MASTERARBEIT

Titel der Masterarbeit

„Stochastic Volatility Models of Financial Markets.
Valuation of European Style Options with Heston Model"

verfasst von

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ABSTRACT

The structure of the Master Thesis is the following:

Chapter 1 describes the fundamental ideas about stochastic process and explains some basic concepts of finance, statistics and mathematical finance. Such as European style options, moneyness of options, volatility, volatility smile, risk-neutral probability and Feyman-Kac theorem.

Chapter 2 demonstrates the frameworks of some famous and widely used and mathematical models such as Black-Scholes model, Vasicek model, Cox-Ingersoll-Ross model (very often called as CIR process), Hull-White model, Heston and Nandi model (which is the only one discrete model presented in this current study) and Bates model (which is stochastic volatility and jump-diffusion model as it is a combination of Merton and Heston models).

Chapter 3 introduces the Heston model with all its important concepts. In particular, in this chapter I present the derivation of the model and give the precise description of it. Additionally I present risk-neutral and partial differential approaches and describe closed-form solution of the model. I also analyze the effect of model parameters on the final results and describe all observed advantages and disadvantages of the Heston model.

Chapter 4 presents the empirical analysis during the periods of economical stability and economical crisis. I describe the data which I have chosen and have used for the calculation and derivation of option prices. I demonstrate the procedure and way of finding the Heston and Balck-Scholes model parameters. Moreover, I present and interpret the results, and compare the Heston and Black-Scholes approaches. Additionally, I do an empirical parameter estimation to prove that it is not a sufficient method of finding model parameters. Thus, only the calibrations method can be used for the finding of Heston model option prices.

Appendix: collects MATLAB-codes which are used for the sensitivity analysis of the Heston model parameters and calculation of option prices with Heston and Black-Scholes models.

ZUSAMMENFASSUNG

Die Struktur der Masterarbeit ist die Folgende:

Kapitel 1 beschreibt die grundlegenden Ideen stochastischer Prozesse und erklärt einige Konzepte des Finanzwesen, der Statistik und der Finanzmathematik, wie z.B. europäische Optionen, Moneyness der Optionen, Volatilität, Volatilitäts Lächeln, risikoneutrale Wahrscheinlichkeit und den Feyman-Kac Satz.
Kapitel 2 zeigt die Rahmenbedingungen einiger berühmter und weit verbreiteter mathematischer Modelle wie das Black-Scholes Modell, das Vasicek Modell, das Cox - Ingersoll - Ross Modell (meistens CIR Prozess genannt), Hull-White Modell, Heston und Nandi Modell, welches das einzige diskrete Modell in dieser aktuellen Studie ist, und Bates Modell, welches ein stochastisches Volatilität - Sprung - Diffusions Modell ist und eine Kombination von Merton und Heston Modelle darstellt.
Kapitel 3 behandelt das Heston Modell mit allen wichtigen Konzepten. Insbesondere präsentiere ich in diesem Kapitel die Ableitung des Modells und beschreibe es detailliert. Ich stelle die risikoneutrale und partielle Differential Ansätze vor, sowie die geschlossene Lösung. Zusätzlich analysiere und beschreibe ich die Auswirkungen der Parameter auf die Modellergebnisse und erwähne alle beobachteten Vor - und Nachteile des Heston Modells.
Kapitel 4 präsentiert die empirische Analyse in Zeiten der wirtschaftlichen Stabilität und der wirtschaftlichen Krise. Ich beschreibe die Daten, die ich für die Berechnung und Herleitung der Optionspreise verwendet habe, und zeige das Verfahren und die Art und Weise wie man die Heston und Balck-Scholes Modell Parameter findet. Zu diesem Zweck präsentiere ich die Ergebnisse, vergleiche die Heston und Black-Scholes Modelle und interpretiere die unterschiedlichen Ansätze. Zusätzlich schätze ich die empirischen Parameter und beweise, dass die Methode nicht ausreichend ist. Somit kann nur das Kalibrierungs- Verfahren für das Heston Modell verwendet werden.

Anhang: beinhaltet MATLAB - Codes, die für die Sensitivitätsanalyse der Heston Modellparameter und die Berechnung der Optionspreise mit Heston und Black-Scholes Modelle verwendet werden.

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INTRODUCTION

The valuation of derivative securities such as options is one of the most central and wildly discussed topics in quantitative finance. For long time many economists tried to develop new methods and models which would help to make the sufficient pricing of derivatives. In the 1970s Fischer Black, Myron Scholes and Robert Merton derived the Black-Scholes model. It was a huge "discovery" for that decade as it presented a simple closed form solution. But unfortunately, it was soon realized that this model also presents some disadvantages which affect negatively the results' precision and fail to explain some important financial issues for example the volatility smile. The most important drawbacks of Black-Schoels model were the assumptions of constant volatility and lognormal returns.

After that time many more stochastic volatility models were derived with the goal to improve the pricing option results. One of very famous short term interest rate stochastic models is the Vasicek model which tried to show that the short-term interest rate is the fundamental source of market uncertainty. Unfortunately, in some cases this model allows the interest rate to take negative value which is unacceptable from an economical point of view. Thus, John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross decided to extend the Vasicek model. They introduced the standard deviation factor for the short-interest rate stochastic differential equation of Vasicek. It was a brilliant idea from both mathematical and economical point of view since the use of root allows only for positive values of risk. Unfortunately, it was not always successful practically. For instance, historical analysis of a Second World War era uncovers negative interest leading to a very complicated volatility determination with the use of complex numbers (due to only positive values of risk accepted into the root).

Many other stochastic volatility and jump diffusion models were presented. Such as Hull-White model, Heston and Nandi model, Bates model and many others. One of the most interesting models was Heston model which disclaims the assumptions of constant volatility and lognormal returns, and simultaneously takes into account the leverage effect and the correlation between asset returns and volatility In comparison to Black-Schoels model, Heston model is a bit more complicated since involvement of numerous parameters makes this model very sensitive to their changes.

A closer look at Heston model helps us understand why it is so important and widely used in the “financial world” This model’s focus is on the impacts of the parameters on the end results. The implementation and analysis of Heston versus Black-Shoels models during periods of economical stability and economical crisis show that the Heston model performs better at all times.

CHAPTER 1 BASIC CONCEPTS

1.1 INTRODUCTION

In the following paragraphs of chapter 1 I present some basic concepts of finance and mathematics which are very important for analysis and description of mathematical models, and complete understanding of Heston model. The goal of this chapter is to introduce the following concepts:

In the first paragraph (§ 1.2) I explain the idea of European style options, present their definition and illustrate them for more detailed description (Berk, DeMarzo, Harford 2012, Hull 2012). Then (§ 1.3), I introduce the meaning of the moneyness of options (Moony, Seon, Weez, and Yoon 2009, Hull 2012, Bauer 2012, Neftci 2008). Afterwards (§ 1.4), I present a paragraph where I explain the basic idea of volatility and describe its main three categories: historical volatility, implied volatility and stochastic volatility (Hull 2012, Forde, Jacquier and Mijatovic 2010, Neftci 2008). Consequently (§ 1.5), in the next paragraph I explain what implied volatility smile means, illustrate the curve which describes it and present the connection between the curve and the moneyness of options (Hull 2012, Kuen-Kwok 2008, Neftci 2008). Therefore (§ 1.6), I talk about the stochastic process (Hull 2012, Doob 1942, Björk 2009, Klein 2011, Rolski, Schmidli, Schmidt & Teugels 1998) where I give some basic concepts about the discrete and continuous times stochastic processes, explain the meaning of Markov chain, Generalized Wiener process and the Brownian motion. Additionally, I present the clear proof of the stock price distribution. Afterwards (§ 1.7), I introduce the paragraph where I describe the Girsanov's theorem, the idea of risk neutral probability and show the strong connection between them (Sundaram 1997, Björk 2009, Klein 2011, Hull 2012, Kuen-Kwok 2008, Chance 2008). In the last paragraph (§ 1.8) I talk about the Feynman-Kac theorem which proved the connection between the SDE and PDE and is the fundamental theorem for a huge amount of financial models (Kuen-Kwok 2008, Jen-Chang Liu and Chau-Chen Yang 2007, Klein 2011).

1.2 EUROPEAN STYLE OPTIONS

"Financial Option: a contract that gives its owner the right but not the obligation to purchase or sell an asset at a fixed price at some future date." (Berk, DeMarzo, Harford, 2012, p.624).

"A Call Option gives the holder the right to buy the underlying asset by a certain date T for a certain price K" (Hull, 2012, p.7).

"A Put Option gives the holder the right to sell the underlying asset by a certain date T for a certain price K" (Hull 2012, p.7).

K: exercise price or strike price.

T: maturity date or expiration date

$\mathbf{S_T}$: price of the underlying asset at maturity T.

The payoff of the call option is:

$$C = (S_T - K)^+ = \begin{cases} S_T - K, & \text{if } S_T > K, \quad (1) \\ 0, & \text{if } S_T \leq K, \quad (2) \end{cases}$$

(1): it is sufficient to exercise the call option and make a profit by selling it at the price S_T and directly buying the stock at the strike price K.

(2): the call option will not be exercised.

The payoff of the out option is:

$$P = (K - S_T)^+ = \begin{cases} K - S_T, & \text{if } K > S_T\,, \quad (1) \\ 0, & \text{if } K \leq S_T\,, \quad (2) \end{cases}$$

(1): it is sufficient to make a profit by buying the put option at the market price S_T and directly exercise it with the exercise price K.

(2): the market price of the asset is higher than its strike price, thus it is better not to exercise the put option.

Hence, the financial contract helps to protect the position of its owner in the market and make his/her assets resist to the market movements and fluctuations with the goal to reduce the loss.

There are two styles of options: European and American. It is sensible to mention one of the most important differences between them which is: the holder of American

style option can exercise it at any time before the maturity, while the European style option can be exercised only on the date T. I my master thesis I focus
on the European style options because one of the Heston and Black-Scholes models assumptions is the model implementation for the European style options.
The following Figures 1.2.1 and 1.2.2 illustrate the European Call and Put Options, respectively.

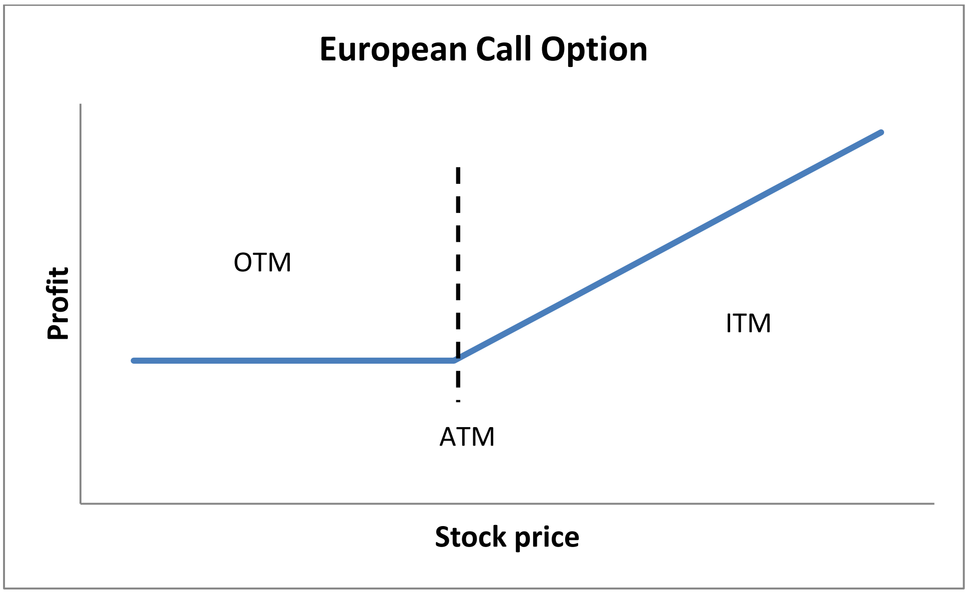

Figure 1.2.1: European Call Option (OTM: out of the money, ITM: in the money, ATM: at the money).

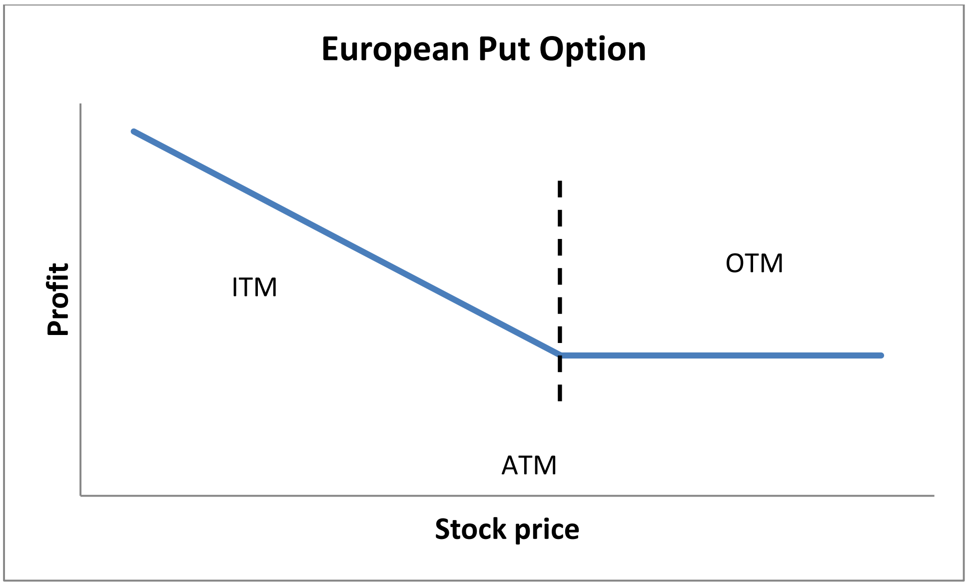

Figure 1.2.2: European Put Option(OTM: out of the money, ITM: in the money, ATM: at the money).

1.3 MONEYNESS OF THE OPTIONS

Moneyness of an option is an indicator which helps the option holder to realize where the option has to be exercised and where not.

$$\mathbf{M_t} = \frac{\mathbf{K}}{\mathbf{S_t}}$$

Moneyness can be categorized in three groups:

ITM (in-the-money): an option is ITM when it is worth to be exercised. For call option, it holds when the stock price is above the strike price $S_t > K$ and for put option when $K > S_t$. In this case is does not mean that exercising option is leading to the profit but it shows if the option is worth to be exercised.

OTM (out-of-the-money) has the opposite meaning of ITM. An option is OTM when it is not worth to be exercised. For call option is holds when the stock price is below the strike price $K > S_t$ and for put options when $S_t > K$.
In this case such options have no intrinsic value, which means that for call option $(S_T - K)^+ = 0$ and for put option $(K - S_T)^+ = 0$.

ATM (at-the-money): a call or put option is ATM when the stock price is equal to the strike price,$S_t = K$.
In paragraph 1.2 the figures 1.2.1 and 1.2.2 represent the European style call and put options, additionally their moneyness.
The following Table 1.3.1 shows all details about the classification of the option moneyness.

Moneyness	Call	Put
<1	ITM	OTM
=1	ATM	ATM
>1	OTM	ITM

Table 1.3.1: Moneyness of options

1.4 VOLATILITY

Volatility is an extent of uncertainty or risk about the magnitude of changes in security values. It presents the level (or degree) of the returns dispersion for a given security. The higher volatility corresponds to the riskier security. In other words, high volatility gives the signal about dramatic value fluctuations. It can be measured by the standard deviation or variance between the security returns and market returns

The volatility can classified into two important sections Historical volatility and Implied volatility.

Historical volatility is a measure of uncertainty which is derived from the past time series prices of the market. It measures the price movements of the underlying asset over the history.

Implied volatility is a magnitude of changes in securities values which is computed from the market price of the derivative contract. Usually, the inverse Black-Scholes formula is used for the calculation of the implied volatility, where the option price is taken from the market and the Black-Scholes formula is rewritten as a closed-form solution for the volatility in terms of market option price. Thus, it gives the volatility of the underlying asset which corresponds to the certain given market option price.

Very often the concept of stochastic volatility is mentioned. The stochastic volatility represents the description of the movements (fluctuation) of the volatility over the time. Usually, the stochastic volatility described as the stochastic process which satisfies a particular stochastic differential equation. In my master thesis I am talking about the stochastic volatility models, one of the most important tasks of these models is to describe the evolution of the volatility over the time.

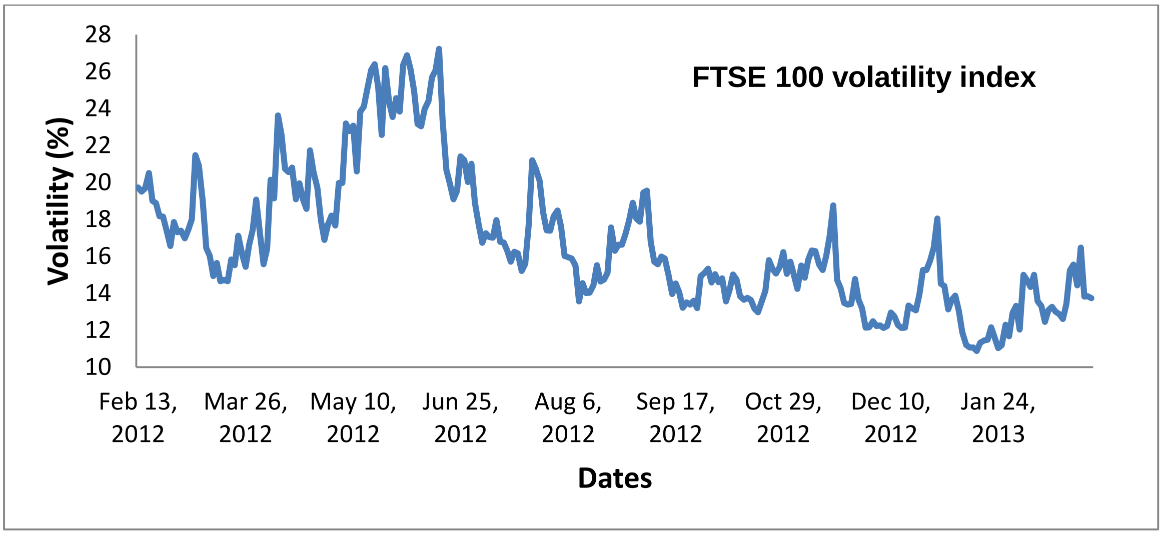

Figure 1.4.1: FTSE 100 volatility index during Feb 13 2012- May 1 2013.

1.5 IMPLIED VOLATILITY SMILE

The implied volatility smile is a curve which is derived from the plot of the strike price (or moneyness) and the implied volatility of a certain amount of options with the same maturity date. A curve gets this name because its shape really reminds a smile. According to the following Figure 1.5.1 we can observe that the volatility increases as the option becomes more in-the-money or out-of-the-money. In fact, in these periods the uncertainty grows and more dramatic movements in the option prices are expected. Consequently, ATM options have lower volatility comparing to the OTM/ITM options. Implied volatility is a invaluably useful tool for the investors which helps them to understand better their positions in the market. Especially, in the cases where the options pricing in the foreign currency markets and equity option markets has to be done. Additionally, the volatility smile shows the bigger option trading demand for OTM/ITM options than for ATM, as it is easier to manipulate with the prices and create arbitrage opportunities.

The following Figure 1.5.1 illustrates the implied volatility smile:

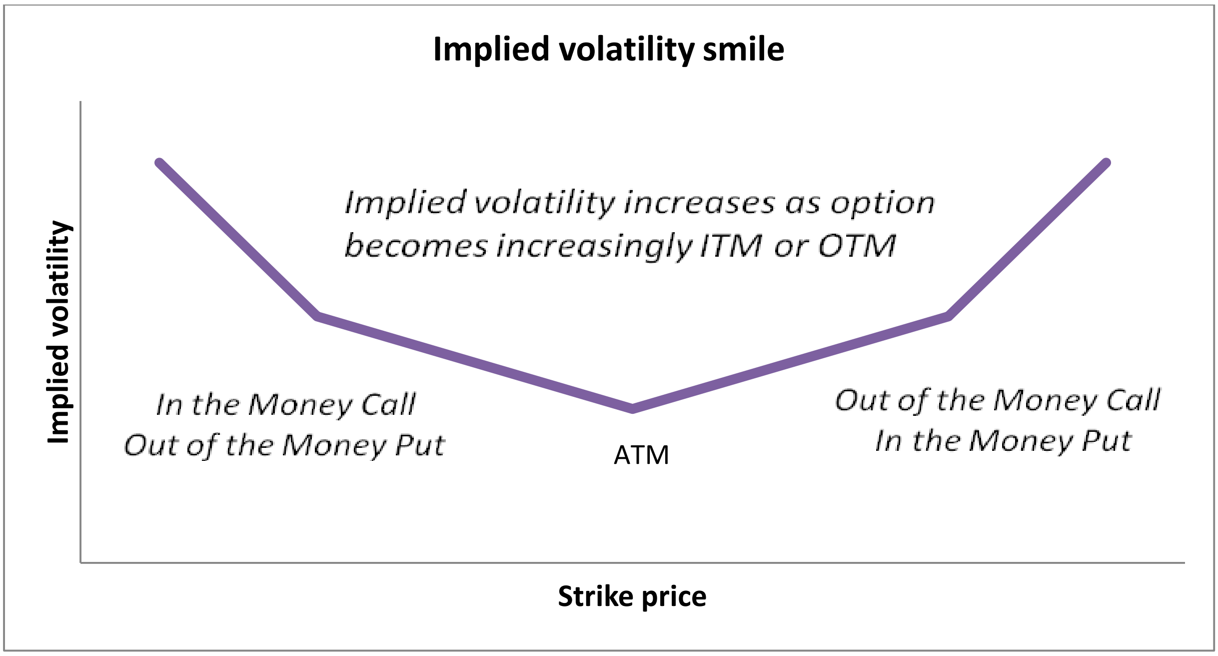

Figure 1.5.1: Implied Volatility smile.

1.6 STOCHASTIC PROCESS

"Any variable whose value changes over time in an uncertain way is said to follow stochastic process." (Hull, 2012, p.280). In other words, a stochastic process describes some random movements of a variable. It can be categorized in discrete time and continuous time stock process:

- ✓ A discrete time stochastic process describes the sequence of random variables known as time series (Markov chain). The values of variables change at the fixed points of the time.
- ✓ Continuous time stochastic processes is a probability process which presents certain probability distributions of some random variables. The values of these variables change continuously over time.

Good examples of stochastic process among many are exchange rate and stock market fluctuations, blood pressure, temperature, Brownian motion, random walk.

A Markov chain is a stochastic process where the past history of variables are irrelevant and only the present value is important for the predicting the future one.
So Markov chain property can be expressed as (Rolski, Schmidli, Schmidt & Teugels 1998, p 269-272):

$$\Pr(X_{n+1} = x | X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \Pr(X_{n+1} = x | X_n = x_n)$$

where $\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}, \dots$ are random variables.

A Wiener process known as Brownian Motion is a particular case of a Markov process, as well, being one of the best known Lévy processes.

"A stochastic process (W_t) is a Brownian motion (Wiener process) if:

The increments are independent, for t $> s$, $W_t - W_s$ is independent of the past. In other words $W_t - W_s$ is independent of the $\sigma - \text{algebra } \mathcal{F}_s$, which is generated by

$$W_u \ , 0 \le u \le s.$$

$W_t - W_s$ is normally distributed with mean 0 and variance t-s:

$$(W_t - W_s) \sim N(0, t = s).$$

The paths are continues, that is,$P(\{\omega : t \to W_t(\omega) \text{continuous}\}) = 1.$"

(Notes of ao. Uni.-Prof. Mag. Dr. Irene Klein, Course "Financial Mathematics", 2011).

where N(x): standard normal distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz$$

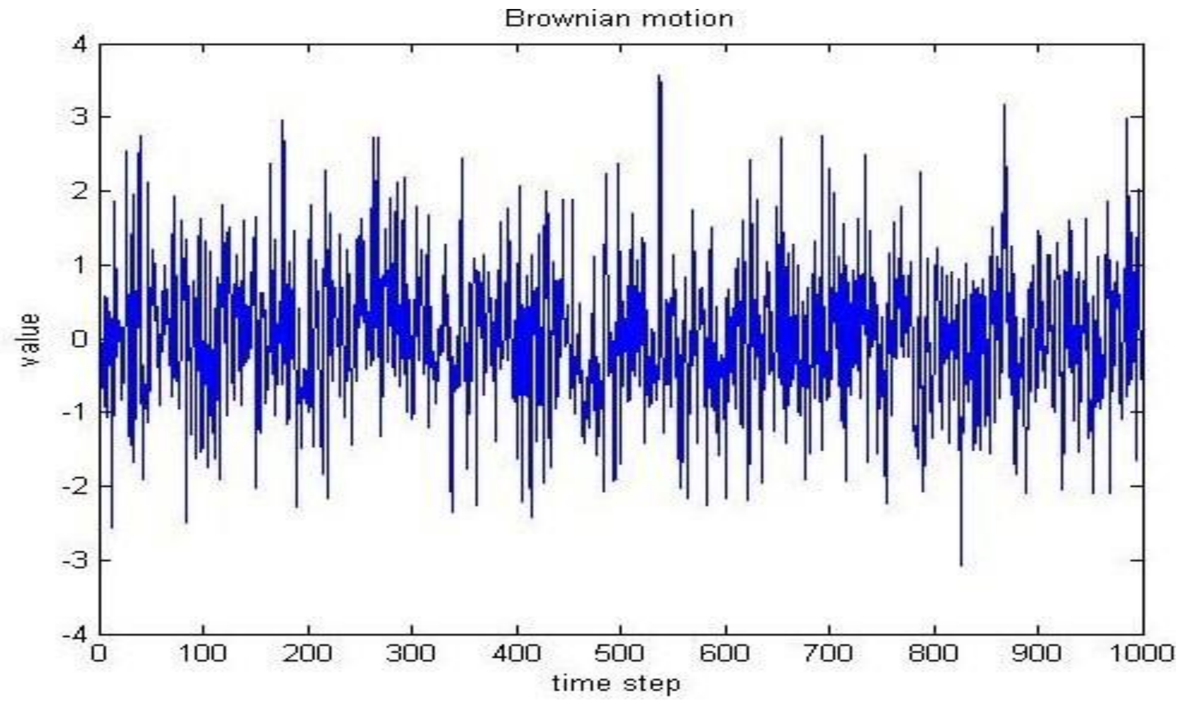

Figure 1.6.1: Brownian motion (number of simulation = 1000)

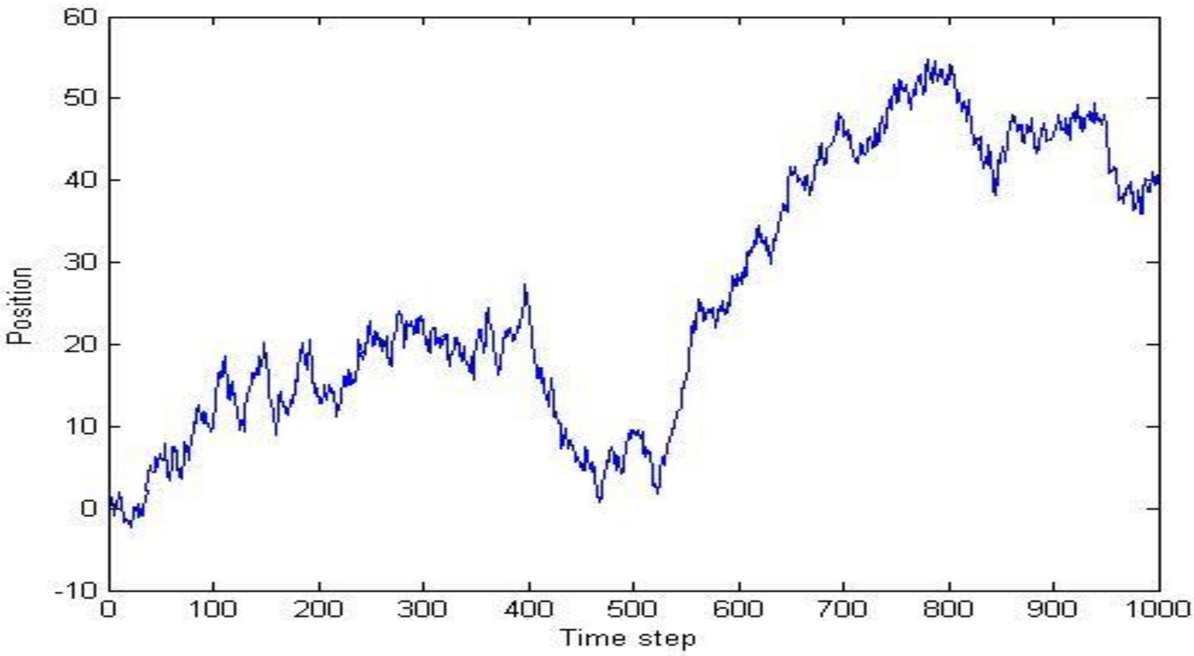

Figure 1.6.2: Trajectory of Brownian motion (number of simulation = 1000)

The Generalized Wiener Process is presented by (Hull, 2012, p.284-285):

$$dX = \nu dt + \xi dz$$

where $\boldsymbol{\nu}$: drift rate per unit of time (constant)

$\boldsymbol{\xi}$: times volatility (constant)

So we say that a variable X follows generalized Wiener process when the X at any time interval $[s, T]$ is normally distributed with mean $\boldsymbol{\nu t}$ and variance $\boldsymbol{\xi^2 t}$:

$$X_t \sim N(\nu t, \xi^2 t)$$

Another important concept that need to be mentioned in this paragraph is the Geometric Brownian Motion (GBM). It is a continuous time stochastic process in which a random variable follows a Brownian motion with drift. So if the stock price S_t follows a GBM if (Hull, 2013, p.286 - 288):

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where $\boldsymbol{\mu}$: constant drift and $\boldsymbol{\sigma}$: constant volatility

I present the detailed solution of the equation (1):

$$dS_t / S_t = \mu dt + \sigma dW_t \tag{2}$$

Let's assume that $f(S_t) = \ln(S_t)$ (3)

by applying the Itō's Lemma to $df(S_t)$ we get:

$$df(S_t) = f'(S_t) dS_t + \frac{1}{2} f''(S_t) dS_t^2$$

substituting the equation (3) it is obvious that

$$d\ln(S_t) = \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} dS_t^2$$

by (1)

[1] $dln(S_t) = \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}(\mu S_t dt + \sigma S_t dW_t)^2 = \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$

Thus,

$$\int_0^t dln(S_t) = \int_0^t (\mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt) \leftrightarrow \ln(S_t) - \ln(S_0) = \mu t + \sigma W_t - \frac{1}{2}\sigma^2 t \leftrightarrow$$

$$S_t = S_0 \exp(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t)$$

<u>Itō's Lemma:</u>

(Yue - Kuen Kwok, 2008, p.83)

Suppose we have Itō's process $dX_t = \mu_t dt + \sigma_t dW_t$.

Then $df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t^2 \frac{\partial f}{\partial x} dW_t$ where $f(t, X_t)$ is twice

differentiable function.

The Itō's process (Yue - Kuen Kwok, 2008, p.83) is a stochastic process which can be represented as:

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \text{ or } dX_t = \mu_t dt + \sigma_t dW_t$$

with $\int_0^T |\mu_t| dt < \infty$ and $\int_0^T \sigma_t^2 dt < \infty$ for all T. It is obvious that the Itō's process has a very similar form with the Generalized Wiener process. The most important difference between them is the dependence of the drift and volatility on time.

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[1] $dtdt = 0$

$dW_t dW_t = dt$

$dtdW_t = 0$

1.7 RISK NEUTRAL PROBABILITY

Risk neutral probability (also called equivalent martingale measure) declares that the derivative price can be estimated as a discounted value of the future payoff $\varphi(T)$ under the risk-neutral measure q^*: Derivative price $= E_T^{q^*}(\varphi(T))$. This estimated derivative price is often called Fair Value. The risk neutral probabilities can be found only in models that do not allow for arbitrage opportunities. Except derivatives pricing, the risk neutral probability helps to simplify the form of complicated models, as in our case the Heston model (which will be discussed in the following chapters). To be able to identify the existence of the risk neutral probability and derive its form it is very important to be acquainted with the Girsanov's Theorem.

Girsanov's Theorem: If W_t is the standard Brownian motion under the real probability measure p and we assume that $W_t^* = W_t + \lambda t$ with $\lambda = \frac{\mu - \nu}{\sigma}$ under the equivalent martingale measure q^* which is defined via the density

$$\frac{dq^*}{dp} = \exp(-vW_T - \frac{1}{2}v^2 T)$$

Then p ~ q^* and the process W_t^* is a standard Brownian motion under q^*.

(Notes of ao. Uni.-Prof. Mag. Dr. Irene Klein, Course "Financial Mathematics", 2011)

Let X be an Ito process with drift μ , volatility σ and $\{W_t\}_{0 \leq t \leq T}$ standard BM:

$$dX_t = \mu dt + \sigma dW_t$$

Assume that we wish to change the drift μ to the drift v.

Define $\lambda = \frac{\mu - \nu}{\sigma}$ and $W_t^* = W_t + \lambda t$ (4)

Thus, $dW_t^* = dW_t + \lambda dt$

Now define $dX_t^* = \nu dt + \sigma dW_t^*$

$$dX_t^* = \nu dt + \sigma dW_t^* = \nu dt + \sigma[dW_t + \lambda dt] = dX_t$$

where $\mu, \nu, \sigma, \lambda$ can be constant or depend on time.

Additionally, Girsanov's Theorem insures that a new probability measure q^* is equivalent to the previous one p.

Hence, define a probability measure q^* via the density:

$$\frac{dq^*}{dp} = \exp\left(-vW_T - \frac{1}{2}v^2 T\right)$$

1.8 FEYNMAN-KAC THEOREM

Before starting talking about the stochastic volatility models, we should be familiar with the Feynman - Kac theorem which establishes the connection between the Partial Differential Equation (PDE) and the Stochastic Differential Equation (SDE). It is a fundamental theorem for a huge amount of financial models as it gives a clear explanation about the derivation of the final solution which is used for valuing options. A lot of economists and mathematicians based their models on the following PDE with the given solution SDE.

Suppose X_t follows the SDE:

$$dX_t = \mu dt + \sigma dW_t \qquad (5)$$

$f(X_t, t)$ is a price of the security at time t with expiration date T and the price at maturity $f(X_T, T) = g(X_T)$. (6)

Additionally, $f(X_t, t)$ follows the PDE:

$$\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2} - r_t f(X_t, t) = 0 \qquad (7)$$

<u>Feynman-Kac Theorem:</u>

If X_t follows SDE (5) and $f(X_t, t)$ satisfies PDE (7) with boundary condition (6) then there exists a unique solution which is presented as:

$$f(X_t, t) = E^Q[\exp\left(-\int_t^T r_u du\right) g(X_T)|\mathcal{F}]$$

The most famous application of this theorem is a Black-Schoels formula where the Black-Scholes equation has the form of PDE (7), $g(X_T) = (S_T - K)^+$ and $f(X_t, t) = C(S_t, t)$: call option which is

$$C = C(S_t, t) = E^Q\left[\exp\left(-\int_t^T r_u du\right)(S_T - K)^+\middle|\mathcal{F}\right]$$

I decided precisely to solve the mentioned above equation:

$$C = C(S_t, t) = E^Q\left[\exp\left(-\int_t^T r_u du\right)(S_T - K)^+\middle|\mathcal{F}\right] =$$

$$= E^Q\left[e^{r(T-t)} S_T\, 1_{\{S_T > K\}}\right] - E^Q[e^{r(T-t)} K\, 1_{\{S_T > K\}}]$$

Assumptions:

$t=0$

$\overline{W_t} = W^* - \sigma t$ (8)

$I_1 = E^Q\left[e^{-rT} S_T\, 1_{\{S_T > K\}}\right]$ and $I_2 = E^Q\left[e^{-rT} K\, 1_{\{S_T > K\}}\right]$

Therefore,

$I_1 = E^Q\left[e^{-rT} S_T\, 1_{\{S_T > K\}}\right] = E^Q\left[S_T^*\, 1_{\{S_T > K\}}\right]$ and $I_2 = E^Q\left[e^{-rT} K\, 1_{\{S_T > K\}}\right]$

where $S_T^* = e^{-rT} S_T$ and $S_T^* = S_0 \exp(\sigma \overline{W_T} + \frac{1}{2}\sigma^2)T)$ (by (8))

$\overline{Q}$ is equivalent probability measure with $\frac{d\overline{Q}}{dQ} = \exp(\sigma W_T^* - \frac{1}{2}\sigma^2 T)$

$\frac{d\overline{Q}}{dQ} = \frac{S_T^*}{S_0}$ and $\overline{W_t} = W^* - \sigma t$ is Standard Brownian Motion according to the Girsanov's theorem.

Thus, I_1 can be expressed as

$$I_1 = E^Q\left[e^{-rT}S_T\, 1_{\{S_T>K\}}\right] = E^Q\left[S_T^*\, 1_{\{S_T>K\}}\right] = E^Q\left[S_0 \frac{d\overline{Q}}{dQ}\, 1_{\{S_T>K\}}\right] = S_0\overline{Q}(S_T > K) =$$

$$= S_0\overline{Q}(S_T^* > e^{-rT}K) = S_0\overline{Q}(S_0 \exp\left(\sigma\overline{W_T} + \left(r + \frac{1}{2}\sigma^2\right)T > K\right))$$

$$= \ S_0\overline{Q}\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} > -\frac{\overline{W_T}}{\sqrt{T}}\right) = S_0N\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = S_0N(d_1)$$

Similarly,

$$I_2 = E^Q\left[e^{-rT}K\, 1_{\{S_T>K\}}\right] = e^{-rT}KQ(S_T > K)$$

$$= e^{-rT}KQ(S_0 \exp\left(\sigma W_T^* + \left(r - \frac{1}{2}\sigma^2\right)T > K\right))$$

$$= e^{-rT}KQ\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} > -\frac{W_T^*}{\sqrt{T}}\right)$$

$$= e^{-rT}KN\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = e^{-rT}KN(d_2)$$

Hence, the well-known Black-Scholes formula is proved:

$$C = C(S_t, t) = E^Q\left[\exp\left(-\int_t^T r_u du\right)(S_T - K)^+ \middle| \mathcal{F}\right] = S_tN(d_1) - e^{-r(T-t)}KN(d_2)$$

CHAPTER 2 BASIC MATHEMATICAL MODELS

2.1 INTRODUCTION

During many dedicates a lot of effort was put into the development and improvement of mathematical models with the goal to produce a solid pricing process of financial instruments in the different compartments of financial world. There is no perfect derivative pricing model at this time time, thus there exists a profit opportunity for portfolios' holders.

Black-Scholes was the first mathematical model which introduced the complete closed-form solution for option pricing Afterwards many models were developed like Heston stochastic volatility model, Bates stochastic volatility and jump-diffusion model, as well, as Heston and Nandi discrete time volatility model.According to the studies of these models and empirical evidences it was shown that Heston, Bates , Heston and Nandi achieve better results than Black-Scholes model, even in the period of economical crises Nevertheless, Black-Scholes is still widely used since it does not demand a necessary calibration to fit market prices and it is easy to implement

Additionally, Heston, Bates and Heston Nandi models cannot be easily ranged or substituted by each other if we categorized the options according to maturity and moneyness. Each of them under some specific circumstances present worse or better results than the others. Consequently, Bates model functions better than any other in the case of short term options, Heston, Heston and Nandi present better approaches in the case of medium or long term options, Black-Scholes provides with efficient approaches for long term in the money options and the worst for short term out of the money. In the periods of high volatility, Black-Scholes and Heston-Nandi show the worst results while Heston and Bates perform pretty well (Moyaert & Petitjean, 2011).

Therefore, there was observed a significant progress in the category of short term models. These models are based on the outline of the fluctuation and evolution of the short term rate. Short term rate models are classified into equilibrium models and no-arbitrage models. Equilibrium model is a model which is produced in one particular market under specific circumstances with certain assumptions with the goal to manage the equilibrium. Therefore, the model is an output of the respective market

system. No-arbitrage model is a model which is implemented in the certain market using the already existing data for calibration (Gairns 2004).
The short term interest rate is very important in the valuation of bonds, options and other derivatives. The Vasicek model and its extension Cox-Ingersoll-Ross model (both equilibrium models) help to describe the movements of short term interest rate and make some forecasts of it. Under some specific circumstances, each of them present worse or better than the others results. Unfortunately, these models cannot be used for the calculation of the long term interest rates. They are mean –reverting processes that compose just a constant line in the long run. Additionally, they do not completely help in the production of the yield curve. Very often in the market yield curve has a very complex shape which cannot be obtained with the help of those models. Thus, there exists one more reason to consider no-arbitrage models (Hull and White model). The Hull-White model can also be seen as the extension of Vasicek model, the only difference is that it assumed that long term mean level that also depends on the time. But unfortunately, the model is not perfect and similarly presents disadvantages.

Consequently, the chapter 2 has the following structure:
In the first paragraph (§ 2.2) I describe the Black-Scholes model, its assumptions and framework. Additionally, I present another version of the model and explain why exactly this version was used for the empirical analysis (Black and Schoels 1973, Hull 2012, Gilli and Schumann 2010, Bakshi and Madan 2000, Klein, 2011). Afterwards (§ 2.3), I describe the first mean-reverting short term interest rate model called Vasicek model (Haugh 2010, Lesniewski 2008, Brigo and Mecurio 2006, Vasicek 1977, Yue - Kuen Kwok 2008, Björk 2009, Bayazit 2004, Gairns 2004, Fouque, Papanicolaou, Sircar and Sølna 2011)). In the next paragraphs I present the extensions of Vasicek model which are Cox-Ingersoll-Ross (CIR) model (§ 2.4) (Cox, Ingersoll and Ross 1985, Björk 2009, Bayazit 2004, Gairns 2004, Fouque, Papanicolaou, Sircar and Sølna 2011)) and Hull-White model (§ 2.5) (Björk 2009, Khan, Aisha, Guan, Eric 2008, Kohn 2005, Hull and White 1987, Bayazit 2004). As well, I introduce one discrete time stochastic volatility model Heston and Nandi model (§ 2.6) which stands out among other financial models because of the GARCH process which involved in the description of model structure (Heston and Nandi

2000). In the last paragraph of this chapter (§ 2.7) I talk about stochastic volatility and jump-diffusion model called Bates model which is a combination of Merton and Heston models. This model except describing the evolution of interest rate and stock price movements takes into account the price jumps during one time interval (Bates 1995, Bates 1996, Bates 2010).

2.2 BLACK-SCHOLES MODEL

Black-Scholes model is one of the most famous and intensively used mathematical models in financial world which estimates European style options and produces the Fair Value of financial instruments. The model was introduced by Fischer Black, Merton and Myron Scholes in 1973. For the first time the complete formula for the option valuation was presented in a closed form which was easy to implement. The model helped to calculate the values of the options over the time. To be able to use Black-Scholes model the following assumptions have been defined:

- The model can be implemented for European style options
- There are no dividend payments
- The stock price follows the geometric Brownian motion with constant drift and volatility
- There are no extra costs such as taxes or transaction costs
- Continuous trading in the market is presented
- No riskless arbitrage opportunity
- No limit in selling or buying any amount of stock
- The interest rate is constant

The stock price S_t for $t \in [0, T]$ follows the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with the unique solution:

$$S_t = S_0 \exp(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t)$$

where

$\mathbf{S_t}$: stock price

$\mathbf{W_t}$: Brownian motion

t: time (in years)

μ: drift rate

σ: volatility of the stock returns

For constant interest rate r, bond $B_t = e^{rt}, B_0 = 1$ satisfies $dB_t = rB_t dt$.

Additionally, Black – Scholes partial differential equation which describes the price of the option during the time has the following form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Terminal condition: $V(S_T, T) = (S_T - K)^+$ for European Call option

$V(S_T, T) = (K - S_T)^+$ for European Put option

where

V(S,t) : price of derivative as a function of stock price and time

K: strike price

r: risk-free interest rate

T: expiry time

How it was mentioned in the previous chapter 1 in paragraph 1.7 the Black-Scholes equation has a unique solution which gives the formula that is used to estimate European style options.

For Call Option: $C(S_t, t) = S_t N(d_1) - e^{-r\tau} K N(d_2)$

For Put Option: $P(S_t, t) = e^{-r\tau} K N(-d_2) - S_t N(-d_1)$

where

C(S,t): price of European call option

P(S,t): price of European put option

$\boldsymbol{\tau = T - t}$: time to maturity

$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right]$ and $d_2 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right] = d_1 - \sigma\sqrt{\tau}$

N(x): standard normal distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2 dz}$$

As well, knowing a single Put option price helps to calculate with the Put-Call parity the Call option price (and opposite):

$$P(S_t, t) - C(S_t, t) = Ke^{-r\tau} - S_t$$

Unfortunately, the Black-Scholes model is not perfect and a huge amount of empirical studies shows that the assumption of constant volatility has negative consequences. One of the most important is that Black-Scholes model cannot explain the volatility smiles and smirks implicit in market options. Hence, it makes no sense to traders and investors. The moneyness of the options is not observed and in this case has not meaning at all.

In 2000 there was presented a bit different version of the Black-Scholes closed form solution which did not contain the standard normal distribution but was built on the probability measures. The probability measures were based on the characteristic function which took in account the logarithmic stock prices. Consequently, the Black-Scholes model with the characteristic function was decomposed as following:

$$C(S_0, \tau) = e^{-q\tau} S_0 P_1 - e^{-r\tau} K P_2$$

where

$\mathbf{q}$: dividend (in this case of this Master thesis it is assumed that $q = 0$)

$\mathbf{P_1}$ and $\mathbf{P_2}$: conditional probabilities dines as the following:

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re(e^{i\varphi \ln(k)} \frac{f(\varphi - i)}{i\varphi f(-i)}) d\varphi \quad \text{and} \quad P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re(e^{i\varphi \ln(k)} \frac{f(\varphi)}{i\varphi}) d\varphi$$

and characteristic function:

$$f(\varphi) = \exp(i\varphi s_0 + i\varphi\tau(r-q) - \frac{1}{2}(i\varphi + \varphi^2)\tau\sigma^2)$$

with $s_\tau = \ln(S_\tau)$, $s_\tau \sim N(S_0 + \tau\left(r - q - \frac{1}{2}\sigma^2\right), \tau\sigma^2)$

In the Chapter 4 will be presented empirical analysis of Heston and Black-Scholes models where will be used the Black-Scholes model with the characteristic function. The main reason of that is the structure of the characteristic function since it is based on the logarithmic stock prices. Theoretically, it is supposed to give better approximation as the calculation of the probability measures are based on the stock prices obtained from the market which do not follow the normal distribution in reality. Thus, the obtained results would get more realistic character. Additionally, this version of Black-Scholes formula has a structure which is more similar to the structure of the Heston model. Therefore, it is interesting enough to compare models with similar framework of the closed form of option pricing formula.

2.3 VASICEK MODEL

For the first time Vasicek model was presented by Oldrich Vasicek in 1977 and was built on the idea of mean reverting interest rate. The model has been characterized as a short rate model and classified to equilibrium models.

The goal of the Vasicek model was to focus on the interest rate movements in the market and describe its evolution. Other important aim was to predict the asset returns and prices over time. This element made the Vasicek model obtain the stochastic investment character. It played an important role in the valuation of interest rate financial instruments.

In Vasicek model the interest rate is described as a following SDE:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$$

with solution :

$$r(t) = r(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s} dW_s$$

and $E[r_t] = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})$

$$Var[r_t] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})$$

consequently $\lim_{t \to \infty} E[r_t] = \theta$

$$\lim_{t \to \infty} Var[r_t] = \frac{\sigma^2}{2\kappa}$$

where

$\mathbf{W_t}$: Wiener process

$\boldsymbol{\sigma}$: volatility of the interest rate which is called instantaneous volatility.

$\boldsymbol{\theta}$: long run normal interest rate which is also called long term mean level. It is an equilibrium value of reversion of interest rate in the long run.

$\boldsymbol{\kappa}$: determines the speed with which the interest rate moves towards its long run normal level. It is called speed of reversion and $\kappa > 0$.

$\boldsymbol{\sigma^2/_{2\kappa}}$**:** long term variance. It shows that in the long run all interest rates will be grouped around the long term mean level with such variance. When the system reaches $\boldsymbol{\sigma^2/_{2\kappa}}$ level then it obtains the stabilized character around $\boldsymbol{\theta}$.

$\boldsymbol{\kappa(\theta - r_t)}$**:** demonstrates the expected instantaneous changes in the interest rate.

The model sometimes expressed as $r_t = a + br_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim N(0, \sigma^2)$, the short rate r_t which is assumed to be the main source of uncertainty in the economy follows an AR(1).

Additionally, Vasicek model gives an exact expression of the value of a zero coupon:

$$B(t,T) = \exp\big(-A(t,T)r_t + D(t,T)\big)$$

where

$$A(t,T) = \frac{1 - e^{-\kappa\tau}}{\kappa}$$

$$D(t,T) = \left(\theta - \frac{\sigma^2}{2\kappa}\right)[A(t,T) - \tau] - \frac{\sigma^2 A(t,T)^2}{4\kappa}$$

Now it is easy to formulate the type for valuing call options with the strike price K, maturity T_0 and the underlying bond maturity T.

$$C = B(t, T)N(d_1) - KB(t, T_0)N(d_2)$$

where d_1 and d_2 are exactly the same like in Black-Scholes formula (paragraph 2.2) Unfortunately, Vasicek model has, as well, some disadvantages. One of the most essential is the possibility of the model to produce negative interest rate something which has no sense from the economical point of view. The bright idea of John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross (among many others), which follows in the next chapter 2.4, helped to fix this drawback.

2.4 COX-INGERSOLL-ROSS (CIR) MODEL

CIR model was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of Vasicek model which belongs to equilibrium sort-rate models. To avoid the possibility of negative interest rate they decided to describe the instantaneous interest rate as a square root diffusion often called as CIR process:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

The parameters are exactly like in Vasicek model. Additionally, $\boldsymbol{\sigma\sqrt{r_t}}$ is a standard deviation factor which helps to avoid the possibility of negative interest rates' existence $\forall\, \kappa, \theta > 0$. A zero interest rate is also rejected if the following condition holds $\boldsymbol{2\kappa\theta} \geq \boldsymbol{\sigma^2}$ (Feller condition). Overall, when the interest rate is close to zero then the $\boldsymbol{\sigma\sqrt{r_t}}$ also approaches zero, this results in reduction of the random shock effects on the interest rate. Therefore, when the interest rate gets close to zero the drift factor $\boldsymbol{\kappa(\theta - r_t)}$ starts dominating the evolution of the rate, consequently it is pushed upwards.

Besides there exists a closed form of the process:

$r_{t+T} = r_t + cY$ where $c = \frac{(1-e^{-\kappa T})\sigma^2}{2\kappa}$ and Y is a non-central Chi-Squared distribution with $4\kappa\theta/\sigma^\wedge 2$ degrees of freedom and non-centrality parameter $2cr_t e^{-\kappa T}$.

As well, the bond may be priced as:

$$B(t, T) = A(t, T)\exp(D(t, T)r_t)$$

where

$$A(t, T) = \left(\frac{2h\exp(\frac{(\kappa + h)\tau}{2})}{2h + (\kappa + h)(\exp(\tau h) - 1)}\right)^{2\kappa\theta/\sigma^2}$$

$$D(t, T) = \frac{2(\exp(\tau)h) - 1)}{2h + (\kappa + h)(\exp((\tau)h) - 1)}$$

$$h = \sqrt{\kappa^2 + 2\sigma^2}$$

Now it is easy to formulate the type for valuing call options with the strike price K, maturity T_0 and the underlying bond maturity T.

$$C = B(t, T)N(d_1) - KB(t, T_0)N(d_2)$$

where d_1 and d_2 are exactly the same like in Black-Scholes formula (chapter 2.2)

Now everything seems to be better, the standard deviation factor looks like a good solution to the Vasicek model problem. But let's go back in time to the period of economical imbalance, for example the period of the Second World War. During that days it is observed that the interest rate was dropped to the lowest level, it even had negative values. If the try to use CIR model was done during that days then a huge problem would have been faced as $\sqrt{r_t}$ accepts only positive numbers. Thus, in addition to apply the model, it would be necessary to switch to the complex numbers something that makes the model be very complicated and inconsiderable from the economical point of view.

2.5 HULL-WHITE MODEL

Hull-White model, which was introduced by John C. Hull and Alan White in 1990, had an extension character of Vasicek model. Comparing to it, the model is considered to be a short rate model but with no-arbitrage character. Nowadays the Hull-White model is still often used in the financial world. It assumes that the short rate is described as mean reverting process which follows the Gaussian distribution.

The short-rate process is presented as:

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t$$

where

$\boldsymbol{\sigma}$: volatility of the short rate

$\boldsymbol{\theta_t}$: is time dependent function which determinates the average direction in which $\mathbf{r}_t$ moves

a : mean reversion rate which plays the controlling role between volatilities' relation

$\mathbf{dW_t}$: Wiener process

$\mathbf{r_t}$ can be computed as:

$r_t = e^{-at}r_0 + ae^{-at}\int_0^t \theta_t e^{as}\,ds + \sigma e^{-at}\int_0^t e^{as}\,dW_s$: normally distributed with

$$E(r_t) = r_o e^{-at} + ae^{-at}\int_0^t \theta_t e^{as}\,ds$$

$$Var(r_t) = \frac{\sigma^2}{2a}[1 - e^{-2at}]$$

The price of bond at time t with maturity at time T is:

$$B(t,T) = A(t,T)e^{-D(t,T)r_t}$$

where $D(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}]$

$$A(t,T) = \exp[-\int_t^T \theta_s\,D(s,T)ds - \frac{\sigma^2}{2a^2}(D(t,T) - T + t) - \frac{\sigma^2}{4a}D(t,T)^2]$$

The parameter θ_t can be computed as:

$$\theta_t = \frac{\partial f}{\partial T}(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

where $f(t,T) = -\frac{\partial \log B(t,T)}{\partial T}$ and

$$\frac{\partial f}{\partial T}(0,T) = \theta_T + \int_0^T \theta_s \, \partial_{TT} D(s,T)ds - \frac{\sigma^2}{2a}\partial_T D(0,T)$$

$$+\frac{\sigma^2}{2a}[(\partial_T D(0,T))^2 + D(0,T)\,\partial_{TT} D(0,T)] + \partial_{TT} D(0,T) r_0$$

$$f(0,T) = \int_0^T \theta_s \, \partial_T D(s,T)ds - \frac{\sigma^2}{2a} D(0,T) + \frac{\sigma^2}{2a} D(0,T)\,\partial_T D(0,T) + \partial_T D(0,T) r_0$$

The price of European option at time t on a pure discounted bond:

$$ZBO = z\{B(t,s)N(zh) - KB(t,T)N[z(h - \sigma_B)]\}$$

Where

$\mathbf{s > T}$: maturity date of the bond

$\mathbf{T > t}$: maturity date of the option

$\mathbf{T}$: maturity date of the option

$\mathbf{K}$: strike price

$$z = \begin{cases} 1, & \text{for call option} \\ -1, & \text{for put option} \end{cases}$$

$$h = \frac{1}{\sigma_B}\ln\frac{B(t,s)}{B(t,T)K} + \frac{\sigma_B}{2}$$

$$\sigma_B^2 = \frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)D(T,S)^2$$

Like in the Vasicek model, the biggest disadvantage of Hull –White model is the possibility of a negative interest rate.

2.6 HESTON AND NANDI MODEL

Heston and Nandi model is discrete time volatility model which was presented for the first time by Steven L. Heston and Saikat Nandi. This model presents an option pricing formula which includes Generalized Autoregressive Conditional Heteroskedasticity (GARCH), where for not only the current but also the historical stock prices are used.

Heston-Nandi model assumes that the log stock price follows a GARCH(p,q) process (accumulated interest or dividends are included)such as :

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}z(t)$$

$$h(t) = \omega + \sum_{i=1}^{p} \beta_i h(t-i\Delta) + \sum_{i=1}^{q} \alpha_i (z(t-i\Delta) - \gamma_i \sqrt{h(t-i\Delta)})^2$$

where

$\boldsymbol{\Delta}$: duration of time

$\mathbf{a_i}, \boldsymbol{\beta_i}, \boldsymbol{\gamma_i}, \boldsymbol{\omega}$: model parameters

$\mathbf{r}$: interest rate fort he time interval Δ

$\mathbf{z(t)}$: shock which is standard normally distributed

$\mathbf{h(t)}$: variance of the long return such that $\mathbf{h(t)} \in [t-\Delta, t]$, also known as return premium.

$\boldsymbol{\sigma} = \sqrt{\mathbf{h(t)}}$: volatility

$\boldsymbol{\lambda}\mathbf{h(t)}$: risk premium which prevents all arbitrage opportunities

$\boldsymbol{\lambda}$: constant

In this master thesis I focus on the first-order case when $p = q = 1$, thus :

$$h(t+\Delta) = \omega + \beta_1 h(t) + \alpha_1 \frac{(\log(S(t)) - \log(S(t-\Delta)) - r - \lambda h(t) - \gamma_1 h(t))^2}{h(t)}$$

and

$$\mathrm{Cov}_{t-\Delta}\left[h(t+\Delta), \log\left(S(t)\right)\right] = -2\alpha_1\gamma_1 h(t)$$

The first order process remains stationary if $\beta_1 + \alpha_1\gamma_1^2 < 1^2$.

where

$\mathbf{a_1}$: demonstrate the kurtosis of the distribution

$\boldsymbol{\gamma_1}$: demonstrates the skewness or distribution asymmetry. The distribution is considered to be symmetric when $\gamma_1 = 0,\ \lambda = 0$.

The variance of the Heston and Nandi model has been presented as the squared-root process of Feller (1952), Cox, Ingersoll and Ross (1985) and Heston (1993):

$$dV = k(\theta - V)dt + \sigma\sqrt{V}dz$$

where z(t) is a Wiener process.

The value of European call option under the risk-neutral probability with strike K that expires at time T can be expressed as:

$$C = e^{-t(T-t)}E_t^*[\max(S(T) - K), 0)]$$

$$= \frac{1}{2}S(t) + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi+1)}{i\varphi}\right]d\varphi$$

$$-Ke^{-r(T-t)}(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi)}{i\varphi}\right]d\varphi)$$

where $E^*[\,.\,]$ is the expected value under the risk-neutral distribution

with the following characteristic function

$$f^*(\varphi) = S(t)^\varphi \exp\left(A(t;T,\varphi) + \textstyle\sum_{i=1}^{p} B_i(t;T,\varphi)h(t+2\Delta - i\Delta\right) +$$

$$\textstyle\sum_{\iota=1}^{q-1} C_i(t;T,\varphi)(z(t+\Delta-i\Delta) - \gamma_i\sqrt{h(t+\Delta-i\Delta)})^2)$$

where

$$A(t;T,\varphi) = A(t+\Delta;T,\varphi) + \varphi r + B_1(t+\Delta;T,\varphi)\omega - \frac{1}{2}\ln(1-2a_1B_1(t+\Delta;T,\varphi))$$

$$B_1(t;T,\varphi) = \varphi\left(\lambda+{\gamma^*}_1\right) - \frac{1}{2}{\gamma^*}_1{}^2 + \beta_1B_1(t+\Delta;T,\varphi) + \frac{1}{2}\left(\frac{\left(\varphi-{\gamma^*}_1\right)^2}{1-2a_1B_1(t+\Delta;T,\varphi)}\right)$$

with $A(T;T,\varphi) = 0, B_1(T;T,\varphi) = 0$, $p = q = 1$ and $\gamma_1^* = \gamma_1 + \lambda + \frac{1}{2}$

One of the biggest advantages, at Heston and Nandi model, is its discrete time formulation. In contrast to the continuous-time models, GARCH model bases the option valuation on the historical observations which make it stand out among others.

2.7 BATES MODEL

Bates model is a stochastic volatility and jump-diffusion model which does not take into account the interest rate risk. It was developed by David S. Bates in 1996. He made a brilliant and effective try to combine Merton and Heston models approaches.

Bates describes the instantaneous variance V_t as the mean-reverting square root process:

$$dV_t = (\alpha - \beta V_t)dt + \sigma\sqrt{V_t}dW_t^2$$

The stock price is given as:

$$\frac{dS_t}{S_t} = (\mu - \lambda\bar{\kappa})dt + \sqrt{V_t}dW_t^1 + kdq$$

with $\mathrm{Cov}(dW_t^1, dW_t^2) = \rho dt$

where

$\boldsymbol{\mu}$: instantaneous expected rate of the foreign currency

q: described the amount of jumps during one year. It is called “Poisson counter with intensity λ with $P(dq = 1) = \lambda dt$

$\mathbf{W_t^1}, \mathbf{W_t^2}$: Wiener process with $dW_t^1dW_t^2 = \rho dt$

σ : volatility of volatility

k: random percentage jump size with $\log(1+\kappa) \sim N(\log(1+\bar{\kappa}) - \frac{1}{2}\delta^2, \delta^2)$

Where the parameters $\bar{\kappa}$, δ determine the distribution of the jumps.

To calculate the option value we need to rewrite our equations in a "risk-neutral" version:

$$\frac{dS_t}{S_t} = (b - \lambda^* \bar{\kappa}^*)dt + \sqrt{V_t}dW_t^1 + \kappa^* dq$$

$$dV_t = (\alpha - \beta^* V_t)dt + \sigma\sqrt{V_t}dW_t^2$$

where all new parameters have the same meaning like in the previous equations.

Additionally, $\lambda^* = \lambda E(1 + \frac{\Delta J_w}{J_w})$

$$\bar{\kappa}^* = \bar{\kappa} + \frac{Cov(k, \Delta J_w/J_w)}{E[1 + \Delta J_w/J_w]}$$

$$\log(1+\kappa^*) \sim N(\log(1+\bar{\kappa}^*) - \frac{1}{2}\delta^2, \delta^2)$$

with

$\mathbf{J_w}$: marginal utility of investor's nominal wealth

$\mathbf{\Delta J_w / J_w}$: random percentage jump conditional on a jump occurring

Now we are ready to formulate the characteristic function of this model, which is

$$\varphi(i\Phi) = E_0{}^*[e^{i\Phi \ln S_T} | S_0, V_0, T] = \exp[i\Phi S_0 + C(T; i\Phi) + D(T; i\Phi)V_0 + \lambda^* T E(i\Phi)]$$

where $E_0{}^*[.]$: expected value under the risk-neutral distribution

$$C(T; z) = bTz - \frac{aT}{\sigma^2}[\rho\sigma z - \beta^* - \gamma(z)] - \frac{2a}{\sigma^2}\ln[1 + [\rho\sigma z - \beta^* - \gamma(z)]\frac{1 - e^{\gamma(z)T}}{2\gamma(z)}]$$

$$\gamma(z) = \sqrt{(\rho\sigma z - \beta)^2 - \sigma^2(z^2 - z)}$$

$$D(T;z) = \frac{z^2 - z}{\gamma(z)\frac{e^{\gamma(z)T} + 1}{e^{\gamma(z)T} - 1} + \beta^* - \rho\sigma z}$$

$$E(z) = (1 + \bar{\kappa}^*)^z e^{1/2\delta^2(z^2 - z)} - 1 - \bar{\kappa}^* z$$

Finally, the value of European call option can be calculated as:

$$C = B_T(FP_1 - KP_2)$$

where

$\mathbf{B_T}$: the price of the bond at time T (at maturity)

$\mathbf{F}$: the forward price on the underlying asset $F = S_t e^{bT}$ with instantaneous risk-free interest rate r and domestic/foreign interest differential $b = r - r^*$ which are known and constant

$\mathbf{K}$: strike price

$\mathbf{P_1}, \mathbf{P_2}$: probability measures

Additionally, there was developed a formula for the direct valuation of call options which is:

$$C = B_T F - B_T K\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{\varphi(i\Phi)e^{-i\Phi \ln K}}{i\Phi(1 - i\Phi)}]d\Phi\}$$

where $\mathrm{Re}[z]$ is the real part of the complex variable.

Therefore, the value of European put option it is possible to be calculated with the help of the put-call parity relation:

$$P = C + B_T K - B_T F$$

According to empirical studies and research papers it is important to be mentioned that the calibration in Bates model produces more precise results for the short term skew and gives more realistic forward – skew dynamics comparing to any other stochastic volatility model. Thus, it explains better the volatility smile which makes the model be an available tool for the participants of the financial markets.

CHAPTER 3 THE HESTON STOCHASTIC VOLATILITY MODEL

3.1 INTRODUCTION TO HESTON MODEL

Heston model is a stochastic volatility model introduced by Steven Heston in 1993. The model takes into account the correlation between the returns on stock and its volatility, describes the movements of the instantaneous variance as the CIR process and presents the close-form formula for pricing European style options. Nowadays, Heston model is one of the most famous and often used stochastic volatility models for pricing European style options, as well as, bond and foreign currency options since it provides with the sufficient approaches and explains volatility smiles. One of the biggest drawbacks is its sensitivity to changes of values in market data and model parameters. According to empirical studies, Heston model generates adequate approximations even during the period of economical crisis that makes it stand out amount other financial models.

In the chapter 3 I present the derivation of the Heston model and its final structure (§ 3.2) (Hull 2012, Heston1993, Fouque, Papanicolaou, Sircar and Sølna 2011). Afterwards (§ 3.3), I explain the derivation of the Heston partial differential equation which describes the stock price performance over time (Heston 2013, Heston 1993, Bauer 2012, Moodley 2005, Fouque, Papanicolaou, Sircar and Sølna 2011)). In the next paragraph (§ 3.4) I summarize the previous paragraphs, present the framework of the Heston model and illustrate the stock price and interest rate movements (Heston 2013, Heston 1993, Bauer 2012, Fouque, Papanicolaou, Sircar and Sølna 2011)). Therefore (§ 3.5), I talk about the risk-neutralized approach of the Heston model which helps to illuminate of the Heston parameters and simplify the model structure that is very helpful for the calibration (Heston 1993, Yuan Yang 2013, Hull 2012, Klein 2011, Moodley 2005). Continuously (§ 3.6), I explain the Heston partial differential approach and closed-form solution of the model (Heston 1993, Heston 2013, Gatheral and Taleb 2006). Afterwards (§ 3.7), I analyze the Heston closed form solutions, do sensitivity analysis of the model parameters with the help of MATLAB and describe their impacts on the stock price distribution and implied volatility smile (Heston 1993, Heston 2013). Then (§ 3.8), I summarize all model advantages and disadvantages (Zhang and Shu 2003, Bauer 2012, Black 1976, Moodley 2005). Finally, in the last paragraph (§ 3.9) I present other version of Heston model called “the little Heston trap”. I explain the fundamental difference between these two interpretations and the reasons why I chose "the Heston little trap" for the empirical analysis (Heston 2013, Albrecher, Mayer, Schoutens and Tistaert 2006, Gilli and Schumann 2010, Kahl and Jäckel 2005.).

3.2 DERIVATION OF HESTON MODEL

Heston model is a stochastic volatility model. Every stochastic volatility model focuses not only on describing the stock movements but also on the evolution of volatility in the financial market. In general, every stochastic volatility mode has the following structure.

$$dS = S\mu(t, S)dt + S\sigma dW_1$$

$$d\sigma = a(t, \sigma)dt + q(t, \sigma)dW_2$$

where S is a stock price and σ its variance, $\mu(t, S)$, $p(t, \sigma)$and $q(t, \sigma)$ are Borel-measurable functions

with

$$dW_1 dW_2 = \rho dt$$

To derive the today very-well known Heston model structure, Heston made the following assumptions:

$$\sigma = \sqrt{V}, a(t, \sigma) = -\beta\sqrt{V} \text{ and } q(t, \sigma) = \delta$$

The stock price S and its instantaneous variance V satisfy the following SDEs:

$$dS_t = \mu_t S_t dt + \sqrt{V_t} S_t dW_t^{(1)}$$

$$d\sqrt{V_t} = -\beta\sqrt{V_t}dt + \delta dW_t^{(2)}$$

$$\text{with } dW_t^{(1)} dW_t^{(2)} = \rho dt$$

where

$\boldsymbol{\mu_t}$: instantaneous drift of stock price return

$\boldsymbol{\sigma}$: volatility of volatility

$\boldsymbol{\rho}$: correlation parameter between the return on stock and its instantaneous volatility

$\mathbf{dW_t^{(1)}}, \mathbf{dW_t^{(2)}}$: standard Brownian motions (Wiener processes).

Thus, it is assumed that the volatility follows the Ornstein-Uhlenbeck process. Now it is defined that $f(x) = x^2$ and $x = \sqrt{V_t}$. Therefore with the help of Ito's lemma $df(x)$ can be presented as:

$$df(\sqrt{V_t}) = f'(\sqrt{V_t})d\sqrt{V_t} + \frac{1}{2}f''(\sqrt{V_t})d\sqrt{V_t}^2 = 2\sqrt{V_t}d\sqrt{V_t} + dV(t)^2 =$$

$$= -2\beta V_t dt + 2\delta\sqrt{V_t}dW^{(2)} + \delta^2 dt \qquad (*)$$

As well, $$df(\sqrt{V_t}) = dV_t \qquad (**)$$

Finally, by (*) and (**) it can be obtained $dV_t = (\delta^2 - 2\beta V_t)dt + 2\delta\sqrt{V_t}dW^{(2)}$

Afterwards it was assumed that $\kappa = 2\beta, \theta = \frac{\delta^2}{2\beta}, \sigma = 2\delta$

Hence,

$$dS_t = \mu_t S_t dt + \sqrt{V_t} S_t dW_t^{(1)}$$

and

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW^{(2)}$$

Three of the most important advantages of the Heston model structure are

- ✓ mean-reverting form which effects on the instantaneous variance in long run and makes it to tend to the average long run variance
- ✓ the possibility of only positive volatility values that makes the volatility process look more realistic as there is no sense for the negative volatility

- ✓ correlation between the stock returns and volatility that makes easier to understand how these two measure effect on each other

3.3 DERIVATION OF HESTON PARTIAL DIFFERENTIAL EQUATION

In order to derive the Heston partial differential equation, we create a riskless portfolio which has the following property: $d\Pi = r\Pi dt$ (***)

The portfolio Π which consists of stock option with value $U(S, V, t)$ and $-\Delta$ units, and another one asset with value U_2 and quantity $-\Delta_1$. Thus,

$$\Pi = U - \Delta_1 S - \Delta_2 U_2$$

The goal is to create the riskless portfolio, therefore the changes in volatility has to be taken into account in order to hedge it with the stock. Hence, the following must hold for the risk-free portfolio:

$$d\Pi = dU - \Delta_1 dS - \Delta_2 dU_2 = r\Pi dt = r(U - \Delta_1 S - \Delta_2 U_2)dt$$

Therefore, using Ito's lemma $d\Pi$ can be calculated as:

$$d\Pi = \left\{\frac{\partial U}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma VS\frac{\partial^2 U}{\partial V\,\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial V^2}\right\}dt$$

$$-\Delta_2\left\{\frac{\partial U_2}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 U_2}{\partial S^2} + \rho\sigma VS\frac{\partial^2 U_2}{\partial V\,\partial S} + \frac{1}{2}\sigma^2 V\frac{\partial^2 U_2}{\partial V^2}\right\}dt + \left\{\frac{\partial U}{\partial S} - \Delta_2\frac{\partial U_2}{\partial S} - \Delta_1\right\}dS$$

$$+\left\{\frac{\partial U}{\partial V} - \Delta_2\frac{\partial U_2}{\partial V}\right\}dV$$

In order to the assumption of riskless portfolio it is chosen that:

$\frac{\partial U}{\partial S} - \Delta_2\frac{\partial U_2}{\partial S} - \Delta_1 = 0$ and $\frac{\partial U}{\partial V} - \Delta_2\frac{\partial U_2}{\partial V} = 0$ (with the goal to eliminate dS and $d\sigma$).

These two equations result in:

$\Delta_2 = \frac{\partial U}{\partial V} / \frac{\partial U_2}{\partial V}$ and $\Delta_1 = \frac{\partial U}{\partial S} - \frac{\partial U}{\partial V} / \frac{\partial U_2}{\partial V} \frac{\partial U_2}{\partial S}$

After these assumptions and equation (***) it is important to mention that:

$$d\Pi = \left\{ \frac{\partial U}{\partial t} + \frac{1}{2} VS^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma VS \frac{\partial^2 U}{\partial V \partial S} + \frac{1}{2} \sigma^2 V \frac{\partial^2 U}{\partial V^2} \right\} dt$$
$$- \Delta_2 \left\{ \frac{\partial U_2}{\partial t} + \frac{1}{2} VS^2 \frac{\partial^2 U_2}{\partial S^2} + \rho\sigma VS \frac{\partial^2 U_2}{\partial V \partial S} + \frac{1}{2} \sigma^2 V \frac{\partial^2 U_2}{\partial V^2} \right\} dt =$$

$$= r\Pi dt = r(U - \Delta_1 S - \Delta_2 U_2)dt$$

The collection of all U terms on the left-hand side and all U_1 terms on the right-hand side leads to the following result:

$$\frac{\frac{\partial U}{\partial t} + \frac{1}{2} VS^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma VS \frac{\partial^2 U}{\partial V \partial S} + \frac{1}{2} \sigma^2 V \frac{\partial^2 U}{\partial V^2} + rS \frac{\partial U}{\partial S} - rU}{\frac{\partial U}{\partial V}}$$

$$= \frac{\frac{\partial U_2}{\partial t} + \frac{1}{2} VS^2 \frac{\partial^2 U_2}{\partial S^2} + \rho\sigma VS \frac{\partial^2 U_2}{\partial V \partial S} + \frac{1}{2} \sigma^2 V \frac{\partial^2 U_2}{\partial V^2} + rS \frac{\partial U_2}{\partial S} - rU_2}{\frac{\partial U_2}{\partial V}}$$

Therefore:

$$\frac{\partial U}{\partial t} + \frac{1}{2} VS^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma VS \frac{\partial^2 U}{\partial V \partial S} + \frac{1}{2} \sigma^2 V \frac{\partial^2 U}{\partial V^2} + rS \frac{\partial U}{\partial S} - rU$$
$$= -(\kappa(\theta - V) - \lambda(S, V, t)) \frac{\partial U}{\partial V}$$

Consequently, without loss of generality it can be assumed that $z(S, V, t) = \kappa(\theta - V) - \lambda(S, V, t)$ which is the volatility premium where $\lambda(S, V, t) = \lambda V$ is called the price of volatility risk and $\kappa(\theta - V)$, λV are drift and volatility function, respectively, for the instantaneous variance (exactly how it has been mentioned before). However, we have assumed no-arbitrage market as we are trying to create the riskless portfolio, thus, the volatility premium is equal to zero.

3.4 DESCRIPTION OF THE HESTON MODEL

In the previous chapters we derived the Heston model and showed how its structure looks like. We showed that the stock price can be demonstrated as the following SDE:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)} \qquad (1)$$

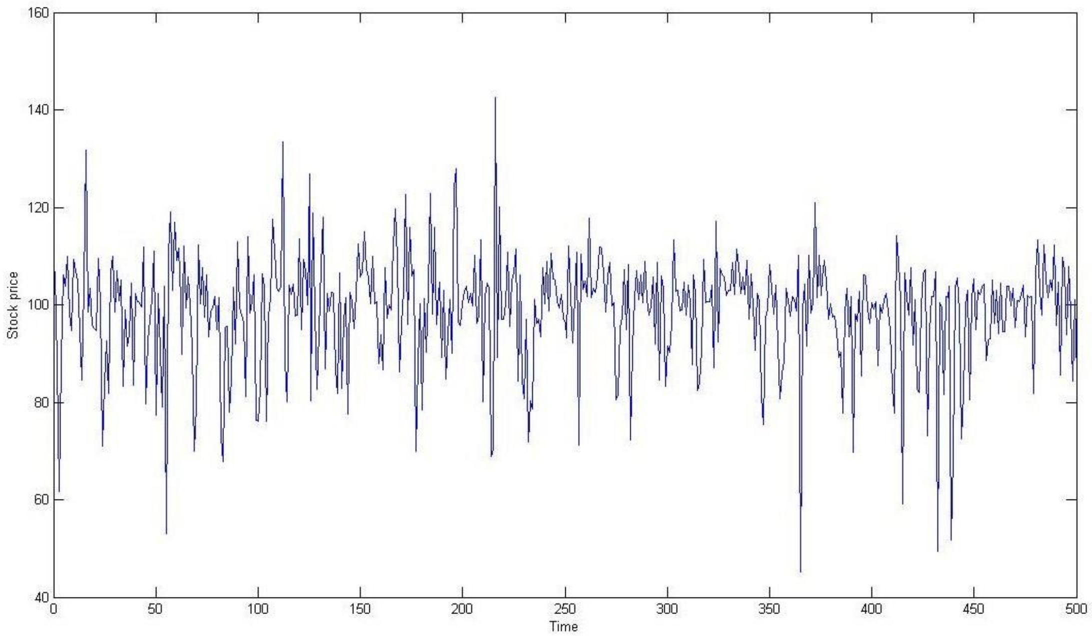

Figure 3.4.1: Stock price under the Heston model with $S_0 = 100, V_0 = 0.07, \mu = 0, k = 3, \theta = 0.03, dt = 0.2, \rho = -0.65$

Where V_t is the instantaneous variance which follows the square root mean reverting process (first used by Cox, Ingersoll & Ross 1985).

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t} W_t^{(2)} \qquad (2)$$

$dW_t^{(1)}$ and $dW_t^{(2)}$ are Wiener processes with

$$dW_t^{(1)} dW_t^{(2)} = \rho dt \qquad (3)$$

where

$\boldsymbol{\mu}$: risk-neural rate of return

$\boldsymbol{\theta}$: long run average of variance; more particular $E[V_t] \to \theta$ as $t \to \infty$

$\mathbf{k}$: rate at witch V_t tends to θ, also often called mean reversion level. It can be described as a speed with which the expected value of instantaneous variance tends to one average level in long run.

$\boldsymbol{\sigma}$: volatility of the volatility determinates the variance of V_t

$\boldsymbol{\rho}$: correlation parameter between the returns on asset price and its instantaneous volatility

$\mathbf{V_0}$: non -random variance at the beginning of period when $t = 0$. In other words, it is a short term at the money variance which can be observed in the market

If the parameters obey the Feller condition: $\mathbf{2k_t\theta} \geq \boldsymbol{\sigma}^2$ then the process V_t is strictly positive.

Additionally, $\mathbf{V_0}, \mathbf{k}, \boldsymbol{\theta}, \boldsymbol{\sigma} > 0$ and $|\boldsymbol{\rho}| < 1$

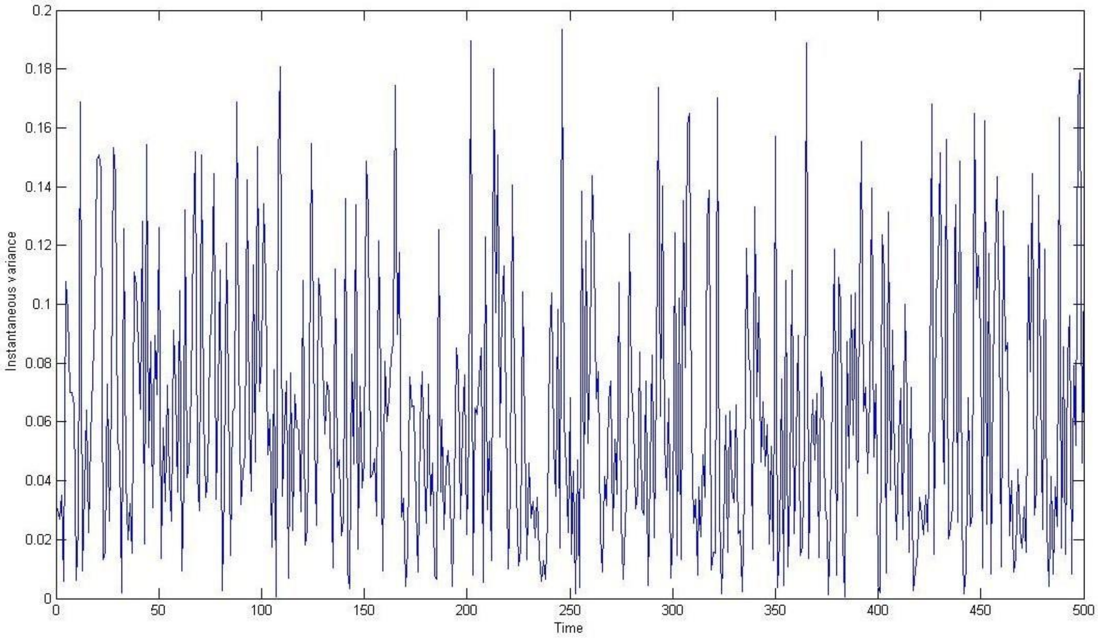

Figure 3.4.2: Instantaneous variance under the Heston model with $S_0 = 100, V_0 = 0.07, \mu = 0, k = 3, \theta = 0.03, dt = 0.2, \rho = -0.65$

(The brave explanation of all parameters of equation (2) was presented in the description of Vasicek model and Cox-Ingersoll-Ross model in the previous chapter 2, §2.3 and §2.4).

3.5 RISK-NEUTRALIZED APPROACH WITH THE HESTON MODEL

The risk-neutralized approach is also known as an Equivalent Martingale Measure (EMM) approach which was presented in the chapter 1, §1.7. One of the reasons why it is so often used in the financial market for pricing derivatives is its ability to produce a new version of the model where the drift is replaced by the risk – free interest rate. This approach is based on the Girsanov's theorem. It claims that the option price is evaluated as an expected discounted value of the future payoff under a new risk-neutralized measure q^*. It can be presented as:

$$\text{Current financial derivative price} = E_t^{q^*}[e^{r\tau}\Phi(T)]$$

where

$\boldsymbol{\Phi(T)}$: future payoff at time T,

$\mathbf{r}$: risk -free interest rate $\in [t, T]$

$\boldsymbol{\tau = T - t}$

Hence

$$dW_t^{*(1)} = dW_t^{(1)} + \frac{\mu - r}{\sigma}dt \quad \text{where} \quad \lambda = \frac{\mu - r}{\sigma}$$

and

$$dW_t^{*(2)} = dW_t^2 + \lambda(S, V, t)dt$$

According to the Girsanov's theorem the following Wiener processes $W_t^{*(1)}$ and $W_t^{*(2)}$ which are independent can be expressed under new EMM q^* as:

$$\frac{dq^*}{dP} = \exp[-\frac{1}{2}\int_0^t (\theta_s^2 + \lambda(S, V, s)^2)ds - \int_0^t \theta_s\, dW_s^{(1)} - \int_0^t \lambda(S, V, t)dW_s{}^{(2)}]$$

where P real market measure.

Thus, under the EMM q^* the Heston model can be presented in the following way:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^{*(1)}$$
$$dV_t = k^*(\theta^* - V_t)dt + \sigma\sqrt{V_t} dW_t^{*(2)}$$

$$dW_t^{*(1)} dW_t^{*(2)} = \rho dt$$

Where

$k^* = \lambda + k$ and $\theta^* = \frac{k\theta}{k+\lambda}$

How it is shown, the parameter λ has been effectively eliminated with the help of risk-neutralized approach .The EMM which has been chosen in this case is not unique, since the market is incomplete, volatility is non-traded measure and the price of the volatility risk $\lambda(S, V, t)$ has no fixed value. Additionally, the new form of the Heston model obtained a simplified character as now it is described with the help of five parameters: $k, \theta, \sigma, V_0, \rho$. This is one of the advantages of finding EMM as the calibration focuses on finding less model parameters. It is especially important to the Heston model since it is very sensitive to the small changes in the values of its parameters (it will be shown later). Accordingly, the option price can be found as:

$$\text{Option Price}^{\,p}(\rho, V_0, \theta, k, \sigma, \lambda) = \text{Option Value}^{q^*}(\rho, V_0, \vartheta^*, k^*, \sigma, 0)$$
$$= \text{Option Value}^{q^*}(\rho, V_0, \vartheta^*, k^*, \sigma)$$

3.6 PDE APPROACH AND CLOSED-FORM SOLUTION

In the previous paragraphs I presented the derivation of Heston model framework and its partial differential equation. In order to describe the closed-form solution it is assumed that the interest rate r is constant, therefore $P(t, T-t) = e^{-r(T-t)}$ which is a price of one unit discount bond at time t (T : maturity of the bond). According to the standard arbitrage arguments any asset with value $U(S,V,t)$ satisfies the following partial differential equation (PDE) :

$$\frac{1}{2}VS^2\frac{\partial^2 U}{\partial S^2}+\rho\sigma VS\frac{\partial^2 U}{\partial S\,\partial V}+\frac{1}{2}\sigma^2 V\frac{\partial^2 U}{\partial V^2}+rS\frac{\partial U}{\partial S}+\{k[\theta-V_t]-\lambda(S,V,t)\}\frac{\partial U}{\partial V}-rU+\frac{\partial U}{\partial t}=0$$

(4)

A European style option with strike price K and maturity at time T satisfies the PDE (4) subject to boundary conditions:

$$U(S,V,T)=\max(0,S-K)$$

$$U(0,V,t)=0$$

$$\frac{\partial U}{\partial S}(\infty,V,t)=1 \qquad (5)$$

$$rS\frac{\partial U}{\partial S}(S,0,t)+k\theta\frac{\partial U}{\partial V}(S,0,t)-rU(S,0,t)+U_t\,(S,0,t)=0$$

$$U(S,\infty,t)=S$$

and can be calculated according to the following formula which satisfies the PDE (4):

$C(S,V,t)=SP_1-Ke^{-r(T-t)}P_2$ (by Girsanov's theorem)

where

$\mathbf{SP_1}$: value of the stock asset

$\mathbf{Ke^{-r(T-t)}P_2}$: value of the strike price payment

Additionally, P_1, P_2 must satisfy the PDEs:

$$\frac{1}{2}V\frac{\partial^2 P_j}{\partial x^2}+\rho\sigma V\frac{\partial^2 P_j}{\partial x\,\partial V}+\frac{1}{2}\sigma^2 V\frac{\partial^2 P_j}{\partial V^2}+\left(r+u_jV\right)\frac{\partial P_j}{\partial x}+\left(a_j-b_jV\right)\frac{\partial P_j}{\partial V}+\frac{\partial P_j}{\partial t}=0 \qquad (6)$$

for $j = 1,2$

$$u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = k\theta, b_1 = k + \lambda - \rho\sigma, b_2 = k + \lambda$$

with $x = \ln[S]$ and terminal condition $P_j(x, V, T; \ln[K]) = 1_{\{x \le \ln[K]\}}$ in order to satisfy the terminal condition in equation (5).

Thus, P_1, P_2 can be interpreted as adjusted or risk-neutralized probabilities such that

$$P_j(x, V, T; \ln[K]) = \Pr(\ln S_T > lnK) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{e^{-i\varphi \ln[K]} f_j(x, V, \tau; \varphi)}{i\varphi}] d\varphi$$

with the characteristic function

$$f_j(x, V, \tau; \varphi) = \exp(C(\tau; \varphi) + D(\tau; \varphi)V + i\varphi x))$$

where

$$C(\tau; \varphi) = r\varphi i\tau + \frac{a}{\sigma^2}[(b_j - \rho\sigma\varphi i + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)]$$

$$D(\tau; \varphi) = \frac{b_j - \rho\sigma\varphi i + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right)$$

$$g_j = \frac{b_j - \rho\sigma\varphi i + d_j}{b_j - \rho\sigma\varphi i - d_j}$$

$$d_j = \sqrt{(\rho\sigma\varphi i - b_j)^2 - \sigma^2(2u_j\varphi i - \varphi^2)}$$

Using the EMM q^* some parameter simplification takes place:

$a = k^*\theta^*, b_1 = k^* - \rho\sigma, b_2 = k^*$ with $k^* = \lambda + k$ and $\theta^* = \frac{k\theta}{k+\lambda}$

And how we can see, the parameter λ is effectively eliminated.

Finally, now it can be claimed bravely that Heston derived the closed form solution for calculation of European style options which is presented above. And with the help of risk-neutralized measures the price of volatility risk was successfully eliminated and new simplified form of parameters was presented. Therefore, we are almost ready to implement the method to value options and present an empirical analysis.

3.7 PARAMETERS IMPORTANCE AND MODEL SENSITIVITY

How it has been shown before in the previous chapters the Heston model structure contains some parameters. Each of them plays a specific role and influences the model performance in a different way. One of the most important mathematical characteristics of this model is its sensitivity to the changes in the values of parameters. Small changes in them can result to the essential modification of the results.

Consequently, how it has been mentioned before, the Heston model allows the correlation ρ between the logarithmic returns on stock and the asset's volatility. It is responsible for the heaviness of the tails, thus affects on the skewness of the distribution. Therefore, if $\rho > 0$ then the volatility will increase as the stock price increases. Additionally, the density is positively skewed which leads to the fat right tail. To show this, I simulated of stock price with the positive correlation 0.95 and presented with the lognormal distribution (red line in the graph).

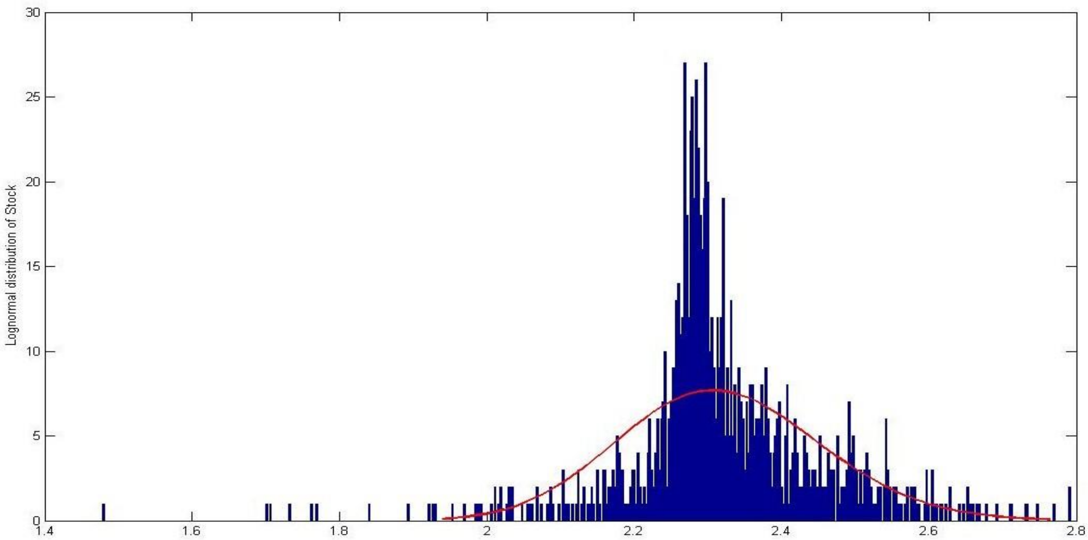

Figure 3.7.1: The effect of the positive ρ on the stock price performance (the simulation has been done 1000 times with $S_0 = 10, V_0 = 0.07, \mu = 0, k = 3, \theta = 0.03, dt = 0.2, \sigma = 0.5, \boldsymbol{\rho = 0.95}$)

Similarly, if $\rho < 0$ then the volatility will increase as the stock price decreases. Thus, we are in the case where the density is negatively skewed, so the fat left tail is presented. The following Figure 3.7.2 illustrates this argument.

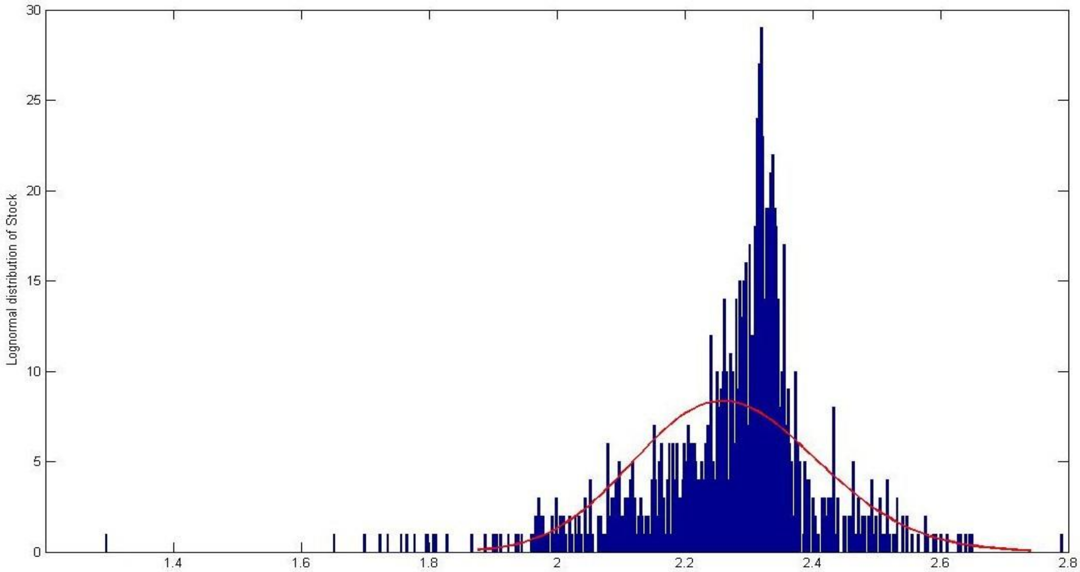

Figure 3.7.2: The effect of the negative ρ on the stock price performance (the simulation has been done 1000 times with $S_0 = 10, V_0 = 0.07, \mu = 0, k = 3, \theta = 0.03, dt = 0.2, \sigma = 0.5, \boldsymbol{\rho = -0.95}$)

The case where $\rho = 0$ the stock distribution obtains the character of the normal distribution. Consequently, no left or no right fait tails can be observed since the distributions behaves more symmetrically.

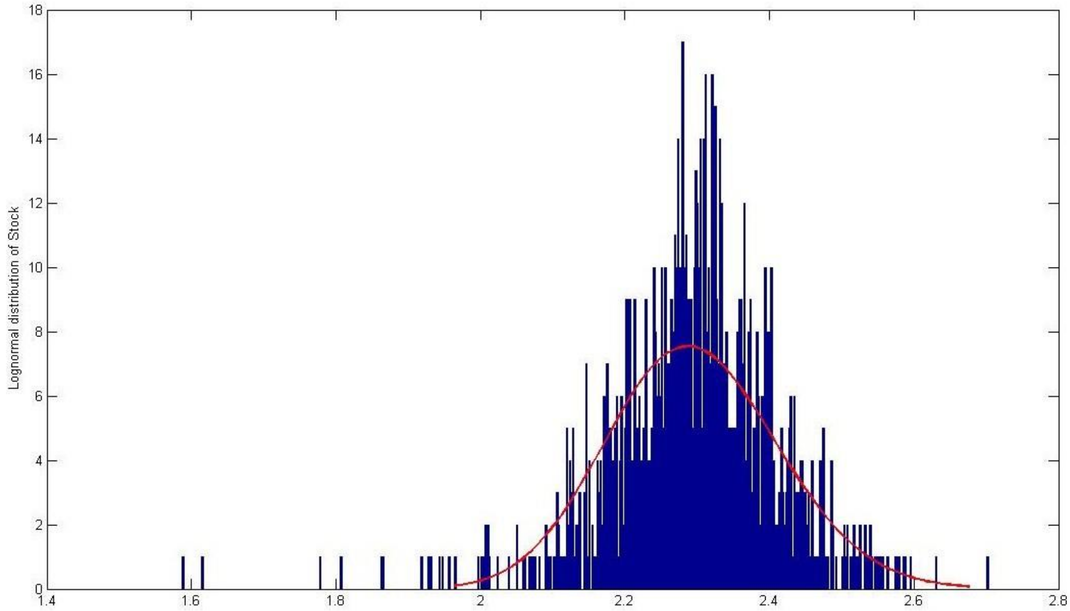

Figure 3.7.3: The effect of ρ=0 on the stock price performance (the simulation has been done 1000 times with $S_0 = 10, V_0 = 0.07, \mu = 0, k = 3, \theta = 0.03, dt = 0.2, \sigma = 0.5, \boldsymbol{\rho} = \mathbf{0}$)

Additionally, the correlation effects on the shape of implied volatility smile and consequently on the slope of the curve. The positive correlation means positive slope of the curve. The negative signals about the negative slope. When the correlation is equal to zero the smile become symmetric.

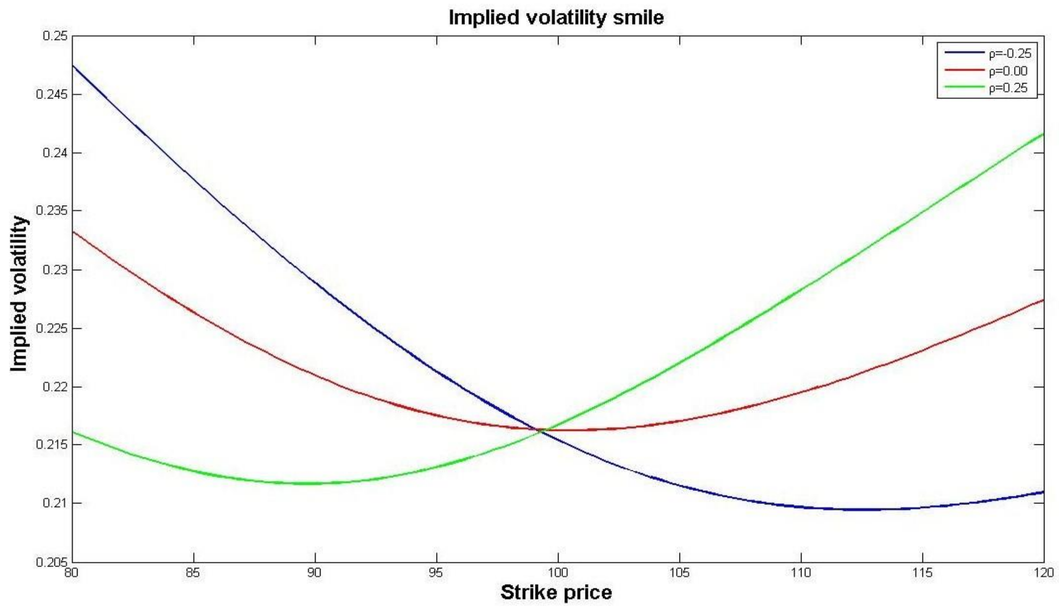

Figure 3.7.4: Effect of the correlation on the implied volatility smile ($S_0 = 100, V_0 = 0.07, K \in \{80.00, 80.40, 80.81, \ldots 120\},\ r = 0.01, k = 3, \theta = 0.03,\ T = 0.5,\ \sigma = 0.5,$ $\boldsymbol{\rho \in \{-0.25, 0, 0.25\}}$)

The volatility of volatility σ effects on the kurtosis of the distribution. High σ signals about the highly dispersed variance, high kurtosis and fat tails on both sides. To show this, there was done a simulation of stock price with the sigma equal to 0.7 and presented with the lognormal distribution (red line in the graph).

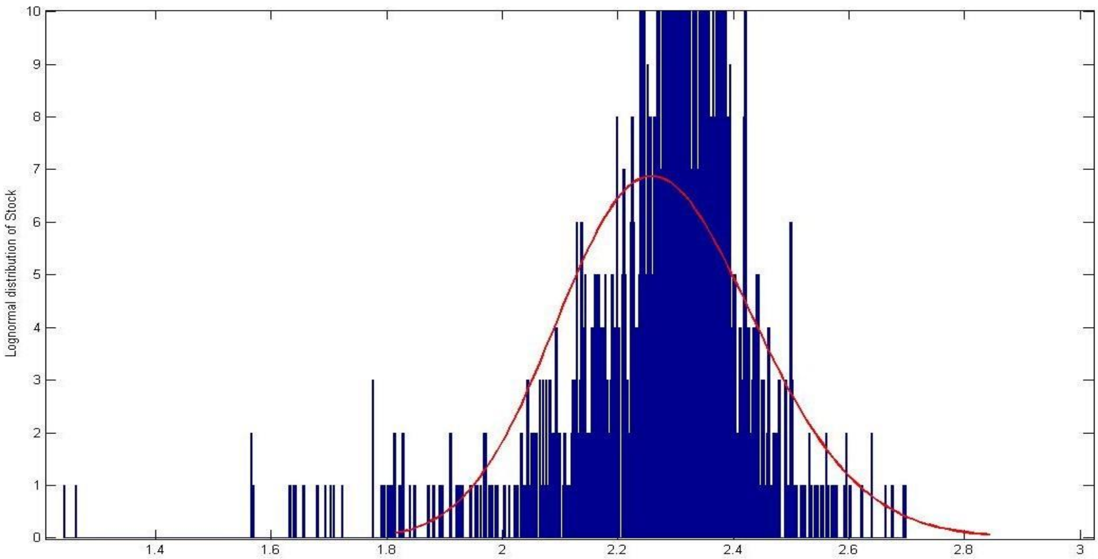

Figure 3.7.5: The effect of the larger σ on the stock price performance (the simulation has been done 1000 times with $S_0 = 10, V_0 = 0.07, \mu = 0, k = 3, \theta = 0.03, dt = 0.2, \boldsymbol{\sigma = 0.7}, \rho = -0.35$)

In the case of the low σ we conclude to the opposite illustration of the distribution. Lowe σ claims about low dispersion of the variance, consequently the kurtosis is lower as well.

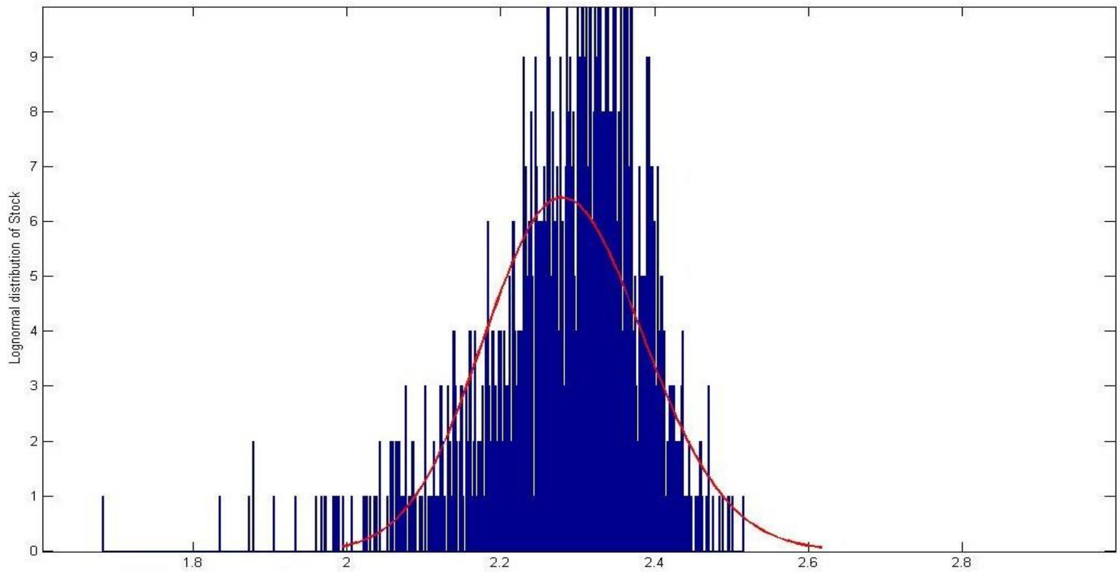

Figure 3.7.6: The effect of the smaller σ on the stock price performance (the simulation has been done 1000 times with $S_0 = 10, V_0 = 0.07, \mu = 0, k = 3, \theta = 0.03, dt = 0.2, \boldsymbol{\sigma = 0.2}, \rho = -0.35$)

When $\sigma = 0$ the logarithmic stock prices are normally distributed:

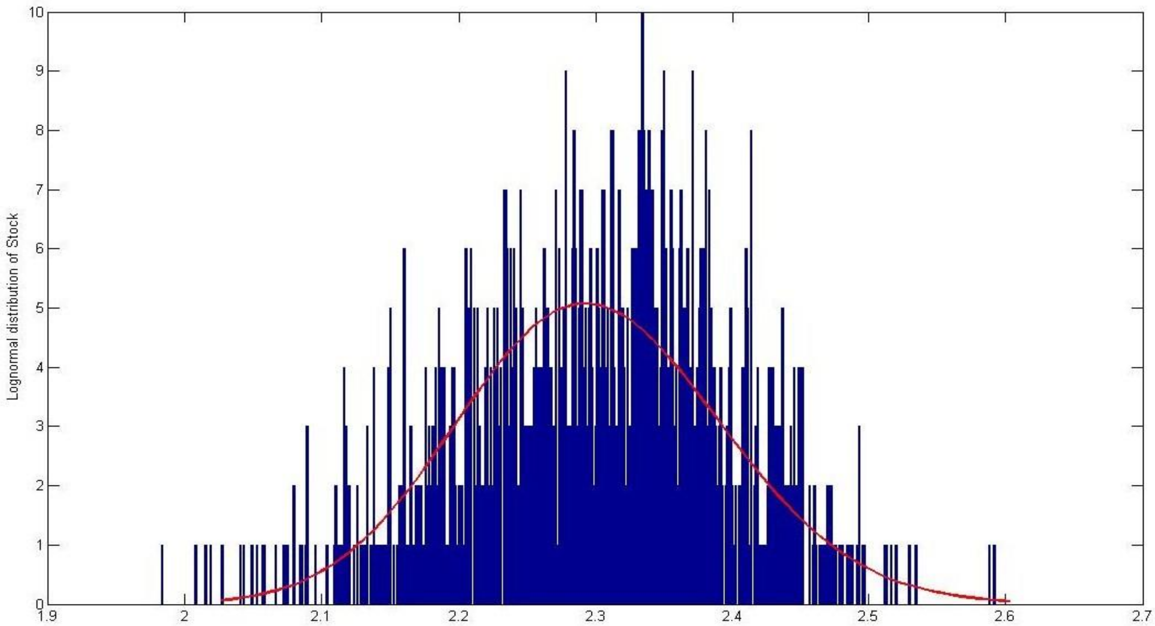

Figure 3.7.7: The effect of the σ = 0 on the stock price performance (the simulation has been done 1000 times with $S_0 = 10, V_0 = 0.07, \mu = 0, k = 3, \theta = 0.03, dt = 0.2, \boldsymbol{\sigma} = \mathbf{0}, \rho = -0.35$).

As well, σ effects on the level of implied volatility smile. Higher volatility of volatility leads to higher level of the smile and more “smiling” curve.

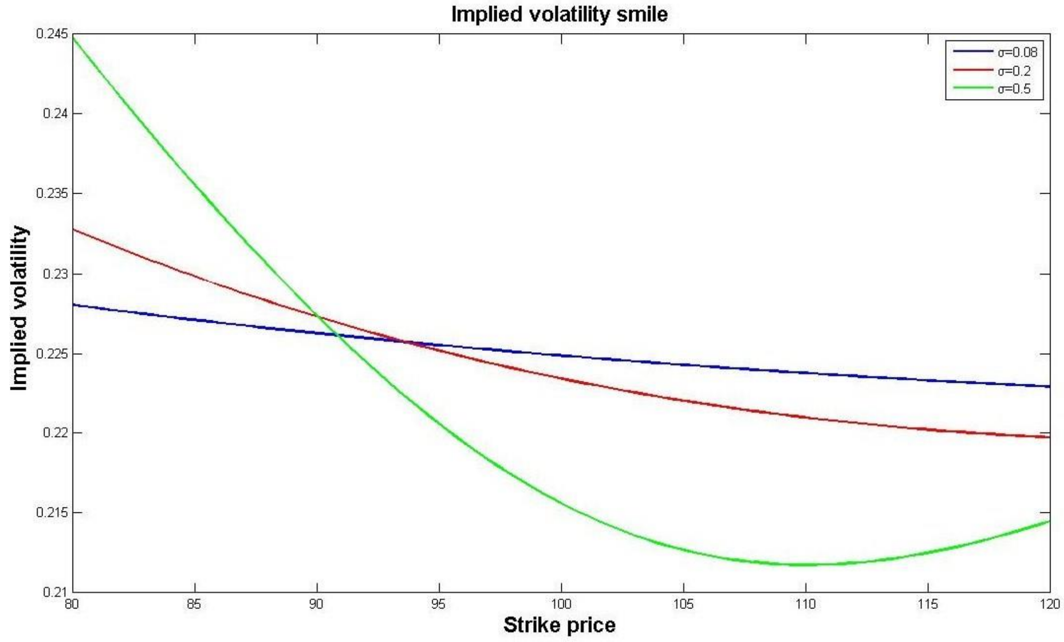

Figure 3.7.8: The effect of σ on the implied volatility smile($S_0 = 100, V_0 = 0.07, K \in \{80.00, 80.40, 80.81, \dots 120\},\ r = 0.01, k = 3, \theta = 0.03,\ T = 0.5,\ \boldsymbol{\sigma} = \{\mathbf{0.01}, \mathbf{0.2}, \mathbf{0.5}\},\ \rho = -0.2$)

Initial variance V_0 and long run variance θ effect on the high of the volatility smile. Increasing any of these parameters leads to upward movement of the curve, and the opposite for decreasing. The Figure 3.7.9 and Figure 3.7.10 clearly illustrate the impacts.

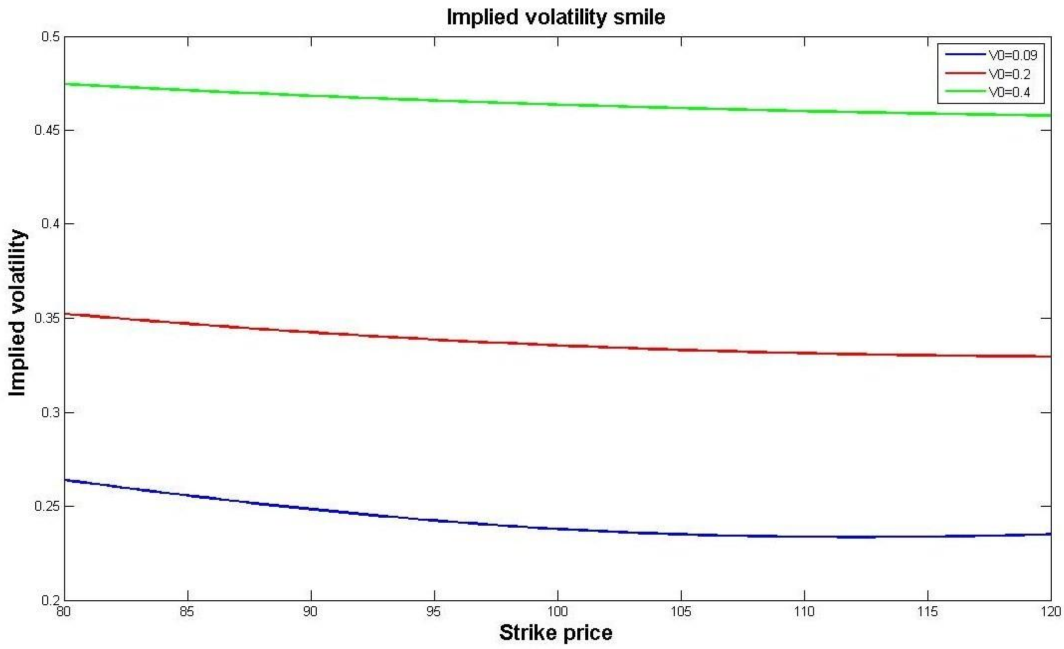

Figure 3.7.9: The effect of V_0 on the level of implied volatility ($S_0 = 100, \mathbf{V_0} = \{\mathbf{0.09}, \mathbf{0.2}, \mathbf{0.4}\}, K \in \{80.00, 80.40, 80.81, \dots 120\}$, $r = 0.01, k = 3, \theta = 0.03$, $T = 0.5$, $\sigma = -0.5$, $\rho = -0.2$)

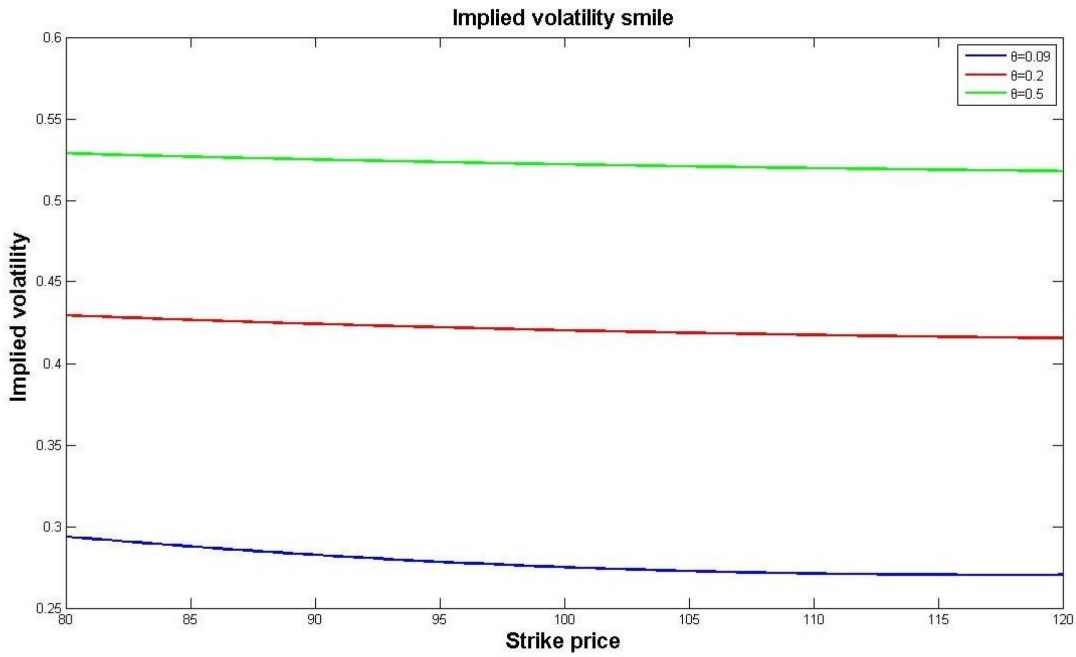

Figure 3.7.10: The effect of θ on the level of implied volatility ($S_0 = 100, V_0 = 0.07, K \in \{80.00, 80.40, 80.81, \dots 120\}$, $r = 0.01, k = 3, \boldsymbol{\theta} = \{\mathbf{0.09}, \mathbf{0.2}, \mathbf{0.5}\}$, $T = 0.5$, $\sigma = -0.5$, $\rho = -0.2$)

The mean reversion κ effects on the level of implied volatility smile in the opposite way comparing to the initial variance or to the long run variance. Lower κ signals about the high level of the curve and more convex shape of the smile.

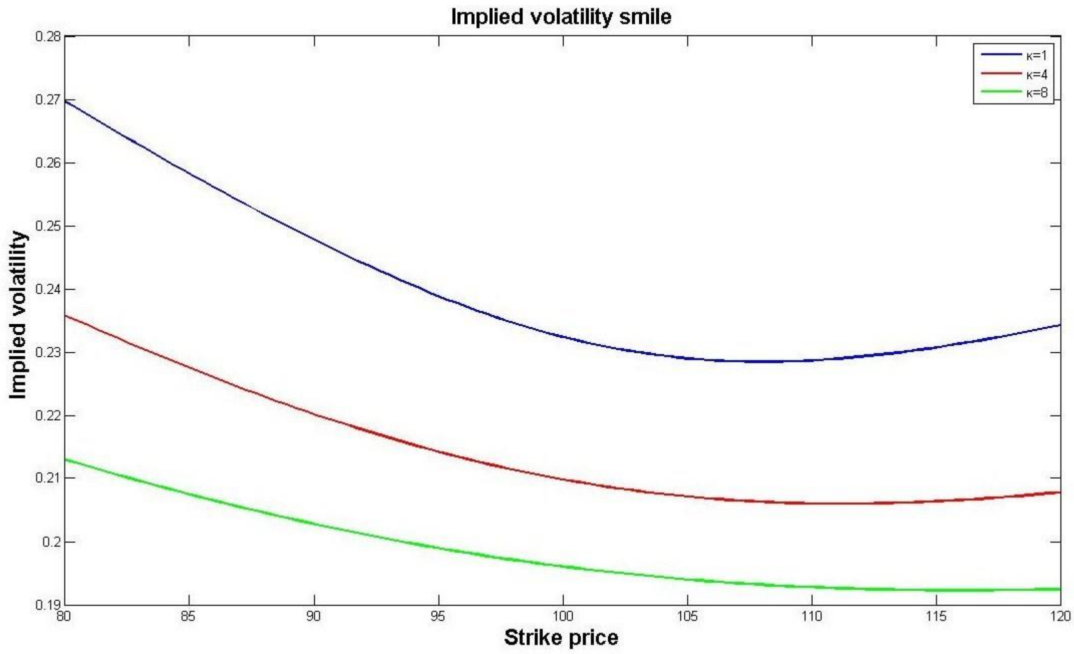

Figure 3.7.11: The effect of κ on the level and curvature of implied volatility smile ($S_0 = 100, V_0 = 0.07, K \in \{80.00, 80.40, 80.81, \dots 120\},\ r = 0.01, \boldsymbol{\kappa} = \{\mathbf{1}, \mathbf{4}, \mathbf{8}\},\ \theta = 0.07,$ $T = 0.5,\ \sigma = -0.5,\ \rho = -0.2$)

3.8 ADVANTAGES AND DISADVANTAGES OF THE HESTON MODEL

Heston model is one of the most popular mathematical and stochastic volatility models which is widely used in the "financial world". The model provides with the closed form solution for pricing European style options. Therefore, it makes the model have a big advantage in its implementation, as the calculations and the model's calibration can be done fast, consequently, with sufficient results. As well, the Heston model takes into account the continuously movement of the volatility, so the model describes the evolution of the instantaneous variance which is expressed with the help of CIR process. This process gives to the model other one very important advantage, which is the mean reverting volatility that takes only positive values. Therefore, the model allows the correlation between the asset returns and volatility. It helps to make conclusions about what happen with one measure if some changes are presented in the value of the other measure. More particular, it takes into the consideration the negative correlation of stock's returns and volatility. This has a simple explanation, when the returns on stock take negative value the risk will increases, thus, the volatility will grow. It also known as leverage effect (Black 1976) Besides, the Heston model has the ability to explain the characteristic of stock price even if it does not follow the Gaussian distribution. Thereupon, the Heston model produces and explains the implied volatility smile which is a very important tool for all market participations. Consequently, it produces an implied volatility surface of option prices. According to empirical studies, the structure of the Heston model is such that even during the period of economical crisis the model composes sufficient approaches of option prices while some of the models fail in this task.

Unfortunately, there is no faultless mathematical models which would work perfectly and have no drawbacks. Even Heston model that has so many advantages presents a lot of weaknesses. The volatility is not observable thus it creates the difficulty to find the proper parameters. The choice of which has to be very precise as the model is very sensitive to the small changes in the value of its parameters. This disadvantage effects, as well, on the model's results, something that is very undesirable. According to empirical studies Heston model is not able to capture the skewness of the derivatives with the short maturity. Other problem which is faced by every continuous time mode is that the data is expressed under the discrete time framework. If Heston model is compared to Black-Scholes model then it is also obvious that the calibration and running the Heston model demands more time than the Black-Scholes. This is one of the most essential advantages why this model is not used by some banks and companies. Additionally, it is impossible to find the risk premium before pricing an option since the market is incomplete. But even those, the Heston model is one of the strongest statistical methods which is characterized as the valuable tool for investors and a great helper for valuing options.

3.9 "THE LITTLE HESTON TRAP"

There exist two interpretations of Heston model. According to the paper of Schoutens, W., Simons E. and Tistaert, J. (2004) the characteristic function of the model can be represented in a bit different way where instead of the parameter d appears -d:

$$f_j(x, V, \tau; \varphi) = \exp(C(\tau; \varphi) + D(\tau; \varphi)V + i\varphi x)$$

where

$$C(\tau; \varphi) = r\varphi i\tau + \frac{a}{\sigma^2}[(b_j - \rho\sigma\varphi i - d)\tau - 2\ln\left(\frac{1 - ge^{-d\tau}}{1 - g}\right)]$$

$$D(\tau; \varphi) = \frac{b_j - \rho\sigma\varphi i - d}{\sigma^2}\left(\frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}\right)$$

$$g = \frac{b_j - \rho\sigma\varphi i - d}{b_j - \rho\sigma\varphi i + d}$$

$$d = \sqrt{(\rho\sigma\varphi i - b_j)^2 + \sigma^2(2u_j\varphi i + \varphi^2)}$$

with risk neutralized probabilities:

$$P_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\omega\log}(K)\, f(\omega - i)}{i\omega f(-i)}\right)d\omega\, , f(-i) = S_0e^{r\tau}$$

and

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\omega\log}(K)\, f(\omega)}{i\omega}\right)d\omega$$

The question is why it exactly happens. This fact is based on the parameter d. How it was shown above the d is presented as the root of the complex number. The root has two solution. So let say there is a case where $d = -c^2$ whith $c \in R^+$ then $\sqrt{d} = ic$ or $\sqrt{d} = -ic$. The original Heston model takes into consideration the solution $\sqrt{d} = ic$ which leads to the discontinuous curve of the function f and the "little Heston trap" takes the root with value $\sqrt{\mathrm{d}} = -\,\mathrm{ic}$ which makes the curve be smooth and avoid the break - points.

According to the empirical studies, it was proved that the “little Heston trap” functions better than the original Heston model, especially in the cases of options with the long time to maturity. Additionally, the Heston model presents problem in Fourier transformation and numerical methods which are used for the calculation of the probability measures and the “little Heston trap” solves them easily. Consequently, there were discussed a lot of topics about these problems but there are not so many empirical studies about the calibration of the “little Heston trap”. Hence, it was interesting to use this interpretation of the model and test how it performs in the calibration procedure.

One negative fact was observed in the “little Heston trap” which is the disability to produce the implied volatility smile in the cases where the option reaches the deep in the moneyness or deep out the moneyness. Usually, in this cases the prices which are produced by the model have a negative values. It results to inability of finding the implied volatility of the option. Consequently, no implied volatility smile can be illustrated. It can be explain with the help of the ‘-d’ in the structure of the “little Heston trap”.

CHAPTER 4 EMPIRICAL ANALYSIS

4.1 DATA SELECTION

For the empirical analysis I considered options on FTSE 100 Index which is one of the most wildly used stock indices in the London Stock Exchange. It contains the biggest 100 companies in the United Kingdom that are indicated with the highest market capitalization. The next figure 4.1.1 illustrates the FTSE 100 Index performance during 2007 – 2014 years.

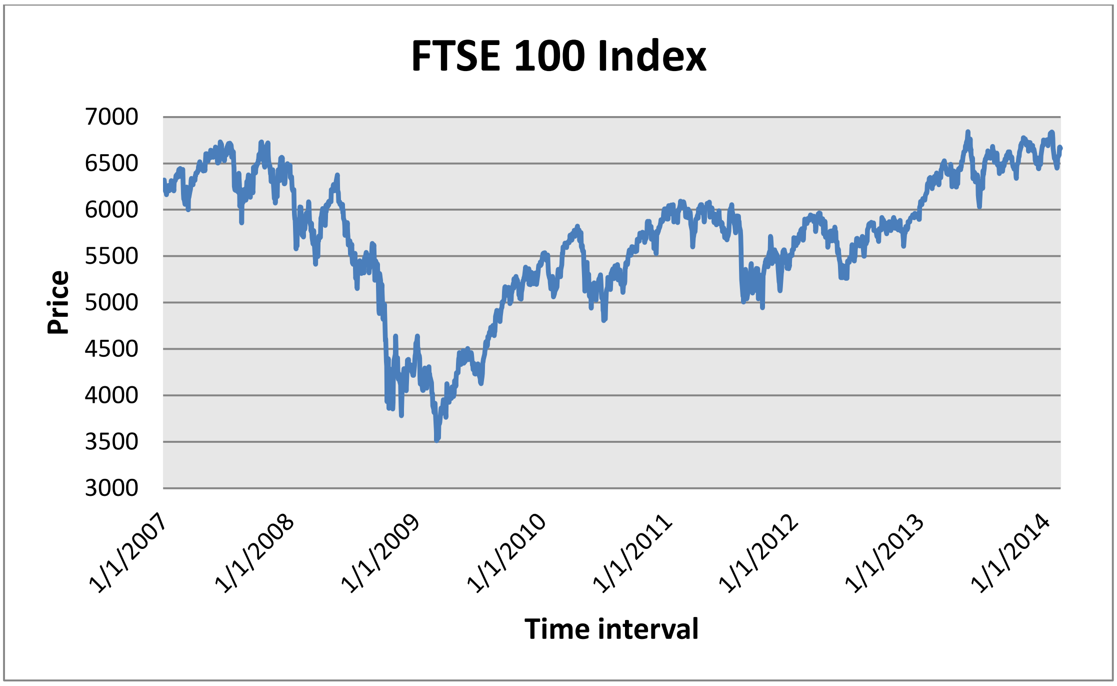

Figure 4.1.1: FTSE 100 Index price chart.

I chosed 80 “dead” European style call options with maturity equal to or less than 1 year, 40 of them during periods February 2007 – February 2008, February 2012 – February 2013 and other 40 options during the crisis periods August 2007 - August 2008, May 2008- May 2009. Each of them presents observations on daily basis. Unfortunately, the Heston model is very sensitive not only to the choice of the initial parameters but also to the movements of the market data. Consequently, it was very important to apply some filtration rules to make the data available for the Heston model calibration. The following filtration was implemented:

- Options' daily observations which mature less than in 7 days were dropped (Moyaert & Petitjen 2011)
- Observations with a price less than 10 were rejected
- Remained options must satisfy the no-arbitrage relation (Merton 1973), which is

$$C_t \geq S_t - Ke^{-r\tau}$$

- Options which present huge changes of stock prices, comparing to the average performance of the stock time series, were excluded.

At the end of the filtration procedure there were remained:

1. 98 daily observations of periods February 2007 – February 2008, February 2012 – February 2013 with the average strike price 5445.4082 £ and average stock price 6106.1638£. Additionally, they were divided into sets according to their maturity.

0.0190 - 0.0770	0.1600 - 0.2000
0.0780 - 0.1210	0.2100 – 0.2520
0.1220- 0.1590	0.2530 – 0.3100

2. 145 daily observations of economical crisis periods August 2007 - August 2008, May 2008- May 2009 with the average strike price 4164.4828£ and average stock price 4773.7883£. Additionally, they were divided into sets according to their maturity.

0.0190 - 0.0770	0.1600 – 0.2203
0.0780 - 0.1210	0.2570 – 0.3022
0.1215 - 0.1590	------------------------

I divided the data into sets according to its maturity and categorized into two parts (stable period and period of economical crisis). The data was divided into sets with the goal to calibrate the Heston model in more sufficient way and get better approximations. Testing the model during different periods would show how the Heston model performs not only during the time interval when the market stability was observed but also during the stress period and compare it to the Black-Scholes model.

The interest rate was assumed to be LDNIBON, UK Interbank Overnight – Middle Rate.

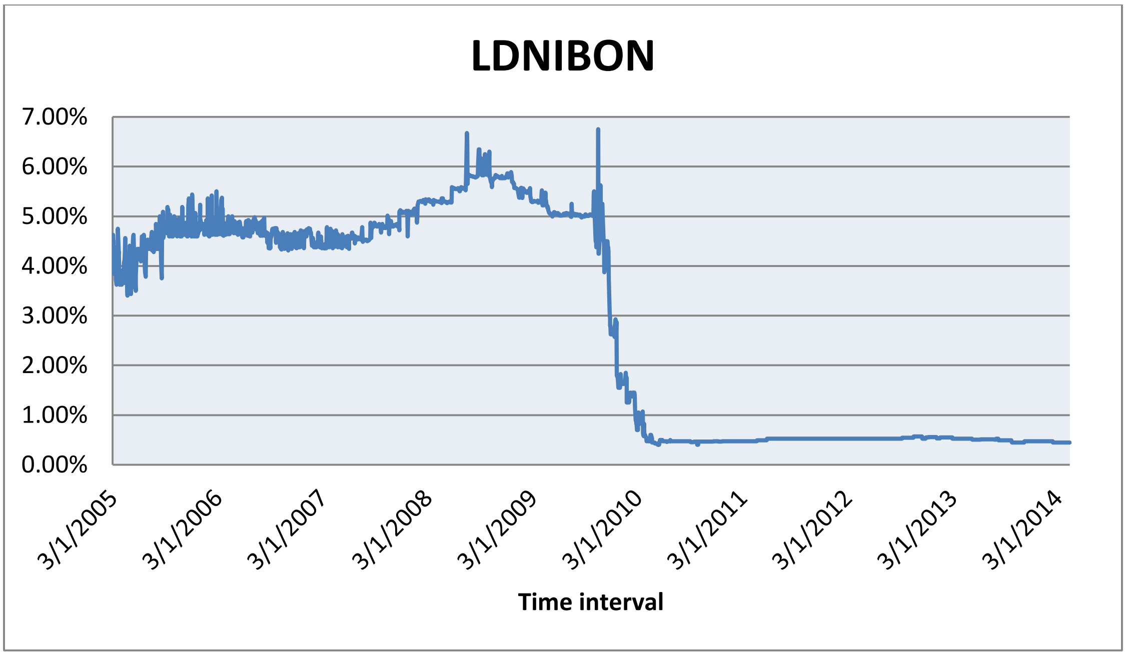

Figure 4.1.2: LDNIBON evolution during 2005 – 2014 period.

4.2 CALIBRATION

How I described in the chapter 3 for the successful Heston model valuation it is necessary to find the values of model's parameters: $\rho, V_0, \theta, \kappa, \sigma$. Unfortunately, the empirical estimation of those parameters is not enough, consequently, the calibration is required to be implemented. In this case the inverse problem structure is presented as the main framework of the calibration process. In order to solve the inverse problem I will be minimize the sum of squares of the differences between the model and the market prices for the sample length N.

$$\min \text{error} = \min \sum_{i=1}^{N} \left(C_i^{\text{Heston}} - C_i^{\text{Market}}\right)^2$$

(Mikhailov and Nögel, 2003, p.76)

Usually, such problems have the non-linear least – squares optimization character.

I assumed that $f(x) = C^{\text{Heston}} - C^{\text{market}}$, thus $\min\left|\left|f(x)\right|\right|_2^2 = \min \sum_{i=1}^{N} f_i(x)^2$.

In order to do the calibration, receive sufficient results and calculate the correct option prices, I will describe and implement in my calibration procedure some other loss functions.

<u>Loss functions</u> (Hydman & Koehler, 2005, p.8):

Mean absolute error: $\text{MAE} = \frac{1}{N}\sum_{i=1}^{N} |C_i^{\text{Heston}} - C_i^{\text{Market}}|$

Mean percentage error: $\text{MPE} = \frac{1}{N}\sum_{i=1}^{N} \frac{C_i^{\text{Heston}} - C_i^{\text{Market}}}{C_i^{\text{Market}}}$

Mean squared error: $\text{MSE} = \frac{1}{N}\sum_{i=1}^{N}\left(C_i^{\text{Heston}} - C_i^{\text{Market}}\right)^2$

Where for all cases N is the number of options' daily observations. In order to deal with calibration and option price calculations, MATLAB 7.12.0.0 (R2011a) and Microsoft Excel (2007) were used in this procedure. More particular, the function lsqnonlin was applied to find the parameters: $\rho, V_0, \theta, \kappa, \sigma$. This function $\text{lsqnonlin}(\text{function}, x_0, \text{lb}, \text{ub})$ starts at the given initial level $x_0 = \{\rho, V_0, \theta, \kappa, \sigma\}$, minimizes sum of squares of the function and produces a new vector x such that $x \in [\text{lb}, \text{ub}]$, where lb is lower bound and ub is upper bound: $0 < \theta < 1, -1 < \rho < 1, 0 < \kappa, 0 < V_0 < 1, 0 < \sigma < 1$ with Feller condition $2k\theta \geq \sigma^2$. In this chapter i assumed $0 < \theta < 1, -1 < \rho < 0,\ 0 < \kappa < 10, 0 < V_0 < 1, 0 < \sigma < 1,$ consequently, $\text{lb} = \{-1, 0, 0, 0, 0\}$ and $\text{ub} = \{0, 1, 1, 10, 1\}$ (Moodley, 2005).

The calibration procedure has been done 5 times (each time with a new vector of initial parameters $x_0 = \{\theta, \rho, \kappa, V_0, \sigma\}$, Table 4.2.1) for every set of data and every section of calibration was applied every time for different loss function described above. Summarizing, the calibration has been done 120 times for the stable periods and 100 times for the stress periods.

№	ρ	V_0	θ	κ	σ
1	-0.4000	0.0400	0.0400	1.1500	0.2000
2	-0.5000	0.1500	0.0500	3.0000	0.5000
3	-0.7571	0.0654	0.0707	0.6067	0.2928
4	-0.2051	0.0231	0.2319	1.3784	1.0000
5	-0.5100	0.3000	0.3000	3.0000	0.5000

Table 4.2.1: Initial parameter values (№ 1 Mert 2011, № 2 Forde, Jacquier & Lee 2011, № 3 Moodley 2005, № 4 Samuelson 1973)

Consequently, that output vector of parameters will be chosen which mostly minimizes the loss function and produces the most correct option prices.

4.3 RESULTS

The error between market and model prices of the option i I calculated as the following:

$$\mathrm{error}_i = \left| \frac{C_i^{\mathrm{Heston}} - C_i^{\mathrm{Market}}}{C_i^{\mathrm{Market}}} \right| \cdot 100\% \ ,$$

the vector of initial parameters $x_0 = \{\rho, V_0, \theta, \kappa, \sigma\}$

Initial parameter 1: $x_0^1 = \{-0.4000, 0.0400, 0.0400, 1.1500, 0.2000\}$

Initial parameter 2: $x_0^2 = \{-0.5000, 0.1500, 0.0500, 3.0000, 0.5000\}$

Initial parameter 3: $x_0^3 = \{-0.7571, 0.0654, 0.0707, 0.6067, 0.2928\}$

Initial parameter 4: $x_0^4 = \{-0.2051, 0.0231, 0.2319, 1.3784, 1.0000\}$

Initial parameter 5: $x_0^5 = \{-0.5100, 0.3000, 0.3000, 3.0000, 0.5000\}$

4.3.1. STABLE PERIOD

In this case I categorized the data into 6 sets according to the maturity, the calibrations was implemented 120 times and the following results were obtained after the minimizing all loss functions:

1. After minimizing $\min \text{error} = \min \sum_{i=1}^{N}\left(C_i^{\text{Heston}} - C_i^{\text{Market}}\right)^2$ there were obtained the following parameters:

MATURITY	ρ	V_0	θ	κ	σ	Initial paramater
0.0190 - 0.0770	-0.4000	0.0403	0.0400	1.1548	0.2000	1
	-0.4517	0.2708	0.0671	3.7725	0.4824	2
	-0.3150	0.7254	0.7750	1.2057	0.1191	3
	-0.5099	0.2998	0.3000	3.0017	0.5000	4
	-0.5045	0.2706	0.2980	3.3351	0.4961	5
0.0780 - 0.1210	-0.4067	0.0454	0.0404	1.1454	0.2011	1
	-0.5000	0.1499	0.0500	3.0009	0.5000	2
	-0.7571	0.0654	0.0707	0.6069	0.2928	3
	-0.4164	0.3678	0.3231	3.7441	0.4811	4
	-0.4970	0.3458	0.3247	3.6305	0.4990	5
0.1220 - 0.1590	-0.4000	0.0400	0.0400	1.1505	0.2000	1
	-0.4994	0.1617	0.0529	2.9968	0.4998	2
	-0.9319	0.0115	0.0005	0.3085	0.8814	3
	-0.0972	0.3345	0.4104	4.9805	0.4106	4
	-0.4802	0.3824	0.3319	3.5288	0.4708	5
0.1600 - 0.2000	-0.3990	0.0392	0.0399	1.2159	0.2007	1
	-0.5000	0.1500	0.0500	3.0000	0.5000	2
	-0.7976	0.0551	0.0510	0.6967	0.4794	3

	-0.5130	0.2924	0.2971	2.9930	0.5052	4
	-0.3610	0.0308	0.0014	4.7515	0.8622	5
0.2100 – 0.2520	-0.4485	0.1356	0.1027	1.1620	0.1435	1
	-0.5000	0.1485	0.0498	3.0079	0.5009	2
	-0.7571	0.0654	0.0707	0.6066	0.2928	3
	-0.6288	0.3930	0.3274	2.2308	0.5801	4
	-0.6289	0.3930	0.3274	2.2308	0.5801	5
0.2530 – 0.3100	-0.2957	0.3487	0.4002	0.9533	0.1454	1
	-0.5576	0.3127	0.1496	2.7035	0.5209	2
	-0.7571	0.0657	0.0707	0.6063	0.2928	3
	-0.6496	0.3404	0.2769	2.2507	0.5985	4
	-0.6485	0.3408	0.2774	2.2551	0.5974	5

Table 4.3.1.1: Heston model parameters after minimizing $\min \text{error} = \min \sum_{i=1}^{N} \left(C_i^{\text{Heston}} - C_i^{\text{Market}} \right)^2$.

2. After minimizing $\text{MSE} = \frac{1}{N} \sum_{i=1}^{N} \left(C_i^{\text{Heston}} - C_i^{\text{Market}} \right)^2$ there were obtained the following parameters:

MATURITY	ρ	V_0	θ	κ	σ	Initial parameter
0.0190 - 0.0770	-0.4000	0.0400	0.0400	1.1519	0.2000	1
	-0.4516	0.2708	0.0671	3.7724	0.4824	2
	-0.3797	0.2585	0.3422	0.5969	0.0051	3
	-0.2877	0.2255	0.4043	1.2368	0.7722	4
	-0.5099	0.2998	0.3000	3.0017	0.5000	5
0.0780 - 0.1210	-0.4067	0.0454	0.0404	1.1454	0.2011	1
	-0.5000	0.1499	0.0500	3.0009	0.5000	2
	-0.7571	0.0654	0.0707	0.6069	0.2928	3
	-0.2051	0.0233	0.2319	1.3828	1.0000	4
	-0.4164	0.3678	0.3231	3.7441	0.4811	5
	-0.3989	0.0324	0.0374	1.4280	0.2702	1
	-0.0200	0.0706	0.0019	4.1547	0.6147	2

0.1220 - 0.1590	-0.9923	0.0113	0.0005	0.2983	0.8837	3
	-0.2056	0.0230	0.2318	1.3698	1.0000	4
	-0.1036	0.3339	0.4107	4.9833	0.4091	5
0.1600 - 0.2000	-0.3990	0.0392	0.0399	1.2159	0.2007	1
	-0.5000	0.1500	0.0500	3.0000	0.5000	2
	-0.7976	0.0551	0.0510	0.6968	0.4794	3
	-0.2051	0.0230	0.2317	1.3891	1.0000	4
	-0.6760	0.0367	0.0017	5.0086	0.8255	5
0.2100 – 0.2520	-0.4496	0.1352	0.1027	1.1662	0.1424	1
	-0.0500	0.0999	0.2646	1.8434	0.4616	2
	-0.7571	0.0654	0.0707	0.6066	0.2928	3
	-0.3482	0.2085	0.5079	1.4315	0.6744	4
	-0.6288	0.3930	0.3274	2.2308	0.5801	5
0.2530 – 0.3100	-0.2775	0.3712	0.4046	0.9870	0.1254	1
	-0.5700	0.2850	0.1434	2.6517	0.5374	2
	-0.7571	0.0657	0.0707	0.6063	0.2928	3
	-0.1636	0.1497	0.3394	1.2720	0.8599	4
	-0.6485	0.3408	0.2774	2.2552	0.5974	5

Table 4.3.1.2: Heston model's parameters after minimizing MSE.

3. After minimizing $\mathrm{MPE} = \frac{1}{N}\sum_{i=1}^{N}\frac{C_i^{\mathrm{Heston}} - C_i^{\mathrm{Market}}}{C_i^{\mathrm{Market}}}$ there were obtained the following parameters:

MATURITY	ρ	V_0	θ	κ	σ	Initial parameter
0.0190 - 0.0770	-0.4000	0.0399	0.0400	1.1709	0.2001	1
	-0.2976	0.4446	0.6327	4.1976	0.3210	2
	-0.7571	0.0664	0.0707	0.6083	0.2928	3
	-0.2333	0.0883	0.2926	1.2862	0.9208	4
	-0.5100	0.2996	0.3000	3.0015	0.5000	5
	-0.9713	0.0225	0.9998	1.2416	0.7338	1
	-0.5228	0.1451	0.0496	3.0318	0.5032	2

0.0780 - 0.1210	-0.7571	0.0654	0.0707	0.6068	0.2928	3
	-0.2051	0.0231	0.2319	1.3790	1.0000	4
	-0.3563	0.4515	0.3996	4.3644	0.4575	5
0.1220 - 0.1590	-0.3990	0.0378	0.0398	1.2571	0.2026	1
	-0.5000	0.1499	0.0500	3.0002	0.5000	2
	-0.9814	0.0079	0.0044	0.8926	0.9998	3
	-0.2084	0.0308	0.2330	1.3133	0.9996	4
	-0.1610	0.2919	0.4112	4.5514	0.4341	5
0.1600 - 0.2000	-0.3797	0.0269	0.0383	2.1368	0.2173	1
	-0.5000	0.1500	0.0500	3.0000	0.5000	2
	-0.7903	0.0574	0.0551	0.7000	0.3974	3
	-0.2040	0.0236	0.2318	1.5078	1.0000	4
	-0.5103	0.2982	0.2993	3.0048	0.5009	5
0.2100 – 0.2520	-0.4122	0.0345	0.0342	1.4202	0.2163	1
	-0.6094	0.2567	0.1350	2.9781	0.4767	2
	-0.7571	0.0654	0.0707	0.6066	0.2928	3
	-0.7588	0.9190	0.9982	2.4425	0.0439	4
	-0.9527	0.8705	0.8804	0.2860	0.9525	5
0.2530 – 0.3100	-0.4176	0.0642	0.0437	1.0905	0.5606	1
	-0.6148	0.1152	0.0380	2.6400	0.5930	2
	-0.9168	0.0148	0.0181	0.4516	0.8282	3
	-0.1398	0.1316	0.3225	1.6464	0.8655	4
	-0.5477	0.2956	0.2931	2.8269	0.5251	5

Table 4.3.1.3: Heston model's parameters after minimizing MPE.

4. After minimizing $\mathrm{MAE} = \frac{1}{N}\sum_{i=1}^{N} |C_i^{\mathrm{Heston}} - C_i^{\mathrm{Market}}|$ there were obtained the following parameters:

MATURITY	**ρ**	V_0	**θ**	**κ**	**σ**	**Initial parameter**
	-0.8801	0.0019	0.0082	1.3731	0.3363	1
	-0.2505	0.6557	0.6679	4.2533	0.2564	2

0.0190 - 0.0770	-0.3797	0.2585	0.3422	0.5969	0.0051	3
	-0.3365	0.3371	0.5474	1.2396	0.5835	4
	-0.5099	0.2997	0.3000	3.0017	0.5000	5
0.0780 - 0.1210	-0.3350	0.1339	0.1523	1.0977	0.1148	1
	-0.5000	0.1498	0.0500	3.0009	0.5001	2
	-0.7571	0.0654	0.0707	0.6069	0.2928	3
	-0.2170	0.0207	0.2161	1.4089	1.0000	4
	-0.2864	0.3824	0.3879	4.3294	0.4359	5
0.1220 - 0.1590	-0.3571	0.0283	0.0273	1.4582	0.6271	1
	-0.5000	0.1499	0.0500	3.0004	0.5001	2
	-0.7573	0.0648	0.0707	0.6057	0.2942	3
	-0.2059	0.0237	0.2320	1.3639	1.0000	4
	-0.0643	0.6100	0.5457	5.0089	0.3580	5
0.1600 - 0.2000	-0.2192	0.0258	0.0253	4.9231	0.5007	1
	-0.5000	0.1485	0.0498	3.0079	0.5009	2
	-0.7568	0.0652	0.0707	0.6560	0.2928	3
	-0.2851	0.0060	0.0589	2.9686	1.0000	4
	-0.5156	0.2868	0.2950	2.9881	0.5088	5
0.2100 – 0.2520	-0.0216	0.3756	0.3847	3.0480	0.1229	1
	-0.0500	0.0999	0.2646	1.8434	0.4616	2
	-0.7571	0.0654	0.0707	0.6066	0.2928	3
	-0.7913	0.4903	0.7549	1.8463	0.3992	4
	-0.6354	0.3352	0.3124	2.4572	0.6034	5
0.2530 – 0.3100	-0.5751	0.1242	0.2201	0.8556	0.3188	1
	-0.5198	0.1435	0.0424	2.9966	0.5235	2
	-0.7572	0.0654	0.0707	0.6060	0.2930	3
	-0.2217	0.2107	0.3569	1.7233	0.8218	4
	-0.9740	0.9189	0.0164	0.1505	0.9733	5

Table 4.3.1.4: Heston model's parameters after minimizing MAE.

The parameters which are in yellow mostly minimize errors between the model and market prices. Summarizing all results, I chosed the following parameters for the option price valuation with the Heston model.

MATURITY	ρ	V_0	θ	κ	σ	AVerror %	min%	max%
0.0190 - 0.0770	-0.3797	0.2585	0.3422	0.5969	0.0051	0.5372	0.0506	0.8847
0.0780 - 0.1210	-0.7571	0.0654	0.0707	0.6069	0.2928	0.3821	0.0550	0.7441
0.1220 - 0.1590	-0.3571	0.0283	0.0273	1.4582	0.6271	0.4277	0.0101	0.8985
0.1600 - 0.2000	-0.2192	0.0258	0.0253	4.9231	0.5007	0.4182	0.0524	0.8497
0.2100 – 0.2520	-0.4496	0.1352	0.1027	1.1662	0.1424	0.4509	0.0477	0.8678
0.2530 – 0.3100	-0.1398	0.1316	0.3225	1.6464	0.8655	0.3694	0.0435	0.9093

Table 4.3.1.5: Final collection of Heston model parameters after minimizing all loss functions.

Now it is everything done to value the options with the Heston model. Additionally, I decided to use the Black-Scholes model and calculate the option prices with its help in order to prove the sufficiency of the Heston model and show which one of the models provides with better results.

At the beginning, for Black-Scholes model it was important to find a parameter $\widehat{\mathrm{var}}_{\mathrm{BS}}$ which is the variance of the stock returns. In this case, I used a maximum likelihood estimator because it gives me the best results and makes these two models to be more competitive:

$$\widehat{\mathrm{var}}_{\mathrm{BS}} = \frac{1}{N-1}\sum_{i=1}^{N}(u_i - \bar{u})^2$$

where

N: the number of observations

$\mathbf{u_i = \log(\frac{S_{i+1}}{S_i})}$: return on stock

$$\bar{u} = \frac{1}{N}\sum_{i=1}^{N} u_i$$

Hence, the following results were obtained:

Maturity	**0.0190 - 0.0770**	**0.0780 - 0.1210**	**0.1220 - 0.1590**	**0.1600 - 0.2000**	**0.2100 – 0.2520**	**0.2530– 0.3100**
$\widehat{var}_{BS}$	0.0001270	0.0000771	0.0000487	0.0001166	0.0001765	0.0005942

Table 4.3.1.6: Variances of stock returns for each set of data.

Consequently, using variance values (Table 4.3.1.6) and initial parameters (Table 4.3.1.5), the call options were priced with the help of Heston and Black-Scholes models, respectively, and the following results were obtained:

HESTON PRICE	B-S PRICE	MARKET PRICE	HESTON ERROR	B-S ERROR
439.4500	447.8200	440.5000	0.2384%	1.6617%
485.9300	486.3200	490.0000	0.8306%	0.7510%
499.0900	496.0800	503.0000	0.7773%	1.3757%
376.3400	369.5200	378.0000	0.4392%	2.2434%
410.5400	409.7400	412.5000	0.4752%	0.6691%
341.2100	336.6800	341.5000	0.0849%	1.4114%
1477.7700	1485.3800	1484.0000	0.4198%	0.0930%
1177.8200	1184.0400	1184.0000	0.5220%	0.0034%
1077.8400	1069.6600	1084.0000	0.5683%	1.3229%
977.8600	984.0900	984.0000	0.6240%	0.0091%
927.8700	920.7700	934.0000	0.6563%	1.4165%
877.8800	875.7900	884.5000	0.7484%	0.9847%
777.9000	775.5800	784.5000	0.8413%	1.1370%
281.5400	289.1700	282.0000	0.1631%	2.5426%
256.5500	260.6000	258.5000	0.7544%	0.8124%
207.8900	198.6400	208.0000	0.0529%	4.5000%

174.4400	168.5400	176.0000	0.8864%	4.2386%
195.8300	187.4900	197.0000	0.5939%	4.8274%
1052.6900	1055.1100	1058.5000	0.5489%	0.3203%
967.7400	968.7900	975.0000	0.7446%	0.6369%
902.6100	895.2700	911.5000	0.9753%	1.7806%
830.7500	837.8700	835.0000	0.5090%	0.3437%
841.9500	847.5300	842.0000	0.0059%	0.6568%
883.8600	882.9100	885.5000	0.1852%	0.2925%
742.4300	734.6300	748.5000	0.8110%	1.8530%
915.8400	911.3400	920.0000	0.4522%	0.9413%
788.7200	786.7800	790.5000	0.2252%	0.4706%
430.5700	435.0000	433.5000	0.6759%	0.3460%
356.4500	349.5800	358.0000	0.4330%	2.3520%
317.5100	320.0500	319.0000	0.4671%	0.3292%
301.4500	307.9300	301.5000	0.0166%	2.1327%
206.2800	198.9700	207.0000	0.3478%	3.8792%
199.2900	191.4500	200.0000	0.3550%	4.2750%
159.6200	156.5000	161.0000	0.8571%	2.7950%
992.2900	998.9600	1000.0000	0.7710%	0.1040%
1079.7800	1075.3300	1086.5000	0.6185%	1.0281%
1138.1800	1132.3900	1146.0000	0.6824%	1.1876%
1115.7100	1107.5200	1125.0000	0.8258%	1.5538%
1023.3900	1031.4300	1023.5000	0.0107%	0.7748%
1022.5600	1030.5700	1023.5000	0.0918%	0.6908%
1021.7300	1029.7000	1023.5000	0.1729%	0.6058%
924.1900	917.7000	928.5000	0.4642%	1.1632%
923.3400	916.9900	928.5000	0.5557%	1.2396%
922.5000	916.3000	928.5000	0.6462%	1.3139%
795.3900	791.7000	797.5000	0.2646%	0.7273%
794.5600	790.6300	797.5000	0.3687%	0.8614%
745.7300	738.8300	752.5000	0.8997%	1.8166%
662.2600	662.5700	663.5000	0.1869%	0.1402%
643.8500	648.5500	645.5000	0.2556%	0.4725%

628.6100	635.9300	632.5000	0.6150%	0.5423%
615.6200	623.0800	616.0000	0.0617%	1.1494%
612.3000	619.5300	616.0000	0.6006%	0.5731%
593.8800	598.2400	597.5000	0.6059%	0.1238%
607.8000	614.6400	608.5000	0.1150%	1.0090%
565.6600	561.7600	568.5000	0.4996%	1.1856%
557.9200	552.3400	560.5000	0.4603%	1.4558%
557.8500	552.2400	560.5000	0.4728%	1.4737%
557.7800	552.1400	560.5000	0.4853%	1.4915%
557.8300	552.1900	560.0000	0.3875%	1.3946%
501.2800	497.8100	503.5000	0.4409%	1.1301%
501.2400	497.7500	503.5000	0.4489%	1.1420%
1155.1000	1154.4100	1165.0000	0.8498%	0.9090%
1147.1100	1142.5100	1151.0000	0.3380%	0.7376%
1221.1600	1226.9800	1230.0000	0.7187%	0.2455%
1015.7900	1019.4600	1021.0000	0.5103%	0.1508%
1015.8900	1019.6800	1019.5000	0.3541%	0.0177%
739.8800	739.2000	740.5000	0.0837%	0.1756%
639.9600	644.3000	644.0000	0.6273%	0.0466%
556.3500	552.8700	557.0000	0.1167%	0.7415%
580.8000	580.2600	580.5000	0.0517%	0.0413%
576.2000	576.1600	578.5000	0.3976%	0.4045%
474.1600	476.7800	475.5000	0.2818%	0.2692%
506.4000	503.1600	510.0000	0.7059%	1.3412%
527.5700	520.7400	527.0000	0.1082%	1.1879%
530.8600	523.8200	533.5000	0.4948%	1.8144%
550.4100	545.3000	552.0000	0.2880%	1.2138%
397.2300	395.8500	399.5000	0.5682%	0.9136%
416.4300	418.8600	419.0000	0.6134%	0.0334%
583.8900	587.1500	589.0000	0.8676%	0.3141%
543.6500	541.6500	547.0000	0.6124%	0.9781%
554.0300	553.2500	555.0000	0.1748%	0.3153%
557.9000	557.6000	560.0000	0.3750%	0.4286%

596.8200	600.4400	600.0000	0.5300%	0.0733%
608.7100	612.6400	609.0000	0.0476%	0.5977%
492.3800	490.5700	496.5000	0.8298%	1.1944%
575.6800	576.4000	576.5000	0.1422%	0.0173%
572.1300	572.3500	574.0000	0.3258%	0.2875%
525.7300	521.6100	530.0000	0.8057%	1.5830%
527.6900	522.0900	529.0000	0.2476%	1.3062%
1177.3900	1177.4200	1183.5000	0.5163%	0.5137%
1117.9100	1117.8500	1120.0000	0.1866%	0.1920%
927.8200	928.1000	931.0000	0.3416%	0.3115%
612.6900	612.9000	617.0000	0.6985%	0.6645%
602.4700	602.7200	608.0000	0.9095%	0.8684%
600.0600	598.7200	600.5000	0.0733%	0.2964%
582.7500	582.9900	583.0000	0.0429%	0.0017%
569.2300	569.2200	569.5000	0.0474%	0.0492%
491.9900	491.9500	494.5000	0.5076%	0.5157%

Table 4.3.1.7: Comparison of Heston and B-S models to Market prices, calculated errors between them.

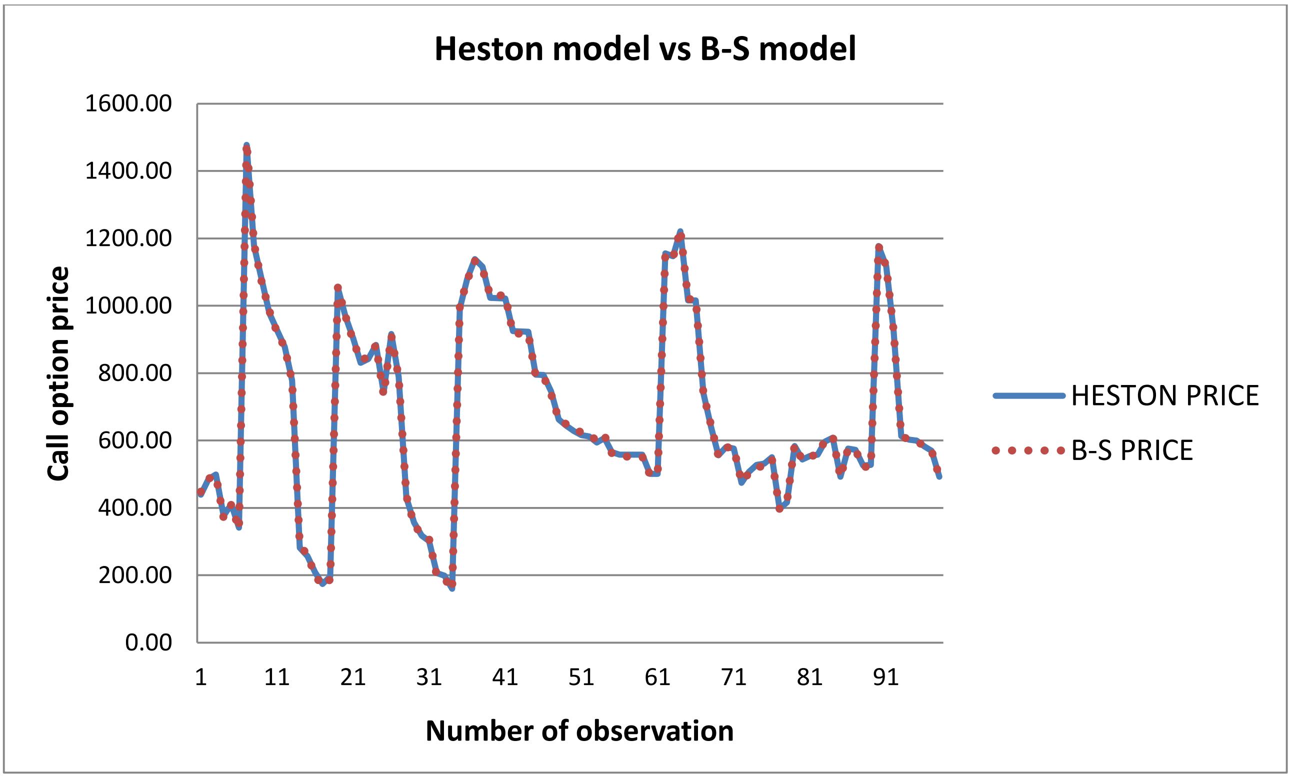

Table 4.3.1.8: Comparison of Heston and Black-Scholes models.

Summarizing all results:

MATURITY	AV HES er	min HES er	max HES er	AV BS er	min BS er	max BS er
0.0190 - 0.0770	0.5376%	0.0529%	0.8864%	1.6667%	0.0034%	4.8274%
0.0780 - 0.1210	0.4756%	0.0059%	0.9753%	1.4628%	0.2925%	4.2750%
0.1220 - 0.1590	0.4447%	0.0107%	0.8997%	0.9759%	0.1040%	1.8166%
0.1600 - 0.2000	0.4181%	0.0517%	0.8498%	0.6026%	0.0177%	1.8144%
0.2100 – 0.2520	0.4508%	0.0476%	0.8676%	0.6450%	0.0173%	1.5830%
0.2530 – 0.3100	0.3693%	0.0429%	0.9095%	0.3792%	0.0017%	0.8684%

Table 4.3.1.9: Average, minimum and maximum errors of Heston and Black-Scholes approximations comparing to market prices.

The above results show that the Heston model performs better than Black-Schole model during the periods of market stability. In some few cases the Black- Scholes produces better prices than the Heston model (Table 4.3.1.7) but the average results proves that the Heston calculates more correct option prices than Black-Scholes (Table 4.3.1.9) .The few cases where the Black-Scholes produces better approaches can be explained in the framework of the calibration. As the calibration helps to find such parameters which would satisfy the whole sample but it does not mean that it would help to find the best approximations for each observation separately.

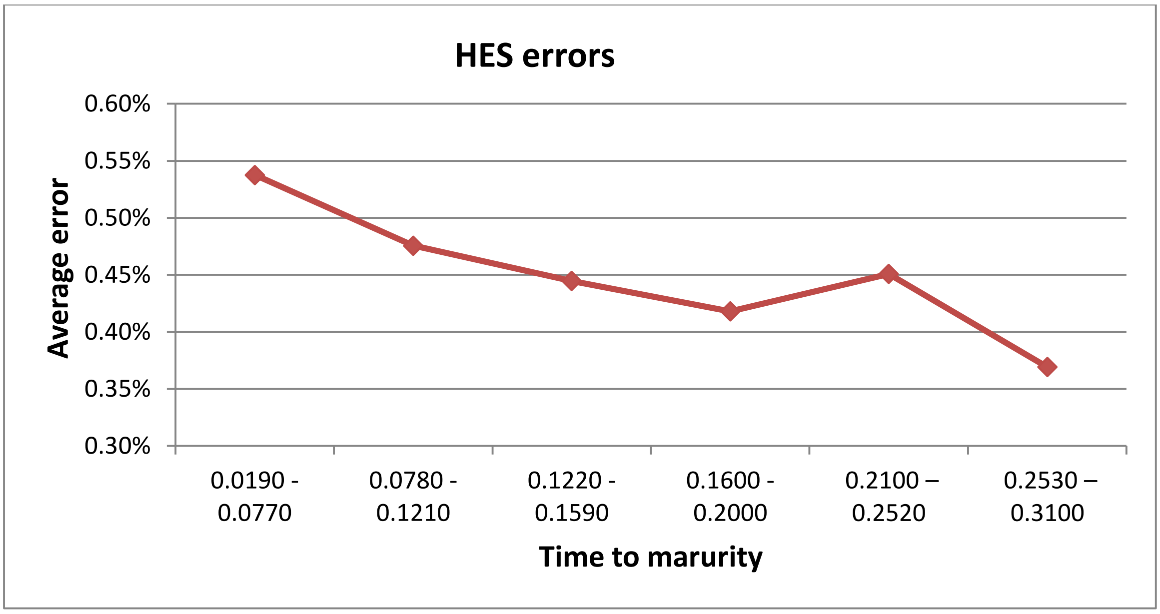

Figure 4.3.1.10: Average Heston model errors according to time to maturity.

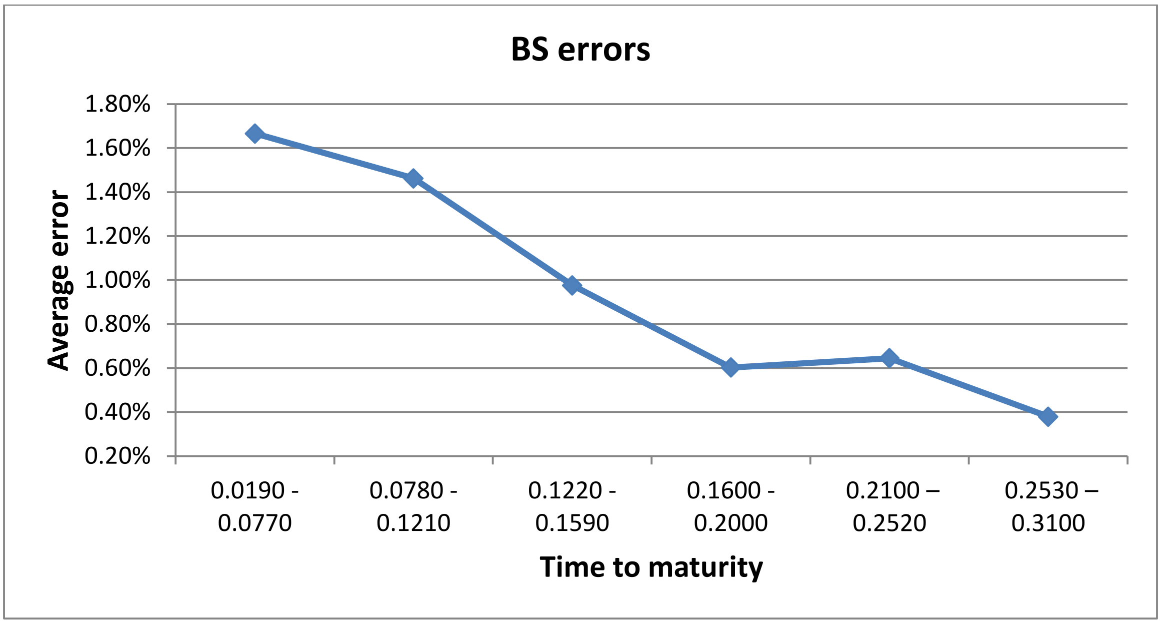

Figure 4.3.1.11: Average Black-Scholes model errors according to time to maturity

Another one very important issue that everyone could observe in the both models results is the changes of the models' errors according to the time to maturity. How it is shown in the above Table 4.3.1.9 and Figures 4.3.1.10 & 4.3.1.11 the options with shorter time to maturity present higher average errors than options with longer time to maturity. It happens due to the fact that unexpected high price jumps in the short time affect more the average performance than in the long time. Every single non-trivial jump in short time can significantly change the whole average performance of the model as it becomes harder to fit it to the data. Thus, the average model errors increasing.

The following graphs are useful for the illustrations of the market prices and obtained models' results. They clearly show that the Heston and Black-Scholes models produce good approximations of the option prices during the period of economical stability.

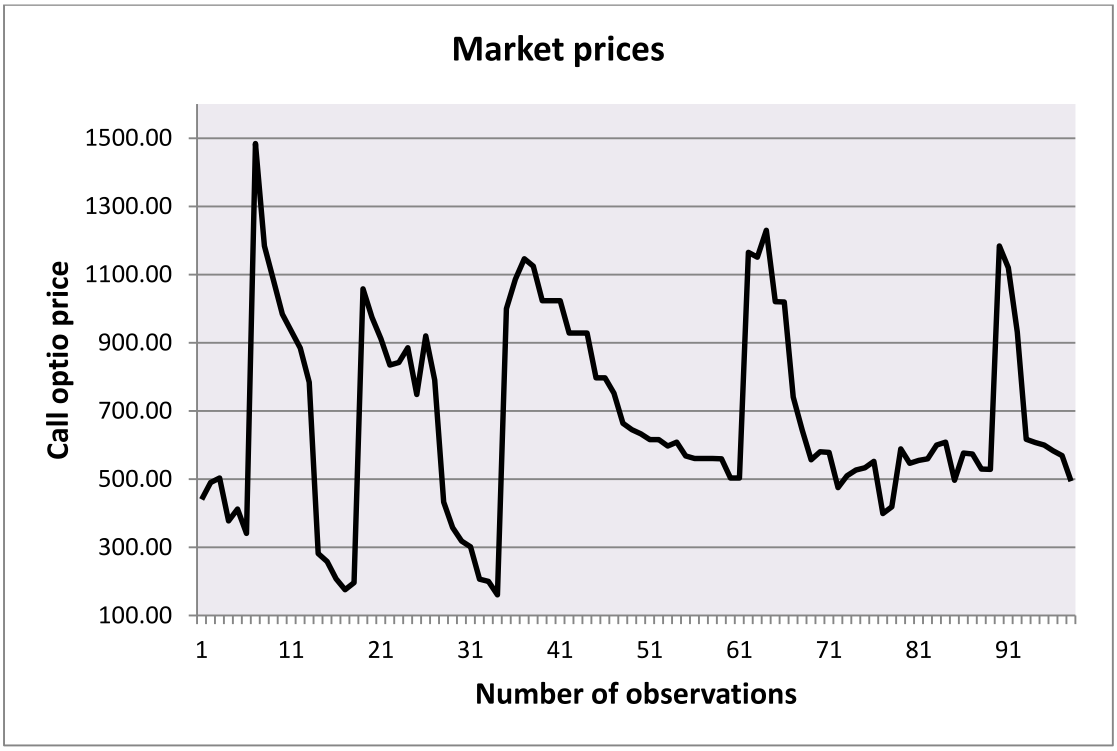

Figure 4.3.1.12: Market option prices.

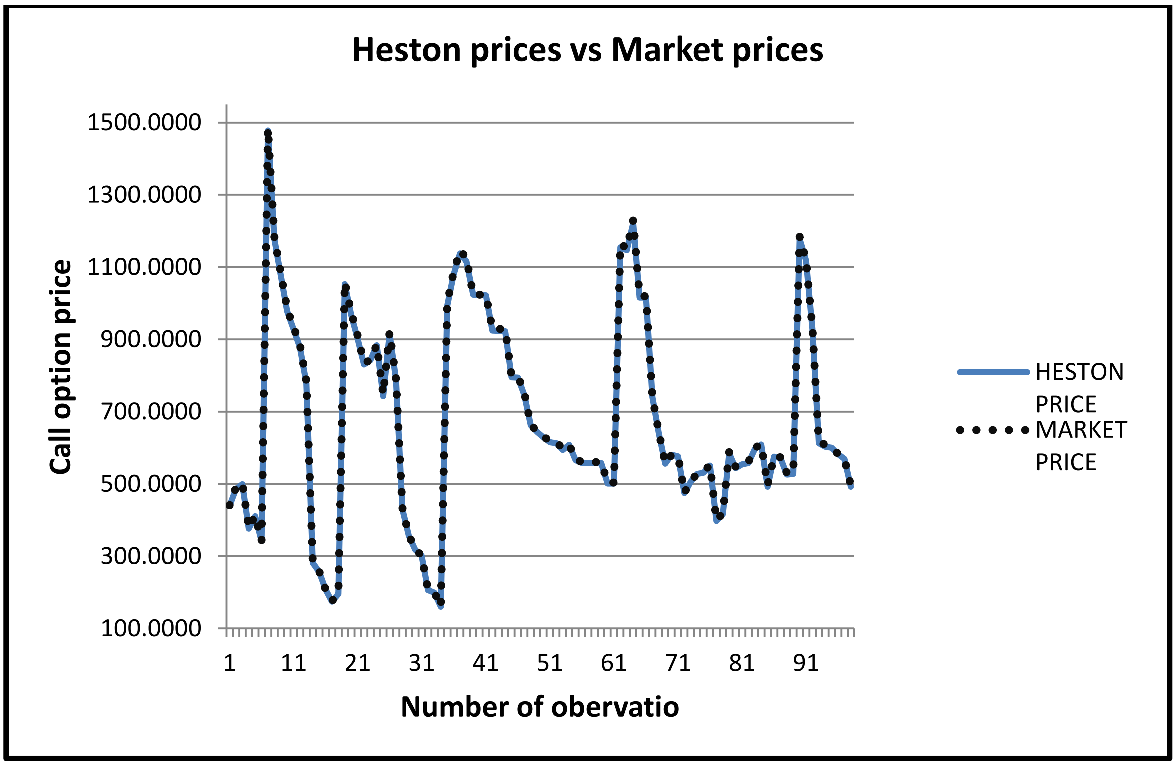

Figure 4.3.1.13: Heston prices vs market prices

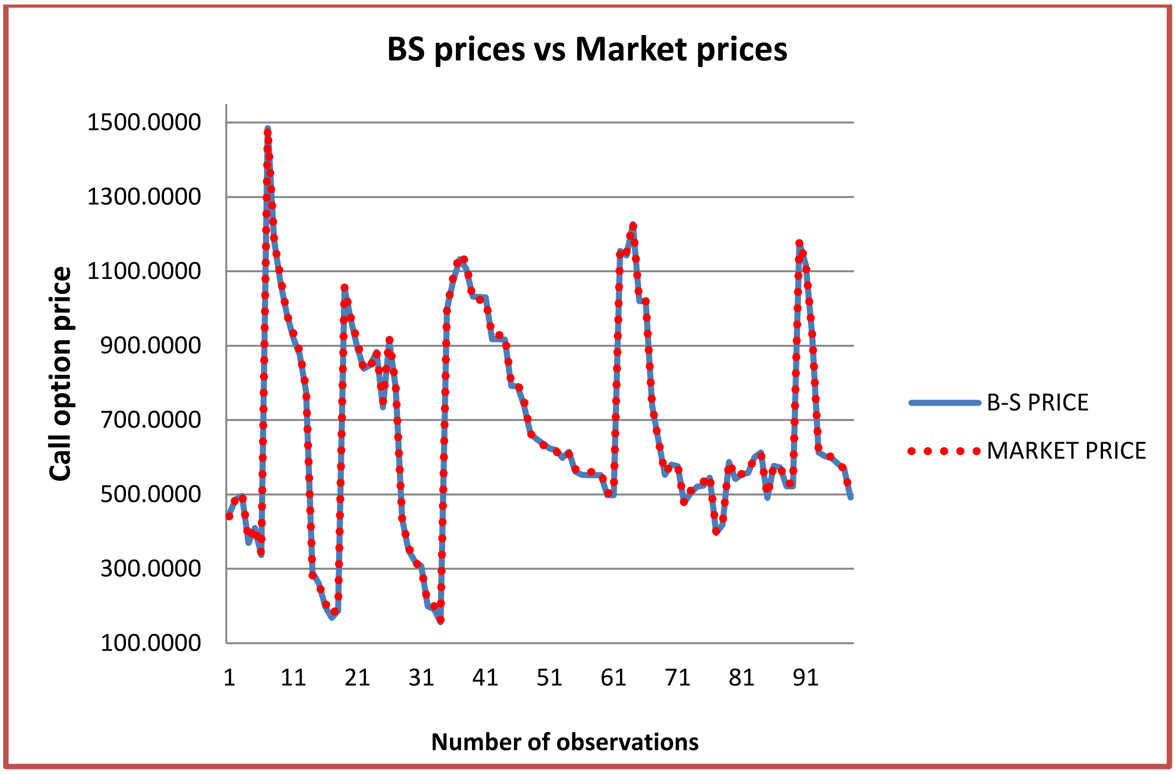

Figure 4.3.14: Black-Scholes prices vs market prices

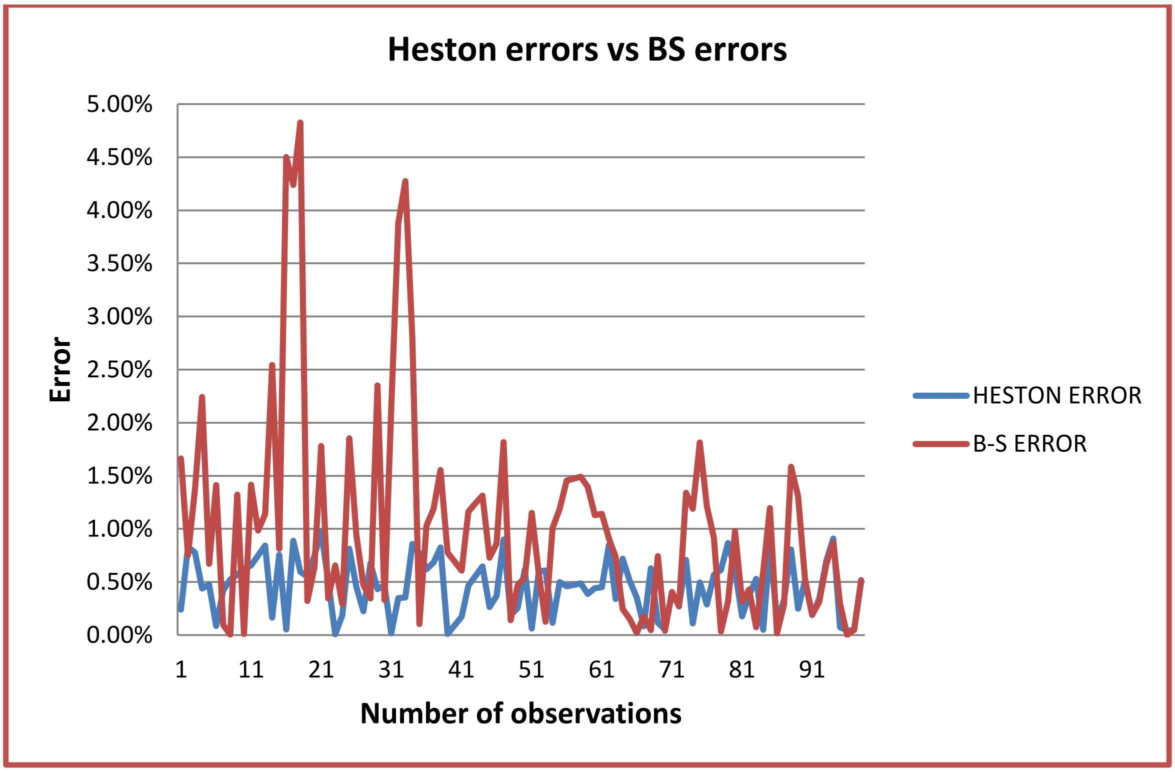

Figure 4.3.1.15: Comparison of Heston errors to BS errors.

The above graph (Figure 4.3.1.15) demonstrates the Heston and Black-Scholes errors. How it is shown the Black-Scholes presents more intensive changes in the errors than the Heston model, especially at the beginning where are introduced the options with shorter time to maturity (it was explained above). The main question here is why the range on the errors of the Heston model differs so much from the errors range of Black-Scholes model. The explanation is very simple. In Black-Scholes model the variance calculation is based on the history of the sample (empirical evidences of stock returns). Thus, the variance straightforwardly depends on the stock movements. Consequently, large movements of stock tend to large changes in volatility. In the Heston model the fundamental item is the calibration which tries to find the parameters which would minimize the errors as much as it is possible and at the same tome to satisfy the whole sample and not single observations (or any amount of them). Thus, the errors of all observations that comprise the sample would be closer to each other than in the Blac-Scholes model. In other words, the errors would be spread inside in the framework of some limits through the whole sample in order to satisfy all observations and not single ones.

4.3.2 PERIOD OF ECONOMICAL CRISIS

(The procedure is exactly the same like in the previous chapter 4.3.1)

In this case I categorized the data into 5 sets according to the time to maturity, the calibrations was implemented 100 times and the following results were obtained after the minimizing all loss functions:

1. After minimizing $\min \text{error} = \min \sum_{i=1}^{N} \left(C_i^{\text{Heston}} - C_i^{\text{Market}}\right)^2$ there were obtained the following parameters:

MATURITY	ρ	V_0	θ	κ	σ	Initial parameter
0.0190 - 0.0770	-0.4040	0.1135	0.0423	0.9933	0.2068	1
	-0.5099	0.2304	0.0513	2.4079	0.5124	2
	-0.6514	0.2312	0.2504	0.7906	0.2564	3
	-0.2124	0.0236	0.2394	1.4221	0.9907	4

	-0.5185	0.4682	0.3000	2.4422	0.5106	5
0.0780 - 0.1210	-0.4689	0.2878	0.0506	1.5309	0.0569	1
	-0.4965	0.1395	0.0495	3.1615	0.5012	2
	-0.7570	0.0650	0.0707	0.6499	0.2930	3
	-0.1801	0.2311	0.3726	1.2504	0.8353	4
	-0.5137	0.2423	0.2918	3.1766	0.5107	5
0.1215 - 0.1590	-0.4000	0.1442	0.0487	1.1396	0.1990	1
	-0.1033	0.9268	0.9162	6.5756	0.0523	2
	-0.7591	0.0721	0.0705	0.5672	0.2994	3
	-0.2073	0.0230	0.2316	1.3377	1.0000	4
	-0.9930	0.0000	0.0000	0.7258	0.9974	5
0.1600 - 0.2203	-0.3999	0.0397	0.0400	1.1838	0.2002	1
	-0.0982	0.9188	0.0105	8.6318	0.9622	2
	-0.6994	0.4008	0.1358	0.8150	0.0786	3
	-0.2219	0.0274	0.2331	1.0861	0.9997	4
	-0.0736	0.5802	0.5748	8.9254	0.3990	5
0.2570 – 0.3022	-0.0172	0.9936	0.9997	9.4091	0.0014	1
	-0.5000	0.1498	0.0500	3.0015	0.5002	2
	-0.7571	0.0654	0.0707	0.6067	0.2928	3
	-0.2061	0.0230	0.2315	1.3661	1.0000	4
	-0.5117	0.2982	0.2990	2.9923	0.5018	5

Table 4.3.2.1: Heston model parameters after minimizing $\min \text{error} = \min \sum_{i=1}^{N} \left(C_i^{Heston} - C_i^{Market} \right)^2$.

2. After minimizing $MSE = \frac{1}{N} \sum_{i=1}^{N} \left(C_i^{Heston} - C_i^{Market} \right)^2$ there were obtained the following parameters:

MATURITY	ρ	V_0	θ	κ	σ	Initial paraneter
0.0190 - 0.0770	-0.4001	0.0411	0.0400	1.1492	0.2001	1
	-0.5132	0.1447	0.0494	1.9810	0.5255	2
	-0.6877	0.2310	0.2029	0.8385	0.2634	3

	-0.2061	0.0277	0.2776	1.3766	0.9999	4
	-0.6917	0.2557	0.4264	1.3673	0.9858	5
0.0780 - 0.1210	-0.4010	0.1087	0.0435	1.1174	0.2013	1
	-0.4748	0.0978	0.0463	3.8287	0.5245	2
	-0.9002	0.0429	0.0438	0.9508	0.9646	3
	-0.1802	0.2861	0.4237	1.2882	0.7721	4
	-0.9987	0.0057	0.0035	8.9954	0.9999	5
0.1215 - 0.1590	-0.1548	0.1831	0.1575	1.2731	0.1658	1
	-0.2801	0.6568	0.8463	5.4837	0.5261	2
	-0.7571	0.0655	0.0707	0.6065	0.2928	3
	-0.2059	0.0230	0.2318	1.3637	1.0000	4
	-0.8566	0.0418	0.0463	2.6573	0.8977	5
0.1600 - 0.2203	-0.3990	0.0364	0.0396	1.4006	0.2029	1
	-0.3127	0.4316	0.0063	8.3472	0.8430	2
	-0.6476	0.1144	0.1376	0.6268	0.0146	3
	-0.2159	0.0218	0.2234	1.2620	1.0000	4
	-0.2218	0.6585	0.6329	9.0451	0.6910	5
0.2570 - 0.3022	-0.0143	0.9914	0.9996	9.5785	0.0016	1
	-0.5000	0.1498	0.0500	3.0015	0.5002	2
	-0.7571	0.0654	0.0707	0.6067	0.2928	3
	-0.2051	0.0231	0.2319	1.3784	1.0000	4
	-0.5115	0.2982	0.2990	2.9927	0.5017	5

Table 4.3.2.2: Heston model's parameters after minimizing MSE.

3. After minimizing $\mathrm{MPE} = \frac{1}{N}\sum_{i=1}^{N} \frac{C_i^{Heston} - C_i^{Market}}{C_i^{Market}}$ there were obtained the following parameters:

MATURITY	ρ	V_0	θ	κ	σ	Initial parameter
0.0190 - 0.0770	-0.4011	0.0615	0.0472	1.1309	0.2013	1
	-0.5097	0.1528	0.0480	2.7145	0.5154	2
	-0.6761	0.2353	0.1801	0.8159	0.2662	3

	-0.2090	0.0222	0.2389	1.3837	1.0000	4
	-0.5188	0.6220	0.3063	2.4788	0.5086	5
0.0780 - 0.1210	-0.6025	0.3203	0.1968	1.0706	0.2742	1
	-0.4989	0.1458	0.0498	3.0561	0.5009	2
	-0.8509	0.0420	0.0630	0.9860	0.4322	3
	-0.1640	0.3339	0.4619	1.2606	0.7199	4
	-0.5232	0.2109	0.2848	3.0144	0.5285	5
0.1215 - 0.1590	-0.1996	0.1595	0.1738	1.0090	0.1605	1
	-0.2143	0.9045	0.9857	6.4202	0.0526	2
	-0.7769	0.1698	0.0710	0.1639	0.3544	3
	-0.2059	0.0230	0.2318	1.3634	1.0000	4
	-0.5520	0.1526	0.2456	3.1662	0.5911	5
0.1600 - 0.2203	-0.3987	0.0371	0.0397	1.4177	0.2014	1
	-0.1757	0.8841	0.0084	8.7444	0.9492	2
	-0.5570	0.2610	0.2957	0.7178	0.6464	3
	-0.2057	0.0230	0.2317	1.3684	1.0000	4
	-0.0835	0.7386	0.7318	8.1395	0.0781	5
0.2570 – 0.3022	-0.0152	0.9888	1.0000	9.5661	0.0017	1
	-0.4779	0.0691	0.0319	4.0852	0.5988	2
	-0.7571	0.0654	0.0707	0.6067	0.2928	3
	-0.2051	0.0231	0.2319	1.3784	1.0000	4
	-0.5104	0.2995	0.2997	2.9978	0.5005	5

Table 4.3.2.3: Heston model's parameters after minimizing MPE.

4. After minimizing $\mathrm{MAE} = \frac{1}{N}\sum_{i=1}^{N} |C_i^{\mathrm{Heston}} - C_i^{\mathrm{Market}}|$ there were obtained the following parameters:

MATURITY	ρ	V_0	θ	κ	σ	Initial parameter
0.0190 - 0.0770	-0.4040	0.1135	0.0423	0.9933	0.2068	1
	-0.5099	0.2304	0.0513	2.4079	0.5124	2
	-0.6514	0.2312	0.2504	0.7906	0.2564	3

	-0.2124	0.0236	0.2394	1.4221	0.9907	4
	-0.5184	0.4725	0.3002	2.4416	0.5102	5
0.0780 - 0.1210	-0.4696	0.2780	0.0479	1.0859	0.0505	1
	-0.4965	0.1395	0.0495	3.1615	0.5012	2
	-0.7570	0.0650	0.0707	0.6499	0.2930	3
	-0.1801	0.2311	0.3726	1.2504	0.8353	4
	-0.5137	0.2423	0.2918	3.1766	0.5107	5
0.1215 - 0.1590	-0.4000	0.1442	0.0487	1.1396	0.1990	1
	-0.1088	0.8840	0.8897	6.5517	0.0567	2
	-0.7591	0.0721	0.0705	0.5672	0.2994	3
	-0.2073	0.0230	0.2316	1.3377	1.0000	4
	-0.9930	0.0000	0.0000	0.7262	0.9974	5
0.1600 - 0.2203	-0.3999	0.0397	0.0400	1.1838	0.2002	1
	-0.0982	0.9188	0.0105	8.6317	0.9622	2
	-0.2390	0.7358	0.7470	0.0998	0.9875	3
	-0.2219	0.0274	0.2331	1.0861	0.9997	4
	-0.0590	0.4510	0.4465	9.1546	0.0186	5
0.2570 – 0.3022	-0.0052	0.9999	0.9982	9.8350	0.0006	1
	-0.5000	0.1498	0.0500	3.0015	0.5002	2
	-0.7571	0.0654	0.0707	0.6067	0.2928	3
	-0.2061	0.0230	0.2315	1.3661	1.0000	4
	-0.5117	0.2983	0.2990	2.9923	0.5018	5

Table 4.3.2.4: Heston model's parameters after minimizing MAE.

The parameters which are in yellow mostly minimize errors between the model and market prices. Summarizing all results, I concluded to choose the following parameters for the option price valuation with the Heston model.

MATURITY	ρ	V_0	θ	κ	σ	AV error %	min %	max %
0.0190 - 0.0770	-0.5188	0.6220	0.3063	2.4788	0.5086	1.1937	0.0088	2.9568
0.0780 - 0.1210	-0.5232	0.2109	0.2848	3.0144	0.5285	1.3488	0.1760	2.8882
0.1215 - 0.1590	-0.5520	0.1526	0.2456	3.1662	0.5911	1.3768	0.0046	2.9696
0.1600 - 0.2203	-0.5570	0.2610	0.2957	0.7178	0.6464	1.4740	0.1281	2.9400
0.2570 – 0.3022	-0.7571	0.0654	0.0707	0.6067	0.2928	2.2621	1.3978	2.8925

Table 4.3.2.5: Final collection of Heston model parameters after minimizing all loss functions.

Additionally, $\widehat{\mathrm{var}}_{\mathrm{BS}}$ parameter of Black Scholes model was calculted exactly in the same way like in the previous chapter 4.3.1. The following results were obtained:

Maturity	0.0190 - 0.0770	0.0780 - 0.1210	0.1215 - 0.1590	0.1600 - 0.2203	0.2570 – 0.3022
$\widehat{var}_{BS}$	0.0000377	0.0008072	0.0008321	0.0012125	0.0000235

Table 4.3.2.6: Variances of stock returns for each set of data.

Consequently, the Heston and Black-Scholes prices were calculated during the economical crisis:

HESTON PRICE	B-S PRICE	MARKET PRICE	HESTON ERROR	B-S ERROR
530.1200	532.7100	531.0000	0.1657%	0.3220%
615.2000	612.5100	615.5000	0.0487%	0.4858%
480.2800	472.6400	485.0000	0.9732%	2.5485%
565.3500	573.0900	567.5000	0.3789%	0.9850%
430.4500	434.8700	439.5000	2.0592%	1.0535%
515.5100	511.1000	520.5000	0.9587%	1.8060%
395.6600	402.5000	400.0000	1.0850%	0.6250%
465.6700	461.6900	474.0000	1.7574%	2.5970%
377.3700	380.3400	378.5000	0.2985%	0.4861%
367.1500	366.9500	368.5000	0.3664%	0.4206%
325.0000	317.1800	329.0000	1.2158%	3.5927%
330.9300	323.2500	331.5000	0.1719%	2.4887%
371.6900	372.0500	375.5000	1.0146%	0.9188%
415.8300	423.6800	428.5000	2.9568%	1.1249%
327.5200	319.6900	335.5000	2.3785%	4.7124%
317.3000	309.9900	325.0000	2.3692%	4.6185%
281.0500	282.8200	288.5000	2.5823%	1.9688%
381.8100	384.9500	382.0000	0.0497%	0.7723%
331.9200	323.9100	336.0000	1.2143%	3.5982%
214.4900	216.1200	219.0000	2.0594%	1.3151%

178.0300	170.6600	183.0000	2.7158%	6.7432%
342.4700	334.5000	342.5000	0.0088%	2.3358%
248.4100	255.4400	250.0000	0.6360%	2.1760%
317.5000	311.3300	318.5000	0.3140%	2.2512%
223.4600	225.9300	229.0000	2.4192%	1.3406%
292.5300	293.5600	295.0000	0.8373%	0.4881%
989.8000	989.7000	992.0000	0.2218%	0.2319%
790.6500	791.0100	801.0000	1.2921%	1.2472%
864.5600	866.0600	870.0000	0.6253%	0.4529%
683.6700	685.1400	688.5000	0.7015%	0.4880%
720.6500	721.8000	725.5000	0.6685%	0.5100%
519.8400	521.3300	521.5000	0.3183%	0.0326%
621.0900	619.2400	634.5000	2.1135%	2.4050%
470.0500	468.0600	479.0000	1.8685%	2.2839%
658.4500	654.5200	670.5000	1.7972%	2.3833%
705.0700	705.9600	714.0000	1.2507%	1.1261%
640.4700	639.2300	648.5000	1.2382%	1.4295%
505.4500	504.6600	514.0000	1.6634%	1.8171%
654.6200	653.0200	660.0000	0.8152%	1.0576%
531.0100	531.3500	540.0000	1.6648%	1.6019%
544.5400	539.7600	545.5000	0.1760%	1.0522%
555.1300	556.1600	569.5000	2.5233%	2.3424%
494.8100	488.4200	502.5000	1.5303%	2.8020%
582.1200	581.3100	583.5000	0.2365%	0.3753%
532.1400	532.8800	539.5000	1.3642%	1.2271%
581.8200	580.9800	587.5000	0.9668%	1.1098%
482.1600	481.6400	496.5000	2.8882%	2.9930%
477.1100	476.3500	480.5000	0.7055%	0.8637%
531.8500	532.7200	544.5000	2.3232%	2.1635%
495.2800	495.0200	501.0000	1.1417%	1.1936%
427.1400	427.7400	438.5000	2.5906%	2.4538%
435.3100	435.8400	441.0000	1.2902%	1.1701%
435.2600	435.8200	441.0000	1.3016%	1.1746%

435.1500	435.7800	441.0000	1.3265%	1.1837%
340.4300	339.7200	347.0000	1.8934%	2.0980%
319.7100	320.4900	326.0000	1.9294%	1.6902%
354.2700	352.9400	361.5000	2.0000%	2.3679%
343.9500	343.6700	346.5000	0.7359%	0.8167%
1056.2700	1056.7200	1062.0000	0.5395%	0.4972%
1067.7100	1068.0200	1070.0000	0.2140%	0.1850%
857.8100	857.8800	875.0000	1.9646%	1.9566%
902.3000	903.0300	907.0000	0.5182%	0.4377%
869.1500	869.5100	880.5000	1.2890%	1.2482%
751.1000	751.4400	758.5000	0.9756%	0.9308%
709.9300	710.8100	714.0000	0.5700%	0.4468%
662.8200	661.8400	669.5000	0.9978%	1.1441%
747.6400	748.2400	755.5000	1.0404%	0.9610%
703.7600	703.6000	723.5000	2.7284%	2.7505%
701.0700	700.8100	705.0000	0.5574%	0.5943%
658.7200	657.8800	664.5000	0.8698%	0.9962%
697.6100	697.1400	707.5000	1.3979%	1.4643%
648.4200	648.1100	667.5000	2.8584%	2.9049%
601.7600	602.5200	616.0000	2.3117%	2.1883%
559.3800	560.0100	576.5000	2.9696%	2.8604%
664.8300	664.5700	672.5000	1.1405%	1.1792%
805.5900	805.0400	808.0000	0.2983%	0.3663%
614.8600	615.1400	628.0000	2.0924%	2.0478%
647.8400	645.0300	650.0000	0.3323%	0.7646%
755.6300	755.1000	762.5000	0.9010%	0.9705%
597.9100	595.5300	607.5000	1.5786%	1.9704%
705.6600	705.1200	717.5000	1.6502%	1.7254%
663.7300	663.3000	668.5000	0.7135%	0.7779%
652.4700	652.2500	652.5000	0.0046%	0.0383%
678.6200	677.9500	681.5000	0.4226%	0.5209%
655.7000	655.1100	673.5000	2.6429%	2.7305%
613.7600	614.2100	625.0000	1.7984%	1.7264%

602.5100	602.7800	609.0000	1.0657%	1.0213%
628.6600	629.3100	636.5000	1.2317%	1.1296%
552.5400	552.2200	565.5000	2.2918%	2.3484%
577.4500	576.9400	582.0000	0.7818%	0.8694%
551.0500	550.7300	555.0000	0.7117%	0.7694%
578.7000	578.0700	592.5000	2.3291%	2.4354%
527.4800	527.9500	539.5000	2.2280%	2.1409%
501.0900	501.4900	513.5000	2.4167%	2.3389%
558.1500	557.7600	563.5000	0.9494%	1.0186%
508.1800	508.6900	521.0000	2.4607%	2.3628%
677.4800	677.7000	681.5000	0.5899%	0.5576%
627.5200	627.5100	636.5000	1.4108%	1.4124%
577.5500	577.3300	593.0000	2.6054%	2.6425%
1071.0800	1071.1800	1077.5000	0.5958%	0.5865%
966.4300	966.3500	971.5000	0.5219%	0.5301%
1019.6500	1019.5800	1027.0000	0.7157%	0.7225%
954.8400	954.9000	960.5000	0.5893%	0.5830%
923.7000	923.7900	931.5000	0.8374%	0.8277%
872.9100	872.8500	894.0000	2.3591%	2.3658%
768.2300	768.3200	791.5000	2.9400%	2.9286%
834.7200	834.6000	836.5000	0.2128%	0.2271%
846.2600	846.1500	853.0000	0.7902%	0.8030%
735.6000	735.6500	749.5000	1.8546%	1.8479%
747.1300	747.2200	763.0000	2.0799%	2.0682%
736.7200	736.7900	741.5000	0.6446%	0.6352%
803.3800	803.4500	810.5000	0.8785%	0.8698%
821.6400	821.6500	845.5000	2.8220%	2.8208%
845.8700	845.8300	850.5000	0.5444%	0.5491%
637.5500	637.5900	654.5000	2.5898%	2.5837%
704.2000	704.0400	720.0000	2.1944%	2.2167%
752.2300	752.2300	767.5000	1.9896%	1.9896%
646.9200	646.9700	664.0000	2.5723%	2.5648%
797.0400	797.0100	803.0000	0.7422%	0.7460%

719.2700	718.6900	723.0000	0.5159%	0.5961%
697.1600	697.1500	714.0000	2.3585%	2.3599%
756.6100	756.5800	758.5000	0.2492%	0.2531%
819.2200	819.2600	823.0000	0.4593%	0.4544%
619.3900	618.8200	637.0000	2.7645%	2.8540%
615.4600	615.4200	629.5000	2.2303%	2.2367%
656.7100	656.6600	672.0000	2.2753%	2.2827%
707.4000	707.5000	715.0000	1.0629%	1.0490%
719.3000	719.3700	733.0000	1.8690%	1.8595%
709.5900	709.6900	710.5000	0.1281%	0.1140%
657.4400	657.3400	670.0000	1.8746%	1.8896%
669.3400	669.2600	688.0000	2.7122%	2.7238%
721.9800	722.0100	724.5000	0.3478%	0.3437%
666.9700	666.8800	670.5000	0.5265%	0.5399%
659.6300	659.5400	667.0000	1.1049%	1.1184%
617.0200	617.1100	628.0000	1.7484%	1.7341%
609.6700	609.7700	624.5000	2.3747%	2.3587%
622.1800	622.2000	641.0000	2.9360%	2.9329%
909.5600	909.5600	933.0000	2.5123%	2.5123%
842.0200	842.0100	864.0000	2.5440%	2.5451%
839.6100	839.6000	853.0000	1.5698%	1.5709%
900.3400	900.3400	922.0000	2.3492%	2.3492%
915.8200	915.8200	939.5000	2.5205%	2.5205%
845.5000	841.3900	865.5000	2.3108%	2.7857%
808.4200	808.4300	832.5000	2.8925%	2.8913%
795.7200	795.7100	807.0000	1.3978%	1.3990%

Table 4.3.2.7: Comparison of Heston and B-S models to Market prices, calculated errors between them.

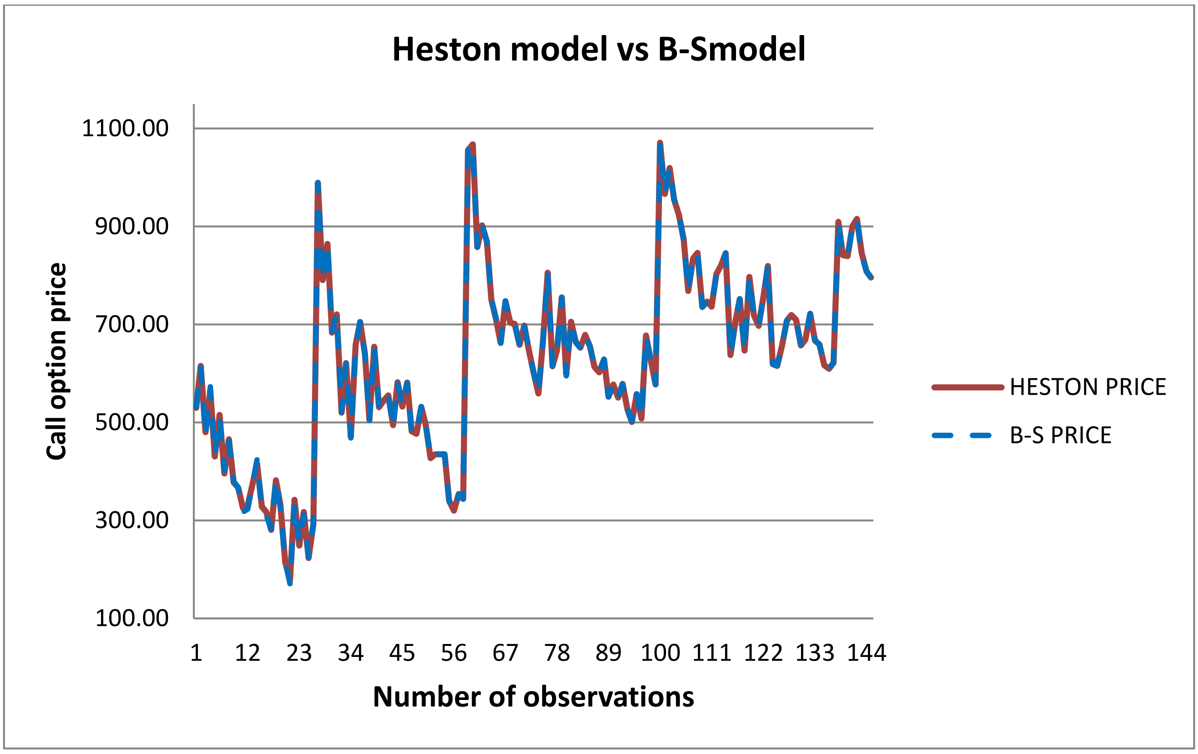

Table 4.3.2.8: Comparison of Heston and Black-Scholes models

Summarizing all results:

MATURITY	AV HES er	min HES er	max HES er	AV BS er	min BS er	max BS er
0.0190 - 0.0770	1.1937%	0.0088%	2.9568%	1.9913%	0.3220%	6.7432%
0.0780 - 0.1210	1.3488%	0.1760%	2.8882%	1.4420%	0.0326%	2.9930%
0.1215 - 0.1590	1.3768%	0.0046%	2.9696%	1.4008%	0.0383%	2.9049%
0.1600 - 0.2203	1.4740%	0.1281%	2.9400%	1.4781%	0.1140%	2.9329%
0.2570 – 0.3022	2.2621%	1.3978%	2.8925%	2.3218%	1.3990%	2.8913%

Table 4.3.2.9: Average, minimum and maximum errors of Heston and Black-Scholes approximations comparing to market prices.

The above results show that the Heston model performs better than Black-Schole model even during the period of economical crisis. The few cases where Black-Scholes produces a bit better results can be explained again with the help of the calibration.

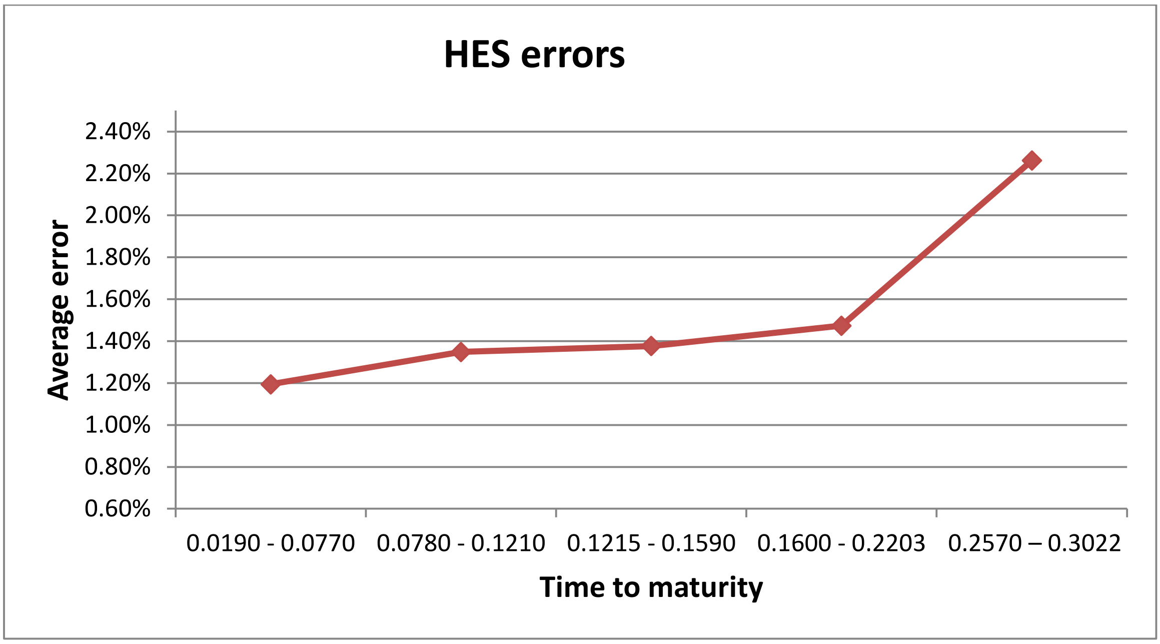

Figure 4.3.2.10: Average Heston model errors.

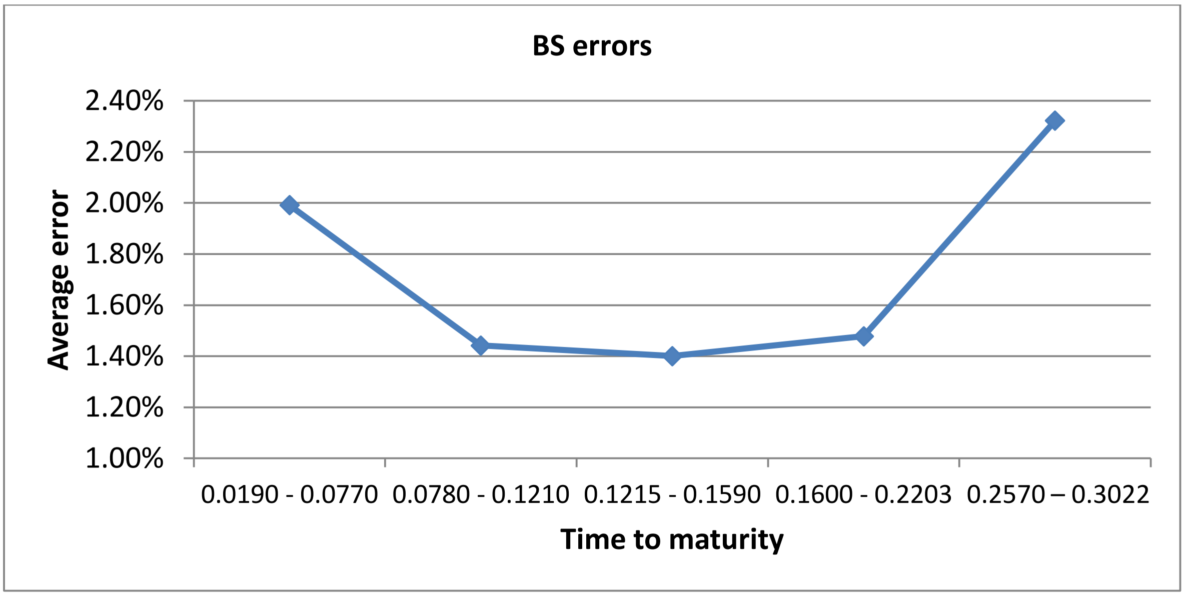

Figure 4.3.2.11: Average Black-Scholes model errors.

The interesting fact which is observed during the period of economical crisis is the opposite evolution of the errors' performance. During the period of economical crisi it is clearly shown that the errors increasing as the time to maturity of options increasing. Of course the main question is why during these 2 periods the errors act so differently.

During the period of economical crisis the market is very unstable, many unexpected jumps in prices are observed and no sufficient forecast can be made. Therefore, the options with long time to maturity have higher probability of more unexpected movements in their price. In the period of economical crisis, when these sensible price movements happen pretty often, the options with short time to maturity are affected less than the options with long time to maturity, as during the short time less unexpected jumps happen, consequently the volatility is lower than the volatility of options with long time to maturity. Thus, it is easier to fit more sufficiently the model in short time, as the average performance of the model errors is affected less than the average errors in the case of options with the long time to maturity.

The following graphs are useful for the illustrations of the market prices and obtained models' results:

Table 4.3.2.12: Market option prices.

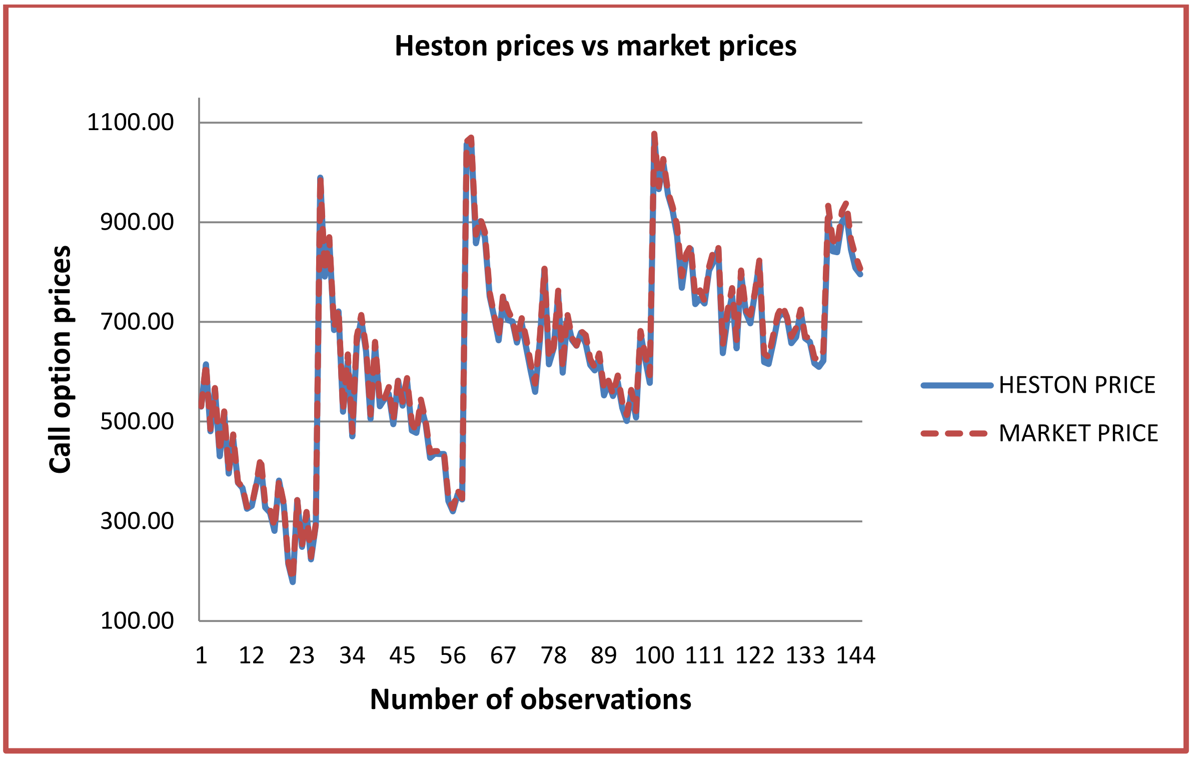

Table 4.3.2.13: Heston prices vs market prices

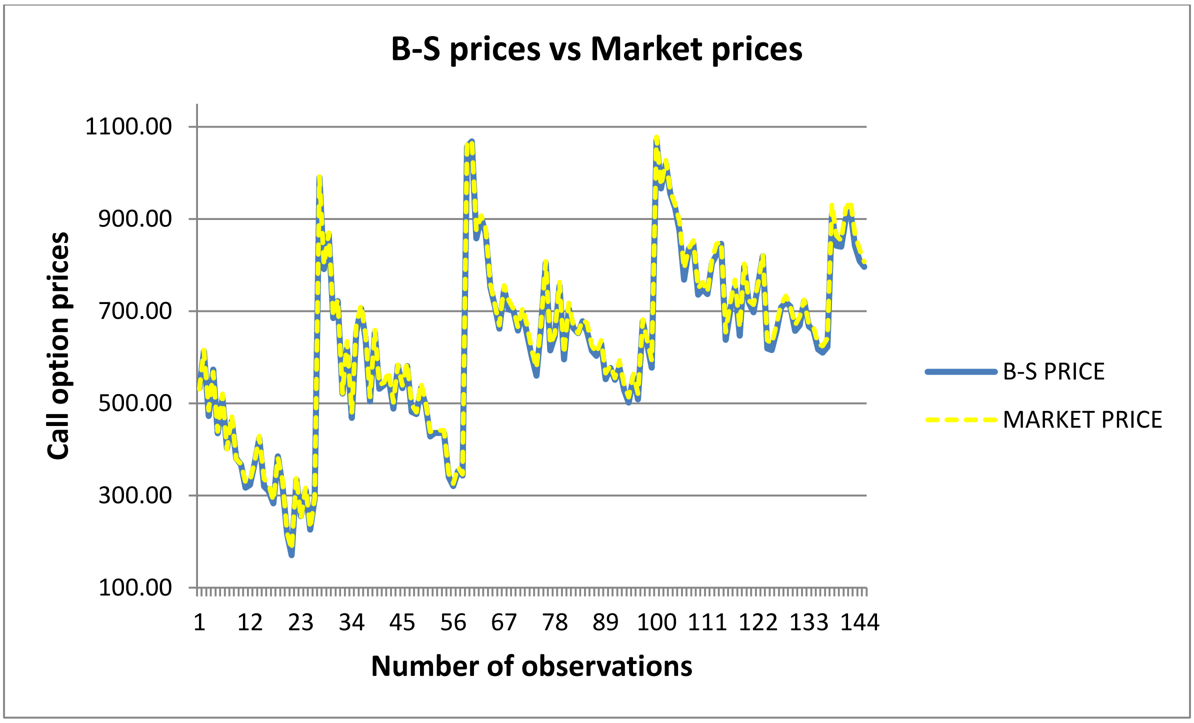

Table 4.3.2.14: Black-Scholes prices vs market prices

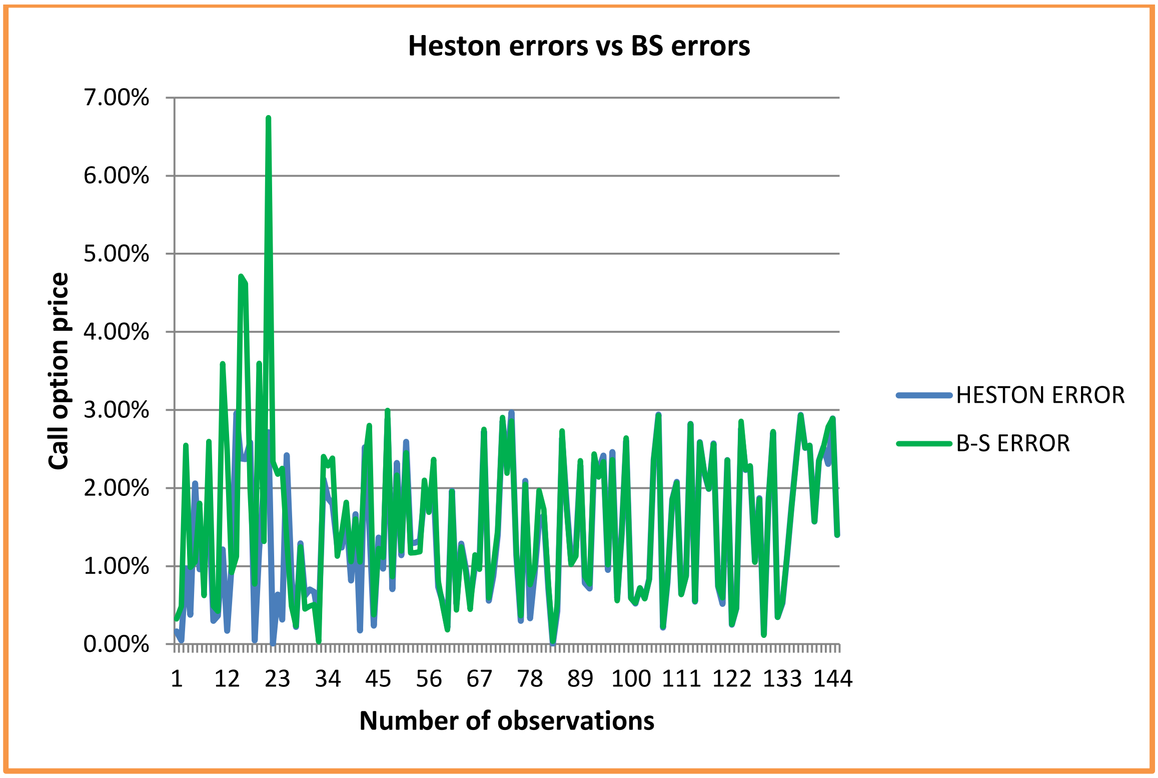

Table 4.3.2.15: Comparison of Heston errors to BS errors

The Table 4.3.2.15 demonstrates the errors' evolution of Heston and Black-Scholes models during the economical crisis. The Heston model presents again more stable level of the errors than the Black-Scholes. But even though, the range of the errors are increased comparing to the stable period. It is explained by the huge jumps of values during the economical crisis. Thus, it is a harder task for calibration to find parameters which would satisfy the whole sample, this fact leads to higher range of the errors in order to satisfy all observations. In the Black-Scholes model, at the beginning of the period big changes in errors are observed. This is derived by the severe market fluctuation during the stress period and short time to maturity character of options. The performance of the errors at the end of the period was explained previously.

4.4 EMPIRICAL PARAMETERS ESTIMATION

After the long calibration procedure which was presented in the last chapter there appeared a question if the Heston parameters can be empirically estimated. Therefore, in this chapter there will be investigated the method where the finding Heston parameters based on the empirical estimation. Which means that there will be done no calibrations and $\rho, V_{0,}, \theta, \kappa, \sigma$ will be found with the help of the data history.

In the first phase there were collected 40 'dead' call options of European style which mature on 9th and 13th February 2013 with time to maturity less or equal to one year. Afterwards, the filtration rules which were described in the previous chapter were used and at the end 81 daily observations were chose. They were divided into six sets according to maturity, exactly like in chapter 4.1

0.0190 - 0.0770	0.1600 - 0.2000
0.0780 - 0.1210	0.2100 – 0.2520
0.1220- 0.1590	0.2530 – 0.3100

The parameters estimation has been done for each set separately. (The following formulas of the current paragraph 4.4 are taken from the book Greene 2012)

In the second phase all the parameters which $k, \theta, \rho, \sigma, V_0$ were calculated according to the following formulas:

I assumed that S_n^i is the stock prices of the option n from the set i, $i \in [1,6], \; N = 81, (\mathrm{N: number\ of\ observations}).$

$$x_n^i = \log\left(\frac{S_{n+1}^i}{S_n^i}\right): \text{ stock returns of observation n from the set i}$$

$$\text{and } \overline{x^i} = \sum_{n=0}^{N}\left(\frac{x_n^i}{N}\right), \; n\epsilon[1, N]$$

$\xi^i = \{\xi_1^i, \xi_2^i, \ldots \xi_N^i\}$: volatilities of FTSE 100 volatility index for the set i, each ξ_j^i corresponds to the observation j from the set i.

$V_0^i = \frac{1}{N}\sum_{n=1}^{N} \xi_n^i$: initial variance of the set i with $V_0^i \in [0,1]$

$\sigma^i = \sqrt{\frac{1}{N}\sum_{n=0}^{N}\left(\xi_n^i - \overline{\xi^i}\right)^2}$: volatility of volatility of the set i and $\overline{\xi^i} = \sum_{n=0}^{N}\left(\frac{\xi_n^i}{N}\right)$ with

$$\sigma^i \in [0,1]$$

$\rho = \frac{\sum_{n=1}^{N}(x_n^i - \overline{x^i})(\xi_n^i - \overline{\xi^i})}{\sqrt{\sum_{n=1}^{N}(x_n^i - \overline{x^i})^2 \sum_{n=1}^{N}(\xi_n^i - \overline{\xi^i})^2}}$: correlation between the log - returns on stock and

index volatility with $|\rho| < 1$

k and θ were found with the help of AR(1) process as the CIR process can be expressed as $\xi_t = \alpha + \beta\xi_{t-1} + \sqrt{\xi_{t-1}}\varepsilon_t$ where $\varepsilon \sim N(0,1)$ and $\alpha = \kappa\theta$, $\beta = 1 - \kappa$ with $\kappa>0$ and $\theta \in [0,1]$.

Consequently, $[\xi_t] = \alpha + \beta E[\xi_{t-1}] + \sqrt{\xi_{t-1}}E[\varepsilon_t] \to \xi_t = \alpha + \beta\xi_{t-1}$. Thus, in this case the Regression Analysis was done and Ordinary Least Squares method was used to estimate α and β.

Consequently, $\hat{\beta} = \frac{\sum_{n=1}^{N}\xi_{n-1}\xi_n - \frac{1}{N}\sum_{n=1}^{N}\xi_{n-1}\sum_{n=0}^{N-1}\xi_n}{\sum_{n=1}^{N}\xi_{n-1}^2 - \frac{1}{N}\left(\sum_{n=1}^{N}\xi_{n-1}\right)^2}$ and

$$\hat{\alpha} = \frac{1}{N}\sum_{n=0}^{N-1}\xi_n - \hat{\beta}\frac{1}{N}\sum_{n=1}^{N}\xi_{n-1}$$

Finally, all parameters were calculated for all sets:

MATURITY	ρ	V_0	θ	κ	σ	Feller condition
0.0190 - 0.0770	-0.62397	0.12771	0.20436	-0.12413	0.00768	-0.05079
0.0780 - 0.1210	-0.17415	0.14953	0.14027	2.19945	0.01019	0.61694
0.1220 - 0.1590	0.10804	0.14357	0.13981	1.79289	0.01594	0.50108
0.1600 - 0.2000	0.41668	0.12277	0.12834	-1.45041	0.00218	-0.37229
0.2100 – 0.2520	0.25965	0.14868	-0.19723	0.03563	0.01578	-0.01430
0.2530 – 0.3100	0.32481	0.15170	0.13555	0.77530	0.00905	0.21010

Table 4.4.1: Calculated Heston parameters.

It is obvious that empirical method cannot be used for calculation of parameters in the Heston model How it is shown in the Table 4.4.1 use of the empirical method leads to inconsistent results with the Heston model's assumption. For example, it is known that κ and θ have to be always positive, in this case there were found some cases were the mean reversion and long-run variance were negative. Additionally, Feller condition is not fulfilled in many cases. Thus, calibration is most sufficient way for the parameters estimation of Heston model.

4.5 CONCLUSION

The main focus of my Master Thesis is Heston model option prices estimation and it's testing during periods of economical crisis and economical stability. Comparing to the calculated Black-Scholes option prices, the main results showed that the Heston model performs very well in all cases. One of the differences is that during the period of economical crisis the errors between the market and model options prices are a bit higher than during the period of economical stability. It can be explained by the high market uncertainty and extremely high and unexpected jumps in option prices during the economical crisis. The following figures 4.5.1 and 4.5.2 summarize and illustrate the results:

MATURITY	AVHES er	minHES er	maxHES er	AVBS er	minBS er	maxBS er
0.0190 - 0.0770	0.5376%	0.0529%	0.8864%	1.6667%	0.0034%	4.8274%
0.0780 - 0.1210	0.4756%	0.0059%	0.9753%	1.4628%	0.2925%	4.2750%
0.1220 - 0.1590	0.4447%	0.0107%	0.8997%	0.9759%	0.1040%	1.8166%
0.1600 - 0.2000	0.4181%	0.0517%	0.8498%	0.6026%	0.0177%	1.8144%
0.2100 - 0.2520	0.4508%	0.0476%	0.8676%	0.6450%	0.0173%	1.5830%
0.2530 - 0.3100	0.3693%	0.0429%	0.9095%	0.3792%	0.0017%	0.8684%

Figure 4.5.1: Average, minimum and maximum errors of Heston and Black-Scholes approximations comparing to market prices during the period of economical stability.

MATURITY	AVHES er	minHES er	maxHES er	AVBS er	minBS er	maxBS er
0.0190 - 0.0770	1.1937%	0.0088%	2.9568%	1.9913%	0.3220%	6.7432%
0.0780 - 0.1210	1.3488%	0.1760%	2.8882%	1.4420%	0.0326%	2.9930%
0.1215 - 0.1590	1.3768%	0.0046%	2.9696%	1.4008%	0.0383%	2.9049%
0.1600 - 0.2203	1.4740%	0.1281%	2.9400%	1.4781%	0.1140%	2.9329%
0.2570 - 0.3022	2.2621%	1.3978%	2.8925%	2.3218%	1.3990%	2.8913%

Figure 4.5.2: Average, minimum and maximum errors of Heston and Black-Scholes approximations comparing to market prices during the period of economical crisis.

The empirical analysis has also showed that the errors behave in a different way during periods of economical crisis and economical stability. They decrease when the time to maturity of options increases during the period of economical stability and increase when the time to maturity, as well, increases during the period of economical crisis. The following figures 4.5.3 and 4.5.4 illustrate the main results:

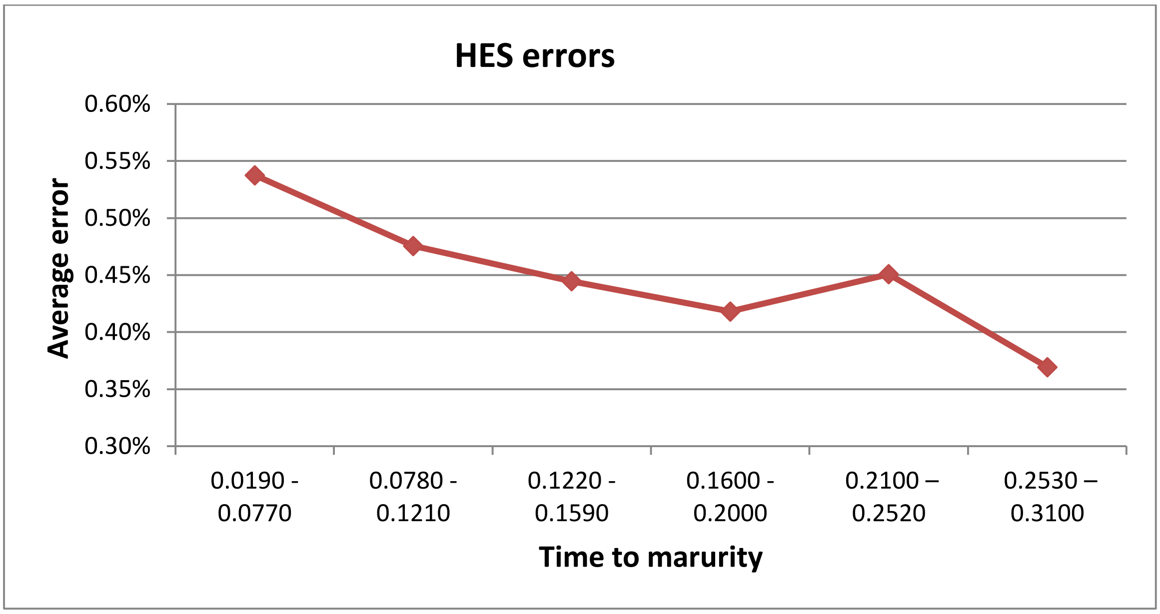

Figure 4.5.3: Average Heston model errors during the period of economical stability.

It happens due to the fact that unexpected high price jumps in the short time affect more the average performance of the model than in the long time. Every single non-trivial jump in short time can significantly change the whole average performance of the model as it becomes harder to fit it to the market data. Thus, the average model error is increasing.

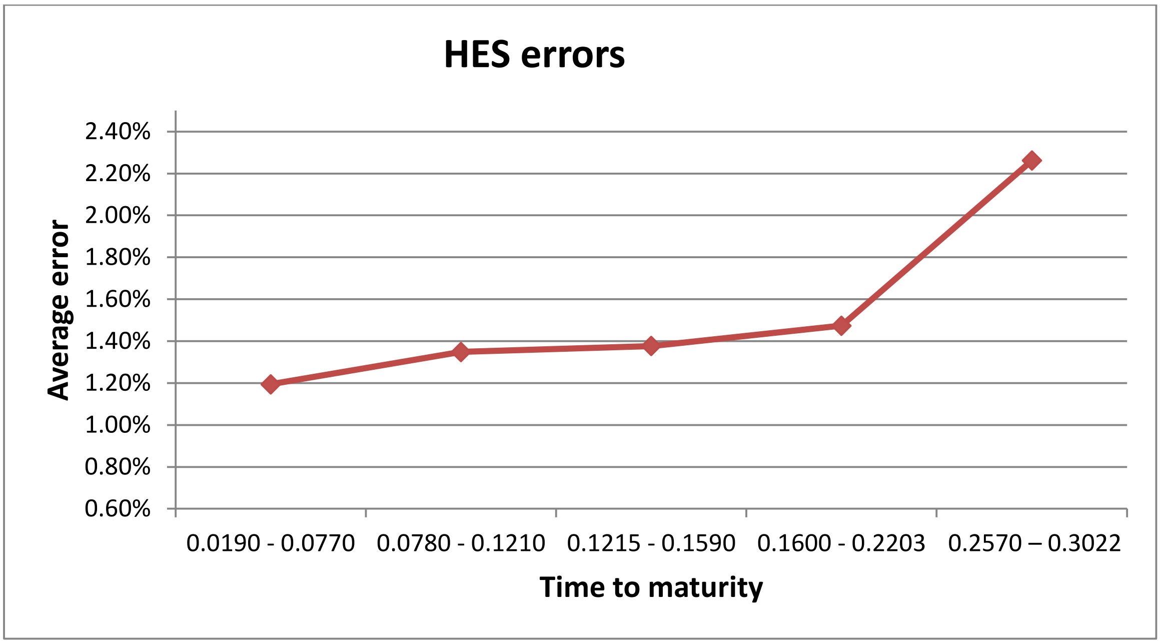

Figure 4.5.4: Average Heston model errors during the period of economical crisis.

The options with short time to maturity are affected less than the options with long time to maturity, since during the short time less unexpected jumps happen; consequently the volatility is lower than the volatility of options with long time to maturity. Thus, it is easier to fit the model more sufficiently in short time to maturity, because the average performance of the model errors is affected less than the average errors in the case of options with the long time to maturity.

The Black-Scholes model performs in a similar way. The difference which was observed between Heston and Black-Scholes is connected to the performance of models' errors. The Heston model errors are distributed similarly in the same range through the whole sample. It can be explained in the framework of the calibration because the main task of the calibration is to find a parameter which would satisfy the whole sample and not every single observation. In Black-Scholes model, the volatility estimation is based on the history of the underlying asset returns. Consequently, at the beginning of the period the model errors present more aggressive jumps. The following figures 4.5.5 and 4.5.6 present the results:

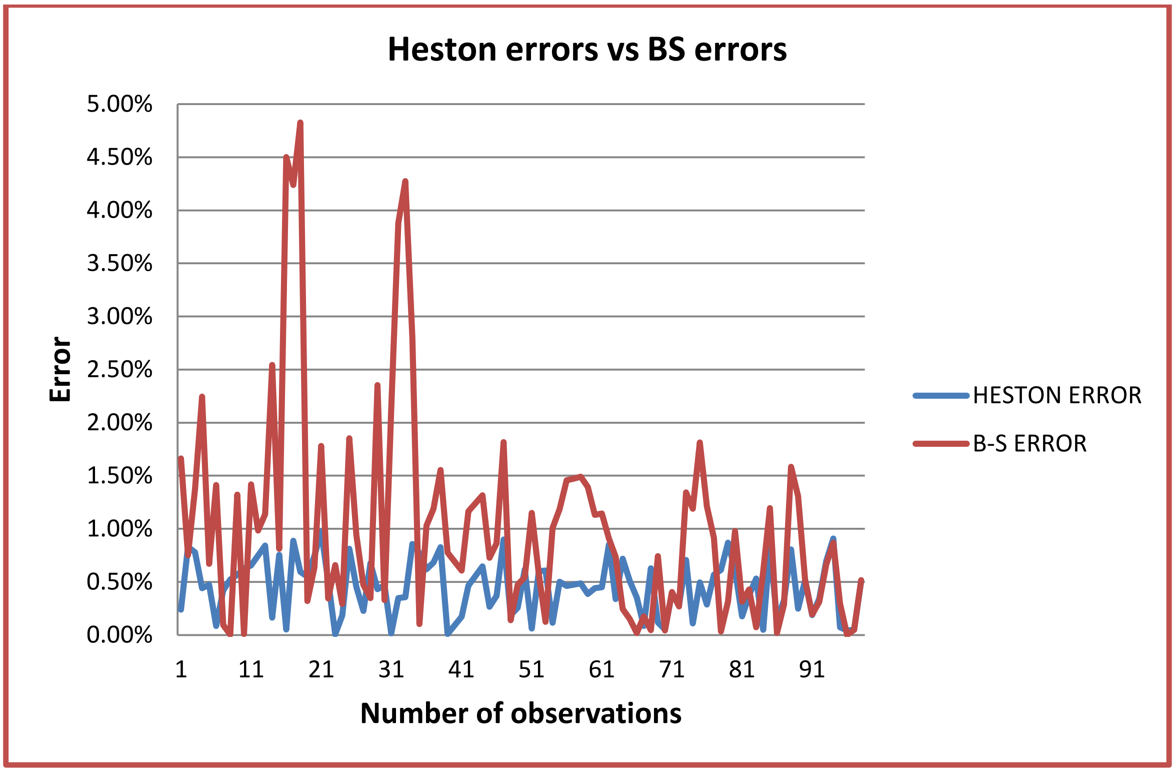

Figure 4.5.5: Comparison of Heston errors to BS errors during the period of economical stability.

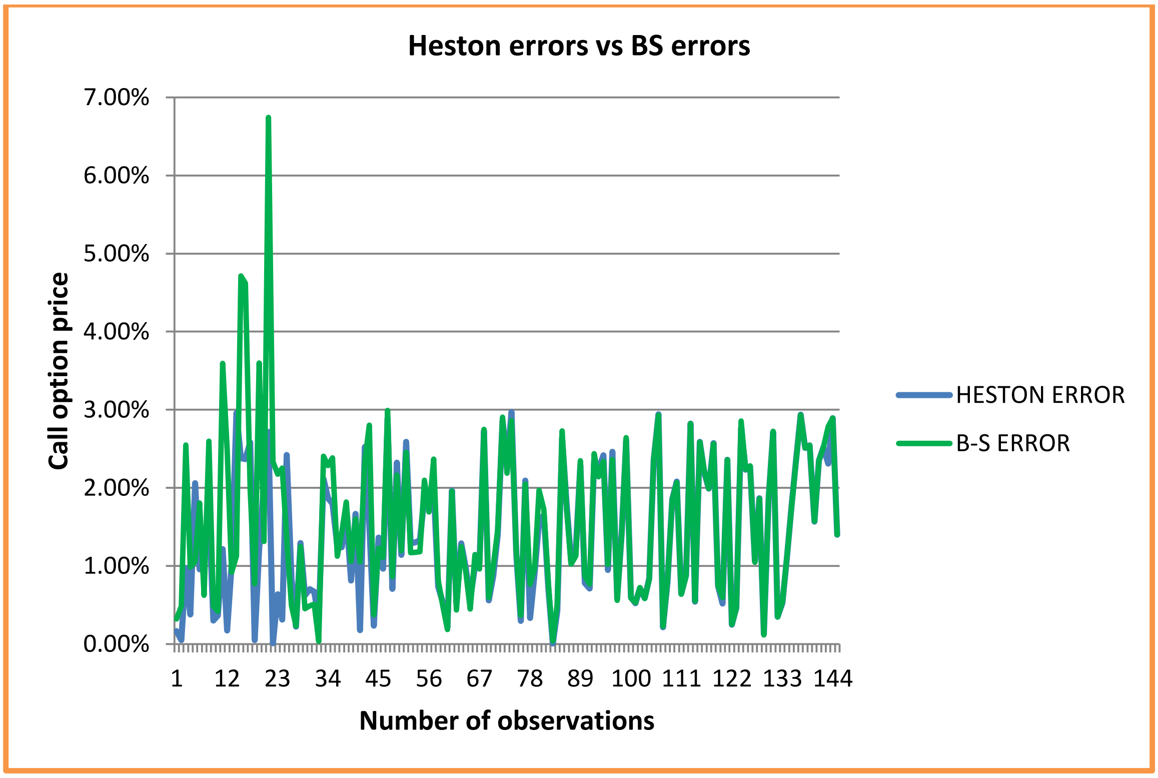

Figure 4.5.6: Comparison of Heston errors to BS errors during the period of economical crisis.

Additionally, I wanted to estimate the model parameters empirically and find out if I could use it for calculation of Heston model option prices. It turned out that this method is not sufficient and produces parameters which contradict the model's assumptions. Moreover, the parameters do not fulfill the Feller condition. The following parameters were obtained:

MATURITY	ρ		θ	κ	σ	Feller condition
0.0190 - 0.0770	-0.62397	0.12771	0.20436	-0.12413	0.00768	-0.05079
0.0780 - 0.1210	-0.17415	0.14953	0.14027	2.19945	0.01019	0.61694
0.1220 - 0.1590	0.10804	0.14357	0.13981	1.79289	0.01594	0.50108
0.1600 - 0.2000	0.41668	0.12277	0.12834	-1.45041	0.00218	-0.37229
0.2100 – 0.2520	0.25965	0.14868	-0.19723	0.03563	0.01578	-0.01430
0.2530 – 0.3100	0.32481	0.15170	0.13555	0.77530	0.00905	0.21010

Figure 4.5.7: Calculated Heston parameters.

For the Heston model calibration I have chosen the "Heston little trap" interpretation. It is the newer version of the Heston model which still is not so wildly calibrated. It was interesting to work with it and to discover if its calibration makes sense and produces good results. Additionally, the "Heston little trap" is easier to implement. Unfortunately, there was found one negative issue about this model's interpretation. Sometimes when there is a case of deep in the money or deep out of the money options the model produces negative prices. Therefore, no implied volatility can be calculated and, consequently, no implied volatility smile can be illustrated. Thus, in chapter 3, in paragraph 3.7 I used the original Heston model for the sensitivity analysis of Heston parameters and illustration of implied volatility smiles.

Finally, I could observe many advantages and disadvantages of Heston model. The most important advantages are the existence of the closed-form solution, the mean-reverting volatility behavior and a great performance even during the period of economical crisis. In addition for all the positive sides, Heston model has many disadvantages. One of the most important is the model sensitivity to data selection and to small changes in the values of parameters. To get good results the data has to get through severe filtration rules and the choice of the initial parameters has to be done very carefully. This is the main reason why so many banks do not use this mode for pricing European style options, as it demands a lot of time for its implementation.

APPENDIX

MATLAB-CODE

Heston model ("the little Heston trap")

```
function call=HestonCall(r,taf, S0, K, rho, v0, theta, k, sigma)
warning off;
N=200;
Prob1=1/2+ 1/ pi*quadl(@Pr1 ,0 ,N ,[] ,[] ,r,taf,S0,k,rho, v0 ,theta, k, sigma);
Prob2=1/2+ 1/ pi*quadl(@Pr2 ,0 ,N ,[] ,[] ,r,taf,S0,k,rho, v0 ,theta, k, sigma);
call = Prob1*S0 - Prob2*K*exp(-r*taf);
function p = Pr1(om, r, taf, S0, K, rho, v0 ,theta,k, sigma)
p = real(exp(-i*om*log(K)).*CharFun(om-i,r,taf,S0,rho, v0 ,theta, k, sigma)./
(i*om*S0*exp(r*taf)));
function q = Pr2(om, r, taf, S0, K, rho, v0, theta, k, sigma)
q = real(exp(-i*om*log(K)).*CharFun(om,r,taf,S0,rho, v0 ,theta, k, sigma)./(i*om));
function HestonCF = CharFun(om,r,taf,S0,rho, v0 ,theta, k, sigma)
d = sqrt ((rho*sigma*i*om-k).^2 + sigma ^2*(i*om + om.^2));
g = (k-rho*sigma*i*om-d)./(k-rho*sigma*i*om+d);
A = i*om.*(log(S0)+r*taf);
B = theta*k/(sigma^2)*((k-rho*sigma*i*om- d)*taf-2*log((1-g.*exp(-d*taf))./(1 - g)));
C = v0/sigma^2*(k-rho*sigma*i*om-d).*(1-exp(-d* taf))./(1-g.*exp(-d*taf));
HestonCF = exp(A + B + C);

function Heston_results=final_price()
warning off ;
rho=…; v0=…; theta=…; k=…; sigma=…;
q=[rho, v0, theta, k, sigma]
v = xlsread('NAME_OF_THE_FILE.xlsx', …);
 c=size(v);
NoOfOptions=c(1,1);
for i =1: NoOfOptions
      Heston_call_prices(i)=HestonCall(v(i,1), v(i,2), v(i,3), v(i,4),  q(1), q(2), q(3), q(4),
q(5));
end
```

```
Heston_results= Heston_call_prices
```

Black-Scholes model (with characteristic function)

```
function CALL=BlackScholesCall(r, taf, S0, K, variance)
warning off;
N=200;
Prob1 = 0.5+1/pi*quad(@Pr1, 0, N, [], [], r, taf, S0, K, variance);
Prob2 = 0.5+1/pi*quad(@Pr2, 0, N, [], [], r, taf, S0, K, variance);
CALL=Prob1*S0-Prob2*K*exp(-r*taf);
function p=Pr1(om, r, taf, S0, K, variance)
p = real(exp(-i*log(K)*om).* CharFun(om-i,r,taf,S0,variance)./(i*om*S0*exp(r*taf)));
function p=Pr2(om,r,taf,S0,K,variance)
p = real(exp(-i*log(K)*om).* CharFun(om,r,taf,S0,variance)./(i*om));
function cf=CharFun(om,r,taf,S0,variance)
cf = exp (i*om*log(S0)+i*taf*r*om-0.5*taf*variance*(i*om+om.^2));

function BS_results = price()
warning off ;
variance=…;
v = xlsread('NAME_OF_THE_FILE.xlsx', …);
 c=size(v);
NoOfOptions=c(1,1);
for i =1: NoOfOptions
     BS_call_prices(i)=BlackScholesCall(v(i,1), v(i,2), v(i,3), v(i,4), variance);
end
BS_results= BS_call_prices
```

Calibration

```
function new_parameters=Calibration(parameter)
warning off ;
%parameter=[rho, v0, theta, k, sigma]
v=xlsread('NAME_OF_THE_FILE.xlsx', …);
```

```
 c=size(v);
NoOfOptions=c(1,1);
for i =1: NoOfOptions
    minimization_of_loss_function(i)=(HestonCall(v(i,1),v(i,2),v(i,3),v(i,4),
parameter(1), parameter(2), parameter(3), parameter(4), parameter(5))-v(i,5));
end
new_parameters= minimization_of_loss_function
```

---

command line: lb=[-1,0,0,0,0]; ub=[0,1,1,10,1]; x0=['vector of initial parameters'];
lsqnonlin(@Calibration, x0, lb, ub)

Simulation of Stock prices

```
function stochastic(r,dt,S0, K, rho, v0, theta, k, sigma)
simulation = 1000;
for i=1:simulation
RN1=randn(1, simulation);
RN2=randn(1, simulation);
S=zeros(1, simulation);
v=zeros(1, simulation);
v=v0+k*(theta-v0)*dt+sigma*sqrt(v0)*RN1*sqrt(dt);
v=abs(v);
S=S0+r*S0*dt+S0*v.^(1/2).*(rho*RN1+sqrt(1-rho^2)*RN2)*sqrt(dt);
end
plot(S)
plot(v)
histfit(S,500,'lognorm')
set(l, 'LineWidth',2);
ylabel(' Lognormal distribution of Stock', 'fontweight', 'bold','fontsize',14);
```

Implied volatility smile

```
function fun=smile('sensitivity_of_parameter')
N=100;
K = linspace(80,120,N);
taf=1/2;
for i = 1:N
price(i)= OriginalHestonCall(r, taf, S0, K(i), rho, v0, theta, k, sigma);
end
for i=1:N
   Volatility(i) = blsimpv(S0, K(i), r, taf, price(i));
end
fun=Volatility;

function effect_on_smile()
A=smile('sensitivity_of_parameter');
B=smile('sensitivity_of_parameter');
C=smile('sensitivity_of_parameter');
x=linspace(80,120,100);
z=plot(x,A,x,B,'r', x,C,'g');
set(z,'LineWidth',2);
title('Implied volatility smile', 'fontweight','bold', 'fontsize',16);
ylabel(' Implied volatility', 'fontweight', 'bold','fontsize',16);
xlabel('Strike price', 'fontweight', 'bold','fontsize',16);
comment = legend('parameter=…', 'parameter=…', 'parameter=…');


function Call=OriginalHestonCall(r, taf, S0, K, rho, v0, theta, k, sigma)
warning off;
Call = S0*Prob(r, taf, S0, K, rho, v0, theta, k, sigma,1)-K*exp(-r*taf)*Prob(r, taf, S0, K, rho, v0, theta, k, sigma, 0)
function integration=Prob(r, taf, S0, K, rho, v0, theta, k, sigma,n)
N=200;
integrand=1/2+1/pi*quadl(@Real_part,0,N,[],[],r,taf,S0,K,rho, v0 ,theta, k, sigma,n);
```

```
function q = Real_part(om, r, taf, S0, K, rho, v0, theta, k, sigma, n)
q = real(exp(-1i*om*log(K)).*CharFun(om,r,taf,S0,rho,v0, theta, k, sigma,n)./(i*om));
function HestonCF=CharFun(om, r, taf, S0, rho, v0, theta, k, sigma, n)
if n ==1
u = 1/2;
b = k - rho*sigma;
else
u = -1/2;
b = k;
end
a = k*theta;
d = sqrt((rho*sigma*om.*i-b).^2-sigma^2*(2*u*om.*i-om.^2));
g = (b-rho*sigma*om*i+d)./(b-rho*sigma*om*i-d);
C = r*om.*i*taf + a/sigma^2.*((b- rho*sigma*om*i + d)*taf-2*log((1-g.*exp(d*taf))./(1-
g)));
D = (b-rho*sigma*om*i+d)./sigma^2.*((1-exp(d*taf))./(1-g.*exp(d*taf)));
HestonCF = exp(C+D*v0+i*om*log(S0));
```

Brownian motion

```
N = 1000;
z =randn(1,N);
plot(z)
x = cumsum(z);
plot(x)
ylabel('position');
xlabel('time step');
title('Position of 1D Particle versus Time');
```

Curriculum Vitae

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Education

- **October 2011- June 2014:** University of Vienna. Master in Quantitative Economics, Management and Finance with specialization in Finance.
- **Language of studying:** English
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- **Language of studying:** English
- **September 2007- February 2011:** University of Crete. Bachelor in Applied Mathematics with specialization in Financial Mathematics.
- **Language of studying:** Greek

Work experience

- **October 2013-until today:** Freelancer in Raiffeisen Bank International. Department of Risk Management, division of Risk Controlling (Country: Austria, Vienna)
- **July 2011-September 2011, November 2011-January 2012:** Employee of Tax Accounting Office "ΝΔ" (Country: Greece)
- **Winter semester 2010-2011:** Assistant in the course of "Introduction to Applied Mathematics I - Ordinary Differential Equations"
- **Spring semester 2009-2010:** Assistant in the course of "Introduction to Applied Mathematics II - Partial Differential Equations"

Language

- Greek (native)
- English (advanced)

- Russian (advanced)
- German (B1 level)

Other Knowledge

- Microsoft Office
- Matlab
- R
- SQL
- Bloomberg
- Thomson Reuters

Hobby

- Playing piano
- Chorus
- Opera
- Theater

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