MASTERARBEIT

Titel der Masterarbeit

On the Structure of Countable Borel Equivalence Relations and the Classification Problem for Torsion-Free Abelian Groups

Verfasser

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to my parents

Abstract (Deutsch)

Das Ziel dieser Masterarbeit ist die Darstellung einiger rezenter Resultate über die Struktur abzählbarer Borel-Äquivalenzrelationen. Die Arbeit beginnt mit der Einführung wesentlicher Voraussetzungen aus der deskriptiven Mengenlehre. §3 gibt einen vollständigen Beweis eines Resultates von Popa, das Rigiditätsphänomene von Property (T) Gruppen beschreibt. Mithilfe dieses Resultates und der in §4 dargestellten Eigenschaften der speziellen linearen Gruppen über den ganzen Zahlen wird in §5 das Resultat von Scot Adams und Alexander S. Kechris bewiesen, welches besagt, dass Continuum-viele unvergleichbare abzählbare Borel-Äquivalenzrelationen existieren. Der Höhepunkt dieser Arbeit ist §6, wo mehrere Resultate von Simon Thomas über die Komplexität des Klassifikationsproblemes für abzählbare torsionsfreie abelsche Gruppen endlichen Ranges dargelegt werden. Der hier beschriebene Beweis folgt der neuen Beweismethode von Samuel Coskey und verwendet Adrian Ioanas Superrigidity Theorem.

Abstract (English)

This thesis provides an exposition of some recent results on the structure of countable Borel equivalence relation. We start by introducing the relevant concepts from descriptive set theory. The main tool for our exploration of the poset of countable Borel equivalence relations is given by Sorin Popa's cocycle superrigidity theorem, which is proved in detail in §3, by means of the ergodic-theoretic methods provided by Alex Furman. Together with the properties of the special linear group over the integers described in §4 (mainly property (T)), these are the tools of the trade for the last two sections. In §5 we show how to directly derive the result of Scot Adams and Alexander S. Kechris on the existence of continuum-many incomparable countable Borel equivalence relations by means of Popa's theorem. We conclude this thesis by giving a detailed sketch of Simon Thomas's proof that the complexity of the classification problem for torsion-free abelian groups of fixed finite rank increases strictly (in the Borel sense) as the rank increases; we furthermore show the result of Thomas that these countably many Borel equivalence relations obtained from this classification problem are not cofinal in the poset of countable Borel equivalence relations. The proof we provide here is by Samuel Coskey, by means of Adrian Ioana's superrigidity theorem.

Overview

Contents

1 Introduction

The purpose of this thesis is twofold; on the one hand, we want to show the reader some results that explain the richness of the partial order of countable Borel equivalence relations when viewed under Borel reducibility. We tried to make this as self-contained as possible while keeping some sensible space-constraints in mind.

The main tool for this exploration will come in the form of Sorin Popa's cocycle superrigidity theorem, as presented in §2.

The second aim is to provide an exposition of various results centered around the classification problem for countable torsion free abelian groups. The main theorem here will certainly be Simon Thomas's result that the «difficulty» of the classification for torsion free abelian groups strictly increases with an increase of rank.

Some words on classification problems The systematic study classification problems set in the context of Borel equivalence relations seems to originate from the seminal paper by L. Stanley and H. Friedman ([FS89]). As a summary, we would like to pick and discuss those results presented therein which are of primary interest for presentation of the theory, since they provide a very natural starting point for the main topics treated in this thesis.

The structural novelty of the discussion certainly lies in viewing the isomorphism relation for a given class of structures in a descriptive setting, whence Borel reductions can be used to discuss the «complexity» of such structures. One of the first structural achievements lies in their definition of a jump operator (customarily called the Friedman–Stanley jump), which allows to construct, starting from any non-trivial Borel equivalence relation, a tower of increasingly complex Borel equivalence relations of height ω_1. However, the main body of their work lies in the description of a number of classes of structures and their corresponding isomorphism relations; their primary concern is whether those are or are not Borel complete or even complete analytic.

Theorem 1 ([FS89, Theorem 5]). *The isomorphism relation on abelian torsion groups is not Borel complete. More precisely, it does not reduce to equality of countable subsets of reals.*

Since equality of countable subsets of reals lies in the lower regions (level 4 or level 2, depending on the concrete definition) of the FS-tower, in this way the incompleteness is shown. The proof analyses the groups from the point of view of their direct decomposition into primary subgroups and in turn codes those subgroups by means of their Ulm invariants.

Furthermore, in [FS89, §2.4] they proceed by direct analysis of the Kurosh–Malcev invariants in order to show that the isomorphism relation on the class of torsion-free abelian groups of finite rank is Borel; we will come back to this soon.

As for [FS89, §2.5], it is conjectured that the isomorphism relation for infinite rank torsion-free abelian groups is in fact Borel complete. In other words,

Conjecture 2 (Friedman–Stanley). *For any countable language $\mathcal{L}$ the following holds:*

$$\cong\restriction_{\mathrm{Mod}(\mathcal{L})} \leq_{\mathrm{B}} \cong\restriction_{TFAG_\infty},$$

where $\cong\restriction_{TFAG_\infty}$ denotes the isomorphism relation on torsion-free abelian groups of infinite rank. At the time of writing this thesis, this seems to still be open. However, the following related result has been proved by G. Hjorth:

Theorem 3 ([Hjo02]). *Start with any countable language $\mathcal{L}$; then for any Borel $X \subseteq \mathrm{Mod}(\mathcal{L})$ such that the resulting isomorphism relation $\cong_{\mathrm{Mod}(X)}$ is also Borel, the following holds:*

$$\cong\restriction_{\mathrm{Mod}(X)} \leq_{\mathrm{B}} \cong\restriction_{TFAG_\infty}$$

We deem it important to emphasize that this directly implies that isomorphism of torsion-free abelian groups of infinite rank $\cong\restriction_{TFAG_\infty}$ is not Borel.

Countable Borel Equivalence Relations Since the classification of hyperfinite Borel equivalence relations in [DJK94], the countable Borel equivalence relations, i.e. those whose classes are all countable, have enjoyed particular scrutiny by the community. One apparent reason is the fact that quite a few problems from other areas of mathematics can be cast in their setting (as for the classification of countable torsion-free abelian groups, which form the example dealt with in this thesis), together with their close relation to group actions and consequently orbit equivalence relations.

For many years it remained an open problem to show that there are any Borel equivalence relations strictly between E_0, the relation of eventual agreement on ω–sequences of zeroes and ones, and E_∞, the universal countable Borel equivalence relation. The definitive settlement in this matter came from [AK00] in the following form.

Theorem 4 (Adams–Kechris)**.** *The partial order of Borel subsets of the reals $\mathbb{R}$ under set inclusion $\subseteq$ can be embedded into the partial order of countable Borel equivalence relations under Borel reducibility (modulo bireducibility). In other words, there exists an assignment*

$$X \mapsto \mathrm{E}_X$$

assigning to each $X \subseteq \mathbb{R}$ a countable Borel equivalence relation E_X such that

$$X \subseteq X' \iff E_X \leq_{\mathrm{B}} E_{X'} .$$

They achieved this by relying heavily (and brilliantly, for that matter) on the superrigidity theorems of R. Zimmer (which, in turn, have deep connections to the superrigidity theory by Margulis and Mostow). More specifically, they prove that considering the groups

$$\mathrm{SO}_7\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$$

for any prime $p \in \mathbb{P}$, the free parts of their induced shift equivalence relation on $\{0,1\}$–sequences are indeed $\leq_{\mathrm{B}}$–incomparable for unequal primes.

Countable Torsion-Free Abelian Groups of Finite Rank As for the classification problem of countable torsion-free abelian groups of finite rank, it was long known that for the rank-one–groups the isomorphism relation is indeed hyperfinite (see the classical result of Baer); Then Hjorth proved in [Hjo99] that the rank-two–problem is not hyperfinite, so strictly above E_0; however, it was not known at the time whether this equivalence relation might be countably universal, so that the complexity of the isomorphism problem might collapse at the second level.

It was not until after the new methods introduced by Adams–Kechris that S. Thomas found a way to directly analyse the complexity for the rank-three–case by using Król's analysis of endomorphism rings of torsion-free abelian groups of rank two ([Tho02a]). In fact, Adams and Kechris themselves had already proved that if the isomorphism relation was restricted to the rigid groups, i.e. those with a most simple automorphism group, then none of the resulting equivalence relations was countably universal; so an ω–tower was obtained. However, the relationship between the rigid groups and the general case was unclear, until Thomas proved the two-three–result and the general case soon afterwards, leading to the following

Theorem 5 ([Tho03])**.** *For any $n \in \omega$ we have*

$$\cong_n <_B \cong_{n+1} .$$

Limitations We remark that we have decided to leave two topics completely out of the thesis (even in the sense of references), although they are situated very close to the theory developed here and certainly reflect deeper insights while furthermore many current developments are even based on them: in the case of Borel equivalence relations, this refers to the concept of `treeability`; for rigidity phenomena of group actions, a different and important reduction notion is given by `orbit equivalence`.

Prerequisites This thesis is written with a set-theoretic reader in mind. The basic concepts of descriptive set theory should be enough to understand most of the material. Furthermore, an introductory chapter is provided by section 2, which introduces all the relevant concepts from descriptive set theory and the theory of Borel equivalence relations. In order to ease the access further, we provide an index as well as an index of symbols at the end. All other concepts are introduced where they are needed, like for example abelian group theory or representation theory. We give a list of the sections which require further prerequisites from the reader:

- Section 3.2 needs some understanding of measurable quotients and the involved relative notions; see [Fur07] or [Gla03].

- Section 6.4 provides only an introduction to the concepts around Ioana's superrigidity theorem, without proving the theorem itself; see [Ioa11] or [Cos13].

- The last paragraph in section 6.4 uses results from the superrigidity theory of Margulis;

- Section 6.5 needs a rather deep result from [Lad79];

- Section 6.5 needs some more results on the structure of abelian groups; those that seem appropriate for the discussion are introduced at the beginning of the section, the rest is referenced to [Fuc73].

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2 Preliminaries

We start out by introducing the basic notions (mostly without providing proofs). We do not pretend the exposition here to be complete by any means; the main concern is introducing language and fundamental results.

Borel Equivalence Relations We introduce and remind the reader of the basic notions; for more details consult [Gao09] or [Kec95].

The main objects of study are Polish spaces X together with an equivalence relation $E \subseteq X \times X$ which is Borel as a subset of the product. In fact, we are mainly concerned with the Borel structure, so that we opt to forget the topology.

Definition 6. *A* `standard Borel space` *is a set X together with a σ–algebra $\mathcal{B}$ such that there exists a Polish topology on X generating $\mathcal{B}$ as Borel sets.*

Naturally we would like to compare the complexity of equivalence relations; for that matter, we define Borel reductions.

Definition 7. *Let E and F be two Borel equivalence relations on their respective spaces X and Y. A Borel function*

$$f : X \to Y$$

is a `reduction from E to F` *if the following holds:*

$$\forall x, y \in X \, xEy \iff f(x)Ff(y)\,.$$

If only implication holds in this last formula, then we say that f is a `homomorphism from E to F`.

In order to make the quasi-order of Borel equivalence relations under reductions into a partial order, we naturally identify bireducible equivalence relations.

Definition 8. *For two Borel equivalence relations E and F, E is* `reducible` *to F (or F* `reduces` *E) if there exists a reduction from E to F, in symbols*

$$E \leq F\,.$$

If both $E \leq F$ and $F \leq E$ hold, we say that the equivalence relations are `bireducible` *or* `equivalent`*.*

We only consider Borel reducibility; we note that one can also consider different reducibility notions, like Borel embeddings or continuous reductions, but these are outside of the scope of this thesis.

We would like to remind the reader of the two classical dichotomy theorems, also observing that there are no other pairs which form a dichotomy for all Borel equivalence relations (see [KL97, §5]).

Theorem 9 (Silver). *For any Borel Equivalence relation E,*
either

1. $E \leq \mathrm{id}(\omega)$

or

2. $\mathrm{id}(2^\omega) \leq E$

Definition 10. *Let E_0 denote the relation of eventual agreement on Cantor space 2^ω.*

Theorem 11 (Glimm–Effros, Harrington–Kechris–Louveau). *For any Borel Equivalence relation E,*
either

1. $E \leq \mathrm{E}_0$

or

2. $\mathrm{E}_0 \leq E$

The original proofs of both the Silver dichotomy as well as the Harrington–Kechris–Louveau dichotomy relied heavily on Gandy–Harrington forcing (or the respective topology depending on your point of view). Since B.D. Miller publications, a different approach using graphs and Borel chromatic numbers is known. We are only interested in `countable` Borel equivalence relations, i.e. those for which all classes are countable. The fundamental and celebrated theorem by Feldman–Moore states that every countable Borel equivalence relation is in fact induced by the Borel action of a countable group.

Theorem 12 (Feldman–Moore). *Let E be a countable Borel equivalence relation on a standard Borel space X. Then there exists a countable group G such that E is induced by a Borel action of G on X.*

Furthermore, we define

Definition 13. *A Borel equivalence relation is called* `finite` *[*`countable`*], if each of its equivalence classes is finite [countable].*

This leads to the definition of hyperfiniteness:

Definition 14. *A Borel equivalence relation is* `hyperfinite`*, if it can be represented as the increasing union of a countable family of finite Borel equivalence relations.*

Theorem 15. *Let E and F be Borel equivalence relations (on standard Borel spaces X and Y respectively).*

1. *If $E \subseteq F$ and F is hyperfinite, then E is also hyperfinite.*
2. *If $E \leq F$ and F is hyperfinite, then E is also hyperfinite.*
3. *Given a Borel set $A \subseteq X$ and a hyperfinite E, then its restriction $E{\restriction}_A$ is also hyperfinite.*

In their seminal paper [DJK94] show that up to Borel reducibility, E_0 is the only non-smooth hyperfinite Borel equivalence relation. They furthermore even characterize hyperfinite relations up to Borel isomorphism.

Some results on group actions

Definition 16. *Given a measure preserving group action of a group G acting on a measure space (X, μ), the system is called* `ergodic`*, if any invariant set is either null or conull.*

The following fact is well-known.

Theorem 17. *The following are equivalent:*

- $G \curvearrowright (X, \mu)$ *is ergodic.*
- *For any G–invariant Borel function $f : X \to Y$ mapping into a standard Borel space Y, it follows that f is constant almost everywhere.*

We end with the following result, which we will need later on. It says that Bernoulli shifts are strongly mixing.

Lemma 18. *For any infinite countable group G, the shift action of G on $\mathrm{Fr}(G \curvearrowright 2^G)$ is strongly mixing (with respect to the usual product measure on 2^G).*

Proof. Recall the definition of the basis consisting of the clopen sets

$$\mathrm{N}_s := \{f \in 2^G \mid f\restriction_{\mathrm{dom}(s)} = s\}\,,$$

for s varying through the maps defined on finite subsets of G and mapping to $\{0,1\}$.

Then we may restrict ourselves to finite unions of these basic open sets, since these finite unions are dense in the measure algebra of $(2^G, mu)$. Assume that

$$A_1 = \bigcup_{i \le k_1} \mathrm{N}_{s_i^1}\,,$$

$$A_2 = \bigcup_{i \le k_2} \mathrm{N}_{s_i^2}\,.$$

Then, except for finitely many group elements γ, the unions of the domains of the basic sets get shifted away from one another, i.e.

$$\gamma\left(\bigcup_{i \le k_1} \mathrm{dom}(s_i^1)\right) \cap \left(\bigcup_{i \le k_2} \mathrm{dom}(s_i^2)\right) = \emptyset\,,$$

so for those γ, the sets $\gamma.A$ and B are really independent, therefore

$$\mu(\gamma.A \cap B) = \mu(\gamma.A)\mu(B) = \mu(A)\mu(B)\,,$$

which completes the argument. ⊣

Representation Theory and Kazhdan's Property (T)

Definition 19. *A* `unitary representation` *of a group G is a pair $(\pi, \mathcal{H})$ consisting of a Hilbert space $\mathcal{H}$ together with a continuous homomorphism*

$$\pi : G \to \mathcal{U}(\mathcal{H})$$

mapping into the unitary group of the hilbert space.

Definition 20. *Let $(\pi, \mathcal{H})$ be a unitary representation of a discrete countable group G. For a subset $K \subseteq G$ and a real $\varepsilon > 0$ a vector ξ is said to be* (K, ε)`-invariant` *if*

$$\sup_{g \in K} \|\pi_x(\xi) - \xi\| < \varepsilon \|\xi\| \,.$$

When the corresponding representation is understood from context, We will denote the set of (K, ε)–invariant vectors by $\mathrm{V}_{K,\varepsilon}$.

Definition 21. *A unitary representation* `has almost invariant vectors` *if* $\mathrm{V}_{K,\varepsilon}$ *is nonempty for every finite $K \subseteq G$ and every $\varepsilon > 0$.*

The basic definition of property (T) is the following.

Definition 22. *A countable discrete group has* `Kazhdan's property (T)` *if every unitary representation which almost has invariant vectors actually has a non-trivial invariant one.*

Property (T) is in fact equivalent to just having a single pair (K, ε) with $K \subseteq G$ finite and $\varepsilon > 0$ such that every unitary representation which has (K, ε)–invariant vectors already has non-trivial invariant ones. For the non-trivial direction, starting with the assumption that G does not have (T), consider all possible admissible pairs (K, ε); for every such pair, there is supposed to exist a unitary representation with non-trivial invariant vectors, but with a (K, ε)–invariant one. Constructing the direct sum of all such representations, it is very easy to see that this new representation certainly possesses almost invariant vectors; however, all invariant ones must be trivial. We will call such a pair as was just described a `Kazhdan pair`.

Theorem 23. *Let G be an arbitrary topological group. Then $(G, \sqrt{2})$ is a Kazhdan pair.*

Proof. Define A as the closed convex hull of

$$\pi(G)\xi_0 \,,$$

where ξ_0 is such that

$$\sup_{g \in G} \|\pi(x)\xi_0 - \xi_0\| < \sqrt{2} \,.$$

Define η_0 as the element in A which is norm-minimal; the G–invariance of A furthermore implies the invariance of η_0. It remains to show that η_0 is non-trivial.

To that end, define

$$\varepsilon := \sqrt{2} - \sup_{g \in G} \|\pi(x)\xi_0 - \xi_0\| \,,$$

which is positive by definition.

From the following simple identity

$$2 - 2\mathrm{Re}\left(\langle \pi(g)\xi_0, \xi_0 \rangle\right) = \|\pi(x)\xi_0 - \xi_0\|^2 \leq (\sqrt{2} - \varepsilon)^2$$

we obtain that for all $g \in G$

$$\mathrm{Re}\left(\langle \pi(g)\xi_0, \xi_0 \rangle\right) \geq \frac{2 - (\sqrt{2} - \varepsilon)^2}{2} = \frac{\varepsilon(2\sqrt{2} - \varepsilon)}{2} > 0 \,.$$

This directly implies for each $\eta \in A$ that

$$\mathrm{Re}\left(\langle \eta - \xi_0 \rangle\right) > 0 \,,$$

so in particular $\eta_0 \neq 0$. $\dashv$

We conclude with a theorem, stating that invariant vectors can be found lying close to the almost invariant ones.

Theorem 24. *If $H \lhd G$ has relative property (T), then for any $\delta > 0$ there exist a finite $K \subseteq G$ and $\varepsilon > 0$ sucht that the following holds: any unitary representation of G with non-empty $\mathrm{V}_{K,\varepsilon}$ has an H–invariant unit vector ξ_0 with*

$$\mathrm{dist}(\xi_0, \mathrm{V}_{K,\varepsilon}) < \delta\,.$$

Proof. Define

$$\Lambda := \{\xi \in \mathcal{H} \mid \forall h \in H\, \pi(h)\xi = \xi\}\,,$$

i.e. the subspace of all H–invariant vectors in $\mathcal{H}$; furthermore let ξ be a $(K, \delta\tilde{\varepsilon})$–invariant vector.

Since H is a normal subgroup, it immediately follows that Λ is G–invariant.

Take the orthogonal decomposition of ξ over Λ, so

$$\xi = \lambda + \lambda'\,,$$

with $\lambda \in \Lambda$ and $\lambda' \in \Lambda^{\perp}$. Since we are dealing with unitary representations, it follows that $\Lambda^{\perp}$ is also G–invariant, so we can consider the subrepresentation by restricting to $\Lambda^{\perp}$; now by definition $\Lambda^{\perp}$ contains no non-trivial invariant vectors and by the fact that $(K, \tilde{\varepsilon})$ is a Kazhdan pair, we know that there exists an $h \in H$ such that

$$\|\pi(h)\lambda' - \lambda'\| \geq \tilde{\varepsilon}\|\lambda'\|\,.$$

By definition we also have

$$\|\pi(h)\lambda' - \lambda'\| = \|\pi(h)\xi - \xi\| < \delta\tilde{\varepsilon}\|\xi\|\,,$$

which gives together

$$\|\xi - \lambda\| = \|\lambda'\| < \delta\|\xi\|$$

thus we have established that using that orthogonal projection onto the space of H–invariant vectors we obtain the one we were looking for; for obtaining unit norm observe that in norming λ we do not obstruct any of the arguments, though the constants might change. $\dashv$

Definition 25. *Let G be a discrete countable group, $H \leq G$ a subgroup. Then we say that H `has relative property (T) in` G (or that the `pair (G,H) has property (T)` if every unitary representation of G which almost has invariant vectors, actually has a non-trivial H–invariant one.*

Projection-valued measures In the case of a locally compact abelian groups G, it turns out that its unitary representations can be described in the form of so called projection-valued measures on the Pontryagin dual $\widehat{G}$.

Definition 26. *A `projection-valued measure` on a locally compact space X is a mapping*

$$E : \mathcal{B}(X) \to \mathrm{Proj}(\mathcal{H}),$$

where $\mathrm{Proj}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$ denotes the space of orthogonal projections, which satisfies the following

1. *$E(\emptyset) = 0$ and $E(X) = \mathrm{id}$;*
2. *$E(B_1 \cap B_2) = E(B_1)E(B_2)$ for all Borel B_1, B_2;*
3. *For a countable sequence of pairwise disjoint Borel sets $(B_n)_{n\in\omega}$, we have*

$$E\left(\bigcup_{n\in\omega} B_n\right) = \sum_{n\in\omega} E(B_n),$$

where the sum converges in the sense of the strong operator topology.

The following classical theorem by Stone, Naimark, Ambrose and Godement gives the desired characterization.

Theorem 27 (SNAG Theorem)**.** *Let $(\pi, \mathcal{H})$ be a unitary representation of a locally compact group G. Then there exists a unique regular projection-valued measure*

$$E_\pi : \mathcal{B}(\widehat{G}) \to \mathrm{Proj}(\mathcal{H})$$

such that

$$\pi(x) = \int_{\widehat{x} \in \widehat{G}} \overline{\widehat{x}(x)} \mathrm{d}E_\pi(\widehat{x}) \, .$$

Cocycles Let us now give the basic definition which is needed for the rigidity theorems and their consequences.

Definition 28. *For G, L locally compact groups and X a standard Borel space, a map $\alpha : G \times X \to L$ is called a* `cocycle` *if the following identity holds:*

$$\alpha(g \cdot h, x) = \alpha(g, h.x) \cdot \alpha(h, x) \; .$$

So a cocycle can be thought as being akin to a group homomorphism, however subject to the caveat of needing to consider the action (and the space acted upon). In case of independence on the second coordinate, we are indeed handed a group homomorphism.

Now the typical way by which cocycles will arise for us is the following: start out with a standard Borel G–space X and a standard Borel H–space Y, for some countable groups G, H. Assume furthermore that the action of H on Y is free, and that we have a homomorphism $f : X \to Y$ between the induced equivalence relations E_G^X and E_H^X at our disposal. As a consequence, we can define a cocycle

$$\alpha : G \times X \to H$$

by setting

$$\alpha(g, x) := \text{«the unique } h \in H \text{ which realises } f(g.x) = h.f(x)\text{»}$$

uniqueness is granted by the freeness of the action, and it is easy to check that this is indeed a cocycle.

We will denote the space of all cocycles from the action $G \curvearrowright X$ into H by $\mathrm{Z}^1(G \curvearrowright X, H)$. Furhtermore, we define an equivalence relation on this space: two cocycles α, β are said to be `cohomologous` if there exists a Borel $\varphi : X \to H$ such that we have for all $g \in G$ and $x \in X$

$$\alpha(g,x) = \varphi(g.x)\, \beta(g,x)\, \varphi(x)^{-1}.$$

The space of cocycles modulo cohomology will be denoted by $\mathrm{H}^1(G \curvearrowright X, H)$. We will always consider $\mathrm{Z}^1(G \curvearrowright X, H)$ to be endowed with the topology of convergence in measure; i.e. we obtain a basis for the topology by considering all finite sets $F \subseteq G$ and all $\varepsilon > 0$ and considering cocycles α, β such that

$$\mu\left(\{x \in X \mid \alpha(g,x) \neq \beta(g,x)\}\right) > 1 - \varepsilon$$

for all $g \in F$.

3 Popa Cocycle Superrigidity

We want to prove a very special case of Popa's Cocycle Superrigidity Theorem. For generalities see subsection 3.4. Let us begin by stating the theorem, which we will use in the coming chapters:

Theorem 29. *Let G be a discrete countable group, $H \lhd G$ a normal subgroup with property T. Let G act by Bernoulli shifts on the space $(X, \mu)^G$, for a probability space (X, μ):*

$$G \curvearrowright (X, \mu)^G .$$

Then for any discrete countable group L, any measurable cocycle $\alpha : G \times X \to L$ is cohomologous to a homorphism, i.e. there exists $\varphi : X \to L$ measurable and $\rho : G \to L$ homomorphism such that

$$\alpha(g, x) = \varphi(g.x)\, \rho(g)\, \varphi(x)^{-1}$$

holds.

We will in fact only prove this for the case where (X, μ) is non-atomic; to see how one can derive the general theorem see the remarks at the end of this chapter in subsection 3.4.

Weak mixing and malleability of Bernoulli actions The following is a property enjoyed by Bernoulli actions, which we will need to get the full strength of Theorem 29.

Lemma 30. *For a non-atomic standard probability space (X, μ), the flip transformation $F(x, y) := (y, x)$ is contained in the path connected component of the identity of the identity in $\mathrm{Aut}(X \times X, \mu \times \mu)$.*

Proof. Since any non-atomic standard probability space is isomorphic to Lebesgue measure on the unit interval $[0, 1]$, we may assume without loss of generality that this is the space in question. Define $T_t : [0, 1] \to$

$\mathrm{Aut}([0,1]^2, m^2)$, where m denotes Lebesgue measure, by

$$T_t(x,y) := \begin{cases} (x,y) & \text{if } x,y \in [0,t] \\ (y,x) & \text{otherwise} \end{cases}$$

Then obviously $T_0 = F$, the flip, and T_1 is the identity. This path is clearly continuous in the strong, and hence also in the weak topology. $\dashv$

From this we obtain the malleability of Bernoulli actions (with non-atomic base space) by observing that the diagonal embedding

$$\mathrm{Aut}(X,\mu) \to \mathrm{Aut}\left((X,\mu)^G\right)$$

is continuous, and its image commutes with G. Furthermore, we note that Bernoulli actions are strongly mixing, as we have seen in Lemma 18.

In fact this holds true even for generalised Bernoulli actions (under the assumption of no finite orbits), which are obtained by substituting the G–product of the space X by the product over an arbitrary countable index set on which G acts (see [Fur07]). We now state the form of the theorem which we will subsequently prove.

Theorem 31. *Start with a discrete countable group G with a normal subgroup $H \lhd G$ with property (T) acts on a standard probability space (X,μ) in a measure preserving way. Assume that the G–action is ergodic and malleable and that furthermore the H–action is weakly mixing.*

Then the action is cocycle-superrigid for any discrete countable group as target.

We first need to introduce a topology on the automorphism group of a measure space.

Definition 32. *For a measure space (X,μ) denote by*

$$\mathrm{Aut}(X,\mu)$$

the `automorphism group of the measure space`; more precisely, the set of all measurable bijections between conull subsets of X which preserve the measure.

We want to endow this group with a topology, so as to make it a topological group. Recall that we can embed

$$\mathrm{Aut}(X,\mu) \to \mathcal{U}\left(\mathrm{L}^2(X,\mu)\right)$$

via the Koopman-representation, thereby identifying each element of the automorphism group with a unitary transformation of the space of square-integrable functions on X. This embedding allows us to use the known topologies on the unitary group in order to topologize $\mathrm{Aut}(X,\mu)$. By the metric completeness of the weak topology on $\mathrm{Aut}(X,\mu)$, which is stated below, we also know that the image under this natural embedding is closed in $\left(\mathrm{L}^2(X,\mu)\right)$.

Definition 33. *Unless stated otherwise,* $\mathrm{Aut}(X,\mu)$ *will always be considered to be endowed with the weak topology, i.e. the topology inherited from* $\mathcal{U}(\mathrm{L}^2(X,\mu))$ *together with the weak operator topology.*

This topology can be viewed as generated by the following family of pseudometrics, one for each measurable set E:

$$d_E(T,S) := \mu\left(T(E) \triangle S(E)\right) .$$

For the sake of completeness, we also quickly show the following important property of the measure automorphism group.

Theorem 34. *For any Lebesgue measure space, the space* $\mathrm{Aut}(X,\mu)$ *endowed with the weak topology is a Polish group.*

Proof. By direct calculation we see that

$$|d_E(T,S) - d_F(T,S)| =$$

$$= |\mu((T(E) \setminus S(E)) \cup (S(E) \setminus T(E))) - \mu((T(F) \setminus S(F)) \cup (S(F) \setminus T(F)))$$
$$\leq 2(|\mu(E) - \mu(F)|) \leq 2\mu(E \triangle F) .$$

So since (X,μ) is a standard Borel space, we have a countable family of sets at our disposal, which are dense in the Borel σ-algebra. Together with the above inequality this means that the space is both metrizable and separable. It is furthermore easily seen that this metric is complete; we leave this to the reader. $\dashv$

3.1 A Local Theorem of Hjorth

The following theorem, which is in principle from [Hjo05] is the core of the proof; it states that if we start from an appropriately ergodic action arising from a property (T) group, then any two cocycles that are (in a specific sense) «close» to each other will already be cohomologous.

Theorem 35 (Hjorth)**.** *Let G be a discrete countable group, $H < G$ a subgroup with property (T), and let $G \curvearrowright (X, \mu)$ be a measure preserving action on a standard probability space such that the restriction to H is ergodic.*

Then for any discrete countable group L, there exists a finite set $K \subseteq G$ and $\varepsilon > 0$ such that for any two cocycles $\alpha, \beta : G \times X \to L$ that satisfy

$$\forall g \in K(\mu(x \in X \mid \alpha(g, x) = \beta(g, x)) > 1 - \varepsilon)$$

there exists a measurable $\varphi : X \to L$ such that α and β are cohomologous as H-cocycles:

$$\forall h \in H(\alpha(h, x) = \varphi(h.x)\beta(h, x)\varphi(x)^{-1}) \, .$$

Proof. As we have shown in Theorem 24, we know that we can find invariant unit vectors lying close to the almost invariant ones; in other words, we know that there exist a finite subset $K \subseteq G$ and $\varepsilon > 0$ such that for any unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ that possesses a unit vector $v_0 \in \mathcal{H}_\pi$ with

$$\inf_{g \in K} | \langle \pi_g(v_0), v_0 \rangle | \geq 1 - \varepsilon$$

there exists an H–invariant unit vector $v \in \mathcal{H}_\pi$ with

$$\|v - v_0\| < \frac{1}{10} \, .$$

Now suppose that $(\alpha, \beta) \in V_{K,\varepsilon}$. The main idea will be to consider a new space given by the product

$$\widetilde{X} := X \times L$$

endowed with the measure $\widetilde{\mu} := \mu \times m_L$, where m_L denotes the counting measure on L. Define

$$G \curvearrowright (\widetilde{X}, \widetilde{\mu})$$
$$g.(x,l) := (g.x, \alpha(g,x) \cdot l \cdot \beta(g,x)^{-1}) \; .$$

This defines a unitary representation π on $L^2(\widetilde{X}, \widetilde{\mu}) =: \mathcal{H}$ by simply composing vectors in $\mathcal{H}$ with the action of an element in G, so

$$\pi_g(v)(x,l) = v\left(g.x, \alpha(g,x) \cdot l \cdot \beta(g,x)^{-1}\right) \; .$$

Let us find a unit vector in $\mathcal{H}$ that suits our needs:

$$F_0 := \chi_{X \times \{e\}} \; ,$$

where e denotes the identity element in L. Remembering that (X, μ) is a probability space clearly shows that this characteristic function is a unit vector. Furthermore we can calculate

$$\begin{aligned}
\langle \pi_g(F_0), F_0 \rangle &= \int_{X \times L} F_0(x,l) \cdot F_0\left(g.x, \alpha(g,x) \cdot l \cdot \beta(g,x)^{-1}\right) \, \mathrm{d}\widetilde{\mu}(x,l) \\
&= \int_{X \times \{e\}} F_0\left(g.x, \alpha(g,x) \cdot l \cdot \beta(g,x)^{-1}\right) \, \mathrm{d}\widetilde{\mu}(x,l) \\
&= \mu(\{x \in X \mid \alpha(g,x) = \beta(g,x)\})
\end{aligned}$$

and for $g \in K$ this last term is by assumption greater than $1 - \varepsilon$. So by property (T) we know that there exists an H-invariant unit vector $F \in \mathcal{H}$ close to F_0.

Take this $F \in \mathcal{H}$ and for each $x \in X$ view it as a function

$$\iota_x : L \to [0, \infty[\quad l \mapsto |F(x,l)|^2 \; .$$

Using Fubini's theorem, we know that ι_x is summable for μ-a.e. $x \in X$, so we can define μ-a.e. the functions

$$\begin{aligned}
w : X &\to [0, \infty[& x &\mapsto \sum_{l \in L} \iota_x(l) \\
p : X &\to [0, \infty[& x &\mapsto \max_{l \in L} \iota_x(l) \\
k : X &\to \omega & x &\mapsto \operatorname{card}\left(\{l \in L \mid \iota_x(l) = p(x)\}\right)
\end{aligned}$$

denoting, respectively, the sum, maximum value and number of points where that maximum value is attained.

These functions w, p, k are by definition certainly measurable, and recalling that F is H-invariant for the representation π, clearly reformulates as being H-invariant for the action $G \curvearrowright (\widetilde{X}, \widetilde{\mu})$ which, by the definition of that action, makes our functions w, p, k H-invariant for the action $G \curvearrowright (X, \mu)$. Hence ergodicity allows us to conclude that they are constant functions for μ-a.e. $x \in X$. Calculating the first is easy and gives $w(x) = 1$; let us call call the others by abuse of notation the same as their constant values, i.e. $p(x) = p, k(x) = k$.

Since

$$p(x) \cdot k(x) \leq w(x)$$

it follows that $p \leq \frac{1}{k}$, so in particular $|F(x, e)|^2 = \iota_x(e) \leq \frac{1}{k}$. This leads to

$$\|F - F_0\|^2 \geq 1 - \frac{1}{\sqrt{k}}$$

which contradicts $\|F - F_0\| < \frac{1}{10}$ unless we take $k = 1$.

This fact gives us the possibility to define a function $\varphi : X \to L$ by the identity

$$|F(x, \varphi(x))|^2 = p \ ,$$

i.e. simply choosing the element where the maximum of ι_x is attained. This choice for constructing the function is furthermore measurable, which stems from the way we were able to define it. Since F is H-invariant, we once more obtain, by turning to the definition of $G \curvearrowright (\widetilde{X}, \widetilde{\mu})$ and looking at the second component, that

$$\forall h \in H \left(\varphi(h.x) = \alpha(h,x) \cdot \varphi(x) \cdot \beta(h,x)^{-1} \right) ,$$

which is clearly equivalent to α and β being H-cohomologous. $\dashv$

3.2 An Untwisting Theorem

Our goal in this section is to show that if we start from a weakly mixing action and are able to establish a connection between to cocycles, which can be described as some sort of untwisting, though depending on two spacial coordinates, then we can deduce that this is already enough to loose the two spacial coordinates and get an untwisting in the sense of cohomology.

The theorems in this section are stated and proved in a more general context than we will need for our main theorem, i.e. with respect to measurable quotients. For the general theorem see subsection 3.4.

A few words on measurable quotients In this subsection, the main objects involved are probability spaces (X, μ) acted upon by discrete countable groups G, together with `measurable quotients` Y; in other words, we are implicitly given a measurable measure preserving surjection

$$\pi : X \to Y$$

which also respects the action (it is understood that Y is a G–space as well). To define the the notion of relative weak mixing, we first recall the notion of fibered product.

Definition 36. *Let $(X_1, \mu_2), (X_2, \mu_2)$ be two standard Borel probability spaces together with a common measurable quotient (Y, ν) and quotient maps p_1, p_2. Then the* `fibred product` *is obtained*

$$X_1 \times_Y X_2 := \{(x_1, x_2) \in X_1 \times X_2 \mid p_1(x_1) = p_2(x_2)\}$$

together with the following measure obtained from the measure disintegration through both p_1 and p_2:

$$(\mu_1 \times_\nu \mu_2)(B) := \int_Y (\mu_{1,y} \times \mu_{2,y})(B)\, \mathrm{d}\nu(y)\,,$$

where $\mu_{1,y}$ and $\mu_{2,y}$ denote the corresponding disintegrations.

We start with the following simple

Lemma 37. *Let (X, μ) be a standard probability space, (M, d) a separable metric space and $\Phi : X \to M$ a Borel map such that for $\mu \times \mu$-almost every (x_1, x_2) we have*

$$d\left(\Phi(x_1), \Phi(x_2)\right) = d_0\,.$$

Then $d_0 = 0$ and there exists $m_0 \in M$ such that the pushfoward measure has the form $\Phi_* \mu = \delta_{m_0}$.

Proof. Towards a contradiction, assume that $d_0 > 0$. Since M is separable, it can be covered by countably many open balls:

$$M = \bigcup_{i<\omega} B_i$$

with $\mathrm{diam}(B_i) < d_0$. Defining $E_i := \Phi^{-1}(B_i)$, these sets have all measure zero by the choice of diameter of the covering balls. But also

$$1 = \sum_{i<\omega} \mu(E_i)\,,$$

obviously a contradiction. Thus it must be that $d_0 = 0$, which implies that $\Phi_* \mu$ the Dirac measure concentrating at some $m_0 \in M$. ⊣

The next Lemma contains the main idea which we want for our goal; if we start with an ergodic action and a relatively weakly mixing quotient, then given two cocycles for the action on the quotient, if we know that we can untwist the into one another by a measurable function on the extension, then we automatically obtain that this untwisting may take place in the quotient. More precisely,

Theorem 38. *Let $G \curvearrowright (X, \mu)$ be an ergodic action, $(X, \mu) \to^p (Y, \nu)$ a measurable quotient which is also relatively weakly mixing. Let L be a discrete countable group, $\alpha, \beta : G \times Y \to L$ two measurable cocycles.*

Let $\Phi : X \to L$ be a measurable function, such that

$$\forall g \in G \, \forall^* x \in X \quad \alpha(g, p(x) = \Phi(g.x)\, \beta(g, p(x))\, \Phi(x)^{-1} .$$

Then Φ descends to Y, i.e. there exists $\varphi : Y \to L$ measurable, with

$$\Phi = \varphi \circ p \quad \textit{a.e. on } X$$

and

$$\alpha(g, y) = \varphi(g.y)\, \beta(g, y)\, \varphi(y)^{-1} .$$

Proof. Since the quotient is relatively weakly mixing, we know that the diagonal action of G on the fibered product $(X \times_Y X, \mu \times_\nu \mu)$ is ergodic. Define $f : X \times_Y X \to [0, \infty)$ by

$$f(x_1, x_2) := d(\Phi(x_1), \Phi(x_2)) .$$

We claim that it is G-invariant: denoting by

$$y := p(x_1) = p(x_2) ,$$

we have for all $g \in G$ and for $\mu \times_\nu \mu$-a.e. (x_1, x_2) that

$$\begin{aligned} f(g.x_1, g.x_2) &= d\left(\Phi(g.x_1), \Phi(g.x_2)\right) \\ &= d\left(\alpha(g, y)\, \Phi(x_1)\, \beta(g, y)^{-1}, \alpha(g, y)\, \Phi(x_2)\, \beta(g, y)^{-1}\right) \\ &= d\left(\Phi(x_1), \Phi(x_2)\right) = f(x_1, x_2) . \end{aligned}$$

Relative weak mixing implies that f is essentially a constant, say d_0, for $(\mu \times_\nu \mu)$-a.e. (x_1, x_2). Applying Lemma 37 to (X, μ_y), we see that

$$\Phi_* \mu_y = \delta_{\varphi(y)}$$

for some measurable $\varphi : Y \to L$. $\dashv$

We shall note the following important corollary, which is of independent interest.

Corollary 39. *Let G be a discrete countable group, together with a probability G–space preserving the measure and two measure preserving extensions:*

$$(X, \mu) \to (X', \mu') \to (Y, \nu) \, .$$

Furthermore assume that the first extension $(X, \mu) \to (X', \mu')$ is relatively weakly mixing.

If every measurable cocycle for $G \curvearrowright (X, \mu)$ targeting any discrete countable group can be untwisted relatively to (Y, ν), then also every cocycle for the intermediate action $G \curvearrowright (X', \mu')$ can be be untwisted relatively to (Y, ν).

Less formally, this means that rigidity is transferred to the intermediate quotient.

Now we have reached the final step, we can take the previous theorem to obtain the following theorem, which will be crucial for getting the global structure theorem for the space of cocycles modulo cohomology.

Theorem 40. *Let $G \curvearrowright (X, \mu)$ be a relatively weakly mixing measure preserving action on a probability space with respect to a quotient (Y, ν). Let $\alpha, \beta : G \times X \to L$ be two measurable cocycles with values in a discrete countable group L. Assume that there exists a map $F : X \times_Y X \to L$ with the following property for all $g \in G$ and μ^2_ν-a.e. $(x_1, x_2) \in X^2_Y$*

$$\alpha(g, x_1) = F(g.x_1, g.x_2) \, \beta(g, x_2) \, F(x_1, x_2)^{-1} \, . \qquad (\star)$$

Then there exists a measurable cocycle on the quotient $\rho : G \times Y \to L$ such that both cocycles α and β are cohomologous to it.

Proof. By the assumption we know that the relation in Equation $\star$ holds on μ^2_y–conull many (x_1, x_2) for ν–conull many $y \in Y$. Using Fubini we get simultaneously the following relations for μ_y^3–a.e. $(x_1, x_2, x_3) \in X^3$ on a ν–conull set of $y \in Y$:

$$\begin{aligned} \alpha(g, x_1) = \; & F(g.x_1, g.x_3)\, \beta(g, x_2)\, F(x_1, x_2)^{-1} \\ & \text{and} \\ \alpha(g, x_3) = \; & F(g.x_3, g.x_3)\, \beta(g, x_2)\, F(x_3, x_2)^{-1} \end{aligned}$$

We can now substitute the first equation into the second one, and by setting

$$\Phi(x_1, x_2, x_3) := F(x_1, x_2) \cdot F(x_3, x_2)^{-1}$$

we get the following identity for μ_ν^3–a.e. (x_1, x_2, x_3)

$$\alpha(g, x_1) = \Phi(g.x_1, g.x_2, g.x_3)\, \alpha(g, x_3)\, \Phi(x_1, x_2, x_3)^{-1} \; .$$

If we now turn to the projection between X_Y^3 and X_Y^2, we obtain a quotient

$$q : (X_Y^3, \mu^3) \to (X_Y^2, \mu^2) \quad (x_1, x_2, x_3) \mapsto (x_1, x_3) \; .$$

This quotient is particularly simple, so we should observe that

$$X_Y^3 \times_{(X^2)} X_Y^3 \cong X_Y^4 \; ,$$

So we can view this fibered product as X_Y^4. Now relative weak mixing for this fibered product simply means ergodicity of the diagonal G-action on X_Y^4, which we have, since our original G–action is weakly mixing relative to (Y, ν).

At this point, Theorem 38 lets us conclude that the map Φ has the following form:

$$\Phi(x_1, x_2, x_3) = f(x_1, x_3) \text{ for some } f : X_Y^2 \to L \; .$$

Thus for μ_ν^3-a.e. x_1, x_2, x_3 we get

$$F(x_1, x_2) = f(x_1, x_3) \cdot F(x_3, x_2) .$$

This in turn allows us to use Fubini to choose a measurable section $s : Y \to X$ such that for ν–a.e. $y \in Y$ for μ_y^2–a.e. (x_1, x_2) we can simply substitute x_3 with $s(y)$. By defining $\varphi, \psi : X \to L$ as

$$\varphi(x) := f(x, s \circ p(x)) \qquad \psi(x) := F(s \circ p(x), x)^{-1}$$

we see that F factors as

$$F(x_1, x_2) = \varphi(x_1) \cdot \psi(x_2)^{-1} .$$

This remarkable factorisation is now plugged into the main assumption of the theorem, resulting in

$$\varphi(g.x_1)^{-1} \cdot \alpha(g, x_1) \cdot \varphi(x_1) = \psi(g.x_2^{-1}) \cdot \beta(g, x_2) \cdot \psi(x_2)$$

for ν–a.e. $y \in Y$ for μ_y^2-a.e. $(x_1, x_2) \in X_Y^2$. Since the two sides of the equation are separated with respect to x_1 and x_2, and the equation holds for μ_y^2-a.e. (x_1, x_2), each is equal to a constant μ_y-a.e., which we use to define a new function $\rho(g, y)$. So finally we have reached our goal, since this function is measurable by the way it was defined and is of course a cocycle between $G \times Y$ and L, since α and β are both cocycles. $\dashv$

For the next section, in order to see that it is enough to work with a normal subgroup, we will also need the following simple

Lemma 41. *Let G be a discrete countable group with a measure preserving action on a probability space (X, μ). Let (Y, ν) be a quotient of (X, μ), and assume that $H < G$ is a subgroup such that the quotient is relatively weakly mixing with respect to the action of H. Then for any cocycle $\alpha : G \times X \to L$ whose restriction to H descends to a cocycle on Y, we are able to extend this on Y to the normaliser of H.*

More precisely, if there exists $\rho : H \times Y \to L$ *such that*

$$\forall h \in H \; \alpha(h,x) = \rho(h, p(x)) \,,$$

then ρ *extends to* $\rho' : \mathrm{N}_G(H) \times Y \to L$ *with*

$$\forall h \in \mathrm{N}_G(H) \; \alpha(h,x) = \rho'(h, p(x)) \,.$$

As always, L can be any discrete countable group, and p denotes the quotient map.

Proof. Start with a fixed $g \in \mathrm{N}_G(H)$ and denote by

$$h^g := ghg^{-1}$$

the corresponding conjugation automorphism of H. Obviously $\alpha(gh,x) = \alpha(h^g g, x)$, and we may split it in two different ways:

$$\begin{aligned} \alpha(gh,x) &= \alpha(g,h.x)\alpha(h,x) = \alpha(g,h.x)\rho(h,p(x)) \\ \alpha(h^g g,x) &= \alpha(h^g, g.x)\alpha(g,x) = \rho(h^g, g.p(x))\alpha(g,x) \end{aligned}$$

So as a consequence we have μ–a.e.

$$\alpha(g,h.x) = \rho(h^g, g.p(x))\alpha(g,x)\rho(h,p(x))^{-1} \,.$$

We may now define $\widehat{\rho} : H \times Y \to L$ by

$$\widehat{\rho}(h,y) = \rho(h^g, g.y) \,,$$

and observe that this is a cocycle. So by fixing g, the function $\Phi(x) := \alpha(g,x)$ satisfies

$$\Phi(h.x) = \widehat{\rho}(h,p(x))\Phi(x)\rho(h,p(x))^{-1}$$

and using Theorem 38, we know that for g in the normaliser this Φ descends to a function on Y, which we shall denote by $\varphi'(g,y)$. It is obvious from the construction that this cocycle agrees with ρ on $H \times Y$. $\dashv$

3.3 Putting it all Together

Once we have the Local Rigidity Theorem, how can we extend this to the whole space? First of all we need to move to a product, so set $(Z, \xi) := (X \times X, \mu \times \mu)$ and define $C := \mathcal{C}_{Aut(Z,\xi)}(G)$, the centraliser of G in $Aut(Z, \xi)$. Denote by C^0 the connected component of the identity in C topologized with the weak topology.

We have shown in Lemma 30 that our Bernoulli actions contain the flip

$$(x, y) \mapsto (y, x)$$

in C^0. Furthermore weak mixing implies that G acts ergodically on (Z, ξ).

Given any measurable cocycle $\alpha : G \times X \to L$, observe that this induces a cocycle $\tilde{\alpha} : G \times Z \to L$ by setting

$$\tilde{\alpha}(g, (x_1, x_2)) := \alpha(g, x_1) .$$

So we simply forget the second coordinate, which, together with the diagonal action on (Z, ξ) certainly makes $\tilde{\alpha}$ a cocycle.

We need the following Lemma in order to see that we can apply Theorem 35.

Lemma 42. *For G and L discrete countable groups and $G \curvearrowright (X, \mu)$ a measure preserving ergodic action it follows that*

$$\mathcal{C}_{\mathrm{Aut}(X,\mu)}(G) \curvearrowright Z^1(G \curvearrowright X, L) \text{ is continuous.}$$

Proof. Since G is discrete, we want to show that given any finite set $F \subseteq G$ and $\varepsilon > 0$ we can find a neighbourhood of the identity in $\mathrm{Aut}(X, \mu)$ such that

$$\forall g \in F \, \forall T \in U \cap \mathcal{C}_{\mathrm{Aut}(X,\mu)}(G) \; \mu\left(\{x \in X \mid \alpha(g, x) = \alpha(g, T(x))\}\right) > 1 - \varepsilon .$$

For each $g \in G$ we can find a finite set $L_g \subseteq L$ such that

$$\mu\left(\{x \in X \mid \alpha(g,x) \in L_g\}\right) > 1 - \frac{\varepsilon}{2}.$$

Defining

$$C_{g,l} := \{x \in X \mid \alpha(g,x) = l\},$$

then there exists a neighbourhood U of the identity in $\mathrm{Aut}(X,\mu)$ that gives for each $T \in U$ and each $g \in F$ that

$$\sum_{l \in L_g} \mu\left(C_{g,l} \triangle T(C_{g,l})\right) < \frac{\varepsilon}{2},$$

since this is exactly what convergence of T to the identity in the weak topology looks like. This allows us to infer that our original claim holds, and remembering that if we restrict T to commute with all of G, we ensure that $\alpha(g, T(x))$ is a cocycle, we obtain the statement of the Lemma. $\dashv$

Since C^0 is connected, the Local Rigidity Theorem tells us that any two cocycles which are in the C^0-orbit in $Z^1(G \curvearrowright X, L)$ are cohomologous, so in particular we get

$$\alpha(g,x_1) = F(g.x_1, g.x_2) \cdot \alpha(g,x_2) \cdot F(x_1,x_2)^{-1}$$

for some measurable $F : Z \to L$. By Theorem 40 the cocycle α is already cohomologous to a homomorphism when restricted to H. We now conclude by using Lemma 41 to see that it is cohomologous to a homomorphism on all of G.

3.4 Generalities

On Theorem 35: This theorem easily generalises to the case where the subgroup $H < G$ is closed with relative property (T) inside a locally compact second countable group G. To generalise the possible target groups to Popa's groups of finite type, a different argument is needed; see [Pop07, Proposition 4.2].

On the other hand, the generalisation of Theorem 24 is not routine; see [Jol05].

For the sake of completeness, we state the Popa superrigidity theorem in its general form:

Theorem 43 (Popa, Furman)**.** *Let G be a locally compact second countable group with a w-normal subgroup H. Assume furthermore that the pair (H, G) has relative property (T) and let $G \curvearrowright (X, \mu)$ be an ergodic measure preserving action on a probability space (X, μ), together with a measurable quotient (Y, ν).*

Assume

1. *$G \curvearrowright (X, \mu)$ is malleable relative to (Y, ν);*

2. *$H \curvearrowright (X, \mu)$ is relatively weakly mixing with respect to (Y, ν).*

Then any cocycle of the action $G \curvearrowright (X, \mu)$ with target group in the class of groups of finite type can be untwisted to a cocycle on the quotient. More precisely, let $\alpha : G \times X \to L$ be a measurable cocycle targeting a group of finite type L. Then there exist $\varphi : G \to X$ measurable and a measurable cocycle on the quotient $\rho : G \times Y \to L$ such that we can represent α as

$$\alpha(g, x) = \varphi(g.x)\rho(g, p(x))\varphi(x)^{-1},$$

with p denoting the quotient map.

4 Some Results on the Structure of $\mathrm{SL}_n(\mathbb{Z})$

We start by introducing some rather well-known facts about $\mathrm{SL}_n(\mathbb{Z})$ (without proof). This chapter intends to give a quick introduction to some concepts involved in handling $\mathrm{SL}_n(\mathbb{Z})$; for the sake of completeness, subsection 4.1 exposes the classical elementary proof of the fact that $\mathrm{SL}_n(\mathbb{Z})$ contains a torsion-free subgroup of finite index. We will need this in section 5 in order to see that the construction presented therein actually produces incomparable equivalence relations. On the other hand, subsection 4.2 gives a sketch of Shalom's new way of proving that $\mathrm{SL}_n(\mathbb{Z})$ has Kazhdan's property (T) for $n \geq 3$. This will be of central relevance to the final two chapters.

A good general references for the structure of linear groups is [Weh73]; the short note [Nic13] provides a particularly easily accessible proof of Selberg's theorem. For the current state of affairs concerning property (T)–groups see the book [BdlHV08]; a lot of background material on representation theory is provided in an extensive appendix, furthermore Shalom's proof is given very detailed treatment. Shalom's orginal paper is [Sha06].

Definition 44. *When the dimension is understood, we use* $\mathrm{e}_{i,j}$ *to denote the matrix with entry* 1 *at position* (i,j) *and zeroes elsewhere.*

Theorem 45. *The elementary matrices*

$$\{\mathrm{id}_n + \mathrm{e}_{i,j} \mid i \neq j\}$$

generate $\mathrm{SL}_n(\mathbb{Z})$ *for* $n \geq 2$.

Definition 46. *A group G is* `boundedly generated` *if there is a finite subset* $(g_i)_{i<n}$ *such that every* $g \in G$ *can be written as*

$$g = g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$$

for some integers $k_i \in \mathbb{Z}$.

We state the following without proof; see e.g. [BdlHV08].

Theorem 47. *The group* $\mathrm{SL}_n(\mathbb{Z})$ *has* `bounded elementary generation`*. In other words, the set of elementary matrices is a finite subset which witnesses that the group is boundedly generated.*

Definition 48. *The minimal* $n \in \omega$ *such that every element* $\mathrm{SL}_n(\mathbb{Z})$ *can be written as a product of at most* n *elementary matrices will be denoted by* $v_n(\mathbb{Z})$*.*

We even have the following

Theorem 49 (Carter–Keller)**.** *For* $n \geq 3$ *we have that*

$$v_n(\mathbb{Z}) \leq \frac{1}{2}(3n^2 - n) + 36 .$$

4.1 A Torsion-Free subgroup of finite index

Our exposition is inspired by the good introduction in [Nic13]; see also [Weh73].

Theorem 50. *The congruence homomorphism*

$$\varphi_m : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})$$

is onto.

Proof. First observe that the congruence homomorphism maps elementary matrices again to elementary matrices. So it is enough to show that $\mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})$ is elementarily generated.

By the Chinese Remainder Theorem, we can decompose

$$\mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z}) \cong \prod \mathbb{Z}/p_i^{s_i} ,$$

where $p_i^{s_i}$ is the decomposition of m into primes. Furthermore notice that in the decomposition we are confronted with local rings. Since elementary generation is preserved under direct products, it suffices to show that it holds for local rings.

Now let R be a local ring and picture a matrix in $\mathrm{SL}_n(R)$. If some first row entry is not in the maximal ideal of R we know that it is invertible. By column swapping we can get this element to the $(1,1)$–position, which allows us to clear the first row and first column. The next step is to be able to transform a diagonal matrix of unit determinant to the identity matrix, via the following sequence of row- and column-operations:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & a \\ 0 & b \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & a \\ (-a-1)b & b \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & a \\ 0 & ab \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\dashv$

Definition 51. *The kernel of the homomorphism φ_m is called the* `principal congruence subgroup of level` *m and denoted by*

$$\Gamma(m) := \ker(\varphi_m) .$$

It is easy to see that $\Gamma(m)$ is a has finite index for any $m \in \omega$. In fact, with a bit more effort one can even compute the index; see [Nic13, Lemma 2.3].

Theorem 52. *If $m \geq 3$ then the group $\Gamma(m)$ is torsion-free.*

Proof. Observe that we have

$$\Gamma(n) \subseteq \Gamma(m) \iff m|n$$

so it suffices to show that $\Gamma(4)$ and for primes $p \geq 3$, $\Gamma(p)$ are torsion-free.

Assume that for a fixed prime p, there is a non-trivial element $X \in \Gamma(p)$ of finite order. We may assume that

$$X^q = \mathrm{id}_n ,$$

after, if necessary, having replaced X by a power of itself. Then

$$-q(X - \mathrm{id}_n) = \sum_{i \geq 2}^{q} (qi)(X - \mathrm{id}_n)^i .$$

Let p^s be such that s is the highest power of p dividing all the entries in $X - \mathrm{id}_n$. In the identity above, the left hand side is divisible by at most p^s for $q \neq p$ and by at most p^{s+1} if $q = p$. Also, the right hand side is divisible by p^{2s}, and if $q = p \geq 3$ even by p^{2s+1}. This implies $q = p = 2$ and $s = 1$.

In fact, $p = 2$ and $s = 1$ means that $\Gamma(2)$ is the only one in the «prime stratum which harbours torsion», and that $\Gamma(4)$, which is its successor in the descending 2-chain, is already free of torsion. Furthermore, $q = 2$ means that torsion elements in $\Gamma(2)$ are of order a power of 2. Since $X^2 \in \Gamma(4)$ for any $X \in \Gamma(2)$, and because $\Gamma(4)$ is torsion-free, we hence know that any non-trivial torsion elements of $\Gamma(2)$ are of order 2. $\dashv$

4.2 A Proof of Property (T) for $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$

We intend to only give a rough sketch of the proof. We follow along the lines of Y. Shalom's new proof, which bears no reference to the fact that $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in $\mathrm{SL}_n(\mathbb{R})$. Our presentation is very close to the one given in [BdlHV08].

Definition 53. *The subset of elementary matrices*

$$\{\mathrm{E}_{i,j}(1) \mid i \leq n,\, j \leq n,\, i \neq j\}$$

of $\mathrm{SL}_n(\mathbb{Z})$ *will be denoted by* Q_n*.*

Property (T) for the pair $(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}^2)$ As a preparation for proving property (T) for the special linear groups over the integers, we need the preparatory result stating that the pair $(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}^2)$ has property (T). We start with a simple lemma.

Lemma 54. *Let ν be a mean (i.e. a finitely additve measure) defined on the Borel sets of* $\mathbb{R}^2 \setminus \{0\} =: X$*.*

Then there exists a Borel set $M \subseteq X$ and a matrix $\gamma \in \{U^{\pm}, L^{\pm}\}$ such that

$$|\nu(\gamma M) - \nu(M)| \geq \frac{1}{4}$$

for the natural linear action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright X$.

Proof. We split X into four subsets in the following way: start with the four lines defined by the equations

$$x = 0 \quad y = 0 \quad x = y \quad x = -y\,.$$

These divide X into eight regions; from these we obtain our four subsets by considering a point and the corresponding point obtained by reflection at the origin as belonging to the same set. We shall call these A, B, C, D. So, for example

$$A = \{(x, y) \in \mathrm{X} \mid 0 \leq y < x \vee x < y \leq 0\}\ ,$$

and the other sets are obtained by successive rotations of by $\frac{\pi}{4}$. By looking directly at how the relevant matrices were defined, we obtain the following four identities:

$$U^+(A \cup B) = A \quad L^+(A \cup B) = B$$

$$U^-(C \cup D) = D \quad L^-(C \cup D) = C\,.$$

Assume towards a contradiction the negation of the statement of the lemma; thus for any Borel $M \subseteq X$ and for all $\gamma \in \{U^\pm, L^\pm\}$ we have

$$|\nu(\gamma M) - \nu(M)| < \frac{1}{4}\,,$$

so in particular taking $A \cup B$ as our Borel set, we obtain

$$\nu(A) = \nu(A \cup B) - \nu(B) = \nu(A \cup B) - \nu(L^+(A \cup B)) < \frac{1}{4}\,,$$

and analogously we obtain the same bound for the measure of B, C, D. But since these four sets form a disjoint partition of X we have reached the contradiction sought for. $\dashv$

Theorem 55. *The pair* $(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ *has property (T). More precisely,*

$$\left(\mathrm{Q}, \frac{1}{10}\right)$$

is a Kazhdan pair for those groups.

Proof. Start with taking $(\pi, \mathcal{H})$ to be a unitary representation for $(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ which furthermore possesses a $(Q, 1/10)$–invariant vector. In order to reach a contradiction for relative property (T) of the inclusion, assume that $\mathcal{H}$ does not have any non-trivial vector invariant under $++Z^2$.

From Theorem 27 we know that there exists a unique projection-valued measure associated to the representation

$$\pi\restriction_{\mathbb{Z}^2}$$

and defined on the Borel sets of the dual group $\hat{\mathbb{Z}}^2$; let us call it E.

We remark that $\hat{\mathbb{Z}}^2$ can be identified with the torus $\mathbb{T}^2$ by associating to a point on the torus

$$(\mathrm{e}^{2\pi\imath x}, \mathrm{e}^{2\pi\imath y}) \in \mathbb{T}^2$$

the character on $\mathbb{Z}^2$ mapping pairs of integers in the following way:

$$\mathbb{Z}^2 \ni (m,n) \mapsto \mathrm{e}^{2\pi\imath(mx+ny)} .$$

Furthermore, the dual action

$$\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \hat{\mathbb{Z}}^2$$

corresponds to the transposed inverse of the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$.

So we obtain, for $z \in \mathbb{Z}^2$ and $\xi_1, \xi_2 \in \mathcal{H}$ that

$$\langle \pi(z)\xi_1, \xi_2 \rangle = \int_{\mathbb{T}^2} \chi(z) \mathrm{d} \langle E(\chi)\xi_1, \xi_2 \rangle .$$

$$E(\gamma B) = \pi(\gamma^{-1}) E(B) \pi(\gamma)$$

Now set
$$\varepsilon := \frac{1}{10}$$
and let ξ be a (Q, ε)–invariant unit vector. Define the following probability measure on $\mathbb{T}^2$ by
$$\mu_\xi(B) := \mathrm{d}E_{\xi,\xi}(B)\,,$$
and since also $\pi\restriction_{\mathbb{Z}^2}$ does not have any non-trivial invariant vectors, we obtain $E(\{0\}) = 0$ and consequently $\mu_\xi(\{0\}) = 0$.

Define
$$X := \left(-\frac{1}{4}, +\frac{1}{4}\right]^2 ;$$
then we have will now show that
$$\mu_\xi(X) \geq 1 - \varepsilon^2\,.$$

To that end, observe that since ξ is (Q, ε)–invariant, we have that for $l \in \{e^\pm, f^\pm\}$
$$\|\pi(l)\xi - \xi\|^2 = \int_{(-\frac{1}{2},+\frac{1}{2})} |e^{\pm 2\pi\imath x} - 1|^2 \mathrm{d}\mu_\xi(x, y) \leq \varepsilon^2\,;$$
it follows that
$$\varepsilon^2 \geq 2\mu_\xi\left(\left\{(x, y) \in \left(-\frac{1}{2}, +\frac{1}{2}\right]^2 \,\Big|\, |x| \geq \frac{1}{4}\right\}\right)$$
by using that for $\frac{1}{4} \leq |t| \leq \frac{1}{2}$ we have
$$|e^{\pm 2\pi\imath t} - 1|^2 = 2 - 2\cos 2\pi t = 4(\sin \pi t)^2 \geq 2\,.$$
As a consequence we obtain both
$$\mu_\xi\left(\left\{(x, y) \in \left(-\frac{1}{2}, +\frac{1}{2}\right]^2 \,\Big|\, |x| \geq \frac{1}{4}\right\}\right) \leq \frac{\varepsilon^2}{2}$$

and

$$\mu_\xi\left(\left\{(x,y)\in\left(-\frac{1}{2},+\frac{1}{2}\right]^2 \;\Big|\; |y|\geq\frac{1}{4}\right\}\right)\leq\frac{\varepsilon^2}{2},$$

which together imply what was claimed.
Define a new probability measure ν by setting

$$\nu(B):=\frac{\mu_\xi(B\cap X)}{\mu_\xi(X)};$$

for this we actually obtain that for any Borel B and any $\gamma\in\{U^\pm,L^\pm\}$ we have

$$|\nu(\gamma B)-\nu(B)|<\frac{1}{4},$$

as we will now see: since by the above we have that

$$E(\gamma B)=\pi(\gamma^{-1})\,E(B)\,\pi(\gamma)$$

we can calculate

$$\begin{aligned}
\left|\mu_\xi(\gamma B)-\mu_\xi(B)\right| &= \left|\left\langle\pi(\gamma^{-1})\,E(B)\,\pi(\gamma)\;\xi,\xi\right\rangle-\langle E(B)\xi,\xi\rangle\right|\\
&\leq\left|\left\langle\pi(\gamma^{-1})\,E(B)\,\pi(\gamma)\;\xi,\xi\right\rangle-\left\langle\pi(\gamma^{-1})E(B)\xi,\xi\right\rangle\right|\\
&\quad+\left|\left\langle\pi(\gamma^{-1})E(B)\xi,\xi\right\rangle-\langle E(B)\xi,\xi\rangle\right|\\
&=\left|\left\langle\pi(\gamma^{-1})E(B)(\pi(\gamma)\xi-\xi),\xi\right\rangle\right|+|\langle E(B)\xi,(\pi(\gamma)\xi-\xi)\rangle|\\
&\leq\|\pi(\gamma^{-1})E(B)\|\cdot\|\pi(\gamma)\xi-\xi\|+\|E(B)\|\cdot\|\pi(\gamma)\xi-\xi\|\\
&\leq\varepsilon+\varepsilon.
\end{aligned}$$

We already know how to infer that

$$0\leq\mu_\xi(B)-\mu_\xi(B\cap X)\leq\varepsilon^2.$$

From this we obtain

$$\begin{aligned}\mu_\xi(\gamma B \cap X) - \mu_\xi(B \cap X) &= (\mu_\xi(\gamma B \cap X) - \mu_\xi(\gamma B)) \\ &\quad + (\mu_\xi(\gamma B) - \mu_\xi(B)) + (\mu_\xi(B) - \mu_\xi(B \cap X)) \\ &\leq 0 + 2\varepsilon + \varepsilon^2 .\end{aligned}$$

Since we can apply this to both B and $\gamma^{-1}B$, we obtain the same for the absolute value, thus

$$|\mu_\xi(\gamma B \cap X) - \mu_\xi(B \cap X)| \leq 2\varepsilon + \varepsilon^2 .$$

Now we are in a position to apply (**), and remembering that we put $\varepsilon = \frac{1}{10}$ we have

$$|\nu(\gamma B) - \nu(B)| \leq \frac{2\varepsilon + \varepsilon^2}{1 - \varepsilon^2} = \frac{21}{99} < \frac{1}{4} ;$$

so the second claim follows.

Since we observed that $\mu_\xi(\{0\}) = 0$, we can consider ν as being a measure on $\mathbb{R}^2 \setminus \{0\}$. Observe that acting with any of $U^\pm, L^\pm$ on X, we do not fall outside of $\left(-\frac{1}{2}, +\frac{1}{2}\right]^2$. Consequently, since $\nu(X) = 1$, we obtain that for every Borel set $B \subseteq \mathbb{R}^2 \setminus \{0\}$ the distance between the ν–measure of B and that of the shift is less than $\frac{1}{4}$, which is a direct contradiction to Lemma 54. $\dashv$

We state the following corollary without proof; it quantitatively relates the existence of (Q, ε)–invariant unit vectors for unitary representations of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}$ to the possibility to find ($\mathbb{Z}$–invariant vectors.

Corollary 56. *Let $(\pi, \mathcal{H})$ be a unitary representation of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}$; assume that there is a $(Q, \varepsilon/20)$–invariant unit vector ξ for some $\varepsilon > 0$.*

Then ξ is $(\mathbb{Z}, \varepsilon)$–invariant.

Proof of the Main Theorem Before we show the main result, we need the following simple fact, which we state without proof; see [BdlHV08, Lemma 4.2.4].

Lemma 57. *There exists an injective homomorphism*

$$\alpha_n : \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \to \mathrm{SL}_n(\mathbb{Z})$$

such that the set

$$\{\mathrm{E}_{i,j}(t)\}_{i \neq j, t \in \mathbb{Z}}$$

is contained in $\alpha_n(\mathbb{Z}^2)$ and

$$\alpha_n(\mathrm{SL}_2(\mathbb{Z})) = \begin{pmatrix} \mathrm{I}_k & 0 & 0 \\ 0 & \mathrm{SL}_2(\mathbb{Z}) & 0 \\ 0 & 0 & \mathrm{I}_{n-k-2} \end{pmatrix}$$

for some $k \leq n - 2$.

Now we are ready for the proof.

Theorem 58 (Shalom). *The group $\mathrm{SL}_n(\mathbb{Z})$ has property (T) if $n \geq 3$.*

In fact, we even obtain a quantitative result:

$$\left(\mathrm{Q}_n, \frac{v_n}{20}\right)$$

is a Kazhdan pair for $\mathrm{SL}_n(\mathbb{Z})$.

Proof. Define

$$T_n := Q_n \cup Q_n^{-1},$$

so T_n consists of all the elementary matrices with 1 in some place outside the diagonal and their inverses. It is sufficient to show that $(T_n, \frac{v_n}{20})$ is a Kazhdan pair, since

$$\|\pi_g(\xi) - \xi\| = \|\pi_{g^{-1}}(\xi) - \xi\|.$$

Now let $(\pi, \mathcal{H})$ be a unitary representation with a unit vector ξ_0 which is $(T_n, \frac{v_n}{20})$–invariant. Take an arbitrary elementary matrix A. Then by

Lemma 57 there exists an embedding

$$\alpha_n : \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \to \mathrm{SL}_n(\mathbb{Z})$$

such that $A \in \alpha_n(\mathbb{Z}^2)$ and furthermore

$$\alpha_n(Q) = T_n \cap \mathrm{Im}(\alpha_n)$$

where Q is as defined before.

Using Corollary 56 we obtain that ξ_0 is $(\alpha_n(\mathbb{Z}^2), \frac{1}{v_n})$–invariant. In particular this means that for any elementary matrix $A \in \mathrm{SL}_n(\mathbb{Z})$ we have

$$\|\pi_A(\xi_0) - \xi_0\| < \frac{1}{v_n}\,.$$

If we now select an arbitrary matrix A in $\mathrm{SL}_n(\mathbb{Z})$, we can represent it as a product of elementary matrices $\{A_i\}_{i<N}$, where $N \leq v_n$. By easy calculation of the bound we get

$$\|\pi_A(\xi) - \xi\| \leq \sum_{i<N} \|\pi_{A_0 \cdots A_{N-i}}(\xi) - \pi_{A_0 \cdots A_{N-(i+1)}}(\xi)\|$$

$$= \sum_{j<N} \|\pi_{A_j}(\xi) - \xi\| \leq \frac{N}{v_n} \leq 1\,,$$

in other words, ξ is $(\mathrm{SL}_n(\mathbb{Z}), 1)$–invariant; from Theorem 23 we now deduce that π has a non-trivial invariant vector. $\dashv$

5 The Structure of Countable Borel Equivalence Relations

This chapter constitutes a partial goal in the journey of this thesis. We will first prove that the structure of countable Borel equivalence relations is very rich. This will follow quite easily from the rigidity result in section 3 together with the considerations in section 4.

The following two lemmata explain why we will opt to work with the free part of shift actions.

Lemma 59. *For any infinite countable group G and μ the standard product measure on 2^G we have that*

$$\mu(\mathrm{Fr}(G \curvearrowright 2^G)) = 1\,.$$

Proof. We show that for each $g \in G$ which is not the identity, it follows that

$$X_g := \{x \in 2^G \mid g.x = x\}$$

has measure zero. Obviously if $x \in X_g$, then x is constant on each right coset of $\langle g \rangle$. Now there are only the following two possibilities for $\langle g \rangle$: either this one-generator group is infinite, or there are infinitely many cosets. $\dashv$

5.1 Intermezzo — Why Invariant Measures Matter

In this short deviation, we would like to present two construction which justify the crucial necessity for the necessity of the existance of an invariant Borel probability measure on a G–space.

We present the result from [DJK94]. The following definition is crucial to their work, and we will shortly see the consequences

Definition 60. *Let E be a countable Borel equivalence relation on a standard Borel space X. We call E `compressible` if there exists $A \subseteq X$ Borel such that*

1. *$X \setminus A$ meets every E-class;*
2. *there exists a function $f : X \to A$ with $\forall x \in X\, xEf(x)$ (the graph is contained in E).*

Theorem 61 ([DJK94, Proposition 11.1]). *Let E be a compressible countable Borel equivalence relation. Assume that E is induced by the free action of a countable group G, and that H is any countable group extending G, i.e. $G \subseteq H$. Then there exists a free action of H which also induces E.*

Since compressibility destroys the existence of a potential invariant measure, this does indeed tell us that there is no way we could hope for getting rid of the assumption of an invariant measure and at the same time recover the acting group (or some kind of meaningful class of inducing groups).

5.2 Producing Continuum-Many Countable Borel Equivalence Relations

We begin with the following simple observation:

Theorem 62. *If G embeds into H, then $E_G \leq_B E_H$.*

Proof. For the sake of notation, suppose that $G \leq H$. Then for each $f \in \mathrm{Fr}(2^G)$, define $f^* \in 2^H$ by setting it to the value of f for arguments from G, and to 0 otherwise.

Then we only need to check that $f^* \in \mathrm{Fr}(2^H)$ since then obviously that the map $f \mapsto f^*$ is a Borel reduction from E_G to E_H, which is obviously true. $\dashv$

We would like to be able to infer the converse in for certain classes of groups – in the most general setting this is certainly not possible.

Definition 63. *If G, H are groups, then a homomorphism $\pi : G \to H$ is called a `virtual embedding` if the kernel* $\ker \pi$ *is finite.*

Using Popa's Cocycle Superrigidity from Theorem 29 we now obtain the following remarkable result:

Theorem 64. *Let S be any countable group, define $G := \mathrm{SL}_3(\mathbb{Z}) \times S$. Assume that H is a countable group and that $H \curvearrowright Y$ is a free Borel action on a standard Borel space Y. Let G act by Bernoulli shifts on 2^G If there exists a Borel homomorphism from E_G to E_H which is not μ-trivial, then there exists a virtual embedding between the groups $\pi : G \to H$.*

Proof. Suppose that $f : \mathrm{Fr}(2^G) \to Y$ is a μ-nontrivial Borel homomorphism from $E_G^{\mathrm{Fr}(2^G)}$ to E_H^Y. Then we can define the associated Borel cocycle $\alpha : G \times \mathrm{Fr}(2^G) \to H$ by selecting as value for $\alpha(g, x)$ the unique $h \in H$ that makes the following identity true:

$$h.f(x) = f(g.x) \ ,$$

as defined in section 2. By Theorem 29, after applying the correct adjustment through a measurable map and possibly disregarding a nullset, we can assume that α does not depend on the second component, which allows for being interpreted as a group homomorphism $\alpha : G \to H$.

Suppose towards a contradiction that $N := \ker \alpha$ is infinite. Since we know by Lemma 18 that the action $G \curvearrowright \mathrm{Fr}(2^G)$ is strongly mixing, it follows that the restricted action of N is also ergodic. By observing that the homomorphism $f : \mathrm{Fr}(2^G) \to Y$ is N–invariant, we can conclude that it must be μ–a.e. constant, which clearly contradicts μ–non-triviality.

Thus the kernel must be finite and we get the desired result. $\dashv$

We now show how to achieve the goal of obtaining many incomparable equivalence relations. First of all define, for each prime p, the groups

$$A_p := \bigoplus_{i<\omega} \mathcal{C}_p$$

as the direct sum of countably many copes of the cyclic groups of order p. For an arbitrary subset of the primes $C \subseteq \mathbb{P}$, define the group

$$G_C := SL_3(\mathbb{Z}) \times \bigoplus_{p \in C} A_p \ .$$

Then the following lemma will lead us to the desired result.

Lemma 65. *For any two subsets of the prime numbers C and D the following holds:*

$$E_{G_C} \leq E_{G_D} \Leftrightarrow C \subseteq D\ .$$

Proof. If we have $C \subseteq D$, then obviously $G_C \leq G_D$ and by Theorem 62 also $E_{G_C} \leq E_{G_D}$.

Conversely, by applying Theorem 64, we know that there exists a virtual embedding $\pi : G_C \to G_D$. As we have seen in subsection 4.1, the linear group $SL_3(\mathbb{Z})$ contains a torsion-free subgroup of finite index. Using this, we see that each of the cyclic groups $\mathcal{C}_p$ for which appear in G_C (so with $p \in C$) necessarily embed into

$$\bigoplus_{q \in D} \mathcal{C}_q\ ,$$

since the torsion-freeness of the finite-index subgroup does not leave any room for the embedding to «cross over» between the coordinates; this obviously implies $p \in D$, as desired. $\dashv$

In order to achieve 2^ω–many incomparable countable Borel equivalence relations, we need to once more glue those constructed here together. This is achieved by identifying a single real with a subset of the primes. Then, given any Borel subset of the reals, we can simply take a disjoint sum of all those reals identified with their corresponding sets of primes, which is summarised in the following

Theorem 66 (Adams–Kechris)**.** *There exist continuum-many incomparable free countable Borel equivalence relations.*

Some thoughts on Essential Freeness We will now show, with very little effort, that the rather long-standing question of whether the universal countable Borel equivalence relation E_∞ can be induced by a free group action has a negative answer.

Definition 67. *A Borel equivalence Relation E on a standard Borel space is* `essentially free` *if there exist a group G with a acting freely on a standard Borel space $G \curvearrowright Y$ such that the thus induced Borel equivalence relation F is bireducible with E, i.e. $E \sim F$.*

The following is a simple fact quoted from [JKL02]:

Theorem 68. *Suppose that E, F and $(E_n)_{n\in\omega}$ are countable Borel equivalence relations. Then we have the following:*

1. *If $E \leq F$ and F is essentially free, then E is also essentially free.*

2. *If $E \subseteq F$ and F is essentially free, then E is also essentially free.*

3. *If for all $n \in \omega$ the relations E_n are essentially free, then $\bigoplus_{n\in\omega} E_n$ is also essentially free.*

In other words, essential freeness is preserved under taking reduced equivalence relations, preserved under subsets and preserved under countable sums.

Theorem 69. *If E is any essentially free Borel equivalence relation, then there exists a countable group G such that*

$$\mathrm{E}_G^{\mathrm{Fr}(2^G)} \not\leq E\,.$$

Proof. Assume that $E = \mathrm{E}_H^Y$ for a suitable standard Borel space Y and a countable group H. Then there exists a finitely generated group H' that does not embed into H; this is possible by means of a classical result of Neumann which grants the existence of uncountably many finitely generated groups. We obtain the promised group by defining it as the product of our favourite linear group together with the free product of H' with the integers, i.e.

$$G := \mathrm{SL}_3(\mathbb{Z}) \times \left(H' \star \mathbb{Z}\right)\,.$$

By construction, G has no nontrivial finite normal subgroups; furthermore, the choice of H' ensures that even G does not embed into H. Using Theorem 64 we arrive at the desired conclusion. $\dashv$

In particular, E_∞ is not essentially free.

We note that in [Tho09] S. Thomas has furthermore constructed continuum-many non–essentially-free countable Borel equivalence relations; like in the Adams–Kechris result, he even gives an embedding from the Borel subsets of the reals under inclusion to a family of non–essentally-free Borel equivalence relations under Borel reducibility.

5.3 Other Ways of Producing Incomparable Borel Equivalence Relations

As stated in the introduction, the original construction of continuum-many incomparables by Adams and Kechris used R. Zimmer's super-rigidity theory. Shortly after that, relying on the same methods, in [Ada02] S. Adams constructed an example of two Borel equivalence relations E, F with $E \subseteq F$ which are nonetheless incomparable.

Theorem 70 (S. Adams)**.** *There exists a pair of countable Borel equivalence relations E, F with $E \subseteq F$ (subsets in the set-theoretic sense) but incomparable in the Borel sense.*

We want to mention the following very natural and interesting example from the original [AK00].

Theorem 71 (Adams–Kechris)**.** *Denote by R_n the equivalence relation induced by the natural action of the general linear group over the integers, i.e.*

$$\mathrm{GL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}\,.$$

Then we have for all $n \in \omega$ that $R_n < R_{n+1}$; in other words we obtain an countable tower of strictly increasing Borel equivalence relations.

With a different and more elementary approach, [HK05] show that incomparables can be produced with the following, leading to a strong result.

Theorem 72 (Hjorth–Kechris)**.** *Let $S \subseteq \mathbb{P}$ be a set of odd primes and consider the following action:*

$$(\star_{p\in S}(\mathcal{C}_p \star \mathcal{C}_p) =: \Gamma_S \curvearrowright \mathrm{Fr}\left(2^{\Gamma_S}\right) ;$$

Then for any two sets of odd primes S_1, S_2, non-inclusion $S \not\subseteq T$ implies that $E_{\Gamma_{S_1}}^{\mathrm{Fr}\left(2^{\Gamma_{S_2}}\right)}$ is $E_{\Gamma_{S_2}}^{\mathrm{Fr}\left(2^{\Gamma_{S_2}}\right)}$–ergodic.

The following result bears mentioning, since it states that we can have an increase in Borel complexity by simply forming disjoint sums; we end our list here.

Theorem 73 ([Tho02b])**.** *There exists a countable equivalence relation E such that*

$$E < E \oplus E < E \oplus E \oplus E < \cdots .$$

Concerning Borel equivalence relations related to the classification problem for torsion-free abelian groups of finite rank, we refer the reader to the next chapter and the notes at the end of it.

6 Torsion-free Abelian Groups of Finite Rank

This chapter represents the culmination of our efforts for this thesis. First of all, the objects of our study will be countable torsion-free abelian groups of finite rank. The rank of an abelian group is defined in analogy to the dimension of vector spaces; in fact, it is easy to see that any torsion-free abelian group of rank n can be viewed as lying inside of $\mathbb{Q}^n$, i.e. the n–dimensional vector space over the rational numbers. This is certainly a Polish space, and the torsion-free abelian groups of rank exactly n are realised as additive subgroups, which furthermore contain n linearly independent elements. This is again a Borel subset, and as such can be viewed as a standard Borel space; we will denote the realisation of the space of countable torsion-free abelian groups of rank n inside of $\mathbb{Q}^n$ by $\mathrm{R}(\mathbb{Q}^n)$; furthermore, denote by $\mathrm{S}(\mathbb{Q}^n)$ the space of all additive subgroups of $\mathbb{Q}^n$.

We will show that the classification problem for countable torsion-free abelian groups is very difficult, at least when the perspective of the Borel setting is advanced. For this, we introduce some notions needed for the study of abelian groups, first and foremost the Kurosh–Malcev invariants. Those will constitute, together with Ioana's superrigidity theorem, the basis of our proofs.

Assume that we are given an isomorphism between two elements of $\mathrm{R}(\mathbb{Q}^n)$. This isomorphism obviously extends to a vector space isomorphism of the whole space $\mathbb{Q}^n$; and of course on the other hand, any vector space isomorphism is a group isomorphism between any abelian subgroups and its image under the map. So we can conclude that the isomorphism relation on the space $\mathrm{R}(\mathbb{Q}^n)$ is realised by the natural group action of $\mathrm{GL}_n(\mathbb{Q})$. By this representation and Theorem 12, we conclude that we are in fact concerning ourselves with a countable Borel equivalence relation.

6.1 Baer's Classification for Rank One Groups

We will soon see that with very little effort it becomes evident that the rank 1 case admits very simple invariants. For the reader interested in other algebraic results concerning those groups, [Fuc73, §85] is a good starting point.

Definition 74. *For an abelian group G, we define for a prime p the* `p-height` *of an elment $g \in G$ as*

$$\mathrm{h}_g(p) := \sup\{n \in \omega \mid \exists h \in G \; nh = g\}.$$

The `characteristic` *of g, denoted by $\chi(g)$, is simply defined to be the function assigning the respective sequence of heights along all the prime numbers, i.e.*

$$\chi(g) := (\mathrm{h}_g(p))_{p \in \mathbb{P}}).$$

Definition 75. *Two characteristics $\chi(g_1), \chi(g_2)$* `belong to the same type` *if both of the following hold:*

1. *$\chi(x_1)(p) = \chi(x_2)(p)$ for cofinitely many primes p;*
2. *if $\chi(x_1)(p) \neq \chi(x_2)(p)$, then both their values are finite.*

This is an equivalence relation among characteristics; we call the equivalence class containing a given characteristic $\chi(x)$ the `type` *of x and denote it by $\tau(x)$.*

Lemma 76. *Every sequence $\lambda : \mathbb{P} \to \omega \cup \{\infty\}$ is a characteristic.*

Proof. Define X as the subgroup of $\mathbb{Q}$ generated by all

$$p^{-k}$$

with $k < \lambda(p)$ and p ranging over all primes. Then the element $1 \in \mathbb{Q}$ will achieve that $\chi(1) = \lambda$. $\dashv$

Observe that for rank-one–groups, the type of two arbitrary (nontrivial) elements belongs to the same type; so in that case we are allowed to define the type of any such element as the `type of the group`.

Theorem 77 (Baer). *Two rank-one torsion-free abelian groups are isomorphic if and only if the are of the same type.*

Proof. Take two groups X, Y and arbitrary nonzero elements x, y. Then the corresponding characteristics differ only on a finite number of indices, say $p_1, \ldots, p_k$. We can now divide x and y by

$$p_1^{\chi(x)(p_1)} \cdots p_k^{\chi(x)(p_k)}$$

and

$$p_1^{\chi(y)(p_1)} \cdots p_k^{\chi(y)(p_k)}$$

respectively, to obtain the elements x' and y' with zeroes in their characteristics at all the indices $p_1, \ldots, p_k$. Of course also $\chi(x') = \chi(y')$; considering the equations

$$mz = nx' \tag{†1}$$

and

$$mz = ny' \tag{†2}$$

this means that Equation †1 has a solution in X if and only if Equation †2 has a solution in Y.

Since these kind of equations have at most one solution in torsion-free groups, this allows us to define an isomorphism simply by sending a solution of Equation †1 to a solution of Equation †2. ⊣

Corollary 78. *There are continuum-many nonisomorphic countable torsion-free groups of rank one.*

Proof. The set of characteristics obviously has cardinality 2^ω, so this is an upper bound. On the other hand, if we consider sequences consisting exclusively of elements of $\{0, \omega\}$, we know that these can be realised as characteristics. And since these sequences certainly represent all distinct types, we also have produced the lower bound 2^ω. ⊣

6.2 Construction of an $\mathrm{SL}_n(\mathbb{Z})$–invariant measure

This section is devoted to the construction of a measure on $\mathrm{S}(\mathbb{Q}^n)$.

Theorem 79 (Hjorth)**.** *There is an ergodic, non-atomic, $\mathrm{SL}_n(\mathbb{Z})$–invariant measure on $\mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$. Furthermore there is an $\mathrm{SL}_n(\mathbb{Z})$–invariant Borel subset, which is conull with respect to this measure and consists entirely of rigid groups.*

Here $\mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$ denotes those subgroups of $\mathbb{Q}^n$ which contain a copy of $\mathbb{Z}^n$.

We start with an outline of how to construct the measure, as defined in [Hjo99]. To begin Hjorth's construction, let $n \geq 2$ and define $\mathcal{S}_n$ as the set of all subgroups of

$$\mathbb{Q}^n/\mathbb{Z}^n\,,$$

furthermore denote by π_n the canonical surjection to that quotient, so

$$\pi_n : \mathbb{Q}^n \to \mathbb{Q}^n/\mathbb{Z}^n\,.$$

The set of groups $\mathcal{S}_n$ are clearly a closed subspace of $2^{\mathbb{Q}^n/\mathbb{Z}^n}$ and thus Polish. Also, we can identify $\mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$ and $\mathcal{S}_n$ via the Borel bijection we obtain by

$$A \mapsto \pi_n(A)\,.$$

Define Γ_n as the dual group of $\mathbb{Q}^n/\mathbb{Z}^n$ (as a discrete group), i.e.

$$\Gamma_n := \{\psi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{R}/\mathbb{Z} \mid \psi \text{ is a homomorphism}\}\,,$$

and equip Γ_n with the topology of pointwise convergence and group structure of pointwise addition.

Furthermore, the fact that $\mathbb{Q}^n/\mathbb{Z}^n$ is a countable discrete torsion group implies that Γ_n is a compact metrizable zero-dimensional space.

Let m_n be Haar measure on Γ_n. Then we can first define a map

$$k_n : \Gamma_n \to \mathcal{S}_n$$

$$\psi \mapsto \ker(\psi)$$

and use this map to push forward to a probability measure $\widehat{\nu}_n$ on $\mathcal{S}_n$, i.e.

$$\widehat{\nu}_n := k_* m_n \,.$$

Turning back, define ν_n as the probability measure on $\mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$ by taking $\widehat{\nu}$ together with the identification described before. This finishes the construction; we are now going to prove the required properties.

Lemma 80. *The measure ν_n is $\mathrm{SL}_n(\mathbb{Z})$–invariant.*

Proof. For any $A \in \mathrm{SL}_n(\mathbb{Z})$ we can define the following map

$$\rho_A : \Gamma_n \to \Gamma_n$$

$$(\rho_A(\psi))(h) := \psi(A^{-1}h) \,.$$

This is clearly a homeomorphism which even respects the group structure, i.e. an endomorphism. We may certainly push forward through this, and by the uniqueness of Haar measure we immediately get

$$\rho_{A*} m = m \,.$$

Now we start calculating; take any measurable $T \subseteq \mathcal{S}_n$, then

$$\begin{aligned}
\nu\left(A(T)\right) &= m\left(\{\psi \in \Gamma_n \mid \ker(\psi) \in A(T)\}\right) \\
&= m\left(\left\{\psi \in \Gamma_n \mid \{A^{-1}h \mid \psi(h) = 0\} \in T\right\}\right) \\
&= m\left(\{\psi \in \Gamma_n \mid \{h \mid \psi(Ah) = 0\} \in T\}\right) \\
&= m\left(\left\{\psi \in \Gamma_n \mid \ker\left(\rho_A^{-1}(\psi)\right) \in T\right\}\right) \\
&= m\left(\rho_A\left(\ker^{-1}(T)\right)\right) \,,
\end{aligned}$$

and by what we have said above, that last term equals $m\left(\rho_A\left(\ker^{-1}(T)\right)\right)$, which is by definition equal to $\nu_n(T)$, as we had set out to find. $\dashv$

Some Remarks on Ergodicity It can easily be seen that Hjorth's measure is not ergodic; to see this, consider ν_n on $\mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$ and any prime $p \in \mathbb{P}$, then

$$\{A \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n) \mid \mathbb{Z}^n \leq p\, A\}$$

is $\mathrm{SL}_n(\mathbb{Z})$–invariant with measure $\frac{1}{p^n}$.

However, Adams and Kechris found a way around this difficulty and used it to derive (from Zimmer's theory) that the classification problem restricted to the rigid groups increases in Borel complexity as the rank increases. We will now give a sketch of how they obtained an ergodic invariant measure from Hjorth's construction. To achieve that, we will consider the ergodic decomposition of the action

$$\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)\,.$$

Define $\mathcal{E}$ to be the standard Borel space of all invariant ergodic measures with respect to this action. Then there is a Borel surjection

$$\pi : \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$$

which is invariant under $\mathrm{SL}_n\mathbb{Z}$, with the additional properties that

$$\forall \varepsilon \in \mathcal{E}\ \varepsilon\left(\{x \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n) \mid \pi(x) = \varepsilon\}\right) = 1$$

and furthermore any $\mathrm{SL}_n(\mathbb{Z})$–invariant measure ν can be represented by

$$\nu(A) = \int e(A) \mathrm{d}(\pi_* \nu)(e)\,.$$

Since $\nu_n\left(S^*(\mathbb{Q}^n, \mathbb{Z}^n)\right) = 1$ there must also be a $\pi_*\nu_n$–conull Borel set $\mathcal{E}' \subseteq \mathcal{E}$ with

$$\forall \varepsilon \in \mathcal{E}'\ \varepsilon(S^*(\mathbb{Q}^n, \mathbb{Z}^n) = 1\,.$$

Claim: There is $\varepsilon' \in \mathcal{E}'$ which is non-atomic.

Proof: Otherwise, for each $\varepsilon \in \mathcal{E}'$ there is a unique $\mathrm{SL}_n(\mathbb{Z})$–orbit θ_ε

with $\varepsilon(\theta_\varepsilon) = 1$, and consequently the set

$$W := \{x \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n) \mid \pi(x) \in \mathcal{E}' \text{ and } x \in \theta_{\pi(x)}\,,$$

and observing that saying $x \in \theta_{\pi(x)}$ is equivalent to saying $\pi(x)([x]_{\mathrm{SL}_n(\mathbb{Z})}) = 1$, we see that W is both Borel and $\mathrm{SL}_n(\mathbb{Z})$–invariant. If we now shift our attention back to the projection, it is clear that

$$x \mathrm{E}^W_{\mathrm{SL}_n(\mathbb{Z})} y \iff \pi(x) = \pi(y)\,,$$

which makes that relation on W smooth; but then W is also ν_n–conull, which contradicts the previous theorem the fact that Hjorth has proved that his measure does not make the equivalence relation smooth when restricting to any conull set. So we have just extended to the following

Theorem 81. *There is an ergodic, non-atomic,* $\mathrm{SL}_n(\mathbb{Z})$*–invariant measure on* $\mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$*. Furthermore there is an* $\mathrm{SL}_n(\mathbb{Z})$*–invariant Borel subset, which is conull with respect to this measure and consists entirely of rigid groups.*

6.3 The Rank Two Case

Some Notes on Amenability We will start by introducing the basic definition of amenability; our interest lies in the proof that under some (very natural) assumptions, non-amenable groups induce non-hyperfinite equivalence relations. For further remarks see [JKL02, §1 and §2].

Definition 82. *For a countable set X, a* `finitely additive probability measure` *is a function* $\varphi : 2^X \to [0, 1]$ *such that*

1. $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ *if* $A \cap B = \emptyset$

2. $\varphi(X) = 1$

Definition 83. *A* `mean` *on X is a positive linear functional* $\bar{\varphi}$ *on* $\ell^\infty(X)$*, the Banach space of bounded real functions on X, with* $\bar{\varphi}(1) = 1$.

This last property makes $\bar{\varphi}$ continuous. Means and finitely additive probability measures can be identified with one another; given any finitely additive probability measure φ, define the corresponding functional as

$$\bar{\varphi} := \int f \mathrm{d}\varphi \,.$$

Conversely, for a positive linear functional $\bar{\varphi}$ we can define a finitely additive measure by setting

$$\varphi(A) := \bar{\varphi}(\chi_A) \,.$$

Thus we do not have to distinguish between the two concepts.

Definition 84. *A countable group G is* `amenable` *if there exists a left-invariant mean $\bar{\varphi}$ on G.*

Examples of amenable groups include all compact groups and by a result of Markov–Kakutani all abelian topological groups. We state the following theorem without proof.

Theorem 85 ([JKL02])**.** *Let G be a countable group acting on a standard Borel probability space (X, μ) in a measure preserving way. Assume furthermore that the action is free on a conull set.*

Then the hyperfiniteness of E_G^X implies the amenability of G.

Completing the Proof Using the observations just presented in the context of amenable groups, we shall now use the construction from the previous section subsection 6.2 together with some observations on amenability to show that the classification problem in the rank two–case is not hyperfinite.

In order to use Theorem 85, we need to ensure that the measure constructed in subsection 6.2 meets the requirements; i.e. we want to find a conull subset where the action is free. The following theorem tells us that there is a conull set whose elements are all rigid groups.

Theorem 86 (Thomas, Hjorth)**.** *Define $\mathcal{X}_n$ to be the Borel subset consisting of the groups $A \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$ which satisfy the following two conditions;*

1. *For any prime $p \in \mathbb{P}$ we have $pA \neq A$.*

2. *For two linearly independent elements $a, b \in A$ it follows that $\tau(a) \neq \tau(b)$.*

Then for all $n \in \omega$ we have that

$$\nu_n \left(\mathcal{X}_n \right) = 1 \, .$$

Proof. First of all, let us show that each $A \in \mathcal{X}_n$ is rigid. Assume that we have an automorphism φ, then clearly

$$\forall a \in A \; \tau(\varphi(a)) = \tau(a) \, .$$

If $a \in A$ is furthermore nonzero, the condition in item 2 of the theorem implies that there is a nonzero $\lambda_a \in \mathbb{Q}$ such that we can represent

$$\varphi(a) = \lambda_a a \, ,$$

and this λ_a must certainly be the same for all $a \in A$. Now we can use the condition in item 1, which leaves no room except for $\lambda_a \in \{-1, +1\}$.

Now take an element $a \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ such that the projection $\pi(a)$ has positive order k in $\mathbb{Q}^n / \mathbb{Z}^n$. Defining

$$\Gamma_n^a := \{ \psi \in \Gamma_n \mid \psi(a) = 0 \} \, ,$$

we get that

$$\nu_n \left(\{ A \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n) \mid a \in A \} \right) = \hat{\nu}(\Gamma_n^a) = \frac{1}{[\Gamma_n : \Gamma_n^a]} = \frac{1}{k} \, ,$$

where the last equality follows from the translation invariance of Haar measure and the rest simply from definition.

Observe now that for two linearly independent elements $a, b \in \mathbb{Q}^n \setminus$

$\mathbb{Z}^n$, the corresponding events

$$\{A \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n) \mid a \in A\}$$

and

$$\{A \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n) \mid b \in A\}$$

the corresponding events are ν_n–independent.

It follows easily that if we fix a prime p, then for ν_n–conull many $A \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$ it holds that $pA \neq A$.

Let us denote by $\{e_i\}_{i<n}$ the elements of the standard basis of $\mathbb{Q}^n$; then we define for each prime p the events

$$\mathrm{F}_p := \{A \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n) \mid \frac{e_0}{p} \in A \text{ and } \frac{e_1}{p} \notin A\}\,.$$

Using what was just said above about events arising from linearly independent elements, we see that all these $\{F_p \mid p \in \mathbb{P}\}$ are also ν_n–independent.

Furthermore, for each prime p we can calculate that

$$\nu_n(\mathrm{F}_p) = \frac{p-1}{p^2}\,.$$

From a classical theorem of Euler we can easily derive that

$$\sum_{p \in \mathbb{P}} \frac{p-1}{p^2} = \infty\,,$$

so that we can apply the Second Borel–Cantelli Lemma to see that

$$\nu_n\left(\bigcap_{n<\omega} \bigcup_{p>n} \mathrm{F}_p\right) = 1\,.$$

Hence the types are unequal, i.e.

$$\tau(e_0) \neq \tau(e_1)$$

in ν_n–conull many $A \in \mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$.

Also observe that $\mathrm{SL}_n(\mathbb{Z})$ acts transitively on the set of ordered pairs of 1–dimensional linear subspaces of $\mathbb{Q}^n$; together with the $\mathrm{SL}_n(\mathbb{Z})$–invariance of ν_n, we obtain that the set of all groups in $\mathrm{S}(\mathbb{Q}^n, \mathbb{Z}^n)$ such that any two linearly independent group elements are not of the same type is ν_n–conull. $\dashv$

But where is the free action we are looking for? We have established the existence of a conull set of rigid subgroups, i.e. the automorphism groups of those groups consist exactly of two elements. Since we are acting by the natural action of the special linear group over the integers, the most natural thing to do is to factor these two elements out of the acting group; so finally, the desired group is the projective special linear group of rank two over the integers. The following facts clarify that this will indeed do.

Theorem 87. *For a countable group G and a normal subgroup Γ, we have that*

$$G/\Gamma \text{ is amenable and } \Gamma \text{ is amenable} \Longrightarrow G \text{ is amenable}.$$

Recall that $\mathrm{SL}_2(\mathbb{Z})$ is not amenable; this can be viewed as stemming from the fact that $\mathrm{SL}_2(\mathbb{Z})$ contains a free subgroup of rank two of finite index, namely the one generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} ;$$

to review the details, consult [Wag93]. Thus, by using Theorem 87 we observe the following for the projective special linear group over the integers of rank two.

Theorem 88. *The group $\mathrm{PSL}_2(\mathbb{Z})$ is not amenable.*

So the action of $\mathrm{PSL}_2(\mathbb{Z})$ gives rise to a non-hyperfinite countable Borel equivalence relation. Using the closure properties of hyperfinite actions summarised in Theorem 15, we conclude that the same holds for the original action of $\mathrm{GL}_2(\mathbb{Z})$.

6.4 All the other Finite Ranks ($n \geq 3$)

Ioana Superrigidity This section is meant as a short introduction to Ioana's superrigidity theorem. We omit the proof, only mentioning that its main tecnique can be thought of threading along the lines of Hjorth's theorem 35, however applied to a different setting.

Definition 89. *A measure preserving action is* `profinite` *if it is the inverse limit of a countable sequence of actions on finite spaces.*

A simple example is the following: assume you are given a countable group G together with an infinite descending chain of finite index subgroups $(G_n)_{n\in\omega}$. For each $n \in \omega$, take μ_n to be the counting measure on the right cosets G/G_n together with the action α_n by left multiplication. The factor maps are obtained by inclusion of cosets. The resulting limit action is ergodic and profinite.

Theorem 90 ([Ioa11]). *Let $G \curvearrowright (X, \mu)$ be a profinite ergodic measure preserving action on a standard probability space (X, μ), together with its invariant factor maps $\pi_n : X \to X_n$ for $n \in \omega$. Then for any cocycle $\alpha : G \times X \to L$ there are $n \in \omega$ and $a \in X_n$ such that the restriction of α to the action*

$$G_a \curvearrowright \pi_n^{-1}(a)$$

is cohomologous to a homomorphism. We remark that obviously profinite actions are far from being weakly mixing; also note that the untwisting is not on the whole space but on one of its «components» obtained from the factor maps of the inverse limit.

Some Abelian Group Theory For two abelian groups G, H, define the `tensor product` as

$$G \otimes H := \mathrm{F}(G \times H)/ \sim$$

where $F(G \times H)$ denotes the free group on $G \times H$ factored by the following relations denoted by $\sim$:

$$(g_1 + g_2, h) - (g_1, h) - (g_2, h)$$

$$(g, h_1 + h_2) - (g, h_1) - (g, h_2)$$

for any $g, g_1, g_2 \in G$ and any $h, h_1, h_2 \in H$. This is the same as the more common tensor product over modules, where each abelian group needs to be viewed as a $\mathbb{Z}$–module (scalar multiplication identities follow implicitly).

From now on we will always view a given countable torsion-free abelian group of rank $n \in \omega$ as an additive subgroup of $\mathbb{Q}^n$, via the natural identification.

Definition 91. *We will denote by* $\mathrm{R}^p(\mathbb{Q}^n)$ *the subset of* $\mathrm{R}(\mathbb{Q}^n)$ *consisting of all* p*–local groups.*

Definition 92. *An abelian group is* `p-local` *if*

$$n \geq 1 \wedge p \in \mathbb{P} \wedge q \neq p \Rightarrow A = q^n A\,.$$

In other words, A *is an* $\mathcal{R}_q$*–module for each prime* q *unequal to* p*.*

For our discussion of the Kurosh–Malcev invariants, it will be important to regard the vector space $\mathbb{Q}^n$ over $\mathbb{Q}$ as a subspace of the n–dimensional space of p–adic numbers $\mathbb{Q}_p^n$ over $\mathbb{Q}_p$. So, fixing p, given the standard basis $(\mathbf{e}_i)_{i \leq n}$ of $\mathbb{Q}_p^n$ over $\mathbb{Q}_p$, we can identify $\mathbb{Q}^n$ with

$$\bigoplus_{i \leq n} \mathbb{Q}\mathbf{e}_i\,.$$

Furthermore, we can extend the action $\mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathbb{Q}^n$ to an action

$$\mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathbb{Q}_p^n\,;$$

this allows us to regard $\mathrm{GL}_n(\mathbb{Q})$ as a subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Definition 93. $\mathrm{PG}(n-1, \mathbb{Q}_p)$ *denotes* `projective space` *over* $\mathbb{Q}_p^n$*, i.e. the space of all* 1*–dimensional subspaces of* $\mathbb{Q}_p^n$ *over* $\mathbb{Q}_p^n$*.*

We introduce the «quasi»-analogues to our relations and provide a discussion of the Kurosh–Malcev invariants.

Definition 94. *For any $A, B \in \mathrm{R}(\mathbb{Q}^n)$ we define the following: A is* `quasi-contained` *in B, or $A \prec_n B$ if there exists a positive $m \in \omega$ such that $mA \leq B$. If furthermore $A \prec_n B$ and $B \prec A$, we write $A \approx_n B$, A and B being* `quasi-equal`.

It follows that $A \approx_n B$ if and only if $A \cap B$ has finite index in both A and B.

Lemma 95. *Quasi-equality $\approx_n$ is a countable Borel equivalence relation on $\mathrm{R}(\mathbb{Q}^n)$.*

Proof. From the definition it follows immediately that $\approx_n$ is a Borel equivalence relation.

To see that it is countable, assume that $A, B \in \mathrm{R}(\mathbb{Q}^n)$ and $A \approx_n B$ with integers $r, s > 0$ that satisfy

$$rA \leq B \quad \bigwedge sB \leq A \ .$$

Then

$$rA \leq B \leq \frac{1}{s}A$$

and consequently

$$[\frac{1}{s}A : rA] = [A : rsA] < \infty \ ,$$

by which only countably many possibilities for B remain. $\dashv$

Definition 96. *Two groups $A, B \in \mathrm{R}(\mathbb{Q}^n)$ are* `quasi-isomorphic`, *$A \sim_n B$, if there is a transformation $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ such that*

$$\varphi(A) \approx_n B \ .$$

Using Lemma 95 we additionally get that for every $n \in \omega$ the relation $\sim_n$ is a countable Borel equivalence relation on $\mathrm{R}(\mathbb{Q}^n)$.

Definition 97. *A linear transformation $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ is a* `quasi-endomorphism` *of $A \in \mathrm{R}(\mathbb{Q}^n)$ if*

$$\varphi(A) \prec_n A \ .$$

Lemma 98. *1. $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ is a quasi-endomorphism of $A \in \mathrm{R}(\mathbb{Q}^n)$ if and only if there exists an integer $m > 0$ with $m\varphi \in \mathrm{End}(A)$.*

2. The collection of quasi-endomorphism of A is a $\mathbb{Q}$-subalgebra of $\mathrm{GL}_n(\mathbb{Q})$.

3. $A \approx_n B$ implies that the two associated algebras of quasi-endomorphisms coincide.

Definition 99. *The collection of all quasi-endomorphisms of $A \in \mathrm{R}(\mathbb{Q}^n)$ is denoted by $\mathrm{QE}(A)$. A linear transformation $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ is a* `quasi-automorphism` *if it is a unit in the $\mathbb{Q}$-algebra $\mathrm{QE}(A)$. We will denote the group of quasi-automorphisms of A by $\mathrm{QAut}(A)$.*

Lemma 100. *The setwise stabiliser of $[A]_{\approx_n}$ in $\mathrm{GL}_n(\mathbb{Q})$ is exactly $\mathrm{QAut}(A)$.*

Proof. First suppose that $\varphi \in \mathrm{QAut}(A)$. Then we know that there is an integer $m > 0$ such that $\psi := m\varphi$ is an endomorphism of A. Since ψ is obviously also a unit in $\mathrm{QE}(A)$ we can conclude that it is a monomorphism. Hence $\psi(A)$ has finite index in A and thus

$$\psi(A) \approx_n A\,.$$

By definition of ψ we also obtain

$$\psi(A) \approx_n \varphi(A)$$

and thus

$$\varphi(A) \approx_n A\,,$$

as desired.

For the converse, we assume that $\varphi \in \mathrm{GL}_n(\mathbb{Q})$ stabilises $[A]_{\approx_n}$. As this means $\varphi(A) \approx_n A$ it follows that for some integer $m > 0$ we have

$$m\varphi(A) \leq A\,.$$

Since additionally $m\varphi$ is a monomorphism, it follows that $m\varphi \in \mathrm{QAut}(A)$ hence $\varphi \in \mathrm{QAut}(A)$. $\dashv$

The Kurosh–Malcev Invariants

Definition 101. *For a prime p define*

$$\mathcal{R}_p := \{\frac{a}{b} \in \mathbb{Q} \mid b \text{ and } p \text{ are relatively prime}\}$$

i.e. the ring of rational numbers with denominator relatively prime to p.

Definition 102. *The* `ring of p-adic integers` *will be denoted by $\mathbb{Z}_p$, the* `field of p-adic numbers` *by $\mathbb{Q}_p$.*

We start by defining two kinds of localisations:

Definition 103. *For a torsion-free abelian group $A \in \mathrm{R}(\mathbb{Q}^n)$, define*

$$A_p := \mathcal{R}_p \otimes A$$

$$A_{\star p} := \mathbb{Z}_p \otimes A\,.$$

Furthermore we shall need the following result, which will lead to the existence of a basis for the following direct-sum–decomposition:

Theorem 104 ([Fuc73, §93.3])**.**

Observe that A_{*p} is a $\mathbb{Z}_p$–submodule of $\mathbb{Q}_p^n$; clearly $A_p \leq A_{\star p}$. Furthermore, we opt to regard both A_p and A_{*p} as lying inside of $\mathbb{Q}_p^n$ (which is certainly a divisible hull for A). Then, for example the elements of A_p can be written as finite sums of products qa, with $q \in \mathcal{R}_p$ and $a \in A$. We remark that these localisation constructions are a way of obtaining an extension of scalars (since $\mathbb{Z}$ clearly embeds into both rings above).

Suppose we have a torsion-free abelian group of rank n. Then the $\mathbb{Z}_p$–module A_{*p} is also of rank n, and by Theorem 104 its reduced part is free. This allows us to conclude that we can represent

$$A_{*p} = \bigoplus_{i \leq k_p} \mathbb{Q}_p v_i \oplus \bigoplus_{i \leq l_p} \mathbb{Z}_p w_i$$

for some $k_p, l_p \in \omega$ such that $k_p + l_p = n$ and all $v_i, w_i \in A_{*p}$.

Now take a maximal independent $\{a_i\}_{1\leq i\leq n}$ in A. We can thus write

$$a_i = \sum_{1\leq j\leq k_p} \alpha_{i,j} v_j + \sum_{1\leq j\leq l_p} \beta_{i,j} w_j \,,$$

where again $\alpha_{i,j} \in \mathbb{Q}_p$ and $\beta_{i,j} \in \mathbb{Z}_p$.

In summary we obtain an $n \times n$–matrix

$$M_p := \begin{pmatrix} \alpha^p_{1,1} & \cdots & \alpha^p_{1,k_p} & \beta^p_{1,1} & \cdots & \beta^p_{1,l_p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha^p_{n,1} & \cdots & \alpha^p_{n,k_p} & \beta^p_{n,1} & \cdots & \beta^p_{n,l_p} \end{pmatrix} ;$$

we furthermore know that these matrices are invertible over $\mathbb{Q}_p$, since the vectors

$$\{v_i, w_j\}_{1\leq i\leq k_p, 1\leq j\leq l_p}$$

form a maximal linear independent system in $A_p^\star$ over $\mathbb{Q}_p$.

It turns out that these matrix sequences together with the sequences of natural numbers $\{l_p\}_{p\in\mathbb{P}}$ and $\{k_p\}_{p\in\mathbb{P}}$ are invariants for the group A; more precisely there is an equivalence relation on the matrices in $\mathrm{M}_n(\mathbb{Q}_p)$ which serve as an invariant for the group. For the details consult [Fuc73, §93]. In fact, even the converse is true: for every sequence of matrices with p–adic entries (with the natural restriction that the last l_p entries always must be p–adic integers in some representant of the class) there exists a torsion-free abelian group of rank n with exactly that sequence as invariant in the construction above.

For our purposes, the main result of Kurosh–Malcev can now be summarized as the following:

Theorem 105 (Kurosh–Malcev)**.** *The map*

$$A \mapsto A_p^\star$$

is a $\mathrm{GL}_n(\mathbb{Q})$*–preserving bijection between the full-rank p–local subgroups of* $\mathbb{Q}^n$ *and the full-rank* $\mathbb{Z}_p$*–submodules of* $\mathbb{Q}_p^n$*. Furthermore, the inverse map is given* *by*

$$A_p^\star \mapsto A_p^\star \cap \mathbb{Q}^n \,.$$

Time for the Harvest We get now get to the core of the proof, which we present in the way given by S. Coskey in [Cos13]. In order to reach a contradiction, we assume now assume that $n \geq 2$ and that there is a reduction from $\cong_{n+1}$ to $\cong_n$. Then we will need to use the Kurosh–Malcev invariants in the following way.

Theorem 106. *The equivalence relation induced by the natural action* $\mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathbb{P}(\mathbb{Q}_p^n)$ *is Borel reducible to* $\approx_n$*.*

Proof. Start with a linear subspace $V \leq \mathbb{Q}_p^n$; then there exists a vector such that

$$V^\perp \oplus \mathbb{Z}_p v$$

is a full-rank submodule of $\mathbb{Q}_p^n$. By Lemma 6.5 we can now identify this module with an element of $\mathrm{R}(\mathbb{Q}^n)$.

To establish that this indeed a reduction; first of all one needs to check that the Kurosh–Malcev-construction is really $\mathrm{GL}_n(\mathbb{Q})$–preserving; then, the following fact completes the argument: assume

$$\dim(V) = \dim(V') = n - 1$$

and

$$V \oplus \mathbb{Z}_p v \text{ and } V' \oplus \mathbb{Z}_p v'$$

are full-rank modules for some v, v'; then the existence of an element of $\mathrm{GL}_n(\mathbb{Q})$ taking W into W' implies that there is an element of $\mathrm{GL}_n(\mathbb{Q})$ taking $V \oplus \mathbb{Z}_p v$ into $V' \oplus \mathbb{Z}_p v'$. $\dashv$

From this and the original assumption we infer that for the subrelation induced by $\mathrm{SL}_{n+1}(\mathbb{Z}) \curvearrowright \mathbb{Q}_p^n$ there exists a countable-to-one homomorphism of equivalence relations f into quasi-isomorphism $\sim_n$. It is furthermore easy to see that $\mathrm{SL}_{n+1}(\mathbb{Z}) \curvearrowright \mathbb{Q}_p^n$ is a profinite ergodic action.

First notice that there are only countably many possibilities for the $\mathbb{Q}$-algebra $\mathrm{QE}(f(x))$.

Hence there exists a Borel subset of positive measure $X' \subseteq X$ and a single fixed $\mathbb{Q}$-subalgebra S of $\mathrm{M}_n(\mathbb{Q})$ such that

$$\forall x \in X' \ \mathrm{QE}\left(f(x)\right) = S \ .$$

By ergodicity we know that the saturation of X' has full measure, i.e.

$$\mu(\mathrm{SL}_m(\mathbb{Z}).X') = 1 \ .$$

For the sake of notation let us assume that this saturation is actually the whole space X. Furthermore, since we may compose f with a function on X whose graph is contained in $E^X_{\mathrm{SL}_m(\mathbb{Z})}$, we are allowed to adjust f as to obtain

$$\forall x \in X \ \mathrm{QE}(f(x)) = S \ ,$$

i.e. the previous identity for the whole space.

That also means that for all $x \in X$ it holds that $\mathrm{QAut}(f(x)) = S^*$, i.e. the group of units of S.

If we pick $x, y \in X$ such that xEy, then also $f(x) \sim_n A(y)$, which by definition means that there is $\varphi \in \mathrm{GL}_n(\mathbb{Q}^n)$ which realises

$$\varphi(f(x)) \approx_n A(y) \ .$$

Conjugating S by φ clearly gives

$$\varphi S \varphi^{-1} = \varphi \mathrm{QE}(f(x)) \varphi^{-1} = \mathrm{QE}(\varphi(f(x))) = \mathrm{QE}(A(y)) = S \, .$$

In other words, φ is in the normalizer of S in $\mathrm{GL}_n(\mathbb{Q})$, which we shall call

$$N := \mathcal{N}_{\mathrm{GL}_n(\mathbb{Q})}(S) \ .$$

Clearly it holds that

$$\varphi([f(x)]_{\approx_n}) = [A(y)]_{\approx_n} ,$$

and by Lemma 100 we know that for each $x \in X$ the stabiliser of $f(x)$ in $\mathrm{GL}_n(\mathbb{Q})$ is simply given by $\mathrm{QAut}(f(x))$, which equals S^*. So we can define

$$H := {}^{N}\!/_{S^*} \quad \text{and for } \varphi \in N \text{ define } \bar{\varphi} := \varphi N . \tag{1}$$

Then, since this action has now become free, we can define the corresponding Borel cocycle

$$\alpha : \mathrm{SL}_m(\mathbb{Z}) \times X \to H$$

$$\alpha(g, x) := \text{the unique } \bar{\varphi} \in H \text{ such that } \varphi([f(x)]_{\approx_n} = [f(g.x)]_{\approx_n} .$$

Using Ioana's superrigidity theorem Theorem 90 we can now deduce that there exists a finite-index subgroup $\Gamma_0 \leq \mathrm{PSL}_{n+1}(\mathbb{Z})$, a positive measure set $X' \subseteq X$ and a homomorphism

$$\rho : \Gamma_0 \to H$$

which is cohomologous to α. By ergodicity this homomorphism must also be nontrivial.

The final word of Margulis By the theorem on normal groups of Margulis, ρ can assumed to be an embedding. Also note that the algebra $S \subseteq \mathrm{M}_n(\mathbb{Q})$ is definable from a vector space basis, we have that $S = \mathbb{S}(\mathbb{Q})$ for an algebraic $\mathbb{Q}$–group lying inside of $\mathrm{M}_n(\mathbb{Q})$. Basic facts from algebraic group theory now imply that also $N = \mathbb{N}(\mathbb{Q})$ and $S^* = \mathbb{K}'(\mathbb{Q})$ for $\mathbb{N}, \mathbb{K}'$ algebraic $\mathbb{Q}$–groups in $\mathrm{M}_n(\mathbb{Q})$. So

$$H = \mathbb{N}(\mathbb{Q}) / \mathbb{K}'(\mathbb{Q}) ,$$

which is furthermore contained in the algebraic $\mathbb{Q}$–group $\mathbb{N}/\mathbb{K}'$, and since for algebraic groups passing to subgroups or quotients decreases the dimension, we have that

$$\dim(\mathbb{N}/\mathbb{K}') \leq \dim(\mathrm{M}_n(\mathbb{Q})) = n^2 < (n+1)^2 - 1 = \dim(\mathrm{PSL}_{n+1}),$$

which is a contradiction due to Margulis superrigidity.

6.5 The Hyperfiniteness of the Quasi-Equality Relation $\approx_n$

The following non-trivial result is from [Lad79]. We will need it in order to show the hyperfiniteness, by first showing that for each prime p the quasi-equality relation restricted to p–local groups is smooth.

Theorem 107 (Lady)**.** *Two groups $A, B \in \mathrm{R}(\mathbb{Q}^n)$ are quasi-equal if and only if both of the following conditions are satisfied:*

1. *$A_p \approx_n B_p$ for all primes p*
2. *$A_p = B_p$ for all but finitely many primes p*

Definition 108. *The restriction of the quasi-equality relation to the space of p-local groups $\mathrm{R}^p(\mathbb{Q}^n)$ will be denoted by $\approx_n^p$.*

The heart of this section will be devoted to proving that for all primes the quasi-equality relation restricted to the corresponding p-local groups is in fact smooth. From this it will follow easily that the quasi-equality relation on the whole space $\mathrm{R}(\mathbb{Q}^n)$ is hyperfinite by directly finding a Borel reduction to a variant of E_0. We tread closely along the lines of [Tho03].

By the definition, we may regard $A_p^\star$ as the subgroup of $\mathbb{Q}_p^n$ consisting of all the finite sums of the form

$$\sum_{i \leq k} \gamma_i a_i \, ,$$

for some finite $k \in \omega$, where each $\gamma_i \in \mathbb{Z}_p$ and each $a_i \in A$.

By there exist natural numbers $k, l \in \omega$ with $k + l = n$ and group elements $(v_i)_{i<k}, (w_j)_{j<l}$ such that we can decompose $A_p^\star$ as

$$A_p^\star = \bigoplus_{i<k} \mathbb{Q}_p v_i \oplus \bigoplus_{j<l} \mathbb{Z}_p w_j \, .$$

Definition 109. *For a group $A \in \mathrm{R}^p(\mathbb{Q}^n)$ we shall call the first part of the decomposition V_A, i.e. for v_i as above, we define*

$$\mathrm{V}_A := \bigoplus_{i<k} \mathbb{Q}_p v_i \, .$$

and the second part will be denoted by L_A,

$$\mathrm{L}_A := \bigoplus_{i<k} \mathbb{Q}_p w_i \, .$$

The next theorem, which is a consequence of the appended lemmas, shows that in order to understand quasi-equality $\approx_n^p$ in $\mathrm{R}^p(\mathbb{Q}^n)$, it is enough to know about V_A.

Theorem 110. *For $A, B \in \mathrm{R}^p(\mathbb{Q}^n)$ we have the following equivalence:*

$$A \approx_n^p B \;\Leftrightarrow\; \mathrm{V}_A = \mathrm{V}_B \, .$$

We now prove two lemmas, which simply split the proof of the theorem.

Lemma 111. *For $A, B \in \mathrm{R}^p(\mathbb{Q}^n)$ we have the following equivalence:*

$$A \approx_n^p B \;\Leftrightarrow\; A_p^\star \approx_n^p B_p^\star \, .$$

Proof. Let us start with the assumption that $A_p^\star \approx_n^p B_p^\star$. Then by we have that $\mathbb{Q}^n$ takes out exactly A and B, i.e. $A = A_p^\star \cap \mathbb{Q}^n$ and $B = B_p^\star \cap \mathbb{Q}^n$. Thus we see that

$$[A : A \cap B] = [A_p^\star \cap \mathbb{Q}^n : (A_p^\star \cap B_p^\star) \cap \mathbb{Q}^n] \leq [A_p^\star : A_p^\star \cap B_p^\star] < \infty \, ,$$

and analogous reasoning gives

$$[B : A \cap B] < \infty\,,$$

hence by we can conclude

$$A \approx^p_n B\,.$$

Conversely, assume that $A \approx^p_n B$ and let $C := A \cap B$. Then also $C^\star_p \leq A^\star_p \cap B^\star_p$, and we can complete the proof by showing that $C^\star_p$ has finite index in both $A^\star_p$ and $B^\star_p$. In order to get there, we first define $F := A/C$ and obviously get the short exact sequence

$$0 \to C \to A \to F \to 0\,,$$

simply because of how these groups were defined. Furthermore, [Fuc73, Theorem 60.2] allows us to conclude that by taking products with $\mathbb{Z}_p$ we still have an exact sequence before us;

$$\mathbb{Z}_p \otimes C \to \mathbb{Z}_p \otimes A \to \mathbb{Z}_p \otimes F \to 0\,.$$

If we now decompose F into a finite direct sum of finite cyclic groups, say

$$F = \bigoplus_{r \leq s} \mathcal{C}_{m_r}\,,$$

[Fuc73, §59] tells us that

$$\mathbb{Z}_p \otimes F \cong \bigoplus_{r<s} \mathbb{Z}_p/(m_r\mathbb{Z}_p)\,.$$

And since this means that $\mathbb{Z}_p \otimes F$ is a finite group, it follows that $C^\star_p$ has finite index in $A^\star_p$; analogous reasoning gives finite index of $C^\star_p$ in $A^\star_p$, which completes the proof. $\dashv$

Lemma 112. *For $A, B \in \mathrm{R}^p(\mathbb{Q}^n)$ we have the following equivalence:*

$$A^\star_p \approx^p_n B^\star_p \;\Leftrightarrow\; \mathrm{V}_A = \mathrm{V}_B\,.$$

Proof. Start with $A_p^\star \approx_n^p B_p^\star$; then for each $v \in \mathrm{V}_A$ we calculate

$$[\mathbb{Q}_p v : B_p^\star \cap \mathbb{Q}_p v] = [A_p^\star \cap \mathbb{Q}_p v : A_p^\star \cap B_p^\star \cap \mathbb{Q}_p v] \leq [A_p^\star : A_p^\star \cap B_p^\star] < \infty\,.$$

Now suppose we have a $v \in \mathrm{V}_A \setminus \mathrm{V}_B$. Then $B_p^\star \cap \mathbb{Q}_p v$ is a proper $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p v$ and so there exists a nonzero $u \in \mathbb{Q}_p v$ such that $B_p^\star \cap \mathbb{Q}_p v = \mathbb{Z}_p u$. But as a consequence

$$\mathbb{Q}_p v/(B_p^\star \cap \mathbb{Q}_p v) \cong \mathbb{Q}_p/\mathbb{Z}_p \cong \mathcal{C}(p^\infty)\,,$$

which is a contradiction. Thus necessarily $\mathrm{V}_A \leq \mathrm{V}_B$, and through analogous arguments $\mathrm{V}_B \leq \mathrm{V}_A$, which means $\mathrm{V}_A = \mathrm{V}_B$.

Now suppose that the components of the decomposition are equal and define

$$\mathrm{V} := \mathrm{V}_A = \mathrm{V}_B\,.$$

Then there exist finitely many elements $(a_i)_{i<l}A_p^\star$, $(b_i)_{i<l}B_p^\star$ such that

$$A_p^\star = V \oplus \bigoplus_{i<l} \mathbb{Z}_p a_i\,,$$

$$B_p^\star = V \oplus \bigoplus_{i<l} \mathbb{Z}_p b_i\,.$$

Observe that we can identify L_A and L_B with the corresponding $\mathbb{Z}_p$–submodules of the l–dimensional $\mathbb{Q}_p$–vector space

$$W := \mathbb{Q}_p^n/V\,.$$

Furthermore there is $t \in \omega$ such that $p^t y_i \in \mathrm{L}_A$ for each $i \leq l$. As a consequence we can bound the index of the intersection by

$$[\mathrm{L}_B : \mathrm{L}_A \cap \mathrm{L}_B] \leq [\mathrm{L}_B : p^t \mathrm{L}_B] = p^{t \cdot l}\,,$$

a fact which proofs finiteness of the index. The same argument establishes

$$[\mathrm{L}_A : \mathrm{L}_A \cap \mathrm{L}_B] < \infty$$

and therefore $A_p^\star \approx_n^p B_p^\star$. $\dashv$

Recall here that the $\mathbb{Q}_p$ vector space $\mathbb{Q}_p^n$ is a separable complete metric space with respect to the metric induced by the p–adic norm. As mentioned in [Kec95], we therefore know that the Effros Borel $\mathrm{F}(\mathbb{Q}_p^n)$ space is a standard Borel space.

By that, it only remains to show that the map taking each group to the first part in the decomposition is a Borel map, i.e.

$$\mathrm{R}^p(\mathbb{Q}_p^n) \to \mathrm{F}(\mathbb{Q}_p^n)$$

$$A \mapsto \mathrm{V}_A$$

is Borel in order to complete the proof of hyperfiniteness of the unrestricted quasi-equality relation for rank n.

First of all, it is easy to see that the map

$$s : (\mathbb{Q}_p^n)^{\leq n} \to \mathrm{F}(\mathbb{Q}_p^n)$$

defined by sending an at most n–long sequence of vector to the spanned $\mathbb{Q}_p$–subspace, is indeed a Borel map. Hence it is enough to show that there exists a Borel map

$$b : \mathrm{R}^p(\mathbb{Q}_p^n) \to (\mathbb{Q}_p^n)^{\leq n}$$

which maps each group A to a basis of V_A.

Definition 113. *A finite sequence of nonzero elements $(a_i)_{i<l}$ in $A \in \mathrm{R}^p(\mathbb{Q}^n)$ is called* `p-independent` *if whenever*

$$\sum_{i<l} n_i a_i \in pA$$

for some $n_i \in \mathbb{Z}$, then $p|n_i$ for all $i < l$. A maximal p–independent such sequence is called a `p-basis`*.*

Let us start with some $A \in \mathrm{R}^p(\mathbb{Q}^n)$. Then we can clearly choose a

p–basis p-basis $(a_1, \dots, a_l)$ of A in a Borel way. Let $P := \langle a_1, \dots, a_l \rangle$ be the subgroup of A generated those basis elements. Then [Fuc73, §32] tells us that A/P is p–divisible and thus a divisible group. Hence we can represent

$$A/P = R \oplus T$$

with T being the torsion subgroup and R the direct sum of $k := n - l$ copies of $\mathbb{Q}$. Furthermore, by it is not difficult to see (as in [Fuc73, §93]) that we have $\dim \mathrm{V}_A = k$.

Continuing, we can again choose — in a Borel way — a sequence $(z_i)_{i<k}$ of elements of A such that

$$(z_1 P, \dots, z_k P$$

is a basis of R (which is isomorphic to $\mathbb{Q}^k$). To complete, we will use both sequences $(z_i)_{i<k}$ and $(a_i)_{i<l}$ to construct a basis, call it $(v_i)_{i<k}$ of V_A.

Fix some $i < k$ and let $t \geq 1$. Suppose inductively that there exist integers $c_s^j \in \omega$ with $j < l$ and $s < t$ that satisfy the following conditions:

1. $c_s^j < p$,

2. There exists $d_t \in A$ such that

$$p^t d_t = z_i + \sum_{i<l} n_t^i a_i\,,$$

   where

$$n_t^i := c_1^i + c_2^i p + \cdots + c_t^i p^{t-1}\,.$$

Since we know that R is divisible, there exists an element $d_{t+1} \in A$ and integers $(c_{t+1}^i)_{i<l}$ such that

$$p d_{t+1} = d_t + \sum_{i<l} c_{t+1}^i a_i\,,$$

and after adjusting our choice of d_{t+1} if necessary, we may assume that

$0 \leq c^i_{t+1} < p$ for $i < l$.

Then obviously

$$p^{t+1} d_{t+1} = z_i + \sum_{i<l} n^i_{t+1} a_i \,.$$

This means that the induction can be completed.

Next, for $j < l$, let

$$\gamma_j := \sum_{i<\omega} c^j_i p^i \in \mathbb{Z}_p \,.$$

Then the corresponding basis element of V_A is

$$v_i = z_i + \sum_{i<l} \gamma_i a_i \in A^\star_p \,.$$

The linear independence of $z_0, \ldots, z_{k-1}$ over $\mathbb{Q}_p$ implies that $v_0, \ldots, v_{l-1}$ are linearly independent as well. This means that it suffices to check that each $v_i \in p^t A^\star_p$ for $i < k$ and $t \geq 1$. To achieve that, fix some $i < k$ and consider (as before) the element

$$v_i = z_i + \sum_{i<l} \gamma_i a_i \in A^\star_p \,.$$

Then

$$p^t d_t = z_i + \sum_{j<l} n^j_t a_j$$

and

$$v_i - p^t d_t = \left(\sum_{r=t+1}^{\infty} c^1_r p^{r-1} \right) a_1 + \ldots + \left(\sum_{r=t+1}^{\infty} c^l_r p^{r-1} \right) a_l \,.$$

Hence

$$p^t e_t = v_i \,,$$

where

$$e_t := d_t + \left(\sum_{r=t+1}^{\infty} c^1_r p^{r-(t+1)} \right) a_1 + \ldots + \left(\sum_{r=t+1}^{\infty} c^l_r p^{r-(t+1)} \right) a_l \,.$$

So in fact, we have just proved the following

Theorem 114. *For each prime p the relation $\approx_n^p$ is smooth.*

From this, we can finally deduce the hyperfiniteness of the unrestricted quasi-equality.

Theorem 115. *The quasi-equality relation $\approx_n$ is hyperfinite for each $n \geq 1$.*

Proof. Define E_0^* as the equivalence relation on $2^{\mathbb{P}\times\omega}$ defined by

$$x\mathrm{E}_0^*y :\Leftrightarrow x(p,n) = y(p,n) \text{ for all but finitely many } (p,n) \in \mathbb{P}\times\omega\,.$$

Clearly this is just another version of E_0, so one can easily show that $\mathrm{E}_0^* \sim \mathrm{E}_0$.

Now being smooth allows us to get, for each prime p, an injective Borel map

$$g_p : \mathrm{R}^p(\mathbb{Q}^n) \to 2^\omega$$

that reduces $\approx_n^p$ to equality, i.e.

$$A \approx_n^p B \Leftrightarrow g_p(A) = g_p(B)\,.$$

Using this, we can define a new Borel map $f : \mathrm{R}(\mathbb{Q}^n) \to 2^{\mathbb{P}\times\omega}$ by

$$f(A)(p,n) := g_p(A_p)(n)\,.$$

Eventually we rejoice since Theorem 107 really tells us that this is a reduction to E_0^*, in other words, $\approx_n$ is hyperfinite. $\dashv$

6.6 Extending the Results to the space of all Finite Ranks

We now shift our attention to the space of countable torsion-free abelian groups of finite rank, without restriction to a fixed rank n. In other words, we take the countable sum over all ranks. The previous result subsection 6.4 that the complexity of the isomorphism relation increases strictly with the rank of the groups certainly implies that none of the relations $(\cong_n)_{n\in\omega}$ are countable universal. However, it might still be possible that these relations are complexity-wise cofinal in the poset of countable Borel equivalence relations. We will show here that this is not the case. This proof was given in [Tho09] and seems to crucially rest on Popa's super-rigidity theorem.

An ergodicity theorem for Bernoulli shifts Before going into the proof, we additionally need a theorem which will help us make use of the fact that quasi-equality $\approx_n$ is hyperfinite for any $n \geq 1$. Before proving the main theorem here, which says that certain Bernoulli shifts are E_0–ergodic, we need the following preparatory characterization of E_0–ergodicity.

We state the following two theorems without proof, only showing how to derive the E_0–ergodicity of shift actions of non-amenable groups from these.

Theorem 116 (Hulanicki). *A countable group G is amenable if and only if the left-regular action*

$$G \curvearrowright \ell^2(G)$$

has non-trivial almost invariant vectors.

Denote by $\mathrm{L}^2_0(X, \mu)$ those closed subspace of those functions $f \in \mathrm{L}^2(X, \mu)$ that satisfy

$$\int_X f \mathrm{d}\mu = 0\,.$$

We will need the following result on E_0–ergodicity:

Theorem 117 ([HK05, §A2,§A3]). *Let G be a a countable group acting in an ergodic measure preserving way a standard Borel probability space (X, μ).*

Then the fact that the corresponding G–action on $\mathrm{L}^2_0(X, \mu)$ has no non-trivial almost invariant sets implies that the induced equivalence relation on X is E_0–ergodic.

Using this we obtain the following

Theorem 118 (Losert–Rindler, Jones–Schmidt). *Let G be a countable group and $\Gamma \leq G$ a non-amenable subgroup. Consider the space 2^G equipped with the usual product measure μ and the shift action of*

$$\Gamma \curvearrowright 2^G .$$

Then the action of

$$\Gamma \curvearrowright \mathrm{L}^2_0(2^G, \mu)$$

has no nontrivial almost invariant vectors. As a consequence, the shift action of Γ on 2^G is E_0-ergodic.

Proof. First, we can view 2^G as a compact abelian group by identification with $(\mathbb{Z}/(2\mathbb{Z}))^G$. As a consequence, μ turns out to be Haar measure. The dual group, i.e. the group of characters $\hat{\mathbb{C}}_2^G$, can be identified with

$$a := \{\chi \in \mathbb{C}_2^G \mid \chi(g) = 0 \text{ for all but finitely many } g \in G\} ,$$

where the associated character is given by

$$\hat{\chi}(x) := (-1)^{\sum_{g \in G} \chi(g) x(g)} .$$

From here on we will work with this identification.

Observe moreover that

$$\int \chi(x) \mathrm{d}\mu(x) = 0$$

for any character χ and that the nonzero characters form a basis for the Hilbert space $\mathrm{L}^2_0(\mathbb{C}_2^G, \mu)$. To see this, first observe that orthonormality can easily be established by direct calculation. To establish that the whole space can be spanned by these, simply note that they separate parts and are also closed under multiplication, which allows us to apply the Stone–Weierstrass theorem.

Let Γ act on our set of characters. As every character can be viewed as a characteristic function of a finite nonempty subset of the group G, this action can in turn be identified with the left translation action of Γ on the finite nonempty subsets o G, which obviously implies that all stabilizers are finite. As a sidenote, we remark that in case of a torsion-free Γ, this action is even free. Enumerate all the orbits of this action of Γ on the characters by $(\mathcal{O}_n)_{n\in\omega}$, and furthermore choose a character from each of these orbits, calling it $\chi_n \in \mathcal{O}_n$.

Now let H_n denote the closed subspace of $\mathrm{L}^2_0(X,\mu)$ with basis all $\chi \in \mathcal{O}_n$. Then we get that

$$\mathrm{L}^2_0(\mathbb{C}_2^G,\mu) = \bigoplus_{n\in\omega} H_n \ ,$$

where furthermore each H_n is Γ–invariant.

Define as Γ_n the elements in Γ that fix χ_n. These are all finite subgroups of Γ.

Consider the quotient

$$\Gamma/\Gamma_n \ ,$$

and let Γ act on these left-cosets by left-translation. Again, shifting our perspective, this gives rise to a unitary action of Γ on

$$\ell^2(\Gamma/\Gamma_n)$$

by

$$\gamma.f(hG_n) := f(\gamma^{-1}hG_n)$$

for an $f \in \ell^2(\Gamma/\Gamma_n)$.

As an action on Hilbert spaces, this is easily isomorphic to the action of $\Gamma \curvearrowright H_n$.

In order to reach a contradiction, let us now assume that the action

$$\Gamma \curvearrowright \mathrm{L}^2_0(\mathbb{C}_2^G,\mu)$$

has nontrivial almost invariant vectors. The strategy is as follows; we first show that this implies that for any (ε, F) there is an $n \in \omega$ such that the action $\Gamma \curvearrowright H_n$ has an (ε, F)–invariant vector. This obviously means that the same is true for the action on $\ell^2(\Gamma/\Gamma_n)$. This will be enough to show that $\ell^2(\Gamma)$ has an (ε, F)–invariant vector, which is a contradiction.

Now for the details. Fix $\varepsilon > 0$ and a finite subset $F \subseteq \Gamma$. Choose a nonzero $x \in \mathrm{L}^2_0(\mathbb{C}_2^G, \mu)$ with

$$\forall \gamma \in F \; \frac{\|\gamma.x - x\|}{\|x\|} < \frac{\varepsilon}{|F|} \; .$$

Represent

$$x = \sum_{i \in \omega} x_i$$

with each $x_i \in H_i$. Then we see

$$\sum_{\gamma \in F} \|\gamma.x - x\|^2 < \varepsilon^2 \cdot \|x\|^2 \; ,$$

and consequently

$$\sum_{\gamma \in F} \sum_{n \in \omega} \|\gamma.x_n - x_n\|^2 < \varepsilon^2 \cdot \sum_{n \in \omega} \|x_n\|^2 \; ;$$

then, by interchanging the order of summation

$$\sum_{n \in \omega} \sum_{\gamma \in F} \|\gamma.x_n - x_n\|^2 < \varepsilon^2 \cdot \sum_{n \in \omega} \|x_n\|^2 \; .$$

So we have, for some $n \in \omega$, that

$$\sum_{\gamma \in F} \|\gamma.x_n - x_n\|^2 < \varepsilon^2 \|x_n\|^2$$

which means that

$$\forall \gamma \in F \; \frac{\|\gamma.x_n - x_n\|^2}{\|x_n\|^2} < \varepsilon^2 \, ,$$

or in other words, that x_n is (ε, F)–invariant.

In order to ascend to $\ell^2(\Gamma)$, first observe that no generality is lost in assuming that the vector we just found has unit norm. Then we get what we need by defining $u \in \ell^2(\Gamma)$ as

$$u(h) := \frac{1}{\sqrt{|\Gamma_n|}} \cdot v(h\Gamma_n) \; .$$

This u is a unit vector and since

$$\forall \gamma \in F \, (\|\gamma.u - u\| = \|\gamma.v - v\| < \varepsilon \, ,$$

we realize that this is the desired almost invariant vector. $\dashv$

Proving the non-Universality We cannot argue that the smooth disjoint union of countable non-universal Borel equivalence relations is again non-universal, since it is still not known whether this is true.

So assume that we have a reduction function

$$f : 2^G \to \bigcup_{n \in \omega} \mathrm{R}(\mathbb{Q}^n)$$

reducing the shift action on the free part of 2^G to the union of the isomorphism relations $\approx_n$ over all finite ranks; we will shortly specify the group G. Let the cocycle $\alpha : G \times \mathrm{Fr}(2^G) \to H$ be defined as before on page 84, i.e. by shifting the classes of quasi-equality $\approx_n$ obtained from the reduction function f.

Now take S to be a simple non-amenable group which does not embed into any of the countably many possibilities for H. This group exists, since we can start with a finitely generated non-amenable group T that does not embed into any of the countably possibilities for H and in turn take S to be a countable simple group into which T embeds.

Defining

$$G := \mathrm{SL}_3(\mathbb{Z}) \times S$$

and we can assume that our cocycle α is already a homomorphism from G to H, since we have shown this to be perfectly fine by Popa's cocycle superrigidty theorem. By the way we chose S, we know that

$$S \leq \ker(\alpha)$$

must hold. Thus for almost all $x \in \mathrm{Fr}(2^G)$ we obtain

$$[A_{g.x}]_{\approx_n} = [A_x]_{\approx_n} .$$

To put it in other words, this means that on a conull set the map

$$f : \mathrm{Fr}(2^G) \to \mathrm{R}(\mathbb{Q}^n)$$

is a Borel homomorphism between the equivalence relation induced by the S-action on $\mathrm{Fr}(2^G)$ and the quasi-equality relation $\approx_n$, which we know to be hyperfinite. By Theorem 118 and using the non-amenability of S, the action

$$S \curvearrowright \mathrm{Fr}(2^G)$$

is E_0-ergodic and hence the homomorphism f is μ–trivial (for the standard product measure), which contradicts its definition.

6.7 Further Results

In a series of papers by S. Coskey, G. Hjorth and S. Thomas culminating in [Tho11] it has been shown, among other results, that for $n \geq 2$ the isomorphism relations restricted to the p–local groups are incomparable for unequal primes:

Theorem 119. *If $p \neq q$ are primes, then for any natural $n \geq 2$ we have that*

$$\cong_n^p \not\leq \cong_n^q \quad \text{and } \cong_n^q \not\leq \cong_n^p .$$

This also means that there are countably many incomparable Borel equivalence relations between $\cong_n$ and $\cong_{n+1}$. With a bit more efford as above, even continuum many unequal equivalence relations can be constructed, lying between isomorphism of rank n and rank $n+1$. An even stronger result is known.

Theorem 120 (Thomas)**.** *Let S, S' be sets of primes and $n \geq 2$. Then the classification problem for the S–local torsion-free abelian groups of rank n is Borel reducible to the classification problem for the S'–local ones if and only if*

$$S \subseteq S' .$$

Here, a group is said to be S–local if it is p–divisible for all primes $p \notin S$.

It seems interesting to point out that for both results, the case where $n \geq 3$ has been solved using Ioana superrigidity, while the rank-2–case still relies on R. Zimmer's results in a critical way.

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