DISSERTATION

Titel der Dissertation

Simple enumeration formulae related to Alternating Sign Matrices, Monotone Triangles and standard Young tableaux

verfasst von

Dipl.-Ing. Lukas Riegler

angestrebter akademischer Grad

Doktor der Naturwissenschaften (Dr. rer. nat)

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Summary

Whenever enumerative combinatorialists are particularly fond of a counting problem, it will tend to have the following three features: First, it is easy to formulate. Second, the solution is a simple closed formula. Third, its proof poses a challenge to our current methods. One of the most intriguing examples of such a problem is given by Alternating Sign Matrices (ASMs), i.e. square matrices with entries $\{0, 1, -1\}$ where in each row and in each column the non-zero entries alternate in sign and sum up to 1. In 1983, a beautiful product formula for the number of $n \times n$-ASMs was conjectured by Mills, Robbins and Rumsey. It remained an open problem for over a decade until the first proof (in an 84-pages paper) by Zeilberger appeared in 1996. The corresponding counting sequence $1, 2, 7, 42, 429, 7436, 218348, \ldots$ (OEIS A005130) appears in the enumeration of a substantial variety of combinatorial objects, some of which are in simple bijection with ASMs, whereas for others no such correspondence is known until today.

In the first part of this thesis we put our focus on Monotone Triangles (MTs), which are in one-to-one correspondence with $n \times n$-ASMs if we fix the bottom row of the MT to be $(1, 2, \ldots, n)$. Fischer derived an operator formula for $\alpha(n; k_1, k_2, \ldots, k_n)$, a polynomial in each k_i which – evaluated at integers $k_1 < k_2 < \cdots < k_n$ – equals the number of MTs with fixed bottom row $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Fischer used the operator formula and additional properties of the polynomial to give an alternative proof of the Refined ASM Theorem in 2007. Since the introduced concepts and methods are the foundation for several subsequent results, we first present the most current version of this proof.

As computer experiments indicated the existence of miraculous identities satisfied by the α-polynomial – for example $\alpha(n; 1, 2, \ldots, n) = \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1)$ – we consider evaluations of the polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ at weakly decreasing sequences $k_1 \geq k_2 \geq \cdots \geq k_n$. As a first result we show that in this case the evaluation can be interpreted as signed enumeration of a new combinatorial object we call Decreasing Monotone Triangle (DMT). We then use this interpretation and the previously introduced method to give a proof of the identity $\alpha(n; 1, 2, \ldots, n) = \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1)$.

Such astonishing identities also appear when evaluating the polynomial at non-monotonous integer sequences. We therefore extend the interpretation of the α-polynomial to $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ by defining another new combinatorial object we call Generalized Monotone Triangle (GMT) –

a joint generalization of both ordinary MTs and DMTs. This result then allowed us to extend the combinatorially understood domain of a simple identity and give a combinatorial interpretation to several conjectural identities, including a surprising generalization of $\alpha(n; 1, 2, \ldots, n) = \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1)$.

After that, we tackle a long-standing conjecture on a refined enumeration of vertically symmetric ASMs (VSASMs). In 2008, Fischer conjectured a product formula for $B_{n,i}$, the number of $(2n+1) \times (2n+1)$-VSASMs where the unique 1 in the left half of the second row is in column $i \in \{1, 2, \ldots, n\}$. Our attempt to prove the conjecture using the introduced method required us to extend the combinatorial interpretation of $B_{n,i}$ to $i \in \{1, 2, \ldots, 2n\}$. We are able to combinatorially explain the symmetry $B_{n,n+i} = B_{n,n+1-i}$, $i = 1, \ldots, n$. However, the derivation of a related system of linear equations leads to a more general conjectural multivariate Laurent polynomial identity, which has so far resisted our approaches. The startling fact about this conjecture is that it is expressible only in terms of elementary mathematical concepts independent from ASMs or related combinatorial objects.

The final part of the thesis is devoted to another remarkable product formula, namely the hook-length formula for counting standard Young tableaux of fixed shape. By applying a sorting algorithm called jeu de taquin, Novelli, Pak and Stoyanovskii gave a bijective proof of the hook-length formula in 1997. We consider a natural extension of jeu de taquin to arbitrary posets, where jeu de taquin transforms each labeling into a (dual) linear extension. In particular, we study jeu de taquin on the so-called double-tailed diamond poset $D_{m,n}$ having two possible linear extensions. The counting problem how many times either of the two linear extensions is obtained leads to an interesting statistic on permutations generalizing right-to-left-minima. We derive an explicit formula for the distribution implying that uniform distribution is obtained if and only if $m \geq n$. In all cases we are also able to explain this result combinatorially by defining appropriate sign-reversing involutions. Finally, we observe that the extended hook-length formula for counting linear extensions on d-complete posets can be applied to provide an answer to a seemingly unrelated question, namely: Given a uniformly random standard Young tableau of fixed shape, what is the expected value of the left-most entry in the second row?

Zusammenfassung

Ein in der Abzählkombinatorik beliebtes Problem zeichnet sich meist durch drei Eigenschaften aus: Erstens lässt sich das Problem leicht verständlich formulieren, zweitens hat die Abzählformel eine einfache, geschlossene Form und drittens stellt das Beweisen der Formel mit den uns zur Verfügung stehenden Methoden eine Herausforderung dar. Eines der faszinierendsten Probleme mit diesen Eigenschaften wird von alternierenden Vorzeichenmatrizen gestellt. Dies sind quadratische Matrizen mit Einträgen aus $\{0, 1, -1\}$, sodass sich in jeder Zeile und in jeder Spalte die von Null verschiedenen Einträge abwechseln und zu 1 aufsummieren. Für die Anzahl der alternierenden Vorzeichenmatrizen mit n Zeilen stellten Mills, Robbins und Rumsey im Jahr 1983 eine Vermutung in Form einer Produktformel auf. Über ein Jahrzehnt später gelang es Zeilberger im Jahr 1996 diese Produktformel (in einem Paper mit 84 Seiten) zu beweisen. Die zugehörige Abzählfolge $1, 2, 7, 42, 429, 7436, 218348, \ldots$ (OEIS A005130) tritt auch bei einer Vielzahl anderer Abzählprobleme auf, wobei nur manche dieser Zusammenhänge bisher bijektiv erklärt werden konnten.

Im ersten Teil der Dissertation behandeln wir monotone Dreiecke, welche mit fixer unterster Zeile $(1, 2, \ldots, n)$ umkehrbar eindeutig den alternierenden Vorzeichenmatrizen mit n Zeilen entsprechen. Fischer leitete eine Operatorformel für das Polynom $\alpha(n; k_1, k_2, \ldots, k_n)$ her, dessen Auswertung an ganzzahligen Stellen $k_1 < k_2 < \cdots < k_n$ gleich der Anzahl an monotonen Dreiecken mit unterster Zeile $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ ist. Unter Anwendung dieser Operatorformel und zusätzlicher Eigenschaften des Polynoms veröffentlichte Fischer im Jahr 2007 einen alternativen Beweis für die verfeinerte Abzählung von alternierenden Vorzeichenmatrizen hinsichtlich der Position des eindeutigen Eintrags 1 in der ersten Zeile. Da die eingeführten Konzepte und Methoden die Grundlage für einige weitere Resultate bilden, präsentieren wir zunächst die aktuellste Version dieses Beweises.

Nachdem Computerexperimente darauf hindeuteten, dass das α-Polynom erstaunliche Identitäten wie zum Beispiel $\alpha(n; 1, 2, \ldots, n) = \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1)$ erfüllt, betrachten wir Auswertungen des Polynoms $\alpha(n; k_1, k_2, \ldots, k_n)$ an ganzzahligen Stellen $k_1 \geq k_2 \geq \cdots \geq k_n$. Als erstes Resultat zeigen wir, dass die Auswertung in diesem Fall eine Interpretation als vorzeichenbehaftete Abzählung eines neuen kombinatorischen Objekts namens „Decreasing Monotone Triangle“ besitzt. Wir wenden daraufhin diese Interpretation und die zuvor eingeführte Methode an, um die Identität $\alpha(n; 1, 2, \ldots, n) = \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1)$ zu beweisen.

Solche eindrucksvollen Identitäten treten auch dann auf, wenn das Polynom an ganzzahligen, aber nicht notwendigerweise monotonen Stellen ausgewertet wird. Daher erweitern wir die Interpretation des α-Polynoms auf $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, indem wir ein weiteres kombinatorisches Objekt namens „Generalized Monotone Triangle" definieren, welches eine Verallgemeinerung von monotonen Dreiecken und „Decreasing Monotone Triangles" darstellt. Dieses Resultat ermöglicht uns einerseits die Domäne, auf der eine einfache Identität bereits kombinatorisch verstanden war, auf $\mathbb{Z}^n$ zu erweitern. Andererseits verleiht das Resultat einigen vermuteten Identitäten – einschließlich einer bewundernswerten Verallgemeinerung von $\alpha(n; 1, 2, \ldots, n) = \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1)$ – eine mögliche kombinatorische Bedeutung.

Im Anschluss betrachten wir eine langjährige Vermutung zu einer verfeinerten Abzählung von vertikal symmetrischen, alternierenden Vorzeichenmatrizen. Im Jahr 2008 vermutete Fischer eine Produktformel für $B_{n,i}$, die Anzahl solcher Matrizen mit $2n+1$ Zeilen, bei denen sich der eindeutige Eintrag 1 in der linken Hälfte der zweiten Zeile in Spalte $i \in \{1, 2, \ldots, n\}$ befindet. Unser Versuch, die Vermutung mit der gleichen Methode zu beweisen, erfordert eine Erweiterung der kombinatorischen Bedeutung von $B_{n,i}$ auf $i \in \{1, 2, \ldots, 2n\}$. Die dadurch entstandene Symmetrie $B_{n,n+i} = B_{n,n+1-i}$, $i = 1, \ldots, n$, können wir bijektiv erklären. Durch Herleitung eines zugehörigen linearen Gleichungssystems lässt sich die Vermutung schließlich auf eine Identität eines multivariaten Laurent-Polynoms reduzieren. Diese neue Vermutung ist unabhängig von alternierenden Vorzeichenmatrizen oder verwandten kombinatorischen Objekten und sollte daher einem großen Publikum zugänglich sein.

Der letzte Teil der Dissertation ist einer weiteren bemerkenswerten Produktformel gewidmet, nämlich der Hakenlängenformel für die Anzahl der „standard Young tableaux" einer fixen Partition. Im Jahr 1997 veröffentlichten Novelli, Pak und Stoyanovskii einen bijektiven Beweis für die Hakenlängenformel unter Anwendung des Sortieralgorithmus „jeu de taquin". Wir betrachten eine naheliegende Verallgemeinerung von „jeu de taquin" auf teilweise geordneten Mengen, welche jedes Labeling in eine (duale) lineare Erweiterung transformiert. Insbesondere untersuchen wir den Sortieralgorithmus auf dem sogenannten „double-tailed diamond" $D_{m,n}$, der zwei mögliche lineare Erweiterungen besitzt. Das Abzählproblem, wie oft man welche der beiden linearen Erweiterungen als Ergebnis der Sortierung erhält, führt zu einer interessanten Statistik auf Permutationen, die Rechts-Links-Minima verallgemeinert. Wir leiten eine explizite Formel für die Verteilung her, die insbesondere eine Gleichverteilung genau im Fall $m \geq n$ impliziert. In allen Fällen können wir die erhaltene Verteilung auch mittels geeigneter vorzeichenumkehrender Involutionen erklären. Schließlich zeigen wir, dass die Hakenlängenformel für lineare Erweiterungen von „d-complete posets" zur Beantwortung einer scheinbar unabhängigen Frage verwendet werden kann: Welchen Wert nimmt der erste Eintrag in der zweiten Zeile bei Betrachtung aller „standard Young tableaux" einer fixen Partition durchschnittlich an?

Chapter 1

Introduction

We start this first chapter by reviewing the amazing story of Alternating Sign Matrices in Section 1.1. In order to avoid distraction, the definitions of all mentioned combinatorial objects are deferred to Section 1.2.

1.1 Timeline

The counting sequence

$$1, 2, 7, 42, 429, 7436, 218348, 10850216, \ldots \tag{1.1.1}$$

has attracted the attention of combinatorialists throughout the past decades. In 1979, it first appeared in a work by Andrews on so-called Descending Plane Partitions (DPPs). He showed that the number of DPPs where no part exceeds n is equal to [And79, Th. 10]

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = \frac{1!\,4!\cdots(3n-2)!}{n!\,(n+1)!\cdots(2n-1)!}, \quad n \geq 1, \tag{1.1.2}$$

i.e. a product formula yielding exactly the numbers in (1.1.1). This already happened before the birth of Alternating Sign Matrices (ASMs), which Robbins encountered when doing computer experiments on a generalization of determinants: Recall that the ordinary determinant of a matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ is

$$\det M = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} m_{i,\sigma(i)}, \tag{1.1.3}$$

and can be computed recursively using Dodgson's condensation algorithm. At the heart of the algorithm is the Desnanot-Jacobi adjoint matrix theorem, which states that

$$\det M \, \det\left(M_{1,n}^{1,n}\right) = \det\left(M_1^1\right) \det\left(M_n^n\right) - \det\left(M_1^n\right) \det\left(M_n^1\right), \quad n \geq 3,$$

whereby subscripts (resp. superscripts) define which rows (resp. columns) are deleted from the matrix. In other words, we can reduce the computation of an $n \times n$-determinant to the computation of $(n-1) \times (n-1)$ and $(n-2) \times (n-2)$-determinants.

Robbin's generalization of the determinant introduces an additional parameter λ. The λ-determinant of a matrix is then inductively defined by

$$\begin{aligned}
\det_\lambda(m_{1,1}) &:= m_{1,1},\\
\det_\lambda\begin{pmatrix} m_{1,1} & m_{1,2}\\ m_{2,1} & m_{2,2}\end{pmatrix} &:= m_{1,1}m_{2,2} + \lambda\, m_{1,2}m_{2,1},\\
\det_\lambda(M) &:= \frac{\det_\lambda\left(M_1^1\right)\det_\lambda\left(M_n^n\right) + \lambda\det_\lambda\left(M_1^n\right)\det_\lambda\left(M_n^1\right)}{\det_\lambda\left(M_{1,n}^{1,n}\right)}, \quad n \geq 3.
\end{aligned}$$

Robbins then conjectured (and proved years later together with Rumsey [RR86, (27)]) that λ-determinants satisfy an analogue of (1.1.3), but instead of summing over all permutations (or equivalently permutation matrices), one has to sum over all Alternating Sign Matrices of size n:

$$\det_\lambda(M) = \sum_{A\in\mathcal{A}_n} \lambda^{I(A)}(1+\lambda^{-1})^{N(A)} \prod_{i,j=1}^{n} m_{i,j}^{a_{i,j}}, \tag{1.1.4}$$

whereby $\mathcal{A}_n$ denotes the set of $n\times n$-ASMs, $N(A)$ the number of (-1)s in the ASM and $I(A) = \sum\limits_{\substack{1\leq i'<i\leq n,\\ 1\leq j<j'\leq n}} a_{i,j}\, a_{i',j'}$ a natural generalization of the inversion number.

If we set $\lambda = -1$ in (1.1.4), then the only summands not vanishing are those with $N(A) = 0$. Since ASMs without (-1)s are exactly the permutation matrices, (1.1.4) is indeed a generalization of (1.1.3). The natural question one may ask at that point is whether the number of summands in (1.1.4) is essentially larger than $n!$, i.e. the number of summands in (1.1.3)? Computer experiments indicated that the number of summands for the first few values of n are exactly given by the sequence in (1.1.1).

In 1983, Mills, Robbins and Rumsey published [MRR83] several conjectures: First of all they conjectured that the number of ASMs of size n is given by

$$A_n := |\mathcal{A}_n| = \prod_{i=0}^{n-1}\frac{(3i+1)!}{(n+i)!}, \tag{1.1.5}$$

which is referred to as ASM Conjecture (resp. ASM Theorem since 1996). Then they also introduced a refinement of this conjecture, namely that the number of ASMs of size n with their first row's unique 1 located in column i is equal to

$$A_{n,i} = \frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{n-1}} A_n, \tag{1.1.6}$$

the Refined ASM Conjecture (resp. Refined ASM Theorem since 1996). Now, if there are supposed to be the same number of ASMs of size n as DPPs with no parts exceeding n, one might expect a combinatorial explanation for this, i.e. a bijection between the two sets. Even though Mills, Robbins and Rumsey were able to conjecturally identify three statistics of DPPs that correspond to the number of (-1)s in the ASM, the inversion number and the position of the unique 1 in the first row [MRR83, Conj. 3], no such bijection in full generality has been found so far (for a bijection restricted to permutation matrices see [Ayy10, Str11]). Recently, a computational proof was given in [BFZJ12] and even more refined by an additional statistic in [BFZJ13]. Back in the 1980s however, trying to prove the equinumerosity of ASMs of size n and DPPs with parts smaller than or equal to n turned out to be a dead end.

It was a connection with a different kind of plane partition that should eventually lead to the first proof of the ASM Conjecture: In 1986, Mills, Robbins and Rumsey conjectured [MRR86, Conj. 1] that the number of totally symmetric self-complementary plane partitions (TSSCPPs) inside a $(2n) \times (2n) \times (2n)$-box is given by the same product formula $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$. Moreover, they observed that these TSSCPPs are in bijective correspondence with certain triangular arrays of integers, which Zeilberger later on called n-Magog triangles. This seemed promising since Mills, Robbins and Rumsey already observed in [MRR83] that ASMs of size n are also in one-to-one correspondence with certain triangular arrays of integers, namely Monotone Triangles (MTs) with bottom row $(1, 2, \ldots, n)$, which Zeilberger later renamed to n-Gog triangles. After Andrews managed to prove [And94, Th. 1] the TSSCPP conjecture in the early 90s, the ASM conjecture was reduced to showing that the number of n-Magog triangles and the number of n-Gog triangles coincides. Even though a bijective proof of this has not been found so far, Zeilberger eventually succeeded in finding a computational proof of it in an 84-pages paper [Zei96a]. But with this first proof of the ASM Conjecture, the story was far from over.

Soon after, a connection with a well-studied model in statistical mechanics was discovered: Kuperberg observed that configurations of the six-vertex model with domain wall boundary condition bijectively correspond to ASMs. This connection allowed him to apply the developed machinery and give an alternative, shorter proof of the ASM Theorem [Kup96]. Afterwards, Zeilberger also applied this method [Zei96b] to prove the Refined ASM Conjecture (1.1.6). For a more detailed account it is highly recommended to consider Bressoud's book [Bre99].

With the two main conjectures resolved, further questions regarding different aspects of ASMs arose: One such aspect is the enumeration of symmetry classes of ASMs, e.g. how many vertically symmetric ASMs of size n are there? As it turned out, the solution of many counting problems involving symmetry classes of ASMs can also be expressed in terms of a product formula (see for example [Kup02, RS06a, RS06b, Oka06]). Another aspect are further refined enumerations of ASMs, e.g. how many ASMs of size n are there if we fix the position of the unique 1 in the first row, first column, last row and last column at the same time? This quadruply refined enumeration of ASMs was recently solved by Ayyer and Romik [AR13] as well as Behrend [Beh13] in 2013.

But also combinatorial objects different from ASMs that are counted by (1.1.1) admit a variety of refined enumerations, generalizations and symmetry properties [Pro01]. One particularly beautiful example for this are the so-called Fully Packed Loop configurations on a square grid and their rotational symmetry due to Wieland [Wie00]. In the present thesis our main focus is put on Monotone Triangles, which bijectively correspond to ASMs of size n if we fix the bottom row to be $(1, 2, \ldots, n)$. In 2006, Fischer derived an operator formula [Fis06] for the number of Monotone Triangles with general bottom row $(k_1, k_2, \ldots, k_n)$, which subsequently allowed her to give an alternative proof of the Refined ASM Theorem [Fis07] in 2007.

1.2 Some combinatorial objects counted by $1, 2, 7, 42, 429, \ldots$

1.2.1 Alternating Sign Matrices

Definition 1.2.1. *An Alternating Sign Matrix (ASM) of size n is an $n \times n$-matrix with entries in $\{0, 1, -1\}$ such that*

- *the sum of the entries in each row and in each column is 1 and*
- *the non-zero entries in each row and in each column alternate in sign.*

Let $\mathcal{ASM}(n)$ denote the set of ASMs of size n and A_n its cardinality. From the definition it follows that permutation matrices are ASMs, but starting from $n = 3$ there are more ASMs than permutation matrices:

Example 1.2.2. *The seven ASMs of size 3 are*

$$\mathcal{ASM}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

and one of the 429 ASMs of size 5 is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

From the definition of ASMs it directly follows that the first row cannot contain a -1 and therefore contains exactly one non-zero entry, namely a 1. Let $A_{n,i}$ denote the number of ASMs of size n where the first row's unique 1 is located in column i. Reflection along the vertical symmetry axis is a bijection between the set of ASMs counted by $A_{n,i}$ and those counted by $A_{n,n+1-i}$, hence

$$A_{n,i} = A_{n,n+1-i}. \tag{1.2.1}$$

If an ASM of size n contains a 1 in the top-left corner, then removing the first row and column yields an ASM of size $n-1$. Conversely, appending such a row and column to an ASM of size $n-1$ gives an ASM of size n with a 1 in the top-left corner (see Figure 1.1). Hence,

$$A_{n,1} = A_{n-1} = \sum_{i=1}^{n-1} A_{n-1,i}. \tag{1.2.2}$$

The definition of ASMs implies that each partial row sum, i.e. $\sum_{j'=1}^{j} a_{i,j'}$, and each partial column sum, i.e. $\sum_{i'=1}^{i} a_{i',j}$, is either 0 or 1. Even though an ASM contains only three different kinds of entries, namely 1s, 0s and -1s, there are six different kinds of entries from the viewpoint of partial row/column sums: If $a_{i,j} = 1$, then the corresponding partial row/column sum is 1. If $a_{i,j} = -1$, then the corresponding partial row/column sum is 0. However, if $a_{i,j} = 0$, then the corresponding partial row sum can be either 0 or 1 and the same is true for the corresponding partial column

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure 1.1: Combinatorial explanation of (1.2.2).

sum, i.e. there are four different kinds of 0s. Given an ASM A, let $z_1(A)$ (resp. $z_2(A)$) denote the number of 0s of A where both the partial row sum and column sum is 0 (resp. 1), and let $z_3(A)$ (resp. $z_4(A)$) denote the number of 0s where the partial row sum is 0 (resp. 1) and the partial column sum is 1 (resp. 0). For example, the ASM of size 5 in Figure 1.1 yields $z_1(A) = z_2(A) = 2$ and $z_3(A) = z_4(A) = 6$. As it turns out, every ASM A satisfies $z_1(A) = z_2(A)$ and $z_3(A) = z_4(A)$: To see this, consider the corresponding corner-sum matrix $C = (c_{i,j})_{0 \le i,j \le n}$ and the height-function matrix $H = (h_{i,j})_{0 \le i,j \le n}$ defined by

$$\begin{aligned} c_{0,j} &= c_{i,0} = 0, \\ c_{i,j} &= \sum_{\substack{1 \le i' \le i \\ 1 \le j' \le j}} a_{i',j'}, \quad i,j \ge 1, \\ h_{i,j} &= i + j - 2c_{i,j}. \end{aligned}$$

For example

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 & 2 & 3 \\ 3 & 2 & 1 & 2 & 3 & 2 \\ 4 & 3 & 2 & 1 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}.$$

Since each row and column of an ASM sums up to 1, the right-most column and bottom row of the corner-sum matrix is $(0, 1, \ldots, n)$, and therefore the right-most column and bottom row of H is $(n, n-1, \ldots, 0)$. Since the first row and column of C is by definition $(0, 0, \ldots, 0)$, the first row and column of H is $(0, 1, \ldots, n)$. The entries of the height-function matrix further satisfy by definition

$$h_{i,j} = h_{i-1,j-1} + 2(1 + c_{i-1,j-1} - c_{i,j}), \quad i,j \ge 1.$$

If $a_{i,j} = 0$ with partial row and partial column sum equal to 0, then $c_{i,j} = c_{i-1,j-1}$ and therefore $h_{i,j} = h_{i-1,j-1} + 2$. If $a_{i,j} = 0$ with partial row and partial column sum equal to 1, then $c_{i,j} = c_{i-1,j-1} + 2$ and therefore $h_{i,j} = h_{i-1,j-1} - 2$. In all other cases $c_{i,j} = c_{i-1,j-1} + 1$ and therefore $h_{i,j} = h_{i-1,j-1}$. Together with the previous observation regarding the border of H, it follows that along each of the diagonals, the number of 0s contributing to $z_1(A)$ has to be equal to the number of 0s contributing to $z_2(A)$ (see Figure 1.2). In particular, $z_1(A) = z_2(A)$. Analogously, one can

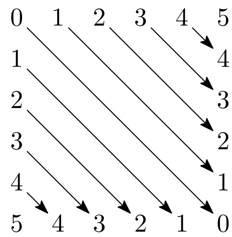


Figure 1.2: Increments by 2 along a diagonal correspond to 0s counted by $z_1(A)$. Decrements correspond to 0s counted by $z_2(A)$.

observe that each anti-diagonal of A has to contain the same number of 0s contributing to $z_3(A)$ as 0s contributing to $z_4(A)$.

Coming back to the question, whether there are essentially more ASMs than permutation matrices: The number of ASMs of size n is trivially bounded by $n! \leq A_n \leq 3^{n^2}$. From the ASM Theorem and Stirling's formula one can deduce that (see Lemma A.1.1)

$$\lim_{n\to\infty} A_n^{\frac{1}{n^2}} = \frac{3\sqrt{3}}{4} \approx 1.299,$$

i.e. an affirmative answer to the question (see also [BF06] for more refined asymptotics).

1.2.2 Monotone Triangles

Definition 1.2.3. *A Monotone Triangle (MT) of size n is a triangular array of integers $(a_{i,j})_{1\leq j\leq i\leq n}$ arranged in the form*

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \iddots & & & & \ddots & \\ a_{n,1} & & \cdots & & \cdots & & a_{n,n} \end{array}$$

such that

- *entries in each row are strictly increasing, i.e. $a_{i,j} < a_{i,j+1}$ and*
- *each entry is weakly between its two bottom neighbours, i.e. $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$.*

To obtain a feasible counting problem one can fix the entries in the bottom row and ask for the number of MTs with this bottom row. Let $\mathcal{MT}(k_1, k_2, \ldots, k_n)$ denote the set of MTs with bottom row $(k_1, k_2, \ldots, k_n)$ and $\alpha(n; k_1, k_2, \ldots, k_n)$ its cardinality.

Example 1.2.4. *The seven MTs with bottom row $(1, 2, 3)$ are*

$$\mathcal{MT}(1,2,3) = \left\{ \begin{smallmatrix} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{smallmatrix}, \begin{smallmatrix} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{smallmatrix}, \begin{smallmatrix} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{smallmatrix}, \begin{smallmatrix} & & 3 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{smallmatrix}, \begin{smallmatrix} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{smallmatrix}, \begin{smallmatrix} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{smallmatrix} \right\},$$

and one of the 429 *MTs with bottom row* $(1, 2, 3, 4, 5)$ *is*

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}.$$

As a result by Mills, Robbins and Rumsey there is the same number of ASMs of size n and MTs with bottom row $(1, 2, \dots, n)$, i.e.

$$\alpha(n; 1, 2, \dots, n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{1.2.3}$$

Proposition 1.2.5 ([MRR83])**.** *The set of Monotone Triangles with bottom row* $(1, 2, \dots, n)$ *is in one-to-one correspondence with the set of Alternating Sign Matrices of size n via the following mapping:*

$$\text{Row } i \text{ of MT contains entry } j. \quad \Leftrightarrow \quad \text{Top } i \text{ entries in column } j \text{ of ASM sum up to 1.} \tag{1.2.4}$$

An example can be seen in Figure 1.3.

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array} \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 1.3: A MT with bottom row $(1, 2, 3, 4, 5)$ and its corresponding ASM of size 5.

Proof. Let us first show that for each ASM A of size n the construction indeed yields a MT M. By definition of ASMs all partial column sums are either equal 0 or 1. Since the entries in each of the first i rows sum up to 1, the i-th row of M contains exactly i different numbers of $\{1, 2, \dots, n\}$. By construction these are ordered in increasing order in each row, so it remains to check that entries along North-East- and South-East-diagonals are weakly increasing. For this, let $c_{i,j}$ count in how many of the first j columns the first i rows sum up to 1. From the definition of ASMs it follows that for each fixed j we have $c_{1,j} \leq c_{2,j} \leq \cdots \leq c_{n,j}$. By construction $c_{i,j}$ is the number of entries $\leq j$ in the i-th row of M and therefore entries along NE-diagonals of M are weakly increasing. If one analogously considers the last j columns of A, then the number of those columns with partial column sum equal to 1 is also weakly increasing row by row. Therefore the number of entries $\geq j$ in M is weakly increasing row by row, thus implying the weak increase along SE-diagonal.

Conversely, let M be a MT with bottom row $(1, 2, \dots, n)$. Then the i-th row of M contains exactly i different integers of $\{1, 2, \dots, n\}$. By construction the corresponding matrix A contains only entries in $\{-1, 0, 1\}$ such that in each column the non-zero entries alternate in sign and sum up to 1. The weak increase along NE-diagonals (resp. SE-diagonals) implies that the number of entries $\leq j$ (resp. the number of entries $\geq j$) in M is weakly increasing row by row. In each row of A the the non-zero entries therefore alternate in sign and sum up to 1, i.e. A is an ASM. $\square$

1.2.3 Descending Plane Partitions

Definition 1.2.6. *A Descending Plane Partition (DPP) is an array* $(D_{i,j})_{\substack{1\leq i\leq r,\\ i\leq j\leq \lambda_i+i-1}}$ *of positive integers arranged in the form*

$$\begin{array}{ccccccc} D_{1,1} & D_{1,2} & D_{1,3} & & \cdots & & D_{1,\lambda_1} \\ & D_{2,2} & D_{2,3} & & \cdots & & D_{2,\lambda_2+1} \\ & & \ddots & & & \iddots & \\ & & & D_{r,r} & \cdots & D_{r,\lambda_r+r-1} & \end{array}$$

such that

- *entries in each row are weakly decreasing, i.e.* $D_{i,j} \geq D_{i,j+1}$*,*
- *entries in each column are strictly decreasing, i.e.* $D_{i,j} > D_{i+1,j}$ *and*
- *the number of entries in each row is strictly less than the first entry in the same row and at least as large as the first entry in the following row, i.e.* $D_{1,1} > \lambda_1 \geq D_{2,2} > \lambda_2 \geq \cdots \geq D_{r-1,r-1} > \lambda_{r-1} \geq D_{r,r} > \lambda_r$*.*

The entries of a DPP are also referred to as parts. Let $\mathcal{DPP}(n)$ denote the set of DPPs where each part is at most n (or equivalently $D_{1,1} \leq n$). The number of rows of the DPP does not have to satisfy any further restriction, in particular $\emptyset \in \mathcal{DPP}(n)$.

Example 1.2.7. *The seven DPPs where* $D_{1,1} \leq 3$ *are*

$$\mathcal{DPP}(3) = \left\{ \emptyset\ ,\ 3\ ,\ 2\ ,\ 3\quad 3\ ,\ 3\quad 2\ ,\ 3\quad 1\ ,\ \begin{array}{cc} 3 & 3 \\ & 2 \end{array} \right\},$$

and one of the 429 *DPPs in* $\mathcal{DPP}(5)$ *is*

$$\begin{array}{cccc} 5 & 4 & 2 & 2 \\ & 3 & 1 & \end{array}.$$

In [BFZJ13] it was shown that ASMs of size n and DPPs with $D_{1,1} \leq n$ are equinumerous with respect to four different statistics on ASMs and DPPs. Finding a bijection between the two sets, however, is still an open problem.

1.2.4 Totally symmetric self-complementary plane partitions

There are two equivalent definitions for plane partitions which are both useful to have in mind.

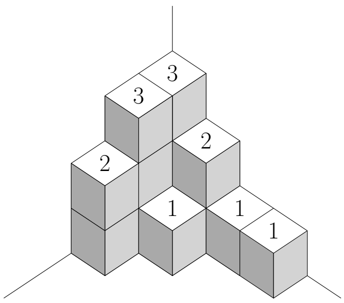
Definition 1.2.8. *A plane partition (PP) is an array* $(P_{i,j})_{\substack{1\leq i\leq r,\\ 1\leq j\leq \lambda_i}}$ *of positive integers arranged in the form*

$$\begin{array}{ccccc} P_{1,1} & P_{1,2} & \cdots & & P_{1,\lambda_1} \\ P_{2,1} & P_{2,2} & \cdots & & P_{2,\lambda_2} \\ \vdots & & & \iddots & \\ P_{r,1} & P_{r,2} & \cdots & P_{r,\lambda_r} & \end{array}$$

such that

- *entries in each row and column are weakly decreasing, i.e.* $P_{i,j} \geq \max\{P_{i,j+1}, P_{i+1,j}\}$ *and*
- *the number of entries in each row is weakly decreasing, i.e.* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$.

A plane partition of n is a plane partition where the sum of all entries is n. On the other hand a plane partition can be thought of as stacks of cubes pushed into the corner of a room (see Figure 1.4). This motivates the following equivalent definition: A plane partition is a finite subset



$$\Longleftrightarrow \quad \begin{matrix} 3 & 2 & 1 & 1 \\ 3 & 1 & & \\ 2 & & & \end{matrix}$$

Figure 1.4: A plane partition of 13.

$P \subseteq \mathbb{N}^3$ such that $(i', j', k') \in P$ whenever $(i, j, k) \in P$ and $1 \leq i' \leq i$, $1 \leq j' \leq j$, $1 \leq k' \leq k$. As a result by MacMahon, the number of plane partitions fitting inside

$$\mathcal{B}(r, s, t) := \{1, 2, \ldots, r\} \times \{1, 2, \ldots, s\} \times \{1, 2, \ldots, t\}$$

is given by

$$\prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{i+j+k-1}{i+j+k-2}.$$

Yet another combinatorial object counted by (1.1.1) is obtained if one imposes additional conditions on the structure of the plane partition:

Definition 1.2.9. *A totally symmetric self-complementary plane partition (TSSCPP) of size* $2n$ *is a plane partition* $T \subseteq \mathcal{B}(2n, 2n, 2n)$ *that is*

- *totally symmetric, i.e.*

$$(i, j, k) \in T \implies (i, k, j), (j, i, k), (j, k, i), (k, i, j), (k, j, i) \in T,$$

- *and self-complementary, i.e.*

$$(i, j, k) \in T \iff (2n+1-i, 2n+1-j, 2n+1-k) \notin T.$$

Let $\mathcal{TSSCPP}(n)$ denote the set of TSSCPPs of size $2n$.

Example 1.2.10. *The seven TSSCPPs of size* 6 *are*

$$\mathcal{TSSCPP}(3) = \left\{ \begin{smallmatrix} 6&6&6&3&3&3\\ 6&6&6&3&3&3\\ 6&6&6&3&3&3\\ 3&3&3&0&0&0\\ 3&3&3&0&0&0\\ 3&3&3&0&0&0 \end{smallmatrix}, \begin{smallmatrix} 6&6&6&4&3&3\\ 6&6&6&3&3&3\\ 6&6&5&3&3&2\\ 4&3&3&1&0&0\\ 3&3&3&0&0&0\\ 3&3&2&0&0&0 \end{smallmatrix}, \begin{smallmatrix} 6&6&6&4&3&3\\ 6&6&6&4&3&3\\ 6&6&4&3&2&2\\ 4&4&3&2&0&0\\ 3&3&2&0&0&0\\ 3&3&2&0&0&0 \end{smallmatrix}, \begin{smallmatrix} 6&6&6&5&5&3\\ 6&5&5&4&3&1\\ 6&5&4&3&2&1\\ 5&4&3&2&1&0\\ 5&3&2&1&1&0\\ 3&1&1&0&0&0 \end{smallmatrix}, \begin{smallmatrix} 6&6&6&5&5&3\\ 6&5&5&3&3&1\\ 6&5&5&3&3&1\\ 5&3&3&1&1&0\\ 5&3&3&1&1&0\\ 3&1&1&0&0&0 \end{smallmatrix}, \begin{smallmatrix} 6&6&6&5&4&3\\ 6&6&5&4&3&2\\ 6&5&4&3&2&1\\ 5&4&3&2&1&0\\ 4&3&2&1&0&0\\ 3&2&1&0&0&0 \end{smallmatrix}, \begin{smallmatrix} 6&6&6&5&4&3\\ 6&6&5&3&3&2\\ 6&5&5&3&3&1\\ 5&3&3&1&1&0\\ 4&3&3&1&0&0\\ 3&2&1&0&0&0 \end{smallmatrix} \right\}.$$

As a result by Andrews [And94] the number of TSSCPPs of size $2n$ is given by $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$, but this equinumerosity with ASMs has not been understood bijectively so far. Striker [Str13] found a natural bijection restricted to the set of permutation matrices.

1.2.5 Gogs & Magogs

Definition 1.2.11. *A (n,k)-Gog trapezoid is an array $(G_{i,j})_{\substack{1\le i\le n,\\ 1\le j\le \min\{k,n+1-i\}}}$ of positive integers arranged in the form*

$$\begin{array}{cccc}
G_{1,1} & \cdots & G_{1,k-1} & G_{1,k} \\
G_{2,1} & \cdots & G_{2,k-1} & G_{2,k} \\
\vdots & & \vdots & \vdots \\
G_{n+1-k,1} & \cdots & G_{n+1-k,k-1} & G_{n+1-k,k} \\
G_{n+2-k,1} & \cdots & G_{n+2-k,k-1} & \\
\vdots & \iddots & & \\
G_{n,1} & & &
\end{array}$$

such that

- *entries in each row are strictly increasing, i.e. $G_{i,j} < G_{i,j+1}$,*
- *entries in each column and each North-East-diagonal are weakly increasing, i.e. $G_{i,j} \le G_{i+1,j} \le G_{i,j+1}$ and*
- *$G_{1,i} = i$, $i = 1,\dots,k$.*

Note that if $k = n$, then these are exactly the Monotone Triangles with bottom row $(1,2,\dots,n)$.

Definition 1.2.12. *A (n,k)-Magog trapezoid is an array $(M_{i,j})_{\substack{1\le i\le \min\{k,n+1-j\},\\ 1\le j\le n}}$ of positive integers arranged in the form*

$$\begin{array}{cccccc}
M_{1,1} & M_{1,2} & \cdots & M_{1,n+1-k} & \cdots & M_{1,n-1} & M_{1,n} \\
M_{2,1} & M_{2,2} & \cdots & M_{2,n+1-k} & \cdots & M_{2,n-1} & \\
\vdots & \vdots & & \vdots & \iddots & & \\
M_{k,1} & M_{k,2} & \cdots & M_{k,n+1-k} & & &
\end{array}$$

such that

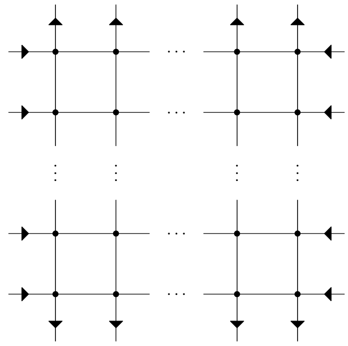
- *entries in each row are weakly increasing, i.e. $M_{i,j} \le M_{i,j+1}$,*
- *entries in each column are weakly decreasing, i.e. $M_{i,j} \ge M_{i+1,j}$ and*
- *entries in the first row are upper-bounded by $M_{1,j} \le j$.*

The objects obtained in the special case $k = n$ each represent the fundamental domain of a TSSCPP of size $2n$, i.e. the objects are in one-to-one correspondence [MRR86] with TSSCPPs of size $2n$. Zeilberger showed [Zei96a] that there is the same number of (n,k)-Gog trapezoids as (n,k)-Magog trapezoids, thus giving the first proof of the ASM Conjecture. In the special case $k = 2$, a bijective proof was found by Cheballah and Biane [CB12]. A more refined conjecture on Gogs and Magogs (including an additional parameter and two statistics) was introduced by Krattenthaler [Kra] and has not been proven so far.

1.2.6 Configurations of the six-vertex model with DWBC

Starting point of the six-vertex model with domain wall boundary condition is the following grid containing n rows of n vertices:

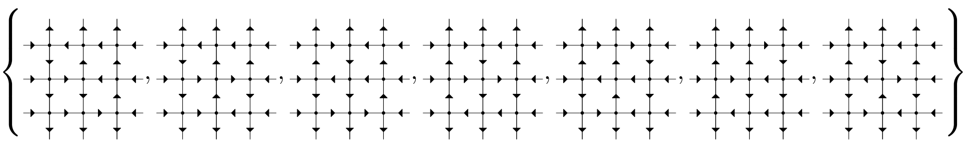
$\mathcal{G}_n$:



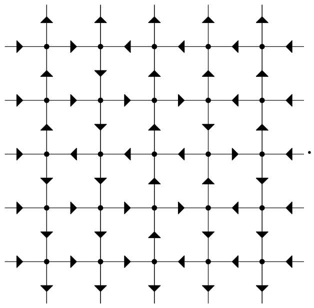
The domain wall boundary condition (DWBC) refers to the fact that the left- and right-most edges along the boundary are directed inwards whereas the top- and bottom-most edges are directed outwards. A configuration of $\mathcal{G}_n$ is an assignment of directions to the remaining edges such that each vertex has in-degree and out-degree two. At a vertex having four undirected edges, there are six ways to choose the directions of the four edges, hence the name six-vertex model. Let $\mathcal{SVM}(n)$ denote the set of configurations of $\mathcal{G}_n$.

Example 1.2.13. *The seven configurations of size* 3 *are*

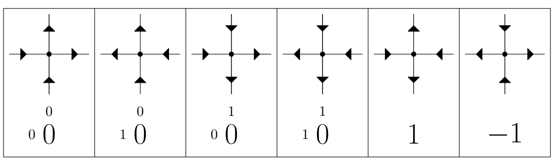
$\mathcal{SVM}(3) =$



and one of the 429 *configurations in* $\mathcal{SVM}(5)$ *is*



The sets $\mathcal{SVM}(n)$ and $\mathcal{ASM}(n)$ are in bijection by identifying the six different kinds of vertices with the six different kinds of entries in an ASM (cf. Subsection 1.2.1) in the following way:

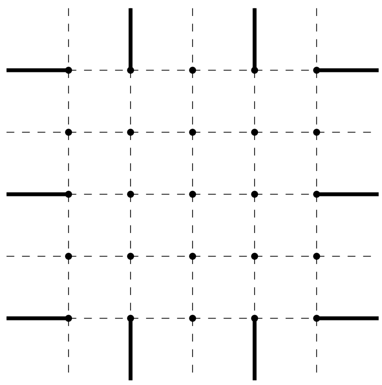


For details on the history and related ice-type models, we refer the reader to [Bax89]. Kuperberg used this connection and applied techniques from statistical mechanics (e.g. determinantal expressions for the partition function of the six-vertex model) to give an alternative proof of the ASM Theorem [Kup96], and Zeilberger soon after [Zei96b] took the same approach to give the first proof of the Refined ASM Theorem. To see how these techniques can be applied, it is recommendable to read [BFZJ12, Section 2.1].

1.2.7 Fully Packed Loop configurations

As in the six-vertex model, one starts with a square grid containing n^2 vertices and $4n$ external edges. Instead of assigning directions to the edges, one considers subgraphs of the grid. Hereby, the boundary condition is that starting with the horizontal, external edge incident with the top-left vertex every other external edge is contained in the subgraph:

$\mathcal{F}_5$:



.

A Fully Packed Loop (FPL) configuration of size n is a subgraph of $\mathcal{F}_n$ such that each vertex has degree 2. An FPL configuration of size 5 is depicted in Figure 1.5. The set of FPL configurations of size n is in one-to-one correspondence with the set of six-vertex configurations of same size (and hence with the set of ASMs of size n) by the following mapping: Partition the set of vertices into odd and even vertices in a chessboard manner, i.e. the top-left vertex is defined to be odd, each vertex adjacent to an odd vertex is even and vice versa each vertex adjacent to an even vertex is odd. At odd vertices the FPL configuration contains exactly the edges directed inwards in the corresponding six-vertex configuration, whereas at even vertices the FPL configuration contains exactly the edges directed outwards (see Figure 1.5).

The degree condition implies that an FPL configuration consists of paths connecting the external edges and may contain interior loops. If one numbers the $2n$ external edges contained in the FPL configuration, one can then associate a so-called link pattern with each FPL configuration that only stores the information which external edges are connected to each other (see Figure 1.6).

One could then refine the enumeration of FPL configurations w.r.t. a fixed link pattern (see for example [CKLN04]). As a result by [Wie00], the enumeration of FPL configurations w.r.t. a fixed link pattern is rotationally invariant. This means that in the example given in Figure 1.6 there is the same number of FPL configurations with the depicted link pattern π as there are for the link pattern $\pi' = \{\{1, 10\}, \{2, 9\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$ (obtained from counter-clockwise rotation by one step). This refined enumeration of FPLs and Wieland gyration also play a central role in the proof of the long-standing open Razumov-Stroganov (ex-)conjecture [CS11].

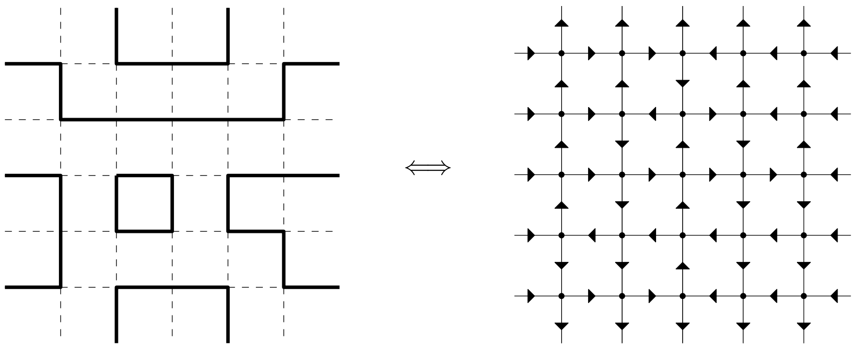


Figure 1.5: An FPL configuration of size 5 and the corresponding configuration of the six-vertex model.

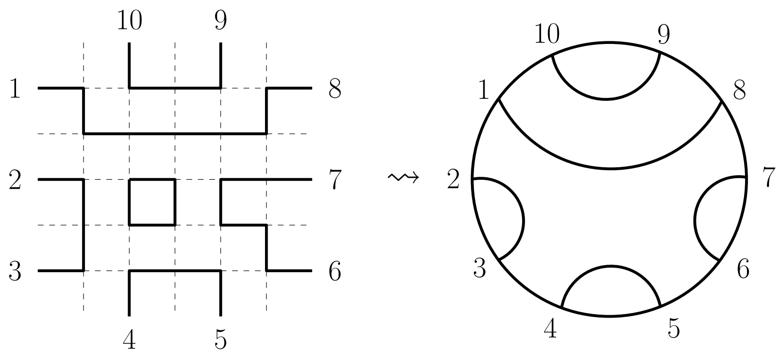
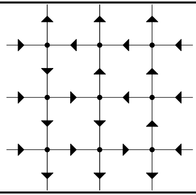
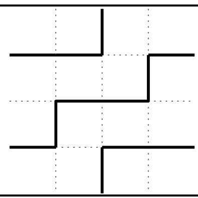
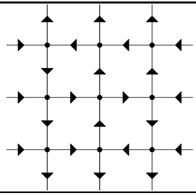
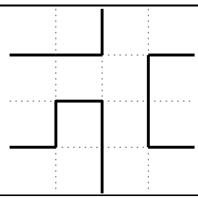
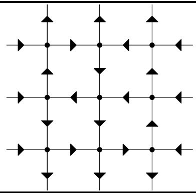
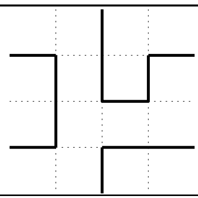
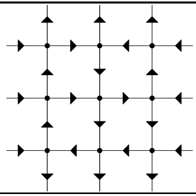
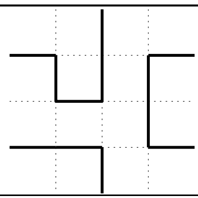
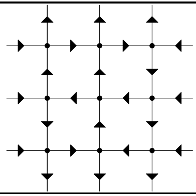
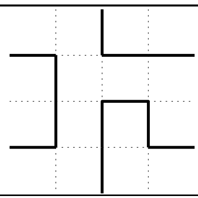
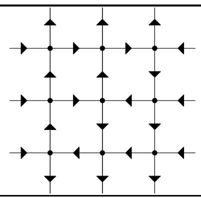
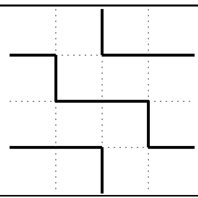
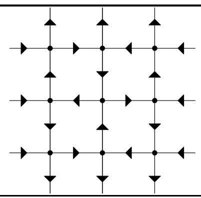
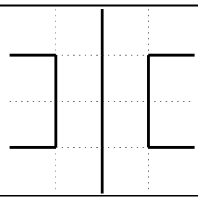


Figure 1.6: Link pattern $\pi = \{\{1,8\},\{2,3\},\{4,5\},\{6,7\},\{9,10\}\}$.

ASM	Monotone Triangle	6-vertex configuration	FPL configuration
$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{array}{ccccc}&&1&&\\&1&&2&\\1&&2&&3\end{array}$		
$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{array}{ccccc}&&1&&\\&1&&3&\\1&&2&&3\end{array}$		
$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{array}{ccccc}&&2&&\\&1&&2&\\1&&2&&3\end{array}$		
$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{array}{ccccc}&&2&&\\&2&&3&\\1&&2&&3\end{array}$		
$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{array}{ccccc}&&3&&\\&1&&3&\\1&&2&&3\end{array}$		
$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{array}{ccccc}&&3&&\\&2&&3&\\1&&2&&3\end{array}$		
$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$	$\begin{array}{ccccc}&&2&&\\&1&&3&\\1&&2&&3\end{array}$		

Chapter 2

Proof of the Refined Alternating Sign Matrix Theorem

2.1 Introduction

In this chapter we present a proof of the Refined Alternating Sign Matrix Theorem:

Theorem 2.1.1 (Refined ASM Theorem)**.** *The number $A_{n,i}$ of ASMs of size n with the first row's unique 1 in column i is given by*

$$A_{n,i} = \frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{n-1}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{2.1.1}$$

The following proof is due to Fischer and is primarily based on the two papers [Fis06] and [Fis07]. Over the years additional insight (for example the observations in [Fis10, Fis11]) helped finding shortcuts and simplifications, which we integrated into this chapter. What makes this proof particularly appealing is that it is at the same time both reasonably short and self-contained (apart from one well-known determinant evaluation due to Andrews [And79, Theorem 3]). The goal of this chapter is to present the complete proof and underlining the self-containment by requiring no prerequisites except for the definition of Alternating Sign Matrices and Monotone Triangles and their bijective correspondence presented in Proposition 1.2.5. Thereby we hope to provide a wide audience with the opportunity to perceive the elegance of this proof.

Let us start by observing that the definition of ASMs is symmetric w.r.t. reflection and rotation. Therefore, $A_{n,i}$ also counts the number of ASMs of size n where the unique 1 in the first (resp. last) column is in row $n+1-i$. From the correspondence (1.2.4) it then follows that $A_{n,i}$ counts the number of MTs with bottom row $(1,2,\dots,n)$ and exactly i entries equal to 1 in the left-most NE-diagonal (resp. exactly i entries equal to n in the right-most SE-diagonal). An example is given in Figure 2.1.

The chapter is structured as follows: In Section 2.2 we observe that for each $n \geq 1$ there exists a unique polynomial $\alpha(n; k_1, k_2, \dots, k_n)$ in the variables $\{k_1, k_2, \dots, k_n\}$ such that the evaluation

$$
\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & \mathbf{1} & & 4 & & & \\
 & & \mathbf{1} & & 3 & & \mathit{5} & & \\
 & \mathbf{1} & & 2 & & 4 & & \mathit{5} & \\
\mathbf{1} & & 2 & & 3 & & 4 & & \mathit{5}
\end{array}
\quad \Longleftrightarrow \quad
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
\mathbf{1} & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & \mathit{1} \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

Figure 2.1: Corresponding statistics in a MT and its ASM-counterpart.

at increasing integers $k_1 < k_2 < \cdots < k_n$ is equal to the number of MTs with bottom row $(k_1, k_2, \dots, k_n)$. Hereby, we show not only the existence of the polynomials, but also provide an explanation how to compute the polynomials recursively. In fact, one can give an explicit formula for $\alpha(n; k_1, k_2, \dots, k_n)$ in terms of an operator formula (Theorem 2.3.1), which first appeared in [Fis06]. We present a proof of this operator formula in Section 2.3. Next, we observe in Section 2.4 that the number of MTs with a fixed number of 1s in the left-most NE-diagonal can be expressed in terms of certain evaluations of $\alpha(n; k_1, k_2, \dots, k_n)$ (an idea which first appeared in [Fis11]). Even though it is not clear how to directly see that these evaluations are equal to the right-hand side of (2.1.1), one can apply properties of $\alpha(n; k_1, k_2, \dots, k_n)$ to derive a linear equation system (LES) they satisfy. This and the fact that the LES has a one-dimensional solution space are shown in Section 2.5. To complete the proof of the Refined ASM Theorem, it then only remains to compute that the numbers on the right-hand side of (2.1.1) also form a solution of this LES and determine the constant factor, which is done in Section 2.6.

2.2 The summation operator

An immediate advantage of considering MTs instead of ASMs is that a generalization of the counting problem has a simple recursive structure [MRR83]: Instead of counting the number of MTs with bottom row $(1, 2, \dots, n)$ we want to know the number of MTs with bottom row $(k_1, k_2, \dots, k_n)$ where the k_i are integers satisfying $k_1 < k_2 < \cdots < k_n$. Suppose we have solved the counting problem for MTs with $n-1$ rows. Given a fixed bottom row $(k_1, k_2, \dots, k_n)$, it follows by definition of MTs that $(l_1, \dots, l_{n-1}) \in \mathbb{Z}^{n-1}$ is an admissible penultimate row if and only if $k_i \leq l_i \leq k_{i+1}$ and $l_i < l_{i+1}$ for all i (see Figure 2.2). The total number of MTs with fixed bottom row $(k_1, k_2, \dots, k_n)$

$$
\begin{array}{ccccccccccccc}
 & & l_1 & & < & & l_2 & & < & \cdots & < & l_{n-1} & \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} \\
k_1 & & & & k_2 & & & & k_3 & \cdots & k_{n-1} & & k_n
\end{array}
$$

Figure 2.2: Bottom and penultimate row of a Monotone Triangle.

can thus be obtained by summing up the number of MTs with bottom row $(l_1, \dots, l_{n-1})$ over all admissible penultimate rows.

The goal of this section is to observe that for each $n \geq 1$ there exists a unique polynomial in n variables – called α-polynomial – counting MTs with n rows and fixed bottom row. More precisely, the evaluation $\alpha(n; k_1, k_2, \dots, k_n)$ at each strictly increasing integer sequence $k_1 < k_2 < \cdots < k_n$ equals the number of MTs with n rows and fixed bottom row $(k_1, k_2, \dots, k_n)$. For instance,

$\alpha(1;k_1) = 1$ and $\alpha(2;k_1,k_2) = k_2 - k_1 + 1$. Let us first remark that if such a polynomial $\alpha(n;k_1,k_2,\ldots,k_n)$ counting MTs exists, it must be unique.

Remark 2.2.1. *A polynomial in $n \geq 1$ variables is uniquely determined by its values on $\{(k_1,k_2,\ldots,k_n) \in \mathbb{Z}^n : k_1 < k_2 < \cdots < k_n\}$: To see this if $n = 1$, first recall (by induction w.r.t. the degree) that a polynomial $f(x) \in \mathrm{R}[x]$ of degree m over an integral domain R has at most m roots. This is because $p(x) = \sum_{k=0}^{m} a_k x^k$ and $p(r) = 0$ imply*

$$p(x) = p(x) - p(r) = \sum_{k=1}^{m} a_k(x^k - r^k) = (x-r)\sum_{k=1}^{m} a_k(x^{k-1} + rx^{k-2} + \cdots + r^{k-1}) = (x-r)q(x),$$

with a polynomial $q(x)$ of degree one less. In particular, two univariate polynomials coinciding on an infinite number of positions have to be identical. Now let $n \geq 2$ and suppose

$$p(k_1,\ldots,k_n) = q(k_1,\ldots,k_n) \quad \text{for all} \quad k_1 < k_2 < \cdots < k_n,\ k_i \in \mathbb{Z}.$$

Then, for fixed integers $k_1 < k_2 < \cdots < k_{n-1}$, the univariate polynomials $k_n \mapsto p(k_1,\ldots,k_n)$ and $k_n \mapsto q(k_1,\ldots,k_n)$ agree, whenever $k_n \in \mathbb{Z}$ and $k_n > k_{n-1}$. From the previous observation, it follows that

$$p(k_1,\ldots,k_{n-1},z) = q(k_1,\ldots,k_{n-1},z) \quad \text{for all} \quad k_1 < k_2 < \cdots < k_{n-1},\ k_i \in \mathbb{Z},\ z \in \mathbb{C}.$$

For each fixed $z \in \mathbb{C}$, the polynomials $p_z(k_1,\ldots,k_{n-1}) := p(k_1,\ldots,k_{n-1},z)$ and $q_z(k_1,\ldots,k_{n-1}) := q(k_1,\ldots,k_{n-1},z)$ are therefore identical by the induction hypothesis. Hence, the two polynomials agree on all positions, which implies that they are identical.

In the following we show how to construct the polynomials $\alpha(n;k_1,k_2,\ldots,k_n)$ inductively. By the previous observations it suffices to define $\alpha(n;k_1,k_2,\ldots,k_n)$, verify that is a polynomial and that it satisfies

$$\alpha(n;k_1,k_2,\ldots,k_n) = \sum_{\substack{(l_1,\ldots,l_{n-1}) \in \mathbb{Z}^{n-1},\\ k_1 \leq l_1 \leq k_2 \leq l_2 \leq \cdots \leq k_{n-1} \leq l_{n-1} \leq k_n,\\ l_i < l_{i+1}}} \alpha(n-1;l_1,\ldots,l_{n-1}) \tag{2.2.1}$$

for all $k_1 < k_2 < \cdots < k_n$, $k_i \in \mathbb{Z}$, $n \geq 2$. For this, we first define a summation operator $\sum\limits_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)}$ for all $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ having two properties:

On the one hand, the summation operator should yield an extension of (2.2.1), i.e. for any function $A(l_1,\ldots,l_{n-1})$ we want

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = \sum_{\substack{(l_1,\ldots,l_{n-1}) \in \mathbb{Z}^{n-1},\\ k_1 \leq l_1 \leq k_2 \leq l_2 \leq \cdots \leq k_{n-1} \leq l_{n-1} \leq k_n,\\ l_i < l_{i+1}}} A(l_1,\ldots,l_{n-1}), \tag{2.2.2}$$

whenever $k_1 < k_2 < \cdots < k_n$, $k_i \in \mathbb{Z}$. On the other hand, we want the summation operator to preserve polynomiality. More precisely, if $A(l_1,\ldots,l_{n-1})$ is a polynomial in each l_i, then

$$(k_1,\ldots,k_n) \mapsto \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) \tag{2.2.3}$$

should be a polynomial function on $\mathbb{Z}^n$. As soon as we have such a summation operator we can obtain the desired polynomials $\alpha(n; k_1, \ldots, k_n)$ inductively by defining $\alpha(1; k_1) := 1$ and

$$\alpha(n; k_1, \ldots, k_n) := \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1; l_1, \ldots, l_{n-1}), \quad (k_1, \ldots, k_n) \in \mathbb{Z}^n. \tag{2.2.4}$$

When constructing all admissible penultimate rows $(l_1, \ldots, l_{n-1})$ of a MT with bottom row $(k_1, \ldots, k_n)$ distinguish between two distinct cases for the values of l_{n-1}, namely $k_{n-1} < l_{n-1} \leq k_n$ and $l_{n-1} = k_{n-1}$ (cf. Figure 2.2). In the former case $(l_1, \ldots, l_{n-1})$ is an admissible penultimate row if and only if $(l_1, \ldots, l_{n-2})$ is an admissible penultimate row of the MT with bottom row $(k_1, \ldots, k_{n-1})$. If $l_{n-1} = k_{n-1}$, however, then the strict increase along rows implies $l_{n-2} < k_{n-1}$. This motivates the following inductive definition of the summation operator and its application to any function $A(l_1, \ldots, l_{n-1})$:

$$\begin{aligned}\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) := & \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1, \ldots, l_{n-2}, l_{n-1}) \\ & + \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-2},k_{n-1}-1)} A(l_1, \ldots, l_{n-2}, k_{n-1}), \quad n \geq 2,\end{aligned} \tag{2.2.5}$$

where $\sum\limits_{()}^{(k_1)} := \mathrm{id}$. To make sense of the summation operator's definition for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ we have to define $\sum_{i=a}^{b}$ also in the case $a > b$, $a, b \in \mathbb{Z}$. The formal equation $\sum_{i=a}^{b} f(i) = \sum_{i \geq a} f(i) - \sum_{i \geq b+1} f(i)$ for $a \leq b$ may motivate the following extension of ordinary sums

$$\sum_{i=a}^{b} f(i) := \begin{cases} 0, & b = a-1, \\ -\sum\limits_{i=b+1}^{a-1} f(i), & b+1 \leq a-1. \end{cases} \tag{2.2.6}$$

For example, in the base case $n = 2$, we then have

$$\sum_{(l_1)}^{(k_1,k_2)} f(l_1) = \sum_{()}^{(k_1)} \sum_{l_1=k_1+1}^{k_2} f(l_1) + \sum_{()}^{(k_1-1)} f(k_1) = \sum_{l_1=k_1}^{k_2} f(l_1), \quad k_1, k_2 \in \mathbb{Z}. \tag{2.2.7}$$

Furthermore it follows inductively from (2.2.5) and (2.2.6) that the summation operator indeed satisfies (2.2.2) for all $k_1 < k_2 < \cdots < k_{n-1} \leq k_n$.

To prove that the defined summation operator also preserves polynomiality, note that (2.2.5) allows us to write the summation operator only in terms of ordinary sums as defined in (2.2.6). It therefore suffices to show that ordinary sums preserve polynomiality. Recall that for every polynomial $p(x)$ there exists a polynomial $q(x)$ such that $q(x+1) - q(x) = p(x)$ – namely if $p(x) = \sum_{j=0}^{n} a_j \binom{x}{j}$, then set $q(x) := \sum_{j=0}^{n} a_j \binom{x}{j+1}$. By telescoping, it follows that

$$\sum_{i=a}^{b} p(i) = q(b+1) - q(a), \quad a \leq b, \quad a, b \in \mathbb{Z}.$$

The crucial observation is that definition (2.2.6) implies the equality also for integers $a > b$. Hence $P(a,b) := q(b+1) - q(a)$ is a polynomial satisfying $P(a,b) = \sum_{i=a}^{b} p(i)$ for all $a, b \in \mathbb{Z}$. By induction, (2.2.4) therefore defines a polynomial in $(k_1, k_2, \ldots, k_n)$ counting MTs with this bottom row.

Moreover, the line of reasoning reveals the following: Whenever we apply the summation operator to polynomials, we can use any other recursive description of the summation operator instead of (2.2.5), as long as it is based on ordinary sums as defined in (2.2.6) and satisfies (2.2.2) for all integers $k_1 < k_2 < \cdots < k_n$. For example, one can also use the recursion building the penultimate row from the left side:

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = \sum_{(l_2,\ldots,l_{n-1})}^{(k_2,\ldots,k_n)} \sum_{l_1=k_1}^{k_2-1} A(l_1,l_2,\ldots,l_{n-1}) + \sum_{(l_2,\ldots,l_{n-1})}^{(k_2+1,k_3,\ldots,k_n)} A(k_2,l_2,\ldots,l_{n-1}), \quad n \geq 2. \tag{2.2.8}$$

In this case, it follows inductively that (2.2.2) is satisfied for all integers $k_1 \leq k_2 < \cdots < k_{n-1} < k_n$.

A third recursion of the summation operator is

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}}^{k_n} A(l_1,\ldots,l_{n-2},l_{n-1}) - \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1}), \quad n \geq 3. \tag{2.2.9}$$

The intuition in this case is that if we allow $(l_1, \ldots, l_{n-2})$ to be any penultimate row of a MT with bottom row $(k_1, \ldots, k_{n-1})$ and independently let $k_{n-1} \leq l_{n-1} \leq k_n$, then not all penultimate rows $(l_1, \ldots, l_{n-1})$ one obtains are admissible. Namely, we have overcounted exactly by those rows where $l_{n-2} = l_{n-1} = k_{n-1}$.

As an example, let us explicitly compute the polynomial $\alpha(3; k_1, k_2, k_3)$ by applying recursion (2.2.9):

$$\begin{aligned}
\alpha(3;k_1,k_2,k_3) &\overset{(2.2.4)}{=} \sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} \alpha(2;l_1,l_2) \overset{(2.2.7)}{=} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \alpha(2;l_1,l_2) - \alpha(2;k_2,k_2) \\
&= \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} (l_2 - l_1 + 1) - 1 \\
&= \sum_{l_1=k_1}^{k_2} \left(\binom{k_3+1}{2} - \binom{k_2}{2} - (k_3-k_2+1)l_1 + (k_3-k_2+1) \right) - 1 \\
&= (k_2-k_1+1)\left(\binom{k_3+1}{2} - \binom{k_2}{2} + (k_3-k_2+1) \right) - (k_3-k_2+1)\left(\binom{k_2+1}{2} - \binom{k_1}{2} \right) - 1 \\
&= \frac{1}{2}\Big(-3k_1 + k_1^2 + 2k_1k_2 - 4k_1k_3 - k_1^2k_2 + k_1^2k_3 + k_1k_2^2 - k_1k_3^2 \\
&\qquad -2k_2^2 + 2k_2k_3 - k_2^2k_3 + k_2k_3^2 + 3k_3 + k_3^2 \Big).
\end{aligned}$$

2.3 Proof of the operator formula for $\alpha(n; k_1, k_2, \dots, k_n)$

The goal of this section is to derive the operator formula for Monotone Triangles (Theorem 2.3.1), which states that the polynomial $\alpha(n; k_1, k_2, \dots, k_n)$ can be obtained by taking a scaled version of the Vandermonde-polynomial $\prod_{1 \le i < j \le n}(k_j - k_i)$ and applying certain operators to it. For further reference, Table 2.1 contains a list of operators that are used throughout the thesis. From the fact that shift operators commute, i.e. $\mathrm{E}_x \mathrm{E}_y f(x, y) = \mathrm{E}_y \mathrm{E}_x f(x, y)$, it follows that all operators in Table 2.1 commute, except for the swap-operator.

$\mathrm{E}_x f(x) := f(x+1)$	(shift operator)
$\Delta_x f(x) := f(x+1) - f(x) = (\mathrm{E}_x - \mathrm{id}) f(x)$	(difference operator / Δ-operator)
$\delta_x f(x) := f(x) - f(x-1) = (\mathrm{id} - \mathrm{E}_x^{-1}) f(x)$	(difference operator / δ-operator)
$\mathrm{V}_{x,y} := \mathrm{id} + \delta_x \Delta_y = \mathrm{E}_x^{-1} + \mathrm{E}_y - \mathrm{E}_x^{-1} \mathrm{E}_y$	(V-operator)
$\mathrm{W}_{x,y} := \mathrm{E}_x \mathrm{V}_{x,y} = \mathrm{E}_x + \Delta_x \Delta_y = \mathrm{id} - \mathrm{E}_y + \mathrm{E}_x \mathrm{E}_y$	(W-operator)
$\mathrm{I}_{x,y} := \mathrm{E}_x^{-1} + \mathrm{E}_y^{-1} - \mathrm{id}$	(I-operator)
$\mathrm{S}_{x,y} f(x, y) := f(y, x)$	(swap operator)

Table 2.1: Overview of operators.

Theorem 2.3.1 (Operator formula for Monotone Triangles)**.** *Let $n \ge 1$. Then*

$$\alpha(n; k_1, k_2, \dots, k_n) = \prod_{1 \le p < q \le n} \mathrm{W}_{k_q, k_p} \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}. \tag{2.3.1}$$

Note that the first product in the operator formula (2.3.1) denotes the usual composition of operators. Since the W-operators commute, there is no need to specify in which order the composition is executed.

Remark 2.3.2. *Let us remark that the operand of* (2.3.1) *can be written as determinant, namely*

$$\prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i} = \det_{1 \le i, j \le n} \binom{k_i}{j-1}. \tag{2.3.2}$$

To see this, take the well-known Vandermonde determinant

$$\prod_{1 \le i < j \le n} (k_j - k_i) = \det_{1 \le i, j \le n} (k_i^{j-1})$$

and perform elementary column operations to obtain

$$\prod_{1 \le i < j \le n} (k_j - k_i) = \det_{1 \le i,j \le n} (p_j(k_i))$$

for any polynomial p_j of degree $j-1$ with leading coefficient 1. Equation (2.3.2) now follows by setting $p_j(x) := x(x-1)\ldots(x-j+1)$ and dividing each column by $(j-1)!$.

The operator formula is trivially true for $n = 1$, since the empty product is defined as 1, and for $n = 2$ the right-hand side of (2.3.1) is

$$\mathrm{W}_{k_2,k_1} \left[\frac{k_2 - k_1}{2-1} \right] = (\mathrm{id} - \mathrm{E}_{k_1} + \mathrm{E}_{k_2} \mathrm{E}_{k_1}) [k_2 - k_1] = k_2 - k_1 + 1,$$

i.e. indeed the number of MTs with bottom row $k_1 < k_2$.

Remark 2.3.3. *Since $\mathrm{W}_{k_q,k_p} = \mathrm{E}_{k_q} \mathrm{V}_{k_q,k_p}$ equation (2.3.1) is equivalent to*

$$\alpha(n; k_1, k_2, \ldots, k_n) = \prod_{1 \le p < q \le n} \mathrm{V}_{k_q,k_p} \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}. \qquad (2.3.3)$$

Even though it is not relevant for the remaining chapter, let us remark that the operator formula written in this form is particularly interesting, because it reveals a connection with a different counting problem: Gelfand-Tsetlin patterns are the same objects as MTs except for the condition of strict increase along rows omitted. Gelfand-Tsetlin patterns with fixed bottom row $(k_1, \ldots, k_n)$ are known to be in bijection with semi-standard Young tableaux of shape $(k_n, \ldots, k_1)$ and largest entry at most n (see [Sta01, p.313]*). From Stanley's hook-content formula* [Sta01, p.374] *one can derive that the number of these objects is exactly the operand $\prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}$.*

Lemma 2.3.4. *Let $B(l_1, \ldots, l_{n-1})$ be a function satisfying $\mathrm{W}_{l_i,l_{i+1}} B(l_1, \ldots, l_{n-1}) \Big|_{l_i = l_{i+1}} = 0$ for all $i = 1, \ldots, n-2$. Then*

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \Delta_{l_1} \ldots \Delta_{l_{n-1}} B(l_1, \ldots, l_{n-1}) = \sum_{r=1}^{n} (-1)^{r-1} B(k_1, \ldots, k_{r-1}, k_{r+1} + 1, \ldots, k_n + 1)$$

holds for $n \ge 1$.

Proof. For $n = 1$ the statement is trivial. If $n = 2$, we have

$$\sum_{(l_1)}^{(k_1,k_2)} \Delta_{l_1} B(l_1) \overset{(2.2.7)}{=} \sum_{l_1=k_1}^{k_2} (B(l_1 + 1) - B(l_1)) = B(k_2 + 1) - B(k_1).$$

For $n \geq 3$ it follows inductively that

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \Delta_{l_1} \ldots \Delta_{l_{n-1}} B(l_1,\ldots,l_{n-1}) \\
\overset{(2.2.9)}{=} &\sum_{l_{n-1}=k_{n-1}}^{k_n} \Delta_{l_{n-1}} \sum_{r=1}^{n-1} (-1)^{r-1} B(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{n-1}+1,l_{n-1}) \\
&- \sum_{r=1}^{n-2} (-1)^{r-1} \Delta_{k_{n-1}} \Delta_{k^*_{n-1}} B(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{n-2}+1,k_{n-1},k^*_{n-1}) \Big|_{k^*_{n-1}=k_{n-1}} \\
= &\sum_{r=1}^{n-1} (-1)^{r-1} B(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{n-1}+1,k_n+1) \\
&+ \sum_{r=1}^{n-1} (-1)^{r} B(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{n-1}+1,k_{n-1}) \\
&+ \sum_{r=1}^{n-2} (-1)^{r} (\mathrm{W}_{k_{n-1},k^*_{n-1}} - \mathrm{E}_{k_{n-1}}) B(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{n-2}+1,k_{n-1},k^*_{n-1}) \Big|_{k^*_{n-1}=k_{n-1}}
\end{aligned}$$

The last summand of the second sum can be added as n-th summand to the first sum. The remaining summands of the second sum cancel with the shifted summands of the third sum and we obtain

$$\begin{aligned}
&\sum_{r=1}^{n} (-1)^{r-1} B(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{n-1}+1,k_n+1) \\
&\quad + \sum_{r=1}^{n-2} (-1)^{r} \,\mathrm{W}_{k_{n-1},k^*_{n-1}}\, B(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{n-2}+1,k_{n-1},k^*_{n-1}) \Big|_{k^*_{n-1}=k_{n-1}}.
\end{aligned}$$

The second sum vanishes by assumption. ☐

Lemma 2.3.5. *Let $m_2,\ldots,m_n$ be non-negative integers and set $m_1 := -1$. Then*

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \prod_{1\leq p<q\leq n-1} \mathrm{W}_{l_q,l_p} \det_{1\leq i,j\leq n-1} \binom{l_i}{m_{j+1}} = \prod_{1\leq p<q\leq n} \mathrm{W}_{k_q,k_p} \det_{1\leq i,j\leq n} \binom{k_i}{m_j+1}.$$

holds for $n \geq 2$.

Proof. Let us first show that

$$B(l_1,\ldots,l_{n-1}) := \prod_{1\leq p<q\leq n-1} \mathrm{W}_{l_q,l_p} \det_{1\leq i,j\leq n-1} \binom{l_i}{m_{j+1}+1}.$$

satisfies the condition of Lemma 2.3.4: Since $\mathrm{W}_{l_i,l_{i+1}} \prod_{1 \le p < q \le n-1} \mathrm{W}_{l_q,l_p}$ is symmetric in l_i and l_{i+1} and $\det_{1 \le i,j \le n-1} \binom{l_i}{m_{j+1}+1}$ is antisymmetric in l_i and l_{i+1}, it follows that $\mathrm{W}_{l_i,l_{i+1}} B(l_1, \ldots, l_{n-1})$ is antisymmetric in l_i and l_{i+1}. In particular, $\mathrm{W}_{l_i,l_{i+1}} B(l_1, \ldots, l_{n-1}) \Big|_{l_i = l_{i+1}} = 0$. From $\Delta_x \binom{x}{k} = \binom{x}{k-1}$ it follows that

$$\Delta_{l_1} \ldots \Delta_{l_{n-1}} B(l_1, \ldots, l_{n-1}) = \prod_{1 \le p < q \le n-1} \mathrm{W}_{l_q,l_p} \det_{1 \le i,j \le n-1} \binom{l_i}{m_{j+1}}.$$

Applying Lemma 2.3.4 therefore yields

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \prod_{1 \le p < q \le n-1} \mathrm{W}_{l_q,l_p} \det_{1 \le i,j \le n-1} \binom{l_i}{m_{j+1}} \\
&\qquad = \sum_{r=1}^{n} (-1)^{r-1} B(k_1, \ldots, k_{r-1}, k_{r+1}+1, \ldots, k_n+1) \\
&\qquad = \sum_{r=1}^{n} (-1)^{r-1} \prod_{\substack{1 \le p < q \le n \\ p,q \ne r}} \mathrm{W}_{k_q,k_p} \left(\det_{1 \le i,j \le n-1} \binom{l_i}{m_{j+1}+1} \Big|_{\substack{(l_1,\ldots,l_{n-1}) \\ =(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_n+1)}} \right).
\end{aligned}$$

Since the r-th summand does not depend on k_r we have $\mathrm{W}_{k_q,k_r} = \mathrm{E}_{k_q}$ and $\mathrm{W}_{k_r,k_p} = \mathrm{id}$. Hence, this is further equal to

$$\begin{aligned}
&\sum_{r=1}^{n} (-1)^{r-1} \prod_{1 \le p < q \le n} \mathrm{W}_{k_q,k_p} \left(\det_{1 \le i,j \le n-1} \binom{l_i}{m_{j+1}+1} \Big|_{(l_1,\ldots,l_{n-1})=(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_n)} \right) \\
&\qquad = \prod_{1 \le p < q \le n} \mathrm{W}_{k_q,k_p} \sum_{r=1}^{n} (-1)^{r-1} \det_{1 \le i,j \le n-1} \binom{l_i}{m_{j+1}+1} \Big|_{(l_1,\ldots,l_{n-1})=(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_n)} \\
&\qquad = \prod_{1 \le p < q \le n} \mathrm{W}_{k_q,k_p} \det_{1 \le i,j \le n} \binom{k_i}{m_j+1},
\end{aligned}$$

where the last assertion is Laplace expansion w.r.t. the first column. □

We are now in the position to prove the operator formula inductively.

Proof of Theorem 2.3.1. Apply (2.2.4), the induction hypothesis, Remark 2.3.2 and Lemma 2.3.5 ($m_j := j-2$) to obtain

$$\begin{aligned}
\alpha(n; k_1, k_2, \ldots, k_n) &= \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1; l_1, l_2, \ldots, l_{n-1}) \\
&= \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \prod_{1 \le p < q \le n-1} \mathrm{W}_{l_q,l_p} \det_{1 \le i,j \le n-1} \binom{l_i}{j-1} \\
&= \prod_{1 \le p < q \le n} \mathrm{W}_{k_q,k_p} \det_{1 \le i,j \le n} \binom{k_i}{j-1} = \prod_{1 \le p < q \le n} \mathrm{W}_{k_q,k_p} \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.
\end{aligned}$$ □

Remark 2.3.6. *Let us observe that the degree of each k_i in the polynomial $\alpha(n; k_1, k_2, \ldots, k_n)$ is exactly $n-1$. Recall that the difference operators decrease the degree of a polynomial by exactly one, since*

$$\Delta_x \sum_{i=0}^{n} a_i \binom{x}{i} = \sum_{i=1}^{n} a_i \binom{x}{i-1} = \sum_{i=0}^{n-1} a_{i+1} \binom{x}{i},$$

and $\delta_x = \mathrm{E}_x^{-1} \Delta_x$, whereas the shift operator E_x leaves the degree invariant. The operator formula (2.3.1) therefore implies that the degree in each k_i is at most $n-1$. Moreover, the operator $\mathrm{W}_{x,y} = \mathrm{id} + \mathrm{E}_y \Delta_x$ acting on polynomials has an inverse, namely $\mathrm{W}_{x,y}^{-1} = \sum_{i \geq 0} (-1)^i \mathrm{E}_y^i \Delta_x^i$ (note that the sum is finite for each polynomial). So, we can apply the inverse operators in (2.3.1) such that only the operand having degree $n-1$ in each k_i remains at the right-hand side. Therefore, the α-polynomial also has to have degree at least $n-1$ in each k_i.

2.4 Expressing $A_{n,i}$ in terms of α-evaluations

There are multiple ways to express $A_{n,i}$ in terms of α-evaluations. One possibility is to observe that the position of the unique 1 in the bottom row of an ASM translates by Proposition 1.2.5 into the unique integer $1 \leq i \leq n$ that is missing in the penultimate row of the corresponding MT. Thus,

$$A_{n,i} = \alpha(n-1; 1, \ldots, i-1, i+1, \ldots, n).$$

This approach is taken in [Fis07] and the proof proceeds by deriving the identity

$$\alpha(n; 1, 2, \ldots, n-1, k) = \sum_{i=1}^{n} A_{n,i} \binom{i-1+k-n}{i-1}.$$

We take a different approach: As explained in Section 2.1, $A_{n,i}$ also counts the number of MTs with bottom row $(1, 2, \ldots, n)$ and precisely i entries equal to 1 in the left-most NE-diagonal (resp. precisely i entries equal to n in the right-most SE-diagonal). The following lemma is a special case of a more recent result in [Fis11] and also allows us to express $A_{n,i}$ in terms of α-evaluations:

Lemma 2.4.1. *Let $1 \leq i \leq n$ and $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$.*

1. *If $k_1 < k_2 < \cdots < k_{n-1} \leq k_n$, then $\delta_{k_n}^{i-1} \alpha(n; k_1, \ldots, k_n)$ counts the number of MTs with bottom row $(k_1, \ldots, k_{n-1}, k_n + 1)$ and precisely i entries equal to $k_n + 1$ in the right-most SE-diagonal (see Figure 2.3). In particular*

$$A_{n,i} = \delta_{k_n}^{i-1} \alpha(n; 1, 2, \ldots, n-1, k_n) \Big|_{k_n = n-1}. \qquad (2.4.1)$$

2. *If $k_1 \leq k_2 < k_3 \cdots < k_n$, then $(-1)^{i-1} \Delta_{k_1}^{i-1} \alpha(n; k_1, \ldots, k_n)$ counts the number of MTs with bottom row $(k_1 - 1, k_2, \ldots, k_n)$ and precisely i entries equal to $k_1 - 1$ in the left-most NE-diagonal (see Figure 2.4). In particular*

$$A_{n,i} = (-1)^{i-1} \Delta_{k_1}^{i-1} \alpha(n; k_1, 2, 3, \ldots, n) \Big|_{k_1 = 2}. \qquad (2.4.2)$$

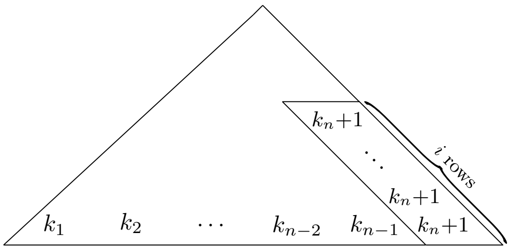


Figure 2.3: $\delta_{k_n}^{i-1}\alpha(n;k_1,\ldots,k_n)$

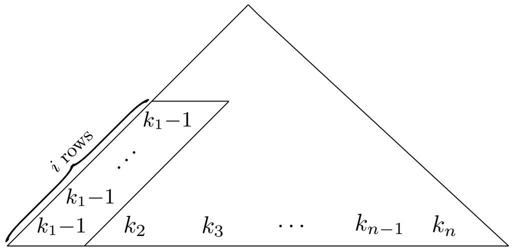


Figure 2.4: $(-1)^{i-1}\Delta_{k_1}^{i-1}\alpha(n;k_1,\ldots,k_n)$

Proof. Recall that for $k_1 < k_2 < \cdots < k_{n-1} \le k_n$ the summation operator satisfies

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1;l_1,\ldots,l_{n-1}) = \sum_{\substack{(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1},\\ k_1\le l_1\le k_2\le l_2\le\cdots\le k_{n-1}\le l_{n-1}\le k_n,\\ l_i<l_{i+1}}} \alpha(n-1;l_1,\ldots,l_{n-1}).$$

Note that the right-hand side counts the number of MTs with bottom row $(k_1,\ldots,k_{n-1},k_n+1)$ having precisely one entry equal to k_n+1 in the right-most SE-diagonal (by replacing the right-most entry k_n in the bottom row with k_n+1). The case $i=1$ now follows from the definition of $\alpha(n;k_1,k_2,\ldots,k_n)$, i.e. (2.2.4).

Recursion (2.2.5) of the summation operator implies

$$\begin{aligned}
&\delta_{k_n}\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1})\\
&= \delta_{k_n}\left(\sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})}\ \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1,\ldots,l_{n-2},l_{n-1}) + \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-2},k_{n-1}-1)} A(l_1,\ldots,l_{n-2},k_{n-1})\right)\\
&= \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-2},k_n).
\end{aligned}$$

With $A(l_1,\ldots,l_{n-1}) := \alpha(n-1;l_1,\ldots,l_{n-1})$ and (2.2.4) we obtain

$$\delta_{k_n}\alpha(n;k_1,\ldots,k_n) = \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \alpha(n-1;l_1,\ldots,l_{n-2},k_n). \tag{2.4.3}$$

According to (2.2.2) the right-hand side of (2.4.3) has a combinatorial interpretation: We sum over all penultimate rows $(l_1,\ldots,l_{n-2})$ of a MT with bottom row $(k_1,\ldots,k_{n-1})$ and append k_n at the right end of each penultimate row (see left side of Figure 2.5). Since $l_1 < l_2 < \cdots < l_{n-2} \le k_n$ we know from the case $i=1$ that $\alpha(n-1;l_1,\ldots,l_{n-2},k_n)$ counts the number of MT with bottom row $(l_1,\ldots,l_{n-2},k_n+1)$ with no further entries equal to k_n+1. Therefore the set of objects counted by the right-hand side of (2.4.3) bijectively corresponds to the set of MT with bottom row $(k_1,\ldots,k_{n-1},k_n+1)$ having precisely two entries equal to k_n+1 in the right-most SE-diagonal

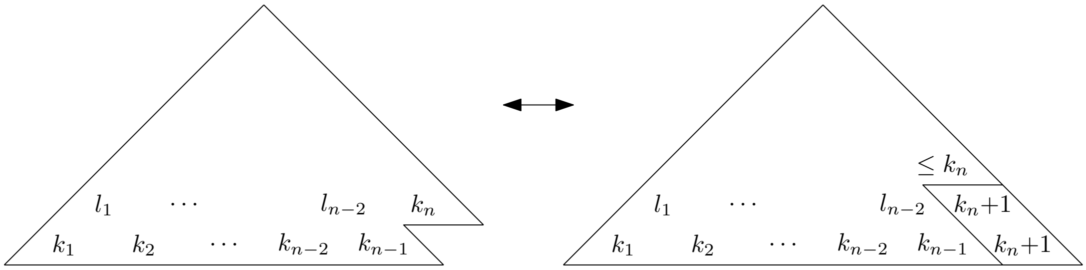


Figure 2.5: Combinatorial interpretation of $\delta_{k_n}\alpha(n; k_1, \ldots, k_n)$ for $k_1 < k_2 < \cdots < k_{n-1} \leq k_n$.

(see Figure 2.5). Iterating (2.4.3) and using the exact same argument shows the claim for all $i = 1, 2, \ldots, n$.

For the second part of the claim we use recursion (2.2.8) which implies

$$
\begin{aligned}
-\Delta_{k_1} & \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) \\
&= -\Delta_{k_1} \left(\sum_{(l_2,\ldots,l_{n-1})}^{(k_2,\ldots,k_n)} \sum_{l_1=k_1}^{k_2-1} A(l_1, l_2, \ldots, l_{n-1}) + \sum_{(l_2,\ldots,l_{n-1})}^{(k_2+1,k_3,\ldots,k_n)} A(n-1; k_2, l_2, \ldots, l_{n-1}) \right) \\
&= \sum_{(l_2,\ldots,l_{n-1})}^{(k_2,\ldots,k_n)} A(k_1, l_2, \ldots, l_{n-1}),
\end{aligned}
$$

and therefore

$$
-\Delta_{k_1}\alpha(n; k_1, \ldots, k_n) = \sum_{(l_2,\ldots,l_{n-1})}^{(k_2,\ldots,k_n)} \alpha(n-1; k_1, l_2, \ldots, l_{n-1}). \tag{2.4.4}
$$

The claim then follows analogously to the first part. $\square$

2.5 A system of linear equations satisfied by $(A_{n,1}, \ldots, A_{n,n})$

The goal of this section is to apply the derived expressions for $A_{n,i}$ to show that $(A_{n,1}, \ldots, A_{n,n})$ satisfies a system of n linear equations, which has a one-dimensional solution space. For this, we need two more identities satisfied by the α-polynomial. The first identity is

$$
\alpha(n; k_1, k_2, \ldots, k_n) = \alpha(n; k_1 + c, k_2 + c, \ldots, k_n + c), \quad c \in \mathbb{Z}, \tag{2.5.1}
$$

which is combinatorially clear for integers $k_1 < k_2 < \cdots < k_n$, since there is the same number of MTs with bottom row $(k_1, k_2, \ldots, k_n)$ as with bottom row $(k_1 + c, k_2 + c, \ldots, k_n + c)$. By Remark 2.2.1, equation (2.5.1) is therefore also true as identity satisfied by the polynomials. The second identity is more surprising and forms the cornerstone in deriving the system of linear equations.

Lemma 2.5.1. *The α-polynomial satisfies the circular shift identity*

$$\alpha(n;k_1,k_2,\ldots,k_n)=(-1)^{n-1}\alpha(n;k_2,k_3,\ldots,k_n,k_1-n) \tag{2.5.2}$$

for all $n\geq 1$.

Note that this identity only makes sense as an identity satisfied by the polynomial $\alpha(n;k_1,\ldots,k_n)$ and has per se no combinatorial interpretation. In fact, Chapter 4 is devoted to giving both sides of the identity a combinatorial meaning for $(k_1,\ldots,k_n)\in\mathbb{Z}^n$. If we believe the circular shift identity for now, then we can derive a system of linear equations satisfied by $(A_{n,1},\ldots,A_{n,n})$.

Lemma 2.5.2. *Let $A_{n,i}$ denote the number of $n\times n$-ASMs where the first row's unique 1 is in column i. Then*

$$A_{n,i}=\sum_{j=i}^{n}\binom{2n-i-1}{j-i}(-1)^{n+j}A_{n,j},\quad i=1,\ldots,n. \tag{2.5.3}$$

Proof.

$$\begin{aligned}
A_{n,i} &\overset{(2.4.2)}{=} (-1)^{i-1}\Delta_{k_1}^{i-1}\alpha(n;k_1,2,3,\ldots,n)|_{k_1=2}\\
&\overset{(2.5.2)}{=} (-1)^{n+i}\Delta_{k_1}^{i-1}\alpha(n;2,3,\ldots,n,k_1-n)|_{k_1=2}\\
&= (-1)^{n+i}\,\mathrm{E}_{k_1}^{-2n+i+1}\,\delta_{k_1}^{i-1}\alpha(n;2,3,\ldots,n,k_1+1)|_{k_1=n-1}\\
&\overset{(2.5.1)}{=} (-1)^{n+i}(\mathrm{id}-\delta_{k_1})^{2n-i-1}\delta_{k_1}^{i-1}\alpha(n;1,2,\ldots,n-1,k_1)|_{k_1=n-1}\\
&= \sum_{j\geq 0}\binom{2n-i-1}{j}(-1)^{n+i+j}\delta_{k_1}^{i+j-1}\alpha(n;1,2,\ldots,n-1,k_1)|_{k_1=n-1}.
\end{aligned}$$

In Remark 2.3.6 we have seen that the α-polynomial has degree $n-1$ in each k_i. Therefore

$$\begin{aligned}
A_{n,i} &= \sum_{j=i}^{n}\binom{2n-i-1}{j-i}(-1)^{n+j}\delta_{k_1}^{j-1}\alpha(n;1,2,\ldots,n-1,k_1)|_{k_1=n-1}\\
&\overset{(2.4.1)}{=} \sum_{j=i}^{n}\binom{2n-i-1}{j-i}(-1)^{n+j}A_{n,j}.
\end{aligned}$$

□

A proof of the circular shift identity was given in [Fis07], and it relies on the following lemma:

Lemma 2.5.3. *Let $Q(X_1,\ldots,X_n)$ be a symmetric polynomial without constant term, i.e. Q is invariant under permuting the variables and $Q(0,\ldots,0)=0$. Then*

$$Q(\Delta_{k_1},\ldots,\Delta_{k_n})\prod_{1\leq i<j\leq n}\frac{k_j-k_i}{j-i}=0. \tag{2.5.4}$$

In particular, $Q(\Delta_{k_1},\ldots,\Delta_{k_n})\,\alpha(n;k_1,\ldots,k_n)=0$.

Proof. By (2.3.2) it is equivalent to show that

$$Q(\Delta_{k_1}, \dots, \Delta_{k_n}) \det_{1 \le i,j \le n} \binom{k_i}{j-1} = 0. \tag{2.5.5}$$

Since Q is a symmetric polynomial without constant term, it suffices to show that

$$\sum_{\pi \in \mathcal{S}_n} \Delta_{k_1}^{m_{\pi(1)}} \dots \Delta_{k_n}^{m_{\pi(n)}} \det_{1 \le i,j \le n} \binom{k_i}{j-1} = 0 \tag{2.5.6}$$

for all n-tuples of non-negative integers $(m_1, \dots, m_n) \neq (0, \dots, 0)$. As $\Delta_x \binom{x}{k} = \binom{x}{k-1}$, this is further equal to

$$\sum_{\pi, \sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \binom{k_1}{\sigma(1) - m_{\pi(1)} - 1} \cdots \binom{k_n}{\sigma(n) - m_{\pi(n)} - 1} = 0.$$

If $\sigma(i) - m_{\pi(i)} - 1 < 0$ for any $1 \le i \le n$, then the corresponding summand vanishes. On the remaining set of summands

$$s(\pi, \sigma) := \operatorname{sgn} \sigma \binom{k_1}{\sigma(1) - m_{\pi(1)} - 1} \cdots \binom{k_n}{\sigma(n) - m_{\pi(n)} - 1}$$

we give a sign-reversing involution:

Let $s(\pi, \sigma)$ be any such summand. Since $0 \le \sigma(i) - m_{\pi(i)} - 1 \le n - 1$ for all $1 \le i \le n$, the existence of a positive m_k implies the existence of $1 \le i < j \le n$ such that $\sigma(i) - m_{\pi(i)} - 1 = \sigma(j) - m_{\pi(j)} - 1$. Let $1 \le i' < j' \le n-1$ be the lexicographically minimal indices with this property and set $\pi' := \pi \circ (i', j')$, $\sigma' := \sigma \circ (i', j')$. Then $s(\pi', \sigma') = -s(\pi, \sigma)$ and $\pi'' = \pi$, $\sigma'' = \sigma$. □

Remark 2.5.4. *From* $\Delta_x = \mathrm{E}_x \, \delta_x$ *and* (2.5.6) *it also follows that*

$$Q(\delta_{k_1}, \dots, \delta_{k_n}) \frac{k_j - k_i}{j - i} = 0 \tag{2.5.7}$$

for every symmetric polynomial $Q(X_1, \dots, X_n)$ *without constant term.*

In the following proof of the circular shift identity we apply Lemma 2.5.3 with the elementary symmetric polynomials, which are defined by

$$e_k(X_1, \dots, X_n) := \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} X_{i_1} X_{i_2} \dots X_{i_k}, \quad k = 1, \dots, n, \tag{2.5.8}$$

and $e_0(X_1, \dots, X_n) := 1$. Observe that for $k \ge 1$ the elementary symmetric polynomials are indeed symmetric polynomials without constant term.

Proof of Lemma 2.5.1. Note that $\mathrm{E}_x f(x)g(x) = (\mathrm{E}_x f(x))(\mathrm{E}_x g(x))$ to obtain

$$\begin{aligned}
\alpha(n;k_2,\dots,k_n,k_1-n) &\overset{(2.5.1)}{=} \alpha(n;k_2+1,\dots,k_n+1,k_1-n+1)\\
&\overset{(2.3.3)}{=} \mathrm{E}_{k_2}\dots\mathrm{E}_{k_n}\mathrm{E}_{k_1}^{-n+1}\prod_{2\le p<q\le n}(\mathrm{id}+\Delta_{k_p}\delta_{k_q})\prod_{p=2}^{n}(\mathrm{id}+\Delta_{k_p}\delta_{k_1})\\
&\qquad\prod_{2\le i<j\le n}\frac{k_j-k_i+j-i}{j-i}\prod_{i=2}^{n}\frac{k_1-k_i+n-i+1}{n-i+1}\\
&= \prod_{2\le p<q\le n}(\mathrm{id}+\Delta_{k_p}\delta_{k_q})\prod_{p=2}^{n}(\mathrm{id}+\Delta_{k_p}\delta_{k_1})\prod_{2\le i<j\le n}\frac{k_j-k_i+j-i}{j-i}\prod_{i=2}^{n}\frac{k_1-k_i-i+1}{i-1}\\
&= (-1)^{n-1}\prod_{2\le p<q\le n}(\mathrm{id}+\Delta_{k_p}\delta_{k_q})\prod_{p=2}^{n}(\mathrm{id}+\Delta_{k_p}\delta_{k_1})\prod_{1\le i<j\le n}\frac{k_j-k_i+j-i}{j-i}.
\end{aligned}$$

To prove the circular shift identity (2.5.2) we therefore have to show that

$$\prod_{2\le p<q\le n}(\mathrm{id}+\Delta_{k_p}\delta_{k_q})\left[\prod_{q=2}^{n}(\mathrm{id}+\Delta_{k_1}\delta_{k_q})-\prod_{p=2}^{n}(\mathrm{id}+\Delta_{k_p}\delta_{k_1})\right]\prod_{1\le i<j\le n}\frac{k_j-k_i+j-i}{j-i}=0.$$

Because of the identities

$$\begin{aligned}
\prod_{q=2}^{n}(\mathrm{id}+\Delta_{k_1}\delta_{k_q}) &= \sum_{r=0}^{n-1}\Delta_{k_1}^{r}e_r(\delta_{k_2},\dots,\delta_{k_n}),\\
\prod_{p=2}^{n}(\mathrm{id}+\Delta_{k_p}\delta_{k_1}) &= \sum_{r=0}^{n-1}\delta_{k_1}^{r}e_r(\Delta_{k_2},\dots,\Delta_{k_n}),
\end{aligned}$$

it is enough to show for all $r=1,\dots,n-1$ that

$$\left[\Delta_{k_1}^{r}e_r(\delta_{k_2},\dots,\delta_{k_n})-\delta_{k_1}^{r}e_r(\Delta_{k_2},\dots,\Delta_{k_n})\right]\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=0.$$

With $X_i := \Delta_{k_i}$ and $Y_i := \delta_{k_i}$ this follows from Lemma 2.5.3 and Remark 2.5.4 as soon as we show the following claim for $1\le r\le n-1$:

$$\begin{aligned}
&X_1^r e_r(Y_2,\dots,Y_n)-Y_1^r e_r(X_2,\dots,X_n)\\
&\quad=\sum_{s=1}^{r}X_1^rY_1^{r-s}(-1)^{r+s}e_s(Y_1,\dots,Y_n)+\sum_{s=1}^{r}X_1^{r-s}Y_1^{r}(-1)^{r+s-1}e_s(X_1,\dots,X_n).
\end{aligned}$$

The case $r=1$ is immediate. For $r\ge 2$ observe that the left-hand side is equal to

$$X_1^re_r(Y_1,\dots,Y_n)-Y_1^re_r(X_1,\dots,X_n)-X_1Y_1\left(X_1^{r-1}e_{r-1}(Y_2,\dots,Y_n)-Y_1^{r-1}e_{r-1}(X_2,\dots,X_n)\right).$$

By induction hypothesis this is further equal to

$$\begin{aligned}
&X_1^re_r(Y_1,\dots,Y_n)-Y_1^re_r(X_1,\dots,X_n)\\
&\quad+\sum_{s=1}^{r-1}X_1^rY_1^{r-s}(-1)^{r+s}e_s(Y_1,\dots,Y_n)+\sum_{s=1}^{r-1}X_1^{r-s}Y_1^{r}(-1)^{r+s-1}e_s(X_1,\dots,X_n). \qquad \square
\end{aligned}$$

We have seen that $(A_{n,1},\dots,A_{n,n})$ satisfies (2.5.3), i.e. a system of n linear equations. However, its solution space is higher-dimensional. This can be remedied by recalling that there are more linear relations between the refined ASM numbers. Namely, by reflection along the vertical symmetry axis the identity $A_{n,i}=A_{n,n+1-i}$ follows combinatorially. If we now replace each entry on the right-hand side of (2.5.3) one arrives at

$$A_{n,i}=\sum_{j=1}^{n+1-i}\binom{2n-i-1}{n-i-j+1}(-1)^{j+1}A_{n,j},\quad i=1,\dots,n. \tag{2.5.9}$$

The goal of the remaining part of the section is to observe that the solution space of (2.5.9) is one-dimensional, i.e. it determines $(A_{n,1},\dots,A_{n,n})$ up to a constant C_n.

Remark 2.5.5. *Whenever binomial coefficients appear in this thesis, we use the definition*

$$\binom{x}{j}:=\begin{cases}\frac{x(x-1)\cdots(x-j+1)}{j!} & \text{if } j\geq 0,\\ 0 & \text{if } j<0,\end{cases} \tag{2.5.10}$$

where $x\in\mathbb{C}$ and $j\in\mathbb{Z}$. While this is for practical reasons the common definition in discrete mathematics [GKP89]*, be aware of two traps: First, the symmetry identity $\binom{n}{k}=\binom{n}{n-k}$ is only true for integers $n\geq 0$. Second, your favourite computer algebra system may use the analytic continuation of the binomial coefficient via Gamma functions. In this case, the computer will tell you that the binomial coefficient $\binom{x}{y}$ with negative integers $y\leq x<0$ is non-zero, namely $(-1)^{x-y}\binom{-y-1}{-x-1}$. When checking results involving such binomial coefficients one should therefore manually define the binomial coefficient with* (2.5.10) *and explicitly set $\binom{x}{j}:=0$ for $j\in\mathbb{Z}^-$.*

Note that (2.5.9) is equivalent to the fact that $(A_{n,1},\dots,A_{n,n})$ is an eigenvector of the matrix $\left(\binom{2n-i-1}{n-i-j+1}(-1)^{j+1}\right)_{1\leq i,j\leq n}$ with eigenvalue 1. The proof in [Fis07, p.262] proceeds by showing that the corresponding eigenspace is 1-dimensional, i.e. that

$$\operatorname{rk}\left(\binom{2n-i-1}{n-i-j+1}(-1)^{j}+\delta_{i,j}\right)_{1\leq i,j\leq n}=n-1.$$

For $n=1$ this is obvious. If $n\geq 2$, let us show that removing the first row and first column yields a matrix of full rank, i.e. we have to show that

$$\det_{1\leq i,j\leq n-1}\left(\binom{2n-i-2}{n-i-j-1}(-1)^{j+1}+\delta_{i,j}\right)\neq 0.$$

This is clear for $n=2$. For $n\geq 3$ consider the matrices

$$\begin{aligned}
B_n&=\left(\binom{2n-i-2}{n-i-j-1}(-1)^{j+1}+\delta_{i,j}\right)_{1\leq i,j\leq n-1},\\
B_n^*&=\left(\binom{i+j}{j-1}(1-\delta_{i,n-1})\right)_{1\leq i,j\leq n-1},\\
R_n&=\left(\binom{n+j-i-1}{j-i}\right)_{1\leq i,j\leq n-1},\\
R_n^{-1}&=\left(\binom{n}{j-i}(-1)^{i+j}\right)_{1\leq i,j\leq n-1}.
\end{aligned}$$

The fact that R_n^{-1} indeed is the inverse of R_n and $R_n^{-1} B_n R_n = B_n^* + I_{n-1}$ are both hypergeometric identities which can be reduced to the Chu–Vandermonde identity (see Lemma A.2.1). Laplace expansion w.r.t. the last row yields

$$\det B_n = \det \left(\binom{i+j}{j-1} + \delta_{i,j} \right)_{1 \le i,j \le n-2}. \tag{2.5.11}$$

This determinant is known to count the number of descending plane partitions with parts strictly smaller than n and is therefore non-zero ([And79, Theorem 3]).

2.6 Proof of the Refined ASM Theorem

Showing that the numbers

$$X_{n,i} := \frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{n-1}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}, \quad i = 1, \ldots, n$$

also solve the LES (2.5.9) is a task of reducing a hypergeometric sum to the Chu–Vandermonde identity (see Lemma A.2.2). Since the solution space is one-dimensional it follows that $(A_{n,1}, \ldots, A_{n,n}) = C_n(X_{n,1}, \ldots, X_{n,n})$ for all $n \ge 1$. To prove the Refined ASM Theorem (Theorem 2.1.1) it only remains to show that $C_n = 1$.

If an ASM of size n has a 1 in the top-left corner, then all other entries in the first row and column are zeroes, and removing the first row and column yields an ASM of size $n-1$. This establishes a one-to-one correspondence between the ASMs counted by $A_{n,1}$ and all ASMs of size $n-1$, hence $A_{n,1} = A_{n-1}$. The hypergeometric identity $X_{n,1} = \sum_{i=1}^{n-1} X_{n-1,i}$ holds as well (see Lemma A.2.2). Since $A_{1,1} = 1 = X_{1,1}$ we have $C_1 = 1$, and for $n \ge 2$ we inductively obtain

$$A_{n,1} = A_{n-1} = \sum_{i=1}^{n-1} A_{n-1,i} = \sum_{i=1}^{n-1} C_{n-1} X_{n-1,i} = X_{n,1},$$

and therefore $C_n = 1$.

Chapter 3

Combinatorial reciprocity for Monotone Triangles

The contents of this chapter essentially consist of the research published in [FR13].

3.1 Introduction

In Section 2.3 we presented a proof of the operator formula for the polynomial $\alpha(n; k_1, k_2, \dots, k_n)$. Recall that the condition that the evaluation at integers $k_1 < k_2 < \cdots < k_n$ should be the number of Monotone Triangles (MTs) with bottom row $(k_1, k_2, \dots, k_n)$ uniquely determined the α-polynomial (Remark 2.2.1). The starting point of the research presented in this chapter were computational experiments indicating the surprising identity

$$\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) = \alpha(n; 1, 2, \dots, n). \tag{3.1.1}$$

Note that so far only the evaluation on the right-hand side of (3.1.1) has a combinatorial meaning, namely the number of MTs with bottom row $(1, 2, \dots, n)$. The first goal of this chapter is to give a combinatorial interpretation to the evaluations of $\alpha(n; k_1, k_2, \dots, k_n)$ at weakly decreasing sequences $k_1 \geq k_2 \geq \cdots \geq k_n$. To this end, we define a new combinatorial object:

Definition 3.1.1. *A Decreasing Monotone Triangle (DMT) of size n is a triangular array of integers $(a_{i,j})_{1 \leq j \leq i \leq n}$ arranged in the form*

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \iddots & & & & \ddots & \\ a_{n,1} & & \cdots & & \cdots & & a_{n,n} \end{array}$$

such that

(D1) entries along NE- and SE-diagonals are weakly decreasing, i.e. $a_{i+1,j} \geq a_{i,j} \geq a_{i+1,j+1}$,

$$
\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & \mathbf{2} & & \mathbf{2} & & & \\
 & & 3 & & \mathbf{2} & & \mathbf{2} & & \\
 & \mathbf{3} & & \mathbf{3} & & 2 & & 2 & \\
6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & \mathbf{3} & & \mathbf{3} & & & \\
 & & \mathbf{3} & & \mathbf{3} & & 2 & & \\
 & \mathbf{3} & & \mathbf{3} & & 2 & & 2 & \\
6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & \mathbf{3} & & \mathbf{3} & & & \\
 & & 3 & & 3 & & 2 & & \\
 & 4 & & 3 & & 2 & & 2 & \\
6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & \mathbf{2} & & \mathbf{2} & & & \\
 & & 4 & & \mathbf{2} & & \mathbf{2} & & \\
 & 5 & & 3 & & 2 & & 2 & \\
6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & \mathbf{3} & & \mathbf{3} & & & \\
 & & 3 & & 3 & & 2 & & \\
 & 5 & & 3 & & 2 & & 2 & \\
6 & & \mathbf{3} & & \mathbf{3} & & 2 & & 1
\end{array}
$$

Figure 3.1: The five Decreasing Monotone Triangles with bottom row $(6, 3, 3, 2, 1)$.

Figure 3.2: Condition (D3') of DMTs.

(D2) each integer appears at most twice in a row and

(D3) two consecutive rows do not both contain the same integer exactly once.

Five DMTs with bottom row $(6, 3, 3, 2, 1)$ are illustrated in Figure 3.1. Before checking that there are no other DMTs with this bottom row, let us state an equivalent definition for DMTs.

Remark 3.1.2. *Even though condition (D3) is more concise, it can be more practical to keep the following equivalent condition in mind (see Figure 3.2):*

(D3') If two adjacent entries in a row are distinct and their interlaced neighbour in the row above is equal to its SW-neighbour (resp. SE-neighbour), then the interlaced neighbour has a left (resp. right) neighbour and is equal to it, i.e.

$$
\begin{aligned}
a_{i,j} = a_{i+1,j} > a_{i+1,j+1} &\implies a_{i,j-1} = a_{i,j}, \\
a_{i+1,j} > a_{i+1,j+1} = a_{i,j} &\implies a_{i,j+1} = a_{i,j}.
\end{aligned}
$$

By condition (D1) entries along each row of a DMT are weakly decreasing and interlaced with their neighbours in the row below. This immediately implies that (D3) and (D3') are equivalent.

If the bottom row of a DMT is $(6, 3, 3, 2, 1)$, condition (D3') implies that the right-most entry of the penultimate row has to be 2, thus its left neighbour has to be 2 too. The second entry has to be 3 and the first entry may be $5, 4$ or 3. Continuing in the same way with these three possible penultimate rows, one obtains that the five DMTs depicted in Figure 3.1 are indeed all DMTs with bottom row $(6, 3, 3, 2, 1)$.

In Section 3.2 we prove that the evaluation of $\alpha(n; k_1, \ldots, k_n)$ at weakly decreasing integer sequences $k_1 \geq k_2 \geq \cdots \geq k_n$ is a signed enumeration of DMTs with bottom row $(k_1, k_2, \ldots, k_n)$. The sign of a DMT is determined by the number of so-called duplicate-descendants in the DMT:

Definition 3.1.3. *A pair of adjacent identical entries in a row is briefly called a pair. A pair is called duplicate-descendant (DD) if it is in the bottom row or if the row below contains the same pair.*

The duplicate-descendants of the DMTs in Figure 3.1 are marked in boldface.

Theorem 3.1.4 ([FR13]). *Let $k_1 \geq k_2 \geq \cdots \geq k_n$ be a sequence of weakly decreasing integers, and let $\mathcal{W}_n(k_1, \ldots, k_n)$ denote the set of DMTs with bottom row $(k_1, \ldots, k_n)$. Then*

$$\alpha(n; k_1, k_2, \ldots, k_n) = (-1)^{\binom{n}{2}} \sum_{A \in \mathcal{W}_n(k_1,\ldots,k_n)} (-1)^{\mathrm{dd}(A)}, \tag{3.1.2}$$

where $\mathrm{dd}(A)$ *denotes the total number of duplicate-descendants of A.*

The example in Figure 3.1 contains four DMTs with an even number of duplicate-descendants and one with an odd number. The assertion of Theorem 3.1.4 hence is that $\alpha(5; 6, 3, 3, 2, 1) = 3$. Theorem 3.1.4 was reproven from a geometric point of view by Jochemko and Sanyal [JS12].

In Section 3.3 we show a correspondence between DMTs with bottom row $(n, n, n-1, n-1, \ldots, 1, 1)$ and a set of ASM-like matrices, which we call 2-ASMs. The notion of 2-ASMs turns out helpful in giving a computational proof of (3.1.1) – the second main contribution of this chapter. In fact, we prove the following stronger result in Section 3.4 and show that (3.1.1) is a consequence:

Theorem 3.1.5 ([FR13]). *The refined ASM-numbers $A_{n,i} = \frac{\binom{n+i-2}{n-1}\binom{2n-1-i}{n-1}}{\binom{3n-2}{n-1}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ satisfy*

$$A_{n,i} = (-1)^{n-1} \alpha(2n-1; n-1+i, n-1, n-1, \ldots, 1, 1) \tag{3.1.3}$$

for $i = 1, \ldots, 2n-1$, $n \geq 1$.

Since both sides of (3.1.3) are polynomials in i of degree $2n-2$, Theorem 3.1.5 implies

$$\alpha(2n-1; n-1+i, n-1, n-1, \ldots, 1, 1) = (-1)^{n-1} \frac{\binom{n+i-2}{n-1}\binom{2n-1-i}{n-1}}{\binom{3n-2}{n-1}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

for arbitrary $i \in \mathbb{C}$, $n \geq 1$.

Finally, having a combinatorial interpretation in terms of DMTs for the left-hand side of (3.1.1) and one in terms of MTs for the right-hand side, the equality demands for a combinatorial explanation. In Section 3.5 a first approach towards a bijective proof is given.

3.2 Decreasing Monotone Triangles

This section is devoted to the proof of Theorem 3.1.4. For this, we require the summation operator explained in Section 2.2 (originally introduced in [Fis06]).

We start by first considering the case that the bottom row $(k_1, \ldots, k_n)$ is weakly decreasing and contains three identical entries, i.e. $k_i = k_{i+1} = k_{i+2}$. By definition, there are no such DMTs, i.e. $\mathcal{W}_n(k_1, \ldots, k_n) = \emptyset$, and thus the right-hand side of (3.1.2) is zero. Let us therefore show that the polynomial $\alpha(n; k_1, \ldots, k_n)$ also vanishes in this case:

Lemma 3.2.1. *Let $n \geq 3$ and $A(l_1, \dots, l_{n-1})$ be a polynomial in each variable satisfying*

$$A(l_1, \dots, l_{i-1}, l_i, l_i, l_i, l_{i+3}, \dots, l_{n-1}) = 0, \quad i = 1, \dots, n-3.$$

The polynomial $B(k_1, \dots, k_n) := \sum\limits_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} A(l_1, \dots, l_{n-1})$ *then satisfies*

$$B(k_1, \dots, k_{i-1}, k_i, k_i, k_i, k_{i+3}, \dots, k_n) = 0, \quad i = 1, \dots, n-2.$$

In particular

$$\alpha(n; k_1, \dots, k_{i-1}, k_i, k_i, k_i, k_{i+3}, \dots, k_n) = 0, \quad i = 1, \dots, n-2. \tag{3.2.1}$$

Proof. The proof is by induction w.r.t. n. In the base case $n = 3$ one obtains

$$\begin{aligned} \sum_{(l_1,l_2)}^{(k_1,k_1,k_1)} A(l_1, l_2) &\overset{(2.2.5)}{=} \sum_{(l_1)}^{(k_1,k_1)} \sum_{l_2=k_1+1}^{k_1} A(l_1, l_2) + \sum_{(l_1)}^{(k_1,k_1-1)} A(l_1, k_1) \\ &\overset{(2.2.7)}{=} \sum_{l_1=k_1}^{k_1} \sum_{l_2=k_1+1}^{k_1} A(l_1, l_2) + \sum_{l_1=k_1}^{k_1-1} A(l_1, k_1) \overset{(2.2.6)}{=} 0. \end{aligned}$$

There are three cases to check for $n \geq 4$ depending on whether there are zero, one or at least two entries to the right of the three identical entries:

- $i = n-2$: Together with the Lemma's assumption it follows that

$$\begin{aligned} B(k_1, \dots, k_{n-3}, k_{n-2}, k_{n-2}, k_{n-2}) &\overset{(2.2.5)}{=} \sum_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-3},k_{n-2},k_{n-2}-1)} A(l_1, \dots, l_{n-2}, k_{n-2}) \\ &\overset{(2.2.9)}{=} -\sum_{(l_1,\dots,l_{n-4})}^{(k_1,\dots,k_{n-3})} A(l_1, \dots, l_{n-4}, k_{n-2}, k_{n-2}, k_{n-2}) = 0. \end{aligned}$$

- $i = n-3$: Applying recursion (2.2.9) yields

$$\begin{aligned} B(k_1, \dots, k_{n-4}, k_{n-3}, k_{n-3}, k_{n-3}, k_n) = &\sum_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-4},k_{n-3},k_{n-3},k_{n-3})} \sum_{l_{n-1}=k_{n-3}}^{k_n} A(l_1, \dots, l_{n-2}, l_{n-1}) \\ &- \sum_{(l_1,\dots,l_{n-3})}^{(k_1,\dots,k_{n-4},k_{n-3},k_{n-3})} A(l_1, \dots, l_{n-3}, k_{n-3}, k_{n-3}). \end{aligned}$$

  Note that $A'(l_1, \dots, l_{n-2}) := \sum\limits_{l_{n-1}=k_{n-3}}^{k_n} A(l_1, \dots, l_{n-2}, l_{n-1})$ satisfies the Lemma's hypothesis. By induction the first sum vanishes, and using (2.2.5) shows that the second sum vanishes too.

- $1 \leq i \leq n-4$: Using recursion (2.2.5) and the induction hypothesis as in the previous case implies that $B(k_1, \dots, k_{i-1}, k_i, k_i, k_i, k_{i+3}, \dots, k_n) = 0$.

In particular, (2.2.4) implies that $\alpha(n; k_1, k_2, \ldots, k_n) = 0$, whenever there are three consecutive identical entries among $(k_1, k_2, \ldots, k_n)$. $\square$

We can now restrict ourselves to the case of weakly decreasing sequences $k_1 \geq k_2 \geq \cdots \geq k_n$ which contain each integer at most twice. Apart from the number of duplicate-descendants, a second statistic on DMTs is of interest:

Definition 3.2.2. *An entry strictly smaller than the South-West-neighbour and strictly larger than the South-East-neighbour is called a newcomer, i.e. an entry $a_{i,j}$ with $1 \leq i < n$ satisfying $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$.*

The proof of Theorem 3.1.4 consists of the following steps: At the heart of the theorem's proof is Lemma 3.2.3 which explains the connection between applying the summation operator $\sum\limits_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)}$ with decreasing integer arguments $k_1 \geq \cdots \geq k_n$, and taking a signed summation over all candidates for penultimate rows of DMTs with bottom row $(k_1, \ldots, k_n)$. In Corollary 3.2.4 we see by induction that $\alpha(n; k_1, \ldots, k_n)$ is the signed summation over all DMTs with bottom row $(k_1, \ldots, k_n)$, where the sign is determined by the total number of pairs and newcomers in the DMT without the bottom row. Finally, we show in Lemma 3.2.5 that the parity of this statistic is equal to the parity of the statistic in Theorem 3.1.4.

Given integers $k_1 \geq k_2 \geq \cdots \geq k_n$, let $\mathcal{P}(k_1, \ldots, k_n)$ denote the set of $(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}$ such that the conditions of DMTs are locally satisfied in the following trapezoid:

$$\begin{array}{ccccccccc} & l_1 & & \cdots & & \cdots & & l_{n-1} & \\ k_1 & & k_2 & & \cdots & & k_{n-1} & & k_n \end{array}$$

More precisely, we require $k_i \geq l_i \geq k_{i+1}$, each integer appearing at most twice in $(l_1, \ldots, l_{n-1})$ and if $k_i > k_{i+1}$, then $l_i = k_i$ implies $l_{i-1} = k_i$ resp. $l_i = k_{i+1}$ implies $l_{i+1} = k_{i+1}$. For example, $\mathcal{P}(6, 3, 3, 2, 1) = \{(3, 3, 2, 2), (4, 3, 2, 2), (5, 3, 2, 2)\}$.

Lemma 3.2.3. *Let $\boldsymbol{k} = (k_1, \ldots, k_n)$ be a weakly decreasing sequence of integers with each integer appearing at most twice. Then, for every polynomial $A(l_1, \ldots, l_{n-1})$ we have*

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) = \sum_{\boldsymbol{l}=(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)} (-1)^{\mathrm{sc}(\boldsymbol{k};\boldsymbol{l})} A(l_1, \ldots, l_{n-1}), \quad n \geq 2, \tag{3.2.2}$$

where the sign-change function $\mathrm{sc}(\boldsymbol{k}; \boldsymbol{l}) := \mathrm{sc}(k_1, \ldots, k_n; l_1, \ldots, l_{n-1})$ counts the total number of pairs and newcomers in $(l_1, \ldots, l_{n-1})$.

Proof. To make the given definitions easier to understand, let us first check the base cases $n = 2, 3$:

$$\sum_{(l_1)}^{(k_1,k_2)} A(l_1) \overset{(2.2.7)}{=} \sum_{l_1=k_1}^{k_2} A(l_1) \overset{(2.2.6)}{=} \begin{cases} A(k_1), & k_1 = k_2, \\ 0, & k_1 = k_2 + 1, \\ -\sum\limits_{l_1=k_2+1}^{k_1-1} A(l_1), & k_1 > k_2 + 1. \end{cases}$$

$$
\begin{array}{ccccccccccccccccc}
 & & l_1 & & > & & l_2 & & & & & & k_2 & & = & & k_2 & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & & & \nearrow & & \backslash\!\backslash & & /\!/ & & \searrow & \\
k_1 & & > & & k_2 & & > & & k_3 & \qquad & k_1 & & > & & k_2 & & > & & k_3
\end{array}
$$

Figure 3.3

$$
\begin{array}{ccccccccccccccccc}
 & & l_1 & & > & & k_2 & & & & & & k_2 & & = & & k_2 & & \\
 & \nearrow & & \searrow & & /\!/ & & \backslash\!\backslash & & & & \nearrow & & \backslash\!\backslash & & /\!/ & & \backslash\!\backslash & \\
k_1 & & > & & k_2 & & = & & k_3 & \qquad & k_1 & & > & & k_2 & & = & & k_3
\end{array}
$$

Figure 3.4

This is in accordance with DMTs with bottom row (k_1, k_2): If on the one hand $k_1 = k_2$, then we have $\mathcal{P}(k_1, k_2) = \{(k_1)\}$. On the other hand, $k_1 > k_2$ implies that $\mathcal{P}(k_1, k_2) = \{(l_1) : k_1 > l_1 > k_2\}$, whereby the entry l_1 is a newcomer. If $n = 3$ and $k_1 > k_2 > k_3$, then

$$\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1, l_2) \overset{(2.2.9)}{=} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} A(l_1, l_2) - A(k_2, k_2) \overset{(2.2.6)}{=} \sum_{l_1=k_2+1}^{k_1-1} \sum_{l_2=k_3+1}^{k_2-1} A(l_1, l_2) - A(k_2, k_2).$$

The situation is depicted in Figure 3.3: either l_1 and l_2 are both newcomers (corresponding to the double sum) or $(l_1, l_2) = (k_2, k_2)$ is a pair (corresponding to the term $-A(k_2, k_2)$). If $k_1 > k_2 = k_3$, then

$$\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1, l_2) \overset{(2.2.9)}{=} \sum_{l_1=k_1}^{k_2} A(l_1, k_2) - A(k_2, k_2) \overset{(2.2.6)}{=} - \sum_{l_1=k_2+1}^{k_1-1} A(l_1, k_2) - A(k_2, k_2).$$

This again corresponds to a signed summation over all elements of $\mathcal{P}(k_1, k_2, k_3)$: either l_1 is a newcomer or $(l_1, l_2) = (k_2, k_2)$ is a pair (see Figure 3.4). If the bottom row is $k_1 = k_2 > k_3$, the claim can be shown analogously. For $n \geq 4$, distinguish between the case $k_{n-1} > k_n$ and $k_{n-1} = k_n$:

Case 1 $(k_{n-1} > k_n)$:
Recursion (2.2.9) of the summation operator, (2.2.6) and the induction hypothesis yield

$$
\begin{aligned}
&\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) \\
&\qquad = - \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_n+1}^{k_{n-1}-1} A(l_1, \ldots, l_{n-1}) - \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1, \ldots, l_{n-3}, k_{n-1}, k_{n-1}) \\
&\qquad = \sum_{(l_1,\ldots,l_{n-2}) \in \mathcal{P}(k_1,\ldots,k_{n-1})} (-1)^{\mathrm{sc}(k_1,\ldots,k_{n-1};l_1,\ldots,l_{n-2})+1} \sum_{l_{n-1}=k_n+1}^{k_{n-1}-1} A(l_1, \ldots, l_{n-1}) \\
&\qquad\quad + \sum_{(l_1,\ldots,l_{n-3}) \in \mathcal{P}(k_1,\ldots,k_{n-2})} (-1)^{\mathrm{sc}(k_1,\ldots,k_{n-2};l_1,\ldots,l_{n-3})+1} A(l_1, \ldots, l_{n-3}, k_{n-1}, k_{n-1}).
\end{aligned}
$$

In order to see that this is indeed equal to the right-hand side of (3.2.2), let us define

$$\mathcal{L}_1 := \mathcal{P}(k_1,\ldots,k_{n-1}) \times \{l_{n-1} \mid k_{n-1} > l_{n-1} > k_n\},$$
$$\mathcal{L}_2 := \mathcal{P}(k_1,\ldots,k_{n-2}) \times \{(k_{n-1},k_{n-1})\},$$

and show

$$\mathcal{P}(k_1,\ldots,k_n) = \mathcal{L}_1 \,\dot{\cup}\, \mathcal{L}_2, \tag{3.2.3}$$

whereby

$$(l_1,\ldots,l_{n-1}) \in \mathcal{L}_1 \implies \operatorname{sc}(\boldsymbol{k};\boldsymbol{l}) = \operatorname{sc}(k_1,\ldots,k_{n-1};l_1,\ldots,l_{n-2}) + 1, \tag{3.2.4}$$
$$(l_1,\ldots,l_{n-1}) \in \mathcal{L}_2 \implies \operatorname{sc}(\boldsymbol{k};\boldsymbol{l}) = \operatorname{sc}(k_1,\ldots,k_{n-2};l_1,\ldots,l_{n-3}) + 1. \tag{3.2.5}$$

We start by observing that the right-hand side of (3.2.3) is contained in the left-hand side, so let $(l_1,\ldots,l_{n-1}) \in \mathcal{L}_1 \cup \mathcal{L}_2$. Condition (D1) of DMTs is clearly satisfied. To see that $(l_1,\ldots,l_{n-1})$ contains each integer at most twice is also immediate except in one case, namely $(l_1,\ldots,l_{n-1}) \in \mathcal{L}_2$ and $l_{n-3} = k_{n-2} = k_{n-1}$. However, $(l_1,\ldots,l_{n-3}) \in \mathcal{P}(k_1,\ldots,k_{n-2})$ and condition (D3') then imply that $k_{n-3} = k_{n-2}$, contradicting the lemma's assumption that $(k_1,\ldots,k_n)$ contains each integer at most twice. Therefore $(l_1,\ldots,l_{n-1})$ satisfies condition (D2). Condition (D3) is again obviously satisfied.

For the reverse direction note that each $(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)$ either satisfies $k_{n-1} > l_{n-1} > k_n$ or $l_{n-1} = k_{n-1}$ (condition (D3') of DMTs implies that $l_{n-1} = k_n$ is not possible if $k_{n-1} > k_n$). In the following we check that in the former case $(l_1,\ldots,l_{n-1}) \in \mathcal{L}_1$ and (3.2.4), whereas in the latter case $(l_1,\ldots,l_{n-1}) \in \mathcal{L}_2$ and (3.2.5).

Case 1.1 $(k_{n-1} > l_{n-1} > k_n)$:

We have to show that $(l_1,\ldots,l_{n-2}) \in \mathcal{P}(k_1,\ldots,k_{n-1})$. Condition (D1) immediately follows from $(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)$. Together with $l_{n-1} < k_{n-1}$ it follows that $l_{n-2} > k_{n-1}$ and therefore condition (D2) and (D3) carry over from $(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)$. Since l_{n-1} is a newcomer and $l_{n-2} \neq l_{n-1}$ we obtain $\operatorname{sc}(k_1,\ldots,k_n;l_1,\ldots,l_{n-1}) = \operatorname{sc}(k_1,\ldots,k_{n-1};l_1,\ldots,l_{n-2}) + 1$.

Case 1.2 $(l_{n-1} = k_{n-1})$:

Since $(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)$ and $k_{n-1} = l_{n-1} > k_n$ condition (D3') implies that $l_{n-2} = k_{n-1}$. It remains to show that $(l_1,\ldots,l_{n-3}) \in \mathcal{P}(k_1,\ldots,k_{n-2})$. Conditions (D1) and (D2) again translate from $(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)$. The only position where condition (D3') does not carry over from $(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)$ is l_{n-3}, namely suppose $k_{n-3} > l_{n-3} = k_{n-2}$. But then condition (D3') implies that $l_{n-2} = k_{n-2}$ and therefore $l_{n-3} = l_{n-2} = l_{n-1}$. This contradicts condition (D2) in $(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)$. The pair $(l_{n-2},l_{n-1}) = (k_{n-1},k_{n-1})$ contributes one sign-change, and thus $\operatorname{sc}(k_1,\ldots,k_n;l_1,\ldots,l_{n-1}) = \operatorname{sc}(k_1,\ldots,k_{n-2};l_1,\ldots,l_{n-3}) + 1$.

Case 2 $(k_{n-1} = k_n)$:
Recursion (2.2.5) and the induction hypothesis yield

$$\begin{aligned}\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_{n-1},k_{n-1})} A(l_1,\ldots,l_{n-1}) &= \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-2},k_{n-1}-1)} A(l_1,\ldots,l_{n-2},k_{n-1})\\ &= \sum_{(l_1,\ldots,l_{n-2})\in\mathcal{P}(k_1,\ldots,k_{n-2},k_{n-1}-1)} (-1)^{\operatorname{sc}(k_1,\ldots,k_{n-2},k_{n-1}-1;l_1,\ldots,l_{n-2})} A(l_1,\ldots,l_{n-2},k_{n-1}).\end{aligned}$$

It remains to observe that

$$\mathcal{P}(k_1,\dots,k_{n-2},k_{n-1}-1)\times\{k_{n-1}\}=\mathcal{P}(k_1,\dots,k_{n-2},k_{n-1},k_{n-1}), \tag{3.2.6}$$
$$\mathrm{sc}(k_1,\dots,k_{n-2},k_{n-1}-1;l_1,\dots,l_{n-2})=\mathrm{sc}(k_1,\dots,k_{n-2},k_{n-1},k_{n-1};l_1,\dots,l_{n-1}). \tag{3.2.7}$$

Equation (3.2.6) follows directly from the definition of DMTs after observing that $k_{n-1}=k_n$ and the lemma's assumption imply that $k_{n-2}>k_{n-1}$. To see (3.2.7), distinguish between $l_{n-2}=k_{n-1}$ and $l_{n-2}>k_{n-1}$: If $l_{n-2}=k_{n-1}$, then l_{n-2} is a newcomer w.r.t. $(k_1,\dots,k_{n-2},k_{n-1}-1)$ on the one hand and (l_{n-2},l_{n-1}) is a pair w.r.t. $(k_1,\dots,k_{n-2},k_{n-1},k_{n-1})$ on the other hand, and therefore both sides of (3.2.7) are equal to $1+\mathrm{sc}(k_1,\dots,k_{n-2};l_1,\dots,l_{n-3})$. If $l_{n-2}>k_{n-1}$, then $k_{n-2}>k_{n-1}$ implies that both sides of (3.2.7) are equal to $\mathrm{sc}(k_1,\dots,k_{n-1};l_1,\dots,l_{n-2})$. □

Let us extend the domain of the sign-changes function sc to DMTs by defining

$$\mathrm{sc}(A):=\sum_{i=1}^{n-1}\mathrm{sc}(a_{i+1,1},\dots,a_{i+1,i+1};a_{i,1},\dots,a_{i,i}),$$

where $A=(a_{i,j})_{1\le j\le i\le n}$ is a DMT with n rows. In other words, $\mathrm{sc}(A)$ is the total number of pairs and newcomers in the top $n-1$ rows of A. Applying Lemma 3.2.3 to $\alpha(n;k_1,\dots,k_n)$ establishes the connection between $\alpha(n;k_1,\dots,k_n)$ and $\mathcal{W}_n(k_1,\dots,k_n)$:

Corollary 3.2.4. *Let $k_1\ge k_2\ge\cdots\ge k_n$. Then*

$$\alpha(n;k_1,\dots,k_n)=\sum_{A\in\mathcal{W}_n(k_1,\dots,k_n)}(-1)^{\mathrm{sc}(A)} \tag{3.2.8}$$

holds for $n\ge1$.

Proof. If $k_i=k_{i+1}=k_{i+2}$ for any $i=1,\dots,n-2$, then we have already seen in Lemma 3.2.1 that the left-hand side is zero. The right-hand side also vanishes since the bottom row of a DMT contains each integer at most twice by definition. The case $n=1$ is also trivial. For $n\ge2$, apply (2.2.4) together with Lemma 3.2.3 and the induction hypothesis to see that

$$\begin{aligned}\alpha(n;k_1,\dots,k_n)&=\sum_{(l_1,\dots,l_{n-1})\in\mathcal{P}(k_1,\dots,k_n)}(-1)^{\mathrm{sc}(k_1,\dots,k_n;l_1,\dots,l_{n-1})}\alpha(n-1;l_1,\dots,l_{n-1})\\&=\sum_{(l_1,\dots,l_{n-1})\in\mathcal{P}(k_1,\dots,k_n)}(-1)^{\mathrm{sc}(k_1,\dots,k_n;l_1,\dots,l_{n-1})}\sum_{A\in\mathcal{W}_{n-1}(l_1,\dots,l_{n-1})}(-1)^{\mathrm{sc}(A)}\\&=\sum_{A\in\mathcal{W}_n(k_1,\dots,k_n)}(-1)^{\mathrm{sc}(A)}.\end{aligned}$$ □

In order to complete the proof of Theorem 3.1.4, it remains to show that the two statistics $\mathrm{sc}(A)$ and $\binom{n}{2}+\mathrm{dd}(A)$ have the same parity.

Lemma 3.2.5. *Each $A\in\mathcal{W}_n(k_1,\dots,k_n)$ satisfies*

$$(-1)^{\mathrm{sc}(A)}=(-1)^{\binom{n}{2}+\mathrm{dd}(A)},\quad n\ge1.$$

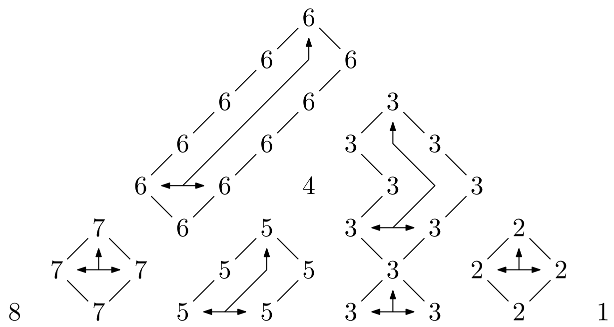


Figure 3.5: A DMT and the correspondence between peaks and base-pairs.

Proof. If one of the top $n-1$ rows of a DMT contains an integer x exactly once, then – by condition (D3) of DMTs – the row below contains x either not at all or twice. In the former case x is a newcomer, and in the latter case let us call x a peak. Let $p(A)$ denote the number of peaks in A, $n(A)$ denote the number of newcomers in A and $t_i(A)$ denote the number of pairs in the i-th row of A. Since every entry in the top $n-1$ rows is either a peak, a newcomer or in a pair, it follows that

$$\binom{n}{2} = p(A) + n(A) + 2\sum_{i=1}^{n-1} t_i(A) \equiv p(A) + n(A) \pmod 2.$$

Let us call a pair (x,x) a base-pair if it is located in the bottom row or the row below contains x exactly once. Note that the set of base-pairs and the set of peaks are in one-to-one correspondence (see Figure 3.5), i.e. the total number of base-pairs is $p(A)$. For the total number of duplicate-descendants (Definition 3.1.3) we therefore obtain

$$\mathrm{dd}(A) = \sum_{i=1}^{n} t_i(A) - p(A) + t_n(A) \equiv \sum_{i=1}^{n-1} t_i(A) - p(A) \pmod 2.$$

Since $\mathrm{sc}(A) = n(A) + \sum_{i=1}^{n-1} t_i(A)$, it follows that

$$\mathrm{sc}(A) - \mathrm{dd}(A) \equiv n(A) + p(A) \equiv \binom{n}{2} \pmod 2.$$

□

3.3 DMTs and 2-ASMs

In this section, we examine the impact of the change from Monotone Triangles to DMTs on the level of matrices. The goal of this section is to show that DMTs with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$ are in one-to-one correspondence with objects we call 2-ASMs.

Definition 3.3.1. *A 2-ASM of size n is a $(2n) \times n$-matrix with entries in $\{0, 1, -1\}$ where*

(2-ASM 1) in each row the non-zero entries alternate in sign and sum up to 1, and

(2-ASM 2) in each column the non-zero entries occur in pairs such that each partial column sum is in $\{0, 1, 2\}$, and each column sums up to 2.

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0\\
0 & 1 & -1 & 1 & 0\\
1 & -1 & 1 & 0 & 0\\
1 & 0 & -1 & 0 & 1\\
0 & 1 & -1 & 0 & 1\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\Longleftrightarrow
\begin{array}{c}
3\\
3 \quad 3\\
4 \quad 3 \quad 3\\
4 \quad 4 \quad 3 \quad 2\\
4 \quad 4 \quad 3 \quad 3 \quad 1\\
5 \quad 4 \quad 4 \quad 3 \quad 1 \quad 1\\
5 \quad 5 \quad 4 \quad 4 \quad 2 \quad 1 \quad 1\\
5 \quad 5 \quad 4 \quad 4 \quad 2 \quad 2 \quad 1 \quad 1\\
5 \quad 5 \quad 4 \quad 4 \quad 3 \quad 2 \quad 2 \quad 1 \quad 1\\
5 \quad 5 \quad 4 \quad 4 \quad 3 \quad 3 \quad 2 \quad 2 \quad 1 \quad 1
\end{array}
$$

Figure 3.6: A 2-ASM of size 5 and the corresponding DMT.

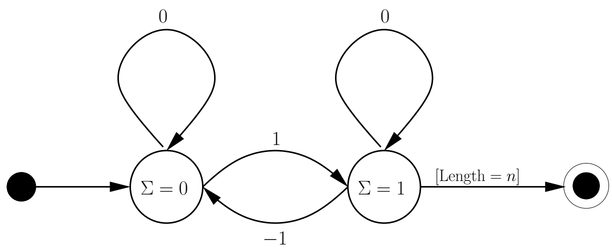


Figure 3.7: ASM-machine generating the rows and columns of ordinary ASMs of size n.

An example of a 2-ASM of size 5 can be seen in Figure 3.6. Before proving that DMTs with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$ are in one-to-one correspondence with 2-ASMs of size n, we consider ASMs and 2-ASMs from another perspective:

An ASM of size n is a $n \times n$-matrix where each row and each column is a word of the ASM-machine depicted in Figure 3.7. The semantics of the machine are the following: When generating a row (resp. column) of an ASM, the initial state is $\Sigma = 0$, meaning that the current partial row sum (resp. column sum) is 0. One may then stay in the state taking the 0-loop or transit to the state $\Sigma = 1$ by taking the edge labelled with 1. In the state $\Sigma = 1$ – i.e. the partial row sum (resp. column sum) is currently equal to 1 – one may either stay in the state by taking the 0-loop or transit back to the state $\Sigma = 0$ taking the edge labelled with -1. As the row sum (resp. column sum) is equal to 1, one has to be in the state $\Sigma = 1$ after n steps.

Analogously, a 2-ASM of size n is a $(2n) \times n$-matrix where each row is a word of the ASM-machine in Figure 3.7 and each column is a word of the 2-ASM-machine in Figure 3.8. It should be clear that this definition of 2-ASMs is equivalent to Definition 3.3.1, but the notion of the 2-ASM-machine and the interpretation of its edges turn out useful in the proof of Lemma 3.4.2. Apart from that, it could be interesting to analyze the combinatorial structures obtained from other modifications of the ASM-machine.

Recall the one-to-one correspondence (Proposition 1.2.5) transforming an ASM $(a_{i,j})_{1 \le i,j \le n}$ into a MT with bottom row $(1, 2 \dots, n)$: The i-th row of the MT contains an entry j, if and only if $b_{i,j} := \sum_{k=1}^{i} a_{k,j} = 1$. In our case let $b_{i,j}$ be the number of entries j in row i of a DMT with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$. Define a $(2n) \times n$-matrix A with entries in $\{0, 1, -1\}$ such

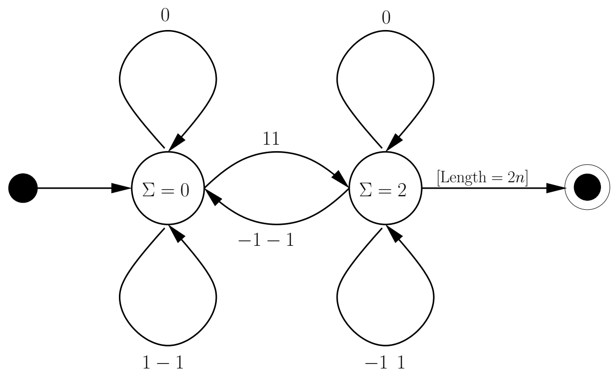


Figure 3.8: 2-ASM-machine generating the columns of 2-ASMs of size n.

that $\sum_{k=1}^{i} a_{k,j} = b_{i,j}$, i.e. $a_{1,j} := b_{1,j}$ and $a_{i,j} := b_{i,j} - b_{i-1,j}$, $2 \leq i \leq 2n$. An example is given in Figure 3.6. Let us show that this is indeed a one-to-one correspondence between 2-ASMs of size n and DMTs with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$.

Proposition 3.3.2. *The condition*

$$\begin{aligned} &\text{row } i \text{ of DMT contains } b_{i,j} \text{ entries equal to } j \quad \iff \\ &\text{top } i \text{ entries in column } j \text{ of 2-ASM sum up to } b_{i,j} \end{aligned} \tag{3.3.1}$$

for all $1 \leq i \leq 2n$, $1 \leq j \leq n$ *establishes a one-to-one correspondence between the set of DMTs with bottom row* $(n, n, n-1, n-1, \dots, 1, 1)$ *and the set of* 2*-ASMs of size* n*.*

Proof. Let $A = (a_{i,j})_{\substack{i=1,\dots,2n,\\ j=1,\dots,n}}$ be a 2-ASM of size n, and $B = (b_{i,j})_{\substack{i=1,\dots,2n,\\ j=1,\dots,n}}$ be the corresponding partial column-sum matrix, i.e. $b_{i,j} = \sum_{k=1}^{i} a_{k,j}$. Since each of the first i rows of A sum up to 1, the i-th row of B sums up to i. From the definition of 2-ASMs it immediately follows that $b_{i,j} \in \{0, 1, 2\}$ and $b_{n,j} = 2$. By putting $b_{i,j}$ entries equal to j in row i in decreasing order one obtains a triangular array T of integers with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$ where each of $\{1, 2, \dots, n\}$ is contained at most twice in each row.

Suppose T violates condition (D3) of DMTs in rows i and $i+1$ for the first time, i.e. $b_{i,j} = b_{i+1,j} = 1$. If $i = 1$, then $a_{1,j} = 1$ and $a_{2,j} = 0$, contradicting condition (2-ASM 2). If $i > 1$, then the minimality of i implies that $b_{i-1,j} \in \{0, 2\}$. In terms of the 2-ASM-machine $b_{i-1,j} = 0$ means that after $i-1$ steps in generating column j of A, one is in state $\Sigma = 0$. But $b_{i,j} = b_{i+1,j} = 1$ implies $a_{i,j} = 1$ and $a_{i,j+1} = 0$ but there is no $(1, 0)$-edge leaving state $\Sigma = 0$, contradiction. Similarly, $b_{i-1,j} = 2$ together with $b_{i,j} = b_{i+1,j} = 1$ translates into trying to take a $(-1, 0)$-edge leaving state $\Sigma = 2$, contradiction.

To prove that $T \in \mathcal{W}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$, it remains to check condition (D1), i.e. the weak decrease along diagonals. Let $c_{i,j} := \sum_{k=1}^{j} b_{i,k}$ denote the total number of entries $\leq j$ in row i of T. From condition (2-ASM 1) it follows that $c_{1,j} \leq c_{2,j} \leq \cdots \leq c_{2n,j}$, which ensures that entries along SE-diagonals of T are weakly decreasing. Analogously, the alternating sign condition of 2-ASMs in rows read from right to left implies the weak increase along NE-diagonals.

To prove that the mapping is indeed a bijection we have to show that the obvious candidate for the inverse mapping yields a 2-ASM for each $T \in \mathcal{W}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$: Given

$T \in \mathcal{W}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$, define the $(2n) \times n$ matrices B and A with $b_{i,j}$ being the number of entries j in row i of T, $a_{1,j} = b_{1,j}$ and $a_{i,j} = b_{i,j} - b_{i-1,j}$ for $2 \leq i \leq 2n$.

Condition (D2) of DMTs implies $b_{i,j} \in \{0, 1, 2\}$ and thus $a_{i,j} \in \{0, 1, -1, 2, -2\}$. If a row of the DMT contains a pair (x, x), then the interlaced neighbour in the row above and below also equals x by condition (D1), and therefore $a_{i,j} \in \{0, 1, -1\}$. Since the total number of entries increases by 1 in each row of T, each row of A sums up to 1. The alternating sign condition (2-ASM 1) follows from the monotonicity of entries along diagonals in T.

It remains to observe that condition (2-ASM 2) is satisfied by A, or equivalently that each column of A is a word of the 2-ASM-machine in Figure 3.8. To check it for column j, consider the entries j in T from top to bottom.

If $b_{i,j} = 0$, then there are no entries j in row i of T. For the number of entries j in row $i+1$, there are two possibilities: Either there is still no entry j in row $i+1$, then $a_{i+1,j} = 0$ (i.e. take the 0-loop in state $\Sigma = 0$), or there exists exactly one entry j. In the latter case we know from condition (D3) that in row $i+2$ there is either no entry j or two entries j. If there is no entry j, then $a_{i+1,j} = 1$ and $a_{i+2,j} = -1$, i.e. take the $(1, -1)$-loop in state $\Sigma = 0$. If there are two entries j, then $a_{i+1,j} = a_{i+2,j} = 1$, i.e. transit to the state $\Sigma = 2$ taking the $(1, 1)$-edge.

If $b_{i,j} = 2$, then there are exactly two entries j in row i of T. Analogously, the row $i+1$ contains either two entries j again, i.e. take the 0-loop at state $\Sigma = 2$, or the row below contains exactly one entry j. In the latter case condition (D3) implies that row $i+2$ contains either two or no entries j, i.e. take the $(-1, 1)$-loop or the $(-1, -1)$-edge.

Since the bottom row of T is $(n, n, n-1, n-1, \dots, 1, 1)$ all columns of A sum up to 2, and therefore A is a 2-ASM. □

3.4 Connections between MTs and DMTs

The starting point for the content of this section was the empirical observation of (3.1.1). The following computational proof of Theorem 3.4.1 applies the methodology used in Chapter 2 to reprove the Refined ASM Theorem.

Theorem 3.4.1. *The numbers*

$$W_{n,i} := \Delta_{k_1}^{i-1} \alpha(2n-1; k_1, n-1, n-1, n-2, n-2, \dots, 1, 1) \Big|_{k_1 = n}$$

are given by

$$W_{n,i} = \sum_{l=1}^{i} \binom{i-1}{l-1} (-1)^{n+i+l-1} A_{n,l} \tag{3.4.1}$$

for all $i = 1, \dots, 2n-1$, $n \geq 1$.

Recall the definition of the shift and difference operators from Table 2.1 (p. 20). Let us first note that Theorem 3.4.1 and Theorem 3.1.5 are equivalent: On the one hand, (3.1.3) implies for

$i = 1, \ldots, 2n-1$ that

$$\begin{aligned} W_{n,i} &= (\mathrm{E}_{k_1} - \mathrm{id})^{i-1} \alpha(2n-1; k_1, n-1, n-1, \ldots, 1, 1) \Big|_{k_1=n} \\ &= \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{i-1-l} \alpha(2n-1; n+l, n-1, n-1, \ldots, 1, 1) \\ &= \sum_{l=1}^{i} \binom{i-1}{l-1} (-1)^{n+i+l-1} A_{n,l}. \end{aligned}$$

Conversely, for $i = 1, \ldots, 2n-1$ we have that

$$\begin{aligned} &\alpha(2n-1; n-1+i, n-1, n-1, \ldots, 1, 1) \\ &\quad = \mathrm{E}_{k_1}^{i-1} \, \alpha(2n-1; k_1, n-1, n-1, \ldots, 1, 1) \Big|_{k_1=n} \\ &\quad = (\Delta_{k_1} + \mathrm{id})^{i-1} \alpha(2n-1; k_1, n-1, n-1, \ldots, 1, 1) \Big|_{k_1=n} \\ &\quad = \sum_{k=0}^{i-1} \binom{i-1}{k} W_{n,k+1} = \sum_{k=0}^{i-1} \binom{i-1}{k} \sum_{l=1}^{k+1} \binom{k}{l-1} (-1)^{n+k+l} A_{n,l} \\ &\quad = \sum_{l=1}^{i} (-1)^{n+l} A_{n,l} \sum_{k=l-1}^{i-1} \binom{i-1}{k} \binom{k}{l-1} (-1)^k. \end{aligned}$$

The inner sum is a Chu–Vandermonde convolution (A.2.1):

$$\begin{aligned} \sum_{k=l-1}^{i-1} \binom{i-1}{k} \binom{k}{l-1} (-1)^k &= \sum_{k=l-1}^{i-1} \binom{i-1}{i-k-1} \binom{k}{k-l+1} (-1)^k \\ &= (-1)^{l-1} \sum_{k} \binom{i-1}{i-k-1} \binom{-l}{k-l+1} = (-1)^{l-1} \binom{i-l-1}{i-l}. \end{aligned}$$

Since $\binom{a-1}{a} = 0$ for all $a \in \mathbb{Z} \setminus \{0\}$ (cf. (2.5.10)), the only summand not vanishing is $l = i$, yielding the equation we claim in Theorem 3.1.5.

Let us also note that (3.1.1) is a consequence of Theorem 3.1.5: On the one hand, Theorem 3.1.5 implies that

$$\alpha(2n+1; n+1, n, n, \ldots, 1, 1) = (-1)^n A_{n+1,1} = (-1)^n A_n = (-1)^n \alpha(n; 1, 2, \ldots, n).$$

On the other hand, since the only admissible penultimate row of a DMT with bottom row $(n+1, n, n, \ldots, 1, 1)$ is $(n, n, n-1, n-1, \ldots, 1, 1)$, Lemma 3.2.3 and (2.2.4) yield

$$\alpha(2n+1; n+1, n, n, \ldots, 1, 1) = (-1)^n \alpha(2n; n, n, n-1, n-1, \ldots, 1, 1).$$

Following the methodology used in Chapter 2, the proof of Theorem 3.4.1 consists of the following steps:

In Lemma 3.4.2 the numbers $W_{n,i}$ are given a combinatorial meaning and shown to satisfy a certain symmetry relation. After that, we derive in Lemma 3.4.3 that $(W_{n,i})_{i=1,\dots,2n-1}$ is an eigenvector of $\left(\binom{n-i}{2n-i-j}(-1)^{n+i}\right)_{1\leq i,j\leq 2n-1}$ with eigenvalue 1. In Lemma 3.4.4 we prove that the corresponding eigenspace is one-dimensional. In Lemma 3.4.6 we show that

$$\left(\sum_{l=1}^{i}\binom{i-1}{l-1}(-1)^{n+i+l-1}A_{n,l}\right)_{i=1,\dots,2n-1}$$

is in the same eigenspace. Finally, we obtain a recursion for $W_{n,1}$ in Lemma 3.4.5, which then lets us inductively derive the constant factor.

The symmetry $A_{n,i}=A_{n+1-i}$ satisfied by the refined ASM numbers is a direct consequence of the involution reflecting an ASM along the horizontal symmetry axis. The idea for proving the symmetry $W_{n,i}=(-1)^{n-1}W_{n,2n-i}$ is to give the numbers a combinatorial interpretation (as signed enumeration) and to construct a one-to-one correspondence between the objects enumerated by $W_{n,i}$ and those enumerated by $W_{n,2n-i}$. In this case a sign-change of $(-1)^{n-1}$ is involved.

Lemma 3.4.2. *The numbers $W_{n,i}$ satisfy the symmetry relation*

$$W_{n,i}=(-1)^{n-1}W_{n,2n-i}$$

for $i=1,\dots,2n-1$, $n\geq 1$.

Proof. Let us start by checking the claim $W_{n,1}=(-1)^{n-1}W_{n,2n-1}$ separately: In Lemma 2.4.1 we showed that $(-1)^{n-1}\Delta_{k_1}^{n-1}\alpha(n;k_1,k_2,\dots,k_n)$ counts the number of MTs with bottom row $(k_2,k_3,\dots,k_n)$ whenever $k_1\leq k_2<k_3<\cdots<k_n$, i.e.

$$(-1)^{n-1}\Delta_{k_1}^{n-1}\alpha(n;k_1,k_2,\dots,k_n)=\alpha(n-1;k_2,k_3,\dots,k_n). \tag{3.4.2}$$

Together with Remark 2.3.6 it follows that (3.4.2) holds as a polynomial identity for all $(k_1,\dots,k_n)\in\mathbb{C}^n$. In particular

$$\begin{aligned}W_{n,2n-1}&=\Delta_{k_1}^{2n-2}\alpha(2n-1;k_1,n-1,n-1,\dots,1,1)\Big|_{k_1=n}\\&=\alpha(2n-2;n-1,n-1,\dots,1,1).\end{aligned}$$

Since the only admissible penultimate row of a DMT with bottom row $(n,n-1,n-1,\dots,1,1)$ is the row $(n-1,n-1,\dots,1,1)$ consisting of $n-1$ pairs, Lemma 3.2.3 implies

$$W_{n,1}=(-1)^{n-1}\alpha(2n-2;n-1,n-1,\dots,1,1)=(-1)^{n-1}W_{n,2n-1}.$$

In the following let $2\leq i\leq 2n-2$. From (2.4.4) and Lemma 3.2.3 it follows for weakly decreasing sequences $k_1\geq k_2\geq\cdots\geq k_n$ that

$$\begin{aligned}&\Delta_{k_1}\alpha(n;k_1,\dots,k_n)\\&\quad=-\sum_{(l_2,\dots,l_{n-1})\in\mathcal{P}(k_2,\dots,k_n)}(-1)^{\mathrm{sc}(k_2,\dots,k_n;l_2,\dots,l_{n-1})}\alpha(n-1;k_1,l_2,\dots,l_{n-1}),\end{aligned} \tag{3.4.3}$$

where $\mathrm{sc}(k_2,\dots,k_n;l_2,\dots,l_{n-1})$ is the total number of pairs and newcomers (Definition 3.2.2) in $(l_2,\dots,l_{n-1})$. Let us define a DMT-trapezoid to be an array of integers $(a_{p,q})_{\substack{1\le p\le j,\\ 1\le q\le n-j+p}}$ arranged in the form

$$\begin{array}{ccccccc} & & a_{1,1} & \cdots & a_{1,n-j+1} & & \\ & \iddots & & & & \ddots & \\ a_{j,1} & & \cdots & & \cdots & & a_{j,n} \end{array}$$

satisfying the same conditions (D1), (D2) and (D3) as DMTs. Let $\mathcal{P}_j(k_1,\dots,k_n;l_1,\dots,l_{n-j+1})$ denote the set of DMT-trapezoids with j rows, bottom row $(k_1,\dots,k_n)$ and top row $(l_1,\dots,l_{n-j+1})$. As an example, the two DMT-trapezoids with bottom row $(7,7,4,1,1)$ and top row $(6,4,2)$ are depicted in Figure 3.9. Given a DMT-trapezoid $\mathcal{T}$ let $\mathrm{sc}(\mathcal{T})$ denote the total number of sign-

$$\begin{array}{ccccccccc} & & 6 & & 4 & & 2 & & \\ & 7 & & 5 & & 3 & & 1 & \\ 7 & & 7 & & 4 & & 1 & & 1 \end{array} \qquad \begin{array}{ccccccccc} & & 6 & & 4 & & 2 & & \\ & 7 & & 4 & & 4 & & 1 & \\ 7 & & 7 & & 4 & & 1 & & 1 \end{array}$$

Figure 3.9: The two DMT-trapezoids contained in $\mathcal{P}_3(7,7,4,1,1;6,4,2)$.

changes in the trapezoid, i.e. the total number of pairs and newcomers contained in all rows except the bottom row. For example, the two DMT-trapezoids depicted in Figure 3.9 contain 5 resp. 3 sign-changes.

Since the Δ-operator is linear and the sequence $(k_1,l_2,\dots,l_{n-1})$ in (3.4.3) is again weakly decreasing, we can apply induction to see that

$$\begin{aligned}&\Delta_{k_1}^{j-1}\alpha(n;k_1,\dots,k_n)\\&\quad=(-1)^{j-1}\sum_{\substack{(l_1,\dots,l_{n-j},\mathcal{T}):\\ \mathcal{T}\in\mathcal{P}_j(k_2,\dots,k_n;l_1,\dots,l_{n-j})}}(-1)^{\mathrm{sc}(\mathcal{T})}\alpha(n-j+1;k_1,l_1,l_2,\dots,l_{n-j})\end{aligned}$$

holds for $2\le j\le n-1$. Together with Corollary 3.2.4 we obtain

$$\begin{aligned}W_{n,i}&=(-1)^{i-1}\sum_{\substack{(l_1,\dots,l_{2n-i-1},\mathcal{T}):\\ \mathcal{T}\in\mathcal{P}_i(n-1,n-1,\dots,1,1;l_1,\dots,l_{2n-i-1})}}(-1)^{\mathrm{sc}(\mathcal{T})}\alpha(2n-i;n,l_1,l_2,\dots,l_{2n-i-1})\\&=\sum_{\substack{(l_1,\dots,l_{2n-i-1},\mathcal{T}):\\ \mathcal{T}\in\mathcal{P}_i(n-1,n-1,\dots,1,1;l_1,\dots,l_{2n-i-1})}}\ \sum_{A\in\mathcal{W}_{2n-i}(n,l_1,l_2,\dots,l_{2n-i-1})}(-1)^{i-1+\mathrm{sc}(\mathcal{T})+\mathrm{sc}(A)},\end{aligned}$$

i.e. $W_{n,i}$ is a signed enumeration of the following partial DMTs: Start with the bottom row $(n-1,n-1,\dots,1,1)$ and construct a DMT-trapezoid with i rows. Then append an entry n at the left end of the top row of the DMT-trapezoid and complete it to the top as a DMT (see Figure 3.10).

Note that condition (D1) of DMTs implies that the right neighbour of the entry n in row $2n-i$ is smaller than n, and by condition (D3') their interlaced neighbour in the row above is also smaller than n. Hence, all entries are in $\{1,2,\dots,n-1\}$ except for the one entry n at the fixed location in row $2n-i$.

Let us now encode the position of the entries smaller than n in a $(2n-1)\times(n-1)$-matrix with entries $\{0,1,-1\}$ as done in Proposition 3.3.2. More precisely, the sum of the top i entries in

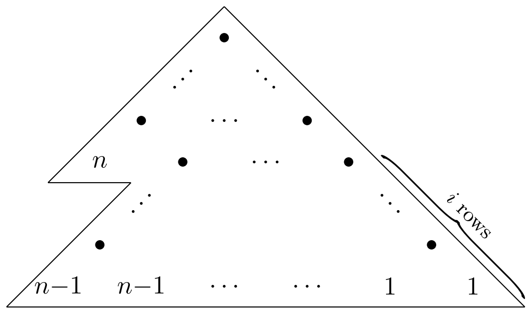


Figure 3.10: Combinatorial interpretation of $W_{n,i}$.

column j of the matrix is equal to the number of entries j in row i of the partial DMT. Conditions (D2) and (D3) of DMTs imply that in each column of the matrix the non-zero entries occur in pairs such that each partial column sum is in $\{0,1,2\}$. Since the bottom row is $(n-1,n-1,\dots,1,1)$ each column sums up to 2. Condition (D1) implies that in all rows the non-zero entries alternate in sign. Moreover, the number of entries smaller than or equal to $n-1$ increases by 1 in each row except row $2n-i$ where it stays the same. As a consequence, the sum of each row in the corresponding matrix is 1 except for row $2n-i$ where the last non-zero entry is a -1 and the row-sum is therefore 0. An example is given in Figure 3.11.

$$
\begin{array}{ccccccccccc}
 & & & & 2 & & & & & & \\
 & & & 2 & & 2 & & & & & \\
 & & 3 & & 2 & & 2 & & & & \\
 & 3 & & 3 & & 2 & & 1 & & & \\
4 & & 3 & & 3 & & 1 & & 1 & & \\
 & 3 & & 3 & & 2 & & 1 & & 1 & \\
3 & & 3 & & 2 & & 2 & & 1 & & 1
\end{array}
\qquad \Longleftrightarrow \qquad
\begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -1 & 1 \\
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}
$$

Figure 3.11: One of the objects counted by $W_{4,3}$ and its corresponding matrix.

Phrased differently, the described matrices are $(2n-1)\times(n-1)$-matrices such that

- all columns are generated by the machine in Figure 3.8 with length $2n-1$,
- the $(2n-i)$-th row is generated by the machine in Figure 3.7 with length $n-1$, but the transition to the final state is located at the state $\Sigma = 0$ (since the last non-zero entry is a -1), and
- all other rows are generated by the machine in Figure 3.7 with length $n-1$.

This establishes a one-to-one correspondence with the partial DMTs occurring in the signed enumeration of $W_{n,i}$ (cf. Proposition 3.3.2). The advantage of this perspective is that the matrices corresponding to $W_{n,i}$ and those corresponding to $W_{n,2n-i}$ are now subject to a simple bijection, namely reflecting the corresponding matrices along the horizontal symmetry axis. It remains to show that this reflection changes the parity of $\mathrm{sc}(\mathcal{T}) + \mathrm{sc}(A)$ by $n-1$.

The total number of sign-changes has an interpretation in terms of the corresponding matrices by counting how often which of the edges in the column-generating machine in Figure 3.8 are taken. On the one hand, the total number of pairs in the partial DMT equals the number of positions in the matrix where the partial column-sum equals 2. This is exactly the number of times an edge leading into the state $\Sigma = 2$ is taken. On the other hand, the number of newcomers in the partial DMT is equal to the number of positions in the matrix where the partial column sum is 1 and the entry below is a -1. Hence, it is given by the total number of times we take the $(-1,-1)$-edge or the $(1,-1)$-edge in Figure 3.8. Since the bottom row contains $n-1$ pairs, we obtain that $\mathrm{sc}(\mathcal{T})+\mathrm{sc}(A)+n-1$ is equal to the total number of edges taken for generating entries in the matrix except for the 0-loop at the state $\Sigma = 0$.

Note that reflecting the matrix along the horizontal symmetry axis means that the columns of the reflected matrix are generated in reverse order, i.e. the number of times the edges are taken is interchanged in the following way: $(1,-1)$-edge $\leftrightarrow$ $(-1,1)$-edge, 0-loop at state $\Sigma = 0 \leftrightarrow$ 0-loop at state $\Sigma = 2$. Hence, the parity of the difference between the number of times we take the two 0-loops gives us the change of $(-1)^{\mathrm{sc}(\mathcal{T})+\mathrm{sc}(A)}$ under the reflection. But the parity of this difference is independent from $\mathcal{T}$ and A. This is because the total number of entries in the matrix is $(2n-1)(n-1)$ and each of the other edges generates two entries of the matrix. Therefore the parity of the difference of the number of times we take the two 0-loops has to be $n-1$. We conclude

$$\begin{aligned} W_{n,i} &= \sum_{\substack{(l_1,\ldots,l_{2n-i-1},\mathcal{T}):\\ \mathcal{T}\in\mathcal{P}_i(n-1,n-1,\ldots,1,1;l_1,\ldots,l_{2n-i-1})}} \sum_{A\in\mathcal{W}_{2n-i}(n,l_1,l_2,\ldots,l_{2n-i-1})} (-1)^{i-1+\mathrm{sc}(\mathcal{T})+\mathrm{sc}(A)} \\ &= (-1)^{n-1} W_{n,2n-i}. \end{aligned}$$ □

In the following, we show that the numbers $W_{n,i}$ satisfy a certain system of linear equations.

Lemma 3.4.3. *Fix $n \geq 1$. The numbers $W_{n,i}$ satisfy the system of linear equations*

$$W_{n,i} = \sum_{j=1}^{2n-1} \binom{n-i}{2n-i-j} (-1)^{n+i} W_{n,j}, \tag{3.4.4}$$

for $i = 1, \ldots, 2n-1$.

For the proof, keep in mind that by (2.5.10) the binomial coefficient $\binom{n-i}{2n-i-j}$ does not necessarily vanish for $i > n$, but does so for $j > 2n-i$.

Proof. The general idea of the proof is similar to Lemma 2.5.2 where we derived a system of linear equations for the refined ASM numbers by applying the circular shift identity (2.5.2). In this proof we additionally require the following identity

$$\alpha(n; k_1, k_2, \ldots, k_n) = \alpha(n; -k_n, -k_{n-1}, \ldots, -k_1), \tag{3.4.5}$$

which is combinatorially clear for integers $k_1 < k_2 < \cdots < k_n$ (reflect MT w.r.t. horizontal symmetry axis and change the sign of each entry). Since both sides of (3.4.5) are polynomials, the identity holds for arbitrary $(k_1, k_2, \ldots, k_n) \in \mathbb{C}^n$ (Remark 2.2.1).

First apply the circular shift identity to obtain

$$\begin{aligned} W_{n,i} &= \Delta_{k_1}^{i-1} \alpha(2n-1; n-1, n-1, \dots, 1, 1, k_1 - 2n + 1) \Big|_{k_1=n} \\ &= \mathrm{E}_{k_1}^{i-n} \, \delta_{k_1}^{i-1} \alpha(2n-1; n-1, n-1, \dots, 1, 1, k_1 - n) \Big|_{k_1=n}. \end{aligned}$$

Then use $\mathrm{E}_x = (\mathrm{id} - \delta_x)^{-1}$, (3.4.5) and Remark 2.3.6 to see that

$$\begin{aligned} W_{n,i} &= \sum_{j \geq 0} \binom{n-i}{j} (-1)^j \delta_{k_1}^{i+j-1} \alpha(2n-1; n-1, n-1, \dots, 1, 1, k_1) \Big|_{k_1=0} \\ &= \sum_{j=0}^{2n-i-1} \binom{n-i}{j} (-1)^j \delta_{k_1}^{i+j-1} \alpha(2n-1; -k_1, -1, -1, \dots, -(n-1), -(n-1)) \Big|_{k_1=0}. \end{aligned}$$

The difference operators satisfy the equation

$$\delta_x^i f(-x) = (-1)^i \Delta_y^i f(y) \Big|_{y=-x} \tag{3.4.6}$$

for any function f and $i \geq 0$: The case $i = 0$ is obvious. For $i \geq 1$ define the operator $\mathrm{N}_x \, f(x) := f(-x)$ and observe that $\delta_x \, \mathrm{N}_x = - \mathrm{N}_x \, \Delta_x$:

$$\delta_x \, \mathrm{N}_x \, f(x) = f(-x) - f(-x+1) = \mathrm{N}_x \, f(x) - \mathrm{N}_x \, f(x+1) = - \mathrm{N}_x \, \Delta_x f(x).$$

Therefore we obtain

$$\delta_x^i f(-x) = \delta_x^i \, \mathrm{N}_x \, f(x) = (-1)^i \, \mathrm{N}_x \, \Delta_x^i f(x) = (-1)^i \Delta_y^i f(y) \Big|_{y=-x}.$$

Together with (2.5.1) it follows that

$$\begin{aligned} W_{n,i} &= \sum_{j=0}^{2n-i-1} \binom{n-i}{j} (-1)^{i-1} \Delta_{k_1}^{i+j-1} \alpha(2n-1; k_1, -1, -1, \dots, -(n-1), -(n-1)) \Big|_{k_1=0} \\ &= \sum_{j=0}^{2n-i-1} \binom{n-i}{j} (-1)^{i-1} \Delta_{k_1}^{i+j-1} \alpha(2n-1; k_1, n-1, n-1, \dots, 1, 1) \Big|_{k_1=n} \\ &= \sum_{j=0}^{2n-i-1} \binom{n-i}{j} (-1)^{i-1} W_{n,i+j} = \sum_{j=1}^{2n-1} \binom{n-i}{j-i} (-1)^{i-1} W_{n,j}. \end{aligned}$$

Using the symmetry shown in Lemma 3.4.2 yields

$$W_{n,i} = \sum_{j=1}^{2n-1} \binom{n-i}{j-i} (-1)^{n+i} W_{n,2n-j} = \sum_{j=1}^{2n-1} \binom{n-i}{2n-i-j} (-1)^{n+i} W_{n,j}. \qquad \square$$

Lemma 3.4.4. *Let $\delta_{i,j}$ denote the Kronecker delta. Then*

$$\operatorname{rk}\left(\binom{n-i}{2n-i-j}(-1)^{n+i}-\delta_{i,j}\right)_{1\le i,j\le 2n-1}=2n-2,$$

holds for $n \ge 1$, i.e. the 1*-eigenspace of $\left(\binom{n-i}{2n-i-j}(-1)^{n+i}\right)_{1\le i,j\le 2n-1}$ is one-dimensional.*

Proof. Note that the n-th row of $S := \left(\binom{n-i}{2n-i-j}(-1)^{n+i}-\delta_{i,j}\right)_{1\le i,j\le 2n-1}$ contains only zeroes. In the following, we show that removing this row and the last column results in a $(2n-2)\times(2n-2)$-matrix S' with non-zero determinant (in fact, it is equal to $(-1)^{n-1}A_{n-1}$, where A_{n-1} denotes the number of ASMs of size $n-1$). The block structure of S' is displayed in Figure 3.12. The

$$\begin{array}{r c}
 & j=1,\dots,2n-2 \\
\begin{array}{r} i=1,\dots,n-1 \\ \\ \\ \\ \\ i=n+1,\dots,2n-1 \end{array} &
\left[\begin{array}{cccc|cccc}
-1 & 0 & \cdots & 0 & & & & \\
0 & -1 & \ddots & \vdots & & \binom{n-i}{2n-i-j}(-1)^{n+i} & & \\
\vdots & \ddots & \ddots & 0 & & & & \\
0 & \cdots & 0 & -1 & & & & \\
\hline
 & & & & 0 & -1 & \cdots & 0 \\
 & \binom{n-i}{2n-i-j}(-1)^{n+i} & & & 0 & 0 & \ddots & \vdots \\
 & & & & \vdots & \ddots & \ddots & -1 \\
 & & & & 0 & \cdots & 0 & 0
\end{array}\right]
\end{array}$$

Figure 3.12: The matrix S' decomposed into four $(n-1)\times(n-1)$-blocks.

determinant of a block matrix $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ with an invertible matrix A and a square matrix D is equal to $\det(A)\det(D-CA^{-1}B)$: This follows from the decomposition (I denotes the identity matrix)

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)=\left(\begin{array}{c|c} A & 0 \\ \hline C & I \end{array}\right)\left(\begin{array}{c|c} I & A^{-1}B \\ \hline 0 & D-CA^{-1}B \end{array}\right)$$

together with the fact that

$$\begin{aligned}\det\left(\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right)&=\det\left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right)=\det\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)=\det\left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & D \end{array}\right)\\ &=\det(A)\det(D).\end{aligned}$$

The block matrices in our case are

$$\begin{aligned} A &= -I_{n-1}, \\ B &= \left(\binom{n-i}{n-i-j+1} (-1)^{n+i} \right)_{1 \le i,j \le n-1}, \\ C &= \left(\binom{-i}{n-i-j} (-1)^{i} \right)_{1 \le i,j \le n-1}, \\ D &= (-\delta_{i+1,j})_{1 \le i,j \le n-1}. \end{aligned}$$

That $C^{-1} = \left(\binom{i-n}{i+j-n} (-1)^{n-i} \right)_{1 \le i,j \le n-1}$ is indeed the inverse matrix of C follows from Chu–Vandermonde convolution (A.2.1):

$$\begin{aligned} \sum_{k=1}^{n-1} \binom{-i}{n-i-k} \binom{k-n}{k+j-n} (-1)^{i+n-k} &= (-1)^{i+j} \sum_{k} \binom{-i}{n-i-k} \binom{j-1}{k+j-n} \\ &= (-1)^{i+j} \binom{j-i-1}{j-i} = \delta_{i,j}. \end{aligned}$$

Note that the binomial coefficients in C vanish, unless $i+j \le n$, i.e. C is an upper-left triangular matrix, whereby the anti-diagonal entry in row i is equal to $(-1)^i$. It follows that $\det(A) = \det(C) = (-1)^{n-1}$. Together with $A^{-1} = -I_{n-1}$, we obtain

$$\begin{aligned} \det(S') &= \det(A) \det(C) \det(C^{-1}D - A^{-1}B) = \det(C^{-1}D + B) \\ &= \det_{1 \le i,j \le n-1} \left(\binom{i-n}{i+j-1-n} (-1)^{n+i+1} + \binom{n-i}{n-i-j+1} (-1)^{n+i} \right). \end{aligned}$$

Since $n-i$ is non-negative, the identity $\binom{n}{k} = \binom{n}{n-k}$ is applicable to the second binomial coefficient. Multiplying the i-th row with $(-1)^{n+i}$ yields

$$\det(S') = (-1)^{\binom{n}{2}} \det_{1 \le i,j \le n-1} \left(\binom{n-i}{j-1} - \binom{i-n}{i+j-1-n} \right).$$

As $(-1)^{\binom{n}{2} + \lfloor \frac{n-1}{2} \rfloor} = (-1)^{n-1}$, switching row i with row $n-i$ for $i = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ gives

$$\det(S') = (-1)^{n-1} \det_{1 \le i,j \le n-1} \left(\binom{i}{j-1} - \binom{-i}{j-i-1} \right).$$

Multiply from the right with the upper-triangular matrix $\left(\binom{j-2}{j-i} \right)_{1 \le i,j \le n-1}$ having determinant 1 and apply Chu–Vandermonde convolution:

$$\begin{aligned} \det(S') &= (-1)^{n-1} \det_{1 \le i,j \le n-1} \left(\sum_{k=1}^{n-1} \left(\binom{i}{k-1} - \binom{-i}{k-i-1} \right) \binom{j-2}{j-k} \right) \\ &= (-1)^{n-1} \det_{1 \le i,j \le n-1} \left(\binom{i+j-2}{j-1} - \binom{j-i-2}{j-i-1} \right) \\ &= (-1)^{n-1} \det_{0 \le i,j \le n-2} \left(\binom{i+j}{i} - \delta_{i,j+1} \right). \end{aligned}$$

In [BFZJ12] it was shown that the determinant of $\left(\binom{i+j}{i} - \delta_{i,j+1}\right)_{0\le i,j\le n-1}$ is equal to the number of Descending Plane Partitions with each part smaller than or equal to n, which is known to be equal to the number of ASMs of size n (see [And79, Theorem 3]). It follows that

$$\det(S') = (-1)^{n-1} A_{n-1} \neq 0. \qquad \square$$

Similar to the combinatorial identity $A_{n,1} = \sum_{i=1}^{n-1} A_{n-1,i}$ satisfied by the refined ASM numbers, $W_{n,1}$ can also be expressed in terms of the numbers $W_{n-1,i}$.

Lemma 3.4.5. *Let $n \geq 2$. Then*

$$W_{n,1} = -\sum_{i=1}^{n-1} \binom{n-1}{i} W_{n-1,i}.$$

Proof. The only penultimate row of a DMT with bottom row $(n, n-1, n-1, \dots, 1, 1)$ is $(n-1, n-1, \dots, 1, 1)$. From Lemma 3.2.3 and the circular shift identity (2.5.1) one therefore obtains

$$\begin{aligned} W_{n,1} &= \alpha(2n-1; n, n-1, n-1, \dots, 1, 1) \\ &= (-1)^{n-1} \alpha(2n-2; n-1, n-1, \dots, 1, 1) \\ &= (-1)^{n} \alpha(2n-2; 2n-1, n-1, n-1, \dots, 2, 2, 1). \end{aligned}$$

The possible penultimate rows of a DMT with bottom row $(2n-1, n-1, n-1, \dots, 2, 2, 1)$ are $(l, n-1, n-1, \dots, 2, 2)$ where l may take any value in $\{n, n+1, \dots, 2n-2\}$. Each such penultimate row consists of $n-2$ pairs and one newcomer (Definition 3.2.2). The claim now follows from Lemma 3.2.3 and the binomial identity $\sum\limits_{k=a}^{b} \binom{k}{a} = \binom{b+1}{a+1}$:

$$\begin{aligned} W_{n,1} &= -\sum_{l=n}^{2n-2} \alpha(2n-3; l, n-1, n-1, \dots, 2, 2) \\ &\stackrel{(2.5.1)}{=} -\sum_{l=n-1}^{2n-3} \alpha(2n-3; l, n-2, n-2, \dots, 1, 1) \\ &= -\sum_{l=n-1}^{2n-3} (\Delta_{k_1} + \mathrm{id})^{l-n+1} \alpha(2n-3; k_1, n-2, n-2, \dots, 1, 1)\Big|_{k_1=n-1} \\ &= -\sum_{l=n-1}^{2n-3} \sum_{i=0}^{l-n+1} \binom{l-n+1}{i} \Delta_{k_1}^{i} \alpha(2n-3; k_1, n-2, n-2, \dots, 1, 1)\Big|_{k_1=n-1} \\ &= -\sum_{i=0}^{n-2} W_{n-1,i+1} \sum_{l=n+i-1}^{2n-3} \binom{l-n+1}{i} = -\sum_{i=1}^{n-1} \binom{n-1}{i} W_{n-1,i}. \qquad \square \end{aligned}$$

The next step is to show that the numbers on the right-hand side of (3.4.1) are also a solution of the LES (3.4.4) with one-dimensional solution space.

Lemma 3.4.6. *Let* $X_{n,i} := \sum_{l=1}^{i} \binom{i-1}{l-1} (-1)^{n+i+l-1} A_{n,l}$. *Then*

$$X_{n,i} = \sum_{j=1}^{2n-1} \binom{n-i}{2n-i-j} (-1)^{n+i} X_{n,j}$$

holds for all $i = 1, \ldots, 2n-1$, $n \geq 1$.

Proof. By (2.1.1), the refined ASM numbers satisfy

$$A_{n,l} = \binom{n+l-2}{n-1} \binom{2n-l-1}{n-1} c_n$$

with a constant c_n independent from l. Hence, it suffices to show that

$$\sum_{j=1}^{2n-1} \binom{n-i}{2n-i-j} \sum_{l=1}^{j} \binom{j-1}{l-1} \binom{n+l-2}{n-1} \binom{2n-l-1}{n-1} (-1)^{i+l+j-1}$$
$$= \sum_{l=1}^{i} \binom{i-1}{l-1} \binom{n+l-2}{n-1} \binom{2n-l-1}{n-1} (-1)^{n+i+l-1}.$$

One way to prove this routinely (see also [PWZ96]) is to apply Chu–Vandermonde convolution to the left-hand side, then identify – under the guidance of C. Krattenthaler's HYP–package [Kra95] – the hypergeometric series on both sides and find out that applying transformation T3207 to the right-hand side twice yields the left-hand side. A computational proof by hand is also included in the appendix (Lemma A.2.3). □

We are now in the position to conclude the proof of Theorem 3.4.1: Since $(W_{n,i})_{i=1,\ldots,2n-1}$ (Lemma 3.4.3) and $(X_{n,i})_{i=1,\ldots,2n-1}$ (Lemma 3.4.6) both satisfy the same system of linear equations with one-dimensional solution space (Lemma 3.4.4), it follows that

$$(X_{n,i})_{i=1,\ldots,2n-1} = C_n (W_{n,i})_{i=1,\ldots,2n-1}, \quad n \geq 1.$$

Let us now show by induction w.r.t. n, that $X_{n,i} = W_{n,i}$ for all $i = 1, \ldots, 2n-1$, $n \geq 1$. The case $n = 1$ is trivial ($X_{1,1} = W_{1,1} = 1$). For $n \geq 2$, it suffices to show that $X_{n,1} = W_{n,1}$, since then $C_n = 1$. Apply Lemma 3.4.5, the induction hypothesis and Chu–Vandermonde convolution to obtain

$$\begin{aligned}
W_{n,1} &= -\sum_{i=1}^{n-1} \binom{n-1}{i} W_{n-1,i} = -\sum_{i=1}^{n-1} \binom{n-1}{i} X_{n-1,i} \\
&= -\sum_{i=1}^{n-1} \binom{n-1}{i} \sum_{l=1}^{i} \binom{i-1}{l-1} (-1)^{n+i+l} A_{n-1,l} \\
&= (-1)^{n-1} \sum_{l=1}^{n-1} A_{n-1,l} \sum_{i=l}^{n-1} \binom{n-1}{n-1-i} \binom{i-1}{i-l} (-1)^{i+l} \\
&= (-1)^{n-1} \sum_{l=1}^{n-1} A_{n-1,l} \sum_{i} \binom{n-1}{n-1-i} \binom{-l}{i-l} \\
&= (-1)^{n-1} \sum_{l=1}^{n-1} A_{n-1,l} = (-1)^{n-1} A_{n-1} = (-1)^{n-1} A_{n,1} = X_{n,1}.
\end{aligned}$$

This concludes the proof of Theorem 3.4.1, or equivalently Theorem 3.1.5, and therefore identity (3.1.1).

3.5 Towards a bijective proof of the identity

Even though we have seen a computational proof of (3.1.1) in the previous section, it is from a combinatorialist's point of view more desirable to find a bijective explanation of the identity. The right-hand side counts the number of MTs with bottom row $(1, 2, \dots, n)$. According to Theorem 3.1.4, the left-hand side of (3.1.1) is a signed enumeration of DMTs with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$:

$$\alpha(2n; n, n, n-1, n-1, \dots, 1, 1) = (-1)^{\binom{2n}{2}} \sum_{A \in \mathcal{W}_{2n}(n,n,n-1,n-1,\dots,1,1)} (-1)^{\mathrm{dd}(A)}.$$

Since there are exactly n pairs in the bottom row, and $\binom{2n}{2}$ has the same parity as n, it follows that (3.1.1) is equivalent to

$$\sum_{A \in \mathcal{W}_{2n}(n,n,n-1,n-1,\dots,1,1)} (-1)^{\overline{\mathrm{dd}}(A)} = \alpha(n; 1, 2, \dots, n), \tag{3.5.1}$$

where $\overline{\mathrm{dd}}(A)$ is the number of pairs (x, x) for which there also exists a pair (x, x) in the row below.

A bijective proof of (3.5.1) could succeed by partitioning the set of DMTs with bottom row $(n, n, \dots, 1, 1)$ into three sets:

$$\mathcal{W}_{2n}(n, n, n-1, n-1, \dots, 1, 1) = S_1 \dot\cup S_2 \dot\cup S_3.$$

Hereby $\overline{\mathrm{dd}}(A)$ should be even for all $A \in S_1 \cup S_2$, whereas $\overline{\mathrm{dd}}(A)$ should be odd for all $A \in S_3$. The DMTs in S_1 should be in bijective correspondence with MTs with bottom row $(1, 2, \dots, n)$, and the DMTs in S_2 and S_3 should also be in one-to-one correspondence, thus cancelling each other out in (3.5.1).

It is plausible that S_1 should contain exactly those DMTs with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$ where the $(2i)$-th row consists of i pairs for all $i = 1, \dots, n$ (see Figure 3.13). Note that this also determines the entries in odd rows. Identifying the entries connected by an edge with one single entry and reflecting the triangle along the vertical symmetry axis yields a MT with bottom row $(1, 2, \dots, n)$. An example is given in Figure 3.14. In fact, this establishes a one-to-one correspondence with MTs with bottom row $(1, 2, \dots, n)$:

The weak increase along diagonals directly translates from MTs to DMTs. The strict increase along rows of MTs corresponds to condition (D2) of DMTs, i.e. each integer is contained at most twice per row. Condition (D3) of DMTs is ensured by the structural restriction, since every even row only consists of pairs. Note that for each $A \in S_1$ we have that $\overline{\mathrm{dd}}(A)$ is exactly twice the number of diagonally adjacent identical entries in the corresponding MT. In particular, it follows that $(-1)^{\overline{\mathrm{dd}}(A)} = 1$ for all $A \in S_1$.

Finding a sign-reversing involution on $\mathcal{W}_{2n}(n, n, n-1, n-1, \dots, 1, 1) \setminus S_1$ to completely understand (3.5.1) in a combinatorial way remains an open problem.

Observe that from the viewpoint of 2-ASMs the subset $S_1 \subseteq \mathcal{W}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$ corresponds to the set of 2-ASMs where the column generation is restricted to the machine in Figure 3.15. The one-to-one correspondence with ASMs of size n is even more obvious from this

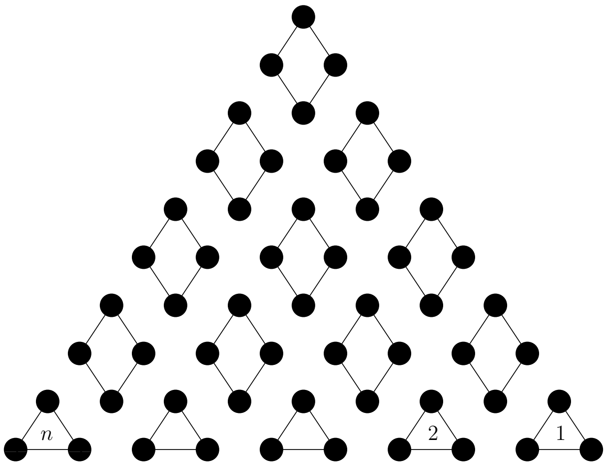


Figure 3.13: Structural restriction of DMTs in S_1: entries connected by an edge must have the same value.

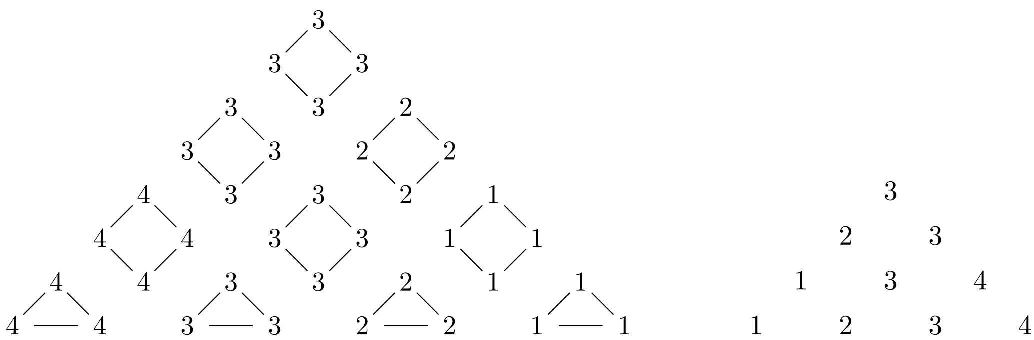


Figure 3.14: A DMT in S_1 and its corresponding MT.

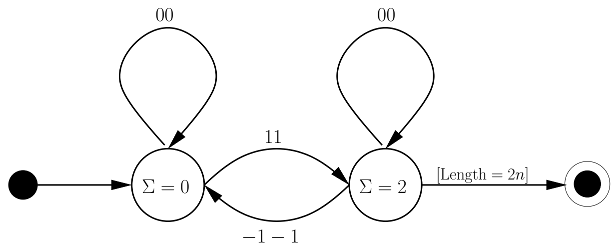


Figure 3.15: Machine generating the columns of those 2-ASMs corresponding to $S_1 \subseteq \mathcal{W}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$.

perspective (cf. Figure 3.7). For all $A \in \mathcal{W}_{2n}(n, n, n-1, n-1, \dots, 1, 1)$, the modified duplicate-descendants statistic $\overline{\mathrm{dd}}$ has an easy translation in terms of the corresponding 2-ASM. Namely, it is equal to the total number of times the 0-edge is taken at the state $\Sigma = 2$ (Figure 3.8) in the generation of all columns. This connection might turn out to be useful in finding a sign-reversing involution.

Another natural question is to ask for the number of DMTs with bottom row $(n, n, n-1, n-1, \dots, 1, 1)$ or equivalently the number of 2-ASMs of size n. However, the first values of the sequence are $1, 2, 7, 58, 1061, 44396$, which are divisible by large prime factors (1061 is actually prime). Hence, one should not expect a nice product formula.

Considering the equation $\alpha(n; 1, 2, \dots, n) = \alpha(2n; n, n, n-1, n-1, \dots, 1, 1)$ it is natural to look for more such identities. The generalization

$$\alpha(n; k_1, \dots, k_n) = \alpha(2n; k_n, k_n, \dots, k_1, k_1)$$

to arbitrary strictly increasing sequences $k_1 < k_2 < \dots < k_n$ does not hold, for example

$$\alpha(2; 1, 4) = 4 \neq 2 = \alpha(4; 4, 4, 1, 1).$$

Yet, the evaluation $\alpha(n; n, n-1, \dots, 1)$ seems to be of interest. Note that for $n = 2m$ the set $\mathcal{W}_{2m}(2m, 2m-1, \dots, 1)$ is empty, and hence Theorem 3.1.4 implies that $\alpha(2m; 2m, 2m-1, \dots, 1) = 0$. However, the situation in the case $n = 2m+1$ seems to be more interesting. The first few values of $\alpha(2m+1; 2m+1, 2m, \dots, 1)$ are displayed in the following table:

$\alpha(3; 3, 2, 1)$	-1
$\alpha(5; 5, 4, 3, 2, 1)$	3
$\alpha(7; 7, 6, 5, 4, 3, 2, 1)$	-26
$\alpha(9; 9, 8, 7, 6, 5, 4, 3, 2, 1)$	646
$\alpha(11; 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$	-45885
$\alpha(13; 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$	9304650

The absolute values of these numbers are known to be the first entries of the sequence of numbers of Vertically Symmetric Alternating Sign Matrices (VSASMs). It is not difficult to see (the argument is provided in Section 5.1) that the set of VSASMs with $2m+1$ rows is in one-to-one correspondence with the set of MTs with bottom row $(2, 4, \dots, 2m)$ leading to Conjecture 3.5.1.

Conjecture 3.5.1. *For $n = 2m + 1$, $m \geq 1$, the equation*

$$\alpha(n; n, n-1, \ldots, 1) = (-1)^m \alpha(m; 2, 4, \ldots, 2m) \tag{3.5.2}$$

holds.

It would be interesting to see, whether similar techniques as presented here (or other techniques) can be applied to prove this conjecture.

Chapter 4

Generalized Monotone Triangles: an extended combinatorial reciprocity theorem

The contents of this chapter appeared in [Rie12].

4.1 Introduction

In the previous chapter we studied the evaluation of the polynomial $\alpha(n; k_1, \ldots, k_n)$ at weakly decreasing sequences $k_1 \geq k_2 \geq \cdots \geq k_n$ and showed that the evaluation can be interpreted as signed enumeration of DMTs. One of the motivations for considering evaluations of $\alpha(n; k_1, \ldots, k_n)$ at non-increasing $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ stems from the connection with ASMs: In Lemma 2.4.1 we observed that the refined ASM numbers can be obtained by applying difference operators to the polynomial, e.g.

$$A_{3,2} = \delta_{k_3} \alpha(3; 1, 2, k_3)\Big|_{k_3=2} = \alpha(3; 1, 2, 2) - \alpha(3; 1, 2, 1). \tag{4.1.1}$$

A second motivation is the circular shift identity

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n), \tag{4.1.2}$$

which plays one of the key roles in the proof of the Refined ASM Theorem given in Chapter 2. Note that if $k_1 < k_2 < \cdots < k_n$, then $k_n > k_1 - n$, i.e. the identity can per se only be understood as identity satisfied by the polynomial. A bijective proof of (4.1.2) could give more combinatorial insight to the Refined ASM Theorem.

In this chapter we first give an interpretation to the evaluation of $\alpha(n; k_1, \ldots, k_n)$ at arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and prove a generalization of Theorem 3.1.4. Then we apply the theorem and combinatorially prove an identity satisfied by $\alpha(n; k_1, \ldots, k_n)$, which already had a combinatorial interpretation in terms of increasing integer arguments. Finally, we consider the circular shift identity in the case $k_1 < k_2 < \cdots < k_n$ and present a first step towards a combinatorial proof.

To state how the evaluation of $\alpha(n; k_1, \ldots, k_n)$ at arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ may be interpreted as signed enumeration, we define a combinatorial object which locally combines the restrictions of ordinary Monotone Triangles and DMTs introduced in Chapter 3.

Definition 4.1.1. *A Generalized Monotone Triangle (GMT) is a triangular array of integers $(a_{i,j})_{1 \le j \le i \le n}$ arranged in the form*

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \iddots & & & & \ddots & \\ a_{n,1} & & \cdots & & \cdots & & a_{n,n} \end{array}$$

satisfying the following three conditions (cf. Figure 4.1):

(G1) Each entry is weakly bounded by its SW- and SE-neighbour, i.e.

$$\min\{a_{i+1,j}, a_{i+1,j+1}\} \le a_{i,j} \le \max\{a_{i+1,j}, a_{i+1,j+1}\}.$$

(G2) If three consecutive entries in a row are weakly increasing, then their two interlaced neighbours in the row above are strictly increasing, i.e.

$$a_{i+1,j} \le a_{i+1,j+1} \le a_{i+1,j+2} \implies a_{i,j} < a_{i,j+1}.$$

(G3) If an entry is strictly larger than its right neighbour and their interlaced neighbour in the row above is equal to its SW-neighbour (resp. SE-neighbour), then the interlaced neighbour has a left (resp. right) neighbour and is equal to it, i.e.

$$\begin{aligned} a_{i,j} = a_{i+1,j} > a_{i+1,j+1} &\implies a_{i,j-1} = a_{i,j}, \\ a_{i+1,j} > a_{i+1,j+1} = a_{i,j} &\implies a_{i,j+1} = a_{i,j}. \end{aligned}$$

As an example (cf. Figure 4.2), let us determine all GMTs with fixed bottom row $(4, 2, 1, 3)$: First, construct all possible penultimate rows (l_1, l_2, l_3). From condition (G1) we know that $(l_1, l_2, l_3) \in \{2, 3, 4\} \times \{1, 2\} \times \{1, 2, 3\}$. Condition (G3) implies that $l_1 \in \{2, 3\}$. If on the one hand $l_1 = 2$, then condition (G3) forces $l_2 = 2$. The right-most entry l_3 could be either 1, 2 or 3, but conditions (G1) and (G2) ensure that a GMT does not have three consecutive equal entries, so $l_3 \in \{1, 3\}$. If on the other hand $l_1 = 3$, then condition (G3) implies that $l_2 = l_3 = 1$. Proceeding in the same way with the three penultimate rows $(2, 2, 1)$, $(2, 2, 3)$ and $(3, 1, 1)$ yields the four GMTs depicted in Figure 4.2.

Remark 4.1.2. *GMTs are a joint generalization of ordinary MTs and DMTs. More precisely:*

1. *If $k_1 < k_2 < \cdots < k_n$, then the set of GMTs and the set of MTs with fixed bottom row $(k_1, k_2, \ldots, k_n)$ coincide: Every GMT with strictly increasing bottom row is by conditions (G1) and (G2) a MT. Conversely, the weak increase along NE- and SE-diagonals of MTs implies condition (G1) of GMTs, the strict increase along rows condition (G2), and the premise of (G3) cannot hold.*

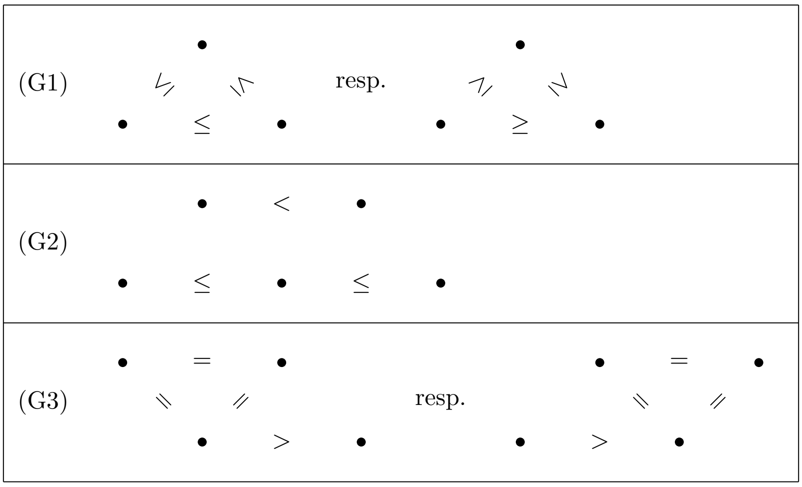


Figure 4.1: Local restrictions of Generalized Monotone Triangles.

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 2 & & 2 & & \\
 & 2 & & 2 & & 1 & \\
4 & & 2 & & 1 & & 3
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 2 & & 3 & & \\
 & 2 & & 2 & & 3 & \\
4 & & 2 & & 1 & & 3
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 3 & & & \\
 & & 2 & & 3 & & \\
 & 2 & & 2 & & 3 & \\
4 & & 2 & & 1 & & 3
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 3 & & 1 & & 1 & \\
4 & & 2 & & 1 & & 3
\end{array}
$$

Figure 4.2: The four GMTs with bottom row $(4, 2, 1, 3)$.

2. *If $k_1 \geq k_2 \geq \cdots \geq k_n$, then the set of GMTs and the set of DMTs with fixed bottom row $(k_1, k_2, \ldots, k_n)$ coincide: Conditions (G1) and (D1) are equivalent for weakly decreasing bottom rows. Conditions (G3) and (D3') are identical. Conditions (G1) and (G2) imply that each integer appears at most twice in a row. Conversely, condition (D2) ensures that the premise of (G2) is never satisfied.*

The main result of this chapter is that the evaluation $\alpha(n; k_1, \ldots, k_n)$ is a signed enumeration of the GMTs with bottom row $(k_1, k_2, \ldots, k_n)$. The sign of a GMT is determined by the following two statistics:

- An entry $a_{i,j}$ is called newcomer if $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$.
- A pair (x, x) of two consecutive equal entries in a row is called sign-changing, if their interlaced neighbour in the row below is also equal to x.

In the following, let $\mathcal{G}_n(k_1, k_2, \ldots, k_n)$ denote the set of GMTs with bottom row $(k_1, k_2, \ldots, k_n)$.

Theorem 4.1.3 ([Rie12])**.** *Let $n \geq 1$ and $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Then*

$$\alpha(n; k_1, k_2, \ldots, k_n) = \sum_{A \in \mathcal{G}_n(k_1, k_2, \ldots, k_n)} (-1)^{\mathrm{sc}(A)},$$

where $\mathrm{sc}(A)$ *is the total number of newcomers and sign-changing pairs in A.*

Consider for example (4.1.1). The theorem implies $\alpha(3; 1, 2, 2) = 2$ and $\alpha(3; 1, 2, 1) = -1$, which corresponds to the three ASMs counted by $A_{3,2}$. Of the four GMTs in Figure 4.2, only the left-most contains an even number of sign-changes, and therefore $\alpha(4; 4, 2, 1, 3) = -2$ by Theorem 4.1.3.

In Remark 4.1.2 we observed that $\mathcal{G}_n(k_1, \ldots, k_n)$ is equal to the set of MTs with bottom row $(k_1, k_2, \ldots, k_n)$ whenever $k_1 < k_2 < \cdots < k_n$. Since $\mathrm{sc}(A) = 0$ for every MT, Theorem 4.1.3 is already known to be true in this case.

If $k_1 \geq k_2 \geq \cdots \geq k_n$ on the other hand, then the set of GMTs and DMTs with this bottom row coincide. The correctness of Theorem 4.1.3 then follows from Corollary 3.2.4 after observing that each pair not contained in the bottom row of a DMT is sign-changing by the weak monotony along diagonals. However, note that general GMTs do not have this property, e.g. $\begin{smallmatrix} & & 2 & & \\ & 2 & & 2 & \\ 1 & & 3 & & 1 \end{smallmatrix}$.

In Section 4.2 we give a straight-forward proof of Theorem 4.1.3 resembling the proof of Theorem 3.1.4 in the case of DMTs. In Section 4.3 we observe a connection with a previously known [Fis12] combinatorial interpretation of $\alpha(n; k_1, \ldots, k_n)$ in the general case $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. This connection allows us to give a shorter, more subtle proof of Theorem 4.1.3. Apart from being a joint generalization of MTs and DMTs, the generalization given here is more reduced in the sense that fewer cancellations occur in the signed enumerations than in previously known generalizations. In Section 4 we apply the theorem to give a combinatorial proof of an identity satisfied by $\alpha(n; k_1, \ldots, k_n)$ and provide a collection of open problems.

4.2 First proof of Theorem 4.1.3

The following proposition establishes a connection between the summation operator and GMTs, which then gives us the means to prove Theorem 4.1.3 inductively.

Given $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, let $\mathcal{P}(k_1, \ldots, k_n)$ denote the set of $(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}$ such that the local restrictions of GMTs (cf. Figure 4.1) are satisfied in the following trapezoid:

$$\begin{array}{ccccccccc} & l_1 & & \cdots & & \cdots & & l_{n-1} & \\ k_1 & & k_2 & & \cdots & & k_{n-1} & & k_n \end{array}$$

For example, $\mathcal{P}(4, 2, 1, 3) = \{(2, 2, 1), (2, 2, 2), (2, 2, 3), (3, 1, 1)\}$.

Proposition 4.2.1. *Let $A(l_1, \ldots, l_{n-1})$ be a polynomial in each variable satisfying*

$$A(l_1, \ldots, l_{i-1}, l_i, l_i, l_i, l_{i+3}, \ldots, l_{n-1}) = 0, \quad i = 1, \ldots, n-3.$$

Then

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) = \sum_{(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)} (-1)^{\mathrm{sc}(\mathbf{k};\mathbf{l})} A(l_1, \ldots, l_{n-1}), \quad n \geq 2,$$

where $\mathrm{sc}(\mathbf{k}; \mathbf{l}) := \mathrm{sc}(k_1, \ldots, k_n; l_1, \ldots, l_{n-1})$ *is the total number of newcomers and sign-changing pairs in* $(l_1, \ldots, l_{n-1})$.

The idea of the proof is to show that both sides satisfy the same recursion. For the summation operator we have already seen the recursions (2.2.5), (2.2.8) and (2.2.9). Before proving Proposition 4.2.1, we first show that the set $\mathcal{P}(k_1, \ldots, k_n)$ can be decomposed in a corresponding way.

Given two sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{Z}^{n-1}$, we write $\mathcal{A} \simeq \mathcal{B}$ iff the two sets only differ by tuples containing three consecutive integers, i.e.

$$(l_1, \ldots, l_{n-1}) \in \mathcal{A} \,\Delta\, \mathcal{B} \implies \exists i : l_i = l_{i+1} = l_{i+2}.$$

Let us further use the notation $(a, b]_{\mathbb{Z}} := \{c \in \mathbb{Z} \mid a < c \leq b\}$ and the analogous notation for open and closed intervals of integers.

Lemma 4.2.2. *Let $n \geq 4$ and $(k_1, \ldots, k_n) \in \mathbb{Z}^n$.*

1. *If $k_{n-1} \leq k_n$, then*

$$\mathcal{P}(k_1, \ldots, k_n) \;\simeq\; \mathcal{P}(k_1, \ldots, k_{n-1}) \times (k_{n-1}, k_n]_{\mathbb{Z}} \,\dot\cup\, \mathcal{P}(k_1, \ldots, k_{n-2}, k_{n-1} - 1) \times \{k_{n-1}\}. \quad (4.2.1)$$

   Moreover, if $l_{n-1} \in (k_{n-1}, k_n]_{\mathbb{Z}}$, then

$$\mathrm{sc}(k_1, \ldots, k_n; l_1, \ldots, l_{n-1}) = \mathrm{sc}(k_1, \ldots, k_{n-1}; l_1, \ldots, l_{n-2}),$$

   and if $l_{n-1} = k_{n-1}$, then

$$\mathrm{sc}(k_1, \ldots, k_n; l_1, \ldots, l_{n-1}) = \mathrm{sc}(k_1, \ldots, k_{n-2}, k_{n-1} - 1; l_1, \ldots, l_{n-2}).$$

2. *If $k_{n-1} > k_n$, then*

$$\mathcal{P}(k_1, \ldots, k_n) \;\simeq\; \mathcal{P}(k_1, \ldots, k_{n-1}) \times (k_n, k_{n-1})_{\mathbb{Z}} \,\dot\cup\, \mathcal{P}(k_1, \ldots, k_{n-2}) \times \{(k_{n-1}, k_{n-1})\}. \quad (4.2.2)$$

   Moreover, if $l_{n-1} \in (k_n, k_{n-1})_{\mathbb{Z}}$, then

$$\mathrm{sc}(k_1, \ldots, k_n; l_1, \ldots, l_{n-1}) = \mathrm{sc}(k_1, \ldots, k_{n-1}; l_1, \ldots, l_{n-2}) + 1,$$

   and if $l_{n-2} = l_{n-1} = k_{n-1}$, then

$$\mathrm{sc}(k_1, \ldots, k_n; l_1, \ldots, l_{n-1}) = \mathrm{sc}(k_1, \ldots, k_{n-2}; l_1, \ldots, l_{n-3}) + 1.$$

Proof.

1. Let us determine which $(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}$ are contained in both sides of (4.2.1). By definition, $l_{n-1} \in [k_{n-1}, k_n]_{\mathbb{Z}}$ on both sides. In the following, distinguish between $k_{n-1} < l_{n-1} \leq k_n$ and $l_{n-1} = k_{n-1}$:

   Case 1.1 ($\mathbf{k_{n-1} < l_{n-1} \leq k_n}$) :

   If $k_{n-2} > k_{n-1}$, then $k_{n-2} \geq l_{n-2} > k_{n-1}$ on both sides:

Left-hand side of (4.2.1)	Right-hand side of (4.2.1)
$\begin{array}{ccccccccc} & & l_{n-2} & & & & l_{n-1} & & \\ & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$>$} & & \rotatebox{45}{$<$} & & \rotatebox{-45}{$\leq$} & \\ k_{n-2} & & > & & k_{n-1} & & \leq & & k_n \end{array}$	$\begin{array}{ccccccc} & & l_{n-2} & & & & l_{n-1} \\ & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$>$} & & & \\ k_{n-2} & & > & & k_{n-1} & & \end{array}$

For fixed (l_{n-2}, l_{n-1}), the restrictions for $(l_1, \dots, l_{n-3})$ are the same on both sides. The entry l_{n-1} does not contribute a sign-change, and the entry l_{n-2} is involved in a sign-change on both sides. Hence, $\mathrm{sc}(k_1, \dots, k_n; l_1, \dots, l_{n-1}) = \mathrm{sc}(k_1, \dots, k_{n-1}; l_1, \dots, l_{n-2})$.

If $k_{n-2} \leq k_{n-1}$, then $k_{n-2} \leq l_{n-2} \leq k_{n-1}$ on both sides:

Left-hand side of (4.2.1)	Right-hand side of (4.2.1)
$\begin{array}{ccccccccc} & & l_{n-2} & & & & l_{n-1} & & \\ & \leq & & \leq & & < & & \leq & \\ k_{n-2} & & \leq & & k_{n-1} & & \leq & & k_n \end{array}$	$\begin{array}{ccccccc} & & l_{n-2} & & & & l_{n-1} \\ & \leq & & \leq & & & \\ k_{n-2} & & \leq & & k_{n-1} & & \end{array}$

For fixed (l_{n-2}, l_{n-1}), the restrictions for $(l_1, \dots, l_{n-3})$ are the same on both sides. The entry l_{n-1} does not contribute a sign-change, and the entry l_{n-2} is involved in a sign-change on the left-hand side if and only if it is on the right-hand side.

Case 1.2 ($\mathbf{l_{n-1} = k_{n-1}}$) :

If $k_{n-2} = k_{n-1}$, then there is no row on the left-hand side with $l_{n-1} = k_{n-1}$, and on the right-hand side this implies $l_{n-3} = l_{n-2} = l_{n-1} = k_{n-1}$:

Left-hand side of (4.2.1)	Right-hand side of (4.2.1)
$\begin{array}{ccccccccc} & & ↯ & & & & k_{n-1} & & \\ & & & & & = & & \leq & \\ k_{n-2} & & = & & k_{n-1} & & \leq & & k_n \end{array}$	$\begin{array}{ccccccccc} & & k_{n-1} & & & & k_{n-1} & & & & k_{n-1} \\ & & & = & & = & & & > & & \\ k_{n-3} & & & & k_{n-1} & & > & & k_{n-1}-1 & & \end{array}$

If $k_{n-2} \leq k_{n-1} - 1$, then $k_{n-2} \leq l_{n-2} < k_{n-1}$ on both sides:

Left-hand side of (4.2.1)	Right-hand side of (4.2.1)
$\begin{array}{ccccccccc} & & l_{n-2} & & & & k_{n-1} & & \\ & \leq & & < & & = & & \leq & \\ k_{n-2} & & < & & k_{n-1} & & \leq & & k_n \end{array}$	$\begin{array}{ccccccc} & & l_{n-2} & & & & k_{n-1} \\ & \leq & & \leq & & & \\ k_{n-2} & & \leq & & k_{n-1}-1 & & \end{array}$

For fixed (l_{n-2}, l_{n-1}), the restrictions for $(l_1, \dots, l_{n-3})$ are the same on both sides. The entry l_{n-1} does not contribute a sign-change, and the entry l_{n-2} is involved in a sign-change on the left-hand side if and only if it is on the right-hand side. It follows that $\mathrm{sc}(k_1, \dots, k_n; l_1, \dots, l_{n-1}) = \mathrm{sc}(k_1, \dots, k_{n-2}, k_{n-1} - 1; l_1, \dots, l_{n-2})$.

If $k_{n-2} > k_{n-1}$, then $k_{n-2} \geq l_{n-2} \geq k_{n-1}$ on both sides:

Left-hand side of (4.2.1)	Right-hand side of (4.2.1)
$\begin{array}{ccccccccc} & & l_{n-2} & & & & k_{n-1} & & \\ & \geq & & \geq & & = & & \leq & \\ k_{n-2} & & > & & k_{n-1} & & \leq & & k_n \end{array}$	$\begin{array}{ccccccc} & & l_{n-2} & & & & k_{n-1} \\ & \geq & & > & & & \\ k_{n-2} & & > & & k_{n-1}-1 & & \end{array}$

For fixed (l_{n-2}, l_{n-1}), the restrictions for $(l_1, \ldots, l_{n-3})$ are the same on both sides. The entry l_{n-2} is involved in a sign-change on both sides (also note the special case $l_{n-2} = k_{n-1}$, where (l_{n-2}, l_{n-1}) is a sign-changing pair on the left-hand side and l_{n-2} a newcomer on the right-hand side).

2. Now let us determine which $(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}$ are contained in both sides of (4.2.2) if $k_{n-1} > k_n$. By definition $l_{n-1} \in [k_{n-1}, k_n)_{\mathbb{Z}}$ on both sides. In the following, distinguish between the cases $l_{n-1} \in (k_{n-1}, k_n)_{\mathbb{Z}}$ and $l_{n-1} = k_{n-1}$:

   Case 2.1 $(\mathbf{k_{n-1} > l_{n-1} > k_n})$:

   If $k_{n-2} > k_{n-1}$, then $k_{n-2} \geq l_{n-2} > k_{n-1}$ on both sides:

Left-hand side of (4.2.2)	Right-hand side of (4.2.2)
$\begin{array}{ccccccccc} & & l_{n-2} & & & & l_{n-1} & & \\ & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$>$} & & \rotatebox{-45}{$>$} & & \rotatebox{45}{$>$} & \\ k_{n-2} & & > & & k_{n-1} & & > & & k_n \end{array}$	$\begin{array}{ccccccc} & & l_{n-2} & & & & l_{n-1} \\ & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$>$} & & & \\ k_{n-2} & & > & & k_{n-1} & & \end{array}$

   For fixed (l_{n-2}, l_{n-1}), the restrictions for $(l_1, \ldots, l_{n-3})$ are the same on both sides. The entry l_{n-1} is a newcomer on the left-hand side, and l_{n-2} is either a newcomer on both sides or in a sign-changing pair with l_{n-3} on both sides. Therefore, $\mathrm{sc}(k_1, \ldots, k_n; l_1, \ldots, l_{n-1}) = \mathrm{sc}(k_1, \ldots, k_{n-1}; l_1, \ldots, l_{n-2}) + 1$.

   If $k_{n-2} \leq k_{n-1}$, then $k_{n-2} \leq l_{n-2} \leq k_{n-1}$ on both sides:

Left-hand side of (4.2.2)	Right-hand side of (4.2.2)
$\begin{array}{ccccccccc} & & l_{n-2} & & & & l_{n-1} & & \\ & \rotatebox{-45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$>$} & & \rotatebox{45}{$>$} & \\ k_{n-2} & & \leq & & k_{n-1} & & > & & k_n \end{array}$	$\begin{array}{ccccccc} & & l_{n-2} & & & & l_{n-1} \\ & \rotatebox{-45}{$\leq$} & & \rotatebox{45}{$\leq$} & & & \\ k_{n-2} & & \leq & & k_{n-1} & & \end{array}$

   For fixed (l_{n-2}, l_{n-1}), the restrictions for $(l_1, \ldots, l_{n-3})$ are the same on both sides. The entry l_{n-1} is a newcomer on the left-hand side, and l_{n-2} is involved in a sign-changing pair on the left-hand side if and only if it is on the right-hand side.

   Case 2.2 $(\mathbf{l_{n-1} = k_{n-1}})$:

   Since $k_{n-1} > k_n$ and $l_{n-1} = k_{n-1}$, we have $l_{n-2} = k_{n-1}$ on both sides of (4.2.2). Distinguish between $k_{n-3} \leq k_{n-2}$ and $k_{n-3} > k_{n-2}$:

   If $k_{n-3} \leq k_{n-2}$, then $k_{n-3} \leq l_{n-3} \leq k_{n-2}$ on the right-hand side:

Left-hand side of (4.2.2)	Right-hand side of (4.2.2)
$\begin{array}{ccccccccccccc} & & l_{n-3} & & & & k_{n-1} & & & & k_{n-1} & & \\ & \rotatebox{-45}{$\leq$} & & & & & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & & \rotatebox{-45}{$>$} & \\ k_{n-3} & & \leq & & k_{n-2} & & & & k_{n-1} & & > & & k_n \end{array}$	$\begin{array}{ccccccccc} & & l_{n-3} & & & & k_{n-1} & & k_{n-1} \\ & \rotatebox{-45}{$\leq$} & & \rotatebox{45}{$\leq$} & & & & & \\ k_{n-3} & & \leq & & k_{n-2} & & & & \end{array}$

On the left-hand side we also have $k_{n-3} \leq l_{n-3} \leq k_{n-2}$, unless $k_{n-2} = k_{n-1}$. In this case $k_{n-3} \leq l_{n-3} < k_{n-2}$ on the left-hand side. However, this missing possibility $l_{n-3} = k_{n-2}$ is remedied by the fact that on the right-hand side we then have $l_{n-3} = k_{n-2} = k_{n-1} = l_{n-2} = l_{n-1}$, i.e. there are three consecutive equal entries. For fixed $(l_{n-3}, l_{n-2}, l_{n-1})$, the restrictions for $(l_1, \ldots, l_{n-4})$ are the same on both sides.

The pair (l_{n-2}, l_{n-1}) contributes a sign-change on the left-hand side, and l_{n-3} is involved in a sign-change on the left-hand side if and only if it is on the right-hand side. Hence, $\mathrm{sc}(k_1, \ldots, k_n; l_1, \ldots, l_{n-1}) = \mathrm{sc}(k_1, \ldots, k_{n-2}; l_1, \ldots, l_{n-3}) + 1$.

If $k_{n-3} > k_{n-2}$, then $k_{n-3} \geq l_{n-3} > k_{n-2}$ on the right-hand side ($n > 4$):

Left-hand side of (4.2.2)	Right-hand side of (4.2.2)
$\begin{array}{ccccccccccccc} & & l_{n-3} & & & & k_{n-1} & & & & k_{n-1} & & \\ & \rotatebox{45}{$\geq$} & & & & & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & & \rotatebox{-45}{$>$} & \\ k_{n-3} & & > & & k_{n-2} & & & & k_{n-1} & & > & & k_n \end{array}$	$\begin{array}{ccccccccc} & & l_{n-3} & & & & k_{n-1} & & k_{n-1} \\ & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$>$} & & & & & \\ k_{n-3} & & > & & k_{n-2} & & & & \end{array}$

On the left-hand side, we also have $l_{n-3} \in (k_{n-2}, k_{n-3}]_{\mathbb{Z}}$, unless $k_{n-2} = k_{n-1}$, where $l_{n-3} \in [k_{n-2}, k_{n-3}]_{\mathbb{Z}}$. However $l_{n-3} = k_{n-2}$ then implies that $l_{n-3} = k_{n-2} = k_{n-1} = l_{n-2} = l_{n-1}$, i.e. there are three consecutive equal entries. For fixed $(l_{n-3}, l_{n-2}, l_{n-1})$, the restrictions for $(l_1, \ldots, l_{n-4})$ are the same on both sides. If $n = 4$, then the argument remains the same with the only difference that $k_1 > l_1$ instead of $k_1 \geq l_1$ on both sides.

The entry l_{n-3} contributes a sign-change on both sides and (l_{n-2}, l_{n-1}) is a sign-changing pair on the left-hand side.

$\square$

Proof of Proposition 4.2.1. If $n = 2$, then $l_1 \in [k_1, k_2]_{\mathbb{Z}}$ if $k_1 \leq k_2$, and $l_1 \in (k_2, k_1)_{\mathbb{Z}}$ if $k_1 > k_2$. In the former case $\mathrm{sc}(k_1, k_2; l_1) = 0$, and in the latter case $\mathrm{sc}(k_1, k_2; l_1) = 1$. The claim now follows from

$$\sum_{(l_1)}^{(k_1,k_2)} A(l_1) \overset{(2.2.7)}{=} \sum_{l_1=k_1}^{k_2} A(l_1) \overset{(2.2.6)}{=} \begin{cases} \sum\limits_{l_1=k_1}^{k_2} A(l_1), & k_1 \leq k_2, \\ 0, & k_1 = k_2 + 1, \\ -\sum\limits_{l_1=k_2+1}^{k_1-1} A(l_1), & k_1 > k_2 + 1. \end{cases}$$

For $n = 3$, it only remains to observe the alternating cases $k_1 < k_2 > k_3$ and $k_1 > k_2 < k_3$. The cases $k_1 \leq k_2 < k_3$ and $k_1 < k_2 \leq k_3$ follow from (2.2.2), the cases $k_1 \geq k_2 > k_3$ and $k_1 > k_2 \geq k_3$ from Lemma 3.2.3, and the case $k_1 = k_2 = k_3$ from Lemma 3.2.1.

In the case $k_1 < k_2 > k_3$, it follows that

$$\mathcal{P}(k_1, k_2, k_3) = [k_1, k_2]_{\mathbb{Z}} \times (k_3, k_2)_{\mathbb{Z}} \cup \{(k_2, k_2)\}.$$

If $(l_1, l_2) \in [k_1, k_2]_{\mathbb{Z}} \times (k_3, k_2)_{\mathbb{Z}}$, then l_2 is a newcomer, and if $(l_1, l_2) = (k_2, k_2)$, then (l_1, l_2) is a sign-changing pair. Either way, we have $\mathrm{sc}(k_1, k_2, k_3; l_1, l_2) = 1$. The summation operator yields the same signed summation:

$$\sum_{(l_1,l_2)}^{(k_1,k_2,k_3)} A(l_1,l_2) \overset{(2.2.9)}{=} \sum_{(l_1)}^{(k_1,k_2)} \sum_{l_2=k_2}^{k_3} A(l_1,l_2) - \sum_{()}^{(k_1)} A(k_2,k_2)$$
$$\overset{(2.2.7)}{=} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} A(l_1,l_2) - A(k_2,k_2) \overset{(2.2.6)}{=} - \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_3+1}^{k_2-1} A(l_1,l_2) - A(k_2,k_2).$$

If $n \geq 4$, we can apply Lemma 4.2.2 by distinguishing between $k_{n-1} \leq k_n$ and $k_{n-1} > k_n$:

Case 1 ($\mathbf{k_{n-1} \leq k_n}$) :

Apply recursion (2.2.5) of the summation operator and the induction hypothesis to obtain

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) \\
&= \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1,\ldots,l_{n-1}) + \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-2},k_{n-1}-1)} A(l_1,\ldots,l_{n-2},k_{n-1}) \\
&= \sum_{(l_1,\ldots,l_{n-2})\in\mathcal{P}(k_1,\ldots,k_{n-1})} (-1)^{\mathrm{sc}(k_1,\ldots,k_{n-1};l_1,\ldots,l_{n-2})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1,\ldots,l_{n-1}) \\
&\quad + \sum_{(l_1,\ldots,l_{n-2})\in\mathcal{P}(k_1,\ldots,k_{n-2},k_{n-1}-1)} (-1)^{\mathrm{sc}(k_1,\ldots,k_{n-2},k_{n-1}-1;l_1,\ldots,l_{n-2})} A(l_1,\ldots,l_{n-2},k_{n-1}).
\end{aligned}$$

By the first part of Lemma 4.2.2 and the proposition's assumption this is further equal to

$$\sum_{(l_1,\ldots,l_{n-1})\in\mathcal{P}(k_1,\ldots,k_n)} (-1)^{\mathrm{sc}(\mathbf{k};\mathbf{l})} A(l_1,\ldots,l_{n-1}).$$

Case 2 ($\mathbf{k_{n-1} > k_n}$) :

Recursion (2.2.9) of the summation operator and the induction hypothesis yield

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) \\
&= - \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} \sum_{l_{n-1}=k_n+1}^{k_{n-1}-1} A(l_1,\ldots,l_{n-1}) - \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1}) \\
&= \sum_{(l_1,\ldots,l_{n-2})\in\mathcal{P}(k_1,\ldots,k_{n-1})} (-1)^{\mathrm{sc}(k_1,\ldots,k_{n-1};l_1,\ldots,l_{n-2})+1} \sum_{l_{n-1}=k_n+1}^{k_{n-1}-1} A(l_1,\ldots,l_{n-1}) \\
&\quad + \sum_{(l_1,\ldots,l_{n-3})\in\mathcal{P}(k_1,\ldots,k_{n-2})} (-1)^{\mathrm{sc}(k_1,\ldots,k_{n-2};l_1,\ldots,l_{n-3})+1} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1}).
\end{aligned}$$

The claim now follows from the second part of Lemma 4.2.2. □

In Lemma 3.2.1 we observed that the α-polynomial fulfills the assumption of Proposition 4.2.1. We are now in the position to give an inductive proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. The result is immediate for $n = 1$. If $n \geq 2$, apply Proposition 4.2.1 and the induction hypothesis to obtain

$$\begin{aligned}
\alpha(n; k_1, \ldots, k_n) &\overset{(2.2.4)}{=} \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1; l_1, \ldots, l_{n-1}) \\
&= \sum_{(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)} (-1)^{\mathrm{sc}(\mathbf{k};\mathbf{l})} \alpha(n-1; l_1, \ldots, l_{n-1}) \\
&= \sum_{(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_n)} (-1)^{\mathrm{sc}(\mathbf{k};\mathbf{l})} \sum_{A \in \mathcal{G}_{n-1}(l_1,\ldots,l_{n-1})} (-1)^{\mathrm{sc}(A)} \\
&= \sum_{A \in \mathcal{G}_n(k_1,\ldots,k_n)} (-1)^{\mathrm{sc}(A)}. \qquad \square
\end{aligned}$$

4.3 Second proof of Theorem 4.1.3

In [Fis12] four different combinatorial extensions of $\alpha(n; k_1, \ldots, k_n)$ to all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ are described. The idea behind all of them is to write the sum in (2.2.1) in terms of ordinary summations, i.e. summations as defined in (2.2.6). In one of the extensions this is achieved by applying the inclusion-exclusion principle in the following way: Let $k_1 < k_2 < \cdots < k_n$ and

$$\begin{aligned}
M &:= \{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \leq l_j \leq k_{j+1} \ \wedge \ l_j < l_{j+1}\}, \\
A &:= \{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \leq l_j \leq k_{j+1}\}, \\
A_i &:= \{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \leq l_j \leq k_{j+1} \ \wedge \ l_{i-1} = k_i = l_i\}, \quad i = 2, \ldots, n-1.
\end{aligned}$$

From $k_i < k_{i+1}$ it follows that $A_i \cap A_{i+1} = \emptyset$, and thus the inclusion-exclusion principle implies for any function $f(\mathbf{l}) := f(l_1, \ldots, l_{n-1})$ that

$$\begin{aligned}
\sum_{\mathbf{l} \in M} f(\mathbf{l}) = \sum_{\mathbf{l} \in A} f(\mathbf{l}) - \sum_{i=2}^{n-1} \sum_{\mathbf{l} \in A_i} f(\mathbf{l}) + \sum_{\substack{2 \leq i_1 < i_2 \leq n-1, \\ i_2 \neq i_1 + 1}} \sum_{\mathbf{l} \in A_{i_1} \cap A_{i_2}} f(\mathbf{l}) \\
- \sum_{\substack{2 \leq i_1 < i_2 < i_3 \leq n-1, \\ i_2 \neq i_1 + 1, \\ i_3 \neq i_2 + 1}} \sum_{\mathbf{l} \in A_{i_1} \cap A_{i_2} \cap A_{i_3}} f(\mathbf{l}) \cdots . \qquad (4.3.1)
\end{aligned}$$

This can be written in terms of ordinary sums as

$$\sum_{p \geq 0} (-1)^p \sum_{\substack{2 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \\ i_{j+1} \neq i_j + 1}} \sum_{l_1 = k_1}^{k_2} \sum_{l_2 = k_2}^{k_3} \cdots \sum_{l_{i_1} - 1 = k_{i_1}}^{k_{i_1}} \sum_{l_{i_1} = k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p} - 1 = k_{i_p}}^{k_{i_p}} \sum_{l_{i_p} = k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1} = k_{n-1}}^{k_n} f(\mathbf{l}). \qquad (4.3.2)$$

We can make sense of (4.3.2) for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ by using the extended definition of ordinary sums (2.2.6). As explained in Section 2.2, the fact that (4.3.2) is a polynomial function in each k_i coinciding with

$$\sum_{\substack{(l_1,\ldots,l_{n-1}) \in \mathbb{Z}^{n-1}, \\ k_1 \le l_1 \le k_2 \le l_2 \le \cdots \le k_{n-1} \le l_{n-1} \le k_n, \\ l_i < l_{i+1}}} f(l_1, \ldots, l_{n-1})$$

whenever $k_1 < k_2 < \cdots < k_n$ ensures that

$$\alpha(n; k_1, \ldots, k_n) = \sum_{p \ge 0} (-1)^p \sum_{\substack{2 \le i_1 < i_2 < \cdots < i_p \le n-1, \\ i_{j+1} \ne i_j + 1}} \tag{4.3.3}$$
$$\sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \cdots \sum_{l_{i_1}-1=k_{i_1}}^{k_{i_1}} \sum_{l_{i_1}=k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p}-1=k_{i_p}}^{k_{i_p}} \sum_{l_{i_p}=k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1}=k_{n-1}}^{k_n} \alpha(n-1; l_1, \ldots, l_{n-1})$$

holds for $(k_1, \ldots, k_n) \in \mathbb{Z}^n$.

As pointed out in [Fis12], we can give (4.3.3) a combinatorial interpretation in terms of a signed enumeration of the following objects: In a triangular array $(a_{i,j})_{1 \le j \le i \le n}$ of integers arranged in the form

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \iddots & & & & \ddots & \\ a_{n,1} & & \cdots & & \cdots & & a_{n,n} \end{array}$$

let us call the entries $a_{i-1,j-1}$ and $a_{i-1,j}$ the parents of $a_{i,j}$. Among the entries $(a_{i,j})_{1<j<i\le n}$, there may be special entries (choosing the special entries in the bottom row corresponds to fixing $(i_1, \ldots, i_p)$ in (4.3.3)). Special entries in the same row must not be adjacent. The requirements for the entries are

1. If $a_{i,j}$ is special, then $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$.
2. If $a_{i,j}$ is not the parent of a special entry and $a_{i+1,j} \le a_{i+1,j+1}$, then $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$.
3. If $a_{i,j}$ is not the parent of a special entry and $a_{i+1,j} > a_{i+1,j+1}$, then $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$. In this case $a_{i,j}$ is called inversion.

Two examples are depicted in Figure 4.3. Let us denote by $\mathcal{T}_n(k_1, \ldots, k_n)$ the set of these objects with bottom row $(a_{n,1}, \ldots, a_{n,n}) = (k_1, \ldots, k_n)$. For $A \in \mathcal{T}_n(k_1, \ldots, k_n)$ let $s(A)$ be the total number of special entries and inversions. Using induction and (4.3.3), one then obtains

$$\alpha(n; k_1, \ldots, k_n) = \sum_{A \in \mathcal{T}_n(k_1,\ldots,k_n)} (-1)^{s(A)}.$$

However, there are cancellations in this signed enumeration, which can be eliminated. To be specific, consider those $A \in \mathcal{T}_n(k_1, \ldots, k_n)$ violating the condition

$$a_{i,j-1} \le a_{i,j} \le a_{i,j+1} \implies a_{i-1,j-1} < a_{i-1,j} \tag{4.3.4}$$

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 2 & & 2 & & \\
 & 2 & & 2 & & 3 & \\
4 & & \overset{*}{2} & & 1 & & 3
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 2 & & 2 & & \\
 & 2 & & \overset{*}{2} & & 3 & \\
4 & & \overset{*}{2} & & 1 & & 3
\end{array}
$$

Figure 4.3: Sign-reversing involution on the set of arrays violating (4.3.4). Special entries are marked with a star on top.

at least once. Given such an array, locate the lexicographically minimal position (i, j) such that $a_{i,j-1} \le a_{i,j} \le a_{i,j+1}$ and $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$. If $a_{i,j}$ is not special, then make it special and vice versa if $a_{i,j}$ is special, then turn it into a non-special entry (see Figure 4.3).

Note that the minimality of i ensures that turning $a_{i,j}$ special is admissible: Suppose a neighbour of $a_{i,j}$ is special, then the row above contains three consecutive equal entries and thus condition (4.3.4) is already violated in the row above. Conversely, turning the special entry $a_{i,j}$ into a non-special one is allowed because of $a_{i,j-1} \le a_{i,j} \le a_{i,j+1}$. Since the number of inversion stays the same and the number of special entries changes by exactly 1, this mapping is a sign-reversing involution. It follows that

$$\alpha(n; k_1, \ldots, k_n) = \sum_{\substack{A \in \mathcal{T}_n(k_1,\ldots,k_n),\\ a_{i,j-1} \le a_{i,j} \le a_{i,j+1} \implies a_{i-1,j-1} < a_{i-1,j}}} (-1)^{s(A)}.$$

Furthermore, observe that in this reduced set an entry $a_{i,j}$ is special if and only if $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$. In fact, we can now establish a one-to-one correspondence with the set of GMTs with bottom row $(k_1, \ldots, k_n)$. For this, let us show that

$$\mathcal{G}_n(k_1, \ldots, k_n) = \{A \in \mathcal{T}_n(k_1, \ldots, k_n) \mid a_{i,j-1} \le a_{i,j} \le a_{i,j+1} \implies a_{i-1,j-1} < a_{i-1,j}\},$$

whereby an entry $a_{i,j}$ is defined to be special if and only if $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$.

Given a GMT, the first condition is satisfied by definition. The second condition follows from (G1). If $a_{i+1,j} > a_{i+1,j+1}$, then $a_{i+1,j} \ge a_{i,j} \ge a_{i+1,j+1}$ by (G1). The third condition then follows from (G1) and (G3) since neither $a_{i+1,j}$ nor $a_{i+1,j+1}$ are special. The additional restriction (4.3.4) is exactly condition (G2).

Conversely, condition (G1) and (G2) are obviously satisfied and condition (G3) is a direct consequence of the third and the first condition. Since special entries correspond to sign-changing pairs and inversions to newcomers, we obtain

$$\alpha(n; k_1, \ldots, k_n) = \sum_{A \in \mathcal{G}_n(k_1,k_2,\ldots,k_n)} (-1)^{\mathrm{sc}(A)}.$$

4.4 A shift-antisymmetry property

Having a combinatorial interpretation of $\alpha(n; k_1, k_2, \ldots, k_n)$ for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, we can try to give a combinatorial interpretation to identities satisfied by the α-polynomial. By way of illustration, take the identity

$$\begin{aligned}&\alpha(n;k_1,\ldots,k_{i-1},k_i,k_i+1,k_{i+2},\ldots,k_n) \\ &\quad=\alpha(n;k_1,\ldots,k_{i-1},k_i,k_i,k_{i+2},\ldots,k_n)+\alpha(n;k_1,\ldots,k_{i-1},k_i+1,k_i+1,k_{i+2},\ldots,k_n).\end{aligned} \tag{4.4.1}$$

A combinatorial proof of this identity in the case that $k_1 < k_2 < \cdots < k_i$ and $k_i+1 < k_{i+2} < \cdots < k_n$ was given in [Fis12]. Using Theorem 4.1.3, we can now give a combinatorial proof for arbitrary $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ by establishing a sign-preserving bijection between the sets

$$\begin{aligned}&\mathcal{G}_n(k_1,\ldots,k_{i-1},k_i,k_i+1,k_{i+2},\ldots,k_n) \\ &\qquad\Longleftrightarrow\ \mathcal{G}_n(k_1,\ldots,k_{i-1},k_i,k_i,k_{i+2},\ldots,k_n)\,\dot\cup\,\mathcal{G}_n(k_1,\ldots,k_{i-1},k_i+1,k_i+1,k_{i+2},\ldots,k_n).\end{aligned}$$

Using the notation from Section 4.2 and

$$\begin{aligned}\mathcal{P}_1 &:= \mathcal{P}(k_1,\ldots,k_{i-1},k_i,k_i,k_{i+2},\ldots,k_n)\\ \mathcal{P}_2 &:= \mathcal{P}(k_1,\ldots,k_{i-1},k_i+1,k_i+1,k_{i+2},\ldots,k_n)\end{aligned}$$

we show this by observing that

$$\mathcal{P}(k_1,\ldots,k_{i-1},k_i,k_i+1,k_{i+2},\ldots,k_n) \simeq \mathcal{P}_1\,\dot\cup\,\mathcal{P}_2, \tag{4.4.2}$$

whereby each row contained in both sides has the same total number of newcomers and sign-changing pairs: Each $(l_1,\ldots,l_{n-1}) \in \mathcal{P}(k_1,\ldots,k_{i-1},k_i,k_i+1,k_{i+2},\ldots,k_n)$ satisfies $l_i \in \{k_i,k_i+1\}$. Let us show that those rows with $l_i = k_i$ correspond to the rows contained in $\mathcal{P}_1$. Symmetrically, those rows with $l_i = k_i+1$ then correspond to $\mathcal{P}_2$.

If $l_i = k_i$, then the restrictions for $(l_1,\ldots,l_{i-1})$ are identical for both $\mathcal{P}(k_1,\ldots,k_{i-1},k_i,k_i+1,k_{i+2},\ldots,k_n)$ and $\mathcal{P}(k_1,\ldots,k_{i-1},k_i,k_i,k_{i+2},\ldots,k_n)$. For the restrictions of (l_{i+1},l_{i+2}) distinguish between $k_i+1 \le k_{i+2}$, $k_i = k_{i+2}$ and $k_i > k_{i+2}$:

If $k_i+1 \le k_{i+2}$, then $k_i+1 \le l_{i+1} \le k_{i+2}$ on both sides and the restrictions for l_{i+2} are the same:

Left-hand side of (4.4.2)	Right-hand side of (4.4.2)
$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$} & \\ k_i & & < & & k_i{+}1 & & \le & & k_{i+2}\end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$<$} & & \rotatebox{-45}{$\le$} & \\ k_i & & = & & k_i & & \le & & k_{i+2}\end{array}$

If $k_i = k_{i+2}$, then each row in $\mathcal{P}(k_1,\ldots,k_{i-1},k_i,k_i+1,k_{i+2},\ldots,k_n)$ with $l_i = k_i$ satisfies $l_i = l_{i+1} = l_{i+2} = k_i$, and $\mathcal{P}(k_1,\ldots,k_{i-1},k_i,k_i,k_{i+2},\ldots,k_n)$ is empty:

Left-hand side of (4.4.2)	Right-hand side of (4.4.2)
$\begin{array}{ccccccccc} & & k_i & & & & k_i & & k_i \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$>$} & & \rotatebox{-45}{$=$} & \quad\rotatebox{45}{$=$} \\ k_i & & < & & k_i{+}1 & & > & & k_i\end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & \lightning & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & & & & \\ k_i & & = & & k_i & & = & & k_i\end{array}$

If $k_i > k_{i+2}$, then $k_i \ge l_{i+1} \ge k_{i+2}$ on both sides and the restrictions for l_{i+2} are the same:

Left-hand side of (4.4.2)	Right-hand side of (4.4.2)
$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$<$} & & \rotatebox{45}{$>$} & & \rotatebox{-45}{$\geq$} & \\ k_i & & < & & k_i+1 & & > & & k_{i+2} \end{array}$	$\begin{array}{ccccccccc} & & k_i & & & & l_{i+1} & & \\ & \rotatebox{45}{$=$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & \\ k_i & & = & & k_i & & > & & k_{i+2} \end{array}$

In this case, the entry l_{i+1} is involved in a sign-change on both sides (also observe the special case $l_{i+1} = k_i$, where l_{i+1} is a newcomer on the left-hand side and in a sign-changing pair on the right-hand side). The restrictions for $(l_{i+3}, \ldots, l_{n-1})$ are clearly the same for both sides. Symmetrically, the set $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \ldots, k_n)$ restricted to $l_i = k_i + 1$ corresponds to $\mathcal{P}(k_1, \ldots, k_{i-1}, k_i{+}1, k_i{+}1, k_{i+2}, \ldots, k_n)$, concluding the combinatorial proof of (4.4.2) for arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$.

The GMTs contained in $\mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i, k_i{+}1, k_{i+2}, \ldots, k_n)$ can therefore be partitioned into those where the i-th entry in the penultimate row equals k_i and those where it equals $k_i{+}1$. The former set is then in one-to-one correspondence with the GMTs in $\mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n)$ and the latter corresponds to the set of GMTs contained in $\mathcal{G}_n(k_1, \ldots, k_{i-1}, k_i{+}1, k_i{+}1, k_{i+2}, \ldots, k_n)$. Together with Theorem 4.1.3, we therefore combinatorially understand identity (4.4.1) for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$.

In [Fis06], a computational proof of the identity (definitions of the operators in Table 2.1, p. 20)

$$(\mathrm{id} + \mathrm{E}_{k_{i+1}} \mathrm{E}_{k_i}^{-1} \mathrm{S}_{k_i,k_{i+1}}) \mathrm{V}_{k_i,k_{i+1}} \alpha(n; k_1, \ldots, k_n) = 0 \tag{4.4.3}$$

was given. For fixed integers $k_1, \ldots, k_{i-1}, k_{i+2}, \ldots, k_n$, let

$$t_n(x, y) := \alpha(n; k_1, \ldots, k_{i-1}, x, y, k_{i+2}, \ldots, k_n).$$

Then (4.4.3) is equivalent to

$$\begin{multline} t_n(k_i - 1, k_{i+1}) + t_n(k_i, k_{i+1} + 1) - t_n(k_i - 1, k_{i+1} + 1) \\ + t_n(k_{i+1}, k_i - 1) + t_n(k_{i+1} + 1, k_i) - t_n(k_{i+1}, k_i) = 0. \end{multline} \tag{4.4.4}$$

Note that if $k_i = k_{i+1} + 1$, then (4.4.4) is equivalent to the identity (4.4.1) which we combinatorially understand. For Gelfand-Tsetlin patterns a shift-antisymmetry property along the lines of (4.4.3) was shown bijectively in [Fis12]. It would be interesting to give a bijective proof of (4.4.4) in the general case using GMTs.

4.5 The circular shift identity

In Section 2.5 we presented a computational proof of the circular shift identity

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n), \quad n \geq 1, \tag{4.5.1}$$

which then allowed us to derive a system of linear equations for the refined ASM numbers. With Theorem 4.1.3 we now have a combinatorial interpretation of both sides in terms of a signed summation. The goal of this section is to describe a first step towards a combinatorial understanding of the identity.

For integers $k_1 < k_2 < \cdots < k_n$, the left-hand side of (4.5.1) counts the number of MTs with bottom row $(k_1, \ldots, k_n)$. The right-hand side is by Theorem 4.1.3 a signed summation over

GMTs with bottom row $(k_2,\dots,k_n,k_1-n)$. There is a reasonable choice for the subset of GMTs corresponding to the MTs:

Given a MT with bottom row $(k_1,\dots,k_n)$, take the left-most NE-diagonal, subtract i from the i-th entry top-down and append the entries as right-most SE-diagonal (see Figure 4.4). By construction, the bottom row then is $(k_2,\dots,k_n,k_1-n)$ and the entries of the right-most SE-diagonal are strictly decreasing. In fact, each of the top $n-1$ entries in the right-most SE-diagonal is a newcomer. Since the remaining entries form a MT, the object one obtains is indeed a GMT with bottom row $(k_2,\dots,k_n,k_1-n)$ and sign $(-1)^{n-1}$.

$$
\begin{array}{ccccccc}
 & & & \mathbf{7} & & & \\
 & & \mathbf{6} & & 7 & & \\
 & \mathbf{5} & & 7 & & 8 & \\
\mathbf{5} & & 7 & & 8 & & 9
\end{array}
\quad\mapsto\quad
\begin{array}{ccccccc}
 & & & \mathbf{6} & & & \\
 & & 7 & & \mathbf{4} & & \\
 & 7 & & 8 & & \mathbf{2} & \\
7 & & 8 & & 9 & & \mathbf{1}
\end{array}
$$

Figure 4.4: A MT with bottom row $(5,7,8,9)$ and its corresponding GMT with bottom row $(7,8,9,1)$.

However, not all GMTs with bottom row $(k_2,\dots,k_n,k_1-n)$ can be obtained in this way. If an entry $a_{i,i}$ in the right-most NE-diagonal is too large (namely $a_{i,i}+i>a_{i+1,1}$ or $a_{i,i}+i=a_{i+1,1}=a_{i,1}$), then increasing it by i and moving it to the left end of the row violates the monotonicity condition of MTs. One therefore needs an argument, why these remaining GMTs cancel each other out. For $n=3$, this cancellation of GMTs not corresponding to MTs can be seen in the following way: First, remove the two GMTs

$$
\begin{array}{ccccc}
 & & k_2-1 & & \\
 & k_2 & & k_2-2 & \\
k_2 & & k_3 & & k_1-3
\end{array}
\quad\Longleftrightarrow\quad
\begin{array}{ccccc}
 & & k_3 & & \\
 & k_3 & & k_3 & \\
k_2 & & k_3 & & k_1-3
\end{array}
,
$$

which have opposite signs, since the left GMT contains two newcomers whereas the right GMT contains one sign-changing pair. What remains are those GMTs

$$
\begin{array}{ccccc}
 & & z & & \\
 & x & & y & \\
k_2 & & k_3 & & k_1-3
\end{array}
$$

with $(x,y,z)\in\mathcal{S}:=\{(x,y,z)\in\mathbb{Z}^3 \mid x\in[k_2,k_3]_{\mathbb{Z}},\ y\in[k_2-1,k_3-1]_{\mathbb{Z}},\ z\in\mathcal{P}(x,y)\}$. We claim that $(x,y,z)\mapsto(y+1,x-1,z)$ defines a sign-reversing involution on $\mathcal{S}$.

Let $(x,y,z)\in\mathcal{S}$. By definition, we then have $y+1\in[k_2,k_3]_{\mathbb{Z}}$ and $x-1\in[k_2-1,k_3-1]_{\mathbb{Z}}$. If $x\le y$, then $x\le z\le y$ and therefore $y+1>z>x-1$, i.e. $(y+1,x-1,z)\in\mathcal{S}$. If $x=y+1$, then $\mathcal{P}(x,y)=\emptyset$. If $x>y+1$, then $y+1\le z\le x-1$, i.e. $(y+1,x-1,z)\in\mathcal{S}$. The mapping is therefore well-defined and a sign-reversing involution, because for each $(x,y,z)\in\mathcal{S}$, the entry y is always a newcomer whereas the top entry z is a newcomer if and only if $x>y$.

It would be interesting to see whether similar arguments can be found to give a combinatorial proof of (4.5.1) in the general case.

4.6 Further conjectures on evaluations of the α-polynomial

In Chapter 3, we showed the surprising identity

$$A_n = \alpha(n; 1, 2, \dots, n) = \alpha(2n; n, n, n-1, n-1, \dots, 1, 1) \tag{4.6.1}$$

computationally and gave initial thoughts on how a bijective proof could succeed. Let us conclude this chapter with a list of related identities – all of them are up to this point conjectured using mathematical computing software. As Theorem 4.1.3 provides a combinatorial interpretation of these identities, bijective proofs are of high interest.

Conjecture 4.6.1. *Let $n \geq 1$. Then*

$$\alpha(n; 2, 4, \dots, 2n) = \alpha(2n; 2n, 2n, 2n-2, 2n-2, \dots, 2, 2). \tag{4.6.2}$$

Conjecture 4.6.2. *Let $n \geq 1$. Then*

$$A_n = \alpha(n+i; 1, 2, \dots, i, 1, 2, \dots, n), \quad i = 0, \dots, n, \tag{4.6.3}$$

$$A_n = (-1)^n \alpha(2n+1; 1, 2, \dots, n+1, 1, 2, \dots, n). \tag{4.6.4}$$

Furthermore, the numbers

$$C_{n,i} := \alpha(2n+1; i, 2, \dots, n+1, 1, 2, \dots, n), \quad i = 1, \dots, 3n+2$$

satisfy the symmetry $C_{n,i} = C_{n,3n+3-i}$.

Conjecture 4.6.3. *Let $n \geq 2$. Then*

$$A_n = \alpha(n+2; 1, 2, \dots, i+1, i, i+1, \dots, n), \quad i = 1, \dots, n-1. \tag{4.6.5}$$

Further computational experiments led to the following interesting conjectural generalization of (4.6.1): Given the sequence $(1, 2, \dots, n)$, take out any subsequence $(i, i+1, \dots, i+k-1)$, duplicate each entry of the subsequence, reverse its order and put it back in. Then the evaluation of the α-polynomial at this modified sequence again yields the ASM numbers:

Conjecture 4.6.4. *Let $n \geq 1$. Then*

$$A_n = \alpha(n+k; 1, \dots, i-1, i+k-1, i+k-1, i+k-2, i+k-2, \dots, i, i, i+k, i+k+1, \dots, n) \tag{4.6.6}$$

holds for all $i = 1, \dots, n-k+1$ and $k = 1, \dots, n$.

Identity (4.6.1) is thus the special case of (4.6.6) where $k = n$. Conjecture 4.6.3 is also a special case of Conjecture 4.6.4 where $k = 2$. To see this, apply (4.4.1) and (3.2.1) twice:

$$\begin{aligned}
&\alpha(n+2; 1, 2, \dots, i-1, i, i+1, i, i+1, i+2, \dots, n) \\
&\quad = \alpha(n+2; 1, 2, \dots, i-1, i, i, i, i+1, i+2, \dots, n) \\
&\qquad + \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i+1, i+2, \dots, n) \\
&\quad = \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i, i+2, \dots, n) \\
&\qquad + \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i+1, i+1, i+2, \dots, n) \\
&\quad = \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i, i+2, \dots, n).
\end{aligned}$$

From Proposition 1.2.5, it follows that $A_{n,i} = \alpha(n-1; 1, 2, \dots, i-1, i+1, \dots, n)$. In the following conjecture we analogously remove the i-th argument from the right-hand side of (4.6.5):

Conjecture 4.6.5. *Let $n \geq 1$. Then*

$$\alpha(n+1;1,2,\ldots,i-1,i+1,i,i+1,\ldots,n) = -\sum_{j=1}^{n}(j-i)A_{n,j}, \quad i=1,\ldots,n-1. \tag{4.6.7}$$

To see the case $i=1$ of (4.6.7), note that the penultimate row $(l_1,\ldots,l_n)$ of a GMT with bottom row $(2,1,2,\ldots,n)$ satisfies $l_1=l_2=1$ by condition (G3). Taking conditions (G1) and (G2) into account, Proposition 4.2.1 implies that

$$\alpha(n+1;2,1,2,\ldots,n) = -\sum_{p=2}^{n}\alpha(n;1,1,2,\ldots,p-1,p+1,\ldots,n).$$

Each penultimate row $(m_1,\ldots,m_{n-1})$ of a GMT with bottom row $(1,1,2,\ldots,p-1,p+1,\ldots,n)$ satisfies $m_1=1, m_2=2,\ldots,m_{p-1}=p-1$. Applying Proposition 4.2.1 again yields

$$\alpha(n+1;2,1,2,\ldots,n) = -\sum_{p=2}^{n}\sum_{j=p}^{n}A_{n,j} = -\sum_{j=2}^{n}(j-1)A_{n,j}.$$

For general i, the set of GMTs with bottom row $(1,2,\ldots,i-1,i+1,i,i+1,\ldots,n)$ can be written as a disjoint union $S_1 \dot\cup S_2 \dot\cup S_3$ by distinguishing between the possible choices for (l_{i-1},l_i,l_{i+1}):

$$S_1:\quad \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i{+}1 & & i{+}1 & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i{-}2 & & i{-}1 & & i{+}1 & & i & & i{+}1 & & i{+}2 & & \cdots & & n \end{array}$$

$$S_2:\quad \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i{+}1 & & i & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i{-}2 & & i{-}1 & & i{+}1 & & i & & i{+}1 & & i{+}2 & & \cdots & & n \end{array}$$

$$S_3:\quad \begin{array}{ccccccccccccccccccc} & l_1 & & \cdots & & l_{i-2} & & i{-}1 & & i & & i & & l_{i+2} & & \cdots & & l_n & \\ 1 & & \cdots & & i{-}2 & & i{-}1 & & i{+}1 & & i & & i{+}1 & & i{+}2 & & \cdots & & n \end{array}$$

For the GMTs in S_3 we have $(l_1,l_2,\ldots,l_{i-2})=(1,2,\ldots,i-2)$. Analogous to the case $i=1$, one obtains that the signed enumeration of GMTs in S_3 is equal to

$$-\sum_{j=i+1}^{n}(j-i)A_{n,j}.$$

Proving that the signed enumeration of GMTs in S_1 and S_2 yields

$$-\sum_{j=1}^{i-1}(j-i)A_{n,j}$$

remains an open problem. The following conjectures are also related to (4.6.5) by removing the $(i-1)$-st argument from the right-hand side.

Conjecture 4.6.6. *Let $n \geq 4$. Then*

$$\alpha(n+1; 1, 3, 4, 3, 4, 5, \dots, n) = \frac{n+4}{2} A_{n-1}. \tag{4.6.8}$$

As an immediate consequence of Theorem 4.1.3 we obtain (the known fact) that the evaluation of $\alpha(n; k_1, \dots, k_n)$ at integral values is integral. Let us observe that the right-hand side of (4.6.8) is an integer too: For n even, this is trivial. Otherwise, observe that vertically symmetric ASMs can only exist for odd size, since the top row of each ASM contains a unique 1. Therefore, reflection along the vertical symmetry axis is a fixed-point-free involution on the set of even-sized ASMs. In particular, the number of even-sized ASMs is even.

Using Krattenthaler's Mathematica package RATE, we were able to find more conjectured formulæ similar to (4.6.8):

Conjecture 4.6.7.

$$\alpha(n+1; 1, 2, 4, 5, 4, 5, \dots, n) = \frac{n^3 + 7n^2 + 10n - 36}{8n - 12} A_{n-1}, \quad n \geq 5,$$
$$\alpha(n+1; 1, 2, 3, 5, 6, 5, 6, \dots, n) = \frac{n^4 + 12n^3 + 53n^2 + 54n - 288}{48n - 72} A_{n-1}, \quad n \geq 6.$$

In general, this leads to the following conjecture:

Conjecture 4.6.8. *Let $n \geq k \geq 4$. Then there exist polynomials $p_k(n)$ and $q_k(n)$ with $\deg p_k - \deg q_k = k - 3$ such that*

$$\alpha(n+1; 1, 2, \dots, k-3, k-1, k, k-1, k, \dots, n) = \frac{p_k(n)}{q_k(n)} A_{n-1}.$$

Chapter 5

Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity

Most of this chapter's content appeared in [FR14].

5.1 Introduction

A Vertically Symmetric Alternating Sign Matrix (VSASM) is an ASM which is invariant under reflection with respect to the vertical symmetry axis (see Figure 5.1). Since the first row of an

$$
\begin{pmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & -1 & 1 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & 1 & -1 & 1 & 0 & 0 \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & -1 & 0 & 0 & 1 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 0 & 0 \\
\mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & -1 & 1 & 0 & -1 & 1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & -1 & 0 & 0 & 1 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\iff
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 5 & & & & & & & & \\
 & & & & & & & \mathbf{4} & & 6 & & & & & & & \\
 & & & & & & \mathbf{3} & & 5 & & 7 & & & & & & \\
 & & & & & \mathbf{2} & & \mathbf{3} & & 7 & & 8 & & & & & \\
 & & & & \mathbf{2} & & \mathbf{3} & & 5 & & 7 & & 8 & & & & \\
 & & & \mathbf{1} & & \mathbf{3} & & \mathbf{4} & & 6 & & 7 & & 9 & & & \\
 & & \mathbf{1} & & \mathbf{3} & & \mathbf{4} & & 5 & & 6 & & 7 & & 9 & & \\
 & \mathbf{1} & & \mathbf{2} & & \mathbf{3} & & \mathbf{4} & & 6 & & 7 & & 8 & & 9 & \\
\mathbf{1} & & \mathbf{2} & & \mathbf{3} & & \mathbf{4} & & 5 & & 6 & & 7 & & 8 & & 9
\end{array}
$$

Figure 5.1: A VSASM and its corresponding MT.

ASM contains a unique 1, it follows that VSASMs can only exist for odd dimensions. Moreover, the alternating sign condition along rows and the vertical symmetry imply that no 0 can occur in the middle column. In a $(2n+1) \times (2n+1)$-VSASM, the $(n+1)$-st column is therefore equal to $(1, -1, 1, \ldots, -1, 1)^T$, and the VSASM is uniquely determined by its first n columns.

The number of $(2n+1)\times(2n+1)$-VSASMs is also given by a beautiful product formula [Kup02], namely

$$B_n = \frac{n!}{2^n(2n)!}\prod_{j=1}^{n}\frac{(6j-2)!}{(2n+2j-1)!}. \tag{5.1.1}$$

As in the case of ordinary ASMs we are also interested in refined enumerations of the VSASM-numbers. The fact that the unique 1 in the first row of a VSASM is always in the middle column implies that the refined enumeration with respect to the first row is trivial. However, it also follows that the second row contains precisely two 1s and one -1. Therefore, a possible refined enumeration of VSASMs is with respect to the unique 1 in the second row that is situated left of the middle column. Let $B_{n,i}$ denote the number of $(2n+1)\times(2n+1)$-VSASMs where the first 1 in the second row is in column i. In [Fis08], it was conjectured that

$$B_{n,i} = \frac{B_{n-1}}{(4n-2)!}\frac{(2n+i-2)!(4n-i-1)!}{(i-1)!(2n-i)!}, \quad i=1,\dots,n. \tag{5.1.2}$$

Another possible refined enumeration is the one with respect to the first column's unique 1. Let $B^*_{n,i}$ denote the number of VSASMs of size $2n+1$ where the first column's unique 1 is located in row i. Razumov and Stroganov showed [RS04] that

$$B^*_{n,i} = \frac{B_{n-1}}{(4n-2)!}\sum_{r=1}^{i-1}(-1)^{i+r-1}\frac{(2n+r-2)!(4n-r-1)!}{(r-1)!(2n-r)!}, \quad i=1,\dots,2n+1. \tag{5.1.3}$$

Interestingly, the conjectured formula (5.1.2) would also imply a particularly simple linear relation between the two refined enumerations, namely

$$B_{n,i} = B^*_{n,i} + B^*_{n,i+1}, \quad i=1,\dots,n.$$

To give a bijective proof of this relation is an open problem. Such a proof would also imply (5.1.2).

If we switch from the world of ASMs to the world of MTs (cf. Proposition 1.2.5), then the property that a $(2n+1)\times(2n+1)$-VSASM is uniquely determined by its first n columns translates into the property that the corresponding MT is uniquely determined by the array of integers formed by the entries $\{1,2,\dots,n\}$. This array consists of $2n$ rows, whereby row i contains $\lceil i/2\rceil$ entries, and the bottom row is $(1,2,\dots,n)$ (see Figure 5.1). Let us therefore define a Halved Monotone Triangle (HMT) to be an array of integers $(a_{i,j})_{\substack{1\le i\le n,\\ 1\le j\le\lceil i/2\rceil}}$ arranged in the form

$$\begin{array}{cccccccc}
 & & & & & & & a_{1,1}\\
 & & & & & & a_{2,1} & \\
 & & & & & a_{3,1} & & a_{3,2}\\
 & & & & a_{4,1} & & a_{4,2} & \\
 & & & a_{5,1} & & a_{5,2} & & a_{5,3}\\
 & & a_{6,1} & & a_{6,2} & & a_{6,3} & \\
 & a_{7,1} & & a_{7,2} & & a_{7,3} & & a_{7,4}\\
a_{8,1} & & a_{8,2} & & a_{8,3} & & a_{8,4} &
\end{array}$$

with strict increase along rows and weak increase along NE- and SE-diagonals. The set of $(2n+1)\times(2n+1)$-VSASMs is then in one-to-one correspondence with the set of HMTs with bottom row $(1,2,\dots,n)$ and no entries larger than n.

The general counting problem of finding the number of HMTs with bottom row $k_1 < k_2 < \cdots < k_{\lceil n/2 \rceil}$ and no entry larger than x was considered in [Fis08]. There it was shown that for each $n \geq 1$ there exists a unique polynomial $\gamma(n, x; k_1, \ldots, k_{\lceil n/2 \rceil})$ in the variables k_i and x such that the evaluation at integers $k_1 < k_2 < \cdots < k_{\lceil n/2 \rceil} \leq x$ yields the number of HMTs with bottom row $(k_1, k_2, \ldots, k_{\lceil n/2 \rceil})$ and no entry larger than x. In particular, the number of $(2n+1) \times (2n+1)$–VSASMs is equal to $\gamma(2n, n; 1, 2, \ldots, n)$. Similar to the α-polynomial for counting MTs with fixed bottom row, the γ-polynomial can also be explicitly expressed in terms of an operator formula [Fis08]:

Theorem 5.1.1 (Operator formula for Halved Monotone Triangles)**.** *Let* $\mathrm{W}_{x,y} := \mathrm{E}_x + \Delta_x \Delta_y$ *and* $\mathrm{I}_{x,y} := \mathrm{E}_x^{-1} + \mathrm{E}_y^{-1} - \mathrm{id}$ *and* $n \geq 1$. *For odd* n *we then have*

$$\gamma(n, x; k_1, \ldots, k_{(n+1)/2}) = \prod_{1 \leq p < q \leq (n+1)/2} \mathrm{W}_{k_q,k_p} \, \mathrm{I}_{k_p,k_q} \prod_{1 \leq i < j \leq (n+1)/2} \frac{(k_j - k_i)(2x + 1 - k_i - k_j)}{(j-i)(j+i-1)},$$

and for even n *we have*

$$\gamma(n, x; k_1, \ldots, k_{n/2}) = \prod_{1 \leq p < q \leq n/2} \mathrm{W}_{k_q,k_p} \, \mathrm{I}_{k_p,k_q} \prod_{1 \leq i < j \leq n/2} \frac{(k_j - k_i)(2x + 2 - k_i - k_j)}{(j-i)(j+i)} \prod_{i=1}^{n/2} \frac{x + 1 - k_i}{i}.$$

In Section 2.4 we expressed the refined ASM numbers $A_{n,i}$ in terms of evaluations of the α-polynomial by considering the position of the unique 1 in the first column of the ASM or equivalently the number of entries 1 in the left-most NE-diagonal of the corresponding MT.

One way to express $B_{n,i}$ is to consider the position of the unique 1 in the left half of the VSASM's penultimate row. In the corresponding HMT with $2n$ rows and bottom row $(1, 2, \ldots, n)$ this translates into the unique integer missing from $\{1, 2, \ldots, n\}$ in row $2n - 2$ (see Figure 5.1), i.e.

$$B_{n,i} = \gamma(2n - 2, n; 1, 2, \ldots, \widehat{i}, \ldots, n).$$

A different way to express $B_{n,i}$ can be obtained by first rotating the VSASM by 90 degrees. The $(n+1)$-st row of the rotated VSASM then is $(1, -1, 1, \ldots, -1, 1)$. From the definition of ASMs, it follows that the vector of partial column sums of the first n rows is $(0, 1, 0, \ldots, 1, 0)$, i.e. the n-th row of the MT corresponding to the rotated VSASM is $(2, 4, \ldots, 2n)$. Since the rotated VSASM is uniquely determined by its first n rows, this establishes a one-to-one correspondence between VSASMs of size $2n + 1$ and MTs with bottom row $(2, 4, \ldots, 2n)$, and thus $B_n = \alpha(n; 2, 4, \ldots, 2n)$. The refined enumeration of VSASMs directly translates into a refined enumeration of MTs with bottom row $(2, 4, \ldots, 2n)$: from the correspondence it follows that $B_{n,i}$ counts MTs with bottom row $(2, 4, \ldots, 2n)$ and exactly $n + 1 - i$ entries equal to 2 in the left-most North-East-diagonal (see Figure 5.2).

In Chapter 2, we explained in general how to count MTs with fixed bottom row $(k_1, k_2, \ldots, k_n)$ and a fixed number of entries k_1 in the left-most NE-diagonal. Applied to our problem, Lemma 2.4.1 yields

$$B_{n,n-i} = (-1)^i \Delta_{k_1}^i \alpha(n; k_1, 4, 6, \ldots, 2n) \Big|_{k_1 = 3}. \tag{5.1.4}$$

The research presented in this chapter started after observing that the numbers $B_{n,i}$ seem to be a

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \Leftrightarrow \begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 2 & & 6 & & 7 & \\ 2 & & 4 & & 6 & & 8 \end{array}$$

Figure 5.2: Upper part of the rotated VSASM and its corresponding MT.

solution of a LES similar to the one satisfied by the refined ASM numbers (cf. Lemma 2.5.2):

$$\begin{aligned} B_{n,n-i} &= \sum_{j=i}^{n-1} \binom{3n-i-2}{j-i} (-1)^{j+n+1} B_{n,n-j}, \qquad & i = -n, -n+1, \ldots, n-1, \\ B_{n,n-i} &= B_{n,n+i+1}, & i = -n, -n+1, \ldots, n-1. \end{aligned} \tag{5.1.5}$$

Here we have to be a bit more precise: $B_{n,i}$ is not yet defined if $i = n+1, n+2, \ldots, 2n$. However, if we use for the moment (5.1.2) to define $B_{n,i}$ for all $i \in \mathbb{Z}$, then $(B_{n,i})_{1 \le i \le 2n}$ is a solution of (5.1.5); in Proposition 5.3.2 we show that the solution space of this LES is also one-dimensional. Coming back to the combinatorial definition of $B_{n,i}$, the goal of this chapter is to show how to naturally extend the combinatorial interpretation of $B_{n,i}$ to $i = n+1, \ldots, 2n$ and to present a conjecture of a completely different flavor which, once it is proven, implies that the numbers are a solution of the LES. The identity analogous to (1.2.2) is

$$B_{n,1} = \sum_{i=1}^{n-1} B_{n-1,i}. \tag{5.1.6}$$

The Chu–Vandermonde summation implies that also the numbers on the right-hand side of (5.1.2) fulfill this identity, and, once the conjecture presented next is proven, (5.1.2) also follows by induction with respect to n.

In order to be able to formulate the conjecture, we recall that the unnormalized symmetrizer $\mathbf{Sym}$ is defined as $\mathbf{Sym}\, p(x_1, \ldots, x_n) := \sum_{\sigma \in \mathcal{S}_n} p(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Conjecture 5.1.2. *For integers $s, t \ge 1$, consider the following rational function in $z_1, \ldots, z_{s+t-1}$*

$$P_{s,t}(z_1, \ldots, z_{s+t-1}) := \prod_{i=1}^{s} z_i^{2s-2i-t+1} (1 - z_i^{-1})^{i-1} \prod_{i=s+1}^{s+t-1} z_i^{2i-2s-t} (1 - z_i^{-1})^{s} \prod_{1 \le p < q \le s+t-1} \frac{1 - z_p + z_p z_q}{z_q - z_p}$$

and let $R_{s,t}(z_1, \ldots, z_{s+t-1}) := \mathbf{Sym}\, P_{s,t}(z_1, \ldots, z_{s+t-1})$. If $s \le t$, then

$$R_{s,t}(z_1, \ldots, z_{s+t-1}) = R_{s,t}(z_1, \ldots, z_{i-1}, z_i^{-1}, z_{i+1}, \ldots, z_{s+t-1})$$

for all $i \in \{1, 2, \ldots, s+t-1\}$.

Note that in fact the following more general statement seems to be true: if $s \le t$, then $R_{s,t}(z_1, \ldots, z_{s+t-1})$ is a linear combination of expressions of the form $\prod_{j=1}^{s+t-1} [(z_j - 1)(1 - z_j^{-1})]^{i_j}$,

$i_j \geq 0$, where the coefficients are non-negative integers. Moreover, it should be mentioned that it is easy to see that $R_{s,t}(z_1, \ldots, z_{s+t-1})$ is in fact a Laurent polynomial: Observe that

$$R_{s,t}(z_1, \ldots, z_{s+t-1}) = \frac{\mathbf{ASym}\left(P_{s,t}(z_1, \ldots, z_{s+t-1}) \prod_{1 \leq i < j \leq s+t-1} (z_j - z_i) \right)}{\prod_{1 \leq i < j \leq s+t-1} (z_j - z_i)}$$

with the unnormalized antisymmetrizer $\mathbf{ASym}\, p(x_1, \ldots, x_n) := \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \, p(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The assertion follows since $P_{s,t}(z_1, \ldots, z_{s+t-1}) \prod_{1 \leq i < j \leq s+t-1} (z_j - z_i)$ is a Laurent polynomial and every antisymmetric Laurent polynomial is divisible by $\prod_{1 \leq i < j \leq s+t-1} (z_j - z_i)$.

We will prove the following two theorems.

Theorem 5.1.3 ([FR14])**.** *Let $R_{s,t}(z_1, \ldots, z_{s+t-1})$ be as in Conjecture 5.1.2. If*

$$R_{s,t}(z_1, \ldots, z_{s+t-1}) = R_{s,t}(z_1^{-1}, \ldots, z_{s+t-1}^{-1})$$

for all $1 \leq s \leq t$, then (5.1.2) *is fulfilled.*

Theorem 5.1.4 ([FR14])**.** *Let $R_{s,t}(z_1, \ldots, z_{s+t-1})$ be as in Conjecture 5.1.2. Suppose*

$$R_{s,t}(z_1, \ldots, z_{s+t-1}) = R_{s,t}(z_1^{-1}, \ldots, z_{s+t-1}^{-1}) \tag{5.1.7}$$

if $t = s$ and $t = s+1$, $s \geq 1$. Then (5.1.7) *holds for all s, t with $1 \leq s \leq t$.*

While we believe that (5.1.2) should probably be attacked with the six vertex model approach (although we have not tried), we also think that the more general Conjecture 5.1.2 is interesting in its own right, given the fact that it only involves very elementary mathematical objects such as rational functions and the symmetric group.

This chapter is organized as follows: As a result independent from the rest of the chapter, we start by giving a shorter proof of Theorem 5.1.1, in which we apply the simplifications derived in Chapter 2. After that we move on to proving Theorem 5.1.3 and Theorem 5.1.4. For this, we first show in Section 5.3 that the solution space of (5.1.5) is one-dimensional, and that the conjectured numbers in (5.1.2) are a solution of this LES and satisfy (5.1.6). Then we present systems of linear equations that generalize the system satisfied by the refined ASM numbers in (2.5.3) and the system in the first line of (5.1.5) when restricting to non-negative i in the latter. Next we use the expression (5.1.4) for $B_{n,i}$ to extend the combinatorial interpretation to $i = n+1, n+2, \ldots, 2n$ and also extend the linear equation system to negative integers i accordingly. In Section 5.7, we justify the choice of certain constants that are involved in this extension. Afterwards we present a first conjecture implying (5.1.2). Finally, we are able to prove Theorem 5.1.3. The proof of Theorem 5.1.4 is given in Section 5.10. It is independent of the rest of the chapter and, at least for our taste, quite elegant. We would love to see a proof of Conjecture 5.1.2 which is possibly along these lines.

5.2 Operator formula for Halved Monotone Triangles

Similar to ordinary MTs, the number of HMTs with n rows, fixed bottom row $(k_1, \ldots, k_{\lceil n/2 \rceil})$ and no entry larger than x can be computed recursively. We have to distinguish between the cases n

odd and n even. If n is odd, then the number of entries in the row above is one less and they have to satisfy the usual restrictions of MTs:

$$
\begin{array}{ccccccccccc}
 & l_1 & & < & & l_2 & & < & \cdots & < & l_{\lceil (n-1)/2 \rceil} & & \\
\rotatebox{45}{$\le$} & & \rotatebox{-45}{$\ge$} & & \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\ge$} & & & & \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\ge$} \\
k_1 & & & & k_2 & & & & k_3 \quad \cdots & & k_{\lceil n/2 \rceil - 1} & & k_{\lceil n/2 \rceil}
\end{array}
$$

However, if n is even, then the row above contains the same number of entries, so that the right-most entry is upper bounded by x:

$$
\begin{array}{ccccccccccccc}
 & l_1 & & < & & l_2 & & < & \cdots & < & l_{\lceil (n-1)/2 \rceil - 1} & < & l_{\lceil (n-1)/2 \rceil} & \le & x \\
\rotatebox{45}{$\le$} & & \rotatebox{-45}{$\ge$} & & \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\ge$} & & & & \rotatebox{45}{$\le$} & & \rotatebox{-45}{$\ge$} & \rotatebox{45}{$\le$} & \\
k_1 & & & & k_2 & & & & k_3 \quad \cdots & & k_{\lceil n/2 \rceil - 1} & & k_{\lceil n/2 \rceil} & &
\end{array}
$$

We proceed as in Section 2.2 and define the γ-polynomials inductively using the summation operator: Let $\gamma(1, x; k_1) := 1$ and for $n \geq 2$

$$
\gamma(n, x; k_1, \ldots, k_{(n+1)/2}) := \sum_{(l_1,\ldots,l_{(n-1)/2})}^{(k_1,\ldots,k_{(n+1)/2})} \gamma(n-1, x; l_1, \ldots, l_{(n-1)/2}), \qquad n \text{ odd}, \tag{5.2.1}
$$

$$
\gamma(n, x; k_1, \ldots, k_{n/2}) := \sum_{(l_1,\ldots,l_{n/2})}^{(k_1,\ldots,k_{n/2},x)} \gamma(n-1, x; l_1, \ldots, l_{n/2}), \qquad n \text{ even}. \tag{5.2.2}
$$

Given integers $k_1 < k_2 < \cdots < k_{\lceil n/2 \rceil} \leq x$, the previous observations and (2.2.2) imply that $\gamma(n, x; k_1, \ldots, k_{\lceil n/2 \rceil})$ is equal to the number of HMTs with n rows, bottom row $(k_1, \ldots, k_{\lceil n/2 \rceil})$ and no entry greater than x, e.g. $\gamma(2, x; k_1) = x - k_1 + 1$.

Lemma 5.2.1. *The operands appearing in Theorem 5.1.1 can each be written as a determinant, namely for odd n*

$$
\prod_{1 \leq i < j \leq (n+1)/2} \frac{(k_j - k_i)(2x + 1 - k_i - k_j)}{(j-i)(j+i-1)} = (-1)^{\binom{(n+1)/2}{2}} \det_{1 \leq i,j \leq (n+1)/2} \binom{k_i + j - x - 2}{2j - 2},
$$

and for even n

$$
\prod_{1 \leq i < j \leq n/2} \frac{(k_j - k_i)(2x + 2 - k_i - k_j)}{(j-i)(j+i)} \prod_{i=1}^{n/2} \frac{x + 1 - k_i}{i} = (-1)^{\binom{(n+2)/2}{2}} \det_{1 \leq i,j \leq n/2} \binom{k_i + j - x - 2}{2j - 1}.
$$

Proof. Let us show the identities

$$
\prod_{1 \leq i < j \leq n} \frac{(k_j - k_i)(k_j + k_i)}{(j-i)(j+i-1)} = \det_{1 \leq i,j \leq n} \binom{k_i + j - 3/2}{2j - 2}, \tag{5.2.3}
$$

and

$$
\prod_{1 \leq i < j \leq n} \frac{(k_j - k_i)(k_j + k_i)}{(j-i)(j+i)} \prod_{i=1}^{n} \frac{k_i}{i} = \det_{1 \leq i,j \leq n} \binom{k_i + j - 1}{2j - 1}. \tag{5.2.4}
$$

The assertions then follow by substituting $k_l \mapsto k_l - x - 1/2$ in the odd case, and $k_l \mapsto k_l - x - 1$ in the even case. Since

$$\binom{k_i + j - 3/2}{2j-2} = \frac{1}{(2j-2)!} \prod_{l=1}^{j-1} (k_i^2 - (l-1/2)^2)$$

we have

$$\det_{1 \le i,j \le n} \binom{k_i + j - 3/2}{2j-2} = \prod_{j=1}^{n} \frac{1}{(2j-2)!} \det_{1 \le i,j \le n} (p_j(k_i^2)),$$

where p_j is a polynomial of degree $j-1$ with leading coefficient 1. From Remark 2.3.2 it follows that

$$\det_{1 \le i,j \le n} \binom{k_i + j - 3/2}{2j-2} = \prod_{j=1}^{n} \frac{1}{(2j-2)!} \prod_{1 \le i < j \le n} (k_j^2 - k_i^2).$$

Equation (5.2.3) is now a consequence of $\prod\limits_{j=1}^{n} \frac{1}{(2j-2)!} = \prod\limits_{1 \le i < j \le n} \frac{1}{(j-i)(j+i-1)}$. The second identity follows similarly with

$$\det_{1 \le i,j \le n} \binom{k_i + j - 1}{2j-1} = \det_{1 \le i,j \le n} \left(\frac{k_i}{(2j-1)!} \prod_{l=1}^{j-1} (k_i^2 - l^2) \right)$$
$$= \prod_{j=1}^{n} \frac{1}{(2j-1)!} \prod_{i=1}^{n} k_i \prod_{1 \le i < j \le n} (k_j^2 - k_i^2)$$

and $\prod\limits_{j=1}^{n} \frac{1}{(2j-1)!} = \frac{1}{n!} \prod\limits_{1 \le i < j \le n} \frac{1}{(j-i)(j+i)}$. □

For the following lemma, recall the definitions of the operators from Table 2.1 (p. 20).

Lemma 5.2.2. *Let $n \ge 1$ odd. Then*

$$\sum_{(l_1,\ldots,l_{(n-1)/2})}^{(k_1,\ldots,k_{(n+1)/2})} \prod_{1 \le p < q \le (n-1)/2} \mathrm{W}_{l_q,l_p} \mathrm{I}_{l_p,l_q} \det_{1 \le i,j \le (n-1)/2} \binom{l_i + j - x - 2}{2j-1}$$
$$= \prod_{1 \le p < q \le (n+1)/2} \mathrm{W}_{k_q,k_p} \mathrm{I}_{k_p,k_q} \det_{1 \le i,j \le (n+1)/2} \binom{k_i + j - x - 2}{2j-2}.$$

Proof. Let us first observe that the function

$$B(l_1, \ldots, l_{(n-1)/2}) := \prod_{1 \le p < q \le (n-1)/2} \mathrm{W}_{l_q,l_p} \mathrm{I}_{l_p,l_q} \det_{1 \le i,j \le (n-1)/2} \binom{l_i + j - x - 2}{2j}$$

fulfills the assumption of Lemma 2.3.4. To see this note that the operator

$$\mathrm{W}_{l_i,l_{i+1}} \prod_{1 \le p < q \le (n-1)/2} \mathrm{W}_{l_q,l_p} \mathrm{I}_{l_p,l_q}$$

is symmetric in the variables l_i and l_{i+1}, and therefore

$$\mathrm{S}_{l_i,l_{i+1}} \mathrm{W}_{l_i,l_{i+1}} B(l_1,\ldots,l_{(n-1)/2}) = -\mathrm{W}_{l_i,l_{i+1}} B(l_1,\ldots,l_{(n-1)/2}).$$

In particular, the assumption of Lemma 2.3.4 is fulfilled, and we obtain

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{(n-1)/2})}^{(k_1,\ldots,k_{(n+1)/2})} \prod_{1\le p<q\le (n-1)/2} \mathrm{W}_{l_q,l_p} \mathrm{I}_{l_p,l_q} \det_{1\le i,j\le (n-1)/2} \binom{l_i+j-x-2}{2j-1} \\
&\quad = \sum_{(l_1,\ldots,l_{(n-1)/2})}^{(k_1,\ldots,k_{(n+1)/2})} \Delta_{l_1}\cdots\Delta_{l_{(n-1)/2}} B(l_1,\ldots,l_{(n-1)/2}) \\
&\quad = \sum_{r=1}^{(n+1)/2} (-1)^{r-1} B(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{(n+1)/2}+1) \\
&\quad = \sum_{r=1}^{(n+1)/2} (-1)^{r-1} \prod_{\substack{1\le p<q\le (n+1)/2,\\ p,q\ne r}} \mathrm{W}_{k_q,k_p} \mathrm{I}_{k_p,k_q} \\
&\qquad\qquad \det_{1\le i,j\le (n-1)/2} \binom{l_i+j-x-2}{2j} \Bigg|_{\substack{(l_1,\ldots,l_{(n-1)/2})\\ =(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{(n+1)/2}+1)}}.
\end{aligned}$$

As the operand in the r-th summand does not contain k_r, we have $\mathrm{W}_{k_r,k_p} = \mathrm{id}$, $\mathrm{W}_{k_q,k_r} = \mathrm{E}_{k_q}$, $\mathrm{I}_{k_r,k_q} = \mathrm{E}_{k_q}^{-1}$ and $\mathrm{I}_{k_p,k_r} = \mathrm{E}_{k_p}^{-1}$. Together with Laplace expansion along the first column it follows that

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{(n-1)/2})}^{(k_1,\ldots,k_{(n+1)/2})} \prod_{1\le p<q\le (n-1)/2} \mathrm{W}_{l_q,l_p} \mathrm{I}_{l_p,l_q} \det_{1\le i,j\le (n-1)/2} \binom{l_i+j-x-2}{2j-1} \\
&\quad = \sum_{r=1}^{(n+1)/2} (-1)^{r-1} \prod_{1\le p<q\le (n+1)/2} \mathrm{W}_{k_q,k_p} \mathrm{I}_{k_p,k_q} \\
&\qquad\qquad \det_{1\le i,j\le (n-1)/2} \binom{l_i+j-x-1}{2j} \Bigg|_{\substack{(l_1,\ldots,l_{(n-1)/2})\\ =(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_{(n+1)/2})}} \\
&\quad = \prod_{1\le p<q\le (n+1)/2} \mathrm{W}_{k_q,k_p} \mathrm{I}_{k_p,k_q} \det_{1\le i,j\le (n+1)/2} \binom{k_i+j-x-2}{2j-2}.
\end{aligned}$$

□

Lemma 5.2.3. *Let $n \ge 2$ even. Then*

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{n/2})}^{(k_1,\ldots,k_{n/2},x)} \prod_{1\le p<q\le n/2} \mathrm{W}_{l_q,l_p} \mathrm{I}_{l_p,l_q} (-1)^{\binom{n/2}{2}} \det_{1\le i,j\le n/2} \binom{l_i+j-x-2}{2j-2} \\
&\quad = \prod_{1\le p<q\le n/2} \mathrm{W}_{k_q,k_p} \mathrm{I}_{k_p,k_q} (-1)^{\binom{n/2+1}{2}} \det_{1\le i,j\le n/2} \binom{k_i+j-x-2}{2j-1}.
\end{aligned}$$

Proof. The function

$$B_x(l_1,\ldots,l_{n/2}) := \prod_{1\le p<q\le n/2} \mathrm{W}_{l_q,l_p}\,\mathrm{I}_{l_p,l_q} \det_{1\le i,j\le n/2} \binom{l_i+j-x-2}{2j-1}$$

fulfills analogously to Lemma 5.2.2 the condition of Lemma 2.3.4. It follows that

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{n/2})}^{(k_1,\ldots,k_{n/2},x)} \prod_{1\le p<q\le n/2} \mathrm{W}_{l_q,l_p}\,\mathrm{I}_{l_p,l_q}(-1)^{\binom{n/2}{2}} \det_{1\le i,j\le n/2} \binom{l_i+j-x-2}{2j-2}\\
&\quad= (-1)^{\binom{n/2}{2}} \sum_{(l_1,\ldots,l_{n/2})}^{(k_1,\ldots,k_{n/2},x)} \Delta_{l_1}\cdots\Delta_{l_{n/2}} B_x(l_1,\ldots,l_{n/2})\\
&\quad= \sum_{r=1}^{n/2} (-1)^{r-1+\binom{n/2}{2}} B_x(k_1,\ldots,k_{r-1},k_{r+1}+1,\ldots,k_{n/2}+1,x+1)\\
&\qquad\qquad + (-1)^{n/2+\binom{n/2}{2}} B_x(k_1,\ldots,k_{n/2}).
\end{aligned}$$

Note that the last summand equals the right-hand side of the equation we want to prove. Let us show that $B_x(l_1,\ldots,l_{n/2-1},x+1)=0$, which implies that all other summands vanish. From Lemma 5.2.1 we obtain

$$\begin{aligned}
&B_x(l_1,\ldots,l_{n/2})\\
&\quad= \prod_{1\le p<q\le n/2} \mathrm{W}_{l_q,l_p}\,\mathrm{I}_{l_p,l_q}(-1)^{\binom{(n+2)/2}{2}} \prod_{1\le i<j\le n/2} \frac{(l_j-l_i)(2x+2-l_i-l_j)}{(j-i)(j+i)} \prod_{i=1}^{n/2} \frac{x+1-l_i}{i}.
\end{aligned}$$

It therefore suffices to show that

$$\prod_{p=1}^{n/2-1} \mathrm{W}_{l_{n/2},l_p}\,\mathrm{I}_{l_p,l_{n/2}}(x+1-l_{n/2}) \prod_{i=1}^{n/2-1} (l_{n/2}-l_i)(2x+2-l_i-l_{n/2})\Big|_{l_{n/2}=x+1} = 0. \tag{5.2.5}$$

Since

$$\begin{aligned}
\mathrm{W}_{y,z}\,\mathrm{I}_{z,y} &= (\mathrm{E}_y\,\mathrm{E}_z - \mathrm{E}_z + \mathrm{id})(\mathrm{E}_z^{-1} + \mathrm{E}_y^{-1} - \mathrm{id})\\
&= \mathrm{E}_y - \mathrm{id} + \mathrm{E}_z^{-1} + \mathrm{E}_z - \mathrm{E}_z\,\mathrm{E}_y^{-1} + \mathrm{E}_y^{-1} - \mathrm{E}_y\,\mathrm{E}_z + \mathrm{E}_z - \mathrm{id}\\
&= 2\Delta_z + \mathrm{E}_z^{-1} - \Delta_z(\mathrm{E}_y + \mathrm{E}_y^{-1})
\end{aligned}$$

equation (5.2.5) holds in particular if

$$(\mathrm{E}_y + \mathrm{E}_y^{-1})^N (x+1-y) \prod_{i=1}^{n/2-1} (y-l_i)(2x+2-l_i-y)\Big|_{y=x+1} = 0, \quad N \in \mathbb{N}_0.$$

From $(\mathrm{E}_y + \mathrm{E}_y^{-1})^N = \sum_{k=0}^{N} \binom{N}{k} \mathrm{E}_y^{2k-N}$ one obtains

$$
\begin{aligned}
(\mathrm{E}_y + \mathrm{E}_y^{-1})^N (x+1-y) \prod_{i=1}^{n/2-1} (y-l_i)(2x+2-l_i-y) \Big|_{y=x+1} \\
&= \sum_{k=0}^{N} \binom{N}{k} (N-2k) \prod_{i=1}^{n/2-1} (x+1+2k-N-l_i)(x+1-l_i-2k+N) \\
&= \sum_{k=0}^{N} \binom{N}{k} (N-2k) \prod_{i=1}^{n/2-1} \left((x+1-l_i)^2-(N-2k)^2\right).
\end{aligned}
$$

The fact that this sum vanishes follows from

$$
\sum_{k=0}^{N} \binom{N}{k} (N-2k)^{2M+1} = 0, \quad N, M \in \mathbb{N}_0,
$$

which holds because of

$$
\sum_{k=0}^{N} \binom{N}{k} (N-2k)^{2M+1} = \sum_{k=0}^{N} \binom{N}{N-k} (N-2(N-k))^{2M+1} = -\sum_{k=0}^{N} \binom{N}{k} (N-2k)^{2M+1}.
$$

□

We are now in the position to prove the operator formula by induction w.r.t. n.

Proof of Theorem 5.1.1.

- $n-1 \to n$, n odd: Applying recursion (5.2.1), the induction hypothesis, Lemma 5.2.1 and Lemma 5.2.2 yields

$$
\begin{aligned}
&\gamma(n, x; k_1, \ldots, k_{(n+1)/2}) \\
&= \sum_{(l_1,\ldots,l_{(n-1)/2})}^{(k_1,\ldots,k_{(n+1)/2})} \gamma(n-1, x; l_1, \ldots, l_{(n-1)/2}) \\
&= \sum_{(l_1,\ldots,l_{(n-1)/2})}^{(k_1,\ldots,k_{(n+1)/2})} \prod_{1 \le p < q \le (n-1)/2} \mathrm{W}_{l_q,l_p} \, \mathrm{I}_{l_p,l_q} (-1)^{\binom{(n+1)/2}{2}} \det_{1 \le i,j \le (n-1)/2} \binom{l_i + j - x - 2}{2j-1} \\
&= \prod_{1 \le p < q \le (n+1)/2} \mathrm{W}_{k_q,k_p} \, \mathrm{I}_{k_p,k_q} (-1)^{\binom{(n+1)/2}{2}} \det_{1 \le i,j \le (n+1)/2} \binom{k_i + j - x - 2}{2j-2} \\
&= \prod_{1 \le p < q \le (n+1)/2} \mathrm{W}_{k_q,k_p} \, \mathrm{I}_{k_p,k_q} \prod_{1 \le i < j \le (n+1)/2} \frac{(k_j - k_i)(2x+1-k_i-k_j)}{(j-i)(j+i-1)}.
\end{aligned}
$$

- $n-1 \to n$, n even: Analogously, apply Lemma 5.2.3 to obtain

$$\begin{aligned}
&\gamma(n,x;k_1,\dots,k_{n/2}) \\
&= \sum_{(l_1,\dots,l_{n/2})}^{(k_1,\dots,k_{n/2},x)} \gamma(n-1,x;l_1,\dots,l_{n/2}) \\
&= \sum_{(l_1,\dots,l_{n/2})}^{(k_1,\dots,k_{n/2},x)} \prod_{1\le p<q\le n/2} \mathrm{W}_{l_q,l_p}\,\mathrm{I}_{l_p,l_q} (-1)^{\binom{n/2}{2}} \det_{1\le i,j\le n/2} \binom{l_i+j-x-2}{2j-2} \\
&= \prod_{1\le p<q\le n/2} \mathrm{W}_{k_q,k_p}\,\mathrm{I}_{k_p,k_q} (-1)^{\binom{(n+2)/2}{2}} \det_{1\le i,j\le n/2} \binom{k_i+j-x-2}{2j-1} \\
&= \prod_{1\le p<q\le n/2} \mathrm{W}_{k_q,k_p}\,\mathrm{I}_{k_p,k_q} \prod_{1\le i<j\le n/2} \frac{(k_j-k_i)(2x+2-k_i-k_j)}{(j-i)(j+i)} \prod_{i=1}^{n/2} \frac{x+1-k_i}{i}.
\end{aligned}$$

□

5.3 The solution space of the LES

The goal of this section is to show that the numbers in (5.1.2) form a solution of the LES (5.1.5) and that the solution space is one-dimensional. Finally we show that the conjectured numbers also satisfy the recursion given in (5.1.6).

Lemma 5.3.1. *Let $n \ge 1$ and*

$$X_{n,i} := \frac{B_{n-1}}{(4n-2)!} \frac{(3n-i-2)!(3n+i-1)!}{(n-i-1)!(n+i)!}. \tag{5.3.1}$$

Then

$$X_{n,i} = \sum_{j=i}^{n-1} \binom{3n-i-2}{j-i} (-1)^{j+n+1} X_{n,j}, \quad i=-n,\dots,n-1.$$

Proof. We have to show that

$$\frac{(3n-i-2)!(3n+i-1)!}{(n-i-1)!(n+i)!} = \sum_{j=i}^{n-1} \binom{3n-i-2}{j-i} (-1)^{j+n+1} \frac{(3n-j-2)!(3n+j-1)!}{(n-j-1)!(n+j)!}.$$

The right-hand side is equal to

$$\begin{aligned}
&(3n-i-2)! \sum_{j=i}^{n-1} (-1)^{j+n+1} \frac{(3n+j-1)!}{(n-j-1)!(n+j)!(j-i)!} \\
&\qquad = \frac{(3n-i-2)!(2n-1)!}{(n-i-1)!} \sum_{j=i}^{n-1} (-1)^{j+n+1} \binom{n-i-1}{j-i} \binom{3n+j-1}{n+j} \\
&\qquad = -\frac{(3n-i-2)!(2n-1)!}{(n-i-1)!} \sum_{j=i}^{n-1} \binom{n-i-1}{n-j-1} \binom{-2n}{n+j}.
\end{aligned}$$

Since $n-i-1 \geq 0$, Chu–Vandermonde convolution yields

$$\sum_j \binom{n-i-1}{n-j-1}\binom{-2n}{n+j} = \binom{-n-i-1}{2n-1} = -\binom{3n+i-1}{2n-1}.$$

□

Proposition 5.3.2. *For fixed $n \geq 1$, the solution space of the following LES*

$$\begin{aligned} Y_{n,i} &= \sum_{j=i}^{n-1} \binom{3n-i-2}{j-i} (-1)^{j+n+1} Y_{n,j}, && i = -n, -n+1, \dots, n-1, \\ Y_{n,i} &= Y_{n,-i-1}, && i = -n, -n+1 \dots, n-1, \end{aligned}$$

in the variables $(Y_{n,i})_{-n \leq i \leq n-1}$ is one-dimensional.

Proof. Since the numbers $X_{n,i}$ from Lemma 5.3.1 establish a solution of the homogeneous system of linear equations, the solution space is at least one-dimensional. As

$$Y_{n,i} = \sum_{j=-n}^{n-1} \binom{3n-i-2}{j-i} (-1)^{j+n+1} Y_{n,-j-1} = \sum_{j=-n}^{n-1} \binom{3n-i-2}{-j-i-1} (-1)^{j+n} Y_{n,j}$$

it suffices to show that the 1-eigenspace of $\left(\binom{3n-i-2}{-j-i-1}(-1)^{j+n}\right)_{-n \leq i,j \leq n-1}$ is 1-dimensional. So, we have to show that

$$\operatorname{rk}\left(\binom{4n-i-1}{2n-i-j+1}(-1)^{j+1} - \delta_{i,j}\right)_{1 \leq i,j \leq 2n} = 2n-1.$$

After removing the first row and column and multiplying each row with -1, we are done as soon as we show that

$$\det\left(\binom{4n-i-1}{2n-i-j+1}(-1)^{j} + \delta_{i,j}\right)_{2 \leq i,j \leq 2n} \neq 0.$$

In Section 2.5 we already came across this determinant and observed that it is non-zero. □

Lemma 5.3.3. *Let $X_{n,i}$ be the numbers defined in* (5.3.1)*. Then*

$$X_{n,n-1} = \sum_{i=0}^{n-2} X_{n-1,i}$$

holds for $n \geq 2$.

Proof. The left-hand side is equal to B_{n-1} by definition. For the right-hand side one obtains

$$\begin{aligned} \sum_{i=0}^{n-2} X_{n-1,i} &= \frac{B_{n-2}}{(4n-6)!} \sum_{i=0}^{n-2} \frac{(3n-i-5)!(3n+i-4)!}{(n-i-2)!(n+i-1)!} \\ &= \frac{B_{n-2}}{\binom{4n-6}{2n-3}} \sum_{i=0}^{n-2} \binom{3n-i-5}{n-i-2}\binom{3n+i-4}{n+i-1}. \end{aligned}$$

Note that the two factors in the sum interchange under substitution $i \mapsto -i-1$. Together with Chu–Vandermonde convolution it follows that

$$\begin{aligned}\sum_{i=0}^{n-2} X_{n-1,i} &= \frac{B_{n-2}}{2\binom{4n-6}{2n-3}} \sum_{i=-n+1}^{n-2} \binom{3n-i-5}{n-i-2}\binom{3n+i-4}{n+i-1} \\ &= -\frac{B_{n-2}}{2\binom{4n-6}{2n-3}} \sum_{i} \binom{-2n+2}{n-i-2}\binom{-2n+2}{n+i-1} \\ &= -\frac{B_{n-2}}{2\binom{4n-6}{2n-3}} \binom{-4n+4}{2n-3} = \frac{B_{n-2}}{2\binom{4n-6}{2n-3}} \binom{6n-8}{2n-3}.\end{aligned}$$

That this is further equal to B_{n-1} follows directly from (5.1.1). □

5.4 A generalized LES for Monotone Triangles

We already observed in (2.4.2) resp. (5.1.4) that the refined ASM numbers resp. the refined VSASM numbers can be expressed as

$$\begin{aligned}A_{n,i+1} &= (-1)^i \Delta_{k_1}^i \alpha(n; k_1, 2, 3, \ldots, n)\Big|_{k_1=2}, \\ B_{n,n-i} &= (-1)^i \Delta_{k_1}^i \alpha(n; k_1, 4, 6, \ldots, 2n)\Big|_{k_1=3},\end{aligned}$$

for all $i = 0, 1, \ldots, n-1$. Let us generalize this by defining

$$C_{n,i}^{(d)} := (-1)^i \Delta_{k_1}^i \alpha(n; k_1, 2d, 3d, \ldots, nd)\Big|_{k_1=d+1}, \quad d \in \mathbb{Z},\ i \geq 0.$$

By Lemma 2.4.1, $C_{n,i}^{(d)}$ is for all $d \geq 1$ equal to the number of MTs with bottom row $(d, 2d, 3d, \ldots, nd)$ and exactly $i+1$ entries equal to d in the left-most NE-diagonal. Note that $C_{n,i}^{(1)} = A_{n,i+1}$ and $C_{n,i}^{(2)} = B_{n,n-i}$. In this section we prove that the numbers $C_{n,i}^{(d)}$ fulfill a certain LES. For $d = 1$, this proves Lemma 2.5.2, while for $d = 2$ it proves the first line of (5.1.5) for non-negative i.

Proposition 5.4.1. *For fixed $n, d \geq 1$ the numbers $(C_{n,i}^{(d)})_{0 \leq i \leq n-1}$ satisfy the following LES*

$$C_{n,i}^{(d)} = \sum_{j=i}^{n-1} \binom{n(d+1)-i-2}{j-i} (-1)^{j+n+1} C_{n,j}^{(d)}, \quad i = 0, \ldots, n-1. \tag{5.4.1}$$

Proof. The main ingredients of the proof are the identities (cf. Section 2.5)

$$\begin{aligned}\alpha(n; k_1, k_2, \ldots, k_n) &= (-1)^{n-1} \alpha(n; k_2, k_3, \ldots, k_n, k_1 - n), \\ \alpha(n; k_1, k_2, \ldots, k_n) &= \alpha(n; k_1 + c, k_2 + c, \ldots, k_n + c), \quad c \in \mathbb{Z}.\end{aligned}$$

Together with $\Delta_x = \mathrm{E}_x \, \delta_x$ and $\mathrm{E}_x^{-1} = (\mathrm{id} - \delta_x)$ we obtain

$$\begin{aligned}
C_{n,i}^{(d)} &= (-1)^i \Delta_{k_1}^i \alpha(n; k_1, 2d, 3d, \dots, nd) \Big|_{k_1=d+1} \\
&= (-1)^{i+n+1} \Delta_{k_1}^i \alpha(n; 2d, 3d, \dots, nd, k_1 - n) \Big|_{k_1=d+1} \\
&= (-1)^{i+n+1} \, \mathrm{E}_{k_1}^{-n-nd+i+2} \, \delta_{k_1}^i \alpha(n; 2d, 3d, \dots, nd, k_1 + d) \Big|_{k_1=nd-1} \\
&= (-1)^{i+n+1} (\mathrm{id} - \delta_{k_1})^{n(d+1)-i-2} \delta_{k_1}^i \alpha(n; d, 2d, \dots, (n-1)d, k_1) \Big|_{k_1=nd-1} \\
&= \sum_{j \geq 0} \binom{n(d+1)-i-2}{j} (-1)^{i+j+n+1} \delta_{k_1}^{i+j} \alpha(n; d, 2d, \dots, (n-1)d, k_1) \Big|_{k_1=nd-1} \\
&= \sum_{j \geq i} \binom{n(d+1)-i-2}{j-i} (-1)^{j+n+1} \delta_{k_1}^{j} \alpha(n; d, 2d, \dots, (n-1)d, k_1) \Big|_{k_1=nd-1}.
\end{aligned}$$

Since applying the δ-operator to a polynomial decreases its degree, and $\alpha(n; k_1, \dots, k_n)$ is a polynomial of degree $n-1$ in each k_i (Remark 2.3.6), it follows that the summands of the last sum are zero whenever $j \geq n$. So, it remains to show that

$$C_{n,j}^{(d)} = \delta_{k_1}^j \alpha(n; d, 2d, \dots, (n-1)d, k_1) \Big|_{k_1=nd-1}. \tag{5.4.2}$$

From Lemma 2.4.1 we know that the right-hand side of (5.4.2) is the number of MTs with bottom row $(d, 2d, \dots, nd)$ and exactly $j+1$ entries equal to nd in the right-most SE-diagonal. Replacing each entry x of the MT by $(n+1)d - x$ and reflecting it along the vertical symmetry axis is a one-to-one correspondence with the objects counted by $C_{n,j}^{(d)}$. □

5.5 The numbers $C_{n,i}^{(d)}$ for negative i

In order to prove (5.1.2), it remains to extend the definition of $C_{n,i}^{(2)}$ to $i = -n, \dots, -1$ in such a way that both the symmetry $C_{n,i}^{(2)} = C_{n,-i-1}^{(2)}$ and the first line of (5.1.5) are satisfied for negative i. Note that the definition of $C_{n,i}^{(2)}$ contains the operator $\Delta_{k_1}^i$ which is per se only defined for $i \geq 0$. The difference operator is (in discrete analogy to differentiation) only invertible up to an additive constant. This motivates the following definitions of right inverse difference operators:

Given a polynomial $p : \mathbb{Z} \to \mathbb{C}$, we define the right inverse difference operators as

$${}^z\Delta_x^{-1} p(x) := -\sum_{x'=x}^{z} p(x') \qquad \text{and} \qquad {}^z\delta_x^{-1} p(x) := \sum_{x'=z}^{x} p(x') \tag{5.5.1}$$

where $x, z \in \mathbb{Z}$ and the extended definition of summation defined in (2.2.6) is used. Recall from Section 2.2 that this definition ensures that polynomiality is preserved, i.e. if $p(i)$ is a polynomial in

i then $(a,b) \mapsto \sum_{i=a}^{b} p(i)$ is a polynomial function on $\mathbb{Z}^2$. The definition of the right inverse difference operators directly implies the following identities:

Proposition 5.5.1. *Let $z \in \mathbb{Z}$ and $p : \mathbb{Z} \to \mathbb{C}$ a function. Then*

1. $\Delta_x \, {}^z\Delta_x^{-1} = \mathrm{id}$ *and* ${}^z\Delta_x^{-1}\Delta_x \, p(x) = p(x) - p(z+1)$,
2. $\delta_x \, {}^z\delta_x^{-1} = \mathrm{id}$ *and* ${}^z\delta_x^{-1}\delta_x \, p(x) = p(x) - p(z-1)$,
3. $\Delta_x = \mathrm{E}_x \, \delta_x$ *and* ${}^z\Delta_x^{-1} = \mathrm{E}_x^{-1} \, \mathrm{E}_z \, {}^z\delta_x^{-1}$,
4. $\Delta_y \, {}^z\Delta_x^{-1} = {}^z\Delta_x^{-1}\Delta_y$ *and* $\delta_y \, {}^z\Delta_x^{-1} = {}^z\Delta_x^{-1}\delta_y$ *for* $y \neq x, z$.

Now we are in the position to define higher negative powers of the difference operators: For $i < 0$ and $\mathbf{z} = (z_i, z_{i+1}, \ldots, z_{-1}) \in \mathbb{Z}^{-i}$ we let

$$\begin{aligned} {}^{\mathbf{z}}\Delta_x^i &:= {}^{z_i}\Delta_x^{-1} \, {}^{z_{i+1}}\Delta_x^{-1} \ldots {}^{z_{-1}}\Delta_x^{-1}, \\ {}^{\mathbf{z}}\delta_x^i &:= {}^{z_i}\delta_x^{-1} \, {}^{z_{i+1}}\delta_x^{-1} \ldots {}^{z_{-1}}\delta_x^{-1}. \end{aligned}$$

After observing that ${}^z\delta_x^{-1} \, \mathrm{E}_x^{-1} = \mathrm{E}_x^{-1} \, \mathrm{E}_z^{-1} \, {}^z\delta_x^{-1}$ we can deduce the following generalization of Proposition 5.5.1 (3) inductively:

$$ {}^{\mathbf{z}}\Delta_x^i = \mathrm{E}_x^i \, \mathrm{E}_{z_i}^{i+2} \, \mathrm{E}_{z_{i+1}}^{i+3} \ldots \mathrm{E}_{z_{-1}}^{1} \, {}^{\mathbf{z}}\delta_x^i. \tag{5.5.2}$$

The right inverse difference operator allows us to naturally extend the definition of $C_{n,i}^{(d)}$. First, let us fix a sequence of integers $\mathbf{x} = (x_j)_{j<0}$ and set $\mathbf{x}_i = (x_i, x_{i+1}, \ldots, x_{-1})$ for $i < 0$. We define

$$C_{n,i}^{(d)} := \begin{cases} (-1)^i \Delta_{k_1}^i \alpha(n; k_1, 2d, 3d, \ldots, nd) \Big|_{k_1 = d+1}, & i = 0, \ldots, n-1, \\ (-1)^i \, {}^{\mathbf{x}_i}\Delta_{k_1}^i \alpha(n; k_1, 2d, 3d, \ldots, nd) \Big|_{k_1 = d+1}, & i = -n, \ldots, -1. \end{cases} \tag{5.5.3}$$

We detail on the choice of $\mathbf{x}$ in Section 5.7.

If $d \geq 1$, it is possible to give a rather natural combinatorial interpretation of $C_{n,i}^{(d)}$ also for negative i, which is based on the observations in Chapter 4. It is of no importance for the rest of the chapter, however, it provides a nice intuition: For non-negative i we already observed that $C_{n,i}^{(d)}$ counts the number of MTs with bottom row $(d, 2d, \ldots, nd)$ and exactly $i+1$ entries equal to d in the left-most NE-diagonal. Equivalently, the quantity $C_{n,i}^{(d)}$ counts partial MT where we cut off the bottom i elements of the left-most NE-diagonal, prescribe the entry $d+1$ in position $i+1$ of the NE-diagonal and the entries $(2d, 3d, \ldots, nd)$ in the bottom row of the remaining array (see Figure 5.3); in fact, in the exceptional case of $d = 1$ we do not require that the bottom element 2 of the truncated left-most NE-diagonal is strictly smaller than its right neighbour.

From (5.5.1) it follows that applying the inverse difference operator has the opposite effect of prolonging the left-most NE-diagonal: if $i < 0$, the quantity $C_{n,i}^{(d)}$ is the signed enumeration of arrays of the shape as depicted in Figure 5.4 subject to the following conditions:

- For the elements in the prolonged NE-diagonal including the entry left of the entry $2d$, we require the following: Suppose e is such an element and l is its SW-neighbour and r its SE-neighbour: if $l \leq r$, then $l \leq e \leq r$; otherwise $r < e < l$. In the latter case, the element contributes a -1 sign.

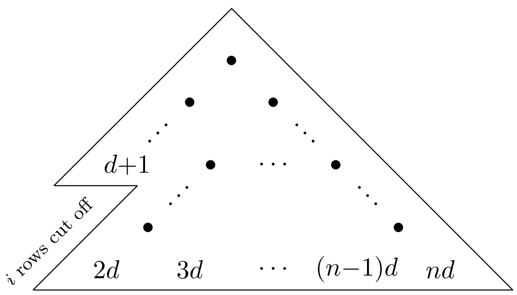


Figure 5.3: $C_{n,i}^{(d)}$ for $i \geq 0$.

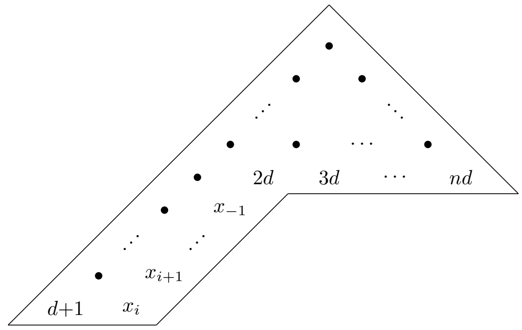


Figure 5.4: $C_{n,i}^{(d)}$ for $i < 0$.

- Inside the triangle, we follow the rules of GMTs (cf. Figure 4.1) as presented in Chapter 4. The total sign is the product of the sign of the GMT and the signs of the elements in the prolonged NE-diagonal.

An example for this combinatorial interpretation of the inverse difference operator is given in Figure 5.5.

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 2 & & 2 & & \\
 & 3 & & 2 & & 3 & \\
4 & & 1 & & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 2 & & 3 & & \\
 & 2 & & 2 & & 3 & \\
4 & & 1 & & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 3 & & & \\
 & & 2 & & 3 & & \\
 & 2 & & 2 & & 3 & \\
4 & & 1 & & & &
\end{array}
$$

Figure 5.5: Combinatorial interpretation of $-\,{}^{1}\Delta_{k_1}^{-1} \alpha(3; k_1, 2, 3)\Big|_{k_1=4} = -1 + 1 + 1 = 1.$

5.6 Extending the LES to negative i

The purpose of this section is the extension of the LES in Proposition 5.4.1 to negative i. This is accomplished with the help of the following lemma which shows that certain identities for $\Delta_{k_1}^{i} \alpha(n; k_1, \ldots, k_n)$, $i \geq 0$, carry over into the world of inverse difference operators.

Lemma 5.6.1. *Let $n, d \geq 1$.*

1. *Suppose $i \geq 0$. Then*

$$(-1)^i \; \Delta_{k_1}^{i} \alpha(n; k_1, 2d, 3d, \ldots, nd)\Big|_{k_1=d+1} = \delta_{k_n}^{i} \alpha(n; d, 2d, \ldots, (n-1)d, k_n)\Big|_{k_n=nd-1}.$$

2. *Suppose $i < 0$, and let $\mathbf{x}_i = (x_i, \ldots, x_{-1})$ and $\mathbf{y}_i = (y_i, \ldots, y_{-1})$ satisfy the relation $y_j = (n+1)d - x_j$ for all j. Then (see Figure 5.6)*

$$(-1)^i \; {}^{\mathbf{x}_i}\Delta_{k_1}^{i} \alpha(n; k_1, 2d, 3d, \ldots, nd)\Big|_{k_1=d+1} = {}^{\mathbf{y}_i}\delta_{k_n}^{i} \alpha(n; d, 2d, \ldots, (n-1)d, k_n)\Big|_{k_n=nd-1}.$$

3. *Suppose $i \geq 0$. Then*

$$\Delta^i_{k_1}\alpha(n;k_1,\ldots,k_n) = (-1)^{n-1}\,\mathrm{E}^{i-n}_{k_1}\,\delta^i_{k_1}\alpha(n;k_2,\ldots,k_n,k_1).$$

4. *Suppose $i < 0$, and let $\mathbf{x}_i = (x_i,\ldots,x_{-1})$ and $\mathbf{y}_i = (y_i,\ldots,y_{-1})$ satisfy the relation $y_j = x_j + j - n + 2$ for all j. Then*

$$^{\mathbf{x}_i}\Delta^i_{k_1}\alpha(n;k_1,\ldots,k_n) = (-1)^{n-1}\,\mathrm{E}^{i-n}_{k_1}\;{}^{\mathbf{y}_i}\delta^i_{k_1}\alpha(n;k_2,\ldots,k_n,k_1).$$

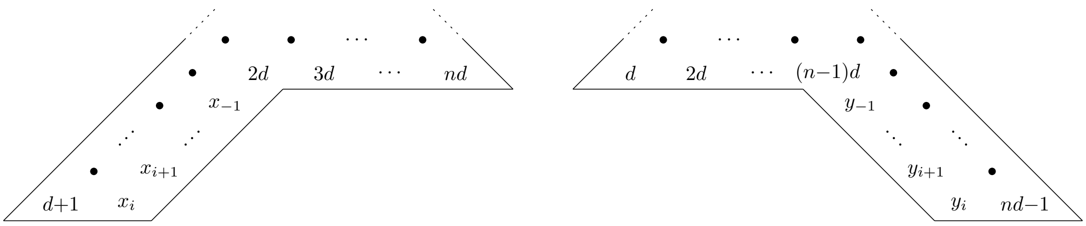


Figure 5.6: Symmetry of inverse difference operators if $y_j = (n+1)d - x_j$.

Proof. For the first part we refer to (5.4.2). Concerning the second part, we actually show the following more general statement: if $r = (n+1)d - l$ and $i \leq 0$, then

$$(-1)^i\;{}^{\mathbf{x}_i}\Delta^i_{k_1}\alpha(n;k_1,2d,3d,\ldots,nd)\Big|_{k_1=l} = {}^{\mathbf{y}_i}\delta^i_{k_n}\alpha(n;d,2d,\ldots,(n-1)d,k_n)\Big|_{k_n=r}, \tag{5.6.1}$$

whereby ${}^{\mathbf{x}_0}\Delta^0_{k_1} :=$ id and ${}^{\mathbf{y}_0}\delta^0_{k_n} :=$ id. We use induction with respect to i:

If $i = 0$, then (5.6.1) follows from the general identity

$$\alpha(n;k_1,\ldots,k_n) = \alpha(n;c-k_n,\ldots,c-k_1),$$

which is combinatorially obvious (cf. (5.4.2)) for $c \in \mathbb{Z}$ and integers $k_1 < \ldots < k_n$, and is therefore an identity satisfied by the polynomials.

If $i < 0$, then, by the definitions of the right inverse operators and the induction hypothesis, we have

$$\begin{aligned}
(-1)^i\;{}^{\mathbf{x}_i}\Delta^i_{k_1}\alpha(n;k_1,2d,3d,\ldots,nd)\Big|_{k_1=l} &= \sum_{k_1'=l}^{x_i}(-1)^{i+1}\;{}^{\mathbf{x}_{i+1}}\Delta^{i+1}_{k_1'}\alpha(n;k_1',2d,3d,\ldots,nd)\\
&= \sum_{k_1'=l}^{x_i}{}^{\mathbf{y}_{i+1}}\delta^{i+1}_{k_n'}\alpha(n;d,2d,\ldots,(n-1)d,k_n')\Big|_{k_n'=(n+1)d-k_1'}\\
&= \sum_{k_1'=(n+1)d-x_i}^{(n+1)d-l}{}^{\mathbf{y}_{i+1}}\delta^{i+1}_{k_n'}\alpha(n;d,2d,\ldots,(n-1)d,k_n')\Big|_{k_n'=k_1'}\\
&= \sum_{k_n'=y_i}^{r}{}^{\mathbf{y}_{i+1}}\delta^{i+1}_{k_n'}\alpha(n;d,2d,\ldots,(n-1)d,k_n')\\
&= {}^{\mathbf{y}_i}\delta^i_{k_n}\alpha(n;d,2d,\ldots,(n-1)d,k_n)\Big|_{k_n=nd-1}.
\end{aligned}$$

The third part follows from the circular shift identity (2.5.2) and $\Delta_x = \mathrm{E}_x \, \delta_x$. The last part is again shown by induction with respect to i. In fact the cases $i = 0$ of the third and last part coincide and can thus be chosen to be the initial case of the induction. If $i < 0$, then the induction hypothesis and (2.2.6) imply

$$\begin{aligned}
{}^{\mathbf{x}_i}\Delta_{k_1}^{i} \alpha(n; k_1, \ldots, k_n) &= - \sum_{l_1=k_1}^{x_i} {}^{\mathbf{x}_{i+1}}\Delta_{l_1}^{i+1} \alpha(n; l_1, k_2, \ldots, k_n) \\
&= - \sum_{l_1=k_1}^{x_i} (-1)^{n-1} \mathrm{E}_{l_1}^{i+1-n} \; {}^{\mathbf{y}_{i+1}}\delta_{l_1}^{i+1} \alpha(n; k_2, \ldots, k_n, l_1) \\
&= \sum_{l_1=x_i+i-n+2}^{k_1+i-n} (-1)^{n-1} \, {}^{\mathbf{y}_{i+1}}\delta_{l_1}^{i+1} \alpha(n; k_2, \ldots, k_n, l_1) \\
&= (-1)^{n-1} \mathrm{E}_{k_1}^{i-n} \; {}^{\mathbf{y}_i}\delta_{k_1}^{i} \alpha(n; k_2, \ldots, k_n, k_1). \qquad \square
\end{aligned}$$

Now we are in the position to generalize Proposition 5.4.1 to negative i.

Proposition 5.6.2. *Let $n, d \geq 1$. For $i < 0$, let $\mathbf{x}_i, \mathbf{z}_i \in \mathbb{Z}^{-i}$ with $z_j = (n+2)(d+1) - x_j - j - 4$ and define*

$$D_{n,i}^{(d)} := \begin{cases} (-1)^i \Delta_{k_1}^{i} \alpha(n; k_1, 2d, 3d, \ldots, nd) \Big|_{k_1=d+1}, & i = 0, \ldots, n-1, \\ (-1)^i \; {}^{\mathbf{z}_i}\Delta_{k_1}^{i} \alpha(n; k_1, 2d, 3d, \ldots, nd) \Big|_{k_1=d+1}, & i = -n, \ldots, -1. \end{cases} \tag{5.6.2}$$

Then

$$C_{n,i}^{(d)} = \sum_{j=i}^{n-1} \binom{n(d+1) - i - 2}{j - i} (-1)^{j+n+1} D_{n,j}^{(d)}.$$

holds for all $i = -n, \ldots, n-1$.

Proof. To simplify notation let us define ${}^{\mathbf{x}_i}\Delta_{k_1}^{i} := \Delta_{k_1}^{i}$ for $i \geq 0$. Since the definition of $C_{n,i}^{(d)}$ and $D_{n,i}^{(d)}$ only differ in the choice of constants for negative i, the fact that the system of linear equations is satisfied for $i = 0, \ldots, n-1$ is Proposition 5.4.1. For $i = -n, \ldots, -1$ first note that, by Lemma 5.6.1, (2.5.1) and $\mathrm{E}_x^{-1} \; {}^{z}\delta_x^{-1} = {}^{z+1}\delta_x^{-1} \, \mathrm{E}_x^{-1}$, we have

$$C_{n,i}^{(d)} = (-1)^{n-1+i} \mathrm{E}_{k_1}^{i-n} \; {}^{\mathbf{y}_i}\delta_{k_1}^{i} \alpha(n; d, 2d, \ldots, (n-1)d, k_1) \Big|_{k_1=1}$$

where $\mathbf{y}_i = (y_i, \ldots, y_{-1})$ with $y_j = x_j + j + 2 - n - d$. This is furthermore equal to

$$(-1)^{n-1+i} \mathrm{E}_{k_1}^{i-n(d+1)+2} \; {}^{\mathbf{y}_i}\delta_{k_1}^{i} \alpha(n; d, 2d, \ldots, (n-1)d, k_1) \Big|_{k_1=nd-1}.$$

Now we use

$$\mathrm{E}_{k_1}^{i-n(d+1)+2} = (\mathrm{id} - \delta_{k_1})^{n(d+1)-i-2} = \sum_{j=0}^{n(d+1)-i-2} \binom{n(d+1)-i-2}{j} (-1)^j \delta_{k_1}^{j}$$

and Proposition 5.5.1 (2) to obtain

$$\sum_{j=0}^{n(d+1)-i-2} \binom{n(d+1)-i-2}{j} (-1)^{n-1+i+j} \, {}^{\mathbf{y}_{i+j}}\delta_{k_1}^{i+j} \alpha(n; d, 2d, \ldots, (n-1)d, k_1) \Big|_{k_1=nd-1}.$$

Since the (ordinary) difference operator applied to a polynomial decreases the degree, the upper summation limit can be changed to $n-1-i$. Together with Lemma 5.6.1 this transforms into

$$\sum_{j=i}^{n-1} \binom{n(d+1)-i-2}{j-i} (-1)^{n-1+j} \, {}^{\mathbf{y}_j}\delta_{k_1}^{j} \alpha(n; d, 2d, \ldots, (n-1)d, k_1) \Big|_{k_1=nd-1}$$
$$= \sum_{j=i}^{n-1} \binom{n(d+1)-i-2}{j-i} (-1)^{n-1} \, {}^{\mathbf{z}_j}\Delta_{k_1}^{j} \alpha(n; k_1, 2d, 3d, \ldots, nd) \Big|_{k_1=d+1},$$

where $z_j = (n+1)d - y_j = (n+2)(d+1) - x_j - j - 4$. □

Now it remains to find an integer sequence $(x_j)_{j<0}$ such that $C_{n,i}^{(2)} = C_{n,-i-1}^{(2)}$ and $C_{n,i}^{(2)} = D_{n,i}^{(2)}$ for negative i.

5.7 How to choose the constants $\mathbf{x} = (x_j)_{j<0}$

In this section, it is shown that $C_{n,i}^{(2)} = C_{n,-i-1}^{(2)}$ if we choose $\mathbf{x} = (x_j)_{j<0}$ with $x_j = -2j+1$, $j<0$. This can be deduced from the following more general result.

Proposition 5.7.1. *Let $x_j = -2j+1$, $j<0$, and set $\mathbf{x}_i = (x_i, x_{i+1}, \ldots, x_{-1})$ for all $i<0$. Suppose $p : \mathbb{Z} \to \mathbb{C}$ and let*

$$c_i := \begin{cases} (-1)^i \Delta_y^i p(y) \Big|_{y=3}, & i \geq 0, \\ (-1)^i \, {}^{\mathbf{x}_i}\Delta_y^i p(y) \Big|_{y=3}, & i < 0, \end{cases}$$

for $i \in \mathbb{Z}$. Then the numbers satisfy the symmetry $c_i = c_{-i-1}$.

Proof. Recall the definition of the right inverse difference operator (5.5.1) and check that $c_0 = c_{-1}$. In the following, we assume $i \geq 1$. Then

$$c_i = (-1)^i (\mathrm{E}_y - \mathrm{id})^i p(y) \Big|_{y=3} = \sum_{d_1=3}^{i+3} \binom{i}{d_1 - 3} (-1)^{d_1+1} p(d_1),$$

and

$$c_{-i-1} = (-1)^{i+1} \, {}^{\mathbf{x}_{-i-1}}\Delta_y^{-i-1} p(y) \Big|_{y=3} = \sum_{d_{i+1}=3}^{2i+3} \sum_{d_i=d_{i+1}}^{2i+1} \cdots \sum_{d_2=d_3}^{5} \sum_{d_1=d_2}^{3} p(d_1). \tag{5.7.1}$$

The situation is illustrated in Figure 5.7. According to (2.2.6), the iterated sum is the signed summation of $(d_1, d_2, \ldots, d_{i+1}) \in \mathbb{Z}^{i+1}$ subject to the following restrictions: We have $3 \leq d_{i+1} \leq 2i+3$, and for $1 \leq j \leq i$ the restrictions are

$$\begin{aligned} d_{j+1} \leq d_j \leq 2j+1 \qquad & \text{if } d_{j+1} \leq 2j+1, \\ d_{j+1} > d_j > 2j+1 \qquad & \text{if } d_{j+1} > 2j+1. \end{aligned} \tag{5.7.2}$$

Note that there is no admissible $(d_1, d_2, \ldots, d_{i+1})$ with $d_{j+1} = 2j + 2$. The sign of $(d_1, d_2, \ldots, d_{i+1})$ is computed as $(-1)^{\#\{1 \le j \le i \colon d_j > 2j+1\}}$.

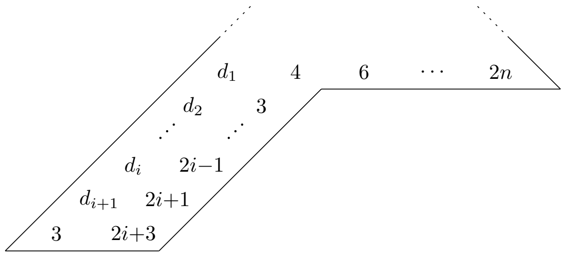


Figure 5.7: Combinatorial interpretation of (5.7.1) if $p(y) = \alpha(n; y, 4, 6, \ldots, 2n)$.

The proof now proceeds by showing that the signed enumeration of $(d_1, \ldots, d_{i+1})$ with fixed d_1 is just $\binom{i}{d_1-3}(-1)^{d_1+1}$. The reversed sequence $(d_{i+1}, d_i, \ldots, d_1)$ is weakly increasing as long as we are in the first case of (5.7.2). However, once we switch from the first case to the second case, the sequence is strictly decreasing afterwards, because $d_{j+1} > 2j + 1$ implies $d_j > 2j + 1 > 2j - 1$. Thus, the sequence splits into two parts: there exists an l, $0 \le l \le i$, with

$$3 \le d_{i+1} \le d_i \le \ldots \le d_{l+1} > d_l > \ldots > d_1.$$

Moreover, it is not hard to see that (5.7.2) implies $d_{l+1} = 2l + 3$ and $d_l = 2l + 2$. The sign of the sequence is $(-1)^l$. Thus it suffices to count the following two types of sequences.

1. $3 \le d_{i+1} \le d_i \le \cdots \le d_{l+2} \le d_{l+1} = 2l + 3$.
2. $d_l = 2l + 2 > d_{l-1} > \cdots > d_2 > d_1 > 3$ and $d_k > 2k + 1$ for $1 < k \le l - 1$; d_1 fixed.

For the first type, this is accomplished by the binomial coefficient $\binom{i+l}{i-l}$.

If $l \ge 1$, then the sequences of the second type are prefixes of Dyck paths in disguise: To see this, consider prefixes of Dyck paths starting in $(0, 0)$ with a steps of type $(1, 1)$ and b steps of type $(1, -1)$. Such a partial Dyck path is uniquely determined by the x-coordinates of its $(1, 1)$-steps. If p_i denotes the position of the i-th $(1, 1)$-step, then the coordinates correspond to such a partial Dyck path if and only if

$$0 = p_1 < p_2 < \cdots < p_a < a + b \quad \text{and} \quad p_k < 2k - 1.$$

In order to obtain our sequences, set $a \mapsto l - 1$, $b \mapsto l + 3 - d_1$ and $p_k \mapsto 2l + 2 - d_{l-k+1}$. By the reflection principle, the number of prefixes of such Dyck paths is

$$\binom{a+b}{b} \frac{a+1-b}{a+1} = \binom{2l+2-d_1}{l+3-d_1} \frac{d_1-3}{l}.$$

If $l = 0$, then $d_1 = d_2 = \ldots = d_{i+1} = 3$ and this is the only case where $d_1 = 3$. Put together, we see that the coefficient of $p(d_1)$ in (5.7.1) is

$$\sum_{l=1}^{i} (-1)^l \binom{i+l}{i-l} \binom{2l+2-d_1}{l+3-d_1} \frac{d_1-3}{l} \tag{5.7.3}$$

if $d_1 \geq 4$. Using standard tools to prove hypergeometric identities, it is not hard to see that this is equal to $\binom{i}{d_1-3}(-1)^{d_1+1}$ if $d_1 \geq 4$ and $i \geq 1$. For instance, C. Krattenthaler's Mathematica package HYP [Kra95] can be applied as follows: After converting the sum into hypergeometric notation, one applies contiguous relation `C16`. Next we use transformation rule `T4306`, before it is possible to apply summation rule `S2101` which is the Chu–Vandermonde summation. For those preferring computational proofs by hand, see Lemma A.2.4 in the appendix. □

In the following, we let $\mathbf{x} = (x_j)_{j<0}$ with $x_j = -2j+1$ and $\mathbf{z} = (z_j)_{j<0}$ with $z_j = (n+2)(d+1)+j-5$. Recall that $\mathbf{x}$ is crucial in the definition of $C_{n,i}^{(d)}$, see (5.5.3), while $\mathbf{z}$ appears in the definition of $D_{n,i}^{(d)}$, see (5.6.2). To complete the proof of (5.1.2), it remains to show

$$C_{n,i}^{(2)} = D_{n,i}^{(2)} \tag{5.7.4}$$

for $i = -n, -n+1, \ldots, -1$, since Proposition 5.6.2 and Proposition 5.7.1 then imply that the numbers $C_{n,i}^{(2)}$, $i = -n, -n+1, \ldots, n-1$, are a solution of the LES (5.1.5). The situation is depicted in Figure 5.8. When trying to proceed as in the proof of Proposition 5.7.1 one eventually ends

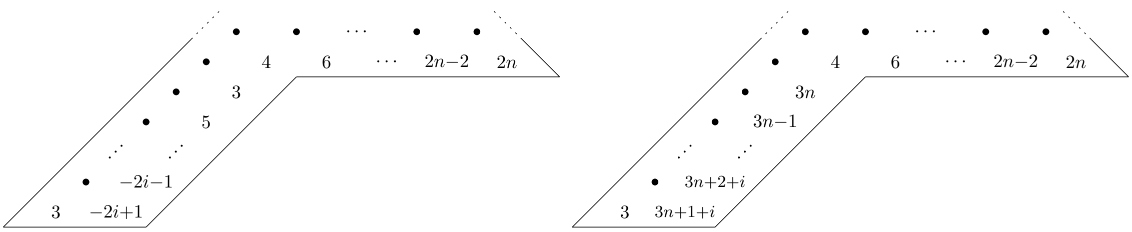


Figure 5.8: Combinatorial interpretation of the open problem (5.7.4).

up with having to show that the refined VSASM numbers $B_{n,i}$ satisfy a different system of linear equations:

$$\sum_{j=0}^{n-1} \left(\binom{3n-i-2}{i+j+1} - \binom{3n-i-2}{i-j} \right) (-1)^j B_{n,n-j} = 0, \quad i = 0, 1, \ldots, n-1. \tag{5.7.5}$$

While computer experiments indicate that this LES uniquely determines $(B_{n,1}, \ldots, B_{n,n})$ up to a multiplicative constant for all $n \geq 1$, it is not clear at all how to derive that the refined VSASM numbers satisfy (5.7.5). We therefore try a different approach in tackling (5.7.4).

The task of the rest of the chapter is to show that (5.7.4) follows from a more general multivariate Laurent polynomial identity and present partial results towards proving the latter.

5.8 The key conjecture

We start this section by showing that the application of the right inverse difference operators ${}^{z}\Delta_{k_1}^{-1}$ to $\alpha(n; k_1, \ldots, k_n)$ can be replaced by the application of a bunch of ordinary difference operators to $\alpha(n+1; k_1, z, k_2, \ldots, k_n)$:

Lemma 5.8.1. *Let $i < 0$ and $\mathbf{x}_i \in \mathbb{Z}^{-i}$. Then*

$$
{}^{\mathbf{x}_i}\Delta^i_{k_j}\alpha(n;k_1,\ldots,k_n) = (-1)^{ij}
$$
$$
\times\,\Delta^{-i}_{k_1}\ldots\Delta^{-i}_{k_{j-1}}\delta^0_{x_i}\delta^1_{x_{i+1}}\ldots\delta^{-i-1}_{x_{-1}}\delta^{-i}_{k_{j+1}}\ldots\delta^{-i}_{k_n}\alpha(n-i;k_1,\ldots,k_j,x_i,x_{i+1},\ldots,x_{-1},k_{j+1},\ldots,k_n)
$$

and

$$
{}^{\mathbf{x}_i}\delta^i_{k_j}\alpha(n;k_1,\ldots,k_n) = (-1)^{(j-1)i+\binom{-i}{2}}
$$
$$
\times\,\Delta^{-i}_{k_1}\ldots\Delta^{-i}_{k_{j-1}}\Delta^{-i-1}_{x_{-1}}\Delta^{-i-2}_{x_{-2}}\ldots\Delta^0_{x_i}\delta^{-i}_{k_{j+1}}\ldots\delta^{-i}_{k_n}\alpha(n-i;k_1,\ldots,k_{j-1},x_{-1},x_{-2},\ldots,x_i,k_j,\ldots,k_n).
$$

Proof. Informally, the lemma follows from the following two facts:

- The quantity ${}^{\mathbf{x}_i}\Delta^i_{k_j}\alpha(n;k_1,\ldots,k_n)$ can be interpreted as the signed enumeration of Monotone Triangle structures of the shape as depicted in Figure 5.9 where the j-th NE-diagonal has been prolonged. Similarly, for ${}^{\mathbf{x}_i}\delta^i_{k_j}\alpha(n;k_1,\ldots,k_n)$, where the shape is depicted in Figure 5.10 and the j-th SE-diagonal has been prolonged.

- The application of the $(-\Delta)$-operator truncates left NE-diagonals, while the δ-operator truncates right SE-diagonals. This idea first appeared in [Fis11].

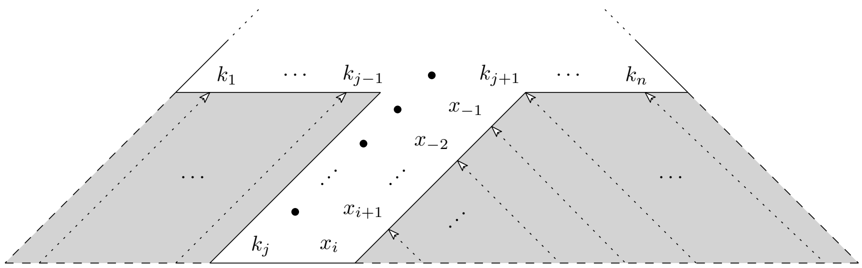


Figure 5.9: ${}^{\mathbf{x}_i}\Delta^i_{k_j}\alpha(n;k_1,\ldots,k_n)$

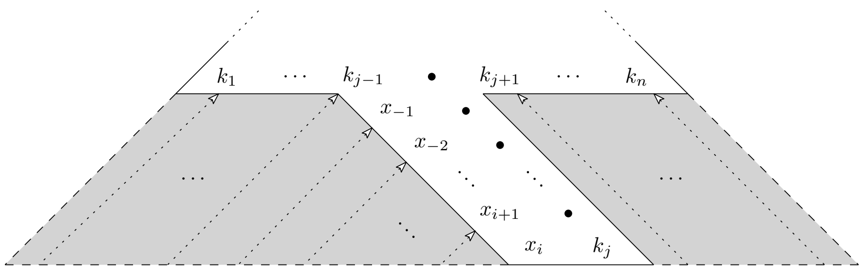


Figure 5.10: ${}^{\mathbf{x}_i}\delta^i_{k_j}\alpha(n;k_1,\ldots,k_n)$

Formally, let us prove the first identity by induction with respect to i. First note that (2.2.5) and (2.2.8) imply

$$(-1)^j \Delta_{k_1} \dots \Delta_{k_{j-1}} \delta_{k_{j+1}} \delta_{k_{j+2}} \dots \delta_{k_n} \sum_{(l_1,\dots,l_n)}^{(k_1,\dots,k_{j-1},k_j,x,k_{j+1},\dots,k_n)} A(l_1,\dots,l_n) \tag{5.8.1}$$
$$= - \sum_{(l_j)}^{(k_j,x)} A(k_1,\dots,k_{j-1},l_j,k_{j+1},\dots,k_n) = {}^{x}\Delta_{k_j}^{-1} A(k_1,k_2,\dots,k_n).$$

Together with (2.2.4) the base case $i = -1$ follows. For the inductive step $i < -1$, apply the induction hypothesis, (5.8.1), (2.2.4) and Proposition 5.5.1 (4) to obtain

$$\begin{aligned}
&{}^{\mathbf{x}_i}\Delta_{k_j}^{i} \alpha(n;k_1,\dots,k_n) \\
&= {}^{x_i}\Delta_{k_j}^{-1} (-1)^{(i+1)j} \Delta_{k_1}^{-i-1} \dots \Delta_{k_{j-1}}^{-i-1} \delta_{x_{i+1}}^{0} \delta_{x_{i+2}}^{1} \dots \delta_{x_{-1}}^{-i-2} \delta_{k_{j+1}}^{-i-1} \dots \delta_{k_n}^{-i-1} \\
&\qquad \alpha(n-i-1;k_1,\dots,k_j,x_{i+1},x_{i+2},\dots,x_{-1},k_{j+1},\dots,k_n) \\
&= (-1)^{ij} \Delta_{k_1}^{-i} \dots \Delta_{k_{j-1}}^{-i} \delta_{x_{i+1}}^{1} \delta_{x_{i+2}}^{2} \dots \delta_{x_{-1}}^{-i-1} \delta_{k_{j+1}}^{-i} \dots \delta_{k_n}^{-i} \\
&\qquad \sum_{(l_1,\dots,l_j,y_{i+1},\dots,y_{-1},l_{j+1},\dots,l_n)}^{(k_1,\dots,k_j,x_i,x_{i+1},\dots,x_{-1},k_{j+1},\dots,k_n)} \alpha(n-i-1;l_1,\dots,l_j,y_{i+1},y_{i+2},\dots,y_{-1},l_{j+1},\dots,l_n) \\
&= (-1)^{ij} \Delta_{k_1}^{-i} \dots \Delta_{k_{j-1}}^{-i} \delta_{x_{i+1}}^{1} \dots \delta_{x_{-1}}^{-i-1} \delta_{k_{j+1}}^{-i} \dots \delta_{k_n}^{-i} \alpha(n-i;k_1,\dots,k_j,x_i,x_{i+1},\dots,x_{-1},k_{j+1},\dots,k_n).
\end{aligned}$$

The second identity can be shown analogously. The sign is again obtained by taking the total number of applications of the Δ-operator into account. □

As mentioned in Section 4.4, the α-polynomial satisfies [Fis06]

$$(\mathrm{id} + \mathrm{E}_{k_{i+1}} \mathrm{E}_{k_i}^{-1} \mathrm{S}_{k_i,k_{i+1}}) \mathrm{V}_{k_i,k_{i+1}} \alpha(n;k_1,\dots,k_n) = 0 \tag{5.8.2}$$

for $1 \le i \le n-1$. This property together with the fact that the degree of $\alpha(n;k_1,\dots,k_n)$ in each k_i is $n-1$ determines the polynomial up to a constant. Next we present a conjecture on general polynomials with property (5.8.2); the goal of the current section is to show that this conjecture implies (5.7.4).

Conjecture 5.8.2. *Let $1 \le s \le t$ and $a(k_1,\dots,k_{s+t-1})$ be a polynomial in $(k_1,\dots,k_{s+t-1})$ with*

$$(\mathrm{id} + \mathrm{E}_{k_{i+1}} \mathrm{E}_{k_i}^{-1} \mathrm{S}_{k_i,k_{i+1}}) \mathrm{V}_{k_i,k_{i+1}} a(k_1,\dots,k_{s+t-1}) = 0 \tag{5.8.3}$$

for $1 \le i \le s+t-2$. Then

$$\prod_{i=1}^{s} \mathrm{E}_{y_i}^{2s+3-2i} \delta_{y_i}^{i-1} \prod_{i=2}^{t} \mathrm{E}_{k_i}^{2i} \delta_{k_i}^{s} a(y_1,\dots,y_s,k_2,\dots,k_t)$$
$$= \prod_{i=2}^{t} \mathrm{E}_{k_i}^{2i} (-\Delta_{k_i})^{s} \prod_{i=1}^{s} \mathrm{E}_{y_i}^{2t+3-2i} (-\Delta_{y_i})^{s-i} a(k_2,\dots,k_t,y_1,\dots,y_s)$$

if $y_1 = y_2 = \dots = y_s = k_2 = k_3 = \dots = k_t$.

Proposition 5.8.3. *Let* $\mathbf{x} = (-2j+1)_{j<0}$ *and* $\mathbf{z} = (3n+j+1)_{j<0}$. *Under the assumption that Conjecture 5.8.2 is true, it follows for all* $-n \le i \le -1$ *that*

1. $^{\mathbf{x}_i}\Delta^i_{k_1} \alpha(n; k_1, 4, 6, \ldots, 2n) \Big|_{k_1 = 3n+2+i} = 0,$

2. $^{\mathbf{x}_i}\Delta^i_{k_1} \alpha(n; k_1, 4, 6, \ldots, 2n) = {}^{\mathbf{z}_i}\Delta^i_{k_1} \alpha(n; k_1, 4, 6, \ldots, 2n)$*; in particular* $C^{(2)}_{n,i} = D^{(2)}_{n,i}$.

Proof. According to Lemma 5.8.1 we have

$$\begin{aligned}
&{}^{\mathbf{x}_i}\Delta^i_{k_1} \alpha(n; k_1, 4, 6, \ldots, 2n) \\
&\quad = (-1)^i \prod_{j=i}^{-1} \mathrm{E}^{-2j+1}_{y_j} \, \delta^{j-i}_{y_j} \prod_{j=2}^{n} \mathrm{E}^{2j}_{k_j} \, \delta^{-i}_{k_j} \alpha(n-i; k_1, y_i, y_{i+1}, \ldots, y_{-1}, k_2, \ldots, k_n) \Big|_{\substack{(y_i, y_{i+1}, \ldots, y_{-1}) = 0, \\ (k_2, \ldots, k_n) = 0}}.
\end{aligned}$$

We set $\overline{y}_j = y_{i+j-1}$ and $s = -i$ to obtain

$$(-1)^s \prod_{j=1}^{s} \mathrm{E}^{2s+3-2j}_{\overline{y}_j} \, \delta^{j-1}_{\overline{y}_j} \prod_{j=2}^{n} \mathrm{E}^{2j}_{k_j} \, \delta^{s}_{k_j} \alpha(n+s; k_1, \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_s, k_2, \ldots, k_n) \Big|_{\substack{(\overline{y}_1, \ldots, \overline{y}_s) = 0, \\ (k_2, \ldots, k_n) = 0}}.$$

By our assumption that Conjecture 5.8.2 is true, this is equal to

$$(-1)^s \prod_{j=2}^{n} \mathrm{E}^{2j}_{k_j} (-\Delta_{k_j})^s \prod_{j=1}^{s} \mathrm{E}^{2n+3-2j}_{\overline{y}_j} (-\Delta_{\overline{y}_j})^{s-j} \alpha(n+s; k_1, k_2, \ldots, k_n, \overline{y}_1, \ldots, \overline{y}_s) \Big|_{\substack{(\overline{y}_1, \ldots, \overline{y}_s) = 0, \\ (k_2, \ldots, k_n) = 0}}.$$

Now we use the properties (2.5.1) and (2.5.2) of the α-polynomial to obtain

$$\begin{aligned}
&(-1)^{n+1} \prod_{j=2}^{n} \mathrm{E}^{2j+n+s}_{k_j} (-\Delta_{k_j})^s \prod_{j=1}^{s} \mathrm{E}^{3n+3-2j+s}_{\overline{y}_j} (-\Delta_{\overline{y}_j})^{s-j} \\
&\qquad\qquad \alpha(n+s; k_2, \ldots, k_n, \overline{y}_1, \ldots, \overline{y}_s, k_1) \Big|_{\substack{(\overline{y}_1, \ldots, \overline{y}_s) = 0, \\ (k_2, \ldots, k_n) = 0}}.
\end{aligned}$$

According to Lemma 5.8.1, this is

$$(-1)^{n+1} \, {}^{\mathbf{w}_i}\delta^i_{k_1} \alpha(n; 4+n-i, 6+n-i, \ldots, 3n-i, k_1)$$

where $\mathbf{w}_i = (3n+3+i, 3n+5+i, \ldots, 3n+1-i)$. Setting $k_1 = 3n+2+i$, the first assertion now follows since $^{x+1}\delta^{-1}_x p(x) = 0$.

For the second assertion we use induction with respect to i. In the base case $i = -1$ note that the two sides differ by $^{3n}\Delta^{-1}_{k_1} \alpha(n; k_1, 4, 6, \ldots, 2n) \Big|_{k_1 = 4}$. By (2.2.6) this is equal to

$$-{}^{3}\Delta^{-1}_{k_1} \alpha(n; k_1, 4, 6, \ldots, 2n) \Big|_{k_1 = 3n+1},$$

which vanishes due to the first assertion. For $i < -1$ observe that

$$
\begin{aligned}
&{}^{\mathbf{x}_i}\Delta^{i}_{k_1} \alpha(n; k_1, 4, 6, \ldots, 2n) \\
&\quad = {}^{-2i+1}\Delta^{-1}_{k_1} \; {}^{\mathbf{x}_{i+1}}\Delta^{i+1}_{k_1} \alpha(n; k_1, 4, 6, \ldots, 2n) \\
&\quad = -\sum_{l_1=k_1}^{-2i+1} {}^{\mathbf{x}_{i+1}}\Delta^{i+1}_{l_1} \alpha(n; l_1, 4, 6, \ldots, 2n) \\
&\quad = -\sum_{l_1=k_1}^{3n+1+i} {}^{\mathbf{z}_{i+1}}\Delta^{i+1}_{l_1} \alpha(n; l_1, 4, 6, \ldots, 2n) + \sum_{l_1=-2i+2}^{3n+1+i} {}^{\mathbf{x}_{i+1}}\Delta^{i+1}_{l_1} \alpha(n; l_1, 4, 6, \ldots, 2n),
\end{aligned}
$$

where we have used the induction hypothesis in the first sum. Now the first sum is equal to the right-hand side in the second assertion, while the second sum is by (2.2.6) just the expression in the first assertion and thus vanishes. □

The problem of proving the refined enumeration formula (5.1.2) is therefore reduced to showing Conjecture 5.8.2. In the following section we explain how this problem can be translated into a multivariate Laurent polynomial identity. We then prove Theorem 5.1.3 by showing that its assumptions imply Conjecture 5.8.2.

5.9 Proof of Theorem 5.1.3

Let $p(x_1, \ldots, x_n)$ be a function in $(x_1, \ldots, x_n)$ and $\mathcal{T} \subseteq \mathcal{S}_n$ a subset of the symmetric group. We define

$$(\mathcal{T} p)(x_1, \ldots, x_n) := \sum_{\sigma \in \mathcal{T}} \operatorname{sgn} \sigma \, p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The unnormalized antisymmetrizer **ASym** as defined in Section 5.1 is obtained in the special case $\mathcal{T} = \mathcal{S}_n$, i.e. $\mathbf{ASym}\, p(x_1, \ldots, x_n) = (\mathcal{S}_n p)(x_1, \ldots, x_n)$. If $\mathcal{T} = \{\sigma\}$, then we write $(\mathcal{T} p)(x_1, \ldots, x_n) = (\sigma p)(x_1, \ldots, x_n)$. A function is said to be antisymmetric if $(\sigma p)(x_1, \ldots, x_n) = \operatorname{sgn} \sigma \cdot p(x_1, \ldots, x_n)$ for all $\sigma \in \mathcal{S}_n$. Before proving Theorem 5.1.3, we need two auxiliary results.

Lemma 5.9.1. *Let $a(z_1, \ldots, z_n)$ be a polynomial in $(z_1, \ldots, z_n)$ with*

$$(\mathrm{id} + \mathrm{E}_{z_{i+1}} \mathrm{E}^{-1}_{z_i} \mathrm{S}_{z_i, z_{i+1}}) \mathrm{V}_{z_i, z_{i+1}} a(z_1, \ldots, z_n) = 0$$

for $1 \le i \le n-1$. Then there exists an antisymmetric polynomial $b(z_1, \ldots, z_n)$ with

$$a(z_1, \ldots, z_n) = \prod_{1 \le p < q \le n} \mathrm{W}_{z_q, z_p} \, b(z_1, \ldots, z_n)$$

where $\mathrm{W}_{x,y} := \mathrm{E}_x \mathrm{V}_{x,y} = \mathrm{id} - \mathrm{E}_y + \mathrm{E}_x \mathrm{E}_y$.

Proof. Denoting $\mathbf{z} := (z_1, \ldots, z_n)$, we have by assumption

$$\mathrm{S}_{z_i, z_{i+1}} \mathrm{W}_{z_i, z_{i+1}} a(\mathbf{z}) = \mathrm{E}_{z_{i+1}} \mathrm{S}_{z_i, z_{i+1}} \mathrm{V}_{z_i, z_{i+1}} a(\mathbf{z}) = -\mathrm{E}_{z_i} \mathrm{V}_{z_i, z_{i+1}} a(\mathbf{z}) = -\mathrm{W}_{z_i, z_{i+1}} a(\mathbf{z}).$$

This implies that

$$c(z_1, \ldots, z_n) := \prod_{1 \le p < q \le n} \mathrm{W}_{z_p, z_q} \, a(z_1, \ldots, z_n)$$

is an antisymmetric polynomial. Now observe that $\mathrm{W}_{x,y} = \mathrm{id} + \mathrm{E}_y \, \Delta_x$ is invertible on $\mathbb{C}[x,y]$, to be more concrete $\mathrm{W}_{x,y}^{-1} = \sum_{i=0}^{\infty} (-1)^i \, \mathrm{E}_y^i \, \Delta_x^i$. Hence, $b(z_1,\ldots,z_n) := \prod_{1 \le p \ne q \le n} \mathrm{W}_{z_p,z_q}^{-1} \, c(z_1,\ldots,z_n)$ is an antisymmetric polynomial with $a(z_1,\ldots,z_n) = \prod_{1 \le p < q \le n} \mathrm{W}_{z_q,z_p} \, b(z_1,\ldots,z_n)$. ☐

Lemma 5.9.2. *Suppose* $\mathrm{Op}(x_1,\ldots,x_n)$ *is a Laurent polynomial and* $a(z_1,\ldots,z_n)$ *is an antisymmetric function. If there exists a non-empty subset* $\mathcal{T}$ *of* $\mathcal{S}_n$ *with* $(\mathcal{T}\,\mathrm{Op})(x_1,\ldots,x_n) = 0$, *then*

$$\left.(\mathrm{Op}(\mathrm{E}_{z_1},\ldots,\mathrm{E}_{z_n}) a(z_1,\ldots,z_n))\right|_{z_1=z_2=\ldots=z_n} = 0.$$

Proof. First observe that the antisymmetry of $a(z_1,\ldots,z_n)$ implies

$$(\mathcal{T}' a)(z_1,\ldots,z_n) = \sum_{\sigma \in \mathcal{T}'} \operatorname{sgn}\sigma \, a(z_{\sigma(1)},\ldots,z_{\sigma(n)}) = |\mathcal{T}'| a(z_1,\ldots,z_n)$$

for any subset $\mathcal{T}' \subseteq \mathcal{S}_n$. Letting

$$\mathrm{Op}(x_1,\ldots,x_n) = \sum_{(i_1,\ldots,i_n) \in \mathbb{Z}^n} c_{i_1,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

we observe that

$$\begin{aligned}
&\left.(\mathrm{Op}(\mathrm{E}_{z_1},\ldots,\mathrm{E}_{z_n}) a(z_1,\ldots,z_n))\right|_{(z_1,\ldots,z_n)=(d,\ldots,d)} \\
&= \sum_{(i_1,\ldots,i_n) \in \mathbb{Z}^n} c_{i_1,\ldots,i_n} a(i_1+d,\ldots,i_n+d) = \frac{1}{|\mathcal{T}^{-1}|} \sum_{(i_1,\ldots,i_n) \in \mathbb{Z}^n} c_{i_1,\ldots,i_n} (\mathcal{T}^{-1} a)(i_1+d,\ldots,i_n+d)
\end{aligned}$$

with $\mathcal{T}^{-1} = \{\sigma^{-1} \mid \sigma \in \mathcal{T}\}$, since $(i_1,\ldots,i_n) \mapsto a(i_1+d,\ldots,i_n+d)$ is also an antisymmetric function. This is further equal to

$$\begin{aligned}
&\frac{1}{|\mathcal{T}|} \sum_{(i_1,\ldots,i_n) \in \mathbb{Z}^n} c_{i_1,\ldots,i_n} \sum_{\sigma \in \mathcal{T}} \operatorname{sgn}\sigma \, a(i_{\sigma^{-1}(1)}+d,\ldots,i_{\sigma^{-1}(n)}+d) \\
&\quad = \frac{1}{|\mathcal{T}|} \sum_{(i_1,\ldots,i_n) \in \mathbb{Z}^n} c_{i_1,\ldots,i_n} \sum_{\sigma \in \mathcal{T}} \operatorname{sgn}\sigma \, \left.\mathrm{E}_{z_1}^{i_{\sigma^{-1}(1)}} \ldots \mathrm{E}_{z_n}^{i_{\sigma^{-1}(n)}} \, a(z_1,\ldots,z_n)\right|_{(z_1,\ldots,z_n)=(d,\ldots,d)} \\
&\quad = \frac{1}{|\mathcal{T}|} \sum_{(i_1,\ldots,i_n) \in \mathbb{Z}^n} c_{i_1,\ldots,i_n} \sum_{\sigma \in \mathcal{T}} \operatorname{sgn}\sigma \, \left.\mathrm{E}_{z_{\sigma(1)}}^{i_1} \ldots \mathrm{E}_{z_{\sigma(n)}}^{i_n} \, a(z_1,\ldots,z_n)\right|_{(z_1,\ldots,z_n)=(d,\ldots,d)} \\
&\quad = \frac{1}{|\mathcal{T}|} \left.[(\mathcal{T}\,\mathrm{Op})(\mathrm{E}_{z_1},\ldots,\mathrm{E}_{z_n})] \, a(z_1,\ldots,z_n)\right|_{(z_1,\ldots,z_n)=(d,\ldots,d)} = 0.
\end{aligned}$$

☐

Now we are in the position to prove Theorem 5.1.3.

Proof of Theorem 5.1.3. In order to prove (5.1.2), it suffices to show that Conjecture 5.8.2 holds under the theorem's assumptions. We set

$$\begin{aligned}\overline{\mathrm{Op}}(z_1,\ldots,z_{s+t-1}) := \prod_{i=1}^{s} z_i^{2s+3-2i}(1-z_i^{-1})^{i-1} \prod_{i=s+1}^{s+t-1} z_i^{2i-2s+2}(1-z_i^{-1})^{s} \\ - \prod_{i=1}^{t-1} z_i^{2i+2}(1-z_i)^{s} \prod_{i=t}^{s+t-1} z_i^{4t+1-2i}(1-z_i)^{s+t-1-i}\end{aligned}$$

and observe that the claim of Conjecture 5.8.2 is that $\overline{\mathrm{Op}}(\mathrm{E}_{z_1},\ldots,\mathrm{E}_{z_{s+t-1}})a(z_1,\ldots,z_{s+t-1})$ vanishes if $z_1=\ldots=z_{s+t-1}$. According to Lemma 5.9.1, there exists an antisymmetric polynomial $b(z_1,\ldots,z_{s+t-1})$ with

$$a(z_1,\ldots,z_{s+t-1}) = \prod_{1\le p<q\le s+t-1} \mathrm{W}_{z_q,z_p}\, b(z_1,\ldots,z_{s+t-1}).$$

Thus, let us deduce that $\mathrm{Op}(\mathrm{E}_{z_1},\ldots,\mathrm{E}_{z_{s+t-1}})b(z_1,\ldots,z_{s+t-1})=0$ if $z_1=\ldots=z_{s+t-1}$ where

$$\mathrm{Op}(z_1,\ldots,z_{s+t-1}) := \overline{\mathrm{Op}}(z_1,\ldots,z_{s+t-1}) \prod_{1\le p<q\le s+t-1}(1-z_p+z_p z_q) \prod_{i=1}^{s+t-1} z_i^{-2-t}.$$

Now, Lemma 5.9.2 implies that it suffices to show $\mathbf{ASym}\,\mathrm{Op}(z_1,\ldots,z_{s+t-1})=0$. Observe that

$$\mathrm{Op}(z_1,\ldots,z_{s+t-1}) = \overline{P}_{s,t}(z_1,\ldots,z_{s+t-1}) - \overline{P}_{s,t}(z_{s+t-1}^{-1},\ldots,z_1^{-1}) \prod_{i=1}^{s+t-1} z_i^{s+t-2}$$

where $\overline{P}_{s,t}(z_1,\ldots,z_{s+t-1}) = P_{s,t}(z_1,\ldots,z_{s+t-1}) \prod_{1\le i<j\le s+t-1}(z_j-z_i)$ and $P_{s,t}(z_1,\ldots,z_{s+t-1})$ is as defined in Conjecture 5.1.2. Furthermore,

$$\begin{aligned}\mathbf{ASym}\,\mathrm{Op}(z_1,\ldots,z_{s+t-1}) = R_{s,t}(z_1,\ldots,z_{s+t-1}) \prod_{1\le i<j\le s+t-1}(z_j-z_i) \\ - R_{s,t}(z_{s+t-1}^{-1},\ldots,z_1^{-1}) \prod_{1\le i<j\le s+t-1}(z_{s+t-j}^{-1}-z_{s+t-i}^{-1}) \prod_{i=1}^{s+t-1} z_i^{s+t-2}\end{aligned}$$

where $R_{s,t}(z_1,\ldots,z_{s+t-1})$ is also defined in Conjecture 5.1.2. Since $R_{s,t}(z_1,\ldots,z_{s+t-1})$ is symmetric we have that $\mathbf{ASym}\,\mathrm{Op}(z_1,\ldots,z_{s+t-1})=0$ follows once it is shown that $R_{s,t}(z_1,\ldots,z_{s+t-1}) = R_{s,t}(z_1^{-1},\ldots,z_{s+t-1}^{-1})$. □

5.10 Proof of Theorem 5.1.4

For integers $s,t\ge 0$, we define the following two rational functions:

$$\begin{aligned} S_{s,t}(z;z_1,\ldots,z_{s+t-2}) &:= z^{2s-t-1}\prod_{i=1}^{s+t-2}\frac{(1-z+z_i z)(1-z_i^{-1})}{z_i-z}, \\ T_{s,t}(z;z_1,\ldots,z_{s+t-2}) &:= (1-z^{-1})^s z^{t-2}\prod_{i=1}^{s+t-2}\frac{1-z_i+z_i z}{(z-z_i)z_i}. \end{aligned} \tag{5.10.1}$$

Based on these two functions, we define two operators on functions f in $s+t-2$ variables that transform them into functions in $(z_1,\ldots,z_{s+t-1})$:

$$\begin{aligned}\mathrm{PS}_{s,t}[f] &:= S_{s,t}(z_1;z_2,\ldots,z_{s+t-1})\cdot f(z_2,\ldots,z_{s+t-1}),\\ \mathrm{PT}_{s,t}[f] &:= T_{s,t}(z_{s+t-1};z_1,\ldots,z_{s+t-2})\cdot f(z_1,\ldots,z_{s+t-2}).\end{aligned}$$

The definitions are motivated by the fact that $P_{s,t}(z_1,\ldots,z_{s+t-1})$ as defined in Conjecture 5.1.2 satisfies the two recursions

$$P_{s,t}=\mathrm{PS}_{s,t}[P_{s-1,t}] \quad \text{and} \quad P_{s,t}=\mathrm{PT}_{s,t}[P_{s,t-1}]. \tag{5.10.2}$$

We also need the following two related operators, which are again defined on functions f in $s+t-2$ variables:

$$\begin{aligned}\mathrm{QS}_{s,t}[f] &:= S_{s,t}(z_{s+t-1}^{-1};z_{s+t-2}^{-1},z_{s+t-3}^{-1},\ldots,z_1^{-1})\cdot f(z_1,\ldots,z_{s+t-2}),\\ \mathrm{QT}_{s,t}[f] &:= T_{s,t}(z_1^{-1};z_{s+t-1}^{-1},z_{s+t-2}^{-1},\ldots,z_2^{-1})\cdot f(z_2,\ldots,z_{s+t-1}).\end{aligned}$$

Note that if we set $Q_{s,t}(z_1,\ldots,z_{s+t-1}):=P_{s,t}(z_{s+t-1}^{-1},\ldots,z_1^{-1})$, then

$$Q_{s,t}=\mathrm{QS}_{s,t}[Q_{s-1,t}] \quad \text{and} \quad Q_{s,t}=\mathrm{QT}_{s,t}[Q_{s,t-1}]. \tag{5.10.3}$$

From the definitions, one can deduce the following commutation properties:

Lemma 5.10.1. *Let s,t be positive integers.*

1. *If $(s,t)\neq(1,1)$, then*

$$\begin{aligned}\mathrm{PS}_{s,t}\circ\mathrm{PT}_{s-1,t}&=\mathrm{PT}_{s,t}\circ\mathrm{PS}_{s,t-1},\\ \mathrm{QS}_{s,t}\circ\mathrm{QT}_{s-1,t}&=\mathrm{QT}_{s,t}\circ\mathrm{QS}_{s,t-1}\,.\end{aligned}$$

2. *If $t\geq 2$, then*

$$\mathrm{PT}_{s,t}\circ\mathrm{QT}_{s,t-1}=\mathrm{QT}_{s,t}\circ\mathrm{PT}_{s,t-1}\,.$$

Proof. Let $f(z_1,\ldots,z_{s+t-3})$ be an arbitrary function. For the left-hand side of the first statement we then obtain

$$\begin{aligned}&\mathrm{PS}_{s,t}\left[\mathrm{PT}_{s-1,t}\left[f(z_1,\ldots,z_{s+t-3})\right]\right]\\ &\quad=\mathrm{PS}_{s,t}\left[T_{s-1,t}(z_{s+t-2};z_1,\ldots,z_{s+t-3})\,f(z_1,\ldots,z_{s+t-3})\right]\\ &\quad=S_{s,t}(z_1;z_2,\ldots,z_{s+t-1})\,T_{s-1,t}(z_{s+t-1};z_2,\ldots,z_{s+t-2})\,f(z_2,\ldots,z_{s+t-2}),\end{aligned}$$

whereas the right-hand side is equal to

$$\begin{aligned}&\mathrm{PT}_{s,t}\left[\mathrm{PS}_{s,t-1}\left[f(z_1,\ldots,z_{s+t-3})\right]\right]\\ &\quad=\mathrm{PT}_{s,t}\left[S_{s,t-1}(z_1;z_2,\ldots,z_{s+t-2})\,f(z_2,\ldots,z_{s+t-2})\right]\\ &\quad=T_{s,t}(z_{s+t-1};z_1,\ldots,z_{s+t-2})\,S_{s,t-1}(z_1;z_2,\ldots,z_{s+t-2})\,f(z_2,\ldots,z_{s+t-2}).\end{aligned}$$

It remains to show the rational function identity

$$\begin{aligned}&T_{s,t}(z_{s+t-1};z_1,\ldots,z_{s+t-2})\,S_{s,t-1}(z_1;z_2,\ldots,z_{s+t-2})\\ &=T_{s-1,t}(z_{s+t-1};z_2,\ldots,z_{s+t-2})\,S_{s,t}(z_1;z_2,\ldots,z_{s+t-1}).\end{aligned}\tag{5.10.4}$$

This follows directly from the definition of the rational functions (5.10.1), since both sides of (5.10.4) are equal to

$$(1-z_{s+t-1}^{-1})^s z_{s+t-1}^{t-2} z_1^{2s-t-1}(1-z_1+z_1z_{s+t-1})/(z_{s+t-1}-z_1) \times \prod_{i=2}^{s+t-2} \frac{(1-z_i+z_iz_{s+t-1})(1-z_1+z_iz_1)(1-z_i^{-1})}{(z_{s+t-1}-z_i)z_i(z_i-z_1)}.$$

The left-hand side of the second equation yields

$$\begin{aligned}&\mathrm{QS}_{s,t}\left[T_{s-1,t}(z_1^{-1};z_{s+t-2}^{-1},\dots,z_2^{-1})\,f(z_2,\dots,z_{s+t-2})\right]\\ &\quad= S_{s,t}(z_{s+t-1}^{-1};z_{s+t-2}^{-1},\dots,z_1^{-1})\,T_{s-1,t}(z_1^{-1};z_{s+t-2}^{-1},\dots,z_2^{-1})\,f(z_2,\dots,z_{s+t-2}),\end{aligned}$$

and the right-hand side is equal to

$$\begin{aligned}&\mathrm{QT}_{s,t}\left[S_{s,t-1}(z_{s+t-2}^{-1};z_{s+t-3}^{-1},\dots,z_1^{-1})\,f(z_1,\dots,z_{s+t-3})\right]\\ &\quad= T_{s,t}(z_1^{-1};z_{s+t-1}^{-1},\dots,z_2^{-1})\,S_{s,t-1}(z_{s+t-1}^{-1};z_{s+t-2}^{-1},\dots,z_2^{-1})\,f(z_2,\dots,z_{s+t-2}).\end{aligned}$$

The second equation is therefore also a consequence of (5.10.4) by replacing $z_i \mapsto z_{s+t-i}^{-1}$. Analogously, the third equation boils down to showing

$$\begin{aligned}&T_{s,t}(z_{s+t-1};z_1,\dots,z_{s+t-2})\,T_{s,t-1}(z_1^{-1};z_{s+t-2}^{-1},\dots,z_2^{-1})\\ &= T_{s,t}(z_1^{-1};z_{s+t-1}^{-1},\dots,z_2^{-1})\,T_{s,t-1}(z_{s+t-1};z_2,\dots,z_{s+t-2}).\end{aligned}$$

By definition (5.10.1) of the rational function both sides are equal to

$$(1-z_{s+t-1}^{-1})^s z_{s+t-1}^{t-2}(1-z_1)^s z_1^{2-t}(1-z_1+z_1z_{s+t-1})/(z_{s+t-1}-z_1) \times \prod_{i=2}^{s+t-2} \frac{(1-z_i+z_iz_{s+t-1})(1-z_i^{-1}+z_i^{-1}z_1^{-1})}{(z_{s+t-1}-z_i)(z_1^{-1}-z_i^{-1})}.$$

□

Moreover, we need the following identities, which follow from the fact that $S_{s,t}(z;z_1,\dots,z_{s+t-2})$ and $T_{s,t}(z;z_1,\dots,z_{s+t-2})$ are symmetric in $z_1,\dots,z_{s+t-2}$ (the symbol $\widehat{z_i}$ indicates that z_i is missing from the argument):

$$\begin{aligned}
\mathbf{Sym}\,\mathrm{PS}_{s,t}[f] &= \sum_{i=1}^{s+t-1} S_{s,t}(z_i;z_1,\dots,\widehat{z_i},\dots,z_{s+t-1})\,\mathbf{Sym}\,f(z_1,\dots,\widehat{z_i},\dots,z_{s+t-1}),\\
\mathbf{Sym}\,\mathrm{PT}_{s,t}[f] &= \sum_{i=1}^{s+t-1} T_{s,t}(z_i;z_1,\dots,\widehat{z_i},\dots,z_{s+t-1})\,\mathbf{Sym}\,f(z_1,\dots,\widehat{z_i},\dots,z_{s+t-1}),\\
\mathbf{Sym}\,\mathrm{QS}_{s,t}[f] &= \sum_{i=1}^{s+t-1} S_{s,t}(z_i^{-1};z_1^{-1},\dots,\widehat{z_i^{-1}},\dots,z_{s+t-1}^{-1})\,\mathbf{Sym}\,f(z_1,\dots,\widehat{z_i},\dots,z_{s+t-1}),\\
\mathbf{Sym}\,\mathrm{QT}_{s,t}[f] &= \sum_{i=1}^{s+t-1} T_{s,t}(z_i^{-1};z_1^{-1},\dots,\widehat{z_i^{-1}},\dots,z_{s+t-1}^{-1})\,\mathbf{Sym}\,f(z_1,\dots,\widehat{z_i},\dots,z_{s+t-1}).
\end{aligned} \tag{5.10.5}$$

We consider words w over the alphabet $\mathcal{A} := \{\mathrm{PS}, \mathrm{PT}, \mathrm{QS}, \mathrm{QT}\}$ and let $|w|_S$ denote the number of occurrences of PS and QS in the word and $|w|_T$ denote the number of occurrences of PT and QT. It is instructive to interpret these words as labelled lattice paths with starting point in the origin, step set $\{(1,0),(0,1)\}$ and labels P, Q. The letters PS and QS correspond to $(1,0)$-steps labelled with P and Q, respectively, while the letters PT and QT correspond to $(0,1)$-steps. With this interpretation, $(|w|_S, |w|_T)$ is the endpoint of the path (see Figure 5.11).

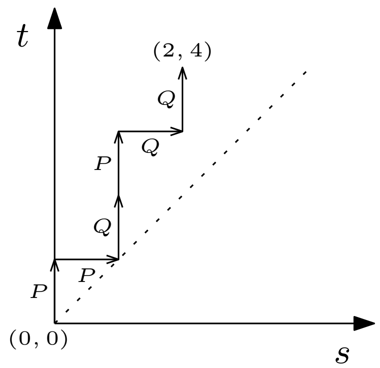


Figure 5.11: Labelled lattice path corresponding to $w = (\mathrm{PT}, \mathrm{PS}, \mathrm{QT}, \mathrm{PT}, \mathrm{QS}, \mathrm{QT})$.

To every word w of length n, we assign a rational function $F_w(z_1, \ldots, z_{n+1})$ as follows: If w is the empty word, then $F_w(z_1) := 1$. Otherwise, if $L \in \mathcal{A}$ and w is a word over $\mathcal{A}$, we set

$$F_{wL} := L_{|wL|_S+1, |wL|_T+1}[F_w].$$

For example, the rational function assigned to w in Figure 5.11 is

$$F_w(z_1, \ldots, z_7) = \mathrm{QT}_{3,5} \circ \mathrm{QS}_{3,4} \circ \mathrm{PT}_{2,4} \circ \mathrm{QT}_{2,3} \circ \mathrm{PS}_{2,2} \circ \mathrm{PT}_{1,2}[1].$$

In this context, Lemma 5.10.1 has the following meaning: on the one hand, we may swap two consecutive steps with the same label, and, on the other hand, we may swap two consecutive $(0,1)$-steps without changing the corresponding rational functions. For example, the rational functions corresponding to the words in Figure 5.11 and Figure 5.12 coincide.

Proof of Theorem 5.1.4. We assume

$$R_{s,t}(z_1, \ldots, z_{s+t-1}) = R_{s,t}(z_1^{-1}, \ldots, z_{s+t-1}^{-1}) \tag{5.10.6}$$

if $t = s$ and $t = s+1$. We show the following more general statement: Suppose w_1, w_2 are two words over $\mathcal{A}$ with $|w_1|_S = |w_2|_S$ and $|w_1|_T = |w_2|_T$, and every prefix w_i' of w_i fulfills $|w_i'|_S \le |w_i'|_T$, $i = 1, 2$. (In the lattice paths language this means that w_1 and w_2 are both prefixes of Dyck paths sharing the same endpoint; there is no restriction on the labels P and Q.) Then

$$\mathbf{Sym}\, F_{w_1} = \mathbf{Sym}\, F_{w_2}. \tag{5.10.7}$$

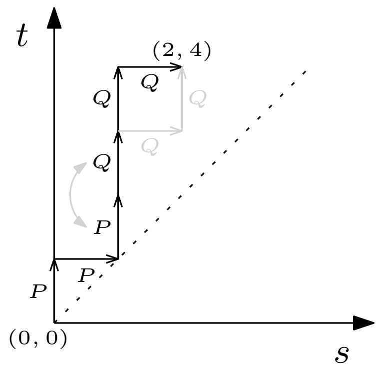


Figure 5.12: Labelled lattice path corresponding to $\tilde{w} = (\mathrm{PT}, \mathrm{PS}, \mathrm{PT}, \mathrm{QT}, \mathrm{QT}, \mathrm{QS})$.

The assertion of the theorem then follows since $F_w = P_{|w|_S+1,|w|_T+1}$ if w is a word over $\{\mathrm{PS}, \mathrm{PT}\}$ (cf. (5.10.2)) and $F_w = Q_{|w|_S+1,|w|_T+1}$ if w is a word over $\{\mathrm{QS}, \mathrm{QT}\}$ (cf. (5.10.3)), and therefore

$$\begin{aligned} R_{s,t}(z_1, \ldots, z_{s+t-1}) &= \mathbf{Sym}\, P_{s,t}(z_1, \ldots, z_{s+t-1}) = \mathbf{Sym}\, Q_{s,t}(z_1, \ldots, z_{s+t-1}) \\ &= \mathbf{Sym}\, P_{s,t}(z_{s+t-1}^{-1}, \ldots, z_1^{-1}) = R_{s,t}(z_1^{-1}, \ldots, z_{s+t-1}^{-1}). \end{aligned}$$

The proof is by induction with respect to the length of the words; there is nothing to prove if the words are empty. Otherwise let w_1, w_2 be two words over $\mathcal{A}$ with $|w_1|_S = |w_2|_S = s - 1$ and $|w_1|_T = |w_2|_T = t - 1$, and every prefix w'_i of w_i fulfills $|w'_i|_S \le |w'_i|_T$, $i = 1, 2$. Note that the induction hypothesis and (5.10.5) imply that $\mathbf{Sym}\, F_{w_i}$ only depends on the last letter of w_i (and on s and t of course). Thus the assertion follows if the last letters of w_1 and w_2 coincide; we assume that they differ in the following.

If $s = t$, then the assumption on the prefixes implies that the last letters of w_1 and w_2 are in $\{\mathrm{PS}, \mathrm{QS}\}$. W.l.o.g. we assume $w_1 = w'_1\,\mathrm{PS}$ and $w_2 = w'_2\,\mathrm{QS}$. By the induction hypothesis and (5.10.5), we have $\mathbf{Sym}\, F_{w_1} = \mathbf{Sym}\, P_{s,s}$ and $\mathbf{Sym}\, F_{w_2} = \mathbf{Sym}\, Q_{s,s}$. The assertion now follows from (5.10.6), since $\mathbf{Sym}\, P_{s,s}(z_1, \ldots, z_{2s-1}) = R_{s,s}(z_1, \ldots, z_{2s-1})$ and $\mathbf{Sym}\, Q_{s,s}(z_1, \ldots, z_{2s-1}) = R_{s,s}(z_1^{-1}, \ldots, z_{2s-1}^{-1})$.

If $s < t$, we show that we may assume that the last letters of w_1 and w_2 are in $\{\mathrm{PT}, \mathrm{QT}\}$: if this is not true for the last letter L_1 of w_i, we may at least assume by the induction hypothesis and (5.10.5) that the penultimate letter L_2 is in $\{\mathrm{PT}, \mathrm{QT}\}$; to be more precise, we require $L_2 = \mathrm{PT}$ if $L_1 = \mathrm{PS}$ and $L_2 = \mathrm{QT}$ if $L_1 = \mathrm{QS}$; now, according to Lemma 5.10.1, we can interchange the last and the penultimate letter in this case. We may therefore assume that $w_1 = w'_1\,\mathrm{PT}$ and $w_2 = w'_2\,\mathrm{QT}$ in the following.

If $t = s + 1$, then the induction hypothesis and (5.10.5) imply – analogously to the case $s = t$ – that $\mathbf{Sym}\, F_{w_1} = \mathbf{Sym}\, P_{s,s+1}$ and $\mathbf{Sym}\, F_{w_2} = \mathbf{Sym}\, Q_{s,s+1}$. The claim $\mathbf{Sym}\, F_{w_1} = \mathbf{Sym}\, F_{w_2}$ again follows from our assumption (5.10.6).

If $s + 1 < t$, then we may assume by the induction hypothesis and (5.10.5) that the penultimate letter of w_1 is QT. According to Lemma 5.10.1, we can interchange the last and the penultimate letter of w_1 and the assertion follows also in this case. $\square$

We conclude this chapter by referring the interested reader to [FR14], where additional remarks on the case $s = 0$ in Conjecture 5.1.2 are described.

Chapter 6

Playing jeu de taquin on d-complete posets

The contents of this chapter appeared in [RN14].

6.1 Introduction

6.1.1 Jeu de taquin on posets

Jeu de taquin (literally translated 'teasing game') is a board game (also known as 15-puzzle) where fifteen square tiles numbered with $\{1, 2, \ldots, 15\}$ are arranged inside a 4×4 square. The goal of the game is to sort the tiles by consecutively sliding a square into the empty spot (see Figure 6.1). In combinatorics the concept of jeu de taquin was originally introduced by Schützenberger [Sch76] on skew standard Young tableaux. Two related operations called promotion and evacuation, which act bijectively on the set of linear extensions of a poset, were also defined by Schützenberger [Sch72]. A modified version of jeu de taquin [NPS97] has an obvious extension to arbitrary posets,

7	11	4	5
8	1	3	13
14	6	12	2
10	15	9	

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 6.1: An initial and final configuration of the board game jeu de taquin.

which we describe first: The goal of jeu de taquin on an n-element poset P is to transform any (bijective) labeling of the poset elements with $[n] := \{1, 2, \ldots, n\}$ into a dual linear extension, i.e. a labeling ι such that $\iota(x) > \iota(y)$ whenever $x <_P y$ in the poset. For this, we first fix a linear extension $\sigma : P \to [n]$ of the poset, which defines the order in which the labels are sorted (see Figure 6.2). The sorting procedure consists of n rounds where after the first i rounds the poset elements $\{\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(i)\}$ have dually ordered labels. To achieve this, we compare in round i the current label of $x := \sigma^{-1}(i)$ with the labels of all poset elements covered by x. If the

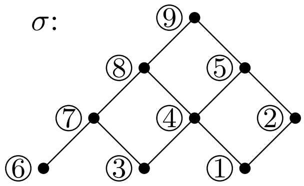


Figure 6.2: Hasse diagram of a poset and a linear extension σ.

current label is the smallest of them, we are done with round i. Else, let y be the poset element with the smallest label. Swap the labels of x and y and repeat with the new label of y. An example can be seen in Figure 6.3. The fact that σ is a linear extension together with the minimality condition in the sorting procedure ensures that after i rounds the poset elements $\{\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(i)\}$ have dually ordered labels. In particular, the sorting procedure transforms each labeling of the poset into a dual linear extension. The question we are interested in is: Given a uniformly random

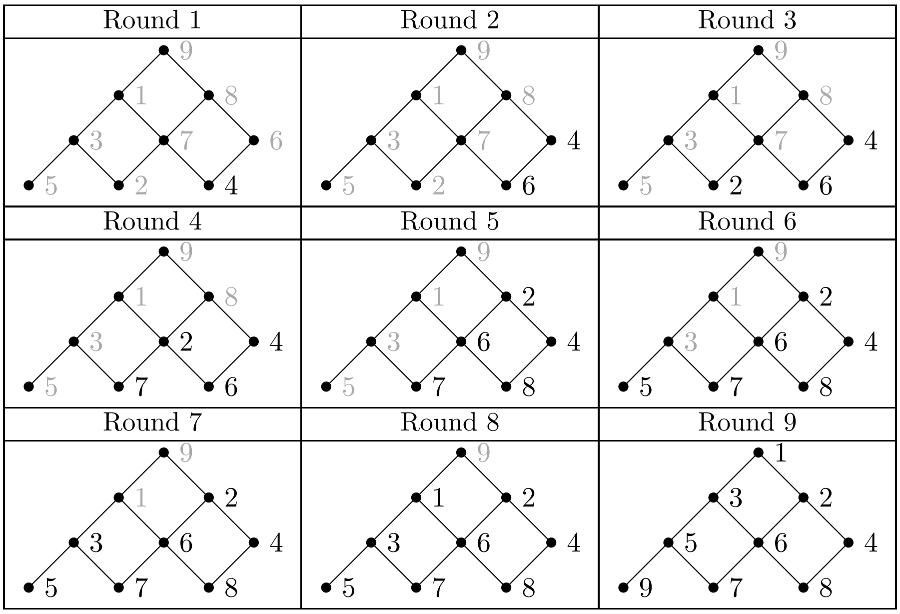


Figure 6.3: Example of jeu de taquin in order σ as given in Figure 6.2.

labeling of the poset elements, does jeu de taquin output a uniformly random dual linear extension of the poset? More specifically, given a poset, is there an order σ such that playing jeu de taquin with all possible labelings yields each dual linear extension equally often? If yes, then jeu de taquin allows us to immediately extend each algorithm for creating uniformly random permutations to an algorithm creating uniformly random linear extensions of the poset.

6.1.2 (Shifted) standard Young tableaux & the hook-length formula

For certain classes of posets we know the answer: Most famously, the Young diagram of an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ can be considered as a poset (see Figure 6.4). A Young tableau

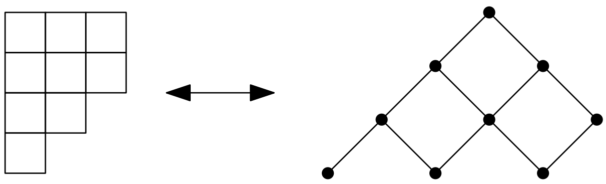


Figure 6.4: Young diagram of the partition $(3, 3, 2, 1)$ and the corresponding poset.

is a bijective filling of the boxes with $[n]$ and thus corresponds to a labeling of the poset. A standard Young tableau is a filling of the boxes where entries in each row (left-to-right) and column (top-down) are strictly increasing. Hence, standard Young tableaux correspond to the dual linear extensions of the respective poset. Novelli, Pak and Stoyanovskii showed [NPS97] that jeu de taquin with column-wise order σ (as in Figure 6.2) yields uniform distribution among standard Young tableaux. Their proof can actually be extended to work for orders different from column-wise order (see also [Sag01]).

A second class of posets where we know the answer corresponds to shifted Young diagrams of strict integer partitions, i.e. partitions where $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ and the boxes of the shifted Young diagram are indented as depicted in Figure 6.5. It was shown by Fischer [Fis01] that row-wise order

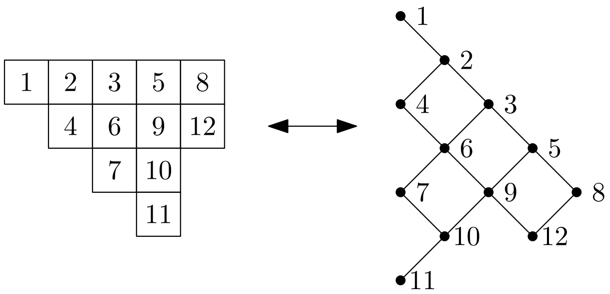
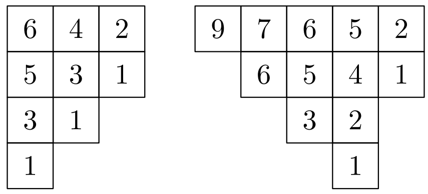


Figure 6.5: A shifted standard Young tableau of shape $(5, 4, 2, 1)$ and the dual linear extension of the corresponding poset.

σ yields uniform distribution among shifted standard Young tableaux (however column-wise order fails for the partition $(4, 3, 2, 1)$).

What both classes have in common is that the number of different standard fillings of fixed shape λ can be obtained with a simple product formula, called hook-length formula: In the case of Young diagrams the hook of a cell consists of all cells to the right in the same row, all cells below in the same column and the cell itself. The hook-length h_c of a cell c is the number of cells in its hook (see Figure 6.6(a)). The number f^λ of standard Young tableaux of fixed shape λ is then given



(a) (b)

Figure 6.6: The hook-lengths of all cells in a Young diagram and a shifted Young diagram.

by [FRT54]

$$f^\lambda = \frac{n!}{\prod\limits_{c\in\lambda} h_c}, \tag{6.1.1}$$

where the product is taken over all cells c in the Young diagram. For shifted Young diagrams, Gansner [Gan78] showed that the number of standard fillings can be obtained with the same hook-length formula (6.1.1). However, the definition of the hooks has to be slightly modified: the shifted hook of a cell contains the same cells as the hook before, and additionally, if the hook contains the left-most cell in row i, then the shifted hook is extended to all cells in row $i+1$ (see Figure 6.6(b)).

6.1.3 d-complete posets

The definition of the hook-lengths can be generalized so that the hook-length formula extends to further classes of posets, called d-complete posets [Pro99].

For $m, n \geq 2$ the poset $D_{m,n}$ consists of $m+n$ elements for which the Hasse diagram is obtained by taking a diamond of four elements and appending a chain of $m-2$ elements at the top element of the diamond and a chain of $n-2$ elements at the bottom element of the diamond (see Figure 6.7). The poset $D_{m,n}$ is referred to as double-tailed diamond and plays a fundamental role as elementary

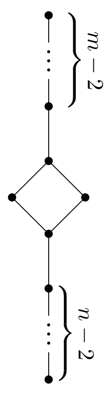


Figure 6.7: Hasse diagram of the double-tailed diamond $D_{m,n}$.

building block in the definition of d-complete posets [Pro99]:

Given a poset P and $k \geq 3$, an interval $[w, z]$ in the poset is called d_k-interval if $[w, z] \cong D_{k-1,k-1}$. An interval $[w, y]$ is called d_k^--interval if $[w, y] \cong D_{k-2,k-1}$ (in the special case $k = 3$ let us abuse notation and say that a d_3^--interval is a diamond with top element removed). A poset P is called d_k-complete if it satisfies the following three conditions:

1. $[w, y]$ is d_k^--interval $\Rightarrow \exists z \in P : [w, z]$ is d_k-interval,
2. $[w, z]$ is d_k-interval $\Rightarrow z$ does not cover an element outside of $[w, z]$ and
3. $[w, z]$ is d_k-interval $\Rightarrow$ there exists no $w' \neq w$ such that $[w', z]$ is d_k-interval.

A poset P is called d-complete if and only if P is d_k-complete for all $k \geq 3$. The posets corresponding to Young diagrams and shifted Young diagrams are examples of d-complete posets (see Figure 6.8). A full classification of d-complete posets can be found in [Pro99].

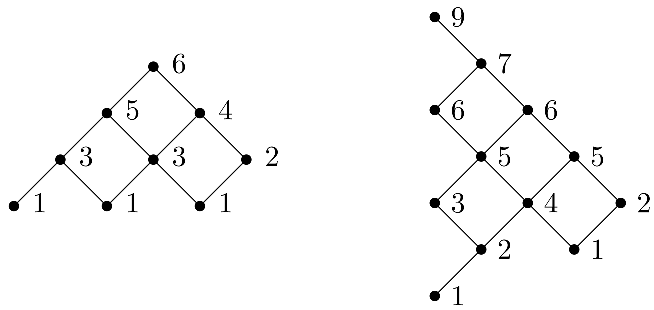


Figure 6.8: Two d-complete posets with assigned hook-lengths.

Recall that for any poset P a map $\sigma : P \to \mathbb{N}$ is called P-partition of n if σ is order-reversing and satisfies $\sum_{x \in P} \sigma(x) = n$. Let $G_P(x)$ denote the corresponding generating function, i.e.

$$G_P(x) := \sum_{n \geq 0} a_n x^n,$$

where a_n denotes the number of P-partitions of n. As a result by R.P. Stanley [Sta11, Theorem 3.15.7] the generating function can be factorized into

$$G_P(x) = \frac{W_P(x)}{(1-x)(1-x^2)\dots(1-x^{|P|})} \tag{6.1.2}$$

with a polynomial $W_P(x)$ such that $W_P(1)$ is the number of linear extensions of P. A poset P is called hook-length poset if there exists a map $h : P \to \mathbb{Z}^+$ such that

$$G_P(x) = \prod_{z \in P} \frac{1}{1 - x^{h(z)}}. \tag{6.1.3}$$

The number f^P of linear extensions of a hook-length poset can be obtained from (6.1.2) and (6.1.3) by taking the limit $x \to 1$:

$$f^P = \frac{|P|!}{\prod_{z \in P} h(z)}. \tag{6.1.4}$$

Every d-complete poset is a hook-length poset [PP]. In fact, d-complete posets were generalized to so-called leaf posets [IT07], which are also hook-length posets. The hook-lengths $h_z := h(z)$ for d-complete posets can be obtained in the following way:

1. Assign all minimal elements of the poset the hook-length 1.
2. Repeat until all elements have their hook-length assigned: Choose a poset element z where all smaller elements have their hook-length assigned. Check whether z is the top element of a d_k-interval $[w, z]$.
   - If no, set $h_z := \#\{y \in P : y \leq z\}$.
   - If yes, set $h_z := h_l + h_r - h_w$, where l and r are the two incomparable elements of the d_k-interval $[w, z]$.

Two examples are given in Figure 6.8. By definition of d-complete posets the procedure is well-defined (there exists at most one d_k-interval with z as top element). Moreover, it is a nice exercise to check that this definition is equivalent to the previous definition of hook-lengths for Young diagrams (which only contain $D_{2,2}$ intervals) and shifted Young diagrams (which additionally contain $D_{3,3}$ intervals along the left rim). As an example compare Figure 6.6 and Figure 6.8.

6.1.4 Jeu de taquin on the double-tailed diamond

Since there is exactly one pair of incomparable elements in the double-tailed diamond $D_{m,n}$, there are two different dual linear extensions T_1 and T_2 of $D_{m,n}$ (see Figure 6.9). For jeu de taquin

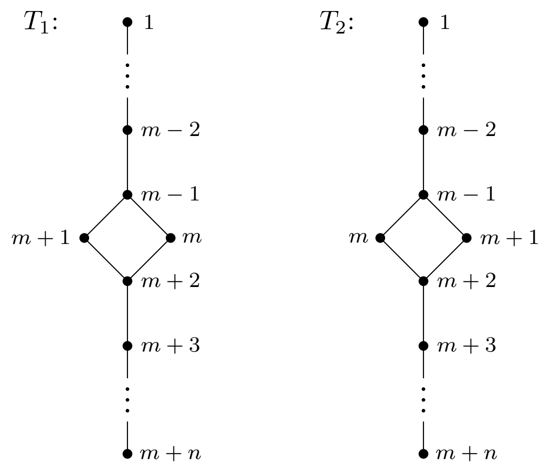


Figure 6.9: The two possible dual linear extension of $D_{m,n}$.

we choose w.l.o.g. the order σ that corresponds to the reverse order of T_1. In Section 6.2 we show that jeu de taquin with all $(m + n)!$ labelings yields T_1 and T_2 equally often if and only if $m \geq n$. We proceed by defining a related statistic on permutations generalizing right-to-left minima. In terms of this statistic we can analyze a refined counting problem, namely counting the number of permutations for which jeu de taquin swaps the order between the labels of the two incomparable elements exactly k times. As it turns out this counting problem has a nice closed solution (Proposition 6.2.2) as well as the resulting difference between the number of permutations yielding T_1 and T_2:

Theorem 6.1.1 ([RN14]). *Let $s^{(1)}_{m,n}$ (resp. $s^{(2)}_{m,n}$) denote the number of permutations in $\mathcal{S}_{m+n}$ which jeu de taquin on $D_{m,n}$ with order σ maps to T_1 (resp. T_2). Then*

$$s^{(1)}_{m,n} - s^{(2)}_{m,n} = (-1)^m \binom{n-1}{m} m!\,n!, \quad m,n \geq 2. \tag{6.1.5}$$

In particular, $s^{(1)}_{m,n} = s^{(2)}_{m,n}$ if and only if $m \geq n$.

What is interesting about the result is that the poset $D_{m,n}$ is d-complete if and only if $m \geq n$. Together with Young diagrams and shifted Young diagrams (two further classes of d-complete posets) this hints towards a connection between d-completeness of a poset and the property that jeu de taquin w.r.t. an appropriate order yields uniform distribution.

In Section 6.3 we give a purely combinatorial proof of Theorem 6.1.1 by constructing an appropriate involution $\Phi_{m,n}$ on $\mathcal{S}_{m+n}$ if $m \geq n$. In the case $m < n$ we identify a set $\mathcal{E}$ of $\binom{n-1}{m} m!\,n!$ exceptional permutations and construct an appropriate involution $\Phi_{m,n}$ on $\mathcal{S}_{m+n} \setminus \mathcal{E}$.

6.1.5 Jeu de taquin on insets

The class of insets (fourth class in the classification of d-complete posets in [Pro99]) can be defined in terms of the shape of its corresponding diagram. For $k \geq 2$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ the inset $P_{k,\lambda}$ is obtained by taking the Young diagram corresponding to λ and adding $k-1$ boxes at the left end of the first row and one box at the left end of the second row (see Figure 6.10). The hook-lengths of the cells in λ can be computed like for Young diagrams. The additional box

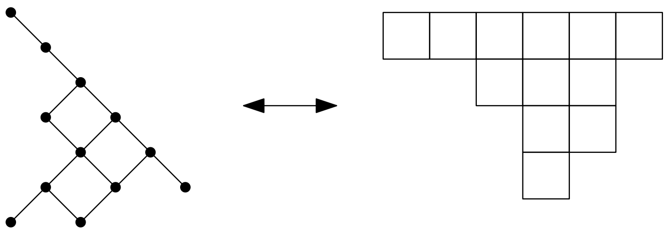


Figure 6.10: The inset $P_{4,(3,2,2,1)}$ and its corresponding box diagram.

in the second row is not the maximum of a double-tailed diamond interval, whereas each of the $k-1$ additional boxes in the first row is the top element of a double-tailed diamond interval. The resulting hook-lengths are depicted in Figure 6.11. The hook-length formula (6.1.4) implies that the number $f^{k,\lambda}$ of standard fillings is given by

$$f^{k,\lambda} = \frac{(n+k)!}{\left(\prod_{c\in\lambda} h_c\right)\left(\prod_{i=1}^{k}(n-\lambda_i+i)\right)}. \tag{6.1.6}$$

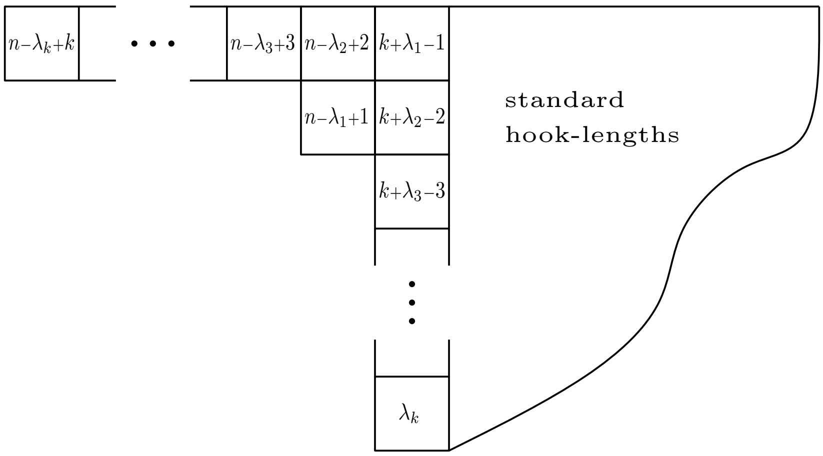


Figure 6.11: The hook-lengths of $P_{k,\lambda}$ with $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

Computational experiments indicate that jeu de taquin on $P_{k,\lambda}$ with row-wise order again yields uniform distribution. Even though we were so far not able to modify the techniques of [NPS97] and [Fis01] to prove that jeu de taquin indeed yields uniform distribution, a quick analysis of insets yields a solution to a different, nice problem:

Fix an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ and consider uniform distribution on the set of standard Young tableaux of shape λ. What is the expected value of the left-most entry in the second row? Three examples are depicted in Figure 6.12. In Section 6.4 we observe that the expected value can

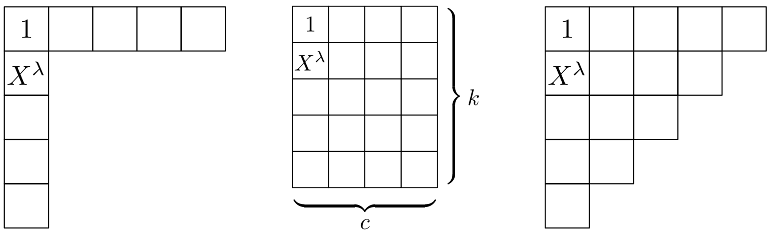


Figure 6.12: What is $\mathbb{E}X^\lambda$ under uniform distribution among standard Young tableaux?

be written in terms of a simple product formula:

Theorem 6.1.2 ([RN14]). *Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. Let $(\Omega, 2^\Omega, P)$ be the probability space containing the f^λ different standard Young tableaux of shape λ and uniform probability measure P. Let $X^\lambda \in \{2, 3, \ldots, \lambda_1 + 1\}$ denote the random variable measuring the left-most entry in the second row. Then*

$$\mathbb{E}X^\lambda = \frac{f^{k,\lambda}}{f^\lambda} = \prod_{i=1}^{k} \frac{n+i}{n+i-\lambda_i}. \tag{6.1.7}$$

Take for example the partition $(3,3,2,1)$ from Figure 6.6(a) which has according to the hook-length formula $\frac{9!}{6\cdot5\cdot4\cdot3\cdot3\cdot2} = 168$ different standard Young tableaux. The corresponding inset has $f^{4,(3,3,2,1)} = 429$ standard fillings. Hence, (6.1.7) tells us that in a standard Young tableau of shape $(3,3,2,1)$ the left-most entry in the second row is on average $\frac{429}{168} \approx 2.554$.

The expected value could also be expressed as a sum of determinants by Aitken's determinant formula for skew standard Young tableaux [Ait43; Sta01, Corollary 7.16.3]. While it should be possible to derive the same product formula (6.1.7) from this expression, the simplicity of the combinatorial argument given in Section 6.4 is somewhat appealing.

A different approach for generating uniformly random linear extensions was taken by Nakada and Okamura [NO10, Nak12]. They generalize a probabilistic algorithm (introduced by Greene, Nijenhuis and Wilf [GNW79]) for generating linear extensions and compute the probability $p(L)$ that a fixed linear extension L is generated by the algorithm. Since they show that $p(L)$ actually does not depend on L their statement not only implies that the algorithm yields uniform distribution among linear extensions but also that the number of linear extensions is given by $\frac{1}{p}$.

6.2 Jeu de taquin on the double-tailed diamond

6.2.1 Reducing the problem to understanding a permutation statistic

For the purpose of this section let us visualize the elements of the double-tailed diamond $D_{m,n}$ as boxes and labelings as fillings of the boxes. Let $B_{i,j}$ denote the box in row i and column j, and given a filling of the boxes let $T_{i,j}$ denote the entry in box $B_{i,j}$ (see Figure 6.13). We perform jeu de taquin

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
				(2, 5)	(2, 6)
					(3, 6)
					(4, 6)
					(5, 6)

8	2	7	3	10	4
				5	11
					1
					9
					6

Figure 6.13: Coordinates of the boxes in $D_{6,5}$ and a filling.

on $D_{m,n}$ with respect to the linear extension σ satisfying $\sigma(B_{2,m-1}) = n$ and $\sigma(B_{1,m}) = n+1$. Given a permutation $\pi = \pi_1\pi_2\dots\pi_{m+n} \in \mathcal{S}_{m+n}$ we start jeu de taquin by assigning $(\pi_1,\pi_2,\dots,\pi_{m+n})$ to the boxes in reverse order of σ (see Figure 6.14).

Let $x_i := x_i(\pi)$ (resp. $y_i := y_i(\pi)$) denote the entry $T_{1,m}$ (resp. $T_{2,m-1}$) after i rounds of jeu de taquin. So, the initial values are $x_0 = \pi_m$ and $y_0 = \pi_{m+1}$, and we know that in the end we have $\{x_{m+n}, y_{m+n}\} = \{m, m+1\}$. As in Theorem 6.1.1 we denote by $s^{(1)}_{m,n}$ the number of permutations $\pi \in \mathcal{S}_{m+n}$ with $x_{m+n}(\pi) = m$ and by $s^{(2)}_{m,n}$ the number of permutations with $x_{m+n}(\pi) = m+1$.

In the first n rounds of jeu de taquin the elements $\{\pi_{m+n}, \pi_{m+n-1}, \dots, \pi_{m+1}\}$ are simply sorted in increasing order (cf. Insertion-Sort algorithm). Therefore

$$x_n(\pi) = \pi_m \quad \text{and} \quad y_n(\pi) = \min\{\pi_{m+1}, \pi_{m+2}, \dots, \pi_{m+n}\}.$$

$$\sigma:\ \begin{array}{|c|c|c|c|c|c|}\hline 11 & 10 & 9 & 8 & 7 & 6 \\ \hline \multicolumn{4}{c|}{} & 5 & 4 \\ \cline{5-6} \multicolumn{5}{c|}{} & 3 \\ \cline{6-6} \multicolumn{5}{c|}{} & 2 \\ \cline{6-6} \multicolumn{5}{c|}{} & 1 \\ \cline{6-6} \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 \\ \hline \multicolumn{4}{c|}{} & \pi_7 & \pi_8 \\ \cline{5-6} \multicolumn{5}{c|}{} & \pi_9 \\ \cline{6-6} \multicolumn{5}{c|}{} & \pi_{10} \\ \cline{6-6} \multicolumn{5}{c|}{} & \pi_{11} \\ \cline{6-6} \end{array}$$

Figure 6.14: Linear extension σ for jeu de taquin and initial filling of the boxes.

In the following we are no longer interested in the exact values of x_i and y_i but only whether $x_i < y_i$ or $x_i > y_i$: If $x_n < y_n$, then $x_n < T_{2,m}$, so nothing happens in the $(n+1)$-st round of jeu de taquin and $x_{n+1} < y_{n+1}$. If on the other hand $x_n > y_n$, then $T_{1,m}$ may or may not be swapped with $T_{2,m}$ and further entries, but in any case $x_{n+1} > y_{n+1}$, since $T_{2,m} > \max\{x_n, y_n\}$ at the start of the round. Therefore, $x_{n+1}(\pi) < y_{n+1}(\pi)$ if and only if $\pi_m = \min\{\pi_m, \pi_{m+1}, \dots, \pi_{m+n}\}$, i.e. if π_m is a right-to-left minimum of π.

For the remaining rounds we observe the following: Before moving π_i at the start of round $m+n+1-i$ the boxes $B_{1,i+1}, \dots, B_{1,m}, B_{2,m-1}$ contain the $m+1-i$ smallest elements of $\{\pi_{i+1}, \pi_{i+2}, \dots, \pi_{m+n}\}$. We have to distinguish between two cases, namely whether π_i is among the $m+1-i$ smallest elements of $\{\pi_i, \pi_{i+1}, \dots, \pi_{m+n}\}$ or not.

If on the one hand $\pi_i > \max\{x_{m+n-i}, y_{m+n-i}\}$, then jeu de taquin first moves π_i to $B_{1,m-1}$ and then swaps π_i with $\min\{x_{m+n-i}, y_{m+n-i}\}$, which – by assumption – changes the order between $T_{1,m}$ and $T_{2,m-1}$. After that π_i may or may not move further, but in any case the order between $x_{m+n-i+1}$ and $y_{m+n-i+1}$ is exactly the opposite of the order between x_{m+n-i} and y_{m+n-i}. If on the other hand $\pi_i < \max\{x_{m+n-i}, y_{m+n-i}\}$, then jeu de taquin moves π_i at most to $B_{1,m}$ or $B_{2,m-1}$ and – by assumption – does not change the order. Thus $x_{m+n-i+1}$ and $y_{m+n-i+1}$ are in the same order as x_{m+n-i} and y_{m+n-i}.

Summed up, we have observed that $x_{n+1}(\pi) < y_{n+1}(\pi)$ if and only if π_m is a right-to-left minimum. After that, the order between $T_{1,m}$ and $T_{2,m-1}$ is kept the same in the round starting with π_i if and only if π_i is among the $m+1-i$ smallest elements of $\{\pi_i, \pi_{i+1}, \dots, \pi_{m+n}\}$. Therefore, we have reduced the problem to understanding a corresponding statistic on permutations.

6.2.2 A generalization of right-to-left-minima

The previous observations motivate the following definition generalizing right-to-left-minima of permutations:

Definition 6.2.1 (RL_k – min). *Let $\pi = \pi_1 \pi_2 \dots \pi_n \in \mathcal{S}_n$. We say that π_i is a* RL_k – min *if and only if π_i is among the k smallest elements of $\{\pi_i, \pi_{i+1}, \dots, \pi_n\}$.*

To solve our counting problem, we need to understand the distribution of

$$c_{m,n}(\pi) := \sum_{i=1}^{m} \left[\pi_i \text{ is } \mathrm{RL}_{m+1-i}\text{-min}\right], \quad \pi = \pi_1 \cdots \pi_{m+n} \in \mathcal{S}_{m+n}, \tag{6.2.1}$$

where the square brackets denote Iverson brackets, i.e. $[\phi] := 1$ if ϕ is true, and 0 otherwise. As it turns out the distribution of $c_{m,n}$ can be expressed in a particularly simple way:

Proposition 6.2.2. *Let*

$$c_{m,n,k} := \#\{\pi \in \mathcal{S}_{m+n} \mid c_{m,n}(\pi) = m - k\}, \quad m, n \geq 1,\ 0 \leq k \leq m.$$

Then

$$c_{m,n,k} = n^k \begin{bmatrix} m+1 \\ k+1 \end{bmatrix} n!, \tag{6.2.2}$$

where $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ *denotes the unsigned Stirling numbers of first kind, i.e. the number of permutations of* s *elements with* t *disjoint cycles.*

Remark 6.2.3. *Note that* $c_{m,n,k}$ *counts the number of* $\pi \in \mathcal{S}_{m+n}$ *for which the order between* $T_{1,m}$ *and* $T_{2,m-1}$ *in jeu de taquin is changed exactly* k *times (with* π_m *contributing to* k *if and only if* π_m *is not a right-to-left minimum). In particular,* $x_{m+n}(\pi) < y_{m+n}(\pi)$ *if and only if* $k = m - c_{m,n}(\pi)$ *is even.*

Proof of Proposition 6.2.2. The proof is split into the two edge cases $k = m$ and $k = 0$ and the case $0 < k < m$.

- $k = 0$: Let us show that

$$c_{m,n}(\pi) = m \quad \iff \quad \{\pi_1, \ldots, \pi_m\} = \{1, \ldots, m\}.$$

Assume $s(\pi_i) := [\pi_i \text{ is } \mathrm{RL}_{m+1-i}\text{-}\min] = 1$ for all $i = 1, \ldots, m$. Then $s(\pi_m) = 1$ implies $1 \in \{\pi_1, \ldots, \pi_m\}$. Suppose there exists $2 \leq k \leq m$ such that $k \in \{\pi_{m+1}, \ldots, \pi_{m+n}\}$. It then follows from $s(\pi_m) = s(\pi_{m-1}) = \cdots = s(\pi_{m+2-k}) = 1$ that none of $\pi_m, \pi_{m-1}, \ldots, \pi_{m+2-k}$ can be greater than k, i.e. $\{\pi_{m+2-k}, \ldots, \pi_m\} = \{1, 2, \ldots, k-1\}$. But this contradicts π_{m+1-k} being a RL_k −min. The reverse direction is obvious. Since there are exactly $m!$ permutations in $\mathcal{S}_{m+1}$ consisting of exactly one cycle, we obtain

$$c_{m,n,0} = m!\, n! = n^0 \begin{bmatrix} m+1 \\ 1 \end{bmatrix} n!.$$

- $k = m$: In this case let us observe that

$$c_{m,n}(\pi) = 0 \quad \iff \quad j \in \{\pi_{m+2-j}, \pi_{m+3-j}, \ldots, \pi_{m+n}\} \text{ for all } j = 1, \ldots, m.$$

Suppose $j \notin \{\pi_{m+2-j}, \pi_{m+3-j}, \ldots, \pi_{m+n}\}$. Then there exists an $i \in \{1, \ldots, m+1-j\}$ such that $\pi_i = j \leq m+1-i$. But this implies that π_i is a RL_{m+1-i} −min, so $c_{m,n}(\pi) > 0$. If we conversely assume that $\{1, 2, \ldots, m+1-i\} \subseteq \{\pi_{i+1}, \ldots, \pi_{m+n}\}$ for all $i = 1, \ldots, m$, it follows that π_i is not among the $m+1-i$ smallest elements of $\{\pi_i, \ldots, \pi_{m+n}\}$, and therefore $c_{m,n}(\pi) = 0$.

Therefore we have exactly n^m possibilities to choose the preimage of $\{1, 2, \ldots, m\}$ and for each such choice the preimages of $\{m+1, \ldots, m+n\}$ can be chosen in any order, i.e.

$$c_{m,n,m} = n^m n! = n^m \begin{bmatrix} m+1 \\ m+1 \end{bmatrix} n!.$$

- $0 < k < m$: We proceed by induction on m. From the recurrence relation $\left[{s+1 \atop t}\right] = s\left[{s \atop t}\right] + \left[{s \atop t-1}\right]$ and the induction hypothesis (resp. the edge cases) it follows that

$$n^k \begin{bmatrix} m+1 \\ k+1 \end{bmatrix} n! = n^k m \begin{bmatrix} m \\ k+1 \end{bmatrix} n! + n^k \begin{bmatrix} m \\ k \end{bmatrix} n! = m\, c_{m-1,n,k} + n\, c_{m-1,n,k-1}.$$

So it only remains to show that

$$m\, c_{m-1,n,k} + n\, c_{m-1,n,k-1} = c_{m,n,k}. \tag{6.2.3}$$

For a bijective proof of (6.2.3) consider the position of 1 in $\pi_1 \dots \pi_m \pi_{m+1} \dots \pi_{m+n}$. Let π' denote the permutation π with $\pi_j = 1$ removed and each number reduced by 1, so that $\pi' \in \mathcal{S}_{m+n-1}$.

If on the one hand $1 \le j \le m$, note that $c_{m-1,n}(\pi') = c_{m,n}(\pi) - 1$ and thus $c_{m-1,n}(\pi') = (m-1) - k$ if and only if $c_{m,n}(\pi) = m - k$. For each $1 \le j \le m$ this establishes a one-to-one correspondence between the permutations $\pi \in \mathcal{S}_{m+n}$ with $\pi_j = 1$ that are counted by $c_{m,n,k}$ and permutations $\pi' \in \mathcal{S}_{m+n-1}$ counted by $c_{m-1,n,k}$.

If on the other hand $m+1 \le j \le m+n$, note that $c_{m-1,n}(\pi') = c_{m,n}(\pi)$ and therefore $c_{m-1,n}(\pi') = (m-1) - (k-1)$ if and only if $c_{m,n}(\pi) = m - k$. For each $m+1 \le j \le m+n$ this is a one-to-one correspondence between the permutations $\pi \in \mathcal{S}_{m+n}$ with $\pi_j = 1$ that are counted by $c_{m,n,k}$ and permutations $\pi' \in \mathcal{S}_{m+n-1}$ counted by $c_{m-1,n,k-1}$.

□

Proof of Theorem 6.1.1. In Remark 6.2.3 we have observed that

$$s^{(1)}_{m,n} - s^{(2)}_{m,n} = \#\{\pi \in \mathcal{S}_{m+n} \mid m - c_{m,n}(\pi) \text{ is even}\} - \#\{\pi \in \mathcal{S}_{m+n} \mid m - c_{m,n}(\pi) \text{ is odd}\}.$$

Together with Proposition 6.2.2 it follows that

$$\begin{aligned} s^{(1)}_{m,n} - s^{(2)}_{m,n} &= \sum_{k=0}^{m} (-1)^k c_{m,n,k} = \sum_{k=1}^{m+1} (-1)^{k-1} n^{k-1} \begin{bmatrix} m+1 \\ k \end{bmatrix} n! \\ &= (-1)^m (n-1)! \sum_{k=0}^{m+1} (-1)^{m+1-k} \begin{bmatrix} m+1 \\ k \end{bmatrix} n^k. \end{aligned}$$

Since

$$(x)_s := x(x-1)\dots(x-s+1) = \sum_{k=0}^{s} (-1)^{s-k} \begin{bmatrix} s \\ k \end{bmatrix} x^k$$

we obtain

$$s^{(1)}_{m,n} - s^{(2)}_{m,n} = (-1)^m (n-1)!(n)_{m+1} = (-1)^m \binom{n-1}{m} m!\, n!.$$ □

So, in particular jeu de taquin yields uniform distribution on the double-tailed diamond $D_{m,n}$ if and only if $m \ge n$.

Let us close this section by noting that the if-direction can also be obtained by a simple inductive argument, which can be extended to general posets: If we play jeu de taquin on $D_{m,m}$ with all permutations where π_1 has a fixed value and stop the sorting procedure before π_1 is moved, then we obtain a (non-uniform) distribution (α, β) with $\alpha + \beta = (2m-1)!$ and α, β independent from π_1. As previously observed π_1 does not change the order between $T_{1,m}$ and $T_{2,m-1}$ if and only if π_1 is a $\mathrm{RL}_m-\min$, i.e. $\pi_1 \in \{1, 2, \ldots, m\}$. After completing jeu de taquin by moving π_1, we therefore obtain the distribution (α, β) if $\pi_1 \in \{1, 2, \ldots, m\}$ and the distribution (β, α) if $\pi_1 \in \{m+1, m+2, \ldots, 2m\}$. In total each of the two standard fillings occurs $m(\alpha + \beta)$ times, i.e. jeu de taquin yields a uniform distribution on $D_{m,m}$. In the same way we can now fix π_1 in jeu de taquin on $D_{m,n}$ with $m > n$. Inductively we obtain a uniform distribution on $D_{m-1,n}$ for each fixed $\pi_1 \in \{1, 2, \ldots, m+n\}$. Since each fixed π_1 either always ($\pi_1 > m$) or never ($\pi_1 \leq m$) changes the order of the entries in the incomparable boxes, we also obtain a uniform distribution on $D_{m,n}$.

A similar argument applies when it comes to playing jeu de taquin on posets with a maximum: Given an n-element poset P and an order σ such that jeu de taquin yields uniform distribution, we can extend this property to the poset P' obtained by adding a maximum element m to P: First it is clear that the total number of (dual) linear extensions remains the same, i.e. $f^P = f^{P'}$. As order for jeu de taquin choose $\sigma'|_P := \sigma|_P$ and $\sigma'(m) := n+1$. Now consider all labelings π of P' where $\pi_m = i$ is fixed. If we play jeu de taquin with all such labelings and stop before moving i we obtain (restricted to P) a uniform distribution among the f^P different dual linear extensions (with entries $[n+1] \setminus \{i\}$). Note that in each dual linear extension of P' the label i has a unique reverse path back to the top. Thus, moving i in the last step of jeu de taquin preserves the uniform distribution. Having a uniform distribution for each $\pi_m = i \in [n+1]$ implies uniform distribution in total.

Since jeu de taquin with row-wise order on Young tableaux yields a uniform distribution [NPS97], it follows from the previous observation that the poset obtained from removing the top row of the inset $P_{k,\lambda}$ has the same property. It remains an open problem to understand why the uniform distribution is also preserved when adding the top row.

6.3 A combinatorial proof of Theorem 6.1.1

In this section we give a bijective proof of Theorem 6.1.1. For this purpose we define the type τ for each permutation $\pi \in \mathcal{S}_{m+n}$ by setting $\tau(\pi) := 1$ if jeu de taquin with input permutation π yields the output tableau T_1, and $\tau(\pi) = -1$ if the output tableau is T_2 (see Figure 6.9 and Figure 6.14). Given two subsets $S_1, S_2 \subseteq \mathcal{S}_{m+n}$ we say that $f : S_1 \to S_2$ is type-inverting if $\tau(\pi) = -\tau(f(\pi))$ for all $\pi \in S_1$. To give a combinatorial proof of Theorem 6.1.1 we define a type-inverting involution $\Phi_{m,n} : \mathcal{S}_{m+n} \to \mathcal{S}_{m+n}$ for all $m \geq n$. In the case $m < n$ we identify a set $\mathcal{E}$ of $\binom{n-1}{m} m!\, n!$ exceptional permutations in $\mathcal{S}_{m+n}$ of the same type. On the remaining set $\mathcal{S}_{m+n} \setminus \mathcal{E}$ we then define a type-inverting involution $\Phi_{m,n}$.

As in Section 6.2 let $x_i(\pi)$ and $y_i(\pi)$ denote the entries $T_{1,m}$ and $T_{2,m-1}$ after i rounds of jeu de taquin. Before giving the formal definition of $\Phi_{m,n}$ let us start by explaining the basic ideas:

First, if the last i entries of π are in the same relative order as the last i entries of π', then $x_i(\pi) < y_i(\pi)$ if and only if $x_i(\pi') < y_i(\pi')$. This means that whether or not $x_i(\pi) < y_i(\pi)$ for $i = n+1, \ldots, n+m$ depends on the relative order of $\pi_{m+n+1-i}$, $\pi_{m+n+2-i}$, $\ldots$, π_{m+n} but not their absolute values.

Second, we have noted in Section 6.2 that π_i does not change the order between $T_{1,m}$ and $T_{2,m-1}$ if and only if π_i is among the $m+1-i$ smallest elements of $\{\pi_i, \pi_{i+1}, \ldots, \pi_{m+n}\}$. This implies that whether or not π_i changes the order only depends on the set of elements $\{\pi_{i+1}, \pi_{i+2}, \ldots, \pi_{m+n}\}$, but not their relative order. In particular, π_1 changes the order between $T_{1,m}$ and $T_{2,m-1}$ if and only if $\pi_1 > m$.

In the case $m = n$ we therefore construct an involution $\Phi_{n,n}$ on $\mathcal{S}_{2n}$ such that the relative order of all entries in π and the relative order of all entries in $\pi' := \Phi_{n,n}(\pi)$ is the same if we exclude π_1 and $\pi'_1 := 2n+1-\pi_1$. In the case $m > n$ we let $\Phi_{m,n}$ fix the first $m-n$ entries of each permutation and apply the type-inverting involution $\Phi_{n,n}$ to the bottom $2n$ entries. If $m < n$ we apply $\Phi_{m,m}$ to the smallest $2m$ entries if $\pi_1 \leq 2m$. Else, we apply $\Phi_{m-1,m-1}$ to the smallest $2(m-1)$ entries if $\pi_2 \leq 2(m-1)$, and so on. Either one of the first m entries is small enough to apply the type-inverting involution $\Phi_{m+1-i,m+1-i}$ or $\pi_1, \pi_2, \ldots, \pi_m$ are all too large. In the latter case we call the permutation exceptional and exclude it from the involution $\Phi_{m,n}$. As it turns out there are exactly $\binom{n-1}{m} m!\, n!$ exceptional permutations all having the same type, thus proving Theorem 6.1.1.

Let us now formally define the involution $\Phi_{m,n}$ in all three cases, prove the correctness and give examples.

6.3.1 Case $m = n$

For $n \in \mathbb{N}$ and $1 \leq t \leq 2n$ define the permutation $\chi_{n,t} \in \mathcal{S}_{2n}$ by

$$1 \leq t \leq n: \qquad \chi_{n,t}(i) := \begin{cases} 2n+1-t & \text{if } i = t, \\ i-1 & \text{if } t < i \leq 2n+1-t, \\ i & \text{otherwise.} \end{cases} \tag{6.3.1}$$

$$n+1 \leq t \leq 2n: \qquad \chi_{n,t}(i) := \begin{cases} 2n+1-t & \text{if } i = t, \\ i+1 & \text{if } 2n+1-t \leq i < t, \\ i & \text{otherwise.} \end{cases} \tag{6.3.2}$$

For all $1 \leq t \leq 2n$ we have

$$\chi_{n,t} \circ \chi_{n,2n+1-t} = \chi_{n,2n+1-t} \circ \chi_{n,t} = \mathrm{id}$$

and $\chi_{n,t}\Big|_{[2n]\setminus t}$ is order-preserving. The desired involution $\Phi_{n,n} : \mathcal{S}_{2n} \to \mathcal{S}_{2n}$ is

$$\Phi_{n,n}(\pi) := \chi_{n,\pi_1} \circ \pi. \tag{6.3.3}$$

An example can be seen in Figure 6.15. As composition of permutations it is clear that $\Phi_{n,n}(\pi) \in$

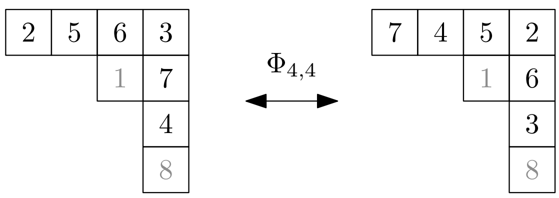


Figure 6.15: The involution $\Phi_{4,4}$ applied to $\pi = 25631748$.

$\mathcal{S}_{2n}$, and from $\chi_{n,\pi_1}(\pi_1) = 2n+1-\pi_1$ it follows that

$$\Phi^2_{n,n}(\pi) = \Phi_{n,n}(\chi_{n,\pi_1} \circ \pi) = \chi_{n,2n+1-\pi_1} \circ \chi_{n,\pi_1} \circ \pi = \pi,$$

i.e. $\Phi_{n,n}$ is an involution. Since χ_{n,π_1} is order-preserving except for π_1 the entries of π and $\pi' := \Phi_{n,n}(\pi)$ have the same relative order except for π_1 and π'_1. Therefore

$$x_{2n-1}(\pi) < y_{2n-1}(\pi) \quad \Longleftrightarrow \quad x_{2n-1}(\pi') < y_{2n-1}(\pi').$$

As $\pi'_1 = 2n + 1 - \pi_1$ exactly one of $\pi_1 > n$ or $\pi'_1 > n$ holds, and thus the two permutations π and π' are of different type, i.e. $\Phi_{n,n}$ is a type-inverting involution on $\mathcal{S}_{2n}$.

6.3.2 Case $m > n$

Given two subsets $A, B \subseteq \mathbb{N}$ with $|A| = |B|$, let $\sigma_{A,B} : A \to B$ denote the unique order-preserving bijection between A and B, i.e. the bijection satisfying $(a_1 < a_2) \to (\sigma_{A,B}(a_1) < \sigma_{A,B}(a_2))$ for all $a_1, a_2 \in A$. Obviously, we have $\sigma_{B,A} \circ \sigma_{A,B} = \sigma_{A,B} \circ \sigma_{B,A} = \mathrm{id}$.

If $m > n$ and $\pi \in \mathcal{S}_{m+n}$ set $A := A^\pi := \{\pi_{m-n+1}, \pi_{m-n+2}, \dots, \pi_{m+n}\}$, $B := \{1, 2, \dots, 2n\}$ and $t := t^\pi := \sigma_{A,B}(\pi_{m-n+1})$. The type-inverting involution $\Phi_{m,n}$ in this case is

$$\Phi_{m,n}(\pi) := \begin{cases} i \mapsto \pi_i & \text{if } 1 \le i \le m-n, \\ i \mapsto \sigma_{B,A} \circ \chi_{n,t} \circ \sigma_{A,B}(\pi_i) & \text{if } m-n+1 \le i \le m+n. \end{cases} \tag{6.3.4}$$

Note that $\Phi_{m,n}(\pi)$ is well-defined and an element of $\mathcal{S}_{m+n}$ (see Figure 6.16 for an example).

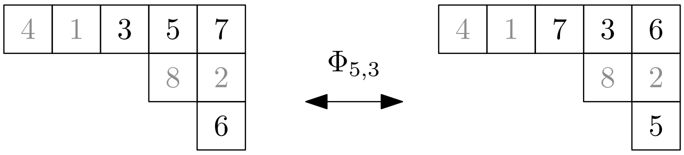


Figure 6.16: The involution $\Phi_{5,3}$ applied to $\pi = 41357826$ with $A^\pi = \{2, 3, 5, 6, 7, 8\}$.

Moreover, we have

$$\Phi_{m,n}(\pi)\Big|_{\{1,\dots,m-n\}} = \pi,$$
$$\Phi_{m,n}(\pi)\Big|_{\{m-n+1,\dots,m+n\}} = \sigma_{B,A} \circ \chi_{n,t} \circ \sigma_{A,B} \circ \pi.$$

Therefore $A^{\Phi_{m,n}(\pi)} = A^\pi$ and $t^{\Phi_{m,n}(\pi)} = \sigma_{A,B}(\sigma_{B,A} \circ \chi_{n,t^\pi} \circ \sigma_{A,B}(\pi_{m-n+1})) = \chi_{n,t^\pi}(t^\pi) = 2n + 1 - t^\pi$. It follows that $\Phi^2_{m,n}(\pi)\Big|_{\{1,\dots,m-n\}} = \pi\Big|_{\{1,\dots,m-n\}}$ and

$$\begin{aligned} \Phi^2_{m,n}(\pi)\Big|_{\{m-n+1,\dots,m+n\}} &= \Phi_{m,n}(\sigma_{B,A} \circ \chi_{n,t^\pi} \circ \sigma_{A,B} \circ \pi)\Big|_{\{m-n+1,\dots,m+n\}} \\ &= \sigma_{B,A} \circ \chi_{n,2n+1-t^\pi} \circ \sigma_{A,B} \circ \sigma_{B,A} \circ \chi_{n,t^\pi} \circ \sigma_{A,B} \circ \pi\Big|_{\{m-n+1,\dots,m+n\}} = \pi\Big|_{\{m-n+1,\dots,m+n\}}, \end{aligned}$$

i.e. $\Phi^2_{m,n} = \mathrm{id}$. As in the case $m = n$ the relative order of the last $2n-1$ entries of π and $\pi' := \Phi_{m,n}(\pi)$ is the same. The entry π_{m-n+1} is among the n smallest elements of $\{\pi_{m-n+1}, \ldots, \pi_{m+n}\}$ if and only if π'_{m-n+1} is not among the n smallest elements of $\{\pi'_{m-n+1}, \ldots, \pi'_{m+n}\}$. Therefore

$$x_{2n}(\pi) < y_{2n}(\pi) \qquad \Longleftrightarrow \qquad x_{2n}(\pi') > y_{2n}(\pi').$$

Since $\pi_i = \pi'_i$ for all $i = 1, \ldots, m-n$, the permutations π and π' are of different type.

6.3.3 Case $m < n$

In the case $m < n$ let us define a subset $\mathcal{E} \subseteq \mathcal{S}_{m+n}$ of exceptional permutations which we exclude from the involution: We say that $\pi = \pi_1 \ldots \pi_{m+n}$ is exceptional if and only if $\pi_i > 2(m+1-i)$ for all $i = 1, \ldots, m$. Note that the number of exceptional permutations is $(n-m)(n-m+1)\ldots(n-1)\,n! = \binom{n-1}{m}\,m!\,n!$. Given $\pi \in \mathcal{S}_{m+n} \setminus \mathcal{E}$, let $k := k^\pi \geq 1$ minimal such that $\pi_k \leq 2(m+1-k)$. Define

$$\Phi_{m,n}(\pi) := \begin{cases} i \mapsto \pi_i & \text{if } \pi_i > 2(m+1-k), \\ i \mapsto \chi_{m+1-k,\pi_k}(\pi_i) & \text{otherwise.} \end{cases} \tag{6.3.5}$$

An example can be seen in Figure 6.17. Note that $\Phi_{m,n}(\pi)$ is well-defined and since $\chi_{m+1-k,\pi_k}(\pi_k) =$

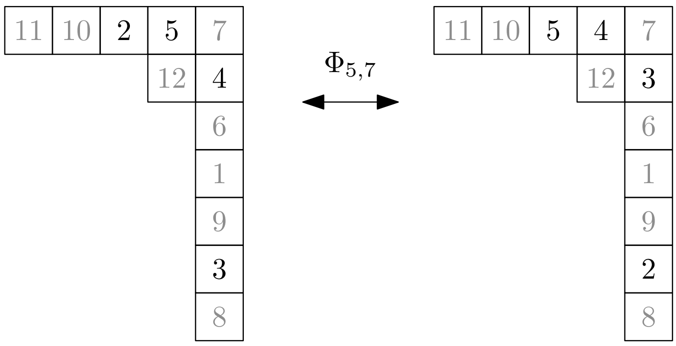


Figure 6.17: The involution $\Phi_{5,7}$ with $k^\pi = 3$.

$2(m+1-k)+1-\pi_k$ we have $k^{\Phi_{m,n}(\pi)} = k^\pi$ and $\Phi_{m,n}(\pi) \in \mathcal{S}_{m+n} \setminus \mathcal{E}$. With $L := \{1 \leq i \leq m+n : \pi_i > 2(m+1-k)\}$ it follows that $\Phi^2_{m,n}(\pi)\Big|_L = \Phi_{m,n}(\pi)\Big|_L = \pi\Big|_L$ and

$$\begin{aligned} \Phi^2_{m,n}(\pi)\Big|_{[m+n]\setminus L} &= \Phi_{m,n}(\chi_{m+1-k,\pi_k} \circ \pi)\Big|_{[m+n]\setminus L} \\ &= \chi_{m+1-k,2(m+1-k)+1-\pi_k} \circ \chi_{m+1-k,\pi_k} \circ \pi\Big|_{[m+n]\setminus L} = \pi\Big|_{[m+n]\setminus L}, \end{aligned}$$

i.e. $\Phi_{m,n}$ is an involution. The relative order of the entries in π and $\pi' := \Phi_{m,n}(\pi)$ is the same except for π_k and π'_k. Since π_k is among the $m+1-k$ smallest elements of $\{\pi_k, \ldots, \pi_{m+n}\}$ if and only if π'_k is not among the $m+1-k$ smallest elements of $\{\pi'_k, \ldots, \pi'_{m+n}\}$ the involution $\Phi_{m,n}$ is type-inverting. We can conclude the proof by noting that all exceptional permutations are of the same type, since $\pi_i > 2(m+1-i)$ for all $i = 1, \ldots, m$ implies that π_i is not among the $m+1-i$ smallest elements of $\{\pi_i, \pi_{i+1}, \ldots, \pi_{m+n}\}$.

6.4 Proof of Theorem 6.1.2 & Examples

The statement of Theorem 6.1.2 can be observed by computing the number $f^{k,\lambda}$ of standard fillings of the inset $P_{k,\lambda}$ in two different ways. On the one hand we can use the hook-length formula for insets (see (6.1.6)). On the other hand we can refine the counting w.r.t. the left-most entry in the second row: In each standard filling of $P_{k,\lambda}$ the $k-1$ left-most entries in the first row are $(T_{1,1}, T_{1,2}, \ldots, T_{1,k-1}) = (1, 2, \ldots, k-1)$. For $i = 0, 1, \ldots, \lambda_1$ let $f_i^{k,\lambda}$ denote the number of standard fillings of $P_{k,\lambda}$ where the left-most entry in the second row is $T_{2,k-1} = k+i$ (see Figure 6.18). Now

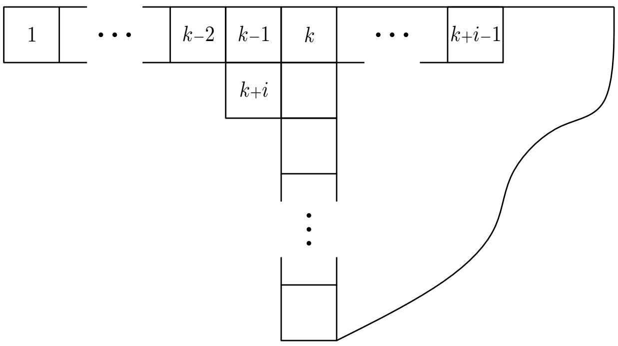


Figure 6.18: Standard fillings counted by $f_i^{k,\lambda}$.

note that for each fixed $i \in \{0, 1, \ldots, \lambda_1\}$ the standard fillings counted by $f_i^{k,\lambda}$ are in one-to-one correspondence with standard Young tableaux of shape λ where the top row starts with $(1, 2, \ldots, i)$. This in turn is equivalent to requiring that the left-most entry in the second row of the standard Young tableau is at least $i+1$. Together with $f^{k,\lambda} = \sum_{i=0}^{\lambda_1} f_i^{k,\lambda}$, (6.1.1) and (6.1.6) we obtain

$$\mathbb{E}X^\lambda = \sum_{i=1}^{\lambda_1+1} \mathbb{P}\{X^\lambda \geq i\} = \sum_{i=0}^{\lambda_1} \frac{f_i^{k,\lambda}}{f^\lambda} = \frac{f^{k,\lambda}}{f^\lambda} = \prod_{i=1}^{k} \frac{n+i}{n+i-\lambda_i}.$$

Let us apply this result to the three families of partitions in Figure 6.12.

Example 6.4.1. *Consider the partition $\lambda = (k, 1^{k-1}) \vdash 2k-1$. From Theorem 6.1.2 we obtain*

$$\mathbb{E}X^\lambda = \prod_{i=1}^{k} \frac{2k-1+i}{2k-1+i-\lambda_i} = \frac{2k}{k} \prod_{i=2}^{k} \frac{2k-1+i}{2k-2+i} = 3 - \frac{1}{k}.$$

Of course, this can also be obtained by the elementary observation that $f^\lambda = \binom{2k-2}{k-1}$ and

$$\mathbb{E}X^\lambda = \sum_{i\geq 1} \mathbb{P}(X^\lambda \geq i) = \frac{1}{\binom{2k-2}{k-1}} \left[\binom{2k-2}{k-1} + \sum_{i\geq 2} \binom{2k-i}{k-1} \right] = 1 + \frac{\binom{2k-1}{k}}{\binom{2k-2}{k-1}} = 3 - \frac{1}{k}.$$

Example 6.4.2. *Fix $c \geq 1$ and consider the partition $\lambda = (c, \dots, c) \vdash kc$. For $k \geq c$ it follows that*

$$\mathbb{E}X^\lambda = \prod_{i=1}^{k} \frac{kc+i}{kc+i-c} = \prod_{i=1}^{c} \frac{k(c+1)+1-i}{kc+1-i} \xrightarrow{k\to\infty} \left(1+\frac{1}{c}\right)^c.$$

Example 6.4.3. *Let $\lambda = (k, k-1, \dots, 1) \vdash \binom{k+1}{2}$ be of staircase shape. From Theorem 6.1.2 it follows that*

$$\begin{aligned}\mathbb{E}X^\lambda &= \prod_{i=1}^{k} \frac{\binom{k+1}{2}+i}{\binom{k+1}{2}+2i-k-1} = \frac{\left(\binom{k+1}{2}+k\right)!\left(\binom{k+1}{2}-k-1\right)!!}{\binom{k+1}{2}!\left(\binom{k+1}{2}+k-1\right)!!} \\ &= \frac{\left(\binom{k+1}{2}+k\right)!!\left(\binom{k+1}{2}-k-1\right)!!}{\binom{k+1}{2}!},\end{aligned}$$

where !! denotes the double factorial, i.e.

$$\begin{aligned}(2n)!! &= (2n)(2n-2)\cdots 2 = 2^n n!, \\ (2n-1)!! &= (2n-1)(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}.\end{aligned}$$

Applying Stirling's formula one can show (Lemma A.3.1) that asymptotically $\mathbb{E}X^\lambda \sim e \approx 2.71828$.

Appendix A

Asymptotics and hypergeometric identities

A.1 Asymptotics of ASMs

Lemma A.1.1. *The number of ASMs of size n satisfies*

$$\lim_{n\to\infty} A_n^{\frac{1}{n^2}} = \frac{3\sqrt{3}}{4}. \tag{A.1.1}$$

Proof. Since $A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$, it is equivalent to show that

$$\lim_{n\to\infty} \frac{1}{n^2} \log\left(\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}\right) = \log\left(\frac{3\sqrt{3}}{4}\right).$$

From Stirling's formula $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ it follows that

$$\log n! = n\log n - n + \frac{1}{2}\log n + c + \alpha_n$$

with a constant c and $\alpha_n \to 0$. Since both

$$\frac{1}{n^2}\sum_{j=0}^{n-1} \log(3j+1) \leq \frac{\log(3n-2)}{n} \quad \text{and} \quad \frac{1}{n^2}\sum_{j=0}^{n-1} \log(n+j) \leq \frac{\log(2n-1)}{n}$$

tend to zero as $n \to \infty$, we obtain

$$\begin{aligned}\frac{1}{n^2} \log\left(\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}\right) &= \frac{1}{n^2}\sum_{j=0}^{n-1} (\log(3j+1)! - \log(n+j)!) \\ &= \frac{1}{n^2}\left(\sum_{j=0}^{n-1} ((3j+1)\log(3j+1) - (n+j)\log(n+j))\right) - \frac{n-1}{n} + 1 + \epsilon_n,\end{aligned}$$

where $\epsilon_n \to 0$. Hence, it suffices to show that

$$S(n) := \sum_{j=0}^{n-1} ((3j+1)\log(3j+1) - (n+j)\log(n+j))$$

satisfies

$$\lim_{n\to\infty} \frac{S(n)}{n^2} = \log\left(\frac{3\sqrt{3}}{4}\right). \tag{A.1.2}$$

The interpretation of lower and upper sums yields for any monotonically increasing function f and integers $a \leq b$

$$\int_{a-1}^{b} f(x)\ \mathrm{dx} \leq \sum_{j=a}^{b} f(j) \leq \int_{a}^{b+1} f(x)\ \mathrm{dx}\,.$$

Set $f(x) := x\log x$ to obtain

$$S(n) = \sum_{j=1}^{n-1} f(3j+1) - \sum_{j=n}^{2n-1} f(j).$$

Together with $\int x\log x\ \mathrm{dx} = \frac{x^2}{2}\log x - \frac{x^2}{4} + c$ it follows that

$$\begin{aligned} S(n) &\geq \int_0^{n-1} f(3x+1)\ \mathrm{dx} - \int_n^{2n} f(x)\ \mathrm{dx} = \frac{1}{3}\int_0^{3n-2} u\log u\ \mathrm{du} - \int_n^{2n} x\log x\ \mathrm{dx} \\ &= \frac{1}{3}\left(\frac{(3n-2)^2}{2}\log(3n-2) - \frac{(3n-2)^2}{4}\right) - \left(\frac{(2n)^2}{2}\log(2n) - \frac{(2n)^2}{4} - \frac{n^2}{2}\log n + \frac{n^2}{4}\right) \end{aligned}$$

and therefore

$$\frac{S(n)}{n^2} \geq \frac{3}{2}\log(3n-2) - \frac{3}{4} - 2\log(2n) + 1 + \frac{1}{2}\log n - \frac{1}{4} - \lambda_n = \log\left(\frac{(3n-2)^{\frac{3}{2}} n^{\frac{1}{2}}}{(2n)^2}\right) - \lambda_n,$$

where $\lambda_n \to 0$. Taking the limit, we obtain

$$\lim_{n\to\infty} \frac{S(n)}{n^2} \geq \log\left(\frac{3\sqrt{3}}{4}\right).$$

Analogously,

$$\begin{aligned} S(n) &\leq \int_1^{n} f(3x+1)\ \mathrm{dx} - \int_{n-1}^{2n-1} f(x)\ \mathrm{dx} = \frac{1}{3}\int_4^{3n+1} u\log u\ \mathrm{du} - \int_{n-1}^{2n-1} x\log x\ \mathrm{dx} \\ &= \frac{1}{3}\left(\frac{(3n+1)^2}{2}\log(3n+1) - \frac{(3n+1)^2}{4} - 8\log 4 + 4\right) \\ &\quad - \left(\frac{(2n-1)^2}{2}\log(2n-1) - \frac{(2n-1)^2}{4} - \frac{(n-1)^2}{2}\log(n-1) + \frac{(n-1)^2}{4}\right) \end{aligned}$$

and thus

$$\frac{S(n)}{n^2} \leq \frac{3}{2}\log(3n+1)-\frac{3}{4}-2\log(2n-1)+1+\frac{1}{2}\log(n-1)-\frac{1}{4}+\mu_n = \log\left(\frac{(3n+1)^{\frac{3}{2}}(n-1)^{\frac{1}{2}}}{(2n-1)^2}\right)+\mu_n,$$

where $\mu_n \to 0$. In the limit we therefore also have

$$\lim_{n\to\infty}\frac{S(n)}{n^2} \leq \log\left(\frac{3\sqrt{3}}{4}\right).$$

In total (A.1.2) and therefore the asymptotics (A.1.1) of ASMs follows. $\square$

A.2 Hypergeometric identities

Apart from the identities $\binom{r}{k} = (-1)^k\binom{k-r-1}{k}$ and $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$, which are true for all $k, m \in \mathbb{Z}$, $r \in \mathbb{C}$, and $\binom{n}{j} = \binom{n}{n-j}$, which only holds for non-negative integers n (if we use definition (2.5.10) of the binomial coefficient), we make extensive use of the Chu–Vandermonde convolution [GKP89, p.169]

$$\sum_{k\in\mathbb{Z}}\binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}, \quad m,n\in\mathbb{Z},\ r,s\in\mathbb{C}. \tag{A.2.1}$$

There are two simple ways to convince oneself of the Chu–Vandermonde convolution: On the one hand there is the generating function approach, i.e. compare the coefficient at z^{m+n} on both sides of $(1+z)^{r+s} = (1+z)^r(1+z)^s$ to obtain

$$\binom{r+s}{m+n} = \sum_{k=0}^{m+n}\binom{r}{k}\binom{s}{m+n-k} = \sum_{k}\binom{r}{m+k}\binom{s}{n-k}.$$

On the other hand the identity is clear from a combinatorial point of view if r and s are non-negative integers (the number of ways to assign $m+n$ balls to $r+s$ slots is the same as the number of ways to assign $m+k$ balls to the first r slots and $n-k$ to the last s slots for any integer k). For fixed integers m and n both sides are polynomials in r and s and therefore the identity also holds for $r, s \in \mathbb{C}$.

Let us also remark that a standard way of dealing with the following identities would be to first write them in generalized hypergeometric function notation and then apply the appropriate transformation and summation formulæ (under guidance of Krattenthaler's HYP package [Kra95]).

Lemma A.2.1. *The matrices*

$$B_n = \left(\binom{2n-i-2}{n-i-j-1}(-1)^{j+1}+\delta_{i,j}\right)_{1\leq i,j\leq n-1},$$

$$B_n^* = \left(\binom{i+j}{j-1}(1-\delta_{i,n-1})\right)_{1\leq i,j\leq n-1},$$

$$R_n = \left(\binom{n+j-i-1}{j-i}\right)_{1\leq i,j\leq n-1},$$

$$R_n^{-1} = \left(\binom{n}{j-i}(-1)^{i+j}\right)_{1\leq i,j\leq n-1}$$

satisfy $R_n R_n^{-1} = I_{n-1}$ *and* $R_n^{-1} B_n R_n = B_n^* + I_{n-1}$.

Proof. The first identity is an immediate consequence of the Chu–Vandermonde convolution:

$$\begin{aligned}\left(R_n R_n^{-1}\right)_{i,j} &= \sum_{k=1}^{n-1} \binom{n+k-i-1}{k-i}\binom{n}{j-k}(-1)^{k+j}\\ &= \sum_k \binom{-n}{k-i}\binom{n}{j-k}(-1)^{i+j} = \binom{0}{j-i}(-1)^{i+j} = \delta_{i,j}.\end{aligned}$$

For the second identity we want to compute

$$\left(R_n^{-1} B_n R_n\right)_{i,j} = \sum_{l=1}^{n-1} \binom{n}{l-i}(-1)^{i+l}(B_n R_n)_{l,j},$$

whereby

$$(B_n R_n)_{l,j} = \left(\sum_{k=1}^{n-1} \binom{2n-l-2}{n-l-k-1}\binom{n+j-k-1}{j-k}(-1)^{k+1}\right) + \binom{n+j-l-1}{j-l}.$$

After observing that

$$\begin{aligned}\sum_{l=1}^{n-1} \binom{n}{l-i}(-1)^{i+l}\binom{n+j-l-1}{j-l} &= (-1)^{i+j}\sum_l \binom{n}{l-i}\binom{-n}{j-l}\\ &= (-1)^{i+j}\binom{0}{j-i} = \delta_{i,j}\end{aligned}$$

it remains to show that

$$\sum_{l=1}^{n-1} \binom{n}{l-i}(-1)^{i+l}\sum_{k=1}^{n-1} \binom{2n-l-2}{n-l-k-1}\binom{n+j-k-1}{j-k}(-1)^{k+1} = \binom{i+j}{j-1}(1-\delta_{i,n-1}).$$

The sum over l evaluates to

$$\begin{aligned}\sum_{l=1}^{n-1} \binom{n}{l-i}\binom{2n-l-2}{n-l-k-1}(-1)^{i+l} &= (-1)^{n+i+k+1}\sum_l \binom{n}{l-i}\binom{-n-k}{n-l-k-1}\\ &= (-1)^{n+i+k+1}\binom{-k}{n-i-k-1} = \binom{n-i-2}{n-i-k-1}.\end{aligned}$$

At this point one must avoid the trap $\binom{n-i-2}{n-i-k-1} = \binom{n-i-2}{k-1}$ set up by $i = n-1$, because in this case $n-i-2 < 0$, and $\binom{n-i-2}{n-i-k-1} = \binom{-1}{-k} = 0$ for all $k \geq 1$. For $1 \leq i \leq n-2$ the claim follows as

$$\begin{aligned}\sum_{k=1}^{n-1} \binom{n+j-k-1}{j-k}\binom{n-i-2}{k-1}(-1)^{k+1} &= (-1)^{j+1}\sum_k \binom{-n}{j-k}\binom{n-i-2}{k-1}\\ &= (-1)^{j+1}\binom{-i-2}{j-1} = \binom{i+j}{j-1}.\end{aligned}$$

□

Lemma A.2.2. *The numbers*

$$X_{n,i} := \frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{n-1}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

satisfy

$$X_{n,i} = \sum_{j=1}^{n} \binom{2n-i-1}{n-i-j+1} (-1)^{j+1} X_{n,j}, \quad i = 1, \ldots, n, \tag{A.2.2}$$

$$X_{n,1} = \sum_{i=1}^{n-1} X_{n-1,i}, \quad n \geq 2. \tag{A.2.3}$$

Proof. After eliminating the common factors on both sides that only depend on n, it remains to show that

$$\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1} = \sum_{j=1}^{n} \binom{2n-i-1}{n-i-j+1}\binom{n+j-2}{n-1}\binom{2n-j-1}{n-1}(-1)^{j+1}. \tag{A.2.4}$$

In order to simplify the right-hand side let us first observe that

$$\binom{2n-i-1}{n-i-j+1}\binom{n+j-2}{n-1}\binom{2n-j-1}{n-1}\binom{n-1}{i-1}$$

is symmetric in i and j: The binomial identity $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$ implies

$$\binom{2n-i-1}{n-i-j+1}\binom{n+j-2}{n-1} = \binom{2n-i-1}{2n-i-j}\binom{2n-i-j}{n-i-j+1} = \binom{2n-i-1}{j-1}\binom{2n-i-j}{n-i-j+1}$$

and

$$\binom{2n-j-1}{n-1}\binom{n-1}{i-1} = \binom{2n-j-1}{i-1}\binom{2n-i-j}{n-i}.$$

Since both $\binom{2n-i-j}{n-i-j+1}$ and $\binom{2n-i-j}{n-i}$ are invariant under switching i and j, the claimed symmetry follows.

After multiplying both sides of (A.2.4) with $\binom{n-1}{i-1}$, applying the symmetry on the right-hand side and cancelling the two identical binomial coefficients on both sides, it only remains to show that

$$\binom{n-1}{i-1} = \sum_{j=1}^{n} \binom{2n-j-1}{n-i-j+1}\binom{n-1}{j-1}(-1)^{j+1}.$$

Indeed the right-hand side is by Chu–Vandermonde convolution equal to

$$(-1)^{n+i} \sum_{j} \binom{-n-i+1}{n-i-j+1}\binom{n-1}{j-1} = (-1)^{n+i}\binom{-i}{n-i} = \binom{n-1}{i-1}.$$

For the second identity observe that

$$\begin{aligned}\sum_{i=1}^{n-1}\binom{n+i-3}{n-2}\binom{2n-i-3}{n-2} &= \sum_{i=1}^{n-1}\binom{n+i-3}{i-1}\binom{2n-i-3}{n-i-1}\\ &= (-1)^n\sum_{i}\binom{-n+1}{i-1}\binom{-n+1}{n-i-1}\\ &= (-1)^n\binom{-2n+2}{n-2} = \binom{3n-5}{n-2}.\end{aligned}$$

The right-hand side of (A.2.3) is therefore $\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n-1+j)!}$, which is further equal to

$$\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n-1+j)!} = \frac{(2n-2)!(2n-1)!}{(3n-2)!(n-1)!}\prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!} = \frac{\binom{2n-2}{n-1}}{\binom{3n-2}{n-1}}\prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!} = X_{n,1}.$$

□

Lemma A.2.3. *The identity*

$$\begin{aligned}&\sum_{j=1}^{2n-1}\binom{n-i}{2n-i-j}\sum_{l=1}^{j}\binom{j-1}{l-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}(-1)^{i+l+j-1}\\ &\quad=\sum_{l=1}^{i}\binom{i-1}{l-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}(-1)^{n+i+l-1}.\end{aligned}$$

holds for all $i=1,\ldots,2n-1$, $n\geq 1$.

Proof. Using $\binom{n}{j}=\binom{n}{n-j}$, $\binom{a}{b}=\binom{b-a-1}{b}(-1)^b$ and Chu–Vandermonde convolution shows that the left-hand side is equal to

$$\begin{aligned}&\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}(-1)^{i+l-1}\sum_{j=l}^{2n-1}\binom{n-i}{2n-i-j}\binom{j-1}{j-l}(-1)^j\\ &\quad=\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}(-1)^{i-1}\sum_{j}\binom{n-i}{2n-i-j}\binom{-l}{j-l}\\ &\quad=\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}\binom{n-i-l}{2n-i-l}(-1)^{i-1}.\end{aligned}$$

Hence, one has to show that

$$\begin{aligned}&\left(\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}\binom{n-i-l}{2n-i-l}\right)_{i=1,\ldots,2n-1}\\ &\qquad=\left(\sum_{l=1}^{i}\binom{i-1}{l-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}(-1)^{n+l}\right)_{i=1,\ldots,2n-1}.\end{aligned}$$

This can be accomplished by multiplying the vectors on both sides with the invertible (lower triangular) matrix $T := \left(\binom{i-1}{j-1}(-1)^j\right)_{1\le i,j\le 2n-1}$ from the left. On the right-hand side this yields the vector

$$\begin{aligned}
&\left(\sum_{j=1}^{2n-1}\binom{i-1}{j-1}(-1)^j\sum_{l=1}^{j}\binom{j-1}{j-l}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}(-1)^{n+l}\right)_{i=1,\ldots,2n-1}\\
&\quad=\left(\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}(-1)^n\sum_j\binom{-l}{j-l}\binom{i-1}{i-j}\right)_{i=1,\ldots,2n-1}\\
&\quad=\left(\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}\binom{i-l-1}{i-l}(-1)^n\right)_{i=1,\ldots,2n-1}\\
&\quad=\left(\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}(-1)^n\right)_{i=1,\ldots,2n-1},
\end{aligned}$$

where the last equality is due to $\binom{j-1}{j} = \delta_{j,0}$, $j \in \mathbb{Z}$. And for the left-hand side one obtains the vector

$$\begin{aligned}
&\left(\sum_{j=1}^{2n-1}\binom{i-1}{j-1}(-1)^j\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}\binom{n-j-l}{2n-j-l}\right)_{i=1,\ldots,2n-1}\\
&\quad=\left(\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}(-1)^l\sum_j\binom{n-1}{2n-j-l}\binom{i-1}{j-1}\right)_{i=1,\ldots,2n-1}\\
&\quad=\left(\sum_{l=1}^{2n-1}\binom{n+l-2}{n-1}\binom{2n-l-1}{n-1}\binom{n+i-2}{2n-l-1}(-1)^l\right)_{i=1,\ldots,2n-1}.
\end{aligned}$$

Since $n+l \ge 2$ we have $\binom{n+l-2}{n-1} = \binom{n+l-2}{l-1}$, and $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$ implies $\binom{2n-l-1}{n-1}\binom{n+i-2}{2n-l-1} = \binom{i-1}{n-l}\binom{n+i-2}{n-1}$. The left-hand side is thus further equal to

$$\begin{aligned}
&\left(\binom{n+i-2}{n-1}\sum_{l=1}^{2n-1}\binom{n+l-2}{l-1}\binom{i-1}{n-l}(-1)^l\right)_{i=1,\ldots,2n-1}\\
&\quad=\left(-\binom{n+i-2}{n-1}\sum_l\binom{-n}{l-1}\binom{i-1}{n-l}\right)_{i=1,\ldots,2n-1}\\
&\quad=\left(-\binom{n+i-2}{n-1}\binom{i-n-1}{n-1}\right)_{i=1,\ldots,2n-1}\\
&\quad=\left(\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}(-1)^n\right)_{i=1,\ldots,2n-1}.
\end{aligned}$$

□

Lemma A.2.4.

$$\sum_{l=j}^{i} \binom{i+l}{i-l} \binom{2l-j-1}{l-j} \frac{(-1)^{l-j}}{l} = \frac{\binom{i}{j}}{j} \tag{A.2.5}$$

for $i, j \geq 1$.

Equation (5.7.3) then follows with the substitution $j \mapsto d_1 - 3$.

Proof. First note that Vandermonde convolution yields for $n, r \geq 0$ and $m \geq q \geq 0$

$$\begin{aligned}
\sum_{0 \leq k \leq r} \binom{r-k}{n} \binom{q+k}{m} &= \sum_{0 \leq k \leq r} \binom{r-k}{r-k-n} \binom{q+k}{q+k-m} \\
&= (-1)^{m+n+q+r} \sum_{0 \leq k \leq r} \binom{-n-1}{r-k-n} \binom{-m-1}{q+k-m} \\
&= (-1)^{m+n+q+r} \binom{-m-n-2}{q+r-m-n} \\
&= \binom{q+r+1}{q+r-m-n} = \binom{q+r+1}{m+n+1}.
\end{aligned}$$

Substituting $m \mapsto j$, $n \mapsto 2l-j-1$, $q \mapsto 0$ and $r \mapsto i+l-1$ gives

$$\binom{i+l}{i-l} = \binom{i+l}{2l} = \sum_{k=0}^{i+l-1} \binom{i+l-k-1}{2l-j-1} \binom{k}{j} = \sum_{k=j}^{i} \binom{i+l-k-1}{2l-j-1} \binom{k}{j}.$$

The left-hand side of (A.2.5) is therefore equal to

$$\sum_{k=j}^{i} \binom{k}{j} \sum_{l=j}^{i} \binom{i+l-k-1}{2l-j-1} \binom{2l-j-1}{l-j} \frac{(-1)^{l-j}}{l}.$$

Apply $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$ and Chu–Vandermonde convolution to see that the inner sum is equal to

$$\begin{aligned}
\sum_{l=j}^{i} \binom{i+l-k-1}{l-j} \binom{i+j-k-1}{l-1} \frac{(-1)^{l-j}}{l} &= \sum_{l=j}^{i} \binom{-i-j+k}{l-j} \binom{i+j-k}{l} \frac{1}{i+j-k} \\
&= \frac{1}{i+j-k} \sum_{l} \binom{-i-j+k}{l-j} \binom{i+j-k}{i+j-k-l} \\
&= \frac{1}{i+j-k} \binom{0}{i-k}.
\end{aligned}$$

So, the only remaining summand is $k = i$ and we indeed obtain the right-hand side of (A.2.5). $\square$

A.3 Asymptotics of $\mathbb{E}X^\lambda$ in the staircase shape

Lemma A.3.1.

$$\lim_{k\to\infty} \frac{\left(\binom{k+1}{2}+k\right)!!\left(\binom{k+1}{2}-k-1\right)!!}{\binom{k+1}{2}!} = e \approx 2.71828.$$

Proof. We show the assertion for $k = 4m$ (the cases $k = 4m+1$, $k = 4m+2$ and $k = 4m+3$ can be observed analogously). The three factorial expressions are then equal to

$$\left(\binom{k+1}{2}+k\right)!! = (2m(4m+3))!! = 2^{m(4m+3)}(m(4m+3))!,$$
$$\left(\binom{k+1}{2}-k-1\right)!! = (2m(4m-1)-1)!! = \frac{(2m(4m-1))!}{2^{m(4m-1)}\,(m(4m-1))!},$$
$$\binom{k+1}{2}! = (2m(4m+1))!.$$

By Stirling's formula we know that $n! \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$. Since the arguments of the two factorials in the numerator and the arguments of the two factorials in the denominator both sum up to $m(12m+1)$, and the respective products of the factorial arguments are both polynomials with leading term $32m^4$, it suffices to show that

$$\lim_{m\to\infty} \frac{2^{m(4m+3)}(m(4m+3))^{m(4m+3)}(2m(4m-1))^{2m(4m-1)}}{2^{m(4m-1)}(m(4m-1))^{m(4m-1)}(2m(4m+1))^{2m(4m+1)}} = e$$

The left-hand side is further equal to

$$\begin{aligned}\lim_{m\to\infty} \frac{(4m+3)^{m(4m+3)}(4m-1)^{m(4m-1)}}{(4m+1)^{2m(4m+1)}} &= \lim_{m\to\infty}\left(1+\frac{2}{4m+1}\right)^{m(4m+3)}\left(1-\frac{2}{4m+1}\right)^{m(4m-1)}\\ &= \lim_{m\to\infty}\left(1+\frac{2}{4m+1}\right)^{4m}\left(1-\frac{4}{(4m+1)^2}\right)^{m(4m-1)}.\end{aligned}$$

Take the logarithm and apply l'Hôpital's rule to see that

$$\lim_{n\to\infty}\left(1+\frac{1}{f(n)}\right)^{g(n)} = e^{\frac{\beta}{\alpha}}$$

for any polynomials f and g with $\deg f = \deg g \geq 1$ and leading coefficients $\mathrm{lc}[f] = \alpha$, $\mathrm{lc}[g] = \beta$. The claim now follows as

$$\lim_{m\to\infty}\left(1+\frac{2}{4m+1}\right)^{4m} = \exp(2),$$
$$\lim_{m\to\infty}\left(1-\frac{4}{(4m+1)^2}\right)^{m(4m-1)} = \exp(-1).$$

□

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Curriculum Vitae
Lukas Riegler

Education:

2005:	Matura with distinction St.Ursula, 1230 Vienna
2005 – 2010:	Studies of ”Technische Mathematik”, Vienna University of Technology
2010:	MSc (Dipl.Ing.) in Mathematics (”Mathematik in den Computerwissenschaften”), Vienna University of Technology, graduated with distinction
since 2010:	PhD Studies in Natural Sciences, Research Area: Mathematics, University of Vienna
since 2012:	Teacher accreditation programme (Mathematics, Computer Science), University of Vienna

Employment:

08/2008, 08/2009:	Internship, IBM Vienna
2008 – 2010:	Teaching Assistant, Exercises: ”Mathematik 1 für Informatik und Wirtschaftsinformatik”, ”Mathematik 2 für Informatik”, Vienna University of Technology
2010 – 2014:	Research Fellow: Combinatorics Group, University of Vienna

Talks and Posters:

2014:	Talk: ”Playing jeu de taquin on d-complete posets”, Seminar AG Diskrete Mathematik, Vienna University of Technology

2013:	Poster: "Generalized monotone triangles - an extended combinatorial reciprocity theorem", FPSAC '13, Paris, France
2012:	Talk: "Refined Enumerations of Vertically Symmetric Alternating Sign Matrices", Seminar AG Diskrete Mathematik, Vienna University of Technology
2012:	Talk: "Combinatorial Reciprocity for Monotone Triangles", FPSAC '12, Nagoya, Japan and SLC 68, Ottrott, France
2011:	Talk: "Decreasing Monotone Triangles and their connection to ordinary Monotone Triangles", Seminar AG Diskrete Mathematik, Vienna University of Technology

Research Publications and Preprints:

2014:	"Playing jeu de taquin on d-complete posets", joint work with Christoph Neumann, arXiv:1401.3619.
2014:	"Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity", joint work with Ilse Fischer, arXiv:1403.0535.
2013:	"Generalized Monotone Triangles: an extended combinatorial reciprocity theorem", DMTCS (2013), p.647-658.
2012:	"Combinatorial Reciprocity for Monotone Triangles", joint work with Ilse Fischer, DMTCS (2012), p.313-324.
2012:	"Generalized Monotone Triangles: an extended Combinatorial Reciprocity Theorem", arXiv:1207.4437.
2011:	"Combinatorial Reciprocity for Monotone Triangles", joint work with Ilse Fischer, J. Combin. Theory Ser. A 120, p.1372-1393.

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