is thus a polarity: that is, a correlation transforming points into lines and lines into points. Or put otherwise, as $X$ runs along the whole of $c$, its polar $p$ turns in $C$ and the two points $c \cdot p$ and $X$, running in opposite directions, meet in $A$ and $B$. To see that not only is $p$ transformed onto $P$ but also $P$ onto $p$ we consider the following. Each point $X$ on $c$ is transformed into for example a line $x$ going through $C$ which intersects $c$ in $Y$. We then have $X \barwedge Y$. If $X$ is $A$, $Y$ is $B$ and if $X$ is $B$, $Y$ is $A$. The correlation is thus an involution.

As the correlation transforms $P_c$ into $CC_p$, the involution includes $P_cC_p$ which is $CP$. It similarly transforms $A_p$ into $AP$ and $B_p$ into $BP$ and therefore also $p = A_pB_p$ into $AP \cdot BP$ as required.

**Notation**: *Analogous to the notation for an involution we can use $(ABC)(Pp)$ to denote a correlation which is also a polarity of pole P and polar p and based on the self-polar triangle ABC.*

[28] See Coxeter [6], p.72.

**Theorem: 4.10 (Chasles)[29]**

*If the polars of the vertices of a triangle do not coincide with the opposite sides, they meet these in three collinear points.*

Proof:

If $PQR$ is the triangle, then its sides $QR$, $RP$ and $PQ$ intersect the polars $p$, $q$ and $r$ of $P$, $Q$ and $R$ respectively in $P_1$, $Q_1$ and $R_1$. The polar of $R_1 = PQ·r$ is $r_1 = (p·q)R$ and define also $P´= PQ·q$, $R´= QR·q$ and polar $p´= (p·q)Q$. By theorem and the polarity we then have $R_1PP´Q \barwedge PR_1QP´ \barwedge pr_1qp´ \barwedge P_1RR´Q$. According to theorem as $Q$ is invariant, $R_1PP´ \barwedge P_1RR´$ (see Fig.4.9). The centre of the perspectivity, $PR·P´R´= Q_1$ lies on the line $R_1P_1$. Therefore $P_1$, $Q_1$ and $R_1$ are collinear.
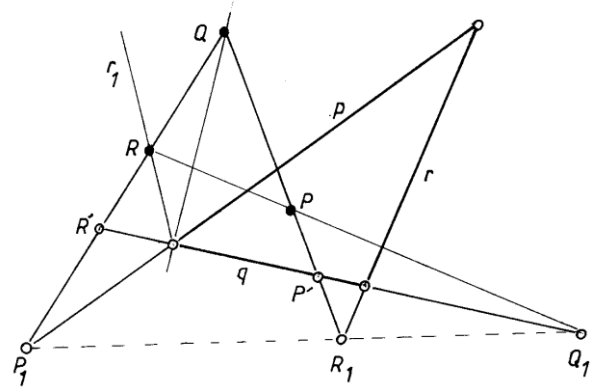


**Fig. 4.9**

The proof fails if $P_1$ or $Q$ lies in $q$. The construction can nevertheless be carried out by permuting the designations of respective points and sides, or alternatively, by interchanging the designations of triangle and trilateral.

We can now present the first step of determining the polar $x$ of any point $X$ not lying in $AP$ $BP$ or $p$.

**Theorem 4.11: Construction of the Polar[30]**

*The polar $x$ of a point $X$ in the polar correlation of $(ABC)(Pp)$ is the line $X_1X_2$ determined by*
$$\{(AP)·[a·(PX)][p·(AX)]\}\{(BP)·[b·(PX)][p·(BX)]\}$$

Proof:

We let:

| | | |
|---|---|---|
| $A_1 = a·PX,$ | $P_1 = p·AX,$ | $X_1 = AP·A_1P_1$ |
| $B_2 = b·PX,$ | $P_2= p·BX,$ | $X_2 = BP·B_2P_2$ |

Applying Chasles's theorem to $PAX$ we see that $AX$, $XP$ and $PA$ intersect the polars $p$, $a$ and $x$ of its vertices in three collinear points, the first two of which are $P_1$ and $A_1$. Therefore x intersects $PA$ in a point on $P_1A_1$ which is $PA·P_1A1 = X_1$.

By applying Chasles's theorem again to $PBX$ we obtain $X_2$.

Note: the construction fails if $X$ lies on $AP$ or $BP$.

Dualising gives us:
$a_1 = A(p·x),$ $p_1 = P(a·x),$ $x_1 = (a·p)(a_1·p_1)$
$b_2 = B(p·x),$ $p_2= P(b·x),$ $x_2 = (b·p)(b_2p_2)$

If we are given a self-polar triangle with a pole and polar and two conjugate points



**Fig. 4.10**

or lines, we are then in a position to form a pencil of polarities (If the two points are conjugates in a particular involution, we can always generate more conjugate points and therefore a whole pencil of lines.) or range of polarities (in the case of two conjugate lines).

Put another way, if $P$ is fixed and we let $p$ turn in a pencil of lines, we generate a pencil of lines which are the polars, $x_1$, $x_2$, etc. of a fixed point $X$. Changing the roles of $P$, $p$ and $X$, $x_1$, $x_2$, etc. we can also fix $p$ and let $x$ turn in a pencil of lines. This then gives us a range of polarities which are the poles $X_1$, $X_2$ etc. of the lines $x_1$, $x_2$, etc.
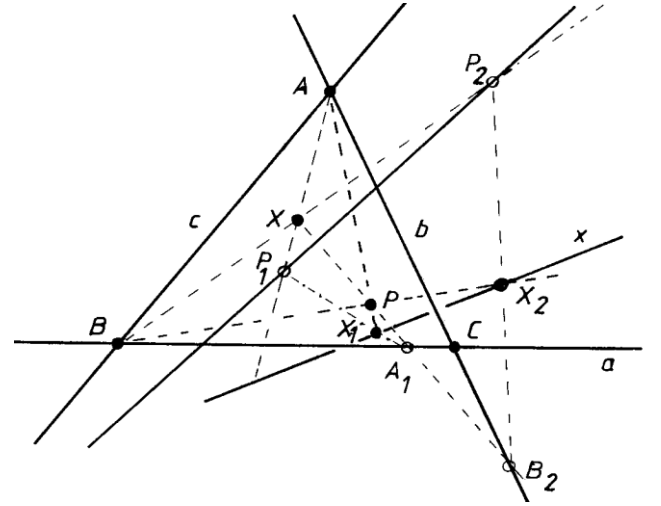
[29] See Coxeter [5], p. 64.
[30] See Coxeter [6], p. 72.

**Theorem 4.12: A Pencil and Range of Polarities in a Self-Dual System[31]**

*The polars of a fixed point X in a pencil of polarities form a pencil of lines excepting X were a point of the common self-polar triangle.*

Proof:

If we let $p$ rotate about a fixed point $P'$ then $Y$ runs up or down $AP$ and $Z$ up or down $BP$ so that:

$$Y \overset{A_1}{\overline{\wedge}} E \overset{P'}{\overline{\wedge}} F \overset{B_1}{\overline{\wedge}} Z$$

This projectivity has an invariant point $P$ when $p$ is the line $P'X$. Then $E$ and $F$ are coincident with $X$ while the line $x$ passes through a fixed point $X$.

An important special case is when $P'$ is on one of the sides of the polar triangle, for example side $a$. (see Fig. 4.11 below) What happens then is that the involution of conjugates on $a$ which is $(BC)(A_pP_a)$ is the same for all of the polarities and the self-polar triangle is no longer unique. The poles $X_1$, $X_2$ etc. projects from $A$ onto $X_a$ on $a$ and the polars of $x$ form a pencil of lines through $A_x$ which is the conjugate of $X_a$ . We could then take the polar triangle $\Delta AA_x X_a$ as a polar triangle. Conversely, if any line $x$ intersects a in $A_x$ , its poles all lie on the fixed line $AX_a$. We can thus conclude:

**A self-dual system of polarities has a line on which the involution of conjugate points is the same for all polarities. The polars of any point $P$ form a pencil of lines through $P' = A_p$ and the poles of any line $p$ form a range of points on $AP_a$ with A being the pole of $a$ for all polarities.**
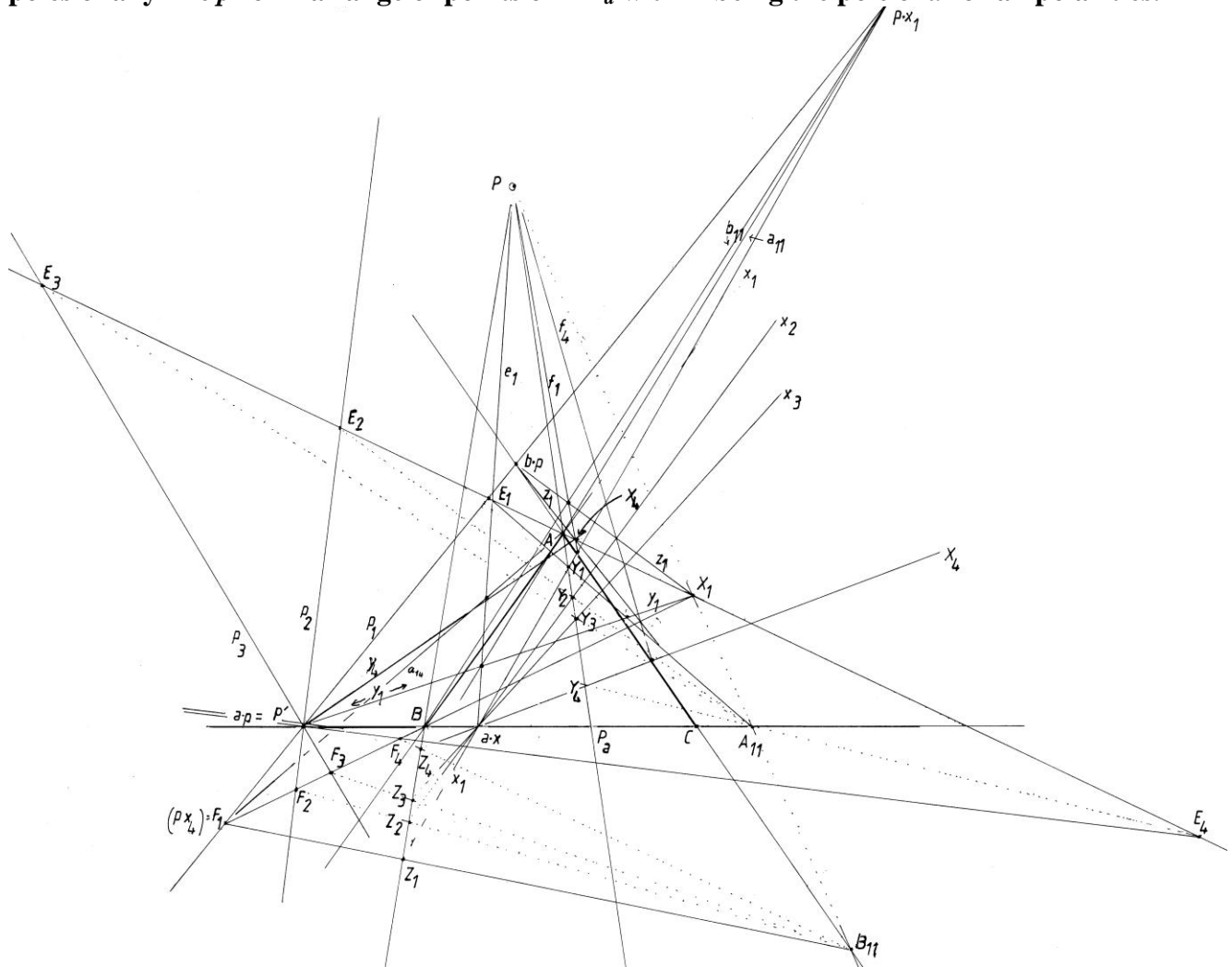


**Fig. 4.11**

---

[31] See Coxeter [6], p.76.

In Fig. 4.11 the following denotation has been used:

$A_{11} = a \cdot PX$,    $E = p \cdot AX$,    $Y = AP \cdot A_{11}E$        $a_{11} = A(p \cdot x)$,   $e_1 = P(a \cdot x)$,   $y = (a \cdot p)(a_1 \cdot e_1)$

$B_{11} = b \cdot PX$,    $F = p \cdot BX$,    $Z = BP \cdot B_2F$        $b_{11} = B(p \cdot x)$,   $f_1 = P(b \cdot x)$,   $z = (b \cdot p)(b_2 f_1)$

$P$ was turned from $P_1$ through to $P_4$ so that $E$ moved up $AX_1$ from $E_1$ and returning from the other direction to $E_4$ thus requiring $Y$ to move down $AP$ form $Y_1$ to $Y_4$. $F$ moved up $BX_1$ from $F_1$ to $F_4$ requiring $Z$ to move up $BP$ from $Z_1$ to $Z_4$. This resulted in $x$ turning in a clockwise direction from $x_1$ to $x_4$ *in the point $a \cdot x$ on $a$!*

In a second step, the poles of $x$ were constructed (this was only possible for $x_1$ and $x_4$ as there would have been too many lines in the drawing). As $x$ turned in a clockwise direction from $x_1$ to $x_4$ in the point $a \cdot x$ on $a$, *X moved up a fixed line* through $A$ from $X_1$ to $X_4$.

The first point of the exercise was to demonstrate the self-dual polarity. Turning $P$ in $a \cdot p$ resulted in another line turning in $a \cdot x$, while X runs up a fixed line and these latter two, being invariant can therefore, together with $A$, be taken as another polar triangle. The second point was to present a construction which would probably only exist on the writing desk of a projective geometer.

The most surprising aspect of the construction though was the accuracy of it. After so many steps, $X$ landed exactly on the fixed line where it was supposed to!

In setting up the construction an ellipse was used together with line $a$, out of which the original polar triangle was constructed along with $p_1$ and $P$. ABC, $P$ and $p_1$ were thus related to a real conic. All other lines and points are genuine constructions.

## Determining Elliptic and Hyperbolic Polarities on a Self-Polar Triangle

There are two different types of polar systems which we will now proceed to characterise.
The polar triangle divides the plane into four regions. The segments bordering the region in which we place the point $P$ can be assigned $+++$ to designate the segments of $a$, $b$ and $c$ respectively which are intersected by the lines $AP$, $BP$ and $CP$ respectively. A moment of consideration allows us to see that any line on the plane intersects the segments of $a$, $b$ and $c$ respectively in one of the possibilities of either 1) $+ + -$ ,$- + +$ , $+ + -$ , or 2) $- - -$.

For example, the placement of $P$ results in $+ + -$ for the respective segments of the lines $a$, $b$ and $c$ intersected by our polar. If we then use the above construction to locate the polar of a new position $P'$ and consider the relative movement of the intersection of the polar with each side and the intersection of the line joining P to the pole of that side:

    1) *a·p* and *a·AP* on *a*        2) *b·p* and *b·BP* on *b*        3) *c·p* and *c·CP* on *c*

Then we find that: in 1 and 2 the pairs move in opposite directions corresponding to the respective + signs for the segments on $a$ and b and in 3 they move in the same direction, corresponding to the $-$ sign for the $c$ segment in which the polar $p$ intersected $c$.

This is shown in the two figures 4.12a and b.

In the first instance, an opposite or hyperbolic sense of movement of the aforementioned points is generated on sides $a$ and $b$ and an elliptic sense of movement is generated on side $c$. This corresponds to the $+ + -$ designation of the polar in the figure.
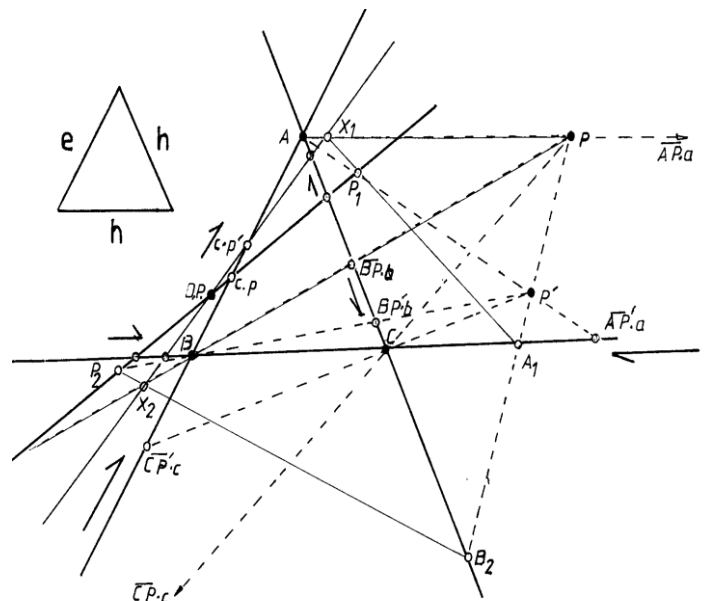


**Fig. 4.12a**

In the second figure, all three senses of movement of the pairs of points are elliptic, corresponding to the $- - -$ of the original position of the polar. The method enables us to pre-plan on which sides we want to have our hyperbolic and on which sides our elliptic polarity, which at the same time are hyperbolic or elliptic involutions of points and lines. This leads to the following definition.
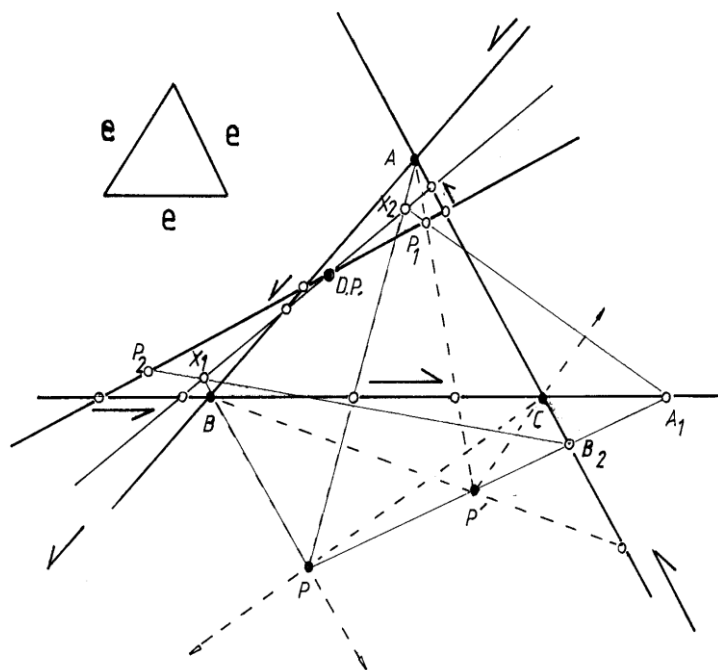
**Fig. 4.12b**

**Definition: *E point, e line, H point, h Line***
*A point lying in a line that is not self-conjugate is an E point or H point according to whether the involution of lines passing through it is elliptic or hyperbolic.*
*A line going through a point that is not self-conjugate is an e line or h line according to whether the involution of points lying in it is elliptic or hyperbolic.*
We can see that in an elliptic polarity all points are *E* points and all lines are *h* lines.
For a hyperbolic polarity, two sides are *h* lines and the third is an *e* line.
In the case of a hyperbolic polarity, every point on an *e* line is an *H* point, being the point in which a pencil of *h* lines lie, but the two *h* lines contain both types of points, including a pair of self-conjugate points, these being the invariant points of the hyperbolic involution on that line. In the case of an elliptic polarity, all points and lines are E points and respectively e lines.
It turns out that the *E* points and *H* points on an *h* line are separated into two segments by the two self-conjugate points. This will have to be proved though.

**Theorem 4.12: An *h* Line Consists of Two Segments[32]**
*An h line consists of two segments; one of E points and one of H points.*
Proof:
Let *o* be an *h* line including the two self-conjugate points *Q* and *R* and *A* be an *H* point (see Fig. 4.13). Let also *P* be one of the self-conjugate points on *a*, the self-conjugate line through *A* and let *a* intersect *o* in $A_1$. Taking any point *X* in the segment *QR/A*, we let its polar *x* intersect *a* in *O* and also *o* in $X_1$ as well as *P'A* in *K* and *PX* in *K'*. As we then have H(*QR,AA₁*) and also H(*QR,XX₁*) then one of the segments contains *A* as well as $X_1$ while the other contains $A_1$ as well as *X*. As the correspondence between the harmonic conjugates with respect to
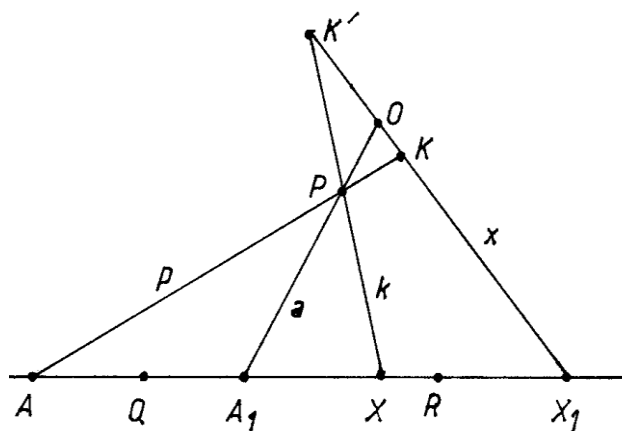
**Fig. 4.13**

[32] See Coxeter [6], p.83.

45

their two fixed points $Q$ and $R$ is opposite, the order of the points on $o$ is either (see Fig. 4.13):

$$QAX_1RXA_1Q \qquad \text{or} \qquad QX_1ARA_1X.$$

Out of this follows that $AX//A_1X_1$. But we can see that when we take $P$ as our perspectivity centre, then $AXA_1X_1 \overline{\overline{\wedge}} KK'OX_1$ .In this case $KK'//OX_1$ and the involution $(KK')(OX_1)$ of conjugate points on $x$ is elliptic. Thus $x$ is an $e$ line and $X$ an $E$ point. We can therefore conclude that $QR/A$ is the segment consisting entirely of $E$ points and $QR/A_1$ the other segment consisting of $H$ points.
If we dualise this we find that the $e$ lines and $h$ lines through an $H$ point are separated by just the two self-conjugate lines.

## How a Hyperbolic Polarity Defines a Conic Section

Given only a self-polar triangle $ABC$ together with any pole $P$ and polar $p$ which are placed so that a hyperbolic polarity is defined, we can construct the unique conic section induced by the polarity (see Fig 4.14 below).
The positioning of $P$ and $p$ has induced a hyperbolic polarity on sides $b$ and $c$ of the polar triangle and defines points $p·b$ and $BP·b$. Using the construction of a harmonic representation of a hyperbolic involution on a line, we can project the four points $A, C, p·b, BP·b$ through $M$ onto the points $A_c, C_c, p·b_c, BP·b_c$ on a 'dummy conic'. The joining lines $A_cC_c$ and $(p·b_c)(BP·b_c)$ of the conjugate points in the involution delivers $O$, their intersection. The tangent lines from $O$ to the dummy conic then give us the invariant points $E_c$ and $F_c$ of the projected involution. Projecting these back down onto $b$ positions $E$ and $F$ and also the tangents $t_1 = BF$ and $t_2 = BE$. Repeating the same for side $c$ results in $G$ and $H$ (and $t_3, t_4$). As we have a self-dual correlation of the plane onto itself, there is a hyperbolic involution on any secant (here $PH$) with $P$ and its intersection with $p = p·PH$ being harmonic conjugates in the involution. This enables us to construct $I$ as the second self-conjugate point (to $H$). The five points thus obtained define the conic. Further points can be obtained by either:
1) repeating the aforesaid for other known points, for example $PF$ or
2) using Pascal's theorem (our 3.14). We can intersect $GH$ with $EF$ to obtain $A$ (already existent) constructing any perspectivity axis $z$ through it. $K = GI·z$ and $L = FI·z$ enable us to construct $J = (HL)(EK)$. Repeating the aforesaid for other sets of five points we can construct any required number of points.
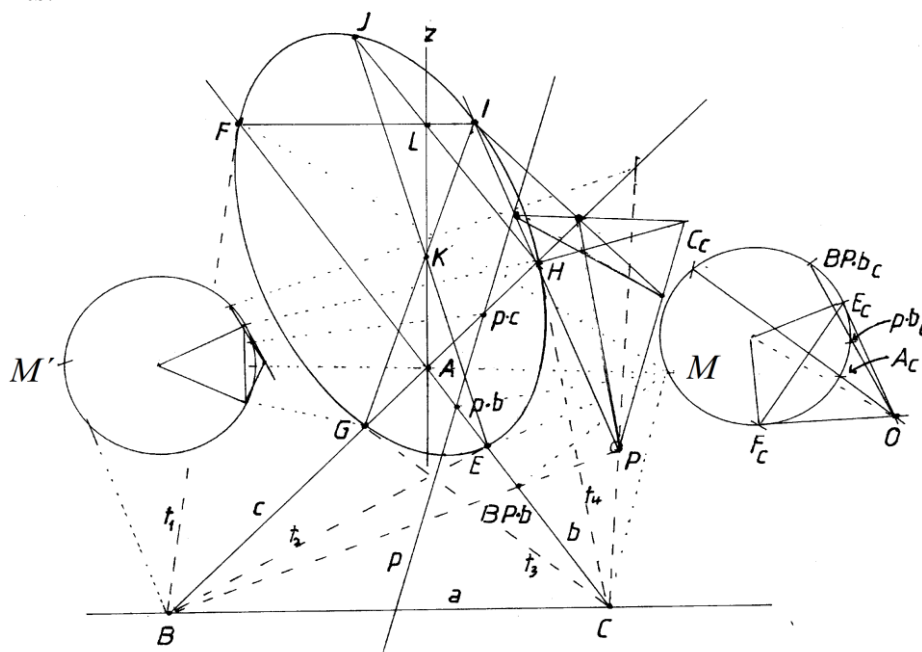


**Fig. 4.14**

The construction is surprisingly simple: the author has taught much more complicated Euclidian constructions to 14 and 15 year olds. The involution concept would only be suitable a couple of

years later though.

We've now shown that: *a hyperbolic polarity induces a real conic section in the plane and conversely, a conic section induces a self-dual hyperbolic polarity of the lines and points in its plane!*

There is still a question to be answered. The significance of the hyperbolic involutions on lines *b* and *c* in Fig. 4.15 should be apparent. What though, is the significance of the elliptic involution on line *a*? This will turn out to be an expression of the imaginary intersection points of line *a* on line *a* with the conic, just as the hyperbolic involutions on *b* and *c* were the expression of the real intersection points of those lines with the conic section (the invariant points of the involutions). Before introducing this though, we'll look more closely at the converse of this chapter: the relationship between a conic and lines and points on the plane.

# Chapter 5    Conics and the Imaginary

**Definition: Conic**
*A conic is the locus of self-conjugate points in a hyperbolic polarity.*

Every hyperbolic polar system defines a conic consisting of all self-conjugate points of the involution generated on the lines of the pencils of the *H* points on the *e* line of the polar triangle, as we saw at the end of the last chapter.

**Theorem 5.1: Harmonic Conjugates on a Line and in a Point**
*Given a hyperbolic involution with invariant points P and Q then two other conjugate points on PQ are harmonic conjugates with respect to P and Q.*
*Given a hyperbolic involution then two conjugate lines p, q are harmonic conjugates with respect to the tangents p and  q.*
Proof:
(See theorem 3.8) If the real line intersects the conic then the real self-conjugate points *P* and *Q* on the line *PQ* are the invariant points (tangent points) of the involution as are the self-conjugate lines p, q the invariant lines (the tangent lines) (see 5.1a, b below).

**Theorem 5.2: Self-Polar Diagonal Triangle of a Quadrilateral or Quadrangle**

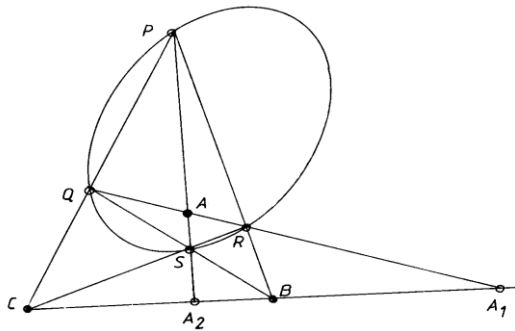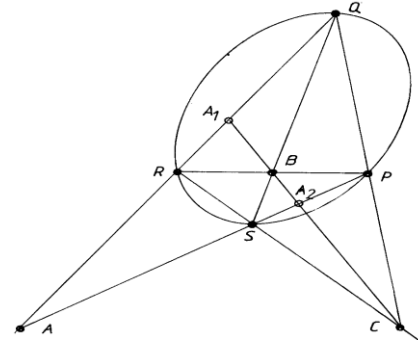| *The diagonal triangle of a quadrangle inscribed in a conic is self polar.* | *The diagonal triangle of a quadrilateral circumscribed on a conic is self polar.* |
|---|---|



**Fig. 5.1a**



**Fig. 5.1b**

Proof:
The diagonal points of the inscribed quadrangle *PQRS* (see Fig. 5.6a) are
$$A = PS{\cdot}QR, \qquad B = QS{\cdot}RP, \qquad C = b\ RS{\cdot}PQ.$$

*BC* intersects *QR* and *PS* in $A_1$ and $A_2$ so that H(*QR*,*AA₁*) and H(*PS*,*AA₂*). These, according to theorem 5.1, are then both harmonic conjugates to *A* and the line *BC* joining them is the polar of *A*, as is *CA* the polar of *B* and *AB* of *C*.
If the four points of the quadrangle are imaginary (see Fig. 5.5) then the diagonal triangle is the real triangle *XST*.
$$X = (A_iB_i){\cdot}(C_iD_i) \quad S = (AC){\cdot}(BD) \quad T = (BC){\cdot}(AD)$$
*XS* is intersected by *BC* and *AD* in points $A_1$ and $A_2$ (as we have H(*XY*,*UU´*) and H(*XZ*,*VV´*) because we constructed them so that this is the case). According to theorem 5.1 point *T* is then the pole of *XS*. Similarly, *S* is the pole of *XT* and *X* the pole of *ST*.

**Theorem 5.3: Inscribed Quadrangle, Circumscribed Quadrilateral, Diagonal Triangle**
*A quadrangle PQRS inscribed in a conic and a quadrilateral pqrs circumscribed on the conic, whereby the quadrilateral consists of the tangents to the quadrangle points, have the same diagonal triangle which is self-polar.*

Proof:

*A* lies on the polars of both *q·r* (= *QR*) and *p·s* (= *PS*) (see Fig. 5.2a and b ). Hence the diagonal triangle side *a* = (*q·r*)(*p·s*) is the polar of *A* and coincides with *BC*. *B* lies on the polars of both *q·s* (= *QS*) and *p·r* (= *RP*). Hence the diagonal triangle side *b* = (*q·r*)(*p·s*) is the polar of *B* and coincides with *AC*. *C* lies on the polars of *s·r* (= *RS*) and *p·s* (= *PS*). Hence the diagonal *c* = (*s·r*)(*p·s*) is the polar of *C* and coincides with *AB*.
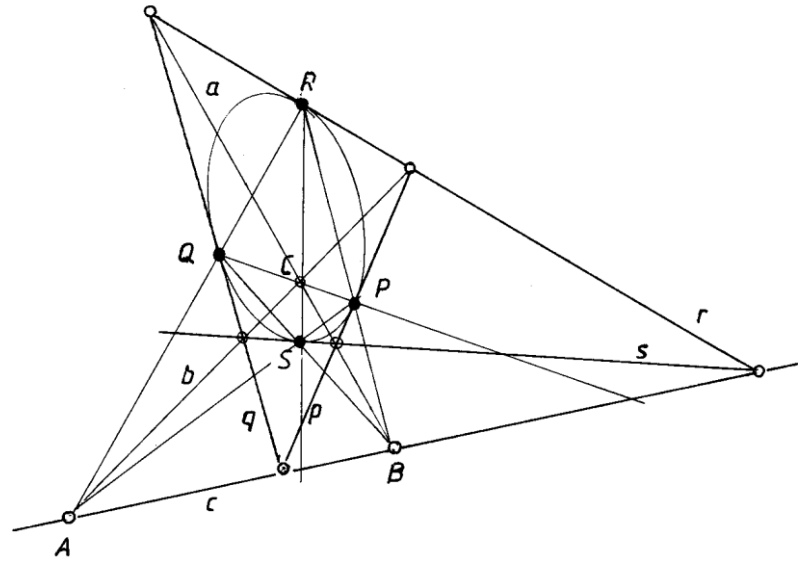


**Fig. 5.2a**

Fig. 5.2b has been included for the sake of completeness

What we have arrived at is really quite astonishing: a conic section induces a self-dual system of polarities in the plane and when the points of a quadrilateral inscribed in the conic section are self conjugate points (the tangents of the circumscribed quadrangle which are self conjugate lines pass through the quadrilateral points) the diagonal triangle/ trilateral of both is the same.
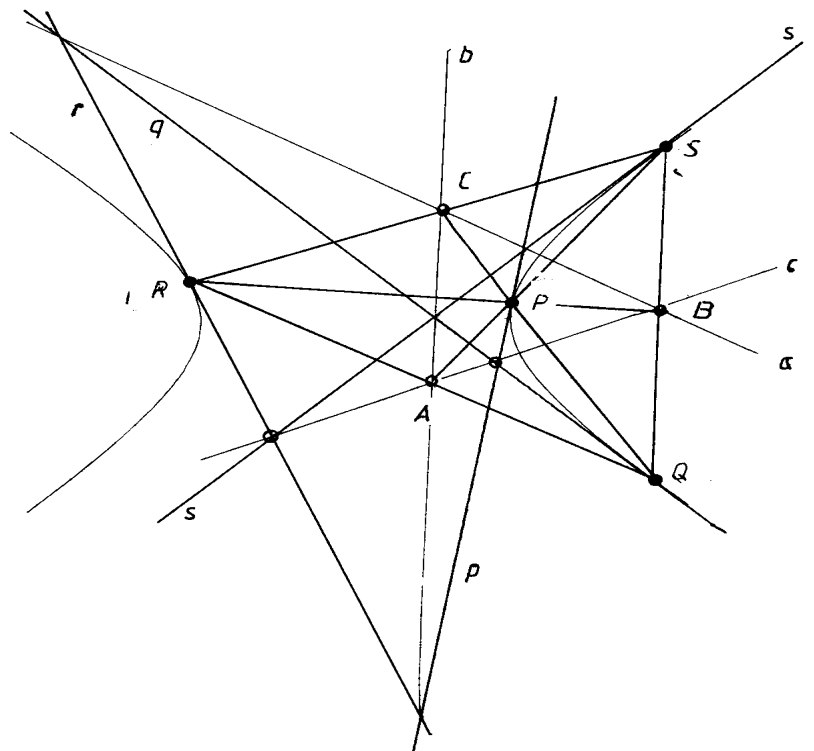


**Fig. 5.2b**

### Theorem 5.4: Seydewitz; Triangle/Trilateral In/Circumscribed in a Conic

| | |
|---|---|
| *If a triangle is inscribed in a conic, a line conjugate to one side of the triangle intersects the other two sides in conjugate points.* | *If a trilateral is circumscribed on a conic, a point conjugate to one point of the trilateral joins the other two points in conjugate lines.* |

Proof:

In triangle $RSX_a$ (see Fig. 5.3a, b) a line $p_p$ conjugate to *RS* is the polar of a point $P_p$ somewhere on *RS*. If $X_aP_p$ meets the conic in *Q*, then according to theorem 5.3, $p_p = AA´$ where $A = X_aS·QR$ and $A´= QS·RX_a$. The conjugate points *A* and *A´* are where the sides $x_1$ and $y_1$ of the triangle intersect $p_p$.

 Due to the constructive importance of this theorem, the four cases have been included.
In Fig. 5.3a for which the proof has been given, the construction can be used to generate an elliptic

involution of points in a line just as Fig. 5.3b gives the construction for generating points in a hyperbolic involution.



**Fig. 5.3a**



**Fig. 5.3b**

For the dual of the proof Fig 5.3c can be used as care was taken to ensure that the nomenclature is the exact dual of that used in Fig. 5.3a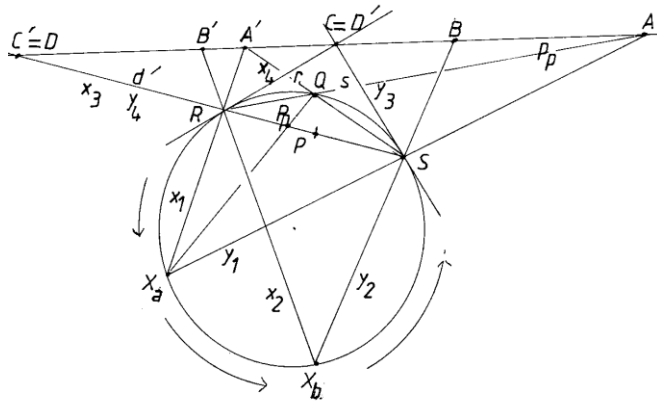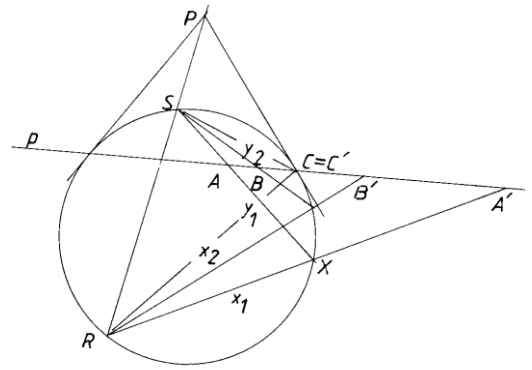. Again, the two versions, one for an elliptic involution of lines and the other for the hyperbolic case have been included.
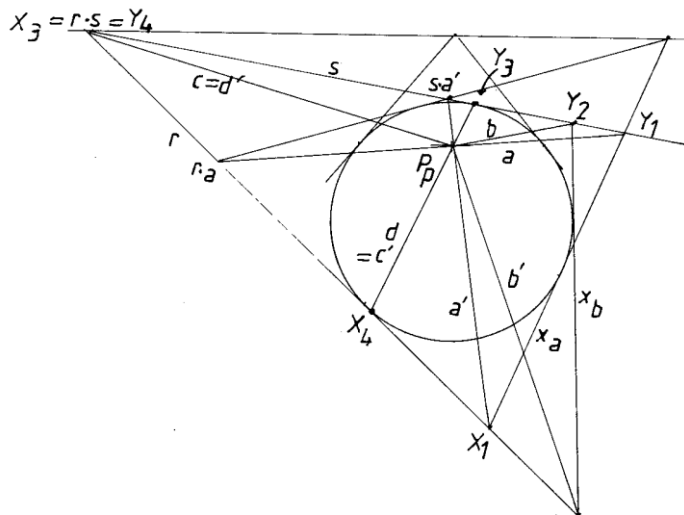


**Fig. 5.3c**
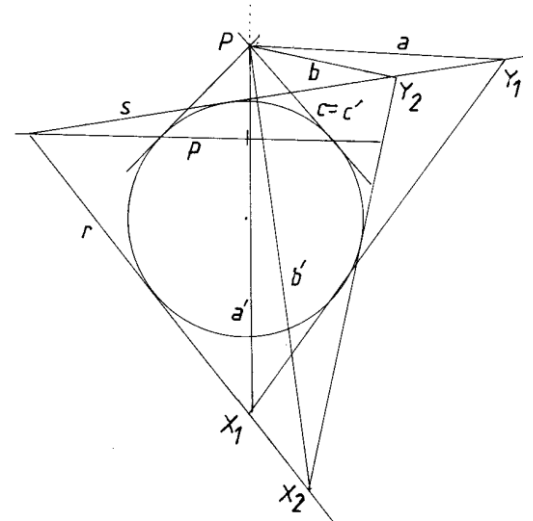


**Fig. 5.3d**

We'll now move on to a projective description of how a conic is generated. Up until now, we have been using von Staudt's definition of a conic: that as the locus of self conjugate points in a hyperbolic polarity. The geometer Josef Steiner gave another: That the conic is the locus of the point of intersection of corresponding lines of two projective, but not perspective pencils. In fact the two definitions can be satisfactorily reconciled.[33] We won't do this but instead move on to give Steiner's construction of a conic from five given points or from five given lines as we'll be using von Staudt's involution construction in the given examples.

**Theorem 5.5: Steiner's Construction of a Conic**

*If lines x, x´ of two pencils A and B are projectively but not perspectively related, the locus of the intersection point x·x´ of the projectively related lines x and x´ of the pencils inscribe a conic.*

*If points X, X´ of two ranges a and b are projectively but not perspectively related, the locus of the joining line X·X´ of the projectively related ranges X and X´ of the ranges circumscribe (envelope) a conic.*

Proof:

The projectivity $x \overline{\wedge} x$ is not a perspectivity as the line $AB$ ($= b$) does not lie on line $BA$ ($= a´$).  (see Fig. 5.9) If  c, c´ and d, d´ and e, e´ are three pairs of corresponding lines then according to the

[33] See Coxeter  [6], pp.87, 88.

fundamental theory (Theorem 3.1) the projectivity is uniquely defined and hence also the conic. The intersection points $c \cdot c'$, $d \cdot d'$, and $e \cdot e'$ are three such points of the conic. As is apparent from the construction, point $C$ corresponds to itself, making line $DC$ perspective to line $EC$. This determines the perspectivity centre $N$ of the two ranges and therefore the unique projectivity between the two pencils through $A$ and $B$. To find further pairs of corresponding lines we just have to find the intersection point $X_1$ of the line $x$ in $CD$, project this onto $X_2$ on $CE$ and this gives us the corresponding line $x'$ of the pencil in $B$.

If any general line $l$ intersected the projective range in more than two points, three for example then the projectivity between $A$ and $B$ would be a perspectivity as it would be determined by these three points with $l$ as the axis of the perspectivity.

This is excluded there that the common line of the two pencils does not correspond to itself. Line $b$ passing through $A$ and $B$ and intersecting $CD$ in $B_1$ corresponds to line $b'$ given by the intersection of line $NB_1$ and line $CE$.
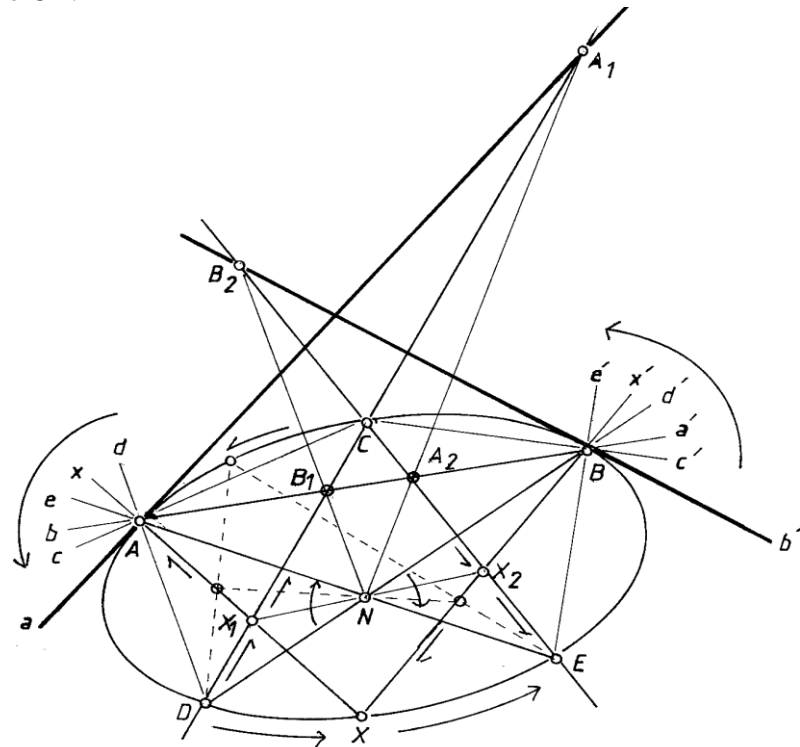


**Fig. 5.4a**

Following the construction, every other line except $a'$ intersects the conic in two points: here both intersection points coincide. We call lines $a$ and $b'$ *tangents*. The point $a \cdot b'$ is called the *pole* of the *polar AB*.

We could consider the construction as a whole complex of movements. As $d$ turns to $x$, to $e$, $b$, $c$ the point $D$ moves to $X$ and on to $E$, $B$, $C$ and $A$ while the points on the fixed line $DC$ run up from $D$ to $X_1$ to $B_1$, C, $A_1$ and over the infinite line and back to $D$. Similarly, the points on the fixed line $CE$ run down that line as the line. Due to the significance of this and its dual construction, the dual construction has been included but without proof (see Fig. 5.4).

The construction gives us more though. If four points, for example $A$, $B$, $C$, $D$ and the tangent in one of them, for example in point $A$, then the line $DC$ immediately gives us point $A_1$. We can choose any point $A_2$ on line $AB$ and construct $A_1A_2$ which intersects $DB$ in $N$. The intersection of $AN$ with $CA_2$ delivers us with the fifth point $E$ and we can continue with the construction as described above.
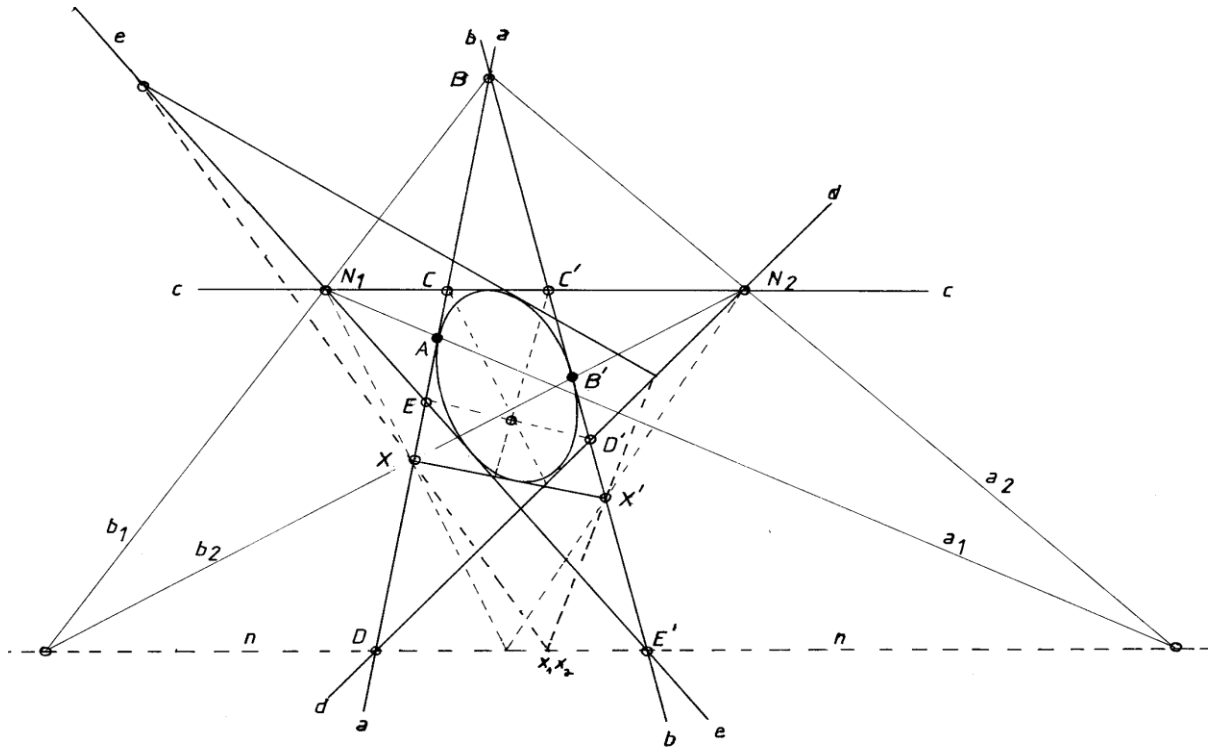
**Fig. 5.4b**

## Theorem 5.6: Three Points with Tangents Determine a Conic
*A conic is determined by three points and the tangents through two of these.*
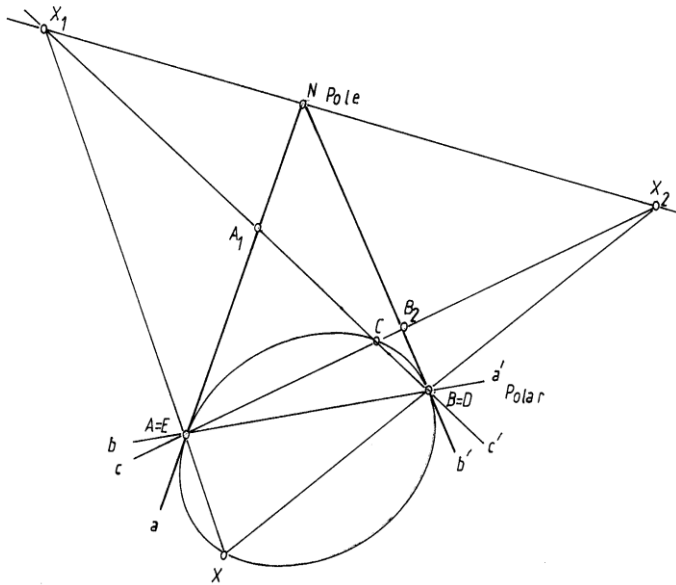


**Fig. 5.5a**
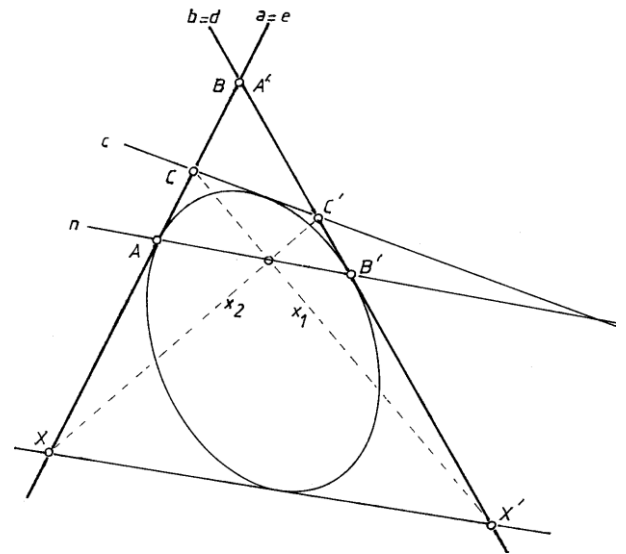


**Fig. 5.5b**

If, instead of five points, we are given three points with the tangents in two of them we proceed as follows: Assume point $E$ of Fig. 5.4 has moved onto $A$ and $D$ has moved onto $B$ (see Fig 5.5a), giving us double points and therefore the tangents $a$ and $b'$ in these points. Line $a$ in $A$ corresponds to $a'$ in $B$ and $b'$ in $B$ to $b$ in $A$. Therefore the two pencils $A$ and $B$ are not perspective to each other. The three lines $a$, $b$ and $c$ in $A$ are then projectively related to $a'$, $b'$ and $c'$ in $B$. The intersection points of $a$, $b$ and $c$ on $c'$ are $A_1$, $B$ and $C$; those of $a'$, $b'$ and $c'$ on $c$ being $A$, $B_2$ and $C$. As $C$ is self-corresponding, the two ranges are perspective to each other with the intersection point $AA_1 \cdot BB_2 = N$ being the perspectivity centre. Any other line $x = XX_1$ in $A$ intersects $c'$ in $X_1$. This point is projected onto $X_2$ on $c$ which then gives us the line $x' = X_2B$ and the Point $x \cdot x'$ on the conic.

52

The corresponding figure to fig. 5.5a is given in fig. 5.5b.

In conclusion we can state the following:

*A second order range is uniquely determined by either 5 points or 4 points with a tangent to one of them or 3 points with tangents in 2 of them.*

*A second order pencil is uniquely determined by either 5 tangents or 4 tangents with a point in one of them or 3 tangents with points in 2 of them.*

Steiner's theorem gives us the possibility of regarding the range of points or pencil of lines on a conic (considered as a one-dimensional object) just as any other one-dimensional range or pencil and consequently delivers us with the possibility of relating it to any other range on a straight line or pencil in a point. Stated otherwise: our whole theory concerning projectivities on a line can be transferred en masse to projectivities on a conic.

At this point we need to widen our definitions to include the elliptic involution on the *e* line.

# Constructing Imaginary Points and Lines[34]

When we consider the picture the repeated direct hyperbolic projectivity gives us (see Fig. 3.3), we see that for two invariant points (or lines) there is an infinite set of possible projectivities (each defined by the positions of *R* and *S*) but which are all characterised by the two invariant points *M* and *N*. If we then relate the points in the two segments in an involution given by (*11´*)(*22´*) we obtain a series of steps of the position of a point progressing in one direction which is related to the point progressing in the other direction (the conjugate in the involution. The number of steps will of course depend on the position of R and S and the projectivity can be either periodic or not but any point *X* and its conjugate *X´* will move along the line in their respective segments 'between' (in the Euclidian sense) the two invariant points *M* and *N*, the movement of both points being 'slowest' when the points are close to either of the two invariant points and greatest when half way between them. The involution then relates exactly half of the points ¼ of the interior points to ¼ of the exterior points: the movement is opposite and the two points `collapse´ into the invariant points.

When we then consider an elliptic projectivity under the same circumstances (see Fig. 3.7), we are confronted by the problem that there are no invariant points. Considering a 90° involution with 10 equal angled steps given by (1 10)(2 11) (1 at 0°, 2 at 10°, 10 at 90°, 11 at 100°) acting on a repeated direct elliptic involution of say 1° we see that as one point races in from infinity, its conjugate crawls along from the middle of this densest concentration of points to the position where the two points are closest. Between the middle of the incolution and each of these two points then lie exactly ¼ of the points of the line. As the first point then moves on towards the middle of the concentration the second point races away from it with ever increasing velocity towards the infinite point before returning from the other 'side'. The movement never 'dies' to a real point but instead 'grows' out of this shortest distance between conjugate points.

Relating the aforesaid to a conic (see Fig. 5.6) and considering our self-dual system, we can say that the sum of the absolute values of the velocities of the points (and the lines) is a maximum at the mid-position of the hyperbolic involution on a line passing through the conic and decreases to a minimum (here zero) at the real intersection points of the conic with the line. Between the maximum and the minimum of the absolute values sum lie ¼ of the interior points (likewise ¼ of the exterior points).

*If our conic were in a Cartesian coordinate system and we were solving for the intersection points*

---

[34] For an excellent introduction to this chapter but using a more algebraic approach see  Kowol [8], pp. 200-225.

*of the conic with the line, we would arrive at the result of x = a $\pm$ ci , y = b $\pm$ di with the mid-point M(a,b) and the two conjugates A(a+c,b+d), A´(a-c,b-d) being in agreement with our construction. (i being equal to the square root of negative 1)*

Apparently imaginary points and lines are intimately connected with elliptic involutions. In the following construction we have intersected the conic (the ellipse) with a set of parallel lines, thereby making a tomography of the effect of the conic on the plane. On each line there is an involution and the smallest distance (the minimum sum of the absolute values of the velocities) between conjugate points is the distance between the arrow heads on any line.
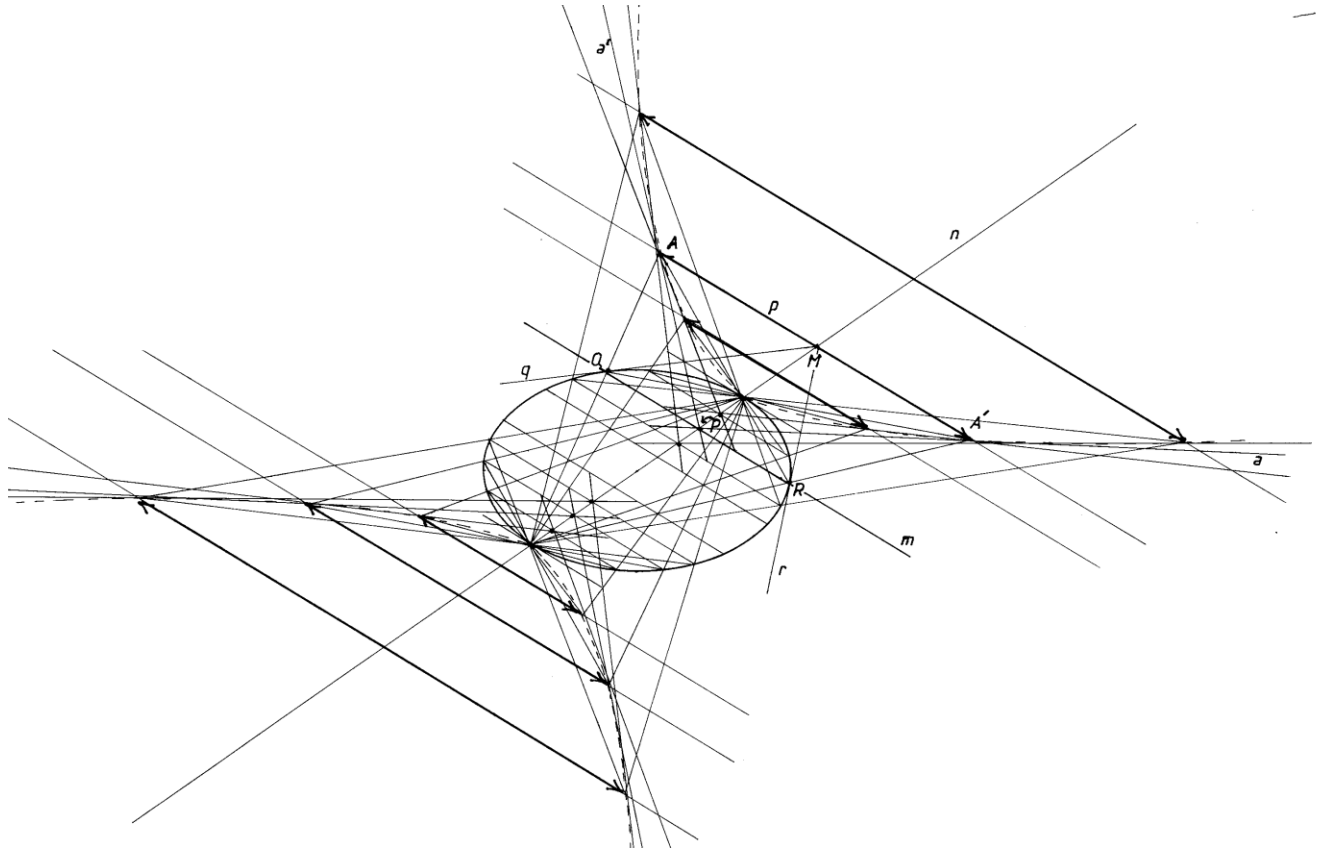


**Fig. 5.6**

For an ellipse, the arrow heads lie on a hyperbola. (Repeating the construction for a hyperbola, we find that the arrow heads lie on an ellipse, whereas for a parabola they lie on a parabola.) We could have intersected the ellipse with a different set of parallel lines and we would again have obtained a set of intersection points; real conic section points for the lines going through the ellipse and imaginary for the lines not intersecting the conic section. Apparently our ellipse covers the whole plane.

 Bringing the above into a more structured form, we can say that every involution includes:
1) the centre of the involution of lines *P*,
2) its polar *p*,
3) conjugate pairs of lines in *P* (the polars) which pass through the conjugate pairs of points in p (the poles).

We never defined what a real point or line are but just assumed they existed and then proceeded to set up the axioms (the basic principles) of where they were (incidence) and then derived further results of how they related to each other (the theorems). We therefore won't attempt to define what imaginary points and lines are but where they are to be found (incidence) and then develop a method of representing them so that we can start working with them and see how they relate to each other. The incidence of imaginary points and lines of course has to conform to the axioms we have already introduced for real points and lines.

We can start with the following:

**Definition: D.E.I.**
*A directed elliptic involution, abbreviated by d.e.i. is a depiction of an elliptic involution of points in a line or lines in a point whereby a particular direction is assigned to the involution.*

**Definition:  Incidence of Imaginary Points in a Real Line and Imaginary lines in a Real Point**

*An imaginary point lies in a real line if the line includes a d.e.i. of the point. The line is said to be the carrier line of the point.*
*An imaginary point lies in an imaginary line if its depiction is a section of the d.e.i. of the line.*

*One and only one real point lies in an imaginary line: the carrier point of the d.e.i. of the line.*

*An imaginary line goes through a real point if the point includes a d.e.i. of the line. The point is said to be the carrier point of the line.*
*An imaginary line goes through an imaginary point if its depiction is a projection of the d.e.i. of the point.*

*One and only one real line goes through an imaginary point: the carrier line of the d.e.i. of the point.*

*An elliptic involution of points on a carrier line, thought of as composed of points, is an expression of an imaginary point. The minimal Euclidian distance between pairs of conjugate points (or minimal sum of velocities) represents uniquely the pair of double points of the involution.*

*An elliptic involution of lines in a carrier point, thought of as composed of lines, is an expression of an imaginary line. The minimal Euclidian angle between pairs of conjugate lines (or minimal sum of angle velocities) represents uniquely the pair of double lines of the involution.*

In order to relate two depictions of different elliptic involutions to each other we need harmonic depiction of the involutions.

**Depicting Imaginary Points and Lines**
From theorem 3.11 and formalising the qualitative description at the start of the chapter we can state that, given a starting point or line, *an elliptic or hyperbolic involution* can be uniquely depicted by a harmonic representation of the involution. If the harmonic representation is given by $(MM´)(NN´)^h$, (that is H($MM´$; $NN´$) ) then providing the line is not the infinite line and we place $M´$ at infinity, the two points $N$ and $N´$ will lie symmetrical to $M$, the middle point of the involution. The two senses of direction are then S($MNM´$) and S($MN´M´$) and are opposite to each other. The involution is then uniquely determined by the two points N and N´ with the absolute value $\left| MN \right| = \left| MN' \right|$.

When we consider the Gaussian plane we can see that for a given pair of Gaussian coordinates of a complex point P($a+ib$) on a Gaussian plane, we can rotate the Gaussian plane 90 degrees about the point $M$ (see Fig. 5.7) and the position of point $P$ maps onto the point of a vector $MN$. Taking this as our new system, our vector $MN$ is then also a depiction of this imaginary point. (The limitations of the Gaussian plane also become apparent: it's a one-dimensional system: one-dimensional with respect to the real numbers and one-dimensional with respect to the imaginary numbers.)



**Fig. 5.7**

The point $M$ of the involution is the conjugate to the point at infinity. The depiction of an imaginary point can be extended to a two-dimensional system. Geometrically, this means that we can then take
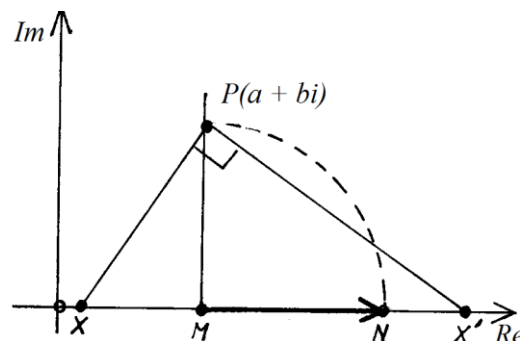
the imaginary point P($a + bi, c + di$) to be a vector with M($a,c$) and N($b,d$) as the beginning and end points of the vector.

(We could call the vectors depicting complex points *Locher vectors* after the mathematician Louis Locher-Ernst (*1906, +1962) who, extending the work done by von Staudt, who showed that an elliptic involution can be interpreted as an expression of an imaginary number, developed the geometrical depiction to include not only that of imaginary (or complex) points but also of lines and, in three dimensions, planes.)

Constructing imaginary points is not difficult when we consider the following:

If A, A´ is a pair of conjugate points of the elliptic involution (see Fig. 5.8), any point on the circumference of circle constructed through the conjugate point pair gives us a projection point whereby A and A´ are 90° apart (as the angle in a semicircle is a right angle). Turning the projection lines in that point and keeping them at right angles to each other generates a right angled involution of points in the line. If we had chosen another point on the circle we would get a different involution of points. But all of them would have the conjugate pair A and A´ in common.
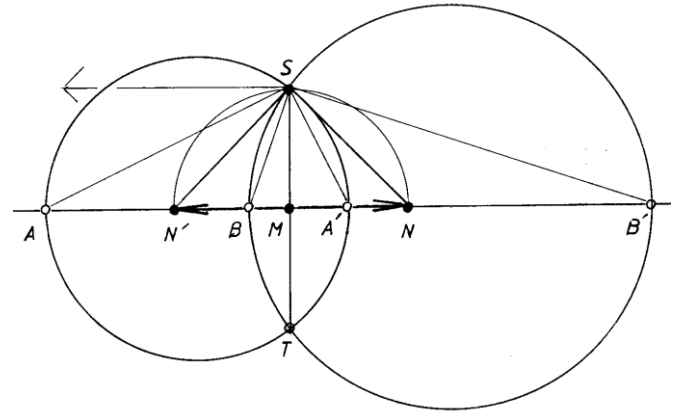


**Fig. 5.8**

Repeating the same for B and B´ we arrive at the same result. From theorem 3.7 we have the assurance that *any two conjugate pairs determine an involution.*

The two circles with diameters │AA´│ and │BB´│ intersect in two points, S and T the common possible projection points. From either of these two points A and A´ are conjugate points as are B and B´. We have determined the projection point for both conjugate pairs and therefore for the whole involution.

As the two pairs of conjugate points have defined the projection of the involution, we can then represent the involution with two other pairs of conjugate points; those of (MM$_\infty$)(NN´), with M$_\infty$ being the point at infinity and conjugate to M. The centre of the involution is then M, the foot of a dropped perpendicular from S, and N and N´ are equally distant from the centre of the involution, i.e. the Euclidean distance between exactly N and N´ is the minimal distance we were looking for and we have a harmonic representation of the involution. We can then represent the conjugate imaginary point with the Locher vectors MN and MN´.

Imaginary Lines

From the aforesaid and considering that the polar of a point passes through the conjugate of that point and vice versa in our self-dual system, we can then extend the depiction of an imaginary point to that of an imaginary line (see Fig. 5.9). The direction of the d.e.i. of lines in pencil S is given by s whereby the lines n and n´ in S are conjugates in the elliptic involution of lines in S. The involution is right angled and induces an involution of points on the line l. The induced involution of points on l is then perspective to the involution of lines in S giving us the imaginary point A on l, as the imaginary line s then passes through imaginary point A on l.
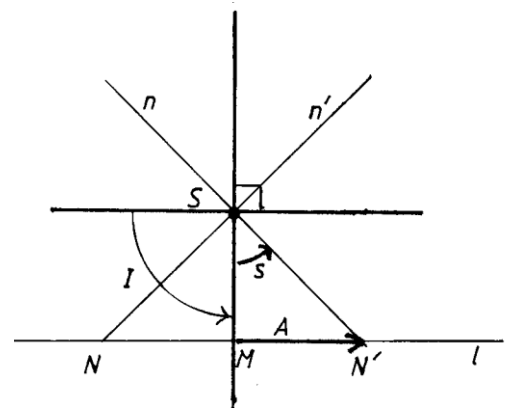


**Fig. 5.9**

The points $N$ and $N'$ are the two intersection points closest to each other on line $l$. Furthermore, as we have seen previously, the centre of an elliptic involution is an interior point of a conic; the one real point through which the imaginary line passes. Apparently we have to consider the $\infty^2$ interior points of the conic as being the real point carriers of the $\infty^2$ imaginary tangentpairs which give us the exterior, imaginary, intersection points of the conic with an exterior line. The $\infty^2$ real exterior points of the conic are then the real point carriers of the $\infty^2$ interior, real, intersection points of the conic with a secant; these being the invariant points of a hyperbolic involution.

We're now in the position to quickly develop the extended, equivalent of school Euclidian geometry where two points define a line and two lines intersect in a point and extend this to triangles and quadrilaterals.

**Application: Joining given imaginary lines and points**

*Two distinct points S and A are incident with one and only one line which goes through both points.* There are three cases to consider:

1) Both points, $S$ and $A$ are real. No problem.

*2) S is real, A is imaginary.* The direction of the d.e.i. on $A$ is given by $(AA')(BB')$. From above we can see that an imaginary line $S(AA')(BB')$ joins $S$ to $A$.

3) Both points are imaginary.
$C$ is the intersection point of the two lines on which the d.e.i.s lie.
We can represent F by the harmonic depiction $(CC')(DD')^{\text{h}}$ of an elliptic involution and $G$ by $(KK')(LL')^{\text{h}}$, both starting from $C$. (see Fig. 5.10)

*Two distinct lines s and a are incident with one and only one point which lies in both lines.* There are three cases to consider:

1) Both lines, $s$ and $a$ are real. No problem

2) $s$ is real, $a$ is imaginary. The direction of the d.e.i. of $g$ is given by $(a \cdot a') \cdot (bb')$. From above we can see that an imaginary point $s(aa')(bb')$ intersects $s$ with $a$.

3) Both lines are imaginary.
$c$ is the joining line of the two points in which the d.e.i.s lie.
We can represent f by the harmonic depiction $(cc')(dd')^{\text{h}}$ of an elliptic involution and g by $(kk')(ll')^{\text{h}}$, both starting from $c$. (see Fig. 5.10)

**Imaginary Fourpoint**
Using the left hand side of number 2, we can now construct an imaginary quadrangle.
If we are given two lines, $e$ with the elliptic involution $(GG')(HH')$ and $f$ with the elliptic involution $(KK')(LL')$ whereby both directions are included, we can construct their centres $M_{\text{e}}$ and $M_{\text{f}}$ and their Locher vectors $A\,i$, $B\,i$, and $C\,i$, $D\,i$, using the method above. (The above construction was carried out using only a set-square and compass. The centre of the circles through the pairs of conjugate points in the involutions was also obtained using the same).We know that six lines pass through the four points. Two of them are the real lines on which the imaginary four points lie; the other four are imaginary lines which will be perspective to the elliptic involutions represented by points $A\,i$, $B\,i$, and $C\,i$, $D\,i$. There that the condition for a perspectivity on the two real lines is that their intersection point $X$ is self-corresponding, we can construct the right angled conjugates $Y$ and $Z$ of $X$ on the respective lines. (see Fig. 5.10) We then construct the unique pair of conjugates $U$, $U'$ and $V$, $V'$ on the respective lines which are harmonic to the pairs of the involutions $X$, $Y$ and $Y$,$Z$ respectively. The line $OU$ divides angle $XOY$ equally, as does $PV$ the angle $XPZ$. $XUYU'$ on the one line has to then be perspective to $XVZV'$ on the other and the perspectivity centre is then easily found by joining respective points. (As both ranges form a harmonic set, the lines $UV$, $YZ$ and $U'V'$ are concurrent.) But this also means that we can interchange $U$ and $U'$ without disturbing the harmonic ratio. The three lines $UV'$, $YZ$ and $VU'$ are then also concurrent. We now have the other four lines: the imaginary conjugate pairs of lines passing through the real points $S$ and $T$.
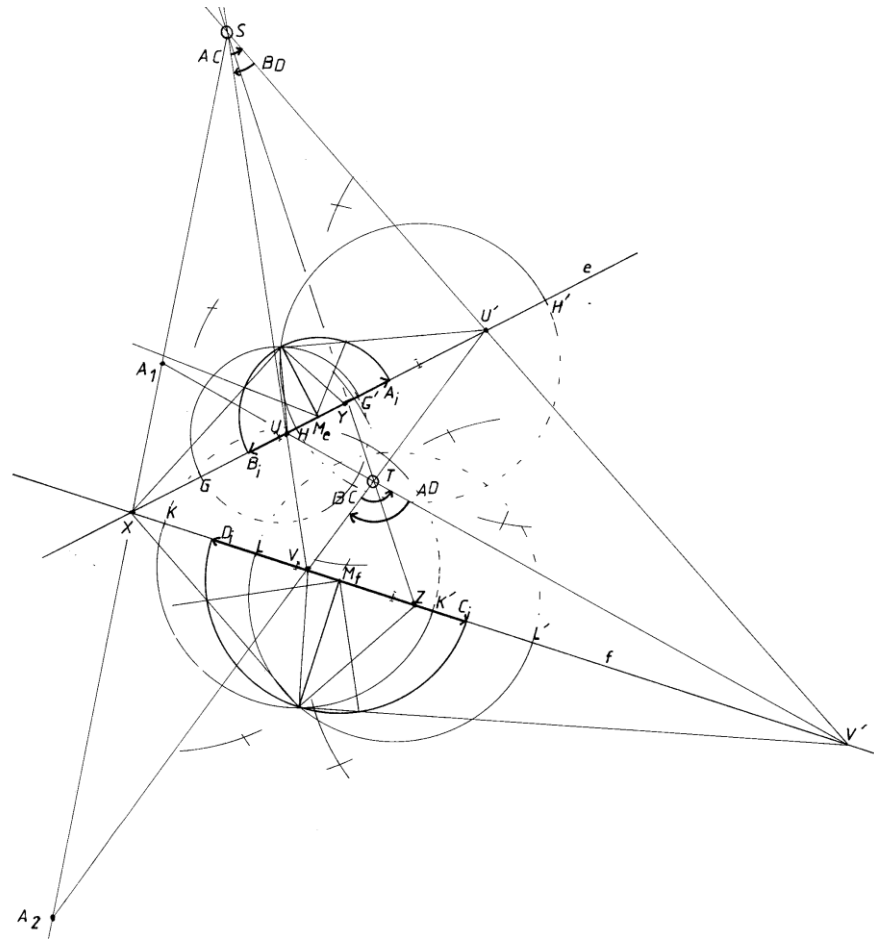
**Fig. 5.10**

Relating the aforesaid to a real conic, (see Fig 5.1) we can state that is comprised of all interior and exterior points and lines which include:

a) the real self-conjugate points and lines (giving the conic *section*) as well as all real points (in the carrier lines) and lines (in the carrier points) of the hyperbolic involutions of real lines in any exterior real point and real points in any interior real line,

b) all real interior carrier points together with the elliptic involution of lines in them (the depiction of the imaginary intersection lines of the conic with an exterior line) and real exterior carrier lines of the elliptic involutions of points in them (the depiction of the imaginary intersection points of the exterior line with the conic.

## Construction Examples for Conics when given Imaginary Points or Lines

When including the constructions for imaginary points and lines by using elliptic involutions of points in a line or lines in a point, the universality of the five points or lines requirement in order to construct the unique conic through the points or touching the lines becomes apparent as the following examples show.

**Example 1**: Given five points, three real points *A*, *B*, *C* as the invariant points on a conic section and two imaginary points *NN´* and *LL´*, we can construct the unique real conic which goes through the five points as well as any other points which are part of the conic.

Construction: (See Fig. 5.11)

In order to obtain a more convincing construction than is presented in some texts and testing the limits of the constructability a stencil ellipse was used and the points *A*, *B*, *C* on the ellipse were predetermined. The result was convincing. In the original covering a complete A4 page, the inaccuracies of the constructed points; pole, self conjugate points $C_1$, $B_1$, $A_1$ were less than 0.5 mm!
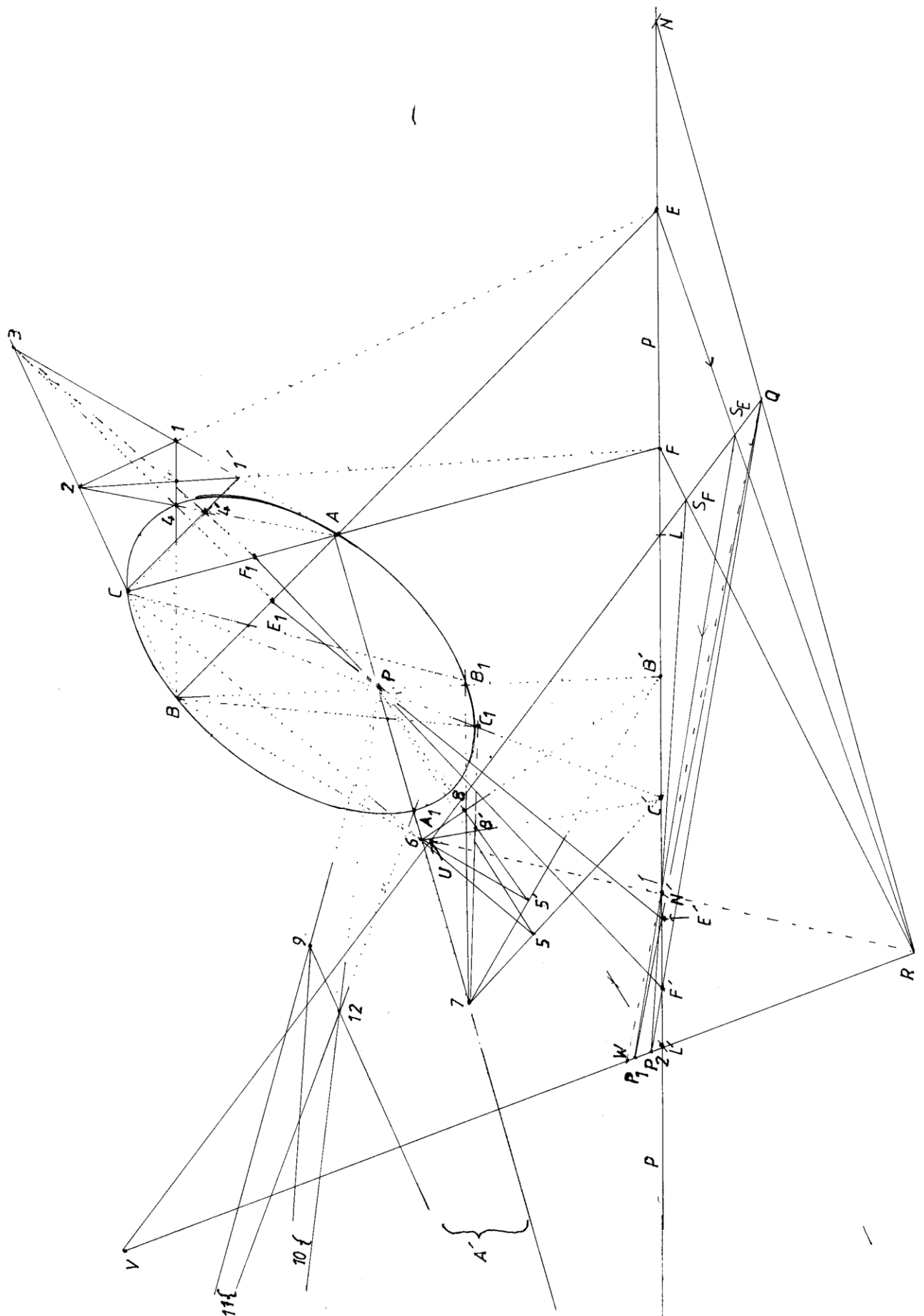
**Fig. 5.11**

The points $E$, $F$ are also part of the involution $(NN´)(LL´)$ on $p$ (the carrier line) depicting the imaginary points. By projecting $N$, $N´$, $L$, $L´$ onto a conic and locating the dummy pole, we could easily find the conjugates of $E$, $F$ in the involution. Alternatively we could use theorem 3.10, Fig. 3.12b (thereby avoiding using a compass) and construct quadrilaterals $P_1QRS$ and $P_2QRS$. These then give us the projectivity:

$$ NN´LL´EF \; \overset{R}{\overline{\wedge}} \; QULVS_E S_F \; \overset{N´}{\overline{\wedge}} \; WRL´VP_1P_2 \; \overset{Q}{\overline{\wedge}} \; N´N´L´LE´F´ $$

1) Locate the pole $P$ as the intersection of the polars of points $E$ and $F$.
   As $BA$ and $CA$ are secants with invariant points $B$, $A$ and $C$, $A$ respectively, the polars of points $E´$ and $F´$ will pass through their harmonic conjugates $E$ and $F$ which, as they lie on polar $p$ are again harmonic conjugates of $E_1$ and $F_1$ on secants $BA$ and $CA$. The partly dotted lines of the quadrilaterals $1234$ and $1234´$ at the top right of the construction give us the points $E_1$ and $F_1$. Joining these to $E´$, $F´$ gives us two polars of $E$ and $F$ which intersect in the pole $P$.
2) Locate the second invariant points of the hyperbolic involution on the polars going through the pole and each of the given points $A$, $B$, $C$. As before, the intersection of the polar $p$ with any secant (here $BB´$ and respectively $CC´$) and pole $P$ are harmonic conjugates to the two secant points (here $BB_1$ and respectively $CC_1$) on the conic.
   Quadrilaterals $5678$ and $5678´$ give us $B_1$ and respectively $C_1$.
3) Either locate $A_1$ using the same method and quadrilateral $9\ 10\ 11\ 12$ or use Pascal's theorem (our 3.4) with any five points and a conveniently, freely chosen perspectivity axis to locate further points or just use the five points on the conic to projectively generate further points.

**Example 2 (dual to example 1)**: Given five tangents, three real tangents $a$, $b$, $c$ as the invariant lines on a conic section and two imaginary lines $kk´$ and $ll´$ in a pole, (see Fig. 5.12) we can construct the unique real conic which goes through the five lines as well as any other lines which are part of the conic.

1) The lines $g$, $f$ are also part of the involution $(kk´)(ll´)$ in $P$ (the carrier point) depicting the imaginary lines. By projecting $k$, $k´$, $l$, $l´$ onto a conic through $P$ giving $K_1$, $K_1´$, $L_1$, $L_1´$ and locating the dummy pole $P_0$ as the intersection $(K_1K_1´)·(L_1L_1´)$, we can easily find the conjugates of $g$, $f$ in the involution. Alternatively we could use The dual of theorem 3.10, Fig. 3.12b (thereby avoiding using a compass)
2) Locate the polar $p$ as the join of the polars of points $f·f´$ and $g·g´$.
   As $a·b$ and $a·c$ are poles with invariant lines $a$, $b$ and $c$, $a$ respectively, the poles of lines $g´$ and $f´$ will lie in their harmonic conjugates $g$ and $f$ which, as they go through pole $P$ are again harmonic conjugates of $g_1$ and $f_1$ in poles $ba$ and $ca$. The lines of the quadrangles $1234$ at the top right of the construction and $1234´$ give us the lines $g_1$ and $f_1$. Intersecting these with $g´$, $f´$ gives us two poles of $g$ and $f$ whose join is the polar $p$.
3) Locate the second invariant line of the hyperbolic involution in the pole lying in the polar and each of the given lines $a$, $b$. As before, the join of the pol $P$ to any pole (here $b·b´$ and polar $p$ are harmonic conjugates of the two tangent lines (here $b·b'$) on the conic. Quadrangle $1´´2´´3´´4´´$ gives us $b´$.
4) Given the development of the construction so far, we can use theorem     w.r.t. pole $p·b$ lying on p and the tangents $b$, $b´$ and construct further tangents. Any line $p_t$ intersects $b$ in $P_t$. The projection through $P$ onto $P_{t1}$ on the conic through $P$ is reflected through $P_0$ onto $P_{t1}´$ which gives us the conjugate line in $P$ to $p_t$. This intersected with $b´$ delivers point $P_t´$and the join $P_tP_t´$ is another tangent to the conic. Repeating the process generates more tangents**.**
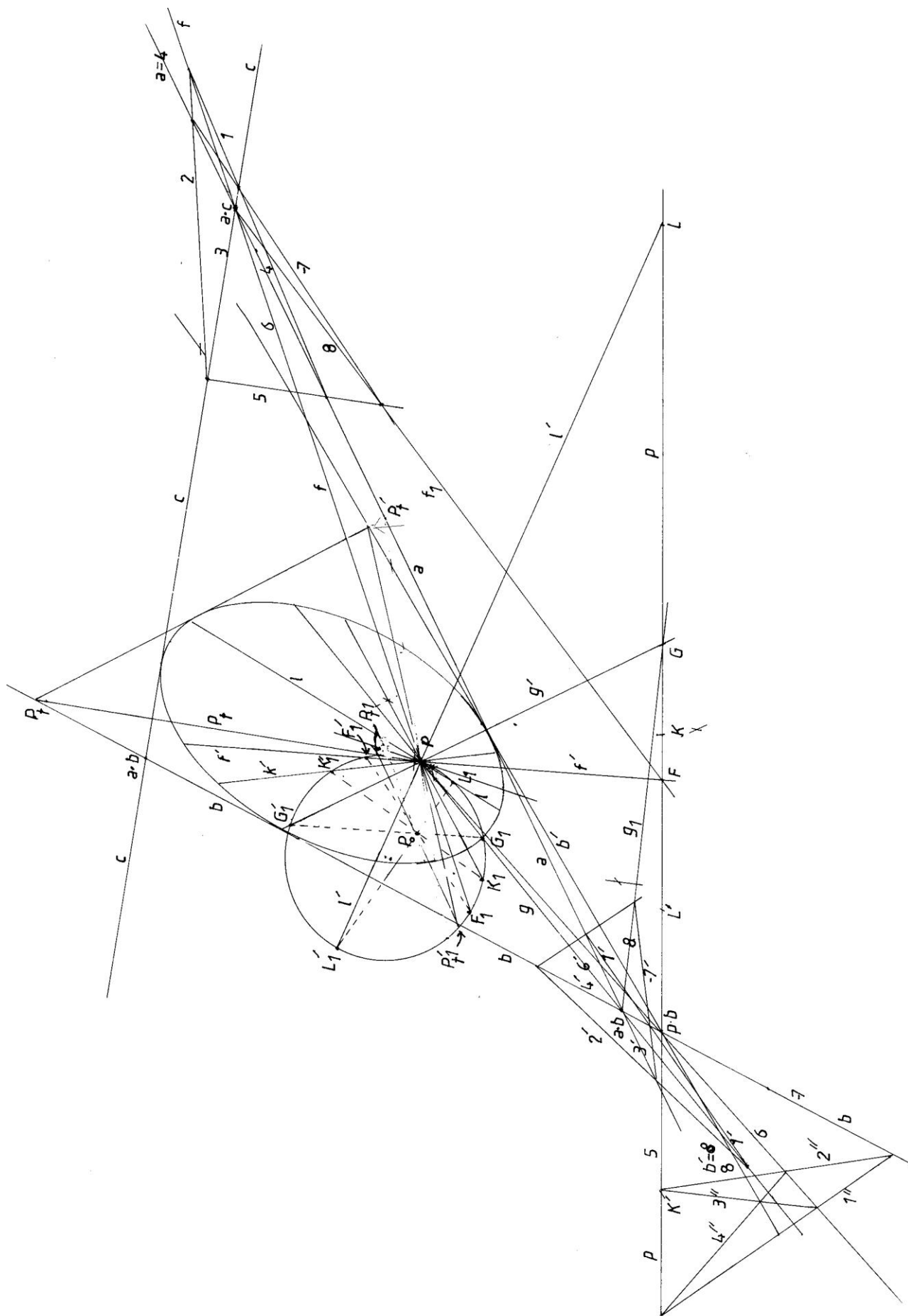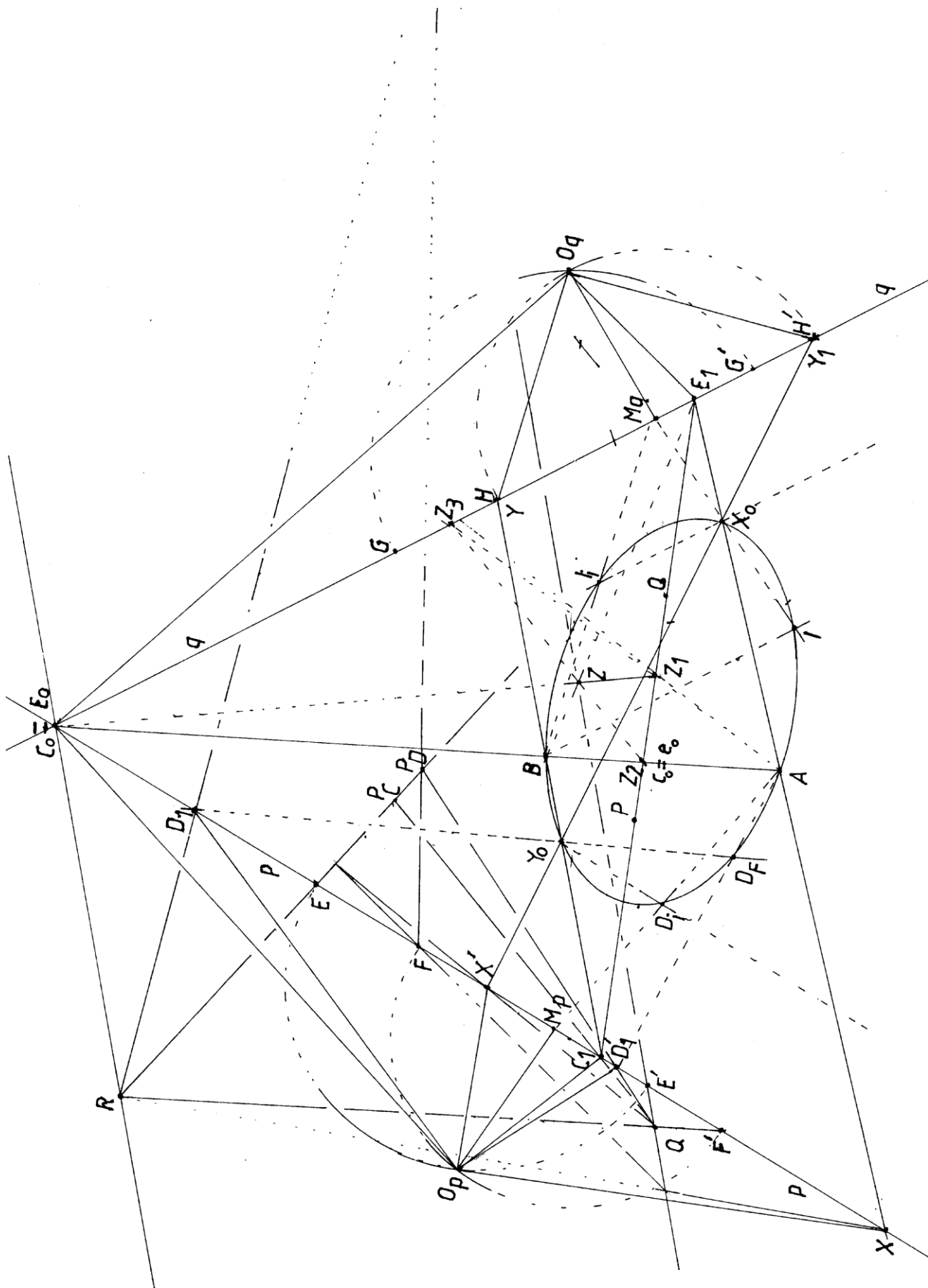
**Fig. 5.12**

**Fig. 5.13**

**Example 3**: Given five points, one real point $A$ on a conic section and two imaginary points $NN'$ and $LL'$ on a line, (see Fig. 5.13) we can construct the unique real conic which goes through the five real points.

1) The involutions expressing the imaginary points are depicted by the points $E$, $E'$, $F$, $F'$ on $p$ and $G$, $G'$, $H$, $H'$ on $q$. The intersection $p \cdot q = C_o E_o$ is a common point of both involutions and is used to coordinate the movement of the two pairs of imaginary points. The conjugates of this point are $C_1$ on $p$ and $E_1$ on $q$., these being obtained by constructing a right-angled involution $O_p$ and $O_q$. At this point in time, joining line $C_1 E_1$ passing through both poles $P$ and $Q$ is the polar of $C_o = E_o$ (the respective polars meet in a point; their poles lie in a line).

2) The point $B$ on $AC_o$ is separated harmonically from $A$ by the polar $C_1 E_1$ and the common pole $C_o = E_o$ .

3) $BC_1$ intersects the conic in a yet to be determined point $Y_o$ which together with $A$ can be thought of as the two fixed points with point $B$ moving around the conic and generating the conjugate points $C_o = BA \cdot p$ and $C_1 = BY_o \cdot p$ of the elliptic involution on $p$. But the same is the case for the yet to be determined point $X_o$ on $X_o Y_o$. The lines $X_o A$ and $X_o Y_o$ intersect $p$ in the conjugates $X$ and $X_1$. There that at this moment $X = E_1 A \cdot p$ its conjugate is easily constructed. Correspondingly the lines $AB$ and $AX_o$ intersect $q$ in conjugates $E_o$ and $E_1$ and therefore the pole $Q$ of $q$ lies on $X_o B$. But the lines $BY_o$ and $X_o Y_o$ intersect $q$ in conjugate pairs $Y$ and $Y_1$ . As $Y = BC_1 \cdot q$ we can construct $Y_1$. $Y_o$ is the intersection point of $X_1 Y_1$ and $BC_1$; $X_o$ the intersection of $AE_1$ and $X_1 Y_1$

**Example 4 (dual to example 3)**: Given five tangents, one real tangents $a$, as the invariant line on a conic section and two pairs of imaginary lines $ee'$ and $ff'$ and $gg'$ and $hh'$ in a pole, we can construct the unique real conic which goes through the five lines as well as any other tangent lines.

1) The involutions expressing the imaginary lines are depicted by the lines $e$, $e'$, $f$, $f'$ in $P$ and $g$, $g'$, $h$, $h'$ in $Q$. The joining line $PQ = co \cdot eo$ is a common line of both involutions and is used to coordinate the movement of the two pairs of imaginary lines. The conjugates of this line are $c_1$ in $P$ and $e_1$ in $Q$., these being obtained by constructing a dummy involution of points on a conic and finding the dummy poles $P_d$ and $Q_d$ as the intersection of the joining lines of respective points. At this point in time, intersection point $c_1 \cdot e_1$ lying in both polars $p$ and $q$ is the pole of $c_o = e_o$ (the respective poles lie in a line; their polars meet in a point).

2) The line $b$ in $a \cdot c_o$ is separated harmonically from $a$ by the polar $c_o \cdot e_o$ and the common pole $C_o = E_o$ .

3) $b \cdot c_1$ joins the conic in a yet to be determined line $y_o$ which together with $a$ can be thought of as the two fixed lines with line $b$ moving around the conic and generating the conjugate lines $c_o = (b \cdot a)P$ and $c_1 = (b \cdot y_o)P$ of the elliptic involution in $P$. But the same is the case for the yet to be determined line $x_o$ in $x_o \cdot y_o$. The points $x_o \cdot a$ and $x_o \cdot y_o$ join $P$ in the conjugates $x$ and $x_1$. There that at this moment $x = (e_1 \cdot a)P$, its conjugate is easily constructed using the dummy pole $P_d$. Correspondingly the points $a \cdot b$ and $a \cdot x_o$ join $Q$ in conjugates $e_o$ and $e_1$ and therefore the polar $q$ of $Q$ goes through $x_o \cdot b$. But the points $b \cdot y_o$ and $x_o \cdot y_o$ join $Q$ in conjugate pairs $y$ and $y_1$ . As $y = (b \cdot c_1)Q$ we can construct $y_1$. $y_o$ is the joining line of $x_1 \cdot y_1$ and $b \cdot c_1$; $x_o$ the joining line of $a \cdot e_1$ and $x_1 \cdot y_1$.
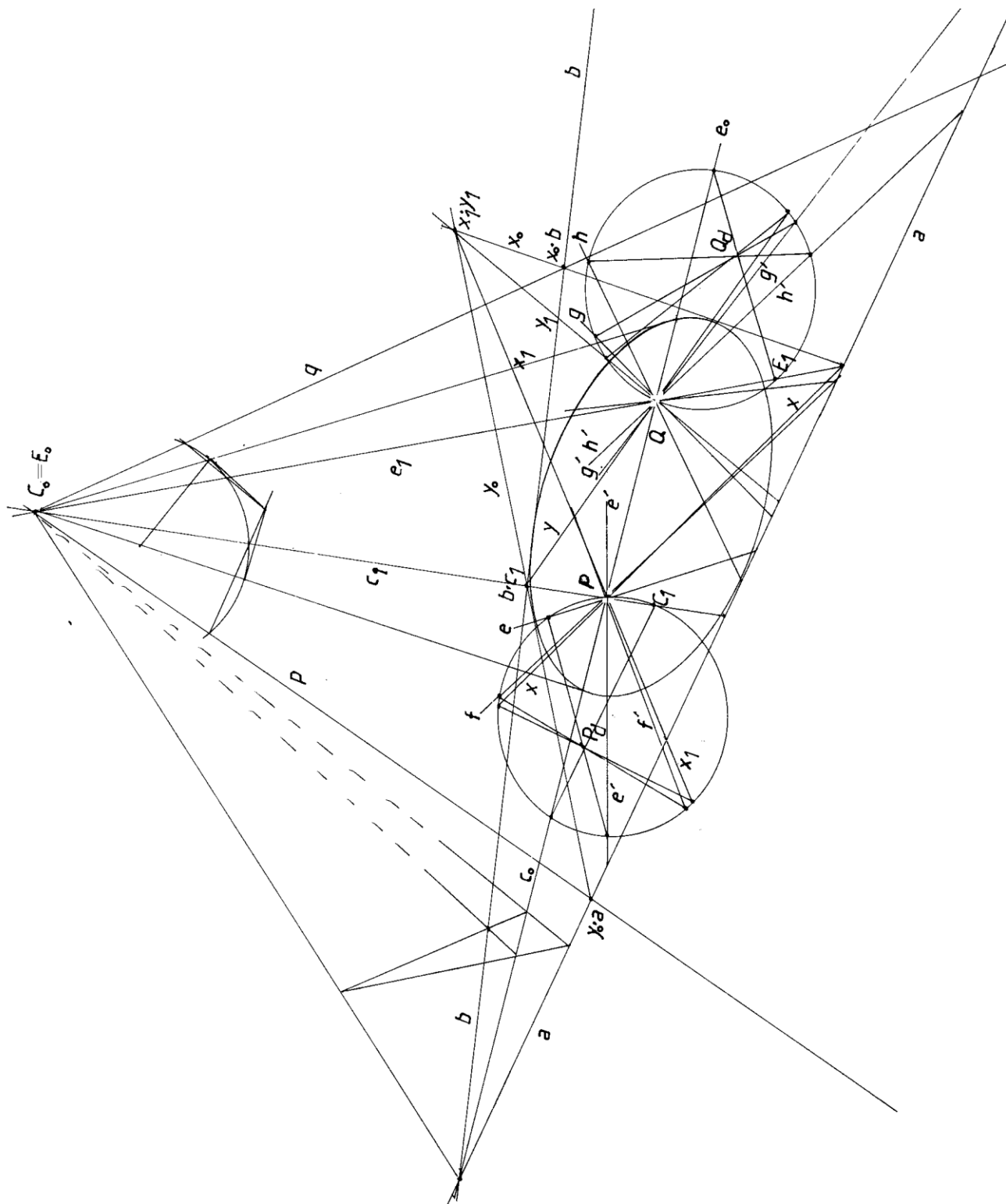
**Fig. 5.14**

# Bibliography

[1]   Cremona, L.: *Elements of Projective Geometry.* Dover Publ., New York 1960

[2]   Chasles, M.: *Traite´ des Sections Coniques.* Gauthier-Villars, Paris 1865

[3]   Baker, H.: *Principles of Geometry vol. 1 and 2.* Univ. Press, Cambridge 1922

[4]   Baker, H.: *An Introduction to Plane Geometry.* Univ. Press, Cambridge 1943

[5]   Coxeter, H.S.M.: *Projective Geometry.*
             Springer-Verlag, New York, 2$^{nd}$ ed. 1987

[6]   Coxeter, H.S.M.: *The Real Projective Plane.* Univ. Press, Cambridge 2$^{nd}$ ed. 1955

[7]   Enriques, F.: *Lezioni di geometria proiettiva.* Teubner Verlag, Leipzig 1903

[8]   Kowol, G. : *Projektive Geometrie und Cayley-Klein Geometrien der Ebene.*
             Birkhäuser Verlag, Basel – Boston – Berlin, 2009

[9]   O'Hara, C. and Ward, D.: *An Introduction to Projective Geometry.*
             Clarendon Press, Oxford 1937

[10]  Robinson, G.: *The Foundations of Geometry.* Univ. of Toronto Press, Toronto 1940

[11]  Veblen, O. and Young, J.: *Projective Geometry, vol. I.*
             Blaisdell, New York Toronto London 1938/1946

[12]  Staudt, C. von : *Geometrie der Lage.* Bauer u. Raspe,  Nürnberg 1847

# Vorwort

Bereits in der Antike hat sich die Geometrie (= Erdvermessung) von einer praktischen Tätigkeit zu einer rein mathematischen Theorie im Sinne axiomatischer Systeme (Euklid) entwickelt. Diese Entwicklung kam mit Hilberts „Grundlagen der Geometrie" zu einem gewissen Abschluss. Die axiomatische Methode bietet die Möglichkeit, fast ausschließlich im Bereich der Logik zu arbeiten. Die undefinierten Elemente "Punkt" und "Gerade" sowie das Zusammentreffen solcher Elemente (= Inzidenz) können als Überreste der ursprünglichen Erdvermessung angesehen werden. Obwohl viel für eine abstrakte, rein logische („weg vom Zeichnerischen") Entwicklung spricht, liegt mein Interesse nicht nur im abstrakt-logischen Bereich, sondern auch darin, durch Konstruktionsbeispiele diese Begriffe zu erfassen und ihnen auch konstruktiv einen Ausdruck zu geben. Um diesen seelischen Reichtum wenigstens anzudeuten, inkludiere ich im Laufe der Darlegung des axiomatischen Aufbaus schließlich die geometrische Darstellung von imaginären Punkten und Geraden in mehreren ausgearbeiteten Konstruktionen, die das Ausgesagte in einen größeren Zusammenhang setzen.

Mein besonderes Interesse gilt Christian von Staudts Deutung einer elliptischen Involution als Darstellung eines imaginären Punktes bzw. einer imaginären Geraden (in der Ebene) und ferner Locher-Ernsts Veranschaulichung derselben, welche es ermöglicht zeichnerisch damit in derselben Dimension zu arbeiten. Zugleich entsteht dabei, worauf in dieser Diplomarbeit besonderer Wert gelegt wird, ein Erfassen von imaginären Punkten und Geraden als ein ganz bestimmter Bewegungsablauf von Punkten einer Geraden bzw. Geraden eines Büschels. Damit lassen sich die imaginären Schnittpunkte von Passanten eines Kegelschnitts bzw. die imaginären Tangenten eines solchen von einem inneren Punkt vorstellungsmäßig ergreifen.

In der Einleitung wird das die projektive Geometrie beherrschende Dualitätsprinzip vorgestellt und der Satz von Desargues bewiesen. Auch werden die Grundbegriffe Perspektivität und Projektivität eingeführt und erste Folgerungen abgeleitet.
Im 1. Kapitel werden die Inzidenz und die harmonische Lage von Punkten einer Geraden bzw. Geraden eines Büschels behandelt.
Das 2. Kapitel ist den Ordnungsaxiomen und dem Stetigkeitsaxiom gewidmet. Die darauf basierenden Begriffe der Trennung und des Bewegungssinns werden ausführlich erläutert.
Im 3. Kapitel wird zunächst der Fundamentalsatz der projektiven Geometrie präsentiert. Es folgt die rein projektive Definition der Kegelschnitte nach Steiner und darauf aufbauend die Sätze von Pappus und Pascal. Sodann wird der für das Weitere grundlegende Begriff der Involution ausgearbeitet. Und zwar als Involution auf einer Geraden, in einem Geradenbüschel und auf einem Kegelschnitt.
Zweidimensionale Projektivitäten und Polaritäten werden im 4. Kapitel ausführlich behandelt. Unter anderem wird gezeigt, dass für letztere Abbildungen stets selbstpolare Dreieck auftreten.

Der Kern der Diplomarbeit folgt in Kapitel 5. Der Text kreist um die zeichnerische Konstruktion von Kegelschnitten anhand imaginärer Punkte und Geraden.

# L E B E N S L A U F

| | |
|---|---|
| <u>Name</u>: | John Peter BOUCHIER |

| | |
|---|---|
| <u>geboren am/in</u>: | **30.05.1957 / Waipiro Bay / Neuseeland (NZ)** |
| <u>Staatsbürgerschaft</u>: | **Niederlande** |

## SCHULBILDUNG:

| | |
|---|---|
| 1963 – 1968 | **St.Mary's Primary School – Auckland / NZ** |
| 1969 – 1975 | **Marcellin College – Auckland** |
| 1975 | **Reifeprüfung** (New Zealand Qualifications Authority) |
| 1976 | **Studium Universität Auckland** / NZ (englische und neuseeländische Literatur, Geschichte) |
| 1982 - 1984 | **Bachelor of Science** (Mathematik, Physik, Geologie) |
| 1985 | **Teachers Training College** – Teaching Diploma |
| 1996 – 1999 | **UNIVERSITÄT Wien** Mathematik und Physik (zusätzliche Kurse) (berufsbegleitend) |
| Ab Oktober 2011 | **UNIVERSITÄT Wien** Fortsetzung des Lehramtsdiploms für Mathematik und Physik (berufsbegleitend) |

## BERUFLICHER WERDEGANG:

| | |
|---|---|
| 1986 | **Mangere College –** Auckland / NZ **Lehrer: Physik** (Oberstufe /Matura), Mathematik |
| 1987 | **Homerton House School** – London **Lehrer:** Mathematik |
| 1988 | **Selwyn College –** Auckland / NZ Lehrer: Mathematik und Statistik Oberstufe |
| 1989 – 1991 | **Onslow College** – Wellington / NZ Lehrer: Mathematik (Oberstufe) |
| 1992 – 1993 | **Englisch-Lehrer in der Erwachsenenbildung** bei diversen Volkshochschulen, BFI, WIFI Wien |
| 1994 – 1996 | **Vienna International School** – Wien Lehrer: Mathematik Oberstufe |
| 1996 – 2001 | **Englisch-Lehrer** bei diversen Volkshochschulen, BFI, WIFI Wien |
| 2001 – 2013 | **Rudolf Steiner Schule Pötzleinsdorf – Wien** Lehrer: Mathematik, Physik, Geographie und Technologie (Oberstufe) |

## WEITERBILDUNG:

| | |
|---|---|
| 1989 | **Goethe Institut –** Wellington / NZ **Zertifikat "Deutsch als Fremdsprache"** |
| 1996 | **Berufsförderungsinstitut Wien** **Österr. Sprachdiplom / Mittelstufe** |
| 1997 | **Berufsförderungsinstitut Wien** |

|  | **CELTA-Certificate (Certificate in English Language Teaching to Adults)** |
| 3.4.2004 – 01.04.2010 | **LEHRERSEMINAR FÜR WALDORFPÄDAGOGIK** |
|  | D – 34131 Kassel |
|  | Pädagogisches Blockstudium Oberstufenlehrer |
|  | Mathematik und Physik |
|  | Berufsbegleitende Ausbildung (21 Vollzeit-Studienwochen) |
|  | abgeschlossen am 24.3.2010 |