DISSERTATION

Titel der Dissertation

"Holomorphic Mappings of Hyperquadrics from $\mathbb{C}^2$ to $\mathbb{C}^3$"

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Abstract

We investigate a specific aspect of a classical problem in the theory of holomorphic mappings between real submanifolds in complex spaces. Poincaré observed that it is in some sense unlikely for two arbitrary given real-analytic real submanifolds to find a holomorphic mapping which sends one into the other. An interesting class studied in this direction is the class of Levi-nondegenerate submanifolds, which was considered in the pioneering works of Cartan, Tanaka and Chern–Moser. Here the simplest examples are Levi-nondegenerate hyperquadrics, which serve as models for Levi-nondegenerate hypersurfaces. After Pinčuk's and Alexander's work dealing with the equidimensional case, Webster's rigidity result constitutes the first step in the study of immersions of spheres contained in complex spaces of different dimensions. More precisely, Webster considered holomorphic maps from the sphere in $\mathbb{C}^n$ to the sphere in $\mathbb{C}^{n+1}$ for $n \geq 3$, and showed that all holomorphic mappings are equivalent to the linear embedding with respect to the groups of automorphisms of the spheres. For $n = 2$ this rigidity fails and Faran proved that there are four classes of holomorphic mappings from the sphere in $\mathbb{C}^2$ to the sphere in $\mathbb{C}^3$ modulo equivalence. More recently, Lebl considered holomorphic mappings from the sphere in $\mathbb{C}^2$ to the hyperquadric of signature $(2, 1)$ in $\mathbb{C}^3$. In this case there are seven classes of holomorphic mappings up to the biholomorphic equivalence mentioned before. With Lebl's results the missing case of mappings of Levi-nondegenerate hyperquadrics in dimension two and three was established.

The present work consists of two parts. In the first part we give a new proof of Faran's and Lebl's results by means of a new CR-geometric approach and classify all holomorphic mappings from the sphere in $\mathbb{C}^2$ to Levi-nondegenerate hyperquadrics in $\mathbb{C}^3$. We use the tools developed by Lamel, which allow us to isolate and study the most interesting class of holomorphic mappings. This family of so-called nondegenerate and transversal maps we denote by $\mathcal{F}$. For $\mathcal{F}$ we introduce a subclass $\mathcal{N}$ of maps which are normalized with respect to the group $\mathcal{G}$ of automorphisms fixing a given point. With the techniques introduced by Baouendi–Ebenfelt–Rothschild and Lamel we deduce a classification of $\mathcal{N}$. This intermediate result is of twofold importance: On the one hand, if we consider the transitive part of the automorphism group of the hyperquadrics, we obtain a complete classification of $\mathcal{F}$ to show Faran's and Lebl's results. On the other hand our classification of $\mathcal{N}$ allows us to prove new topological results for $\mathcal{F}$, which yield the second part of our work. We demonstrate that from a topological point of view there is a major difference between the class of mappings of the spheres and mappings of the sphere in $\mathbb{C}^2$ to the hyperquadric with signature $(2, 1)$ in $\mathbb{C}^3$. In the first case $\mathcal{F}$ modulo the groups of automorphisms is discrete in contrast to the second case where this property fails to hold. Furthermore we study some basic properties such as freeness and properness of the action of $\mathcal{G}$ on $\mathcal{F}$. Finally we obtain a structural result for a particularly interesting subset of $\mathcal{F}$ using the real-analytic version of the local slice theorem for free and proper actions.

Zusammenfassung

In dieser Arbeit untersuchen wir einen Aspekt eines klassischen Problems in der Theorie holomorpher Abbildungen zwischen reellen Teilmannigfaltigkeiten in komplexen Räumen. Poincaré bemerkte, dass es in einem gewissen Sinne unwahrscheinlich ist, für zwei beliebig gegebene reell-analytische reelle Teilmannigfaltigkeiten eine holomorphe Abbildung zu finden, welche die eine in die andere überführt. Unter diesem Gesichtspunkt wurde vor allem eine interessante Klasse, nämlich Levi-nichtdegenerierte Teilmannigfaltigkeiten, von Cartan, Tanaka und Chern–Moser studiert. Als einfache Beispiele treten hierbei Levi-nichtdegenerierte Hyperquadriken auf, die als Modelle für Levi-nichtdegenerierte Hyperflächen dienen. Nachdem Pinčuk und Alexander den equidimensionalen Fall behandelt hatten, gelang es Webster mit seinem Rigiditätssatz Immersionen von Sphären in komplexen Räumen unterschiedlicher Dimension zu beschreiben. Webster betrachtete holomorphe Abbildungen zwischen der Sphäre in $\mathbb{C}^n$ und der Sphäre in $\mathbb{C}^{n+1}$ für $n \geq 3$ und konnte zeigen, dass alle holomorphen Abbildungen äquivalent zur linearen Einbettung sind bezüglich der Gruppen der Automorphismen der Sphären. Für $n = 2$ gibt es keine solche Rigidität, denn Faran konnte zeigen, dass es vier Klassen von holomorphen Abbildungen von der Sphäre in $\mathbb{C}^2$ und der Sphäre in $\mathbb{C}^3$, modulo Äquivalenz, gibt. Kürzlich studierte Lebl holomorphe Abbildungen von der Sphäre in $\mathbb{C}^2$ und der Hyperquadrik mit Signatur $(2,1)$ in $\mathbb{C}^3$. Er bewies, dass es sieben Klassen von holomorphen Abbildungen bezüglich der vorher beschriebenen biholomorphen Äquivalenz gibt. Dieses Resultat von Lebl vollendete die Klassifizierung holomorpher Abbildungen zwischen Levi-nichtdegenerierten Hyperquadriken in den Dimensionen zwei und drei.

Die vorliegende Arbeit besteht aus zwei Teilen. Im ersten Teil wird ein ein neuer Beweis von Faran's und Lebl's Resultat mittels eines neuen CR-geometrischen Zugangs gegeben. Wir klassifizieren alle holomorphen Abbildungen von der Sphäre in $\mathbb{C}^2$ und Levi-nichtdegenerierten Hyperquadriken in $\mathbb{C}^3$. Dazu werden Resultate von Lamel verwendet, die es uns erlauben unsere Untersuchungen auf eine spezielle Klasse von holomorphen Abbildungen einzuschränken. Diese Familie von sogenannten nichtdegenerierten und transversalen Abbildungen werden wir mit $\mathcal{F}$ bezeichnen. Für $\mathcal{F}$ geben wir eine Unterklasse $\mathcal{N}$ von Abbildungen an, die, bezüglich der Gruppe $\mathcal{G}$ von Automorphismen welche einen gegebenen Punkt fixieren, normalisiert sind. Vermöge der Techniken von Baouendi–Ebenfelt–Rothschild und Lamel erhalten wir eine Klassifikation von $\mathcal{N}$, welche von doppelter Bedeutung ist. Einerseits erhalten wir eine vollständige Klassifizierung von $\mathcal{F}$ und reproduzieren die Resultate von Faran und Lebl, wenn wir den transitiven Teil der Automorphismen der Hyperquadriken verwenden. Andererseits erlaubt es unsere Klassifikation von $\mathcal{N}$ neue topologische Resultate für $\mathcal{F}$ im zweiten Teil der Arbeit zu beweisen. Wir zeigen, dass es von einem topologischen Standpunkt aus gesehen einen bedeutenden Unterschied zwischen der Klasse der Abbildungen der Sphären und der Abbildungen zwischen der Sphäre in $\mathbb{C}^2$ und der Hyperquadrik mit Signatur $(2,1)$ gibt. Im ersten Fall ist $\mathcal{F}$ modulo der Gruppen der Automorphismen diskret, im Gegensatz zum zweiten Fall. Weiters studieren wir Eigenschaften wie Freiheit und Eigentlichkeit der Aktion von $\mathcal{G}$ auf $\mathcal{F}$. Schließlich erhalten wir ein strukturelles Resultat für eine interessante Teilmenge von $\mathcal{F}$, bei dem wir eine reell-analytische Version des lokalen Slice-Theorems für freie und eigentliche Aktionen verwenden.

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1 Introduction and Results

Poincaré [Poi07] asked whether for two given real-analytic real hypersurfaces in $\mathbb{C}^2$ one can find holomorphic mappings sending one into the other. He also gave an intuitive answer, originally for biholomorphisms, that for two given arbitrary real-analytic hypersurfaces in general it is in some sense unlikely to find holomorphic mappings sending locally one hypersurface into the other. We also note that for real-analytic mappings of real-analytic hypersurfaces Poincaré's question is trivial by the real-analytic Implicit Function Theorem.

Considerable work was done classifying Levi-nondegenerate hypersurfaces of $\mathbb{C}^N, N \geq 2$ up to biholomorphisms: In $\mathbb{C}^2$, this "biholomorphic equivalence problem" was solved by Cartan [Car33, Car32] and for $N \geq 2$ by Tanaka [Tan62] and Chern–Moser [CM74].

In the class of strictly pseudoconvex hypersurfaces Poincaré's question is answered by this classification of Levi-nondegenerate hypersurfaces and results by Pinčuk [Pin74] and Alexander [Ale74, Ale77]. They proved that any holomorphic self-mapping of a strictly pseudoconvex hypersurface in $\mathbb{C}^N$ is necessarily an automorphism. This implies that a holomorphic mapping of two biholomorphically equivalent strictly pseudoconvex hypersurfaces $M_1, M_2 \in \mathbb{C}^N$ is given by the composition of the biholomorphism sending M_1 to M_2 and an automorphism of M_2. Hence we note that the class of holomorphic mappings between two arbitrary given strictly pseudoconvex hypersurfaces is small in some sense.

For $N' > N$ and a mapping $H : \mathbb{C}^N \to \mathbb{C}^{N'}$ we refer to the number $N' - N$ as *codimension*. If we consider holomorphic mappings in high codimension the situation changes drastically. Here models of Levi-nondegenerate hypersurfaces, i.e., hyperquadrics received reasonable attention. For $k \in \mathbb{N}$ we denote the *hyperquadric* $\mathbb{S}_k^N$ *of signature* $(k, N-k)$ in $\mathbb{C}^N$ by

$$\mathbb{S}_k^N \coloneqq \big\{(z_1, \ldots, z_N) \in \mathbb{C}^N : \ |z_1|^2 + \ldots + |z_k|^2 - |z_{k+1}|^2 - \ldots - |z_N|^2 = 1\big\}, \tag{1.1}$$

and write $\mathbb{S}^N \coloneqq \mathbb{S}_N^N$ for the *sphere* in $\mathbb{C}^N$. Studying holomorphic mappings of hyperquadrics it is natural to introduce the following equivalence relation, see [Far82, §2] and [Leb11a, sections 3.4-3.5]: We consider the homogeneous model $\hat{\mathbb{S}}_k^N$ of $\mathbb{S}_k^N$ given by

$$\hat{\mathbb{S}}_k^N \coloneqq \big\{(z_1, \ldots, z_N, t) \in \mathbb{C}^{N+1} : \ |z_1|^2 + \ldots + |z_k|^2 - |z_{k+1}|^2 - \ldots - |z_N|^2 - |t|^2 = 0\big\}. \tag{1.2}$$

Let us denote by $SU(N-k, k+1)$ the special unitary group with respect to the Hermitian form in $\mathbb{C}^{N+1}$ with signature $(N-k, k+1)$ induced by the quadratic form which occurs in (1.2). The group of automorphisms of $\hat{\mathbb{S}}_k^N$ is $SU(N-k, k+1)/K$, where K is the subgroup of $SU(N-k, k+1)$ consisting of diagonal matrices with all entries being equal to ζ a $(N+1)$-root of unity, see e.g. [BER00, §2].

Let $V \subset \mathbb{C}^N$ be an open neighborhood of $p \in \mathbb{S}_k^N$. Any holomorphic mapping $H : V \to \mathbb{C}^{N'}$ which satisfies $H(V \cap \mathbb{S}_k^N) \subset \mathbb{S}_{k'}^{N'}$ can be identified with a CR-mapping $\hat{H} : \hat{V} \subset \mathbb{C}^{N+1} \to \mathbb{C}^{N'+1}$ for some open neighborhood $\hat{V}$ of $\hat{p} \in \hat{\mathbb{S}}_k^N$ satisfying $\hat{H}(\hat{V} \cap \hat{\mathbb{S}}_k^N) \subset \hat{\mathbb{S}}_{k'}^{N'}$. We say that two holomorphic mappings H_1, H_2, which both satisfy $H_m : \mathbb{C}^N \supset V_m \to \mathbb{C}^{N'}$, where V_m is a neighborhood of $p_m \in \mathbb{S}_k^N$, such that $H_m(V_m \cap \mathbb{S}_k^N) \subset \mathbb{S}_{k'}^{N'}$ for $m = 1, 2$, are *equivalent* if there exist matrices $U \in SU(N-k, k+1)$ and $U' \in SU(N'-k', k'+1)$ such that $\hat{H}_2 = U' \circ \hat{H}_1 \circ U$. We give other ways of defining the equivalence relation for holomorphic mappings of hyperquadrics in Definition 2.26, Definition 2.27 and Definition 6.3 below.

If $N' \geq 2N$ D'Angelo [D'A88] shows that there are infinitely many quadratic mappings from $\mathbb{S}^N$ to $\mathbb{S}^{N'}$ which are not equivalent. In small codimensions the family of holomorphic mappings is less richer. Webster [Web79] proved that for holomorphic mappings between the spheres in $\mathbb{C}^N$ and $\mathbb{C}^{N+1}$, where $N \geq 3$, there is only one equivalence class, namely the one containing the linear embedding. Faran [Far86] and Huang [Hua99] extended this result to holomorphic mappings of $\mathbb{S}^N$ to $\mathbb{S}^{N'}$ with $N \geq 3$ and $N' \leq 2N-2$. The case of mappings from $\mathbb{S}^N$ to $\mathbb{S}^{2N-1}$ for $N \geq 3$ is covered by Huang–Ji [HJ01], where they show that there exist two classes of mappings which are not equivalent.

We would like to point out the study of Poincaré's question in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ if $N' < N$ is trivial for non-constant mappings of spheres, since there are none. This can be seen as in [Leb11a, Proposition 3.1.4]: if we let $N, N' \in \mathbb{N}$ be arbitrary, $p \in \mathbb{S}^N$ and $U \subset \mathbb{C}^N$ be an open and connected neighborhood of p such that a holomorphic mapping $H: U \to \mathbb{C}^{N'}$ satisfies $H(U \cap \mathbb{S}^N) \subset \mathbb{S}^{N'}$ and $p' = H(p)$. Let $\mathbb{B}^N \coloneqq \{(z_1, \ldots, z_N) \in \mathbb{C}^N : |z_1|^2 + \ldots + |z_N|^2 < 1\}$ denote the *ball* in $\mathbb{C}^N$, which possesses $\mathbb{S}^N$ as its boundary. Then H considered as a mapping $H: U \cap \mathbb{B}^N \to V \cap \mathbb{B}^{N'}$ for a sufficiently small open and connected neighborhood $V \subset \mathbb{C}^{N'}$ of p' is a proper mapping, see [D'A93, Chapter 1, Lemma 1], which implies that $H^{-1}(q')$ is a compact subset of $\mathbb{B}^N$ for $q' \in V \cap \mathbb{B}^{N'}$. The Rank Theorem, see e.g. [Rud76, Theorem 9.32], yields that $H^{-1}(q')$ is a complex variety at least of dimension $N' - N > 0$.

The situation differs again if we consider holomorphic mappings between hyperquadrics with signature $(\ell, N-\ell)$ and $(\ell', N'-\ell')$ with $3 \leq N' < N$ and $0 < N' - \ell' < \ell'$. For this purpose we write $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and let $F: \mathbb{C}^N \to \mathbb{C}^{N'}$ be the holomorphic mapping given by

$$z \mapsto \big(h_1(z), \ldots, h_{N'-\ell'}(z), 0, \ldots, 0, 1, h_1(z), \ldots, h_{N'-\ell'}(z)\big),$$

for some holomorphic functions $h_j, j = 1, \ldots, N' - \ell'$. Note that the constant 1 in the definition of F occurs in the ℓ'-th component. Then F sends $\mathbb{S}^N_\ell$ to $\mathbb{S}^{N'}_{\ell'}$. A similar construction works for the case $0 < N' - \ell' = \ell'$ to obtain non-constant holomorphic mappings from $\mathbb{S}^N_\ell$ to $\mathbb{S}^{N'}_{\ell'}$.

Moreover there are no holomorphic mappings from $\mathbb{S}^N_\ell$ with $N - \ell > 0$ to $\mathbb{S}^{N'}$, since there are no complex varieties contained in $\mathbb{S}^{N'}$.

In order to discuss the case of holomorphic mappings between hyperquadrics in $\mathbb{C}^2$ and $\mathbb{C}^3$ in a more detailed exposition we introduce the hypersurface $\mathbb{S}^3_\varepsilon$, which for $\varepsilon = \pm 1$ is given by

$$\mathbb{S}^3_\pm \coloneqq \big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : \ |z_1|^2 + |z_2|^2 \pm |z_3|^2 = 1\big\},$$

so that $\mathbb{S}^3 = \mathbb{S}^3_+$. In fact, $\mathbb{S}^2$ and $\mathbb{S}^3_\varepsilon$ are the only Levi-nondegenerate hyperquadrics in $\mathbb{C}^2$ and $\mathbb{C}^3$, respectively, since every Levi-nondegenerate hyperquadric can be mapped biholomorphically to one of these hypersurfaces as is well known, see e.g. the argument given in Remark 2.8 below.

Before we discuss the most interesting case of holomorphic mappings of $\mathbb{S}^2$ to $\mathbb{S}^3_\varepsilon$ we briefly discuss the cases of mappings from $\mathbb{S}^3_-$ to $\mathbb{S}^3_-$ and $\mathbb{S}^3_-$ to $\mathbb{S}^2$. In the first case we refer to [BH05, Theorem 1.6], which says that under equivalence the only holomorphic mappings in this case are linear embeddings or the mapping is of the form $z \mapsto (1, h(z), h(z))$, for $z \in \mathbb{S}^3_-$ and some holomorphic function h. In the second case, where we map $\mathbb{S}^3_-$ to $\mathbb{S}^2$, there are no non-constant holomorphic mappings, as we argued above or we can verify directly from the mapping equation.

Faran [Far82] classified holomorphic mappings between balls in $\mathbb{C}^2$ and $\mathbb{C}^3$ with certain boundary regularity. Below we formulate the Main Theorem of [Far82] in terms of mappings between spheres disregarding regularity issues.

Theorem 1.1 (Faran [Far82]). *Let $p \in \mathbb{S}^2$, $U \subset \mathbb{C}^2$ be an open and connected neighborhood of p and $F : U \to \mathbb{C}^3$ a non-constant holomorphic mapping satisfying $F(U \cap \mathbb{S}^2) \subset \mathbb{S}^3$. Then F is equivalent to exactly one of the following maps:*

(i) $F_1(z, w) = (z, w, 0)$
(ii) $F_2(z, w) = (z, zw, w^2)$
(iii) $F_3(z, w) = (z^2, \sqrt{2}zw, w^2)$
(iv) $F_4(z, w) = (z^3, \sqrt{3}zw, w^3)$

Faran's proof consists of giving a characterization of so-called planar maps from $\mathbb{C}^2$ to $\mathbb{C}^3$ which send complex lines to complex planes and uses Cartan's method of moving frames.

Cima–Suffridge [CS89] approached Faran's Theorem via a reflection principle deduced in [CS83] by the same authors, which contains some inconsistencies when using certain degeneracy conditions. More recently Ji [Ji10] gave a new proof of Faran's Theorem based on Huang's study [Hua99] of the Chern–Moser operator and several preceding articles [HJ01, Hua03, HJX06, CJX06]. In [Ji10] a small fixable mistake in the case distinction leads to the wrong mapping at the very end of the article.

Lebl [Leb11b] classified mappings sending $\mathbb{S}^2$ to $\mathbb{S}^3_-$, using a classification result for quadratic maps and Faran's approach:

Theorem 1.2 (Lebl [Leb11b]). *Let $p \in \mathbb{S}^2$, $U \subset \mathbb{C}^2$ be an open and connected neighborhood of p and $L : U \to \mathbb{C}^3$ a non-constant holomorphic mapping satisfying $L(U \cap \mathbb{S}^2) \subset \mathbb{S}^3_-$. Then L is equivalent to exactly one of the following maps:*

(i) $L_1(z, w) = (z, w, 0)$
(ii) $L_2(z, w) = \left(z^2, \sqrt{2}w, w^2\right)$
(iii) $L_3(z, w) = \left(\frac{1}{z}, \frac{w^2}{z^2}, \frac{w}{z^2}\right)$
(iv) $L_4(z, w) = \frac{\left(z^2+\sqrt{3}zw+w^2-z, w^2+z-\sqrt{3}w-1, z^2-\sqrt{3}zw+w^2-z\right)}{w^2+z+\sqrt{3}w-1}$
(v) $L_5(z, w) = \frac{\left(\sqrt[4]{2}(zw-\mathrm{i}\,z), w^2-\sqrt{2}\,\mathrm{i}\,w+1, \sqrt[4]{2}(zw+\mathrm{i}\,z)\right)}{w^2+\sqrt{2}\,\mathrm{i}\,w+1}$
(vi) $L_6(z, w) = \frac{\left(2w^3, z(z^2+3), \sqrt{3}w(z^2-1)\right)}{3z^2+1}$
(vii) $L_7(z, w) = \left(1, \ell(z, w), \ell(z, w)\right)$*, for an arbitrary holomorphic function $\ell : \mathbb{C}^2 \to \mathbb{C}$*

Let us now state our results and outline some intermediate steps in our work. The first and main part of this work is to provide a new proof of Theorem 1.1 and Theorem 1.2. The following Theorem is based on a very different approach than the one of Faran or Lebl and is independent of their proofs.

Theorem 1.3 (Main Theorem). *Let $p \in \mathbb{S}^2$, $U \subset \mathbb{C}^2$ be an open and connected neighborhood of p and $H : U \to \mathbb{C}^3$ a non-constant holomorphic mapping satisfying $H(U \cap \mathbb{S}^2) \subset \mathbb{S}^3_\varepsilon$. Then H is equivalent to exactly one of the following maps:*

(i) $H_1^{\varepsilon}(z,w) = (z,w,0)$
(ii) $H_2^{\varepsilon}(z,w) = \left(z^2, \frac{(1-\varepsilon+z(1+\varepsilon))w}{\sqrt{2}}, w^2\right)$
(iii) $H_3^{\varepsilon}(z,w) = \left(z, \frac{(1-\varepsilon+z^2(1+\varepsilon))w}{2z}, \frac{(1-\varepsilon+z(1+\varepsilon))w^2}{2z}\right)$
(iv) $H_4^{\varepsilon}(z,w) = \frac{\left(4z^3,(3(1-\varepsilon)+(1+3\varepsilon)w^2)w,\sqrt{3}(1-\varepsilon+2(1+\varepsilon)w+(1-\varepsilon)w^2)z\right)}{1+3\varepsilon+3(1-\varepsilon)w^2}$

Additionally for $\varepsilon=-1$ we have:
(v) $H_5(z,w) = \left(\frac{(2+\sqrt{2}z)z}{1+\sqrt{2}z+w}, w, \frac{(1+\sqrt{2}z-w)z}{1+\sqrt{2}z+w}\right)$
(vi) $H_6(z,w) = \frac{\left((1-w)z,1+w-w^2,(1+w)z\right)}{1-w-w^2}$
(vii) $H_7(z,w) = \big(1,h(z,w),h(z,w)\big)$ *for some non-constant holomorphic function $h: \mathbb{C}^2 \to \mathbb{C}$*

Further, H_3^- is equivalent to L_3, H_4^- to L_6, H_5 to L_4 and H_6 to L_5.

Before we give some more details and results for the proof of Theorem 1.3 let us mention some features of our approach. One advantage of our chosen method is, that we prove Faran's and Lebl's result in a unified manner, i.e., we treat mapping from $\mathbb{S}^2 \to \mathbb{S}^3$ and $\mathbb{S}^2 \to \mathbb{S}^3_-$ in the same way and use the same techniques for both situations.
Another aspect of our proof is to be in some sense computationally effective, meaning that our technique allows us to give explicit formulas for the automorphisms which bring an arbitrary mapping to one of the mappings listed in the Main Theorem. Moreover we provide a list of biholomorphic invariants associated to each mapping of the Main Theorem which also implies that all the maps in the Main Theorem are not equivalent to each other. Thus we think we can provide a new proof of Faran's and Lebl's results which is easier to verify and more elementary. Nevertheless our proof is long, technical and features some huge computations.
Now we provide some details of our proof: We introduce the class $\mathcal{F}_2$, which consists of germs of 2-nondegenerate transversal mappings. These notions are defined below in Definition 3.6 and Definition 3.1 respectively. For this class we give a normal form and denote the set of normalized mappings by $\mathcal{N}_2$. Then we prove a first local characterization in terms of automorphisms fixing a given point. The following theorem is formulated for holomorphic mappings in $\mathcal{N}_2$ from $\mathbb{H}^2$ to $\mathbb{H}^3_{\varepsilon}$, which are biholomorphic images of $\mathbb{S}^2$ and $\mathbb{S}^3_{\varepsilon}$ except one point and defined below in Definition 2.4.

Theorem 1.4. *The set $\mathcal{N}_2$ consists of the following mappings, where $s \geq 0$:*

$$
\begin{aligned}
G_1^{\varepsilon}(z,w) &:= \frac{\big(z(1+\mathrm{i}\,\varepsilon\, w), \sqrt{2}z^2, w\big)}{1-w^2},\\
G_{2,s}^{\varepsilon}(z,w) &:= \frac{\big(z-2\,\varepsilon\, sz^2+\mathrm{i}\big(\varepsilon-s^2\big)zw+2sw^2, 2\big(z^2+s^2w^2\big), w\big(1-2\,\varepsilon\, sz-\mathrm{i}(\varepsilon+s^2)w\big)\big)}{1-2\,\varepsilon\, sz-\mathrm{i}\big(\varepsilon+s^2\big)w-4\,\mathrm{i}\, szw-4\,\varepsilon\, s^2w^2},\\
G_{3,s}^{\varepsilon}(z,w) &:= \Big(16\,\varepsilon\, z+24\,\mathrm{i}\, zw+8\,\varepsilon\, sw^2+16z^3+8\,\mathrm{i}\,\varepsilon\, sz^2w+3\big(s^2-3\,\varepsilon\big)zw^2+2\,\mathrm{i}\, sw^3,\\
&\qquad 32\,\varepsilon\, z^2-8w^2+16sz^3+8\,\mathrm{i}\, z^2w-4\,\varepsilon\, szw^2-2\,\mathrm{i}\,\varepsilon\, w^3,\\
&\qquad w\big(16\,\varepsilon-8\,\mathrm{i}\, w+16z^2-8\,\mathrm{i}\,\varepsilon\, szw-\big(s^2+\varepsilon\big)w^2\big)\Big)\\
&\quad \Big/\Big(16\,\varepsilon-8\,\mathrm{i}\, w+16z^2-24\,\mathrm{i}\,\varepsilon\, szw-\big(9s^2+17\,\varepsilon\big)w^2+32\,\mathrm{i}\,\varepsilon\, z^2w+12szw^2+4\,\mathrm{i}\, w^3\Big).
\end{aligned}
$$

Each map in $\mathcal{N}_2$ is not equivalent to any different map of $\mathcal{N}_2$ with respect to automorphisms fixing 0.

For $\varepsilon = -1$ we have the following picture of $\mathcal{N}_2$ according to Theorem 1.4:

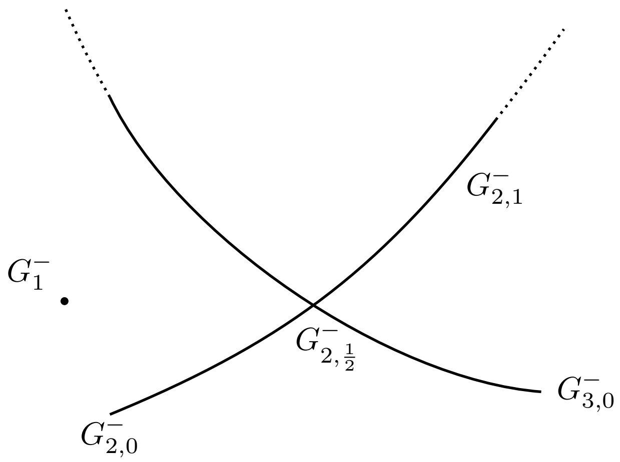

Figure 1: Picture of $\mathcal{N}_2$ for $\varepsilon = -1$

We choose certain values for s and define the following mappings:

$$\mathcal{G}^\varepsilon_1(z,w) := G^\varepsilon_{2,0}(z,w), \qquad \mathcal{G}^\varepsilon_2(z,w) := G^\varepsilon_{2,1/2}(z,w), \qquad \mathcal{G}^\varepsilon_3(z,w) := G^\varepsilon_{2,1}(z,w), \qquad \mathcal{G}^\varepsilon_4(z,w) := G^\varepsilon_{3,0}(z,w). \tag{1.3}$$

Non-isotropic automorphisms which we apply to the mappings $G^\varepsilon_{k,s}$ allow us to reduce the parameter s to finitely many values in the sense of the following theorem.

Theorem 1.5. *For $m = 2, 3$ and $1 \leq k \leq 4$ let $G^\varepsilon_{m,s}$ be as in Theorem 1.4 and $\mathcal{G}^\varepsilon_k$ as in (1.3).*
For $\varepsilon = +1$ we have:

(i) *For every $s \geq 0$ the mapping $G^+_{2,s}$ is equivalent to $\mathcal{G}^+_1$.*
(ii) *For every $s \geq 0$ the mapping $G^+_{3,s}$ is equivalent to $\mathcal{G}^+_4$.*

For $\varepsilon = -1$ we have:

(iii) *For every $0 \leq s < \frac{1}{2}$ the mapping $G^-_{2,s}$ is equivalent to $\mathcal{G}^-_1$.*
(iv) *For every $s > \frac{1}{2}$ the mapping $G^-_{2,s}$ is equivalent to $\mathcal{G}^-_3$.*
(v) *The mappings $\mathcal{G}^-_1, \mathcal{G}^-_2$ and $\mathcal{G}^-_3$ are pairwise not equivalent to each other.*
(vi) *For every $0 \leq s \neq 2$ the mapping $G^-_{3,s}$ is equivalent to $\mathcal{G}^-_4$ and $G^-_{3,2} = \mathcal{G}^-_2$.*

The mapping G^ε_1 is not equivalent to any of the mappings $\mathcal{G}^\varepsilon_k$.

The second part of our work, which is heavily based on the first part, deals with topological aspects of holomorphic mappings in our setting to provide new and profound insights into the topological and real-analytic structure of the set of holomorphic maps and the moduli space.
We denote the equivalence relation used in Theorem 1.3 by $\sim$, then by Theorem 1.4 and Theorem 1.5 the following result holds true:

Theorem 1.6. *The quotient space $\mathcal{F}_2/_\sim$ is discrete for $\varepsilon = +1$ and not discrete for $\varepsilon = -1$.*

The above result was not known before and stands in contrast to the case of the group of germs of real-analytic CR-diffeomorphisms fixing a point $p \in M$, denoted by $\mathrm{Aut}_p(M,p)$, for a germ of a real-analytic CR-submanifold (M,p) in $\mathbb{C}^N$. Assuming some nondegeneracy conditions for certain (M,p) it is shown that $\mathrm{Aut}_p(M,p)$ admits a Lie group structure, see [BER97], [BER99a], [BRWZ04], [Kow05], [KZ05], [LM07], [LMZ08] and [JL13].

Next we study the action of the group of automorphisms fixing a given point on the set of holomorphic maps. Let us denote by $G := \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0) \times \mathrm{Aut}_0(\mathbb{H}^2, 0)$, the direct product of the stability groups of $\mathbb{H}^2$ and $\mathbb{H}^2_\varepsilon$ respectively, with elements $g = (g_1, g_2) \in G$. We write $\mathfrak{F}_2 \subset \mathcal{F}_2$ for the set where the action of G on $\mathcal{F}_2$ given by $G \times \mathcal{F}_2 \to \mathcal{F}_2, (g_1, g_2, h) \mapsto g_1 \circ h \circ g_2^{-1}$, has only trivial stabilizers. Then the following results holds:

Theorem 1.7. *The mapping $N : G \times \mathfrak{F}_2 \to \mathfrak{F}_2$ given by*

$$N(\phi', \phi, H) := \phi' \circ H \circ \phi^{-1},$$

is a free and proper left action.

Based on this result we obtain the following result concerning the topological and real-analytic structure of $\mathfrak{F}_2$, where $\Pi : \mathfrak{F}_2 \to \mathfrak{N}_2$ denotes the normalization map induced by the mapping N and $\mathfrak{N}_2 \subset \mathcal{N}_2$ denotes a set of representatives of the quotient $\mathfrak{F}_2 /G$:

Theorem 1.8. (i) *If $\varepsilon = +1$ then $\Pi : \mathfrak{F}_2 \to \mathfrak{F}_2 /G$ is a real-analytic principal fibre bundle with structure group G.*

(ii) *If $\varepsilon = -1$ then locally $\mathfrak{F}_2$ is mapped to $G \times \mathfrak{N}_2$ via locally real-analytic diffeomorphisms. In particular $\mathfrak{F}_2$ is not a smooth manifold.*

This theorem allows us to obtain the following result for the different topologies we can associate to $\mathfrak{N}_2$. All relevant notions are introduced in section 9.

Theorem 1.9. *The quotient topology on $\mathfrak{N}_2$ coincides with the induced topology of $\mathfrak{F}_2$, which carries the topology induced by the jet space $J^3_0(\mathbb{H}^2, \mathbb{H}^3_\varepsilon)$.*

We organize this work as follows: In section 2 we compute all relevant automorphisms and introduce the precise notion of equivalence. The following section 3 introduces all biholomorphic invariants we use in order to obtain a class $\mathcal{F}_2$ of interesting mappings, more precisely 2-nondegenerate transversal mappings. For this class of mappings, we compute a normal form in subsection 4.1 and obtain $\mathcal{N}_2 \subset \mathcal{F}_2$, the set of normalized mappings with respect to the stability groups. We also discuss different suitable normal forms with respect to the stability group and their effects on the classification. For $\mathcal{N}_2$ we compute a jet parametrization in section 5 and after some desingularization it turns out that $\mathcal{N}_2$ consists of one separated mapping and two one-parameter families of mappings, denoted by C_1 and C_2. Then in section 6 we use the non-isotropic part of the automorphism groups to see how the families C_1 and C_2 are recovered from finitely many normalized mappings. For this purpose we study mappings at points, where the degeneracy is higher than at generic points in subsection 6.3. In section 7 we treat the case of degenerate mappings such that we are able to complete the proof of the Main Theorem in subsection 8.1. Finally in section 9 we consider topological questions related to Theorem 1.4, which provides topological information about $\mathcal{F}_2$ as well as the quotient spaces. The main effort is to prove that the application of the stability group gives a proper action on $\mathfrak{F}_2$ and that $\mathfrak{N}_2$ at least contains some manifold structure.

Our computations are carried out with *Mathematica 7.0.1.0* [Wol08].

2 Preliminaries

We start this section with a well-known fact concerning the *complexification* of real-analytic equations, see e.g. [D'A93, Chapter 1, Proposition 1]. For $U \subset \mathbb{C}^N$ we introduce the set $\bar{U}$ consisting of the conjugated elements of U.

Theorem 2.1 (Complexification). *Let $U \subset \mathbb{C}^N$ be open and connected and let $F : U \times \bar{U} \subset \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ be a holomorphic function such that $F(z, \bar{z}) = 0$ for all $z \in U$, then $F(z, \chi) = 0$ for all $(z, \chi) \in U \times \bar{U}$.*

Proof. First we set $V := \{(z, \bar{z}) : z \in U\} \subset \mathbb{C}^{2N}$ and choose coordinates $(z, \chi) = (x + \mathrm{i}\, y, u + \mathrm{i}\, v)$ in $\mathbb{C}^N \times \mathbb{C}^N$. Then $V = \{(z, \chi) \in \mathbb{C}^{2N} : \chi = \bar{z}\}$ such that V is given by $2N$ real defining functions

$$\begin{aligned} \rho_{2j+1}(x, y, u, v) &= u - x, \quad 0 \leq j \leq N - 1, \\ \rho_{2j}(x, y, u, v) &= v + y, \quad 1 \leq j \leq N. \end{aligned}$$

After linearly changing coordinates to $(\widetilde{x} + \mathrm{i}\, \widetilde{y}, \widetilde{u} + \mathrm{i}\, \widetilde{v}) = (\widetilde{z}, \widetilde{\chi}) := \varphi(z, \chi) = \big(\mathrm{i}(\chi - z), z + \chi\big)$ we have

$$\begin{aligned} \widetilde{\rho}_{2j+1}(\widetilde{x}, \widetilde{y}, \widetilde{u}, \widetilde{v}) &= \widetilde{y}, \quad 0 \leq j \leq N - 1, \\ \widetilde{\rho}_{2j}(\widetilde{x}, \widetilde{y}, \widetilde{u}, \widetilde{v}) &= \widetilde{v}, \quad 1 \leq j \leq N. \end{aligned}$$

as local defining functions for V. Our assumption $F|_V \equiv 0$ then becomes $\widetilde{F}(\widetilde{x}, \widetilde{u}) = 0$ for all $(\widetilde{x}, \widetilde{u}) \in \mathbb{R}^{2N}$ and $\widetilde{F} = F \circ \varphi^{-1}$. If we write $\widetilde{F}(\widetilde{z}, \widetilde{\chi}) = \sum_{\alpha,\beta} \widetilde{F}_{\alpha\beta} \widetilde{z}^\alpha \widetilde{\chi}^\beta$ we have

$$0 = \widetilde{F}(\widetilde{x}, \widetilde{u}) = \sum_{\alpha,\beta} \widetilde{F}_{\alpha\beta} \widetilde{x}^\alpha \widetilde{u}^\beta,$$

which implies $\widetilde{F}_{\alpha\beta} = 0$ for all α, β, hence $F_{\alpha\beta} = 0$ and the claim is proved. □

Definition 2.2 (Normal coordinates). For $n, n' \geq 1$ we denote by $Z = (z_1, \ldots, z_n, w) \in \mathbb{C}^{n+1}$ and $Z' = (z', w') = (z'_1, \ldots, z'_{n'}, w') \in \mathbb{C}^{n'+1}$ coordinates in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{n'+1}$ respectively.
We consider the complexification of a real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$, denoted by $\mathcal{M}$, where we write $\chi := \bar{z}$ and $\tau := \bar{w}$. Coordinates $(z, w) \in \mathbb{C}^{n+1}$ are called *normal coordinates* near 0, if there is $U \subset \mathbb{C}^{n+1}$ a neighborhood of 0 such that

$$\mathcal{M} \cap \{U \times \bar{U}\} = \big\{(z, w, \chi, \tau) \in U \times \bar{U} : w = Q(z, \chi, \tau)\big\},$$

where $Q : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ is holomorphic in a neighborhood of 0 satisfying

$$\tau = Q(z, 0, \tau) = Q(0, \chi, \tau), \qquad w = Q\big(z, \chi, \bar{Q}(\chi, z, w)\big).$$

Before we introduce our prototype example of hypersurfaces given in normal coordinates we need the following definition.

Definition 2.3. For $z, \zeta \in \mathbb{C}^n$ we define

$$\langle z, \zeta \rangle_k \coloneqq z_1\zeta_1 + \ldots + z_k\zeta_k - z_{k+1}\zeta_{k+1} - \ldots - z_n\zeta_n, \qquad ||z||_k^2 \coloneqq \langle z, \bar{z} \rangle_k.$$

In $\mathbb{C}^2$ we denote for $\varepsilon = \pm 1$

$$\langle z, \zeta \rangle_\varepsilon \coloneqq z_1\zeta_1 + \varepsilon\, z_2\zeta_2, \qquad ||z||_\varepsilon^2 \coloneqq \langle z, \bar{z} \rangle_\varepsilon.$$

The standard real euclidean inner product in $\mathbb{C}^n$ is denoted by

$$\langle z, \zeta \rangle \coloneqq z_1\zeta_1 + \ldots + z_n\zeta_n, \qquad ||z||^2 \coloneqq \langle z, \bar{z} \rangle.$$

Definition 2.4. For $(z, w) \in \mathbb{C}^{n+1}$ and $k \geq \frac{n}{2}$ we define

$$\rho_k(z, w, \bar{z}, \bar{w}) \coloneqq \operatorname{Im} w - ||z||_k^2,$$

and

$$\mathbb{H}_k^{n+1} \coloneqq \big\{ (z, w) \in \mathbb{C}^{n+1} : \ \rho_k(z, w, \bar{z}, \bar{w}) = 0 \big\}.$$

If $k = n$ we write $\mathbb{H}^{n+1} \coloneqq \mathbb{H}_{n+1}^n$. In $\mathbb{C}^2$ and $\mathbb{C}^3$ we denote for $\varepsilon = \pm 1$

$$\begin{aligned} \mathbb{H}^2 &\coloneqq \big\{ (z, w) \in \mathbb{C}^2 : \ \operatorname{Im} w = |z|^2 \big\}, \\ \mathbb{H}_\varepsilon^3 &\coloneqq \big\{ (z', w') \in \mathbb{C}^3 : \ \operatorname{Im} w' = ||z'||_\varepsilon^2 \big\}, \end{aligned}$$

respectively. Further we write $\mathbb{H}^3 \coloneqq \mathbb{H}_+^3$.

Remark 2.5. We also write $p_0 = (z_0, w_0) = (r_0 e^{\mathrm{i}\,\theta_0}, v_0 + \mathrm{i}\, r_0^2) \in \mathbb{H}^2$ with $r_0 \geq 0$, $0 \leq \theta_0 < 2\pi$ and $v_0 \in \mathbb{R}$ and identify $\mathbb{H}^2$ with the subset $\mathfrak{H}^2 \subset \mathbb{R}^3$ given by

$$\mathfrak{H}^2 \coloneqq \big\{ p_0 = (r_0, \theta_0, v_0) \in \mathbb{R}^3 : r_0 \geq 0, 0 \leq \theta_0 < 2\pi, v_0 \in \mathbb{R} \big\}, \tag{2.1}$$

using a slight abuse of notation.

Definition 2.6 (Cayley-Transformation)**.** We define the following biholomorphism T_N sending $\mathbb{C}^N \setminus \{z_N = -1\}$ to $\mathbb{C}^N \setminus \{z_N = -\mathrm{i}\}$:

$$T_N(z_1, \ldots, z_N) \coloneqq \Big(z_1, \ldots, z_{N-1}, \mathrm{i}(1 - z_N) \Big) / (1 + z_N). \tag{2.2}$$

The inverse T_N^{-1} of T_N maps $\mathbb{C}^N \setminus \{z_N = -\mathrm{i}\}$ to $\mathbb{C}^N \setminus \{z_N = -1\}$ and is given by:

$$T_N^{-1}(z_1, \ldots, z_N) = \Big(2z_1, \ldots, 2z_{N-1}, 1 + \mathrm{i}\, z_N \Big) / (1 - \mathrm{i}\, z_N). \tag{2.3}$$

Remark 2.7. Let $M \coloneqq \mathbb{S}_k^N$ be a hyperquadric with signature $(k, N - k)$ from (1.1) given in coordinates $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and let $p \in M$. Then we decompose $\mathbb{C}^N = p^\perp \oplus \mathbb{C}p$, where $p^\perp \coloneqq \{ v \in \mathbb{C}^N :$

$\langle p, \bar{v}\rangle_k = 0\}$. In this decomposition we obtain new coordinates ξ for M with $(\xi_1, \ldots, \xi_{N-1}) \in p^\perp$ and $\xi_N \in \mathbb{C}p$.

Let $H : \mathbb{C}^N \to \mathbb{C}^{N'}$ be a holomorphic mapping defined in a neighborhood U of $p \in M$ with $H(U \cap M) \subset M'$ and $H(p) = q'$, where $M' = \mathbb{S}^{N'}_{\ell'}$ for some $\ell' \in \mathbb{N}$. Then we decompose $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ with respect to p and q' as described above. If we consider $\hat{H} := T_{N'} \circ H \circ T_N^{-1}$ we possibly need to shrink U to avoid the poles at $-\mathrm{i}\, p$ and $-q'$ respectively. Moreover in this coordinates $\hat{H}$ satisfies $\hat{H}(0) = 0$ and $\hat{H}(U \cap \mathbb{H}^{k-1}_N) \subset \mathbb{H}^{\ell'-1}_{N'}$.

Remark 2.8. We call a hypersurface M given by $M = \{z \in \mathbb{C}^N : \langle z, A\bar{z}\rangle = 1\}$, where A is an $N \times N$-Hermitian matrix, *Levi-nondegenerate* if A has no zero eigenvalue. The *signature* of M is a pair of natural numbers, the first one is the number of positive eigenvalues of A and the second one is the number of negative eigenvalues of A.

Let M be a Levi-nondegenerate hyperquadric in $\mathbb{C}^{n+1}$ with signature $(k+1, n-k)$ and fix $p_0 \in M$, then $M^* := M \setminus \{p_0\}$ is mapped to $\mathbb{H}^{n+1}_k$ as follows: First we apply a linear change of coordinates to M^* such that M^* is mapped into $\mathbb{S}^{n+1}_k$ for some $k \in \mathbb{N}$ from (1.1). Then we map $\mathbb{S}^N_k$ according to Remark 2.7 outside a point $q_0 \in \mathbb{S}^{n+1}_k$, biholomorphically to $\mathbb{H}^{n+1}_k$ via T_{n+1}, where we may have to vary the definition of T_{n+1} by permuting the variables $(z_1, \ldots, z_{n+1})$. Further if $n = 2k$ we possibly need to apply an automorphism of $\mathbb{H}^{n+1}_k$ of the form $(z_1, \ldots, z_n, w) \mapsto (z_{k+1}, \ldots, z_n, z_1, \ldots, z_k, -w)$.

Definition 2.9 (Notation)**.** (i) Let $h : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function given by $h(z, w) = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha w^\beta$, defined near 0. We write for the complex conjugate of h

$$\bar{h}(\bar{z}, \bar{w}) := \overline{h(z, w)} = \sum_{\alpha,\beta} \bar{a}_{\alpha\beta} \bar{z}^\alpha \bar{w}^\beta.$$

For derivatives of h with respect to z or w we write

$$h_{z^\alpha w^\beta}(0) := \alpha!\beta! a_{\alpha\beta}.$$

For $n \geq 1$, a holomorphic mapping $H : \mathbb{C}^{n+1} \to \mathbb{C}^{n'+1}$ defined near 0 with components $H = \big(f_1, \ldots, f_{n'}, g\big)$ is given by a power series as follows:

$$H(z, w) = \sum_{\alpha,\beta} \frac{H_{z^\alpha w^\beta}(0)}{\alpha!\beta!} z^\alpha w^\beta,$$

where

$$H_{z^\alpha w^\beta}(0) = \big(f_{1 z^\alpha w^\beta}(0), \ldots, f_{n' z^\alpha w^\beta}(0), g_{z^\alpha w^\beta}(0)\big).$$

(ii) For $H = (f_1, \ldots, f_{n'}, g)$ a holomorphic mapping of $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n'+1}$ near 0 we denote

$$\Delta(\alpha_1, \beta_1; \ldots; \alpha_{n'}, \beta_{n'}) := \begin{vmatrix} f_{1 z^{\alpha_1} w^{\beta_1}}(0) & \cdots & f_{1 z^{\alpha_{n'}} w^{\beta_{n'}}}(0) \\ \vdots & & \vdots \\ f_{n' z^{\alpha_1} w^{\beta_1}}(0) & \cdots & f_{n' z^{\alpha_{n'}} w^{\beta_{n'}}}(0) \end{vmatrix}. \tag{2.4}$$

(iii) Let $H : \mathbb{C}^{n+1} \to \mathbb{C}^{n'+1}$ be a holomorphic mapping defined at $p \in \mathbb{C}^{n+1}$ and $\alpha \in \mathbb{N}^{n+1}$. We denote by $j_p^k H$ the *k-jet of H at p* defined as

$$j_p^k H \coloneqq \left(\frac{\partial^{|\alpha|} H}{\partial Z^\alpha}(p) : |\alpha| \leq k \right).$$

We denote by J_p^k the collection of all k-jets at p. We write $J_p^k(M,p;M',p')$ for the collection of all k-jets at p of mappings, which send $(M,p) \subset (\mathbb{C}^N, p)$ to $(M',p') \subset (\mathbb{C}^{N'}, p')$.

2.1 Tangent Spaces

In this section we follow [BER99b, §1.2]. Let $Z = (z_1, \dots, z_N) \in \mathbb{C}^N$ be coordinates in $\mathbb{C}^N$. We identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ by setting $x_j \coloneqq \mathrm{Re}(z_j)$ and $y_j \coloneqq \mathrm{Im}(z_j)$ for $1 \leq j \leq N$. Let M be a smooth real submanifold of codimension d in C^N. For $p \in M$ we let $\rho = (\rho_1, \dots, \rho_d) : \mathbb{C}^N \to \mathbb{R}^d$ be a smooth real-valued mapping defined in a neighborhood $U \subset \mathbb{C}^N$ of p such that $M \cap U = \{Z \in U : \rho(Z) = 0\}$. We write $\rho(Z, \bar{Z})$ instead of $\rho(Z)$ to indicate that ρ is in general not holomorphic.
We define the *real tangent space $T_p\mathbb{C}^N$ of $\mathbb{C}^N$ at p* by

$$T_p\mathbb{C}^N \coloneqq \left\{ X = \sum_{j=1}^N a_j \frac{\partial}{\partial x_j}\bigg|_p + b_j \frac{\partial}{\partial y_j}\bigg|_p \;:\; a_j, b_j \in \mathbb{R} \right\}. \tag{2.5}$$

Then $X \in T_p\mathbb{C}^N$ is called *tangent to M at p* if

$$(X\rho)(p,\bar{p}) = \sum_{j=1}^N a_j \frac{\partial \rho}{\partial x_j}(p,\bar{p}) + b_j \frac{\partial \rho}{\partial y_j}(p,\bar{p}) = 0. \tag{2.6}$$

We write T_pM for the *real tangent space of M at p* which consists of all real vectors $X \in T_p\mathbb{C}^N$ which are tangent to M at p. T_pM is a $2N - d$-dimensional real vector space.
If we allow $a_j, b_j \in \mathbb{C}$ in (2.5) and (2.6) we obtain complex vector spaces denoted by $\mathbb{C}T_p\mathbb{C}^N$ and $\mathbb{C}T_pM$ respectively. We introduce a real-linear mapping $J : T_p\mathbb{C}^N \to T_p\mathbb{C}^N$

$$J\left(\frac{\partial}{\partial x_j}\bigg|_p \right) = \frac{\partial}{\partial y_j}\bigg|_p, \quad J\left(\frac{\partial}{\partial y_j}\bigg|_p \right) = -\frac{\partial}{\partial x_j}\bigg|_p, \qquad 1 \leq j \leq N. \tag{2.7}$$

By linearly extending J to $\mathbb{C}T_p\mathbb{C}^N$ we obtain a complex-linear mapping again denoted by $J : \mathbb{C}T_p\mathbb{C}^N \to \mathbb{C}T_p\mathbb{C}^N$. We call the maximal subspace of T_pM which is invariant under J the complex tangent space given by $T_p^cM \coloneqq T_pM \cap JT_pM$. As above we consider $\mathbb{C}T_p^cM$ and extend J to an operator $\mathbb{C}T_p^cM \to \mathbb{C}T_p^cM$. Then we decompose $\mathbb{C}T_p^cM$ into a direct sum of subspaces consisting of the eigenspaces of J according to its eigenvalues $\pm\,\mathrm{i}$. We set

$$\mathcal{V}_p \coloneqq \{X \in \mathbb{C}T_pM : J(X) = -\mathrm{i}\, X\}, \tag{2.8}$$

to obtain $\mathbb{C}T_p^cM = \mathcal{V}_p \oplus \bar{\mathcal{V}}_p$. Then we can write $\mathbb{C}T_pM = \mathbb{C}T_p^cM \oplus \mathcal{N}_pM$, where $\mathcal{N}_pM$ is the orthogonal complement of $\mathbb{C}T_p^cM$ in $\mathbb{C}T_pM$.
M is called *CR-submanifold* if the dimension of $\mathcal{V}_p$ is locally constant. If M is a hypersurface, then T_p^cM is an $N-1$-dimensional complex vector space for every $p \in M$.
A smooth complex vector field X on $U \subset M$ is a smooth mapping defined in an open neighborhood $U \subset M$ of $p \in M$ such that $X(q) \in \mathbb{C}T_qM$ for all $q \in U$. In coordinates a complex vector field X can be expressed as follows:

$$X = \sum_{j=1}^{N} a_j(z,\bar{z})\frac{\partial}{\partial z_j} + b_j(z,\bar{z})\frac{\partial}{\partial \bar{z}_j}, \tag{2.9}$$

where a_j, b_j are smooth complex-valued functions defined in U. Then according to the above decomposition of $\mathbb{C}T_p^cM$ we can write tangent vectors $v \in \mathcal{V}_p$ as $v = \sum_{j=1}^N b_j(p,\bar{p})\frac{\partial}{\partial \bar{z}_j}$, which we refer to as an antiholomorphic tangent vector. The space of antiholomorphic tangent vectors is denoted by $T_p^{0,1}\mathbb{C}^N$. Similar for so called holomorphic tangent vectors $\bar{v}$, given by $\bar{v} = \sum_{j=1}^N a_j(p,\bar{p})\frac{\partial}{\partial z_j}$ such that $\bar{v} \in \bar{\mathcal{V}}_p$, we denote the space of holomorphic tangent vectors by $T^{1,0}\mathbb{C}^N$. Then we have $\mathcal{V}_p = T^{0,1}\mathbb{C}^N \cap \mathbb{C}T_pM$ and $\bar{\mathcal{V}}_p = T^{1,0}\mathbb{C}^N \cap \mathbb{C}T_pM$.
If M is CR, then vector fields L with the property that $L(p) \in \mathcal{V}_p$ for $p \in M$ are called *CR-vector fields.*

2.2 Segre Sets

We need to introduce the so-called Segre sets, which arise in studying holomorphic mappings of real-analytic submanifolds using a "reflection principle"-argument, as for example in the proof of Lemma 5.5 below. The definition is based on [BER99b, Proposition 10.4.1].

Definition 2.10 (Segre mappings)**.** For M a real-analytic hypersurface in $\mathbb{C}^N$ we choose normal coordinates near $0 \in M$ as in Definition 2.2. Let $p \in \mathbb{C}^{n+1}$ be sufficiently close to 0. We define

$$v_p^1 : \mathbb{C}^n \to \mathbb{C}^{n+1}, \quad v_p^1(z) \coloneqq \big(z, Q(z,\bar{p})\big), \tag{2.10}$$

the *first Segre mapping* v_p^1 of M at p. Let $\ell \geq 2$ then for $1 \leq j \leq \ell$ we write $z^j = (z_1^j, \ldots, z_n^j) \in \mathbb{C}^n$ to define

$$v_p^\ell : \mathbb{C}^{\ell n} \to \mathbb{C}^{n+1}, \quad v_p^\ell(z^1, \ldots, z^\ell) \coloneqq \Big(z^\ell, Q\big(z^\ell, \bar{v}_p^{\ell-1}(\bar{z}^{\ell-1}, \ldots, \bar{z}^1)\big)\Big), \tag{2.11}$$

the *ℓ-th (iterated) Segre map* v_p^ℓ of M at p.

Definition 2.11 (Segre sets)**.** Let $\ell \geq 1$ and $p \in \mathbb{C}^{n+1}$ sufficiently close to M. We call the image of v_p^ℓ the *ℓ-th Segre set* $\mathcal{S}_p^\ell$ of M at p.

Example 2.12. For the complexification of $\mathbb{H}_k^{n+1}$ we have

$$Q(z,\chi,\tau) = Q_k(z,\chi,\tau) \coloneqq \tau + 2\,\mathrm{i}\langle z,\chi\rangle_k, \tag{2.12}$$

such that

$$\mathcal{S}_0^1 = \{(z,0) \in \mathbb{C}^{n+1} : z \in \mathbb{C}^n\}, \tag{2.13}$$

$$\mathcal{S}_0^2 = \Big\{\big(z, 2\,\mathrm{i}\langle z,\chi\rangle_k\big) \in \mathbb{C}^{n+1} : z,\chi \in \mathbb{C}^n\Big\}, \tag{2.14}$$

since Q_k is defined on $\mathbb{C}^{2n+1}$.

To see the relevance of the Segre sets in the following theorem we introduce the *generic rank* $Rk(F)$ of a mapping F as in [BER03, §1]: Let $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ be given by $F = (F_1, \ldots, F_{N'})$, where each $F_j : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ is a formal power series and write $z = (z_1, \ldots, z_N)$ for coordinates of $\mathbb{C}^N$. The generic rank $Rk(F)$ of F is defined as the largest number $s \in \mathbb{N}$ such that there is an $s \times s$-minor of the Jacobi matrix $\frac{\partial F}{\partial z}$ which does not vanish identically as a formal power series in z.

Theorem 2.13 ([BER03, Theorem 1.1])**.** *Let $M \subset \mathbb{C}^N$ be a real-analytic and generic submanifold of codimension d with $0 \in M$. The following statements hold:*

(i) *The generic rank $Rk\big(v_0^k\big)$ of v_0^k is an increasing function of $k \geq 1$ and is independent of the choice of holomorphic coordinates for $\mathbb{C}^N$ and the defining function for M.*

(ii) *There exists $k_0 \in \mathbb{N}$ with $k_0 \leq d+1$, such that*

$$Rk\big(v_0^j\big) = Rk\big(v_0^{j+1}\big), \qquad \forall j \geq k_0,$$

and

$$Rk\big(v_0^{j-1}\big) < Rk\big(v_0^j\big), \qquad 2 \leq j \leq k_0.$$

(iii) *The following statements are equivalent:*
- *M is of finite type at 0.*
- *$Rk\big(v_0^{k_0}\big) = N$.*

Remark 2.14. If $v_0^{k_0}$ is of generic rank N we note that the second condition in (iii) is equivalent to the statement that $\mathcal{S}_0^{k_0}$ contains an open set of $\mathbb{C}^N$, i.e., the Segre set provides a uniqueness set for holomorphic functions.
If we consider the complexification of $\mathbb{H}_k^{n+1} \subset \mathbb{C}^{n+1}$, then $Rk\big(v_0^1\big) = n$ and $Rk\big(v_0^2\big) = n+1$. The rank is full outside of $\{(z,\chi) \in \mathbb{C}^{2n} : z = 0\}$. Note that in order to get $Rk\big(v_0^2\big) = n+1$ it is enough to set $\chi_2 = \ldots = \chi_n = 0$ in $\mathcal{S}_0^2$.

2.3 Automorphisms

Since automorphisms play a crucial role in our study of mappings of hyperquadrics we provide a rather self-contained presentation of the computation of the well-known automorphism group $\mathrm{Aut}(\mathbb{H}_k^{n+1})$.
First we compute the infinitesimal CR-automorphisms of $\mathbb{H}_k^{n+1}$ as described in [Bel02, §2-3], which surveys the well-known method used in several previous works, e.g. in [Bel79]. Then we show a jet determination result for isotropies of $\mathbb{H}_k^{n+1}$ following the method introduced in [BER97], from which, together with the infinitesimal CR-automorphisms, we are able to compute all isotropies of $\mathbb{H}_k^{n+1}$.

In this section we fix $k, n \in \mathbb{N}$, write $M = \mathbb{H}_k^{n+1}$ and skip the subscript in Definition 2.4 for the defining function of M. Moreover we are going to complexify ρ and write $\chi = \bar{z}$ and $\tau = \bar{w}$. We denote the set of infinitesimal CR-automorphisms $\mathfrak{hol}(M, 0)$ by

$$\mathfrak{hol}(M,0) := \left\{ X = \sum_{j=1}^{n+1} a_j(Z) \frac{\partial}{\partial z_j} : a_j : (\mathbb{C}^{n+1}, 0) \to \mathbb{C} \text{ holomorphic}, \mathrm{Re}(X) \text{ is tangent to } M \text{ near } 0 \right\}.$$

As in [Sta91, Theorem 2.2, Corollary 2.4] we can show that $\mathfrak{hol}(M, 0)$ is a Lie algebra and germs of flows of germs of vector fields in $\mathfrak{hol}(M, 0)$ generate germs of automorphisms of M.

We assign the weight 1 to the z-coordinate and weight 2 to the w-coordinate to turn $\mathfrak{hol}(M, 0)$ into a graded Lie algebra. Then $\mathfrak{hol}(M, 0)$ is given by

$$\mathfrak{hol}(M,0) = \bigoplus_{m \geq -2} \mathfrak{hol}_m(M,0),$$

where $\mathfrak{hol}_m(M, 0)$ contains all vector fields of $\mathfrak{hol}(M, 0)$ which are weighted homogeneous of order m. Note that here $\frac{\partial}{\partial z}$ has weight -1 and $\frac{\partial}{\partial w}$ has weight -2. The collection of weighted homogeneous vector fields of order m is denoted by $\mathfrak{g}_m$ and for $X_m \in \mathfrak{g}_m$ we write

$$X_m = f_{m+1}(z,w) \frac{\partial}{\partial z} + g_{m+2}(z,w) \frac{\partial}{\partial w}, \tag{2.15}$$

where $f_{m+1}(z, w)$ and $g_{m+2}(z, w)$ are homogeneous polynomials of weighted order $m + 1$ and $m + 2$ respectively.

We compute $\mathfrak{hol}_m(M, 0)$ for $-2 \leq m \leq 2$ in the following manner: We let $X_m \in \mathfrak{g}_m$, then $X_m \in \mathfrak{hol}_m(M, 0)$ if for $\rho(z, w, \chi, \tau)$ a complexified weighted homogeneous defining function for $(M, 0)$ there exists a complex-analytic function $A_m(z, w, \chi, \tau)$, weighted homogeneous of order m, such that

$$\frac{1}{2}\big(X_m \rho(z,w,\chi,\tau) + \bar{X}_m \rho(z,w,\chi,\tau)\big) = A_m(z,w,\chi,\tau)\rho(z,w,\chi,\tau), \tag{2.16}$$

for all (z, w, χ, τ) near 0.

On the other hand for $m \geq -2$ we let X_m be given by (2.15) such that $X_m \in \mathfrak{hol}_m(M, 0)$ if

$$\begin{aligned} \mathrm{Re}(X_m)\rho\big|_M \equiv 0 \Leftrightarrow & \big(-2\,\mathrm{i}\langle f_{m+1}, \bar{z}\rangle_k + g_{m+2} - 2\,\mathrm{i}\langle \bar{f}_{m+1}, z\rangle_k + \bar{g}_{m+2}\big)\big|_M \equiv 0 \\ \Leftrightarrow & \,\mathrm{Re}(\mathrm{i}\, g_{m+2} + 2\langle f_{m+1}, \bar{z}\rangle_k)\big|_M \equiv 0. \end{aligned}$$

Thus we give the following definition.

Definition 2.15. For $H = (f, g)$ a holomorphic mapping from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}^{n+1}, 0)$ we define the *Chern-Moser operator L for M* as

$$L(f,g) := \mathrm{Re}(ig + 2\langle f, \bar{z}\rangle_k)\big|_M.$$

The following lemma is crucial when treating vector fields of weighted homogeneous order $m \geq 3$.

Lemma 2.16 (Chern–Moser [CM74]). *Let $H = (f, g)$ be a mapping from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}^{n+1}, 0)$. If*

$$f(0) = g(0) = f_{z_j}(0) = g_{z_j}(0) = f_w(0) = g_w(0) = g_{z_j z_k}(0) = g_{w^2}(0) = 0,$$

for $1 \leq j, k \leq n+1$, then $L(f, g) \equiv 0$ has the unique solution $(f, g) \equiv 0$.

Proof. See [CM74, Lemma 2.1]. □

We denote by $I_{n,k} \coloneqq I_{k,n-k}$ the $(n \times n)$-diagonal matrix with 1 in the first k diagonal entries and -1 for the rest of the diagonal and note that $\langle z, \zeta \rangle_k = \langle z, I_{n,k} \zeta \rangle$.

Theorem 2.17 (Infinitesimal CR-Automorphisms of M). *The space $\mathfrak{hol}(M, 0)$ of infinitesimal CR-automorphisms of M is generated by the following vector fields:*

- $X_{-2} = \frac{\partial}{\partial w}$
- $X_{-1} = \langle a, \frac{\partial}{\partial z} \rangle + 2\,\mathrm{i} \langle \bar{a}, z \rangle_k \frac{\partial}{\partial w}$
- $X_0^1 = \langle z, \frac{\partial}{\partial z} \rangle + 2w \frac{\partial}{\partial w}$
- $X_0^2 = \langle Hz, \frac{\partial}{\partial z} \rangle$
- $X_1 = \langle b, z \rangle \langle z, \frac{\partial}{\partial z} \rangle + \frac{\mathrm{i}\, w}{2} \langle \bar{b}, \frac{\partial}{\partial z} \rangle_k + \langle b, z \rangle w \frac{\partial}{\partial w}$
- $X_2 = w \langle z, \frac{\partial}{\partial z} \rangle + w^2 \frac{\partial}{\partial w}$,

where $a, b \in \mathbb{C}^n$ and $H = (h_{\ell m})_{1 \leq \ell, m \leq n} \in \mathbb{C}^n \times \mathbb{C}^n$ satisfies

$$I_{n,k} \, {}^t\bar{H} = -H I_{n,k}, \tag{2.17}$$

which means that H is a skew-Hermitian matrix with respect to the Hermitian form $(z, \zeta) \mapsto \langle z, \bar{\zeta} \rangle_k$, i.e., $\langle Hz, \bar{z} \rangle_k = -\langle z, \bar{H} \bar{z} \rangle_k$.

Proof. We note that if we consider (2.16) for $m \in \{-2, -1\}$ then $A_m = 0$. We have $X_{-2} = A \frac{\partial}{\partial w}$ for $A \in \mathbb{C}$, which implies $A \in \mathbb{R}$.
For $X_{-1} \in \mathfrak{g}_{-1}$ we write

$$X_{-1} = \left\langle a, \frac{\partial}{\partial z} \right\rangle + \langle b, z \rangle \frac{\partial}{\partial w},$$

where $a, b \in \mathbb{C}^n$. In (2.16) all we have to consider are the coefficients of z_j for $1 \leq j \leq n$, which give

$$b_j = \begin{cases} 2\,\mathrm{i}\, \bar{a}_j, & 1 \leq j \leq k, \\ -2\,\mathrm{i}\, \bar{a}_j, & k+1 \leq j \leq n. \end{cases}$$

We note that for $m \geq 0$ there are no pure z-terms occurring as coefficients of $\frac{\partial}{\partial w}$ in X_m, since there are none at the right-hand side of (2.16). Next for $m = 0$ we have $A_0 = A \in \mathbb{R}$ and

$$X_0 = \left\langle Hz, \frac{\partial}{\partial z} \right\rangle + cw \frac{\partial}{\partial w},$$

where $H = (h_{rs})_{1 \leq r, s \leq n}$ for $h_{rs}, c \in \mathbb{C}$. In (2.16) we have if we compare the coefficients of w, that $c = A$ and if we consider for $1 \leq j \leq n$ the coefficients of $z_j \chi_j$ we obtain $2\,\mathrm{Re}(h_{jj}) = A$. Hence we obtain X_0^1

and the vector field X_0 reduces to $X_0 = \langle Hz, \frac{\partial}{\partial z}\rangle$ with $\mathrm{Re}(h_{jj}) = 0$ for $1 \leq j \leq n$. The equation which H has to satisfy is

$$\sum_{s=1}^{k}\sum_{\substack{r=1\\r\neq s}}^{n} h_{rs}z_r\chi_s - \sum_{s=k+1}^{n}\sum_{\substack{r=1\\r\neq s}}^{n} h_{rs}z_r\chi_s + \sum_{s=1}^{k}\sum_{\substack{r=1\\r\neq s}}^{n} \bar{h}_{rs}z_s\chi_r - \sum_{s=k+1}^{n}\sum_{\substack{r=1\\r\neq s}}^{n} \bar{h}_{rs}z_s\chi_r = 0,$$

which implies if $r \neq s$ that

$$\begin{aligned}
h_{rs} &= -\bar{h}_{sr}, \quad \{(r,s) : 1 \leq r,s \leq k\} \cup \{(r,s) : k+1 \leq r,s \leq n\},\\
h_{rs} &= \bar{h}_{sr}, \quad \{(r,s) : k+1 \leq r \leq n, 1 \leq s \leq k\} \cup \{(r,s) : 1 \leq r \leq k, k+1 \leq s \leq n\}.
\end{aligned}$$

Then we note that we obtain the same system of equations if we consider the components of the equation given in (2.17) resulting in the vector field X_0^2.
If $m = 1$ we have $A_1(z,\chi) = \sum_{j=1}^{n} b_j z_j + \bar{b}_j\chi_j$ for $b_j \in \mathbb{C}$. Next we let $a(z) = \big(a_1(z), \ldots, a_n(z)\big)$, where for $1 \leq \ell \leq n$ the function $a_\ell(z)$ is a holomorphic polynomial in z of degree 2 with coefficients $a_{\ell\alpha} \in \mathbb{C}$ for $\alpha \in \mathbb{N}^n$ with $|\alpha| = 2$ and $c(z) = \sum_{j=1}^{n} c_j z_j$ for $c_j \in \mathbb{C}$. For $X_1 \in \mathfrak{g}_1$ we let $d \in \mathbb{C}^n$ and write

$$X_1 = \left\langle a(z) + dw, \frac{\partial}{\partial z}\right\rangle + c(z)w\frac{\partial}{\partial w}.$$

On the right-hand side of (2.16) there are monomials $z_j z_m \chi_j$ and $z_j\chi_j\chi_m$ for $1 \leq j,m \leq n$, thus $a_\ell(z) = z_\ell\widetilde{a}_\ell(z) \coloneqq z_\ell\sum_{m=1}^{n} a_{\ell m}z_m$. If we compare $z_\ell\chi_\ell$ in (2.16) we obtain for $1 \leq \ell \leq n$:

$$\widetilde{a}_\ell(z) + \bar{\widetilde{a}}_\ell(\chi) = A_1(z,\chi),$$

hence $a_{\ell m} = b_m$ for $1 \leq \ell, m \leq n$. Considering $w\chi_j$ for $1 \leq j \leq n$ we obtain if $j \leq k$ that $d_j = \frac{\mathrm{i}\,\bar{b}_j}{2}$ and if $j > k$ we have $d_j = -\frac{\mathrm{i}\,\bar{b}_j}{2}$. Finally the coefficients $z_j w$ for $1 \leq j \leq n$ show $c_j = b_j$ to get X_1, since the remaining coefficients $\chi_j\tau$ and $z_j\tau$ do not give new equations.
If $m = 2$ in (2.16) we have

$$A_2(z,w,\chi,\tau) = A(z) + Bw + \bar{A}(\chi) + \bar{B}\tau,$$

where $A(z)$ is a holomorphic polynomial of degree 2 in z. Further $X_2 \in \mathfrak{g}_2$ has the following form:

$$X_2 = \left\langle c(z) + wd(z), \frac{\partial}{\partial z}\right\rangle + \big(e(z)w + hw^2\big)\frac{\partial}{\partial w},$$

where $c(z)$ is a holomorphic polynomial of degree 3 in z, $d(z)$ is linear in z, $e(z)$ a holomorphic polynomial of degree 2 in z and $h \in \mathbb{C}$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we write $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$. Then the monomials $z^\alpha\tau$ for $|\alpha| = 2$ only occur on the right-hand side of (2.16) hence $A(z) = 0$. There are only monomials involving w or τ on the right-hand side of (2.16) which implies $c(z) = 0$. Since the terms involving z on the right-hand side of (2.16) are of the form $z_j\chi_j$ we obtain $e(z) = 0$. With the same argument we obtain that $d(z) = (d_1z_1, \ldots, d_nz_n)$ where $d_j \in \mathbb{C}$. Comparing $w\tau$ shows $B \in \mathbb{R}$, w^2 gives $h = B$

and the coefficients of $z_j w \chi_j$ imply $d_j = B$. The remaining coefficients τ^2 and $z_j\chi_j\tau$ do not give new equations and we obtain X_2.
To treat the case $m \geq 3$ we apply Lemma 2.16 to obtain $X_m = 0$, which finishes the proof. $\square$

Definition 2.18. We denote the collection of local real-analytic CR-diffeomorphisms $\mathrm{Aut}(M,0)$ of $(M,0)$ by

$$\mathrm{Aut}(M,0) \coloneqq \{H : (\mathbb{C}^{n+1},0) \to \mathbb{C}^{n+1} : H \text{ holomorphic}, H(M) \subset M, \det(H'(0)) \neq 0\},$$

and the group of *isotropies* or *stability group* $\mathrm{Aut}_0(M,0)$ of $(M,0)$ by

$$\mathrm{Aut}_0(M,0) \coloneqq \{H \in \mathrm{Aut}(M,0) : H(0) = 0\}.$$

Elements of the subgroup of $\mathrm{Aut}(M,0)$ generated by the flows of the vector fields X_{-2} and X_{-1} from Theorem 2.17 are referred to as *translations*.

We prove the following well-known theorem ([Bel90], [ES97], [BER98]) with the approach as in [BER97], where a more general result is shown. We follow the algorithm given in [BER97, §6].

Theorem 2.19 (Jet Determination for $\mathrm{Aut}_0(M,0)$)**.** *If $G, H \in \mathrm{Aut}_0(M,0)$ with $j_0^2 G = j_0^2 H$, then $G \equiv H$.*

Proof. We let $H = (f_1, \ldots, f_n, g) \in \mathrm{Aut}_0(M,0)$. We use the notation introduced in the beginning of subsection 2.3 and write $\langle z, \chi\rangle = \sum_{j=1}^n \sigma_j z_j \chi_j$, where $\sigma_j = +1$ for $1 \leq j \leq k$ and $\sigma_j = -1$ for $k+1 \leq j \leq n$. Since H maps $(M,0)$ to $(M,0)$ we have the following equation after setting $w = \tau + 2\,\mathrm{i}\langle z,\chi\rangle$:

$$g(z, \tau + 2\,\mathrm{i}\langle z,\chi\rangle) - \bar{g}(\chi,\tau) = 2\,\mathrm{i}\sum_{j=1}^n \sigma_j f_j(z, \tau + 2\,\mathrm{i}\langle z,\chi\rangle)\bar{f}_j(\chi,\tau). \tag{2.18}$$

If we set $\chi, \tau = 0$ we obtain $g(z,0) = 0$ such that we need to require $\det(f_z(0)) \neq 0$ and $g_w(0) \neq 0$. We also write $z = (z_1, z')$ and $\chi = (\chi_1, \chi')$. Our computations are devoted to prove the dependence of $H(z, 2\,\mathrm{i}\,\sigma_1 z_1 \chi_1)$ on $j_0^2 H$, since Remark 2.14 implies that H only depends on $j_0^2 H$ and the jet determination is proved.
Setting $\chi', \tau = 0$ in (2.18) we get

$$g(z, 2\,\mathrm{i}\,\sigma_1 z_1 \chi_1) = 2\,\mathrm{i}\sum_{j=1}^n \sigma_j f_j(z, 2\,\mathrm{i}\,\sigma_1 z_1\chi_1)\bar{f}_j(\chi_1, 0). \tag{2.19}$$

In the remaining part of the proof we deduce the dependence of $f_j(z, 2\,\mathrm{i}\,\sigma_1 z_1\chi_1)$ on $j_0^2 H$. First we differentiate (2.18) with respect to z_ℓ for $1 \le \ell \le n$ to obtain

$$\begin{aligned} & g_{z_\ell}(z, \tau + 2\,\mathrm{i}\langle z, \chi\rangle) + 2\,\mathrm{i}\,\sigma_\ell \chi_\ell g_w(z, \tau + 2\,\mathrm{i}\langle z, \chi\rangle) \\ &= 2\,\mathrm{i} \sum_{j=1}^{n} \sigma_j \bar{f}_j(\chi, \tau) \Big(f_{j z_\ell}(z, \tau + 2\,\mathrm{i}\langle z, \chi\rangle) + 2\,\mathrm{i}\,\sigma_\ell \chi_\ell f_{jw}(z, \tau + 2\,\mathrm{i}\langle z, \chi\rangle) \Big). \end{aligned} \tag{2.20}$$

If we set $z' = 0, \tau = -2\,\mathrm{i}\,\sigma_1 z_1 \chi_1$ in (2.20) and conjugate the result we deduce for $1 \le \ell \le n$ the following equation:

$$\sigma_\ell z_\ell \bar{g}_\tau(\chi_1, 0) = \sum_{j=1}^{n} \sigma_j \bar{f}_j(\chi, -2\,\mathrm{i}\,\sigma_1 z_1 \chi_1) \Big(\bar{f}_{j\chi_\ell}(\chi_1, 0) - 2\,\mathrm{i}\,\sigma_\ell z_\ell \bar{f}_{j\tau}(\chi_1, 0) \Big),$$

hence the theorem follows if we prove the dependence of $\bar{f}_j, \bar{f}_{j\chi_\ell}, \bar{f}_{j\tau}$ and $\bar{g}_\tau$ at $(\chi_1, 0)$ on $j_0^2 H$ since it is possible to invert the matrix $\big((\bar{f}_{j\chi_\ell} - 2\,\mathrm{i}\,\sigma_\ell z_\ell \bar{f}_{j\tau})(\chi_1, 0)\big)_{j,\ell=1,\dots,n}$ for $(z, \chi_1) \in \mathbb{C}^{n+1}$ near 0.
First we set $z, \chi', \tau = 0$ in (2.20) for each $\ell = 1, \dots, n$ to obtain the following system:

$$\begin{pmatrix} \sigma_1 \chi_1 g_w(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = J \begin{pmatrix} \bar{f}_1(\chi_1, 0) \\ \vdots \\ \bar{f}_n(\chi_1, 0) \end{pmatrix}, \tag{2.21}$$

where J denotes the following $n \times n$-matrix

$$J := \begin{pmatrix} \sigma_1(f_{1z_1}(0) + 2\,\mathrm{i}\,\sigma_1 \chi_1 f_{1w}(0)) & \cdots & \sigma_n(f_{nz_1}(0) + 2\,\mathrm{i}\,\sigma_1 \chi_1 f_{nw}(0)) \\ \sigma_1 f_{1z_2}(0) & \cdots & \sigma_n f_{nz_2}(0) \\ \vdots & & \vdots \\ \sigma_1 f_{1z_n}(0) & \cdots & \sigma_n f_{nz_n}(0) \end{pmatrix},$$

which is invertible for χ_1 near 0. Thus from (2.21) it follows that $\bar{f}_j(\chi_1, 0)$ depends on $j_0^1 H$ for $1 \le j \le n$.
Next we differentiate (2.20) with respect to χ_m for $1 \le m \le n$ and set $z, \chi', \tau = 0$ to get

$$\sigma_m z_m \delta_{m\ell} \left(g_w(0) - 2\,\mathrm{i} \sum_{j=1}^{n} \sigma_j f_{jw}(0) \bar{f}_j(\chi_1, 0) \right) = \sum_{j=1}^{n} \sigma_j \Big(f_{jz_\ell}(0) + 2\,\mathrm{i}\,\sigma_\ell \delta_{1\ell} \chi_\ell f_{jw}(0) \Big) \bar{f}_{j\chi_m}(\chi_1, 0),$$

where we write δ_{jk} for the Kronecker delta. The fact that J is invertible implies that $\bar{f}_{j\chi_m}(\chi_1, 0)$ for $1 \le j, m \le n$ depends on $j_0^1 H$.

If we differentiate (2.20) with respect to τ and take $z, \chi', \tau = 0$ we obtain

$$\begin{aligned} & g_{z_\ell w}(0) + 2\,\mathrm{i}\,\sigma_\ell \chi_\ell \delta_{1\ell} g_{w^2}(0) - 2\,\mathrm{i} \sum_{j=1}^{n} \sigma_j \Big(f_{j z_\ell w}(0) + 2\,\mathrm{i}\,\sigma_\ell \chi_\ell \delta_{1\ell} f_{j w^2}(0) \Big) \\ = {} & 2\,\mathrm{i} \sum_{j=1}^{n} \sigma_j \Big(f_{j z_\ell}(0) + 2\,\mathrm{i}\,\sigma_\ell \delta_{1\ell} \chi_\ell f_{jw}(0) \Big) \bar{f}_{j\tau}(\chi_1, 0), \end{aligned}$$

which determines $\bar{f}_{j\tau}(\chi_1, 0)$ for $1 \le j \le n$ by $j_0^2 H$.
Finally we differentiate (2.18) with respect to τ and set $z, \chi', \tau = 0$ to get

$$\bar{g}_\tau(\chi_1, 0) = g_w(0) - 2\,\mathrm{i} \sum_{j=1}^{n} \sigma_j f_{jw}(0) \bar{f}_j(\chi_1, 0),$$

and the dependence of $\bar{g}_\tau(\chi_1, 0)$ on $j_0^1 H$, which is the missing piece in the proof of the theorem. □

Theorem 2.20 (Automorphisms of M)**.** Aut$(M, 0)$ *is generated by the following mappings:*

- $H_{-2}(z, w) = (z, w + r)$
- $H_{-1}(z, w) = \big(z + a, w + \mathrm{i}\,||a||_k^2 + 2\,\mathrm{i}\langle \bar{a}, z\rangle_k\big)$
- $H_0^1(z, w) = (\lambda z, \sigma \lambda^2 w)$
- $H_0^2(z, w) = (Uz, w)$
- $H_1(z, w) = \frac{(z + bw, w)}{1 - 2\,\mathrm{i}\langle \bar{b}, z\rangle_k - \mathrm{i}\,||b||_k^2 w}$
- $H_2(z, w) = \frac{(z, w)}{1 + sw}$,

where $\sigma = \pm 1, r, s, \lambda \in \mathbb{R}, a, b \in \mathbb{C}^n$ *with* $\lambda > 0$ *and* $U = (u_{\ell m})_{1 \le \ell, m \le n} \in \mathbb{C}^n \times \mathbb{C}^n$ *satisfies*

$$\sigma I_{n,k} = U I_{n,k}\, {}^t\bar{U}, \tag{2.22}$$

which says that U *is a unitary matrix with respect to the Hermitian form* $(z, \zeta) \mapsto \langle z, \bar{\zeta}\rangle_k$*, i.e.,* $||Uz||_k^2 = \sigma ||z||_k^2$*.*
The case $\sigma = -1$ *only appears if* $n = 2k$ *and we define the following automorphism* π *of* M *given by*

$$\pi(z, w) \coloneqq (\tilde{\pi}(z), -w) \coloneqq (z_{k+1}, \ldots, z_n, z_1, \ldots, z_k, -w), \tag{2.23}$$

such that each matrix U *satisfying* (2.22) *with* $\sigma = -1$ *can be written as* $U = V \circ \tilde{\pi}$*, where the matrix* V *satisfies* (2.22) *with* $\sigma = +1$*.*

Remark 2.21. The real dimension of $\mathrm{Aut}(\mathbb{H}_k^N)$ is $(N+1)^2 - 1$ and the real dimension of $\mathrm{Aut}_0(\mathbb{H}_k^N, 0)$ is $N^2 + 1$. The real dimension of the group of translations of $\mathbb{H}_k^N$ is $2N - 1$.

Proof. The proof consists of two parts: In the first part we obtain some automorphisms from infinitesimal CR-automorphisms. In the second part we compute all automorphisms by using some of the isotropies we deduced in the first part of the proof. We take the notation as in the proof of Theorem 2.19.
To obtain a mapping H_m or H_m^ℓ as in the statement of the theorem, we integrate the corresponding vector field X_m or X_m^ℓ from Theorem 2.17. We use the notation for vector fields $X_m \in \mathfrak{hol}_m(M, 0)$ from

(2.15). Then we have to solve for $H_m(t) \coloneqq \big(z_m(t), w_m(t)\big) : \mathbb{R} \to \mathbb{C}^{n+1}$ satisfying

$$\begin{aligned}\dot{z}_m(t) &= f_{m+1}\big(z_m(t), w_m(t)\big)\\ \dot{w}_m(t) &= g_{m+2}\big(z_m(t), w_m(t)\big),\end{aligned}$$

with initial value $\big(z_m(0), w_m(0)\big) = (z, w) \in M$. Here we use the convention $\dot{\psi}(t) = \frac{d\psi(t)}{dt}$. Then the solution $H_m(t)$ is an automorphism of M depending on $(z, w) \in M$ for sufficiently small $t \in \mathbb{R}$ near 0 by the Fundamental Theorem of ODEs.
To obtain H_{-2} and H_{-1} we integrate X_{-2} and X_{-1} and reparametrize $a \in \mathbb{C}^n$ for $m = -1$. Next H_0^1 is obtained directly from X_0^1, after setting $\lambda \coloneqq e^t$. For X_0^2 we integrate to obtain $U = \exp(Ht)$, where H is the matrix from X_0^2 satisfying (2.17). Let us write H again for Ht. Then we have

$$\exp\left(-{}^t\bar{H}\right) = \exp(I_{n,k} H I_{n,k}) = I_{n,k} \exp(H) I_{n,k},$$

where we used (2.17) for the first and $I_{n,k}^2 = I_{n,n}$, which is the usual identity matrix in $\mathbb{C}^n$, and the definition of the matrix exponential for the second equality. Next we have

$$\exp\left(-{}^t\bar{H}\right) = \left(\exp\left({}^t\bar{H}\right)\right)^{-1} = \left(\overline{\exp\left({}^tH\right)}\right)^{-1} = \left({}^t\overline{\exp\left(H\right)}\right)^{-1},$$

which shows if we set $U \coloneqq \exp(H)$, that $({}^t\bar{U})^{-1} = I_{n,k} U I_{n,k}$ and (2.22) with $\sigma = +1$ follows.
In order to integrate X_1 we have to reduce the following system of differential equations:

$$\begin{aligned}\dot{z}_j(t) &= z_j(t)\langle b, z(t)\rangle + \frac{\mathrm{i}\,\bar{b}_j w(t)}{2}, && 1 \le j \le k,\\ \dot{z}_j(t) &= z_j(t)\langle b, z(t)\rangle - \frac{\mathrm{i}\,\bar{b}_j w(t)}{2}, && k+1 \le j \le n,\\ \dot{w}(t) &= \langle b, z(t)\rangle w(t).\end{aligned}$$

From the last equation we get $\langle b, z(t)\rangle = \frac{\dot{w}(t)}{w(t)}$, which we substitute into the other equations. For $1 \le j \le n$ we multiply the j-th equation with b_j. Then we sum up all the resulting n equations to obtain

$$\langle b, \dot{z}(t)\rangle = \langle b, z(t)\rangle \frac{\dot{w}(t)}{w(t)} + \frac{\mathrm{i}}{2}||b||_k^2 w(t).$$

After again substituting the formula for $\langle b, z(t)\rangle$ we obtain the equation

$$\left(\frac{1}{w(t)}\right)^{\cdot\cdot} = -\frac{\mathrm{i}\,||b||_k^2}{2},$$

which can be solved for $w(t)$. The other components $z_j(t)$ can now be obtained from the equations after the first substitution, which give H_1. Finally integrating X_2 directly gives H_2. This completes the study of flows of infinitesimal CR-automorphisms of M and the first part of the proof.
In the second part of the proof we want to show the list of automorphisms in Theorem 2.20 is exhaustive.

The transitive part of $\mathrm{Aut}(M,0)$ is given by H_{-2} and H_{-1} due to dimensional reasons. In the remaining parts of the proof we show that any given isotropy belongs to the group generated by H_0^1, H_0^2, H_1 and H_2.

Let $H \in \mathrm{Aut}_0(M,0)$ be given by $H = (f,g) = (f_1,\ldots,f_n,g)$. We assign the weight 1 to z and 2 to w and consider the weighted homogeneous expansion of H given by

$$H(z,w) = \sum_{\nu \geq 1} H^\nu(z,w), \qquad H^\nu = (f_1^\nu, \ldots, f_n^\nu, g^\nu),$$

where each H^ν is weighted homogeneous of order ν with respect to (z,w). The mapping H has to satisfy the same equation as in (2.18). Then we collect terms of weighted order $\kappa \geq 1$ to obtain

$$g^\kappa(z, \tau + 2\,\mathrm{i}\langle z,\chi\rangle) - \bar{g}^\kappa(\chi,\tau) = 2\,\mathrm{i}\sum_{j=1}^n \sigma_j \left(\sum_{\mu+\nu=\kappa} f_j^\mu(z,\tau+2\,\mathrm{i}\langle z,\chi\rangle)\bar{f}_j^\nu(\chi,\tau)\right). \tag{2.24}$$

For the rest of the proof we investigate the cases $\kappa = 1,\ldots,5$. For $\kappa = 1$ we obtain $g(z,0) = 0$. If $\kappa = 2$ we set $A = (a_j^i)_{1\leq i,j\leq n}$, an invertible complex $n\times n$-matrix, and write $f^1(z,w) = Az$ and $g^2(z,w) = bw$, where $b \in \mathbb{C}\setminus\{0\}$, then (2.24) becomes

$$b(\tau + 2\,\mathrm{i}\langle z,\chi\rangle) - \bar{b}\tau = 2\,\mathrm{i}\sum_{j=1}^n \sigma_j \sum_{i,m=1}^n a_j^i \bar{a}_j^m z_i \chi_m, \tag{2.25}$$

which implies $b \in \mathbb{R}\setminus\{0\}$ and after scaling the above equation with $1/|b|$ and setting $c_j^k := a_j^k/\sqrt{|b|}, C = (c_j^k)_{1\leq j,k\leq n}$ we obtain the equation

$$\sigma||z||^2 = ||Cz||^2,$$

where $\sigma = \mathrm{sgn}(b) = \pm 1$. We note that the scaling we performed corresponds to an application of an isotropy S to M given by $S(z,w) := (\frac{z}{\sqrt{|b|}}, \frac{w}{|b|})$, which is of the form as H_0^1 in the theorem. The term σ in H_0^1 comes from the fact that $\sigma||z||^2 = \mathrm{Im}(\sigma w)$ in the above equation. The matrix C is of the form as H_0^2 in the theorem and satisfies (2.22). Thus after composing H with S and C^{-1} we obtain that $H(z,w) = \big(z + O(2), w + O(3)\big)$, where $O(m)$ stands for terms in z,w of weighted order at least $m \geq 1$. To see the last claims of the theorem one proceeds similar with π as we did with S and we note that $\sigma = -1$ can only occur if $n = 2k$, since the signature of the Hermitian form $(z,\zeta) \mapsto \langle z, \bar{\zeta}\rangle$ is invariant under isomorphisms.

Next, for $\kappa = 3$ we take

$$f_j^1(z,w) = z_j, \qquad f_j^2(z,w) = \sum_{1\leq \ell\leq m\leq n} a_j^{\ell m} z_\ell z_m + b_j w, \qquad g^3(z,w) = \sum_{j=1}^n c_j z_j w, \tag{2.26}$$

where $a_j^{\ell m}, b_j, c_j \in \mathbb{C}$, and plug them into (2.24). If we compare the coefficients of $z_j\tau$ we obtain $c_j = 2\,\mathrm{i}\,\sigma_j \bar{b}_j$ and if we collect terms of the form $z_j\chi_\ell\chi_m$ we obtain the following equation after removing the common factor $\sigma_j z_j$:

$$0 = \sum_{1\le \ell \le m \le n} \bar{a}_j^{\ell m} \chi_\ell \chi_m + \sum_{k=1}^{n} 2\,\mathrm{i}\,\sigma_k b_k \chi_k \chi_j,$$

which implies that $a_j^{\ell m} = 0$ if both $\ell, m \neq j$ and otherwise we have after conjugation

$$\begin{aligned} a_j^{\ell j} &= 2\,\mathrm{i}\,\sigma_\ell \bar{b}_\ell, && \ell < j, \\ a_j^{jm} &= 2\,\mathrm{i}\,\sigma_m \bar{b}_m, && m \geq \ell. \end{aligned}$$

Hence the weighted homogeneous expansion of H is of the following form

$$\begin{aligned} f_j(z,w) &= z_j \left(1 + 2\,\mathrm{i} \sum_{j=1}^{n} \sigma_j \bar{b}_j z_j \right) + b_j w + O(3), \\ g(z,w) &= w \left(1 + 2\,\mathrm{i} \sum_{j=1}^{n} \sigma_j \bar{b}_j z_j \right) + O(4). \end{aligned}$$

The weighted homogeneous expansion of H_1 which we obtained in the first part of the proof gives the same expansion for terms up to weighted order 2 for the first n components and weighted order 3 in the last component. Thus $H\big(H_1^{-1}(z,w)\big) = \big(z + O(3), w + O(4)\big)$, which is the form of H we assume in the remaining cases $\kappa \geq 4$.
For $\kappa = 4$ we write

$$f_j^3(z,w) = \sum_{|\alpha|=3} a_j^\alpha z^\alpha + \sum_{m=1}^{n} b_j^m z_m w, \qquad g^4(z,w) = cw^2 + \sum_{|\beta|=2} d_\beta z^\beta w,$$

where $a_j^\alpha, b_j^m, c, d_\beta \in \mathbb{C}$, and $f_j^1(z,w)$ as in (2.26). In (2.24) if we consider the coefficients of $z^\alpha \chi_j$ and $\chi^\beta \tau$ we obtain $a_j^\alpha, d_\beta = 0$ for all $\alpha \in \mathbb{N}^3$ with $|\alpha| = 3, \beta \in \mathbb{N}^2$ with $|\beta| = 2$ and $1 \leq j \leq n$. The coefficient of τ^2 implies $c \in \mathbb{R}$. Considering terms of the form $z^\gamma \chi^\delta$ with $|\gamma|, |\delta| = 2$ we end up after dividing by $\langle z, \chi \rangle$ at the equation

$$c\langle z, \chi\rangle = \sum_{j=1}^{n} \sigma_j \sum_{m=1}^{n} b_j^m z_m \chi_j,$$

which implies $b_j^m = 0$ for $j \neq m$ and $c = b_j^j$ for $1 \leq j \leq n$. Then the homogeneous expansion of H is given by

$$\begin{aligned} f_j(z,w) &= z_j(1 + cw) + O(4), \\ g(z,w) &= w(1 + cw) + O(5), \end{aligned}$$

which is the same expansion as for H_2 with $s = c \in \mathbb{R}$, which we obtained in the first part of the proof and hence $H(H_2^{-1}(z,w)) = \big(z + O(4), w + O(5)\big)$.

If $\kappa = 5$ we take in (2.24)

$$f_j^4(z,w) = a_j w^2 + \sum_{|\alpha|=2} b_j^\alpha z^\alpha w + \sum_{|\beta|=4} c_j^\beta z^\beta, \qquad g^5(z,w) = \sum_{j=1}^n d_j z_j w^2 + \sum_{|\gamma|=2} e_\gamma z^\gamma w,$$

where $a_j, b_j^\alpha, c_j^\beta, d_j, e_\gamma \in \mathbb{C}$, and $f_j^1(z,w)$ as in (2.26) which we plug into (2.24). If we consider coefficients of terms of the form $\chi^\gamma w$ for $|\gamma| = 2$ we obtain $e_\gamma = 0$, terms of the form $z^\beta \chi_j$ for $|\beta| = 4$ give $c_j^\beta = 0$ and coefficients of terms of the form $z^\delta \chi^\varepsilon$ for $|\delta| = 2, |\varepsilon| = 3$ imply $a_j = 0$. Then we immediately get $f^4, g^5 \equiv 0$ and we have $H(z,w) = \big(z + O(5), w + O(6)\big)$. Next we apply Theorem 2.19 to this mapping, which gives $H \equiv \mathrm{id}_{\mathbb{C}^{n+1}}$, hence we have found all elements of $\mathrm{Aut}(M,0)$ and the list in the statement of Theorem 2.20 contains all elements of $\mathrm{Aut}(M,0)$, which completes the proof. □

We obtain the following corollary from Theorem 2.20:

Corollary 2.22. *Let $\phi \in \mathrm{Aut}(M,0)$, then there exists a unique translation t and isotropy σ of $(M,0)$ such that $\phi = t \circ \sigma$.*

Proof. We let $\phi \in \mathrm{Aut}(M,0)$ with $\phi(0) = p \in M$. According to Theorem 2.20 there exists a unique translation t with $t(0) = p$ such that $\sigma := t^{-1} \circ \phi$ satisfies $\sigma(0) = 0$ and is an automorphism of M, hence $\sigma \in \mathrm{Aut}_0(M,0)$ is exactly one of the isotropies listed in Theorem 2.20, which implies $\phi = t \circ \sigma$. □

We set $n = 1$ in Theorem 2.20 to obtain the automorphisms of $\mathbb{H}^2$. We compose and reparametrize isotropies and translations accordingly to obtain biholomorphic mappings given in the following definition.

Definition 2.23 (Automorphisms of $\mathbb{H}^2$)**.** (i) We write $\mathbb{R}^+ := \{x \in \mathbb{R} : \ x > 0\}$, denote the unit sphere in $\mathbb{C}$ by $\mathbb{S}^1 := \{e^{\mathrm{i}\, t} : 0 \leq t < 2\pi\}$ and set $\Gamma := \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1 \times \mathbb{C}$. Then we parametrize $\mathrm{Aut}_0(\mathbb{H}^2, 0)$ via Γ and write for $\gamma = (\lambda, r, u, c) \in \Gamma$:

$$\sigma_\gamma(z,w) := \frac{(\lambda u(z + cw), \lambda^2 w)}{1 - 2\,\mathrm{i}\,\bar{c} z + (r - \mathrm{i}\,|c|^2) w}. \tag{2.27}$$

(ii) We define for $p_0 = (z_0, w_0) \in \mathbb{H}^2$ the following mapping which form the translations of $\mathbb{H}^2$:

$$t_{p_0} : \mathbb{H}^2 \to \mathbb{H}^2, \quad t_{p_0}(z,w) := (z + z_0, w + w_0 + 2\,\mathrm{i}\,\overline{z_0} z), \tag{2.28}$$

with inverse given by

$$t_{p_0}^{-1} : \mathbb{H}^2 \to \mathbb{H}^2, \quad t_{p_0}^{-1}(z,w) := (z - z_0, w - \overline{w_0} - 2\,\mathrm{i}\,\overline{z_0} z). \tag{2.29}$$

To get automorphisms of $\mathbb{H}^3_\varepsilon$ we set $n = 2$ in Theorem 2.20. We are going to describe how to parametrize the 2×2-matrix U given by

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with complex components. We let I_ε be the 2×2-diagonal matrix with 1 in the first and ε in the second diagonal entry. Then by Theorem 2.20 the matrix U is an automorphism of $\mathbb{H}^3_\varepsilon$ if $U I_\varepsilon {}^t\bar{U} = \sigma I_\varepsilon$, where $\sigma = \pm 1$ if $\varepsilon = -1$, which gives the following system

$$\begin{aligned}
&|a|^2 + \varepsilon\,|b|^2 = \sigma,\\
&|d|^2 + \varepsilon\,|c|^2 = \sigma,\\
&\bar{a}c + \varepsilon\,\bar{b}d = 0.
\end{aligned}$$

The last equation says that $(d,c) = \alpha(\bar{a}, -\varepsilon\,\bar{b})$ for $\alpha \in \mathbb{C}$ and using the other two equations we obtain $|\alpha| = 1$. After some reparametrization we obtain

$$U = \begin{pmatrix} ua & -\varepsilon\, ub \\ \bar{b} & \bar{a} \end{pmatrix},$$

for $|u| = 1$ and $|a|^2 + \varepsilon\,|b|^2 = \sigma$.

Definition 2.24 (Automorphisms of $\mathbb{H}^3_\varepsilon$)**.** (i) We define for $\sigma = \pm 1$ if $\varepsilon = -1$

$$\mathcal{S}^2_{\varepsilon,\sigma} := \left\{a' \in \mathbb{C}^2 : ||a'||^2_\varepsilon = \sigma\right\},$$

and let

$$U' := \begin{pmatrix} u'a'_1 & -\varepsilon\, u'a'_2 \\ \bar{a}'_2 & \bar{a}'_1 \end{pmatrix}, \qquad u' \in \mathbb{S}^1, \quad a' = (a'_1, a'_2) \in \mathcal{S}^2_{\varepsilon,\sigma}. \tag{2.30}$$

We set $\Gamma' := \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1 \times \mathcal{S}^2_{\varepsilon,\sigma} \times \mathbb{C}^2$ to parametrize $\mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ via Γ' and write for $\gamma' = (\lambda', r', u', a', c') \in \Gamma'$:

$$\sigma'_{\gamma'}(z', w') := \frac{\left(\lambda' U'\,{}^t(z' + c'w'), \sigma\lambda'^2 w'\right)}{1 - 2\,\mathrm{i}\langle \bar{c}', z'\rangle_\varepsilon + \left(r' - \mathrm{i}\,||c'||^2_\varepsilon\right) w'}. \tag{2.31}$$

(ii) We define for $p'_0 = (z'_0, w'_0) = (z_0'^1, z_0'^2, w'_0) \in \mathbb{H}^3_\varepsilon$ the following mapping is a translations of $\mathbb{H}^3_\varepsilon$:

$$t'^{-1}_{p'_0} : \mathbb{H}^3_\varepsilon \to \mathbb{H}^3_\varepsilon, \quad t'^{-1}_{p'_0}(z', w') := \left(z' + z'_0, w' + w'_0 + 2\,\mathrm{i}\langle \bar{z}'_0, z'\rangle_\varepsilon\right). \tag{2.32}$$

with inverse given by

$$t'_{p'_0} : \mathbb{H}^3_\varepsilon \to \mathbb{H}^3_\varepsilon, \quad t'_{p'_0}(z', w') := \left(z' - z'_0, w' - \overline{w'_0} - 2\,\mathrm{i}\langle \bar{z}'_0, z'\rangle_\varepsilon\right). \tag{2.33}$$

Remark 2.25. For $\varepsilon = -1$ in the definition of U' in (2.30) we emphasize that we also allow for $|a'_1|^2 - |a'_2|^2 = -1$. We define the following matrix V', which also belongs to the group of isotropies of $\mathbb{H}^3_-$, as follows:

$$V' := \begin{pmatrix} b'_1 & b'_2 & 0 \\ \bar{b}'_2 & \bar{b}'_1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.34}$$

with $|b'_1|^2 - |b'_2|^2 = -1$. If we take $b'_1 = 0$ and set $b'_2 = 1$ in V' we obtain the following automorphism π' of $\mathbb{H}^3_-$ as in (2.23):

$$\pi'(z'_1, z'_2, w') := (z'_2, z'_1, -w'). \tag{2.35}$$

If we do not mention otherwise we take $\sigma = +1$ in the definition of σ' and use π' separately.
Next, if we set $a'_2 = 0$ in U' of (2.30), we define the following automorphisms $U_2(v)$ and $U'_3(v_1, v_2)$ of $\mathbb{H}^2$ and $\mathbb{H}^3_\varepsilon$ respectively:

$$U_2(v) := \begin{pmatrix} 1/v & 0 \\ 0 & 1 \end{pmatrix}, \qquad U'_3(v_1, v_2) := \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U'_3(v) := U'_3\Big(v, v^2\Big), \tag{2.36}$$

where $|v| = 1 = |v_1| = |v_2|$ and we sometimes skip arguments in U_2, U'_3 and U_3.

2.4 Equivalence Relations

We distinguish if we apply isotropies or translations to mappings. Roughly speaking isotropies are easier to work with, since they do not move points as translations. Composing a mapping with translations may have the consequence that the resulting mapping does look different in a certain way if we move the base point.
We are going to introduce families of mappings by composing a mapping with translations depending on some parameter set P_0. Then for a mapping which is defined locally, P_0 depends on the neighborhood where the mapping is defined. Since at some point we only treat mappings which are defined everywhere in $\mathbb{C}^2$ outside some complex-analytic set we only give definitions for this particular family of mappings. In the case of composing mappings with isotropies we use the language of germs to have all parameters of the isotropies available.

Definition 2.26 (Local equivalence). (i) Let $G, H : (\mathbb{H}^2, 0) \to (\mathbb{H}^3_\varepsilon, 0)$ be germs of holomorphic mappings. We let $(\gamma, \gamma') \in \Gamma \times \Gamma'$ to define

$$H_{\gamma,\gamma'}(z, w) := \Big(\sigma'_{\gamma'} \circ H \circ \sigma_\gamma\Big)(z, w) \tag{2.37}$$

and

$$O_0(H) := \Big\{H_{\gamma,\gamma'} : (\gamma, \gamma') \in \Gamma \times \Gamma'\Big\}, \tag{2.38}$$

which we call the *isotropic orbit* of H. We say G is *isotropically equivalent* to H if $G \in O_0(H)$.

(ii) We will refer to the elements of $\Gamma \times \Gamma'$ as *standard parameters*. In the case where we take standard parameters $(\gamma, \gamma') \in \Gamma \times \Gamma'$ such that $\sigma_\gamma = \mathrm{id}_{\mathbb{C}^2}$ and $\sigma'_{\gamma'} = \mathrm{id}_{\mathbb{C}^3}$, we say the standard parameters are *trivial*.

For the first part of the next definition we follow [Hua99, Section 4] for mappings which are defined everywhere on $\mathbb{H}^2$. In subsection 6.1 below we give an equivalence relation for mappings defined in an open set of $\mathbb{H}^2$.

Definition 2.27 (Global equivalence). (i) Let $U \subset \mathbb{C}^2$ be a neighborhood of $\mathbb{H}^2$ such that $H : U \to \mathbb{C}^3$ is a holomorphic mapping with $H(\mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$. Then we define for $(z, w) \in \mathbb{H}^2$ and $p_0 \in \mathbb{H}^2$:

$$H_{p_0}(z, w) := \Big(t'_{H(p_0)} \circ H \circ t_{p_0}\Big)(z, w). \tag{2.39}$$

(ii) Let H be as above, $(z, w) \in \mathbb{H}^2$ and $(\gamma, \gamma') \in \Gamma \times \Gamma'$. Then we define for $p_0 \in \mathbb{H}^2$ the following mapping:

$$H_{p_0,\gamma,\gamma'}(z, w) := \Big(\sigma'_{\gamma'} \circ t'_{H(p_0)} \circ H \circ t_{p_0} \circ \sigma_\gamma\Big)(z, w).$$

As above in the case where we take standard parameters $(\gamma, \gamma') \in \Gamma \times \Gamma'$ such that $\sigma_\gamma = \mathrm{id}_{\mathbb{C}^2}$ and $\sigma'_{\gamma'} = \mathrm{id}_{\mathbb{C}^3}$ and $p_0 = 0$, we say the standard parameters and p_0 are *trivial*.

We note that if the standard parameters are chosen to be trivial in $H_{p_0,\gamma,\gamma'}$ we obtain a mapping H_{p_0} as given in (2.39) and for $H_{0,\gamma,\gamma'}$ we can use the notation as given in (2.37).

Letting p_0 vary in $\mathbb{H}^2$ and (γ, γ') in $\Gamma \times \Gamma'$ we consider the following definition in the sense of germs of mappings $H_{p_0,\gamma,\gamma'}$:

(iii) We define the *orbit* of H as

$$O(H) := \big\{H_{p_0,\gamma,\gamma'} : p_0 \in \mathbb{H}^2, (\gamma, \gamma') \in \Gamma \times \Gamma'\big\}. \tag{2.40}$$

For $G : U \to \mathbb{C}^3$ a holomorphic mapping sending $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$ for $U \subset \mathbb{C}^2$ a neighborhood of 0, we say G is *equivalent* to H if $G \in O(H)$ after possibly shrinking U.

Definition 2.28 (Degree). For a rational, holomorphic mapping $H : \mathbb{C}^N \to \mathbb{C}^{N'}$ given by $H = (P_1, \ldots, P_{N'})/Q$, where $P_1, \ldots, P_{N'}$ and Q are polynomial and complex-valued we say H is *reduced* if $P_1, \ldots, P_{N'}$ and Q do not possess any common factor. Then the degree $\deg H$ of a reduced rational map H is defined as

$$\deg H := \max\big((\deg P_k)_{k=1,\ldots,N'}, \deg Q\big).$$

3 First Properties

In this section we introduce the equation a mapping from $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$ has to satisfy and deduce some basic properties to obtain some invariants of the mappings. From this we get a first, rough classification of mappings as well as a class of mappings which we are going to study more extensively.

Assumption

According to Remark 2.7 and Definition 2.4 our starting point is for $U \subset \mathbb{C}^2$ an open and connected neighborhood of 0 we have given a mapping $H : U \to \mathbb{C}^3$ with $H(U \cap \mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$ and $H(0) = 0$. The components of H are denoted by $H = (f, g) = (f_1, f_2, g)$.
The condition that H maps $\mathbb{H}^2$ into $\mathbb{H}^3_\varepsilon$ can be expressed via a so called *mapping equation* which is given as follows:

$$\operatorname{Im}\big(g(z,w)\big) = |f_1(z,w)|^2 + \varepsilon\,|f_2(z,w)|^2, \tag{3.1}$$

if $\operatorname{Im} w = |z|^2$ for $(z,w) \in U$. In order to work with such an equation in a more convenient way we write (3.1) as

$$g(z,w) - \bar{g}(\bar{z},\bar{w}) = 2\,\mathrm{i}\Big(f_1(z,w)\bar{f}_1(\bar{z},\bar{w}) + \varepsilon\, f_2(z,w)\bar{f}_2(\bar{z},\bar{w})\Big), \tag{3.2}$$

if $w - \bar{w} = 2\,\mathrm{i}\, z\bar{z}$. After expressing w in the last equation and plugging the result into (3.2) we apply Theorem 2.1. By setting $\chi \coloneqq \bar{z}$ and $\tau \coloneqq \bar{w}$ we obtain the following equation:

$$g(z,\tau + 2\,\mathrm{i}\, z\chi) - \bar{g}(\chi,\tau) = 2\,\mathrm{i}\Big(f_1(z,\tau+2\,\mathrm{i}\, z\chi)\bar{f}_1(\chi,\tau) + \varepsilon\, f_2(z,\tau+2\,\mathrm{i}\, z\chi)\bar{f}_2(\chi,\tau)\Big), \tag{3.3}$$

which holds for all $(z,\chi,\tau) \in \mathbb{C}^3$ sufficiently close to 0. We refer to this equation as *complexified mapping equation.*
Some easy facts can be deduced directly from (3.3): If we evaluate at $(z,\chi,\tau) = (z,0,0)$ we obtain $g(z,0) = 0$. Moreover differentiating (3.3) with respect to z and χ and evaluating the result at 0 we have

$$g_w(0) = f_{1z}(0)\bar{f}_{1\chi}(0) + \varepsilon\, f_{2z}(0)\bar{f}_{2\chi}(0) = |f_{1z}(0)|^2 + \varepsilon\,|f_{2z}(0)|^2, \tag{3.4}$$

which implies $g_w(0) \in \mathbb{R}$.

3.1 Transversality of Mappings

This section is devoted to introduce a well-known first-order biholomorphic invariant for mappings.

Definition 3.1 (Transversality)**.** Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic, real hypersurfaces and $U \subset \mathbb{C}^N$ be a neighborhood of $p \in M$. A holomorphic mapping $H : \mathbb{C}^N \to \mathbb{C}^{N'}$ with $H(U \cap M) \subset M'$

is called *transversal to M' at $H(p)$* if

$$T_{H(p)}M' + dH(T_p\mathbb{C}^N) = T_{H(p)}\mathbb{C}^{N'}. \tag{3.5}$$

Remark 3.2. (i) In view of (3.5) it is easy to observe that transversality is invariant under biholomorphic changes of coordinates: Let H be a mapping as in Definition 3.1. We assume w.l.o.g. that $p = 0$ and $H(0) = 0$. Let ψ and ψ' be biholomorphisms of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ sending $(M, 0)$ to $(\widetilde{M}, \widetilde{p})$ and $(M', 0)$ to $(\widetilde{M}', \widetilde{p}')$ respectively. For the induced mapping $\widetilde{H} = \psi' \circ H \circ \psi^{-1} : (\widetilde{M}, \widetilde{p}) \to (\widetilde{M}', \widetilde{p}')$ we consider (3.5). Then we note that $T_{\widetilde{p}'}\widetilde{M}'$ and $d\widetilde{H}(T_{\widetilde{p}}\mathbb{C}^N)$ are related to $T_0 M'$ and $dH(T_0\mathbb{C}^N)$ via the Jacobian matrix of ψ',which shows that $\widetilde{H}$ is transversal to $\widetilde{M}'$ at $\widetilde{p}'$, hence again transversal.

(ii) When dealing with submanifolds there also exists the notion of the so called CR-transversality of a mapping H. We use the notation from subsection 2.1 and let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be real-analytic real submanifolds of codimension d and d' respectively and H a holomorphic mapping sending locally M to M'. Let $p \in M$, then H is called CR-transversal to M' at $H(p)$ if

$$T^{1,0}_{H(p)}M' + dH(T^{1,0}_p\mathbb{C}^N) = T^{1,0}_{H(p)}\mathbb{C}^{N'}.$$

It can be shown that a mapping of real submanifolds which is CR-transversal is transversal in the sense of Definition 3.1, if we allow for submanifolds instead of hypersurfaces in this definition. The converse is in general not true, but if we deal with mappings of hypersurfaces these notions coincide, see [ER06, §5].

We give some characterizations for transversality of a mapping which will be useful for our purpose.

Lemma 3.3 ([ER06, Theorem 5.2])**.** *Let $(M, p) \subset \mathbb{C}^{N+1}$ and $(M', p') \subset \mathbb{C}^{N'+1}$ be germs of connected, real-analytic, real hypersurfaces given in coordinates $Z = (z_1, \dots, z_{N+1}) \in \mathbb{C}^{N+1}$ and $Z' = (z'_1, \dots, z'_{N'+1}) \in \mathbb{C}^{N'+1}$ by ρ and ρ' defining functions for M and M' respectively. Let $H : (M, p) \to (M', p')$ be a germ of a holomorphic mapping. Then the following statements are equivalent:*

(i) *H is transversal to M' at p'.*

(ii) *There exists a holomorphic function $A : (\mathbb{C}^{2N+2}, p) \to \mathbb{C}$ such that the following equation holds:*

$$\rho'\big(H(Z), \bar{H}(\zeta)\big) = A(Z, \zeta)\rho(Z, \zeta), \tag{3.6}$$

with $A(p, \bar{p}) \neq 0$.

(iii) *If we choose normal coordinates as in Definition 2.2 with $p = p' = 0$ we have $\frac{\partial g}{\partial w}(0) \neq 0$.*

Proof. To prove the lemma we first change to normal coordinates $(z, w) = (z_1, \dots, z_N, w) \in \mathbb{C}^{N+1}$ and $(z', w') = (z'_1, \dots, z'_{N'}, w') \in \mathbb{C}^{N'+1}$ centered at $p = p' = 0$ as in Definition 2.2 and write $H = (f, g) = (f_1, \dots, f_{N'}, g) : (M, 0) \to (M', 0)$. By Remark 3.2 H is transversal to M' at 0 if and only if H is transversal to M' at p'.

To prove the lemma we set $p = p' = 0$ and show (i) $\Leftrightarrow$ (iii) and then we prove (ii) $\Leftrightarrow$ (iii). The first equivalence is proved by verifying what (3.5) means under the assumptions of the lemma. For this purpose we write

$$\mathbb{C}T_0M = \mathbb{C}T_0^cM \oplus \mathcal{N}_0M = T_0^{1,0}M \oplus T_0^{0,1}M \oplus \mathcal{N}_0M, \tag{3.7}$$

where we use the definitions from subsection 2.1 such that

$$\begin{aligned} T_0^{1,0}M &= \left\langle \frac{\partial}{\partial z_j}, j = 1, \ldots, N \right\rangle_{\mathbb{C}} \cap \mathbb{C}T_0M, \\ T_0^{0,1}M &= \left\langle \frac{\partial}{\partial \bar{z}_j}, j = 1, \ldots, N \right\rangle_{\mathbb{C}} \cap \mathbb{C}T_0M, \\ \mathcal{N}_0M &= \left\langle \frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}} \right\rangle_{\mathbb{C}}. \end{aligned}$$

Using (3.7) and since M is of hypersurface type we have

$$\mathbb{C}T_0\mathbb{C}^N = \mathbb{C}T_0M + J\mathbb{C}T_0M = \mathbb{C}T_0M \oplus J\mathcal{N}_0M. \tag{3.8}$$

Note that an analogous decomposition holds if we replace M by M'. Next we complexify M and M' and if we use (3.8), the definition of transversality from (3.5) is equivalent to

$$\mathbb{C}T_0M' + dH(J\mathcal{N}_0M) = \mathbb{C}T_0\mathbb{C}^{N'}, \tag{3.9}$$

where we take coordinates $(z, w, \bar{z}, \bar{w}) \in \mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$ and $(z', w', \bar{z}', \bar{w}') \in \mathbb{C}^{N'+1} \times \mathbb{C}^{N'+1}$ for the complexification of M and M' respectively, where $(z, w) \in \mathbb{C}^{N+1}$ and $(z', w') \in \mathbb{C}^{N'+1}$ are normal coordinates for M and M' respectively. Then $\frac{\partial}{\partial w} - \frac{\partial}{\partial \bar{w}} \in J\mathcal{N}_0M$ corresponds to the vector $\mathcal{N} :=$ $(0, 1, 0, -1) \in \mathbb{C}^N \times \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}$. Since we are working with normal coordinates we can deduce similar to (3.4) that $\frac{\partial g}{\partial w}(0)$ is a real matrix and $g(z, 0) = 0$.
To complete this part of the proof we consider the expression $dH(J\mathcal{N}_0M)$ and compute the tangent vector $X := dH|_0\mathcal{N}$ as follows:

$$X = \left(\frac{\partial(f, g, \bar{f}, \bar{g})}{\partial(z, w, \bar{z}, \bar{w})}(0) \right) \mathcal{N} = \begin{pmatrix} \frac{\partial f}{\partial z}(0) & \frac{\partial f}{\partial w}(0) & 0 & 0 \\ 0 & \frac{\partial g}{\partial w}(0) & 0 & 0 \\ 0 & 0 & \frac{\partial \bar{f}}{\partial \bar{z}}(0) & \frac{\partial \bar{f}}{\partial \bar{w}}(0) \\ 0 & 0 & 0 & \frac{\partial g}{\partial w}(0) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial w}(0) \\ \frac{\partial g}{\partial w}(0) \\ -\frac{\partial \bar{f}}{\partial \bar{w}}(0) \\ -\frac{\partial g}{\partial w}(0) \end{pmatrix},$$

which in normal coordinates is the following vector:

$$X = \left(\frac{\partial f}{\partial w}(0) \frac{\partial}{\partial z'} + \frac{\partial g}{\partial w}(0) \frac{\partial}{\partial w'} - \frac{\partial \bar{f}}{\partial \bar{w}}(0) \frac{\partial}{\partial \bar{z}'} - \frac{\partial g}{\partial w}(0) \frac{\partial}{\partial \bar{w}'} \right).$$

The part of X which is not in $\mathbb{C}T_0M'$ is given by $\frac{\partial g}{\partial w}(0)\left(\frac{\partial}{\partial w'} - \frac{\partial}{\partial \bar{w}'}\right)$. Thus (3.9) is satisfied if and only if $\frac{\partial g}{\partial w}(0) \neq 0$, which completes the proof of the equivalence of (i) and (ii).

Next we show (ii) $\Leftrightarrow$ (iii): Since we have given normal coordinates near 0 we write

$$\rho(z,w,\chi,\tau) = w - Q(z,\chi,\tau),$$
$$\rho'(z',w',\chi',\tau') = w' - Q'(z',\chi',\tau'),$$

where $\tau = Q(z,0,\tau) = Q(0,\chi,\tau)$ and $\tau' = Q'(z',0,\tau') = Q'(0,\chi',\tau')$. Because H maps M to M' there exists a nontrivial holomorphic function $A : \mathbb{C}^{2(N+1)} \to \mathbb{C}$ such that

$$g(z,w) - Q'\big(f(z,w), \bar{f}(\chi,\tau), \bar{g}(\chi,\tau)\big) = A(z,w,\chi,\tau)\big(w - Q(z,\chi,\tau)\big), \tag{3.10}$$

for all $(z,w,\chi,\tau) \in \mathbb{C}^{2N+2}$ near 0. Then we differentiate the previous equation (3.10) with respect to w and evaluate at 0. By the normality condition of Q and Q' we have $Q'_{z'}(0) = Q_z(0) = 0$ to obtain $g_w(0) = A(0)$, which proves the equivalence of (ii) and (iii). $\square$

Remark 3.4. (i) Lemma 3.3 shows that if H is not transversal to M' at $H(q)$ if and only if there exists a holomorphic function A satisfying $A(q,\bar{q}) = 0$. The set $\{q \in M : A(q,\bar{q}) = 0\}$ defines a proper, real-analytic subset of M and hence we say H *is transversal to* M' *outside a proper, real-analytic subset of* M if H is transversal to M' at $H(p)$ for some $p \in M$. Otherwise we say H is *nontransversal*.

(ii) Lemma 3.3 (iii) together with (3.4) shows that a transversal mapping H from $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$ is immersive.

We see in the next proposition what happens if we study mappings from $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$. It turns out that for $\varepsilon = -1$ there are mappings which need not be transversal, in contrast to the case $\varepsilon = +1$.

Proposition 3.5 ([BER07, Theorem 1.1]). *Let $U \subset \mathbb{C}^2$ be an open, connected neighborhood of* 0 *and $H : U \to \mathbb{C}^3$ a non-constant holomorphic mapping with $H(U \cap \mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$. Then we have the following two mutually exclusive statements:*

(i) *H is transversal to $\mathbb{H}^3_\varepsilon$ outside a proper, real-analytic subset of $U \cap \mathbb{H}^2$.*

(ii) *The mapping satisfies $H(U) \subset \mathbb{H}^3_\varepsilon$.*

Furthermore (ii) can only appear if $\varepsilon = -1$.

Proof. By our assumptions there exists a holomorphic function $a : \mathbb{C}^4 \to \mathbb{C}$ such that

$$g(z,w) - \bar{g}(\chi,\tau) - 2\,\mathrm{i}\big(f_1(z,w)\bar{f}_1(\chi,\tau) + \varepsilon\, f_2(z,w)\bar{f}_2(\chi,\tau)\big) = a(z,w,\chi,\tau)\big(w - \tau - 2\,\mathrm{i}\, z\chi\big), \tag{3.11}$$

for all $(z,w,\chi,\tau) \in \mathbb{C}^4$ near 0. We have the possibility that $a \equiv 0$ to obtain (ii) for $\varepsilon = -1$, since if $\varepsilon = +1$ we would have $H \equiv 0$. If $a \not\equiv 0$ then we divide the function a sufficiently often by the defining function of $\mathbb{H}^2$ to obtain a holomorphic function $A : \mathbb{C}^4 \to \mathbb{C}$ satisfying

$$a(z,w,\chi,\tau) = A(z,w,\chi,\tau)(w - \tau - 2\,\mathrm{i}\, z\chi)^m, \tag{3.12}$$

for $m \geq 0$ and $A|_{\mathbb{H}^2} \not\equiv 0$. If $m = 0$ we are in the business of (ii) of Lemma 3.3 to obtain (i), since the proper, real-analytic set of points $q \in \mathbb{H}^2$ where H is not transversal to $\mathbb{H}^3_\varepsilon$ at $q \in \mathbb{H}^2$ is given by

$A(q,\bar{q}) = 0$ according to Remark 3.4 (i).
The rest of the proof consists of showing that $m \geq 1$ is possible only if $\varepsilon = -1$ and H satisfies the property given in (ii). For this purpose we change coordinates to assume $A(0) \neq 0$, let $m \geq 1$ and replace a in (3.11) by (3.12) to obtain

$$g(z,w) - \bar{g}(\chi,\tau) - 2\,\mathrm{i}\big(f_1(z,w)\bar{f}_1(\chi,\tau) + \varepsilon\, f_2(z,w)\bar{f}_2(\chi,\tau)\big) = A(z,w,\chi,\tau)\big(w - \tau - 2\,\mathrm{i}\, z\chi\big)^k, \quad (3.13)$$

for $k \geq 2$ and $A(0) \neq 0$.
If we set $\chi = \tau = 0$ we obtain

$$g(z,w) = A(z,w,0,0)w^k,$$

and using this in (3.13) we get

$$\begin{aligned} &A(z,w,0,0)w^k - \bar{A}(\chi,\tau,0,0)\tau^k - 2\,\mathrm{i}\big(f_1(z,w)\bar{f}_1(\chi,\tau) + \varepsilon\, f_2(z,w)\bar{f}_2(\chi,\tau)\big) \\ &= A(z,w,\chi,\tau)\big(w - \tau - 2\,\mathrm{i}\, z\chi\big)^k, \end{aligned} \quad (3.14)$$

Next we differentiate (3.14) with respect to z and χ and evaluate at 0 to obtain

$$f_{1z}(0)\bar{f}_{1\chi}(0) + \varepsilon\, f_{2z}(0)\bar{f}_{2\chi}(0) = 0, \quad (3.15)$$

since $k \geq 2$.
For $\varepsilon = +1$ we obtain $f_z(0) = \big(f_{1z}(0), f_{2z}(0)\big) = 0$ and since we always have $g_z(0) = 0$ as noted in section 3, the rank of the Jacobian of H is at most 1, which means that H is not immersive. We conclude that there is no non-immersive mapping if $\varepsilon = +1$ similar as in the proof of [Leb11b, Theorem 1.2] as follows: Having the rank of the Jacobian of H at most 1 means that outside a complex-analytic set in $U \subset \mathbb{C}^2$ the mapping H sends a neighborhood of $p \in \mathbb{C}^2$ into a complex 1-dimensional subset of $\mathbb{C}^3$. It is possible to find $p \in \mathbb{H}^2$, such that H sends an open neighborhood of p in $\mathbb{H}^2$ to $\mathbb{H}^3$ which implies that there is no non-immersive H, since $\mathbb{H}^3$ does not contain complex-analytic sets.
From now on we treat the case $\varepsilon = -1$. In view of (3.15) we say the vector $f_z(0)$ is trivial if at least one of the components is 0. Hence if $f_z(0)$ is trivial we have that H is not immersive and we conclude (ii) as in the proof of [Leb11b, Theorem 1.2]: We proceed as above where $\varepsilon = +1$ and note that outside a complex-analytic set in $U \subset \mathbb{C}^2$ the mapping H sends a neighborhood of $p \in \mathbb{H}^2$ to a 1-dimensional subset in $\mathbb{C}^3$. Then we observe that H maps a neighborhood $V \subset \mathbb{C}^2$ of $p \in \mathbb{H}^2$ into $\mathbb{H}^3_-$, since if the image of H in $\mathbb{H}^3_-$ is less than two real-dimensional, then the preimage of such a point would give a complex-analytic set in $\mathbb{H}^2$ according to the rank theorem, which is not possible.
Let us assume $f_z(0)$ is nontrivial for the rest of the proof. Then we proceed by setting $z = \tau = 0$ in (3.14), differentiate with respect to χ and evaluate at $\chi = 0$ to obtain

$$-2\,\mathrm{i}\big(f_1(0,w)\bar{f}_{1\chi}(0) - f_2(0,w)\bar{f}_{2\chi}(0)\big) = A_\chi(0,w,0,0)w^k. \quad (3.16)$$

Differentiating (3.16) with respect to w and evaluating at 0 gives

$$f_{1w}(0)\bar{f}_{1\chi}(0) - f_{2w}(0)\bar{f}_{2\chi}(0) = 0, \tag{3.17}$$

since $k \geq 2$, which implies the vector $f_w(0) = \big(f_{1w}(0), f_{2w}(0)\big)$ is a multiple of $f_z(0)$. At the end of the proof we need the w^{k-1}-coefficient in (3.16), which satisfies

$$f_{1w^{k-1}}(0)\bar{f}_{1\chi}(0) - f_{2w^{k-1}}(0)\bar{f}_{2\chi}(0) = 0. \tag{3.18}$$

Next we take $z = \chi = 0$ in (3.14), differentiate with respect to τ and evaluate at $\tau = 0$ to get

$$-2\,\mathrm{i}\big(f_1(0,w)\bar{f}_{1\tau}(0) - f_2(0,w)\bar{f}_{2\tau}(0)\big) = A_\tau(0,w,0,0)w^k - kA(0,w,0,0)w^{k-1}. \tag{3.19}$$

Then we differentiate (3.19) $k-1$-times with respect to w and evaluate at 0 to obtain

$$-2\,\mathrm{i}\big(f_{1w^{k-1}}(0)\bar{f}_{1\tau}(0) - f_{2w^{k-1}}(0)\bar{f}_{2\tau}(0)\big) = -k!A(0). \tag{3.20}$$

Since we already know from (3.17) that $f_w(0)$ is a multiple of $f_z(0)$ we substitute $\bar{f}_\chi(0)$ into (3.20) and use (3.18) to obtain $A(0) = 0$, a contradiction. □

3.2 Degeneracy of Mappings

The next biholomorphic invariant we need is the well-known (finite) degeneracy for mappings. This invariant was used by among others Faran [Far82], Cima–Suffridge [CS83] and Forstnerič [For89] to extend proper holomorphic mappings, which are smooth up to the boundary of their domain, holomorphically past the boundary. This section is based on [Lam01, Section 2.5].

Definition 3.6 (Degeneracy)**.** Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be generic, real-analytic submanifolds of codimension d and d' respectively and denote $n := N - d$ and $n' := N' - d'$. For $p \in M, p' \in M'$ and $U \subset \mathbb{C}^N$ a neighborhood of p we let $H : U \to \mathbb{C}^{N'}$ be a holomorphic mapping satisfying $H(U \cap M) \subset M'$. We choose coordinates Z and Z' centered at p and p' for M and M' respectively. In the complexification of M and M' we write $\zeta := \bar{Z}$ and $\zeta' := \bar{Z}'$. For $\rho' = (\rho'_1, \ldots, \rho'_{d'})$ a defining function for M' near p' we denote for $1 \leq j \leq d'$ the complex gradient $\rho'_{j,Z'}(Z', \bar{Z}')$ of ρ'_j with respect to Z' by defining

$$\rho'_{j,Z'}(Z', \zeta') := \left(\frac{\partial \rho'_j(Z', \zeta')}{\partial z'_1}, \ldots, \frac{\partial \rho'(Z', \zeta')}{\partial z'_{N'}}\right).$$

For $L_1, \ldots, L_n$ a basis of CR-vector fields for M near p, as defined in subsection 2.1, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we denote $L^\alpha := L_1^{\alpha_1} \cdots L_n^{\alpha_n}$. Then we define for $k \geq 0$ and $q \in M$ near p the following vector spaces after possibly shrinking U:

$$E'_k(q) := \operatorname{span}_{\mathbb{C}} \left\{ L^\alpha \rho'_{j,Z'}\big(H(Z), \bar{H}(\zeta)\big)\Big|_{(Z,\zeta)=(q,\bar{q})} : 0 \leq |\alpha| \leq k, 1 \leq j \leq d' \right\} \subset \mathbb{C}^{N'}. \tag{3.21}$$

Since for $k \geq 0$ the $E'_k(q)$ form an ascending chain of vector spaces in $\mathbb{C}^{N'}$, there exists a minimal $k_0 \geq 0$

such that $E'_k(q) = E'_{k_0}(q)$ for all $k \geq k_0$ and $E'_{k_0-1}(q) \subsetneq E'_{k_0}(q)$ in a neighborhood of $q \in M$. We set

$$s(q) \coloneqq N' - \dim_{\mathbb{C}} E'_{k_0}(q),$$

called the *degeneracy* of H at q and H is called $\big(k_0, s(q)\big)$*-degenerate* at $q \in M$.
If $s = s(q)$ is constant in a neighborhood of $p \in M$ we say H is *constantly* (k_0, s)*-degenerate* near $p \in M$ and s is called *constant degeneracy* of H.
If for some $q \in M$ we have $s(q) = 0$, then $E'_{k_0}(q) = \mathbb{C}^{N'}$ which means that H is of constant degeneracy $s = 0$ near q and H is called k_0*-nondegenerate*.

Lemma 3.7. *Definition 3.6 is independent of the choices of a basis of CR-vector fields and the defining function.*

Proof. Let $\widetilde{L} = (\widetilde{L}_1, \ldots, \widetilde{L}_n)$ be another basis of CR-vector fields for M, such that $\widetilde{L} = A(Z,\zeta)L$ for an invertible matrix $A(Z,\zeta) = (a_{jk}(Z,\zeta))_{j,k=1,\ldots,n}$ in a neighborhood of $p \in M$. Then $\widetilde{L}^\alpha$ is a linear combination of L^β for $|\beta| \leq |\alpha|$. Thus if we denote by $\widetilde{E}'_k(q)$ the subspace as given in (3.21) where we use $\widetilde{L}$ instead of L, we obtain that $\widetilde{E}'_k(q)$ consists of linear combinations of vectors in $E'_k(q)$. Hence if we interchange roles of $\widetilde{L}$ and L we have $\widetilde{E}'_k(q) = E'_k(q)$.
To see the independence of Definition 3.6 of the defining function we let $\widetilde{\rho}'$ be another defining function for M'. Then we have $\widetilde{\rho}'(Z',\zeta') = B(Z',\zeta')\rho'(Z',\zeta')$ for an invertible matrix $B(Z',\zeta') = (b_{jk}(Z',\zeta'))_{j,k=1,\ldots,d'}$ near $p' \in M'$. For $\ell = 1, \ldots, d'$ we compute

$$\widetilde{\rho}'_{\ell,Z'}(Z',\zeta') = \sum_{k=1}^{d'} b_{\ell k,Z'}(Z',\zeta')\rho'_k(Z',\zeta') + \sum_{k=1}^{d'} b_{\ell k}(Z',\zeta')\rho'_{k,Z'}(Z',\zeta').$$

Then in $L^\alpha \widetilde{\rho}'_{\ell,Z'}(H(Z),\bar{H}(\zeta))$ the first sum vanishes if we restrict to M and in the second sum we obtain terms of the form $L^\beta \rho'_{\ell,Z'}(H(Z),\bar{H}(\zeta))$ for $|\beta| \leq |\alpha|$. Again if we write $\widetilde{E}'_k(q)$ for the subspace given by (3.21) where we use $\widetilde{\rho}'$ instead of ρ', we obtain, after interchanging $\widetilde{\rho}'$ and ρ' in the previous consideration, $\widetilde{E}'_k(q) = E'_k(q)$. $\square$

Example 3.8. For $U \subset \mathbb{C}^2$ an open set containing $\mathbb{S}^2$ we consider the mapping $F_4 : U \to \mathbb{S}^3$ of Theorem 1.1 and choose coordinates $Z = (z_1, z_2) \in \mathbb{C}^2$ and $Z' = (z'_1, z'_2, z'_3) \in \mathbb{C}^3$. We write $\zeta = \bar{Z}$ and $\zeta' = \bar{Z}'$, such that S^2, the complexification of $\mathbb{S}^2$, is given by $\rho(Z,\zeta) = Z\zeta - 1$ and the complexification of $\mathbb{S}^3$ is given by $\rho'(Z',\zeta') = Z'\zeta' - 1$. Then

$$\varphi(Z,\zeta) \coloneqq \rho'_{Z'}\big(F_4(Z), \bar{F}_4(\zeta)\big) = \Big(\zeta_1^3, \sqrt{3}\zeta_1\zeta_2, \zeta_2^3\Big),$$

and we take $L = z_2\frac{\partial}{\partial\zeta_1} - z_1\frac{\partial}{\partial\zeta_2}$ as a basis for the CR-vector fields. We note that $L^k\varphi = 0$ for $k \geq 4$ and compute

$$\big\{L\varphi, L^2\varphi, L^3\varphi\big\} = \Big\{\Big(3z_2\zeta_1^2, \sqrt{3}(z_2\zeta_2 - z_1\zeta_1), -3z_1\zeta_2^2\Big), \Big(6z_2^2\zeta_1, -2\sqrt{3}z_1z_2, 6z_1^2\zeta_2\Big), \Big(6z_2^3, 0, -6z_1^3\Big)\Big\}.$$

Then the set for S^2, where F_4 is 3-nondegenerate is given by

$$\{(Z,\zeta)\in S^2 : \det(\varphi, L\varphi, L^2\varphi)=0\} = \{(Z,\zeta)\in S^2 : \zeta_1\zeta_2 = 0\}.$$

In the following we show that the notion of degeneracy is invariant under biholomorphic changes of coordinates in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$.

Lemma 3.9 ([Lam01, Lemma 14])**.** *Definition 3.6 is independent of the choices of holomorphic coordinates in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$.*

Proof. Let Ψ and Ψ' be biholomorphisms of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively, such that $\widetilde{Z} := \Psi(Z)$ and $\widetilde{Z}' := \Psi'(Z')$ are holomorphic coordinates for $\widetilde{M}$ and $\widetilde{M}'$ near $\widetilde{p}$ and $\widetilde{p}'$ respectively.
A change of coordinates in $\mathbb{C}^N$ has the consequence that the CR-vector fields L are mapped to $\widetilde{L} = \Psi_* L$, which form a basis of CR-vector fields for $\widetilde{M}$. Similar as we did in the proof of Lemma 3.7 we can show that $\widetilde{E}'_k(\widetilde{q}) = E'_k(q)$, where $\widetilde{E}'_k(\widetilde{q})$ is obtained from (3.21) if we write $\widetilde{H} := H\circ\Psi^{-1}$, use $\widetilde{Z}$ as coordinates for $\widetilde{M}$ and $\widetilde{L}$ as a basis of CR-vector fields near $\widetilde{p}$.
To show the invariance of a change of coordinates in $\mathbb{C}^{N'}$ we have $\widetilde{\rho}'(\widetilde{Z}',\widetilde{\zeta}') := \rho'(\Psi'^{-1}(\widetilde{Z}'), \bar{\Psi}'^{-1}(\widetilde{\zeta}'))$ as defining function for $\widetilde{M}'$ and we compute for $1\leq \ell\leq d'$:

$$\widetilde{\rho}'_{\ell,\widetilde{Z}'}(\widetilde{Z}',\widetilde{\zeta}') = \Big(\rho'_\ell(\Psi'^{-1}(\widetilde{Z}'), \bar{\Psi}'^{-1}(\widetilde{\zeta}'))\Big)_{\widetilde{Z}'} = \rho'_{\ell,Z'}(\Psi'^{-1}(\widetilde{Z}'), \bar{\Psi}'^{-1}(\widetilde{\zeta}'))\frac{\partial\Psi'^{-1}}{\partial\widetilde{Z}'}(\widetilde{Z}').$$

If we plug in $H(Z)$ for $\widetilde{Z}'$ we set $\widetilde{H} := \Psi'^{-1}\circ H$ and note that the CR-vector fields of L annihilate $\frac{\partial\Psi'^{-1}}{\partial\widetilde{Z}'}(H(Z))$. Again we obtain $\widetilde{E}'_k(\widetilde{q}) = E'_k(q)$ if we choose $\widetilde{\rho}'$ in (3.21). ☐

In order to obtain a more global view on the concept of degeneracy we define the following degeneracy.

Definition 3.10. Let $M\subset\mathbb{C}^N$ and $M'\subset\mathbb{C}^{N'}$ be generic, real-analytic submanifolds and $U\subset\mathbb{C}^N$ be a neighborhood of $p\in M$. Let $H: U\to\mathbb{C}^{N'}$ be a holomorphic mapping satisfying $H(U\cap M)\subset M'$ and fix $V\subset U$ a neighborhood of $p\in M$ such that $\overline{V\cap M}\subset U$. The number

$$s_H(V) := \min_{q\in\overline{V\cap M}} s(q),$$

is called *generic degeneracy* in $V\subset\mathbb{C}^N$ a neighborhood of $p\in M$.

Note that H is of constant degeneracy $s_H(V)$ near $p\in V$. The following lemma shows that H having degeneracy s_H happens generically in U.

Lemma 3.11 ([Lam01, Lemma 22])**.** *Let $M\subset\mathbb{C}^N$ and $M'\subset\mathbb{C}^{N'}$ be generic, real-analytic submanifolds and $U\subset\mathbb{C}^N$ a neighborhood of $p\in M$. Let $H: U\to\mathbb{C}^{N'}$ be a holomorphic mapping satisfying $H(U\cap M)\subset M'$ and fix $V\subset U$ a neighborhood of $p\in M$. Then H is constantly $\big(k_0, s_H(V)\big)$-degenerate outside a proper, real-analytic subset of $V\cap M$ for some $k_0\in\mathbb{N}$.*

Proof. In order to show the claim we prove that the set

$$X := \big\{q\in V\cap M : s(q) > s_H(V)\big\} \subsetneq V\cap M,$$

is a real-analytic subset in V by giving real-analytic defining functions for X. We go back to Definition 3.6 and define for $1 \leq \ell \leq d'$ and $1 \leq k \leq N'$ the following real-analytic functions:

$$\varphi_{\ell,k}(Z,\zeta) \coloneqq \rho'_{\ell z'_k}\big(H(Z), \bar{H}(\zeta)\big). \tag{3.22}$$

Being of constant degeneracy $s_H = s_H(V)$ in $V \cap M$ means the following: There exist $t = t_H \coloneqq N' - s_H$ multi-indices $\beta^1, \ldots, \beta^t \in \mathbb{N}^n$ and numbers $m_1, \ldots, m_t \in \mathbb{N}$ with $1 \leq m_r \leq d'$ such that, after possibly permuting components in Z', the following vectors with real-analytic entries

$$v_j(Z,\zeta) \coloneqq \Big(L^{\beta^j}\varphi_{m_j,1}(Z,\zeta), \ldots, L^{\beta^j}\varphi_{m_j,t}(Z,\zeta)\Big) \in \mathbb{C}^t, \qquad 1 \leq j \leq t \tag{3.23}$$

form a basis of $\mathbb{C}^t$. Since s_H is the smallest possible degeneracy in $V \cap M$ the set X of points $q \in V \cap M$, where the degeneracy $s(q)$ is bigger than s_H is given by

$$X = \big\{q \in V \cap M : \det\big(v_1(q,\bar{q}), \ldots, v_t(q,\bar{q})\big) = 0\big\},$$

which is a proper and real-analytic subset.
The number k_0 is given by the maximal length of the β^r for $1 \leq r \leq t$ from above. □

Lemma 3.11 shows if we take a smaller neighborhood $W \subset V$ in Definition 3.10 then $s_H(V) = s_H(W)$. Hence we skip the argument in $s_H(V)$ and write s_H from now on.

Remark 3.12. For nondegenerate mappings and mappings of degeneracy equal to 1 we can deduce jet parametrizations which we are going to give in two of the following sections below. In the case of the constantly 1-degenerate mappings we mention the following easy fact for this purpose:
In the proof of Lemma 3.11 we had for a mapping H of constant degeneracy s, that the vectors v_j given in (3.23) form a basis of $\mathbb{C}^t$, where $t = N' - s$. We furthermore have that this set of vectors satisfies the following equations: For any $\gamma \in \mathbb{N}^n, t+1 \leq k \leq N'$ and $1 \leq \ell \leq d'$ the determinant of the matrix

$$\begin{pmatrix} L^{\beta^1}\varphi_{m_1,1} & \cdots & L^{\beta^1}\varphi_{m_1,t} & L^{\beta^1}\varphi_{m_1,k} \\ \vdots & & \vdots & \vdots \\ L^{\beta^t}\varphi_{m_t,1} & \cdots & L^{\beta^t}\varphi_{m_t,t} & L^{\beta^t}\varphi_{m_t,k} \\ L^{\gamma}\varphi_{\ell,1} & \cdots & L^{\gamma}\varphi_{\ell,t} & L^{\gamma}\varphi_{\ell,k} \end{pmatrix} \tag{3.24}$$

restricted to points in M vanishes.

Next we obtain bounds for the generic degeneracy s_H and k_0 adapted to our setting.

Proposition 3.13 ([Lam01, Lemma 23–24])**.** *Let $U \subset \mathbb{C}^2$ be a neighborhood of $p \in \mathbb{H}^2$ and $H : U \to \mathbb{C}^3$ a holomorphic mapping with components $H = (f_1, f_2, g)$ and $H(U \cap \mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$ which is transversal to $\mathbb{H}^3_\varepsilon$ outside a proper real-analytic subset of $\mathbb{H}^2$. There exists a proper, real-analytic subset X of $U \cap \mathbb{H}^2$ such that after shrinking U and performing a change of coordinates in $U \setminus X$ the following two mutually exclusive statements hold:*

(i) *H is 2-nondegenerate, such that $f_{1z}(0)f_{2z^2}(0) - f_{2z}(0)f_{1z^2}(0) \neq 0$.*
(ii) *H is constantly $(1,1)$-degenerate, such that $f_{1z}(0)f_{2z^k}(0) - f_{2z}(0)f_{1z^k}(0) = 0$, for all $k \geq 2$.*

Proof. By Lemma 3.11 we have (k_0, s_H)-degeneracy outside a proper, real-analytic subset of $\mathbb{H}^2$. By Remark 3.2 and Lemma 3.9 after a change of coordinates we assume that 0 is a point where H is constantly (k_0, s_H)-degenerate and transversal to $\mathbb{H}^3_\varepsilon$. This change of coordinates is performed via composing H with translations such that 0 gets mapped to a point q where H is constantly (k_0, s_H)-degenerate and transversal to $\mathbb{H}^3_\varepsilon$, i.e., we consider the mapping $t'_{H(q)} \circ H \circ t_q$ from (2.28) and (2.33) instead of H. At this point it is possible that we need to shrink U.

Before we give estimates for k_0 and s_H we introduce some notation first:

Let $(Z, \zeta) = (z, w, \chi, \tau)$ and $(Z', \zeta') = (z'_1, z'_2, w', \chi'_1, \chi'_2, \tau')$ be coordinates of the complexification of $\mathbb{H}^2$ and $\mathbb{H}^3_\varepsilon$ and

$$\rho'(Z', \zeta') := w' - \tau' - 2\,\mathrm{i}(z'_1\chi'_1 + \varepsilon\, z'_2\chi'_2)$$

a defining function for the complexification of $\mathbb{H}^3_\varepsilon$. A basis of (0,1)-vector fields of the complexification of $\mathbb{H}^2$ is given by

$$L := \frac{\partial}{\partial \chi} - 2\,\mathrm{i}\, z \frac{\partial}{\partial \tau}. \tag{3.25}$$

Next we define for $k \geq 0$

$$v_k(Z, \zeta) := L^k \rho'_{Z'}\big(H(Z), \bar{H}(\zeta)\big) = L^k \Big(-2\,\mathrm{i}\, \bar{f}_1(\chi, \tau), -2\,\mathrm{i}\, \varepsilon\, \bar{f}_2(\chi, \tau), 1\Big), \tag{3.26}$$

and $u_k := v_k(0, 0)$. Further let us define the subspaces $E'_k := \mathrm{span}_{\mathbb{C}}\{u_m : 0 \leq m \leq k\}$.

Then we have $u_0 = (0, 0, 1)$ and $u_\ell = -2\,\mathrm{i}\big(\bar{f}_{1\chi^\ell}(0), \varepsilon\, \bar{f}_{2\chi^\ell}(0), 0\big)$ for $\ell \geq 0$. Since H is transversal at 0 we have by Remark 3.4 (ii) $u_1 \neq 0$ such that u_0 and u_1 are linearly independent. Consequently $E'_0 \subsetneq E'_1$ and $\dim_{\mathbb{C}} E'_k \geq 2$ for $k \geq 1$, which implies $k_0 \geq 1$ and $0 \leq s_H \leq 1$. We are left with two cases:

If $s_H = 0$, then $k_0 \geq 2$. In order to show $k_0 = 2$ we prove as in [BER99b, Lemma 11.5.4]

$$\Delta(z) := f_{1z}(z, 0) f_{2z^2}(z, 0) - f_{1z^2}(z, 0) f_{2z}(z, 0) \not\equiv 0, \tag{3.27}$$

which says that Δ vanishes on a proper, complex-analytic set of $\mathbb{C}$. Since $\mathbb{H}^2$ does not contain any complex-analytic sets we obtain that H is 2-nondegenerate outside a proper real-analytic subset of $U \cap \mathbb{H}^2$ satisfying the linear independence condition in (i).

We show (3.27) by assuming the converse $\Delta \equiv 0$ and write

$$\big(f_1(z, 0), f_2(z, 0)\big) = \left(\sum_{k \geq 1} a_k z^k, \sum_{\ell \geq 1} b_\ell z^\ell\right),$$

where $(a_k, b_\ell) = \big(f_{1z^k}(0)/k!, f_{2z^\ell}(0)/\ell!\big)$. Then we have

$$\Delta(z) = \sum_{m \geq 3} \left(\sum_{k=1}^{m-2} k(m-k)(m-k-1)\Big(a_k b_{m-k} - a_{m-k} b_k\Big)\right) z^{m-3}.$$

Considering the coefficients z^α for $\alpha \geq 0$ in $\Delta \equiv 0$ we inductively obtain that there exists $A_k \in \mathbb{C}$ such that $(a_k, b_k) = A_k(a_1, b_1)$ for all $k \geq 2$. This implies

$$\big(f_1(z,0), f_2(z,0)\big) = \big(f_{1z}(0), f_{2z}(0)\big)h(z),$$

for some holomorphic function $h : \mathbb{C} \to \mathbb{C}$. Then we have $E'_1 = E'_k$ for $k \geq 2$, hence $k_0 = 1$, a contradiction.
Finally we consider the case $s_H = 1$, where we must have $\dim_{\mathbb{C}} E'_k = 2$. This is already achieved for $k = 1$, which means that H is $(1,1)$-degenerate outside a proper, real-analytic set of $U \cap \mathbb{H}^2$. Furthermore, since then $E'_1 = E'_k$ for $k \geq 2$, we obtain the condition

$$f_{1z}(0)f_{2z^k}(0) - f_{2z}(0)f_{1z^k}(0) = 0, \qquad \forall k \geq 2, \tag{3.28}$$

which completes the proof. □

Remark 3.14. We let $H = (f_1, f_2, g)$ be as in Proposition 3.13. According to Definition 3.6 and (3.26) we note that the set N of points in $\mathbb{H}^2$, where H is not 2-nondegenerate, is given by

$$N := \left\{p \in \mathbb{H}^2 : Lf_1(p)L^2f_2(p) - Lf_2(p)L^2f_1(p) = 0\right\}.$$

Remark 3.15. The conditions for H given in Proposition 3.13 (i) and (ii) are invariant under applications of isotropies or appropriate translations as in (2.37) or (2.39), if one assumes that the parameter occurring in (2.39) belongs to a sufficiently small neighborhood of 0. Since translations are not needed at this point of our investigations we will discuss them in a subsequent chapter in more detail.

3.3 Initial Classification and the Class $\mathcal{F}_2$

We are going to use the invariants we introduced in the previous section to obtain a first classification of mappings.

Proposition 3.16. *Let $U \subset \mathbb{C}^2$ be an open and connected neighborhood of 0 and $H : U \to \mathbb{C}^3$ a non-constant holomorphic mapping given by $H = (f_1, f_2, g)$ with $H(U \cap \mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$ and $H(0) = 0$. Then, after possibly shrinking U, changing coordinates or composing H with automorphisms, one of the following mutually exclusive statements holds:*

(i) *H is transversal to $\mathbb{H}^3_\varepsilon$ and 2-nondegenerate at 0 and we can assume $H(0) = 0$, $g_w(0) = |f_{1z}(0)|^2 + \varepsilon\,|f_{2z}(0)|^2 > 0$ and $f_{1z}(0)f_{2z^2}(0) - f_{2z}(0)f_{1z^2}(0) \neq 0$.*

(ii) *H is equal to the linear embedding $(z,w) \mapsto (z,0,w)$.*

(iii) *For $\varepsilon = -1$: H is a mapping of the form $(z,w) \mapsto (h(z,w), h(z,w), 0)$ for some non-constant holomorphic function $h : U \to \mathbb{C}$ with $h(0) = 0$.*

Definition 3.17. We assign to the mappings from *Proposition* 3.16 (i) the following notation: For a neighborhood $U \subset \mathbb{C}^2$ of 0 let us denote the set $\mathcal{F}_2(U)$ of holomorphic mappings $H = (f_1, f_2, g)$ with

$H(U \cap \mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$, which satisfy $H(0) = 0$, using the notation of (2.4)

$$\Delta(1,0;2,0) = f_{1z}(0) f_{2z^2}(0) - f_{2z}(0) f_{1z^2}(0) \neq 0 \tag{3.29}$$

and

$$g_w(0) > 0. \tag{3.30}$$

We denote by $\mathcal{F}_2$ the set of germs H, such that $H \in \mathcal{F}_2(U)$ for some $U \subset \mathbb{C}^2$ a neighborhood of 0.

Proof of Proposition 3.16. We apply Proposition 3.5 to obtain that either H is transversal to $\mathbb{H}^3_\varepsilon$ outside a proper, real-analytic set of $U \cap \mathbb{H}^2$ or for $\varepsilon = -1$ we have H maps a neighborhood $U \subset \mathbb{C}^2$ of 0 to $\mathbb{H}^3_-$. We assume the first condition for H and apply Proposition 3.13, such that after performing a change of coordinates via translations and possibly shrinking U as in the beginning of the proof of Proposition 3.13, that H is transversal to $\mathbb{H}^3_\varepsilon$ at 0 and either 2-nondegenerate or $(1,1)$-degenerate near 0. By Lemma 3.3 transversality to $\mathbb{H}^3_\varepsilon$ at 0 is equivalent to $g_w(0) \neq 0$. For $\varepsilon = +1$ by (3.4) we immediately have $g_w(0) > 0$. If $\varepsilon = -1$ and we have $g_w(0) < 0$ we compose H with the automorphism π' from (2.35).
If we assume H is transversal to $\mathbb{H}^3_\varepsilon$ at 0 and 2-nondegenerate near 0, then we immediately obtain (i) by (3.27) in the proof of Proposition 3.13.
If we assume H is transversal to $\mathbb{H}^3_\varepsilon$ at 0 and $(1,1)$-degenerate near 0 we also have the property which is given in (3.28) and we refer to Theorem 7.2 in section 7 below to obtain (ii).
To finish the proof we need to treat the case if $\varepsilon = -1$ and H maps a neighborhood $U \subset \mathbb{C}^2$ to $\mathbb{H}^3_-$. Here the following mapping equation holds for all $(z, w, \chi, \tau) \in W$ for some neighborhood $W \subset \mathbb{C}^4$ of 0:

$$g(z,w) - \bar{g}(\chi,\tau) - 2\,\mathrm{i}\big(f_1(z,w)\bar{f}_1(\chi,\tau) - f_2(z,w)\bar{f}_2(\chi,\tau)\big) = 0. \tag{3.31}$$

Setting $\chi = 0 = \tau$ we obtain $g(z,w) = 0$ such that (3.31) reduces to

$$f_1(z,w)\bar{f}_1(\chi,\tau) = f_2(z,w)\bar{f}_2(\chi,\tau), \tag{3.32}$$

for $(z,w,\chi,\tau) \in \mathbb{C}^4$. Next we either apply [D'A93, Chapter 3, Proposition 3] or we proceed as follows: Differentiation of (3.32) gives $|f_{1z}(0)| = |f_{2z}(0)|, |f_{1w}(0)| = |f_{2w}(0)|$ and $f_{1z}(0)\bar{f}_{1\tau}(0) - f_{2z}(0)\bar{f}_{2\tau}(0) = 0$. These equations together imply that the Jacobi matrix of H is of rank 1 near 0. This means at a generic point p_0 near 0 the mapping $f : (z,w) \mapsto \big(f_1(z,w), f_2(z,w)\big)$ sends a full neighborhood W of p_0 into an irreducible complex- analytic curve C of $\mathbb{C}^2$. We proceed as in the proof of [Leb11b, Theorem 1.2] and apply an automorphism of $\mathbb{H}^3_\varepsilon$ as U'_3 from (2.36) to $(z,w) \mapsto (f(z,w), 0)$, such that the image of H is contained in the complex variety given by $\big\{(z'_1, z'_2, w') \in \mathbb{C}^3 : z'_1 = z'_2, w' = 0\big\}$. Thus H is equivalent to the map $(z,w) \mapsto (h(z,w), h(z,w), 0)$ for some holomorphic function $h : \mathbb{C}^2 \to \mathbb{C}$ with $h(0) = 0$. $\square$

4 Isotropical Equivalence of Mappings in $\mathcal{F}_2$

In this section we provide a normal form for mappings in the class $\mathcal{F}_2$, which was defined in Definition 3.17. Note that the conditions for H to belong to $\mathcal{F}_2$ given in Definition 3.17 are preserved if we apply isotropies which are fixing 0 to H.

4.1 Normal Form $\mathcal{N}_2$

Proposition 4.1. *Let $H \in \mathcal{F}_2$. Then there exist automorphisms $\sigma \in \mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\sigma' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ such that $\widehat{H} := \sigma' \circ H \circ \sigma$ satisfies $\widehat{H}(0) = 0$ and the following conditions:*

(i) $\widehat{H}_z(0) = (1, 0, 0)$
(ii) $\widehat{H}_w(0) = (0, 0, 1)$
(iii) $\widehat{f}_{2z^2}(0) = 2$
(iv) $\widehat{f}_{2zw}(0) = 0$
(v) $\widehat{f}_{1w^2}(0) = |\widehat{f}_{1w^2}(0)| \geq 0$
(vi) $\mathrm{Re}\big(\widehat{g}_{w^2}(0)\big) = 0$
(vii) $\mathrm{Re}\big(\widehat{f}_{2z^2w}(0)\big) = 0$

Definition 4.2. We refer to the equations given in Proposition 4.1 as *normalization conditions.* A holomorphic mapping of $\mathcal{F}_2$ satisfying the normalization conditions is called a *normalized mapping.* The set of normalized mappings is denoted by $\mathcal{N}_2$.

Proof of Proposition 4.1. For $H \in \mathcal{F}_2$ we proceed as follows: We normalize H in 6 steps. In each step we apply certain isotropies to H in order to normalize some coefficients of H and to obtain a partial normal form for H, which is used in the subsequent steps. At some points it is necessary to renormalize to preserve some already achieved normalized coefficients of H.
We write $H = (f, g) = (f_1, f_2, g)$. We introduce the following notation: For $k \geq 1$, in the k-th step if we apply isotropies $\sigma_k \in \mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\sigma'_k \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ we write $H_k := \sigma'_k \circ H_{k-1} \circ \sigma_k$ with components $H_k = (f^k, g^k) = (f^k_1, f^k_2, g^k)$. We set $H_0 := H$.
We start by defining $H_1 := (\lambda' U' f, \lambda'^2 g)$, where $\lambda' > 0$ and U' is a 2×2-matrix as in (2.30). We compute

$$
\begin{aligned}
H_{1z}(0) &= \Big(\lambda'\big(a'_1 f_{1z}(0) - \varepsilon\, a'_2 f_{2z}(0)\big), \lambda'\big(\bar{a}'_2 f_{1z}(0) + \bar{a}'_1 f_{2z}(0)\big), 0\Big),\\
g^1_w(0) &= \lambda'^2 g_w(0).
\end{aligned}
$$

Since we assume $g_w(0) > 0$ we can choose $\lambda' > 0$ to obtain $g^1_w(0) = 1$, which gives one equation of (ii) from our desired normalization conditions. Next we set

$$
a'_1 := \frac{\bar{f}_{1\chi}(0)}{\sqrt{g_w(0)}}, \qquad a'_2 := -\frac{\bar{f}_{2\chi}(0)}{\sqrt{g_w(0)}},
$$

to obtain by (3.4) that $(a'_1, a'_2) \in \mathcal{S}^2_{\varepsilon,\sigma}$ and $f^1_z(0) = (1, 0)$, which is (i).

In the second step we apply the isotropy of $\mathbb{H}^3_\varepsilon$ from (2.31) involving $c' = (c'_1, c'_2) \in \mathbb{C}^2$ and define

$$H_2 := \frac{\left(f^1_1 + c'_1 g^1, f^1_2 + c'_2 g^1, g^1\right)}{1 - 2\,\mathrm{i}(\bar{c}'_1 f^1_1 + \varepsilon\, \bar{c}'_2 f^1_2) - \mathrm{i}\,||c'||^2_\varepsilon g^1}.$$

We verify that $H_{2z}(0) = (1, 0, 0)$ and $g^2_w(0) = 1$ and compute that $f^2_w(0) = (c'_1 + f^1_{1w}(0), c'_2 + f^1_{2w}(0))$, such that we can achive $f^2_w(0) = (0, 0)$, which gives the normalization condition (ii).
Next we define

$$H_3(z,w) := \frac{\left(f^2_1\left(\frac{(z+cw,w)}{1-2\,\mathrm{i}\,\bar{c}z-\mathrm{i}\,|c|^2 w}\right) + \hat{c}g^2\left(\frac{(z+cw,w)}{1-2\,\mathrm{i}\,\bar{c}z-\mathrm{i}\,|c|^2 w}\right), f^2_2\left(\frac{(z+cw,w)}{1-2\,\mathrm{i}\,\bar{c}z-\mathrm{i}\,|c|^2 w}\right), g^2\left(\frac{(z+cw,w)}{1-2\,\mathrm{i}\,\bar{c}z-\mathrm{i}\,|c|^2 w}\right)\right)}{1 - 2\,\mathrm{i}\,\bar{\hat{c}} f^2_1\left(\frac{(z+cw,w)}{1-2\,\mathrm{i}\,\bar{c}z-\mathrm{i}\,|c|^2 w}\right) - \mathrm{i}\,|\hat{c}|^2 g^2\left(\frac{(z+cw,w)}{1-2\,\mathrm{i}\,\bar{c}z-\mathrm{i}\,|c|^2 w}\right)}, \tag{4.1}$$

where $c, \hat{c} \in \mathbb{C}$. It holds that $H_{3z}(0) = (1, 0, 0), H_{3w}(0) = (c + \hat{c}, 0, 1)$, such that we need to set $\hat{c} = -c$ to fulfill the normalization conditions from the previous steps. We note that the condition for H_3 from Definition 3.17 given by $f^3_{1z}(0) f^3_{2z^2}(0) - f^3_{2z}(0) f^3_{1z^2}(0) \neq 0$ reduces to $f^2_{2z^2}(0) \neq 0$, such that $f^3_{2zw}(0) = c f^2_{2z^2}(0) + f^2_{2zw}(0)$, which implies that we can achieve (iv), i.e., $f^3_{2zw}(0) = 0$, for the remaining steps.
In the fourth step we define

$$H_4(z,w) := \left(\lambda' d' v' f^3_1(\lambda z, \lambda^2 w), \lambda' \bar{d}' f^3_2(\lambda z, \lambda^2 w), \lambda'^2 g^3(\lambda z, \lambda^2 w)\right),$$

where $\lambda, \lambda' > 0$ and $|d'| = |v'| = 1$. We compute $H_{4z}(0) = (d'v'\lambda\lambda', 0, 0), H_{4w}(0) = (0, 0, (\lambda\lambda')^2)$ and $f^4_{2zw}(0) = 0$ and set $v' = 1/d'$ and $\lambda' = 1/\lambda$ such that all normalization conditions we obtained so far are satisfied by H_4. Then we have $f^4_{2z^2}(0) = \lambda \bar{d}' f^3_{2z^2}(0) \neq 0$ since $H_4 \in \mathcal{F}_2$. Hence we can find d' and $\lambda > 0$ to get $f^4_{2z^2}(0) = 2$, which is (iii).
In the fifth step we define

$$H_5(z,w) := \left(e'u' f^4_1(uz, w), \bar{e}' f^4_2(uz, w), g^4(uz, w)\right),$$

where $|u| = |u'| = |e'| = 1$. We have $H_{5z}(0) = (uu'e', 0, 0), H_{5w}(0) = (0, 0, 1), f^5_{2z^2}(0) = 2\bar{e}'u^2$ and $f^5_{2zw}(0) = 0$. To preserve the so far obtained normalization conditions we set $e' = u^2$ and $u' = 1/u^3$. Then we calculate $f^5_{1w^2}(0) = f^4_{1w^2}(0)/u$, such that we can normalize $f^5_{1w^2}(0) \geq 0$ with the standard parameter u, which is (v).
In the last step we define

$$H_6(z,w) := \frac{\left(f^5_1\left(\frac{(z,w)}{1+rw}\right), f^5_2\left(\frac{(z,w)}{1+rw}\right), g^5\left(\frac{(z,w)}{1+rw}\right)\right)}{1 + r' g^5\left(\frac{(z,w)}{1+rw}\right)},$$

where $r, r' \in \mathbb{R}$. Then we verify that all normalization conditions from the previous steps are satisfied by H_6 and we obtain that $g^6_{w^2}(0) = -2(r + r') + g^5_{w^2}(0)$ and $f^6_{2z^2w}(0) = -(2r + r') + f^5_{2z^2w}(0)$. Hence we can find unique $r, r' \in \mathbb{R}$ such that $\mathrm{Re}\big(g^6_{w^2}(0)\big) = \mathrm{Re}\big(f^6_{2z^2w}(0)\big) = 0$. These conditions are the missing normalization conditions (vi) and (vii). The isotropies $\sigma \in \mathrm{Aut}_0(\mathbb{H}^2,0)$ and $\sigma' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon,0)$ in the proposition consist of the appropriate composition of the isotropies we used in each of the 6 normalization steps. $\square$

Remark 4.3. It is possible to obtain explicit formulas for the standard parameters of the isotropies we used in the normalization procedure of Proposition 4.1. One possibility is to keep track of the applications of isotropies in each step and each relevant coefficient in the previous proof. Alternatively we consider $\widehat{H} := \sigma' \circ H \circ \sigma$, where we use all standard parameters in σ and σ' with the notation of (2.27) and (2.31). Then we have to compute the coefficients of $\widehat{H}$ we want to normalize and solve the resulting equations for the standard parameters. The first equations are the following:

$$\widehat{H}_z(0) = \left(u\lambda\lambda' \begin{pmatrix} u'a_1' & -\varepsilon\, u'a_2' \\ \bar{a}_2' & \bar{a}_1' \end{pmatrix} \begin{pmatrix} f_{1z}(0) \\ f_{2z}(0) \end{pmatrix}, 0\right) = \big(1,0,0\big), \tag{4.2}$$

$$\begin{aligned}\widehat{H}_w(0) &= \left(u\lambda\lambda' \begin{pmatrix} u'a_1' & -\varepsilon\, u'a_2' \\ \bar{a}_2' & \bar{a}_1' \end{pmatrix} \begin{pmatrix} c_1'\lambda g_w(0) + \lambda f_{1w}(0) + cuf_{1z}(0) \\ c_2'\lambda g_w(0) + \lambda f_{2w}(0) + cuf_{2z}(0) \end{pmatrix}, \lambda^2\lambda'^2 g_w(0)\right) \\ &= (0,0,1),\end{aligned} \tag{4.3}$$

which can be solved using (3.4) by

$$a_1' = \frac{\bar{f}_{1\chi}(0)}{uu'||f_z(0)||_\varepsilon}, \qquad a_2' = -\frac{\bar{f}_{2\chi}(0)}{uu'||f_z(0)||_\varepsilon}, \tag{4.4}$$

such that $a' = (a_1', a_2') \in \mathcal{S}^2_{\varepsilon,\sigma}$ and we obtain

$$c_1' = \frac{-cuf_{1z}(0) - \lambda f_{1w}(0)}{\lambda g_w(0)}, \qquad c_2' = \frac{-cuf_{2z}(0) - \lambda f_{2w}(0)}{\lambda g_w(0)}, \qquad \lambda' = \frac{1}{\lambda\sqrt{g_w(0)}}, \tag{4.5}$$

since we require $\lambda, g_w(0) > 0$. For the following equations we use the notation for 2×2-determinants of coefficients of H we introduced in (2.4). Then we use (3.4) as well as the formulas for the standard parameters for a', c' and λ' to obtain the following equation:

$$\widehat{f}_{2zw}(0) = \frac{u^2u'\lambda}{g_w(0)^2}\Big(cug_w(0)\Delta(1,0;2,0) + \lambda\big(g_{zw}(0)\Delta(0,1;1,0) + g_w(0)\Delta(1,0;1,1)\big)\Big) = 0, \tag{4.6}$$

which has a unique solution $c \in \mathbb{C}$, since $g_w(0) > 0$ and $\Delta(1,0;2,0) \neq 0$, given by

$$c = -\frac{\lambda\big(g_{zw}(0)\Delta(0,1;1,0) + g_w(0)\Delta(1,0;1,1)\big)}{ug_w(0)\Delta(1,0;2,0)}. \tag{4.7}$$

Then using the representations for a', λ' and equation (3.4):

$$\widehat{f}_{2z^2}(0) = \frac{u^3u'\lambda\Delta(1,0;2,0)}{||f_z(0)||^2_\varepsilon} = 2, \tag{4.8}$$

the unique solution is given by

$$\lambda = \frac{2||f_z(0)||_\varepsilon^2}{|\Delta(1,0;2,0)|}, \qquad u' = \frac{\bar{\Delta}(1,0;2,0)}{u^3|\Delta(1,0;2,0)|}, \tag{4.9}$$

since $\Delta(1,0;2,0) \neq 0$. Then, using all the previously determined standard parameters, we compute $\widehat{f}_{1w^2}(0) = T_1\big(j_0^2 H\big)/u$, where $T_1\big(j_0^2 H\big) \in \mathbb{C}$ is a real-analytic function in $j_0^2 H$, which does not depend on u. Thus there is a u with $|u| = 1$ and $0 \le \operatorname{Arg} u < 2\pi$, such that $\widehat{f}_{1w^2}(0) = |\widehat{f}_{1w^2}(0)| \ge 0$. Finally we consider the following coefficients, where $\lambda > 0$ is given by (4.9)

$$\operatorname{Re}\big(\widehat{g}_{w^2}(0)\big) = -\,2r - 2r'\lambda^2 g_w(0) + T_2\big(j_0^2 H\big) = 0, \tag{4.10}$$

$$\operatorname{Re}\big(\widehat{f}_{2z^2w}(0)\big) = -\,2r - r'\lambda^2 g_w(0) + T_3\big(j_0^3 H\big) = 0, \tag{4.11}$$

where $T_2\big(j_0^2 H\big), T_3\big(j_0^3 H\big) \in \mathbb{R}$ are real-analytic functions in $j_0^2 H$ and $j_0^3 H$ respectively and both do not depend on r or r', we can uniquely solve for the real parameters r and r'.

Remark 4.4. (i) We need an explicit expression of $|\widehat{f}_{1w^2}(0)|$ for later purposes. To this end we either consider the coefficient $f^3_{1w^2}(0)$ of H_3 from (4.1) or we compute $\widehat{f}_{1w^2}(0)$ of $\widehat{H}$ from Remark 4.3, using the definitions for a' from (4.4), c', λ' from (4.5) and the equation from (3.4). In this case we obtain the following term still containing some standard parameters:

$$\widehat{f}_{1w^2}(0) = \frac{-\lambda\big(\bar{f}_{1\chi}(0) S_1 + \varepsilon\, \bar{f}_{2\chi}(0) S_2\big)}{u g_w(0)^2},$$

where for $\ell = 1, 2$ we have

$$\begin{aligned} S_\ell = {} & \lambda^2\big(g_{w^2}(0) f_{\ell w}(0) - g_w(0) f_{\ell w^2}(0)\big) + cu\lambda\big(g_{w^2}(0) f_{\ell z}(0) + 2 g_{zw}(0) f_{\ell w}(0) - 2 g_w(0) f_{\ell zw}(0)\big) \\ & + c^2 u^2 \big(2 g_{zw}(0) f_{\ell z}(0) - g_w(0) f_{\ell z^2}(0)\big). \end{aligned}$$

If we use the definition for c from (4.7) we obtain the same formula as for the coefficient $f^3_{1w^2}(0)$ of H_3, given by

$$\widehat{f}_{1w^2}(0) = \frac{\lambda^3}{u} S\big(j_0^2 H\big),$$

where $S\big(j_0^2 H\big)$ is an explicitly given, real-analytic function in $j_0^2 H$ not depending on any standard parameter. Then we have

$$|\widehat{f}_{1w^2}(0)| = \lambda^3 |S\big(j_0^2 H\big)|, \tag{4.12}$$

and we note that in order to compute $|\widehat{f}_{1w^2}(0)|$ it is only necessary to compute the standard parameters a', c', λ' and c, λ.

(ii) Further inspection of T_3 in (4.11) shows that the coefficients of H at 0 of order 3 occurring in T_3 are $f_{z^3}(0)$ and $H_{z^2w}(0)$.

(iii) Uniqueness of the choice of isotropies in the proof of Proposition 4.1 or of the standard parameters in Remark 4.3 cannot be achieved in general, since for the latter case in the equation $\widehat{f}_{1w^2}(0) = T_1\big(j_0^2 H\big)/u$ it may occur that $T_1 = 0$. In this case the standard parameter u appears as a free parameter. For a discussion concerning the stabilizer of mappings see Lemma 5.18 below.

Proposition 4.5. *Let $H \in \mathcal{N}_2$. Then necessarily the derivatives of H satisfy the following equations:*

(i) $f_{1z^k}(0) = 0 \quad (k \geq 2)$

(ii) $f_{1zw}(0) = \frac{\mathrm{i}\,\varepsilon}{2}$

(iii) $\mathrm{Im}(g_{w^2}(0)) = 0$

(iv) $\mathrm{Im}(g_{w^3}(0)) = 0$

(v) $f_{2z^3}(0) = -3\,\mathrm{i}\,\varepsilon\, f_{1z^2w}(0)$

(vi) $g_{z^kw}(0) = 0 \quad (k \geq 1)$

(vii) $f_{1z^2w}(0) = 2g_{zw^2}(0)$

(viii) $f_{1zw^2}(0) = \frac{1}{4}\big(-1 + 2\,\mathrm{Re}(g_{w^3}(0))\big)$

(ix) $g_{zw^2}(0) = 2\,\mathrm{i}\,|f_{1w^2}(0)|$

Proof. The conditions are simply verified by differentiating (3.3) assuming the normalization conditions given in Proposition 4.1. We list which coefficients we consider and normalization conditions we use.

Differentiation of (3.3) with respect to z and evaluating the result at $(z,\chi,\tau) = (0,\chi,0)$ gives $\chi = \bar{f}_1(\chi,0)$ assuming the normalization conditions for the 1-jet of H at 0, hence (i) holds.

If we differentiate (3.3) twice with respect to τ and evaluate the result at 0 we obtain, using $H_w(0) = (0,0,1)$, that $\mathrm{Im}(g_{w^2}(0)) = 0$, which is the statement of (iii). In a similar way we obtain (iv), when differentiating three times with respect to τ.

Differentiation of (3.3) with respect to τ and evaluating the result at $(z,\chi,\tau) = (0,\chi,0)$ shows $\bar{g}_\tau(\chi,0) = 1$, again by $H_w(0) = (0,0,1)$, which implies (vi).

To get (ii) we differentiate (3.3) twice with respect to z and χ, evaluate at 0 and use $H_z(0) = (1,0,0)$, $f_{2z^2}(0) = 2$ and $g_{w^2}(0) = 0 = f_{1z^2}(0)$.

Differentiation of (3.3) twice with respect to z and once with respect to τ and χ and evaluating at 0 gives (vii) if we use $f_{1w}(0) = 0 = f_{2zw}(0)$, $H_{z^2}(0) = (0,2,0)$, $H_z(0) = (1,0,0)$ as well as (ii).

Taking derivatives of (3.3) three times with respect to z and twice with respect to χ and evaluate at 0 we use $H_z(0) = (1,0,0), f_{1z^3}(0) = 0$ and $f_{2z^2}(0) = 2$ to get (v).

If we differentiate (3.3) twice with respect to z and χ and once with respect to τ, evaluate at 0 and use $H_z(0) = (1,0,0), H_w(0) = (0,0,1), H_{z^2}(0) = (0,2,0), f_{1zw}(0) = \frac{\mathrm{i}\,\varepsilon}{2}, f_{2zw}(0) = 0$ and $\mathrm{Re}(f_{2z^2w}(0)) = 0$, we obtain (viii) according to (iv).

Finally, to obtain (ix) we differentiate (3.3) twice with respect to τ and once with respect to z, evaluate at 0 and use $f_{1z}(0) = 0 = f_{1w}(0)$. $\square$

Remark 4.6. We summarize the conditions for the 3-jet of $H \in \mathcal{N}_2$ at 0 by collecting the normalization conditions from Proposition 4.1 and their consequences given in Proposition 4.5:

(i) $H(0) = 0$

(ii) $H_z(0) = (1,0,0)$

(iii) $H_w(0) = (0,0,1)$

(iv) $H_{z^2}(0) = (0,2,0)$

(v) $H_{zw}(0) = (\frac{\mathrm{i}\,\varepsilon}{2},0,0)$

(vi) $H_{w^2}(0) = (|f_{1w^2}(0)|, f_{2w^2}(0), 0)$

(vii) $H_{z^3}(0) = (0, 12\,\varepsilon\,|f_{1w^2}(0)|, 0)$

(viii) $H_{z^2w}(0) = (4\,\mathrm{i}\,|f_{1w^2}(0)|, \mathrm{i}\,\mathrm{Im}(f_{2z^2w}(0)), 0)$

(ix) $H_{zw^2}(0) = \Big(\frac{1}{4}\big(-1 + 2\,\mathrm{Re}(g_{w^3}(0))\big), f_{2zw^2}(0), 2\,\mathrm{i}\,|f_{1w^2}(0)|\Big)$

(x) $H_{w^3}(0) = \Big(f_{1w^3}(0), f_{2w^3}(0), \mathrm{Re}\big(g_{w^3}(0)\big)\Big)$

We would like to point out the differences to the normalization used in [Ji10, Lemma 2.2], which is the normalization obtained by Huang [Hua03, Lemma 3.2]. In Huang's normal form a normalized mapping $\widehat{H}$ fulfills $\widehat{f}_{1w^2}(0) = 0$ assuming the original mapping H satisfies $f_{1zw}(0) \neq 0$, which is a consequence of having the so-called "geometric rank" equal to 1. The concept of this invariant is introduced in [Hua03, Definition 2.1]. We note that here a mapping of geometric rank 1 is 2-nondegenerate at 0 if we consider the normalized mapping in the sense of [Ji10] at the end of [Ji10, §3]. On the other hand if we start with a mapping in $\mathcal{N}_2$, then the geometric rank is 1, since it is of the form as in [Ji10, Lemma 2.1] with nontrivial condition (3) from [Ji10, Lemma 2.1].
Moreover in Huang's normal form the coefficient $\widehat{f}_{2zw}(0)$ is still present, which we require to be 0, since in our considerations the standard parameter c from $\mathrm{Aut}_0(\mathbb{H}^2, 0)$ is linear in $\widehat{f}_{2zw}(0)$ and has a nonzero coefficient, see (4.6).

4.2 Homeomorphic Variations of Normal Forms

In this section we investigate what happens, when we consider different admissible normal forms with respect to isotropies. The question is then, how does the resulting normal form differ from $\mathcal{N}_2$, given in Definition 4.2?

Definition 4.7. For $p \in \mathbb{C}^N$ and $p' \in \mathbb{C}^{N'}$ we denote by

$$\mathcal{H}(p; p') := \{H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p') : \ H \text{ holomorphic}\},$$

the *set of germs of holomorphic mappings from* $(\mathbb{C}^N, p)$ *to* $(\mathbb{C}^{N'}, p')$.
For $(M, p) \subset \mathbb{C}^N$ and $(M', p') \subset \mathbb{C}^{N'}$ germs of real-analytic hypersurfaces we denote by

$$\mathcal{H}(M, p; M', p') := \{H \in \mathcal{H}(p; p') : H(M \cap U) \subset M' \text{ for some neighborhood } U \text{ of } p\},$$

the *set of germs of holomorphic mappings from* (M, p) *to* (M', p') and denote $\mathcal{H}(M, p) := \mathcal{H}(M, p; M, p)$.

Definition 4.8. For $(M, p) \subset \mathbb{C}^N$ a germ of a real-analytic hypersurface we denote by

$$\mathrm{Aut}_p(M, p) := \{H \in \mathcal{H}(M, p) : |H'(p)| \neq 0\},$$

the *group of local automorphisms of* (M, p) or the *group of isotropies of* (M, p) *fixing* p.

Remark 4.9. For $G, H \in \mathcal{H}(M, p; M', p')$ the relation $G \sim H :\Leftrightarrow \exists (\phi, \phi') \in \mathrm{Aut}_p(M, p) \times \mathrm{Aut}_{p'}(M', p') : G = \phi' \circ H \circ \phi^{-1}$ defines an equivalence relation in $\mathcal{H}(M, p; M', p')$. The equivalence classes in $\mathcal{H}(M, p; M', p')/_\sim$ are denoted by $[F] := \{G \in \mathcal{H}(M, p; M', p') : G \sim F\}$.

Definition 4.10. (i) A proper subset $\mathcal{N} \subsetneq \mathcal{F} \subset \mathcal{H}(M, p; M', p')$ is called *normal form for* $\mathcal{F}$, if for each $[F] \in \mathcal{F}/_\sim$, there exists a unique representative $G \in \mathcal{N} \cap [F]$. We denote the mapping which assigns to each $H \in \mathcal{F}$ the representative $G \in \mathcal{N} \cap [H]$ as $\pi : \mathcal{F} \to \mathcal{N}$.

(ii) A normal form $\mathcal{N}$ for $\mathcal{F}$ is called *admissible* if $\pi : \mathcal{F} \to \mathcal{N}$ is continuous.

Remark 4.11. The uniqueness of the representative $F \in \mathcal{N} \cap [F]$ in Definition 4.10 (i) is no restriction: Assume we have another representative $F \neq G \in \mathcal{N}$ in the class $[F]$, then G is equivalent to F, hence

it suffices to choose only one element from the set of all representatives which belong to $\mathcal{N} \cap [F]$.
There exist admissible normal forms for $\mathcal{F}_2$, since $\mathcal{N}_2$ is an admissible normal form for $\mathcal{F}_2$. Thus for any admissible normal form $\mathcal{N}$ there is a unique element $\widehat{H} \in \mathcal{N}$ in each orbit of some – not necessarily admissible – normal form $\mathcal{N}'$. So we can always restrict ourselves to admissible normal forms.

The main theorem of this section is the following result for holomorphic mappings from $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$ belonging to $\mathcal{F}_2$.
For the discussion which topology we associate to $\mathcal{N}_2$ we refer to the beginning of section 9.

Theorem 4.12. *Let $\mathcal{N}$ be an admissible normal form for $\mathcal{F}_2$. Then $\mathcal{N}$ is homeomorphic to $\mathcal{N}_2$.*

Proof. Let us denote by $\pi : \mathcal{F}_2 \to \mathcal{N}$ the continuous mapping as in Definition 4.10 (i). We note that the class $\mathcal{N}_2$ from Definition 4.2 is an admissible normal form for $\mathcal{F}_2$ as in Definition 3.17: For $H \in \mathcal{F}_2$ the standard parameters $(\gamma, \gamma') \in \Gamma \times \Gamma'$ such that $\phi'_{\gamma'} \circ H \circ \phi_\gamma \in \mathcal{N}_2$ depends continuously on H, more precisely on $j_0^3(H)$, as can be seen in Remark 4.3. In this case we denote the corresponding continuous mapping by $\pi_2 : \mathcal{F}_2 \to \mathcal{N}_2$. Hence we have the following diagram:

$$\begin{tikzcd}
 & \mathcal{F}_2 \arrow[dl, shift right, "\pi"'] \arrow[dr, shift left, "\pi_2"] & \\
\mathcal{N} \arrow[ur, shift right, "\mathrm{incl}"'] \arrow[rr, "\psi"'] & & \mathcal{N}_2 \arrow[ul, shift left, "\mathrm{incl}_2"]
\end{tikzcd}$$

Figure 2: Diagram for admissible normal forms

The mapping $\mathrm{incl} : \mathcal{N} \to \mathcal{F}_2$ is the inclusion mapping, which is given by $\mathrm{incl}(H) := H$ for all $H \in \mathcal{N}$ and analogously for incl_2. In the diagram $\psi : \mathcal{N} \to \mathcal{N}_2$ is given as follows: Let $H \in \mathcal{N}$, then $\psi(H) := F \in \mathcal{N}_2$, where $F \in \mathcal{N}_2 \cap [H]$. By the uniqueness of the choice of representatives in each orbit of elements of $\mathcal{N}$ and $\mathcal{N}_2$ respectively and since both $\mathcal{N}$ and $\mathcal{N}_2$ are normal forms, we obtain that ψ is a bijective mapping. Further since $\psi = \pi_2 \circ \mathrm{incl}$ and $\psi^{-1} = \pi \circ \mathrm{incl}_2$ are compositions of continuous mappings, we obtain that ψ is a homeomorphism. $\square$

Example 4.13. Starting with $\mathcal{N}_2$ we can construct different admissible normal forms $\mathcal{N}$ as follows: We fix a pair of isotropies $(\phi_0, \phi'_0) \in \mathrm{Aut}_0(\mathbb{H}^2, 0) \times \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ and consider the isotropies $(\phi, \phi') \in \mathrm{Aut}_0(\mathbb{H}^2, 0) \times \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ from the proof of Proposition 4.1 or Remark 4.3, such that $\pi_2 : \mathcal{F}_2 \to \mathcal{N}_2$ is given by $\pi_2(H) := \hat{\phi}' \circ H \circ \hat{\phi}$, denoted by $\widehat{H}$. We define $\phi := \hat{\phi} \circ \phi_0$ and $\phi' := \phi'_0 \circ \hat{\phi}'$, to obtain for any $F \in \mathcal{F}_2$,

$$\phi' \circ F \circ \phi = \phi'_0 \circ \hat{\phi}' \circ F \circ \hat{\phi} \circ \phi_0 = \phi'_0 \circ \widehat{F} \circ \phi_0,$$

where $\widehat{F} \in \mathcal{N}_2$. Hence we define

$$\mathcal{N} := \left\{ \phi_0' \circ \widehat{F} \circ \phi_0 : \widehat{F} \in \mathcal{N}_2 \right\}.$$

Since $\hat{\phi}$ and $\hat{\phi}'$ depend continuously on $F \in \mathcal{F}_2$, the mapping $\pi : \mathcal{F}_2 \to \mathcal{N}$ given by $\pi(F) := \phi' \circ F \circ \phi$ is continuous, such that $\mathcal{N}$ is an admissible normal form.

5 Mappings in $\mathcal{N}_2$

Let us recall Theorem 1.4.

Theorem 5.1. *The set $\mathcal{N}_2$ consists of the following mappings, where we denote for $H = (f_1, f_2, g) \in \mathcal{N}_2$ the parameter $s := f_{1w^2}(0) \geq 0$:*

$$
\begin{aligned}
G_1^\varepsilon(z,w) &:= \Big(2z(2+\mathrm{i}\,\varepsilon\,w), 4z^2, 4w\Big)/(4-w^2),\\
G_{2,s}^\varepsilon(z,w) &:= \Big(4z - 4\,\varepsilon\,sz^2 + \mathrm{i}(\varepsilon - s^2)zw + sw^2, 4z^2 + s^2w^2, w(4 - 4\,\varepsilon\,sz - \mathrm{i}(\varepsilon + s^2)w)\Big)\\
&\qquad /\Big(4 - 4\,\varepsilon\,sz - \mathrm{i}(\varepsilon + s^2)w - 2\,\mathrm{i}\,szw - \varepsilon\,s^2w^2\Big),\\
G_{3,s}^\varepsilon(z,w) &:= \Big(256\,\varepsilon\,z + 96\,\mathrm{i}\,zw + 64\,\varepsilon\,sw^2 + 64z^3 + 64\,\mathrm{i}\,\varepsilon\,sz^2w - 3(3\,\varepsilon - 16s^2)zw^2 + 4\,\mathrm{i}\,sw^3,\\
&\qquad 256\,\varepsilon\,z^2 - 16w^2 + 256sz^3 + 16\,\mathrm{i}\,z^2w - 16\,\varepsilon\,szw^2 - \mathrm{i}\,\varepsilon\,w^3,\\
&\qquad w(256\,\varepsilon - 32\,\mathrm{i}\,w + 64z^2 - 64\,\mathrm{i}\,\varepsilon\,szw - (\varepsilon + 16s^2)w^2)\Big)\\
&\qquad /\Big(256\,\varepsilon - 32\,\mathrm{i}\,w + 64z^2 - 192\,\mathrm{i}\,\varepsilon\,szw - (17\,\varepsilon + 144s^2)w^2 + 32\,\mathrm{i}\,\varepsilon\,z^2w + 24szw^2 + \mathrm{i}\,w^3\Big).
\end{aligned}
$$

Each mapping in $\mathcal{N}_2$ is not isotropically equivalent to any different mapping in $\mathcal{N}_2$.
We write $G_{k,s}^\varepsilon = \big(f_{1k,s}^\varepsilon, f_{2k,s}^\varepsilon, g_{k,s}^\varepsilon\big)$ for the components.

Remark 5.2. (i) The mappings listed in Theorem 1.4 are equivalent to the mappings given in Theorem 5.1 via an application of dilations: We either apply automorphisms of the form $(z,w) \mapsto (2z, 4w)$ or $(z,w) \mapsto (\sqrt{2}z, 2w)$. In the case of the degree-3-mapping we also scale the parameter s by $s \mapsto s/4$.

(ii) Since G_k^ε are rational, these maps are globally defined. More precisely we denote the zero set of the denominator of G_k^ε in $\mathbb{H}^2$ by Q_k^ε, which may depend on s. Then each of the above mappings G_k^ε is actually defined in $V_k^\varepsilon := \mathbb{H}^2 \backslash Q_k^\varepsilon$ and sends V_k^ε to $\mathbb{H}^3_\varepsilon$. Note there is an open neighborhood U_k^ε of 0, which depends on s and is contained in V_k^ε.

(iii) In the definition of mappings listed in Theorem 5.1, which depend on the parameter $s \geq 0$, we could formally allow $s \in \mathbb{C}$. Then a small computation shows that we obtain mappings belonging to $\mathcal{F}_2$ only if we assume $s \in \mathbb{R}$.

The family of mappings $G_{3,s}^\varepsilon$ in Theorem 5.1 is not of degree 3 for each $s \geq 0$: If we set $\varepsilon = -1$ and $s = 1/2$ in $G_{3,s}^\varepsilon$ the denominator and the numerator of each component is divisible by $16\,\mathrm{i} - 8\,\mathrm{i}\,z + w$, resulting in a mapping of degree 2, which coincides with $G_{2,1/2}^-$. The following lemma shows that this is the only possibility.

Lemma 5.3. *The mapping $G_{3,s}^\varepsilon$ from Theorem 5.1 is of degree 2 if and only if $\varepsilon = -1$ and $s = \frac{1}{2}$ in $G_{3,s}^\varepsilon$.*

Proof. The necessary direction can be verified directly. The other direction is proved as follows: We let H denote an arbitrary rational mapping of degree 2 with $H(0) = 0$ defined in a sufficiently small neighborhood $U \subset \mathbb{C}^2$ of 0. We require H to be holomorphic in U. Then H is of the form $H =$

$(p_1, p_2, p_3)/q$, where for $1 \leq j \leq 3$ the terms p_j and q are polynomials of degree 2 given by

$$\begin{aligned} p_j(z,w) &= a_j z + b_j w + c_j z^2 + d_j zw + e_j w^2, \\ q(z,w) &= 1 + a_4 z + b_4 w + c_4 z^2 + d_4 zw + e_4 w^2, \end{aligned}$$

where each element of $\Lambda_m \coloneqq \{a_m, b_m, c_m, d_m, e_m\}$ is a complex number for $1 \leq m \leq 4$. We denote by Λ the collection of all Λ_m. If we compare the 3-jets of H and $G^{\varepsilon}_{3,s}$ and solve for the elements of Λ we obtain

$$H(z,w) = \frac{\Big(16z - 16s\,\varepsilon\, z^2 + 5\,\mathrm{i}\,\varepsilon\, zw + 4sw^2, 16z^2 - \varepsilon\, w^2, w\big(16 - 16\,\varepsilon\, sz - 3\,\mathrm{i}\,\varepsilon\, w\big)\Big)}{16 + 16\,\varepsilon\, sz + 3\,\mathrm{i}\,\varepsilon\, w + 8\,\mathrm{i}\, szw + (1 + 8\,\varepsilon\, s^2)w^2}.$$

Comparing the $f_{1z^3w}(0)$-coefficients of H and $G^{\varepsilon}_{3,s}$ we find a solution if and only if $\varepsilon = -1$ and $s = 1/2$. Then we observe with these choices the mapping H coincides with $G^{-}_{3,1/2}$. □

Remark 5.4. Clearly the formulas for $G^{\varepsilon}_{k,s}$ depend on our choices for the normalization conditions in Proposition 4.1, but we can say more: For mappings in $\mathcal{N}_2$ we prove Theorem 5.1 by only using isotropies of the source and target hypersurface to obtain 2 families of mappings parametrized by a nonnegative real number. In Theorem 4.12 we proved, that the picture we obtain from Theorem 5.1 is intrinsic. More precisely we have shown that whenever we consider a reasonable normal form $\mathcal{N}$, given in Definition 4.10, then $\mathcal{N}$ is homeomorphic to $\mathcal{N}_2$. In particular, it is not possible to reduce to finitely many mappings by considering only isotropies.
Moreover by Proposition 4.1 we observe that Theorem 5.1 gives a complete description of $\mathcal{N}_2$, such that $\mathcal{F}_2 = \bigcup_{k=1}^3 O_0(G^{\varepsilon}_k)$.
The proof of Theorem 5.1 is based on the following lemmas. After stating them, we show how Theorem 5.1 is deduced from these lemmas.
In the first lemma we obtain a so called *jet parametrization* for $H \in \mathcal{N}_2$ at 0 along the second Segre set. In order to simplify our formulas we introduce the following notation:

$$A_{k\ell} \coloneqq f_{1z^k w^\ell}(0), \quad B_{k\ell} \coloneqq f_{2z^k w^\ell}(0), \quad C_{k\ell} \coloneqq g_{z^k w^\ell}(0), \quad D_\ell \coloneqq D_{0\ell}, \tag{5.1}$$

for $k, \ell \geq 0$ and $D \in \{A, B, C\}$. In the list of coefficients of a mapping $H \in \mathcal{F}_2$ we gave in Remark 4.6, there are still some unknown coefficients belonging to J^4_0. These remaining coefficients we denote by

$$j \coloneqq (A_2, B_2, B_{21}, B_{12}, A_3, B_3, C_3, A_{22}, B_{22}, C_{22}, A_{13}, B_{13}, C_{13}, A_4, B_4, C_4)\,. \tag{5.2}$$

We refer to the coefficients $D_{k\ell}$ we listed in (5.2) as components of j. We set $N_0 \coloneqq 16$ and define the following set:

$$J \coloneqq \big\{ j \in \mathbb{C}^{N_0} : A_2 \geq 0, C_3 \in \mathbb{R}, B_{21} \in \mathrm{i}\,\mathbb{R} \big\} \subset \mathbb{C}^{N_0}. \tag{5.3}$$

We consider j from (5.2) as variable for $J \subset \mathbb{C}^{N_0}$.
The following lemma is based on [Lam01, Proposition 25, Corollary 26–27].

Lemma 5.5 (Jet Parametrization)**.** *Let $H \in \mathcal{N}_2$. Then there exists an explicitly computable, rational mapping Ψ satisfying*

$$H(z, 2\,\mathrm{i}\, z\chi) = \Psi\big(z, \chi, j\big) \tag{5.4}$$

for all $(z, \chi) \in \mathbb{C}^2$ sufficiently near 0. The formula for Ψ is given in Appendix A, where we scaled $j \in J$ for simplification.

Remark 5.6. In order to compute Ψ in Lemma 5.5 we only need to assume the nondegeneracy of H, but to simplify expressions we require $H \in \mathcal{N}_2$.

The approach we take in the next lemmas follows the line of thought of [BER97, Proposition 2.11–3.1,§6]. The following two lemmas are Proposition 2.11 and Proposition 3.1 from [BER97] adapted to our setting. We restrict to $\mathcal{N}_2$ to make computations easier to handle.

Lemma 5.7 ([BER97, Proposition 2.11])**.** *There exists a $\mathbb{C}^3$-valued function $\Phi(z, \chi, \Lambda)$, which is holomorphic in a neighborhood of $0 \times 0 \times J_0^4$ in $\mathbb{C} \times \mathbb{C} \times J_0^4$ and a germ at 0 of a nontrivial function $A(z)$, such that for a fixed $\Lambda_0 \in J_0^4$, satisfying the normalization conditions from Proposition 4.1, the following equivalence holds:*
There exists $H \in \mathcal{N}_2$ with

$$\left(\frac{\partial^{|\alpha|} H}{\partial Z^\alpha}(0)\right)_{|\alpha| \leq 4} = \Lambda_0, \tag{5.5}$$

if and only if all of the following properties are satisfied:

(i) *The map $(z, w) \mapsto \Phi\left(z, \frac{w}{A(z)}, \Lambda_0\right)$ extends to a function $\widehat{H}_{\Lambda_0}(z, w)$, which is holomorphic in a full neighborhood of 0 in $\mathbb{C}^2$.*
(ii) *We have $\left(\frac{\partial^{|\alpha|} \widehat{H}_{\Lambda_0}}{\partial Z^\alpha}(0)\right)_{|\alpha| \leq 4} = \Lambda_0$.*
(iii) *We have $\widehat{H}_{\Lambda_0}(\mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$.*

If (i), (ii) and (iii) hold, then the unique mapping $H \in \mathcal{N}_2$, which satisfies (5.5) is given by $H(Z) = \widehat{H}_{\Lambda_0}(Z)$.

Proof. The proof is the same as in [BER97, Proposition 2.11] and uses the jet parametrization for $\mathcal{N}_2 \subset \mathcal{F}_2$ from Lemma 5.5 instead of [BER97, Lemma 2.8]. □

It is possible to give conditions which are equivalent to (i), (ii) and (iii) of Lemma 5.7 by means of an explicit system of equations.

Lemma 5.8 ([BER97, Proposition 3.1])**.** *We use the notation as in Lemma 5.7. There exists a function $G(z, \Lambda)$, which is holomorphic in a neighborhood of $0 \times J_0^4$ in $\mathbb{C}^2 \times J_0^4$, such that Lemma 5.7 (i) holds for a fixed $\Lambda_0 \in J_0^4$ if and only if $\Phi(z, \frac{w}{A(z)}, \Lambda_0) \equiv G(z, w, \Lambda_0)$.*
The following equivalences also hold true:

(i) *There exist functions $a_k, k \in \mathbb{N}$, holomorphic in J_0^4, such that Lemma 5.7 (i) holds if and only if $a_k(\Lambda_0) = 0$ for $k \in \mathbb{N}$.*

(ii) *There exist functions $b_k, 1 \leq k \leq K$, holomorphic in J_0^4, such that if Lemma 5.7 (i) is satisfied, then Lemma 5.7 (ii) holds if and only if $b_k(\Lambda_0) = 0$ for $1 \leq k \leq K$.*

(iii) *There exist functions $c_k, k \in \mathbb{N}$, holomorphic in $J_0^4 \times J_0^4$, such that if Lemma 5.7 (i) is satisfied, then Lemma 5.7 (iii) holds if and only if $c_k(\Lambda_0, \bar{\Lambda}_0) = 0$ for $k \in \mathbb{N}$.*

Proof. The proof is the same as in [BER97, Proposition 3.1] and uses Lemma 5.7. □

The proof of Lemma 5.8 in [BER97, Proposition 3.1] explains how to obtain the equations for (i) – (iii) of Lemma 5.8. By using the approach of [BER97, Proposition 3.1] we give the following lemma, which guarantees that Lemma 5.8 (i) and (ii) hold. We give the resulting mappings here, instead of listing the equations of (i) and (ii). We refer to this step as "desingularization".

Lemma 5.9. *Let $H \in \mathcal{N}_2$ and Ψ be given as in Lemma 5.5. If*

$$\psi(z, w) \coloneqq \Psi\Big(z, \frac{w}{2\,\mathrm{i}\,z}, j\Big) \tag{5.6}$$

is holomorphic for $(z, w) \in \mathbb{C}^2$ near 0 and $j_0^4\psi = j_0^4 H$, then $\psi \in \{\psi_1, \ldots, \psi_5\}$ is of at most degree 3 and depends on $A_2, B_2, B_{21}, A_{22}, B_{22}$ and C_{22} satisfying $A_2 \geq 0$ and $\mathrm{Re}(B_{21}) = 0$, whenever these parameters are present in ψ. The concrete formulas for $(\psi_k)_{k=1,\ldots,5}$ are listed in Appendix C.

Next we show Lemma 5.8 (iii), which gives condition (iii) of Lemma 5.8, based on [BER97, Proposition 3.1]. Again we give the resulting maps, instead of the defining equations.

Lemma 5.10. *Let $U \subset \mathbb{C}^2$ be a sufficiently small neighborhood of 0 and $\psi \in \{\psi_1, \ldots, \psi_5\}$ from Lemma 5.9 satisfies $\psi(U \cap \mathbb{H}^2) \subset \mathbb{H}_\varepsilon^3$. Then $\psi \in \{G_1^\varepsilon, G_{2,s}^\varepsilon, G_{3,s}^\varepsilon\}$ from Theorem 5.1, where $s \coloneqq A_2 \geq 0$.*

Next we describe how to prove Theorem 5.1 from the previously stated lemmas, which might also be viewed as an easy proof of Lemma 5.7 and Lemma 5.8.

Proof of Theorem 5.1. Let $H \in \mathcal{N}_2$ and $U \subset \mathbb{C}^2$ be a sufficiently small neighborhood of 0. As in Definition 2.4 we write

$$\rho(z, w, \chi, \tau) \coloneqq w - \tau - 2\,\mathrm{i}\,z\chi,$$

for a defining function of the complexification of $\mathbb{H}^2$. For the parametrization of $\mathcal{S}_0^2$ we write as in (2.12)

$$Q(z, \chi, \tau) \coloneqq \tau + 2\,\mathrm{i}\,z\chi,$$

such that the second Segre set $\mathcal{S}_0^2$ of $\mathbb{H}^2$ at 0 is given as the image of $v_0^2(z, \chi) \coloneqq (z, 2\,\mathrm{i}\,z\chi)$, for $(z, \chi) \in U$ by (2.14). Then a point $(z_0, w_0) \in U$ is contained in $\mathcal{S}_0^2$ if and only if for some $\chi \in \mathbb{C}$ near 0 we have $w_0 = 2\,\mathrm{i}\,z_0\chi$.

Since v_0^2 is of rank 2 outside of the complex variety

$$X \coloneqq \big\{(z, \chi) \in U : z = 0\big\} \subset \mathbb{C}^2,$$

it follows that $\mathcal{S}_0^2$ contains an open set $V \subset U \setminus X$ of $\mathbb{C}^2$. From Lemma 5.5 we know after scaling the variable $j \in J$ from (5.2), that

$$H\big(v_0^2(z,\chi)\big) = \Psi(z,\chi,j) = \sum_{k,\ell} \Psi_{k\ell}(j) z^k \chi^\ell, \tag{5.7}$$

holds, where we have written Ψ in the Taylor expansion with coefficients $\Psi_{k\ell}(j) \in \mathbb{C}^3$ depending on $j \in J$. Then for $(z,w) \in V$ we have:

$$H(z,w) = H\left(v_0^2\left(z, \frac{w}{2\,\mathrm{i}\,z}\right)\right) = \Psi\left(z, \frac{w}{2\,\mathrm{i}\,z}, j\right) = \sum_{\alpha,\beta} \widehat{\Psi}_{\alpha\beta}(j) z^\alpha w^\beta, \tag{5.8}$$

where $\widehat{\Psi}_{\alpha\beta}(j) \in \mathbb{C}^3$. On the right-hand side of (5.8) there may occur terms as w^{ℓ_0}/z^{k_0} for $k_0 \geq 0$, but since the left-hand side of (5.8) is required to be holomorphic in a neighborhood of 0, (5.8) yields equations $\widehat{\Psi}_{\alpha\beta}(j) = 0$ for $\alpha < 0$. Equivalently from (5.7), we obtain equations

$$\Psi_{k\ell}(j) = 0, \qquad \ell > k. \tag{5.9}$$

We examine these equations for j in the proof of Lemma 5.9 to end up with $\Psi\big(z, w/(2\,\mathrm{i}\,z), j\big)$ being one of 5 holomorphic mappings $\widehat{\psi}_1(z,w), \ldots, \widehat{\psi}_5(z,w)$, defined in a neighborhood of 0 and given in Appendix C. Moreover (5.8) can only hold if

$$j_0^4 H(z,w) = j_0^4 \Psi\Big(z, \frac{w}{2\,\mathrm{i}\,z}, j\Big) = j_0^4 \widehat{\psi}_k(z,w),$$

for each $1 \leq k \leq 5$. We carry out these computations in the last part of the proof of Lemma 5.9, which yield H being one of the holomorphic mappings $\psi_1, \ldots, \psi_5$ according to Lemma 5.9 listed in Appendix C.

Since we require H being a mapping of $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$ and j was an arbitrary variable in J so far, we have to ensure ψ_k sends $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$ for $1 \leq k \leq 5$. This last step is carried out in Lemma 5.10 and we end up with the mappings $G_1^\varepsilon, G_{2,s}^\varepsilon$ and $G_{3,s}^\varepsilon$ as in Theorem 5.1, where $s = f_{1w^2}(0)$. The last claim, that the maps we listed in Theorem 5.1 are not isotropically equivalent is proved in Theorem 5.19 below. $\square$

The rest of the section is devoted to the proofs of Lemma 5.5, Lemma 5.9 and Lemma 5.10 and to give a jet determination result deduced from the jet parametrization.

5.1 Jet Parametrization

Proof of Lemma 5.5. We need to carry out the following steps: From the mapping equation we can determine H along the germ of the second Segre set $\mathcal{S}_0^2$ of $\mathbb{H}^2$ near 0 in terms of the 2-jet of H evaluated along the germ of the conjugated version of the first Segre set $\bar{\mathcal{S}}_0^1 = \{(\chi, 0) : \chi \in \mathbb{C}\}$ of $\mathbb{H}^2$ near 0. In a similar way we obtain formulas for the 2-jet of H along $\mathcal{S}_0^1$ depending on $j \in J$. In both steps it is essential that we assume 2-nondegeneracy. The resulting representation of H gives the desired mappings Ψ depending on j. Now we present the detailed version of the proof.

Using the notation as in the proof of Proposition 3.13 we start by computing

$$\Phi_{r+1}(z,w,\chi,\tau) := L^r \rho'\big(H(z,w),\bar{H}(\chi,\tau)\big), \qquad 0 \le r \le 2,$$

to obtain

$$\begin{aligned}
\Phi_1(z,w,\chi,\tau) &:= g(z,w) - \bar{g}(\chi,\tau) - 2\,\mathrm{i}(f_1(z,w)\bar{f}_1(\chi,\tau) + \varepsilon\, f_2(z,w)\bar{f}_2(\chi,\tau)),\\
\Phi_2(z,w,\chi,\tau) &:= -\,\bar{g}_\chi(\chi,\tau) + 2\,\mathrm{i}\, z\bar{g}_\tau(\chi,\tau) - 2\,\mathrm{i}\Big(f_1(z,w)\big(\bar{f}_{1\chi}(\chi,\tau) - 2\,\mathrm{i}\, z\bar{f}_{1\tau}(\chi,\tau)\big)\\
&\qquad + \varepsilon\, f_2(z,w)\big(\bar{f}_{2\chi}(\chi,\tau) - 2\,\mathrm{i}\, z\bar{f}_{2\tau}(\chi,\tau)\big)\Big),\\
\Phi_3(z,w,\chi,\tau) &:= -\,\bar{g}_{\chi^2}(\chi,\tau) + 4\,\mathrm{i}\, z\bar{g}_{\chi\tau}(\chi,\tau) + 4z^2\bar{g}_{\tau^2}(\chi,\tau)\\
&\qquad - 2\,\mathrm{i}\Big(f_1(z,w)\big(\bar{f}_{1\chi^2}(\chi,\tau) - 4\,\mathrm{i}\, z\bar{f}_{1\chi\tau}(\chi,\tau) - 4z^2\bar{f}_{1\tau^2}(\chi,\tau)\big)\\
&\qquad + \varepsilon\, f_2(z,w)\big(\bar{f}_{2\chi^2}(\chi,\tau) - 4\,\mathrm{i}\, z\bar{f}_{2\chi\tau}(\chi,\tau) - 4z^2\bar{f}_{2\tau^2}(\chi,\tau)\big)\Big).
\end{aligned} \tag{5.10}$$

We introduce the following variables for expressions which occur in Φ_j for $1 \le j \le 3$:

$$\begin{aligned}
(Z,\zeta) &:= (z,w,\chi,\tau) \in \mathbb{C}^4, \qquad (Z',\zeta') := \big(H(z,w),\bar{H}(\chi,\tau)\big) \in \mathbb{C}^6,\\
W &:= \left(\frac{\partial^{|\beta|}}{\partial\zeta^\beta}\bar{H}(\chi,\tau)\right)_{1\le|\beta|\le 2} \in \mathbb{C}^{15}.
\end{aligned}$$

By a slight abuse of notation we obtain $\Phi_j(Z,\zeta,Z',\zeta',W) = 0$ for $1 \le j \le 3$ when restricted to $\mathbb{H}^2$, i.e., setting $Z = (z, \tau + 2\,\mathrm{i}\, z\chi)$. Further if we write $\Phi := (\Phi_1,\Phi_2,\Phi_3)$ we have

$$\det\left(\frac{\partial\Phi}{\partial Z'}(0)\right) = \varepsilon\Big(\bar{f}_{2\chi}(0)\bar{f}_{1\chi^2}(0) - \bar{f}_{1\chi}(0)\bar{f}_{2\chi^2}(0)\Big) = -\,\varepsilon \neq 0, \tag{5.11}$$

since we assumed $H \in \mathcal{N}_2 \subset \mathcal{F}_2$. Hence we can explicitly solve the system given in (5.10) for Z' near 0 as follows. We denote by $B(z,\chi,\tau)$ the matrix

$$\begin{pmatrix}
\bar{f}_1(\chi,\tau) & \varepsilon\,\bar{f}_2(\chi,\tau) & -\frac{\mathrm{i}}{2}\\
\bar{f}_{1\chi}(\chi,\tau) - 2\,\mathrm{i}\, z\bar{f}_{1\tau}(\chi,\tau) & \varepsilon\big(\bar{f}_{2\chi}(\chi,\tau) - 2\,\mathrm{i}\, z\bar{f}_{2\tau}(\chi,\tau)\big) & 0\\
\bar{f}_{1\chi^2}(\chi,\tau) - 4\,\mathrm{i}\, z\bar{f}_{1\chi\tau}(\chi,\tau) - 4z^2\bar{f}_{1\tau^2}(\chi,\tau) & \varepsilon\big(\bar{f}_{2\chi^2}(\chi,\tau) - 4\,\mathrm{i}\, z\bar{f}_{2\chi\tau}(\chi,\tau) - 4z^2\bar{f}_{2\tau^2}(\chi,\tau)\big) & 0
\end{pmatrix},$$

thus we have for all $(z,\chi,\tau) \in \mathbb{C}^3$ near 0 the following identity

$$H(z,\tau + 2\,\mathrm{i}\, z\chi) = \frac{1}{2\,\mathrm{i}} B^{-1}(z,\chi,\tau)\begin{pmatrix}
-\bar{g}(\chi,\tau)\\
-\bar{g}_\chi(\chi,\tau) + 2\,\mathrm{i}\, z\bar{g}_\tau(\chi,\tau)\\
-\bar{g}_{\chi^2}(\chi,\tau) + 4\,\mathrm{i}\, z\bar{g}_{\chi\tau}(\chi,\tau) + 4z^2\bar{g}_{\tau^2}(\chi,\tau)
\end{pmatrix}. \tag{5.12}$$

If we evaluate (5.12) at $\tau = 0$ we obtain a formula for H along $\mathcal{S}_0^2$ depending on the 2-jet of $\bar{H}$ along $\bar{\mathcal{S}}_0^1$. So to finish our computations we need to find formulas for $j^2_{(\chi,0)}\bar{H}$. To this end we introduce the vector field S tangent to $\mathbb{H}^2$ defined as

$$S \coloneqq \frac{\partial}{\partial w} + \frac{\partial}{\partial \tau},$$

such that $S^k H(z, \tau + 2\,\mathrm{i}\, z\chi) = H_{w^k}(z, \tau + 2\,\mathrm{i}\, z\chi)$ for $k \in \mathbb{N}$. Applying S and S^2 to (5.12) and setting $\chi = 0$ and $\tau = 0$ we obtain formulas for $H_w(z, 0)$ and $H_{w^2}(z, 0)$ respectively, which are rational and depend on $j \in J$. After conjugating these expressions we obtain the components of $j^2_{(\chi,0)}\bar{H}$ as rational function of j, which consists of components of $j^4_0 H$, see (5.2).

The resulting mapping is denoted by Ψ and depends on $j \in J$. In order to get rid of powers of 2 in formulas we scale j as follows:

$$\begin{aligned}&(A_2, B_2, B_{12}, A_3, B_3, C_3, A_{22}, B_{22}, A_{13}, B_{13}, C_{13}, A_4, B_4, C_4) \mapsto \\ &\left(\frac{A_2}{2}, \frac{B_2}{2}, \frac{B_{12}}{4}, \frac{A_3}{4}, \frac{B_3}{4}, \frac{C_3}{2}, \frac{A_{22}}{2}, \frac{B_{22}}{2}, \frac{A_{13}}{8}, \frac{B_{13}}{8}, \frac{C_{13}}{4}, \frac{A_4}{8}, \frac{B_4}{8}, \frac{C_4}{4}\right).\end{aligned} \tag{5.13}$$

The numerator of the components of H are polynomials of highest degree $(3, 8)$ in (z, χ) and are homogeneous in z. The components of H have the same denominator, which is a polynomial of highest degree $(3, 9)$ in (z, χ). The complete expression is listed in Appendix A. □

5.2 Desingularization

We introduce the following relation:

Definition 5.11. For $J_1, J_2 \subset J$ from (5.3) we denote variables $j_1 \in J_1$ and $j_2 \in J_2$ as in (5.2) respectively. We set $\Psi_1(z, \chi) \coloneqq \Psi(z, \chi, j_1)$ and $\Psi_2(z, \chi) \coloneqq \Psi(z, \chi, j_2)$, where Ψ is given in Lemma 5.5. We say that Ψ_1 is a *special case* of Ψ_2, if $J_1 \subset J_2$.

More geometrically this means that the variety given by the defining equations for Ψ_1 is contained in the variety generated by the defining equations for Ψ_2.

Proof of Lemma 5.9. As described in the proof of Theorem 5.1, in (5.7) we expand the mapping $\Psi\big(z, \chi, j\big)$ from (5.4) into a power series

$$\Psi(z, \chi, j) = \sum_{k,\ell} \Psi_{k\ell}(j) z^k \chi^\ell,$$

around 0. For the components we write

$$\Psi_{k\ell}(j) = \Big(\Psi^1_{k\ell}(j), \Psi^2_{k\ell}(j), \Psi^3_{k\ell}(j)\Big) \in \mathbb{C}^3,$$

and then we set

$$\Psi_{k\ell}(j) = 0, \qquad \forall\ \ell > k, \tag{5.14}$$

as in (5.9), which are obtained by the expansion given in (5.8). These equations allow us to obtain conditions for $j \in J$. Each solution of an equation from (5.14) corresponds to considering maps as in (5.4), but instead $j \in J$ we have $j \in J'$, where J' is a subvariety of J. This means that we gradually restrict the space of possible mappings in $\mathcal{F}_2$. In the following we describe which coefficients $\Psi_{k\ell}$ we consider and which components of j we can eliminate from equations given as in (5.14).

We start considering $\Psi^3_{34} = 0 = \Psi^1_{34} = \Psi^3_{45} = \Psi^1_{45}$, which determine the following components of j:

$$
\begin{aligned}
A_3 &= \frac{\mathrm{i}}{2}\Big(6A_2^3 + 3\,\varepsilon\, B_{12} - A_2\big(6B_2 + \varepsilon(-3 + C_3)\big)\Big)\\
B_3 &= \frac{\varepsilon}{10}\Big(-18\,\mathrm{i}\,\varepsilon\, A_2^4 + 15\,\mathrm{i}\, A_2 B_{12} - 2B_2\big(9\,\varepsilon\, B_{21} + 4\,\mathrm{i}(-3 + C_3)\big) + 3\,\mathrm{i}\, A_2^2(-3 + 6\,\varepsilon\, B_2 - 6\,\mathrm{i}\,\varepsilon\, B_{21} + C_3)\Big)\\
A_4 &= \frac{\varepsilon}{5}\Big(-324\,\varepsilon\, A_2^5 - 15A_2^2(-\varepsilon\, A_{22} + 2B_{12} + \varepsilon\, C_{13}) + 5\big(-3\,\varepsilon\, A_{22}B_2 + \mathrm{i}\, B_{13} + B_{12}\big(-6\,\mathrm{i}\, B_{21}\\
&\quad + \varepsilon(-6 + C_3)\big) + 3\,\varepsilon\, B_2 C_{13}\big) + A_2\big(-5\,\mathrm{i}\, A_{13} + 30\,\mathrm{i}\, B_{21} + 10\,\mathrm{i}\, B_{21}C_3 - 5\,\varepsilon(6B_{21}^2 + (-5 + C_3)C_3)\\
&\quad + 5\,\mathrm{i}\, C_4 + 3B_2(44 + 48\,\mathrm{i}\,\varepsilon\, B_{21} - 18C_3 + 15\,\mathrm{i}\, C_{22})\big) + 3A_2^3(-34 + 108\,\varepsilon\, B_2 - 28\,\mathrm{i}\,\varepsilon\, B_{21} + 28C_3\\
&\quad - 15\,\mathrm{i}\, C_{22})\Big)\\
B_4 &= \frac{\varepsilon}{20}\Big(3060\,\varepsilon\, A_2^6 - 45\,\varepsilon\, B_{12}^2 + 2B_2\big(-40\,\mathrm{i}\, A_{13} + 102\,\varepsilon\, B_{21}^2 + 5\,\mathrm{i}\, B_{21}(-33 + 23C_3)\\
&\quad + \varepsilon(-42 - 30B_{22} + 78C_3 - 28C_3^2) + 40\,\mathrm{i}\, C_4\big) + 180A_2^3(-\varepsilon\, A_{22} + B_{12} + \varepsilon\, C_{13})\\
&\quad + 20A_2\big(9\,\varepsilon\, A_{22}B_2 + \mathrm{i}\, B_{13} + 6\,\mathrm{i}\, B_{12}B_{21} + 2\,\varepsilon\, B_{12}C_3 - 3B_2(B_{12} + 3\,\varepsilon\, C_{13})\big)\\
&\quad + A_2^2\big(60\,\mathrm{i}\, A_{13} + 900\,\varepsilon\, B_2^2 + 150\,\mathrm{i}\, B_{21} - 290\,\mathrm{i}\, B_{21}C_3 + \varepsilon(9 - 24B_{21}^2 + 60B_{22} - 106C_3 + 61C_3^2)\\
&\quad - 60\,\mathrm{i}\, C_4 + 12B_2(-79 - 117\,\mathrm{i}\,\varepsilon\, B_{21} + 63C_3 - 65\,\mathrm{i}\, C_{22})\big) - 12A_2^4(-69 + 330\,\varepsilon\, B_2 - 97\,\mathrm{i}\,\varepsilon\, B_{21}\\
&\quad + 73C_3 - 45\,\mathrm{i}\, C_{22}) + 240\,\mathrm{i}\, B_2^2 C_{22}\Big)
\end{aligned}
$$

Then we consider $\Psi^2_{34} = 0$ to obtain two cases, either

(i) Case A: $B_{12} = \frac{2\,\varepsilon\, A_2}{5}\Big(6A_2^2 + 5B_2 + 6\,\mathrm{i}\, B_{21} + \varepsilon(3 - C_3)\Big)$, or

(ii) Case B: $B_2 = A_2^2$.

Next we assume one of the expressions for B_{12} or B_2 respectively for Ψ and consider another equation from (5.14) in order to solve for further components of j in terms of the remaining elements. It turns out that each of the remaining equations of the system given in (5.14) has more than one possible solution, resulting in a case distinction, when we solve one equation. In Appendix B we give two diagrams of this elimination process for case A and case B respectively. In these diagrams we keep track of all the equations $\Psi_{k\ell}(j) = 0$ we consider, which components of j we are able to determine and which holomorphic expressions we obtain in the end. Now we describe the diagrams in a more detailed way:

Let us write $\gamma := (A_2, C_3, B_{21}, C_4, A_{13}, B_{13}, C_{13}, A_{22}, B_{22}, C_{22})$. In case A Ψ still depends on the variables γ and B_2 and in case B Ψ depends on the variables γ and B_{12}. Since both cases are treated in the same way we write Λ for the set of the remaining variables in Ψ with components denoted by $(D_1, \ldots, D_{11})$.

Inductively we start considering equations $\Psi^j_{k\ell} = 0$, which determine further variables $D_{m_1}, \ldots, D_{m_n} \in \Lambda$, where $1 \leq m_j \leq 11$ for $1 \leq j \leq n$. Each determined variable D_{m_j} corresponds to a case E_{rs_i}. It turns out that we have $0 \leq r \leq 7$ and $1 \leq s_i \leq 13$, where $r = 0$ corresponds to the starting node from case A or B. The notation for E_{rs_i} is chosen in a way such that the first index r indicates the number

of nodes one has to pass in order to get from the starting node, i.e., case A or case B from above, to E_{rs_i}.
Let us denote by E some already achieved case, starting with case A or case B. In the diagram such an induction step is displayed as in the following Figure 3:

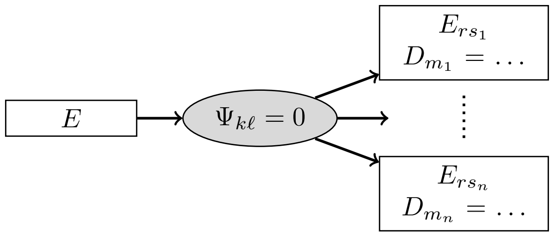

Figure 3: Diagram for new cases

Now we take all parameters from the preceding cases of E_{rs_i}, plug them into Ψ and denote the resulting rational mapping by $\varphi(z, \chi)$. Then we have several possibilities:

(i) If $\varphi(z, \frac{w}{2\,\mathrm{i}\,z})$ is holomorphic near 0 we do not consider further equations. Then we have the possibility that φ is a special case of a holomorphic mapping φ' from some other case, which is indicated in Figure 4 or φ is not a special case of any of the occurring mappings in the diagrams, which is indicated in Figure 5.

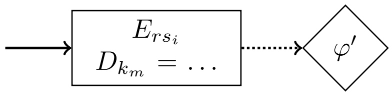

Figure 4: Diagram for special cases of holomorphic maps

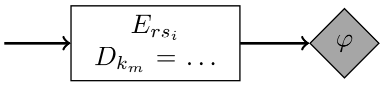

Figure 5: Diagram for new holomorphic maps

(ii) If $\varphi(z, \frac{w}{2\,\mathrm{i}\,z})$ is not holomorphic, we either proceed with another induction step as shown in Figure 3 or we recognize that the mapping φ is a special case of a mapping φ'' from some case $E_{r''s_i''}$. We indicate this situation as $E_{rs_i} \subset E_{r''s_i''}$, which is shown in the following Figure 6.

$$\longrightarrow \boxed{\begin{array}{c} E_{rs_i} \subset E_{r''s_i''} \\ D_{k_m} = \ldots \end{array}}$$

Figure 6: Diagram for special cases of maps

The complete case distinction is carried out in Appendix B, where we denote the cases E_{rs_i} by "Ars_i" and "Brs_i" for case A and B respectively. As mentioned above after at most 7 steps the process terminates, which means, that after setting $\chi = \frac{w}{2\,\mathrm{i}\,z}$ in $\Psi(z, \chi, j)$ we obtain a holomorphic expression. It turns out that we obtain 5 rational, holomorphic mappings, which we denote by $\widehat{\psi}_k(z, w)$ for $1 \leq k \leq 5$, as can be seen in the diagrams and is indicated in Figure 5. We point out that these mappings include all $H \in \mathcal{N}_2$ by construction. The formulas for $\widehat{\psi}_k$ are given in Appendix C.

We write $\widehat{\psi}_k = (\widehat{\psi}_k^1, \widehat{\psi}_k^2, \widehat{\psi}_k^3)$ and proceed by verifying $j_0^4 \widehat{\psi}_k = j_0^4 H$. For some $\widehat{\psi}_k$ this allows us to determine further parameters from (5.2). We also have to take into account, that we scaled $j \in J$, when we compare the coefficients of $\widehat{\psi}_k$ and the initial coefficients of H. Whenever we have expressed one component of j in terms of the remaining components we use this expression for the subsequent computations.

First we treat $\widehat{\psi}_1$ and consider $\widehat{\psi}^3_{1w^3}(0) = \frac{C_3}{2}$ to obtain

$$C_3 = 3(1 + 2\,\mathrm{i}\,\varepsilon\, B_{21}).$$

Next we consider $\widehat{\psi}^1_{1z^2w^2}(0) = \frac{A_{22}}{2}$ to get

$$C_{13} = \frac{3A_{22}}{2}.$$

Then we inspect $\widehat{\psi}^2_{1z^2w^2}(0) = \frac{B_{22}}{2}$ which gives

$$C_4 = A_{13} + 18B_{21} - 6\,\mathrm{i}\,\varepsilon\big(1 - B_{21}^2\big).$$

Verifying the normalization conditions we obtain $\mathrm{Re}(B_{21}) = 0$ and we end up with the mapping ψ_1 as claimed, which still depends on B_{21}, A_{22}, B_{22} and C_{22} and is given in Appendix C.

For $\widehat{\psi}_2$ we start with considering $\widehat{\psi}^1_{2zw^2}(0) = \frac{A_{13}}{8}$ to obtain

$$A_{13} = -10B_{21} + \mathrm{i}\,\varepsilon(4 + B_{22}) + C_4 - 2\,\mathrm{i}\,\varepsilon\, A_2(A_{22} - C_{13}) + 2A_2^2(6\,\mathrm{i} - C_{22}),$$

such that $\widehat{\psi}_2$ is independent of B_{22} and C_4. Then we compute $\widehat{\psi}^3_{2zw^3}(0) = \frac{C_{13}}{4}$ to get

$$C_{13} = \frac{3}{2}\Big(A_{22} + A_2\big(2\,\mathrm{i}\, B_{21} + \varepsilon(4 - \mathrm{i}\, C_{22})\big)\Big).$$

The rest of the coefficients are already in the correct form and the normalization conditions give $A_2 \geq 0$

and $\operatorname{Re}(B_{21}) = 0$. The resulting mapping is denoted by ψ_2, depends on A_2, B_{21}, A_{22} and C_{22} and is given in Appendix C.
The maps $\widehat{\psi}_k$ for $k = 3, 4, 5$ already satisfy $j_0^4 \widehat{\psi}_k = j_0^4 H$ and by verifying the normalization conditions we obtain for $k = 3, 5$ that $A_2 \geq 0$ and additionally for $k = 3$ that $\operatorname{Re}(B_{21}) = 0$. Finally we denote $\psi_k = \widehat{\psi}_k$ for $k = 3, 4, 5$. The mapping ψ_3 depends on A_2, B_2 and B_{21}, ψ_4 on B_2 and C_{22} and ψ_5 depends on A_2 and C_{22}. All these mappings are given in Appendix C. □

5.3 Reduction to One-Parameter-Families of Mappings

In order to achieve the normalization condition $f_{1w^2}(0) \in \mathbb{R}$ for rational mappings $H = (f_1, f_2, g)$ of a certain form, instead of using the parameter u' from $\operatorname{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$, we mention a simple observation in the following lemma.

Lemma 5.12. *Let G_2 be the following rational mapping of degree 2:*

$$\begin{aligned} G_2(z,w) = &\Big(a_1 z + a_2 e^{-\mathrm{i}\,\theta} z^2 + a_3 zw + a_4 e^{\mathrm{i}\,\theta} w^2, b_1 z^2 + b_2 e^{2\,\mathrm{i}\,\theta} w^2, w(1 + c_1 e^{-\mathrm{i}\,\theta} z + c_2 w)\Big)/ \\ &\Big(1 + d_1 e^{-\mathrm{i}\,\theta} z + d_2 w + d_3 e^{-\mathrm{i}\,\theta} zw + d_4 w^2\Big), \end{aligned}$$

where $a_k, b_\ell, c_\ell, d_k \in \mathbb{C}$ for $1 \leq k \leq 4$ and $\ell = 1, 2$, $\theta \in \mathbb{R}$ and let G_3 be a rational mapping of degree 3 of the following form:

$$\begin{aligned} G_3(z,w) = &\Big(a_1 e^{2\,\mathrm{i}\,\theta} z + a_2 e^{2\,\mathrm{i}\,\theta} zw + a_3 e^{3\,\mathrm{i}\,\theta} w^2 + a_4 z^3 + a_5 e^{\mathrm{i}\,\theta} z^2 w + a_6 e^{2\,\mathrm{i}\,\theta} zw^2 + a_7 e^{3\,\mathrm{i}\,\theta} w^3, \\ &e^{\mathrm{i}\,\theta}(b_1 e^{\mathrm{i}\,\theta} z^2 + b_2 e^{3\,\mathrm{i}\,\theta} w^2 + b_3 z^3 + b_4 e^{\mathrm{i}\,\theta} z^2 w + b_5 e^{2\,\mathrm{i}\,\theta} zw^2 + b_6 e^{3\,\mathrm{i}\,\theta} w^3), \\ &w\big(c_1 e^{2\,\mathrm{i}\,\theta} + c_2 e^{\mathrm{i}\,\theta} z + c_3 e^{2\,\mathrm{i}\,\theta} w + c_4 z^2 + c_5 e^{\mathrm{i}\,\theta} zw + c_6 e^{2\,\mathrm{i}\,\theta} w^2\big)\Big)/ \\ &\Big(e^{2\,\mathrm{i}\,\theta} + d_1 e^{2\,\mathrm{i}\,\theta} w + d_2 z^2 + d_3 e^{\mathrm{i}\,\theta} zw + d_4 e^{2\,\mathrm{i}\,\theta} w^2 + d_5 z^2 w + d_6 e^{\mathrm{i}\,\theta} zw^2 + d_7 e^{2\,\mathrm{i}\,\theta} w^3\Big), \end{aligned}$$

where $a_k, b_\ell, c_m, d_k \in \mathbb{C}$ for $1 \leq k \leq 7$, $1 \leq \ell \leq 6$ and $1 \leq m \leq 8$, $\theta \in \mathbb{R}$.
Then, after setting $v = e^{-\mathrm{i}\,\theta}$ in (2.36) and considering

$$\check{G}_k \coloneqq U_3'(v) \circ G_k \circ U_2(v),$$

for $k = 2, 3$, we obtain $\check{G}_k$ is independent of θ.

Proof of Lemma 5.10. We plug ψ_k into the complexified version of the mapping equation (3.3) and compare coefficients with respect to z, χ and τ. We list the monomials $z^k \chi^\ell \tau^m$ we consider in the mapping equation and which of the remaining coefficients of H in ψ_k we are able to determine. Whenever B_{21} is present in ψ_k we write $B_{21} = \mathrm{i}\, b_{21}$, where $b_{21} \in \mathbb{R}$. Moreover we recall that $A_2 \geq 0$.
We start with ψ_1 in which we have the terms b_{21}, A_{22}, B_{22} and C_{22}. The coefficient of $\chi^2\tau^2$ yields $C_{22} = 0$ and $\chi\tau^3$ gives $A_{22} = 0$. We write $B_{22} = \operatorname{Re}(B_{22}) + \mathrm{i} \operatorname{Im}(B_{22})$ to get from τ^4

$$\operatorname{Re}(B_{22}) = 2\big(1 - 3\,\varepsilon\, b_{21} + b_{21}^2\big).$$

The coefficient of τ^6gives the following equation:

$$\operatorname{Im}(B_{22})^2 + 4b_{21}^2\big(1 - 2\,\varepsilon\, b_{21}\big)^2 = 0.$$

Thus $\operatorname{Im}(B_{22}) = 0$ and either $b_{21} = 0$ or $b_{21} = \varepsilon/2$. The first case $b_{21} = 0$ results in G_1^ε and from the second case if $b_{21} = \varepsilon/2$ we obtain $G_{2,0}^\varepsilon$.

Next, we insert ψ_2 into (3.3), which depends on A_2, b_{21}, A_{22} and C_{22}. The coefficient of $\chi^2\tau^2$ gives

$$C_{22} = 2\,\mathrm{i}\,\varepsilon\, A_2^2,$$

and the coefficient of $z\chi^2\tau^2$ shows

$$A_{22} = 2\big(5\,\mathrm{i}\, A_2 b_{21} + 3A_2^3\big).$$

The coefficient of τ^4 yields two cases: Either $b_{21} = -\frac{3A_2^2}{2}$ or $b_{21} = \frac{\varepsilon + A_2^2}{2}$.
Assuming the first case $b_{21} = -\frac{3A_2^2}{2}$, we obtain from the coefficient of τ^6 either $A_2 = 0$, which results in G_1^ε, or

$$1 + 12\,\varepsilon\, A_2^2 + 48A_2^4 + 64\,\varepsilon\, A_2^6 = 0,$$

which, since $A_2 \geq 0$, has the only solution if we take $\varepsilon = -1$ and $A_2 = 1/2$. This choice of parameters gives $G_{2,1/2}^-$.
In the second case $b_{21} = \frac{\varepsilon + A_2^2}{2}$ we immediately obtain the mapping $G_{2,s}^\varepsilon$, where we set $s = A_2 \geq 0$.

If we handle ψ_3, which depends on A_2, B_2 and b_{21}, we first consider the coefficient of $\chi^2\tau^2$ in (3.3) to get $B_2 = A_2^2$. Then the coefficient of τ^4 yields two cases:
The first one is $b_{21} = \frac{A_2^2}{2}$. If we consider the coefficient of $\chi\tau^3$ we obtain $A_2 = 0$ and thus the mapping G_1^ε.
The second case is $b_{21} = \frac{\varepsilon + A_2^2}{2}$ which again gives $G_{2,s}^\varepsilon$ after setting $s = A_2 \geq 0$.

Treating ψ_4, which depends on B_2 and C_{22}, we proceed as follows: The coefficient of $\chi^2\tau^2$ shows $C_{22} = 2\,\mathrm{i}\,\varepsilon\,\bar{B}_2$ and τ^4 gives $B_2 = \frac{e^{\mathrm{i}\,t}}{4}$ for $t \in \mathbb{R}$. In order to get rid of $e^{\mathrm{i}\,t}$ in ψ_4 we apply $U_2(v)$ and $U_3'(v)$ from (2.36) as in Lemma 5.12 with

$$v = \frac{2e^{-\frac{\mathrm{i}\,t}{2}}}{1 - \varepsilon + \mathrm{i}(1 + \varepsilon)} \in \mathbb{S}^1,$$

to ψ_4, which does not affect the normalization. The resulting mapping is $G_{3,0}^\varepsilon$.

Finally we deal with ψ_5 in which the terms A_2 and C_{22} occur. We write $C_{22} = \operatorname{Re}(C_{22}) + \mathrm{i}\operatorname{Im}(C_{22})$ and consider the coefficient of $\chi^2\tau^2$ to obtain $\operatorname{Im}(C_{22}) = -\frac{1}{2}$ and $\operatorname{Re}(C_{22}) = 0$. We end up with the mapping $G_{3,s}^\varepsilon$ after setting $s = A_2 \geq 0$, which completes the proof of the lemma. $\square$

5.4 Jet Determination

In this section we provide a jet determination result based on Theorem 5.1, with the consequence that we do not need to consider all elements of the 4-jet of a mapping at 0 as in Lemma 5.5 when we compare two mappings and would like to show that they coincide.

Corollary 5.13 (Jet determination for $\mathcal{F}_2$)**.** *Let $U \subset \mathbb{C}^2$ be a neighborhood of 0 and $H : U \to \mathbb{C}^3$ a holomorphic mapping. We denote the components of H by $H = (f_1, f_2, g)$ and write $f = (f_1, f_2)$. Further let Λ be the collection of $j_0^2 H$ and the coefficients $f_{z^2w}(0)$. If for $H_1, H_2 \in \mathcal{F}_2$ the coefficients belonging to Λ coincide, we have $H_1 \equiv H_2$.*

Proof. We note that $\mathcal{N}_2$ is the collection of the mappings $G_1^\varepsilon, G_{2,s}^\varepsilon$ and $G_{3,s}^\varepsilon$ from Theorem 5.1. The only parameter left in elements of $\mathcal{N}_2$ is $s = f_{1w^2}(0)$. Let $H_1, H_2 \in \mathcal{N}_2$, then we need to verify that if the coefficients which belong to Λ coincide, this yields $H_1 \equiv H_2$.
If $s = 0$ in some H_1 or H_2, then the mappings H_1 and H_2 already differ considering the elements of Λ if we look at the coefficients $f_{2w^2}(0)$ and $f_{2z^2w}(0)$.
If $s \neq 0$, the coefficient $f_{1w^2}(0)$ yields that we may have $G_{2,s}^\varepsilon = G_{3,t}^\varepsilon$ for some $s, t \geq 0$. According to Lemma 5.3 this is only possible if and only if $t = s = 1/2$ and $\varepsilon = -1$. In this case we have $G_{2,1/2}^- \equiv G_{3,1/2}^-$. Next we note the following: In order to be able to apply Theorem 5.1 to a mapping $H \in \mathcal{F}_2$ we need to compose H with isotropies according to Proposition 4.1. We see from the proof of Proposition 4.1 and Remark 4.4 that the standard parameters used to normalize H precisely depend on the elements of Λ as well as $g_{z^2w}(0)$ and $f_{z^3}(0)$. To show the dependence of $g_{z^2w}(0)$ on $j_0^2 f$ we take derivatives of (3.3) twice with respect to z and once with respect to τ and evaluate at 0 to obtain

$$g_{z^2w}(0) = 2\,\mathrm{i}\Big(f_{1z^2}(0)\bar{f}_{1w}(0) + \varepsilon\, f_{2z^2}(0)\bar{f}_{2w}(0)\Big).$$

To get rid of the dependence of $f_{z^3}(0)$ we consider the system of equations in (5.10) and set $w = \tau + 2\,\mathrm{i}\, z\chi$ and $(\chi, \tau) = 0$. Then due to the 2-nondegeneracy of H we can solve for $f(z, 0)$, which then depends on elements of $j_0^2 H$. This completes the proof of the jet determination. $\square$

Example 5.14. The following example shows that we cannot do better than Corollary 5.13 and have to consider coefficients of order 3: For $t \in \mathbb{R}$ the family of mappings $H_t = (f_{1,t}, f_{2,t}, g_t)$ given by

$$H_t(z, w) \coloneqq \left(\frac{(1 + (\mathrm{i}\,\varepsilon - t)w)z}{1 - (\mathrm{i}\,\varepsilon + t)w}, \frac{2z^2}{1 - (\mathrm{i}\,\varepsilon + t)w}, w\right),$$

sends $\mathbb{H}^2$ into $\mathbb{H}_\varepsilon^3$ and has the property that $j_0^2 H_t$ is independent of t, but $\mathrm{Re}\big((f_{2,t})_{z^2w}(0)\big) = 4t$. These mappings are all isotropically equivalent to $G_{2,0}^\varepsilon(z, w)$ by an application of isotropies of the form $(z, w) \mapsto (z, w)/(1 + tw)$ and $(z_1', z_2', w') \mapsto (z_1', z_2', w')/(1 - tw')$ and dilations $(z, w) \mapsto (2z, 4w)$ and $(z_1', z_2', w') \mapsto (z_1'/2, z_2'/2, w'/4)$.

5.5 Isotropic Stabilizers

We need to introduce some notation concerning group actions.

Definition 5.15. Let X be a set and G a group with unit element e. A *(left) action* $\alpha : G \times X \to X$ *of* G *on* X is a map, which satisfies:

(i) $\alpha(e, x) = x$ for all $x \in X$,

(ii) $\alpha\big(g_1, \alpha(g_2, x)\big) = \alpha(g_1 g_2, x)$, for all $g_1, g_2 \in G$ and $x \in X$.

We write $\alpha(g, x) = g \cdot x$ for $g \in G$ and $x \in X$.

The *stabilizer* $\mathrm{stab}_G(x)$ of x is defined by $\mathrm{stab}_G(x) \coloneqq \{g \in G : g \cdot x = x\}$. An action of G on X is called *free* if for all $x \in X$ we have $\mathrm{stab}_G(x) = \{e\}$, i.e., all stabilizers are trivial.

Lemma 5.16. *The mapping* $N : \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0) \times \mathrm{Aut}_0(\mathbb{H}^2, 0) \times \mathcal{F}_2 \to \mathcal{F}_2$ *given by*

$$N(\phi', \phi, H) \coloneqq \phi' \circ H \circ \phi^{-1},$$

is a left action.

Proof. For $\phi_1, \phi_2 \in \mathrm{Aut}_0(\mathbb{H}^2, 0), \phi'_1, \phi'_2 \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ and $H \in \mathcal{F}_2$ we have to show that

$$N\big(\phi'_1, \phi_1, N(\phi'_2, \phi_2, H)\big) = N(\phi'_1 \circ \phi'_2, \phi_1 \circ \phi_2, H).$$

Indeed, we have

$$\begin{aligned} N\big(\phi'_1, \phi_1, N(\phi'_2, \phi_2, H)\big) &= N\big(\phi'_1, \phi_1, \phi'_2 \circ H \circ \phi_2^{-1}\big) \\ &= \phi'_1 \circ \Big(\phi'_2 \circ H \circ \phi_2^{-1}\Big) \circ \phi_1^{-1} \\ &= (\phi'_1 \circ \phi'_2) \circ H \circ (\phi_1 \circ \phi_2)^{-1} = N(\phi'_1 \circ \phi'_2, \phi_1 \circ \phi_2, H), \end{aligned}$$

which proves the claim. □

Definition 5.17. Let N be the action given in Lemma 5.16 and define $G \coloneqq \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0) \times \mathrm{Aut}_0(\mathbb{H}^2, 0)$. For a mapping $H \in \mathcal{F}_2$ we call $\mathrm{stab}_0(H) \coloneqq \mathrm{stab}_G(H)$ the *isotropic stabilizer for* H.

We prove the following fact about the isotropic stabilizers of mappings in $\mathcal{N}_2$ from Theorem 5.1.

Lemma 5.18. *We set* $\mathcal{E} \coloneqq \{G^\varepsilon_1, G^\varepsilon_{2,0}, G^\varepsilon_{3,0}\}$. *If* $H \in \mathcal{N}_2 \setminus \mathcal{E}$, *then the isotropic stabilizer* $\mathrm{stab}_0(H)$ *of* H *is trivial. Furthermore we have* $\mathrm{stab}_0(G^\varepsilon_1) = \mathrm{stab}_0(G^\varepsilon_{2,0})$ *is homeomorphic to* $\mathbb{S}^1$ *and* $\mathrm{stab}_0(G^\varepsilon_{3,0})$ *is homeomorphic to* $\mathbb{Z}_2$.

Proof. We let $H = (f, g) = (f_1, f_2, g) \in \mathcal{N}_2$ satisfy the conditions we collected in Remark 4.6. We write $s \coloneqq |f_{1w^2}(0)| \geq 0, x \coloneqq f_{2w^2}(0) \in \mathbb{C}$ and $y \coloneqq \mathrm{Im}\big(f_{2z^2w}(0)\big) \in \mathbb{R}$. By Corollary 5.13 we only need to consider coefficients in $j_0^2 H$ and $f_{z^2w}(0)$. We let $(\sigma', \sigma) \in G$ with the notation from (2.27), (2.30) and (2.31) respectively and consider the equation

$$\sigma' \circ H \circ \sigma = H. \tag{5.15}$$

The coefficients of order 1, which are $f_z(0)$ and $H_w(0)$, are given as follows:

$$U'\ {}^t(u\lambda\lambda', 0) = (1, 0), \tag{5.16}$$

$$U'\ {}^t(uc + \lambda c_1', \lambda c_2', \lambda\lambda') = (0, 0, 1). \tag{5.17}$$

These equations imply $\lambda' = 1/\lambda, a_2' = c_2' = 0, a_1' = 1/(uu')$ and $c_1' = -uc/\lambda$. Assuming these standard parameters we consider the coefficients of order 2, which are $f_{z^2}(0), H_{zw}(0)$ and $H_{w^2}(0)$, given by:

$$(0, 2u'u^3\lambda) = (0, 2), \tag{5.18}$$

$$\left(-r - \lambda^2 r' + \frac{\mathrm{i}\,\varepsilon\,\lambda^2}{2}, 2u'u^3\lambda c, 0\right) = \left(\frac{\mathrm{i}\,\varepsilon}{2}, 0, 0\right), \tag{5.19}$$

$$\big(\lambda^2(\lambda s + \mathrm{i}\,\varepsilon\, uc)/u, uu'\lambda(\lambda^2 x + 2u^2c^2), -2(r + \lambda^2 r')\big) = (s, x, 0). \tag{5.20}$$

The second component of (5.19) implies $c = 0$. If we assume this value for c we obtain for the third order terms $f_{z^2w}(0)$ the following equation:

$$\big(4\,\mathrm{i}\, u\lambda^3 s, u'u^3\lambda(-4r - 2\lambda^2 r' + \mathrm{i}\,\lambda^2 y)\big) = (4\,\mathrm{i}\, s, \mathrm{i}\, y). \tag{5.21}$$

The second component of (5.18) shows $\lambda = 1$. Furthermore we obtain from the third component of (5.20) that $r' = -r$ and since from the second component of (5.18) we get $u'u^3 = 1$, we obtain from the second component of (5.21) that $r = 0$. The equation from $f_{2z^2}(0)$ given by $u'u^3 = 1$ uniquely determines u'. The remaining equation from the first component of (5.20), which comes from the coefficient $f_{1w^2}(0)$, is $s/u = s$.

If $s > 0$ we obtain that $u = 1$ and hence all standard parameters are trivial, which proves the first claim of the lemma.

If $s = 0$, then $H \in \mathcal{E}$, since elements in $\mathcal{E}$ are the only maps satisfying $f_{1w^2}(0) = 0$ in the list of mappings from Theorem 5.1. It is easy to check that the isotropic stabilizers of the maps G_1^ε and $G_{2,0}^\varepsilon$ consist precisely of the isotropies $\sigma(z, w) = (uz, w)$ and $\sigma'(z_1', z_2', w') = (z_1'/u, z_2'/u^2, w')$ with $|u| = 1$. If we consider $G_{3,0}^\varepsilon$ in (5.15), then we obtain that $\sigma(z, w) = (\delta z, w)$ and $\sigma'(z_1', z_2', w') = (\delta z_1', z_2', w')$, where $\delta = \pm 1$, are the only elements of $\mathrm{stab}_0(G_{3,0}^\varepsilon)$, which proves the last claim of the lemma. □

With a similar procedure as in the previous Lemma 5.18 we obtain the following result:

Theorem 5.19. *Let $G, H \in \mathcal{N}_2$ and $\sigma \in \mathrm{Aut}_0(\mathbb{H}^2, 0), \sigma' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ such that*

$$\sigma' \circ H \circ \sigma = G, \tag{5.22}$$

then $G = H$. If G or H does not belong to $\mathcal{E}$, then $\sigma = \mathrm{id}_{\mathbb{C}^2}$ and $\sigma' = \mathrm{id}_{\mathbb{C}^3}$.

Remark 5.20. The above Theorem 5.19 says that the isotropic orbit of a given normalized map does not intersect the isotropic orbit of a different normalized map.

Proof of Theorem 5.19. Let $H = (f_1, f_2, g)$ and $G = (\hat{f}_1, \hat{f}_2, \hat{g})$ be as in the hypothesis. In the same way as in the proof of Lemma 5.18 we consider the equations (5.16) to (5.21) and solve for standard parameters. We write $s \coloneqq |f_{1w^2}(0)| \geq 0$ and $\hat{s} \coloneqq |\hat{f}_{1w^2}(0)| \geq 0$. As before a coefficient where the left-hand side of (5.22) may differ from the right-hand side, is the w^2-coefficient of the first component of (5.22), which gives $s/u = \hat{s}$. Note that all standard parameters except u are uniquely determined. We have to consider two cases:

If $\hat{s} > 0$, then $s > 0$, which implies $u = 1$ is the only possibility. This gives $\sigma = \mathrm{id}_{\mathbb{C}^2}$ and $\sigma' = \mathrm{id}_{\mathbb{C}^3}$ and hence $G = H$. The same conclusion holds if we assume $s > 0$.

If $\hat{s} = 0$, then also $s = 0$ and from the equations (5.16)–(5.21) we obtain that G and H agree up to order 3 such that Corollary 5.13 implies $G = H$. $\square$

6 Global Equivalence of Mappings in $\mathcal{N}_2$

In this section we prove the main parts of the Main Theorem by getting rid of the parameter s in Theorem 5.1. For this purpose we compose the mappings $G^{\varepsilon}_{k,s}$ with translations depending on a parameter p_0 to obtain mappings denoted by $\mathcal{G}^{\varepsilon}_{k,p_0}$. These mappings are in general not elements of $\mathcal{N}_2$, hence we have to renormalize, that means we have to compose $\mathcal{G}^{\varepsilon}_{k,p_0}$ with appropriate isotropies, such that we end up with a normalized mapping, denoted by $\widetilde{\mathcal{G}}^{\varepsilon}_{k}$. Using Theorem 5.1 the mappings $\widetilde{\mathcal{G}}^{\varepsilon}_{k}$ are again of the form as $G^{\varepsilon}_{k,s}$, with the difference that $s = s(p_0)$ depends on the parameter p_0 of the translations. This new free parameter p_0 suffices to completely reduce the one-parameter-family of mappings $G^{\varepsilon}_{k,s}$ to finitely many mappings.

6.1 Equivalence Revisited

We start by adapting Definition 2.27 for the global equivalence relation. The mappings $\mathcal{G}^{\varepsilon}_{k}$ are not defined everywhere in $\mathbb{H}^2$. Composing $\mathcal{G}^{\varepsilon}_{k}$ with translations, depending on a parameter p_0, lead to restrictions of the parameter space for p_0.

Remark 6.1. We are dealing with mappings $H : \mathbb{C}^2 \setminus X \to \mathbb{C}^3$ with X being a complex-analytic set in $\mathbb{C}^2$ and $0 \notin X$. We denote by Y the proper, real-analytic set $Y := \mathbb{H}^2 \cap X$. Then we suppose that $H(\mathbb{H}^2 \setminus Y) \subset \mathbb{H}^3_{\varepsilon}$ and $H(0) = 0$.
For $p, p_0 \in \mathbb{H}^2$ we define

$$H_{p_0}(p) := \Big(t'_{H(p_0)} \circ H \circ t_{p_0}\Big)(p), \qquad \text{if } t_{p_0}(p) \notin Y, \tag{6.1}$$

for translations as in (2.28) and (2.33) respectively. For $p \in \mathbb{H}^2$ we set

$$U_p := \{q_0 \in \mathbb{H}^2 \setminus Y : t_{q_0}(p) \notin Y\},$$

such that each U_p is open in $\mathbb{H}^2$ and $0 \in U_p$ if and only if $p \notin Y$. We denote by $Y_p := \mathbb{H}^2 \setminus U_p$ such that $Y_0 = Y$.
For $V \subsetneq \mathbb{H}^2 \setminus Y$ an open neighborhood of 0 the set $U := \bigcap_{p\in V} U_p$ contains an open and connected neighborhood $\emptyset \neq W \subset U$ of 0. Thus if we write $\widehat{H}(p, p_0) := H_{p_0}(p)$ the domain of $\widehat{H}$ consists of the nontrivial set $V \times U$. Now the following definition makes sense.

Definition 6.2. Let X be a complex-analytic set in $\mathbb{C}^2$ and $0 \notin X$ and denote by Y the proper, real-analytic set $Y := \mathbb{H}^2 \cap X$. Let $H : \mathbb{C}^2 \setminus X \to \mathbb{C}^3$ be a holomorphic mapping, such that $H(\mathbb{H}^2 \setminus Y) \subset \mathbb{H}^3_{\varepsilon}$ and $H(0) = 0$. Consider $Z \in V \subset \mathbb{H}^2$ and $p_0 \in U \subset \mathbb{H}^2$ sufficiently small open and connected neighborhoods of 0 from above. Then we define

$$H_{p_0}(Z) := \Big(t'_{H(p_0)} \circ H \circ t_{p_0}\Big)(Z) = \Big(f_{1,p_0}, f_{2,p_0}, g_{p_0}\Big)(Z). \tag{6.2}$$

From now on we consider H_{p_0} as germs of mappings and refer to $p_0 \in U$, depending on the neighborhood V on which H_{p_0} is defined, as *admissible* parameter of the translations.
In case we are dealing with the mappings $\mathcal{G}^{\varepsilon}_{k}$ from Definition 6.6, we write $\big(f^{\varepsilon}_{1k,p_0}, f^{\varepsilon}_{2k,p_0}, g^{\varepsilon}_{k,p_0}\big)$ for the

components of $\mathcal{G}^{\varepsilon}_{k,p_0}$.

Since the mapping H_{p_0} is fixing 0 for all admissible p_0 such that, if we use Definition 2.26, we can apply isotropies to H_{p_0}.

Definition 6.3 (Equivalence revisited)**.** Let H be as in Definition 6.2, $\sigma_\gamma \in \mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\sigma'_{\gamma'} \in \mathrm{Aut}_0(\mathbb{H}^\varepsilon_3, 0)$ as in Definition 2.23 and Definition 2.24. Then for admissible $p_0 \in \mathbb{H}^2$ we define

$$H_{p_0,\gamma,\gamma'}(Z) := \Big(\sigma'_{\gamma'} \circ t'_{H(p_0)} \circ H \circ t_{p_0} \circ \sigma_\gamma\Big)(Z). \tag{6.3}$$

We say a mapping F, defined in a neighborhood of 0, is *equivalent* to H, if there exist an admissible $p_0 \in \mathbb{H}^2, \sigma_\gamma \in \mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\sigma'_{\gamma'} \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$, such that $F = H_{p_0,\gamma,\gamma'}$ after possibly shrinking the neighborhoods involved. Analogous to (2.40) we define the *orbit* $O(H)$ of H:

$$O(H) := \big\{H_{p_0,\gamma,\gamma'} : p_0 \in \mathbb{H}^2 \textit{ admissible}, (\gamma, \gamma') \in \Gamma \times \Gamma'\big\}. \tag{6.4}$$

We write $\widetilde{H} := H_{p_0,\gamma,\gamma'}$ and denote the components of $\widetilde{H}$ by $\widetilde{H} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{g})$. If we have $H = \mathcal{G}^\varepsilon_k$ we write $\widetilde{\mathcal{G}}^\varepsilon_k = \big(\widetilde{f}^\varepsilon_{1k}, \widetilde{f}^\varepsilon_{2k}, \widetilde{g}^\varepsilon_k\big)$ for the components.

Lemma 6.4. *The relation defined in Definition 6.3 is an equivalence relation.*

Proof. Throughout this proof we write σ_k for isotropies and t_k for translations of the according hypersurfaces, where $k \in \mathbb{N}$.

Reflexivity of the relation is clear, for symmetry we note that if G is equivalent to H, then we have $G = \sigma_1 \circ t_1 \circ H \circ t_2 \circ \sigma_2$ which we rewrite as $H = t_1^{-1} \circ \sigma_1^{-1} \circ G \circ \sigma_2^{-1} \circ t_2^{-1}$. By Corollary 2.22 we write $(t_1^{-1} \circ \sigma_1^{-1})^{-1} = t_3 \circ \sigma_3$ and $\sigma_2^{-1} \circ t_2^{-1} = t_4 \circ \sigma_4$, such that $H = \sigma_3^{-1} \circ t_3^{-1} \circ G \circ t_4 \circ \sigma_4$, i.e., $H \in O(G)$. To show transitivity we proceed similar: Let G be equivalent to H and H be equivalent to F, i.e, $G = \sigma_1 \circ t_1 \circ H \circ t_2 \circ \sigma_2$ and $H = \sigma_3 \circ t_3 \circ F \circ t_4 \circ \sigma_4$. Thus $G = (\sigma_1 \circ t_1 \circ \sigma_3 \circ t_3) \circ F \circ (t_4 \circ \sigma_4 \circ t_2 \circ \sigma_2)$ and by Corollary 2.22 we write $(\sigma_1 \circ t_1 \circ \sigma_3 \circ t_3)^{-1} = t_5 \circ \sigma_5$ and $t_4 \circ \sigma_4 \circ t_2 \circ \sigma_2 = t_6 \circ \sigma_6$, such that $G = \sigma_5^{-1} \circ t_5^{-1} \circ F \circ t_6 \circ \sigma_6$, which shows $G \in O(F)$. ☐

In the next lemma we observe that the equivalence relation we give in Definition 6.3 covers the most general case of an equivalence relation in our setting. More precisely we have the following result:

Lemma 6.5. *For $G, H \in \mathcal{F}_2$ we write $G \in [H]$ if there exists $\phi \in \mathrm{Aut}(\mathbb{H}^2, 0)$ and $\phi' \in \mathrm{Aut}(\mathbb{H}^3_\varepsilon, 0)$ such that $G = \phi' \circ H \circ \phi^{-1}$. If $G \in [H]$, then G is equivalent to H in the sense of Definition 6.3.*

Proof. We let $G, H \in \mathcal{F}_2$ and $\phi \in \mathrm{Aut}(\mathbb{H}^2, 0)$ and $\phi' \in \mathrm{Aut}(\mathbb{H}^3_\varepsilon, 0)$ such that $G = \phi' \circ H \circ \phi^{-1}$. By Corollary 2.22 we write $\phi^{-1} = t_1 \circ \sigma_1$ and $\phi'^{-1} = t_2 \circ \sigma_2$, where for $k = 1, 2$ σ_k is an isotropy and t_k is a translation. Hence we have

$$G = \sigma_2^{-1} \circ t_2^{-1} \circ H \circ t_1 \circ \sigma_1,$$

which means that $G \in O(H)$. ☐

We recall the mappings given in (1.3).

Definition 6.6. Let $G^{\varepsilon}_{2,s}(Z)$ and $G^{\varepsilon}_{3,s}(Z)$ be as in Theorem 5.1.
We define

$$\mathcal{G}^{\varepsilon}_1(Z) := G^{\varepsilon}_{2,0}(Z), \qquad \mathcal{G}^{\varepsilon}_2(Z) := G^{\varepsilon}_{2,1/2}(Z), \qquad \mathcal{G}^{\varepsilon}_3(Z) := G^{\varepsilon}_{2,1}(Z),$$
$$\mathcal{G}^{\varepsilon}_4(Z) := G^{\varepsilon}_{3,0}(Z).$$

We denote the components of $\mathcal{G}^{\varepsilon}_k$ by $\left(f^{\varepsilon}_{1k}, f^{\varepsilon}_{2k}, g^{\varepsilon}_k\right)$.

Remark 6.7.
(i) Throughout this section we only need $\mathcal{G}^{\varepsilon}_1(Z)$ and $\mathcal{G}^{\varepsilon}_3(Z)$ for $\varepsilon = -1$. We define the case $\varepsilon = +1$ to keep the notation more consistent.
(ii) We would like to point out, that the value of s to define $\mathcal{G}^{\varepsilon}_1(Z)$, i.e., $s = \frac{1}{2}$, depends on the choices of our normalization conditions we made in Proposition 4.1 and how we scaled the elements of J in (5.13), see also Theorem 4.12. It turns out that we would only need to require $s > \frac{1}{2}$.
(iii) We recall the notation for $\mathfrak{H}^2$ from (2.1) in Remark 2.8. We note that we apply unitary matrices $U'_3(e^{-\mathrm{i}\,\theta_0})$ and $U_2(e^{-\mathrm{i}\,\theta_0})$ to $\mathcal{G}^{\varepsilon}_{1,p_0}$ as in Lemma 5.12 such that from now on we only need to deal with $\mathcal{G}^{\varepsilon}_{1,p_0}$ which is now independent of θ_0. Also we set $\theta_0 = v_0 = 0$ in $\mathcal{G}^{\varepsilon}_{4,p_0}$, which only depends on $r_0 \geq 0$. This parameter will suffice to reduce to finitely many mappings.

Now we have introduced all relevant notions to recall and make Theorem 1.5 sensible.

Theorem 6.8. *For $m = 2, 3$ and $1 \leq k \leq 4$ let $G^{\varepsilon}_{m,s}$ be as in Theorem 5.1 and $\mathcal{G}^{\varepsilon}_k$ as in Definition 6.6. The following statements hold if we use the equivalence relation of Definition 6.3:*
For $\varepsilon = +1$ we have:
(i) *For every $s \geq 0$ the mapping $G^{+}_{2,s}$ is equivalent to $\mathcal{G}^{+}_1$.*
(ii) *For every $s \geq 0$ the mapping $G^{+}_{3,s}$ is equivalent to $\mathcal{G}^{+}_4$.*
For $\varepsilon = -1$ we have:
(iii) *For every $0 \leq s < \frac{1}{2}$ the mapping $G^{-}_{2,s}$ is equivalent to $\mathcal{G}^{-}_1$.*
(iv) *For every $s > \frac{1}{2}$ the mapping $G^{-}_{2,s}$ is equivalent to $\mathcal{G}^{-}_3$.*
(v) *The mappings $\mathcal{G}^{-}_1, \mathcal{G}^{-}_2$ and $\mathcal{G}^{-}_3$ are pairwise not equivalent to each other.*
(vi) *For every $0 \leq s \neq \frac{1}{2}$ the mapping $G^{-}_{3,s}$ is equivalent to $\mathcal{G}^{-}_4$ and $G^{-}_{3,1/2} = \mathcal{G}^{-}_2$.*
The mapping G^{ε}_1 is not equivalent to any of the mappings $\mathcal{G}^{\varepsilon}_k$.

Remark 6.9. The equivalence relation of (6.3) gives a finer description of $\mathcal{N}_2$ and Theorem 6.8 shows that $\mathcal{N}_2$ is given by finitely many orbits $O(\mathcal{G}^{\varepsilon}_k)$.

The rest of this chapter is devoted to prove Theorem 6.8.

6.2 Admissible Sets for Translations

In this section we give the definition for the admissible sets for translations. We think of the mappings $\mathcal{G}^{\varepsilon}_k$ if we deal with H in the following considerations.

Definition 6.10. Let H from Definition 6.2 be given by $H = P/Q$, where P, Q are polynomials such that $P(0) = 0$ and $Q(0) \neq 0$. Let $H_{p_0} =: P_{p_0}/Q_{p_0}$ be given by (6.2). For fixed $Z \in \mathbb{H}^2$ we define the proper, real-analytic set

$$D_H(Z) \coloneqq \{p_0 \in \mathbb{H}^2 : Q_{p_0}(Z) = 0\},$$

and write $D_H \coloneqq D_H(0)$. If $H = \mathcal{G}_k^\varepsilon$ we write $D_H = D_k^\varepsilon$.

We take a closer look at rational mappings H_{p_0} given by $H_{p_0} = P_{p_0}/Q_{p_0}$. We already know that H_{p_0} is defined in a sufficiently small neighborhood of 0 and makes sense. If we expand H_{p_0} into a power series around 0, all denominators of the Taylor coefficients consist of powers of $Q_{p_0}(0)$. Hence D_H defines the set $Y_0 = Y$ from Remark 6.1. Since $D_H = \{p \in \mathbb{H}^2 : Q(p) = 0\}$ we give the following Definition.

Definition 6.11. Let H from Definition 6.2 be given by $H = P/Q$, where P, Q are polynomials such that $P(0) = 0$ and $Q(0) \neq 0$. Then we define A_H as *the set of admissible parameters for H* or *admissible set of H* by

$$A_H \coloneqq \mathbb{H}^2 \setminus D_H = \{p \in \mathbb{H}^2 : Q(p) \neq 0\}.$$

If $H = \mathcal{G}_k^\varepsilon$ we write $A_H = A_k^\varepsilon$.

Next we give another observation, which provides some positivity condition if $\varepsilon = -1$.

Remark 6.12. The mappings $H \in \mathcal{F}_2$ we are dealing with depend on $\varepsilon = \pm 1$. We write $H = (f_1, f_2, g)$. If we consider $H_{p_0} = (f_{1,p_0}, f_{2,p_0}, g_{p_0})$ as in (6.2), then it may happen for some choices of p_0, that $g_{p_0 w}(0) < 0$ if $\varepsilon = -1$, as pointed out in the proof of Proposition 3.16. In this remark we describe explicitly which isotropies we need to apply to these mappings, such that the resulting mapping $\hat{H} = (\hat{f}_1, \hat{f}_2, \hat{g})$ satisfies $\hat{g}_w(0) > 0$. Consequently, if we consider H_{p_0} we can always restrict ourselves to parameters of the translations p_0, such that in H_{p_0} we have $g_{p_0 w}(0) > 0$.

Let us denote by $H^- = (f_1^-, f_2^-, g^-)$ the mapping H, where we set $\varepsilon = -1$ and we have $g_w^-(0) < 0$ and by $H^+ = (f_1^+, f_2^+, g^+)$ the mapping H, where we set $\varepsilon = -1$ and we have $g_w^+(0) > 0$. If we want to normalize H^- as in Proposition 4.1 we first compose H^- with π' from (2.35), such that

$$\widehat{H}^- \coloneqq \pi' \circ H^- = \left(\widehat{f_1^-}, \widehat{f_2^-}, \widehat{g^-}\right) \coloneqq \left(f_2^-, f_1^-, -g^-\right),$$

satisfies $\widehat{g_w^-}(0) > 0$. For H^+ we keep the components as they are and write $\widehat{H}^+ \coloneqq H^+$ with components $(\widehat{f_1^+}, \widehat{f_2^+}, \widehat{g}^+) = (f_1^+, f_2^+, g^+)$ for consistency.

For the normalization of $\widehat{H}^\varepsilon$ we proceed as in the proof of Proposition 4.1 by first deriving the parameter $a' = (a'_1, a'_2) \in \mathcal{S}^2_{-,\sigma}$ from U' given in (2.30). For $\widehat{H}^\varepsilon$ in order to satisfy the normalization condition $\widehat{f_z^\varepsilon}(0) = (1, 0)$ we obtain the matrix U'^{ε} with standard parameters $a'^{\varepsilon} = (a_1'^{\varepsilon}, a_2'^{\varepsilon}) \in \mathcal{S}^2_{-,\sigma}$. Since we flipped f_1 and f_2 in H^- we have that

$$\begin{aligned} a_1'^- &= a_2'^+, \\ a_2'^- &= a_1'^+, \end{aligned}$$

such that $|a_1'^-|^2 - |a_2'^-|^2 = -1$, i.e., $\sigma = -1$. Summing up the steps we carried out so far, we apply a matrix V'^-, as we defined in (2.34), to H^-, which belongs to the group of isotropies of $\mathbb{H}^3_-$ and is given by

$$V'^{-} := \begin{pmatrix} a_2'^{+} & a_1'^{+} & 0 \\ \bar{a}_1'^{+} & \bar{a}_2'^{+} & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and we apply a matrix V'^{+} to H^{+} given by

$$V'^{+} := \begin{pmatrix} a_1'^{+} & a_2'^{+} & 0 \\ \bar{a}_2'^{+} & \bar{a}_1'^{+} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

such that $V'^{-t}(\widehat{f}_1^-, \widehat{f}_2^-) = V'^{+t}(\widehat{f}_1^+, \widehat{f}_2^+)$. Thus after this first normalization step the mappings agree and we assume without loss of generality that the mappings H satisfies $g_w^-(0) > 0$.

Definition 6.13. Let H from Definition 6.2 be given by $H = (f_1, f_2, g)$. The real-analytic set N_H of admissible points p_0, such that $H_{p_0}(Z)$ does not satisfy condition (3.29) from the definition for the class $\mathcal{F}_2$

$$N_H := \big\{p_0 \in A_H : f_{1,p_0 z}(0) f_{2,p_0 z^2}(0) - f_{2,p_0 z}(0) f_{1,p_0 z^2}(0) = 0\big\}. \tag{6.5}$$

The proper, real-analytic set T_H of admissible points p_0, where $H_{p_0}(Z)$ does not satisfy condition (3.30) is given by

$$T_H := \big\{p_0 \in A_H : g_{p_0 w}(0) = 0\big\},$$

if we take into account Remark 6.12. We define

$$S_H := T_H \cup N_H, \qquad W_H := \complement S_H \cap A_H. \tag{6.6}$$

If we deal with the mappings $\mathcal{G}_k^\varepsilon$ we write subscripts S_k^ε and W_k^ε and the same for N_H and T_H. If we write S_2^ε or S_3^ε, we set $S_2^+ = S_3^+ = \emptyset$.

Remark 6.14. According to Proposition 3.5 and Proposition 3.13 it is possible to compose the mapping with translations to obtain the conditions (3.30) and (3.29), which define the class $\mathcal{F}_2$. So we may also exclude those points of $\mathbb{H}^2$, which belong to T_H or N_H, such that H_{p_0} as defined in Definition 6.2 satisfies (3.30) and (3.29).

The connection of N_H and T_H with the sets associated to H, where H is not 2-nondegenerate or transversal is given in the following lemma.

Lemma 6.15. *Let H from Definition 6.2 be given by $H = (f_1, f_2, g)$. Let N denote the set of points $p \in \mathbb{H}^2$ where H is not 2-nondegenerate at p and let T be the set of points $q \in \mathbb{H}^2$ where H is not transversal to $\mathbb{H}^3_\varepsilon$ at $H(q)$. Then $N = N_H$ and $T = T_H$.*

Proof. To show the first equality we note according to Remark 3.14 the set N where $H = (f_1, f_2, g)$ is not 2-nondegenerate in $\mathbb{H}^2$ is given by

$$N = \{p = (p_1, p_2) \in \mathbb{H}^2 : \Big(f_{1z}(p) + 2\,\mathrm{i}\,\bar{p}_1 f_{1w}(p)\Big)\Big(f_{2z^2}(p) + 4\,\mathrm{i}\,\bar{p}_1 f_{2zw}(p) - 4\bar{p}_1^2 f_{2w^2}(p)\Big) - \Big(f_{2z}(p) + 2\,\mathrm{i}\,\bar{p}_1 f_{2w}(p)\Big)\Big(f_{1z^2}(p) + 4\,\mathrm{i}\,\bar{p}_1 f_{1zw}(p) - 4\bar{p}_1^2 f_{1w^2}(p)\Big) = 0\}.$$

Then if we consider H_{p_0} and compute N_H we obtain the same equation as the one defining N above. For the second equality we recall that the set T of $p \in \mathbb{H}^2$ where H is not transversal to $\mathbb{H}^3_\varepsilon$ at $H(p)$ according to Lemma 3.3 is described in Remark 3.4. More precisely there exists a real-analytic function $A : \mathbb{C}^4 \to \mathbb{R}$ satisfying

$$\operatorname{Im}\big(g(z,w)\big) - \big(|f_1(z,w)|^2 + \varepsilon\,|f_2(z,w)|^2\big) = A(z,w,\bar{z},\bar{w})\big(\operatorname{Im} w - |z|^2\big), \tag{6.7}$$

such that H is transversal to $\mathbb{H}^3_\varepsilon$ at $H(p)$ if and only if $A(p,\bar{p}) \neq 0$. If we consider (6.7) for H_{p_0} with admissible $p_0 \in \mathbb{C}^2$ and evaluate at $(z,w) = (0,0)$ we again obtain equation (6.7) but for the variable p_0 instead of $p = (z,w)$. Thus by Lemma 3.3 (ii) $\Leftrightarrow$ (iii) we have shown that for a mapping H as above it holds that $T = T_H$. □

6.3 Mappings of Higher (Non-)Degeneracy

In this section we want to see what happens if we translate mappings to points, where they are not 2-nondegenerate, i.e., where $H_{p_0} \notin \mathcal{F}_2$. We already computed in Example 3.8, that Faran's map F_4 of Theorem 1.1 is not 2-nondegenerate at 0, but 3-nondegenerate at 0. Using translations we provide some examples of mappings of higher (non-)degeneracy at 0, which are then used in the following sections to prove Theorem 6.8. This subsection concludes with a collection of some monomial maps.

Definition 6.16. Let $H \in \mathcal{F}_2$. We call H *deficient* at $p \in \mathbb{H}^2$ if H is not 2-nondegenerate at p. We refer to such p as *deficient point* for H. If H is 2-nondegenerate at $p \in \mathbb{H}^2$ we say H is *nondeficient* at p, which we call a *nondeficient point* for H.

Remark 6.17. By Proposition 3.16 the set of deficient points is a proper, real-analytic subset of $\mathbb{H}^2$.

For a mapping $H \in \mathcal{F}_2$ we have the following possibilities: Either there exists $p \in \mathbb{H}^2$ such that H is deficient at p or H is nondeficient everywhere in $\mathbb{H}^2$.
In the first case we consider H_p and compose with isotropies fixing 0. Then we try to normalize the resulting mapping according to some different normalization conditions than we introduced in Proposition 4.1.
In the second case if H is nondeficient everywhere in $\mathbb{H}^2$ we may try to normalize H with respect to different normalization conditions as given in Proposition 4.1 by composing H with isotropies fixing some appropriate point $q \in \mathbb{H}^2$. Here the mapping H belongs to $\mathcal{F}_2 \setminus \mathcal{N}_2$.
At this point we refer to Lemma 6.15, which gives another way how to find deficient points.
The following example gives a mapping, which is nondeficient everywhere:

Example 6.18. We consider the mapping $H := G_1^+$ such that $A_H = \mathbb{H}^2 \setminus \{(0, \pm 1)\}$ is the admissible set. Then we need to compute N_H, which is given by the following equation if we take $p_0 = (r_0 e^{\mathrm{i}\,\theta_0}, v_0 +$

$\mathrm{i}\, r_0^2), r_0 \geq 0, , \theta_0, v_0 \in \mathbb{R}$ instead of (z,w):

$$\frac{(1+\mathrm{i}\, v_0 + r_0^2)(1+v_0-\mathrm{i}\, r_0^2)(1-v_0+\mathrm{i}\, r_0^2)}{(1+(r_0^2-\mathrm{i}\, v_0)^2)^3} = 0,$$

which admits no solution in A_H.

In the following paragraphs we deduce some mappings of higher (non-)degeneracy at 0.

Example 6.19. For $H := G_1^-$ we consider H_{p_0}. The admissible set A_H consists of points $(0,\pm 1) \in \mathbb{H}^2$ such that $N_H = \{q_0 \in \mathbb{H}^2 : q_0 = (e^{\mathrm{i}\,\theta_0}, \mathrm{i}), \theta_0 \in \mathbb{R}\}$. We choose $p_0 = (1, \mathrm{i})$ in H_{p_0} to obtain that 0 is a deficient point for H_{p_0}. We compose H_{p_0} with isotropies fixing 0 which results in a mapping $G := \widetilde{H}$, if we use the notation from Definition 6.3. We write $G = (f,g) = (f_1, f_2, g)$ and consider the following normalization conditions:

(i) $f_w(0) = (0,1)$
(ii) $f_{1z^2}(0) = 2\sqrt{2}$
(iii) $f_{1zw}(0) = 0$
(iv) $f_{w^2}(0) = (0,0)$
(v) $f_{1z^2w}(0) = 0$
(vi) $f_{2w^3}(0) = 0$

These equations are satisfied if we choose the following standard parameters:

$$c = \frac{\mathrm{i}}{2}, \quad a_1' = \sqrt{2}\,\mathrm{i}, \quad a_2' = 1, \quad \lambda' = 2, \quad c_1' = \frac{1}{2},$$

and the rest trivially. After an application of the automorphism π' of $\mathbb{H}^3_-$ from (2.35) the resulting mapping is equal to

$$(z,w) \mapsto (w, \sqrt{2} z^2, \mathrm{i}\, w^2).$$

We note that this mapping is $(2,1)$-degenerate and not transversal at 0.

Example 6.20. For $H := \mathcal{G}_1^+$ we consider H_{p_0}, where $A_H = \mathbb{H}^2$. Then we obtain $N_H = \{p_0 = (2, 4\,\mathrm{i})\}$. After composing H_{p_0} with elements of $\mathrm{Aut}_0(\mathbb{H}^2)$ and $\mathrm{Aut}_0(\mathbb{H}^3)$ we denote $G := \widetilde{H}$, using the notation from Definition 6.3. We impose the following normalization conditions when we write $G = (f,g) = (f_1, f_2, g)$:

(i) $f_z(0) = (1,0)$
(ii) $G_w(0) = (0,0,1)$
(iii) $f_{zw}(0) = (0,1)$
(iv) $f_{2w^2}(0) = 0$
(v) $g_{w^2}(0) = 0$
(vi) $f_{2zw^2}(0) = 0$

They can be achieved if we take the following standard parameters except the trivial ones $r, c_1', r' = 0$:

$$c = \mathrm{i}, \quad \lambda = 4, \quad u' = -\,\mathrm{i}, \quad a_1' = a_2' = -\frac{\mathrm{i}}{\sqrt{2}}, \quad \lambda' = \frac{1}{4\sqrt{2}}, \quad c_2' = -\frac{\mathrm{i}}{4}.$$

The resulting mapping is of the form

$$(z,w) \mapsto \frac{(z, zw, w)}{1-w^2},$$

which is $(1,1)$-degenerate and transversal at 0.

Example 6.21. Next we consider the mapping $H := G^-_{2,\frac{1}{\sqrt{3}}}$. For H_{p_0} we find $p_0 = (\sqrt{3}, 3\,\mathrm{i}) \in N_H$ and then renormalize with respect to the same normalization conditions as in Example 6.20. The following nontrivial standard parameters provide that $\widetilde{H}$ satisfies the above normalization conditions:

$$\lambda = 3^{\frac{3}{4}}, \quad u' = -\mathrm{i}, \quad a'_1 = \frac{\sqrt{3}(1+\mathrm{i})}{2}, \quad a'_2 = -\frac{1+\mathrm{i}}{2}, \quad \lambda' = \frac{3^{\frac{3}{4}}}{2\sqrt{2}}, \quad c'_2 = -\frac{\mathrm{i}}{2},$$

to obtain the mapping

$$(z,w) \mapsto \frac{(z, zw, w)}{1+w^2}, \tag{6.8}$$

which is $(1,1)$-degenerate and transversal at 0.

Remark 6.22. We note Example 6.20 and Example 6.21 show that the mapping

$$(z,w) \mapsto \frac{(z, zw, w)}{1-\varepsilon\, w^2},$$

is equivalent to $\mathcal{G}^+_1$ for $\varepsilon = +1$ and to $G^-_{2,\frac{1}{\sqrt{3}}}$ for $\varepsilon = -1$.

Example 6.23. We let $H := G^-_{2,\frac{\sqrt{5}}{4}}$ and take H_{p_0} to find $q_0 = \left(\frac{376+32\,\mathrm{i}}{89\sqrt{5}}, -\frac{512+320\,\mathrm{i}}{89}\right) \in N_H \cap T_H$. We apply isotropies fixing 0 to H_{q_0} and denote the resulting mapping by $G = (f,g) = (f_1, f_2, g)$. We normalize the mapping according to the following conditions:

(i) $f_z(0) = (1,1)$
(ii) $f_w(0) = (0,0)$
(iii) $f_{z^2}(0) = (0,0)$
(iv) $f_{1zw}(0) = 0$
(v) $g_{w^2}(0) = 2$
(vi) $f_{1zw^2}(0) = 0$

when we use the following standard parameters:

$$c = -\frac{2+199\,\mathrm{i}}{2848\sqrt{5}}, \quad r = \frac{1223}{2048}, \quad c'_1 = \frac{1276-3243\,\mathrm{i}}{22304\sqrt{5}}, \quad c'_2 = \frac{11484+29187\,\mathrm{i}}{55760},$$
$$a'_1 = \frac{30613535492-20104041651\,\mathrm{i}}{353339968\sqrt{3485}}, \quad a'_2 = -\frac{11384417567-3593306283\,\mathrm{i}}{353339968\sqrt{697}}, \quad \lambda' = \frac{32\sqrt{697}}{89},$$
$$u' = \frac{538504992958+544496189479\,\mathrm{i}}{342480284921\sqrt{5}}, \quad r' = -\frac{756545275}{32444416},$$

and the remaining standard parameters are chosen trivially. The resulting mapping is of the form

$$(z,w) \mapsto \left(z, \frac{z}{1+w}, \frac{w^2}{1+w}\right). \tag{6.9}$$

This mapping is (1,1)-degenerate and not transversal at 0. If we apply translations $t_{p_0}^{-1}$ and ${t'}_{q_0}^{-1}$, where $p_0 = (0,1)$ and $q_0 = (0,0,2)$, to the map from (6.9) we obtain

$$(z,w) \mapsto \left(z, \frac{z}{w}, \frac{1+w^2}{w}\right). \tag{6.10}$$

Example 6.24. For $H \coloneqq \mathcal{G}_4^-$ we consider H_{p_0} to obtain that $q_0 = \left(\frac{4}{\sqrt{3}}, \frac{16\,\mathrm{i}}{3}\right) \in N_H$. With this choice for p_0 we scale H_{q_0} via dilations given by $(z,w) \mapsto (\sqrt{3}z, 3w)$ and $(z_1', z_2', w') \mapsto \left(\frac{11z_1'}{27}, \frac{11z_2'}{27}, \frac{121w'}{729}\right)$. Then we compose the resulting mapping $G = (f,g) = (f_1, f_2, g)$ with isotropies fixing 0 and we impose the following normalization conditions:

(i) $f_z(0) = (0, \sqrt{3})$
(ii) $G_w(0) = (0,0,-3)$
(iii) $f_{2zw}(0) = 0$
(iv) $g_{w^2}(0) = 0$
(v) $f_{1z^2w}(0) = 0$
(vi) $f_{1z^3}(0) = 24$

which are achieved if we take the following standard parameters:

$$c = \mathrm{i}, \quad \lambda = \frac{8}{3}, \quad a_1' = \frac{14}{11}, \quad a_2' = -\frac{5\sqrt{3}}{11}, \quad \lambda' = \frac{3\sqrt{3}}{8}, \quad c_1' = \frac{3\sqrt{3}\,\mathrm{i}}{8},$$

and the remaining parameters are chosen trivially. The resulting mapping is given by

$$(z,w) \mapsto \frac{(4z^3, \sqrt{3}(1-w^2)z, -(3+w^2)w)}{1+3w^2},$$

which is 3-nondegenerate and transversal at 0.

Example 6.25. In this example we consider $H \coloneqq \mathcal{G}_4^+$ and again H_{p_0}. Here we take $p_0 = (4(1 + 2\sqrt{2}/3)^{1/2}, 16(1+2\sqrt{2}/3)\,\mathrm{i})$ such that 0 is a deficient point for H_{p_0}. First we compose H_{p_0} with the following dilations

$$\begin{aligned}
(z,w) &\mapsto \left((9+6\sqrt{2})^{\frac{1}{2}} z, 3(3+2\sqrt{2})w\right),\\
(z_1', z_2', w') &\mapsto \left(\frac{(23+20\sqrt{2})z_1'}{27}, \frac{(23+20\sqrt{2})z_2'}{27}, \frac{(1329+920\sqrt{2})w'}{729}\right),
\end{aligned}$$

to remove some common factors. Then we compose the resulting mapping with isotropies fixing 0 and denote this mapping by G. Next we consider the same normalization conditions as in Example 6.24 except we need to require $G_w(0) = (0,0,3)$ and use the following nontrivial standard parameters:

$$\begin{aligned}
&c = (1+\sqrt{2})\,\mathrm{i}, \quad \lambda = \frac{8\sqrt{2}}{3}, \quad u' = -1,\\
&a_1' = \frac{(75\sqrt{2}-154)}{271}, \quad a_2' = -\frac{5\sqrt{3}(11+14\sqrt{2})}{271}, \quad \lambda' = \frac{3}{8}\sqrt{\frac{51}{2} - 18\sqrt{2}},\\
&c_1' = \frac{-21\,\mathrm{i}(3987-2760\sqrt{2})^{\frac{1}{2}}}{1084\sqrt{2}}, \quad c_2' = \frac{45\,\mathrm{i}(40-23\sqrt{2})}{4336}.
\end{aligned}$$

The resulting mapping is given by

$$(z,w) \mapsto \frac{(4z^3, \sqrt{3}(1+w^2)z, (3-w^2)w)}{1-3w^2},$$

which is 3-nondegenerate and transversal at 0.

Remark 6.26. Example 6.24 and Example 6.25 show that the mapping

$$(z,w) \mapsto \frac{(4z^3, \sqrt{3}(1+\varepsilon\, w^2)z, (3\,\varepsilon - w^2)w)}{1-3\,\varepsilon\, w^2}, \tag{6.11}$$

is equivalent to $\mathcal{G}_4^{\varepsilon}$. For $\varepsilon = +1$ after applying the Cayley-Transformation to this mapping we obtain the mapping F_4 from Theorem 1.1 when we interchange the second and third component.

For the next mapping we cannot proceed as in the previous examples and we normalize differently at 0.

Example 6.27. We prove that $H := G_{2,s}^{-}$ with $0 \leq s \leq 1/2$ admits no admissible points, where H fails to be 2-nondegenerate. For admissible points $p = (r_0 e^{\mathrm{i}\,\theta_0}, v_0 + \mathrm{i}\, r_0^2) \in A_H$ the set N_H is given by

$$N_H = \{p \in A_H : -4r_0 s + e^{\mathrm{i}\,\theta_0}\big(4 + (1-3s^2)(r_0^2 + \mathrm{i}\, v_0)\big) = 0, 0 \leq s \leq 1/2\}.$$

Splitting up the defining equation for N_H into real and imaginary part we obtain the following system:

$$\begin{pmatrix} \cos(\theta_0) & \sin(\theta_0) \\ \sin(\theta_0) & -\cos(\theta_0) \end{pmatrix} \begin{pmatrix} 4 + r_0^2(1-3s^2) \\ v_0(1-3s^2) \end{pmatrix} - \begin{pmatrix} 4r_0 s \\ 0 \end{pmatrix} = 0,$$

which does not admit any solution for $0 \leq s < 1/2$ and any $p \in \mathbb{H}^2$ and if $s = 1/2$ the solution of the above system does not belong to A_H.

We set $H := \mathcal{G}_2^{-}$ and compose H with isotropies fixing 0 to obtain the mapping $\widetilde{H} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{g})$. We consider the following normalization conditions at 0:

(i) $\widetilde{f}_z(0) = (1,0)$

(ii) $\widetilde{G}_w(0) = (0,0,1)$

(iii) $\widetilde{f}_{2z^2}(0) = 2\sqrt{2}$

(iv) $\widetilde{f}_{1w^2}(0) = 0$

(v) $\widetilde{g}_{w^2}(0) = 0$

(vi) $\mathrm{Re}(\widetilde{f}_{1z^2w}(0)) = 0$

These conditions are achieved with the following nontrivial standard parameters

$$c = -\frac{\mathrm{i}}{2\sqrt{2}}, \quad \lambda = \sqrt{2}, \quad \lambda' = \frac{1}{\sqrt{2}}, \quad c_1' = \frac{1}{\sqrt{2}}, \quad c_2' = \frac{\mathrm{i}}{4},$$

which results in the following mapping given by

$$(z,w) \mapsto \left(\frac{z(1+\sqrt{2}z - \mathrm{i}\, w)}{1+\sqrt{2}z}, \frac{z(\sqrt{2}z - \mathrm{i}\, w)}{1+\sqrt{2}z}, w \right). \tag{6.12}$$

We note that in contrast to the normalization conditions of Proposition 4.1 we have here $\widetilde{f}_{1w^2}(0) = 0$, but $\widetilde{f}_{2zw}(0) = -\mathrm{i}$ and a different scaling of $\widetilde{f}_{2z^2}(0)$.

Example 6.28. We complete this section by mentioning some monomial maps we have found during our studies of mappings of hyperquadrics of $\mathbb{C}^2$ and $\mathbb{C}^3$. A monomial map is a mapping, with the property that each of its components consists of monomials.
Additional examples of monomial mappings from $\mathbb{S}^2 \to \mathbb{S}^3_-$ besides L_2 and L_3 of Theorem 1.2 are given as follows:

$$(z,w) \mapsto \frac{\left(1, z^2, \sqrt{2}z\right)}{w^2}, \qquad (z,w) \mapsto \left(\frac{\sqrt{2}}{w}, \frac{z^2}{w^2}, \frac{1}{w^2}\right),$$

which are both equivalent to G_1^-. Further there is an example of a monomial mapping of $\mathbb{H}^2 \to \mathbb{H}^3_-$ of degree 3 given by

$$(z,w) \mapsto \left(\sqrt{3}zw, 2z^3, w^3\right),$$

which is equivalent to $\mathcal{G}_4^-$.

6.4 Renormalization

We introduce the following notation for some particular mappings:

Definition 6.29. We define

$$\begin{aligned}
\mathcal{H}_1^\varepsilon(z,w) &:= \left(\frac{z(1+\mathrm{i}\,\varepsilon\, w)}{1-\mathrm{i}\,\varepsilon\, w}, \frac{2z^2}{1-\mathrm{i}\,\varepsilon\, w}, w\right),\\
\mathcal{H}_2^-(z,w) &:= \left(\frac{z(1+\sqrt{2}z-\mathrm{i}\,w)}{1+\sqrt{2}z}, \frac{z(\sqrt{2}z-\mathrm{i}\,w)}{1+\sqrt{2}z}, w\right),\\
\mathcal{H}_3^-(z,w) &:= \left(z, \frac{z}{w}, \frac{1+w^2}{w}\right),\\
\mathcal{H}_4^\varepsilon(z,w) &:= \frac{(4z^3, \sqrt{3}(1+\varepsilon\, w^2)z, (3\,\varepsilon - w^2)w)}{1-3\,\varepsilon\, w^2}.
\end{aligned}$$

We write $\mathcal{H}_k^\varepsilon = \left(f_{1k}^\varepsilon, f_{2k}^\varepsilon, g_k^\varepsilon\right)$. The notation for sets associated to $\mathcal{G}_k^\varepsilon$ from subsection 6.2 carries over to the maps $\mathcal{H}_k^\varepsilon$.
The mapping $\mathcal{H}_1^\varepsilon$ is equivalent to $G_{2,0}^\varepsilon$, by scaling in $\mathbb{C}^2$ and $\mathbb{C}^3$ with the following maps:

$$(z,w) \mapsto (2z, 4w), \qquad (z_1', z_2', w') \mapsto (z_1'/2, z_2'/2, w'/4).$$

The map $\mathcal{H}_2^-$ is the one from (6.12), $\mathcal{H}_3^-$ is the map (6.10) and $\mathcal{H}_4^-$ is taken from (6.11).

We observe that each mapping $H := \mathcal{H}_k^\varepsilon$ belongs to some orbit $O(G)$, where $G \in \mathcal{N}_2$, although $H \notin \mathcal{F}_2$. Nevertheless in this section we consider H_{p_0} for appropriate $p_0 \in \mathbb{H}^2$, such that $H_{p_0} \in \mathcal{F}_2$, see Definition 6.13 and Remark 6.14. The sets W_H are given below. Then we normalize $H_{p_0} \in \mathcal{F}_2$ and we consider mappings $\widetilde{H} := \widetilde{\mathcal{H}}_k^\varepsilon$ from (6.3) and standard parameters of the isotropies according to Proposition 4.1 to achieve $\widetilde{H} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{g}) \in \mathcal{N}_2$. Applying Theorem 5.1 to $\widetilde{H}$, the mapping $\widetilde{H}$ coincides with one of the families of mappings $G_{k,s}^\varepsilon$ with the difference, that in this particular case $s = s(p_0)$ is a

function, depending on p_0. We know from Theorem 5.1 and Proposition 4.1 that $s = |\widetilde{f}_{1w^2}(0)|$ up to a scaling factor.
In this section we compute the expressions $s(p_0)$ and some of the standard parameters which are needed for this purpose. Then we analyze the image of the function $s(p_0)$ for appropriate p_0 which allows to determine the orbits of the mappings $\mathcal{H}_k^\varepsilon$. Then the reduction of the one-parameter-families of mappings $G_{k,s}^\varepsilon$ and the proof of Theorem 6.8 is completed.
In the next proposition we list the sets D_k^ε from Definition 6.10.

Proposition 6.30. *Let $\mathcal{H}_{k,p_0}^\varepsilon$, where $1 \leq k \leq 4$, be as in Definition 6.2 for the maps from Definition 6.29. Then for $\varepsilon = +1$ we have $D_1^+ = D_4^+ = \emptyset$ and for $\varepsilon = -1$ we compute the following nontrivial sets D_k^-:*

$$\begin{aligned}
D_1^- &= \left\{p_0 \in \mathfrak{H}^2 : 1 + \mathrm{i}\, v_0 - r_0^2 = 0\right\},\\
D_2^- &= \left\{p_0 \in \mathfrak{H}^2 : 1 + \sqrt{2} r_0 e^{\mathrm{i}\,\theta_0} = 0\right\},\\
D_3^- &= \left\{p_0 \in \mathfrak{H}^2 : v_0 + \mathrm{i}\, r_0^2 = 0\right\},\\
D_4^- &= \left\{r_0 \geq 0 : 1 - 3r_0^4 = 0\right\},
\end{aligned}$$

Proof. For $\varepsilon = -1$ the computations of D_k^- and the nontriviality are straightforward. For $\varepsilon = +1$ we obtain $D_1^+ = \left\{p_0 \in \mathfrak{H}^2 : 1 - \mathrm{i}\, v_0 + r_0^2 = 0\right\} = \emptyset$ and $D_4^+ = \left\{r_0 \geq 0 : 1 + 3r_0^4 = 0\right\} = \emptyset$. ◻

Proposition 6.31. *Let $\mathcal{H}_{k,p_0}^\varepsilon$, where $1 \leq k \leq 4$, be as in Definition 6.2 for the maps from Definition 6.29. The sets N_k^ε and T_k^ε from Definition 6.13 are given as follows:*
For $\varepsilon = +1$:

$$\begin{aligned}
N_1^+ &= \left\{p_0 \in \mathfrak{H}^2 : 1 + v_0^2 - 2\,\mathrm{i}\, r_0^2 v_0 - r_0^4 = 0\right\},\\
T_1^+ &= \emptyset,\\
N_4^+ &= \left\{r_0 \geq 0 : r_0(1 - r_0^4) = 0\right\},\\
T_4^+ &= \emptyset.
\end{aligned}$$

For $\varepsilon = -1$:

$$\begin{aligned}
N_1^- &= \emptyset,\\
T_1^- &= \left\{p_0 \in A_1^- : 1 - 6r_0^2 + v_0^2 + r_0^4 = 0\right\},\\
N_2^- &= \emptyset,\\
T_2^- &= \left\{p_0 \in A_2^- : e^{\mathrm{i}\,\theta_0} + \sqrt{2} r_0 (1 + e^{2\,\mathrm{i}\,\theta_0}) = 0\right\},\\
N_3^- &= \left\{p_0 \in A_3^- : r_0 = 0\right\},\\
T_3^- &= \left\{p_0 \in A_3^- : -1 + v_0^2 + r_0^4 = 0\right\},\\
N_4^- &= \left\{r_0 \in A_4^- : r_0 = 0\right\},\\
T_4^- &= \left\{r_0 \in A_4^- : 1 - 14r_0^4 + r_0^8 = 0\right\}.
\end{aligned}$$

Proof. It is straightforward to get the nontrivial sets N_k^ε and T_k^ε. To show the triviality of T_k^+ for $k = 1, 4$ we write $H_k := \mathcal{H}_{k,p_0}^+$ with components (H_k^1, H_k^2, H_k^3) and compute $H_{kw}^3(0)$:

$$H_{1w}^3(0) = \frac{1 + 6r_0^2 + v_0^2 + r_0^4}{1 + 2r_0^2 + v_0^2 + r_0^4},$$
$$H_{4w}^3(0) = \frac{3\big(1 + 14r_0^4 + r_0^8\big)}{(1 + 3r_0^4)^2}.$$

To show triviality of N_k^- for $k = 1, 2$ we note in Example 6.27 we have shown $H := G_{2,s}^-$ with $0 \le s \le 1/2$ admits no admissible points, where H fails to be 2-nondegenerate. By Lemma 3.9 this must also hold for $\mathcal{H}_1^-$ and $\mathcal{H}_2^-$, since they are equivalent to $G_{2,0}^-$ and $G_{2,\frac{1}{2}}^-$ respectively. □

Remark 6.32. According to Remark 6.12 we can assume the mapping $\mathcal{H}_{k,p_0}^\varepsilon$ satisfies $g_{kp_0w}^\varepsilon(0) > 0$, which is given by the following expressions:

$$g_{1p_0w}^\varepsilon(0) = \frac{1 + 6\,\varepsilon\, r_0^2 + v_0^2 + r_0^4}{1 + 2\,\varepsilon\, r_0^2 + v_0^2 + r_0^4}, \tag{6.13}$$
$$g_{2p_0w}^-(0) = \frac{1 + \sqrt{2}r_0(e^{-\mathrm{i}\,\theta_0} + e^{\mathrm{i}\,\theta_0})}{(1 + \sqrt{2}r_0e^{-\mathrm{i}\,\theta_0})(1 + \sqrt{2}r_0e^{\mathrm{i}\,\theta_0})}, \tag{6.14}$$
$$g_{3p_0w}^-(0) = \frac{-1 + v_0^2 + r_0^4}{v_0^2 + r_0^4}, \tag{6.15}$$
$$g_{4p_0w}^\varepsilon(0) = \frac{3(\varepsilon + 14r_0^4 + \varepsilon\, r_0^8)}{(1 + 3\,\varepsilon\, r_0^4)^2}. \tag{6.16}$$

One can verify that these expressions make sense since we assume $p_0 \in W_k^\varepsilon$.

As already mentioned we apply Proposition 4.1 to obtain $\widetilde{\mathcal{H}}_k^\varepsilon \in \mathcal{N}_2$ for some maps in order to compute the coefficient $\widetilde{f}_{1w^2}(0)$ in the following lemma. For this purpose we provide explicit computations of some of the standard parameters.

Lemma 6.33. *Using the notation from* (6.3)*, for* $1 \le k \le 4$ *we set* $H_k := \mathcal{H}_k^\varepsilon$ *and consider* $\widetilde{H}_k$ *with components* $\widetilde{H}_k = (\widetilde{f}_{1k}, \widetilde{f}_{2k}, \widetilde{g}_k)$*. Moreover in* $\widetilde{H}_k$ *we let* $p_0 \in W_k^\varepsilon \subset \mathbb{H}^2$*. If we assume* $\widetilde{H}_k \in \mathcal{N}_2$*, then* $\Sigma_k^\varepsilon(p_0) := \frac{|\widetilde{f}_{1kw^2}(0)|^2}{4}$ *is given as follows:*

$$\Sigma_1^\varepsilon(p_0) = \frac{r_0^2\big(1 + 2\,\varepsilon\, r_0^2 + v_0^2 + r_0^4\big)}{(1 - 2\,\varepsilon\, r_0^2 + v_0^2 + r_0^4)^2},$$
$$\Sigma_2^-(p_0) = \frac{1}{16},$$
$$\Sigma_3^-(p_0) = \frac{(1 - v_0^2)^2 + 2(1 + v_0^2)r_0^4 + r_0^8}{16r_0^4},$$
$$\Sigma_4^\varepsilon(p_0) = \frac{(1 + \varepsilon\, r_0^4)^2(1 - 34\,\varepsilon\, r_0^4 + r_0^8)^2}{1728r_0^4(1 - \varepsilon\, r_0^4)^4}.$$

We write $s_k^\varepsilon(p_0) := \sqrt{\Sigma_k^\varepsilon(p_0)}$.

Proof. First we take a look at the expression $\widetilde{f}_{1w^2}(0)$ from Remark 4.4 for an arbitrary normalized mapping. In order to achieve $\widetilde{f}_{1kw^2}(0) = |\widetilde{f}_{1kw^2}(0)|$ we need to compute $a' = (a'_1, a'_2), c' = (c'_1, c'_2), \lambda'$ and c, λ according to the formulas given in Remark 4.3. This is possible since we have for $p_0 \in W_k^\varepsilon$ that $H_k \in \mathcal{F}_2$. For each $1 \le k \le 4$ we denote the corresponding standard parameters in $\widetilde{H}_k$ by

$$a'_k = (a'_{1k}, a'_{2k}) \in \mathcal{S}^2_{\varepsilon,\sigma}, \quad \hat{c}'_k = (c'_{1k}, c'_{2k}) \in \mathbb{C}^2, \quad \lambda'_k > 0, \quad c_k \in \mathbb{C}, \quad \lambda_k > 0,$$

where all of these expressions depend on p_0. For $k = 1, 4$ they further depend on $\varepsilon = \pm 1$ and if $k = 2, 3$ we set $\varepsilon = -1$, but we suppress this dependence notationally. We denote the collection of these standard parameters for a fixed k by Ω_k. The parameters a'_k are given in (4.4) and $\hat{c}'_k$ as well as λ'_k are given in (4.5). Then c_k is computed according to (4.7) and λ_k is of the form as in (4.9). All these expressions in Ω_k are given in Appendix D for the mappings $\widetilde{H}_k$. We recall that $\widetilde{H}_1$ is independent of θ_0 as we described in Remark 6.7 (iii). We note that it is crucial to assume $p_0 \in W_k^\varepsilon$ such that the elements of Ω_k make sense.
If we put all these parameters into $\widetilde{f}_{1kw^2}(0)$ we obtain

$$\widetilde{f}_{1kw^2}(0) = \frac{S'\big(j_0^2 H_k\big)}{u_k},$$

where S' is a rational function in the coefficients of H_k at 0 up to order 2 according to Remark 4.4. Thus, instead of computing u_k explicitly at this point, we have

$$\Sigma_k^\varepsilon(p_0) = \frac{|S'\big(j_0^2 H_k\big)|^2}{4},$$

which yields the expressions in the statement of the lemma. □

Remark 6.34. It turns out that Σ_2^- is constant equal to $\frac{1}{16}$, which means the mapping $\mathcal{G}_2^-$ is a fixed point of the renormalization map given by $H \mapsto \sigma' \circ t'_{H(p_0)} \circ H \circ t_{p_0} \circ \sigma$ considered as a mapping from $\mathcal{N}_2$ to $\mathcal{N}_2$.

In order to achieve equivalence of $G^\varepsilon_{m,s}$ and $\mathcal{G}^\varepsilon_k$ via choices of translations in $s = s^\varepsilon_k$ we restrict ourselves to subsets of W^ε_k which we give in the following Definition.

Definition 6.35. For $\varepsilon = +1$ we define:

$$\mathcal{W}_1^+ := \left\{p_0 \in \mathfrak{H}^2 : v_0 = 0 = \theta_0, 0 < r_0 < 1\right\}, \tag{6.17}$$
$$\mathcal{W}_4^+ := \left\{r_0 \in \mathbb{R}^+ : 0 < r_0 < -1 + \sqrt{2}\right\}, \tag{6.18}$$

and the following sets for $\varepsilon = -1$:

$$\mathcal{W}_1^- := \left\{p_0 \in \mathfrak{H}^2 : v_0 = 0 = \theta_0, 0 < r_0 < -1 + \sqrt{2}\right\}, \tag{6.19}$$
$$\mathcal{W}_3^- := \left\{p_0 \in \mathfrak{H}^2 : v_0 = 0 = \theta_0, 0 < r_0 < 1\right\} \tag{6.20}$$
$$\mathcal{W}_4^- := \left\{r_0 \in \mathbb{R}^+ : r_0 > 1, r_0 \neq \sqrt{2 + \sqrt{3}}\right\}. \tag{6.21}$$

If we write $\mathcal{W}_3^\varepsilon$, we set $\mathcal{W}_3^+ = \emptyset$.

Next we derive some properties of the functions Σ_k^ε.

Lemma 6.36. *We set $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ and define*

$$\begin{aligned}
&R_1^+ := \mathbb{R}_0^+, && R_4^+ := \mathbb{R}_0^+, \\
&R_1^- := [0, 1/16), && R_3^- := \mathbb{R}_0^+ \setminus [0, 1/16], && R_4^- := \mathbb{R}_0^+ \setminus \{1/16\}.
\end{aligned}$$

Then Σ_k^ε from Lemma 6.33 has the following properties:

(i) *For $p_0 \in W_1^-$ we have $0 \leq \Sigma_1^-(p_0) < 1/16$ and for $p_0 \in W_3^-$ we have $\Sigma_3^-(p_0) > 1/16$.*

(ii) *$\Sigma_k^\varepsilon : \mathcal{W}_k^\varepsilon \to R_k^\varepsilon$ is a bijection for $\varepsilon = +1$ if $k = 1, 4$ and for $\varepsilon = -1$ if $k = 1, 3, 4$.*

Proof. The first statement in (i) holds, since for $p_0 \in W_1^-$ we have

$$\Sigma_1^-(p_0) - \frac{1}{16} = -\frac{(1 - 6r_0^2 + v_0^2 + r_0^4)^2}{16(1 + 2r_0^2 + v_0^2 + r_0^4)^2} < 0,$$

and the second statement, since for $p_0 \in W_3^-$ we have

$$\Sigma_3^-(p_0) - \frac{1}{16} = \frac{(1 - v_0^2 - r_0^4)^2}{64 r_0^4} > 0.$$

To show (ii) we compute the derivatives of Σ_k^ε restricted to $\mathcal{W}_k^\varepsilon$ with respect to r_0: Then we have Σ_1^ε is strictly increasing, Σ_3^- is strictly decreasing in $\mathcal{W}_3^-$. The function Σ_4^+ is strictly decreasing in $\mathcal{W}_4^+$ and Σ_4^- is strictly increasing in $\mathcal{W}_4^-$. We note that $\Sigma_4^-\big(\sqrt{2+\sqrt{3}}\big) = 1/16$. □

Since we only consider equivalence with respect to isotropies in Theorem 5.1 we give the following lemma.

Lemma 6.37. *Consider the mappings G_1^ε and $G_{2,s}^\varepsilon$ from Theorem 5.1. In the sense of Definition 6.3 the mapping G_1^ε is not equivalent to $G_{2,s}^\varepsilon$ for any $s \geq 0$.*

Proof. Let us denote $H := G_{1,p_0}^\varepsilon$ as in (6.2) with components (f_1, f_2, g) and $\hat{H} := G_{2,s}^\varepsilon$ with components $(\hat{f}_1, \hat{f}_2, \hat{g})$. Then we compute S for H from (6.6) given by

$$S = \left\{p_0 \in \mathfrak{H}^2 : \left(4 - v_0^2 - 2\,\mathrm{i}\, v_0 r_0^2 + r_0^4\right)\left(2\,\varepsilon + \mathrm{i}\, v_0 + r_0^2\right) = 0\right\}.$$

Moreover we have for $p_0 \in W = \complement S \cap \mathbb{H}^2$

$$g_w(0) = \frac{4\Big(4 + 4\,\varepsilon\, r_0^2 + v_0^2 + r_0^4\Big)}{\left(4 - v_0^2\right)^2 + 2r_0^4\left(4 + v_0^2\right) + r_0^8} > 0. \tag{6.22}$$

Then for $p_0 \in W$ we have $H \in \mathcal{F}_2$ and we compute standard parameters to normalize the mapping according to Proposition 4.1. First we introduce the expression

$$R := \sqrt{\frac{4+4\,\varepsilon\, r_0^2+v_0^2+r_0^4}{(4-v_0^2)^2+2r_0^4(4+v_0^2)+r_0^8}},$$

which is the square root of (6.22), such that the standard parameters for the normalization are given by

$$\begin{aligned}
c_1' :=\; & -(4+r_0^4+2\,\mathrm{i}\, r_0^2 v_0 - v_0^2)\Big(cu\big(8\,\varepsilon + r_0^6 - r_0^4(6\,\varepsilon+\mathrm{i}\, v_0)+4\,\mathrm{i}\, v_0 - 2\,\varepsilon\, v_0^2 - \mathrm{i}\, v_0^3 \\
& - r_0^2(12-4\,\mathrm{i}\,\varepsilon\, v_0 - v_0^2)\big) + \lambda r_0 e^{\mathrm{i}\,\theta_0}\big(4\,\mathrm{i} + 4\,\varepsilon\, v_0 + 4\,\mathrm{i}\,\varepsilon\, r_0^2 + \mathrm{i}\, v_0^2 - 2r_0^2 v_0 - \mathrm{i}\, r_0^4\big)\Big) \\
& /\Big(2\lambda(4-v_0^2-2\,\mathrm{i}\, r_0^2 v_0 + r_0^4)(4+4\,\varepsilon\, r_0^2+v_0^2+r_0^4)\Big), \\
c_2' :=\; & \frac{2r_0 e^{\mathrm{i}\,\theta_0}\big(4-v_0^2+2\,\mathrm{i}\, r_0^2 v_0 + r_0^4\big)\big(cu(-4+v_0^2+r_0^4)+\mathrm{i}\,\lambda r_0 e^{\mathrm{i}\,\theta_0}(\mathrm{i}\, v_0 - r_0^2)\big)}{\lambda\big(4-v_0^2-2\,\mathrm{i}\, r_0^2 v_0 + r_0^4\big)\big(4+4\,\varepsilon\, r_0^2+v_0^2+r_0^4\big)}, \\
\lambda' :=\; & \frac{1}{\lambda R}, \\
a_1' :=\; & \frac{8-4\,\mathrm{i}\,\varepsilon\, v_0 - 12\,\varepsilon\, r_0^2 - 2v_0^2 - 4\,\mathrm{i}\, r_0^2 v_0 + \mathrm{i}\,\varepsilon\, v_0^3 - 6r_0^4 + \varepsilon\, r_0^2 v_0^2 + \mathrm{i}\,\varepsilon\, r_0^4 v_0 + \varepsilon\, r_0^6}{uu' R(4-v_0^2+2\,\mathrm{i}\, r_0^2 v_0 + r_0^4)^2}, \\
a_2' :=\; & \frac{4r_0 e^{-\mathrm{i}\,\theta_0}(-4+v_0^2+r_0^4)}{uu' R(4-v_0^2+2\,\mathrm{i}\, r_0^2 v_0 + r_0^4)^2}, \\
c :=\; & \frac{\mathrm{i}\,\lambda r_0 e^{\mathrm{i}\,\theta_0}}{u(2\,\varepsilon + \mathrm{i}\, v_0 + r_0^2)}, \\
u' :=\; & \frac{(4-v_0^2-2\,\mathrm{i}\, r_0^2 v_0 + r_0^4)^2\sqrt{4+v_0^2+4\,\varepsilon\, r_0^2+r_0^4}}{u^3(2+\mathrm{i}\,\varepsilon\, v_0 + \varepsilon\, r_0^2)(4-v_0^2-+2\,\mathrm{i}\, r_0^2 v_0 + r_0^4)^2}, \\
\lambda :=\; & \frac{\sqrt{4+v_0^2+4\,\varepsilon\, r_0^2+r_0^4}}{2}.
\end{aligned}$$

The resulting normalized mapping is denoted by $\widetilde{H} \in \mathcal{N}_2$, according to Definition 6.3, and we want to solve $\widetilde{H} = \hat{H}$. First we compare coefficients belonging to the 2-jet at 0. Using Remark 4.4 we have $\widetilde{f}_{1w^2}(0) = 0$ and $\hat{f}_{1w^2}(0) = s/2$, thus we need to require $s = 0$ such that the 2-jets of the mappings at 0 coincide. Considering higher order derivatives we discover $\widetilde{f}_{2z^2w}(0) \in \mathbb{R}$, depending on r and r' and is of the form as in (4.11). The standard parameter u is not present in the coefficient $\widetilde{f}_{2z^2w}(0)$, see Lemma 5.18. On the other hand we have $\hat{f}_{2z^2w}(0) = \frac{\mathrm{i}\,\varepsilon}{2}$, hence G_1^ε and $G_{2,s}^\varepsilon$ are not equivalent for any $s \geq 0$. □

Remark 6.38. We can also compute the remaining standard parameters in the normalization in the proof of Lemma 6.37, which are given by

$$\begin{aligned}
r' :=\; & -\frac{4r\big((4-v_0^2)^2+2(4+v_0^2)r_0^4+r_0^8\big)+v_0\big(-48+8v_0^2+v_0^4+2(12+v_0^2)r_0^4+r_0^8\big)}{4(4+v_0^2+4\,\varepsilon\, r_0^2+r_0^4)^2}, \\
r :=\; & -\frac{v_0}{4}.
\end{aligned}$$

Then if we compose G_1^ε with translations and renormalize as in Definition 6.3, the resulting mapping is

G_1^ε, which again shows that $O(G_1^\varepsilon) = O_0(G_1^\varepsilon)$.

The following Lemma is stated in [Mey06, Lemma 2.1].

Lemma 6.39. *Let $H \in \mathcal{F}_2$ and $\phi \in \mathrm{Aut}(\mathbb{H}^2, 0)$ and $\phi' \in \mathrm{Aut}(\mathbb{H}^3_\varepsilon, 0)$, then $\widetilde{H} \coloneqq \phi' \circ H \circ \phi$ satisfies $\deg \widetilde{H} = \deg H$.*

Proof. Let $F : \mathbb{C}^2 \to \mathbb{C}^3$ be a reduced rational mapping of maximal degree $k \geq 1$ satisfying $F(0) = 0$ and $F = (f_1, f_2, f_3)/p$ with $p(0) \neq 0$. Further let $L : \mathbb{C}^2 \to \mathbb{C}^2$ and $L' : \mathbb{C}^3 \to \mathbb{C}^3$ be linear rational mappings satisfying $L(0) = L'(0) = 0$ which are given by $L = (\ell_1, \ell_2)/q$ and $L' = (\ell'_1, \ell'_2, \ell'_3)/q'$ with $q(0), q'(0) \neq 0$. Let $\widetilde{F} \coloneqq L' \circ F \circ L$, then we claim that $\deg \widetilde{F} \leq \deg F$ after removing common factors. For convenience we write for coordinates $z = (z_1, z_2) \in \mathbb{C}^2$. Let $h \in \{f_1, f_2, f_3, p\}$ and write with mulitindex notation $h(z) = \sum_{|\alpha| \leq k} a_\alpha z^\alpha$. Then we have

$$h(L(z)) = \sum_{|\alpha|\leq k} a_\alpha \frac{\ell(z)^\alpha}{q(z)^{|\alpha|}} = \frac{1}{q(z)^k} \sum_{|\alpha|\leq k} a_\alpha q(z)^{k-|\alpha|} \ell(z)^\alpha \eqqcolon \frac{h_L(z)}{q(z)^k},$$

where each monomial of h_L is of maximal degree k. Note that $\widehat{F} \coloneqq F(L)$ satisfies $\widehat{F}(0) = 0$ and is of the form $\widehat{F} = \widehat{f}/\widehat{p} = (\widehat{f}_1, \widehat{f}_2, \widehat{f}_3)/\widehat{p}$. Hence if we consider the m-th component of $\widehat{F}$ we have $\deg(\widehat{f}_m/\widehat{p}) \leq k$. Let $\widehat{h} \in \{\widehat{\ell}_1, \widehat{\ell}_2, \widehat{\ell}_3, \widehat{q}\}$ where we write $\widehat{h}(z) = \sum_{|\beta|\leq 1} b_\beta z^\beta$. Then we have

$$\widehat{h}(\widehat{F}(z)) = \sum_{|\beta|\leq 1} b_\beta \frac{\widehat{f}(z)^\beta}{\widehat{p}(z)^\beta} = \frac{1}{\widehat{p}(z)} \sum_{|\beta|\leq 1} b_\beta \widehat{p}(z)^{1-|\beta|} \widehat{f}(z)^\beta \eqqcolon \frac{\widehat{h}_F(z)}{\widehat{p}(z)},$$

where each monomial of $\widehat{h}_F$ is of maximal degree k. We set $\widetilde{F} \coloneqq L'(\widehat{F})$, which satisfies $\widetilde{F}(0) = 0$ and is of the form $\widetilde{F} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3)/\widetilde{p}$. Thus if we consider the m-th component of $\widetilde{F}$ we obtain that $\deg(\widetilde{f}_m/\widetilde{p}) \leq k$, which proves the claim.
To prove the lemma let $H \in \mathcal{F}_2$. Then from Theorem 5.1 it follows that there exists $G \in \mathcal{N}_2$ with $2 \leq \deg G \leq 3$ and isotropies $\psi \in \mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\psi' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ such that $H = \psi' \circ G \circ \psi$. In particular H is a rational mapping of some degree m. Hence by the above claim we obtain $m \leq \deg G$. If we rewrite the representation of H by G, i.e., $G = \psi'^{-1} \circ H \circ \psi^{-1}$, we get $\deg G \leq m$, which shows $\deg H = \deg G$.
If we let $\widetilde{H}$ be as in the hypothesis and H as in the previous paragraph, we have

$$\widetilde{H} = \phi' \circ H \circ \phi = \phi' \circ \psi' \circ G \circ \psi \circ \phi,$$

and we set $\varphi \coloneqq \psi \circ \phi$ and $\varphi' \coloneqq \phi' \circ \psi'$. Since φ and φ' are of the form as L and L' above, we are in the situation as above and we can argue as we did for H and G before to obtain $\deg \widetilde{H} = \deg H$. □

We now summarize all the previous results to give a proof of Theorem 6.8.

Proof of Theorem 6.8. The last claim of the theorem concerning G_1^ε is proved in Lemma 6.37 and Lemma 6.39. Let us consider for $1 \leq k \leq 4$ the mappings $\mathcal{H}_k^\varepsilon$ from Definition 6.29, $\mathcal{H}_{k,p_0}^\varepsilon$ from (6.2) and $\widetilde{\mathcal{H}}_k^\varepsilon$ as in (6.3) and $p_0 \in W_k^\varepsilon$ according to Remark 6.14. We note that according to subsection 6.3

and Definition 6.29 the map $\mathcal{H}_1^\varepsilon$ is equivalent to $\mathcal{G}_1^\varepsilon$, $\mathcal{H}_2^-$ is equivalent to $\mathcal{G}_2^-$, $\mathcal{H}_3^-$ is equivalent to $G^-_{2,\frac{\sqrt{5}}{4}}$ and $\mathcal{H}_4^\varepsilon$ is equivalent to $\mathcal{G}_4^\varepsilon$.
Since $p_0 \in W_k^\varepsilon$ we have $\mathcal{H}_{k,p_0}^\varepsilon \in \mathcal{F}_2$. In Lemma 6.33 we explicitly derive standard parameters such that $\widetilde{\mathcal{G}}_k^\varepsilon$ satisfies (i)–(iv) of the normalization conditions in Proposition 4.1 and we compute the expressions

$$s_k^\varepsilon(p_0) := \frac{|\widetilde{f}^\varepsilon_{1k,p_0w^2}(0)|}{2}.$$

We start investigating the case of mappings of degree 2, which are $\widetilde{\mathcal{H}}_1^\varepsilon, \widetilde{\mathcal{H}}_2^\varepsilon$ and $\widetilde{\mathcal{H}}_3^\varepsilon$ since the degree does not change if we apply automorphisms according to Lemma 6.39. We recall that these mappings originate from $G_{2,s}^\varepsilon$ which is of degree 2. According to Theorem 5.1 we have

$$\widetilde{\mathcal{H}}_k^\varepsilon \in \{G_1^\varepsilon, G_{2,s}^\varepsilon\},$$

where $s = s_k^\varepsilon(p_0)$ holds. By Lemma 6.37 the only possible case is $\widetilde{\mathcal{H}}_k^\varepsilon = G_{2,s}^\varepsilon$, where $s = s_k^\varepsilon(p_0)$. We obtain (i) by Lemma 6.36 (ii) since the mappings $G_{2,s}^+$ are equivalent to $\mathcal{H}_1^+$ for $s \geq 0$. In the case of $\varepsilon = -1$ we obtain again by Lemma 6.36 (ii) that $G_{2,s}^-$ is equivalent to $\mathcal{H}_1^-$ if $0 \leq s < 1/4$ and $G_{2,s}^-$ is equivalent to $\mathcal{H}_3^-$ if $s > 1/4$, proving (iii) and (iv). Finally applying Lemma 6.36 (i) shows that $\mathcal{G}_1^-, \mathcal{G}_2^-$ and $\mathcal{G}_3^-$ are pairwise not equivalent, which proves (v).
Next we treat the case of mappings of degree 3, i.e., $G_{3,s}^+$ for $s \geq 0$ and $G_{3,s}^-$ for $1/4 \neq s \geq 0$ by Lemma 5.3. According to Lemma 6.39 we have

$$\deg\big(\widetilde{\mathcal{H}}_4^\varepsilon\big) = 3.$$

Thus by Theorem 5.1 it holds that $\widetilde{\mathcal{H}}_4^\varepsilon = G_{3,s}^\varepsilon$, where $s = s_4^\varepsilon(p_0)$ and satisfies $s \neq 1/4$ for $\varepsilon = -1$. Then we restrict the parameter space of the translations to $p_0 = (r_0, \mathrm{i}\, r_0^2) \in \mathbb{H}^2$ for $r_0 \geq 0$ to obtain if $s \neq 1/4$ for $\varepsilon = -1$ or if $s \geq 0$ for $\varepsilon = +1$ from Lemma 6.36 (ii) that $G_{3,s}^\varepsilon$ is equivalent to $\mathcal{H}_4^\varepsilon$, which proves (ii) and (vi) of Theorem 6.8. In the exceptional case $t := |f^-_{13w^2}(0)| = 1/2$ in $G_{3,t}^-$ we have $G_{3,1/2}^- = \mathcal{G}_2^-$ as in (vi). □

Remark 6.40. In the proof of Theorem 6.8 we avoid to compute all standard parameters such that $\widetilde{\mathcal{H}}_k^\varepsilon \in \mathcal{N}_2$. In Proposition 4.1 and Remark 4.3 we have shown that we can achieve $\widetilde{H} \in \mathcal{N}_2$ for any $H \in \mathcal{F}_2$. In Appendix D we give the standard parameters appearing in the mapping $\widetilde{\mathcal{H}}_k^\varepsilon$, such that $\widetilde{\mathcal{H}}_k^\varepsilon \in \mathcal{N}_2$ for $k = 1, 3, 4$. For the mapping $\widetilde{\mathcal{H}}_2^-$ we proceed differently and make use of Theorem 4.12 and Example 4.13. We define the admissible normal form $\mathcal{N} := \{\sigma' \circ H \circ \sigma : H \in \mathcal{N}_2\}$, where σ and σ' are the isotropies we used in Example 6.23 to show equivalence of $\mathcal{H}_2^-$ and $G^-_{2,\sqrt{5}/4}$, which by Theorem 6.8 is equivalent to $\mathcal{G}_2^- = G^-_{2,1/2}$. Then we renormalize $\widetilde{\mathcal{H}}_2^-$ with respect to $\mathcal{N}$, i.e., we require $\widetilde{\mathcal{H}}_2^- \in \mathcal{N}$ and list the appropriate standard parameters for this inclusion in Appendix D. Then analogously as for $\mathcal{N}_2$ we obtain that $\widetilde{\mathcal{H}}_2^- = \mathcal{H}_2^-$ for all admissible $p_0 \in \mathbb{H}^2$.

7 The Class of Degenerate Mappings

In order to complete the classification in Proposition 3.16 we need to study the following class of holomorphic mappings.

Definition 7.1. For a neighborhood $U \subset \mathbb{C}^2$ of 0 let us denote the set $\mathcal{F}_1(U)$ of holomorphic mappings $H = (f_1, f_2, g)$ with $H(U \cap \mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$, which are constantly $(1,1)$-degenerate, transversal at 0 and satisfy $H(0) = 0$. By Proposition 3.13 we have

$$f_{1z}(0) f_{2z^k}(0) - f_{2z}(0) f_{1z^k}(0) = 0, \qquad \forall k \geq 2, \tag{7.1}$$

and by Lemma 3.3 we obtain $g_w(0) > 0$. We denote by $\mathcal{F}_1$ the set of germs H, such that $H \in \mathcal{F}_1(U)$ for some $U \subset \mathbb{C}^2$ a neighborhood of 0.

The following theorem shows the missing claim (ii) in Proposition 3.16.

Theorem 7.2. *Let $H \in \mathcal{F}_1$, then in the sense of Definition 2.26 we have H is equivalent to the mapping $(z, w) \mapsto (z, 0, w)$.*

Remark 7.3.

(i) We prove the theorem by proceeding as in the nondegenerate case: First we compose the degenerate mapping with automorphisms in order to fix some coefficients and then we compute a jet parametrization which gives the linear embedding $(z, w) \mapsto (z, 0, w)$.

(ii) A different way to prove the theorem is to refer to [ES10, Theorem 1.1] to obtain that the image of H is contained in a 2-dimensional hyperplane and conclude directly that H is equivalent to a linear embedding. Yet another alternative for $\varepsilon = +1$ can be found in [Far82, Lemma 1.7].

(iii) Note that Theorem 7.2 together with Proposition 3.13 implies that a mapping which is $(1,1)$-degenerate in an open, dense subset of its domain in $\mathbb{H}^2$ is already $(1,1)$-degenerate everywhere in its domain in $\mathbb{H}^2$.

Proposition 7.4. *Let $H \in \mathcal{F}_1$. Then there exist automorphisms $\sigma \in \mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\sigma' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ such that $\widetilde{H} \coloneqq \sigma' \circ H \circ \sigma$ satisfies $\widetilde{H}(0) = 0$ and the following conditions:*

(i) $\widetilde{H}_z(0) = (1, 0, 0)$

(ii) $\widetilde{H}_w(0) = (0, 0, 1)$

(iii) $\widetilde{f}_{2z^2}(0) = 0$

(iv) $\mathrm{Re}\big(\widetilde{g}_{w^2}(0)\big) = 0$

Definition 7.5. We write $\mathcal{N}_1$ for the set of holomorphic mapping of $\mathcal{F}_1$ satisfying the conditions given in Proposition 7.4.

Proof. We start by setting $u = 1 = u' = \lambda$ and $c = 0 = r'$ in the definitions of the isotropies from Definition 2.23 and Definition 2.24. Next we proceed as in Proposition 4.1 and consider the following coefficients of $\widetilde{H}$ together with the conditions we impose on them.

$$\widetilde{H}_z(0) = \left(\lambda' \begin{pmatrix} a_1' & -\varepsilon\, a_2' \\ \bar{a}_2' & \bar{a}_1' \end{pmatrix} \begin{pmatrix} f_{1z}(0) \\ f_{2z}(0) \end{pmatrix}, 0 \right) = \big(1, 0, 0\big), \tag{7.2}$$

$$\widetilde{H}_w(0) = \left(\lambda' \begin{pmatrix} a_1' & -\varepsilon\, a_2' \\ \bar{a}_2' & \bar{a}_1' \end{pmatrix} \begin{pmatrix} c_1' g_w(0) + f_{1w}(0) \\ c_2' g_w(0) + f_{2w}(0) \end{pmatrix}, \lambda'^2 g_w(0) \right) = (0, 0, 1). \tag{7.3}$$

Considering the first two equations of (7.2) we set

$$a_1' = \frac{f_{1z}(0)}{||f_z(0)||_\varepsilon}, \qquad a_2' = \frac{f_{2z}(0)}{||f_z(0)||_\varepsilon},$$

and the third equation of (7.3) gives

$$\lambda' = \frac{1}{\sqrt{g_w(0)}},$$

such that the corresponding equations are satisfied if we use (3.4). By setting

$$c_1' = -\frac{f_{1w}(0)}{g_w(0)}, \qquad c_2' = -\frac{f_{2w}(0)}{g_w(0)},$$

we have fixed the 1-jet of H at 0 such that $a' = (a_1', a_2') \in \mathcal{S}^2_{\varepsilon,\sigma}$, $\lambda' > 0$ and $c_1', c_2' \in \mathbb{C}$. With the choices for a' we obtain

$$\widetilde{f}_{2z^2}(0) = \lambda'\Big(a_2' f_{1z^2}(0) + \bar{a}_1' f_{2z^2}(0)\Big) = 0,$$

since we assumed (7.1) with $k = 2$. Finally we solve

$$\mathrm{Re}\big(\widetilde{g}_{w^2}(0)\big) = -2r + {\lambda'}^2\,\mathrm{Re}\big(g_{w^2}(0)\big) = 0,$$

for $r \in \mathbb{R}$ and we are done. $\square$

Proposition 7.6. *Let $H \in \mathcal{N}_1$. Then necessarily the coefficients of H satisfy the following equations:*

(i) $f_{1z^k}(0) = 0 \quad (k \geq 2)$

(ii) $\mathrm{Im}\big(g_{w^2}(0)\big) = 0$

(iii) $g_{z^k w}(0) = 0 \quad (k \geq 1)$

(iv) $f_{1w^2}(0) = 0$

(v) $f_{1w}(z, 0) = 0$

Proof. The conditions are simply verified by differentiating (3.3) assuming the conditions on H given in Proposition 7.4:

Differentiation of (3.3) with respect to z and evaluating the result at $(z, \chi, \tau) = (0, \chi, 0)$ gives $\chi = \bar{f}_1(\chi, 0)$ if we assume the conditions on the 1-jet of H at 0. Differentiating k-times this equation for $k \geq 2$ gives (i).

If we differentiate (3.3) twice with respect to τ and evaluate the result at 0 we obtain using $H_w(0) = (0, 0, 1)$ that $\mathrm{Im}(g_{w^2}(0)) = 0$ which is (ii).

Differentiation of (3.3) with respect to τ and evaluating the result at $(z, \chi, \tau) = (0, \chi, 0)$ shows, again if we use $H_w(0) = (0, 0, 1)$, $\bar{g}_\tau(\chi, 0) = 1$ and thus (iii).

In order to obtain (v) we first show $f_{1zw}(0) = 0$: Here we differentiate (3.3) twice with respect to z and twice with respect to χ and evaluate at 0. If we use $H_z(0) = (1, 0, 0)$ and $H_{z^2}(0) = (0, 0, 0)$ we obtain the desired condition.

Next if we differentiate (3.3) twice with respect to z and three times with respect to χ we obtain (iv)

when evaluated at 0. Here we need to use $H_z(0) = (1,0,0), H_{z^2}(0) = (0,0,0)$ and $f_{1zw}(0) = 0$.
We get the last condition by combining two equations. The first equation is obtained by differentiating (3.3) twice with respect to z and the second one by differentiating with respect to z and τ. Both equations we evaluate at $(0,\chi,0)$ and use $H_z(0) = (1,0,0), f_w(0) = (0,0), H_{z^2}(0) = (0,0,0), f_{1zw}(0) = 0 = g_{zw}(0)$ and $f_{1w^2}(0) = 0 = g_{w^2}(0)$ to get the following equations:

$$\begin{aligned}\bar{f}_2(\chi,0)\big(f_{2zw}(0) + \mathrm{i}\,\chi f_{2w^2}(0)\big) &= 0,\\ \bar{f}_{1\tau}(\chi,0) + \varepsilon\,\bar{f}_2(\chi,0)\big(f_{2zw}(0) + 2\,\mathrm{i}\,\chi f_{2w^2}(0)\big) &= 0,\end{aligned}$$

from which we conclude $\bar{f}_{1\tau}(\chi,0) = 0$ and (v). □

Remark 7.7. We summarize the conditions we obtained for the 2-jet of $H \in \mathcal{N}_1$ at 0 from Proposition 7.4 and Proposition 7.6.

(i) $H(0) = 0$
(ii) $H_z(0) = (1,0,0)$
(iii) $H_w(0) = (0,0,1)$
(iv) $H_{z^2}(0) = (0,0,0)$
(v) $H_{zw}(0) = (0, f_{2zw}(0), 0)$
(vi) $H_{w^2}(0) = (0, f_{2w^2}(0), 0)$

Lemma 7.8 ([Lam01, Proposition 30])**.** *Let $H \in \mathcal{N}_1$. Then after applying an automorphism of $\mathbb{H}^2$ we have $H(z, 2\,\mathrm{i}\,z\chi) = (z, 0, 2\,\mathrm{i}\,z\chi)$ for all $(z,\chi) \in \mathbb{C}^2$ near 0.*

Proof. We proceed as in the proof of [Lam01, Proposition 30]. We additionally assume the conditions for the coefficients of H from Remark 7.7. The $(1,1)$-degeneracy, more precisely equation (7.1), allows us to solve for $H(z, 2\,\mathrm{i}\,z\chi)$ in three equations according to the functions Φ_1, Φ_2, Φ_3 defined as follows. The function Φ_1 simply is (3.3):

$$\Phi_1(z,w,\chi,\tau) \coloneqq g(z,w) - \bar{g}(\chi,\tau) - 2\,\mathrm{i}\Big(f_1(z,w)\bar{f}_1(\chi,\tau) + \varepsilon\, f_2(z,w)\bar{f}_2(\chi,\tau)\Big).$$

For Φ_2 we take derivatives of (3.3) with respect to the vector field $L = \frac{\partial}{\partial\chi} - 2\,\mathrm{i}\,z\frac{\partial}{\partial\tau}$ as in the proof of Proposition 3.13:

$$\begin{aligned}\Phi_2(z,w,\chi,\tau) \coloneqq L\rho'\big(H(z,w), \bar{H}(\chi,\tau)\big) &= \bar{g}_\chi(\chi,\tau) - 2\,\mathrm{i}\,z\bar{g}_\tau(\chi,\tau)\\ &\quad - 2\,\mathrm{i}\Big(f_1(z,w)\big(\bar{f}_{1\chi}(\chi,\tau) - 2\,\mathrm{i}\,z\bar{f}_{1\tau}(\chi,\tau)\big) + \varepsilon\, f_2(z,w)\big(\bar{f}_{2\chi}(\chi,\tau) - 2\,\mathrm{i}\,z\bar{f}_{2\tau}(\chi,\tau)\big)\Big).\end{aligned}$$

For the function Φ_3 we use the $(1,1)$-degeneracy: We write

$$\rho'(z_1', z_2', w', \chi_1', \chi_2', \tau') \coloneqq w' - \tau' - 2\,\mathrm{i}\big(z_1'\chi_1' + \varepsilon\, z_2'\chi_2'\big),$$

and as in the proof of Lemma 3.11 in (3.22) we define for $k = 1,2$

$$\varphi_k(z,w,\chi,\tau) \coloneqq \rho'_{z_k'}\Big(H(z,w), \bar{H}(\chi,\tau)\Big), \qquad \varphi_3(z,w,\chi,\tau) \coloneqq \rho'_{w'}\Big(H(z,w), \bar{H}(\chi,\tau)\Big),$$

In our case we have

$$\begin{aligned}
&\varphi_1(z,w,\chi,\tau) = -2\,\mathrm{i}\,\bar f_1(\chi,\tau), \qquad \varphi_2(z,w,\chi,\tau) = -2\,\mathrm{i}\,\varepsilon\,\bar f_2(\chi,\tau),\\
&\varphi_3(z,w,\chi,\tau) = 1.
\end{aligned}$$

Then we set

$$\begin{aligned}
\bar\Phi_3(z,w,\chi,\tau) &:= L\varphi_1(z,w,0,w)\varphi_2(z,w,\chi,\tau) - L\varphi_2(z,w,0,w)\varphi_1(z,w,\chi,\tau) \qquad (7.4)\\
&= -4\,\varepsilon\Big(\big(\bar f_{1\chi}(0,w) - 2\,\mathrm{i}\,z\bar f_{1\tau}(0,w)\big)\bar f_2(\chi,\tau) - \big(\bar f_{2\chi}(0,w) - 2\,\mathrm{i}\,z\bar f_{2\tau}(0,w)\big)\bar f_1(\chi,\tau)\Big).
\end{aligned}$$

After barring the previous expression $\bar\Phi_3$ we get an equation denoted by Φ_3. Then we restrict Φ_k to $\mathbb{H}^2$ to obtain

$$\Phi_k\big(z,\tau+2\,\mathrm{i}\,z\chi,\chi,\tau\big) = 0, \qquad 1 \le k \le 3. \tag{7.5}$$

Let us give an argument why we choose Φ_3 in the above form:
We refer to Remark 3.12. We let $\gamma \in \mathbb{N}^n$, then in our case the determinant of (3.24) becomes

$$\left|\begin{pmatrix} L\varphi_1 & L\varphi_2 \\ L^\gamma\varphi_1 & L^\gamma\varphi_2 \end{pmatrix}(z,w,\chi,\tau)\right|, \tag{7.6}$$

for $(z,w,\chi,\tau) \in \mathbb{H}^2$, because $L^\gamma\varphi_3 = 0$ for $\gamma \ge 1$. Points of the form $p = (z,w,0,w)$ belong to the complexified version of $\mathbb{H}^2$, such that the vanishing of the determinant in (7.6) at p becomes

$$L\varphi_1(z,w,0,w)L^\gamma\varphi_2(z,w,0,w) - L\varphi_2(z,w,0,w)L^\gamma\varphi_1(z,w,0,w) = 0,$$

for all $\gamma \ge 0$, which is equivalent to the equation in (7.4) being 0, since for $k = 1,2$

$$L^\gamma\varphi_k(z,w,0,w) = \frac{\partial^\gamma}{\partial\chi^\gamma}\bigg|_{\chi=0} \bar f_k(\chi, w-2\,\mathrm{i}\,z\chi).$$

We proceed setting $\tau = 0$ in (7.5) which yields the following system of equations using the conditions of Proposition 7.4 and Proposition 7.6:

$$\begin{pmatrix} 2\,\mathrm{i}\,\chi & 2\,\mathrm{i}\,\varepsilon\,\bar f_2(\chi,0) & 1 \\ 2\,\mathrm{i} & 2\,\mathrm{i}\,\varepsilon(\bar f_{2\chi}(\chi,0) - 2\,\mathrm{i}\,z\bar f_{2\tau}(\chi,0)) & 0 \\ 0 & -1 & 0 \end{pmatrix} H(z,2\,\mathrm{i}\,z\chi) + \begin{pmatrix} 0 \\ 2\,\mathrm{i}\,z \\ 0 \end{pmatrix} = 0.$$

Solving the above equation and applying the automorphism $(z,w) \mapsto (-z,w)$ of $\mathbb{H}^2$ shows the claim. $\square$

Proof of Theorem 7.2. Let $H \in \mathcal{F}_1$, then we apply automorphisms according to Proposition 7.4 to obtain a mapping $\widetilde H \in \mathcal{N}_1$. Then we use Lemma 7.8 for $\widetilde H$ and as in the proof of Theorem 5.1 we set $\chi = \frac{w}{2\,\mathrm{i}\,z}$, which implies that H is equivalent to the linear embedding given by $L(z,w) := (z,0,w)$. $\square$

8 Classification of Mappings

In this section we give the proof of our Main Theorem by bringing together all the previously deduced steps.

8.1 Proof of the Main Theorem

Proof of the Main Theorem. Let $U \subset \mathbb{C}^2$ be an open and connected neighborhood of $p \in \mathbb{S}^2$ and $H : U \to \mathbb{C}^3$ a holomorphic mapping with $H(U \cap \mathbb{S}^2) \subset \mathbb{S}^3_\varepsilon$. At some point of the proof it may occur that we have to shrink U. By abuse of notation we denote the resulting neighborhood again by U. According to Remark 2.7 we change coordinates to obtain $p = (0,1)$ and $H(0,1) = (0,1,0) \in \mathbb{S}^3_\varepsilon$. Then we use the biholomorphisms T_3 and T_2^{-1} from (2.2) and (2.3) respectively to define

$$S_1(H) := T_3 \circ V \circ H \circ T_2^{-1},$$

where V is a unitary matrix, which interchanges the second and the third component of H. We obtain a holomorphic mapping $S_1(H) : U \to \mathbb{C}^3$, which satisfies $S_1(H)(0) = 0$ and maps $W \cap \mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$, where W is a sufficiently small and open neighborhood of 0.
By Proposition 3.16, $S_1(H)$ is either H_1^ε or H_7, after changing coordinates to obtain mappings from $\mathbb{S}^2$ to $\mathbb{S}^3_\varepsilon$, or belongs to $\mathcal{F}_2$. This class of mappings is introduced in Definition 3.17.
Next we define

$$S_2(H) := \sigma_1' \circ H \circ \sigma_1,$$

where $\sigma_1 \in \mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\sigma_1' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$. If $S_1(H) \in \mathcal{F}_2$ we consider $S_2(S_1(H))$ and choose the appropriate isotropies such that $S_2(S_1(H)) \in \mathcal{N}_2$ according to Proposition 4.1. In Theorem 5.1 we obtain that

$$S_2(S_1(H)) \in \{G_1^\varepsilon, G_{2,s}^\varepsilon, G_{3,s}^\varepsilon\},$$

where $G_{2,s}^\varepsilon, G_{3,s}^\varepsilon$ are two one-parameter families of mappings, which both depend on a real parameter $s \geq 0$. We obtain from the proof of Theorem 6.8 that the parameter s of the mappings listed in Theorem 5.1 depends on admissible $p_0 \in \mathbb{H}^2$. In Lemma 6.37 we conclude that for every $s \geq 0$ the mapping $G_{2,s}^\varepsilon$ is not equivalent to G_1^ε. Furthermore by Lemma 6.39 for $s \neq \frac{1}{2}$ it holds that $G_{3,s}^\varepsilon$ is not equivalent to G_1^ε or $G_{2,s}^\varepsilon$, since the degree of the mappings do not agree. The classification of mappings $G_{2,s}^\varepsilon$ and $G_{3,s}^\varepsilon$ is carried out in Theorem 6.8. Then we note that in Lemma 6.5 we conclude that the equivalence relation we use is the most general equivalence relation in our setting, i.e., mappings which are not equivalent with respect to our equivalence relation, cannot be equivalent with respect to any other equivalence relation using composition of automorphisms, as described in Lemma 6.5.
In order to prove equivalence to the mappings listed in the Main Theorem we recall Definition 6.3 and

introduce the following mapping:

$$S_3(H) := \sigma_2' \circ \left(t'_{H(p_0)} \circ H \circ t_{p_0}\right) \circ \sigma_2, \tag{8.1}$$

where $p_0 \in \mathbb{H}^2, \sigma_2 \in \mathrm{Aut}_0(\mathbb{H}^2, 0), \sigma_2' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$. In these considerations we may use the parameter $p_0 \in \mathbb{H}^2$ to guarantee $S_3(H) \in \mathcal{F}_2$, see subsection 6.2, Proposition 6.30 and Proposition 6.31.
We show that $S_1^{-1}(G_1^\varepsilon)$ is equivalent to H_2^ε by composing G_1^ε with dilations $(z,w) \mapsto (\sqrt{2}z, 2w)$ and then we apply S_1^{-1}, which results in the mapping H_2^ε.
Next we handle the one-parameter families of mappings $G_{2,s}^\varepsilon, G_{3,s}^\varepsilon$:
If $S_2(S_1(H)) = G_{2,s}^\varepsilon$, according to Theorem 6.8, $S_3(G_{2,s}^\varepsilon)$ is equivalent to either $\mathcal{G}_1^\varepsilon, \mathcal{G}_2^-$ or $\mathcal{G}_3^-$ when using appropriate standard parameters and choices for p_0. Note that by Theorem 6.8 (v) we have $\mathcal{G}_1^-, \mathcal{G}_2^-$ and $\mathcal{G}_3^-$ are pairwise not equivalent to each other.
It holds that $S_1^{-1}(\mathcal{H}_1^\varepsilon)$, where $\mathcal{H}_1^\varepsilon$ from Definition 6.29 is equivalent to $\mathcal{G}_1^\varepsilon$, agrees with H_3^ε.
For $\varepsilon = -1$ we have $S_1^{-1}(\mathcal{H}_2^-)$, where $\mathcal{H}_2^-$ from Definition 6.29 is equivalent to $\mathcal{G}_2^-$, is the mapping H_5.
Further $S_1^{-1}(\mathcal{H}_3^-)$, where $\mathcal{H}_3^-$ from Definition 6.29 is equivalent to $\mathcal{G}_3^-$, is equivalent to H_6. We apply the isotropy $(z_1', z_2', w') \mapsto (z_1'/2, \mathrm{i}\, z_2'/2, w'/4)$ and then S_1^{-1} to the map in (6.10) to obtain H_6.
In the case $S_2(S_1(H)) = G_{3,s}^\varepsilon$, Theorem 6.8 yields that $S_3(G_{3,s}^\varepsilon)$ is equivalent to the mapping $\mathcal{G}_4^\varepsilon$. If we consider $S_1^{-1}(\mathcal{H}_4^\varepsilon)$, where $\mathcal{H}_4^\varepsilon$ from Definition 6.29 is equivalent to $\mathcal{G}_4^\varepsilon$, we obtain H_4^ε.
It remains to prove the last statement of the Main Theorem. We show equivalence of $S_1(L_3)$ and $\mathcal{G}_1^-$, $S_1(L_4)$ and $\mathcal{G}_2^-$, $S_1(L_5)$ and $\mathcal{G}_3^-$ and finally equivalence of $S_1(L_6)$ and $\mathcal{G}_4^-$. We keep the notation for the equivalence relation from (8.1).
We start by showing the first equivalence by considering $S_3(S_1(L_3))$ and defining

$$u' = -1, \quad \lambda = \frac{1}{2}, \quad a_1' = -1, \quad c_2' = \frac{\mathrm{i}}{2},$$

and the rest of the occurring parameters trivially. Then we have $S_3(S_1(L_3)) = \mathcal{G}_1^-$.
In the case of the mapping $S_1(L_4)$ we define

$$\begin{aligned} &p_0 = (2, 4\,\mathrm{i}), \quad c = \frac{11\,\mathrm{i}}{4}, \quad u = -1, \quad \lambda = 3, \quad \lambda' = \frac{2}{3\ 3^{3/4}}, \\ &a_1' = -\frac{2}{3^{1/4}} - \frac{3^{1/4}}{8}, \quad a_2' = -\frac{2}{3^{1/4}} + \frac{3^{1/4}}{8}, \quad c_1' = -\frac{\mathrm{i}(272 - 5\sqrt{3})}{144}, \quad c_2' = \frac{\mathrm{i}(272 + 5\sqrt{3})}{144}, \end{aligned}$$

and the rest of the parameters trivially. With these choices we obtain $S_3(S_1(L_4)) = \mathcal{G}_2^-$.
Next we want to see that $S_1(L_5)$ is equivalent to $\mathcal{G}_3^-$. We define the following parameters for $S_3(S_1(L_5))$

$$\begin{aligned} &p_0 = \left(\sqrt{2}, -1 + 2\,\mathrm{i}\right), \quad c = \frac{4 + 3\,\mathrm{i}}{8\sqrt{5}}, \quad u = -\frac{1 - 2\,\mathrm{i}}{\sqrt{5}}, \quad \lambda = \frac{1}{\sqrt{2}}, \quad r = \frac{1}{8}, \quad r' = 3\sqrt{2}, \\ &\lambda' = 4\ 2^{1/4}, \quad u' = -\frac{2 - 11\,\mathrm{i}}{5\sqrt{5}}, \quad a_1' = \frac{-1 + 7\,\mathrm{i}}{5}, \quad a_2' = \frac{-4 + 3\,\mathrm{i}}{5}, \\ &c_1' = -\frac{1 - 5\,\mathrm{i}}{2^{3/4}}, \quad c_2' = -\frac{\mathrm{i}}{2^{3/4}}, \end{aligned}$$

and the remaining parameters we choose trivially. Then we have $S_3(S_1(L_5)) = G^-_{2,\frac{\sqrt{5}}{4}}$, which, since

$\sqrt{5}/4 > 1/2$, is equivalent to $\mathcal{G}_3^-$ by Theorem 6.8.
Finally we consider $S_1(L_6)$ and we want to see that this mapping is equivalent to H_4^-. Here we note that after a linear change of coordinates, L_6 is the same mapping as H_4^-, which we know is equivalent to $\mathcal{G}_3^-$. The change of coordinates in $\mathbb{C}^2$ and $\mathbb{C}^3$ is performed via the following unitary matrices V_1 and V_2 respectively given by

$$V_1 = \begin{pmatrix} 0 & \mathrm{i} \\ 1 & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This completes the proof of the Main Theorem. □

8.2 Invariants of Mappings

To show for the collection of maps listed in Main Theorem that two different maps are not equivalent to each other, we argued that any application of automorphisms to mappings in $\mathcal{F}_2$ by composition is of the form as given in the proof of the Main Theorem, see Lemma 6.5. In this section we want to give some invariants with respect to automorphisms of the hyperquadrics to distinguish the maps listed in the Main Theorem from each other.
We start with the easy cases: H_7 is not equivalent to any other map of the list, since it is not immersive. Also H_1^ε cannot be equivalent to any other map, since the map is $(1,1)$-degenerate everywhere and the mappings $H_2^\varepsilon, H_3^\varepsilon, H_4^\varepsilon, H_5$ and H_6 have points in its domain, where the map is 2-nondegenerate, see Proposition 3.16.
Next we note by Lemma 6.39 that H_4^ε is not equivalent to any other map in the list. It remains to distinguish mappings of degree 2. First we treat the case $\varepsilon = +1$. Here we have H_2^+ is equivalent to G_1^+, which is 2-nondegenerate everywhere, see Example 6.18. The map H_2^+ is equivalent to $\mathcal{G}_1^+ = G_{2,0}^+$, which has points in its domain, where the map is not 2-nondegenerate, see Proposition 6.31. Thus H_1^+ and H_2^+ are not equivalent.
Next we consider the case $\varepsilon = -1$. First we note according to Proposition 6.31 the map H_3^-, which is equivalent to $\mathcal{G}_1^- = G_{2,0}^-$, and H_5, which is equivalent to $\mathcal{G}_2^- = G_{2,1/2}^-$, are 2-nondegenerate everywhere. The maps H_2^- and H_6, which are equivalent to G_1^- and $\mathcal{G}_3^- = G_{2,1}^-$ respectively, also by Proposition 6.31, do contain points in their domains, where the maps are not 2-nondegenerate. Example 6.19 shows that G_1^- is equivalent to a map, which has $(2,1)$-degenerate points in its domain and a similar computation shows that G_1^- does not contain any point in the domain where the map is $(1,1)$-degenerate. We computed in Example 6.21, that there is a mapping which has $(1,1)$-degenerate points in its domain and is equivalent to H_6 by Theorem 6.8. Thus the maps H_2^- and H_6 are not equivalent to any other map of the list. Next we make the following observation concerning the isotropic stabilizer of isotropically equivalent mappings:

Remark 8.1. We set $G := \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon) \times \mathrm{Aut}_0(\mathbb{H}^2)$. If we let $H \in \mathcal{N}_2$ and $F = \varphi' \circ H \circ \varphi$, where $(\varphi', \varphi) \in G$, it is a well-known fact that

$$\operatorname{stab}_0(F) = \left\{(\varphi' \circ \sigma' \circ \varphi'^{-1}, \varphi \circ \sigma \circ \varphi^{-1}) \in G : (\sigma', \sigma) \in \operatorname{stab}_0(H)\right\}.$$

It remains to distinguish H_3^- from H_5. First we observe that $G_{2,0}^-$ has a nontrivial isotropic stabilizer according to Lemma 5.18. On the other hand again by Lemma 5.18 the map $G_{2,1/2}^-$ has a trivial isotropic stabilizer. By the above Remark 8.1 this property implies that any map belonging to the isotropic orbit of $G_{2,1/2}^-$ cannot have a nontrivial isotropic stabilizer. In Lemma 6.33 we concluded that $O(G_{2,1/2}^-) = O_0(G_{2,1/2}^-)$. Hence any map to which $G_{2,1/2}^-$ is equivalent must have a trivial isotropic stabilizer. Thus H_3^- and H_5 are not equivalent with respect to automorphisms preserving the hyperquadrics $\mathbb{S}^2$ and $\mathbb{S}^3_\varepsilon$. This completes the alternative proof, that none of the maps listed in the Main Theorem are equivalent to each other.
Note that the above considerations give a way to decide to which mapping a given mapping is equivalent without performing a normalization with respect to isotropies.

8.3 Algorithm for the Classification and Overviews

In the proof of the Main Theorem we describe an algorithm based on [BER97, §6] to decide for a given mapping H from $\mathbb{S}^2$ to $\mathbb{S}^3_\varepsilon$ with which of the mappings we listed in the Main Theorem the mapping H coincides after a series of applications of changes of coordinates and automorphisms. We want to summarize all the steps we need to carry out and keep track of the automorphisms we use for the normalization procedure. An overview is given in Figure 7 below.
According to Remark 2.7 we first change variables and compose H with the Cayley-Transformation to obtain $H(0) = 0$ and H maps an open neighborhood of 0 in $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$. Then as in Proposition 3.16 we need to verify if H is transversal or not.
If H is nontransversal, then it is equivalent to $(z, w) \mapsto (h(z, w), h(z, w), 0)$ for some holomorphic function $h : \mathbb{C}^2 \to \mathbb{C}$ with $h(0) = 0$. Here we basically apply a diagonal matrix to H as we did in the proof of Proposition 3.16.
If H is transversal, the Main Theorem shows that H necessarily is rational of maximal degree 3. Next we inspect where the mapping is nondegenerate or degenerate in its domain.
If H is degenerate at every point of its domain, then H is equivalent to the linear embedding $(z, w) \mapsto (z, 0, w)$ according to Theorem 7.2.
If H is nondegenerate at some point p in a neighborhood of 0, then we need to compose H with translations and consider H_p according to Definition 6.2. This step is carried out as in the beginning of the proof of Proposition 3.13, see also Lemma 6.15 for explicit arguments for the invariance of transversality and degeneracy under translations. We note that it is possible to make use of Remark 6.12 at this moment.
Then we normalize the mapping H_p, such that the conditions of Proposition 4.1 are satisfied. We denote the resulting mapping by $\widetilde{H} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{g})$. All the automorphisms we have used so far are isotropies and are given explicitly in the proof of Proposition 4.1.
Next we consider Theorem 5.1. In both cases $\varepsilon = \pm 1$ we have that $\widetilde{H}$ is one of the mappings

$G_1^{\varepsilon}(z,w), G_{2,s_0}^{\varepsilon}(z,w)$ and $G_{3,s_0}^{\varepsilon}(z,w)$ given in Theorem 5.1, where $s_0 := \widetilde{f}_{1w^2}(0) \geq 0$. We recall the jet determination result given in Corollary 5.13. This shows, in order to decide to which mapping $\widetilde{H}$ is equivalent, we only need to compare some coefficients of $\widetilde{H}$ and G_k^{ε} of at most order 3. For $\varepsilon = -1$ we need to appeal to Theorem 6.8 and decide according to the value of s_0 to which orbit the mapping $\widetilde{H}$ does belong. If $\varepsilon = +1$, by Theorem 6.8 and Lemma 6.37, there are only three orbits. To give explicit automorphisms we mention that in Theorem 6.8 the equivalence relation is defined in Definition 6.3. The standard parameters are chosen according to Proposition 4.1 and the necessary parameters $p_0 \in \mathbb{H}^2$ of the translations are among those given in Definition 6.35.

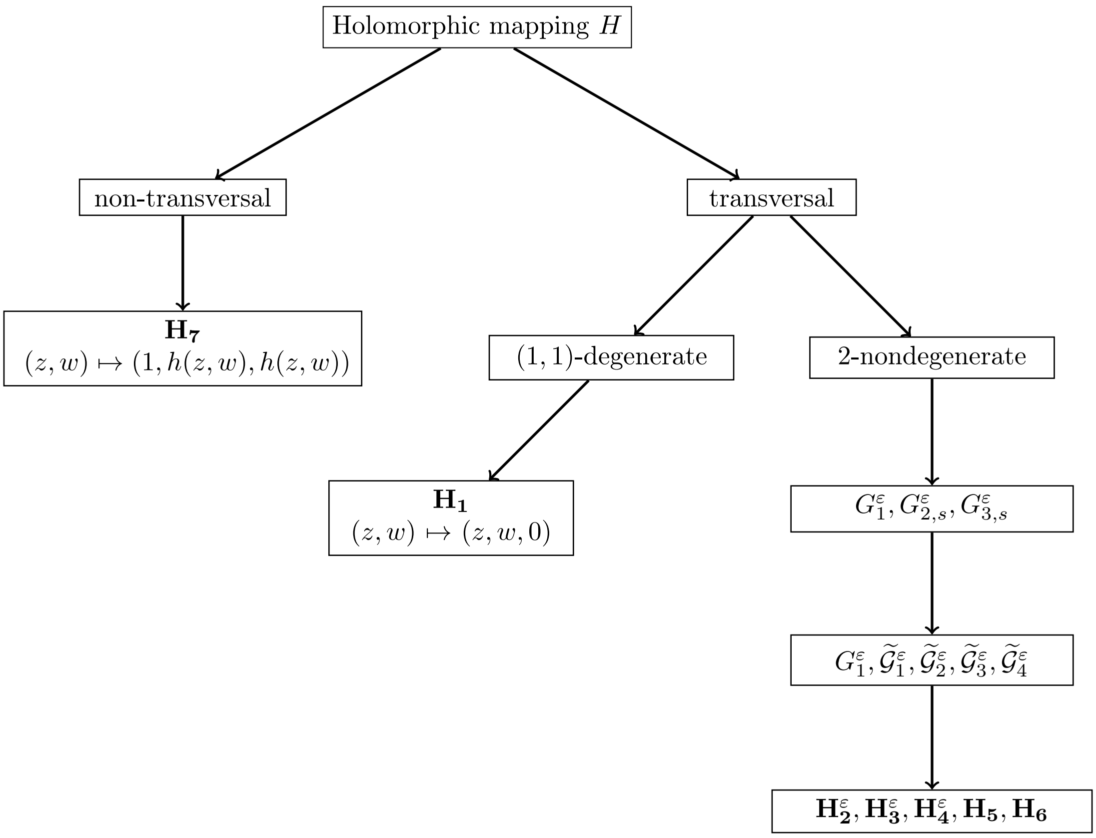

Figure 7: Overview of the classification

In the following table we list all nontrivial mappings we obtained in our classification, i.e., mappings which belong to $\mathcal{F}_2$. We also recall the nontrivial mappings from Theorem 1.1 and Theorem 1.2.

$G^{\varepsilon}_{k,s}$	**Notation**	**Value of ε**	**Equivalent Maps from $\mathbb{H}^2$ to $\mathbb{H}^3_{\varepsilon}$**	**Equivalent Maps from $\mathbb{S}^2$ to $\mathbb{S}^3_{\varepsilon}$**
G^{ε}_1		$\pm$	$(z,w) \mapsto \left(\frac{(1+w^2)z}{(1-w^2)(w+\mathrm{i}\varepsilon)}, \frac{\sqrt{2}z^2}{1-w^2}, \frac{w}{1-w^2}\right)$	$H^{\varepsilon}_2(z,w) = \left(z^2, \frac{(1-\varepsilon+(1+\varepsilon)z)w}{\sqrt{2}}, w^2\right)$
		$+$		$F_3(z,w) = \left(z^2, \sqrt{2}zw, w^2\right)$
		$-$	$(z,w) \mapsto \left(w, \sqrt{2}z^2, \mathrm{i}w^2\right)$	$L_2(z,w) = \left(z^2, \sqrt{2}w, w^2\right)$ $(z,w) \mapsto \left(\frac{\sqrt{2}}{w}, \frac{z^2}{w^2}, \frac{1}{w^2}\right)$ $(z,w) \mapsto \frac{(1,z^2,\sqrt{2}z)}{w^2}$
$G^{\varepsilon}_{2,0}$	$\mathcal{G}^{\varepsilon}_1$	$\pm$	$\mathcal{H}^{\varepsilon}_1(z,w) = \left(\frac{z(1+\mathrm{i}\varepsilon w)}{1-\mathrm{i}\varepsilon w}, \frac{2z^2}{1-\mathrm{i}\varepsilon w}, w\right)$	$H^{\varepsilon}_3(z,w) = \left(z, \frac{(1-\varepsilon+(1+\varepsilon)z^2)w}{2z}, \frac{(1-\varepsilon+(1+\varepsilon)z)w^2}{2z}\right)$
		$+$	$(z,w) \mapsto \frac{(z,zw,w)}{1-w^2}$	$F_2(z,w) = \left(z, zw, w^2\right)$
		$-$		$L_3(z,w) = \left(\frac{1}{z}, \frac{w^2}{z^2}, \frac{w}{z^2}\right)$
$G^{-}_{2,\frac{1}{2}}$	$\mathcal{G}^{-}_2$	$-$	$\mathcal{H}^{-}_2(z,w) = \left(\frac{(1+\sqrt{2}z-\mathrm{i}w)z}{1+\sqrt{2}z}, \frac{(\sqrt{2}z-\mathrm{i}w)z}{1+\sqrt{2}z}, w\right)$	$H_5(z,w) = \left(\frac{(2+\sqrt{2}z)z}{1+\sqrt{2}z+w}, w, \frac{(1+\sqrt{2}z-w)z}{1+\sqrt{2}z+w}\right)$ $L_4(z,w) = \frac{\left(z^2+\sqrt{3}zw+w^2-z, w^2+z-\sqrt{3}w-1, z^2-\sqrt{3}zw+w^2-z\right)}{w^2+z+\sqrt{3}w-1}$
$G^{-}_{2,1}$	$\mathcal{G}^{-}_3$	$-$	$\mathcal{H}^{-}_3(z,w) = \left(z, \frac{z}{w}, \frac{1+w^2}{w}\right)$ $(z,w) \mapsto \left(z, \frac{z}{1+w}, \frac{w^2}{1+w}\right)$ $(z,w) \mapsto \frac{(z,zw,w)}{1+w^2}$	$H_6(z,w) = \frac{\left((1-w)z, 1+w-w^2, (1+w)z\right)}{1-w+w^2}$ $L_5(z,w) = \frac{\left(\sqrt[4]{2}(zw-\mathrm{i}z), w^2-\sqrt{2}\mathrm{i}w+1, \sqrt[4]{2}(zw+\mathrm{i}z)\right)}{w^2+\sqrt{2}\mathrm{i}w+1}$
$G^{\varepsilon}_{3,0}$	$\mathcal{G}^{\varepsilon}_4$	$\pm$	$\mathcal{H}^{\varepsilon}_4(z,w) = \frac{\left(4z^3, \sqrt{3}(1+\varepsilon w^2)z, (3\varepsilon-w^2)w\right)}{1-3\varepsilon w^2}$	$H^{\varepsilon}_4(z,w) = \frac{\left(4z^3, (3(1-\varepsilon)+(1+3\varepsilon)w^2)w, \sqrt{3}(1-\varepsilon+2(1+\varepsilon)w+(1-\varepsilon)w^2)z\right)}{1+3\varepsilon+3(1-\varepsilon)w^2}$
		$+$		$F_4(z,w) = \left(z^3, \sqrt{3}zw, w^3\right)$
		$-$	$(z,w) \mapsto \left(\sqrt{3}zw, 2z^3, w^3\right)$	$L_6(z,w) = \frac{\left(2w^3, z(z^2+3), \sqrt{3}w(z^2-1)\right)}{3z^2+1}$

Figure 8: Overview of maps in $\mathcal{F}_2$

9 Topological Aspects

The goal of this section is to clarify some topological questions which arise in the study of holomorphic mappings from $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$. First we treat the relation of the different topologies we can associate to $\mathfrak{N}_2 \subset \mathcal{N}_2$ and the question of local triviality of $\mathfrak{F}_2 \subset \mathcal{F}_2$, for some appropriate subsets $\mathfrak{N}_2 \subset \mathfrak{F}_2$. Then we also treat the question of connectedness of $\mathcal{F}_2$ and Hausdorffness of the quotient space of $\mathcal{F}_2$ with respect to automorphisms.

We have the natural inductive limit topology of uniform convergence on compact sets τ_C, the induced topology from the jet space τ_J and the quotient topology τ_Q on $\mathcal{N}_2$. First we review the well-known fact that $\tau_C = \tau_J$, which follows from the jet parametrization for $\mathcal{F}_2$. When considering τ_Q we show that $\mathfrak{F}_2$ is a principal fibre bundle with respect to isotropies, which then implies $\tau_Q = \tau_J$ on $\mathfrak{N}_2$.

Throughout this introduction we follow [BER97]. Let us recall Definition 4.7.

Definition 9.1. For $p \in \mathbb{C}^N$ and $p' \in \mathbb{C}^{N'}$ we denote by

$$\mathcal{H}(p;p') \coloneqq \{H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p') : \ H \text{ holomorphic}\},$$

the *set of germs of holomorphic mappings from* $(\mathbb{C}^N, p)$ *to* $(\mathbb{C}^{N'}, p')$.

For $(M,p) \subset \mathbb{C}^N$ and $(M',p') \subset \mathbb{C}^{N'}$ germs of real-analytic hypersurfaces we denote by

$$\mathcal{H}(M,p;M',p') \coloneqq \{H \in \mathcal{H}(p;p') : H(M \cap U) \subset M' \text{ for some neighborhood } U \text{ of } p\},$$

the *set of germs of holomorphic mappings from* (M,p) *to* (M',p').

Definition 9.2. For $K \subset \mathbb{C}^N$ a compact neighborhood of $p \in \mathbb{C}^N$ we denote the Frechét space $\mathcal{H}_K(p;p')$ of germs of holomorphic mappings, defined in a neighborhood of K, which map $p \in \mathbb{C}^N$ to $p' \in \mathbb{C}^{N'}$. The topology for $\mathcal{H}_K(p;p')$ is given by uniform convergence on compact sets.

We equip $\mathcal{H}(p;p')$ with the inductive limit topology, denoted by τ_C, with respect to Frechét spaces $\mathcal{H}_K(p;p')$, where K is some compact neighborhood of p in $\mathbb{C}^N$. Then for $H, H_n \in \mathcal{H}(p;p')$ we say that H_n converges to H, if there exists $K \subset \mathbb{C}^N$ a compact neighborhood of p, such that each H_n is holomorphic in a neighborhood of K and H_n converges uniformly to H on K.

For $\mathcal{H}(M,p;M',p') \subset \mathcal{H}(p;p')$ we consider the induced topology τ_C of $\mathcal{H}(p;p')$.

Based on Definition 2.9 (iii) we give the following definition.

Definition 9.3 (Jet Space)**.** We denote by $J^k_{p,p'}$ the collection of all k-jets at p of germs of holomorphic mappings from $(\mathbb{C}^N, p)$ to $(\mathbb{C}^{N'}, p')$. We set $J^k_p \coloneqq J^k_{p,p}$.

Let $(M,p) \subset (\mathbb{C}^N, p)$ and $(M',p') \subset (\mathbb{C}^{N'}, p')$ be germs of real-analytic hypersurfaces. For $k \in \mathbb{N}$ we denote by $J^k_q(M,p;M',p')$ the *space of k-jets of* $\mathcal{H}(M,p;M',p')$ *at* q or the *k-jet space of* $\mathcal{H}(M,p;M',p')$ *at* q. We write $J^k_q(M,p) \coloneqq J^k_q(M,p;M,p)$ and $J^k_0(M;M') \coloneqq J^k_0(M,0;M',0)$.

We denote by $G^k_p(M,p) \subset J^k_p(M,p)$ the *space of k-jets of* $\mathrm{Aut}_p(M,p)$ *at* p.

Remark 9.4. Note that $J^k_p(M,p;M',p') \subset J^k_{p,p'}$. Then $J^k_{p,p'}$ can be identified with the space of germs of holomorphic polynomial mappings from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$ up to degree k, which map $p \in \mathbb{C}^N$ to $p' \in \mathbb{C}^{N'}$. Thus $J^k_{p,p'}$ can be identified with some $\mathbb{C}^K$, where $K \coloneqq N'\binom{N+k}{N}$, such that the topology for $J^k_{p,p'}$, denoted by τ_J, is induced by the natural topology of $\mathbb{C}^K$. We refer to the topology τ_J as *topology of* *the jet space.*

Definition 9.5 (Jet Parametrization)**.** We say $\mathcal{F} \subset \mathcal{H}(M,p;M',p')$ admits a *jet parametrization for $\mathcal{F}$ of order k* if the following properties hold:
There exists a mapping $\Psi : \mathbb{C}^N \times \mathbb{C}^K \supset U \to \mathbb{C}^{N'}$, where U is an open neighborhood of $\{p\} \times J_p^k(M,p;M',p')$, which is holomorphic in the first N variables, real-analytic in the remaining K variables, such that $F(Z) = \Psi(Z, j_p^k F)$, for all $F \in \mathcal{F}$.

Remark 9.6. (i) If $\mathcal{F} \subset \mathcal{H}(M,p;M',p')$ admits a jet parametrization of some order k, then $\tau_C = \tau_J$, which follows from the real-analyticity in the last K variables.

(ii) In our situation, where $\mathcal{F} = \mathcal{F}_2$ we have by Corollary 5.13 that $K = K_0 := 15$, where the following coefficients of $H = (f,g) = (f_1,f_2,g) \in \mathcal{F}_2$ are involved:

$$J(H) := \{f_z(0), H_w(0), f_{z^2}(0), H_{zw}(0), H_{w^2}(0), f_{z^2w}(0)\}\,.$$

Hence by Theorem 5.1 we identify $\mathcal{F}_2$ with a subset $\mathfrak{J}_2 \subset \mathbb{C}^{K_0}$, given by

$$\mathfrak{J}_2 := \{J(H) : H \in \mathcal{F}_2\},$$

and the topology we use in the sequel for $\mathcal{F}_2$ is τ_J.

9.1 Properties of the Normalization Map restricted to $\mathfrak{F}_2$

In the following definition we use the notation from Definition 5.15.

Definition 9.7. Let X, Y be topological spaces. A continuous map $f : X \to Y$ is called proper if f is closed and for each $y \in Y$ the preimage $f^{-1}(y)$ is compact.
An action α of G, a topological group, on X, a topological space, is called *proper* if the associated map $\alpha'(g,x) := \big(x, \alpha(g,x)\big)$ is a proper map in the sense defined in the previous paragraph.

Let us recall the notation from Lemma 5.18, where we set $\mathcal{E} := \{G_1^\varepsilon, G_{2,0}^\varepsilon, G_{3,0}^\varepsilon\}$.

Definition 9.8. We define $\mathfrak{N}_2 := \mathcal{N}_2 \setminus \mathcal{E}$ and $\mathfrak{F}_2 := \bigcup_{H \in \mathfrak{N}_2} O_0(H)$.

We aim for the following theorem.

Theorem 9.9. *The mapping $N : \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0) \times \mathrm{Aut}_0(\mathbb{H}^2, 0) \times \mathfrak{F}_2 \to \mathfrak{F}_2$ given by*

$$N(\phi', \phi, H) := \phi' \circ H \circ \phi^{-1},$$

is a free and proper action.

From Lemma 5.18 it is easy to see that N is a free action. To show the properness in Theorem 9.9 we use the following well-known characterization of properness in the case of free actions, whose proof can be found in e.g. [tD87].

Lemma 9.10 ([tD87, Proposition 3.20])**.** *Let G be a topological group acting freely on a topological space X via the action $\alpha : G \times X \to X$. Then the following statements are equivalent:*

(i) *G acts properly.*
(ii) *Let $\alpha' : G \times X \to X \times X$ be given by $\alpha'(g,x) := \big(x, \alpha(g,x)\big)$. The image $C \subset X \times X$ of α' is closed and the map $\varphi_\alpha : C \to G$, given by $\varphi_\alpha\big(x, \alpha(g,x)\big) := g$ is continuous.*

Remark 9.11. For $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ a germ of a holomorphic mapping, for which we assume that $F \in \mathcal{F}_2$ and the jet $J(F) \subset j_0^3 F$ is of the form as in Remark 4.6, we write $F = (f^1, f^2, f^3)$ for the components and denote derivatives of F at 0 by $f^k_{\ell m} := f^k_{z^\ell w^m}(0)$. Here as usual we write (z, w) for coordinates in $\mathbb{C}^2$.

The following lemma is useful in this context.

Lemma 9.12. *For $n \in \mathbb{N}$ we let $H_n, H \in \mathfrak{N}_2$ and $\phi_n \in \mathrm{Aut}_0(\mathbb{H}^2, 0), \phi'_n \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ such that*

$$\phi'_n \circ H_n \circ \phi_n^{-1} \to H \qquad (n \to \infty),$$

then $\phi_n \to \mathrm{id}_{\mathbb{C}^2}, \phi'_n \to \mathrm{id}_{\mathbb{C}^3}$ and $H_n \to H$ as $n \to \infty$.

Proof. We assume for $H_n = (h_n^1, h_n^2, h_n^3)$ and $H = (h^1, h^2, h^3)$ to be given as in Remark 9.11, where in H_n the coefficients depend on $n \in \mathbb{N}$. We write $s_n := |h^1_{n02}| \in \mathbb{R}^+, x_n := h^2_{n02} \in \mathbb{C}$ and $y_n := \mathrm{Im}\big(h^2_{n21}\big)$. To each ϕ_n and ϕ'_n we associate $\gamma_n \in \Gamma$ and $\gamma'_n \in \Gamma'$ respectively, where we use the notation for the parametrization of $\mathrm{Aut}_0(\mathbb{H}^2, 0)$ as in Definition 2.23 and for $\mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ as in Definition 2.24 respectively. According to Theorem 5.1 we have that H_n depends on $s_n > 0$. Let us denote $\Xi := \Gamma \times \Gamma' \times \mathbb{R}^+$ and write $\xi_n = (\gamma_n, \gamma'_n, s_n) \in \Xi$. We define $\Psi_n := \phi'_n \circ H_n \circ \phi_n^{-1}$, which depends on $\xi_n \in \Xi$. For components of Ψ_n, we write $\Psi_n = (\psi_n^1, \psi_n^2, \psi_n^3)$ and $\psi_n = (\psi_n^1, \psi_n^2)$.
Note in the following the similarity with the equations we considered in the proof of Lemma 5.18. We start considering the first order terms of Ψ_n. We set

$$U'_n := \begin{pmatrix} u'_n a'_{1n} & -\varepsilon\, u'_n a'_{2n} \\ \bar{a}'_{2n} & \bar{a}'_{1n} \end{pmatrix},$$

where $|a'_{1n}|^2 + \varepsilon\,|a'_{2n}|^2 = 1$ and $u'_n \in \mathbb{S}^1$ for all $n \in \mathbb{N}$. We have

$$\psi_{nz}(0) = U'_n\ {}^t(u_n \lambda_n \lambda'_n, 0), \tag{9.1}$$
$$\Psi_{nw}(0) = \lambda_n \lambda'_n \Big(U'_n\ {}^t(u_n c_n + \lambda_n c'_{1n}, \lambda_n c'_{2n}), \lambda_n \lambda'_n\Big). \tag{9.2}$$

Since $\psi^3_{nw}(0) \to 1$ we obtain

$$\lambda_n \lambda'_n \to 1, \qquad (n \to \infty), \tag{9.3}$$

which implies if we consider (9.1), since $\psi_{nz}(0) \to (1, 0)$ as $n \to \infty$, that

$$u_n u'_n a'_{1n} \to 1, \tag{9.4}$$
$$a'_{2n} \to 0. \tag{9.5}$$

Because of $a'_n = (a'_{1n}, a'_{2n}) \in \mathcal{S}^2_{\varepsilon,\sigma}$ from Definition 2.24 (i), we have

$$|a'_{1n}| \to 1, \qquad (n \to \infty). \tag{9.6}$$

If we consider the first two components in (9.2) we obtain since $\psi_{nw}(0) \to (0,0)$, as $n \to \infty$, and by (9.5) and (9.6) that

$$u_n c_n + \lambda_n c'_{1n} \to 0, \tag{9.7}$$

$$c'_{2n} \to 0, \tag{9.8}$$

if $n \to \infty$. Next we consider the second order terms of Ψ_n.

$$\psi_{nz^2}(0) = 2u_n\lambda_n\lambda'_n U'_n \begin{pmatrix} 2\,\mathrm{i}\big(\bar{c}_n + u_n\lambda_n\bar{c}'_{1n}\big) \\ u_n\lambda_n \end{pmatrix}, \tag{9.9}$$

where the left-hand side of (9.9), $\psi_{nz^2}(0)$, must converge to $(0,2)$ as $n \to \infty$. After applying ${U'_n}^{-1}$ we rewrite the second components of (9.9) as

$$2u_n^2\lambda_n^2\lambda'_n = a'_{1n}\Big(-\frac{\bar{a}'_{2n}\psi^1_{nz^2}(0)}{u'_n a'_{1n}} + \psi^2_{nz^2}(0)\Big), \tag{9.10}$$

where the absolute value of the right-hand side of (9.10) according to (9.5) and (9.6) converges to 2 when $n \to \infty$. Taking the absolute value of the left-hand side of (9.10) implies that

$$\lambda_n \to 1, \qquad (n \to \infty), \tag{9.11}$$

which together with (9.3) shows

$$\lambda'_n \to 1, \qquad (n \to \infty). \tag{9.12}$$

Further inspection of (9.10) gives

$$\frac{u_n^2}{a'_{1n}} \to 1, \qquad (n \to \infty). \tag{9.13}$$

Next we consider

$$\psi_{nzw}(0) = \frac{\mathrm{i}}{2}\lambda_n\lambda'_n U'_n \begin{pmatrix} T_1(\gamma_n, \gamma'_n) \\ 4\lambda_n\Big(c'_{2n}\big(\bar{c}_n + u_n\lambda_n\bar{c}'_{1n}\big) - \mathrm{i}\,u_n^2 c_n\Big) \end{pmatrix}, \tag{9.14}$$

where the real-analytic function $T_1 : \Gamma \times \Gamma' \to \mathbb{C}$ does not depend on $a'_n \in \mathcal{S}^2_{\varepsilon,\sigma}$ and u'_n. The left-hand side of (9.14) has to converge to $\big(\frac{\mathrm{i}\varepsilon}{2}, 0\big)$ and we rewrite the second component of (9.14) as

$$4\lambda_n\Big(c'_{2n}\big(\bar{c}_n + u_n\lambda_n\bar{c}'_{1n}\big) - \mathrm{i}\,u_n^2 c_n\Big) = \frac{-2\,\mathrm{i}}{\lambda_n\lambda'_n u'_n}\Big(-\bar{a}'_{2n}\psi^1_{nzw}(0) + u'_n a'_{1n}\psi^2_{nzw}(0)\Big). \tag{9.15}$$

Taking the limit, we know from (9.5), (9.6) and (9.11), (9.12), that the right-hand side of (9.15) converges to 0 and if we also use (9.7) we obtain that

$$c_n \to 0, \qquad (n \to \infty), \tag{9.16}$$

such that (9.7) implies

$$c'_{1n} \to 0, \qquad (n \to \infty). \tag{9.17}$$

Next we compute

$$\psi^3_{nw^2}(0) = 2\lambda_n^2 {\lambda'_n}^2 \Big(-(r_n + \lambda_n^2 r'_n) + \mathrm{i}\Big(c_n \bar{c}_n + \varepsilon\, \lambda_n^2 c'_{2n} \bar{c}_{2n'} + \lambda_n \bar{c}'_{1n} \big(2u_n c_n + \lambda_n c'_{1n} \big) \Big) \Big). \tag{9.18}$$

We let $n \to \infty$ and take all the previously obtained limits of the sequences $c'_n = (c'_{1n}, c'_{2n}) \in \mathbb{C}^2, c_n$ and λ_n, λ'_n, then we have since $\psi^3_{nw^2}(0) \to 0$, that

$$r_n + \lambda_n^2 r'_n \to 0, \qquad (n \to \infty). \tag{9.19}$$

Next we consider

$$\psi_{nw^2}(0) = \lambda_n \lambda'_n U'_n \begin{pmatrix} \lambda_n^3 s_n + T_2(\gamma_n, \gamma'_n) \\ \lambda_n^3 x_n + T_3(\gamma_n, \gamma'_n) \end{pmatrix}, \tag{9.20}$$

where $T_2 : \Gamma \times \Gamma' \to \mathbb{C}$ and $T_3 : \Gamma \times \Gamma' \to \mathbb{C}$ are real-analytic functions and T_2 is given by

$$\begin{aligned} T_2(\gamma_n, \gamma'_n) = {} & 2(u_n c_n + c'_{1n} \lambda_n)\big(\mathrm{i}\,|c_n|^2 - r_n - \lambda_n^2 r'_n\big) + 2\,\mathrm{i}\,\lambda_n \bar{c}'_{1n} (u_n c_n + \lambda_n c'_{1n})(2u_n c_n + \lambda_n c'_{1n}) \\ & + \mathrm{i}\,\varepsilon\, \lambda_n^2 \big(u_n c_n (1 + 2|c'_{2n}|^2) + 2\lambda_n c'_{1n} |c'_{2n}| \big), \end{aligned}$$

such that $T_2(\gamma_n, \gamma'_n) \to 0$ as $n \to \infty$. Then the first component of (9.20) becomes

$$\lambda_n^3 s_n + T_2(\gamma_n, \gamma'_n) = \frac{1}{\lambda_n \lambda'_n u'_n} \Big(\bar{a}'_{1n} \psi^1_{nw^2}(0) + \varepsilon\, u'_n a'_{2n} \psi^2_{nw^2}(0) \Big). \tag{9.21}$$

Since $\big(\psi^1_{nw^2}(0), \psi^2_{nw^2}(0)\big) \to (|h^1_{02}|, h^2_{02}) \in \mathbb{R}^+ \times \mathbb{C}$, if we let $n \to \infty$ we obtain that $\bar{a}'_{1n}/u'_n \to 1$ and $s_n \to |h^1_{02}|$. Then (9.4) shows $u_n \to 1$ and (9.13) gives $a'_{1n} \to 1$, hence $u'_n \to 1$.
Finally we consider

$$\psi_{nz^2w}(0) = \lambda_n \lambda'_n U'_n \begin{pmatrix} -4\,\mathrm{i}\, u_n^2 \lambda_n^3 s_n + T_4(\gamma_n, \gamma'_n) \\ -2\,\varepsilon\, u_n^2 \lambda_n (2r_n + \lambda_n^2 r'_n) + \mathrm{i}\,\varepsilon\, u_n^2 \lambda_n^3 y_n + 12 u_n^3 \lambda_n^2 c_n s_n + T_5(\gamma_n, \gamma'_n) \end{pmatrix}, \tag{9.22}$$

where $T_4 : \Gamma \times \Gamma' \to \mathbb{C}$ and $T_5 : \Gamma \times \Gamma' \to \mathbb{C}$ are real-analytic functions and T_5 is given by

$$\begin{aligned} T_5(\gamma_n, \gamma'_n) = \quad & 2\,\mathrm{i}\,\varepsilon\,\lambda_n\Big(4\,\mathrm{i}\,\bar c_n c'_{2n}(\bar c_n + 2u_n\lambda_n \bar c'_{1n}) + 2c_n u_n^2(5\bar c_n + 3u_n\lambda_n \bar c'_{1n}) \\ & + u_n^2\lambda_n^2\big(|c'_{1n}|^2 + 3\,\varepsilon\,|c'_{2n}|^2 + 4\,\mathrm{i}\,\bar c'_{1n}c'_{2n}\big)\Big), \end{aligned}$$

hence $T_5(\gamma_n, \gamma'_n) \to 0$, if $n \to \infty$. If we consider the second component of (9.22) we obtain, since $(\psi^1_{nz^2w}(0), \psi^2_{nz^2w}(0)) \to (4\,\mathrm{i}\,|h^1_{02}|, \mathrm{i}\,h^2_{21}) \in \mathrm{i}\,\mathbb{R} \times \mathrm{i}\,\mathbb{R}$, that $2r_n + r'_n \to 0$ as $n \to \infty$. Hence by (9.19) we get $r_n \to 0$ and $r'_n \to 0$.

To sum up we obtain $\phi_n \to \mathrm{id}_{\mathbb{C}^2}$ and $\phi'_n \to \mathrm{id}_{\mathbb{C}^3}$, as $n \to \infty$, which completes the proof. □

Proof of Theorem 9.9. First we observe that N is a continuous map from $\mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0) \times \mathrm{Aut}_0(\mathbb{H}^2, 0) \times \mathfrak{F}_2$ to $\mathfrak{F}_2$, since the image of N consists of rational mappings, which depend real-analytically on the jets of the isotropies and the mapping.

Next we show the freeness of N: For any $H \in \mathfrak{F}_2$ and $\phi \in \mathrm{Aut}_0(\mathbb{H}^2, 0), \phi' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ we have to show that if $\phi' \circ H \circ \phi^{-1} = H$, this implies $\phi = \mathrm{id}_{\mathbb{C}^2}$ and $\phi' = \mathrm{id}_{\mathbb{C}^3}$.

By Lemma 5.18 we obtain that N restricted to $\mathfrak{N}_2$ is a free action. Next we assume the general case $H \in \mathfrak{F}_2$ and consider the equation $\phi' \circ H \circ \phi^{-1} = H$. We can write $H = \widehat{\phi}' \circ \widehat{H} \circ \widehat{\phi}^{-1}$, where $\widehat{H} \in \mathfrak{N}_2$ and $\widehat{\phi} \in \mathrm{Aut}_0(\mathbb{H}^2, 0), \widehat{\phi}' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ are unique according to Lemma 5.18. Then we have

$$\phi' \circ H \circ \phi^{-1} = H \quad \Longleftrightarrow \quad \widehat{\phi}'^{-1} \circ \phi' \circ \widehat{\phi}' \circ \widehat{H} \circ \widehat{\phi}^{-1} \circ \phi^{-1} \circ \widehat{\phi} = \widehat{H}.$$

Since N acts freely on $\mathfrak{N}_2$ we obtain that $\widehat{\phi}^{-1} \circ \phi^{-1} \circ \widehat{\phi} = \mathrm{id}_{\mathbb{C}^2}$ and $\widehat{\phi}'^{-1} \circ \phi' \circ \widehat{\phi}' = \mathrm{id}_{\mathbb{C}^3}$, which shows the freeness of the action.

To show the properness of N we prove (ii) of Lemma 9.10 using Lemma 9.12. We let the mapping $N' : \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0) \times \mathrm{Aut}_0(\mathbb{H}^2, 0) \times \mathfrak{F}_2 \to \mathfrak{F}_2 \times \mathfrak{F}_2$ be given by $N'(\phi', \phi, H) := \big(H, N(\phi', \phi, H)\big)$. Then we know from Proposition 4.1 that the image C_N of N' agrees with $\mathfrak{F}_2 \times \mathfrak{F}_2$, which is closed in $\mathfrak{F}_2 \times \mathfrak{F}_2$.

Next we let the mapping $\varphi_N : C_N \to \mathrm{Aut}_0(\mathbb{H}^2, 0) \times \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ be given by $\varphi_N\big(H, N(\phi, \phi', H)\big) := (\phi, \phi')$. To show the continuity of φ_N we let $(H_n)_{n\in\mathbb{N}} \in \mathfrak{F}_2$ be a sequence of mappings with

$$H_n \to H \in \mathfrak{F}_2, \tag{9.23}$$

$$\phi'_n \circ H_n \circ \phi_n^{-1} \to \check{H} \in \mathfrak{F}_2\,. \tag{9.24}$$

Using Proposition 4.1 we assume w.l.o.g. $H \in \mathfrak{N}_2$. Moreover we can write by Proposition 4.1 $\check{H} = \phi' \circ \widehat{H} \circ \phi^{-1}$ for $\widehat{H} \in \mathfrak{N}_2$. Then we need to conclude that $\phi_n \to \phi, \phi'_n \to \phi'$ and $H = \widehat{H}$, which implies the continuity of φ_N.

For each $n \in \mathbb{N}$ we write $H_n = \widehat{\phi}'_n \circ \widehat{H}_n \circ \widehat{\phi}_n^{-1}$, where $\widehat{H}_n \in \mathfrak{N}_2$. If we substitute the above representations of H_n and $\check{H}$ into (9.24) we obtain

$$\phi'^{-1} \circ \phi'_n \circ \widehat{\phi}'_n \circ \widehat{H}_n \circ \widehat{\phi}_n^{-1} \circ \phi_n^{-1} \circ \phi \to \widehat{H} \in \mathfrak{N}_2\,.$$

By Lemma 9.12 we have $\phi^{-1} \circ \phi_n \circ \widehat{\phi}_n \to \mathrm{id}_{\mathbb{C}^2}$ and $\phi'^{-1} \circ \phi'_n \circ \widehat{\phi}'_n \to \mathrm{id}_{\mathbb{C}^3}$. Since $H_n \to H \in \mathfrak{N}_2$ Lemma 9.12 shows that $\widehat{\phi}_n \to \mathrm{id}_{\mathbb{C}^2}$ and $\widehat{\phi}'_n \to \mathrm{id}_{\mathbb{C}^3}$, we obtain $\phi_n \to \phi$ and $\phi'_n \to \phi'$ as required. □

9.2 On the Real-Analytic Structure of $\mathfrak{F}_2$

Let us recall the description of $\mathfrak{F}_2$ given in Remark 9.6 (ii), for $\mathfrak{N}_2$ we proceed similar.

Lemma 9.13. *Let $\Pi : \mathfrak{F}_2 \to \mathfrak{N}_2$ be given by $\Pi(H) := \phi' \circ H \circ \phi^{-1}$, where $\phi \in \mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\phi' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ are the unique isotropies according to Proposition 4.1 and Lemma 5.18. For $k = 2, 3$ we write*

$$M_{k,\varepsilon} := \{\Pi^{-1}(G^\varepsilon_{k,s}) : s > 0\}.$$

Then $M_{k,\varepsilon}$ is a real-analytic real submanifold of $\mathfrak{F}_2$ of real dimension 16.

Proof of Lemma 9.13. For fixed $k = 2, 3$, $s > 0$ and $\delta > 0$ we write

$$G_{\delta,s} := \{G^\varepsilon_{k,t} : t \in B_\delta(s) \cap \mathbb{R}^+\}. \tag{9.25}$$

To prove the lemma we show that for every $s_0 \in \mathbb{R}^+$ and sufficiently small $\delta_0 > 0$ there exists a local real-analytic parametrization for $M_{\delta_0,s_0} := \Pi^{-1}(G_{\delta_0,s_0})$.
We abbreviate $M := M_{\delta_0,s_0}$ from now on. As noted in Remark 9.6 we identify $\mathcal{F}_2$ with the set $\mathfrak{J}_2 \subset \mathbb{C}^{K_0}$. Theorem 5.1 implies that for each $H \in M$ there exist $\phi \in \mathrm{Aut}_0(\mathbb{H}^2, 0), \phi' \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0), k \in \{2, 3\}$ and $s_1 \in B_{\delta_0}(s_0) \cap \mathbb{R}^+$, such that $H = \phi' \circ G^\varepsilon_{k,s_1} \circ \phi$. This fact is used to describe M locally via parametrizations as follows: For $s > 0$ sufficiently near s_0 let F_s be a mapping as in Remark 9.11, which depends real-analytically on $s := |f^1_{02}|$. For the remaining coefficients in $J(F_s)$ we write $x := f^2_{02}$ and $y := \mathrm{Im}\big(f^2_{21}\big)$, where we suppress the dependence on s notationally.
We use the real version of the notation for the parametrization of $\mathrm{Aut}_0(\mathbb{H}^2, 0)$ as in Definition 2.23 and for $\mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ as in Definition 2.24. Here we denote the set of real parameters of $\mathrm{Aut}_0(\mathbb{H}^2, 0)$ by Γ and of $\mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ by Γ'. Let us denote $\Xi := \Gamma \times \Gamma' \times \mathbb{R}^+ \subset \mathbb{R}^{N_0}$, where $N_0 := 16$. For $\xi \in \Xi$ we write $\xi = (\gamma, \gamma', s)$ and define the mapping

$$\Psi : \Xi \to \mathfrak{J}_2, \qquad \Psi(\xi) := J(\phi'_{\gamma'} \circ F_s \circ \phi_\gamma), \tag{9.26}$$

where we use the notation as in (2.27) and (2.31) for ϕ_γ and $\phi'_{\gamma'}$ respectively and suppress the dependence on ε. We set $\check{\Psi}(z, w) := \big(\phi'_{\gamma'} \circ F_s \circ \phi_\gamma\big)(z, w)$ with components $\check{\Psi} = (\check{\psi}^1, \check{\psi}^2, \check{\psi}^3)$ and $\check{\psi} := (\check{\psi}^1, \check{\psi}^2)$. The holomorphic mapping $\check{\Psi}$ is defined in a small neighborhood $U \subset \mathbb{C}^2$ of 0 and satisfies $\check{\Psi}(\mathbb{H}^2 \cap U) \subset \mathbb{H}^3_\varepsilon$.
By Theorem 5.1 and the real-analytic dependence of the isotropies on the standard parameters, we note that Ψ and $\check{\Psi}$ are real-analytic in $\xi \in \Xi$. We make the following assumptions and consider w.l.o.g. that ξ_0 is chosen in such a way that $\phi_\gamma = \mathrm{id}_{\mathbb{C}^2}$ and $\phi'_{\gamma'} = \mathrm{id}_{\mathbb{C}^3}$. Consequently we write $O(2)$ for terms involving standard parameters of the isotropies which vanish to second order at ξ_0. Moreover since we only consider $a'_1 \in \mathbb{C}$ near 1 and $a' = (a'_1, a'_2) \in \mathcal{S}^2_{\varepsilon,\sigma}$ from Definition 2.24, we substitute

$\bar{a}'_1 = (1 - \varepsilon\, |a'_2|^2)/a'_1$ into Ψ, which is then given by the following expressions:

$$
\begin{aligned}
\check{\psi}_z(0) =& \big(uu'\lambda\lambda' a'_1, u\lambda\lambda'\bar{a}'_2\big),\\
\check{\Psi}_w(0) =& \Big(u'\lambda\lambda' a'_1(uc + \lambda c'_1), \lambda^2\lambda' c'_2/a'_1, \lambda^2\lambda'^2\Big) + O(2),\\
\check{\psi}_{z^2}(0) =& \Big(2\,\mathrm{i}\, uu'\lambda\lambda'(\mathrm{i}\,\varepsilon\, u\lambda a'_2 + 2(\bar{c} + u\lambda\bar{c}'_1)a'_1), 2u^2\lambda^2\lambda'/a'_1\Big) + O(2),\\
\check{\Psi}_{zw}(0) =& \Big(-\frac{1}{2}uu'\lambda\lambda' a'_1(2(r + \lambda^2 r') - \mathrm{i}\,\varepsilon\,\lambda^2), u\lambda^2\lambda'\left(\frac{\mathrm{i}\,\varepsilon}{2}\lambda\bar{a}'_2 + 2uc/a'_1\right), 2\,\mathrm{i}\,\lambda^2\lambda'^2(\bar{c} + u\lambda\bar{c}'_1)\Big) + O(2),\\
\check{\Psi}_{w^2}(0) =& \Big(u'\lambda^3\lambda'\big(a'_1(\mathrm{i}\,\varepsilon\, uc + \lambda s) - \varepsilon\,\lambda a'_2 x\big),\\
& \lambda^4\lambda'\big(x/a'_1 + \bar{a}'_2 s\big), -2\lambda^2\lambda'^2(r + \lambda^2 r')\Big) + O(2),\\
\check{\psi}_{z^2w}(0) =& \Big(-uu'\lambda^3\lambda'\Big(4a'_1\big(-\mathrm{i}\, u\lambda s + \varepsilon(\bar{c} + u\lambda\bar{c}'_1)\big) + \mathrm{i}\,\varepsilon\, u\lambda a'_2 y\Big),\\
& u^2\lambda^2\lambda'\Big(\Big(-2(2r + \lambda^2 r') + 12\,\varepsilon\, u\lambda cs + \mathrm{i}\,\lambda^2 y\Big)/a'_1 + 4\,\mathrm{i}\,\lambda^2\bar{a}'_2 s\Big)\Big) + O(2).
\end{aligned}
$$

In a first step we show that for given $\xi_0 \in \Xi$ the Jacobian of Ψ with respect to ξ evaluated at ξ_0, denoted by $\Psi_\xi(\xi_0)$, is of full rank N_0. But instead of considering the real equations of Ψ, we conjugate Ψ and compute the Jacobian of the system

$$\Phi \coloneqq (\Psi, \overline{\Psi}) \in \mathbb{C}^{2K_0},$$

with respect to $\xi = (u, \lambda, c, r, u', a'_1, a'_2, \lambda', c'_1, c'_2, r', s; \bar{c}, \bar{a}'_2, \bar{c}'_1, \bar{c}'_2) \in \mathbb{C}^{N_0}$ and evaluate at

$$\xi_0 = (1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, s_0; 0, 0, 0, 0) \in \mathbb{R}^{N_0}, \tag{9.27}$$

denoted by $\Phi_\xi(\xi_0)$. We bring the transpose of $\Phi_\xi(\xi_0)$ into echelon form, where we denote the resulting matrix by $\varphi = (\varphi^1, \ldots, \varphi^{N_0})$, each $\varphi^j = (\varphi^j_1, \ldots, \varphi^j_{2K_0}) \in \mathbb{C}^{2K_0}, 1 \leq j \leq N_0$, such that $\operatorname{rank}\big(\Phi_\xi(\xi_0)\big) = \operatorname{rank}(\varphi)$. In the following we suppress the evaluation of Φ at ξ_0 notationally and perform elementary row operations. The matrix given by

$$
\begin{aligned}
(\varphi^1, \ldots, \varphi^{11}) \coloneqq \quad & \Big(\Phi_u, \Phi_{\bar{a}'_2}, \Phi_{c'_1}, \Phi_{c'_2}, \Phi_\lambda, \Phi_{\bar{c}}, \Phi_{a'_1}, \Phi_{r'}, \Phi_c, \Phi_{a'_2}, \Phi_s\Big)\\
& - \Big(0, 0, 0, \Phi_u, \Phi_u, 0, \Phi_u, 0, \Phi_{c'_1}, \mathrm{i}\,\varepsilon\,/2\Phi_{\bar{c}}, 0\Big),
\end{aligned}
$$

is in row echelon form, with constant nonzero entries in the main diagonal. Each 0 in the last vector above represents $0 \in \mathbb{C}^{2K_0}$. Next we define

$$
\begin{aligned}
\varphi^{12} \coloneqq& \Phi_{\lambda'} + \Phi_u/3 - \Phi_\lambda - \Phi_{a'_1}/3 - \mathrm{i}\,\varepsilon\,/8\Phi_{r'} + 10s_0/3\Phi_s,\\
\varphi^{13} \coloneqq& \Phi_{u'} - \Phi_u/3 - 2/3\Phi_{a'_1} - 2/3\Phi_s,
\end{aligned}
$$

which are of the following form:

$$
\begin{aligned}
\varphi^{12} &= (0,\dots,0,\varphi^{12}_{12},\dots,\varphi^{12}_{2K_0})\\
&= \left(0,\dots,0,\frac{-2(4x-5s_0x')}{3},2\,\mathrm{i}\,\varepsilon,\frac{8\,\mathrm{i}\,s_0}{3},\frac{2\,\mathrm{i}(3\,\varepsilon-3y+5s_0y')}{3},-\frac{1}{3},\varphi^{12}_{17},\dots\varphi^{12}_{2K_0}\right)\\
\varphi^{13} &= (0,\dots,0,\varphi^{13}_{12},\dots,\varphi^{13}_{2K_0}) = \left(0,\dots,0,\frac{2x-s_0x'}{3},0,-\frac{8\,\mathrm{i}\,s_0}{3},-\frac{\mathrm{i}\,s_0y'}{3},-\frac{2}{3},\varphi^{13}_{17},\dots\varphi^{13}_{2K_0}\right).
\end{aligned}
$$

Then we define

$$
\varphi^{14} := \Phi_r - \Phi_{r'}, \qquad \varphi^{15} := \Phi_{\bar{c}_2'}, \qquad \varphi^{16} := \Phi_{\bar{c}_1'},
$$

and compute

$$
\varphi^{14} = -2(e_{15}+e_{2K_0}), \qquad \varphi^{15} = e_{19}, \qquad \varphi^{16} = -2e_{24} + \mathrm{i}\,\varepsilon\, e_{26} - 12\,\varepsilon\, s e_{2K_0},
$$

where for $j \in \mathbb{N}$ we denote by e_j the j-th unit vector in $\mathbb{R}^{2K_0}$. We have to consider several cases. If $\varphi^{12}_{12} \neq 0$, we consider $\tilde{\varphi}^{13} := \varphi^{13} - \varphi^{13}_{12}\varphi^{12}/\varphi^{12}_{12}$, such that $\tilde{\varphi}^{13}_{13}$ is a multiple of $-2x+s_0x'$. If $\tilde{\varphi}^{13}_{13} \neq 0$, then $\varphi = (\varphi^1,\dots,\varphi^{12},\tilde{\varphi}^{13},\varphi^{14},\varphi^{15},\varphi^{16})$ is in echelon form. If $\tilde{\varphi}^{13}_{13} = 0$, then $x = Cs_0^2$, where $C \in \mathbb{C}\setminus\{0\}$ and we have $\tilde{\varphi}^{13}_{14} \neq 0$, which again implies that $\varphi = (\varphi^1,\dots,\varphi^{12},\tilde{\varphi}^{13},\varphi^{14},\varphi^{15},\varphi^{16})$ is in echelon form.
Next we treat $\varphi^{12}_{12} = 0$. First we consider the trivial case. If $x = 0$, then since $s_0 > 0$, we have $x' = 0$ and $\varphi = (\varphi^1,\dots,\varphi^{16})$ is in echelon form. Now we assume $x \neq 0$ which implies $s_0, x' \neq 0$ and we solve $\varphi^{12}_{12} = 0$. The solution is given by $x = Cs_0^{4/5}$, where $C \in \mathbb{C}\setminus\{0\}$ and $\varphi = (\varphi^1,\dots,\varphi^{11},\varphi^{13},\varphi^{12},\varphi^{14},\varphi^{15},\varphi^{16})$ is in echelon form.
We sum up that in all cases the Jacobian $\Phi_\xi(\xi_0)$ of the system Φ evaluated at ξ_0 is of full rank N_0, hence we conclude that Ψ from (9.26) is a real-analytic locally regular mapping if we choose $\delta_0 > 0$ sufficiently small in M.
For Ψ to be a local parametrization of $\mathfrak{J}_2$ it remains to show that for each sufficiently small neighborhood $U \subset \Xi \subset \mathbb{R}^{N_0}$ of ξ_0, there exists a neighborhood $W \subset \mathbb{C}^{K_0}$ of $\Psi(\xi_0) = F_{s_0}$, such that $\Psi(U) = W \cap M$. We have

$$
\Psi(U) = \{J(H) : \exists \xi = (\gamma,\gamma',t) \in U : H = \phi'_{\gamma'} \circ F_t \circ \phi_\gamma\},
$$

and with the notation of (9.25) for $\delta > 0$ we have

$$
M = \Pi^{-1}(F_{\delta,s_0}) = \{H \in \mathfrak{F}_2 : \exists (\gamma,\gamma',s) \in \Gamma \times \Gamma' \times B_\delta(s_0) \cap \mathbb{R}^+ : \phi'_{\gamma'} \circ H \circ \phi_\gamma^{-1} = F_s\}.
$$

By Remark 9.6 (ii) and since for each $H \in M$ we can write $H = {\phi'}^{-1}_{\gamma'} \circ F_s \circ \phi_\gamma$ we obtain $\Psi(U) \subset M$. We assume that there exists $U \subset \Xi$ a neighborhood of ξ_0, such that for any neighborhood W of $\Psi(\xi_0) = F_{s_0}$ we have $\Psi(U) \neq W \cap M$. We choose open, connected neighborhoods $(W_n)_{n\in\mathbb{N}}$ of F_{s_0} with $\bigcap_n W_n = \{F_{s_0}\}$ and $\Psi(U) \neq W_n \cap M$ for all $n \in \mathbb{N}$. There exists a sequence of mappings $(H_n)_{n\in\mathbb{N}} \in \mathfrak{F}_2$ such that $H_n \in W_n \cap M$ and $H_n \notin \Psi(U)$. We write $H_n = {\phi'}_{\gamma_n'} \circ F_{s_n} \circ \phi^{-1}_{\gamma_n}$, and conclude by Lemma 9.12 that $(\gamma_n,\gamma_n',s_n) \to \xi_0$ in Ξ. Thus eventually $H_n \in \Psi(U)$ for large enough $n \in \mathbb{N}$, which completes the proof of the lemma. □

Definition 9.14. (i) For a manifold M and a Lie group G acting on M via $(g, m) \mapsto g \cdot m$, we denote by $\pi : M \to M/G$ the canonical projection given by $\pi(m) = G \cdot m := \{g \cdot m : g \in G\}$ for $m \in M$.

(ii) Let a group G act on two sets X, Y: We call a map $\phi : X \to Y$ *equivariant with respect to* G if $\phi(g \cdot x) = g \cdot \phi(x)$ for all $x \in X$ and $g \in G$.

(iii) For G a real-analytic Lie group acting on M a real-analytic manifold we say the action $\alpha : G \times M \to M$ of G on M is *real-analytic*, if the map $(g, m) \to g \cdot m$ is a real-analytic map between real-analytic manifolds.

Remark 9.15. (i) By [BER97, Corollary 1.2] the groups $\mathrm{Aut}_0(\mathbb{H}^2, 0)$ and $\mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ are totally real, closed, real-analytic submanifolds of $G^2_0(\mathbb{H}^2, 0) \subset J^2_0(\mathbb{H}^2, 0)$ and $G^2_0(\mathbb{H}^3_\varepsilon, 0) \subset J^2_0(\mathbb{H}^3_\varepsilon, 0)$ respectively, which correspond to the jet spaces of holomorphic mappings with nonvanishing Jacobian determinant at 0. Hence $G := \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0) \times \mathrm{Aut}_0(\mathbb{H}^2, 0)$ is a real-analytic real Lie group. Since the pair $(\phi, \phi') \in G$ depends real-analytically on $\Gamma \times \Gamma'$, we obtain for M being one of the real-analytic submanifolds given in Lemma 9.13, that $N : G \times M \to M$ is a real-analytic action.

(ii) We let a group G act on $G \times M$ via $h \cdot (g, m) := (h \cdot g, m)$.

Definition 9.16. Let M be a real-analytic manifold and G a real-analytic Lie group acting real-analytically on M. A *real-analytic principal fibre bundle with structure group* G is a triple (π, M, X), where $\pi : M \to X$ is a real-analytic map, which satisfies the following property:
For every $x \in X$ there exists an open neighborhood U of x in X and a real-analytic diffeomorphism $\phi : \pi^{-1}(U) \to G \times U$, such that

(i) $\pi = \mathrm{pr}_U \circ \phi$ on $\pi^{-1}(U)$, where $\mathrm{pr}_U : G \times U \to U$ is the projection on the second factor.

(ii) ϕ is equivariant with respect to G.

We call M the *total space*, X the *base space* and ϕ is called a *local trivialization* of the bundle.

Theorem 9.17 (Local trivialization)**.** *Let M be a real-analytic manifold equipped with a real-analytic action $G \times M \to M$, where G is a real-analytic Lie group. If the action is free and proper, then the triple $(\pi, M, M/G)$ is a real-analytic principal fibre bundle with structure group G, i.e., M/G has a unique real-analytic manifold structure, such that $\pi : M \to M/G$ is a real-analytic submersion.*

Proof. See e.g. [vdB10, Theorem 13.5] for the smooth version of this theorem. □

The proof of the above Theorem 9.17 is based on the following result. We call a set $V \subset M$ G-invariant if $g \cdot V \subset V$ for all $g \in G$.

Lemma 9.18 (Local Slice-Theorem for free and proper actions)**.** *Let M be a real-analytic manifold equipped with a free and proper real-analytic action $G \times M \to M$, where G is a real-analytic Lie group. Then for each $m \in M$ there exists a real-analytic submanifold $S \subset M$ with $m \in S$ such that $(g, s) \mapsto g \cdot s$ is a real-analytic diffeomorphism from $G \times S$ onto an open G-invariant neighborhood $U \subset M$ of m. A submanifold as S above is called a slice for the action of G at m.*

Proof. See e.g. [vdB10, Lemma 13.7] for the smooth version of this lemma. □

Remark 9.19. For proper smooth actions of non-compact Lie groups the first proof of the local Slice-Theorem was given in [Pal61, 2.2.2 Proposition], where references treating compact Lie groups are included. In the real-analytic setting a global Slice-Theorem was proved by [HHK96, section VI] and [IK00, Theorem 0.6]. In both works the action is assumed to be proper. In [vdB10, sections 11–13] and [Lee13, Theorem 21.10] smooth versions of Lemma 9.18 and Theorem 9.17 are treated. To obtain the statements in the real-analytic category the proofs of [vdB10] need to be slightly modified.

Definition 9.20. Let X, Y be topological spaces and $\pi : X \to Y$ a surjective mapping. Then π is called *quotient map* if it satisfies the following property: A set $U \subset Y$ is open in Y if and only if $\pi^{-1}(U)$ is open in X. We call the topology on Y induced by π the *quotient topology* τ_Q *on* Y, where a set $U \subset Y$ is open in Y if $\pi^{-1}(U)$ is open in X.

Remark 9.21. We note the following well-known fact about the quotient topology τ_Q: Let $\pi : X \to Y$ be as in Definition 9.20, then τ_Q is unique. More precisely, if τ is a topology for Y such that π is a quotient map, then we have $\tau_Q = \tau$. We also have that if $f : X \to Y$ is a surjective, continuous and open or closed mapping then f is a quotient map.

Theorem 9.22 (Structure of $\mathfrak{F}_2$)**.** *We define* $G \coloneqq \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0) \times \mathrm{Aut}_0(\mathbb{H}^2, 0)$.

(i) *If* $\varepsilon = +1$ *then* $\Pi : \mathfrak{F}_2 \to \mathfrak{F}_2 /G$ *is a real-analytic principal fibre bundle with structure group* G.

(ii) *If* $\varepsilon = -1$ *then locally* $\mathfrak{F}_2$ *is mapped to* $G \times \mathfrak{N}_2$ *via locally real-analytic diffeomorphisms. In particular* $\mathfrak{F}_2$ *is not a smooth manifold.*

(iii) *The quotient topology* τ_Q *on* $\mathfrak{F}_2 /G \simeq \mathfrak{N}_2$ *agrees with the topology* τ_J *induced by the jet space.*

Proof. To prove (i) we note that by Lemma 9.13 the set $\mathfrak{F}_2$ is a real-analytic manifold and from Theorem 9.17 the conclusion in (i) follows.
Next we show (ii): For k=1,2 we set

$$N_k \coloneqq \{G^-_{k+1,s} : s > 0\},$$

and $N_0 \coloneqq N_1 \cap N_2 = \{G^-_{2,1/2}\}$. The corresponding preimages are denoted by $M_k \coloneqq \Pi^{-1}(N_k) \subset \mathfrak{F}_2$, such that $M_0 \coloneqq M_1 \cap M_2 = \Pi^{-1}(N_0)$. We set $M \coloneqq M_1 \cup M_2$. By Lemma 9.13 for $k = 1, 2$ we have that M_k is a real-analytic submanifold of $\mathfrak{F}_2$, hence by Theorem 9.17 locally M_k is real-analytically diffeomorphic to $G \times S_k$, where S_k is a slice for the action of G according to Lemma 9.18 such that $\dim_{\mathbb{R}}(S_k) = \dim_{\mathbb{R}}(M_k) - \dim_{\mathbb{R}}(G) = 1$, by Lemma 9.13 and Remark 2.21. Since $\dim_{\mathbb{R}}(N_k) = 1$ and it is possible to uniquely normalize any element in the slice S_k by Proposition 4.1, we obtain that S_k can be mapped to N_k via real-analytic diffeomorphisms. Hence locally we have that M_k is real-analytically diffeomorphic to $G \times N_k$ for $k = 1, 2$.
In order to prove (ii) we show that if we let $U_0 \subset \mathfrak{F}_2$ be a sufficiently small open neighborhood of N_0 there exists a real-analytic diffeomorphism $\phi : U_0 \to V_0$ such that $\phi(U_0 \cap M_0) = (G \times N_0) \cap V_0$, where V_0 is an open neighborhood of $N_0' \coloneqq \{\mathrm{id}\} \times N_0 \subset G \times M$ and $\mathrm{id} = (\mathrm{id}_{\mathbb{C}^2}, \mathrm{id}_{\mathbb{C}^3})$. By Lemma 9.18 for $k = 1, 2$ there exists an open neighborhood $U_k \subset \mathfrak{F}_2$ of N_0 and a real-analytic diffeomorphism $\phi_k : U_k \to V_k$ such that $\phi_k(U_k \cap M_k) = (G \times N_k) \cap V_k$, where V_k is an open neighborhood of $N_0' \subset G \times M$. Moreover

$\phi_k(U_k \cap N_k) = (\{\mathrm{id}\} \times N_k) \cap V_k$ and ϕ_k is equivariant with respect to G. We define

$$\phi : U_0 \to V_0, \quad \phi(x) := \begin{cases} \phi_1(x), & x \in U_0 \cap U_1, \\ \phi_2(x), & x \in U_0 \cap U_2, \end{cases}$$

where $V_0 = V_1 \cup V_2$ is an open neighborhood of N_0'. We define $\widetilde{U} := U_1 \cap U_2 \cap U_0 \subset \mathfrak{F}_2$, an open neighborhood of N_0. Then we have $\phi\big|_{\widetilde{U}} = \phi_1\big|_{\widetilde{U}}$, which implies that the mapping $\phi\big|_{\widetilde{U}}$ is a real-analytic diffeomorphism. Furthermore, since

$$\mathrm{image}\big(\phi_1\big|_{\widetilde{U}\cap M}\big) = (G \times N_0) \cap \widetilde{V} = \mathrm{image}\big(\phi_2\big|_{\widetilde{U}\cap M}\big),$$

where $\widetilde{V}$ is an open neighborhood of $N_0' \subset G \times M$, the mapping ϕ locally maps M_0 real-analytically diffeomorphic to $G \times N_0$. Finally the last statement of (ii) follows from Theorem 9.17, since if $\mathfrak{F}_2$ would be a smooth manifold, then the quotient $\mathfrak{N}_2$ needs to be a smooth manifold, which is not the case.
To prove (iii) we use Remark 9.21 and prove that $\Pi : \mathfrak{F}_2 \to \mathfrak{N}_2$ is a surjective, continuous and closed mapping with respect to τ_J to obtain $\tau_Q = \tau_J$.
Surjectivity is clear from Proposition 4.1 and Theorem 5.1. To show continuity of Π with respect to τ_J we either refer to Remark 4.11 and Theorem 4.12 or we proceed similar as in the proof of Lemma 9.13 and use Lemma 9.12. We let $(H_n)_{n\in\mathbb{N}}$ be a sequence of mappings in $\mathfrak{F}_2$ and $H \in \mathfrak{F}_2$, such that $H_n \to H$, then we need to conclude that $\Pi(H_n) \to \Pi(H)$. W.l.o.g. we assume $H \in \mathfrak{N}_2$, hence $\Pi(H) = H$ by Lemma 5.18. We have $\Pi(H_n) = \phi_n' \circ H_n \circ \phi_n \in \mathfrak{N}_2$, where $(\phi_n', \phi_n) \in G$ are the isotropies according to Proposition 4.1. Assume $\phi_n' \circ H_n \circ \phi_n \to \widehat{H} \in \mathfrak{N}_2$, then by Lemma 9.12 we obtain $\phi_n' \to \mathrm{id}_{\mathbb{C}^3}, \phi_n \to \mathrm{id}_{\mathbb{C}^2}$ and since $H_n \to H$ we get $\widehat{H} = H$.
We are left by proving the closedness of Π with respect to τ_J: Let $C \subset \mathfrak{F}_2$ be a closed subset. We need to show that $\Pi(C) \subset \mathfrak{N}_2$ is a closed subset. To prove this statement we let $H_n \in \Pi(C)$ for $n \in \mathbb{N}$, forming a sequence of mappings in $\mathfrak{N}_2$ such that $H_n \to H_0$, where $H_0 \in \mathfrak{N}_2$. To show the closedness of $\Pi(C)$ we need to conclude that $H_0 \in \Pi(C)$. By Theorem 5.1 there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the mappings H_n and H_0 are of the same form. More precisely we can write $H_n = G^{\varepsilon}_{k,s_n}$ and $H_0 = G^{\varepsilon}_{k,s_0}$ for $s_0, s_n \in \mathbb{R}^+$ and we have $s_n \to s_0$. Next we consider the elements of the orbits $\Pi^{-1}(H_n)$ in C. Let $G_n \in \Pi^{-1}(H_n) \cap C$ for $n \in \mathbb{N}$ be a sequence of maps with $G_n \to G_0$. By what we have shown in (i) and (ii) of Theorem 9.22 we have $G_0 \in \Pi^{-1}(H_0) \subset \mathfrak{F}_2$. Since $(G_n)_{n\in\mathbb{N}}$ is a convergent sequence in the closed set C we obtain $G_0 \in C$, which implies $H_0 = \Pi(G_0) \in \Pi(C)$. □

9.3 Basic Topological Properties of $\mathcal{F}_2$

Finally we show the following result concerning the connectedness of $\mathcal{F}_2$ which follows from Theorem 5.1.

Theorem 9.23. *The set $\mathcal{F}_2$ consists of $\frac{5+\varepsilon}{2}$ connected components.*

Proof. We denote by $c(X)$ the number of connected components of a topological space X and observe that for $\varepsilon = -1$ we have $c(\mathcal{N}_2) = 2$ and for $\varepsilon = +1$ we have $c(\mathcal{N}_2) = 3$. We use the notation for the parametrization of $\mathrm{Aut}_0(\mathbb{H}^2, 0)$ as in Definition 2.23 and for $\mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ as in Definition 2.24. We denote $\Xi := \Gamma \times \Gamma' \times \mathbb{R}^+_0 \subset \mathbb{R}^{N_0}$, where $N_0 := 16$ and for $\xi \in \Xi$ we write $\xi = (\gamma, \gamma', s)$. The set Ξ is connected, since for mappings in $\mathcal{F}_2$ and $\varepsilon = -1$ we only consider isotropies as in (2.31) with $\sigma = +1$. Since we consider τ_J as the topology on $\mathcal{F}_2$ and $\mathcal{N}_2$, which is induced by the topology of some $\mathbb{C}^K$, we have that connectedness is the same as path-connectedness. Clearly each isotropic orbit of a fixed mapping is connected. Also any isotropic orbit $O_0(H)$ of a fixed mapping $H \in \mathcal{N}_2$ is closed, since if we let $(G_n)_{n\in\mathbb{N}}$ be a sequence in $O_0(H)$ and write $G_n = \phi'_n \circ H \circ \phi_n$. Then we obtain for $G_n \to F \in \mathcal{F}_2$, where $F = \phi' \circ \widehat{F} \circ \phi$ with $\widehat{F} \in \mathcal{N}_2$. Lemma 9.12 can be adapted, such that the conclusion $\widehat{F} = H$, i.e., $F \in O_0(H)$ holds for maps in $\mathcal{N}_2$.

First we show that $C := O_0(G_1^\varepsilon)$ is a connected component of $\mathcal{F}_2$. We denote $\widehat{\mathcal{F}_2} := \complement C \subset \mathcal{F}_2$ and $\widehat{\mathcal{N}_2} := \mathcal{N}_2 \setminus \{G_1^\varepsilon\}$. By Lemma 6.5 and Lemma 6.37 we have $O(G_1^\varepsilon) = C$, consisting of all maps in $\mathcal{F}_2$ which are equivalent to G_1^ε. Assume there exists a continuous path $p : [0,1] \to \mathcal{F}_2$ with $p(0) \in C$ and $p(1) \in \widehat{\mathcal{F}_2}$, i.e., $p(1)$ is isotropically equivalent to a mapping of $\widehat{\mathcal{N}_2}$. Thus there exists $t_0 \in [0,1]$ such that $p(t) \in C$ for all $t \le t_0$ and $p(t) \in \widehat{\mathcal{F}_2}$ for all $t > t_0$. Hence there exists a sequence $(H_n)_{n\in\mathbb{N}}$ of mappings in $\widehat{\mathcal{F}_2}$, such that $H_n \to p(t_0) \in C$. Again by Lemma 9.12, if we write $\widehat{H}_n \in \widehat{\mathcal{N}_2}$ for the normalized mapping associated to H_n, this would imply that $\widehat{H}_n \to G_1^\varepsilon$, which is not possible.

Next we want to show that if $\varepsilon = +1$ we have $c(\widehat{\mathcal{F}_2}) = 2$. We observe that $\pi_2 : \mathcal{F}_2 \to \mathcal{N}_2$ is a continuous and surjective mapping, hence we obtain $c(\widehat{\mathcal{F}_2}) \ge 2$. Otherwise if $\widehat{\mathcal{F}_2}$ is assumed to be connected, then $\pi_2(\widehat{\mathcal{F}_2}) = \widehat{\mathcal{N}_2}$ would be connected, which is not the case. For $k = 2,3$ we denote $C_k := \{G_{k,s}^+ : s \ge 0\}$ and the corresponding preimage $\widehat{C}_k := \pi_2^{-1}(C_k)$. By Lemma 5.3 and Lemma 6.39, the set $\widehat{C}_3$ only consists of mappings of degree 3. Thus we have $\widehat{C}_2 \cap \widehat{C}_3 = \emptyset$ and hence by Theorem 6.8 a partition $\widehat{\mathcal{F}_2} = \widehat{C}_2 \cup \widehat{C}_3$ of connected sets, since this decomposition also holds if we consider translations, instead of isotropies, as in Definition 6.3 by Lemma 6.39 and by the fact that C is a connected component of $\mathcal{F}_2$. Since for $k = 2,3$ the set $\widehat{C}_k$ is homeomorphic to Ξ, defined at the beginning of the proof, we obtain the connectedness of $\widehat{C}_k$, which proves the theorem for $\varepsilon = +1$.

To prove the statement for $\varepsilon = -1$ we need to show that $\widehat{\mathcal{F}_2}$ is connected. Let $H_0, H_1 \in \widehat{\mathcal{F}_2}$ and for $k = 0,1$ associate $(\gamma_k, \gamma'_k, s_k) \in \Xi$ to H_k such that H_k is isotropically equivalent to $G^-_{\ell_k,s_k} \in \widehat{\mathcal{N}_2}$ for some $\ell_k = 2,3$. Since $\widehat{\mathcal{N}_2}$ is connected we can find a continuous path $p_s : [0,1] \to \mathbb{R}_0^+$ connecting s_0 and s_1. Moreover we can find continuous paths $(p_\gamma, p_{\gamma'}) : [0,1]^2 \to \Gamma \times \Gamma'$ connecting (γ_0, γ'_0) and (γ_1, γ'_1). The corresponding mapping $P := \phi'_{p_{\gamma'}} \circ G^-_{p_\ell,p_s} \circ \phi_{p_\gamma}$, where $p_\ell = 2,3$, describes a continuous path $P : [0,1] \to \widehat{\mathcal{F}_2}$ which connects H_0 and H_1. □

We consider the equivalence relation induced by Definition 2.26 and Definition 6.3, i.e., we allow translations and normalize via isotropies twice. More precisely we say that $F, G \in \mathcal{F}_2$ are equivalent and write $F \sim G$ if F is isotropically equivalent to some $\widetilde{F} \in \mathcal{N}_2$ according to Definition 2.26 and where $\widetilde{F}$ is equivalent to $\widetilde{G} \in \mathcal{N}_2$ according to Definition 6.3, where G is isotropically equivalent to $\widetilde{G}$ as in Definition 2.26. We note that by Lemma 6.5 the equivalence relation $\sim$ constitutes the most general equivalence relation in our setting. Then we have the following result for the quotient topology of the quotient space with respect to $\sim$.

Theorem 9.24. *Let $\sim$ denote the equivalence relation given by Definition 2.26 and Definition 6.3 defined above. Then $\mathcal{F}_2/_\sim$ is discrete if $\varepsilon = +1$ and is not discrete if $\varepsilon = -1$.*

Proof. We set $X := \mathcal{F}_2/_\sim$ consisting of elements denoted by $[F]$ for $F \in \mathcal{F}_2$. We equip X with the

quotient topology such that the canonical projection $\pi : \mathcal{F}_2 \to X$ is continuous.
For $\varepsilon = +1$ we have $X = \{G_1^+, G_{2,0}^+, G_{3,0}^+\}$ by Theorem 6.8. For $H \in X$ we have $\pi^{-1}(H) = O(H)$, which we have shown in the proof of Theorem 9.23 is a connected component of $\mathcal{F}_2$, hence open. Thus X carries the discrete topology.
To prove the statement if $\varepsilon = -1$ we write $H_0 \coloneqq G_{2,1/2}^- \in \mathcal{N}_2$ and $H_1 \coloneqq G_{3,0}^- \in \mathcal{N}_2$. For $k = 0, 1$ let $U_k \in X$ be an open neighborhood of $[H_k]$, then $V_k \coloneqq \pi^{-1}(U_k)$ is an open neighborhood of H_k in $\mathcal{F}_2$. According to Theorem 5.1 and Theorem 6.8 there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of mappings in $\mathcal{F}_2$, where each $G_n \in [H_1]$ and $G_n \to H_0$ in $\mathcal{F}_2$ as $n \to \infty$. Thus there exists $N \in \mathbb{N}$ such that $G_n \in V_0 \cap V_1$ for all $n \geq N$, which shows $[H_1] \in U_0 \cap U_1$ and completes the proof. $\square$

Appendix A: Formula for Jet Parametrization

In Lemma 5.5 we have the following formulas: Denote $\Psi = (f_1, f_2, g)$. We order the monomials by degree and by assigning the weight 1 to z and the weight 2 to the variable χ. The numerator of $f_1(z, 2\,\mathrm{i}\, z\chi)$ is the following expression:

$$
\begin{aligned}
&2\,\varepsilon\, z + 6A_2 z\chi + \mathrm{i}\, C_{22} z^3 + 4\,\mathrm{i}\,\varepsilon\, B_{21} z^2\chi + 6\,\varepsilon\, B_2 z\chi^2 + \Big(2\,\varepsilon\, A_2 + A_{22} - C_{13}\Big) z^3\chi \\
&- 2\Big(3\,\mathrm{i}\, A_3 + 3\,\varepsilon\, B_{12} + A_2(7\,\varepsilon - 3\,\mathrm{i}\, B_{21})\Big) z^2\chi^2 + 2A_2B_2 z\chi^3 \\
&+ \Big(6A_2^2 + \mathrm{i}\, A_{13} + \varepsilon(-1 - 2B_{21}^2 + B_{22} + C_3) - \mathrm{i}\, C_4 - 2\,\mathrm{i}\, B_2C_{22}\Big) z^3\chi^2 \\
&- 2\Big(5A_2^2 + 4\,\mathrm{i}\,\varepsilon\, B_3 + 4A_2B_{12} + B_2(6 - 2\,\mathrm{i}\,\varepsilon\, B_{21})\Big) z^2\chi^3 \\
&+ \Big(-A_4 - 2A_{22}B_2 + B_{12} + 2A_3B_{21} + \mathrm{i}\,\varepsilon(4A_3 + B_{13} - 4B_{12}B_{21}) \\
&\qquad + A_2(5 + 4\,\varepsilon\, B_2 - 4\,\mathrm{i}\,\varepsilon\, B_{21} - 2B_{21}^2 + B_{22} + 3C_3) + 2B_2C_{13}\Big) z^3\chi^3 \\
&+ 2\,\mathrm{i}\Big(B_2(4A_3 + \mathrm{i}\,\varepsilon\, B_{12}) + A_2\big(-5B_3 + B_2(5\,\mathrm{i}\,\varepsilon + B_{21})\big)\Big) z^2\chi^4 \\
&+ \Big(2\,\mathrm{i}\, B_3 + 2\,\mathrm{i}\, A_3B_{12} + \mathrm{i}\, A_2\big(4A_3 + B_{13} + B_{12}(-6\,\mathrm{i}\,\varepsilon - 4B_{21})\big) + 2A_2^2(5\,\varepsilon - 2B_2 - \mathrm{i}\, B_{21}) \\
&\qquad + \varepsilon(-B_4 + 2B_{12}^2 + 4B_3B_{21}) + B_2\big(-2\,\mathrm{i}\, A_{13} - 6\,\mathrm{i}\, B_{21} + \varepsilon(2 - B_{22} + 2C_3) + 2\,\mathrm{i}\, C_4\big) + \mathrm{i}\, B_2^2C_{22}\Big) z^3\chi^4 \\
&- 2A_2^2B_2 z^2\chi^5 \\
&+ \Big(4A_2^3 + 2A_4B_2 + A_{22}B_2^2 + 3A_2^2B_{12} + 5B_2B_{12} + 4\,\mathrm{i}\,\varepsilon\, B_3B_{12} - \mathrm{i}\,\varepsilon\, B_2B_{13} - 2A_3\big(B_3 + B_2(\mathrm{i}\,\varepsilon + B_{21})\big) \\
&\qquad - A_2\big(6\,\varepsilon\, B_2^2 + B_4 - 2B_{12}^2 + B_3(-8\,\mathrm{i}\,\varepsilon - 4B_{21}) + B_2(-4 + 8\,\mathrm{i}\,\varepsilon\, B_{21} + B_{22} + 2C_3)\big) - B_2^2C_{13}\Big) z^3\chi^5 \\
&- 2\,\mathrm{i}\, B_2\Big(A_3B_2 - A_2B_3\Big) z^2\chi^6 \\
&+ \Big(-2\,\varepsilon\, B_3^2 + B_2(4\,\mathrm{i}\, B_3 + \varepsilon\, B_4 - 2\,\mathrm{i}\, A_3B_{12}) - \mathrm{i}\, A_2\big(2A_3B_2 - 4B_3B_{12} + B_2(6\,\mathrm{i}\,\varepsilon\, B_{12} + B_{13})\big) \\
&\qquad + 2A_2^2\big(-B_2^2 + 2\,\mathrm{i}\, B_3 + B_2(\varepsilon - 2\,\mathrm{i}\, B_{21})\big) + B_2^2(3\,\varepsilon + \mathrm{i}\, A_{13} - 3\,\varepsilon\, C_3 - \mathrm{i}\, C_4)\Big) z^3\chi^6 \\
&+ \Big(B_2\big(-A_4B_2 + 2A_3(-\mathrm{i}\,\varepsilon\, B_2 + B_3)\big) + 3A_2^2B_2B_{12} + A_2\big(-2B_3^2 + B_2(4\,\mathrm{i}\,\varepsilon\, B_3 + B_4) \\
&\qquad - B_2^2(-3 + C_3)\big)\Big) z^3\chi^7 \\
&+ 2\,\mathrm{i}\, A_2B_2\Big(-A_3B_2 + A_2B_3\Big) z^3\chi^8
\end{aligned}
$$

The numerator of $f_2(z, 2\,\mathrm{i}\, z\chi)$ is equal to the following formula:

$$
\begin{aligned}
&2\,\varepsilon\, z^2 + 2A_2 z^3 + 6A_2 z^2\chi + \Big(-1 + C_3\Big) z^3\chi + 4\,\varepsilon\, B_2 z^2\chi^2 - \Big(2\,\mathrm{i}\, A_3 + 6A_2(\varepsilon + B_2) + \varepsilon\, B_{12}\Big) z^3\chi^2 \\
&- 4A_2B_2 z^2\chi^3 + \Big(-4A_2^2 - 2\,\mathrm{i}\,\varepsilon\, B_3 - A_2B_{12} + B_2(1 + 4\,\mathrm{i}\,\varepsilon\, B_{21} - 3C_3)\Big) z^3\chi^3 - 6\,\varepsilon\, B_2^2 z^2\chi^4 \\
&+ \Big(2\,\mathrm{i}\, B_2(A_3 + \mathrm{i}\,\varepsilon\, B_{12}) + A_2\big(6B_2^2 - 2\,\mathrm{i}\, B_3 + 4B_2(\varepsilon + \mathrm{i}\, B_{21})\big)\Big) z^3\chi^4 - 2A_2B_2^2 z^2\chi^5 \\
&+ B_2\Big(4A_2^2 - 2A_2B_{12} + B_2(-3 - 4\,\mathrm{i}\,\varepsilon\, B_{21} + 3C_3)\Big) z^3\chi^5
\end{aligned}
$$

$$+ B_2^2\Big(2\,\mathrm{i}\,A_3 + 3\,\varepsilon\,B_{12} + 2A_2(\varepsilon - B_2 - 2\,\mathrm{i}\,B_{21})\Big)z^3\chi^6 + B_2^2\Big(2\,\mathrm{i}\,\varepsilon\,B_3 + 3A_2B_{12} - B_2(-3+C_3)\Big)z^3\chi^7$$
$$- 2\,\mathrm{i}\,B_2^2\Big(A_3B_2 - A_2B_3\Big)z^3\chi^8$$

The numerator of $g(z, 2\,\mathrm{i}\,z\chi)$ is equal to the following formula:

$$4\,\mathrm{i}\,\varepsilon\,z\chi + 12\,\mathrm{i}\,A_2z\chi^2 - 2C_{22}z^3\chi + \Big(4\,\mathrm{i} - 8\,\varepsilon\,B_{21}\Big)z^2\chi^2 + 12\,\mathrm{i}\,\varepsilon\,B_2z\chi^3 + 2\,\mathrm{i}\Big(4\,\varepsilon\,A_2 + A_{22} - C_{13}\Big)z^3\chi^2$$
$$+ 12\Big(A_3 - \mathrm{i}\,\varepsilon\,B_{12} + A_2(-\mathrm{i}\,\varepsilon - B_{21})\Big)z^2\chi^3 + 4\,\mathrm{i}\,A_2B_2z\chi^4$$
$$+ 2\Big(8\,\mathrm{i}\,A_2^2 - A_{13} - \mathrm{i}\,\varepsilon(2 + 2B_{21}^2 - B_{22} - 2C_3) + C_4 + 2B_2C_{22}\Big)z^3\chi^3$$
$$- 4\Big(2\,\mathrm{i}\,A_2^2 - 4\,\varepsilon\,B_3 + 4\,\mathrm{i}\,A_2B_{12} + B_2(3\,\mathrm{i} + 2\,\varepsilon\,B_{21})\Big)z^2\chi^4$$
$$+ 2\Big(-\mathrm{i}\,A_4 - 2\,\mathrm{i}\,A_{22}B_2 - \varepsilon\,B_{13} - 2A_3(\varepsilon - \mathrm{i}\,B_{21}) + 4\,\varepsilon\,B_{12}B_{21} + A_2\big(4\,\varepsilon\,B_{21} - 2\,\mathrm{i}\,B_{21}^2$$
$$+ \mathrm{i}(-2 + B_{22} + 4C_3)\big) + 2\,\mathrm{i}\,B_2C_{13}\Big)z^3\chi^4$$
$$- 4\Big(B_2(4A_3 + \mathrm{i}\,\varepsilon\,B_{12}) + A_2\big(-5B_3 + B_2(\mathrm{i}\,\varepsilon + B_{21})\big)\Big)z^2\chi^5$$
$$+ 2\Big(-2A_3B_{12} - A_2\big(2A_3 + B_{13} + B_{12}(-4\,\mathrm{i}\,\varepsilon - 4B_{21})\big) + 2A_2^2(-4\,\mathrm{i}\,B_2 + B_{21})$$
$$- \mathrm{i}\,\varepsilon\big(B_4 - 2(B_{12}^2 + 2B_3B_{21})\big) + B_2\big(2A_{13} + 2B_{21} - \mathrm{i}\,\varepsilon(-2 + B_{22}) - 2C_4\big) - B_2^2C_{22}\Big)z^3\chi^5$$
$$- 2\,\mathrm{i}\Big(-2A_4B_2 - A_{22}B_2^2 - 2A_2^2B_{12} - 2B_2B_{12} - 4\,\mathrm{i}\,\varepsilon\,B_3B_{12} + \mathrm{i}\,\varepsilon\,B_2B_{13} + 2A_3\big(B_3 + B_2(\mathrm{i}\,\varepsilon + B_{21})\big)$$
$$+ A_2\big(4\,\varepsilon\,B_2^2 + B_4 - 2B_{12}^2 + B_3(-4\,\mathrm{i}\,\varepsilon - 4B_{21}) + B_2(-2 + B_{22} + 4C_3)\big) + B_2^2C_{13}\Big)z^3\chi^6$$
$$+ 4B_2\Big(A_3B_2 - A_2B_3\Big)z^2\chi^7$$
$$- 2\Big(A_{13}B_2^2 + 2A_2^2B_3 + 2B_2B_3 - 2A_3B_2B_{12} - A_2(2A_3B_2 - 4B_3B_{12} + B_2B_{13})$$
$$+ \mathrm{i}\,\varepsilon(2B_3^2 - B_2B_4 + 2B_2^2C_3) - B_2^2C_4\Big)z^3\chi^7$$
$$- 2\,\mathrm{i}\Big(A_4B_2^2 - 2A_3B_2B_3 + A_2(2B_3^2 - B_2B_4)\Big)z^3\chi^8$$

The denominator of H is of the following form:

$$2\,\varepsilon + 6A_2\chi + \mathrm{i}\,C_{22}z^2 + \Big(2 + 4\,\mathrm{i}\,\varepsilon\,B_{21}\Big)z\chi + 6\,\varepsilon\,B_2\chi^2 + \Big(6\,\varepsilon\,A_2 + A_{22} - C_{13}\Big)z^2\chi$$
$$- 6\Big(\mathrm{i}\,A_3 + \varepsilon\,B_{12} + A_2(\varepsilon - \mathrm{i}\,B_{21})\Big)z\chi^2 + 2A_2B_2\chi^3 - \mathrm{i}\,\varepsilon\,C_{22}z^3\chi$$
$$+ \Big(12A_2^2 + \mathrm{i}\,A_{13} - 2\,\mathrm{i}\,B_{21} + \varepsilon(-3 - 2B_{21}^2 + B_{22} + 3C_3) - \mathrm{i}\,C_4 - 2\,\mathrm{i}\,B_2C_{22}\Big)z^2\chi^2$$
$$- 2\Big(2A_2^2 + 4\,\mathrm{i}\,\varepsilon\,B_3 + 4A_2B_{12} + B_2(3 - 2\,\mathrm{i}\,\varepsilon\,B_{21})\Big)z\chi^3 + \Big(\varepsilon(-A_{22} + C_{13}) + 2\,\mathrm{i}\,A_2(\mathrm{i} + \varepsilon\,B_{21} - C_{22})\Big)z^3\chi^2$$
$$+ \Big(-A_4 - 2A_{22}B_2 + 3B_{12} + 2A_3B_{21} + \mathrm{i}\,\varepsilon(4A_3 + B_{13} - 4B_{12}B_{21})$$
$$+ A_2(-2 - 10\,\mathrm{i}\,\varepsilon\,B_{21} - 2B_{21}^2 + B_{22} + 6C_3) + 2B_2C_{13}\Big)z^2\chi^3$$
$$+ 2\,\mathrm{i}\Big(B_2(4A_3 + \mathrm{i}\,\varepsilon\,B_{12}) + A_2\big(-5B_3 + B_2(\mathrm{i}\,\varepsilon + B_{21})\big)\Big)z\chi^4$$

$$
\begin{aligned}
&-\Big(-1+2A_2^2(8\,\varepsilon-\mathrm{i}\,B_{21})-2B_{21}^2+B_{22}+C_3+\mathrm{i}\,\varepsilon\big(A_{13}-B_{21}(-1+C_3)-C_4\big)\\
&\quad+2A_2(\mathrm{i}\,A_3+A_{22}+\varepsilon\,B_{12}-C_{13})\Big)z^3\chi^3\\
&-\Big(-4\,\mathrm{i}\,B_3-2\,\mathrm{i}\,A_3B_{12}-\mathrm{i}\,A_2\big(4A_3+B_{13}+B_{12}(-13\,\mathrm{i}\,\varepsilon-4B_{21})\big)+4A_2^2(3B_2+2\,\mathrm{i}\,B_{21})\\
&\quad+\varepsilon\big(B_4-2(B_{12}^2+2B_3B_{21})\big)+B_2\big(2\,\mathrm{i}\,A_{13}+2\,\mathrm{i}\,B_{21}+\varepsilon(-8+B_{22})-2\,\mathrm{i}\,C_4\big)-\mathrm{i}\,B_2^2C_{22}\Big)z^2\chi^4\\
&-\Big(16A_2^3+2A_2^2B_{12}+\mathrm{i}\big(\mathrm{i}\,\varepsilon\,A_4+B_{13}-3B_{12}B_{21}-\mathrm{i}\,\varepsilon\,B_{12}C_3+A_3(3+C_3)\big)+A_2\big(2\,\mathrm{i}\,A_{13}+3\,\mathrm{i}\,B_{21}\\
&\quad-\mathrm{i}\,B_{21}C_3+\varepsilon(2\,\mathrm{i}\,B_3-6B_{21}^2+3B_{22}+8C_3)-2\,\mathrm{i}\,C_4+2B_2(6+\mathrm{i}\,\varepsilon\,B_{21}-\mathrm{i}\,C_{22})\big)\Big)z^3\chi^4\\
&+\Big(2A_4B_2+A_{22}B_2^2+10A_2^2B_{12}+3B_2B_{12}+4\,\mathrm{i}\,\varepsilon\,B_3B_{12}-\mathrm{i}\,\varepsilon\,B_2B_{13}-2A_3\big(B_3+B_2(4\,\mathrm{i}\,\varepsilon+B_{21})\big)\\
&\quad-A_2\big(6\,\varepsilon\,B_2^2+B_4-2B_{12}^2+B_3(-16\,\mathrm{i}\,\varepsilon-4B_{21})+B_2(-10+2\,\mathrm{i}\,\varepsilon\,B_{21}+B_{22}+6C_3)\big)-B_2^2C_13\Big)z^2\chi^5\\
&-2\,\mathrm{i}\,B_2\Big(A_3B_2-A_2B_3\Big)z\chi^6\\
&-\Big(2A_3^2-2B_2-B_4+B_{12}^2+2B_3B_{21}+A_3\big(-\mathrm{i}\,\varepsilon\,B_{12}+2A_2(-2\,\mathrm{i}\,B_2+B_{21})\big)+B_2B_{22}+6B_2C_3\\
&\quad+2A_2^2\big(\mathrm{i}\,B_3+B_2(8\,\varepsilon+\mathrm{i}\,B_{21})-2B_{21}^2+B_{22}+4C_3\big)+A_2\big(-2A_4-2A_{22}B_2-2B_{12}+\varepsilon(-2B_2B_{12}\\
&\quad+3\,\mathrm{i}\,B_{13}-10\,\mathrm{i}\,B_{12}B_{21})+B_{12}C_3+2B_2C_{13}\big)+\mathrm{i}\,\varepsilon\big(B_3(1+C_3)+B_2(B_{21}(-5+C_3)\\
&\quad-B_2C_{22})\big)\Big)z^3\chi^5\\
&+\Big(12\,\mathrm{i}\,A_2^2B_3-2\,\varepsilon\,B_3^2+B_2(4\,\mathrm{i}\,B_3+\varepsilon\,B_4-2\,\mathrm{i}\,A_3B_{12})-\mathrm{i}\,A_2\big(12A_3B_2-4B_3B_{12}+B_2(3\,\mathrm{i}\,\varepsilon\,B_{12}+B_{13})\big)\\
&\quad+B_2^2\big(3\,\varepsilon(1-C_3)+\mathrm{i}\,A_{13}-\mathrm{i}\,C_4\big)\Big)z^2\chi^6\\
&+\Big(-\mathrm{i}\,B_3B_{12}-\mathrm{i}\,B_2B_{13}+A_2^2(2B_2B_{12}-2\,\mathrm{i}\,B_{13}+7\,\mathrm{i}\,B_{12}B_{21})+A_3\big(-2\,\varepsilon\,B_3-\mathrm{i}\,A_2B_{12}+B_2\big(4\,\varepsilon\,B_{21}\\
&\quad+2\,\mathrm{i}(-1+C_3)\big)\big)+\mathrm{i}\,A_2\big(2A_{13}B_2+2\,\mathrm{i}\,B_2^2+B_3-\mathrm{i}\,\varepsilon\big(3B_4-4(B_{12}^2+2B_3B_{21})\big)-B_3C_3+B_2\big(2\,\varepsilon\,B_3\\
&\quad-B_{21}(-7+C_3)+\mathrm{i}(\varepsilon(B_{22}+8C_3)+2\,\mathrm{i}\,C_4)\big)\big)+\varepsilon\,B_2\big(A_{22}B_2+B_{12}(-4+C_3)-B_2C_{13}\big)\Big)z^3\chi^6\\
&+\Big(-A_4B_2^2+2A_3B_2(-2\,\mathrm{i}\,\varepsilon\,B_2+B_3)+A_2\big(-2B_3^2+B_2(4\,\mathrm{i}\,\varepsilon\,B_3+B_4)\big)\Big)z^2\chi^7\\
&+\Big(A_2^2\big(2\,\mathrm{i}\,B_2B_3+2B_4-3(B_{12}^2+2B_3B_{21})\big)-A_2\big(2A_4B_2+6B_2B_{12}+6\,\mathrm{i}\,\varepsilon\,B_3B_{12}+\mathrm{i}\,\varepsilon\,B_2B_{13}\\
&\quad+2\,\mathrm{i}\,A_3B_2(B_2+3\,\mathrm{i}\,B_{21})-B_2B_{12}C_3\big)+B_2\big(B_4+\mathrm{i}\,\varepsilon\big(3A_3B_{12}+B_3(-3+C_3)\big)\\
&\quad-B_2(3-\mathrm{i}\,\varepsilon\,A_{13}+C_3+\mathrm{i}\,\varepsilon\,C_4)\big)\Big)z^3\chi^7\\
&+\Big(-\varepsilon\,A_4B_2^2+A_2\big(B_3(2\,\varepsilon\,B_3-5\,\mathrm{i}\,A_2B_{12})+B_2\big(\varepsilon\,B_4+\mathrm{i}\,B_3(-5+C_3)\big)\big)-\mathrm{i}\,A_3B_2\big(-2\,\mathrm{i}\,\varepsilon\,B_3-5A_2B_{12}\\
&\quad+B_2(-5+C_3)\big)\Big)z^3\chi^8\\
&+2\Big(A_3B_2-A_2B_3\Big)^2z^3\chi^9
\end{aligned}
$$

Appendix B: Case A and B

In the proof of Lemma 5.9 the following diagrams occur:

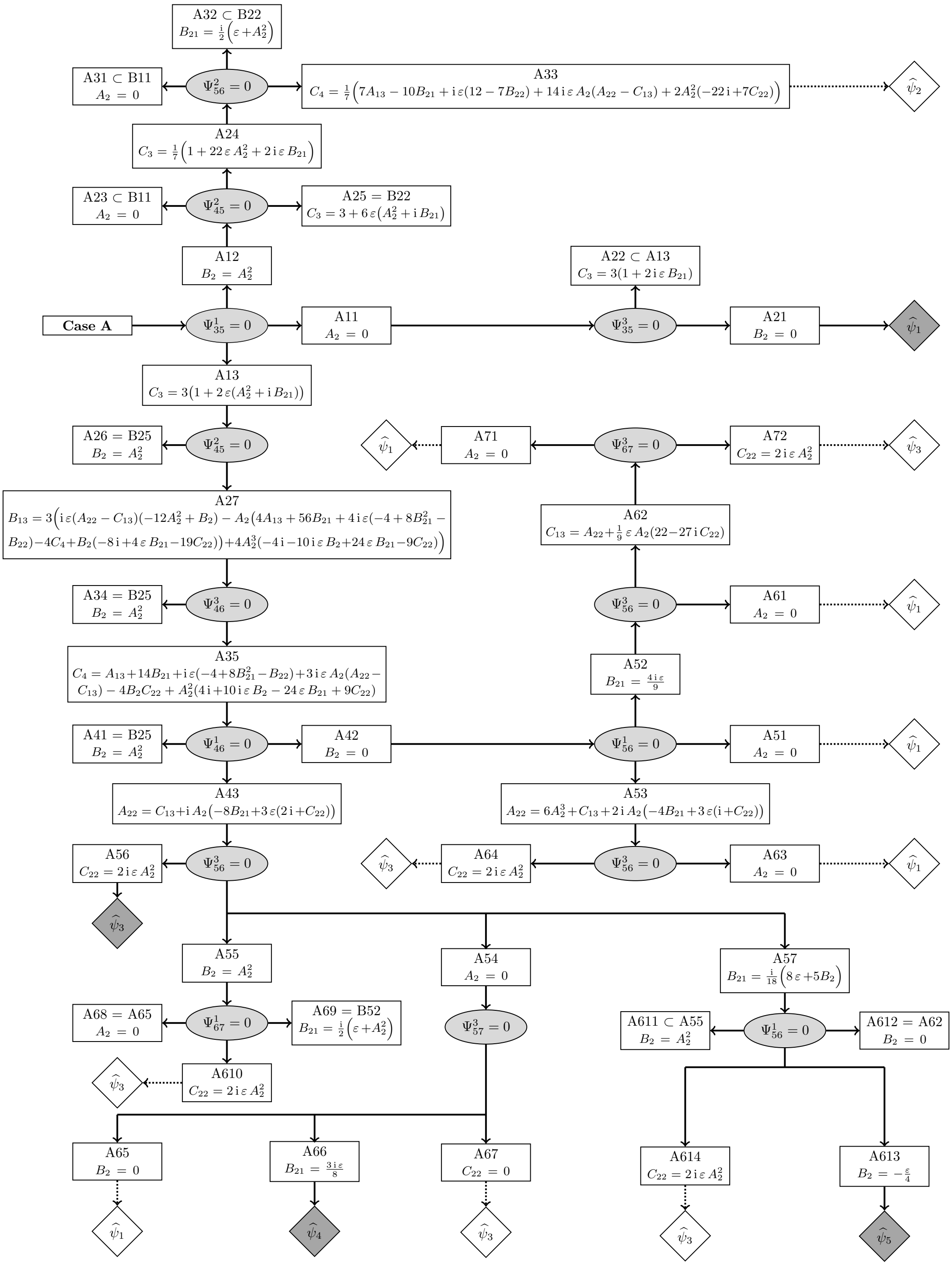

Figure 9: Diagram for Case A

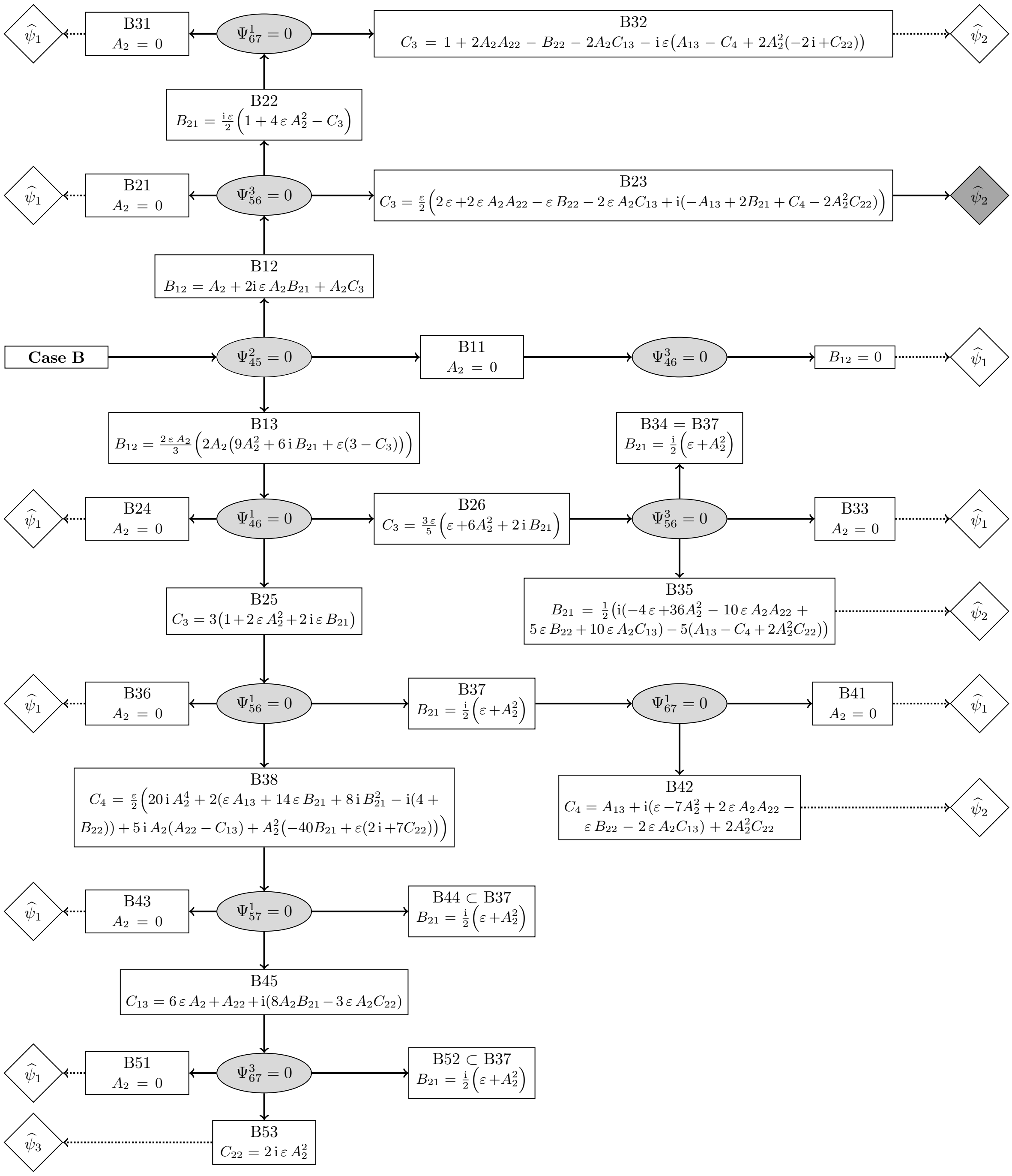

Figure 10: Diagram for Case B

Appendix C: Formulas for ψ_k and $\widehat{\psi}_k$

In Lemma 5.9 we have the following formulas:

$$
\begin{aligned}
\psi_1(z,w) =&\Big(2z\Big(8+8B_{21}w+4\,\mathrm{i}\,\varepsilon\, C_{22}z^2+\mathrm{i}\,\varepsilon\, A_{22}zw+(4+12\,\mathrm{i}\,\varepsilon\, B_{21}-4B_{21}^2-B_{22})w^2\Big),\\
&8z^2\Big(2-\mathrm{i}(\varepsilon+3\,\mathrm{i}\,B_{21})w\Big),2w\Big(8-4(\mathrm{i}\,\varepsilon-2B_{21})w+4\,\mathrm{i}\,\varepsilon\, C_{22}z^2+\mathrm{i}\,\varepsilon\, A_{22}zw\\
&+(2+6\,\mathrm{i}\,\varepsilon\, B_{21}-4B_{21}^2-B_{22})w^2\Big)\Big)/\\
&\Big(16-8(\mathrm{i}\,\varepsilon-2B_{21})w+8\,\mathrm{i}\,\varepsilon\, C_{22}z^2+2\,\mathrm{i}\,\varepsilon\, A_{22}zw+2(2\,\mathrm{i}\,\varepsilon\, B_{21}-4B_{21}^2-B_{22})w^2\\
&-4C_{22}z^2w-A_{22}zw^2-\big(14B_{21}-\mathrm{i}\,\varepsilon(4-10B_{21}^2-B_{22})\big)w^3\Big)\\
\psi_2(z,w) =&\Big(2\Big(16z+16B_{21}zw+4A_2w^2+8\,\mathrm{i}\,\varepsilon\, C_{22}z^3+2\big(\mathrm{i}\,\varepsilon\, A_{22}+A_2(8\,\mathrm{i}-6\,\varepsilon\, B_{21}-3C_{22})\big)z^2w\\
&-\big(A_2A_{22}-4(1+2\,\mathrm{i}\,\varepsilon\, B_{21}+B_{21}^2)+\mathrm{i}\,A_2^2(6B_{21}+\varepsilon\, C_{22})\big)zw^2+2\,\mathrm{i}\,A_2(\varepsilon+2A_2^2+\mathrm{i}\,B_{21})w^3\Big),\\
&4\Big(8z^2+2A_2^2w^2+8\,\varepsilon\, A_2z^3-4(\mathrm{i}\,\varepsilon-3B_{21})z^2w+2A_2(2+3\,\varepsilon\, A_2^2+4\,\mathrm{i}\,\varepsilon\, B_{21})zw^2\\
&+\mathrm{i}\,A_2^2(\varepsilon+2A_2^2+\mathrm{i}\,B_{21})w^3\Big),2w\Big(16-8(\mathrm{i}\,\varepsilon-2B_{21})w+8\,\mathrm{i}\,\varepsilon\, C_{22}z^2+2\big(\mathrm{i}\,\varepsilon\, A_{22}+A_2(4\,\mathrm{i}\\
&-6\,\varepsilon\, B_{21}-3C_{22})\big)zw-\Big(A_2A_{22}+4B_{21}(\mathrm{i}\,\varepsilon-B_{21})+\mathrm{i}\,A_2^2\big(6B_{21}-\varepsilon(4\,\mathrm{i}-C_{22})\big)\Big)w^2\Big)\Big)/\\
&\Big(32-16(\mathrm{i}\,\varepsilon-2B_{21})w+16\,\mathrm{i}\,\varepsilon\, C_{22}z^2+4\big(\mathrm{i}\,\varepsilon\, A_{22}-3A_2(2\,\varepsilon\, B_{21}+C_{22})\big)zw\\
&-2\Big(A_2A_{22}+4(1+3\,\mathrm{i}\,\varepsilon\, B_{21}-B_{21}^2)+\mathrm{i}\,A_2^2\big(6B_{21}-\varepsilon(12\,\mathrm{i}-C_{22})\big)\Big)w^2-8C_{22}z^2w\\
&-2\Big(A_{22}+A_2\big(10\,\mathrm{i}\,B_{21}+\varepsilon(8-\mathrm{i}\,C_{22})\big)\Big)zw^2+\big(\mathrm{i}\,\varepsilon\, A_2A_{22}-12B_{21}+4\,\mathrm{i}\,\varepsilon(1-2B_{21}^2)\\
&+A_2^2(12\,\mathrm{i}-14\,\varepsilon\, B_{21}-C_{22})\big)w^3\Big)\\
\psi_3(z,w) =&\Big(4z-4\,\varepsilon\, A_2z^2+2\,\mathrm{i}(\varepsilon+\mathrm{i}\,B_{21})zw+A_2w^2,4z^2+w^2B_2,2w(2-2\,\varepsilon\, A_2z-B_{21}w)\Big)/\\
&\Big(4-4\,\varepsilon\, A_2z-2B_{21}w-2\,\mathrm{i}\,A_2zw-(1+2\,\varepsilon\, A_2^2+2\,\mathrm{i}\,\varepsilon\, B_{21})w^2\Big)\\
\psi_4(z,w) =&\Big(z\Big(256+96\,\mathrm{i}\,\varepsilon\, w+128\,\mathrm{i}\,\varepsilon\, C_{22}z^2-(5-32\,\mathrm{i}\,\varepsilon\, B_2C_{22})w^2\Big),\\
&4\Big(64z^2+16B_2w^2+4\,\mathrm{i}\,\varepsilon\, z^2w+\mathrm{i}\,\varepsilon\, B_2w^3\Big),\\
&w\Big(256-32\,\mathrm{i}\,\varepsilon\, w+128\,\mathrm{i}\,\varepsilon\, C_{22}z^2+(3+32\,\mathrm{i}\,\varepsilon\, B_2C_{22})w^2\Big)\Big)/\\
&\Big(256-32\,\mathrm{i}\,\varepsilon\, w+128\,\mathrm{i}\,\varepsilon\, C_{22}z^2-(13-32\,\mathrm{i}\,\varepsilon\, B_2C_{22})w^2-64C_{22}z^2w-(\mathrm{i}\,\varepsilon-16B_2C_{22})w^3\Big)\\
\psi_5(z,w) =&\Big(256z+96\,\mathrm{i}\,\varepsilon\, zw+64A_2w^2+128\,\mathrm{i}\,\varepsilon\, C_{22}z^3+64\,\mathrm{i}\,A_2z^2w-(5-48\,\varepsilon\, A_2^2+8\,\mathrm{i}\,C_{22})zw^2\\
&+4\,\mathrm{i}\,\varepsilon\, A_2w^3,256z^2-16\,\varepsilon\, w^2+256\,\varepsilon\, A_2z^3+16\,\mathrm{i}\,\varepsilon\, z^2w-16A_2zw^2-\mathrm{i}\,w^3,\\
&w\Big(256-32\,\mathrm{i}\,\varepsilon\, w+128\,\mathrm{i}\,\varepsilon\, C_{22}z^2-64\,\mathrm{i}\,A_2zw+(3-16\,\varepsilon\, A_2^2-8\,\mathrm{i}\,C_{22})w^2\Big)\Big)/\\
&\Big(256-32\,\mathrm{i}\,\varepsilon\, w+128\,\mathrm{i}\,\varepsilon\, C_{22}z^2-192\,\mathrm{i}\,A_2zw-(13+144\,\varepsilon\, A_2^2+8\,\mathrm{i}\,C_{22})w^2-64C_{22}z^2w\\
&+8\,\varepsilon\, A_2(-1+8\,\mathrm{i}\,C_{22})zw^2-\varepsilon(\mathrm{i}+4C_{22})w^3\Big)
\end{aligned}
$$

We have $\widehat{\psi}_k = \psi_k$ for $k = 3, 4, 5$.

$$
\begin{aligned}
\widehat{\psi}_1(z,w) = &\Big(2z\Big(8\,\varepsilon + 8\,\varepsilon\,B_{21}w + 4\,\mathrm{i}\,C_{22}z^2 - 2\,\mathrm{i}(A_{22}-C_{13})zw + (\varepsilon - \mathrm{i}\,A_{13} + 2\,\varepsilon\,B_{21}^2 - \varepsilon\,B_{22} - \varepsilon\,C_3 \\
&+ \mathrm{i}\,C_4)w^2\Big), 4z^2\Big(4\,\varepsilon + \mathrm{i}(1-C_3)w\Big), 2w\Big(8\,\varepsilon - 4(\mathrm{i} - 2\,\varepsilon\,B_{21})w + 4\,\mathrm{i}\,C_{22}z^2 \\
&- 2\,\mathrm{i}(A_{22}-C_{13})zw - \big(\mathrm{i}\,A_{13} - 2\,\varepsilon\,B_{21}^2 - \varepsilon(2 - B_{22} - 2C_3) - \mathrm{i}\,C_4\big)w^2\Big)\Big)/ \\
&\Big(16\,\varepsilon - 8(\mathrm{i} - 2\,\varepsilon\,B_{21})w + 8\,\mathrm{i}\,C_{22}z^2 - 4\,\mathrm{i}(A_{22}-C_{13})zw - 2\big(\mathrm{i}\,A_{13} - 2\,\mathrm{i}\,B_{21} - 2\,\varepsilon\,B_{21}^2 \\
&- \varepsilon(3 - B_{22} - 3C_3) - \mathrm{i}\,C_4\big)w^2 - 4\,\varepsilon\,C_{22}z^2w + 2\,\varepsilon(A_{22}-C_{13})zw^2 + \big(\varepsilon\,A_{13} + 2\,\mathrm{i}\,B_{21}^2 \\
&+ \varepsilon\,B_{21}(1-C_3) + \mathrm{i}(1 - B_{22} - C_3 + \mathrm{i}\,\varepsilon\,C_4)\big)w^3\Big) \\
\widehat{\psi}_2(z,w) = &\Big(32\,\mathrm{i}\,z + 32\,\mathrm{i}\,B_{21}zw + 8\,\mathrm{i}\,A_2w^2 - 16\,\varepsilon\,C_{22}z^3 + 8\big(\varepsilon(A_{22}-C_{13}) + A_2(2 - 3\,\mathrm{i}\,C_{22})\big)z^2w \\
&+ 2\big(\varepsilon\,A_{13} + 2\,\varepsilon\,B_{21} + 4\,\mathrm{i}\,B_{21}^2 - \mathrm{i}\,B_{22} - \varepsilon\,C_4 + 4\,\mathrm{i}\,A_2(A_{22}-C_{13}) + 6\,\varepsilon\,A_2^2C_{22}\big)zw^2 \\
&+ A_2\big(\mathrm{i}\,A_{13} + 6\,\mathrm{i}\,B_{21} + \varepsilon\,B_{22} - \mathrm{i}\,C_4 - 2\,\varepsilon\,A_2(A_{22}-C_{13}) + A_2^2(4 + 2\,\mathrm{i}\,C_{22})\big)w^3, \\
&32\,\mathrm{i}\,z^2 + 8\,\mathrm{i}\,A_2^2w^2 + 32\,\mathrm{i}\,\varepsilon\,A_2z^3 - 4\big(\mathrm{i}(A_{13} - 2B_{21} - \mathrm{i}\,\varepsilon\,B_{22} - C_4) - 2\,\varepsilon\,A_2(A_{22}-C_{13}) \\
&+ 2A_2^2(6 + \mathrm{i}\,C_{22})\big)z^2w - 4A_2\big(\mathrm{i}\,B_{22} - \varepsilon(A_{13} + 2B_{21} - C_4) - 2\,\mathrm{i}\,A_2(A_{22}-C_{13}) \\
&+ 2\,\varepsilon\,A_2^2(3\,\mathrm{i} - C_{22})\big)zw^2 + A_2^2(\mathrm{i}\,A_{13} + 6\,\mathrm{i}\,B_{21} + \varepsilon\,B_{22} - \mathrm{i}\,C_4 - 2\,\varepsilon\,A_2(A_{22}-C_{13}) \\
&+ A_2^2(4 + 2\,\mathrm{i}\,C_{22}))w^3, 4w\Big(8\,\mathrm{i} + 4(\varepsilon + 2\,\mathrm{i}\,B_{21})w - 4\,\varepsilon\,C_{22}z^2 + 2\big(\varepsilon(A_{22}-C_{13}) \\
&+ A_2(4 - 3\,\mathrm{i}\,C_{22})\big)zw + \big(2(\varepsilon + \mathrm{i}\,B_{21})B_{21} + \mathrm{i}\,A_2(A_{22}-C_{13}) + 2\,\varepsilon\,A_2^2(2\,\mathrm{i} + C_{22})\big)w^2\Big)\Big)/ \\
&\Big(32\,\mathrm{i} + 16(\varepsilon + 2\,\mathrm{i}\,B_{21})w - 16\,\varepsilon\,C_{22}z^2 + 8\big(\varepsilon(A_{22}-C_{13}) + A_2(6 - 3\,\mathrm{i}\,C_{22})\big)zw \\
&+ 2\Big(4\,\mathrm{i}\,B_{21}^2 + \mathrm{i}\,B_{22} - \varepsilon\big(A_{13} - 2B_{21} - C_4 - 2A_2^2(6\,\mathrm{i} + C_{22})\big)\Big)w^2 - 8\,\mathrm{i}\,C_{22}z^2w \\
&+ 4\Big(\mathrm{i}(A_{22}-C_{13}) + A_2\big(2B_{21} + \varepsilon(2\,\mathrm{i} + C_{22})\big)\Big)zw^2 + \Big(2\,\mathrm{i}\,B_{21} + A_{13}(\mathrm{i} - \varepsilon\,B_{21}) \\
&+ \mathrm{i}\,B_{21}B_{22} - \mathrm{i}\,C_4 - \varepsilon(2B_{21}^2 - B_{22} - B_{21}C_4) - 2\,\mathrm{i}\,A_2B_{21}(A_{22}-C_{13}) \\
&+ 2A_2^2\big(6 - \mathrm{i}\,C_{22} + \varepsilon\,B_{21}(2\,\mathrm{i} - C_{22})\big)\Big)w^3\Big)
\end{aligned}
$$

Appendix D: Standard Parameters

In the proof of Lemma 6.33 and Remark 6.40 we compute the following standard parameters:
Here we display the standard parameters for $\widetilde{\mathcal{G}}_1^\varepsilon$. First we define the following expression, which is the square root of (6.13):

$$R_1 := \sqrt{\frac{1+6\,\varepsilon\,r_0^2+v_0^2+r_0^4}{1+2\,\varepsilon\,r_0^2+v_0^2+r_0^4}}.$$

$$c'_{11} := \frac{(\varepsilon+\mathrm{i}\,v_0+r_0^2)(c_1u_1(-1-v_0^2+2r_0^2(2\,\varepsilon-\mathrm{i}\,v_0)+r_0^4)-2\,\mathrm{i}\,\varepsilon\,r_0\lambda_1)}{\lambda_1(\varepsilon-\mathrm{i}\,v_0+r_0^2)(1+6\,\varepsilon\,r_0^2+v_0^2+r_0^4)}$$

$$c'_{21} := \frac{2\,\mathrm{i}\,r_0(\varepsilon+\mathrm{i}\,v_0+r_0^2)(2c_1u_1(4\,\mathrm{i}+\varepsilon\,v_0)-\varepsilon\,r_0\lambda_1)}{\lambda_1(\varepsilon-\mathrm{i}\,v_0+r_0^2)(1+6\,\varepsilon\,r_0^2+v_0^2+r_0^4)}$$

$$\lambda'_1 := (\lambda_1R_1)^{-1}$$

$$a'_{11} := \frac{(1+v_0^2-2\,\mathrm{i}\,r_0^2(v_0-2\,\mathrm{i}\,\varepsilon)-r_0^4)}{u_1u'_1R_1(\varepsilon+\mathrm{i}\,v_0+r_0^2)^2}, \qquad a'_{21} := -\frac{4r_0(1+\mathrm{i}\,\varepsilon\,v_0)}{u_1u'_1R_1(\varepsilon+\mathrm{i}\,v_0+r_0^2)^2}$$

$$c_1 := \frac{\mathrm{i}\,r_0\lambda_1(4\,\varepsilon\,r_0^2+(\mathrm{i}\,\varepsilon+v_0)^2+r_0^4)}{u_1(-\,\varepsilon+\mathrm{i}\,v_0+r_0^2)(1+6\,\varepsilon\,r_0^2+v_0^2+r_0^4)}$$

$$u'_1 := \frac{R_1(1+\mathrm{i}\,\varepsilon\,v_0-\varepsilon\,r_0^2)(\varepsilon-\mathrm{i}\,v_0+r_0^2)^2}{u_1^3(\varepsilon-\mathrm{i}\,v_0+r_0^2)^2(1+6\,\varepsilon\,r_0^2+v_0^2+r_0^4)}, \qquad u_1 := \frac{(\varepsilon-\mathrm{i}\,v_0+r_0^2)}{(-\,\varepsilon+\mathrm{i}\,v_0+r_0^2)}\Big/\sqrt{\frac{1+2\,\varepsilon\,r_0^2+v_0^2+r_0^4}{1-2\,\varepsilon\,r_0^2+v_0^2+r_0^4}}$$

$$\lambda_1 := \frac{16+r_0^4+v_0^2+24r_0^2\,\varepsilon}{4\sqrt{16+r_0^4+v_0^2-8r_0^2\,\varepsilon}}$$

$$r'_1 := \frac{-4r_1(1+2v_0^2-2r_0^4+v_0^4+2r_0^4v_0^2+r_0^8)}{(1+6\,\varepsilon\,r_0^2+v_0^2+r_0^4)^3}$$

$$r_1 := -\frac{-v_0\left((1-v_0^2)^2+8\,\varepsilon\,r_0^2(1+v_0)^2+2r_0^4(7+v_0^2)+8\,\varepsilon\,r_0^6+r_0^8\right)}{4(1-2\,\varepsilon\,r_0^2+v_0^2+r_0^4)^2}$$

We give the standard parameters for $\widetilde{\mathcal{G}}_2^-$ in the following paragraphs. First we introduce the following expression, which is the square root of (6.14), to simplify formulas:

$$R_2 := \left(\frac{1+\sqrt{2}r_0(e^{-\,\mathrm{i}\,\theta_0}+e^{\mathrm{i}\,\theta_0})}{(1+\sqrt{2}r_0e^{-\,\mathrm{i}\,\theta_0})(1+\sqrt{2}r_0e^{\mathrm{i}\,\theta_0})}\right)^{1/2},$$

also we introduce the following expression:

$$S_2 := \frac{(1+\sqrt{2}(e^{-\,\mathrm{i}\,\theta_0}+e^{\mathrm{i}\,\theta_0})r_0+2r_0^2)^2(2(e^{-\,\mathrm{i}\,\theta_0}+e^{\mathrm{i}\,\theta_0})r_0+\sqrt{2}(1+2r_0^2))^2}{(1+\sqrt{2}r_0e^{-\,\mathrm{i}\,\theta_0})^4(1+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)^4(1+\sqrt{2}(e^{-\,\mathrm{i}\,\theta_0}+e^{\mathrm{i}\,\theta_0})r_0)^2}.$$

$$c'_{12} \coloneqq \frac{(e^{\mathrm{i}\,\theta_0}+\sqrt{2}r_0)(-c_2u_2(1+3r_0^2+2e^{2\,\mathrm{i}\,\theta_0}r_0^2+2\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0(1+r_0^2)-\mathrm{i}\,v_0)+\mathrm{i}\,e^{\mathrm{i}\,\theta_0}r_0(1+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)\lambda_2)}{(1+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)(e^{\mathrm{i}\,\theta_0}+\sqrt{2}r_0+\sqrt{2}e^{2\,\mathrm{i}\,\theta_0}r_0)\lambda_2}$$

$$c'_{22} \coloneqq \frac{(e^{\mathrm{i}\,\theta_0}+\sqrt{2}r_0)(c_2u_2(-r_0(3r_0+2e^{2\,\mathrm{i}\,\theta_0}r_0+2\sqrt{2}e^{\mathrm{i}\,\theta_0}(1+r_0^2))+\mathrm{i}\,v_0)+\mathrm{i}\,e^{\mathrm{i}\,\theta_0}r_0(1+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)\lambda_2)}{(1+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)(e^{\mathrm{i}\,\theta_0}+\sqrt{2}r_0+\sqrt{2}e^{2\,\mathrm{i}\,\theta_0}r_0)\lambda_2}$$

$$\lambda'_2 \coloneqq (\lambda_2R_2)^{-1}$$

$$a'_{12} \coloneqq \frac{1+3r_0^2+2e^{-2\,\mathrm{i}\,\theta_0}r_0^2+2\sqrt{2}e^{-\,\mathrm{i}\,\theta_0}r_0(1+r_0^2)+\mathrm{i}\,v_0}{u_2u'_2R_2(1+\sqrt{2}e^{-\,\mathrm{i}\,\theta_0}r_0)^2}$$

$$a'_{22} \coloneqq -\frac{3r_0^2+2e^{-2\,\mathrm{i}\,\theta_0}r_0^2+2\sqrt{2}e^{-\,\mathrm{i}\,\theta_0}r_0(1+r_0^2)+\mathrm{i}\,v_0}{u_2u'_2R_2(1+\sqrt{2}e^{-\,\mathrm{i}\,\theta_0}r_0)^2}$$

$$u'_2 \coloneqq \frac{e^{\mathrm{i}\,\theta_0}(\sqrt{2}r_0+\sqrt{2}e^{-2\,\mathrm{i}\,\theta_0}r_0+e^{-\,\mathrm{i}\,\theta_0}(1+2r_0^2))(2r_0+2e^{-2\,\mathrm{i}\,\theta_0}r_0+\sqrt{2}e^{-\,\mathrm{i}\,\theta_0}(1+2r_0^2))}{(1+\sqrt{2}e^{-\,\mathrm{i}\,\theta_0}r_0)^4(e^{-\,\mathrm{i}\,\theta_0}+\sqrt{2}r_0+\sqrt{2}e^{-2\,\mathrm{i}\,\theta_0}r_0)S_2u_2^3}$$

$$u_2 \coloneqq \frac{2S_2(1+\sqrt{2}r_0e^{-\,\mathrm{i}\,\theta_0})^4(1+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)^4(1+\sqrt{2}r_0e^{-\,\mathrm{i}\,\theta_0}+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)}{(1+\sqrt{2}r_0e^{-\,\mathrm{i}\,\theta_0}+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0+2r_0^2)(\sqrt{2}+2r_0e^{-\,\mathrm{i}\,\theta_0}+2e^{\mathrm{i}\,\theta_0}r_0+2r_0^2)^3}$$

$$\lambda_2 \coloneqq \frac{\sqrt{2}S_2(1+\sqrt{2}r_0e^{-\,\mathrm{i}\,\theta_0})^4(1+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)^4(1+\sqrt{2}r_0e^{-\,\mathrm{i}\,\theta_0}+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0)^2}{(1+\sqrt{2}r_0e^{-\,\mathrm{i}\,\theta_0}+\sqrt{2}e^{\mathrm{i}\,\theta_0}r_0+2r_0^2)^2(\sqrt{2}+2r_0e^{-\,\mathrm{i}\,\theta_0}+2e^{\mathrm{i}\,\theta_0}r_0+2r_0^2)^2}$$

The remaining parameters c_2, r_2 and r'_2 are set to 0.
We give the standard parameters for $\widetilde{\mathcal{G}}_3^-$ in the following paragraphs. Before we define an expression, which is the square root of (6.15), to simplify the subsequent formulas.

$$R_3 \coloneqq \sqrt{\frac{-1+v_0^2+r_0^4}{v_0^2+r_0^4}}$$

$$c'_{13} \coloneqq -\frac{c_3u_3(v_0^2+r_0^4)}{\lambda_3(-1+v_0^2+r_0^4)}, \qquad c'_{23} \coloneqq \frac{(r_0^2+\mathrm{i}\,v_0)(c_3u_3(v_0-\mathrm{i}\,r_0^2)-r_0\lambda_3)}{\lambda_3(r_0^2-\mathrm{i}\,v_0)(-1+v_0^2+r_0^4)}$$

$$\lambda'_3 \coloneqq \Big(\lambda_3R_3\Big)^{-1}$$

$$a'_{13} \coloneqq 1/(u_3u'_3R_3), \qquad a'_{23} \coloneqq \frac{\mathrm{i}(r_0^2-\mathrm{i}\,v_0)}{u_3u'_3R_3(r_0^2+\mathrm{i}\,v_0)^2}$$

$$c_3 \coloneqq \frac{\mathrm{i}\,\lambda_3}{2u_3r_0}, \qquad \lambda_3 \coloneqq \frac{-1+v_0^2+r_0^4}{2r_0}$$

$$u'_3 \coloneqq \frac{\mathrm{i}(r_0^2-\mathrm{i}\,v_0)}{u_3^3(r_0^2+\mathrm{i}\,v_0)^2}, \qquad u_3 \coloneqq -\frac{\mathrm{i}(1-v_0^2-2\,\mathrm{i}\,v_0r_0^2+r_0^4)}{\sqrt{(1-v_0^2)^2+2r_0^4(1+v_0^2)+r_0^8}}$$

The remaining parameters r_3 and r'_3 are set to 0.
For the last mapping $\widetilde{\mathcal{G}}_4^\varepsilon$ we obtain the following standard parameters:

$$R_4 \coloneqq \sqrt{3}\sqrt{\frac{\varepsilon+14r_0^4+\varepsilon\,r_0^8}{1+3\,\varepsilon\,r_0^4}}$$

$$c'_{14} \coloneqq \frac{4c_4 r_0^2 u(-1+r_0^4\,\varepsilon) - 8\,\mathrm{i}\,r_0^5\,\varepsilon\,\lambda_4}{(14r_0^4+\varepsilon+r_0^8\,\varepsilon)\lambda_4}, \qquad c'_{24} \coloneqq \frac{c_4 u_4(-1+3r_0^8+14r_0^4\,\varepsilon) - 8\,\mathrm{i}\,r_0^3\,\varepsilon\,\lambda_4}{\sqrt{3}(14r_0^4+\varepsilon+r_0^8\,\varepsilon)\lambda_4}$$

$$\lambda'_4 \coloneqq \Big(\lambda_4 R_4\Big)^{-1}$$

$$a'_{14} \coloneqq \frac{-12r_0^2(-1+r_0^4\,\varepsilon)}{u_4 u'_4 R_4(1+3r_0^4\,\varepsilon)^2}, \qquad a'_{24} \coloneqq -\sqrt{3}\frac{1-3r_0^8-14r_0^4\,\varepsilon}{u_4 u'_4 R_4(1+3r_0^4\,\varepsilon)^2}$$

$$c_4 \coloneqq \frac{\mathrm{i}\,r_0^3(-7-26r_0^8+9r_0^{16}-36r_0^4\,\varepsilon+60r_0^{12}\,\varepsilon)\lambda_4}{u_4(-19r_0^4-38r_0^{12}+9r_0^{20}-(1+74r_0^8-123r_0^{16})\,\varepsilon)}$$

$$\lambda_4 \coloneqq \left(4\sqrt{3}r_0\left|\frac{\varepsilon-r_0^4}{1+14\,\varepsilon\,r_0^4+r_0^8}\right|\right)^{-1}, \qquad u'_4 \coloneqq \frac{\operatorname{sgn}(r_0^4-\varepsilon)}{u_4^3\operatorname{sgn}(1+r_0^8+14r_0^4\,\varepsilon)}$$

$$u_4 \coloneqq \left(\frac{1-\varepsilon}{2}\right)\left(\frac{\operatorname{sgn}(-1-33r_0^4+33r_0^8+r_0^{12})}{\operatorname{sgn}(1-14r_0^4+r_0^8)}\right) + \left(\frac{1+\varepsilon}{2}\right)\operatorname{sgn}(-1+34r_0^4-34r_0^{12}+r_0^{16})$$

The remaining parameters r_4 and r'_4 are taken to be 0.

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June 28, 2010 **Prize of the NAWI-Club**, *University of Vienna, Vienna, Austria*.

Popularization of Mathematics

July 11, 2013 **Kinderuni 2013**, *University of Vienna, Vienna, Austria*.

July 13, 2012 **Kinderuni 2012**, *University of Vienna, Vienna, Austria*.