MASTERARBEIT

Titel der Masterarbeit

"Low regularity geometry on semi-Riemannian manifolds"

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Contents

Introduction

This thesis is about different aspects of low regularity geometry on semi-Riemannian manifolds and is roughly split into four chapters. The first chapter offers a brief introduction to the theory of distributions (in the sense of Laurent Schwartz) on general manifolds which is mainly based on the book [**GKOS01**].

In the second chapter we will look at singular objects on a manifold with a smooth semi-Riemannian metric. The smooth metric allows us to effectively deal with functions (and tensor fields) of the "lowest" regularity, i.e., distributions. In particular we are interested in studying distributions with support in a (closed, semi-Riemannian) hypersurface: As a first step we follow [**Sta11**] to give a nice description of the pullback of the Dirac delta distribution onto a hypersurface (Thm. 2.2.3) as a so-called single-layer distribution. Furthermore we derive a jump formula for the exterior derivative of a function with a jump discontinuity across the hypersurface in terms of single-layer distributions (Ex. 2.3.9).

Next we want to move on to derivatives of the delta distribution, which is done by defining multilayer distributions (as in [**Wag10**], see also (2.3.1)). Following [**Wag10**] we study their relationship with the delta distribution and derive formulas for their normal derivatives (Thm. 2.3.11) and multiplication with smooth functions (Thm. 2.3.12). As the final result of the second chapter it is shown that any distribution with support in a hypersurface can be written as a sum of multilayer distributions, which is a generalization of the well-known fact that every distribution supported in a point can be written as a sum of derivatives of delta distributions.

For the third chapter we no longer assume the existence of a smooth semi-Riemannian metric on our manifold but instead study distributional metrics (or, more generally, distributional geometry). Such metrics have been studied e.g. in [**LM07**], [**MS93**], [**GT87**] and are of particular interest in physics, mainly general relativity, especially considering that many physically relevant spacetimes must be singular by the Penrose-Hawking singularity theorems (see e.g. [**Wal84**], section 9.5). However, due to the impossibility of multiplication of distributions it is not possible to deal with merely distributional metrics in any meaningfull way and we have to assume a certain minimal regularity like at least local square integrability for a connection (see Prop. 3.3.3) or a specific Sobolev regularity and non-degeneracy condition for the metric (Prop. 3.4.3). For this reason there is a short introduction to Sobolev spaces on manifolds in the beginning of the third chapter including a brief discussion on which Sobolev spaces form algebras with continuous multiplication.

Now the main focus will be the definition of the various curvature quantities and derivation of jump formulas for them, that is, how the Riemann and Ricci tensor and the scalar curvature look like for a metric that suffers a jump discontinuity across a hypersurface (see (3.3.11), (3.3.12) and (3.4.4)). One possible application of those jump formulas to the Einstein field equations is outlined in section 3.6.

As a mathematical side note we also take a short look at the compatibility of this distributional approach to generalized geometry and a Colombeau theoretic approach, following the recent paper [**SV09**]. We see that both approaches are indeed equivalent for a certain class of distributional metrics (section 3.5).

Finally, the fourth and last chapter focuses on the geometry induced on a general (i.e. potentially null) hypersurface by a given connection or metric on our manifold. Again this was done in [**MS93**] and more recently by [**LM07**] and is of interest when studying solutions of the Einstein field equations, as there are several exact solutions where general hypersurfaces appear, e.g. Gödel's universe (where there are no hypersurfaces without boundary that are spacelike everywhere, see [**HE73**], section 5.7) or the Kerr solution (the stationary limit surface is timelike everywhere except at two points, where it is null, see [**HE73**], section 5.6).

As a substitute for the normal unit vector field (that is only available in the nowhere null case) we will use a so-called rigging vector field (see Def. 4.1.2). This leads to a projected connection on the hypersurface and a generalization of the second fundamental form (Def. 4.2.4) and the Gauss and Codazzi equations ((4.2.7) and (4.2.12)).

If the rigging vector field can be chosen nowhere null (we will show that this is always the case for a time-oriented Lorentzian manifold), then the inverse of the metric restricted to the hypersurface can be inverted again to induce a metric (and thus a metric connection) on the hypersurface. In Prop. 4.3.7 we calculate the difference between this metric connection and the projected connection. In the end we show that they coincide in the case of a nowhere null hypersurface if we choose the normal unit vector field as our rigging and that the generalized Gauss and Codazzi equations reduce to the well-known standard expressions. As always we try to keep things very general by not assuming smoothness of the connection/metric but just the regularity really needed to make sense of occurring products and traces.

CHAPTER 1

Distributions on manifolds

To begin with we will briefly summarize the most important notations that will be used throughout this work. As is common in differential geometry all our manifolds will be assumed to be smooth, Hausdorff and second countable. We will not, however, assume further properties like orientability or connectedness without explicitly saying so. In general our manifold will be denoted with X, the letter M will be used for submanifolds of X and E for vector bundles over X. The space of smooth sections shall be denoted by $\Gamma(X, E)$, smooth sections with compact support by $\Gamma_c(X, E)$ or $\mathcal{D}(X, E)$. The space $\Gamma\left(X, T_q^p X\right)$ of (p, q)-tensor fields on X will sometimes be denoted by $\mathcal{T}_q^p(X)$, vector fields by $\mathfrak{X}(X)$ and q-forms by $\Omega^q(X)$. The main source for this chapter will be [**GKOS01**].

1.1. The space of densities

Analogous to the definition of distributions on an open subset of $\mathbb{R}^n$ the space of distributions on a manifold X is defined as the dual space of certain smooth 'test objects'. However, if one wants to preserve the embedding of smoooth functions into distributions via integration, the space of these test objects has to be the space of compactly supported sections of the *volume bundle* over X, rather than $\mathcal{D}(X)$ itself, as there is no canonical way of defining $\int_X f\phi$ for $f \in \mathcal{C}^\infty(X)$ and $\phi \in \mathcal{D}(X)$.[1]

Before we actually define the volume bundle over a manifold, we will quickly review the cocycle approach to vector bundles.

Given a vector bundle (E, π, X) with atlas $(U_\alpha, \Psi_\alpha)_{\alpha \in I}$ (that is $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ a fiber respecting diffeomorphism, $U_\alpha \subset X$ a chart domain, V some finite dimensional vector space) one obtains a family of transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ via $\Psi_\alpha \circ \Psi_\beta^{-1}(x, v) = (x, \psi_{\alpha\beta}(x) v)$ which satisfies (since the different charts have to be compatible)

$$(1.1.1) \qquad \psi_{\alpha\beta} \cdot \psi_{\beta\gamma} = \psi_{\alpha\gamma} \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \quad \text{and}$$

$$(1.1.2) \qquad \psi_{\alpha\alpha} = id_V \qquad \text{on } U_\alpha.$$

On the other hand, one can show that for any family $\{\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)\}$ (where the U_α form an open cover of X) satisfying the cocycle conditions (1.1.1) and (1.1.2), there exists a unique (up to isomorphism) vector bundle (E, π, X) over X having those $\psi_{\alpha\beta}$ as its transition functions (for details see [**Mic08**], 8.3).

This description of vector bundles allows a very elegant definition of the so-called $q-volume\ bundle$ via its transition functions.

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[1] Of course defining $\mathcal{D}'(X)$ as the dual space of $\mathcal{D}(X)$ is still a valid option, but then the regular objects in $\mathcal{D}'(X)$ will no longer be smooth functions but sections of the volume bundle instead (this definition is used, for example, in [**Die72**]). Further options are also discussed in [**GKOS01**], chapter 3.1.

DEFINITION 1.1.1 (The q-volume bundle). Let X be a manifold with atlas $(U_\alpha, \phi_\alpha)_{\alpha \in I}$. Then for $q \in \mathbb{R}$ the one-dimensional real vector bundle $\mathrm{Vol}^q (X)$ given by the cocycle

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R} \setminus \{0\} = GL(1, \mathbb{R})$$

(1.1.3) $$\psi_{\alpha\beta}(x) = \left| \det D \left(\phi_\alpha \circ \phi_\beta^{-1} \right) (\phi_\beta(x)) \right|^{-q} = \left| \det D \left(\phi_\beta \circ \phi_\alpha^{-1} \right) (\phi_\alpha(x)) \right|^q$$

is called the q-volume bundle over X. The space $\Gamma(X, \mathrm{Vol}^q (X))$ of smooth sections of $\mathrm{Vol}^q (X)$ is called the space of q-densities on X.

REMARK 1.1.2. Clearly the $\psi_{\alpha\beta}$ defined in (1.1.3) satisfy (1.1.2) (since $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ and of course $\det D(id_{\mathbb{R}^n}) = 1$) and (1.1.1):

$$\psi_{\alpha\beta}(x)\psi_{\beta\gamma}(x) = \left| \det \left(D \left(\phi_\alpha \circ \phi_\beta^{-1} \right) (\phi_\beta(x)) \ D \left(\phi_\beta \circ \phi_\gamma^{-1} \right) (\phi_\gamma(x)) \right) \right|^{-q} =$$
$$= \left| D \left(\phi_\alpha \circ \phi_\gamma^{-1} \right) (\phi_\gamma(x)) \right|^{-q}.$$

Also, if X is an orientable manifold we have $\mathrm{Vol}^1 (X) \cong \Lambda^n T^* X$ (the vector bundle of exterior n-forms) since both vector bundles have the same transition functions as $\left| \det D \left(\phi_\alpha \circ \phi_\beta^{-1} \right) (\phi_\beta(x)) \right| = \det D \left(\phi_\alpha \circ \phi_\beta^{-1} \right) (\phi_\beta(x)) > 0$ for any oriented atlas $(U_\alpha, \phi_\alpha)_{\alpha \in I}$ of X.

We will mostly be dealing with the special case $q = 1$ and are going to write $\mathrm{Vol}^1 (X) \equiv \mathrm{Vol}(X)$ as well as speak of the volume bundle and its densities (instead of the 1-volume bundle and 1-densities). Next, we show that the volume bundles are trivial.

PROPOSITION 1.1.3. *The vector bundle* $\mathrm{Vol}^q (X)$ *is trivial, i.e. there exists a vector bundle isomorphism between* $\mathrm{Vol}^q (X)$ *and* $X \times \mathbb{R}$.

PROOF. Let (U_α, Ψ_α) be a vector bundle chart for $\mathrm{Vol}^q (X)$. For $x \in U_\alpha$ we set $s_\alpha(x) := \Psi_\alpha^{-1}(x, 1)$, then $s_\alpha : U_\alpha \to \mathrm{Vol}^q (X)$ is smooth and $\pi \circ s_\alpha(x) = \pi \circ \Psi_\alpha^{-1}(x, 1) = \mathrm{pr}_1(x, 1) = x$, so $s_\alpha(x)$ is in the fiber over x (which we will denote by F_x). Now let χ_α be a partition of unity subordinate to the U_α's and define $s : X \to \mathrm{Vol}^q (X)$ by $s(x) := \sum \chi_\alpha(x) s_\alpha(x)$ (using the vector space structure on F_x) to obtain an element in $\Gamma(X, \mathrm{Vol}^q (X))$.

Next we want to show that s does not become zero anywhere. We will do this by showing that for any given U_α the function $s|_{U_\alpha}$ is positive (in the sense that $\mathrm{pr}_2(\Psi_\alpha(s(x))) > 0$ for all $x \in U_\alpha$). We have

$$\mathrm{pr}_2(\Psi_\alpha(s(x))) = \sum_{\{\beta : x \in U_\beta\}} \chi_\beta(x) \, \mathrm{pr}_2(\Psi_\alpha(s_\beta(x))) = \sum_{\{\beta : x \in U_\beta\}} \chi_\beta(x) \, \mathrm{pr}_2\left(\Psi_\alpha \circ \Psi_\beta^{-1}(x, 1) \right) =$$
$$= \sum_{\{\beta : x \in U_\beta\}} \chi_\beta(x) \, \psi_{\alpha\beta}(x) \, 1 = \sum_{\{\beta : x \in U_\beta\}} \chi_\beta(x) \left| \det D \left(\phi_\alpha \circ \phi_\beta^{-1} \right) (\phi_\beta(x)) \right|^{-q} > 0,$$

as all the terms are non-negative and there has to be at least one β with $\chi_\beta(x) > 0$.

Finally we obtain $E \cong X \times \mathbb{R}$ by defining $f : X \times \mathbb{R} \to E$ as $f(x, v) := v\, s(x)$ and noting that this commutes with the projections π and pr_1, is fiber linear, smooth and bijective, hence a vector bundle isomorphism. $\square$

One notes, however, that there is no canonical trivialization (we used a partition of unity in the construction of f).

Given a section $\mu \in \Gamma(X, \mathrm{Vol}(X))$ we will denote its local components with respect to a given vector bundle chart (U_α, Ψ_α) by μ^α, i.e.

$$\mu^\alpha := \mathrm{pr}_2 \circ \Psi_\alpha \circ \mu|_{U_\alpha} \in \mathcal{C}^\infty (U_\alpha).$$

Next we will briefly introduce the concept of integration of densities. The integral of a compactly supported density $\mu \in \Gamma_c \left(X, \text{Vol} \left(X \right) \right)$ is defined analogous to the integration of an n-form on an orientable manifold by

$$\int_X \mu = \sum_\alpha \int_{U_\alpha} \chi_\alpha \mu := \sum_\alpha \int_{\psi_\alpha (U_\alpha)} \chi_\alpha \left(\psi_\alpha^{-1}(x) \right) \mu^\alpha \left(\psi_\alpha^{-1}(x) \right) dx$$

for a given atlas (U_α, ψ_α) and a subordinate partition of unity χ_α. That this is indeed well-defined is also shown completely analogous to the orientable case, thus we will omit the proof (which can be found in [Mic08], 10.3).

Last but not least we have to define a topology on our densities. We will do this very generally by equipping the spaces $\Gamma \left(X, E \right)$ and $\Gamma_c \left(X, E \right)$ for any given vector bundle E over X with suitable locally convex topologies.

DEFINITION 1.1.4 (Convergence in $\Gamma(X, E)$). A net $(u_\iota)_{\iota \in I} \subset \Gamma \left(X, E \right)$ is said to converge to $u \in \Gamma \left(X, E \right)$ if for all charts (V_α, ψ_α) of X the net $(u_\iota|_{V_\alpha})$ converges to $u|_{V_\alpha}$ in $\Gamma \left(V_\alpha, E \right)$, where we define the topology on $\Gamma \left(V_\alpha, E \right)$ so that the map ψ_α (together with the corresponding vector bundle chart Ψ_α) induces a homeomorphism between $\Gamma \left(V_\alpha, E \right)$ and $\mathcal{C}^\infty \left(\psi_\alpha \left(V_\alpha \right), \mathbb{R}^{\dim E} \right)$ with the usual locally convex topology (that is uniform convergence of all derivatives on compact subsets) via $u \mapsto u^\alpha \circ \psi_\alpha^{-1}$.

PROPOSITION 1.1.5. *Let E be a vector bundle over X. Then $\Gamma \left(X, E \right)$ is a Fréchet space.*

PROOF. We will only give a sketch of the proof, further details can be found in [Die72], XVII, 2. First we note that the topology defined above is induced by the seminorms

$$(1.1.4) \qquad p_{s,m,\alpha} \left(u \right) = \sum_{j=1}^{\dim E} \tilde{p}_{s,m,\alpha} \left(\Psi_\alpha^j \circ u|_{V_\alpha} \circ \psi_\alpha^{-1} \right),$$

where the $\tilde{p}_{s,m,\alpha}$ are chosen to be a (countable and separating) basis of seminorms for $\mathcal{C}^\infty \left(\psi_\alpha \left(V_\alpha \right) \right)$. Now by second countability X possesses a countable atlas and thus this familiy of seminorms is countable. Clearly it also separates points, so $\Gamma \left(X, E \right)$ is Hausdorff. Finally completeness follows in a very straightforward way from completeness of the spaces $\mathcal{C}^\infty \left(\psi_\alpha \left(V_\alpha \right) \right)$. $\square$

DEFINITION 1.1.6 (Topology on $\Gamma_c(X, E)$). The topology on $\Gamma_c \left(X, E \right)$ is defined to be the inductive limit topology with respect to the spaces $\Gamma_{K_m} \left(X, E \right) := \{u \in \Gamma \left(X, E \right) : \text{supp} \, u \subset K_m \}$, where the K_m are to be an exhaustive sequence of compact sets for X. That is a net $(u_\iota)_{\iota \in I}$ converges to u in $\Gamma_c \left(X, E \right)$ if and only if there exists an m such that $\bigcup_{\iota \in I} \text{supp} \, u_\iota \cup \text{supp} \, u \subset K_m$ and $u_\iota \to u$ in $\Gamma \left(X, E \right)$. We write

$$\Gamma_c \left(X, E \right) = \varinjlim \Gamma_{K_m} (X, E).$$

The spaces $\Gamma_{K_m} \left(X, E \right)$ are closed subspaces[2] of the Frechet space $\Gamma \left(X, E \right)$ and thus themselves Frechet, which makes $\Gamma_c \left(X, E \right)$ an (LF)-space.

The following proposition will show that $\Gamma_c \left(X, E \right)$ is a Montel space, i.e., a barrelled topological vector space where every closed and bounded set is compact.

PROPOSITION 1.1.7. *The space $\Gamma_c \left(X, E \right)$ is Montel.*

PROOF. First we show that $\Gamma_c \left(X, E \right)$ has the Heine-Borel property, i.e., every closed bounded subset is compact. Let $B \subset \Gamma_c \left(X, E \right)$ be bounded and closed. Then there exists a compact set $K \subset X$ such that $B \subset \Gamma_K \left(X, E \right)$ is bounded and closed in $\Gamma_K \left(X, E \right)$ (by the properties of the inductive limit topology). Let $(u_\alpha)_{\alpha \in I}$ be a net in $B \subset \Gamma_K \left(X, E \right)$ and choose an atlas of E

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[2]If $(u_m)_{m \geq 0}$ is a sequence in $\Gamma_K \left(X, E \right)$ converging to some $u \in \Gamma \left(X, E \right)$ then for any chart domain $U_\alpha \subset X \setminus K$ the components u_m^α satisfy $u_m^\alpha = 0$ for all m and thus $u^\alpha = 0$, so $u|_{X \setminus K} = 0$.

such that $U_i \cap K \neq \emptyset$ only for finitely many i, which will be called $i_1, \ldots, i_k$. Clearly the net $u_\alpha^{i_1} = \Psi_{i_1} \circ u_\alpha \circ \psi_{i_1}^{-1} \in \mathcal{C}^\infty (\psi_{i_1} (U_{i_1}))$ is bounded (because B is) and thus has a convergent subnet $u_{\alpha_1}^{i_1} \to u^{i_1}$ ($\mathcal{C}^\infty (\Omega)$ is a Montel space, see [**Edw65**], 8.4.7.). Replacing the original net u_α with the subnet u_{α_1} and repeating the process for i_2 to i_k one obtains a subnet u_β of u_α such that u_β^i converges to u^i for all i (since u_β^i is the constant 0 net for $i \neq i_1, \ldots, i_k$), which implies $u_\beta \to u$ in $\Gamma (X, E)$, thus in $\Gamma_K (X, E)$. Because B is closed one has $u \in B$, showing that B is compact.

Thus $\Gamma_c (X, E)$ is a separated barrelled topological vector space (because it is the strict inductive limit of Frechet spaces) which satisfies the Heine-Borel property, hence Montel.   □

REMARK 1.1.8. One may replace the net $(u_\alpha)_{\alpha \in I}$ in the proof of the above Proposition by a sequence $(u_n)_{n \in \mathbb{N}}$ because $\Gamma_K (X, E)$ is metrizable.

1.2. (Tensor-)Distributions

Finally we are ready to give the definition of distributions on a manifold.

DEFINITION 1.2.1 (Distributions on a manifold). Let X be a manifold. The space of *distributions* on X is defined as

$$\mathcal{D}' (X) := \Gamma_c (X, \mathrm{Vol}\,(\mathrm{X}))'.$$

More generally, given a vector bundle E over X, one may define *E-valued distributions* by setting

$$\mathcal{D}' (X, E) := \Gamma_c (X, E^* \otimes \mathrm{Vol}\,(\mathrm{X}))',$$

where E^* denotes the dual bundle of E. In particular we obtain the spaces of so-called *tensor distributions*

$$\mathcal{D}'\mathcal{T}_q^p(X) \equiv \mathcal{D}' \left(X, T_q^p X \right) = \Gamma_c \left(X, T_p^q X \otimes \mathrm{Vol}\,(\mathrm{X}) \right)'$$

as $T_q^p X^* = T_p^q X$.

If U is an open subset of X, the restriction $T|_U \in \mathcal{D}' (U, E|_U) \equiv \mathcal{D}' (U, E)$ of $T \in \mathcal{D}' (X, E)$ to U is defined by $\langle T|_U, \phi \rangle = \langle T, \tilde{\phi} \rangle$, where $\tilde{\phi} \in \Gamma_c (X, E^* \otimes \mathrm{Vol}\,(\mathrm{X}))$ is the extension of $\phi \in \Gamma_c \left(U, E|_U^* \otimes \mathrm{Vol}\,(\mathrm{U}) \right) = \Gamma_c (U, E^*|_U \otimes \mathrm{Vol}\,(\mathrm{X})|_U)$ by zero.

PROPOSITION 1.2.2. *With the restriction operation described above $\mathcal{D}' (X, E)$ is a fine sheaf. In particular if $(U_\lambda)_{\lambda \in \Lambda}$ is an open covering of X one has that*

(i) if $u, v \in \mathcal{D}' (X, E)$ and $u|_{U_\lambda} = v|_{U_\lambda}$ for all $\lambda \in \Lambda$ then $u = v$ and

(ii) if one has a family $(u_\lambda)_{\lambda \in \Lambda}$ of distributions $u_\lambda \in \mathcal{D}' (U_\lambda, E)$ satisfying

$$u_\lambda|_{U_\lambda \cap U_\mu} = u_\mu|_{U_\lambda \cap U_\mu} \quad \text{for all } \mu, \lambda \in \Lambda \text{ with } U_\lambda \cap U_\mu \neq \emptyset$$

then there exists some $u \in \mathcal{D}' (X, E)$ with $u|_{U_\lambda} = u_\lambda$ for all $\lambda \in \Lambda$.

The support of a distribution $u \in \mathcal{D}' (X, E)$ is defined as the complement of the largest open set U for which $u|_U$ vanishes. The space of E-valued distributions on X with compact support will be denoted by $\mathcal{E}' (X, E)$.

Given an atlas (U_α, Ψ_α) of E it is sometimes useful to identify the space $\mathcal{D}' (X, E)$ of E-valued distributions with families $(T^\alpha)_\alpha$ of distributions $T^\alpha = \left(T^{\alpha,1}, \ldots, T^{\alpha,n} \right) \in \mathcal{D}' (\psi_\alpha (U_\alpha))^n = \mathcal{D}' (\psi_\alpha (U_\alpha), \mathbb{R}^n)$ (where $n = \dim (E^* \otimes \mathrm{Vol}\,(\mathrm{X})) = \dim E$) satisfying certain transformation laws. To make those explicit, let (Ψ, U) be a vector bundle chart of $E^* \otimes \mathrm{Vol}\,(\mathrm{X})$ (and (ψ, U) the corresponding chart on X) and denote by $\Psi_* : \Gamma_c (U, E^* \otimes \mathrm{Vol}\,(\mathrm{X})) \to \mathcal{D} (\psi (U), \mathbb{R}^n)$ the map $T \mapsto \Psi \circ T \circ \psi^{-1}$. Because Ψ_* is continuous one obtains its adjoint map $(\Psi_*)' : \mathcal{D}' (\psi (U), \mathbb{R}^n) \to \mathcal{D}' (U, E)$ by setting $\langle (\Psi_*)' T, u \rangle := \langle T, \Psi_* u \rangle$. Now we may state the following proposition (the proof of which can be found in [**GKOS01**], p. 235):

PROPOSITION 1.2.3. *The space $\mathcal{D}'(X, E)$ of E-valued distributions on X can be identified with families $(T^\alpha)_\alpha$ of distributions $T^\alpha \in \mathcal{D}'\left(\psi_\alpha(U_\alpha), \mathbb{R}^{\dim E}\right)$ that satisfy*

$$((\Psi_\alpha)_*)' T^\alpha = ((\Psi_\beta)_*)' T^\beta \quad \text{on } U_\alpha \cap U_\beta.$$

There are two more useful identifications presented in [**GKOS01**], p. 237: First we consider the bilinear map

$$(1.2.1) \qquad \tilde{\beta} : \mathcal{D}'(X) \times \Gamma(X, E) \to \mathcal{D}'(X, E)$$

$$\left\langle \tilde{\beta}(T, z), \phi \right\rangle := \left\langle T, \operatorname{tr}_E \otimes id_{\text{Vol}(X)} (z \otimes \phi) \right\rangle,$$

where tr_E denotes the trace operation on $E \otimes E^*$ (we will omit the index E in the future if it is clear which space is meant). This map induces a linear map $\beta : \mathcal{D}'(X) \otimes \Gamma(X, E) \to \mathcal{D}'(X, E)$. Next we look at

$$(1.2.2) \qquad \gamma : \mathcal{D}'(X) \otimes \Gamma(X, E) \to L_{\mathcal{C}^\infty(X)}(\Gamma(X, E^*); \mathcal{D}'(X))$$

$$\gamma(T, z)(x) = \operatorname{tr}(z \otimes x) T.$$

THEOREM 1.2.4. *Both β and γ induce $\mathcal{C}^\infty(X)$-isomorphisms, i.e.*

$$\mathcal{D}'(X, E) \cong \mathcal{D}'(X) \otimes \Gamma(X, E) \cong L_{\mathcal{C}^\infty(X)}(\Gamma(X, E^*); \mathcal{D}'(X)).$$

It is worth noting that the second isomorphism is but a special case of a very general algebraic result, namely $E^* \otimes_A F \cong \operatorname{Hom}_A(E, F)$ for A−modules E and F, where either E or F is a finitely generated projective module (see [**Bou98**], p. 271).

In particular the previous theorem shows that

$$\mathcal{D}'\mathcal{T}_q^p(X) \cong \mathcal{D}'(X) \otimes_{\mathcal{C}^\infty} \mathcal{T}_q^p(X) \cong L_{\mathcal{C}^\infty(X)}\left(\left(\mathcal{T}_1^0(X)\right)^p \otimes \left(\mathcal{T}_0^1(X)\right)^q; \mathcal{D}'(X)\right),$$

thus locally $T|_{U_\alpha} \in \mathcal{D}'\left(U_\alpha, T_q^p X\right)$ can be written as

$$T|_{U_\alpha} = (T^\alpha)^{i_1 \ldots i_p}_{j_1 \ldots j_q} \, \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}$$

with local coefficients $(T^\alpha)^{i_1 \ldots i_p}_{j_1 \ldots j_q} \in \mathcal{D}'(U_\alpha)$.

Similarly to the case of distributions on $\mathbb{R}^n$ one has a natural embedding Φ of the spaces $\Gamma(X, E)$ into $\mathcal{D}'(X, E)$ via

$$(1.2.3) \qquad \langle \Phi(f), \phi \rangle = \int_X (\operatorname{tr}_E \otimes id_{\text{Vol}(X)})(f \otimes \phi) \quad \forall \phi \in \Gamma_c(X, E^* \otimes \text{Vol}(X)).$$

Of course, one may extend this embedding to locally integrable sections of E.

The next two propositions deal with the density of smooth objects in $\mathcal{D}'(X, E)$. First we show that compactly supported sections are strongly dense (see [**Sch66**], Theorem XV).

PROPOSITION 1.2.5 (Strong density of compactly supported sections). *The space $\Gamma_c(X, E)$ of compactly supported sections is dense in $\mathcal{D}'(X, E)$ with respect to the strong topology.*

PROOF. First we note that by Proposition 1.1.7 $\Gamma_c(X, E)$ is a Montel space, thus reflexive (see [**Edw65**], 8.4.7). Now let $\phi \in \mathcal{D}'(X, E) \setminus \overline{\Gamma_c(X, E)}$ then, by Hahn-Banach (see [**Rud73**], Theorem 3.5) there exists an $\omega \in \mathcal{D}'(X, E)' = \Gamma_c(X, E \otimes \text{Vol}(X))'' = \Gamma_c(X, E \otimes \text{Vol}(X))$ such that $\omega(\phi) = \langle \phi, \omega \rangle = 1$ but $\omega(g) = \langle g, \omega \rangle = \int_X g\omega = 0$ for all $g \in \Gamma_c(X, E)$ and thus $\omega \equiv 0$, which is a contradiction. □

A second result concerning the density of smooth objects is obtained by using the local description of E-valued distributions given in Proposition 1.2.3 and noting that $\mathcal{D}(\Omega)$ is weakly sequentially dense in $\mathcal{D}'(\Omega)$ for any open subset $\Omega \subset \mathbb{R}^n$ (which follows from a convolution argument):

PROPOSITION 1.2.6 (Weak sequential density of smooth sections). *The space $\Gamma(X, E)$ of smooth sections is weakly sequentially dense in $\mathcal{D}'(X, E)$, i.e. for all $T \in \mathcal{D}'(X, E)$ there exists a sequence $(T_n)_{n \in \mathbb{N}} \subset \Gamma(X, E)$ such that*

$$\langle T_n, \phi \rangle \to \langle T, \phi \rangle \quad \text{for all } \phi \in \Gamma_c(X, E^* \otimes \text{Vol}(X)).$$

A detailed proof can be found in [**GKOS01**], p. 241. Note that this also implies that $\Gamma_c(X, E)$ is weakly sequentially dense in $\mathcal{D}'(X, E)$ since it is weakly sequentially dense in $\Gamma(X, E)$ (this can be easily shown by using appropriate cut-off functions). When talking about density of smooth sections we will generally refer to the result above and not to Proposition 1.2.5. Similarly the standard topology on $\mathcal{D}'(X, E)$ will be the topology of weak sequential convergence. The above proposition now allows the unique continuous extension of continuous operators on $\Gamma_c(X, E)$ to $\mathcal{D}'(X, E)$.

CHAPTER 2

Distributions with support in a hypersurface

In this chapter we are going to introduce the concept of single-layer distributions for semi-Riemannian submanifolds $M \subset X$ and show how the pullback of the delta distribution by a submersion h satisfying $M = h^{-1}(0)$ can be understood as such a single-layer distribution. Next we will move on to hypersurfaces and define multilayer distributions for which we will derive formulas for multiplication with smooth functions as well as their normal derivatives. Last we show that each distribution on X with support in a hypersurface can be written as the sum of such multilayers.

To be able to do this we are going to need some more results about distributions on manifolds concerning their interaction with the additional structure given by a semi-Riemannian metric tensor. These will be presented in the next section.

2.1. Distributions on semi-Riemannian manifolds

Given a metric one defines the pointwise norm of an element $u \in \Gamma\left(X, T_q^p X\right)$ by setting $\|u(p)\|^2 := \left|u^{i_1 \dots i_{p+q}}(p) \, u_{i_1 \dots i_{p+q}}(p)\right|$ (where indices are lowered/raised by the metric). For $(0, q)$-tensor fields one may equivalently express the pointwise norm through $\|\omega(p)\|^2 := \left|\left(g^{-1}\right)^{\otimes q}(p)\left(\omega(p), \omega(p)\right)\right|$, where

$$\left(g^{-1}\right)^{\otimes q}(p) : \left(T_p X^*\right)^{\otimes q} \times \left(T_p X^*\right)^{\otimes q} \to \mathbb{R} \quad \left(v_1 \otimes \cdots \otimes v_q, v_1' \otimes \dots v_q'\right) \mapsto \prod_{i=1}^q g^{-1}(p)(v_i, v_i').$$

One has the following nice relation between the locally convex topology on $\Gamma\left(X, T_q^p X\right)$ and the pointwise norm defined above. This metric allows an alternative description of the topology on the space of $(p, q)-$tensor fields:

PROPOSITION 2.1.1. *The topology on $\Gamma\left(X, T_q^p X\right)$ defined in Def.* 1.1.4 *is induced by the seminorms*

$$(2.1.1) \qquad\qquad p_{K,s}(u) := \sup_{x \in K, l \leq s} \left\|\nabla^{(l)} u(x)\right\|,$$

where $K \subset X$ is compact and $s \in \mathbb{N}_0$.

More details regarding this can be found in [CBDMDB82], Section VI. B. 8.

2.1.1. Canonical identification of $\mathcal{D}'(X, E)$ with $\mathcal{D}\left(X, E^*\right)'$. For a general manifold X there is no canonical way of identifying, for instance, the spaces $\mathcal{D}'(X)$ and $\mathcal{D}(X)'$ (as there is no canonical trivialization of $\mathrm{Vol}(X)$, see Prop. 1.1.3). However, given a metric g on X, one may define a canonical *volume density* $\hat{\Omega}_g \in \Gamma(X, \mathrm{Vol}(X))$ via the local representations

$$(2.1.2) \qquad\qquad pr_2 \circ \Psi_\alpha \circ \hat{\Omega}_g\Big|_{U_\alpha} \equiv \hat{\Omega}_g^\alpha := \sqrt{\left|\det g_{ij}^\alpha\right|} \in \mathcal{C}^\infty\left(U_\alpha\right),$$

where we choose U_α and Ψ_α so that (U_α, Ψ_α) is a vector bundle chart and g_{ij}^α are the local components of the tensor field g with respect to a corresponding chart (U_α, ϕ_α) on the manifold.

PROPOSITION 2.1.2. *The local representations $\hat{\Omega}_g^\alpha$ defined in* (2.1.2) *indeed give a well-defined global section $\hat{\Omega}_g$ of $\mathrm{Vol}(X)$.*

PROOF. We have to show that

$$\Psi_\alpha^{-1} \circ \left(id_{U_\alpha \cap U_\beta} \times \hat{\Omega}_g^\alpha \Big|_{U_\beta} \right) = \hat{\Omega}_g \Big|_{U_\alpha \cap U_\beta} = \Psi_\beta^{-1} \circ \left(id_{U_\alpha \cap U_\beta} \times \hat{\Omega}_g^\beta \Big|_{U_\alpha} \right),$$

that is

$$\hat{\Omega}_g^\alpha(p) = \psi_{\alpha\beta} \left(\hat{\Omega}_g^\beta(p) \right) = \left| \det D \left(\phi_\beta \circ \phi_\alpha^{-1} \right) (\phi_\alpha(p)) \right| \hat{\Omega}_g^\beta(p)$$

for all $p \in U_\alpha \cap U_\beta$. If we denote the components of ϕ_α by x^i and those of ϕ_β by y^i, we have

$$\det g_{ij}^\alpha = \det g \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \det g \left(\sum_k \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k}, \sum_l \frac{\partial y^l}{\partial x^j} \frac{\partial}{\partial y^l} \right) = \det \sum_{k,l} \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} g_{kl}^\beta.$$

Now we note that $\sum_{k,l} A_{ki} B_{kl} A_{lj} = \sum_k A_{ki} (BA)_{kj} = \left(A^T B A \right)_{ij}$ and thus obtain for $A_{ij} = \frac{\partial y^i}{\partial x^j}$ and $B = g^\beta$

$$\sqrt{|\det g^\alpha|} = \sqrt{|\det A^T \det g^\beta \det A|} = |\det A| \sqrt{|\det g^\beta|}.$$

But since $A(p) = D \left(\phi_\beta \circ \phi_\alpha^{-1} \right) (\phi_\alpha(p)) = \psi_{\alpha\beta}(p)$ this finishes the proof. $\square$

REMARK 2.1.3. On an oriented manifold (of dimension n) we can identify $\mathrm{Vol}(X) \cong \Omega^n(X)$ and obtain $\hat{\Omega}_g = \sqrt{|\det g|} \, dx^1 \wedge \cdots \wedge dx^n$.

Since $\hat{\Omega}_g$ is nowhere zero, we can use it to identify $\mathrm{Vol}(X) \cong X \times \mathbb{R}$ (analogous to the proof of Proposition 1.1.3) and thus $\mathcal{C}^\infty(X) = \Gamma(X, X \times \mathbb{R}) \cong \Gamma(X, \mathrm{Vol}(X))$ via $\phi \mapsto \phi \hat{\Omega}_g$.

So

$$\mathcal{D}'(X, E) = \Gamma_c \left(X, E^* \otimes \mathrm{Vol}(X) \right)' \cong \Gamma_c \left(X, E^* \otimes (X \times \mathbb{R}) \right)' \cong \Gamma_c \left(X, E^* \right)'$$

and, more specifically,

$$\mathcal{D}' \left(X, T_q^p X \right) \cong \Gamma_c \left(X, T_p^q X \right)'.$$

This identification will be used implicitly throughout this second chapter and allows us to simplify some calculations. For instance, the embedding $\mathcal{T}_q^p(X) \hookrightarrow \mathcal{D}' \left(X, T_q^p X \right)$ given in equation (1.2.3) reduces to

$$(2.1.3) \qquad \langle T, \phi \rangle \equiv \left\langle \Phi(T), \phi \otimes \hat{\Omega}_g \right\rangle = \int_X (\mathrm{tr} \otimes id_{\mathrm{Vol}(X)})(T \otimes \phi \otimes \hat{\Omega}_g) = \int_X \mathrm{tr}(T \otimes \phi) \hat{\Omega}_g$$

for $T \in \mathcal{T}_q^p(X)$ and $\phi \in \mathcal{D} \left(X, T_p^q X \right)$.

2.1.2. Covariant derivatives. Next we will study the extension of covariant derivatives to distributions. To do so we will need the contraction operators $C_b^a : \mathcal{T}_q^p(X) \to \mathcal{T}_{q-1}^{p-1}(X)$ given by $C_b^a(T)_{j_1\ldots j_{q-1}}^{i_1\ldots i_{p-1}} = T_{j_1\ldots j_{b-1} r j_{b+1}\ldots j_q}^{i_1\ldots i_{a-1} r i_{a+1}\ldots i_p}$ (where we use the summation convention) and that the trace operator $\mathrm{tr} : \mathcal{T}_q^p(X) \otimes \mathcal{T}_p^q(X) = \mathcal{T}_{p+q}^{p+q}(X) \to \mathcal{C}^\infty(X)$ can be written as a combination of such contractions ($\mathrm{tr} = C_1^1 \circ \cdots \circ C_p^p \circ C_1^{p+1} \cdots \circ C_q^{p+q}$). It is also worth mentioning that the contraction operators can of course be continuously extended to $C_b^a : \mathcal{D}' \left(X, T_q^p X \right) \to \mathcal{D}' \left(X, T_{q-1}^{p-1} X \right)$.

First recall that if we denote the Levi-Civita connection on X by ∇ and the Christoffel symbols by Γ, the components of $\nabla T \in \mathcal{T}_{q+1}^p(X)$ for $T \in \mathcal{T}_q^p(X)$ are given by

$$(2.1.4) \quad (\nabla T)_{j_1\ldots j_q r}^{i_1\ldots i_p} = \partial_r \left(T_{j_1\ldots j_q}^{i_1\ldots i_p} \right) + \Gamma_{rk}^{i_1} T_{j_1\ldots j_q}^{k\ldots i_p} + \cdots + \Gamma_{rk}^{i_p} T_{j_1\ldots j_q}^{i_1\ldots k} - \Gamma_{rj_1}^k T_{k\ldots j_q}^{i_1\ldots i_p} - \cdots - \Gamma_{rj_q}^k T_{j_1\ldots k}^{i_1\ldots i_p},$$

where the Γ_{ij}^k can be computed using the metric:

$$(2.1.5) \qquad\qquad \Gamma_{ij}^k = \frac{1}{2} g^{km} \left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij} \right).$$

Now we are ready to prove the next proposition.

PROPOSITION 2.1.4. *Let (X, g) be a semi-Riemannian manifold with Levi-Civita connection ∇. Then $\nabla : \mathcal{T}_q^p(X) \to \mathcal{T}_{q+1}^p(X)$ can be extended uniquely to a continuous operator $\nabla : \mathcal{D}'(X, T_q^p X) \to \mathcal{D}'(X, T_{q+1}^p X)$ and one has*

$$(2.1.6) \qquad \langle \nabla T, \phi \rangle = - \left\langle T, C_{p+1}^{q+1}(\nabla \phi) \right\rangle \quad \text{for } T \in \mathcal{D}'(X, T_q^p X), \phi \in \mathcal{D}(X, T_p^{q+1} X).$$

PROOF. First let U be a chart domain and $T \in \Gamma(U, T_q^p U)$, $\phi \in \mathcal{D}(U, T_p^{q+1} U)$ with components $T_{j_1 \dots j_q}^{i_1 \dots i_p}$ and $\phi_{l_1 \dots l_p}^{k_1 \dots k_{q+1}}$, respectively. Then, using (2.1.3) and (2.1.4),

$$\langle \nabla T, \phi \rangle = \int_U \operatorname{tr}(\nabla T \otimes \phi) \, \hat{\Omega}_g = \int_{\mathbb{R}^n} \sqrt{|\det g|} \phi_{i_1 \dots i_p}^{j_1 \dots j_q r} \partial_r \left(T_{j_1 \dots j_q}^{i_1 \dots i_p} \right) d^n x +$$

$$+ \int_{\mathbb{R}^n} \phi_{i_1 \dots i_p}^{j_1 \dots j_q r} \left(\Gamma_{rk}^{i_1} T_{j_1 \dots j_q}^{k \dots i_p} + \cdots + \Gamma_{rk}^{i_p} T_{j_1 \dots j_q}^{i_1 \dots k} - \Gamma_{rj_1}^k T_{k \dots j_q}^{i_1 \dots i_p} - \cdots - \Gamma_{rj_q}^k T_{j_1 \dots k}^{i_1 \dots i_p} \right) \sqrt{|\det g|} d^n x =$$

$$= - \int_{\mathbb{R}^n} \sqrt{|\det g|} \partial_r \left(\phi_{i_1 \dots i_p}^{j_1 \dots j_q r} \right) T_{j_1 \dots j_q}^{i_1 \dots i_p} d^n x - \int_{\mathbb{R}^n} \partial_r \left(\sqrt{|\det g|} \right) \phi_{i_1 \dots i_p}^{j_1 \dots j_q r} T_{j_1 \dots j_q}^{i_1 \dots i_p} d^n x +$$

$$+ \int_{\mathbb{R}^n} T_{j_1 \dots j_q}^{i_1 \dots i_p} \left(\Gamma_{ri_1}^k \phi_{k \dots i_p}^{j_1 \dots j_q r} + \cdots + \Gamma_{ri_p}^k \phi_{i_1 \dots k}^{j_1 \dots j_q r} - \Gamma_{rk}^{j_1} \phi_{i_1 \dots i_p}^{k \dots j_q r} - \cdots - \Gamma_{rk}^{j_q} \phi_{i_1 \dots i_p}^{j_1 \dots k r} \right) \sqrt{|\det g|} d^n x.$$

To simplify the second integral, we note that $\frac{1}{\sqrt{|\det g|}} \partial_r \sqrt{|\det g|} = \Gamma_{jr}^j$ (see Lemma 2.1.5 below). Collecting everything, one finally obtains

$$\langle \nabla T, \phi \rangle = - \int_{\mathbb{R}^n} \sqrt{|\det g|} T_{j_1 \dots j_q}^{i_1 \dots i_p} \left(\partial_r \left(\phi_{i_1 \dots i_p}^{j_1 \dots j_q r} \right) + \Gamma_{jr}^j \phi_{i_1 \dots i_p}^{j_1 \dots j_q r} \right) d^n x +$$

$$+ \int_{\mathbb{R}^n} \sqrt{|\det g|} T_{j_1 \dots j_q}^{i_1 \dots i_p} \left(\Gamma_{ri_1}^k \phi_{k \dots i_p}^{j_1 \dots j_q r} + \cdots + \Gamma_{ri_p}^k \phi_{i_1 \dots k}^{j_1 \dots j_q r} - \Gamma_{rk}^{j_1} \phi_{i_1 \dots i_p}^{k \dots j_q r} - \cdots - \Gamma_{rk}^{j_q} \phi_{i_1 \dots i_p}^{j_1 \dots k r} \right) d^n x =$$

$$= - \int_{\mathbb{R}^n} \sqrt{|\det g|} T_{j_1 \dots j_q}^{i_1 \dots i_p} C_{p+1}^{q+1} \left(\partial_{i_{p+1}} \left(\phi_{i_1 \dots i_p}^{j_1 \dots j_q j_{q+1}} \right) - \Gamma_{i_{p+1} i_1}^k \phi_{k \dots i_p}^{j_1 \dots j_q j_{q+1}} \cdots - \Gamma_{i_{p+1} i_p}^k \phi_{i_1 \dots k}^{j_1 \dots j_q j_{q+1}} \right) +$$

$$+ \sqrt{|\det g|} T_{j_1 \dots j_q}^{i_1 \dots i_p} C_{p+1}^{q+1} \left(\Gamma_{i_{p+1} k}^{j_{q+1}} \phi_{i_1 \dots i_p}^{j_1 \dots j_q k} + \Gamma_{i_{p+1} k}^{j_1} \phi_{i_1 \dots i_p}^{k \dots j_q j_{q+1}} + \cdots + \Gamma_{i_{p+1} k}^{j_q} \phi_{i_1 \dots i_p}^{j_1 \dots k j_{q+1}} \right) d^n x =$$

$$= - \int_U \operatorname{tr} \left(T \otimes C_{p+1}^{q+1}(\nabla \phi) \right) \hat{\Omega}_g = - \left\langle T, C_{p+1}^{q+1}(\nabla \phi) \right\rangle.$$

That this formula also holds true for $T \in \Gamma(X, T_q^p X)$ follows by using a partition of unity and writing $\phi = \sum_j \chi_j \phi$.

This shows that $\nabla : \Gamma(X, T_q^p X) \to \Gamma(X, T_{q+1}^p X)$ is continuous (with respect to the topology on $\mathcal{D}'(X, T_q^p X)$), thus existence and uniqueness follows from density of $\Gamma(X, T_q^p X)$ in $\mathcal{D}'(X, T_q^p X)$ and the extension is given by (2.1.6). $\qquad\square$

LEMMA 2.1.5. *Let (X, g) be a semi-Riemannian manifold. Then*

$$\Gamma_{jr}^j = \frac{1}{\sqrt{|\det g|}} \partial_r \sqrt{|\det g|}.$$

PROOF. First we use (2.1.5) to calculate

$$\Gamma_{jr}^j = \frac{1}{2} g^{ij} (\partial_r g_{ij} + \partial_i g_{jr} - \partial_j g_{ri}) = \frac{1}{2} g^{ij} \partial_r g_{ij} = \frac{1}{2} \operatorname{tr} \left(g^{-1} \partial_r g \right).$$

Now, using Jacobi's formula $\frac{d}{dt} \det A(t) = \operatorname{tr}(\operatorname{adj}(A(t)) A'(t)) = \det A(t) \operatorname{tr}(A(t)^{-1} A'(t))$, one obtains

$$\frac{1}{\sqrt{|\det g|}} \partial_r \sqrt{|\det g|} = \frac{1}{2} \frac{1}{|\det g|} \partial_r |\det g| = \frac{1}{2} \frac{1}{\det g} \partial_r \det g = \Gamma_{jr}^j. \qquad\square$$

COROLLARY 2.1.6. *For $\mathbf{v} \in \mathfrak{X}(X)$ the unique continuous extension of the map $\nabla_{\mathbf{v}}$ to $\mathcal{D}'(X, T_q^p X)$ satisfies*

$$\langle \nabla_{\mathbf{v}} T, \phi \rangle = - \left\langle T, C_{p+1}^{q+1} \nabla (\phi \otimes \mathbf{v}) \right\rangle \quad \text{for } T \in \mathcal{D}'(X, T_q^p X), \phi \in \mathcal{D}(X, T_p^q X).$$

PROOF. For $T \in \mathcal{T}_p^q(X)$ we have

$$\langle \nabla_{\mathbf{v}} T, \phi \rangle = \int_X \operatorname{tr}\left(\nabla_{\mathbf{v}} T \otimes \phi\right) \hat{\Omega}_g = \int_X \operatorname{tr}\left(\nabla T \otimes (\phi \otimes \mathbf{v})\right) \hat{\Omega}_g = \langle \nabla T, \phi \otimes \mathbf{v} \rangle.$$

Now the claim follows immediately from Proposition 2.1.4. $\qquad\square$

For $T \in \mathcal{D}'(U)$ one can also obtain an analogous formula for the kth-partial derivatives.

PROPOSITION 2.1.7. *If $U \subset X$ is a chart domain and $T \in \mathcal{D}'(U)$, we have*

$$\left\langle \partial_j^k T, \phi \right\rangle = (-1)^k \left\langle T, \frac{1}{\sqrt{|\det g|}} \partial_j^k \left(\phi \sqrt{|\det g|} \right) \right\rangle, \quad \forall \phi \in \mathcal{D}(U).$$

PROOF. For $T \in \mathcal{C}^\infty(U)$ we have

$$\left\langle \partial_j^k T, \phi \right\rangle = \int_{\mathbb{R}^n} \partial_j^k T \, \phi \, \sqrt{|\det g|} d^n x = (-1)^k \int_{\mathbb{R}^n} T \, \partial_j^k \left(\phi \sqrt{|\det g|} \right) d^n x =$$

$$= (-1)^k \left\langle T, \frac{1}{\sqrt{|\det g|}} \partial_j^k \left(\phi \sqrt{|\det g|} \right) \right\rangle.$$

$\qquad\square$

2.1.3. Pullbacks. For semi-Riemannian manifolds X one can define the pullback $h^* T \in \mathcal{D}'(X)$ of a distribution $T \in \mathcal{D}'(\Omega)$ by a submersion $h : X \to \Omega \subset \mathbb{R}^m$ completely analogous to the case of distributions on open subsets of $\mathbb{R}^n$ (see [**FJ98**], Thm. 7.2.2):

$$\langle h^* T, \phi \rangle := \langle T, \phi_h \rangle \quad \forall \phi \in \mathcal{D}(X),$$

with

$$\phi_h\left(t^1, \ldots, t^m\right) := \frac{\partial^m}{\partial t^1 \ldots \partial t^m} \int_{\{x \in X : h^i(x) < t^i\}} \phi(x) \hat{\Omega}_g \in \mathcal{D}(\Omega).$$

We may also use the familiar notation $T \circ h$ for $h^* T$, which is justified by the Lemma below.

LEMMA 2.1.8. *The map $h^* : \mathcal{D}'(\Omega) \to \mathcal{D}'(X)$ is the continuous extension of the pullback of smooth functions.*

PROOF. By [**FJ98**], Thm. 7.2.2 this is true if X is an open subset of $\mathbb{R}^n$. Let (ψ, U) be a chart in X, $\phi \in \mathcal{D}(U)$ and set $\Phi := \phi \circ \psi^{-1} \in \mathcal{D}(\psi(U))$, $\mathbf{h} := h \circ \psi^{-1} : \psi(U) \to \Omega$. Then $\mathbf{h}$ is again a submersion and for $T \in \mathcal{D}(\Omega)$ we have

$$\langle T \circ h, \phi \rangle = \int_{\psi(U)} \sqrt{|\det g_{ij}|} \Phi(x) T \circ \mathbf{h}(x) dx = \left\langle T \circ \mathbf{h}, \sqrt{|\det g_{ij}|} \Phi \right\rangle_{\mathbb{R}^n} =$$

$$= \left\langle T, \left(\sqrt{|\det g_{ij}|} \Phi \right)_{\mathbf{h}} \right\rangle_{\mathbb{R}^m} = \langle T, \phi_h \rangle$$

because

$$\left(\sqrt{|\det g_{ij}|} \Phi \right)_{h \circ \psi^{-1}}(t) = \frac{\partial^m}{\partial t^1 \ldots \partial t^m} \int_{\{x \in \psi^{-1}(U) : h^i(\psi^{-1}(x)) < t^i\}} \sqrt{|\det g_{ij}|} \Phi(x) dx =$$

$$= \frac{\partial^m}{\partial t^1 \ldots \partial t^m} \int_{\{y \in U : h^i(y) < t^i\}} \phi(y) \hat{\Omega}_g = \phi_h(t).$$

This shows that $T \circ h = h^* T$ in $\mathcal{D}'(U)$ for all chart domains $U \subset X$ and thus $T \circ h = h^* T$ for all $T \in \mathcal{D}(\Omega)$ and hence for all $T \in \mathcal{D}'(\Omega)$ by continuity. $\qquad\square$

Note that Lemma 2.1.8 implies that the usual rules of computation remain valid. In particular, the chain rule holds, i.e.,

$$(2.1.7) \qquad \partial_j \left(h^* T \right) = \sum_{k=1}^{m} h^* \left(\partial_k T \right) \partial_j h^k.$$

2.2. Single-layer distributions

DEFINITION 2.2.1 (Single-layer distribution). Let M be a closed semi-Riemannian submanifold of a semi-Riemannian manifold X and $T \in \mathcal{D}' \left(M \right)$. The *single-layer distribution* $S_M \left(T \right) \in \mathcal{D}' \left(X \right)$ on M with density T is defined by

$$\langle S_M \left(T \right), \phi \rangle := \langle T, \phi|_M \rangle, \quad \phi \in \mathcal{D} \left(X \right).$$

REMARK 2.2.2. The above indeed defines a distribution on X: First, M being closed gives $\phi|_M \in \mathcal{D} \left(M \right)$ for all $\phi \in \mathcal{D} \left(X \right)$, so $\langle S_M \left(T \right), \phi \rangle \in \mathbb{R}$ is well-defined. Second, convergence of $\phi_\alpha \to \phi$ in $\mathcal{D} \left(X \right)$ clearly implies $\phi_\alpha|_M \to \phi|_M$ in $\mathcal{D} \left(M \right)$, so $S_M \left(T \right)$ is continuous.

The concept of single-layer distributions may be used to give an alternative formula for the pullback of the Dirac delta distribution on $\mathbb{R}^m$ by a smooth submersion $h : X \to \mathbb{R}^m$ (see [Sta11]):

THEOREM 2.2.3. *Let X be a semi-Riemannian manifold and $h : X \to \mathbb{R}^m$ a submersion such that $M := h^{-1} \left(0 \right)$ is a $(n - m$ dimensional$)$ semi-Riemannian submanifold of X. Then*

$$\delta_{\mathbb{R}^m} \circ h = S_M \left(\sqrt{m!} \, \left\| dh^1 \wedge \cdots \wedge dh^m \right\|^{-1} \right).$$

PROOF. By definition of the pullback of a distribution by a submersion

$$\langle \delta_{\mathbb{R}^m} \circ h, \phi \rangle = \langle \delta_{\mathbb{R}^n}, \phi_h \rangle = \phi_h(0) = \left. \left(\frac{\partial^m}{\partial t^1 \dots \partial t^m} \int_{\{h^i(x) < t^i\}} \phi(x) \hat{\Omega}_g \right) \right|_{t=0}.$$

Now let $y \in X$ and choose a neighborhood U of y and a smooth map $\psi' = \left(x^{m+1}, \dots, x^n \right) : U \to \mathbb{R}^{m-n}$ such that $\psi := \left(h^1, \dots, h^m, \psi' \right) = (x^1, \dots, x^n)$ is a chart on U (this is always possible because h is a submersion). In this chart we have for $\phi \in \mathcal{D} \left(U \right)$

$$(2.2.1) \quad \phi_h(0) = \left. \left(\frac{\partial^m}{\partial t^1 \dots \partial t^m} \int_{\psi(U) \cap \{x : x^i < t^i\}} \left(\phi \circ \psi^{-1} \right) (x) \sqrt{|\det g_{ij}|} dx \right) \right|_{t=0} =$$

$$= \int_{\{x' : (0, x') \in \psi(U)\}} \left(\phi \circ \psi^{-1} \right) (0, \xi') \sqrt{|\det g_{ij} (0, x')|} dx'.$$

Next, we evaluate $\langle S_M \left(f \right), \phi \rangle = \int_M f \, \phi|_M \, \hat{\Omega}_{\tilde{g}}$ for $f \in \mathcal{C}^\infty (M \cap U)$, where $\tilde{g}$ denotes the induced metric on M. Clearly, using the chart ψ' on $U \cap M$, we have $\tilde{g} = \sum_{i,j=m+1}^n g_{ij}|_{M \cap U} \, dx^i \otimes dx^j$, thus for $f \in \mathcal{C}^\infty \left(M \cap U \right)$

$$(2.2.2) \quad \langle S_M \left(f \right), \phi \rangle = \int_{M \cap U} \phi|_M \, f \, \hat{\Omega}_{\tilde{g}} = \int_{\psi'(M \cap U)} f \left(x' \right) \phi \left(0, x' \right) \sqrt{|\det \left((g_{ij})_{i,j > m} \right) (0, x')|} dx'.$$

Finally, on U, we have

$$(2.2.3)$$

$$\left\| dh^1 \wedge \cdots \wedge dh^m \right\|^2 = \left\| dx^1 \wedge \cdots \wedge dx^m \right\|^2 = \left| \left(g^{-1} \right)^{\otimes m} \left(dx^1 \wedge \cdots \wedge dx^m, dx^1 \wedge \cdots \wedge dx^m \right) \right|^2 =$$

$$= \left| \sum_{\pi, \sigma \in \mathcal{S}_m} \mathrm{sgn}\pi \, \mathrm{sgn}\sigma \prod_{j=1}^m g^{-1} \left(dx^{\pi(j)}, dx^{\sigma(j)} \right) \right| = m! \left| \det(g^{ij})_{1 \le i,j \le m} \right|.$$

Showing that

$$(2.2.4) \qquad |\det\left((g_{ij})_{m<i,j\leq n}\right)| = \left|\det\left((g^{ij})_{1\leq i,j\leq m}\right)\right| \left|\det\left(g_{ij}\right)_{1\leq i,j\leq n}\right|$$

finishes the proof: Note that this equality implies in particular $\left|\det\left((g^{ij})_{1\leq i,j\leq m}\right)\right| \neq 0$ on U because both $\det g_{ij}$ and $\det\left((g_{ij})_{i,j>m}\right) = \det \tilde{g}_{ij}$ are non-zero (M and X are both semi-Riemannian), thus $\left\|dh^1 \wedge \cdots \wedge dh^m\right\|^{-1} \in \mathcal{C}^\infty(U)$. The claim now follows directly from setting

$$f = \sqrt{m!}\, \left\|dh^1 \wedge \cdots \wedge dh^m\right\|^{-1}\Big|_{U_y \cap M}$$

in (2.2.2), using (2.2.3) and (2.2.4) and comparing the result to (2.2.1).

Equation (2.2.4) holds true for every invertible $n \times n$ matrix, a proof can be found in [**Sta11**], appendix A.

$$\square$$

2.3. Multilayer distributions

In the previous chapter we dealt with the concept of single-layer distributions $S_M(T) \in \mathcal{D}'(X)$ for $T \in \mathcal{D}'(M)$ on closed semi-Riemannian submanifolds $M \subset X$ of *arbitrary* dimension. To define multilayer distributions $L_M^{(k)}(T) \in \mathcal{D}'(X)$, however, our closed submanifold M has to be a *hypersurface* that admits a *normal unit vector field*[1] $\mathbf{n} \in \Gamma\left(M, TX|_M\right)$, as one sets

$$(2.3.1) \qquad \left\langle L_M^{(k)}(T), \phi \right\rangle := (-1)^k \left\langle T, \left(\nabla_{\mathbf{n}}^k \phi\right)|_M \right\rangle, \quad \phi \in \mathcal{D}(X).$$

Note that for $\mathbf{v} \in \mathfrak{X}(X)$ the expression $\nabla_{\mathbf{v}} \phi(p)$ only depends on $\mathbf{v}(p)$ and thus $\left(\nabla_{\mathbf{n}}^k \phi\right)(p)$ is well-defined for $p \in M$ by taking any extension of $\mathbf{n}$ to some open neighborhood of p (a concrete extension of $\mathbf{n}$ will be given in Remark 2.3.7). Also, these multilayers are a generalization of the single-layer distributions defined before since $L_M^{(0)}(T) = S_M(T)$. The aim of this section is to establish formulas for the covariant derivatives of these multilayers, closely following the paper [**Wag10**] by P. Wagner.

REMARK 2.3.1. The existence of a normal unit vector field to a semi-Riemannian hypersurface is equivalent to the orientability of the normal bundle since this gives by definition a smooth map that assigns to each $p \in M$ an orientation of $\{p\} \times T_p M^\perp$ (see [**O'N83**], p. 198), i.e., a nonzero vector in $T_p M^\perp$ which we may normalize to have unit length. If X is orientable this is also equivalent to orientability of the hypersurface $M \subset X$ itself (see [**O'N83**], p. 189). However, if X is not orientable this is not sufficient: As an example we may look at the Möbius strip for X and $S^1 \subset X$, where clearly S^1 itself is orientable but does not admit a unit normal vector field when viewed as a hypersurface in the Möbius strip.

EXAMPLE 2.3.2. Let $X = \mathbb{R}^n$ with the standard metric, $M = \{0\} \times \mathbb{R}^{n-1} \cong \mathbb{R}^{n-1}$ and $T \in \mathcal{D}'(M)$. To calculate $L_M^{(k)}(T)$ let $\phi = \phi_1 \otimes \tilde{\phi} \in \mathcal{D}(X)$, then

$$(-1)^k \left\langle L_M^{(k)}(T), \phi \right\rangle = \left\langle T, \left(\nabla_{\mathbf{n}}^k \phi\right)|_M \right\rangle = \left\langle T, \left(\phi_1^{(k)} \otimes \tilde{\phi}\right)|_M \right\rangle = \phi_1^{(k)}(0)\left\langle T, \tilde{\phi} \right\rangle =$$
$$= (-1)^k \left\langle \delta^{(k)} \otimes T, \phi \right\rangle.$$

By density of $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{R}^{n-1})$ in $\mathcal{D}(\mathbb{R}^n)$ we conclude

$$L_M^{(k)}(T) = \delta^{(k)} \otimes T.$$

Of course it is also possible to define multilayer tensor-distributions.

---

[1] *A vector field* $\mathbf{n} \in \Gamma\left(M, TX|_M\right)$ *is called unit normal vector field for* M *if* $\mathbf{n}(p) \in T_p M^\perp$ *for all* $p \in M$ *and* $g(\mathbf{n}, \mathbf{n}) = 1$.

DEFINITION 2.3.3. Let $T \in \mathcal{D}'\left(M, T_q^p X\big|_M\right)$. Then

$$\left\langle \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}), \phi \right\rangle := (-1)^k \left\langle T, \left(\nabla_{\mathbf{n}}^k \phi\right)\big|_M \right\rangle, \quad \phi \in \mathcal{D}\left(X, T_p^q X\right)$$

defines the multilayer (tensor-)distribution of order k with density T as an element of $\mathcal{D}'\left(X, T_q^p X\right)$.

2.3.1. Introduction of 'canonical' coordinates. To simplify further calculations involving multilayer distributions we first introduce special coordinates $\psi = \left(x^1, \ldots, x^n\right)$ around each $p \in M \subset X$ with the following properties:

(1) The map $\psi' = \left(x^2, \ldots, x^n\right)$ is a chart around p in M,
(2) the first coordinate measures the arc length of geodesics orthogonal to M (i.e. $x^1\left(c(t)\right) = t\,|c'(0)|$ for all geodesics c with $c'(0) \in TM^\perp$ and $g\left(c'(0), \mathbf{n}\right) \geq 0$)
(3) and $\mathbf{n} = \frac{\partial}{\partial x^1}$ holds.

To this end, we use the normal exponential map

$$\exp^\perp: NM = \bigsqcup_{p \in M} T_p M^\perp \to X,$$

$$\nu \mapsto c_\nu(1).$$

Similarly to the exponential map itself, $\exp^\perp$ is smooth (where it is defined) and the following holds (proofs can be found in e.g. [**O'N83**], p.199):

LEMMA 2.3.4. *Let $M \subset X$ be a semi-Riemannian submanifold and $p \in M$. Then there exists a neighborhood $U_p \subset X$ of p in X and a neighborhood $\tilde{U}$ of $0_p \in NM$, such that $\exp^\perp: \tilde{U} \to U_p$ is a diffeomorphism.*

THEOREM 2.3.5. *Let $M \subset X$ be a semi-Riemannian submanifold. Then M has a normal neighborhood in X, i.e., a neighborhood that is the diffeomorphic image under $\exp^\perp$ of a neighborhood of $\{0_p\,|\, p \in M\} \subset NM$.*

Now, for $q \in U_p$ as in Lemma 2.3.4 we may set

$$x^1(q) := g\left(\left(\exp^\perp\right)^{-1}(q), \mathbf{n}\right)$$

and, for any fixed chart ϕ on $U_p \cap M$ (w.l.o.g. $U_p \cap M$ is a chart domain of M),

$$\psi'(q) := \left(\phi \circ \pi \circ \left(\exp^\perp\right)^{-1}\right)(q).$$

THEOREM 2.3.6. *The map $\psi: U_p \to \mathbb{R}^n$, $q \mapsto \psi(q) := \left(x^1(q), \psi'(q)\right)$ with x^1, ψ' as above is smooth and has the properties* (1) *to* (3)*. Furthermore we may shrink U_p such that the image of ψ is a rectangle, i.e., such that $\psi(U_p) = U \times V$ for some $U \subset \mathbb{R}$ with $0 \in U$ and $V \subset \mathbb{R}^{n-1}$.*

PROOF. Clearly, the components of ψ are compositions of smooth maps, thus ψ is itself $\mathcal{C}^\infty$. It is easily checked that the inverse of ψ is given by $\left(x^1, x^2, \ldots, x^n\right) \mapsto \exp^\perp\left(x^1 \mathbf{n}_{\phi^{-1}(x^2, \ldots, x^n)}\right)$, which is smooth, making ψ a diffeomorphism. For $q \in U_p \cap M$, we have $\exp^\perp(0_q) = q$ and thus, taking into account Lemma 2.3.4, $\psi'(q) = \phi\left(\pi\left(0_q\right)\right) = \phi(q)$, so $\psi'|_{U_p \cap M}$ is a chart in M. To show (2), let $\nu = |\nu|\,\mathbf{n} \in TM^\perp$ and $t \in \mathbb{R}$ such that $c_\nu(t) \in U_p$, then

$$x^1\left(c_\nu(t)\right) = x^1\left(c_{t\nu}(1)\right) = g\left(t\nu, \mathbf{n}\right) = t\,|\nu|.$$

Now let $q \in M \cap U_p$. Then, from $\psi\left(c_{\mathbf{n}_q}(t)\right) = (t, \phi(q))$ one obtains

$$\frac{\partial}{\partial x^1}\bigg|_q = (T_q \psi)^{-1}(e_1) = T_{(0, \phi(q))} \psi^{-1}(e_1) = \frac{d}{dt}\bigg|_0 \psi^{-1}(t, \phi(q)) = \frac{d}{dt}\bigg|_0 c_{\mathbf{n}_q}(t) = \mathbf{n}_q,$$

and so $\frac{\partial}{\partial x^1}\big|_M = \mathbf{n}$. $\qquad\square$

REMARK 2.3.7. This also shows that for any $p \in M$ the vector field $q \mapsto \frac{\partial}{\partial x^1}\big|_q$ provides a smooth extension of $\mathbf{n}$ to an open neighborhood U_p of p in X. Extending $\mathbf{n}$ in this way, one gets $\mathbf{n} \in \mathfrak{X}(U_p)$. In fact, this even gives $\mathbf{n} \in \mathfrak{X}(U)$ for some open $U \supset M$: For two charts (U_p, ψ) and $(U_{\tilde{p}}, \tilde{\psi})$ such that U_p and $U_{\tilde{p}}$ are contained in a normal neighborhood of M (in the sense of Thm. 2.3.5) one has

$$\left(\psi \circ \tilde{\psi}^{-1}\right)\left(x^1, \ldots, x^n\right) = \psi\left(\exp^{\perp}\left(x^1 \mathbf{n}_{\tilde{\phi}^{-1}(x^2, \ldots, x^n)}\right)\right) = \left(x^1, \left(\phi \circ \tilde{\phi}^{-1}\right)\left(x^2, \ldots, x^n\right)\right)$$

for all $\left(x^1, \ldots, x^n\right) \in \tilde{\psi}(U_p \cap U_{\tilde{p}})$. This implies

$$e_1 = D\left(\psi \circ \tilde{\psi}^{-1}\right)\left(\tilde{\psi}(q)\right)(e_1) = T_q \psi \left(T_q \tilde{\psi}\right)^{-1}(e_1)$$

for all $q \in U_p \cap U_{\tilde{p}}$ which proves that $\frac{\partial}{\partial x^1}\big|_q = \frac{\partial}{\partial \tilde{x}^1}\big|_q$.

Furthermore since $\psi(p) = (0, \phi(p))$ there exist $U \subset \mathbb{R}$ with $0 \in U$ and $V \subset \mathbb{R}^{n-1}$ with $\phi(p) \in V$ such that $U \times V \subset \psi(U_p)$. Now replacing U_p with $\psi^{-1}(U \times V)$ shows the last claim.

The next proposition shows another nice property of these new coordinates.

PROPOSITION 2.3.8. *Let $p \in M$ and (U_p, ψ) be canonical coordinates around p. Then*

(2.3.2) $$g_{1j} = \delta_{1j} \quad \text{and} \quad g^{1j} = \delta_{1j} \quad \text{on } U_p.$$

PROOF. Clearly (2.3.2) holds on M since $\partial_1 = \mathbf{n} \perp \partial_k \in TM$ for all $k \neq 1$. Now letting $q \in U_p \cap M$, we will show that the g_{1j} are constant along the geodesic $c_{\mathbf{n}_q}$. First note that

$$g_{11}(c_{\mathbf{n}_q}(t)) = g\left(\partial_1\big|_{c_{\mathbf{n}_q}(t)}, \partial_1\big|_{c_{\mathbf{n}_q}(t)}\right) = g\left(c'_{\mathbf{n}_q}(t), c'_{\mathbf{n}_q}(t)\right) = g(\mathbf{n}_q, \mathbf{n}_q) = 1,$$

so g_{11} is indeed constant along $c_{\mathbf{n}_q}$. As seen in the proof of Theorem 2.3.6 one has $\psi\left(c_{\mathbf{n}_q}(t)\right) = (t, \phi(q))$. Because this is a geodesic we have

(2.3.3) $$0 = \ddot{\psi}^i + \Gamma^i_{11}\dot{\psi}^1\dot{\psi}^1 = 0 + \Gamma^i_{11} = g^{im}\partial_1 g_{m1}$$

since by equation (2.1.5)

$$\Gamma^i_{11} = \frac{1}{2}g^{im}\left(\partial_1 g_{1m} + \partial_1 g_{1m} - \partial_m g_{11}\right) = g^{im}\partial_1 g_{m1}$$

and g_{11} is constant along $c_{\mathbf{n}_q}$. Now multiplying (2.3.3) from the left with g_{ki} and summing over i implies $\partial_1 g_{k1} = 0$, so g_{k1} is constant along $c_{\mathbf{n}_q}$ for all k.

Finally, let $q \in U_p$ be arbitrary, set $t := x^1(q)$ and choose $\tilde{q} \in U_p \cap M$ such that $\psi(\tilde{q}) = (0, \psi'(q))$ (note that $(0, \psi'(q))$ is in the image of ψ since the image is a rectangle), then $q = c_{\mathbf{n}_{\tilde{q}}}(t)$ because $\psi\left(c_{\mathbf{n}_{\tilde{q}}}(t)\right) = \left(t, \psi'\left(c_{t\mathbf{n}_{\tilde{q}}}(1)\right)\right) = \left(x^1(q), \psi'(q)\right)$ since

$$\psi'\left(c_{\mathbf{n}_{\tilde{q}}}(t)\right) = \left(\phi \circ \pi \circ \left(\exp^{\perp}\right)^{-1}\right)(c_{t\mathbf{n}_{\tilde{q}}}(1)) = \phi \circ \pi(t\mathbf{n}_{\tilde{q}}) = \phi(\tilde{q}) = \psi'(\tilde{q}) = \psi'(q)$$

by the definition of $\tilde{q}$. This finishes the proof of the first equation in (2.3.2). The second one follows immediately from the first by noting that

$$\delta_{1j} = g_{1m}g^{mj} = \delta_{1m}g^{mj} = g^{1j}. \qquad \square$$

Finally, we will give a nice example showing the usefulness of both multilayer distributions and canonical coordinates.

EXAMPLE 2.3.9 (A jump formula). Let us assume that the hypersurface M is given as the zero set of a smooth submersion $h: X \to \mathbb{R}$. We want to derive a formula for the exterior derivative of a smooth function on $X \setminus M$ with a jump discontinuity along M, that is we want to calculate $d(f \cdot H \circ h)$ where $H: \mathbb{R} \to \mathbb{R}$ denotes the Heaviside function and $f \in \mathcal{C}^{\infty}(X)$. On $X \setminus M$ one has obviously $d(f \cdot H \circ h)|_{X \setminus M} = (df \cdot H \circ h)|_{X \setminus M}$.

Now let U be a neighborhood around some $p \in M$ and ψ canonical coordinates on U such that $h \circ \psi^{-1}$ w.l.o.g. maps $(\mathbb{R}_+ \times \mathbb{R}^{n-1}) \cap \psi(U)$ onto $\mathbb{R}_+$ (otherwise replace $\mathbf{n}$ by $-\mathbf{n}$, i.e., change the orientation of the normal bundle) and $\phi \in \mathcal{D}(U, TU)$, then (using that $\operatorname{div}(\phi) \circ \psi^{-1} = \frac{1}{\sqrt{|\det g_{ij}|}} \partial_k(\phi^k \sqrt{|\det g_{ij}|})$ locally) one obtains by Prop. 2.1.4:

$$\langle d(f \cdot H \circ h), \phi \rangle = -\langle f \cdot H \circ h, C_1^1 \nabla \phi \rangle = -\int_{\psi(U)} (\operatorname{div}(\phi) f (H \circ h)) \circ \psi^{-1} \sqrt{|\det g_{ij}|} d^n x =$$

$$= -\int_{x^1 \geq 0} f \circ \psi^{-1} \partial_k \left(\phi^k \sqrt{|\det g_{ij}|} \right) d^n x = \int_{\mathbb{R}^{n-1}} \left(f \circ \psi^{-1} \phi^1 \sqrt{|\det g_{ij}|} \right) \Big|_{x^1=0} d^{n-1} x +$$

$$+ \int_{x^1 \geq 0} \partial_k(f \circ \psi^{-1}) \phi^k \sqrt{|\det g_{ij}|} d^n x = \langle S_M(f|_M \eta), \phi \rangle + \langle df \, H \circ h, \phi \rangle,$$

where $\eta \in \Gamma(M, T^* X|_M)$ denotes the canonical normal one-form given by $\eta(\mathbf{n}) = 1$ and $\eta(\mathbf{v}) = 0$ for $\mathbf{v} \in \Gamma(M, TM)$ and the last equality follows from Prop. 2.3.8. Noting that $S_M(f|_M \eta)|_{X \setminus M} = 0$ shows

$$(2.3.4) \qquad d(f \cdot H \circ h) = S_M(f|_M \eta) + H \circ h \cdot df.$$

2.3.2. Relations between the multilayers $\mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(1)$ and the pullbacks $\delta^{(\mathrm{k})} \circ x^1$. Now, using coordinates as described above, we want to derive relations between the multilayer distributions $\mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(1)$ and the pullbacks $\delta^{(\mathrm{k})} \circ x^1$, generalizing Theorem 2.2.3.

DEFINITION 2.3.10. Let U be an open neighborhood of M such that $\mathbf{n} \in \mathfrak{X}(U)$ (see Remark 2.3.7). The tensor field $W := \nabla \mathbf{n} \in \mathcal{T}_1^1(U)$ is called the *Weingarten map*. Its trace shall be denoted by $\chi := \operatorname{tr} W \in \mathcal{C}^\infty(U)$.

Choosing an open set V such that $M \subset V \subset \overline{V} \subset U$ (such a V exists because X is metrizable and thus normal) and a partition of unity subordinate to $\{U, X \setminus \overline{V}\}$ one may extend $\chi|_V$ to a smooth function $\tilde{\chi}$ on all of X. This extension can be used to define the product χT for $T \in \mathcal{D}'(X, T_q^p X)$ with $\operatorname{supp} T \subset M$ by $\langle \chi T, \phi \rangle := \langle \tilde{\chi} T, \phi \rangle = \langle T, \tilde{\chi} \phi \rangle$ which is independent of both the choice of V and the extension $\tilde{\chi}$ since $\langle T, \phi \rangle$ only depends on the values of ϕ in a neighborhood of $\operatorname{supp} T$. Similarly one can also define $\nabla_{\mathbf{n}} T := \nabla_{\tilde{\mathbf{n}}} T$ where $\tilde{\mathbf{n}} \in \mathfrak{X}(X)$ and $\tilde{\mathbf{n}}|_V = \mathbf{n}|_V$ for some open neighborhood V of M.

These definitions together with the observation that $\operatorname{supp} \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}) \subset M$ (since for any $\phi \in \mathcal{D}(X \setminus M)$ one has $\left\langle \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}), \phi \right\rangle = \langle T, \nabla_{\mathbf{n}}^k \phi|_M \rangle = \langle T, 0 \rangle = 0$) enable us to state (and prove) the following theorem.

THEOREM 2.3.11. *Let (X, g) be a semi-Riemannian manifold and $M \subset X$ a semi-Riemannian hypersurface that admits a normal vector field. Let (U, ψ) be an open neighborhood in X with canonical coordinates $\psi = (x^1, \ldots, x^n)$ and set $\chi_k := (\chi + \nabla_{\mathbf{n}})^k 1$. Then the following formulas hold*

$$(2.3.5) \qquad \forall T \in \mathcal{D}'\left(M, T_q^p X|_M\right) : \ \mathrm{L}_{\mathrm{M}}^{(\mathrm{k}+1)}(\mathrm{T}) = (\nabla_{\mathbf{n}} + \chi) \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T})$$

$$(2.3.6) \qquad \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(1) = (\partial_1 + \chi)^k (\delta \circ x^1) = \sum_{j=0}^{k} \binom{k}{j} \chi_{k-j} \left(\delta^{(j)} \circ x^1\right) \quad \text{in } \mathcal{D}'(U)$$

$$(2.3.7) \qquad \delta^{(\mathrm{k})} \circ x^1 = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} L_M^{(j)}\left(\chi_{k-j}|_M\right) \quad \text{in } \mathcal{D}'(U).$$

PROOF. To show equation (1.1.3), let $\phi \in \mathcal{D}\left(X, T_p^q X\right)$ and note that by Corollary 2.1.6

$$\left\langle \nabla_{\tilde{\mathbf{n}}} \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}), \phi \right\rangle = -\left\langle \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}), C_{p+1}^{q+1} \nabla\left(\phi \otimes \tilde{\mathbf{n}}\right) \right\rangle = -\left\langle \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}), C_{p+1}^{q+1}\left(\phi \otimes \nabla \tilde{\mathbf{n}} + \nabla \phi \otimes \tilde{\mathbf{n}}\right) \right\rangle =$$

$$= -\left\langle \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}), \phi \operatorname{tr} \nabla \tilde{\mathbf{n}} \right\rangle - \left\langle \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}), \nabla_{\tilde{\mathbf{n}}} \phi \right\rangle = -\left\langle \tilde{\chi} \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}), \phi \right\rangle + \left\langle \mathrm{L}_{\mathrm{M}}^{(\mathrm{k}+1)}(\mathrm{T}), \phi \right\rangle.$$

Since $\nabla_{\mathbf{n}} \phi = \mathbf{n}(\phi) = \partial_1 \phi$ for all $\phi \in \mathcal{D}(U)$ and thus by density for all $\phi \in \mathcal{D}'(U)$, the first equality in (2.3.6) now follows directly from Theorem 2.2.3 (with $h = x^1$, $\|dh\|^2 = g^{11} = 1$) and (1.1.3):

$$\mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(1) = \left(\partial_1 + \chi\right)^k S_M(1) = \left(\partial_1 + \chi\right)^k \left(\delta \circ x^1\right).$$

The second equality is shown by induction. Using the chain rule (2.1.7) we have

$$\mathrm{L}_{\mathrm{M}}^{(\mathrm{k}+1)}(1) = \left(\partial_1 + \chi\right) \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(1) = \left(\partial_1 + \chi\right) \sum_{j=0}^k \binom{k}{j} \chi_{k-j}\left(\delta^{(j)} \circ x^1\right) =$$

$$= \sum_{j=0}^k \binom{k}{j} \chi_{k+1-j}\left(\delta^{(j)} \circ x^1\right) + \sum_{j=0}^k \binom{k}{j} \chi_{k-j} \partial_1\left(\delta^{(j)} \circ x^1\right) = \chi_{k+1}\left(\delta \circ x^1\right) + \left(\delta^{(k+1)} \circ x^1\right) +$$

$$+ \sum_{j=1}^k \left(\binom{k}{j} + \binom{k}{j-1}\right) \chi_{k+1-j}\left(\delta^{(j)} \circ x^1\right) = \sum_{j=0}^{k+1} \binom{k+1}{j} \chi_{k+1-j}\left(\delta^{(j)} \circ x^1\right).$$

Now let $\phi \in \mathcal{D}(U)$. Then by Corollary 2.1.7

$$\left\langle \delta^{(k)} \circ x^1, \phi \right\rangle = \left\langle \partial_1^k\left(\delta \circ x^1\right), \phi \right\rangle = (-1)^k \left\langle S_M(1), \frac{1}{\sqrt{|\det g|}} \partial_1^k\left(\phi \sqrt{|\det g|}\right) \right\rangle =$$

$$= (-1)^k \sum_{j=0}^k \binom{k}{j} \left\langle 1, \left.\left(\partial_1^j \phi \chi_{k-j}\right)\right|_M \right\rangle = \left\langle \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} L_M^{(j)}\left(\left.\chi_{k-j}\right|_M\right), \phi \right\rangle,$$

where we have used $\partial_1^j \sqrt{|\det g|} = \chi_j \sqrt{|\det g|}$, which follows immediately from Lemma 2.1.5:

$$\frac{1}{\sqrt{|\det g|}} \partial_1 \sqrt{|\det g|} = \Gamma_{l1}^l = \operatorname{tr} \nabla \partial_1 = \chi$$

by induction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

2.3.3. Normal derivatives and multiplication with smooth functions.

It follows from equation (2.3.5) that $\nabla_{\mathbf{n}} \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}) = L_M^{(k+1)}(T) - \chi \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T})$, so if we want to better understand the normal derivatives of multilayers we should study products of the form $\psi \cdot \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T})$ for $\psi \in \mathcal{C}^\infty(X)$ and $T \in \mathcal{D}'(M)$. Considering the simple example $X = \mathbb{R}$ and $M = \{0\}$ it is immediately obvious that $\psi \cdot \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}) \neq \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}\left(\left.\psi\right|_M \mathrm{T}\right)$ for $k \geq 1$ as

$$\psi \cdot L_M^{(1)}(1) = \psi \cdot \delta' = \psi(0)\delta' - \psi'(0)\delta = L_M^{(1)}(\psi(0)) - S_M(\psi'(0)) \neq L_M^{(1)}(\psi(0)).$$

The next theorem is going to show that such a product can always be expressed as a sum of multilayers.

THEOREM 2.3.12. *Let* $\psi \in \mathcal{C}^\infty(X)$ *and* $T \in \mathcal{D}'(M)$. *Then*

$$(2.3.8) \qquad\qquad \psi \cdot \mathrm{L}_{\mathrm{M}}^{(\mathrm{k})}(\mathrm{T}) = \sum_{j=0}^k (-1)^j \binom{k}{j} L_M^{(k-j)}\left(\left.\nabla_{\mathbf{n}}^j \psi\right|_M T\right).$$

PROOF. Let $\phi \in \mathcal{D}(X)$, then

$$\left\langle \psi \cdot L_M^{(k)}(T), \phi \right\rangle = \left\langle L_M^{(k)}(T), \psi\phi \right\rangle = (-1)^k \left\langle T, \nabla_{\mathbf{n}}^k (\psi\phi)\big|_M \right\rangle =$$

$$= \left\langle T, \sum_{j=0}^k \binom{k}{j} (-1)^j \nabla_{\mathbf{n}}^j \psi\big|_M (-1)^{k-j} \nabla_{\mathbf{n}}^{k-j} \phi\big|_M \right\rangle =$$

$$= \sum_{j=0}^k (-1)^j \binom{k}{j} \left\langle \nabla_{\mathbf{n}}^j \psi\big|_M T, (-1)^{k-j} \nabla_{\mathbf{n}}^{k-j} \phi\big|_M \right\rangle = \sum_{j=0}^k (-1)^j \binom{k}{j} \left\langle L_M^{(k-j)} \left(\nabla_{\mathbf{n}}^j \psi\big|_M T \right), \phi \right\rangle.$$

Using this together with equation (2.3.5) we obtain the following formula for the normal derivative of $L_M^{(k)}(T)$:

$$(2.3.9) \quad \nabla_{\mathbf{n}} L_M^{(k)}(T) = L_M^{(k+1)}(T) - \chi L_M^{(k)}(T) = L_M^{(k+1)}(T) - \tilde{\chi} L_M^{(k)}(T) =$$

$$= L_M^{(k+1)}(T) - \sum_{j=0}^k (-1)^j \binom{k}{j} L_M^{(k-j)} \left(\nabla_{\mathbf{n}}^j \tilde{\chi}\big|_M T \right) = L_M^{(k+1)}(T) - \sum_{j=0}^k (-1)^j \binom{k}{j} L_M^{(k-j)} \left(\nabla_{\mathbf{n}}^j \chi\big|_M T \right).$$

$\square$

It is possible to derive somewhat similar expressions for the covariant derivative $\nabla L_M^{(k)}(T)$, the calculations for this can be found in [**Wag10**]. We will, however, stop here and instead turn our attention to the relationship between multilayers and distributions supported in hypersurfaces.

2.4. Multilayers and distributions supported in a hypersurface

It is clear from the definition that $\operatorname{supp} L_M^{(k)}(T) \subset M$ for all $T \in \mathcal{D}'(M)$. Our goal in this section is going to be to prove that every $T \in \mathcal{D}'(X)$ with $\operatorname{supp} T \subset M$ can be written as a sum of multilayers. To do this we will make use of an analogous result for distributions on $\mathbb{R}^n$ which is in some sense a generalization of the well-known fact that every distribution with support in a single point can be written as a sum of derivatives of delta distributions.

THEOREM 2.4.1. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ with $\operatorname{supp} T \subset \{0\} \times \mathbb{R}^k$. Then there exist distributions $T_{\mathbf{q}} \in \mathcal{D}'(\mathbb{R}^k)$ such that*

$$T = \sum_{\mathbf{q} \in \mathbb{N}_0^{n-k}} \partial_1^{q_1} \dots \partial_{n-k}^{q_{n-k}} \overline{T}_{\mathbf{q}},$$

where $\left\langle \overline{T}_q, \phi\left(x^1, x^2, \dots, x^n\right) \right\rangle := \left\langle T_q, \phi\left(0, \dots, 0, x^{n-k+1}, \dots, x^n\right) \right\rangle$ is called the extension of T_q to $\mathbb{R}^n$ and the sum is locally finite. Furthermore the supports of the $T_{\mathbf{q}}$ are contained in the support of T, they depend continuously on T and are unique.

A proof of this can be found in [**Sch66**], Thm. XXVI. Now we will use Proposition 1.2.3 to prove a version of Thm. 2.4.1 concerning distributions on semi-Riemannian manifolds with support in a hypersurface.

THEOREM 2.4.2. *Every distribution $T \in \mathcal{D}'(X)$ with support contained in a closed, semi-Riemannian hypersurface $M \subset X$ with unit vector field $\mathbf{n}$ admits a unique decomposition as a locally finite sum of normal derivatives of extensions to X of distributions defined on M:*

$$(2.4.1) \qquad T = \sum_{k=0}^{\infty} \nabla_{\mathbf{n}}^k \overline{T}_k; \quad T_k \in \mathcal{D}'(M),$$

where $\overline{T}_k$ is defined locally via $\left(\overline{T}_k\right)^\alpha := \overline{\left(T_k\right)^\alpha}$ where the index α is used to denote the local components of a distribution with respect to a chart (U_α, ψ_α).

PROOF. We show this locally using Theorem 2.4.1 and Proposition 1.2.3. Covering X with chart domains U_α such that we have canonical coordinates on U_α whenever $U_\alpha \cap M \neq \emptyset$ and choosing a partition of unity χ_α subordinate to these chart domains we see that T is equal to the (locally finite) sum of the $\chi_\alpha T \in \mathcal{E}'(U_\alpha)$ thus it suffices to show (2.4.1) for $T \in \mathcal{E}'(U_\alpha)$ with $\operatorname{supp} T \subset M \cap U_\alpha$. If $U_\alpha \cap M = \emptyset$ this is trivial since $T = 0$. If $U_\alpha \cap M \neq \emptyset$, there exist $T_k^\alpha \in \mathcal{E}'(\psi_\alpha(U_\alpha) \cap (\{0\} \times \mathbb{R}^{n-1}))$ such that $T^\alpha = \sum_k \partial_1^k \overline{T_k^\alpha}$. Defining $T_k \in \mathcal{E}'(M)$ to be the unique distribution satisfying $\langle T_k, \phi \rangle = \langle T_k^\alpha, \phi \circ \psi_\alpha^{-1} \rangle$ for $\phi \in \mathcal{D}(U_\alpha \cap M)$ and $T_k|_{M \setminus \psi_\alpha^{-1}(\operatorname{supp} T_k^\alpha)} = 0$ (this is indeed well-defined because $\phi \circ \psi_\alpha^{-1} \in \mathcal{D}(\psi_\alpha(U_\alpha) \cap (\{0\} \times \mathbb{R}^{n-1}) \setminus \operatorname{supp} T_k^\alpha)$ for $\phi \in \mathcal{D}((M \cap U_\alpha) \setminus \psi_\alpha^{-1}(\operatorname{supp} T_k^\alpha))$) we obtain

$$\left(\sum_k \nabla_{\mathbf{n}}^k \overline{T}_k \right)^\alpha = \sum_k \left(\nabla_{\mathbf{n}}^k \overline{T}_k \right)^\alpha = \sum_k \partial_1^k \left(\overline{T}_k \right)^\alpha = \sum_k \partial_1^k \overline{T_k^\alpha} = T^\alpha$$

and thus $T = \sum_k \nabla_{\mathbf{n}}^k \overline{T}_k$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This is not yet exactly what we want to show since in general $\mathrm{L}_M^{(k)}(\mathrm{T}) \neq \nabla_{\mathbf{n}}^k \overline{T}$. First we note that the extension $\overline{T}$ of $T \in \mathcal{D}'(M)$ itself coincides with $S_M(T)$ since we have locally (in normal coordinates (ψ_α, U_α)) that

$$\left\langle \overline{T}^\alpha, \phi(x^1, x^2, \ldots, x^n) \right\rangle = \left\langle T^\alpha, \phi(0, x^2, \ldots, x^n) \right\rangle = \left\langle T^\alpha, \phi|_{\psi_\alpha(M \cap U)} \right\rangle = \langle S_M(T)^\alpha, \phi \rangle$$

for all $\phi \in \mathcal{D}(\psi_\alpha(U_\alpha))$. However, the higher derivatives $\nabla_{\mathbf{n}}^k \overline{T} = \nabla_{\mathbf{n}}^k S_M(T)$ do no longer coincide with the multilayers $\mathrm{L}_M^{(k)}(\mathrm{T})$ as is obvious from the formulas in the previous chapters. But, using (2.3.9), one has

$$\nabla_{\mathbf{n}}^k S_M(T) = \nabla_{\mathbf{n}}^{k-1} \left(L_M^{(1)}(T) - S_M(\chi|_M T) \right) =$$

$$= \nabla_{\mathbf{n}}^{k-2} \left(L_M^{(2)}(T) - 2 L_M^{(1)}(\chi|_M T) + S_M\left((\nabla_{\mathbf{n}}\chi + \chi^2)|_M T \right) \right) = \ldots = \mathrm{L}_M^{(k)}(\mathrm{T}) + \sum_{j=0}^{k-1} L_M^{(j)}(T_{(j)})$$

where the $T_{(j)}$ are distributions on M given by the product of T with some linear combination of products of χ and its normal derivatives up to order $k-1$. This now shows that every decomposition of the form (2.4.1) gives rise to a decomposition $T = \sum_{k=0}^\infty \mathrm{L}_M^{(k)}(\mathrm{T}_k)$, albeit with different (but still unique) T_k. Thus we have shown the following:

COROLLARY 2.4.3. *Every $T \in \mathcal{D}'(X)$ with $\operatorname{supp} T \subset M$ can uniquely be written as a locally finite sum of multilayers, i.e.*

$$(2.4.2) \qquad\qquad\qquad\qquad T = \sum_{k=0}^\infty \mathrm{L}_M^{(k)}(\mathrm{T}_k)$$

with $T_k \in \mathcal{D}'(M)$ for all k.

CHAPTER 3

Distributional geometry

In the previous part we studied singular objects and their interaction with the (smooth) geometry of our manifold, in particular describing the special properties of distributions supported in a (closed, semi-Riemannian) hypersurface (with a normal unit vector field) and deriving formulas for the covariant derivative of tensor distributions (see equation (2.1.4)) and the exterior derivative of a smooth function suffering a jump discontinuity along a hypersurface (Example 2.3.9).

Now we want to study what happens if the geometry (i.e., the metric tensor, connection,...) of our manifold is itself distributional. This will offer some generalizations, but also some restrictions of the previous results - for example the concept of multilayer distributions is unavailable because their definition relies on the trivialization of the volume bundle using the smooth metric and the existence of a normal unit vector field.

The main goal of this part is to introduce a suitable concept of a distributional metric and derive necessary regularity conditions for such a metric to give rise to a (distributional) Levi-Civita connection that allows the definition of a Riemann curvature tensor distribution. Along the way we will derive some jump formulas concerning discontinuities along a hypersurface which will in the end allow us to deduce some regularity conditions for a vacuum spacetime, i.e., a distributional metric satisfying **Ric** $= 0$, suffering such a jump discontinuity. We will mainly follow [**LM07**].

At first, however, we will briefly summarize some important results concerning local Sobolev spaces on manifolds that will be needed later on.

3.1. Local Sobolev spaces on manifolds

The local Sobolev spaces $W_{\mathrm{loc}}^{k,p}(X)$ (for $k \in \mathbb{N}$, $1 \leq p < \infty$) on X are defined as the subspaces of $\mathcal{D}'(X)$ containing all distributions $T \in \mathcal{D}'(X)$ whose local representations $T^\alpha \in \mathcal{D}'(\psi_\alpha(U_\alpha))$ belong to $W_{\mathrm{loc}}^{k,p}(\psi_\alpha(U_\alpha))$ for an atlas $\{(U_\alpha, \psi_\alpha)\}_\alpha$, i.e., that satisfy $\partial^\beta T^\alpha \in L^p(\Omega)$ for all β with $|\beta| \leq k$ and $\Omega \subset \psi_\alpha(U_\alpha)$ relatively compact or equivalently $\phi T^\alpha \in W_{\mathrm{loc}}^{k,p}(\mathbb{R}^n)$ for all $\phi \in \mathcal{D}(\psi_\alpha(U_\alpha))$. Note that this is well-defined because for bounded open subsets U, V of $\mathbb{R}^n$ the pullback $\phi^* f$ of a function $f \in W^{k,p}(V)$ under a diffeomorphism $\phi : U \to V$ is in $W^{k,p}(U)$ (see [**AF03**], Thm. 3.41), so $T^\alpha \in W_{\mathrm{loc}}^{k,p}(\psi_\alpha(U_\alpha \cap U_\beta))$ if and only if $T^\beta \in W_{\mathrm{loc}}^{k,p}(\psi_\beta(U_\alpha \cap U_\beta))$.

The topology of $W_{\mathrm{loc}}^{k,p}(X)$ can be described by the (countable) family of seminorms given by

$$(3.1.1) \qquad\qquad p_{\beta,\alpha,\Omega_m^\alpha}(T) := \left\| \partial^\beta T^\alpha \right\|_{L^p(\Omega_m^\alpha)}$$

where $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq k$, the (U_α, ψ_α) form a (countable) atlas for X and the Ω_m^α are an exhaustion of $\psi_\alpha(U_\alpha)$ by relatively compact sets. The topology induced by these seminorms does not depend on the choice of the atlas. By countability of the basis of seminorms the $W_{\mathrm{loc}}^{k,p}(X)$ are metrizable and if X is compact, $W_{\mathrm{loc}}^{k,p}(X)$ is normable (because there exists a finite atlas for X), however in general there exists no canonical norm, i.e. different choices of the atlas lead to different (but equivalent) norms.

As for distributions, many of the results concerning (local) Sobolev spaces on open subsets of $\mathbb{R}^n$ carry over to the corresponding function spaces on manifolds, for instance $W_{\mathrm{loc}}^{k,p}(X)$ is a Frechet

space (in particular complete), $H_{\text{loc}}^k(X) := W_{\text{loc}}^{k,2}(X)$ is hilbertizable (for compact X), $\mathcal{C}^\infty(X)$ and $\mathcal{D}(X)$ are dense in $W_{\text{loc}}^{k,p}(X)$ and so on. For proofs of the first two claims see [**CP82**], Chapter 2, Thm 2.19. (actually they only deal with the spaces $H_{\text{loc}}^k(X)$ but the proofs for general $p \in \mathbb{N}$ are completely analogous); density of $\mathcal{C}^\infty(X)$ follows easily from density of $\mathcal{D}(\Omega)$ in $W_{\text{loc}}^{k,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ by approximating locally and gluing those approximations together using a partition of unity, density of $\mathcal{D}(X)$ then follows by using appropriate cut-off functions over an exhaustion of compact sets of X.

It is also worth noting that $W_{\text{loc}}^{k,p}(X) \subset W_{\text{loc}}^{k,1}(X)$ (because $L_{\text{loc}}^p(\Omega) \subset L_{\text{loc}}^1(\Omega)$, which is easily seen using Hölder's inequality).

Another construction that will be of interest to us later are local Sobolev spaces on manifolds with boundary and the corresponding (generalizations of the) trace theorems. As a quick reminder we will now state the exact version of the trace theorem we want to generalize to manifolds.

THEOREM 3.1.1 (Trace theorem). *Let $\mathbb{R}_+^n = \left\{ x \in \mathbb{R}^n : x^1 \geq 0 \right\}$ and $k \geq 1$ then the restriction operator*

$$\gamma : \mathcal{C}\left(\mathbb{R}_+^n\right) \cap W^{k,p}\left(\mathbb{R}_+^n\right) \to \mathcal{C}\left(\partial \mathbb{R}_+^n\right)$$
$$\gamma(u) := u|_{\partial u}$$

can be extended uniquely to a bounded operator

$$\gamma : W^{k,p}\left(\mathbb{R}_+^n\right) \to W^{k-1,p}\left(\partial \mathbb{R}_+^n\right).$$

PROOF. This follows immediately from collecting some of the results in [**AF03**]. First we note that by Theorem 5.21 there exists a bounded extension operator $E : W^{k,p}\left(\mathbb{R}_+^n\right) \to W^{k,p}(\mathbb{R}^n)$. Next, by the Sobolev embedding theorem 4.12 there exists a trace $\iota : W^{k,p}(\mathbb{R}^n) \to W^{k-1,q}\left(\partial \mathbb{R}_+^n\right)$ for all $\infty > q \geq p$ if $p \geq n$. If $p < n$ then ι exists for $p \leq q \leq \frac{(n-1)p}{n-p}$ since either $n - p < \dim \partial \mathbb{R}_+^n = n - 1$, i.e. $p > 1$, or $p = 1$. All in all setting $\gamma = \iota \circ E : W^{k,p}\left(\mathbb{R}_+^n\right) \to W^{k-1,p}\left(\partial \mathbb{R}_+^n\right)$ proves the theorem. $\square$

REMARK 3.1.2. Note that the trace theorem remains true for $\Omega = B_r((0, x')) \cap \mathbb{R}_+^n$ ($r > 0$ and $x' \in \mathbb{R}^{n-1}$) instead of $\mathbb{R}_+^n$ and $\partial \mathbb{R}_+^n \cap B_r(x')$ instead of $\partial \mathbb{R}_+^n$ since $\mathbb{R}_+^n \cong \Omega$ and $\partial \mathbb{R}_+^n \cap B_r((0, x')) \cong \partial \mathbb{R}_+^n$.

In the following the letter $\overline{X}$ will be used to denote a manifold with boundary and X itself for $\overline{X} \setminus \partial \overline{X}$.

DEFINITION 3.1.3 (Local Sobolev spaces on $\overline{X}$). A distribution $T \in \mathcal{D}(X)$ is said to be in $W_{\text{loc}}^{k,p}\left(\overline{X}\right)$ if $T \in W_{\text{loc}}^{k,p}(X)$ and for every chart (U_α, ψ_α) of $\overline{X}$ at the boundary and $\phi \in \mathcal{D}(\psi_\alpha(U_\alpha))$ one has $\phi T^\alpha \in W^{k,p}\left(\mathbb{R}_+^n\right)$. The topology is again given by seminorms of the form (3.1.1).

Now we want to generalize Theorem 3.1.1 as illustrated for the spaces $H_{\text{loc}}^k\left(\overline{X}\right)$ in [**CP82**], Chapter 2, Cor. 4.5.

THEOREM 3.1.4 (Trace at the boundary). *The operator $\gamma : \mathcal{C}^\infty\left(\overline{X}\right) \to \mathcal{C}^\infty(\partial X)$ of restriction to ∂X uniquely extends to a continuous linear operator γ from $W_{\text{loc}}^{k,p}\left(\overline{X}\right)$ to $W_{\text{loc}}^{k-1,p}(\partial X)$.*

PROOF. Let (U_α, ψ_α) be an atlas for $\overline{X}$ and $u \in \mathcal{C}^\infty\left(\overline{X}\right)$. Then there is an atlas for ∂X consisting of charts of the form $\left(U_\alpha \cap \partial X, \psi_\alpha|_{U_\alpha \cap \partial X}\right)$. W.l.o.g. $\psi_\alpha(U_\alpha) = B_r((0, x')) \cap \mathbb{R}_+^n$ for some $r > 0$ and $x' \in \mathbb{R}^{n-1}$. Let $v = u|_{\partial X} \in \mathcal{C}^\infty(\partial X)$ then

$$v^\alpha = v \circ \left(\psi_\alpha|_{U_\alpha \cap \partial X}\right)^{-1} = u \circ \left(\psi_\alpha|_{U_\alpha \cap \partial X}\right)^{-1} = u^\alpha|_{\psi_\alpha(U_\alpha \cap \partial X)} = u^\alpha|_{\psi_\alpha(U_\alpha) \cap \partial \mathbb{R}_+^n}$$

and thus $\|v^\alpha\|_{W^{k-1,p}\left(\Omega_m^\alpha \cap \partial \mathbb{R}_+^n\right)} \le \|\tilde{u^\alpha}\|_{W^{k,p}(\Omega_m^\alpha)}$ for all charts ψ_α at the boundary for $\Omega_m^\alpha = B_{r-\frac{1}{m}}((0,x')) \cap \mathbb{R}_+^n$ by Rem. 3.1.2. This shows that $\gamma : \mathcal{C}^\infty\left(\overline{X}\right) \to \mathcal{C}^\infty\left(\partial X\right)$ is continuous as an operator from $W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$ to $W_{\mathrm{loc}}^{k-1,p}\left(\partial X\right)$, so it can be extended to all of $W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$ by density of $\mathcal{C}^\infty\left(\overline{X}\right)$ (which will be shown in Prop. 3.1.5 below). $\qquad\square$

PROPOSITION 3.1.5. *The space of smooth functions $\mathcal{C}^\infty\left(\overline{X}\right)$ is dense in the local Sobolev spaces $W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$.*

PROOF. We follow the outline given in [**CP82**], p. 111. First we note that every manifold $\overline{X}$ with boundary can be embedded in the so called double of $\overline{X}$, which is a smooth manifold without boundary (for details see e.g. [**Mun66**], Def. 5.10.) and which we will denote by M.

Next we want to construct a continuous extension operator $E' : W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right) \to W_{\mathrm{loc}}^{k,p}(M)$: Choosing a partition of unity χ_α subordinate to an atlas (U_α, ψ_α) of M we may define an extension of $u \in W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$ by setting $E'u|_{U_\alpha} = u|_{U_\alpha}$ if $U_\alpha \subset X$, $E'u|_{U_\alpha} = 0$ if $U_\alpha \subset M \setminus \overline{X}$ and finally if $U_\alpha \cap \partial X \ne \emptyset$, we set $(E'u)^\alpha = E(\phi\, u^\alpha)$, where $\phi \in \mathcal{D}\left(\psi_\alpha\left(U_\alpha \cap \overline{X}\right)\right)$ is chosen such that $\phi \equiv 1$ on $\mathrm{supp}\,\chi_\alpha \cap \overline{X}$, $\phi \equiv 0$ on $\left(\psi_\alpha\left(U_\alpha \cap \overline{X}\right)_{m_\alpha}\right)^c$ for a suitable m_α and E is the extension operator from the proof of Thm. 3.1.1. Clearly continuity of E implies continuity of E' since

$$\left\|\partial^\beta E\left(\phi u^\alpha\right)\right\|_{L^p(\Omega_m^\alpha)} = \left\|\partial^\beta E\left(\phi u^\alpha\right)\right\|_{L^p(\mathbb{R}^n)} \le C \left\|\partial^\beta\left(\phi u^\alpha\right)\right\|_{L^p\left(\mathbb{R}_+^n\right)} =$$
$$= C \left\|\partial^\beta\left(\phi u^\alpha\right)\right\|_{L^p\left(\Omega_m^\alpha \cap \mathbb{R}_+^n\right)} \le C_\phi \left\|\partial^\beta\left(u^\alpha\right)\right\|_{L^p\left(\Omega_m^\alpha \cap \mathbb{R}_+^n\right)}.$$

Now let $u \in W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$ then there exist $u_j \in \mathcal{C}^\infty(M)$ such that $u_j \to Eu$ in $W_{\mathrm{loc}}^{k,p}(M)$, but then $u_j|_{\bar{X}} \in \mathcal{C}^\infty\left(\overline{X}\right)$ and $u_j|_{\overline{X}} \to u$ in $W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$. $\qquad\square$

There are of course more general theorems concerning traces (e.g. using fractional Sobolev spaces, see [**AF03**], Thm. 7.39) that can be generalized to manifolds in a completely analogous way to the one shown in Theorem 3.1.4, but the theorem stated here and its generalization to (p,q)-tensor distributions discussed below is sufficient for our purpose.

REMARK 3.1.6. For $u \in \mathcal{C}^\infty\left(\overline{X}\right)$ and $v \in W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$ one has $\gamma(uv) = u|_{\partial X}\,\gamma(v)$ since approximating v in $W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$ by a sequence v_n of smooth functions gives $\gamma(uv) = \lim_{n\to\infty}(\gamma(uv_n)) = \lim_{n\to\infty} uv_n|_{\partial X} = u|_{\partial X}\,\gamma(v)$ (note that $uv_n \to uv$ in $W_{\mathrm{loc}}^{k,p}\left(\overline{X}\right)$ since $u \in L_{\mathrm{loc}}^\infty\left(\overline{X}\right)$ and thus $\gamma(uv_n) \to uv$ in $W_{\mathrm{loc}}^{k-1,p}\left(\partial X\right)$).

So far we have only looked at Sobolev *functions* on manifolds. Now we want to generalize our observations to Sobolev *tensor fields*. Luckily this does not pose any difficulties. Using the identification $\mathcal{D}'\left(X, T_s^r X\right) \cong L_{\mathcal{C}^\infty(X)}\left(\left(\mathcal{T}_1^0\right)^r \otimes \left(\mathcal{T}_0^1\right)^s; \mathcal{D}'(X)\right)$ provided in Theorem 1.2.4 we say that $T \in \mathcal{D}'\left(X, T_s^r X\right)$ is in $W_{\mathrm{loc}}^{k,p}\left(X, T_s^r X\right)$ if and only if $T\left(\omega_1, \ldots, \omega_r, \mathbf{u}_1, \ldots, \mathbf{u}_s\right) \in W_{\mathrm{loc}}^{k,p}(X)$ for all $\omega_i \in \mathcal{T}_1^0(X)$ and $\mathbf{u}_i \in \mathfrak{X}(X)$. The topology is induced by the following notion of convergence: A net T_ι converges to T in $W_{\mathrm{loc}}^{k,p}\left(X, T_s^r X\right)$ if $T_\iota\left(\omega_1, \ldots, \omega_r, \mathbf{u}_1, \ldots, \mathbf{u}_s\right) \to T\left(\omega_1, \ldots, \omega_r, \mathbf{u}_1, \ldots, \mathbf{u}_s\right)$ in $W_{\mathrm{loc}}^{k,p}(X)$ for all $\omega_i \in \mathcal{T}_1^0(X)$ and $\mathbf{u}_i \in \mathfrak{X}(X)$. This is of course equivalent to the local description: $T \in \mathcal{D}'\left(X, T_s^r X\right)$ is in $W_{\mathrm{loc}}^{k,p}\left(X, T_s^r X\right)$ if and only if all the components $(T^\alpha)_i$ of the (vector bundle) chart representations $T^\alpha \in \mathcal{D}'\left(\psi_\alpha(U_\alpha), \mathbb{R}^{n^{r+s}}\right)$ are in $W_{\mathrm{loc}}^{k,p}(\psi_\alpha(U_\alpha))$ and of course one may also characterize convergence locally.

PROPOSITION 3.1.7. *Let $\overline{X}$ be a manifold with boundary. Then there exists a continuous trace operator $\gamma : W_{\mathrm{loc}}^{k,p}\left(\overline{X}, T_s^r \overline{X}\right) \to W_{\mathrm{loc}}^{k-1,p}\left(\partial X, T_s^r \overline{X}\big|_{\partial X}\right)$.*

PROOF. The goal is to use Thm. 3.1.4 to define

$$(3.1.2) \qquad \gamma(T)\left(\omega_1, \ldots, \omega_r, \mathbf{u}_1, \ldots, \mathbf{u}_s\right) := \gamma\left(T\left(\tilde{\omega}_1, \ldots, \tilde{\omega}_r, \tilde{\mathbf{u}}_1, \ldots, \tilde{\mathbf{u}}_s\right)\right) \in L_{\mathrm{loc}}^1(\partial X)$$

for $\omega_i \in \Gamma\left(\partial X, T_1^0 \overline{X}\big|_{\partial X}\right)$ and $\mathbf{u}_i \in \Gamma\left(\partial X, T_0^1 \overline{X}\big|_{\partial X}\right)$ with smooth extensions $\tilde{\omega}_i \in \mathcal{T}_1^0\left(\overline{X}\right)$ and $\tilde{\mathbf{u}}_i \in \mathfrak{X}\left(\overline{X}\right)$. First we show that there exist appropriate extensions $\tilde{\omega}_i \in \mathcal{T}_1^0\left(\overline{X}\right)$ and $\tilde{\mathbf{u}}_i \in \mathfrak{X}\left(\overline{X}\right)$: Let $p \in \partial X$. By setting $\tilde{f} = \chi\, f \circ \psi|_{U \cap \partial X}^{-1} \circ pr_{\mathbb{R}^{n-1}} \circ \psi \in \mathcal{C}^\infty\left(\overline{X}\right)$ for a chart (U, ψ) of $\overline{X}$ around p (where χ is an appropriate cut-off) we see that we can locally extend smooth functions $f \in \mathcal{C}^\infty\left(\partial X\right)$. Now $\mathbf{u} \in \Gamma\left(\partial X, T_0^1 \overline{X}\big|_{\partial X}\right)$ is locally given by $\mathbf{u} = \sum_{i=2}^n u^i \partial_i$ in adapted coordinates and setting $\tilde{\mathbf{u}} = \sum_{i=2}^n \tilde{u}^i \partial_i$ gives the desired extension around p. Of course the same argument works for $\omega \in \Gamma\left(\partial X, T_1^0 \overline{X}\big|_{\partial X}\right)$.

The operator given by (3.1.2) is then obviously continuous but we have to show that it is well-defined, i.e., independent of the choice of the extensions. W.l.o.g. let $T \in W_{\mathrm{loc}}^{k,p}\left(\overline{X}, T\overline{X}\right)$. By linearity it suffices to show that $\gamma\left(T\left(\tilde{\omega}\right)\right) = 0$ if $\tilde{\omega}|_{\partial X} = 0$, which we will do locally. Choose some chart $\psi = \left(x^1, \ldots, x^n\right)$ at the boundary then $\tilde{\omega} = \tilde{\omega}_j\, dx^j$ and $T\left(\tilde{\omega}\right) = \tilde{\omega}_j\, T\left(dx^j\right)$ and thus Remark 3.1.6 gives $\gamma\left(T\left(\tilde{\omega}\right)\right) = \tilde{\omega}_j|_{\partial X}\, \gamma\left(T\left(dx^j\right)\right) = 0$. $\qquad\qquad\square$

We are also going to need some results concerning whether the product of two functions in $L_{\mathrm{loc}}^\infty(X) \cap W_{\mathrm{loc}}^{k,p}(X)$ is again in $L_{\mathrm{loc}}^\infty(X) \cap W_{\mathrm{loc}}^{k,p}(X)$. For this it is obviously sufficient to look at the spaces $L_{\mathrm{loc}}^\infty(\Omega) \cap W_{\mathrm{loc}}^{k,p}(\Omega)$ (or $L^\infty(\Omega) \cap W^{k,p}(\Omega)$) for open balls $\Omega = B_r(x) \subset \mathbb{R}^n$ ($x \in \mathbb{R}^n$, $r > 0$). We will equip $A^{k,p}(\Omega) := L^\infty(\Omega) \cap W^{k,p}(\Omega)$ with the norm $\|f\|_{A^{k,p}} := \|f\|_\infty + \|f\|_{W^{k,p}}$ such that both embeddings $A^{k,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $A^{k,p}(\Omega) \hookrightarrow W^{k,p}(\Omega)$ are continuous. Then the following holds.

PROPOSITION 3.1.8. *Let $p \geq \frac{n}{k}$ and $\Omega = B_r(x) \subset \mathbb{R}^n$ an open ball. Then there exists a constant $K > 0$ such that $\left(A^{k,p}(\Omega), K\, \|.\|_{A^{k,p}}\right)$ is a Banach algebra with unit ($e \equiv 1 \in A^{k,p}(\Omega)$) and if either $|f| \geq c$ for some constant $c > 0$ almost everywhere then f is invertible in $A^{k,p}(\Omega)$.*

PROOF. Clearly $A^{k,p}(\Omega)$ is a Banach space, it remains to show the algebra property. For $p > \frac{n}{k}$ or $p = 1$ and $n = k$ there exists a $K > 0$ such that $\left(W^{k,p}(\Omega), K\, \|.\|_{W^{k,p}}\right)$ is a Banach algebra (see Thm. 4.39 in [**AF03**], Ω satisfies the cone condition because balls satisfy the uniform $\mathcal{C}^\infty$-regularity condition 4.10) and thus

$$\|fg\|_{A^{k,p}} = \|fg\|_\infty + \|fg\|_{W^{k,p}} \leq \|f\|_\infty \|g\|_\infty + K\, \|f\|_{W^{k,p}} \|g\|_{W^{k,p}} \leq (1 + K)\, \|f\|_{A^{k,p}} \|g\|_{A^{k,p}}$$

hence $\left(A^{k,p}(\Omega), (K+1)\, \|.\|_{A^{k,p}}\right)$ is a Banach algebra.

Now let $p = \frac{n}{k}$ and $p \geq 2$. First we are going to show that the Leibniz rule

$$\tag{3.1.3} \partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f\, \partial^{\alpha-\beta} g$$

remains valid for $f, g \in A^{k,p}(\Omega)$ and $|\alpha| \leq k$. Let f_ε be a sequence of smooth functions converging to f in the $W^{k,p}$-norm (such functions exist by density) then (3.1.3) holds for $f_\varepsilon g$. We have $f_\varepsilon g \to fg$ in $L^p(\Omega)$ (since $\|f_\varepsilon g - fg\|_p \leq \|g\|_\infty \|f_\varepsilon - f\|_p$) and thus in $\mathcal{D}'(\Omega)$ and $\partial^\alpha(f_\varepsilon g) \to \partial^\alpha(fg)$ in $\mathcal{D}'(\Omega)$. Since Ω is bounded and $p \geq 2$ there exists a continuous embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$ by Hölder's inequality implying $\left\|\partial^\beta f_\varepsilon \partial^{\alpha-\beta} g\right\|_1 \leq \left\|\partial^\beta f_\varepsilon\right\|_2 \left\|\partial^{\alpha-\beta} g\right\|_2 \leq C\left\|\partial^\beta f_\varepsilon\right\|_p \left\|\partial^{\alpha-\beta} g\right\|_p$ and so $\partial^\beta f_\varepsilon \partial^{\alpha-\beta} g \to \partial^\beta f\, \partial^{\alpha-\beta} g$ in $L^1(\Omega)$ (and hence in $\mathcal{D}'(\Omega)$) showing that (3.1.3) holds in $\mathcal{D}'(\Omega)$ for $f, g \in A^{k,p}(\Omega)$.

This reduces our task to proving that $\left\|\partial^\beta f\, \partial^{\alpha-\beta} g\right\|_p \leq K\, \|f\|_{A^{k,p}} \|g\|_{A^{k,p}}$ for all $|\alpha| \leq k$ and $\beta \leq \alpha$. For $\beta = 0$ we have $\|f\, \partial^\alpha g\|_p \leq \|f\|_\infty \|\partial^\alpha g\| \leq \|f\|_{A^{k,p}} \|g\|_{A^{k,p}}$. The same holds for $\alpha - \beta = 0$ so we may assume $k - 1 \geq |\beta| \geq 1$ and $k - 1 \geq |\alpha - \beta| \geq 1$ implying $n > (k - |\beta|)\, p$ and $n > (k - |\alpha - \beta|)\, p$. Since

$$\frac{n - (k - |\beta|)\, p}{n} + \frac{n - (k - |\alpha - \beta|)\, p}{n} = \frac{|\alpha|\, p}{n} \leq \frac{kp}{n} = 1$$

this shows that there exist $1 \leq r, r' < \infty$ with $\frac{1}{r} + \frac{1}{r'} = 1$ such that

$$p \leq rp \leq \frac{np}{n - (k - |\beta|)\, p} \quad \text{and} \quad p \leq r'p \leq \frac{np}{n - (k - |\alpha - \beta|)\, p}.$$

Now using Hölder's inequality gives

$$\int_\Omega \left| \partial^\beta f(x)\, \partial^{\alpha - \beta} g(x) \right|^p dx \leq \left(\int_\Omega \left| \partial^\beta f(x) \right|^{rp} dx \right)^{\frac{1}{r}} \left(\int_\Omega \left| \partial^{\alpha - \beta} g(x) \right|^{r'p} dx \right)^{\frac{1}{r'}} =$$
$$= \left\| \partial^\beta f \right\|_{rp}^p \left\| \partial^{\alpha - \beta} g \right\|_{r'p}^p$$

if $\partial^\beta f \in L^{rp}(\Omega)$ and $\partial^{\alpha - \beta} g \in L^{r'p}(\Omega)$. By the Sobolev embedding Theorem (Thm. 4.12, C in [**AF03**]) $W^{k', p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \leq q \leq \frac{np}{n - k'p}$ if $k'p < n$ and $n - k'p < n$. Applying this to q equal to rp (or $r'p$) and k' equal to $k - |\beta|$ (or $k - |\alpha - \beta|$) gives

$$\left\| \partial^\beta f \right\|_{rp}^p \left\| \partial^{\alpha - \beta} g \right\|_{r'p}^p \leq K \left\| \partial^\beta f \right\|_{W^{k - |\beta|, p}}^p \left\| \partial^{\alpha - \beta} g \right\|_{W^{k - |\alpha - \beta|, p}}^p \leq K \left\| f \right\|_{W^{k, p}}^p \left\| g \right\|_{W^{k, p}}^p$$

which shows that indeed $\partial^\beta f \in L^{rp}(\Omega)$ and $\partial^{\alpha - \beta} g \in L^{r'p}(\Omega)$ and that

$$\left\| \partial^\beta f\, \partial^{\alpha - \beta} g \right\|_p \leq K \left\| \partial^\beta f \right\|_{W^{k - |\beta|, p}} \left\| \partial^{\alpha - \beta} g \right\|_{W^{k - |\alpha - \beta|, p}} \leq K \left\| f \right\|_{W^{k, p}} \left\| g \right\|_{W^{k, p}} \leq$$
$$\leq K \left\| f \right\|_{A^{k, p}} \left\| g \right\|_{A^{k, p}}.$$

Because Ω is bounded the function e given by $e(x) = 1$ for all $x \in \Omega$ is in $A^{k, p}(\Omega)$ for any k and p. Considering invertibility we note that clearly $|f| \geq c$ implies $\frac{1}{f} \in L^\infty(\Omega) \subset L^p(\Omega)$. Now for smooth f we can express $\partial^\alpha \left(\frac{1}{f} \right)$ (for $1 \leq |\alpha| \leq k$) as a sum of products of the form $K \frac{\partial^{\beta_1} f \ldots \partial^{\beta_m} f}{f^j}$ for some $j \in \mathbb{N}$ and $|\beta_i| \geq 1$ with $\sum_{i=1}^m \beta_i = \alpha$. We have $\partial^{\beta_1} f_1 \partial^{\beta_2} f_2 \ldots \partial^{\beta_m} f_m \in L^p(\Omega)$ for $f_i \in A^{k, p}(\Omega)$ because it is one of the terms appearing in $\partial^\alpha (f_1 f_2 \ldots f_m)$ and the proof of the algebra property shows that all those terms belong to $L^p(\Omega)$ and so $K\, \partial^{\beta_1} f \ldots \partial^{\beta_m} f\, f^j \in L^p(\Omega) \cdot L^\infty(\Omega) \subset L^p(\Omega)$. It only remains to show that the rules of differentiation used to calculate $\partial^\alpha \left(\frac{1}{f} \right)$ remain valid under our weaker regularity assumptions. We already know when the Leibniz rule holds so we only have to concern ourselves with establishing the chain rule $\partial_i (F \circ f) = F' \circ f\, \partial_i f$, where $F(x) = \frac{1}{x^n}$ $(x \in \mathbb{R})$ for some $n \in \mathbb{N}$. It is shown in [**GT98**], Lemma 7.5 that the chain rule holds for $F \in \mathcal{C}^1(\mathbb{R})$ with $F' \in L^\infty(\mathbb{R})$ if $\partial_i f$ is at least locally integrable. Unfortunately, $x \mapsto \frac{1}{x^n}$ is not continuous on $\mathbb{R}$ but one can always choose a function F satisfying the above requirements that is equal to $x \mapsto \frac{1}{x^n}$ on $\mathbb{R} \setminus (-c, c)$ implying $\partial_i \left(\frac{1}{f^n} \right) = -\frac{1}{f^{2n}} \partial_i (f^n)$ for all $f \in W^{k, p}(\Omega) \cap L^\infty(\Omega)$ with $|f| \geq c$. In particular this shows that $\frac{1}{f^n} \in W^{1, p}(\Omega)$, which means that we may use both the chain rule and the Leibniz rule to calculate higher order derivatives of $\frac{1}{f}$ in the usual way. $\qquad \square$

3.2. (Tensor-)Distributions on orientable manifolds

It has already been pointed out in Remark 1.1.2 that for an orientable manifold the volume bundle is isomorphic to the vector bundle of exterior n-forms on TX which implies that $\Gamma_c(X, \mathrm{Vol}(X)) \cong \Omega_c^n(X)$ and thus

$$\mathcal{D}'(X) \cong (\Omega_c^n(X))'.$$

This description is particularly useful when dealing with Lie derivatives of distributions with respect to smooth vector fields. We are going to show that, given $\mathbf{u} \in \mathfrak{X}(X)$,

$$(3.2.1) \qquad\qquad \langle L_\mathbf{u} T, \phi \rangle = - \langle T, L_\mathbf{u} \phi \rangle$$

holds for all $T \in \mathcal{C}^\infty(X)$ and $\phi \in \Omega_c^n(X)$ and can be used to extend $L_\mathbf{u}$ to $\mathcal{D}'(X)$, generalizing the 'integration by parts formula' $\langle \partial_j u, \phi \rangle = - \langle u, \partial_j \phi \rangle$ for distributions on $\mathbb{R}^n$.

There are two important results needed to prove (3.2.1) that both concern k-forms on X, the first one being Stokes' theorem.

THEOREM 3.2.1 (Stokes' Theorem). *Let X be a smooth, oriented n-dimensional manifold with boundary and $\phi \in \Omega_c^{n-1}(X)$. Then*

$$\int_X d\phi = \int_{\partial X} \phi.$$

If $\partial X = \emptyset$ (i.e. X is a manifold without boundary), then the right-hand side vanishes.

The second one is a relationship between the interior product, Lie derivative and exterior derivative known as Cartan's formula.

For $\mathbf{u} \in \mathfrak{X}(X)$ the interior product $i_{\mathbf{u}}$ of a k-form $\phi \in \Omega^k(X)$ is the $(k-1)$-form given by

$$i_{\mathbf{u}}\phi(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) = \phi(\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}).$$

Clearly $i_{\mathbf{u}}\phi$ has compact support if either ϕ or $\mathbf{u}$ are compactly supported.

PROPOSITION 3.2.2 (Cartan's formula). *For any $\mathbf{u} \in \mathfrak{X}(X)$ and $\phi \in \Omega^k(X)$,*

$$(3.2.2) \qquad\qquad L_{\mathbf{u}}\phi = i_{\mathbf{u}}(d\phi) + d(i_{\mathbf{u}}\phi).$$

A proof of both Stokes' theorem and Cartan's formula may be found in [**Lee97**], Thm. 14.9 and Prop. 18.13 respectively. Now we have all the necessary tools to prove (3.2.1).

THEOREM 3.2.3. *Let X be orientable and $\mathbf{u} \in \mathfrak{X}(X)$. Then the Lie derivative $L_{\mathbf{u}} : \mathcal{C}^{\infty}(X) \to \mathcal{C}^{\infty}(X)$ satisfies*

$$(3.2.3) \qquad\qquad \langle L_{\mathbf{u}}T, \phi \rangle = - \langle T, L_{\mathbf{u}}\phi \rangle$$

for all $T \in \mathcal{C}^{\infty}(X)$ and $\phi \in \Omega_c^n(X)$ and $L_{\mathbf{u}}$ can be extended uniquely to $\mathcal{D}'(X)$ by (3.2.3).

PROOF. As usual it suffices to show (3.2.3) for $T \in \mathcal{C}^{\infty}(X)$ which implies continuity of $L_{\mathbf{u}}$ and the claim then follows by density of $\mathcal{C}^{\infty}(X)$. Now let $T \in \mathcal{C}^{\infty}(X)$, then

$$\langle L_{\mathbf{u}}T, \phi \rangle = \int_X L_{\mathbf{u}}(T)\phi = \int_X L_{\mathbf{u}}(T \cdot \phi) - T \cdot L_{\mathbf{u}}\phi = - \langle T, L_{\mathbf{u}}\phi \rangle + \int_X i_{\mathbf{u}}(d(T\phi)) + d(i_{\mathbf{u}}(T\phi))$$

by the derivation property of $L_{\mathbf{u}}$ and Cartan's formula. Now we note that $T\phi \in \Omega^n(X)$ and thus $d(T\phi) = 0$ and apply Stokes' theorem to the remaining term to obtain

$$\int_X i_{\mathbf{u}}(d(T\phi)) + d(i_{\mathbf{u}}(T\phi)) = 0,$$

showing (3.2.3). $\square$

REMARK 3.2.4. If $\overline{X}$ is a manifold with (nonempty) boundary and $T \in W_{\mathrm{loc}}^{1,p}(\overline{X})$, then

$$(3.2.4) \qquad\qquad \langle T, L_{\mathbf{u}}\phi \rangle = - \int_{\overline{X}} \mathbf{u}(T)\phi + \int_{\partial \overline{X}} T\, i_{\mathbf{u}}\phi.$$

Note that the right hand side of the equation is well-defined for $T \in W_{\mathrm{loc}}^{1,p}(\overline{X})$ because this implies $\mathbf{u}(T) \in L_{\mathrm{loc}}^p(X)$. For $T \in \mathcal{C}^{\infty}(\overline{X})$ equation (3.2.4) follows as in the proof of Thm. 3.2.3, for $T \in W_{\mathrm{loc}}^{1,p}(\overline{X})$ it follows by density of smooth functions and continuity of both sides in the $W_{\mathrm{loc}}^{1,p}$ topology.

3.3. Distributional connections

We will henceforth assume our manifold X to be *orientable* (this is necessary because we will need the results of section 3.2). To start with we will generalize the concept of a smooth connection on a manifold.

DEFINITION 3.3.1 (Distributional connection). An operator $\nabla : \mathfrak{X}(X) \times \mathfrak{X}(X) \to \mathcal{D}'(X, TX)$ is called a distributional connection if it satisfies

$$\nabla_{f\mathbf{u}+\mathbf{v}}\mathbf{w} = f\,\nabla_{\mathbf{u}}\mathbf{w} + \nabla_{\mathbf{v}}\mathbf{w},$$
$$\nabla_{\mathbf{u}}(\lambda\mathbf{v} + \mathbf{w}) = \lambda\,\nabla_{\mathbf{u}}\mathbf{v} + \nabla_{\mathbf{u}}\mathbf{w} \quad \text{and}$$
$$\nabla_{\mathbf{u}}(f\,\mathbf{v}) = f\,\nabla_{\mathbf{u}}\mathbf{v} + \mathbf{u}(f)\,\mathbf{v} \quad \text{(Leibniz rule)}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}(X)$, $f \in \mathcal{C}^\infty(X)$ and $\lambda \in \mathbb{R}$, where $\nabla_{\mathbf{u}}\mathbf{v}$ is the usual notation for $\nabla(\mathbf{u}, \mathbf{v})$.

The next proposition shows that analogously to a smooth connection a distributional connection can be extended to take arbitrary $(p, q)-$tensor fields as second input.

PROPOSITION 3.3.2. *Any distributional connection can be extended to a map* $\nabla : \mathfrak{X}(X) \times \mathcal{T}_q^p(X) \to \mathcal{D}'(X, T_q^p X)$.

PROOF. Just as in the smooth case one first sets $\nabla_{\mathbf{u}} f := \mathbf{u}(f)$ to obtain an operator $\nabla : \mathfrak{X}(X) \times \mathcal{C}^\infty(X) \to \mathcal{D}'(X)$. Next, we define $\nabla : \mathfrak{X}(X) \times \mathcal{T}_1^0(X) \to \mathcal{D}'(X, T_1^0 X)$ via

$$(3.3.1) \qquad\qquad (\nabla_{\mathbf{u}}\omega)(\mathbf{v}) := \mathbf{u}(\omega(\mathbf{v})) - \omega(\nabla_{\mathbf{u}}\mathbf{v}),$$

which uses the fact that any one-form $\omega \in \mathcal{T}_1^0(X) \cong L_{\mathcal{C}^\infty(X)}(\mathfrak{X}(X); \mathcal{C}^\infty(X))$ can be extended to $\omega : \mathcal{D}'(X, T_0^1 X) \to \mathcal{D}'(X)$ via $\omega(\mathbf{u}) := \mathbf{u}(\omega)$ for $\mathbf{u} \in \mathcal{D}'(X, T_0^1 X) \cong L_{\mathcal{C}^\infty(X)}(\mathcal{T}_1^0(X); \mathcal{D}'(X))$. Finally, we set

$$(\nabla_{\mathbf{u}} T)(\omega_1, \ldots, \omega_p, \mathbf{v}_1, \ldots, \mathbf{v}_q) := \nabla_{\mathbf{u}}(T(\omega_1, \ldots, \omega_p, \mathbf{v}_1, \ldots, \mathbf{v}_q)) -$$
$$- \sum_{i=1}^p T(\omega_1, \ldots, \nabla_{\mathbf{u}}\omega_i, \ldots, \omega_p, \mathbf{v}_1, \ldots, \mathbf{v}_q) - \sum_{i=1}^q T(\omega_1, \ldots, \omega_p, \mathbf{v}_1, \ldots, \nabla_{\mathbf{u}}\mathbf{v}_i, \ldots, \mathbf{v}_q)$$

for $T \in \mathcal{T}_q^p(X)$. $\qquad\qquad\square$

One major difference between smooth and distributional connections is that while a smooth connection can be extended to a map from $\mathcal{D}'(X, TX) \times \mathcal{D}'(X, TX)$ to $\mathcal{D}'(X, TX)$ this is not possible for a distributional connection. In fact, to be able to, for example, define a Riemann curvature tensor distribution based on a distributional connection one already has to assume a higher regularity of the connection itself. To elaborate on this: The problematic parts in the definition of the Riemann tensor $(R(\mathbf{u}, \mathbf{v})\mathbf{w} = \nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w} - (\nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w}))$ are the terms of the form $\nabla_{\mathbf{u}}(\nabla_{\mathbf{v}}\mathbf{w})$. For them to be well-defined the connection has to be extendable to $\mathfrak{X}(X) \times \operatorname{Im}\nabla$, so we want to identify a (maximal) suitable subspace $E(X, TX)$ of $\mathcal{D}'(X, TX)$ (where E stands for some function space) which allows the extension of any connection with values in $E(X, TX)$ to $\mathfrak{X}(X) \times E(X, TX)$. It turns out that a suitable space E is the space of locally square integrable sections of TM, i.e. $L_{\text{loc}}^2(X, TX)$.

PROPOSITION 3.3.3. *Let* ∇ *be a distributional connection. If* $\nabla_{\mathbf{u}}\mathbf{v} \in L_{\text{loc}}^2(X, TX)$ *for all* $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(X)$, *there exists an extension* $\nabla : \mathfrak{X}(X) \times L_{\text{loc}}^2(X, TX) \to \mathcal{D}'(X, TX)$. *This extension is given by*

$$(3.3.2) \qquad\qquad (\nabla_{\mathbf{u}}\mathbf{v})(\omega) = \mathbf{u}(\omega(\mathbf{v})) - \mathbf{v}(\nabla_{\mathbf{u}}\omega) \quad \text{in } \mathcal{D}'(X)$$

for $\mathbf{u} \in \mathfrak{X}(X)$, $\mathbf{v} \in L_{\text{loc}}^2(X, TX)$ *and* $\omega \in \mathcal{T}_1^0$.

In this case ∇ *is called an* L_{loc}^2-*connection and we may also write* $\nabla \in L_{\text{loc}}^2(X)$.

PROOF. First one notes that (3.3.2) holds for $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(X)$. The first term, $\mathbf{u}(\omega(\mathbf{v}))$, is well-defined in $\mathcal{D}'(X)$ for $\mathbf{v} \in \mathcal{D}'(X)$ because ω is smooth and can therefore be extended to take distributional arguments (as in the proof of Prop. 3.3.2) and $\mathbf{u}(f) = L_{\mathbf{u}}(f)$ for smooth functions which shows that $\mathbf{u}$ also can be extended to take distributions as input (see Thm. 3.2.3). The second

term is a bit more problematic. By (3.3.1) one has $\nabla_{\mathbf{u}}\omega \in E\left(X, T_1^0 X\right)$ for a connection with values in $E\left(X, TX\right)$. The problem of the definition of $\mathbf{v}\left(\tilde{\omega}\right)$ for $\tilde{\omega} \in E\left(X, T_1^0 X\right)$, $\mathbf{v} \in E\left(X, TX\right)$ now corresponds directly to the question if the product of two functions $f, g \in E\left(U\right)$ for some open $U \subset \mathbb{R}^n$ exists in $\mathcal{D}'\left(U\right)$. Observing that this is the case for $E = L_{\mathrm{loc}}^2$ (the product of two L_{loc}^2-functions is in $L_{\mathrm{loc}}^1 \subset \mathcal{D}'\left(U\right)$) proves the theorem. $\square$

Now we may define the distributional Riemann tensor of an L_{loc}^2-connection.

DEFINITION 3.3.4. The distributional Riemann tensor of an L_{loc}^2-connection is the tensor distribution $\mathbf{Riem} \in \mathcal{D}'\left(X, T_3^1 X\right)$ defined by

$$(3.3.3) \qquad \mathbf{Riem}\left(\mathbf{u}, \mathbf{v}\right)\mathbf{w} := \nabla_{[\mathbf{u}, \mathbf{v}]}\mathbf{w} - \left(\nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w}\right) \quad \text{in } \mathcal{D}'\left(X, TX\right)$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}\left(X\right)$.

REMARK 3.3.5. If $\nabla \in W_{\mathrm{loc}}^{k,p}\left(X\right)$ for $k \geq 1$ and $p > \frac{n}{k}$ (or $p = 1$ and $k \geq n$) then $\mathbf{Riem} \in W_{\mathrm{loc}}^{k-1,p}\left(X, T_3^1 X\right)$: First note that $W_{\mathrm{loc}}^{k,p}\left(X\right) \subset L_{\mathrm{loc}}^2\left(X\right)$ (for $p \geq 2$ this follows from Hölder's inequality, for $p < 2$ from the Sobolev embedding, see [AF03], Thm. 4.12, A). For $\mathbf{v} \in W_{\mathrm{loc}}^{k,p}\left(X, TX\right)$ and smooth $\mathbf{u}, \omega$ we have $\mathbf{u}\left(\omega\left(\mathbf{v}\right)\right) \in W_{\mathrm{loc}}^{k-1,p}\left(X\right)$ and $\mathbf{v}\left(\nabla_{\mathbf{u}}\omega\right) \in W_{\mathrm{loc}}^{k,p}\left(X\right)$ (since $W_{\mathrm{loc}}^{k,p}\left(X\right)$ is an algebra for $p > \frac{n}{k}$ or $p = 1$ and $k \geq n$, see Prop. 3.1.8 or Thm. 4.39 in [AF03]) and thus $\nabla_{\mathbf{u}}\mathbf{v} \in W_{\mathrm{loc}}^{k-1,p}\left(X, TX\right)$ by (3.3.2). This shows that $\mathbf{Riem} \in W_{\mathrm{loc}}^{k-1,p}\left(X, T_3^1 X\right)$.

Based on this we can also define the Ricci curvature tensor distribution (using the extension of the contraction operation to tensor distributions) as

$$(3.3.4) \qquad\qquad \mathbf{Ric} := C_3^1 \mathbf{Riem} \in \mathcal{D}'\left(X, T_2^0 X\right),$$

where the order of the one-form inputs is done in the conventional way of defining the local components $\mathbf{Riem}_{jkl}^i$ by $\mathbf{Riem}\left(\partial_k, \partial_l\right)\partial_j = \mathbf{Riem}_{jkl}^i \partial_i$. In other words, one has locally

$$(3.3.5) \qquad\qquad \mathbf{Ric}\left(\mathbf{u}, \mathbf{v}\right) = dx^j\left(\mathbf{Riem}\left(\mathbf{u}, \partial_j\right)\mathbf{v}\right) \in \mathcal{D}'\left(U\right)$$

for $\mathbf{u}, \mathbf{v} \in \mathfrak{X}\left(U\right)$ and, as in the smooth case, (3.3.5) remains true when replacing the ∂_j and dx^j with an arbitrary local frame E_j and corresponding dual frame E^j.

3.3.1. A jump formula for distributional connections. One of the most basic types of singularities is a simple jump discontinuity, so as a next step we want to study the behavior of $\nabla_{\mathbf{u}}\mathbf{v}$ if both ∇ and $\mathbf{v}$ suffer a jump discontinuity along a hypersurface $M \subset X$. We will need to make some further (minor) assumptions about M, namely we want M to split X into two parts, which will be denoted by X^+ and X^-, such that $X = X^+ \cup X^-$, $X^+ \cap X^- = M$ and $X^{\pm}$ are smooth manifolds with boundary $\partial X^{\pm} = M$. We will choose the orientation on the boundary M to be the one induced from X^-.

Before going into detail about the regularity assumptions on ∇ and $\mathbf{v}$ we will define the Dirac measure on the hypersurface M as a special 1-form distribution δ_M on X.

DEFINITION 3.3.6. The Dirac measure $\delta_M \in \mathcal{D}'\left(X, T_1^0 X\right)$ on the hypersurface M is the one-form distribution given by

$$(3.3.6) \qquad\qquad \langle \delta_M\left(\mathbf{u}\right), \phi \rangle = \int_M j^*\left(i_{\mathbf{u}}\phi\right), \quad \mathbf{u} \in \mathfrak{X}\left(X\right), \phi \in \Omega^n\left(X\right),$$

where $j^* : \Omega^{n-1}\left(X\right) \to \Omega^{n-1}\left(M\right)$ is the pullback induced by the inclusion $j : M \hookrightarrow X$ (we may omit j^* in the future for notational simplicity).

REMARK 3.3.7. Clearly $\delta_M\left(\mathbf{u}\right)$ only depends on $\mathbf{u}|_M$ (if $\mathbf{u}|_M = 0$ then also $i_{\mathbf{u}}\phi = C_1^1\left(\mathbf{u} \otimes \phi\right)$ and thus $j^*\left(i_{\mathbf{u}}\phi\right)$ are zero on M) which implies that $\operatorname{supp}\delta_M \subset M$. Furthermore $\delta_M\left(\mathbf{u}\right) = 0$ if $\mathbf{u}|_M \in \Gamma\left(M, TM\right)$: Let $p \in M$ and $\left(U, \psi = \left(x^1, \ldots, x^n\right)\right)$ be coordinates around p adapted

to the hypersurface[1], then $\frac{\partial}{\partial x^j}\big|_M \in \Gamma\left(U \cap M, TM|_{U \cap M}\right)$ for $j = 2, \ldots, n$ and we can write $\mathbf{u}|_M = \sum_{j=2}^n u^j \frac{\partial}{\partial x^j}\big|_M$. Now for $\phi = f\, dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(U)$ one has $i_{\partial_j}\phi(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) = f\left(dx^1 \wedge \cdots \wedge dx^n\right)(\partial_j, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) = 0$ for $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \Gamma(M, TM)$ and $2 \le j \le n$ and thus $\delta_M(\partial_j) = 0$ for $j = 2, \ldots, n$.

At first glance this definition looks rather unwieldy but the following proposition shows that it is just a generalization of the one-form distribution $S_M(\eta)$ that appeared in the jump formula (2.3.4) in Example 2.3.9.

PROPOSITION 3.3.8. *Let X be an oriented semi-Riemanian manifold and $M \subset X$ be a closed semi-Riemannian hypersurface with the properties described above and let $\mathbf{n} \in \Gamma\left(M, TM^\perp\right)$ denote the unit normal vector field on M, then*

$$\delta_M = S_M(\eta),$$

where $\eta \in \Gamma\left(M, T^*X|_M\right)$ denotes the canonical one-form given by $\eta(\mathbf{n}) = 1$ and $\eta(\mathbf{v}) = 0$ for $\mathbf{v} \in \Gamma(M, TM)$.

PROOF. First we note that using the definition of S_M and taking into account the identifications (1.2.1) and (1.2.2) one has that

$$L_{\mathcal{C}^\infty(X)}\left(\mathfrak{X}(X); \mathcal{D}'(X)\right) \ni \left(\mathbf{u} \mapsto S_M(\eta(\mathbf{u}))\right) = \mathbf{u} \mapsto \tilde{\eta}(\mathbf{u})\, S_M(1) \cong S_M(1) \otimes \tilde{\eta} \cong$$
$$\cong \beta\left(S_M(1) \otimes \tilde{\eta}\right) = S_M(\eta) \in \mathcal{D}'\left(X, T_1^0 X\right)$$

for any smooth extension $\tilde{\eta}$ of η to X, hence showing $\delta_M = S_M(\eta)$ is equivalent to showing

$$(3.3.7) \qquad\qquad \delta_M(\mathbf{u}) = S_M(\eta(\mathbf{u})) \quad \text{for all } \mathbf{u} \in \mathfrak{X}(X).$$

Also the support of both $S_M(\eta)$ and δ_M is contained in M so it suffices to show their equality locally around M. To do this let $p \in M$ and (U, ψ) be canonical coordinates around p (see subsection 2.3.1). Then every $\phi \in \Omega_c^n(U)$ is equal to $\phi = f\sqrt{\left|\det\left(g_{ij}\right)_{1 \le i,j \le n}\right|}\, dx^1 \wedge \cdots \wedge dx^n$ for a unique $f \in \mathcal{D}(U)$ (because X is semi-Riemannian and orientable $\Gamma_c(X, \mathrm{Vol}(X)) = \Omega_c^n(X)$ and $\hat{\Omega}_g = \sqrt{|\det g|}\, dx^1 \wedge \cdots \wedge dx^n$ is a basis), every $\mathbf{u} \in \mathfrak{X}(U)$ can be written as $\mathbf{u} = u^i\, \partial_{x^i}$ and $\eta = dx^1$ on $U \cap M$ (because $\mathbf{n} = \frac{\partial}{\partial x^1}$ in canonical coordinates). Taking into account that the definition of the single-layer distribution is based on using $\hat{\Omega}_g$ to identify $\Gamma_c(X, \mathrm{Vol}(X))$ and $\mathcal{D}(X)$ we have to show that

$$(3.3.8) \qquad\qquad \langle \delta_M(\mathbf{u}), \phi \rangle = \langle S_M(\eta(\mathbf{u})), f \rangle$$

for all $\mathbf{u} \in \mathfrak{X}(U), \phi \in \Omega_c^n(U)$. This allows us to calculate

$$j^*\left(i_{\mathbf{u}}\left(dx^1 \wedge \cdots \wedge dx^n\right)\right) = j^*\left(i_{\mathbf{u}}\left(dx^1\right) \wedge dx^2 \wedge \cdots \wedge dx^n\right) - j^*\left(dx^1 \wedge i_{\mathbf{u}}\left(dx^2 \wedge \cdots \wedge dx^n\right)\right) =$$
$$= u^1\big|_M\, j^*\left(dx^2 \wedge \cdots \wedge dx^n\right) - 0 = \eta(\mathbf{u})\ dx^2 \wedge \cdots \wedge dx^n\big|_M.$$

Thus the left hand side of (3.3.8) becomes

$$\int_{M \cap U} j^*\left(i_{\mathbf{u}}\phi\right) = \int_{M \cap U} f|_M\,\sqrt{\left|\det\left(g_{ij}|_M\right)_{1 \le i,j \le n}\right|}\,\eta(\mathbf{u})\ dx^2 \wedge \cdots \wedge dx^n\big|_M =$$
$$= \int_{M \cap U} f|_M\,\sqrt{\left|\det\left(g_{ij}|_M\right)_{2 \le i,j \le n}\right|}\,\eta(\mathbf{u})\ dx^2 \wedge \cdots \wedge dx^n\big|_M = \int_{M \cap U} f|_M\,\eta(\mathbf{u})\,\hat{\Omega}_{g_M} =$$
$$= \langle S_M(\eta(\mathbf{u})), f \rangle,$$

where we have used that $g_{1j} = \delta_{1j}$ (cf. Prop. 2.3.8). $\qquad\qquad\qquad\qquad\qquad\qquad \square$

---

[1]Unfortunately we can no longer use the canonical coordinates defined in subsection 2.3.1 as we no longer have a smooth metric on X. Instead, our standard coordinates (U, ψ) around a point $p \in M \subset X$ are chosen such that $\left(U \cap M, \psi|_{U \cap M}\right)$ is a chart in M and $\psi(U \cap M) = \psi(U) \cap \left(\{0\} \times \mathbb{R}^{n-1}\right)$. We are going to call such coordinates adapted to the hypersurface.

To deal with tensor fields that have higher regularity away from the hypersurface we will need some more notations. For $T \in \mathcal{D}'\left(X, T_s^r X\right)$ we say that $T \in W_{\text{loc}}^{k,p}\left(X^{\pm}, T_s^r X^{\pm}\right)$ if $T|_{X^{\pm} \setminus M}$ is locally integrable and can be extended to some $T^{\pm} \in W_{\text{loc}}^{k,p}\left(X^{\pm}, T_s^r X^{\pm}\right)$ (note that $T_s^r X|_p = T_s^r X^{\pm}|_p$ for all $p \in X^{\pm}$ even at the boundary). For $T \in L_{\text{loc}}^1\left(X^{\pm}, T_s^r X^{\pm}\right)$ we define the regular part of T as

$$(3.3.9) \qquad\qquad T^{\text{reg}} := \begin{cases} T^+ & \text{on } X^+ \\ T^- & \text{on } X^- \end{cases}.$$

Note that this is a well-defined locally integrable tensor field on all of X that depends only on $T|_{X \setminus M}$ since M is a set of measure zero in X.

We are now going to assume that ∇ is an L_{loc}^2-connection on X such that $\nabla \in L_{\text{loc}}^2\left(X^{\pm}\right) \cap W_{\text{loc}}^{1,p}\left(X^{\pm}\right)$ (i.e. $\nabla_{\mathbf{u}} \mathbf{v} \in L_{\text{loc}}^2\left(X^{\pm}, TX^{\pm}\right) \cap W_{\text{loc}}^{1,p}\left(X^{\pm}, TX^{\pm}\right)$ for all $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(X)$) for some $p \geq 1$. Then ∇ induces connections $\nabla^{\pm}$ of $L_{\text{loc}}^2 \cap W_{\text{loc}}^{1,p}$ regularity on $X^{\pm}$ and $\nabla_{\mathbf{u}}^{\pm} \mathbf{v}$ is the vector field in $L_{\text{loc}}^2\left(X^{\pm}\right) \cap W_{\text{loc}}^{1,p}\left(X^{\pm}\right)$ uniquely determined by

$$\nabla_{\mathbf{u}}^{\pm} \mathbf{v}\big|_{X^{\pm} \setminus M} = \nabla_{\mathbf{u}} \mathbf{v}\big|_{X^{\pm} \setminus M} \quad \text{for } \mathbf{u}, \mathbf{v} \in \Gamma\left(X^{\pm}, TX^{\pm}\right) = \Gamma\left(X^{\pm}, TX|_{X^{\pm}}\right).$$

We have $\left(\nabla_{\mathbf{u}} \mathbf{v}\right)^{\pm} = \nabla_{\mathbf{u}^{\pm}}^{\pm} \mathbf{v}^{\pm}$ and for $\mathbf{u} \in \mathfrak{X}(X)$ and $\mathbf{v} \in \mathfrak{X}(X) \cap W_{\text{loc}}^{k,p}\left(X^{\pm}\right)$ this is in $W_{\text{loc}}^{k-1,p}\left(X^{\pm}\right)$ because $\mathbf{u}(f) \in W_{\text{loc}}^{k-1,p}(X)$ if $f \in W_{\text{loc}}^{k,p}(X)$ and $\mathbf{u} \in \mathfrak{X}(X)$ (see the proof of Prop. 3.3.3) and thus $\left(\nabla_{\mathbf{u}} \mathbf{v}\right)^{\text{reg}}$ is at least $L_{\text{loc}}^p(X)$.

Given a vector field $\mathbf{v} \in \mathcal{D}'(X, TX) \cap W_{\text{loc}}^{1,s}\left(X^{\pm}, TX|_{X^{\pm}}\right)$ for some $\infty > s \geq 1$ we define its jump across the hypersurface M by

$$[\mathbf{v}]_M := \gamma_{X^+}\left(\mathbf{v}^+\right) - \gamma_{X^-}\left(\mathbf{v}^-\right) \in L_{\text{loc}}^s\left(M, TX|_M\right),$$

where $\gamma_{X^{\pm}}$ denotes the trace operation from Prop. 3.1.7.

There is one last notation to be introduced before we can state our jump formula: For $\mathbf{u} \in \mathfrak{X}(X)$ the rather intuitive expression $[\mathbf{v}]_M \delta_M(\mathbf{u})$ is used to denote the distributional vector field on X defined by

$$\left([\mathbf{v}]_M \delta_M(\mathbf{u})\right)(\omega) := \phi \mapsto \int_M [\mathbf{v}]_M\left(\omega|_M\right) i_{\mathbf{u}} \phi \quad \text{for } \omega \in \Gamma\left(X, T_1^0 X\right)$$

Similarly $[f]_M \delta_M(\mathbf{u}) := \phi \mapsto \int_M [f]_M i_{\mathbf{u}} \phi$ for a function $f \in W_{\text{loc}}^{1,s}\left(X^{\pm}\right) \cap L_{\text{loc}}^2(X)$.

THEOREM 3.3.9 (Jump formula for a singular connection). *Let ∇ be an L_{loc}^2-connection satisfying $\nabla^{\pm} \in L_{\text{loc}}^2\left(X^{\pm}\right) \cap W_{\text{loc}}^{1,p}\left(X^{\pm}\right)$ for some $\infty > p \geq 1$ and $\mathbf{v} \in W_{\text{loc}}^{1,s}\left(X^{\pm}\right) \cap L_{\text{loc}}^2(X)$. Then*

$$(3.3.10) \qquad\qquad \nabla_{\mathbf{u}} \mathbf{v} = \left(\nabla_{\mathbf{u}} \mathbf{v}\right)^{\text{reg}} + [\mathbf{v}]_M \delta_M(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathfrak{X}(X).$$

PROOF. By Prop. 3.3.3 one has $\left(\nabla_{\mathbf{u}} \mathbf{v}\right)(\omega) = \mathbf{u}\left(\omega\left(\mathbf{v}\right)\right) - \mathbf{v}\left(\nabla_{\mathbf{u}} \omega\right)$ for locally square integrable $\mathbf{v}$ and smooth one-forms ω. Now let $\phi \in \Omega^n(X)$, then by Thm. 3.2.3 we get

$$\langle \mathbf{u}\left(\omega\left(\mathbf{v}\right)\right), \phi \rangle = -\langle \omega\left(\mathbf{v}\right), L_{\mathbf{u}} \phi \rangle = -\int_X \omega\left(\mathbf{v}\right) L_{\mathbf{u}} \phi = -\int_{X^-} \omega\left(\mathbf{v}^-\right) L_{\mathbf{u}} \phi - \int_{X^+} \omega\left(\mathbf{v}^+\right) L_{\mathbf{u}} \phi =$$

$$= \int_{X^-} \mathbf{u}\left(\omega\left(\mathbf{v}^-\right)\right) \phi + \int_{X^+} \mathbf{u}\left(\omega\left(\mathbf{v}^+\right)\right) \phi - \int_M \omega\left(\gamma_{X^-}\left(\mathbf{v}^-\right)\right) i_{\mathbf{u}} \phi + \int_M \omega\left(\gamma_{X^+}\left(\mathbf{v}^+\right)\right) i_{\mathbf{u}} \phi =$$

$$= \int_{X^-} \mathbf{u}\left(\omega\left(\mathbf{v}^-\right)\right) \phi + \int_{X^+} \mathbf{u}\left(\omega\left(\mathbf{v}^+\right)\right) \phi + \langle\left([\mathbf{v}]_M \delta_M\right)(\mathbf{u})(\omega), \phi \rangle$$

using (3.2.4), where the last plus in the second line comes from the fact that we choose the orientation on M as the orientation on the boundary induced from X^- and not from X^+. Since the second term of (3.3.2) is always at least $L_{\text{loc}}^1(X)$ one obtains

$$\langle \mathbf{v}\left(\nabla_{\mathbf{u}} \omega\right), \phi \rangle = \int_X \mathbf{v}\left(\nabla_{\mathbf{u}} \omega\right) \phi = \int_{X^-} \mathbf{v}^-\left(\nabla_{\mathbf{u}}^- \omega\right) \phi + \int_{X^+} \mathbf{v}^+\left(\nabla_{\mathbf{u}}^+ \omega\right) \phi.$$

Putting everything together gives

$$\langle (\nabla_{\mathbf{u}} \mathbf{v})(\omega), \phi \rangle = \int_{X^-} \left(\mathbf{u} \left(\omega \left(\mathbf{v}^- \right) \right) - \mathbf{v}^- \left(\nabla_{\mathbf{u}}^- \omega \right) \right) \phi + \int_{X^+} \left(\mathbf{u} \left(\omega \left(\mathbf{v}^+ \right) \right) - \mathbf{v}^+ \left(\nabla_{\mathbf{u}}^+ \omega \right) \right) \phi +$$
$$+ \langle ([\mathbf{v}]_M \, \delta_M)(\mathbf{u})(\omega), \phi \rangle = \langle (\nabla_{\mathbf{u}} \mathbf{v})^{\mathrm{reg}}(\omega), \phi \rangle + \langle ([\mathbf{v}]_M \, \delta_M)(\mathbf{u})(\omega), \phi \rangle,$$

proving (3.3.10). □

REMARK 3.3.10. This formula is actually very similar to the one in Example 2.3.9, just the notation is a bit different: Let $f \cdot H \circ h$ be as in Example 2.3.9 and ∇ be the (smooth) metric connection. Then (2.3.4) and Prop. 3.3.8 give

$$\nabla_{\mathbf{u}} (f \cdot H \circ h) = d (f \cdot H \circ h)(\mathbf{u}) = S_M \left(f|_M \, \eta \right)(\mathbf{u}) + H \circ h \cdot df(\mathbf{u}) =$$
$$= f|_M \, \delta_M (\mathbf{u}) + (\nabla_{\mathbf{u}} (f \cdot H \circ h))^{\mathrm{reg}} = [f \cdot H \circ h]_M \, \delta_M (\mathbf{u}) + (\nabla_{\mathbf{u}} (f \cdot H \circ h))^{\mathrm{reg}}.$$

The previous result allows us to derive jump formulas for both Riemann and Ricci curvature.

COROLLARY 3.3.11. *Let ∇ be an L^2_{loc}-connection on X satisfying $\nabla^{\pm} \in W^{1,p}_{\mathrm{loc}} (X^{\pm})$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}(X)$. Then*

$$(3.3.11) \qquad \mathbf{Riem}(\mathbf{u}, \mathbf{v}) \, \mathbf{w} = (\mathbf{Riem}(\mathbf{u}, \mathbf{v}) \, \mathbf{v})^{\mathrm{reg}} - [\nabla_{\mathbf{v}} \mathbf{w}]_M \, \delta_M (\mathbf{u}) + [\nabla_{\mathbf{u}} \mathbf{w}]_M \, \delta_M (\mathbf{v})$$

and locally

$$(3.3.12) \qquad \mathbf{Ric}(\mathbf{u}, \mathbf{v}) = (\mathbf{Ric}(\mathbf{u}, \mathbf{v}))^{\mathrm{reg}} - \left[dx^i \left(\nabla_{\partial x^i} \mathbf{v} \right) \right]_M \, \delta_M (\mathbf{u}) + \left[dx^j \left(\nabla_{\mathbf{u}} \mathbf{v} \right) \right]_M \, \delta_M \left(\partial x^j \right)$$

for any chart $\psi = \left(x^1, \ldots, x^n \right)$.

PROOF. Clearly $\nabla_{\mathbf{u}} \mathbf{v} \in L^2_{\mathrm{loc}}(X) \cap W^{1,p}_{\mathrm{loc}}(X^{\pm})$ by the regularity assumptions on ∇. Thus we may apply Thm. 3.3.9 to the definition of the Riemann tensor to obtain

$$\mathbf{Riem}(\mathbf{u}, \mathbf{v}) \, \mathbf{w} = \nabla_{[\mathbf{u}, \mathbf{v}]} \mathbf{w} - (\nabla_{\mathbf{u}} \nabla_{\mathbf{v}} \mathbf{w} - \nabla_{\mathbf{v}} \nabla_{\mathbf{u}} \mathbf{w}) = \nabla_{[\mathbf{u}, \mathbf{v}]} \mathbf{w} - (\nabla_{\mathbf{u}} \nabla_{\mathbf{v}} \mathbf{w})^{\mathrm{reg}} - [\nabla_{\mathbf{v}} \mathbf{w}]_M \, \delta_M (\mathbf{u}) +$$
$$+ (\nabla_{\mathbf{v}} \nabla_{\mathbf{u}} \mathbf{w})^{\mathrm{reg}} + [\nabla_{\mathbf{u}} \mathbf{w}]_M \, \delta_M (\mathbf{v}) = (\mathbf{Riem}(\mathbf{u}, \mathbf{v}) \, \mathbf{w})^{\mathrm{reg}} - [\nabla_{\mathbf{v}} \mathbf{w}]_M \, \delta_M (\mathbf{u}) + [\nabla_{\mathbf{u}} \mathbf{w}]_M \, \delta_M (\mathbf{v}).$$

The second claim follows immediately from the first using (3.3.5):

$$\mathbf{Ric}(\mathbf{u}, \mathbf{v}) = dx^j \left(\mathbf{Riem}(\mathbf{u}, \partial_j) \, \mathbf{v} \right) =$$
$$= (\mathbf{Ric}(\mathbf{u}, \mathbf{v}))^{\mathrm{reg}} - dx^i \left([\nabla_{\partial x^i} \mathbf{v}]_M \right) \delta_M (\mathbf{u}) + dx^j \left([\nabla_{\mathbf{u}} \mathbf{v}] \right) \delta_M \left(\partial x^j \right). \quad □$$

Note that the non-regular parts of (3.3.10), (3.3.11) and (3.3.12) are supported in the hypersurface M and are thus only relevant locally around M. Furthermore in coordinates (U, ψ) adapted to the hypersurface $\delta_M \left(\partial x^j \right) = 0$ for $j = 2, \ldots, n$ (since in this case $\partial x^j|_M \in \Gamma \left(U \cap M, TM|_{U \cap M} \right)$) so in this case the jump formula for the Ricci tensor reads simply

$$(3.3.13) \qquad \mathbf{Ric}(\mathbf{u}, \mathbf{v}) = (\mathbf{Ric}(\mathbf{u}, \mathbf{v}))^{\mathrm{reg}} + \left[dx^1 \left(\nabla_{\mathbf{u}} \mathbf{v} \right) \right]_M \, \delta_M \left(\partial x^1 \right) - \left[dx^i \left(\nabla_{\partial x^i} \mathbf{v} \right) \right]_M \, \delta_M (\mathbf{u}).$$

3.4. Distributional metrics and scalar curvature

In the following we are going to study distributional metrics on manifolds, in particular we will determine what regularity is needed for such a metric to give rise to a (distributional) Levi-Civita connection.

DEFINITION 3.4.1 (Distributional metric). A distributional metric is a tensor distribution $g \in \mathcal{D}' \left(X, T^0_2 X \right)$ that satisfies

$$g(\mathbf{u}, \mathbf{v}) = g(\mathbf{v}, \mathbf{u}) \quad \forall \mathbf{u}, \mathbf{v} \in \mathfrak{X}(X) \text{ and}$$
$$g(\mathbf{u}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathfrak{X}(X) \implies \mathbf{u} = 0,$$

i.e., that is symmetric and non-degenerate.

Note that this non-degeneracy requirement is weaker than that for smooth metrics, where one demands non-degeneracy in every point $p \in X$, because it is not possible to talk about pointwise properties of tensor distributions. Additionally this definition does not require g to have constant signature (again this condition can not be formulated in the distributional case). In classical semi-Riemannian geometry any metric g induces a (unique) Levi-Civita connection ∇ (that is a torsion free connection satisfying $\nabla g = 0$) via the Koszul formula

(3.4.1)
$$2g\left(\nabla_{\mathbf{u}}\mathbf{v}, \mathbf{w}\right) = \mathbf{u}\left(g\left(\mathbf{v}, \mathbf{w}\right)\right) + \mathbf{v}\left(g\left(\mathbf{w}, \mathbf{u}\right)\right) - \mathbf{w}\left(g\left(\mathbf{u}, \mathbf{v}\right)\right) - g\left(\mathbf{u}, [\mathbf{v}, \mathbf{w}]\right) + g\left(\mathbf{v}, [\mathbf{w}, \mathbf{u}]\right) + g\left(\mathbf{w}, [\mathbf{u}, \mathbf{v}]\right).$$

We will quickly review the duality between vector fields and one-forms induced by a smooth metric. Given a vector field $\mathbf{u} \in \mathfrak{X}(X)$ one defines $\mathbf{u}^{\flat} \in \mathcal{T}_1^0(X)$ by $\mathbf{u}^{\flat}(\mathbf{v}) := g(\mathbf{u}, \mathbf{v})$ for all $\mathbf{v} \in \mathfrak{X}(X)$. Locally $\mathbf{u}^{\flat}$ is given by $\left(\mathbf{u}^{\flat}\right)_i = g_{ij}\mathbf{u}^j$. The map $\flat : \mathfrak{X}(X) \to \mathcal{T}_1^0(X)$ is a $\mathcal{C}^{\infty}(X)-$linear isomorphism whose inverse $\sharp : \mathcal{T}_1^0(X) \to \mathfrak{X}(X)$ is locally given by $\left(\omega^{\sharp}\right)^i = g^{ij}\omega_j$ where g^{ij} is the inverse matrix of g_{ij}. Now given $\mathbf{u}$ and $\mathbf{v}$ the Koszul formula actually only defines the one-form $\left(\nabla_{\mathbf{u}}\mathbf{v}\right)^{\flat}$, but by the above this immediately also gives the vector field $\nabla_{\mathbf{v}}\mathbf{u}$ and thus the connection ∇ - at least as long as g is smooth.

In the distributional case (3.4.1) and non-degeneracy still implies uniqueness of ∇ (under the requirements that $\nabla g = 0$ and that ∇ is torsion free) and and shows the existence of a map $\nabla^{\flat} : \mathfrak{X}(X) \times \mathfrak{X}(X) \to \mathcal{D}'\left(X, T_1^0 X\right)$, however, this does not give existence of $\nabla : \mathfrak{X}(X) \times \mathfrak{X}(X) \to \mathcal{D}'\left(X, T_0^1 X\right)$ because $\flat$ is no longer an isomorphism. So to give rise to a distributional Levi-Civita connection a distributional metric must be regular enough for the inverse g^{ij} to exist (in the smooth case this follows from non-degeneracy of g in every point $p \in X$ which we do not have if g is merely distributional) and allow products of the form $g^{ij}\omega_j$ for $\omega \in \operatorname{Im}\nabla^{\flat}$. If we furthermore want the induced connection to be L_{loc}^2 the products $g^{ij}\omega_j$ have to be in $L_{\mathrm{loc}}^2(X)$.

DEFINITION 3.4.2. A locally integrable metric g is said to be uniformly non-degenerate if for any chart $\left(U, \psi = \left(x^1, \ldots, x^n\right)\right)$ there exists a constant $c_K > 0$ for every compact set $K \subset U$ such that

$$\left|\det\left(g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right)\right| \geq c_K$$

almost everywhere on K.

PROPOSITION 3.4.3. *Let g be a distributional metric on X. If $g \in W_{\mathrm{loc}}^{1,2}(X) \cap L_{\mathrm{loc}}^{\infty}(X)$ and uniformly non-degenerate, then $g^{ij} \in L_{\mathrm{loc}}^{\infty}(U)$ and $g^{ij}\left(\nabla_{\mathbf{u}}^{\flat}\mathbf{v}\right)_j \in L_{\mathrm{loc}}^2(U)$ for all $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(X)$ and all chart domains U. In this case the local distributional vector fields $g^{ij}\left(\nabla_{\mathbf{u}}^{\flat}\mathbf{v}\right)_j \frac{\partial}{\partial x^i}$ define a global distributional vector field $\nabla_{\mathbf{u}}\mathbf{v} \in L_{\mathrm{loc}}^2\left(X, T_1^0 X\right)$.*

PROOF. First we note that for $g \in W_{\mathrm{loc}}^{1,2}(X)$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}(X)$ we have that $\left(\nabla_{\mathbf{u}}^{\flat}\mathbf{v}\right)(\mathbf{w}) \in L_{\mathrm{loc}}^2(X)$ since

$$2\left(\nabla_{\mathbf{u}}^{\flat}\mathbf{v}\right)(\mathbf{w}) = \mathbf{u}\left(g\left(\mathbf{v}, \mathbf{w}\right)\right) + \mathbf{v}\left(g\left(\mathbf{w}, \mathbf{u}\right)\right) - \mathbf{w}\left(g\left(\mathbf{u}, \mathbf{v}\right)\right) - g\left(\mathbf{u}, [\mathbf{v}, \mathbf{w}]\right) -$$
$$- g\left(\mathbf{v}, [\mathbf{w}, \mathbf{u}]\right) + g\left(\mathbf{w}, [\mathbf{u}, \mathbf{v}]\right)$$

and thus $\nabla_{\mathbf{u}}^{\flat}\mathbf{v} \in L_{\mathrm{loc}}^2\left(X, T_1^0 X\right)$. Next we want to determine the regularity of g^{ij}: Let $K \subset U$ compact, then uniform non-degeneracy implies that $|\det g_{ij}| \geq c_K$ almost everywhere on K so $g^{ij} = \frac{1}{\det g_{ij}}C_{ij}$ (by Cramer's rule where C_{ij} denotes the matrix of cofactors) almost everywhere on K, giving $g^{ij} \in L_{\mathrm{loc}}^{\infty}(U)$. This shows that $g^{ij}\left(\nabla_{\mathbf{u}}^{\flat}\mathbf{v}\right)_j \in L_{\mathrm{loc}}^2(U)$ for all $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(X)$ and all chart domains U which cover X. That they are indeed local representations of a global L_{loc}^2-vector field on X follows in the same way as one shows that local expressions of the form $g^{ij}\omega_j \frac{\partial}{\partial x^i}$ define a global vector field for smooth metrics and one-forms. $\qquad\square$

We will refer to a metric that satisfies the requirements of Prop. 3.4.3 as Geroch-Traschen- or gt-regular since metrics with this regularity were first introduced by Geroch and Traschen in [**GT87**] (actually they assumed local boundedness of the inverse g^{ij} instead of uniform non-degeneracy of g_{ij} but both are equivalent if one already assumes that g_{ij} is locally bounded). It is easy to verify that (as in the smooth case) the connection given by the Koszul formula indeed satisfies $\nabla g = 0$ and $T = 0$, where $T(\mathbf{u}, \mathbf{v}) := \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u}, \mathbf{v}]$ is the torsion of g.

LEMMA 3.4.4. *Any gt-regular metric g satisfies*

$$g^{ij} \in W_{\text{loc}}^{1,2}(U) \cap L_{\text{loc}}^{\infty}(U)$$

for every chart (U, ψ) with $U \subset K$ locally compact.

PROOF. To show this one first proves that the product of two functions $f, g \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ is again in $W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ for any bounded subset $\Omega \subset \mathbb{R}^n$ (unfortunately this case is in general not going to be covered by Prop. 3.1.8). It is obvious that $fg \in L^{\infty}(\Omega) \subset L^2(\Omega)$, to show $\partial_k(fg) \in L^2(\Omega)$ one uses $\partial_k(fg) = g\partial_k f + f\partial_k g$ which remains valid for $f, g \in W^{1,2}(\Omega)$ by the same argument as in Prop. 3.1.8. Since the right hand side is locally square integrable we have $\partial_k(fg) \in L^2(\Omega)$.

Because C_{ij} is merely a linear combination of products of certain entries of the matrix $(g_{ij})_{1 \leq i,j \leq n}$ the above implies $C_{ij} \in W_{\text{loc}}^{1,2}(U) \cap L_{\text{loc}}^{\infty}(U)$. The same holds true for $\det g_{ij}$. It remains to show that $\frac{1}{f} \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ for $f \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ with $|f| \geq c$. Clearly $\frac{1}{f} \in L^{\infty}(\Omega) \subset L^2(\Omega)$ and $\partial_k\left(\frac{1}{f}\right) = -\frac{\partial_k f}{f^2} \in L^{\infty}(\Omega) \cdot L^2(\Omega) \subset L^2(\Omega)$ (that this formula holds follows again by the same argument as in Prop. 3.1.8). $\square$

To summarize, we have so far shown that a sufficiently regular metric induces an L_{loc}^2-connection which in turn allows the definition of the Riemann and Ricci curvature tensors. Now we want to investigate if a gt-regular metric allows the definition of the scalar curvature via the usual formula of $\mathbf{R} = g^{ij}\,\mathbf{Ric}\,(\partial x^i, \partial x^j)$. At first sight this appears to be problematic because while g^{ij} has nice regularity the Ricci curvature tensor is in general not even L_{loc}^1. However, looking a bit closer one quickly finds an alternative expression for $\mathbf{R}$ in the smooth case which can be directly generalized to the distributional one as it will only involve products of the form $L_{\text{loc}}^2 \cdot L_{\text{loc}}^{\infty} \subset L_{\text{loc}}^2$. First one calculates

$$(3.4.2) \quad \mathbf{Ric}\,(\partial_i, \partial_j) = dx^k\,(\mathbf{Riem}\,(\partial_i, \partial_k)\,\partial_j) = (\nabla_{\partial_k}(\nabla_{\partial_i}\partial_j) - \nabla_{\partial_i}(\nabla_{\partial_k}\partial_j))\,(dx^k) +$$
$$+ dx^k\,(\nabla_{[\partial_i, \partial_k]}\partial_j) = dx^k\,(\nabla_{[\partial_i, \partial_k]}\partial_j) - \partial_i\,(dx^k\,(\nabla_{\partial_k}\partial_j)) + (\nabla_{\partial_k}\partial_j)\,(\nabla_{\partial_i}dx^k) +$$
$$+ \partial_k\,(dx^k\,(\nabla_{\partial_i}\partial_j)) - (\nabla_{\partial_i}\partial_j)\,(\nabla_{\partial_k}dx^k)$$

using equation (3.3.2) from Prop. 3.3.3. Now for an L_{loc}^2-connection the first, the third and the fifth term are at least L_{loc}^2 and can thus be multiplied with $g^{ij} \in L_{\text{loc}}^{\infty}(X)$. Concerning the second and fourth term we see that for a smooth connection

$$(3.4.3) \qquad g^{ij}\,\partial_i\,(dx^k\,(\nabla_{\partial_k}\partial_j)) = \partial_i\,(g^{ij}\,dx^k\,(\nabla_{\partial_k}\partial_j)) - \partial_i g^{ij}\,dx^k\,(\nabla_{\partial_k}\partial_j)$$

and that although the left hand side does not make sense for a merely gt-regular metric the right hand side does since $g^{ij} \in W_{\text{loc}}^{1,2}$ by Lemma 3.4.4 above. Using (3.4.3) to define $g^{ij}\,\partial_i\,(dx^k\,(\nabla_{\partial_k}\partial_j))$ and $g^{ij}\,\partial_k\,(dx^k\,(\nabla_{\partial_i}\partial_j))$ gives the following Proposition.

PROPOSITION 3.4.5. *Let g be a gt-regular metric on X. Then there exists a well-defined scalar curvature distribution $\mathbf{R}$. Locally, $\mathbf{R}$ is given by*

$$\mathbf{R} = g^{ij}\mathbf{Ric}\,(\partial x^i, \partial x^j)$$

where the product is to be understood as discussed above.

Finally we will briefly study how additional regularity of g influences the regularity of ∇.

PROPOSITION 3.4.6. *Let* $g \in W_{\text{loc}}^{k,p}\left(X, T_2^0 X\right) \cap L_{\text{loc}}^{\infty}\left(X, T_2^0 X\right)$ *with* $p \geq \frac{n}{k}$ *be a uniformly non-degenerate metric on* X. *Then* ∇ *as in* (3.4.1) *is a* $W_{\text{loc}}^{k-1,p}$-*connection.*

PROOF. The proof follows the same general steps used in showing that a gt-regular metric induces an L_{loc}^2-connection and in showing $W_{\text{loc}}^{1,2}$-regularity of g^{ij} in Lemma 3.4.4. First we note that by Prop. 3.1.8 $W_{\text{loc}}^{k,p}\left(X\right) \cap L_{\text{loc}}^{\infty}\left(X\right)$ is an algebra so $\det g_{ij} \in W_{\text{loc}}^{k,p}\left(X\right) \cap L_{\text{loc}}^{\infty}\left(X\right)$ and $C_{ij} \in W_{\text{loc}}^{k,p}\left(X\right) \cap L_{\text{loc}}^{\infty}\left(X\right)$. Since $\left|\det g_{ij}\right| \geq c$ (on compact subsets of the given chart domain) we have that $\frac{1}{\det g_{ij}} \in W_{\text{loc}}^{k,p}\left(X\right) \cap L_{\text{loc}}^{\infty}\left(X\right)$ (by Prop. 3.1.8) and thus $g^{ij} \in W_{\text{loc}}^{k,p}\left(X\right) \cap L_{\text{loc}}^{\infty}\left(X\right)$. Next we use that ∇ can be expressed explicitly in terms of the metric via $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$ with $\Gamma_{ij}^k = \frac{1}{2}g^{km}\left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right)$ (which can be shown completely analogous to the smooth case). This implies that ∇ has the same regularity as Γ, so $\nabla \in W_{\text{loc}}^{k-1,p}\left(X\right)$ since terms of the form $g^{km}\partial_r g_{ij}$ are in $W_{\text{loc}}^{k-1,p}\left(X\right)$ (since $\partial^\alpha\left(g^{km}\partial_r g_{ij}\right)$ is a linear combination of terms appearing in $\partial^{\alpha+e_r}\left(g^{km}g_{ij}\right)$ and those are all in $L_{\text{loc}}^p(X)$ for $|\alpha| \leq k-1$ by the proof of Prop. 3.1.8). $\qquad\square$

REMARK 3.4.7. In many cases the L_{loc}^{∞} regularity requirement in the previous proposition is actually superfluous. For instance, for $p > \frac{n}{k}$ one has $W^{k,p}\left(\Omega\right) \subset \mathcal{C}\left(\Omega\right) \subset L_{\text{loc}}^{\infty}\left(\Omega\right)$ (by the Sobolev embedding, see e.g. [AF03] Thm. 4.12) and so $g \in W_{\text{loc}}^{k,p}\left(X\right)$ already implies $g \in L_{\text{loc}}^{\infty}\left(X\right)$.

3.4.1. Jump formula for the scalar curvature. We will again consider the situation outlined in subsection 3.3.1, that is we have a hypersurface $M \subset X$, such that $X = X^+ \cup X^-$ where $X^\pm$ are manifolds with boundary and $\partial X^\pm = X^+ \cap X^- = M$. We want to derive a jump formula for the scalar curvature across the hypersurface similar to the ones for Riemannian and Ricci curvature given in equations (3.3.11) and (3.3.12).

PROPOSITION 3.4.8. *Let* $g \in \mathcal{C}\left(X, T_2^0 X\right)$ *be a uniformly non-degenerate metric on* X *such that* $g^\pm \in W_{\text{loc}}^{2,p}\left(X^\pm\right) \cap W_{\text{loc}}^{1,2}\left(X^\pm\right)$ *for some* $p \geq \frac{n}{2}$. *Then the scalar curvature* $\mathbf{R}$ *is well-defined (i.e.* g *is gt-regular) and*

(3.4.4) $$\mathbf{R} = \mathbf{R}^{\text{reg}} + \left[\left(g^{kj}dx^1 - g^{1j}dx^k\right)\left(\nabla_{\partial_{x^k}}\partial x^j\right)\right]_M \delta_M\left(\partial x^1\right)$$

for coordinates $\left(x^1, \ldots, x^n\right)$ *adapted to the hypersurface (see Footnote 1 on page 31).*

PROOF. Since g is continuous on X one has $g \in L_{\text{loc}}^{\infty}\left(X\right)$. To show that $g \in W_{\text{loc}}^{1,2}\left(X\right)$ we first note that $W_{\text{loc}}^{2,p}\left(\Omega\right) \subset W_{\text{loc}}^{1,2}\left(\Omega\right)$ for $p \geq \frac{n}{2}$ and $\Omega \subset \mathbb{R}^n$: For $p \geq 2$ this follows from $L_{\text{loc}}^p\left(\Omega\right) \subset L_{\text{loc}}^q\left(\Omega\right)$ for all $1 \leq q \leq p$ which is an easy consequence of Hölder's inequality. So it only remains to consider the case $1 \leq p < 2$ which implies $n = 1$, $n = 2$ or $n = 3$. For $n = 1$ one has $p > n$ or $p = n = 1$ and in both cases $W^{1,p}\left(\Omega\right) \subset L^2\left(\Omega\right)$ by the Sobolev embedding ([AF03] Thm. 4.12, A). If $n = 2$ or $n = 3$, then $\frac{n}{2} \leq p < 2 \leq n$ and $p \leq 2 \leq \frac{2p}{2-p}$ (or $p \leq 2 \leq 3 = \frac{3}{2}3/3 - \frac{3}{2} \leq \frac{3p}{3-p}$) and again $W^{1,p}\left(\Omega\right) \subset L^2\left(\Omega\right)$ by one of the Sobolev embeddings ([AF03] Thm. 4.12, C).

It remains to show that $g \in W_{\text{loc}}^{1,2}\left(X\right)$. To do this, let $\Omega \subset \mathbb{R}^n$ and set $\Omega^+ := \Omega \cap \mathbb{R}_+^n$ and $\Omega^- := \Omega \cap \mathbb{R}_-^n$ and let $f \in \mathcal{C}\left(\Omega\right)$ such that $f^\pm \in W_{\text{loc}}^{1,p}\left(\Omega^\pm\right)$ for some $1 \leq p \leq \infty$. We are going to show that $\partial_i f = \left(\partial_i f\right)^{\text{reg}} \in L_{\text{loc}}^p\left(\Omega\right)$ (one notes that this only holds for continuous f, for example the Heaviside function $H \in L_{\text{loc}}^1\left(\mathbb{R}\right)$ satisfies $\partial H = \delta$ but $\left(\partial H\right)^{\text{reg}} = 0$). Let $\phi \in \mathcal{D}\left(\Omega\right)$, then

$$-\langle\partial_i f, \phi\rangle = \langle f, \partial_i \phi\rangle = \int_{\mathbb{R}^n} f(x)\partial_i\phi(x)dx = \int_{\mathbb{R}_+^n} f^+(x)\,\partial_i\phi(x)dx + \int_{\mathbb{R}_-^n} f^-(x)\,\partial_i\phi(x)dx =$$

$$= -\int_{\mathbb{R}_+^n} \partial_i f^+(x)\,\phi(x)dx - \int_{\mathbb{R}^{n-1}}\left(f^+\phi\right)(0, x')\,dx' - \int_{\mathbb{R}_-^n} \partial_i f^-(x)\,\phi(x)dx +$$

$$+ \int_{\mathbb{R}^{n-1}}\left(f^-\phi\right)(0, x')\,dx' = -\int_{\mathbb{R}_+^n} \partial_i f^+(x)\,\phi(x)dx - \int_{\mathbb{R}_-^n} \partial_i f^-(x)\,\phi(x)dx = -\langle\left(\partial_i f\right)^{\text{reg}}, \phi\rangle,$$

where the two integrals over $\mathbb{R}^{n-1}$ cancel because $f^+|_{\partial\Omega^+\cap\partial\mathbb{R}^n_+} = f^-|_{\partial\Omega^-\cap\partial\mathbb{R}^n_-}$ by continuity of f. This shows that $g \in W^{1,p}_{\mathrm{loc}}(X)$, which immediately implies gt-regularity of g.

Now we are ready to prove (3.4.4). By the definition of the scalar curvature (see (3.4.2) and (3.4.3)) we have

$$\mathbf{R} = g^{ij}\mathbf{Ric}\,(\partial_i,\partial_j) = g^{ij}dx^k\left(\nabla_{[\partial_i,\partial_k]}\partial_j\right) - \partial_i\left(g^{ij}\,dx^k\left(\nabla_{\partial_k}\partial_j\right)\right) + \partial_i g^{ij}\,dx^k\left(\nabla_{\partial_k}\partial_j\right) +$$
$$+ g^{ij}\left(\nabla_{\partial_k}\partial_j\right)\left(\nabla_{\partial_i}dx^k\right) + \partial_k\left(g^{ij}\,dx^k\left(\nabla_{\partial_i}\partial_j\right)\right) - \partial_k g^{ij}\,dx^k\left(\nabla_{\partial_i}\partial_j\right) - g^{ij}\left(\nabla_{\partial_i}\partial_j\right)\left(\nabla_{\partial_k}dx^k\right)$$

If we look at the regularity of all the terms appearing in this sum, we see that only the second and fifth term are not at least $W^{1,2}_{\mathrm{loc}}$ and thus not equal to their regular parts. Looking at the proof of Thm. 3.3.9, where we derived $\mathbf{u}\left(\omega\left(\mathbf{v}\right)\right) = \left(\mathbf{u}\left(\omega\left(\mathbf{v}\right)\right)\right)^{\mathrm{reg}} + \left[\omega\left(\mathbf{v}\right)\right]_M\delta_M\left(\mathbf{u}\right)$ for $\omega \in \mathcal{T}^0_1\left(X\right)$, $\mathbf{u} \in \mathfrak{X}\left(X\right)$ and $\mathbf{v} \in W^{1,s}_{\mathrm{loc}}\left(X^\pm,TX^\pm\right)\cap L^2_{\mathrm{loc}}\left(X,TX\right)$, yields the jump formula

$$\partial_i\left(g^{ij}\,dx^k\left(\nabla_{\partial_k}\partial_j\right)\right) = \left(\partial_i\left(g^{ij}\,dx^k\left(\nabla_{\partial_k}\partial_j\right)\right)\right)^{\mathrm{reg}} + \left[g^{ij}dx^k\left(\nabla_{\partial_k}\partial_j\right)\right]_M\delta_M\left(\partial_i\right)$$

for the second and fifth term. We obtain

$$g^{ij}\mathbf{Ric}\,(\partial_i,\partial_j) = g^{ij}\left(\mathbf{Ric}\,(\partial_i,\partial_j)\right)^{\mathrm{reg}} - \left[g^{ij}dx^k\left(\nabla_{\partial_k}\partial_{jx^j}\right)\right]_M\delta_M\left(\partial_i\right) +$$
$$+ \left[g^{ij}dx^k\left(\nabla_{\partial_i}\partial_j\right)\right]_M\delta_M\left(\partial_k\right) = \mathbf{R}^{\mathrm{reg}} + \left[g^{kj}dx^1\left(\nabla_{\partial_k}\partial_j\right) - g^{1j}dx^k\left(\nabla_{\partial_k}\partial_j\right)\right]_M\delta_M\left(\partial x^1\right),$$

where we used that $\delta_M\left(\partial x^j\right) = 0$ for $j = 2,\ldots,n$ in adapted coordinates by Remark 3.3.7. $\qquad\square$

REMARK 3.4.9. One would like to simply use the jump formula (3.3.12) (or (3.3.13) to be more precise) for the Ricci curvature (we have $\nabla^\pm \in W^{1,p}_{\mathrm{loc}}\left(X^\pm\right)$ by Prop. 3.4.6) to obtain

$$\mathbf{R} = g^{ij}\mathbf{Ric}\left(\partial x^i,\partial x^j\right) = g^{ij}\left(\mathbf{Ric}\left(\partial x^i,\partial x^j\right)\right)^{\mathrm{reg}} + g^{ij}\left[dx^1\left(\nabla_{\partial x^i}\partial x^j\right)\right]_M\delta_M\left(\partial x^1\right) -$$
$$-g^{ij}\left[dx^k\left(\nabla_{\partial x^k}\partial x^j\right)\right]_M\delta_M\left(\partial x^i\right) = \mathbf{R}^{\mathrm{reg}} + \left[g^{kj}dx^1\left(\nabla_{\partial x^k}\partial x^j\right) - g^{1j}dx^k\left(\nabla_{\partial x^k}\partial x^j\right)\right]_M\delta_M\left(\partial x^1\right).$$

Unfortunately this reasoning is a bit problematic because $g^{ij}\mathbf{Ric}\left(\partial x^i,\partial x^j\right)$ is only a convenient notation for the scalar curvature, the actual definition of $\mathbf{R}$ is more complicated (albeit equivalent to this multiplication for smooth metrics). Also the terms $g^{ij}\left[dx^1\left(\nabla_{\partial x^i}\partial x^j\right)\right]_M\delta_M\left(\partial x^1\right)$ are a priori not well-defined distributions (but are of course to be understood in the sense of $\left[g^{ij}dx^1\left(\nabla_{\partial x^i}\partial x^j\right)\right]_M\delta_M\left(\partial x^1\right)$).

3.5. Compatibility with Colombeau theory

A somewhat different approach to low regularity metrics is provided by Colombeau theory. One of the main problems in the previous discussions was to make sense of nonlinear operations involving the metric which necessitated the requirement of at least gt-regularity for g. An alternative way to deal with those difficulties is to use a nonlinear theory of generalized functions (in the sense of J.F. Colombeau) instead of classical distribution theory. In this section we will briefly discuss connections and differences between the two approaches, following [**SV09**].

To do so we first need to review some of the most basic facts about the special Colombeau algebra on manifolds, more details can be found in [**GKOS01**], specifically chapter 1.2 for $\mathbb{R}^n$ and chapter 3.2 for the manifold case. Let $\mathcal{P}(X)$ be the space of linear differential operators on X. We define the space $\mathcal{E}_M(X) \subset \mathcal{C}^\infty(X)^{(0,1)}$ of *moderate nets* by

$$\mathcal{E}_M(X) := \left\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X)^{(0,1)} : \forall\,\mathrm{compact}\,K\,\forall P \in \mathcal{P}(X)\exists N \in \mathbb{N} : \sup_{p\in K}|Pu_\varepsilon(p)| = O(\varepsilon^{-N})\right\}$$

and the space $\mathcal{N}(X)$ of *negligible nets* by

$$\mathcal{N}(X) := \left\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X)^{(0,1)} : \forall\,\mathrm{compact}\,K\,\forall P \in \mathcal{P}(X)\forall m \in \mathbb{N} : \sup_{p\in K}|Pu_\varepsilon(p)| = O(\varepsilon^m)\right\}.$$

The special Colombeau algebra is the quotient algebra

$$\mathcal{G}(X) = \frac{\mathcal{E}_M(X)}{\mathcal{N}(X)}.$$

Alternatively, one may describe $\mathcal{G}(X)$ locally: A net $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X)^{(0,1)}$ is moderate respectively negligible if and only if its chart representations $(u_\varepsilon^\alpha)_\varepsilon = \left(u_\varepsilon \circ \psi_\alpha^{-1}\right)_\varepsilon \in \mathcal{C}^\infty(\psi_\alpha(U_\alpha))^{(0,1)}$ are moderate respectively negligible for every chart (U_α, ψ_α). So given any $u \in \mathcal{G}(X)$ we may use this to define corresponding $u^\alpha \in \mathcal{G}(\psi_\alpha(U_\alpha))$ and these u^α determine u uniquely which allows us to identify $\mathcal{G}(X)$ with the set of all families $(u^\alpha)_\alpha$ with $u^\alpha \in \mathcal{G}(\psi_\alpha(U_\alpha))$ satisfying

$$u^\alpha\big|_{\psi_\alpha(U_\alpha \cap U_\beta)} = u^\beta\big|_{\psi_\beta(U_\alpha \cap U_\beta)} \circ \psi_\beta \circ \psi_\alpha^{-1}$$

for all α, β with $U_\alpha \cap U_\beta \neq \emptyset$ (see [**GKOS01**], Prop. 3.2.3).

This algebra allows both an embedding $\sigma : \mathcal{C}^\infty(X) \to \mathcal{G}(X)$ of smooth functions ($\sigma(f)$ is the equivalence class of the constant net $(f)_\varepsilon$) and an embedding $\iota : \mathcal{D}'(X) \to \mathcal{G}(X)$ of distributions: For an open subset $\Omega \subset \mathbb{R}^n$ the embedding of distributions is done via convolution with some $\rho \in \mathcal{S}(\mathbb{R}^n)$ with unit integral that satisfies $\int_{\mathbb{R}^n} \rho(x)x^\alpha dx = 0$ for all $|\alpha| \geq 1$. Given such a function ρ we define $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(\frac{x}{\varepsilon})$. If T is a distribution on $\mathbb{R}^n$ with compact support we simply set $\iota(T) := [(T * \rho_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$ and we have $\iota|_{\mathcal{C}_c^\infty} = \sigma$.

For $T \in \mathcal{D}'(X)$ we choose a (countable) atlas (U_α, ψ_α), a partition of unity $\{\chi_\alpha\}$ subordinate to the U_α and functions $\zeta_\alpha \in \mathcal{D}(U_\alpha)$ with $|\zeta_\alpha| \leq 1$ and $\zeta_\alpha \equiv 1$ on an open neighborhood of $\text{supp}(\chi_\alpha)$ in U_α and set

(3.5.1) $$\iota(T)_\varepsilon = \sum_\alpha \zeta_\alpha \cdot \left(\left(\chi_\alpha \circ \psi_\alpha^{-1}\, T^\alpha\right) * \rho_\varepsilon\right) \circ \psi_\alpha,$$

where $T^\alpha \in \mathcal{D}'(\psi_\alpha(U_\alpha))$ denotes the chart representation of T, see [**GKOS01**], Thm. 3.2.10. Again one has $\iota|_{\mathcal{C}^\infty} = \sigma$.

This embedding ι allows us to view distributions as elements in $\mathcal{G}(X)$, however, it depends on the choice of the mollifier ρ and the partition of unity χ_α. Next we want to try to associate a distribution to a given generalized function u by taking the distributional limit (which need not exist) of a representative u_ε.

DEFINITION 3.5.1. An element $u \in \mathcal{G}(X)$ is called associated with 0 (denoted by $u \approx 0$) if $u_\varepsilon \to 0$ in $\mathcal{D}'(X)$ for one (and hence any) representative $(u_\varepsilon)_\varepsilon$. Two generalized functions $u, v \in \mathcal{G}(X)$ are associated, $u \approx v$, if $u - v \approx 0$.

This gives an equivalence relation $\approx$ on $\mathcal{G}(X)$ that is strictly weaker than equality (see [**GKOS01**], 1.2.68 for examples). If $u \in \mathcal{G}(X)$ with $u \approx \iota(T)$ for some $T \in \mathcal{D}'(X)$ we may suppress the embedding and simply write $u \approx T$. Some nice properties of association are listed in [**GKOS01**], Prop. 3.2.12 and 3.2.14.

Of course we are also going to need spaces of generalized sections of vector bundles E over X, which will be denoted by $\mathcal{G}(X, E)$ and can be defined analogously to $\mathcal{G}(X)$. A more convenient description for our purposes is, however, given by

$$\mathcal{G}(X, E) = \mathcal{G}(X) \otimes \Gamma(X, E)$$

or

$$\mathcal{G}(X, E) = L_{\mathcal{C}^\infty(X)}(\Gamma(X, E^*); \mathcal{G}(X)).$$

Note that we used the analogous descriptions for $\mathcal{D}'(X, E)$ (see Thm. 1.2.4) quite heavily in the previous sections. For generalized tensor fields we may simply write $\mathcal{G}_s^r(X)$ instead of $\mathcal{G}(X, T_s^r X)$. Given a distributional tensor field $T \in \mathcal{D}'(X, T_s^r X)$ we want to define its embedding $\iota(T) \in \mathcal{G}_s^r(X)$: This is done in the same way as for the scalar case. Given a vector bundle atlas (U_α, Ψ_α) and a

corresponding manifold atlas (U_α, ψ_α) we can (by Prop. 1.2.3) identify T with the family of its chart representations $T^\alpha := \left((T^\alpha)^{j_1,\dots,j_r}_{i_1,\dots,i_s} \right)_{1 \le i_k, j_l \le n} \in \mathcal{D}'\left(\psi_\alpha(U_\alpha), \mathbb{R}^{n^{r+s}} \right)$ and define $\left(\chi_\alpha \circ \psi_\alpha^{-1} T^\alpha \right) * \rho_\varepsilon$ componentwise. Then

$$(3.5.2) \qquad \iota(T)_\varepsilon := \sum_\alpha \zeta_\alpha \cdot \left(\left(\chi_\alpha \circ \psi_\alpha^{-1} T^\alpha \right) * \rho_\varepsilon \right) \circ \Psi_\alpha,$$

where the ζ_α and χ_α are chosen as in (3.5.1).

Now we are ready to define the concept of a generalized metric in the Colombeau sense.

DEFINITION 3.5.2. A generalized $(0, 2)$-tensor field $g \in \mathcal{G}_2^0(X)$ is called generalized metric on X if it is symmetric and $\det g_{ij} \in \mathcal{G}(U_\alpha)$ is invertible in $\mathcal{G}(U_\alpha)$ for every chart (U_α, ψ_α).

REMARK 3.5.3. Invertibility of $\det g_{ij}$ in $\mathcal{G}(U_\alpha)$ is equivalent to invertibility of $\det g_{ij}^\alpha$ in $\mathcal{G}(\psi_\alpha(U_\alpha))$ which in turn is equivalent to the following condition (see [**GKOS01**], Thm. 1.2.5.):

$$\forall \text{ compact } K \subset \psi_\alpha(U_\alpha) \exists m \in \mathbb{N}, \varepsilon_0 > 0 : \inf_{p \in K} |\det g_\varepsilon^\alpha| \ge \varepsilon^m \, \forall \varepsilon < \varepsilon_0.$$

Given a generalized metric g we may use the usual formulas on a representative $(g_\varepsilon)_\varepsilon$ to obtain an inverse metric $g^{-1} \in \mathcal{G}_0^2(X)$, a generalized Levi-Civita connection $\nabla : \mathfrak{X}(X) \times \mathfrak{X}(X) \to \mathcal{G}_0^1(X)$ and all the curvature quantities (e.g., the Riemann- and Ricci tensor as well as the scalar curvature). A natural question to ask is if these operations commute with the embedding ι up to association for gt-regular metrics (i.e., if $\iota(g)^{-1} \approx g^{-1}$, $\mathbf{Riem}(\iota(g)) \approx \mathbf{Riem}(g), \dots$) and, more fundamentally, whether $\iota(g)$ even is a generalized metric for any gt-regular and uniformly non-degenerate distributional metric g.

To proceed we will need to use a mollifier with special properties in the embedding ι:

DEFINITION 3.5.4 (Admissible mollifier). A net $(\rho_\varepsilon)_\varepsilon$ of smooth functions is called *admissible* if

(1) $\mathrm{supp}\,(\rho_\varepsilon) \subset B_\varepsilon(0)$ and $\int \rho_\varepsilon(x) dx = 1 \quad$ for all $\varepsilon \in (0, 1]$,
(2) it is moderate, i.e., $\forall \alpha \in \mathbb{N}^n \exists m \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} |\partial^\alpha \rho_\varepsilon(x)| = O(\varepsilon^{-m})$,
(3) it has finally vanishing moments, i.e.,

$$\forall j \in \mathbb{N} \exists \varepsilon_0 \in (0, 1] : \int x^\alpha \rho_\varepsilon(x) dx = 0 \quad \text{for all } 1 \le |\alpha| \le j \text{ and all } \varepsilon \le \varepsilon_0$$

(4) the negative parts have arbitrarily small L^1-norm, i.e.,

$$\forall \eta > 0 \exists \varepsilon_0(\eta) \in (0, 1] : \int |\rho_\varepsilon(x)| \, dx \le 1 + \eta \quad \text{for all } \varepsilon \le \varepsilon_0(\eta)$$

The existence of such admissible nets is shown in the appendix of [**SV09**]. It can also be shown that given an admissible net $(\rho_\varepsilon)_\varepsilon$ one obtains an embedding $\iota_\rho : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n)$ through componentwise convolution and thus $\iota_\rho : \mathcal{D}'(X) \to \mathcal{G}(X)$ via (3.5.1) (although $(\rho_\varepsilon)_\varepsilon$ does not satisfy the original conditions imposed on the net used in the embedding ι given by (3.5.1) the relevant properties of ι_ρ follow analogously - on $\mathbb{R}^n$ this is again outlined in the appendix of [**SV09**], the generalization to manifolds proceeds in literally the same way as in [**GKOS01**], Thm. 3.2.10).

Because $\mathrm{supp}\,(\rho_\varepsilon) \subset B_\varepsilon(0)$, $\int \rho_\varepsilon(x) dx = 1$ and $\|\rho_\varepsilon\|_{L^1} \le C$ for ε small, ρ_ε is a mollifier and the embedding ι_ρ inherits many nice properties from the various convergence theorems regarding convolutions of the form $f * \rho_\varepsilon$ that can be found, e.g., in [**Eva98**], p. 630 (note that while Evans uses a special mollifier the proofs only use those three properties).

PROPOSITION 3.5.5. *If $T \in W^{k,p}_{\mathrm{loc}}(X, T^r_s X)$ (for some $1 \le p < \infty$), then $\iota_\rho(T)_\varepsilon \in W^{k,p}_{\mathrm{loc}}(X, T^r_s X)$ and $\iota_\rho(T)_\varepsilon \to T$ pointwise almost everywhere and in $W^{k,p}_{\mathrm{loc}}(X)$ as $\varepsilon \to 0$.*

PROOF. We know that $f * \rho_\varepsilon \in W_{\text{loc}}^{k,p}(\Omega)$ and $f * \rho_\varepsilon \to f$ pointwise a.e. and in $W_{\text{loc}}^{k,p}(\Omega)$ for $f \in W_{\text{loc}}^{k,p}(\Omega)$ with compact support (see [**Eva98**], p. 630 and use $\partial_i(f * \rho_\varepsilon) = (\partial_i f) * \rho_\varepsilon$). This shows that

$$\left(\zeta_\alpha \cdot \left(\left(\chi_\alpha \circ \psi_\alpha^{-1} T^\alpha \right) * \rho_\varepsilon \right) \circ \Psi_\alpha \right) \circ \Psi_\alpha^{-1} = \zeta_\alpha \circ \psi_\alpha^{-1} \cdot \left(\left(\chi_\alpha \circ \psi_\alpha^{-1} T^\alpha \right) * \rho_\varepsilon \right) \to \zeta_\alpha \circ \psi_\alpha^{-1} \cdot \chi_\alpha \circ \psi_\alpha^{-1} T^\alpha$$

in $W^{k,p}(\psi_\alpha(U_\alpha), \mathbb{R}^{n^{r+s}})$ and thus each term in (3.5.2) converges to $\zeta_\alpha \cdot \chi_\alpha T = \chi_\alpha T$ pointwise a.e. and in $W_{\text{loc}}^{k,p}(X, T_s^r X)$ (by one of the equivalent the definitions of $W_{\text{loc}}^{k,p}$-convergence of tensor fields, see section 3.1). Because the sum is locally finite we only have to consider a finite number of terms for each compact $K \subset X$, proving the claim. $\square$

COROLLARY 3.5.6. *If $T \in W_{\text{loc}}^{k,p}(X, T_s^r X)$ (for some $1 \le p < \infty$), then*

$$\iota_\rho(T)_\varepsilon(\mathbf{u}_1, \ldots, \mathbf{u}_s, \omega_1, \ldots, \omega_r) - \iota_\rho(T(\mathbf{u}_1, \ldots, \mathbf{u}_s, \omega_1, \ldots, \omega_r))_\varepsilon \to 0$$

in $W_{\text{loc}}^{k,p}(X)$ for all $\omega_1, \ldots, \omega_r \in \mathcal{T}_1^0(X)$ and $\mathbf{u}_1, \ldots, \mathbf{u}_s \in \mathfrak{X}(X)$.

PROOF. By Prop. 3.5.5 both $\iota_\rho(T)_\varepsilon(\mathbf{u}_1, \ldots, \mathbf{u}_s, \omega_1, \ldots, \omega_r)$ and $\iota_\rho(T(\mathbf{u}_1, \ldots, \mathbf{u}_s, \omega_1, \ldots, \omega_r))_\varepsilon$ converge to $T(\mathbf{u}_1, \ldots, \mathbf{u}_s, \omega_1, \ldots, \omega_r)$ in $W_{\text{loc}}^{k,p}(X)$, so their difference converges to zero. $\square$

PROPOSITION 3.5.7. *If $T \in L_{\text{loc}}^\infty(X, T_s^r X)$, then*

$$\iota_\rho(T)_\varepsilon(\mathbf{u}_1, \ldots, \mathbf{u}_s, \omega_1, \ldots, \omega_r) - \iota_\rho(T(\mathbf{u}_1, \ldots, \mathbf{u}_s, \omega_1, \ldots, \omega_r))_\varepsilon \to 0$$

uniformly on compact sets.

PROOF. Let $\psi_\alpha = (x^1, \ldots, x^n)$ be a manifold chart, then

$$(T^\alpha)_{i_1, \ldots, i_s}^{j_1, \ldots, j_r} = \left(T(\partial x^{i_1}, \ldots, \partial x^{i_s}, dx^{j_1}, \ldots, dx^{j_r}) \right)^\alpha$$

for the vector bundle chart $\Psi_\alpha = T_s^r \psi_\alpha$, thus

$$T(\mathbf{u}_1, \ldots, \mathbf{u}_s, \omega_1, \ldots, \omega_r)^\alpha = (T^\alpha)_{i_1, \ldots, i_s}^{j_1, \ldots, j_r} (u^\alpha)^{i_1} \ldots (u^\alpha)^{i_s} (\omega^\alpha)_{j_1} \ldots (\omega^\alpha)_{j_r}.$$

Now let $\Omega \subset \mathbb{R}^n$, $f \in L_{\text{loc}}^\infty(\Omega)$ with compact support in Ω and $g \in \mathcal{C}^\infty(\Omega)$. Then $(f * \rho_\varepsilon) \cdot g - (fg) * \rho_\varepsilon \to 0$ uniformly on Ω: We have

$$((f * \rho_\varepsilon) \cdot g - (fg) * \rho_\varepsilon)(x) = \int_{\mathbb{R}^n} f(x-y)(g(x) - g(x-y))\rho_\varepsilon(y)dy \le$$

$$\le \|f\|_{L^\infty(\text{supp} f + B_\varepsilon(0))} \|\rho_\varepsilon\|_{L^1} \sup_{y \in B_\varepsilon(0)} |f(x) - f(x-y)| = C \sup_{y \in B_\varepsilon(0)} |g(x) - g(x-y)|$$

since $\|\rho_\varepsilon\|_{L^1}$ is bounded by the fourth requirement in Def. 3.5.4. Furthermore $\text{supp}\,(f * \rho_\varepsilon) \cdot g - (fg) * \rho_\varepsilon \subset \text{supp} f + B_\varepsilon(0)$ is a compact subset of Ω for ε small and thus $(f * \rho_\varepsilon) \cdot g - (fg) * \rho_\varepsilon \to 0$ uniformly by uniform continuity of g on compact subsets.

To illustrate that this proves the claim, let w.l.o.g. $T \in L_{\text{loc}}^\infty(X, TX)$. From the above we get that $\left(\left(\chi_\alpha \circ \psi_\alpha^{-1} T^\alpha \right) * \rho_\varepsilon \right)^j (\omega^\alpha)_j - \left(\chi_\alpha \circ \psi_\alpha^{-1} \left((T^\alpha)^j (\omega^\alpha)_j \right) \right) * \rho_\varepsilon \to 0$ in $L_{\text{loc}}^\infty(\psi_\alpha(U_\alpha))$ and thus each summand appearing in the difference $\iota(T)_\varepsilon(\omega) - \iota(T(\omega))_\varepsilon$ converges to zero uniformly. Since the sum is locally finite we immediately obtain that $\iota(T)_\varepsilon(\omega) - \iota(T(\omega))_\varepsilon \to 0$ uniformly on compact sets. $\square$

We also get the following result regarding positivity and invertibility of $\iota_\rho(f)_\varepsilon$.

PROPOSITION 3.5.8. *Let $f \in L_{\text{loc}}^\infty(X)$, $f > 0$ a.e. and locally uniformly bounded, i.e., for every compact set $K \subset X$ there exists a constant C_K such that $f(x) \ge C_K > 0$ a.e. on K. Then for any admissible mollifier $(\rho_\varepsilon)_\varepsilon$ we have:*

(1) *The net $\iota_\rho(f)_\varepsilon$ is locally uniformly bounded, i.e.,*

(3.5.3) $\qquad \forall L \subset X$ *compact* $\exists C_L' \exists \varepsilon_0(L) : \iota_\rho(f)_\varepsilon(x) \ge C_L' > 0 \ \forall x \in L, \ \forall \varepsilon \le \varepsilon_0(L)$

(2) *If, in addition, $f \in W^{1,2}_{\mathrm{loc}}(X)$ then $\frac{1}{\iota_\rho(f)_\varepsilon} \to \frac{1}{f}$ in $W^{1,2}_{\mathrm{loc}}$ for $\varepsilon \to 0$.*

PROOF. Let $L \subset X$ be compact, then $\iota_\rho(f)_\varepsilon|_L$ is a finite sum of terms of the form $\zeta_\alpha \cdot \left(\left(\chi_\alpha \circ \psi_\alpha^{-1} f^\alpha \right) * \rho_\varepsilon \right) \circ \psi_\alpha$ and since $\zeta_\alpha \cdot \left(\left(\chi_\alpha \circ \psi_\alpha^{-1} f^\alpha \right) * \rho_\varepsilon \right) \circ \psi_\alpha = \left(\left(\chi_\alpha \circ \psi_\alpha^{-1} f^\alpha \right) * \rho_\varepsilon \right) \circ \psi_\alpha$ for ε small enough it suffices to show (3.5.3) for each term which is in turn equivalent to showing the estimate for functions $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, $f > 0$ with compact support.

Let f be such a function and $L, K \subset \mathbb{R}^n$ be compact with $K^\circ \supset L$ and ε so small that $L + B_\varepsilon(0) \subset K$. For $\rho_\varepsilon^+ := \max(\rho_\varepsilon, 0)$ and $\rho_\varepsilon^- := -\min(\rho_\varepsilon, 0)$ conditions one and four in Def. 3.5.4 give $1 = \|\rho_\varepsilon^+\|_{L^1} - \|\rho_\varepsilon^-\|_{L^1}$ and $1 + \eta \geq \|\rho_\varepsilon^+\|_{L^1} + \|\rho_\varepsilon^-\|_{L^1}$, i.e.,

$$\left\| \rho_\varepsilon^+ \right\|_{L^1} \geq 1 \text{ and } \left\| \rho_\varepsilon^- \right\|_{L^1} \leq \frac{\eta}{2}$$

for $\varepsilon \leq \varepsilon_0(\eta)$. Now we may estimate

$$f * \rho_\varepsilon(x) = f * \left(\rho_\varepsilon^+ - \rho_\varepsilon^- \right)(x) \geq f * \rho_\varepsilon^+(x) - \left\| f * \rho_\varepsilon^- \right\|_{L^\infty(L)} \geq f * \rho_\varepsilon^+(x) - \|f\|_{L^\infty(K)} \left\| \rho_\varepsilon^- \right\|_{L^1(L)}$$

and using $f|_K \geq C_K$ gives $f * \rho_\varepsilon^+(x) \geq C_K \int \rho_\varepsilon^+(x) dx = C_K \|\rho_\varepsilon^+\|_{L^1} \geq C_K$, so altogether

$$\iota_\rho(f)_\varepsilon = f * \rho_\varepsilon(x) \geq C_K - \|f\|_{L^\infty(K)} \frac{\eta}{2} \geq \frac{C_K}{2}$$

for η small enough, showing (3.5.3).

Regarding the second claim we first note that (3.5.3) shows that $\iota_\rho(f)_\varepsilon$ is invertible in $W^{1,2}_{\mathrm{loc}}(X) \cap L^\infty_{\mathrm{loc}}(X)$ and that $\partial_j \left(\frac{1}{f} \right) = -\frac{\partial_j f}{f^2}$ (see Lem. 3.4.4). Now the convergence follows from $\frac{1}{f} - \frac{1}{\iota_\rho(f)_\varepsilon} = \frac{1}{f \, \iota_\rho(f)_\varepsilon} \left(\iota_\rho(f)_\varepsilon - f \right) \to 0$ in $L^2_{\mathrm{loc}}(X)$ and $-\frac{\partial_j f}{f^2} + \frac{\partial_j(\iota_\rho(f)_\varepsilon)}{(\iota_\rho(f)_\varepsilon)^2} = \frac{1}{f^2 \, (\iota_\rho(f)_\varepsilon)^2} \left(\partial_j(\iota_\rho(f)_\varepsilon) - \partial_j f \right) \to 0$ in $L^2_{\mathrm{loc}}(X)$ (since $\iota_\rho(f)_\varepsilon \to f$ in $W^{1,2}_{\mathrm{loc}}(X)$ by Prop. 3.5.5 and $\frac{1}{f \, \iota_\rho(f)_\varepsilon}$ is essentially bounded on compact sets for ε small). $\qquad\square$

Coming back to the issue of compatibility between distributional geometry and generalized geometry in the Colombeau sense we will now take a closer look at the determinant of a given distributional metric g and its embedding $\iota_\rho(g)$.

PROPOSITION 3.5.9. *Let g be a gt-regular metric on X. Then*

$$W^{1,2}_{\mathrm{loc}}(U_\alpha) \cap L^\infty_{\mathrm{loc}}(U_\alpha) \ni \det \left(\iota_\rho(g)_\varepsilon \right)_{ij} \to \det g_{ij} \text{ in } W^{1,2}_{\mathrm{loc}} \text{ and pointwise a.e}$$

for all chart domains U_α of X. In particular, $\det(\iota_\rho(g)_{ij}) \approx \det g_{ij}$.

PROOF. Pointwise convergence almost everywhere follows from pointwise a.e. convergence $\iota_\rho(g)_\varepsilon \to g$. By Lemma 3.4.4 the space $W^{1,2}_{\mathrm{loc}}(X) \cap L^\infty_{\mathrm{loc}}(X)$ is an algebra and the Leibniz rule holds, so

$$\|\partial_i(fg)\|_{L^2} \leq \|g\|_{L^2} \|\partial_i f\|_{L^2} + \|\partial_i g\|_{L^2} \|f\|_{L^2} \leq \|g\|_{W^{1,2}} \|f\|_{W^{1,2}},$$

showing that multiplication of functions in $W^{1,2}_{\mathrm{loc}}(X) \cap L^\infty_{\mathrm{loc}}(X)$ is continuous with respect to the $W^{1,2}_{\mathrm{loc}}$-topology. Since $(\iota_\rho(g)_\varepsilon)_{ij} = \iota_\rho(g)_\varepsilon(\partial x^i, \partial x^j) \in \mathcal{C}^\infty(U)$ converges to $g_{ij} = g(\partial x^i, \partial x^j) \in W^{1,2}_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U)$ in $W^{1,2}_{\mathrm{loc}}(U)$ (by Prop. 3.5.5) this immediately gives convergence of the determinant in $W^{1,2}_{\mathrm{loc}}(U)$ (and thus also in $\mathcal{D}'(U)$). $\qquad\square$

Next we have to investigate which additional conditions we have to impose on a gt-regular metric g to ensure that $\det \iota_\rho(g)$ is invertible in $\mathcal{G}(U)$, i.e., that $\iota_\rho(g)$ is a generalized metric. Unfortunately it turns out that gt-regularity alone is not sufficient and one needs a more complex stability condition given in [**SV09**]:

DEFINITION 3.5.10 (Stability condition for gt-regular metrics). Let g be a gt-regular metric on X, U be a chart domain and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $(g_{ij})_{1 \le ij \le n}$. For every compact $K \subset U$ let

$$\mu_K := \min_{1 \le i \le n} \operatorname*{ess\,inf}_{x \in K} \left| \lambda^i(x) \right|$$

be the (essential) absolute infimum of the eigenvalues of $(g_{ij})_{1 \le i,j \le n}$ on K (where we of course define $\operatorname{ess\,inf}_{x \in K} f(x) := \operatorname{ess\,inf}_{y \in \psi(K)} f(\psi^{-1}(y))$). We call g *stable* if for every compact K with $K \subset U$ for some chart domain U there exists a tensor field $A^K \in \mathcal{C}(U, T_2^0 U)$ such that

$$(3.5.4) \qquad\qquad \operatorname*{ess\,sup}_{x \in K} \left| g_{ij}(x) - A_{ij}^K(x) \right| \le \tilde{C}_K < \frac{\mu_K}{2n}$$

for all $1 \le i, j \le n$.

LEMMA 3.5.11. *Let $f \in L_{\mathrm{loc}}^\infty(X)$, U a relatively compact chart domain and $L \subset U$ be compact. Suppose that there exists a function $f^L \in \mathcal{C}(L)$ such that $\left\| f - f^L \right\|_{L^\infty(L)} \le c_L$. Then*

$$(3.5.5) \quad \forall \, \mathrm{compact} \, K \subset L^o \, \forall \sigma > 0 \exists \varepsilon_0 \, (K, \sigma): \quad \left\| f - \iota_\rho(f)_\varepsilon \right\|_{L^\infty(K)} \le 2c_L + \sigma \quad \forall \varepsilon \le \varepsilon_0(K, \sigma).$$

PROOF. We have

$$(3.5.6) \quad \left\| f - \iota_\rho(f)_\varepsilon \right\|_{L^\infty(K)} \le \left\| f - f^L \right\|_{L^\infty(K)} + \left\| f^L - \iota_\rho(f^L)_\varepsilon \right\|_{L^\infty(K)} + \left\| \iota_\rho(f^L)_\varepsilon - \iota_\rho(f)_\varepsilon \right\|_{L^\infty(K)}.$$

By the assumption and $K \subset L$ the first term is bounded by c_L and by an argument similar to the one in Prop. 3.5.5 (for continuous functions $f * \rho_\varepsilon \to f$ uniformly on compact sets, see [Eva98], p. 630) the second term converges to zero from above. We will show show below that $\left\| \iota_\rho(f^L)_\varepsilon - \iota_\rho(f)_\varepsilon \right\|_{L^\infty(K)} \to c_L$ for $\varepsilon \to 0$, so altogether

$$\left\| f - \iota_\rho(f)_\varepsilon \right\|_{L^\infty(K)} \le c_L + F_\varepsilon \text{ with } F_\varepsilon \to c_L,$$

which shows (3.5.5).

It remains to prove the convergence of the third term in (3.5.6): If we define $g := f^L - f$, then $\left\| \iota_\rho(f^L)_\varepsilon - \iota_\rho(f)_\varepsilon \right\|_{L^\infty(K)} = \left\| \iota_\rho(g)_\varepsilon \right\|_{L^\infty(K)}$ by linearity of ι_ρ. Now let $p \in K$, then

$$\iota_\rho(g)_\varepsilon(p) \overset{(3.5.1)}{=} \sum_{\alpha \in I} \zeta_\alpha(p) \cdot \left(\left(\chi_\alpha \circ \psi_\alpha^{-1} \, g^\alpha \right) * \rho_\varepsilon \right) (\psi_\alpha(p)),$$

where $I := \{\alpha : \psi_\alpha(U) \cap \left(\operatorname{supp} \chi_\alpha \circ \psi_\alpha^{-1} + B_\varepsilon(0) \right) \ne \emptyset \}$ is finite since U is relatively compact. Because the sum is finite we have

$$(3.5.7) \quad |\iota_\rho(g)_\varepsilon(p)| \le \int_{B_\varepsilon(0)} \sum_{\alpha \in I} |\zeta_\alpha(p)| \chi_\alpha(\psi_\alpha^{-1}(\psi_\alpha(p) - y)) \, |g^\alpha(\psi_\alpha(p) - y)| \, |\rho_\varepsilon(y)| \, dy \overset{|\zeta_\alpha| \le 1}{\le}$$

$$\le \|\rho_\varepsilon\|_{L^1} \operatorname*{ess\,sup}_{y \in B_\varepsilon(0)} \sum_{\alpha \in I} \left| \chi_\alpha(\psi_\alpha^{-1}(\psi_\alpha(p) - y)) \, g^\alpha(\psi_\alpha(p) - y) \right|.$$

Now for $y \in B_\varepsilon(0)$ and ε small enough we have $\psi_\alpha(p) - y \in \psi_\alpha(L)$ (because $p \in K$ and $\psi_\alpha(K) \subset \psi_\alpha(L^o)$ is compact) and thus

$$\operatorname*{ess\,sup}_{y \in B_\varepsilon(0)} |g^\alpha(\psi_\alpha(p) - y)| \le \operatorname*{ess\,sup}_{x \in L} \left| f^L(x) - f(x) \right| \le c_L,$$

so (3.5.7) becomes

$$|\iota_\rho(g)_\varepsilon(p)| \le \|\rho_\varepsilon\|_{L^1} \, c_L \operatorname*{ess\,sup}_{y \in B_\varepsilon(0)} \sum_{\alpha \in I} \chi_\alpha(\psi_\alpha^{-1}(\psi_\alpha(p) - y)).$$

By Def. 3.5.4, (4) we have $\|\rho_\varepsilon\|_{L^1} \to 1$ for $\varepsilon \to 0$ and

$$\operatorname*{ess\,sup}_{y \in B_\varepsilon(0)} \sum_{\alpha \in I} \chi_\alpha(\psi_\alpha^{-1}(\psi_\alpha(p) - y)) \to \sum_{\alpha \in I} \chi_\alpha(\psi_\alpha^{-1}(\psi_\alpha(p) - 0)) = 1$$

for $\varepsilon \to 0$ by continuity, thus $\left\| \iota_\rho(f^L)_\varepsilon - \iota_\rho(f)_\varepsilon \right\|_{L^\infty(K)} \to c_L$.

$\square$

Now we are finally ready to show that $\iota_\rho(g)$ is indeed a generalized metric for any stable, gt-regular metric g on X.

PROPOSITION 3.5.12. *Let g be a stable, gt-regular metric on X and ρ an admissible mollifier. Then for all chart domains U_α of X and all $K \subset U_\alpha$ compact there exists $\varepsilon'_0(K)$ such that $\left\{ \|\iota_\rho(g_{ij})_\varepsilon\|_{L^\infty(K)} : \varepsilon \leq \varepsilon'_0(K) \right\}$ is bounded and for*

(3.5.8) $\qquad \forall \, compact \, K \subset U_\alpha \exists C'_K \exists \varepsilon_0(K) : \ |\det \iota_\rho(g)_\varepsilon(x)| \geq C'_K > 0 \, \forall x \in K, \forall \varepsilon \leq \varepsilon_0(K)$

In particular, $\iota_\rho(g) \in \mathcal{G}_2^0(X)$ is a generalized metric.

PROOF. It clearly suffices to show that this holds for some compact exhaustion $(K_m)_{m \in \mathbb{N}}$ of U_α and since $U_\alpha \cong \psi_\alpha(U_\alpha) \subset \mathbb{R}^n$ we may choose the K_m such that $\overline{K_m^o} = K_m$ and $\psi_\alpha(\partial K_m)$ has Lebesgue measure zero which implies that $\operatorname{ess\,sup}_{x \in K_m} f(x) = \operatorname{ess\,sup}_{x \in K_m^o} f(x) = \sup_{x \in K_m^o} f(x) = \sup_{x \in K_m} f(x)$ (and similarly the essential infimum coincides with the infimum) for any continuous function f on U_α. Now let $K = K_m$ for some $m \in \mathbb{N}$ and choose $L \subset U_\alpha$ compact with $K \subset L^o$. Then the stability condition (3.5.4) and Lem. 3.5.11 give

$$\|g_{ij} - \iota_\rho(g_{ij})_\varepsilon\|_{L^\infty(K)} \leq 2\tilde{C}_L + \sigma,$$

which shows boundedness of $\|\iota_\rho(g_{ij})_\varepsilon\|_{L^\infty(K)}$ in ε for ε small. Since $\tilde{C}_L < \frac{\mu_L}{2n}$ we may choose σ so small that $2\tilde{C}_L + \sigma < \frac{\mu_L}{n}$, i.e., $\|g_{ij} - \iota_\rho(g_{ij})_\varepsilon\|_{L^\infty(K)} < \frac{\mu_L}{n}$ for all $1 \leq i, j \leq n$ and ε small. For $x \in K$ let $\lambda^1(x) \geq \lambda^2(x) \geq \cdots \geq \lambda^n(x)$ be the ordered eigenvalues of $g(x)$ and let $\lambda^1_\varepsilon(x) \geq \lambda^2_\varepsilon(x) \geq \cdots \geq \lambda^n_\varepsilon(x)$ be the ordered eigenvalues of $(\iota_\rho(g_{ij})_\varepsilon)_{ij}(x)$. Now by Weyl's Perturbation Theorem ([Bha02], Cor. III.2.6) we have

$$\max_{1 \leq i \leq n} \left| \lambda^i(x) - \lambda^i_\varepsilon(x) \right| \leq \|g(x) - \iota_\rho(g)_\varepsilon(x)\| \leq \operatorname*{ess\,sup}_{x \in K} \|g(x) - \iota_\rho(g)_\varepsilon(x)\|$$

for a.e. $x \in K$. Since the operator norm on $n \times n$ matrices (with respect to the euclidean norm on $\mathbb{R}^n$) satisfies $\|A\| \leq n \max_{1 \leq i,j \leq n} |A_{ij}|$ we get

$$\max_{1 \leq i \leq n} \left| \lambda^i(x) - \lambda^i_\varepsilon(x) \right| \leq n \max_{1 \leq i,j \leq n} \|g_{ij} - \iota_\rho(g_{ij})_\varepsilon\|_{L^\infty(K)} <$$
$$< \mu_L - \eta \leq \mu_K - \eta = \min_{1 \leq i \leq n} \operatorname*{ess\,inf}_{x \in K} \left| \lambda^i(x) \right| - \eta$$

for some small $\eta > 0$. So

$$\left| \lambda^i(x) - \lambda^i_\varepsilon(x) \right| < \left| \lambda^i(x) \right| - \eta \text{ for all } 1 \leq i \leq n$$

almost everywhere on K for ε small. This shows that the absolute values of all eigenvalues of $(\iota_\rho(g_{ij})_\varepsilon)_{ij}$ are greater than η almost everywhere on K and hence

$$|\det \iota_\rho(g_{ij})_\varepsilon(x)| \geq \inf_{x \in K} |\det \iota_\rho(g_{ij})_\varepsilon(x)| = \operatorname*{ess\,inf}_{x \in K} |\det \iota_\rho(g_{ij})_\varepsilon(x)| \geq \eta^n > 0$$

on K by continuity of $\iota_\rho(g_{ij})_\varepsilon$. Now by Prop. 3.5.7 we have that $\iota_\rho(g_{ij})_\varepsilon - (\iota_\rho(g)_\varepsilon)_{ij} \to 0$ uniformly on K and thus $\det \iota_\rho(g_{ij})_\varepsilon - \det \iota_\rho(g)_\varepsilon \to 0$ uniformly on K showing that $\|\det \iota_\rho(g)_\varepsilon\|_{L^\infty(K)} \geq C'_K$ for some C'_K and ε small enough.

That $\iota_\rho(g)$ is a generalized metric then follows immediately from Rem. 3.5.3 and the fact that we can always choose an atlas consisting of relatively compact chart domains. $\qquad \square$

As a consequence we obtain the stability of the inverse for stable, gt-regular metrics.

PROPOSITION 3.5.13 (Stability of the inverse). *Let g be a stable, gt-regular metric on X and ρ an admissible mollifier. Then $(\iota_\rho(g)_\varepsilon)^{-1} \in \Gamma(X, T_0^2 X) \cap W_{\mathrm{loc}}^{1,2}(X, T_0^2 X)$ for ε small enough*

and for all chart domains U_α of X and all $K \subset U_\alpha$ compact there exists $\varepsilon_0'(K)$ such that $\left\{ \left\| \left((\iota_\rho(g)_\varepsilon)^{-1} \right)^{ij} \right\|_{L^\infty(K)} : \varepsilon \leq \varepsilon_0'(K) \right\}$ is bounded. Furthermore

$$(\iota_\rho(g)_\varepsilon)^{-1} \to g^{-1} \ \text{in } W^{1,2}_{\text{loc}}(X, T_0^2 X) \text{ and pointwise a.e.}$$

In particular, $(\iota_\rho(g))^{-1} \approx g^{-1}$.

PROOF. Let U_α be a chart domain. Then on U_α the components $\iota_\rho(g)_\varepsilon^{ij} = \left((\iota_\rho(g)_\varepsilon)^{-1} \right)^{ij}$ are given by $\iota_\rho(g)_\varepsilon^{ij} = \frac{1}{\det \iota_\rho(g)_\varepsilon} C_{ij}^\varepsilon$ (where C_{ij}^ε denotes the matrix of cofactors) which combined with (3.5.8) immediately gives local uniform boundedness of $\iota_\rho(g)_\varepsilon^{ij}$ and pointwise convergence almost everywhere. Now $\det \iota_\rho(g)_\varepsilon \to \det g$ in $W^{1,2}_{\text{loc}}(U_\alpha)$ by Prop. 3.5.9 and thus $\frac{1}{\det \iota_\rho(g)_\varepsilon} \to \frac{1}{\det g}$ in $W^{1,2}_{\text{loc}}$ by the same argument as in the proof of the second claim of Prop. 3.5.8 (note that $\frac{1}{\det \iota_\rho(g)_\varepsilon}$ is bounded on compact sets by (3.5.8) for ε small and $\frac{1}{\det g} \in L^\infty_{\text{loc}}(U_\alpha)$ by uniform non-degeneracy, which is exactly what is needed). The claim then follows from continuity of multiplication of functions in $W^{1,2}_{\text{loc}}(X) \cap L^\infty_{\text{loc}}(X)$ with respect to the $W^{1,2}_{\text{loc}}$-topology (see Prop. 3.5.9). $\square$

From the stability of the inverse it easily follows that the Christoffel symbols as well as all the common curvature quantities are stable. Regarding the Christoffel symbols we have:

COROLLARY 3.5.14 (Stability of the Christoffel symbols). *Let g be a stable, gt-regular metric on X and ρ an admissible mollifier. Then*

$$\Gamma_{kl}^i \left[\iota_\rho(g)_\varepsilon \right] \to \Gamma_{kl}^i [g] \ \text{in } L^2_{\text{loc}}(U_\alpha)$$

for all chart domains U_α and so $\Gamma_{kl}^i \left[\iota_\rho(g) \right] \approx \Gamma_{jk}^i [g]$.

PROOF. This follows immediately from $\Gamma_{kl}^i[g] = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$, Prop. 3.5.13, $\iota_\rho(g)_\varepsilon \to g$ in $W^{1,2}_{\text{loc}}$ and Lemma 3.5.15 below (taking $f = \frac{\partial g_{mk}}{\partial x^i}$ and $h = g^{im}$). $\square$

LEMMA 3.5.15. *Let $L^p_{\text{loc}}(\Omega) \ni f_\varepsilon \to f$ in L^p_{loc} for some p and $h_\varepsilon \in L^\infty_{\text{loc}}(\Omega)$ be locally uniformly bounded, i.e., for all $K \subset \Omega$ compact there exists $\varepsilon_0'(K)$ such that $\left\{ \|h_\varepsilon\|_{L^\infty(K)} : \varepsilon \leq \varepsilon_0'(K) \right\}$ is bounded. If $h_\varepsilon \to h \in L^\infty_{\text{loc}}(\Omega)$ pointwise almost everywhere, then*

$$f_\varepsilon h_\varepsilon \to fh \ \text{in } L^p_{\text{loc}}(\Omega).$$

PROOF. Let $K \subset \Omega$ be compact. We estimate

$$\|f_\varepsilon h_\varepsilon - fh\|_{L^p(K)} \leq \|h_\varepsilon\|_{L^\infty(K)} \|f_\varepsilon - f\|_{L^p(K)} + \|fh - fh_\varepsilon\|_{L^p(K)}.$$

The first term converges to zero since $\|h_\varepsilon\|_{L^\infty(K)} \leq C_K$ for all ε small enough. That the second term converges to zero as well follows by dominated convergence since $fh_\varepsilon \to fh$ pointwise a.e. and $|f(x)h_\varepsilon(x)| \leq C_K |f(x)|$ on K and $f \in L^p(K)$. $\square$

Finally we will investigate the Riemann tensors $\mathbf{Riem}_{\mathcal{D}'}(g)$ and $\mathbf{Riem}_{\mathcal{G}}(\iota_\rho(g))$ induced by g and $\iota_\rho(g)$ respectively.

THEOREM 3.5.16 (Compatibility of the Riemann curvature). *Let g be a stable, gt-regular metric on X and ρ an admissible mollifier. Then*

$$\mathbf{Riem}_{\mathcal{G}}(\iota_\rho(g))_\varepsilon \to \mathbf{Riem}_{\mathcal{D}'}(g) \ \text{in } \mathcal{D}'\left(X, T_3^1 X \right).$$

So if we denote the space of all stable, gt-regular metrics on X by $\mathcal{D}'_{gt}(X, T_2^0 X)$, the following diagram commutes.

$$\begin{CD}
\mathcal{D}'_{gt}\left(X, T_2^0 X\right) @>{\iota_\rho}>> \operatorname{im}\left(\iota_\rho\right) \subset \mathcal{G}_2^0(X) \\
@V{\mathbf{Riem}_{\mathcal{D}'}}VV @VV{\mathbf{Riem}_{\mathcal{G}}}V \\
\mathcal{D}'\left(X, T_3^1 X\right) @<<{\approx}< \mathcal{G}_3^1(X)
\end{CD}$$

PROOF. This follows immediately from

$$\mathbf{Riem}_{jkl}^i = \partial_l \Gamma_{kj}^i - \partial_k \Gamma_{lj}^i + \Gamma_{lm}^i \Gamma_{kj}^m - \Gamma_{km}^i \Gamma_{lj}^m$$

and Cor. 3.5.14 by continuity of $\partial : \mathcal{D}' \to \mathcal{D}'$ and multiplication from $L_{\text{loc}}^2 \times L_{\text{loc}}^2 \to L_{\text{loc}}^1 \to \mathcal{D}'$. $\quad\square$

The same holds for the Ricci curvature an the scalar curvature: Since $\mathbf{Ric}_{ij} = \mathbf{Riem}_{ikj}^k$ we clearly have $\mathbf{Ric}_{\mathcal{G}}(\iota_\rho(g)) \approx \mathbf{Ric}_{\mathcal{D}'}(g)$. Regarding the scalar curvature we first rewrite equations (3.4.2) and (3.4.3) in terms of the Christoffel symbols. Since $\nabla_{\partial_i} dx^k = -\Gamma_{im}^k dx^m$ and $[\partial_i, \partial_j] = 0$, equation (3.4.2) becomes

$$\mathbf{Ric}_{ij} = -\partial_i \Gamma_{kj}^k + \Gamma_{kj}^m \partial_m(-\Gamma_{il}^k dx^l) + \partial_k \Gamma_{ij}^k - \Gamma_{ij}^m \partial_m(-\Gamma_{kl}^k dx^l) = -\partial_i \Gamma_{kj}^k - \Gamma_{kj}^m \Gamma_{im}^k + \partial_k \Gamma_{ij}^k + \Gamma_{ij}^l \Gamma_{kl}^k$$

and (3.4.3) is equivalent to

$$g^{ij} \, \partial_i \Gamma_{kj}^k = \partial_i \left(g^{ij} \Gamma_{kj}^k\right) - \left(\partial_i g^{ij}\right) \Gamma_{kj}^k.$$

This shows that

$$\mathbf{R} = g^{ij} \left(-\Gamma_{kj}^m \Gamma_{im}^k + \Gamma_{ij}^l \Gamma_{kl}^k\right) - \partial_i \left(g^{ij} \Gamma_{kj}^k\right) + \left(\partial_i g^{ij}\right) \Gamma_{kj}^k + \partial_k \left(g^{ij} \Gamma_{ij}^k\right) - \left(\partial_k g^{ij}\right) \Gamma_{ij}^k$$

and thus $\mathbf{R}_{\mathcal{G}}(\iota_\rho(g))_\varepsilon \to \mathbf{R}_{\mathcal{D}'}(g)$ in $\mathcal{D}'(X)$ by similar convergence arguments as in the proof of Thm. 3.5.16.

Altogether, this section shows that for stable, gt-regular metrics it does not matter whether we approach the induced geometry on a manifold from a purely distributional or a Colombeau theoretic viewpoint.

3.6. Einstein equations in vacuum

After the mathematical side note of the previous section we will use this section to discuss some applications of the various jump formulas derived in subsection 3.3.1 and 3.4.1. Those jump formulas are of particular importance in general relativity where they allow us to deduce certain minimal regularities of the spacetime metric from the regularities of the stress-energy tensor describing the density of energy and momentum in our spacetime.

In general relativity the Einstein field equations (EFE) relate the curvature (and thus the smooth metric) of the spacetime to the stress-energy tensor T_{ij} and are given by

$$(3.6.1) \qquad \mathbf{Ric}_{ij} - \frac{1}{2} g_{ij} \mathbf{R} + g_{ij} \Lambda = \frac{8\pi G}{c^4} T_{ij},$$

where c denotes the speed of light in vacuum, G is the gravitational and Λ the cosmological constant. In vacuum one has $T_{ij} = 0$ and it is possible to rewrite the Einstein field equations in a much simpler form, namely

$$(3.6.2) \qquad \mathbf{Ric}_{ij} - g_{ij} \Lambda = 0.$$

This is an immediate consequence of the following Proposition.

PROPOSITION 3.6.1. *The Einstein field equations* (3.6.1) *are equivalent to*

$$(3.6.3) \qquad \mathbf{Ric}_{ij} - g_{ij} \Lambda = \frac{8\pi G}{c^4} \left(T_{ij} - \frac{1}{2} T g_{ij}\right),$$

where $T := g^{ij}T_{ij}$.

PROOF. This follows easily from some simple calculations. First we contract (3.6.1) with the metric g^{ij} to obtain

$$4\Lambda - \mathbf{R} = \mathbf{R} - 2\mathbf{R} + 4\Lambda = g^{ij}\mathbf{Ric}_{ij} - \frac{1}{2}g^{ij}g_{ij}\mathbf{R} + g^{ij}g_{ij}\Lambda = g^{ij}\frac{8\pi G}{c^4}T_{ij} = \frac{8\pi G}{c^4}T,$$

which immediately gives

$$\mathbf{Ric}_{ij} - g_{ij}\Lambda = \frac{8\pi G}{c^4}T_{ij} - \frac{1}{2}g_{ij}(4\Lambda - \mathbf{R}) = \frac{8\pi G}{c^4}\left(T_{ij} - \frac{1}{2}g_{ij}T\right). \qquad \square$$

If one assumes $\Lambda = 0$[2] the EFE in vacuum reduce further to the Ricci flat condition

$$\mathbf{Ric} = 0,$$

however, in the following discussion we will not need this assumption.

As mentioned very briefly in the beginning of this chapter our goal is to study necessary conditions for low regularity metrics to satisfy (3.6.2), in particular concerning jumps across a hypersurface. First we note that (3.6.2) is a well-defined equation for any gt-regular metric (which is not true for the original equation (3.6.1) since it contains the product $g_{ij}\mathbf{R}$ of a non-smooth function with a distribution). In the following let $g \in \mathcal{C}\left(X, T_2^0 X\right)$ be a uniformly non-degenerate metric on X such that $g^\pm \in W_{\mathrm{loc}}^{2,p}(X^\pm)$ for some $p \geq \frac{n}{2}$ (as in Prop. 3.4.4), then the induced connection ∇ is an L_{loc}^2-connection satisfying $\nabla^\pm \in W_{\mathrm{loc}}^{1,p}(X^\pm)$ and the jump formula (3.3.13) for the Ricci curvature in coordinates (U, ψ) adapted to the hypersurface holds. Now (3.6.2) gives

$$(3.6.4) \qquad \left[dx^1\left(\nabla_{\partial_i}\partial_j\right)\right]_M \delta_M(\partial_1) - \left[dx^k\left(\nabla_{\partial_k}\partial_j\right)\right]_M \delta_M(\partial_i) = g_{ij}\Lambda - (\mathbf{Ric}_{ij})^{\mathrm{reg}} \in L_{\mathrm{loc}}^1(U).$$

PROPOSITION 3.6.2. *Let $f \in L_{\mathrm{loc}}^1(M)$ and $\mathbf{u} \in \mathfrak{X}(X)$ such that $\mathbf{u}|_M \notin \Gamma(M, TM)$. Then the distribution $f\,\delta_M(\mathbf{u})$ (given by $\phi \mapsto \int_M f\, i_\mathbf{u}\phi$ for $\phi \in \Omega^n(X)$, cf. Def. 3.3.6) is in $L_{\mathrm{loc}}^1(X)$ if and only if $\mathrm{supp}\, f \cap \mathrm{supp}\, \mathbf{u}|_M = \emptyset$, i.e., if and only if $f\delta_M(\mathbf{u}) = 0$.*

PROOF. Let $p \in M$ and (U, ψ) be adapted coordinates around p with U relatively compact. By Rem. 3.3.7 it suffices to show that $f\delta_M(\partial_1) \in L_{\mathrm{loc}}^1(U)$ if and only if $f = 0$ for all $f \in L_{\mathrm{loc}}^1(M \cap U)$. Assume that $f\delta_M(\partial_1) = g \in L_{\mathrm{loc}}^1(U)$ but $f \neq 0$ and choose $h \in \mathcal{D}(U)$ such that $\int_M f\, h\, dx^2 \wedge \cdots \wedge dx^n \neq 0$. For $\phi := h\, dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(U)$ this implies $\langle f\delta_M(\partial_1), \phi\rangle = \int_M f\, h\, dx^2 \wedge \cdots \wedge dx^n \neq 0$. Let $\tilde{\eta}_\varepsilon\left(x^1, x^2, \ldots, x^n\right) := \eta_\varepsilon(x_1)$, where $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ is defined by $\eta_\varepsilon(x) := \eta(x/\varepsilon)$ with

$$\eta(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}.$$

Then $\psi^*\tilde{\eta}_\varepsilon \in \mathcal{C}^\infty(U)$, $\psi^*\tilde{\eta}_\varepsilon|_M = \frac{1}{e}$, $\psi^*\tilde{\eta}_\varepsilon \leq \frac{1}{e} \in L^1(U)$ and $\psi^*\tilde{\eta}_\varepsilon \to 0$ pointwise almost everywhere, then

$$0 \neq \frac{1}{e}\int_M f\, i_{\partial_1}\phi = \langle f\,\delta_M(\partial_1), \psi^*\tilde{\eta}_\varepsilon\, \phi\rangle = \int_X g\, \psi^*\tilde{\eta}_\varepsilon\, \phi \to 0$$

by dominated convergence, giving a contradiction. The other direction is obvious. $\qquad \square$

This shows that (3.6.4) requires that the singular part of the Ricci curvature vanishes, which is equivalent to $\left[dx^1\left(\nabla_{\partial_i}\partial_j\right)\right]_M = 0$ for $i = 2, \ldots, n$ (because for those i, $\delta_M(\partial_i) = 0$) and $j = 1, \ldots, n$ and $\sum_{k=2}^n \left[dx^k\left(\nabla_{\partial_k}\partial_j\right)\right]_M = 0$ for all $j = 1, \ldots, n$.

A sufficient condition for the vanishing of the singular part of the Ricci curvature is the vanishing of the singular part $[\nabla_\mathbf{u}\mathbf{w}]_M \delta_M(\mathbf{v}) - [\nabla_\mathbf{v}\mathbf{w}]_M \delta_M(\mathbf{u})$ of the Riemann tensor $\mathbf{Riem}(\mathbf{u}, \mathbf{v})\mathbf{w}$ (as given by the jump formula (3.3.11)), which is equivalent to $[\nabla_\mathbf{u}\mathbf{w}]_M = 0$ for all $\mathbf{u}, \mathbf{w} \in \mathfrak{X}(X)$ with

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[2]Experiments indicate that the actual value is approximately $\Lambda \approx 1.7 \times 10^{-121}$ in Planck units, i.e., units where $G = c = \hbar = 1$ (see [BS11]).

$\mathbf{u}|_M \in \Gamma(M, TM)$. We are going to see a little bit later (in Prop. 3.6.5) that those two conditions are even equivalent under certain additional assumptions (namely the hypersurface being nowhere null). For now we will study some other equivalent conditions to the vanishing of the singular part of the Riemann tensor.

LEMMA 3.6.3. *Let $\gamma : W^{k,p}_{\mathrm{loc}}(\overline{X}) \to W^{k-1,p}_{\mathrm{loc}}(\partial X)$ be the trace operator from Thm. 3.1.4, $f \in W^{k+1,p}_{\mathrm{loc}}(\overline{X})$ and $\mathbf{u} \in \mathfrak{X}(\overline{X})$ such that $\mathbf{u}|_{\partial X} \in \Gamma(\partial X, T(\partial X))$. Then $\gamma(\mathbf{u}(f)) = \mathbf{u}|_{\partial X}(\gamma(f))$. In particular for $f \in W^{2,p}_{\mathrm{loc}}(X^{\pm})$ and $2 \le i \le n$ one has $\partial_i([f]_M) = [\partial_i f]_M$ in coordinates adapted to the hypersurface M.*

PROOF. First we consider the case that $f \in \mathcal{C}^\infty(\overline{X})$. Let $p \in \partial X$ and (U, ψ) a chart at the boundary around p, then any $\mathbf{u} \in \mathfrak{X}(\overline{X})$ such that $\mathbf{u}|_{\partial X} \in \Gamma(\partial X, T(\partial X))$ can be written as $\mathbf{u} = \sum_{i=2}^n u^i \frac{\partial}{\partial x^i}$ and

$$\mathbf{u}(f)(p) = \sum_{i=2}^n u^i(p) D_i(f \circ \psi^{-1})(\psi(p)) = \sum_{i=2}^n u^i(p) D_i\left(f \circ \psi^{-1}\big|_{\{0\} \times \mathbb{R}^{n-1}}\right)(\psi(p)) =$$

$$= \sum_{i=2}^n u^i(p) D_i\left(f|_{\partial X} \circ (\psi|_{\partial X})^{-1}\right)(\psi(p)) = \mathbf{u}|_{\partial X}(f|_{\partial X})(p),$$

so $\gamma(\mathbf{u}(f)) = \mathbf{u}(f)|_{\partial X} = \mathbf{u}|_{\partial X}(f|_{\partial X}) = \mathbf{u}|_{\partial X}(\gamma(f))$. Now if $f \in W^{k+1,p}_{\mathrm{loc}}(\overline{X})$ we can choose $f_\varepsilon \in \mathcal{C}^\infty(\overline{X})$ such that $f_\varepsilon \to f$ in $W^{k+1,p}_{\mathrm{loc}}(\overline{X})$ and thus

$$\gamma(\mathbf{u}(f)) = \lim_{\varepsilon \to 0} \gamma(\mathbf{u}(f_\varepsilon)) = \lim_{\varepsilon \to 0} \mathbf{u}|_{\partial X}(\gamma(f_\varepsilon)) = \mathbf{u}|_{\partial X}(\gamma(f))$$

in $\mathcal{D}'(\partial X)$. $\qquad\square$

PROPOSITION 3.6.4. *The following are equivalent:*

(1) *The singular part of the Riemann tensor vanishes, i.e., $[\nabla_{\mathbf{u}} \mathbf{w}]_M = 0$ for all $\mathbf{u}, \mathbf{w} \in \mathfrak{X}(X)$ with $\mathbf{u}|_M \in \Gamma(M, TM)$,*
(2) *$[\partial_1 g_{ij}]_M = 0$ for $i, j = 2, \ldots, n$ in adapted coordinates (U, ψ)*
(3) *$[\mathbf{u}(g(\mathbf{v}, \mathbf{w}))]_M = 0$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}(X)$ such that $\mathbf{v}|_M, \mathbf{w}|_M \in \Gamma(M, TM)$.*

PROOF. To show that the first two statements are equivalent we note that (1) is equivalent to $[\Gamma^k_{ij} \partial_k]_M = [\Gamma^k_{ij}]_M \partial_k|_M = 0$ in adapted coordinates for $1 \le j \le n$ and $2 \le i \le n$. Now for $2 \le i, j \le n$ we have
(3.6.5)
$$[\Gamma^k_{ij}]_M = \frac{1}{2} g^{km}\big|_M [\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}]_M = -\frac{1}{2} g^{km}\big|_M [\partial_m g_{ij}]_M = -\frac{1}{2} g^{k1}\big|_M [\partial_1 g_{ij}]_M,$$

where we used Lemma 3.6.3 and continuity of g. This shows that $[\partial_1 g_{ij}]_M = 0$ for all $2 \le i, j \le n$ if and only if $[\Gamma^k_{ij}]_M = 0$ by uniform non-degeneracy of g, proving that (1) implies (2). For (2) $\implies$ (1) note that (again by Lemma 3.6.3) (2) is equivalent to $[\partial_m g_{ij}]_M = 0$ for all m and all $2 \le i, j \le n$ immediately giving $[\Gamma^k_{ij}]_M = 0$ for $2 \le i, j \le n$ by (3.6.5) and that for $j \ge 2$, $[\Gamma^k_{1j}]_M = \frac{1}{2} g^{km}\big|_M [\partial_1 g_{jm} - \partial_m g_{1j}]_M = \frac{1}{2} g^{k1}\big|_M [\partial_1 g_{j1} - \partial_1 g_{1j}]_M = 0$.

That (2) is equivalent to (3) is obvious (again taking into account that $[\partial_m g_{ij}]_M = \partial_m [g_{ij}]_M = 0$ for all $m \ne 1$ and $i, j \ge 2$ anyway). $\qquad\square$

Our next goal is to show that for nowhere null hypersurfaces the singular part of the Ricci tensor vanishes if and only if the singular part of the Riemann tensor vanishes. To do so we need the so-called normal one-form on the hypersurface η defined in section 4.1 below and the vector field $\mathbf{n} = \eta^\sharp \in W^{k,p}_{\mathrm{loc}}(X, TX)$ (for $g \in W^{k,p}_{\mathrm{loc}}(X)$) that is metrically equivalent to η (and also introduced in section 4.1). The hypersurface M is called *nowhere null* if $g(p)(\mathbf{n}_p, \mathbf{n}_p) \ne 0$ (or equivalently if

$\mathbf{n}_p \notin T_pM$, see Rem. 4.4.2) for all $p \in M$ (see Def. 4.4.1). Some additional properties of nowhere null hypersurfaces will be discussed in section 4.4 but will not be needed here.

PROPOSITION 3.6.5. *If the hypersurface M is nowhere null, then the singular part of the Ricci tensor vanishes if and only if the singular part of the Riemann tensor vanishes.*

PROOF. That the vanishing of the singular part of the Riemann tensor implies the vanishing of the singular part of the Ricci tensor is obvious because the Ricci tensor is a contraction of the Riemann tensor, so it remains to show the other direction.

Since the hypersurface is nowhere null we have $g(\mathbf{n}, \mathbf{n}) \neq 0$ on M. Given $p \in M$ let $\left(U, \left(x^1, \ldots, x^n\right)\right)$ be adapted coordinates around p. Because the $\frac{\partial}{\partial x^i}\big|_p \in T_pM$ are a basis of T_pM for $2 \leq i \leq n$ for $p \in M \cap U$ and $\mathbf{n}_p \notin T_pM$ (see Rem. 4.4.2) we have that $\left(\mathbf{n}(dx^1)\right)(p) \neq 0$ and thus $\mathbf{n}(dx^1) \neq 0$ in some neighborhood V of p by continuity of $\mathbf{n}$ (clearly $\mathbf{n}$ has the same regularity as g). So the ∂_1-component of $\mathbf{n}$ is non-zero on V, which shows that $\left\{\mathbf{n}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right\}$ forms a basis of TV. Its dual basis $\left\{\omega^{(1)}, \ldots, \omega^{(n)}\right\}$ satisfies $\omega^{(1)}\big|_M = \frac{1}{g(\mathbf{n},\mathbf{n})}\eta\big|_M$ (since $\eta(\mathbf{n}) = g(\mathbf{n}, \mathbf{n})$ and $\eta|_M(\mathbf{u}) = 0$ for $\mathbf{u} \in \mathfrak{X}(M)$ by definition, see (4.1.1)). Looking at (3.3.12) and replacing the basis $\{\partial_1, \ldots, \partial_n\}$ with $\left\{\mathbf{n}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right\}$ and $\left\{dx^1, \ldots, dx^n\right\}$ with $\left\{\omega^{(1)}, \ldots, \omega^{(n)}\right\}$ shows that on V the singular part of the Ricci tensor becomes

$$\mathbf{Ric}(\mathbf{u}, \mathbf{v})^{\mathrm{sing}} = -\left(\left[\omega^{(1)}\left(\nabla_{\mathbf{n}}\mathbf{v}\right)\right]_M + \sum_{i=2}^{n}\left[\omega^{(i)}\left(\nabla_{\partial x^i}\mathbf{v}\right)\right]_M\right)\delta_M\left(\mathbf{u}\right) + \left[\omega^{(1)}\left(\nabla_{\mathbf{u}}\mathbf{v}\right)\right]_M \delta_M\left(\mathbf{n}\right)$$

since $\delta_M(\partial x^j) = 0$ for $2 \leq j \leq n$ by Rem. 3.3.7. Now let $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(X)$ with $\mathbf{u}|_M, \mathbf{v}|_M \in \Gamma(M, TM)$, then $\delta_M(\mathbf{u}) = 0$ and we get

$$\mathbf{Ric}(\mathbf{u}, \mathbf{v})^{\mathrm{sing}} = \left[\omega^{(1)}\left(\nabla_{\mathbf{u}}\mathbf{v}\right)\right]_M \delta_M\left(\mathbf{n}\right)$$

By Prop. 3.6.2 this vanishes if and only if

$$(3.6.6) \qquad \left[\omega^{(1)}\left(\nabla_{\mathbf{u}}\mathbf{v}\right)\right]_M = \left[g(\nabla_{\mathbf{u}}\mathbf{v}, (\omega^{(1)})^\sharp)\right]_M = 0.$$

Using the Koszul formula (3.4.1), continuity of g and that

$$(3.6.7) \qquad \left[\mathbf{u}(g(\mathbf{v}, \mathbf{w}))\right]_M \overset{Rem.\,3.1.6}{=} \omega^{(1)}(\mathbf{u})\big|_M \left[\mathbf{n}\left(g\left(\mathbf{v}, \mathbf{w}\right)\right)\right]_M$$

since $\left[\mathbf{u}(g(\mathbf{v}, \mathbf{w}))\right]_M = 0$ for $\mathbf{u} \in \mathfrak{X}(X)$ with $\mathbf{u}|_M \in \mathfrak{X}(M)$ and $\mathbf{u} - \omega^{(1)}(\mathbf{u})\mathbf{n}\big|_M \in \mathfrak{X}(M)$, we have

$$2\left[g\left(\nabla_{\mathbf{u}}\mathbf{v}, \mathbf{w}\right)\right]_M \overset{(3.4.1)}{=} \left[\mathbf{u}\left(g\left(\mathbf{v}, \mathbf{w}\right)\right)\right]_M + \left[\mathbf{v}\left(g\left(\mathbf{w}, \mathbf{u}\right)\right)\right]_M - \left[\mathbf{w}\left(g\left(\mathbf{u}, \mathbf{v}\right)\right)\right]_M - \left[g\left(\mathbf{u}, [\mathbf{v}, \mathbf{w}]\right)\right]_M$$
$$+ \left[g\left(\mathbf{v}, [\mathbf{w}, \mathbf{u}]\right)\right]_M + \left[g\left(\mathbf{w}, [\mathbf{u}, \mathbf{v}]\right)\right]_M$$
$$\overset{(3.6.7)}{=} \omega^{(1)}(\mathbf{u})\big|_M \left[\mathbf{n}\left(g\left(\mathbf{v}, \mathbf{w}\right)\right)\right]_M + \omega^{(1)}(\mathbf{v})\big|_M \left[\mathbf{n}\left(g\left(\mathbf{w}, \mathbf{u}\right)\right)\right]_M$$
$$- \omega^{(1)}(\mathbf{w})\big|_M \left[\mathbf{n}\left(g\left(\mathbf{u}, \mathbf{v}\right)\right)\right]_M$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}(X)$. This gives

$$2\left[g(\nabla_{\mathbf{u}}\mathbf{v}, (\omega^{(1)})^\sharp)\right]_M = \omega^{(1)}(\mathbf{u})\big|_M \left[\mathbf{n}\left(g\left(\mathbf{v}, (\omega^{(1)})^\sharp\right)\right)\right]_M +$$
$$+ \omega^{(1)}(\mathbf{v})\big|_M \left[\mathbf{n}\left(g((\omega^{(1)})^\sharp, \mathbf{u})\right)\right]_M - \omega^{(1)}((\omega^{(1)})^\sharp)\big|_M \left[\mathbf{n}\left(g\left(\mathbf{u}, \mathbf{v}\right)\right)\right]_M.$$

Now for $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(X)$ with $\mathbf{u}|_M, \mathbf{v}|_M \in \Gamma(M, TM)$ equation (3.6.6) is equivalent to

$$\left[\mathbf{n}\left(g\left(\mathbf{u}, \mathbf{v}\right)\right)\right]_M = 0$$

since $\omega^{(1)}(\mathbf{u})\big|_M = \omega^{(1)}(\mathbf{v})\big|_M = 0$ and $\omega^{(1)}((\omega^{(1)})^\sharp) = \frac{1}{g(\mathbf{n},\mathbf{n})}\eta(\frac{1}{g(\mathbf{n},\mathbf{n})}\mathbf{n}) = \frac{1}{g(\mathbf{n},\mathbf{n})} \neq 0$ on M. So we have shown that $\left[\mathbf{w}\left(g\left(\mathbf{u}, \mathbf{v}\right)\right)\right]_M = 0$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}(X)$ with $\mathbf{u}|_M, \mathbf{v}|_M \in \Gamma(M, TM)$ which is equivalent to the vanishing of the singular part of the Riemann tensor by Prop. 3.6.4. $\square$

CHAPTER 4

Geometry induced on a hypersurface

In this last part we are going to study how a given L^2_{loc}-connection (or gt-regular metric) on X can induce a connection (or metric) on a hypersurface $M \subset X$. As in the previous section X is assumed to be oriented and we further assume that $X = X^+ \cup X^-$ where $X^\pm$ are manifolds with boundary, $\partial X^\pm = M$ (implying that M is oriented and we choose the orientation induced from X^-) and $X^+ \cap X^- = M$. In particular we will be studying the difference between null and non-null hypersurfaces and are going to derive the Gauss and Codazzi equations. We are mainly going to follow [LM07] and [MS93].

4.1. Normal form and rigging vector fields

First we are going to look at ways in which an $(L^2_{\text{loc}}$-)connection ∇ on X induces a connection on M. For this we need projection operators $(.)^{\shortparallel} : \mathcal{D}'(M, TX|_M) \to \mathcal{D}'(M, TM)$ for vector fields and $(.)_{\shortparallel} : \mathcal{D}'(M, T^*X|_M) \to \mathcal{D}'(M, T^*M)$ for one-forms. These are defined with the help of a normal form $\eta \in \mathcal{T}^0_1(X)$ for the hypersurface M, that is a one-form η such that for all $p \in M$

$$\text{(4.1.1)} \qquad \eta(\mathbf{u})(p) \begin{cases} = 0 & \mathbf{u}(p) \in T_pM \\ \neq 0 & \mathbf{u}(p) \notin T_pM \end{cases}.$$

The next proposition shows the existence of such an object (pointwise existence of $\eta|_p$ is of course obvious, but it is a priori not clear that there exists a smooth object with the desired pointwise properties).

PROPOSITION 4.1.1. *Let X be an orientable manifold and $M \subset X$ an orientable hypersurface. Then there exists a one-form $\eta \in \mathcal{T}^0_1(X)$ satisfying* (4.1.1).

PROOF. Choose some Riemannian metric g on X, then there exists a normal unit vector field $\mathbf{n}$ on M (by Rem. 2.3.1). Now define $\eta(\mathbf{u}) = g(\mathbf{n}, \mathbf{u})$ for $\mathbf{u} \in \mathfrak{X}(X)$. Then $\eta \in \mathcal{T}^0_1(X)$ and $\eta(\mathbf{u})(p) = 0$ for $\mathbf{u}(p) \in T_pM$ and $\eta(\mathbf{u})(p) \neq 0$ for $\mathbf{u}(p) \notin T_pM$ since $0 \neq \mathbf{n}(p) \in T_pM^\perp$. $\qquad\square$

Note that while (4.1.1) does not determine η completely it does determine $\eta|_p$ for $p \in M$ up to a scalar multiplicative factor $\sigma(p)$.[1]

Now there are two different ways to obtain vector fields defined on M from the normal form η. The first is via metric equivalence of vector fields and one-forms: Given a gt-regular metric g we can define a normal vector field $\mathbf{n} \in \mathcal{D}'(X, TX)$ as $\mathbf{n} := \eta^\sharp$ (see the beginning of Section 3.4 for details). Note that the regularity for $\mathbf{n}$ will be the same as for the inverse metric g^{ij}. The second does not require a metric and instead uses duality to define so called rigging vector fields ℓ:

DEFINITION 4.1.2 (Rigging vector field). A section $\ell \in \Gamma(M, TX|_M)$ is called a *rigging vector field* (or simply a *rigging*) for M if

$$\ell(p) \notin T_pM \quad \forall p \quad \text{and} \quad \eta(\ell) \equiv 1.$$

---

[1]Given two normal one-forms η_1 and η_2 one has $\eta_1|_p = \sigma(p)\,\eta_2|_p$ for $\sigma(p) = {}^{\eta_1(\partial_{x^1})(p)}/_{\eta_2(\partial_{x^1})(p)}$.

Note that the second requirement can be easily achieved from the first one by replacing ℓ with $\frac{1}{\eta(\ell)}\ell$ (or η with $\frac{1}{\eta(\ell)}\eta$). A similar argument as in Prop. 4.1.1 (setting $\ell = \mathbf{n}$ for any unit normal vector field with respect to some Riemannian metric g on X) shows that such a rigging vector field always exists. It is, however, highly non unique: there exist lots of different and linearly independent riggings (for example adding an arbitrary vector field $\mathbf{u} \in \mathfrak{X}(M)$ to any given rigging ℓ produces a new rigging). So when using such a rigging to define objects on the hypersurface (e.g. an induced connection, ...) we will have to take note on whether these depend on the rigging and if so, how.

Since $\dim T_p M = n - 1$ we may use $\ell_p \equiv \ell(p)$ to decompose $T_p X$ in a direct sum

$$(4.1.2) \qquad\qquad T_p X = \langle \ell_p \rangle \oplus T_p M,$$

where $\langle \ell_p \rangle$ denotes the vector space generated by ℓ_p. This means that we can decompose any vector field $\mathbf{u} \in \Gamma(M, TX|_M)$ (and any distribution $\mathbf{u} \in \mathcal{D}'(M, TX|_M)$) in a part tangential to the hypersurface and a part proportional to ℓ, we write

$$(4.1.3) \qquad\qquad \mathbf{u} = \eta(\mathbf{u})\ell + \mathbf{u}^{\shortparallel},$$

where $\mathbf{u}^{\shortparallel} \in \mathfrak{X}(M)$ (or $\mathcal{D}'(M, TM)$, respectively) is called the *rigging projection* of $\mathbf{u}$ (note that $\mathbf{u}^{\shortparallel}$ depends on the rigging, so we will sometimes write $\mathbf{u}_\ell^{\shortparallel}$ to highlight this dependence).

The choice of a rigging ℓ also allows a decomposition of the cotangent space $T_p^* X$ at $p \in M$ similar to that of the tangent space given in equation (4.1.2): The main problem here is that $T_p^* M$ is a priori not a subset of $T_p^* X$, however, we may identify

$$T_p^* M \cong \left\{ \omega \in T_p^* X : \omega(\ell) = 0 \right\}$$

to obtain

$$(4.1.4) \qquad\qquad T_p^* X = \langle \eta_p \rangle \oplus T_p^* M.$$

Again this allows the decomposition of any one-form $\omega \in \mathcal{D}'(M, T^*X|_M)$ as

$$(4.1.5) \qquad\qquad \omega = \omega(\ell)\eta + \omega_{\shortparallel},$$

where $\omega_{\shortparallel} \in \mathcal{D}'(M, T^*M)$ is called the *normal projection* of ω. Note that $\omega_{\shortparallel}$ is independent of the rigging ℓ since $\langle \omega_{\shortparallel}, \mathbf{u} \rangle = \langle \omega, \mathbf{u} \rangle$ for $\mathbf{u} \in \mathfrak{X}(M)$.

4.2. The projected connection and the second fundamental form

In the following it will be important keep in mind when the vector fields used are in $\mathfrak{X}(M)$ and when in $\mathfrak{X}(X)$. To better distinguish between those two we will from now on generally use the letters $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for objects in $\mathfrak{X}(M)$ and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ for $\mathfrak{X}(X)$ (or $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ if we want to specify that they are extensions of $\mathbf{u}, \mathbf{v}$). Similarly we will use $\omega, \theta \in \Gamma\left(M, T_1^0 M\right)$ for one forms on M and $\zeta, \xi \in \Gamma\left(X, T_1^0 X\right)$ for one forms on X.

Given a $W_{\text{loc}}^{k,p}$-connection ∇ on X we can define an operator $\underline{\nabla} : \Gamma(M, TX|_M) \times \Gamma(M, TX|_M) \to W_{\text{loc}}^{k-1,p}(M, TM)$ on M locally via

$$(4.2.1) \qquad\qquad \left(\underline{\nabla}_{\mathbf{u}} \mathbf{v}\right)(p) := \left(\gamma\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}\right)\right)^{\shortparallel}(p) \quad \text{for } \mathbf{u}, \mathbf{v} \in \Gamma(M, TX|_M),$$

where $\tilde{\mathbf{u}}, \tilde{\mathbf{v}} \in \mathfrak{X}(X)$ such that $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}|_{U \cap M} = \mathbf{u}, \mathbf{v}|_{U \cap M}$ for some open $U \subset X$ (with $p \in U$) and γ is the trace operator from Prop. 3.1.7 (to be precise, in the language of subsection 3.3.1, $\gamma(T) := \gamma^+(T^+)$ for $T \in W_{\text{loc}}^{1,p}(X, T_s^r X)$ which is equal to $\gamma^-(T^-)$ since they coincide for smooth tensor fields).

LEMMA 4.2.1. *Let ∇ be a $W_{\text{loc}}^{k,p}$-connection (for some $k \geq 1$) on X. Then the operator $\underline{\nabla}$ given by (4.2.1) is well-defined.*

PROOF. First we show that there exist appropriate extensions $\tilde{\mathbf{u}} \in \mathfrak{X}(X)$: By setting $\tilde{f} = \chi \, f \circ \psi|_{U \cap M}^{-1} \circ pr_{\mathbb{R}^{n-1}} \circ \psi \in \mathcal{C}^{\infty}(X)$ for an adapted chart ψ around p (where χ is an appropriate cut-off) we see that we can locally extend smooth functions $f \in \mathcal{C}^{\infty}(M)$. Now $\mathbf{u} \in \mathfrak{X}(M)$ is locally given by $\mathbf{u} = \sum_{i=2}^{n} u^i \partial_i$ in adapted coordinates and setting $\tilde{\mathbf{u}} = \sum_{i=2}^{n} \tilde{u^i} \partial_i$ gives the desired extension around p.

Next we have to check that $\gamma(\nabla_{\mathbf{x}} \mathbf{y})$ only depends on $\mathbf{x}|_M$ and $\mathbf{y}|_M$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{X}(X)$. In coordinates adapted to the hypersurface we have

$$\gamma(\nabla_{\mathbf{x}} \mathbf{y}) = \mathbf{x}\left(y^i\right)\big|_M \, \gamma(\partial_i) + x^i y^j\big|_M \, \gamma\left(\nabla_{\partial_j} \partial_i\right)$$

by the Leibniz rule and $\mathcal{C}^{\infty}(M)$-linearity of the connection operator in the first argument (see Def. 3.3.1) and Rem. 3.1.6 as well as Lemma 3.6.3. Since $\mathbf{x}|_M \in \mathfrak{X}(M)$ we have $\mathbf{x}\left(y^i\right)\big|_M = \sum_{i=2}^{n} x^i\big|_M \, \partial_i\left(y^i\right)\big|_M = \sum_{i=2}^{n} x^i\big|_M \, \partial_i\left(y^i\big|_M\right)$ showing that this also only depends on $\mathbf{x}|_M$ and $\mathbf{y}|_M$.   □

Since $\mathfrak{X}(M) \subset \Gamma(M, TX|_M)$ we may restrict $\underline{\nabla}$ to obtain an operator from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to $W_{\text{loc}}^{k-1,p}(M, TM)$ (which we will again denote by $\underline{\nabla}$). This gives us a connection on M.

PROPOSITION 4.2.2. *The operator* $\underline{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to W_{\text{loc}}^{k-1,p}(M, TM)$ *defines a* $W_{\text{loc}}^{k-1,p}$-*connection on* M, *the so-called* projected connection.

PROOF. The required properties, i.e., that $\underline{\nabla}$ is $\mathcal{C}^{\infty}(M)-$linear in the first and $\mathbb{R}$-linear in the second argument and satisfies the Leibniz rule follow immediately from those same properties of ∇ (and Remark 3.1.6).   □

REMARK 4.2.3. If ∇ is torsion free then the projected connection $\underline{\nabla}$ is torsion free as well since

$$\underline{\nabla}_{\mathbf{u}} \mathbf{v} - \underline{\nabla}_{\mathbf{v}} \mathbf{u} = (\gamma(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}} - \nabla_{\tilde{\mathbf{v}}} \tilde{\mathbf{u}}))'' = (\gamma([\tilde{\mathbf{u}}, \tilde{\mathbf{v}}]))'' = ([\tilde{\mathbf{u}}, \tilde{\mathbf{v}}]|_M)'' = [\mathbf{u}, \mathbf{v}]$$

for all $\mathbf{u}, \mathbf{v} \in \mathfrak{X}(M)$ by the properties of the Lie bracket on submanifolds. However, even if ∇ is the Levi-Civita connection for some smooth metric g and the pullback of g is actually a metric on M the projected connection $\underline{\nabla}$ need not be the Levi-Civita connection associated with the pullback metric j^*g on M: Obviously $\underline{\nabla}$ depends on the choice of the rigging vector field ℓ and, by (4.1.3) and (4.2.1),

$$(4.2.2) \quad g\left(\nabla_{\mathbf{x}} \mathbf{y}, \mathbf{z}\right)|_M = g\left(\nabla_{\mathbf{x}} \mathbf{y}|_M, \mathbf{z}|_M\right) = g\left(\left(\nabla_{\mathbf{x}} \mathbf{y}|_M\right)'' + \eta\left(\nabla_{\mathbf{x}} \mathbf{y}|_M\right) \ell, \mathbf{z}|_M\right) =$$
$$= j^*g\left(\underline{\nabla}_{\mathbf{x}|_M} \mathbf{y}|_M, \mathbf{z}|_M\right) + \eta\left(\nabla_{\mathbf{x}} \mathbf{y}|_M\right) g\left(\ell, \mathbf{z}|_M\right)$$

for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathfrak{X}(X)$ with restrictions in $\mathfrak{X}(M)$. So we have

$$\underline{\nabla}_{\mathbf{u}}(j^*g)(\mathbf{v}, \mathbf{w}) = \mathbf{u}(g(\mathbf{v}, \mathbf{w})) - j^*g(\underline{\nabla}_{\mathbf{u}} \mathbf{v}, \mathbf{w}) - j^*g(\mathbf{v}, \underline{\nabla}_{\mathbf{u}} \mathbf{w}) \overset{(4.2.2)}{=} \mathbf{u}(g(\mathbf{v}, \mathbf{w})) - g(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}, \tilde{\mathbf{w}})|_M +$$
$$+ \eta\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}|_M\right) g(\ell, \mathbf{w}) - g(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{w}}, \tilde{\mathbf{v}})|_M + \eta\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{w}}|_M\right) g(\ell, \mathbf{v}) = (\nabla_{\tilde{\mathbf{u}}} g)(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})|_M +$$
$$+ \eta\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}|_M\right) g(\ell, \mathbf{w}) + \eta\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{w}}|_M\right) g(\ell, \mathbf{v}) \overset{\nabla g = 0}{=} \eta\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}|_M\right) g(\ell, \mathbf{w}) + \eta\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{w}}|_M\right) g(\ell, \mathbf{v})$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$. This shows that $\underline{\nabla}(j^*g) = 0$ if $g(\ell, \mathbf{v}) = \ell^{\flat}(\mathbf{v}) = 0$ (where $\ell^{\flat} = g_{ij} \ell^j dx^i$) for all $\mathbf{v} \in \mathfrak{X}(M)$ that is if $\ell^{\flat} \propto \eta|_M$ (since $\ell^{\flat}(p) \neq 0$ it satisfies the conditions (4.1.1) and thus is proportional to $\eta|_M$ by uniqueness of the normal form up to a scalar multiple). If $\ell^{\flat} \not\propto \eta$ then $\underline{\nabla}(j^*g)$ need not be zero since in general $\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}|_M \notin \mathfrak{X}(M)$ even if $\tilde{\mathbf{u}}|_M, \tilde{\mathbf{v}}|_M \in \mathfrak{X}(M)$ (in some special cases it may still be zero, e.g., for hyperplanes in $\mathbb{R}^n$ with the standard euclidean metric).

Next we introduce the second fundamental form of the hypersurface.

DEFINITION 4.2.4 (Second fundamental form). Let ∇ be a $W_{\text{loc}}^{k,p}$- connection (with $k \geq 1$) on X. The second fundamental form on M is the tensor field $K \in W_{\text{loc}}^{k-1,p}\left(M, T_2^0 M\right)$ given by

$$K\left(\mathbf{u}, \mathbf{v}\right) := \gamma\left(\left(\nabla_{\tilde{\mathbf{u}}} \eta\right)(\tilde{\mathbf{v}})\right)$$

for $\mathbf{u}, \mathbf{v} \in \mathfrak{X}\left(M\right)$, where $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ are (local) extensions of $\mathbf{u}, \mathbf{v}$.

That this is well-defined follows from a similar argument to the one in Lemma 4.2.1.

PROPOSITION 4.2.5. *The second fundamental form satisfies*

$$(4.2.3) \qquad\qquad \gamma\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}\right) = \underline{\nabla}_{\mathbf{u}} \mathbf{v} - K\left(\mathbf{u}, \mathbf{v}\right) \ell$$

for all $\mathbf{u}, \mathbf{v} \in \mathfrak{X}\left(M\right)$ *and*

$$(4.2.4) \qquad\qquad K\left(\mathbf{u}, \mathbf{v}\right) - K\left(\mathbf{v}, \mathbf{u}\right) = \eta\left(\gamma\left(T\left(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}\right)\right)\right),$$

where $T \in W_{\text{loc}}^{k,p}\left(X, T_2^0 X\right)$ *denotes the torsion tensor field given by* $T\left(\mathbf{x}, \mathbf{y}\right) = \nabla_{\mathbf{x}} \mathbf{y} - \nabla_{\mathbf{y}} \mathbf{x} - [\mathbf{x}, \mathbf{y}]$. *If* ∇ *is torsion free then* K *is symmetric.*

PROOF. To show (4.2.3) it suffices to show that $K\left(\mathbf{u}, \mathbf{v}\right) = -\eta\left(\gamma\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}\right)\right)$ by (4.1.3) and (4.2.1). We have

$$(4.2.5) \qquad K\left(\mathbf{u}, \mathbf{v}\right) = \gamma\left(\left(\nabla_{\tilde{\mathbf{u}}} \eta\right)(\tilde{\mathbf{v}})\right) = \gamma\left(\tilde{\mathbf{u}}\left(\eta(\tilde{\mathbf{v}})\right)\right) - \gamma\left(\eta(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}})\right) = 0 - \eta\left(\gamma\left(\nabla_{\tilde{\mathbf{u}}} \tilde{\mathbf{v}}\right)\right)$$

by Lem. 3.6.3 since $\mathbf{u}, \mathbf{v} \in \mathfrak{X}\left(M\right)$ implies $\eta\left(\tilde{\mathbf{v}}\right)|_M = \eta\left(\mathbf{v}\right) = 0$.

Concerning (4.2.4) we note that $\eta\left(\gamma\left([\tilde{\mathbf{u}}, \tilde{\mathbf{v}}]\right)\right) = \eta\left([\mathbf{u}, \mathbf{v}]\right) = 0$ since the Lie bracket of two vector fields in $\mathfrak{X}\left(M\right)$ is well-defined and again in $\mathfrak{X}\left(M\right)$. This and (4.2.5) immediately gives (4.2.4). □

REMARK 4.2.6. That the expression $\eta\left(\gamma\left(T\left(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}\right)\right)\right)$ does not depend on the extensions of $\mathbf{u}$ and $\mathbf{v}$ is clear since the left hand side of (4.2.4) does not depend on them. However, there is a more general way to see that this must be true. For $T \in \Gamma\left(X, T_s^0 X\right)$ the pullback tensor field $j^* T \in \Gamma\left(M, T_s^0 M\right)$ is defined by

$$j^* T\left(\mathbf{v}_1, \ldots, \mathbf{v}_s\right)\left(p\right) = T\left(p\right)\left(\mathbf{v}_1\left(p\right), \ldots, \mathbf{v}_s\left(p\right)\right)$$

which is equal to $\gamma\left(T\left(\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_s\right)\right)$ for any (local) extensions $\tilde{\mathbf{v}}_1, \ldots \tilde{\mathbf{v}}_s$ of $\mathbf{v}_1, \ldots, \mathbf{v}_s$. This shows that the pullback map is continuous with respect to the $W_{\text{loc}}^{k,p}$-topology on $\Gamma\left(X, T_s^0 X\right)$ and the $W_{\text{loc}}^{k-1,p}$-topology on $\Gamma\left(M, T_s^0 M\right)$ for all $k \geq 1$ and thus extends to all of $W_{\text{loc}}^{k,p}\left(X, T_s^0 X\right)$. Clearly one still has that $j^* T\left(\mathbf{v}_1, \ldots, \mathbf{v}_s\right) = \gamma\left(T\left(\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_s\right)\right)$ for $T \in W_{\text{loc}}^{k,p}\left(X, T_s^0 X\right)$ and in particular the right hand side is independent of the chosen extensions.

4.2.1. Gauss and Codazzi equations. Next we want to study the curvature of M. On the one hand, if ∇ is at least in $W_{\text{loc}}^{1,2}\left(X\right)$, the projected connection $\underline{\nabla} \in L_{\text{loc}}^2\left(M\right)$ (see (4.2.1) and Prop. 4.2.2) on M gives rise to a Riemann curvature tensor $\underline{\mathbf{Riem}} \in \mathcal{D}'\left(M, T_3^1 M\right)$ on M.

On the other hand, if ∇ is at least in $W_{\text{loc}}^{k,p}\left(X\right)$ for some $k \geq 2$ and $p > \frac{n}{k}$ (or $p = 1$ and $k \geq n$), we can use the Riemann tensor of ∇ on X to obtain $\gamma(\mathbf{Riem}) \in W_{\text{loc}}^{k-2,p}\left(M, T_3^1 X|_M\right)$ (since $\mathbf{Riem}$ itself is at least in $W_{\text{loc}}^{k-1,p}\left(X\right)$ by Rem. 3.3.5) and by $T_3^0 X|_M \subset T_3^0 M$ (because $TM \subset TX|_M$) we may restrict it to get $\gamma(\mathbf{Riem}) \in W_{\text{loc}}^{k-2,p}\left(M, T_3^0 M \otimes T_0^1 X|_M\right)$.

Before we can relate these two objects with each other, we have to investigate in which sense an analogue to (4.2.3) holds for $\mathbf{v} \in W_{\text{loc}}^{k-1,p}\left(M, TM\right)$. For $\nabla \in W_{\text{loc}}^{k,p}\left(X\right)$ for some $k \geq 2$ and $p > \frac{n-1}{k-1}$ (or $p = 1$ and $k \geq n$) we may extend $\underline{\nabla} \in W_{\text{loc}}^{k-1,p}(M)$ to

$$\underline{\nabla}: \mathfrak{X}\left(M\right) \times W_{\text{loc}}^{k-1,p}\left(M, TM\right) \to W_{\text{loc}}^{k-2,p}\left(M, TM\right)$$

by setting $\left(\underline{\nabla}_{\mathbf{u}} \mathbf{v}\right)\left(\omega\right) = \mathbf{u}\left(\mathbf{v}\left(\omega\right)\right) - \mathbf{v}\left(\underline{\nabla}_{\mathbf{u}} \omega\right)$ for $\omega \in \Gamma\left(M, T_1^0 M\right)$ (see Prop. 3.3.3 and Rem. 3.3.5) and $K: \mathfrak{X}\left(M\right) \times \mathfrak{X}\left(M\right) \to W_{\text{loc}}^{k-1,p}\left(M\right)$ to

$$K: \mathfrak{X}\left(M\right) \times W_{\text{loc}}^{k-1,p}\left(M, TM\right) \to W_{\text{loc}}^{k-1,p}\left(M\right)$$

by $K\left(\mathbf{u}, v^i \partial_i\right) = v^i\, K\left(\mathbf{u}, \partial_i\right)$.

PROPOSITION 4.2.7. *Let $\nabla \in W^{k,p}_{\text{loc}}(X)$ for some $k \geq 2$ and $p > \frac{n-1}{k-1}$ (or $p = 1$ and $k \geq n$). If $\mathbf{v} \in W^{k-1,p}_{\text{loc}}(M, TM)$ and there exists a $\tilde{\mathbf{v}} \in W^{k,p}_{\text{loc}}(X, TX)$ such that $\gamma(\tilde{\mathbf{v}}) = \mathbf{v}$ then for $\mathbf{u} \in \mathfrak{X}(M)$,*

$$(4.2.6) \qquad\qquad \gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{v}}\right) = \underline{\nabla}_{\mathbf{u}}\mathbf{v} - K\left(\mathbf{u}, \mathbf{v}\right)\ell.$$

PROOF. First we note that $p > \frac{n-1}{k-1}$ implies also $p > \frac{n}{k}$ (from $pk - p > n - 1$ one immediately obtains $pk > n$ since $p \geq 1$), so the regularity assumptions on ∇ guarantee that both $W^{k,p}_{\text{loc}}(X)$ and $W^{k-1,p}_{\text{loc}}(M)$ are algebras (with continuous multiplication) and so if $f_n \to f$ in $W^{k,p}_{\text{loc}}(X)$ and $g \in W^{k,p}_{\text{loc}}(X)$ then $\partial_i f_n \to \partial_i f$ in $W^{k-1,p}_{\text{loc}}(X)$ and $f_n g \to fg$ in $W^{k,p}_{\text{loc}}(X)$.

Now choose a sequence $\tilde{\mathbf{v}}_n \in \mathfrak{X}(X)$ such that $\tilde{\mathbf{v}}_n \to \tilde{\mathbf{v}}$ in $W^{k,p}_{\text{loc}}(X)$ and let $\tilde{\mathbf{u}} \in \mathfrak{X}(X)$. For every $\omega \in \Gamma\left(X, T^0_1 X\right)$ the function $\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{v}}_n\right)(\omega) = \tilde{\mathbf{u}}\left(\tilde{\mathbf{v}}_n(\omega)\right) - \tilde{\mathbf{v}}_n\left(\nabla_{\tilde{\mathbf{u}}}\omega\right)$ converges to $\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{v}}\right)(\omega)$ in $W^{k-1,p}_{\text{loc}}(X)$. Thus $\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{v}}_n\right) \to \gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{v}}\right)$ in $W^{k-2,p}_{\text{loc}}\left(M, TX|_M\right)$. Looking at the right hand side of (4.2.6) the same arguments show that $\underline{\nabla}_{\mathbf{u}}\mathbf{v}_n \to \underline{\nabla}_{\mathbf{u}}\mathbf{v}$ in $W^{k-2,p}_{\text{loc}}(M)$ (note that $\mathbf{v}_n = \gamma(\tilde{\mathbf{v}}_n) \to \gamma(\tilde{\mathbf{v}}) = \mathbf{v}$ in $W^{k-1,p}_{\text{loc}}(M)$) and $K(\mathbf{u}, \mathbf{v}_n) = \gamma\left(\left(\nabla_{\tilde{\mathbf{u}}}\eta\right)(\tilde{\mathbf{v}}_n)\right) \to \gamma\left(\left(\nabla_{\tilde{\mathbf{u}}}\eta\right)(\tilde{\mathbf{v}})\right) = K(\mathbf{u}, \mathbf{v})$ in $W^{k-1,p}_{\text{loc}}(M)$, which together with the fact that (4.2.6) holds for smooth $\mathbf{v}$ (see (4.2.3)) proves the proposition. $\qquad\square$

Now we are ready to prove the following proposition.

PROPOSITION 4.2.8. *Let $\nabla \in W^{k,p}_{\text{loc}}(X)$ for some $k \geq 2$ and $p > \frac{n-1}{k-1}$ (or $p = 1$ and $k \geq n$). Then*

$$\gamma(\mathbf{Riem})(\mathbf{u}, \mathbf{v})\,\mathbf{w} = \underline{\mathbf{Riem}}(\mathbf{u}, \mathbf{v})\,\mathbf{w} + K(\mathbf{v}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right) - K(\mathbf{u}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right)$$
$$(4.2.7) \qquad\qquad + \left(\underline{\nabla}_{\mathbf{u}}K\right)(\mathbf{v}, \mathbf{w})\,\ell - \left(\underline{\nabla}_{\mathbf{v}}K\right)(\mathbf{u}, \mathbf{w})\,\ell$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$.

PROOF. We first calculate

$$(4.2.8) \quad \gamma\left(\nabla_{\tilde{\mathbf{u}}}\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}}\right) = \gamma\left(\nabla_{\tilde{\mathbf{u}}}\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}} - \eta\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}}\right)\tilde{\ell} + \eta\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}}\right)\tilde{\ell}\right)\right) =$$

$$\overset{(4.2.6)}{=} \underline{\nabla}_{\mathbf{u}}\left(\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}} - \eta\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}}\right)\tilde{\ell}\right)\right) - K\left(\mathbf{u}, \gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}} - \eta\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}}\right)\tilde{\ell}\right)\right)\ell + \gamma\left(\nabla_{\tilde{\mathbf{u}}}\left(\eta\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}}\right)\tilde{\ell}\right)\right) =$$

$$\overset{(4.2.5)}{\underset{(4.2.6)}{=}} \underline{\nabla}_{\mathbf{u}}\underline{\nabla}_{\mathbf{v}}\mathbf{w} - K\left(\mathbf{u}, \underline{\nabla}_{\mathbf{v}}\mathbf{w}\right)\ell - \gamma\left(\nabla_{\tilde{\mathbf{u}}}\left(-\eta\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}}\right)\tilde{\ell}\right)\right) =$$

$$\overset{(4.2.5)}{=} \underline{\nabla}_{\mathbf{u}}\underline{\nabla}_{\mathbf{v}}\mathbf{w} - K\left(\mathbf{u}, \underline{\nabla}_{\mathbf{v}}\mathbf{w}\right)\ell - \mathbf{u}\left(K(\mathbf{v}, \mathbf{w})\right)\ell - K(\mathbf{v}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right),$$

where we used that $\gamma(fg) = \gamma(f)\,\gamma(g)$ for $f, g \in W^{k,p}_{\text{loc}}(X)$ (which can be shown by approximating f and g by smooth functions) and that $\gamma\left(-\eta\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{w}}\right)\right) = K(\mathbf{v}, \mathbf{w})$ (see (4.2.5)). Now

$$\gamma(\mathbf{Riem})(\mathbf{u}, \mathbf{v})\,\mathbf{w} = \gamma\left(\mathbf{Riem}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})\,\tilde{\mathbf{w}}\right)$$

$$\overset{(4.2.8)}{=} -\underline{\nabla}_{\mathbf{u}}\underline{\nabla}_{\mathbf{v}}\mathbf{w} + K\left(\mathbf{u}, \underline{\nabla}_{\mathbf{v}}\mathbf{w}\right)\ell + \mathbf{u}\left(K(\mathbf{v}, \mathbf{w})\right)\ell + K(\mathbf{v}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right)$$

$$+ \underline{\nabla}_{\mathbf{v}}\underline{\nabla}_{\mathbf{u}}\mathbf{w} - K\left(\mathbf{v}, \underline{\nabla}_{\mathbf{u}}\mathbf{w}\right)\ell - \mathbf{v}\left(K(\mathbf{u}, \mathbf{w})\right)\ell - K(\mathbf{u}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right) + \gamma\left(\nabla_{[\tilde{\mathbf{u}}, \tilde{\mathbf{v}}]}\tilde{\mathbf{w}}\right)$$

$$\overset{(4.2.9)}{\underset{(4.2.6)}{=}} \underline{\mathbf{Riem}}(\mathbf{u}, \mathbf{v})\,\mathbf{w} - K\left([\mathbf{u}, \mathbf{v}], \mathbf{w}\right)\ell + \left(\underline{\nabla}_{\mathbf{u}}K\right)(\mathbf{v}, \mathbf{w})\ell + K\left(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}\right)\ell$$

$$- \left(\underline{\nabla}_{\mathbf{v}}K\right)(\mathbf{u}, \mathbf{w})\ell - K\left(\underline{\nabla}_{\mathbf{v}}\mathbf{u}, \mathbf{w}\right)\ell + K(\mathbf{v}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right) - K(\mathbf{u}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right)$$

$$\overset{(4.2.10)}{=} \underline{\mathbf{Riem}}(\mathbf{u}, \mathbf{v})\,\mathbf{w} + K(\mathbf{v}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right) - K(\mathbf{u}, \mathbf{w})\,\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right)$$

$$+ \left(\underline{\nabla}_{\mathbf{u}}K\right)(\mathbf{v}, \mathbf{w})\ell - \left(\underline{\nabla}_{\mathbf{v}}K\right)(\mathbf{u}, \mathbf{w})\ell,$$

where we used the fact that

$$(4.2.9) \qquad \mathbf{u}\left(K(\mathbf{v}, \mathbf{w})\right) - K\left(\mathbf{v}, \underline{\nabla}_{\mathbf{u}}\mathbf{w}\right) = \left(\underline{\nabla}_{\mathbf{u}}K\right)(\mathbf{v}, \mathbf{w}) + K\left(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}\right)$$

for the third and

$$(4.2.10) \qquad [\mathbf{u}, \mathbf{v}] = \gamma\left([\tilde{\mathbf{u}}, \tilde{\mathbf{v}}]\right)'' = \gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{v}}\right)'' - \gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\mathbf{u}}\right)'' = \underline{\nabla}_{\mathbf{u}}\mathbf{v} - \underline{\nabla}_{\mathbf{v}}\mathbf{u}$$

by Rem. 4.2.3 for the last equality. □

A somewhat similar formula may be derived for $\gamma\left(\mathbf{Riem}\right)(\mathbf{u}, \mathbf{v})\ell$ (again $\mathbf{u}, \mathbf{v}$ are assumed to be in $\mathfrak{X}(M)$): Using

$$\gamma\left(\nabla_{\tilde{\mathbf{u}}}\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right) = \gamma\left(\nabla_{\tilde{\mathbf{u}}}\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell} - \eta(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell})\tilde{\ell} + \eta(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell})\tilde{\ell}\right)\right)$$

$$\overset{(4.2.6)}{\underset{(4.1.3)}{=}} \underline{\nabla}_{\mathbf{u}}\left(\underline{\nabla}_{\mathbf{v}}\ell\right) - K\left(\mathbf{u}, \underline{\nabla}_{\mathbf{v}}\ell\right)\ell + \eta(\gamma(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}))\,\gamma(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell})) + \mathbf{u}(\eta(\gamma(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell})))\ell$$

$$\overset{(4.2.5)}{\underset{(4.2.6)}{=}} \underline{\nabla}_{\mathbf{u}}\left(\underline{\nabla}_{\mathbf{v}}\ell\right) + \eta(\gamma(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}))\underline{\nabla}_{\mathbf{u}}\ell + K\left(\mathbf{u}, \underline{\nabla}_{\mathbf{v}}\ell\right)\ell + \eta(\gamma(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}))\,\eta(\gamma(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}))\ell$$

$$(4.2.11) \qquad\qquad + \mathbf{u}(\eta(\gamma(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell})))\ell$$

we obtain

$$\gamma\left(\mathbf{Riem}\right)(\mathbf{u}, \mathbf{v})\ell = \gamma\left(\nabla_{[\tilde{\mathbf{u}}, \tilde{\mathbf{v}}]}\tilde{\ell}\right) - \gamma\left(\nabla_{\tilde{\mathbf{u}}}\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right) + \gamma\left(\nabla_{\tilde{\mathbf{v}}}\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right)$$

$$\overset{(4.2.1)}{\underset{(4.1.3)}{=}} \underline{\nabla}_{[\mathbf{u}, \mathbf{v}]}\ell + \eta\left(\gamma\left(\nabla_{[\tilde{\mathbf{u}}, \tilde{\mathbf{v}}]}\tilde{\ell}\right)\right)\ell - \gamma\left(\nabla_{\tilde{\mathbf{u}}}\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right) + \gamma\left(\nabla_{\tilde{\mathbf{v}}}\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right)$$

$$\overset{(4.2.11)}{=} \underline{\mathbf{Riem}}\left(\mathbf{u}, \mathbf{v}\right)\ell + \eta\left(\gamma\left(\nabla_{[\tilde{\mathbf{u}}, \tilde{\mathbf{v}}]}\tilde{\ell}\right)\right)\ell - \eta\left(\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right)\right)\underline{\nabla}_{\mathbf{u}}\ell + \eta\left(\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right)\right)\underline{\nabla}_{\mathbf{v}}\ell$$

$$(4.2.12) \qquad\qquad + \left\{\mathbf{v}(\eta(\gamma(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}))) - \mathbf{u}(\eta(\gamma(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}))) + K\left(\mathbf{v}, \underline{\nabla}_{\mathbf{u}}\ell\right) - K\left(\mathbf{u}, \underline{\nabla}_{\mathbf{v}}\ell\right)\right\}\ell,$$

where $\underline{\mathbf{Riem}}\left(\mathbf{u}, \mathbf{v}\right)\ell$ is defined in the obvious way, namely

$$\underline{\mathbf{Riem}}\left(\mathbf{u}, \mathbf{v}\right)\ell = \underline{\nabla}_{[\mathbf{u}, \mathbf{v}]}\ell - \left(\underline{\nabla}_{\mathbf{u}}\underline{\nabla}_{\mathbf{v}}\ell - \underline{\nabla}_{\mathbf{v}}\underline{\nabla}_{\mathbf{u}}\ell\right).$$

The equations (4.2.7) and (4.2.12) provide a generalization of the well-known Gauss and Codazzi equations to general hypersurfaces. We will later see that in the case of non-null hypersurfaces (4.2.7) reduces to the usual Gauss and Codazzi equations and (4.2.12) becomes equivalent to (4.2.7).

4.3. Geometry on a hypersurface induced by a metric

In the previous section, where we discussed the projected connection and the second fundamental form we did not need any metric on X, we only used the connection ∇. Now given a (sufficiently regular) metric g on X we of course have its Levi-Civita connection ∇ on X and thus get a projected connection $\underline{\nabla}$ on M. However, now this is not necessarily the only natural connection on M associated with ∇.

From now on we will assume that we have a uniformly non-degenerate metric g on X that is at least of $W^{k+1,p}_{\text{loc}}(X)$-regularity for some $k \geq 2$ and $p > \frac{n-1}{k-1}$ (or $p = 1$ and $n = k$).

REMARK 4.3.1. While some of the following results may be true under weaker regularity assumptions, the above hypotheses guarantee a number of important things:

(1) The Levi-Civita connection ∇ on X has $W^{k,p}_{\text{loc}}(X)$-regularity (see Prop. 3.4.6) and all the results of the previous section hold, in particular $\underline{\nabla}$ is well-defined and has $W^{k-1,p}_{\text{loc}}(M)$-regularity,

(2) the spaces $W^{k+1,p}_{\text{loc}}(X)$, $W^{k,p}_{\text{loc}}(X)$, $W^{k,p}_{\text{loc}}(M)$ and $W^{k-1,p}_{\text{loc}}(M)$ are algebras with continuous multiplication (see Prop. 3.1.8),

(3) by the Sobolev embedding ([**AF03**], Thm. 4.12 A) we have both

$$W^{k+1,p}_{\text{loc}}(X) \subset W^{k,p}_{\text{loc}}(X) \hookrightarrow \mathcal{C}(X)$$

and
$$W_{\text{loc}}^{k,p}\left(M\right) \subset W_{\text{loc}}^{k-1,p}\left(M\right) \hookrightarrow \mathcal{C}\left(M\right),$$
(4) g is gt-regular (since $W_{\text{loc}}^{k+1,p}\left(X\right) \subset W_{\text{loc}}^{1,2}\left(X\right)$, see Rem. 3.3.5) and

(5) the metric is invertible and satisfies $g^{-1} \in W_{\text{loc}}^{k+1,p}\left(X, T_0^2 X\right)$ (which follows immediately from the algebra property of $W_{\text{loc}}^{k+1,p}\left(X\right)$, uniform non-degeneracy of g and the characterization of invertible elements in $W_{\text{loc}}^{k+1,p}\left(X\right)$).

Note that the uniform non-degeneracy requirement can be replaced with the usual pointwise non-degeneracy (clearly uniform non-degeneracy implies pointwise non-degeneracy, the converse follows from continuity of $|\det g_{ij}|$ because continuous functions attain their minimum on compact sets).

On X the metric g offers a way to identify $W_{\text{loc}}^{k,p}\left(X, T_s^r X\right)$ and $W_{\text{loc}}^{k,p}\left(X, T_r^s X\right)$. This is done using the $\mathcal{C}^\infty\left(X\right)-$linear isomorphism

$$\flat_X : W_{\text{loc}}^{k,p}\left(X, TX\right) \to W_{\text{loc}}^{k,p}\left(X, T_1^0 X\right)$$
$$\mathbf{x} \mapsto (\mathbf{y} \mapsto g(\mathbf{x}, \mathbf{y}))$$

and its inverse

$$\sharp_X : W_{\text{loc}}^{k,p}\left(X, T_1^0 X\right) \to W_{\text{loc}}^{k,p}\left(X, TX\right)$$
$$\zeta \mapsto \left(\xi \mapsto g^{-1}(\zeta, \xi)\right)$$

Locally $\mathbf{x}^{\flat_X}$ is given by $\left(\mathbf{x}^{\flat_X}\right)_i = g_{ij}\,\mathbf{x}^j$ and $\zeta^{\sharp_X}$ by $\left(\zeta^{\sharp_X}\right)^i = g^{ij}\zeta_j$ where $\left(g^{ij}\right)_{1\le i,j\le n}$ is the matrix inverse of $(g_{ij})_{1\le i,j\le n}$. Our next goal will be to define analogous maps on M.

From g we obtain the trace $\gamma(g) \in W_{\text{loc}}^{k,p}\left(M, T_2^0 X\big|_M\right)$ and, by restriction, a $(0,2)-$tensor field $g_M \in W_{\text{loc}}^{k,p}\left(M, T_2^0 M\right)$ on M. In general g_M will not be non-degenerate and thus not provide a metric on M (the special case where g_M actually is a metric on M is treated in section 4.4 below), in particular the matrix $\left((g_M)_{ij}\right)_{1\le i,j\le n-1} = (\gamma(g_{ij}))_{1\le i,j\le n-1}$ (for coordinates adapted to the hypersurface, see footnote 1 on page 31) will not be invertible. Regardless, it still provides a $\mathcal{C}^\infty(M)$-linear map

$$\flat_M : \mathfrak{X}\left(M\right) \to \Gamma\left(M, T_1^0 M\right)$$
$$\mathbf{u} \mapsto (\mathbf{v} \mapsto \gamma(g)\left(\mathbf{u}, \mathbf{v}\right)).$$

Clearly this definition can be extended to $\flat_M : W_{\text{loc}}^{k-1,p}\left(M, TM\right) \to W_{\text{loc}}^{k-1,p}\left(M, T_1^0 M\right)$.

While this map does not have an inverse in general, there is a natural way to define a map $\sharp_M : \Gamma\left(M, T_1^0 M\right) \to \mathfrak{X}\left(M\right)$ by using the trace of $g^{-1} \in W_{\text{loc}}^{k+1,p}\left(X, T_0^2 X\right)$. Given a one form $\omega \in \Gamma\left(M, T_1^0 M\right)$ we may use the rigging to interpret ω as an element $\omega' \in \Gamma\left(M, T_1^0 X\big|_M\right)$ via $\omega'\left(\mathbf{x}\right) := \omega\left(\mathbf{x} - \eta(\mathbf{x})\ell\right)$ for $\mathbf{x} \in \Gamma\left(M, TX\big|_M\right)$. With this we can define $\sharp_M$ by

$$\sharp_M : \Gamma\left(M, T_1^0 M\right) \to \mathfrak{X}\left(M\right)$$
$$\omega \mapsto \left(\theta \mapsto \gamma\left(g^{-1}\right)\left(\omega', \theta\right)\right).$$

Note that $\omega^{\sharp_M}$ depends on the rigging since the inclusion map $\Gamma\left(M, T_1^0 M\right) \to \Gamma\left(M, T_1^0 X\big|_M\right)$ does. As was the case for $\flat_M$ we may extend $\sharp_M$ to a map $\sharp_M : W_{\text{loc}}^{k-1,p}\left(M, T_1^0 M\right) \to W_{\text{loc}}^{k-1,p}\left(M, TM\right)$.

In the future we may simply write $\flat, \sharp$ for either $\flat_X, \sharp_X$ or $\flat_M, \sharp_M$ if it is clear which one is meant. We can use this to define the so-called normal vector field $\mathbf{n} \in W_{\text{loc}}^{k+1,p}\left(X, TX\right)$ by $\mathbf{n} := \eta^{\sharp_X}$ (this definition has actually already been mentioned in the paragraph before Def. 4.1.2). Now we are ready to investigate necessary and sufficient conditions for the degeneracy/non-degeneracy of g_M and $\gamma(g^{-1})\big|_{T*M}$, where $\gamma(g^{-1})\big|_{T*M} \in W_{\text{loc}}^{k,p}\left(M, T_0^2 M\right)$ denotes the $(2,0)$-tensor field on M given by $\gamma(g^{-1})\big|_{T*M}(\omega, \eta) := \gamma(g^{-1})(\omega', \eta')$ (for $\omega, \eta \in \mathcal{T}_1^0(M)$).

THEOREM 4.3.2. *The following equivalences hold:*

(1) $\gamma(g)$ *is degenerate at* $p \in M$ *if and only if* $\mathbf{n}_p$ *is null (that is iff* $g(p)(\mathbf{n}_p, \mathbf{n}_p) = 0$)
(2) $\gamma(g^{-1})\big|_{T^*M}$ *is degenerate at* $p \in M$ *if and only if* ℓ_p *is null (that is iff* $g(p)(\ell_p, \ell_p) = 0$)

PROOF. First we observe that it makes sense to talk about pointwise properties since all the occurring objects are at least continuous by Rem. 4.3.1. To show (1) note that $g(p)(\mathbf{n}_p, \mathbf{u}_p) = \eta_p(\mathbf{u}_p) = 0$ for all $\mathbf{u}_p \in T_p M$. Now if $\mathbf{n}_p$ is null we have $\eta_p(\mathbf{n}_p) = 0$ and thus $\mathbf{n}_p \in T_p M$ (by definition of η one has $\eta_p(\mathbf{x}_p) \neq 0$ for $\mathbf{x}_p \notin T_p M$) and g_M is degenerate at p. On the other hand if g_M is degenerate at p, then there exists a vector $\mathbf{v}_p \in T_p M \setminus \{0\} \subset T_p X \setminus \{0\}$ such that $g(p)(\mathbf{v}_p, \mathbf{u}_p) = 0$ for all $\mathbf{u}_p \in T_p M$ and so $\mathbf{v}_p^\flat\big|_{T_p M} = 0$ giving $\mathbf{v}_p^\flat = \mathbf{v}_p^\flat(\ell_p)\, \eta_p$ (note that $\mathbf{v}_p^\flat(\ell_p) \neq 0$ since $\mathbf{v}_p \neq 0$). This shows that $\mathbf{n}_p = \frac{1}{\mathbf{v}_p^\flat(\ell_p)} \mathbf{v}_p \in T_p M$ and thus is a null vector.

The proof of (2) uses a similar argument: We have $g^{-1}(p)(\ell_p^\flat, \omega_p) = 0$ for all $\omega_p \in T_p^* M$. Now if ℓ_p is null, then $0 = g(p)(\ell_p, \ell_p) = \ell_p^\flat(\ell_p)$ showing that $\ell_p^\flat \in T_p^* M$ and thus $\gamma(g^{-1})\big|_{T^*M}(p)$ is degenerate. On the other hand if $\gamma(g^{-1})\big|_{T^*M}$ is degenerate at p, there exists a form $\xi_p \in T_p^* M \setminus \{0\} \subset T_p^* X \setminus \{0\}$ such that $g^{-1}(p)(\xi_p, \omega_p) = 0$ for all $\omega_p \in T_p^* M$ and so $\xi_p^\sharp = \xi_p^\sharp(\eta_p)\, \ell_p$ showing that $0 = g^{-1}(p)(\ell_p^\flat, \ell_p^\flat) = \ell_p(\ell_p^\flat) = \ell_p^\flat(\ell_p) = g(p)(\ell_p, \ell_p)$ since $(\ell_p^\flat)^\sharp = \ell_p$. $\qquad\square$

REMARK 4.3.3. In general the operations $\flat$ and $\sharp$ do not commute with the projections $(.)^{\shortparallel}$ and $(.)_{\shortparallel}$, i.e., $(\mathbf{x}^{\shortparallel})^\flat \neq (\mathbf{x}^\flat)_{\shortparallel}$ and $(\zeta_{\shortparallel})^\sharp \neq (\zeta^\sharp)^{\shortparallel}$ for $\mathbf{x} \in \Gamma(M, TX|_M)$ and $\zeta \in \Gamma(M, T_1^0 X|_M)$. In fact (using that $\mathbf{x}^{\shortparallel} = \mathbf{x} - \eta(\mathbf{x})\ell$) one has on M

$$\left(\mathbf{x}^{\shortparallel}\right)^\flat(\mathbf{u}) = g_M\left(\mathbf{x}^{\shortparallel}, \mathbf{u}\right) = g(\mathbf{x}, \mathbf{u}) - \eta(\mathbf{x})g(\ell, \mathbf{u}) \neq g(\mathbf{x}, \mathbf{u}) - g(\ell, \mathbf{x})g(\mathbf{n}, \mathbf{u}) =$$
$$= \mathbf{x}^\flat(\mathbf{u}) - \ell(\mathbf{x}^\flat)\eta(\mathbf{u}) = (\mathbf{x}^\flat)_{\shortparallel}(\mathbf{u}),$$

where we have simply written g instead of $\gamma(g)$ since the trace is simply the restriction of g to M by continuity of g. An analogous calculation can be done regarding $(\zeta_{\shortparallel})^\sharp$ and $(\zeta^\sharp)^{\shortparallel}$. Furthermore, $(\mathbf{n}^{\shortparallel})^\flat = -g(\mathbf{n}, \mathbf{n})(\ell^\flat)_{\shortparallel}$ and $(\ell^\flat)_{\shortparallel}(\mathbf{n}^{\shortparallel}) = 1 - g(\mathbf{n}, \mathbf{n})\, g(\ell, \ell)$:

The first equation follows immediately from the calculation above by replacing $\mathbf{x}$ with $\mathbf{n}$ and noting that $g(\mathbf{n}, \mathbf{u}) = \eta(\mathbf{u}) = 0$ for all $\mathbf{u} \in \mathfrak{X}(M)$. Regarding the second equation we have

$$(\ell^\flat)_{\shortparallel}(\mathbf{n}^{\shortparallel}) = \ell^\flat(\mathbf{n} - \eta(\mathbf{n})\ell) = \eta(\ell) - g(\mathbf{n}, \mathbf{n})\ell^\flat(\ell) = 1 - g(\mathbf{n}, \mathbf{n})\, g(\ell, \ell).$$

Theorem 4.3.2 shows that whether g_M is degenerate or not solely depends on the given hypersurface and the metric g on X (since η_p is uniquely determined up to a scalar multiple for all $p \in M$), but whether $\gamma(g^{-1})\big|_{T^*M}$ is degenerate or not depends on the choice of the rigging.

PROPOSITION 4.3.4. *For every* $p \in M$ *there exists a rigging* ℓ *(depending on* p*) that is non-null in a neighborhood of* p.

PROOF. Choose an arbitrary rigging ℓ. If ℓ_p is not null we are finished. If ℓ_p is null we can find a vector field $\mathbf{u} \in \mathfrak{X}(M)$ such that $g(p)(\ell_p, \mathbf{u}_p) \neq 0$ or $g(p)(\mathbf{u}_p, \mathbf{u}_p) \neq 0$: Otherwise $g(p)(\mathbf{x}_p, \mathbf{x}_p) = 0$ for all $\mathbf{x}_p \in T_p X$ (since $\mathbf{x} = \mathbf{x}^{\shortparallel} + \eta(\mathbf{x})\ell$ for all $\mathbf{x} \in \Gamma(M, TX|_M)$ by (4.1.3)) and thus g is zero in p by the polarization identity, which gives a contradiction to g being a metric on X. Setting $\ell' = \ell + \mathbf{u}$ we see that $g(p)(\ell_p', \ell_p') = 2g(p)(\ell_p, \mathbf{u}_p) + g(p)(\mathbf{u}_p, \mathbf{u}_p)$ which can be assumed to be non-zero (if $2g(\ell, \mathbf{u}) + g(\mathbf{u}, \mathbf{u}) = 0$ at p one can simply replace $\mathbf{u}$ with $-\mathbf{u}$ so that both terms have the same sign). By continuity ℓ' is non-null in a neighborhood of p. $\qquad\square$

Unfortunately it is in general not possible to simply glue these local non-null riggings together with a partition of unity to obtain a rigging field ℓ that is non-null everywhere on M since being non-null at p is not a convex condition, i.e., $\mathbf{u}_p + \mathbf{v}_p$ may be null even if $\mathbf{u}_p$ and $\mathbf{v}_p$ are both not null. However, there is a very important (because physically relevant) special case:

PROPOSITION 4.3.5. *If g is a continuous metric on X and (X, g) is a time-oriented Lorentzian manifold*[2] *and $M \subset X$ a hypersurface, then there exists a rigging $\ell \in \Gamma(M, TX|_M)$ such that $g(p)(\ell_p, \ell_p) > 0$ and ℓ_p is future directed*[3] *for all $p \in M$. In particular, ℓ is nowhere null.*

PROOF. We first show that there exists a basis of future-directed unit timelike ($\mathbf{u}_p \in T_P X$ is called timelike if $g(p)(\mathbf{u}_p, \mathbf{u}_p) < 0$) vectors for $T_p X$ for every $p \in X$: Let $e_1, \dots, e_n$ be an orthonormal basis for $T_p X$ such that e_1 is timelike. Then $e_1, 2e_1 + e_2, \dots, 2e_1 + e_n$ are n timelike and linearly independent vectors in $T_p X$. By normalizing and multiplying with minus one if necessary we obtain a basis of future-directed unit timelike vectors.

Now let $p \in M$, then $T_p M \subset T_p X$ is a $(n-1)$-dimensional subspace of $T_p X$ and we may choose some future directed unit timelike vector $\mathbf{u}_p \notin T_p M$. If $\tilde{\mathbf{u}}^p \in \Gamma(M, TX|_M)$ is any smooth extension of $\mathbf{u}_p$ there exists a neighborhood U_p of p in M such that $\tilde{\mathbf{u}}^p|_{U_p}$ is still future-directed timelike (by continuity of g and $\mathbf{t}$). Because M is second countable we may choose a countable subcover $\{U_i\}_{i \in \mathbb{N}} \subset \{U_p : p \in M\}$ (so for every i there exists a $p_i \in M$ such that $U_i = U_{p_i}$). Now let $\{\chi_i\}_{i \in \mathbb{N}}$ be a partition of unity subordinate to the cover $\{U_i\}_{i \in \mathbb{N}}$ and set

$$\ell = \sum_{i \in \mathbb{N}} \chi_i \, \tilde{\mathbf{u}}^{pi}.$$

Then ℓ_p is future-directed timelike for all $p \in M$ since the $(\chi_i \tilde{\mathbf{u}}^{p_i})(p)$ are future-directed timelike (if they are non-zero because $\operatorname{supp} \chi_i \subset U_{p_i}$) and the sum of future-directed timelike vectors is again future directed timelike (see [Nab92], Lemma 1.4.3). $\qquad\square$

From now on we are going to assume that a nowhere null rigging vector field ℓ is given. By Thm. 4.3.2 this implies that $\gamma(g^{-1})|_{T^* M} \in W^{k,p}_{\mathrm{loc}}(M, T^2_0 M)$ is pointwise non-degenerate on all of M which in turn shows that there is a $(0,2)$-tensor field $\overline{g} \in W^{k,p}_{\mathrm{loc}}(M, T^0_2 M)$ with

$$(4.3.1) \qquad (\overline{g}_{ij})_{1 \le i,j \le n-1} = \left(\left(\gamma(g^{-1})\big|_{T^* M} \right)^{ij} \right)^{-1}_{1 \le i,j \le n-1}$$

for all charts $(U_\alpha, \psi_\alpha = (x^1, \dots, x^{n-1}))$ on M (since the regularity of $\gamma(g^{-1})|_{T^* M}$ is high enough, see also Rem. 4.3.1). Clearly $\overline{g}$ is also pointwise non-degenerate and symmetric (the inverse of a symmetric matrix is again symmetric) and thus provides a metric on M.

DEFINITION 4.3.6 (The metric connection on M). The Levi-Civita connection associated with $\overline{g}$ is called the metric connection on M and denoted by $\overline{\nabla}$.

The next proposition will deal with the differences between the metric connection $\overline{\nabla}$ defined above and the projected connection $\underline{\nabla}$ from section 4.2. First we note that both the connections have $W^{k-1,p}_{\mathrm{loc}}(M)$-regularity (see Prop. 3.4.6 for $\overline{\nabla}$ and Rem. 4.3.1 for $\underline{\nabla}$) and depend on the rigging vector field ℓ.

PROPOSITION 4.3.7. *The operator $\overline{\nabla}$ is a metric connection on M and while $\underline{\nabla}$ is torsion free as well it need not be a metric connection. We have*

$$(4.3.2) \qquad\qquad \overline{\nabla} = \underline{\nabla} + F,$$

where $F \in W^{k-1,p}_{\mathrm{loc}}(M, T^1_2 M)$ is defined by

$$(4.3.3) \qquad \overline{g}(F(\mathbf{u}, \mathbf{v}), \mathbf{w}) := \frac{1}{2} \left\{ (\underline{\nabla}_{\mathbf{u}} \overline{g})(\mathbf{v}, \mathbf{w}) + (\underline{\nabla}_{\mathbf{v}} \overline{g})(\mathbf{u}, \mathbf{w}) - (\underline{\nabla}_{\mathbf{w}} \overline{g})(\mathbf{u}, \mathbf{v}) \right\}.$$

---

[2]A Lorentzian manifold (i.e. a semi-Riemannian manifold with signature $n-1$) is called time orientable if there exists some smooth vector field $\mathbf{t} \in \mathfrak{X}(X)$ such that $\mathbf{t}$ is timelike everywhere, i.e., that satisfies $g(p)(\mathbf{t}_p, \mathbf{t}_p) < 0$ for all $p \in X$ (see [Rin09], 10.4.2).

[3]A timelike vector $\mathbf{u}_p \in T_p X$ is called future directed if $g(p)(\mathbf{u}_p, \mathbf{t}_p) > 0$.

PROOF. That $\underline{\nabla}$ is torsion free was shown in Remark 4.2.3. It only remains to prove (4.3.2). This is equivalent to showing

$$\overline{g}\left(\overline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}\right) = \overline{g}\left(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}\right) + \overline{g}\left(F(\mathbf{u}, \mathbf{v}), \mathbf{w}\right).$$

Since $\overline{\nabla}$ is the Levi-Civita connection associated with $\overline{g}$ it satisfies the Koszul formula (3.4.1), i.e.,

$$2\overline{g}\left(\overline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}\right) = \mathbf{u}\left(\overline{g}\left(\mathbf{v}, \mathbf{w}\right)\right) + \mathbf{v}\left(\overline{g}\left(\mathbf{w}, \mathbf{u}\right)\right) - \mathbf{w}\left(\overline{g}\left(\mathbf{u}, \mathbf{v}\right)\right) - \overline{g}\left(\mathbf{u}, [\mathbf{v}, \mathbf{w}]\right) + \overline{g}\left(\mathbf{v}, [\mathbf{w}, \mathbf{u}]\right)$$

$$(4.3.4) \qquad\qquad + \overline{g}\left(\mathbf{w}, [\mathbf{u}, \mathbf{v}]\right).$$

By Prop. 4.2.2, we have

$$\mathbf{u}\left(\bar{g}(\mathbf{v}, \mathbf{w})\right) = \left(\underline{\nabla}_{\mathbf{u}}\bar{g}\right)(\mathbf{v}, \mathbf{w}) + \bar{g}(\mathbf{v}, \underline{\nabla}_{\mathbf{u}}\mathbf{w}) + \overline{g}(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}),$$

so (4.3.4) becomes

$$2\overline{g}\left(\overline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}\right) = \left(\underline{\nabla}_{\mathbf{u}}\bar{g}\right)(\mathbf{v}, \mathbf{w}) + \left(\underline{\nabla}_{\mathbf{v}}\bar{g}\right)(\mathbf{u}, \mathbf{w}) - \left(\underline{\nabla}_{\mathbf{w}}\bar{g}\right)(\mathbf{u}, \mathbf{v})$$

$$+ \bar{g}(\mathbf{v}, \underline{\nabla}_{\mathbf{u}}\mathbf{w}) + \bar{g}(\mathbf{w}, \underline{\nabla}_{\mathbf{v}}\mathbf{u}) - \bar{g}(\mathbf{u}, \underline{\nabla}_{\mathbf{w}}\mathbf{v}) + \overline{g}(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}) + \overline{g}(\underline{\nabla}_{\mathbf{v}}\mathbf{w}, \mathbf{u}) - \overline{g}(\underline{\nabla}_{\mathbf{w}}\mathbf{u}, \mathbf{v})$$

$$- \overline{g}(\mathbf{u}, [\mathbf{v}, \mathbf{w}]) + \overline{g}(\mathbf{v}, [\mathbf{w}, \mathbf{u}]) + \overline{g}(\mathbf{w}, [\mathbf{u}, \mathbf{v}])$$

$$\overset{(4.3.3)}{=} 2\overline{g}\left(F(\mathbf{u}, \mathbf{v}), \mathbf{w}\right) + \bar{g}(\mathbf{w}, \underline{\nabla}_{\mathbf{v}}\mathbf{u}) + \overline{g}(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}) + \overline{g}(\mathbf{w}, [\mathbf{u}, \mathbf{v}])$$

$$= 2\overline{g}\left(F(\mathbf{u}, \mathbf{v}), \mathbf{w}\right) + \bar{g}(\mathbf{w}, \underline{\nabla}_{\mathbf{v}}\mathbf{u}) - \overline{g}(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}) + 2\overline{g}(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}) + \overline{g}(\mathbf{w}, [\mathbf{u}, \mathbf{v}])$$

$$= 2\overline{g}\left(F(\mathbf{u}, \mathbf{v}), \mathbf{w}\right) + 2\overline{g}(\underline{\nabla}_{\mathbf{u}}\mathbf{v}, \mathbf{w}),$$

where we used several times that the $\underline{\nabla}$ is torsion free, i.e., that $\underline{\nabla}_{\mathbf{u}}\mathbf{v} - \underline{\nabla}_{\mathbf{v}}\mathbf{u} - [\mathbf{u}, \mathbf{v}] = 0$ (see Rem. 4.2.3). $\qquad\square$

4.4. Nowhere null hypersurfaces

Again we assume $g \in W_{\text{loc}}^{k+1,p}\left(X, T_2^0 X\right)$ (for some $k \geq 2$ and $p > \frac{n-1}{k-1}$ or $p = 1$ and $k = n$, see Rem. 4.3.1). We have shown in Theorem 4.3.2 that $\gamma(g)$ is a metric on M if and only if $\mathbf{n} \in W_{\text{loc}}^{k+1,p}\left(X, TX\right) \subset \mathcal{C}\left(X, TX\right)$ is nowhere null on M. This leads to the following definition.

DEFINITION 4.4.1 (Nowhere null hypersurfaces). Let g be a continuous metric on X, $M \subset X$ a hypersurface with normal form η and normal vector $\mathbf{n} = \eta^{\sharp}$. Then M is called *nowhere null* if $g_p\left(\mathbf{n}_p, \mathbf{n}_p\right) \neq 0$ for every $p \in M$.

REMARK 4.4.2. By (4.1.1) we have that M being nowhere null is equivalent to $\mathbf{n}\left(p\right) \notin T_p M$ for all $p \in M$ (since $g_p\left(\mathbf{n}_p, \mathbf{n}_p\right) = \eta(\mathbf{n})(p) \neq 0$ iff $\mathbf{n}_p \notin T_p M$.

As implied by the title of this section we are now going to have a closer look at nowhere null hypersufaces. Since $\mathbf{n}$ is nowhere null we may normalize our normal one form η to satisfy $\left|g^{-1}\left(\eta, \eta\right)\right| = |g\left(\mathbf{n}, \mathbf{n}\right)| = 1$ on M (the definition of the normal one form only determines $\eta|_M$ up to a scalar multiple). From now on we will always assume that we have this normalization.

If we define

$$(4.4.1) \qquad\qquad \ell = \text{sgn}(\mathbf{n})\,\gamma(\mathbf{n}) \in W_{\text{loc}}^{k,p}\left(M, TX|_M\right),$$

where $\text{sgn}(\mathbf{n}) := \text{sgn}(g(\mathbf{n}, \mathbf{n}))$ is either constant one or minus one (so $\ell = \pm\,\mathbf{n}|_M$), then ℓ satisfies all the requirements of a rigging vector field except smoothness (see Def. 4.1.2).

Fortunately this is not a problem since the smoothness assumption on ℓ was merely convenient, not necessary. In fact, all results of the previous section still hold for a rigging $\ell \in W_{\text{loc}}^{k,p}\left(M, TX|_M\right)$ that satisfies $\ell = \gamma(\tilde{\ell})$ for some $\tilde{\ell} \in W_{\text{loc}}^{k+1,p}\left(X, TX\right)$. That this regularity is sufficient follows from the fact that $\nabla_{\mathbf{x}}\mathbf{y} \in W_{\text{loc}}^{k-1,p}$ for $\mathbf{x}$ smooth and $\mathbf{y} \in W_{\text{loc}}^{k,p}$ for any $W_{\text{loc}}^{k,p}$-connection with k large enough (see Rem. 3.3.5), so in particular $\nabla_{\mathbf{x}}\tilde{\ell} \in W_{\text{loc}}^{k,p}(X, TX)$ for all $\mathbf{x} \in \mathfrak{X}(X)$ and $\underline{\nabla}_{\mathbf{u}}\ell \in W_{\text{loc}}^{k-1,p}(M, TX|_M)$ for all $\mathbf{u} \in \mathfrak{X}(M)$. Using this and looking, e.g., at (4.2.12) we see that

the terms of the form $\underline{\nabla}_{\mathbf{u}} \left(\gamma(\nabla_{\tilde{\mathbf{v}}} \tilde{\ell})^{\shortparallel} \right)$ in $\underline{\mathbf{Riem}} \left(\mathbf{u}, \mathbf{v} \right) \ell$ are well-defined since $\gamma \left(\nabla_{\tilde{\mathbf{v}}} \tilde{\ell} \right) \in W_{\mathrm{loc}}^{k-1,p}(M)$ (thus $\gamma \left(\nabla_{\tilde{\mathbf{v}}} \tilde{\ell} \right)^{\shortparallel} = \gamma \left(\nabla_{\tilde{\mathbf{v}}} \tilde{\ell} \right) - \eta \left(\gamma \left(\nabla_{\tilde{\mathbf{v}}} \tilde{\ell} \right) \right) \ell \in W_{\mathrm{loc}}^{k-1,p}(M)$, as $W_{\mathrm{loc}}^{k-1,p}(M)$ is an algebra) and $\underline{\nabla}$ is a $W_{\mathrm{loc}}^{k-1,p}$-connection on M. Similar considerations show that all the arguments used in the derivation of (4.2.12) and the other results of the previous sections remain valid.

Remark 4.2.3 shows that the following statement holds (since $\ell \propto \mathbf{n}$).

PROPOSITION 4.4.3. *The projected connection $\underline{\nabla}$ is the Levi-Civita connection induced by the metric g_M on M.*

Our next goal is to show that the two metrics g_M and $\overline{g} = \left(\gamma(g^{-1})\big|_{T^*M} \right)^{-1}$ on M are equal. One way to do this is to show that $\flat_M$ is the inverse of $\sharp_M$ (implying that the inverse of g_M is just $\gamma(g^{-1})\big|_{T^*M}$), which will be an immediate consequence of the next lemma.

LEMMA 4.4.4. *If ℓ is given by (4.4.1), the operations $\flat$ and $\sharp$ do commute with the projections $(.)^{\shortparallel}$ and $(.)_{\shortparallel}$, i.e.,*

(4.4.2) $$(\mathbf{x}^{\shortparallel})^{\flat_M} = (\mathbf{x}^{\flat_X})_{\shortparallel} \text{ for } \mathbf{x} \in \Gamma \left(M, TX \right)$$

and

(4.4.3) $$(\zeta_{\shortparallel})^{\sharp_M} = (\zeta^{\sharp_X})^{\shortparallel} \text{ for } \zeta \in \Gamma \left(M, T_1^0 X \right)$$

(contrary to the general case outlined in Rem. 4.3.3).

PROOF. The calculations in Rem. 4.3.3 show that for $p \in M$ equality of $(\mathbf{x}^{\shortparallel})^{\flat}(p)$ and $(\mathbf{x}^{\flat})_{\shortparallel}(p)$ is equivalent to $\eta(\mathbf{x})(p)g(p)(\ell_p, \mathbf{u}_p) = g(p)(\ell_p, \mathbf{x}_p)g(p)\left(\mathbf{n}_p, \mathbf{u}_p \right)$ for all $\mathbf{u} \in \mathfrak{X}(M)$ which is obviously true for $\ell \propto \mathbf{n}$. Regarding $(\zeta_{\shortparallel})^{\sharp}$ and $(\zeta^{\sharp})^{\shortparallel}$ we calculate

(4.4.4) $$(\zeta_{\shortparallel})^{\sharp}(\omega)(p) \overset{(4.1.5)}{=} \left(g^{-1} \right)(p) \left(\zeta_p - (\ell(\zeta))(p)\eta_p, \omega_p \right) =$$
$$= \left(g^{-1} \right)(p)\left(\zeta_p, \omega_p \right) - (\ell(\zeta))(p) \gamma \left(g^{-1} \right)(p) \left(\eta_p, \omega_p \right)$$

(note that $\zeta_{\shortparallel}(p)$ is equal to the $\zeta'_{\shortparallel}(p)$ appearing in the definition of $\sharp_M$ since $\ell(\zeta_{\shortparallel})(p) = 0$) and

(4.4.5) $$(\zeta^{\sharp})^{\shortparallel}(\omega)(p) \overset{(4.1.3)}{=} \left(\zeta^{\sharp} - \eta \left(\zeta^{\sharp} \right) \ell \right)(\omega)(p) =$$
$$= g^{-1}(p)(\zeta_p, \omega_p) - (\ell(\omega))(p) \, g^{-1}(p)(\zeta_p, \eta_p),$$

for $\omega \in \Gamma \left(M, T_1^0 M \right)$. Obviously (4.4.4) equals (4.4.5) if $\ell \propto \eta^{\sharp}$. $\qquad \square$

PROPOSITION 4.4.5. *If $\mathbf{n}$ is nowhere null and ℓ is as in (4.4.1), then $g_M = \overline{g}$ (where $\overline{g}$ is the metric on M defined in (4.3.1)).*

PROOF. Since g_M is a metric $\flat_M : \mathfrak{X}(M) \to \Gamma \left(M, T_1^0 M \right)$ is an isomorphism hence invertible. Now every $\omega \in \Gamma \left(M, T_1^0 M \right)$ can be written as $\zeta_{\shortparallel}$ for some $\zeta \in \Gamma \left(M, T_1^0 X \right)$ (set $\zeta(\mathbf{x}) = \omega \left(\mathbf{x} - \eta(\mathbf{x})\ell \right)$ for $\mathbf{x} \in \Gamma(M, TX)$) and using Lemma 4.4.4 we obtain $\omega = \zeta_{\shortparallel} = \left((\zeta^{\sharp_X})^{\flat_X} \right)_{\shortparallel} \overset{(4.4.2)}{=} \left((\zeta^{\sharp_X})^{\shortparallel} \right)^{\flat_M} \overset{(4.4.3)}{=} \left((\zeta_{\shortparallel})^{\sharp_M} \right)^{\flat_M} = \left(\omega^{\sharp_M} \right)^{\flat_M}$, so $\flat_M^{-1}(\omega) = \omega^{\sharp_M}$. $\qquad \square$

COROLLARY 4.4.6. *The projected connection $\underline{\nabla}$ and the metric connection $\overline{\nabla}$ coincide.*

PROOF. This follows from the uniqueness of the Levi-Civita connection: From Def. 4.3.6 we know that $\overline{\nabla}$ is the Levi-Civita connection induced by $\overline{g}$ and by Prop. 4.4.3 it is also the Levi-Civita connection induced by g_M, but since $\overline{g} = g_M$ (by Prop. 4.4.5) $\overline{\nabla}$ has to be equal to $\underline{\nabla}$. $\qquad \square$

Now we are going to investigate the generalized Gauss and Codazzi equations (4.2.7) and (4.2.12).

PROPOSITION 4.4.7. *Let $M \subset X$ is a nowhere null hypersurface and $\ell = \mathrm{sgn}(\mathbf{n})\,\gamma(\mathbf{n})$, then*

$$(4.4.6) \quad \gamma(g)\left(\gamma(\mathbf{Riem})\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\mathbf{t}\right) = \gamma(g)\left(\underline{\mathbf{Riem}}\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\mathbf{t}\right) +$$
$$+ \mathrm{sgn}(\mathbf{n})\left\{K\left(\mathbf{v},\mathbf{w}\right)K(\mathbf{u},\mathbf{t}) - K\left(\mathbf{u},\mathbf{w}\right)K(\mathbf{v},\mathbf{t})\right\}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{t} \in \mathfrak{X}(M)$ and

$$(4.4.7) \qquad\qquad \gamma(g)\left(\gamma(\mathbf{Riem})\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\gamma(\mathbf{n})\right) = \left(\underline{\nabla}_{\mathbf{u}}K\right)\left(\mathbf{v},\mathbf{w}\right) - \left(\underline{\nabla}_{\mathbf{v}}K\right)\left(\mathbf{u},\mathbf{w}\right),$$

i.e., the usual Gauss and Codazzi equations.

PROOF. Since $\underline{\nabla}$ is the connection associated with $\gamma(g)$ (see Prop. 4.4.3) the $(1,3)$-tensor field $\underline{\mathbf{Riem}}$ is the Riemann tensor of the semi-Riemannian submanifold $(M,\gamma(g))$. Using

$$\gamma(\mathbf{Riem})\,(\mathbf{u},\mathbf{v})\,\mathbf{w} \stackrel{(4.2.7)}{=} \underline{\mathbf{Riem}}\,(\mathbf{u},\mathbf{v})\,\mathbf{w} + K\left(\mathbf{v},\mathbf{w}\right)\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right) - K\left(\mathbf{u},\mathbf{w}\right)\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right) +$$
$$(4.4.8) \qquad\qquad + \left(\underline{\nabla}_{\mathbf{u}}K\right)\left(\mathbf{v},\mathbf{w}\right)\ell - \left(\underline{\nabla}_{\mathbf{v}}K\right)\left(\mathbf{u},\mathbf{w}\right)\ell$$

and that

$$(4.4.9) \qquad\qquad \gamma(g)(\ell,\mathbf{t}) \stackrel{(3.1.2)}{=} \gamma\left(g(\tilde{\ell},\tilde{\mathbf{t}})\right) \propto \gamma(g(\mathbf{n},\tilde{\mathbf{t}})) = \gamma(\eta(\tilde{\mathbf{t}})) \stackrel{(3.1.2)}{=} \eta(\mathbf{t}) = 0$$

for all $\mathbf{t} \in \mathfrak{X}(M)$ since $\ell = \mathrm{sgn}(\mathbf{n})\,\gamma(\mathbf{n})$ (see (4.4.1)), we get

$$(4.4.10) \quad \gamma(g)\left(\gamma(\mathbf{Riem})\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\mathbf{t}\right) = \gamma(g)\left(\underline{\mathbf{Riem}}\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\mathbf{t}\right) - K\left(\mathbf{u},\mathbf{w}\right)\gamma(g)\left(\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right),\mathbf{t}\right) +$$
$$+ K\left(\mathbf{v},\mathbf{w}\right)\gamma(g)\left(\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right),\mathbf{t}\right)$$

for $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{t} \in \mathfrak{X}(M)$. Since

$$\gamma(g)\left(\gamma\left(\nabla_{\tilde{\mathbf{u}}}\mathbf{n}\right),\mathbf{t}\right) \stackrel{(3.1.2)}{=} \gamma\left(g(\nabla_{\tilde{\mathbf{u}}}\mathbf{n},\tilde{\mathbf{t}})\right) \stackrel{\nabla g=0}{=} \gamma\left(\tilde{\mathbf{u}}(g(\mathbf{n},\tilde{\mathbf{t}}))\right) - \gamma\left(\eta(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{t}})\right) =$$
$$\stackrel{Lem.\,3.6.3}{\underset{(3.1.2)}{=}} \mathbf{u}(\eta(\mathbf{t})) - \eta(\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{t}}\right)) = -\eta(\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\mathbf{t}}\right)) \stackrel{(4.2.5)}{=} K\left(\mathbf{u},\mathbf{t}\right)$$

and $\ell = \mathrm{sgn}(\mathbf{n})\,\gamma(\mathbf{n})$ (see (4.4.1)) equation (4.4.10) becomes

$$\gamma(g)\left(\gamma(\mathbf{Riem})\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\mathbf{t}\right) = \gamma(g)\left(\underline{\mathbf{Riem}}\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\mathbf{t}\right) +$$
$$+ \mathrm{sgn}(\mathbf{n})\left\{K\left(\mathbf{v},\mathbf{w}\right)K(\mathbf{u},\mathbf{t}) - K\left(\mathbf{u},\mathbf{w}\right)K(\mathbf{v},\mathbf{t})\right\},$$

which is just the usual Gauss equation.

Regarding the Codazzi equation we observe that

$$(4.4.11) \quad 2\gamma(g)\left(\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right),\gamma(\mathbf{n})\right) \stackrel{\mathrm{sgn}(\mathbf{n})\equiv\pm 1}{\underset{(3.1.2)}{=}} 2\mathrm{sgn}(\mathbf{n})\,\gamma\left(g\left(\nabla_{\tilde{\mathbf{u}}}\mathbf{n},\mathbf{n}\right)\right) =$$
$$= \mathrm{sgn}(\mathbf{n})\,\gamma\left(\mathbf{u}\left(g(\mathbf{n},\mathbf{n})\right) - \left(\nabla_{\tilde{\mathbf{u}}}g\right)(\mathbf{n},\mathbf{n})\right) = 0$$

on M, that $\underline{\mathbf{Riem}}(\mathbf{u},\mathbf{v})\mathbf{w} \in \mathfrak{X}(M)$ and thus

$$(4.4.12) \qquad\qquad \gamma(g)\left(\underline{\mathbf{Riem}}\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\gamma(\mathbf{n})\right) = \mathrm{sgn}(\mathbf{n})\,\gamma(g)\left(\underline{\mathbf{Riem}}\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\ell\right) \stackrel{(4.4.9)}{=} 0$$

and that

$$(4.4.13) \qquad\qquad\qquad\qquad \gamma(g)(\ell,\gamma(\mathbf{n})) = \eta(\mathbf{n}) = 1$$

(by the same argument as in (4.4.9)). Using this and (4.4.8) gives

$$\gamma(g)\left(\mathbf{Riem}\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\gamma(\mathbf{n})\right) \stackrel{(4.4.13)}{=} \gamma(g)\left(\underline{\mathbf{Riem}}\,(\mathbf{u},\mathbf{v})\,\mathbf{w},\gamma(\mathbf{n})\right) + \left(\underline{\nabla}_{\mathbf{u}}K\right)\left(\mathbf{v},\mathbf{w}\right) - \left(\underline{\nabla}_{\mathbf{v}}K\right)\left(\mathbf{u},\mathbf{w}\right) +$$
$$+ K\left(\mathbf{v},\mathbf{w}\right)\gamma(g)\left(\gamma\left(\nabla_{\tilde{\mathbf{u}}}\tilde{\ell}\right),\gamma(\mathbf{n})\right) - K\left(\mathbf{u},\mathbf{w}\right)\gamma(g)\left(\gamma\left(\nabla_{\tilde{\mathbf{v}}}\tilde{\ell}\right),\gamma(\mathbf{n})\right)$$
$$\stackrel{(4.4.11)}{\underset{(4.4.12)}{=}} \left(\underline{\nabla}_{\mathbf{u}}K\right)\left(\mathbf{v},\mathbf{w}\right) - \left(\underline{\nabla}_{\mathbf{v}}K\right)\left(\mathbf{u},\mathbf{w}\right),$$

which shows the Codazzi equation (4.4.7).   □

It is easy to see that the other equation (4.2.12) from subsection 4.2.1 for $\gamma\left(\mathbf{Riem}\right)\left(\mathbf{u},\mathbf{v}\right)\ell$ does not provide any additional information:

For $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathfrak{X}(M)$ we have

$$\text{sgn}(\mathbf{n})\,\gamma(g)\left(\gamma\left(\mathbf{Riem}\right)\left(\mathbf{u},\mathbf{v}\right)\ell,\mathbf{w}\right)\overset{(4.4.1)}{\underset{(3.1.2)}{=}}\gamma\left(g(\mathbf{Riem}(\tilde{\mathbf{u}},\tilde{\mathbf{v}})\mathbf{n},\tilde{\mathbf{w}})\right)=$$

$$=-\gamma\left(g(\mathbf{Riem}(\tilde{\mathbf{u}},\tilde{\mathbf{v}})\tilde{\mathbf{w}},\mathbf{n})\right)=-\gamma(g)\left(\gamma\left(\mathbf{Riem}\right)\left(\mathbf{u},\mathbf{v}\right)\mathbf{w},\mathbf{n}\right)$$

by one of the symmetries of the Riemann tensor (sometimes called pair-interchange symmetry) and similarly for $\mathbf{w}=\mathbf{n}\notin\mathfrak{X}(M)$

$$\text{sgn}(\mathbf{n})\,\gamma(g)\left(\gamma\left(\mathbf{Riem}\right)\left(\mathbf{u},\mathbf{v}\right)\ell,\mathbf{n}\right)=\gamma\left(g(\mathbf{Riem}(\tilde{\mathbf{u}},\tilde{\mathbf{v}})\mathbf{n},\mathbf{n})\right)=-\gamma\left(g(\mathbf{Riem}(\tilde{\mathbf{u}},\tilde{\mathbf{v}})\mathbf{n},\mathbf{n})\right),$$

so $\gamma(g)\left(\gamma\left(\mathbf{Riem}\right)\left(\mathbf{u},\mathbf{v}\right)\ell,\mathbf{n}\right)=0$. This shows that $\gamma\left(\mathbf{Riem}\right)\left(\mathbf{u},\mathbf{v}\right)\ell$ is uniquely determined by knowing $\gamma\left(\mathbf{Riem}\right)\left(\mathbf{u},\mathbf{v}\right)\mathbf{w}$ for all $\mathbf{w}\in\mathfrak{X}(M)$ since $\gamma(g)\in W^{k,p}_{\text{loc}}(M,T^0_2X\big|_M)$ provides an isomorphism $\flat$ between $\Gamma(M,TX\big|_M)$ and $\Gamma(M,T^0_1X\big|_M)$.

Abstract

This thesis is about different aspects of low regularity geometry on semi-Riemannian manifolds and is roughly split into four chapters. The first chapter offers a brief introduction to the theory of distributions (in the sense of Laurent Schwartz) on manifolds.

In the second chapter we look at singular objects on a manifold with a smooth semi-Riemannian metric. The smooth metric allows us to effectively deal with functions (and tensor fields) of the "lowest" regularity, i.e., distributions. In particular we are interested in studying distributions with support in a (semi-Riemannian) hypersurface.

In the third chapter we no longer assume the existence of a smooth semi-Riemannian metric on our manifold but instead study distributional metrics (or, more generally, distributional geometry). However, this is rather hopeless without assuming some higher (Sobolev) regularity. For this reason we have included a short introduction to Sobolev spaces on manifolds in the beginning of the third chapter. The main focus of this chapter lies on deriving jump formulas for the various curvature quantities, that is, how the Riemann and Ricci tensor and the scalar curvature look like for a metric that suffers a jump discontinuity across a hypersurface. Of course the reason why this is of a particular interest lies in physics, mainly general relativity, where such formulas might find an application due to the Einstein field equations. As a mathematical side note we also take a short look at the compatibility of this distributional approach to generalized geometry and a Colombeau theoretic approach and see that those two are indeed equivalent for a certain class of distributional metrics.

Finally, the fourth and last chapter focuses on the geometry induced on a general (i.e. potentially null) hypersurface by a given connection or metric on our manifold. As a substitute for the normal unit vector field (that is only available in the nowhere null case) one may use a so-called rigging vector field. This leads to a generalization of the second fundamental form and the Gauss and Codazzi equations. Finally we show that all our results reduce to the well-known standard expressions in the case of a nowhere null hypersurface. Again we try to keep things very general by not assuming smoothness of the connection/metric but just the regularity really needed to make sense of occurring products and traces.

Zusammenfassung

Die vorliegende Masterarbeit behandelt verschiedene Aspekte niedrig regulärer Geometrie auf semi-riemannschen Mannigfaltigkeiten und besteht im Wesentlichen aus vier Kapiteln. Das erste gibt eine kurze Einführung in die Theorie der Distributionen (im Sinn von Laurent Schwartz) auf Mannigfaltigkeiten.

Im zweiten Kapitel betrachten wir singuläre Objekte auf einer Mannigfaltigkeit mit einer glatten semi-riemannschen Metrik. Dies erlaubt es uns, erfolgreich Funktionen (und Tensorfelder) von "niedrigster" Regularität, d.h., Distributionen, zu behandeln. Insbesondere interessieren wir uns dabei für Distributionen mit Träger in einer (semi-riemannschen) Hyperfläche.

Im dritten Kapitel nehmen wir nicht mehr an, dass auf unserer Mannigfaltigkeit eine glatte semi-riemannsche Metrik gegeben ist, sondern beschäftigen uns stattdessen damit, was passiert, wenn die Metrik selbst distributionell ist. Es stellt sich heraus, dass dies ziemlich aussichtslos ist, wenn man nicht zumindest etwas bessere Regularität verlangt. Daher beinhaltet dieses Kapitel auch eine kurze Einführung zu Sobolevräumen auf Mannigfaltigkeiten. Das Hauptaugenmerk des dritten Kapitels liegt allerdings auf der Herleitung von Sprungformeln für diverse Krümmungsgrößen, d.h., darauf, wie Riemann- und Riccitensor sowie die Skalarkrümmmung für eine entlang der Hyperfläche unstetige Metrik aussehen. Solche Sprungformeln sind zum Beispiel in der Physik, insbesondere in der allgemeinen Relativitätstheorie, aufgrund der Einsteinschen Feldgleichungen von Bedeuntung. Als mathematische Randbemerkung betrachten wir auch noch kurz den Zusammenhang zwischen diesem distributionellen Zugang zu generalisierter Geometrie und einem Colombeau theoretischen Zugang und zeigen, dass diese Zugänge für eine bestimmte Klasse distributioneller Metriken wirklich äquivalent sind.

Das vierte und letzte Kapitel beschäftigt sich mit der Geometrie, die auf einer allgemeinen Hyperfläche durch die Geometrie der gegebenen Mannigfaltigkeit induziert wird. Als Ersatz für das Normalvektorfeld, das bei nirgends lichtartigen Hyperflächen zur Verfügung steht, kann man ein so genanntes Riggingvektorfeld verwenden. Dies führt zu einer Verallgemeinerung der zweiten Fundamentalform sowie der Gauss- und Codazzi-Gleichungen. Schlussendlich zeigen wir, dass sich alle unsere Resultate im Fall einer nirgends lichtartigen Hyperfläche wieder auf die wohlbekannten reduzieren. Wieder versuchen wir unsere Aussagen so allgemein wie möglich zu halten, indem wir keine Glattheit sondern nur die für die auftretenden Produkte und Spuren benötigte Regularität des Zusammenhangs/der Metrik voraussetzen.

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Curriculum Vitae

Education

2012 - Present	**Master of Science in Mathematics**, *University of Vienna*. Thesis on low regularity geometry on semi-Riemannian manifolds. Expected in summer 2014.
2009 - 2013	**B.Sc. in Physics**, *University of Vienna*. Undergraduate thesis: Symmetrien des Wasserstoffatoms (Symmetries of the hydrogen atom). Passed with distinction.
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Experience

Spring 2014	**Tutor**, *University of Vienna*. Tutorial sessions to Analysis for Physicists II.
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July 2010	**Guide at IYPT**, 23rd *International Young Physicists Tournament*, Vienna. Guide of the team from Belarus.
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