DISSERTATION

Titel der Dissertation

Topics in enumeration and the theory of digraphs

Verfasser

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Contents

Abstract

We review the algorithm of Novelli, Pak and Stoyanowskii, introduce the distribution vector of the algorithm depending on its defining order, and derive, that some of its properties (namely the complexity, the exchange numbers, the drop function and the signed exit numbers) depend solely on the distribution vector rather than the defining order. We consider the algorithm in settings different from Young diagrams. For the double-tailed diamond and the case with an additional cell in row 1 we can describe the distribution vector directly and with the help of distribution matrices, which we introduce. For general insets we conjecture various properties of the distribution vectors and matrices and derive some of their structure.

We use a direct fibre product to construct highly arc transitive digraphs which are counterexamples for a conjecture by Cammeron, Praeger and Wormald. This product also generalises other ad hoc constructions of highly arc transitive digraphs.

Kurzzusammenfassung

Wir besprechen den Algorithmus von Novelli, Pak und Stoyanowskii, führen Verteilungsvektoren für den Algorithmus in Abhängigkeit seiner definierenden Ordnung ein, und leiten ab, dass einige seiner Eigenschaften (nämlich die Komplexität, die Vertauschungszahlen, die "Drop"-Funktion und die "signed exit"-Zahlen) nur vom Verteilungsvektor und nicht von der definierenden Ordnung abhängen. Wir betrachten den Algorithmus nicht nur auf Young Diagrammen sondern auch in anderen Situationen. Für den zweifach-geschwänzten Diamanten und den Fall mit einer zusätzlichen Zelle in Zeile 1 können wir den Verteilungsvektor sowohl direkt als auch unter Verwendung von Verteilungsmatritzen (die wir einführen) beschreiben. Für allgemeine Insets vermuten wir einige Eigenschaften der Verteilungsvektoren und -matritzen und leiten einige Aussagen über ihre Struktur ab.

Wir verwenden ein direktes Faserprodukt um hochgradig bogentransitive Digraphen zu konstrutieren, die Gegenbeispiele für eine Vermutung von Cammeron, Praeger und Wormald sind. Dieses Produkt verallgemeinert einige ad hoc Konstruktionen anderer hochgradig bogentransitiver Digarphen.

Acknowledgement

Preface

The work towards the present thesis started with the investigation of highly arc transitive (HAT) digraphs which continued considerations from my master thesis. A digraph is HAT if its automorphism group acts transitively on the set of its s-arcs for every $s \in \mathbb{N}$. I used a fibre product to construct HAT digraphs which are counterexamples to a conjecture by Cameron, Praeger and Wormald [**CPW93**]. This fibre product generalises other ad hoc constructions of highly arc transitive digraphs which I would not have expected to be related. In an independent work [**DMŠ11**] by DeVos, Mohar and Šámal isomorphic counterexamples were constructed using so called templates. Chapter 2 consists of my work on HAT digraphs done in [**Neu13**].

I was looking for digraphs with interesting properties (like flag-transitivity, which would have yielded further interesting examples of HAT digraphs), when a talk by Krattenthaler motivated me to consider the digraph with fillings of a certain partition as vertices and jeu de taquin steps as directed edges. This very digraph has none of the properties, that I was looking for, but it was the starting point for the considerations leading to the complexity theorems [**NS13**] and yielded a part of a proof of the Δ-Theorem [**NR14**]. In this way the considerations leading to the results and conjectures in Chapter 1 were triggered.

Chapter 1 treats distribution vectors of the jeu de taquin. Using the jeu de taquin under a certain order one can sort a filling of a diagram (given by a permutation) to a standard filling (i.e. a filling that grows along rows and columns). The distribution vector can be understood as a property of such an order, it counts for every standard filling, how many fillings are mapped to it. From this point of view, the famous bijection of Novelli, Pak and Stoyanowskii [**NPS97**] proves that all components of the distribution vector of the row-wise order of a Young diagram are equal. In Section 2 this algorithm is recapitulated.

The above mentioned talk by Krattenthaler contained a conjecture that the inherent sorting algorithm has the same complexity for both the row- and column-wise order. I worked on this conjecture in cooperation with Sulzgruber. We found, that the complexity and some other properties depend on the distribution vector and since the distribution

vectors of the row- and column-wise orders agree the conjecture follows. This considerations are available as preprint [**NS13**] and form Section 3.

Riegler considered the jeu de taquin on d-complete posets (as introduced by Proctor [**Pro99**]). The smallest case of a d-complete poset is the so called double-tailed diamond. The Δ-Theorem describes the distribution vectors of the double-tailed diamond. Currently we know three (very different) proofs of the Δ-Theorem – one by Riegler, one by myself and one in joint work between us. The latter can be found in [**NR14**] and we give it here in Section 4.

My proof of the Δ-Theorem uses distribution matrices which are introduced in Section 5. Distribution matrices describe how the distribution vector changes, if a top cell is added to the diagram. There is some knowledge about the structure of the distribution matrices but only little is known about their spectral properties. Indeed a strong relation between distribution vectors and the eigenvectors of distribution matrices is conjectured. There are also mysterious conjectures about the eigenvalues of distribution matrices – they appear to be integers, most of which are consecutive.

CHAPTER 1

Jeu de taquin

1. Introduction

The first part of this thesis is devoted to several aspects of the "Jeu de taquin". Schützenberger [**Sch72**] introduced this operation motivated by Robinson's correspondence [**Rob38**] and Knuth's generalisation [**Knu70**] of the Robinson-Schensted correspondence [**Sch61**]. Its name actually comes from the French name of what is known as the 15-Puzzle.

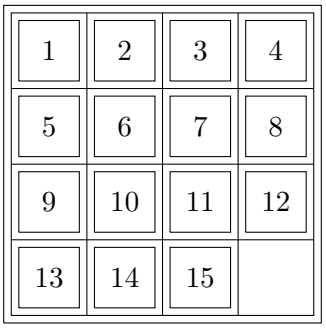


FIGURE 1. 15-Puzzle

In the present work we do not consider any of the original motivations of Schützenberger et. al. to investigate partitions and the jeu de taquin. We are going to introduce distribution vectors and will focus on surprising observations concerning them. First we notice that there are a couple of statistics (such as the complexity or the drop functions) of the jeu de taquin sorting algorithm that depend only on the resulting distribution vector rather than the order defining the algorithm. Then we observe that we can explain the distribution vectors in certain situations and recognise that they seem (conjecturally) to have a close relation to the eigenvectors of the distribution matrices which we are going to introduce as well.

The outline of Chapter 1 chronological. We start by recalling and introducing the required notions for the rest of Section 1. We continue by recapitulating the bijection of Novelli, Pak and Stoyanovskii in Section 2. In Section 3 we present the complexity theorems (joint work with Sulzgruber). In Section 4 we present a proof of the Δ-Theorem

(joint work with Riegler). We start Section 5 by motivating distribution matrices, and using them to reprove the Δ-Theorem. Moreover, we present various conjectures concerning the spectrum of distribution matrices and prove some of them for some special cases. We close the section and chapter with results on the structure of distribution matrices and distribution vectors.

1.1. Partitions, diagrams and standard fillings. We begin with a couple of definitions to introduce the objects which we will face throughout the entire first chapter.

DEFINITION 1.1 (Partition). *Let $n \in \mathbb{N}$. A sequence $\lambda := (\lambda_1, \dots, \lambda_m)$ with*

$$\sum_{i=1}^{m} \lambda_i = n \qquad \text{and}$$
$$\text{for all } i \in \{1, \dots, m-1\} : \ \lambda_i \geq \lambda_{i+1}$$

is called **partition** *of n and denoted by $\lambda \vdash n$. We use superscript notation, i.e. we write entries which appear more than once only once with their multiplicity as superscript, e.g. $(4,3,3,3,3,2,2,1) = (4,3^4,2^2,1)$.*

We identify a partition with its Young diagram as shown in Figure 2.

DEFINITION 1.2 (Young diagram). *Let $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ be a partition. A* **Young diagram** *consists of n cells arranged in m rows such that the i^{th} row contains λ_i cells. We address each cell with the vector $\binom{j}{k}$ of its coordinates, where $1 \leq j \leq m$ and $1 \leq k \leq \lambda_j$, i.e. j denotes the row and i the column. We say that λ is the* **shape** *of the Young diagram.*

In an abuse of notation we also call the Young diagram λ to indicate that it is of shape λ. Formally we understand a Young diagram as a set of cells $\binom{i}{j}$ distinguished by their coordinates. Note that we will denote functions of cells $\binom{i}{j}$ as functions of their coordinates i.e. $f(i,j) := f(\binom{i}{j})$.

The conjugate partition can be understood intuitively by reflecting the corresponding Young diagram diagonally. Figure 2 shows the Young diagrams of the partition $\lambda = (4,3^2,2^3,1^2)$ and its conjugate $\lambda' = (8,6,3,1)$.

There are various ways to formalise conjugate partitions. For completeness we give one way to calculate its entries.

DEFINITION 1.3 (Conjugate partition). *Let $\lambda \vdash n$ be a partition. The conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$ is given by the entries*

$$\lambda'_i := |\{\lambda_j \in \lambda \mid \lambda_j \geq i\}| .$$

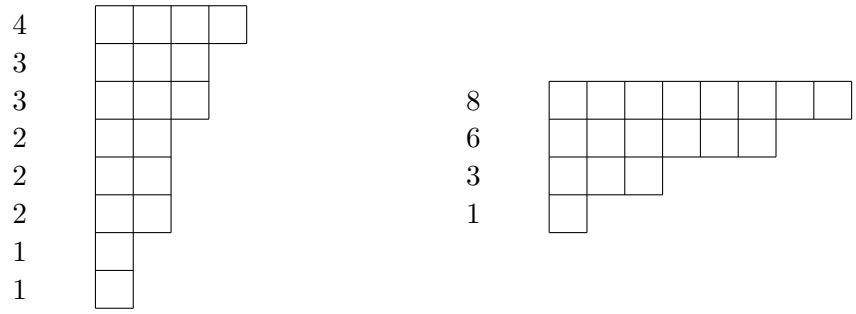


FIGURE 2. A partition and its conjugate

Note that for a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ we have $m = \lambda'_1$. Hence, we will always refer to the number of entries of λ by λ'_1. And vice verse we use λ_1 for the number of entries of λ' as we did in Definition 1.3.

The second kind of diagram that we will consider is the double-tailed diamond, which was introduced by Proctor [**Pro99**]. It consists of two neighbouring columns such that the lowest cell of the left column is in the same row as the second cell of the right column.

DEFINITION 1.4 (Double-tailed diamond). *Let $m \geq 1$ and $n \geq 2$ be integers. The* **double-tailed diamond** $(m; 1^n)$ *is the set of cells*

$$(m; 1^n) := \{\tbinom{3-m}{0}, \ldots, \tbinom{2}{0}\} \cup \{\tbinom{1}{1}, \ldots, \tbinom{n}{1}\}.$$

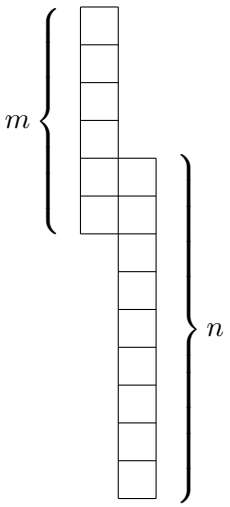


FIGURE 3. The double-tailed diamond $(m; 1^n)$

The double-tailed diamond is a special case of an inset. Insets were also introduced by Proctor [**Pro99**]. An inset can intuitively be understood as a double-tailed diamond

where the right column was replaced by a Young diagram. Insets are the third class of diagrams which we will consider.

DEFINITION 1.5 (Inset). *Let $m \geq 1$, $n \in \mathbb{N}$ and $\lambda \vdash n$. The* **inset** $(m;\lambda)$ *is the set of cells*

$$(m;\lambda) := \{\tbinom{3-m}{0}, \ldots, \tbinom{2}{0}\} \cup \lambda.$$

(Remember that we identified partitions and Young diagrams, hence the above union is a set of cells.)

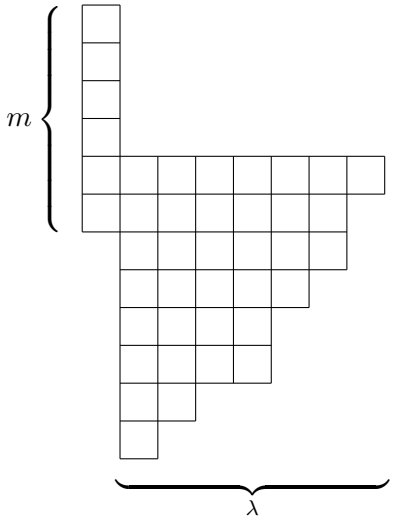


FIGURE 4. The inset $(6;7,6^2,5,4^2,2,1)$

If we know the partition λ we will write insets without the brackets of λ, i.e.

$$(m;\lambda_1,\lambda_2,\ldots,\lambda_{\lambda'_1}) := (m;(\lambda_1,\lambda_2,\ldots,\lambda_{\lambda'_1})).$$

Now, it becomes clear, why we chose to denote the double-tailed diamond by $(m;1^n)$ which is now a special case of our notation for insets.

For convenience we define the neck and body of an inset.

DEFINITION 1.6. *Given an inset $I = (m;\lambda)$. The first column without its bottom cell is the* **neck** *of I and denoted by* $\mathrm{NECK}(I)$*, i.e.*

$$\mathrm{NECK}(m;\lambda) := \{\tbinom{3-m}{0}, \ldots, \tbinom{1}{0}\}.$$

The rest of the inset is called the **body** *of I and denoted by* $\mathrm{BODY}(I)$*, i.e.*

$$\mathrm{BODY}(I) := I \setminus \mathrm{NECK}(I).$$

For any of these diagrams we can consider a skew version which cuts out a part at the upper left.

Definition 1.7 (Skew-diagram). *Let P and Q be diagrams of any of the above kinds such that $Q \subset P$. The* **skew-diagram**

$$P/Q := P \setminus Q$$

is the set-theoretical difference of the sets of cells.

For insets this definition makes only sense if the lengths of the leftmost columns of P and Q agree. In this case (and canonically for Young diagrams) we call Q an **upper left part** of P.

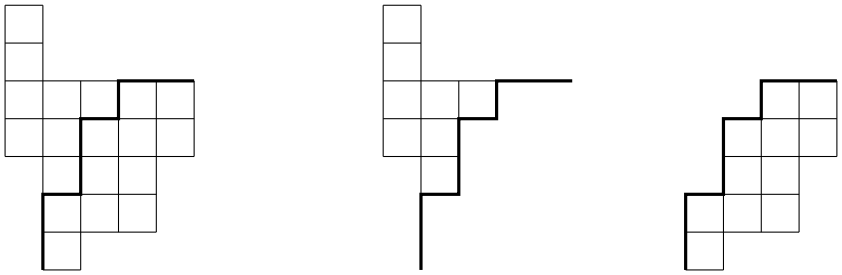


Figure 5. The insets $(4; 4^2, 3^2, 1)$, $(4; 2, 1^2)$ and their skew-inset

Remark 1.8. We will use skew diagrams of an inset $(1; \lambda)$ and a Young diagram (i) with one line, i.e. $(1; \lambda)/(i)$.

There is a canonical partial order $\prec$ on the cells which we get if we let the cells be covered by their lower and right neighbours, i.e. a cell is minimal if it is a top left corner and maximal if it is a bottom right corner. We call a cell that is minimal in this sense a **top cell** – this mixture of nomenclature appears to be a little counter intuitive, but neither calling top cells differently nor reversing the order are reasonable options.

Definition 1.9 (Tableau-order). *The covering relation given by*

$$\begin{array}{lll} \binom{i}{j} & \prec\cdot & \binom{i+1}{j} \qquad \text{and} \\ \binom{i}{j} & \prec\cdot & \binom{i}{j+1} \end{array}$$

for all $i, j \in \mathbb{Z}$ extends to a partial order $\prec$ on $\mathbb{Z}^2$. We call the restriction of $\prec$ to any subset of $\mathbb{Z}^2$ (especially to any of the above diagrams) the **tableau-order** *and denote it by $\prec$ as well.*

Sometimes we will wish to address certain parts of diagrams. Usually, we want these parts to be what is known as convex polyominos. Since we are not treating the theory of polyominos here, we will use the term cell set instead.

DEFINITION 1.10 (convex cell set). *Let P be some diagram and I a subset of its cells. I is called a* **convex cell set** *if for two cells of I which are either in the same row or same column also the cells in between are cells of I, i.e. if*

$$\begin{aligned} &\text{for cells } \tbinom{i}{k}, \tbinom{i}{l} \in I \text{ with } k < l \quad \text{also} \quad \tbinom{i}{k+1}, \tbinom{i}{k+2}, \ldots, \tbinom{i}{l-1} \in I \quad \text{and} \\ &\text{for cells } \tbinom{k}{j}, \tbinom{l}{j} \in I \text{ with } k < l \quad \text{also} \quad \tbinom{k+1}{j}, \tbinom{k+2}{j}, \ldots, \tbinom{l-1}{j} \in I. \end{aligned}$$

Note that all diagrams which we treat in this thesis are convex cell sets as well. Moreover, with the tableau-order they become posets. Unlike the usual drawings of posets which grow bottom-up, the tableau-order grows from top-left to bottom-right. We will need a notation for the cells covering or being covered by a fixed cell.

DEFINITION 1.11 (Neighbours). *Let P be one of the above diagrams and $x \in P$ be a cell. We define the set of* **inneighbours** *of x*

$$\mathrm{N}_P^-(x) := \{y \in P \mid y \prec\cdot\, x\},$$

the set of **outneighbours** *of x*

$$\mathrm{N}_P^+(x) := \{y \in P \mid x \prec\cdot\, y\}$$

and the set of **neighbours** *of x*

$$\mathrm{N}_P(x) := \mathrm{N}_P^-(x) \cup \mathrm{N}_P^+(x).$$

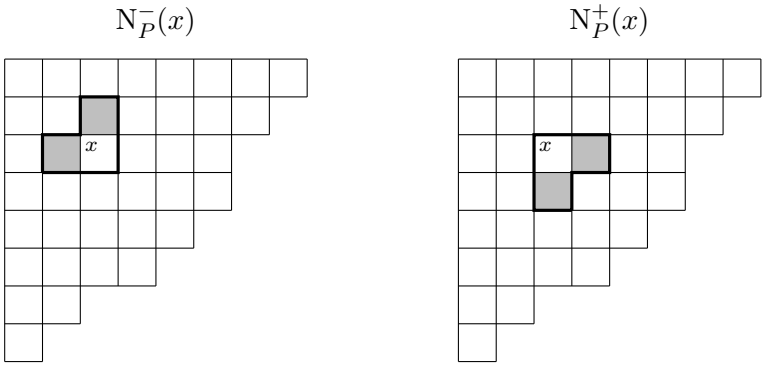


FIGURE 6. Inneighbours and outneighbours

We will consider the set of linear orders on the cells and its subset of linear extensions of $\prec$. These orders can be identified with bijective maps from the set of cells to a set of integers.

DEFINITION 1.12 (Filling). *Let P be one of the above diagrams. A* **filling** *F is a bijective map*

$$F : P \to \{N, N+1, \ldots, n-1, n\}.$$

If $P = \lambda \vdash n$ is a Young diagram, the lower bound is $N = 1$ and the filling is also called **tabloid**. *If P is a double-tailed diamond $(m; 1^n)$ or an inset $(m; \lambda)$ with $\lambda \vdash n$, the lower bound is $N = 1 - m$. We denote the set of fillings of P with $\mathrm{T}(P)$. For a cell $x \in P$ its image $F(x)$ is called the* **entry** *of x.*

If the cell $x = \binom{i}{j}$ is given by its coordinates, we write fillings as functions of the coordinates, i.e. $F(i, j) := F(x)$.

DEFINITION 1.13 (Standard filling). *Let P be one of the above diagrams. A* **standard filling** *S is a filling with the* **standard property**

$$\text{for all } x, y \in P : \ x \prec y \text{ implies } S(x) < S(y). \tag{1.1}$$

Standard fillings of Young diagrams $P = \lambda$ are also called **standard Young tableaux**. *We denote the set of standard fillings of P by $\mathrm{SYT}(P)$. We extend this notion to standard fillings of convex cell sets I meaning that the restriction $S|_I$ has the standard property for cells $x, y \in I$. For convenience we will often say that a filling* **is standard** *(respectively* **is standard on** *a convex cell set) rather than saying that it has the standard property (on a convex cell set).*

3	2	11	14	10	12	18	13
9	7	17	6	4	5		
1	16	15					
8							

1	2	4	6	10	12	13	18
3	5	7	11	14	17		
8	15	16					
9							

FIGURE 7. A filling (left) and a standard filling (right)

Note that the standard property can be described by the local property

$$\text{for all } \ x \in P : \ F(x) < \min F(\mathrm{N}_P^+(x)),$$

i.e. the entries grow along rows and columns. This can easily be seen from the fact that this local property imposes the desired inequality along the covering relations which define $\prec$.

The set of standard fillings is in an obvious one-to-one correspondence with the linear extensions of the tableau order.

DEFINITION 1.14 (S-order). *Given a diagram P and a standard filling $S \in \mathrm{SYT}(P)$. The* **S-order** *$\prec_S$ is given by*

$$x \prec_S y \text{ if and only if } S(x) < S(y) \tag{1.2}$$

for cells $x, y \in P$.

REMARK 1.15. We will be interested in two canonical special cases, namely the **row-wise** and the **column-wise order**. These can be intuitively understood as ordering the cells lexicographically by their indices – for the row-wise order first by the first coordinate, for the column-wise order first by the second coordinate. We will always denote the standard filling corresponding to the row-wise order by R and the row-wise order by $\prec_R$. Analogously we will always denote C and $\prec_C$ respectively in the column-wise case.

1	5	8	11	13	15	17	18
2	6	9	12	14	16		
3	7	10					
4							

1	2	3	4	5	6	7	8
9	10	11	12	13	14		
15	16	17					
18							

FIGURE 8. C (left) and R (right) for $\lambda = (8, 6, 3, 1)$

1.2. Sorting algorithms – the jeu de taquin. We are going to define sorting algorithms that transform fillings to standard fillings. Every standard filling defines such an algorithm. These algorithms process a sequence of steps, each of which exchanges the entries of two neighbouring cells.

DEFINITION 1.16 (Jeu de taquin step). *Given a diagram P and a cell $x \in P$. A* **jeu de taquin step** *is a map $\sigma_x : \mathrm{T}(P) \to \mathrm{T}(P)$ that exchanges the entry of x with the minimal entry from $\mathrm{N}_P^+(x)$. We denote jeu de taquin steps as transpositions, i.e. multiplicative*

$$\sigma_x F := (F(x), \min F(\mathrm{N}_P^+(x)))F.$$

Since for a given filling F every cell $x \in P$ is uniquely identified by its entry $F(x)$ we may use the convention

$$\sigma_{F(x)} F := \sigma_x F.$$

Note that we are abusing notation here. σ_x is not a transposition but is a function that applies a transposition depending on the argument, i.e. σ_x without an argument does not define a transposition. An example of a jeu de taquin step is given in Figure 9.

If we want to use jeu de taquin steps to create a standard filling, it makes sense to perform it only, if it exchanges $F(x)$ with a smaller entry.

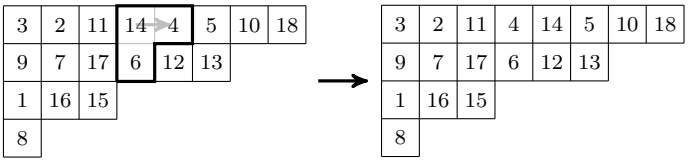


FIGURE 9. A jeu de taquin step $\sigma_{14} = \sigma_{\binom{1}{4}}$

DEFINITION 1.17. *Given a diagram P, a filling $F \in \mathrm{T}(P)$ and an entry b. Let x be the cell containing b, i.e. $F(x) = b$. We say that b is* **in order in** *F if*

$$b < \min F(\mathrm{N}_P^+(x)).$$

As jeu de taquin we understand the procedure of taking an entry and performing jeu de taquin steps with it until it is in order (which must happen since the diagram has a bottom right border).

DEFINITION 1.18 (Jeu de taquin). *Given a diagram P, a filling $F \in \mathrm{T}(P)$ and an entry $b \in F(P)$. A* **jeu de taquin** *is a map $\mathrm{jdt}_b : \mathrm{T}(P) \to \mathrm{T}(P)$ that processes jeu de taquin steps σ_b until a filling is reached such that b is in order. We can understand the jeu de taquin as permutation of the entries, hence, we again denote it multiplicative*

$$\mathrm{jdt}_b F := \sigma_b^{r(b,F)} F,$$

where $r(b, F)$ is the minimal, non-negative integer such that b is in order in $\sigma_b^{r(b,F)} F$. For $F(x) = b$ we can again use the conventions

$$\begin{aligned} \mathrm{jdt}_x F &:= \mathrm{jdt}_b F \qquad \textit{and} \\ r(x, F) &:= r(b, F). \end{aligned}$$

Moreover, if $\mathrm{jdt}_b F(y) = b$ we say that b **dropped** *from x to y.*

Note that we used the fact that b uniquely defines x for given F already for the well-definedness of $r(b, F)$ and not only for the mentioned convention. Actually, $r(b, F)$ may as well be zero. Further note, that in performing a jeu de taquin the definition of $r(b, F)$ guarantees that we do not run into a situation where the transposition is not defined (because there are no outneighbours).

Consider a diagram P, a filling $F \in \mathrm{T}(P)$ and an upper left part Q of P such that the restriction $F|_{P/Q}$ of the filling to the skew diagram is standard. We choose a maximal cell x of Q (i.e. x is bottom right) and play the jeu de taquin with its entry $F(x)$. Because the jeu de taquin step exchanges an entry with the smaller entry of the outneighbours, the convex cell set, on which $\mathrm{jdt}_x F$ is standard, did grow by the cell x, i.e. $\mathrm{jdt}_x F|_{P/(Q\setminus\{x\})}$ is standard. This motivates the following definition.

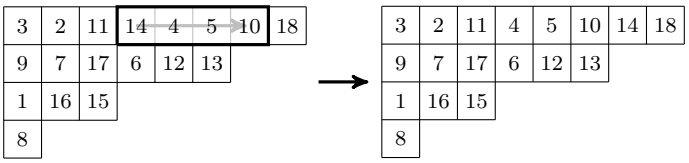


FIGURE 10. A jeu de taquin $\mathrm{jdt}_{14} = \mathrm{jdt}_{\binom{1}{4}}$

DEFINITION 1.19 ($\prec_S$-algorithm[1]). *Given a diagram P and a standard filling $S \in \mathrm{SYT}(P)$. The* **$\prec_S$-algorithm** JDT_S *is given by*

$$\mathrm{JDT}_S := \mathrm{jdt}_{x_{n-N+1}} \mathrm{jdt}_{x_{n-N}} \cdots \mathrm{jdt}_{x_1},$$

where $x_{n-N+1} \prec_S x_{n-N} \prec_S \cdots \prec_S x_1$ are the cells of the diagram.

REMARK 1.20. It is a bit unlucky that the cells are indexed reverse to the tableau-order. But since the $\prec_S$-algorithm processes the cells in this reverse order, and the standard property is irrevocable, we have to reverse the indices somewhere – and I believe that this is the least uncomfortable place to do so.

From the above it is clear, that every $\prec_S$-algorithm applied to any filling has a standard filling as output, i.e.

$$\mathrm{JDT}_S : \mathrm{T}(P) \to \mathrm{SYT}(P).$$

From here on we will address JDT_R as the **row-wise algorithm** and analogously JDT_C as the **column-wise algorithm** (remember Remark 1.15). Figure 11 gives an example of the application of the column-wise algorithm.

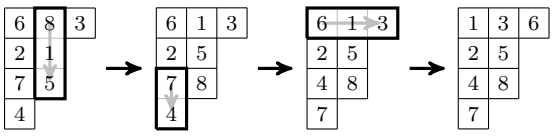


FIGURE 11. Application of the column-wise algorithm

1.3. Distribution vectors. We devote a section on its own to the definition of the distribution vector, because this notion is the central object of the investigations in Chapter 1.

DEFINITION 1.21 (Distribution vector). *Given a diagram P and an algorithm defining standard filling $S \in \mathrm{SYT}(P)$. Let $U \in \mathrm{SYT}(P)$ and let $\mathrm{z}_U(P, S)$ count the fillings*

[1] spoken: S-order-algorithm

$F \in \mathrm{T}(P)$ that are sorted to U by the $\prec_S$-algorithm, i.e.

$$\mathrm{z}_U(P,S) := |\{F \in \mathrm{T}(P) \mid \mathrm{JDT}_S F = U\}|\,.$$

The **distribution vector** $\mathbf{z}(P,S)$ *collects these multiplicities, i.e.*

$$\mathbf{z}(P,S) := (\mathrm{z}_U(P,S))_{U \in \mathrm{SYT}(P)}\,.$$

If P is an inset (or double-tailed diamond), all standard fillings must agree on the the entries of the first $m-1$ cells (since these are liniearly ordered). Hence, we can as well specify and index the distribution vector of an inset $P = (m;\lambda)$ with standard fillings $S, U \in \mathrm{SYT}(1;\lambda)$ of its body.

We need one more useful notation to describe a uniform distribution. For $k \in \mathbb{N}$ let $\mathbf{1}_k$ be the vector of dimension k consisting only of 1's, i.e.

$$\mathbf{1}_k := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},\ \dim(\mathbf{1}_k) = k.$$

If there is no question about k, it is unknown, or there is no need to specify it, we will write $\mathbf{1}$ rather than $\mathbf{1}_k$.

Definition 1.22 (Uniform distribution). *Given a diagram P and a standard filling (of the body) $S \in \mathrm{SYT}(P)$ (respectively $S \in \mathrm{SYT}(\mathrm{BODY}(P))$). We say that the $\prec_S$-algorithm is* **uniformly distributed** *if all components $\mathrm{z}_U(P,S)$ of the distribution vector agree, i.e. if there is a constant $c(P)$ such that*

$$\mathbf{z}(P,S) = c(P)\,\mathbf{1}.$$

Note that the **matrix of multiplicities** that arises if one understands S as index, is not called distribution matrix! If we speak of a distribution, we address a property of a specific algorithm, and hence we fix S. The matrix of multiplicities for Young diagrams was studied by Fischer [**Fis02**] and found to be symmetric.

2. The Novelli-Pak-Stoyanovskii bijection

2.1. The hook-length formula. Frame, Robinson and Thrall [**FRT54**] gave a surprisingly easy formula for the number of standard Young tableaux of a given shape λ. We define

$$f_\lambda := |\mathrm{SYT}(\lambda)|\,.$$

Definition 2.1 (hook-length). *Let $\lambda \vdash n$ be a Young diagram and $x \in \lambda$ be a cell. We call the set $\mathbf{a}_\lambda(x)$ (and its size $a_\lambda(x)$) of cells to the right of x the* **arm** *of x. The set $\mathbf{l}_\lambda(x)$ (and its size $l_\lambda(x)$) of cells below x is called the* **leg** *of x. The union of the*

singleton $\{x\}$ *with the arm and leg of* x *is the* **hook** $\mathbf{h}_\lambda(x)$ *of* x*. The size* $h_\lambda(x)$ *of the hook of* x *is the* **hook-length** *of* x*.*

If $x = \binom{i}{j}$ is given by its coordinates, we write the above as a functions of the coordinates and get

$$\begin{aligned} a_\lambda(i,j) := a_\lambda(\tbinom{i}{j}) &= \lambda_i - j, \\ l_\lambda(i,j) := l_\lambda(\tbinom{i}{j}) &= \lambda'_j - i \quad \text{and} \\ h_\lambda(i,j) := h_\lambda(\tbinom{i}{j}) &= a_\lambda(i,j) + l_\lambda(i,j) + 1. \end{aligned}$$

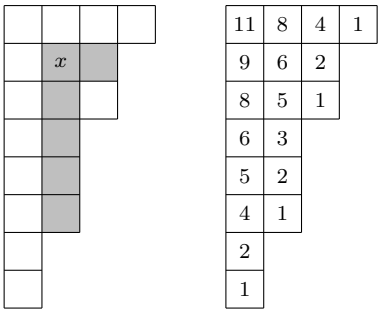


FIGURE 12. The hook of x and the hook-lengths

We formulate the famous hook-length formula as theorem.

THEOREM 2.2 (hook-length formula). *Let* $\lambda \vdash n$ *be a Young diagram, then the number of standard Young tableaux of shape* λ *is*

$$f_\lambda = \frac{n!}{\prod_{x\in\lambda} h_\lambda(x)}.$$

There are various proofs for this result. One outstanding proof was provided by Novelli, Pak and Stoyanovskii [**NPS97**] using the concept of the jeu de taquin. We recapitulate their proof in the following section.

2.2. The bijection. Novelli, Pak and Stoyanovskii introduced objects called hook tableaux to construct a bijective proof of the hook length formula.

DEFINITION 2.3 (Hook tableau). *Let* $\lambda \vdash n$ *define a Young diagram. A* **hook tableau** *is a map* $H : \lambda \to \mathbb{Z}$ *such that for every* $x \in \lambda$

$$H(x) \in \{-l_\lambda(x), -l_\lambda(x)+1, \ldots, a_\lambda(x)\}. \tag{2.1}$$

We denote the set of hook tableaux of shape λ *by* $\mathrm{H}(\lambda)$*.*

Obviously, the hook tableaux of shape λ are counted by

$$|\mathrm{H}(\lambda)| = \prod_{x\in\lambda} h_\lambda(x).$$

The proof gives a bijection between tabloids and pairs of a standard Young tableau and a hook tableau (all of the same shape λ). Since there are as many tabloids as permutations we get

$$n! = |\mathrm{T}(\lambda)| = |\mathrm{SYT}(\lambda) \times \mathrm{H}(\lambda)| = f_\lambda \prod_{x\in\lambda} h_\lambda(x)$$

and the hook-length formula follows.

Before we investigate the bijection, we need some useful nomenclature. Later on we will define the reverse procedure for the jeu de taquin for which the notions are similar enough to introduce them right here. The set of cells which have $F(x)$ as entry at some point during the (backward) jeu de taquin (b)jdt$_x$ is called **(reverse) path**. The (backward) jeu de taquin steps are canonically (depending on their direction) called **north/east/south/west steps**. If $y \in \mathbf{a}(x)$ we say x is **west of** y and y is **east of** x. Analogously, if $y \in \mathbf{l}(x)$ we say that x is **north of** y and y is **south of** x. We say that x is **weakly west/... of** y if it is west/... or $x = y$. A cell x is **(weakly) west/... of** the (reverse) path p if there is a cell $y \in p$ such that x is (weakly) west/... of y. If a path p has cells in rows where also the path q has cells, we say that p is **(weakly) west/east of** q if in these rows all cells of p are (weakly) west/east of some cell in q. Analogously, if a path p has cells in columns where also the path q has cells, we say that p is **(weakly) north/south of** q if in these columns all cells of p are (weakly) north/south of some cell in q. Note the asymmetry of these notions for paths.

Example 2.4. Figure 13 shows a black path p and the grey path q

$$\begin{aligned} p &= \{\tbinom{2}{2}, \tbinom{3}{2}, \tbinom{3}{3}, \tbinom{3}{4}, \tbinom{3}{5}, \tbinom{4}{5}, \tbinom{5}{5}\} \qquad \text{and} \\ q &= \{\tbinom{2}{2}, \tbinom{2}{3}, \tbinom{3}{3}, \tbinom{3}{4}, \tbinom{3}{5}, \tbinom{3}{6}, \tbinom{3}{7}, \tbinom{3}{8}\} \end{aligned}$$

and cells x, y and z (we will always print the names of cells in the upper left corner to distinguish them from entries). Note that two such paths cannot simultaneously appear in the NPS-algorithm since they start in the same cell and are different, nevertheless they are suitable examples for the above definitions.

The cell x is east, west, weakly south and weakly north of p but neither south nor north of it. x is south, west, weakly north and weakly east of q. Moreover, x is north of z.

The cell y is east of p and south of q. Moreover, y is east of z.

The cell z is south and west of p and south of q. It is also south of x and west of y but x and y are in none of these relations.

The path p is west and weakly south of q, but q is only weakly east and weakly north of p because the cell $\binom{2}{2}$ is not east of p.

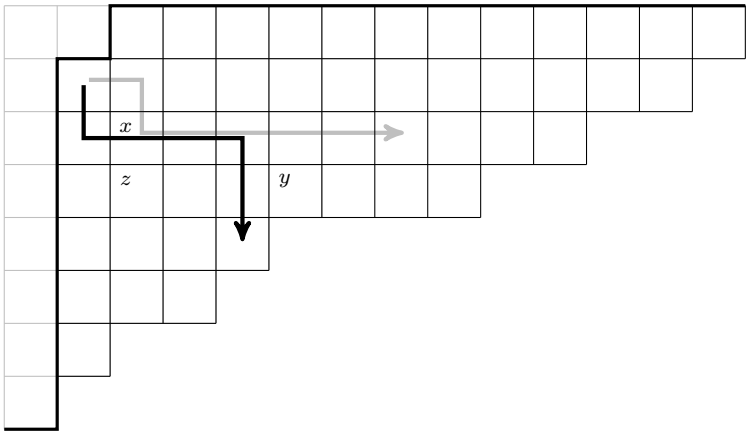


FIGURE 13. Some paths and cells

Prepared with these notions we are going to have a look at the actual algorithm that defines the bijection. Usually, the algorithm is described in a way such that it produces a sequence of pairs of a tabloid and a hook-tableau both of shape λ. We will right here take a step towards understanding the jeu de taquin from the point of view of distribution matrices by adding a cell in every step such that we get a sequence of pairs of skew tableaux. As usual we use the column-wise algorithm.

ALGORITHM 2.5 (Novelli-Pak-Stoyanovskii algorithm). We start with a tabloid $T \in \mathrm{T}(\lambda)$. Let $x_n \prec_C x_{n-1} \prec_C \cdots \prec_C x_1$ be the cells of λ in column-wise order and let

$$\mu_k := \{x_{k+1}, x_{k+2}, \ldots, x_n\}$$

for $0 \leq k \leq n$ be Young diagrams (where we mean $\mu_n = \emptyset$). We consider the skew-diagrams

$$\nu_k := \lambda/\mu_k.$$

We will construct a sequence of pairs (U_k, H_k) of an injection $U_k : \nu_k \to \{1, \ldots, n\}$ and a map $H_k : \nu_k \to \mathbb{Z}$ satisfying (2.1). For $0 < k < n$ we call U_k an **intermediate filling**[2] and H_k an **intermediate hook tableau** (we do not give a name to H_0 and U_0

[2]We defined fillings as bijections, nevertheless we use the term here for injections to avoid an unnecessary complicated nomenclature.

since they are empty). The pair $(U, H) := (U_n, H_n)$ is the output of the algorithm. For $1 \leq k \leq n$ we are going to define operations

$$\begin{aligned} \mathrm{add}_{x_k}^T &: U_{k-1} \mapsto U_k \qquad \text{and} \\ \mathrm{track}_{x_k}^T &: H_{k-1} \mapsto H_k. \end{aligned}$$

We define $\mathrm{add}_{x_k}^T$ as follows: First we add the cell x_k with the entry $T(x_k)$ to U_{k-1}, i.e. we build the map

$$u_k : \nu_k \to \{1, 2, \ldots, n\}, x \mapsto \begin{cases} T(x) & \text{if } x = x_k \\ U_{k-1}(x) & \text{otherwise.} \end{cases}$$

Then we play the jeu de taquin with the new cell, i.e.

$$U_k = \mathrm{jdt}_{x_k} u_k.$$

Figure 14 illustrates an application of $\mathrm{add}_{\binom{2}{2}}^T$.

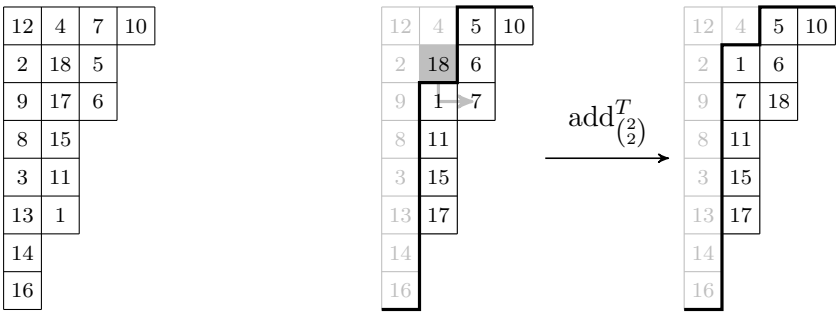


FIGURE 14. A tabloid T and the operator $\mathrm{add}_{\binom{2}{2}}^T$ in the column-wise algorithm

We need to store some tracking information in order to be able to reverse the procedure later on. To do so, we apply $\mathrm{track}_{x_k}^T : H_{k-1} \mapsto H_k$ as soon as $\mathrm{add}_{x_k}^T$ was performed. Let $x_k = \binom{i}{j}$ and $\binom{i'}{j'}$ be the cell where the entry $T(x_k)$ came to rest. We construct H_k from H_{k-1} by shifting the $i' - i$ northern cells of the westernmost column by one to the north, subtracting 1 from their entries and adding a cell with entry $j' - j$ at position $\binom{i'}{j}$, i.e.

$$H_k : \nu_k \to \mathbb{Z}, \binom{r}{s} \mapsto \begin{cases} H_{k-1}(r+1, s) - 1 & \text{if } i \leq r < i' \text{ and } s = j \\ j' - j & \text{if } r = i' \text{ and } s = j \\ H_{k-1}(r, s) & \text{otherwise.} \end{cases}$$

We illustrate this step in Figure 15 where the grey path illustrates the way taken by the new entry and the grey cells are shifted northwards.

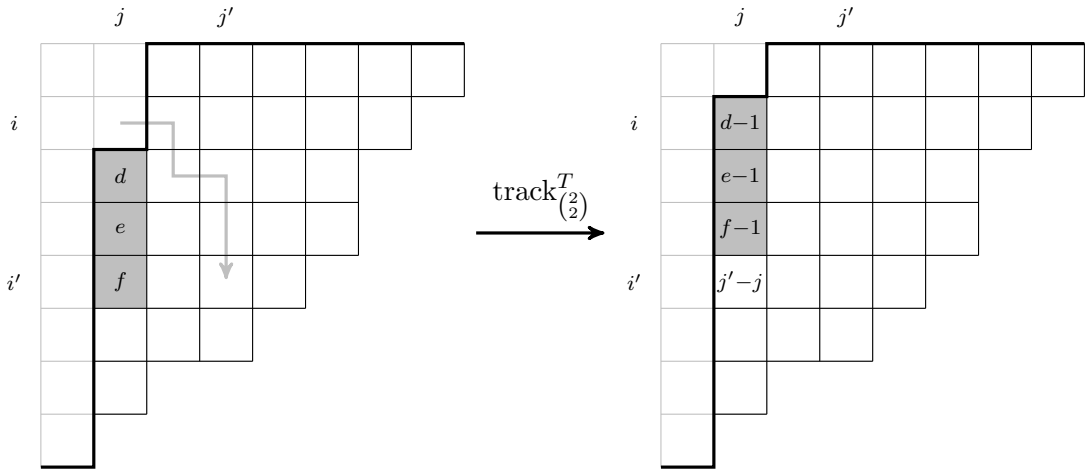


FIGURE 15. Modifying the intermediate hook tableau

The n^{th} cell that is added is $x_n = \binom{1}{1}$. After it was processed, the algorithm terminates with a pair (U, H) of a standard Young tableau U and a hook-tableau H both of shape λ. The algorithm induces a function that we denote by

$$\mathrm{NPS} : \mathrm{T}(\lambda) \to \mathrm{SYT}(\lambda) \times \mathrm{H}(\lambda), T \mapsto (U, H).$$

Novelli, Pak and Stoyanovskii [**NPS97**] showed that this map is a bijection. We will recapitulate the proof adapted to our notation roughly following Sagan [**Sag01**]. Before we do so, we give an example of all steps of an algorithm in Figure 16.

It is not difficult to see that all H_k have the property (2.1). Namely an entry $H_k(x)$ in the (intermediate) hook tableau is calculated by starting with a number $\alpha \in \{0, 1, \ldots, a\}$ where a is the arm of a cell somewhere in the leg of x (and hence $\alpha \leq a \leq a(x)$) and then subtracting β times 1, where $0 \leq \beta \leq l(x)$. It immediately follows that

$$-l(x) \leq \alpha - \beta \leq a(x).$$

The proof of bijectivity runs by giving the inverse

$$\mathrm{SPN} : \mathrm{SYT}(\lambda) \times \mathrm{H}(\lambda) \to \mathrm{T}(\lambda),$$

which reverses the algorithm step by step. First we define the procedure reversing the jeu de taquin.

DEFINITION 2.6 (backward jeu de taquin step). *Given a diagram P and a cell $x \in P$ and a filling $F \in \mathrm{T}(P)$. Let $b = F(x)$. A* **backward jeu de taquin step** *is a map $\theta_x : \mathrm{T}(P) \to \mathrm{T}(P)$ that exchanges b with the largest entry from its inneighbours. Like jeu*

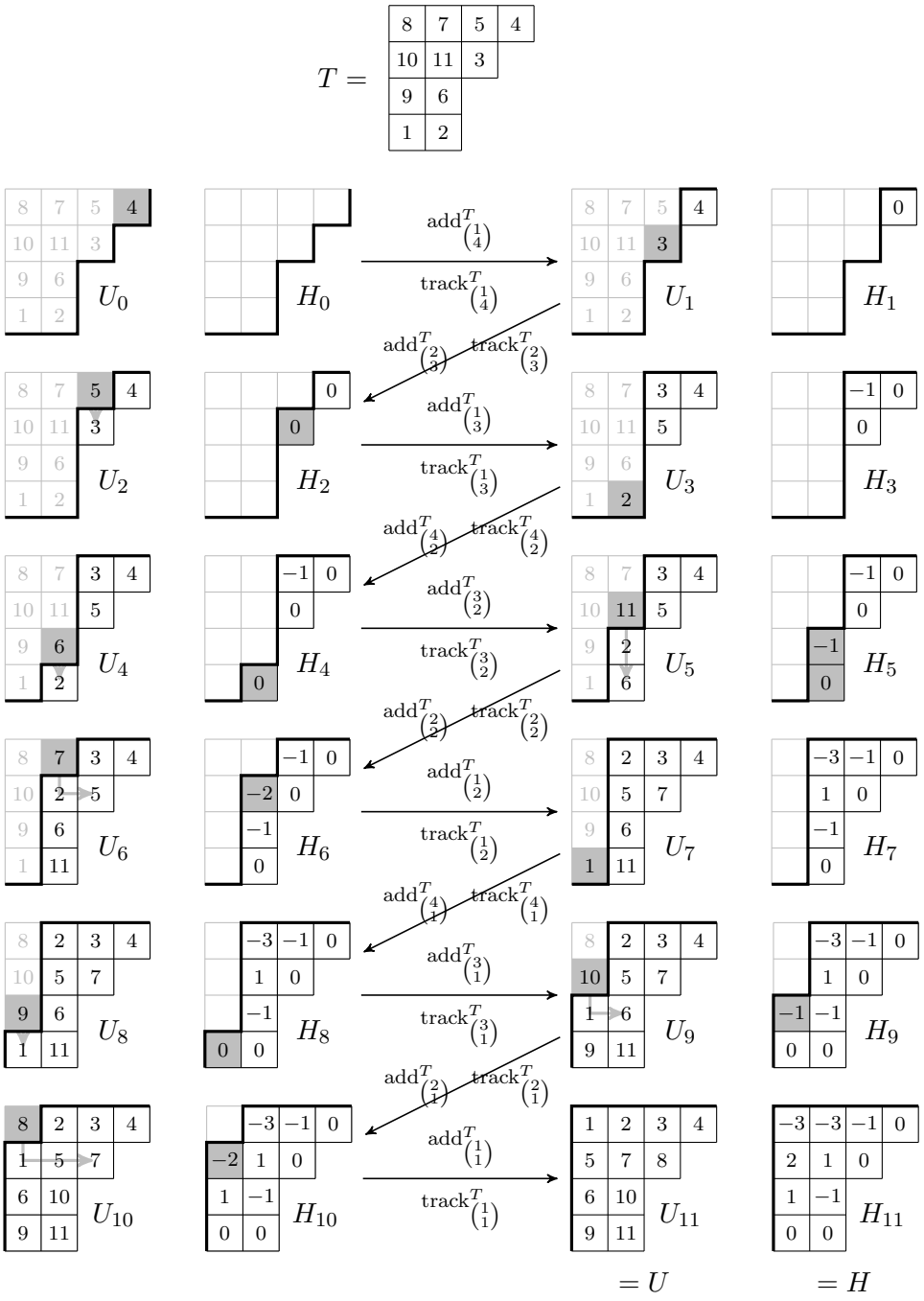


FIGURE 16. An evaluation of the column-wise algorithm

de taquin steps we write backward jeu de taquin steps as transpositions, i.e. multiplicative

$$\theta_x\, F := (b, \min F(\mathrm{N}_P^-(x)))\, F.$$

Since for a given filling F every cell $x \in P$ is uniquely identified by its entry b we may again use the convention

$$\theta_b\, F := \theta_x\, F.$$

We compose backward jeu de taquin steps to a backward slide.

DEFINITION 2.7 (backward jeu de taquin). *Given a diagram P and a cell $x \in P$ and a filling $F \in \mathrm{T}(P)$. Let again $b = F(x)$. The* **backward jeu de taquin** *is a map* $\mathrm{bjdt}_b : \mathrm{T}(P) \to \mathrm{T}(P)$ *that performs backward jeu de taquin steps θ_b until b reaches a top cell. We can understand the backward jeu de taquin as permutation of the entries hence we again denote it multiplicative*

$$\mathrm{bjdt}_b\, F := \theta_B^{s(b,F)}\, F,$$

where $s(b, F)$ is the minimal, non-negative integer such that the cell $(\theta_B^{s(b,F)}\ F)^{-1}(b)$ containing b is a top cell. Again, we can use the conventions

$$\begin{aligned} \mathrm{bjdt}_x\, F &:= \mathrm{bjdt}_b\, F \qquad \textit{and} \\ s(x, F) &:= s(b, F). \end{aligned}$$

Unlike the jeu de taquin the backward jeu de taquin does not move an entry to somewhere in the diagram, but it moves it to a top cell. The choice of the inneighbour ensures, that if the tableau was standard then after the slide the only cell violating the standard property is the one in which the entry b ends up. Moreover, the jeu de taquin and backward jeu de taquin are inverse procedures in the following sense.

LEMMA 2.8. *Let P be a (skew) diagram, $x \in P$ a $\prec$-minimal cell, $F \in \mathrm{T}(P)$ a filling such that the only cell violating the standerd property is x (i.e. $F|_{P\setminus\{x\}}$ is standard). Let x have the entry $b = F(x)$, then*

$$F = \mathrm{bjdt}_b\, \mathrm{jdt}_b\, F.$$

PROOF. To see this, it is enough to see that "inside a standard convex cell set" the backward jeu de taquin step reverses a jeu de taquin step. Consider the cells $\binom{i}{j}$, $\binom{i-1}{j+1}$ and $\binom{i}{j+1}$ with entries $b = F(i, j)$, $a = F(i-1, j+1)$ and $c = F(i, j+1)$. Suppose there is a jeu de taquin east step

$$\sigma_b\, F = (b, c)\, F.$$

I.e. $a < c < b$. Then in $(b, c)\, F$ the entry of $\binom{i}{j}$ will be c and hence larger than the entry of $\binom{i-1}{j+1}$ which is still a, hence

$$\theta_b\, (b, c)\, F = (b, c)(b, c)\, F = F.$$

Figure 17 shows this situation. Analogously, the same holds for a south step and of course also if not both neighbours are in the diagram. □

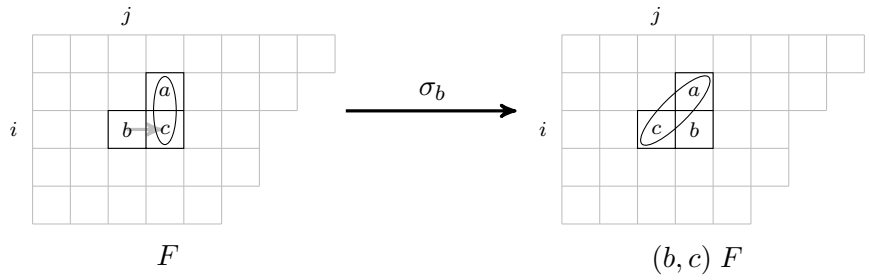


FIGURE 17. A jeu de taquin step shifts the location for the order relation

We can use the backward jeu de taquin to assign to every entry of a standard filling of any (skew) diagram a unique path to a northwestern corner. It is clear, that after a backward jeu de taquin step (a, b) of the entry a the rest of the reverse path of a is, what would have been the reverse path of b. Hence, the collection of backward jeux de taquin of a standard filling defines a binary forest on the set of cells with the $\prec$-minimal cells as roots (see Figure 18 (left)).

DEFINITION 2.9 (path-order). *Let P be a diagram and $S \in \mathrm{SYT}(P)$. Let $B_1, \ldots, B_s$ be the binary trees in the above mentioned forest with roots $\mathrm{rt}(B_k) = \binom{i_k}{j_k}$. For $x \in P$ we denote the tree containing x by $B_S(x)$. Moreover, if x is the root of $B_S(x)$ we call $B_S(x)$ the x-**tree**. We impose a left-to-right order $\prec^{l\text{-}t\text{-}r}$ on the binary trees, i.e.*

$$B_i \prec^{l\text{-}t\text{-}r} B_j \text{ if and only if } j_i < j_j.$$

On the vertices of the binary trees B_i we impose the usual in-order $\prec^{in}$, which is given as follows: consider the cell $x \in \mathcal{V}(B_i)$, all cells in the south subtree are $\prec^{in}$ smaller than x and all cells in the west subtree are $\prec^{in}$-greater than x (here $\mathcal{V}$ is the function returning the vertex set of a (directed) graph). For details on the in-order we refer to [**Knu97**].

The path-order $\prec_S^p$ is a linear order on the cells, that sorts the cells first by their binary tree and then according to the in-order, i.e.

$$x \prec_S^p y \text{ if and only if } \begin{cases} x \prec^{in} y & \text{if } B_S(x) = B_S(y) \\ B_S(x) \prec^{l\text{-}t\text{-}r} B_S(y) & \text{else.} \end{cases}$$

As example the 1-based indices of the cells under $\prec_S^p$ corresponding to the standard filling on the left side of the same figure can be found in Figure 18 (right).

We are now well prepared to reverse the NPS-algorithm.

ALGORITHM 2.10 (inverse Novelli-Pak-Stoyanovskii algorithm). We proceed by inversing every step of the NPS-algorithm. Hence, we have to explain the operations $\mathrm{slideout}_x^{(U,H)}$ and $\mathrm{backtrack}_x^{(U,H)}$ which prospectively reverse add_x^T and track_x^T.

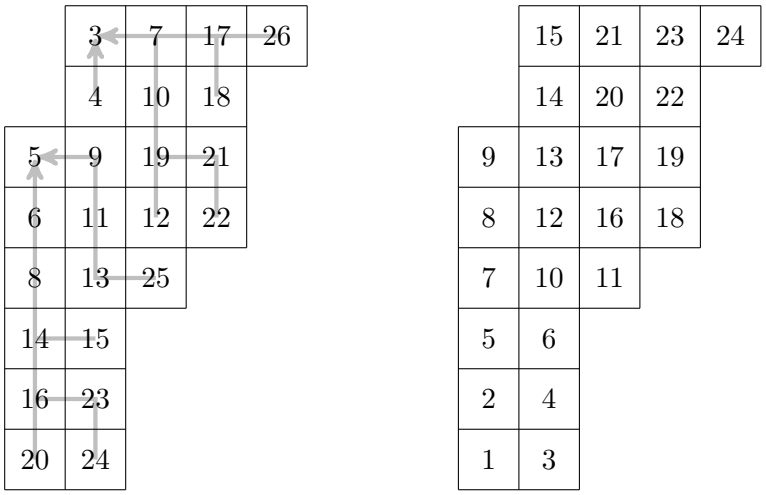


FIGURE 18. A standard filling S and the corresponding indices of $\prec_S^p$

If we know at some point of the NPS-algorithm the pair (U_k, H_k) and want to determine the pair (U_{k-1}, H_{k-1}) we have to figure out to which cell the entry $T(x_k)$ has dropped. Naïvely, these could be the cells described as follows: If $x_k = \binom{i}{j}$, we look for all cells $\binom{i_l}{j}$ with $H_{k+1}(i_l, j) \geq 0$, one of the cells which are $H_{k+1}(i_l, j)$ to the east from there must be the correct one. We collect them in the set of **candidate cells**

$$\mathcal{C}_{x_k} = \{\tbinom{i_l}{j_l} \mid H_{k+1}(i_l, j) \geq 0, j_l = j + H_{k+1}(i_l, j)\}.$$

From this set we choose the $\prec_{U_k}^p$-maximal cell $y = \binom{i'}{j'}$ and set

$$u_k = \mathrm{bjdt}_y\, U_k.$$

Finally, $\mathrm{slideout}_{x_k}^{(U,H)}$ deletes the cell x_k to get U_{k-1} and we record $T(x_k) = U_k(y)$.

It is easy to reverse $\mathrm{track}_{x_k}^T$ by some operation $\mathrm{backtrack}_{x_k}^{(U,H)}$ that deletes the cell $\binom{i'}{j}$ and shifts the cells $\binom{i}{j}, \ldots, \binom{i'-1}{j}$ of the intermediate hook tableau H_k one to the south and increases their entries by 1, as illustrated in Figure 19.

Again the grey path (with the white arrow-tip) indicates the backward-path and the grey cells are the ones shifted to the south with incremented entries.

The inverse Novelli-Pak-Stoyanovskii algorithm induces a map

$$\mathrm{SPN} : \mathrm{SYT}(\lambda) \times \mathrm{H}(\lambda) \to \mathrm{T}(\lambda).$$

We reverse our example from Figure 16 in Figure 20.

A priori it is unclear why this should work for three reasons. Using the above notation we formulate these problems:

(P1) the cell y could be no vertex of the x_k-tree,

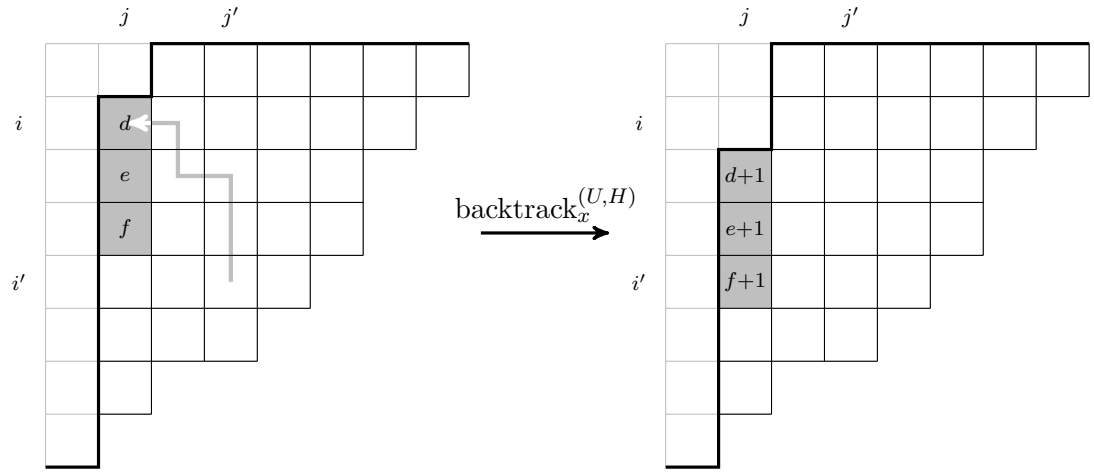


FIGURE 19. Reversing the modification of the intermediate hook tableau

(P2) the entries which are increased and moved southwards by backtrack$_{x_k}^{(U,H)}$ could violate (2.1) and

(P3) $T(x_k)$ could have been sorted to a cell in $\mathcal{C}_{x_k}$ other than y.

If we can overcome these problems it is clear by the definitions of the algorithms and Lemma 2.8 that $\mathrm{NPS}^{-1} = \mathrm{SPN}$ and the proof is complete.

In the current setting we face at most two binary trees. To be more precise, if we add the top cell of a column or a cell in the easternmost column, there is only one root and hence only one tree, otherwise there are two roots and hence two trees. Anyway, the tree containing the smallest entries under $\prec_{U_{x_k}}^{p}$ is the x_k-tree. We will later on resolve problem (P1) by proving that $\mathcal{C}_{x_k} \subset \mathcal{V}(B_{U_k}(x))$. But this would also resolve problem (P2) in the following way.

LEMMA 2.11. *Using the above notation and requiring that* $\mathcal{C}_{x_k} \subset \mathcal{V}(B_{U_k}(x_k))$ *we have that* backtrack$_{x_k}^{(U,H)}(H_k)$ *satisfies* (2.1).

PROOF. We use reverse induction on k. $H = H_n$ satisfies (2.1) and it is clear, that so does H_m when x_n is in the first row. We prove that backtrack$_{x_k}^{(U,H)}(H_k)$ satisfies 2.1 if H_k does.

Suppose for contradiction that backtrack$_{x_k}^{(U,H)}(H_k)$ violates (2.1). Then there must be a cell $\binom{i}{j}$ in the westernmost column of ν_{k-1} – actually it must be one of the cells that were shifted to the south – with

$$\mathrm{backtrack}_{x_k}^{(U,H)}(H_k)(i,j) \notin \{-l(i,j), \ldots, a(i,j)\}.$$

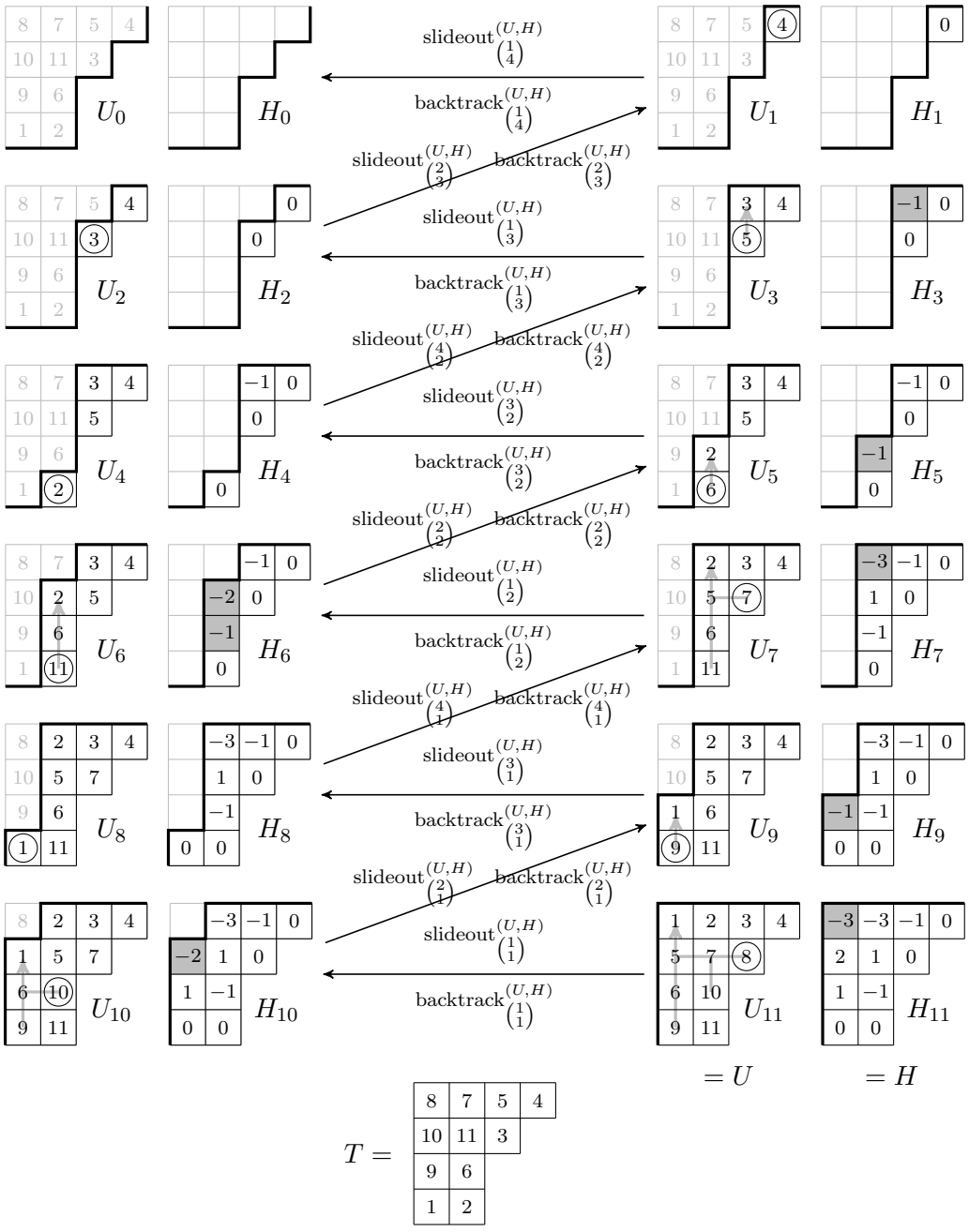


FIGURE 20. Reversing the column-wise algorithm

It is impossible that the lower bound is violated since

$$\begin{aligned}\text{backtrack}^{(U,H)}_{x_k}(H_k)(i,j) = H_k(i-1,j)+1 &\geq -l(i-1,j)+1 = \\ &= -(l(i,j)+1)+1 = -l(i,j).\end{aligned}$$

Hence, the upper bound must be violated, i.e.

$$\text{backtrack}_{x_k}^{(U,H)}(H_k)(i,j) = H_k(i-1,j) + 1 > a(i,j) \geq 0.$$

Hence, $H_k(i-1,j)$ is non-negative and, hence,

$$z = \binom{i-1}{j+H_k(i-1,j)} \in \mathcal{C}_{x_k}.$$

Since we required $\mathcal{C}_{x_k} \subset \mathcal{V}(B_{U_k}(x_k))$, there is a reverse path p from z to x_k (this is illustrated in Figure 21). And since $H_k(i-1,j) \geq a(i,j)$, every backward-path starting in a row southern of the i^{th} row must join p with a north step. Hence, for all the cells z' in these rows we have $z' \prec_{U_k}^{p} z$ and hence none of them is y. But then the cell $\binom{i-1}{j}$ is none of the cells shifted to the south – which it must have been. □

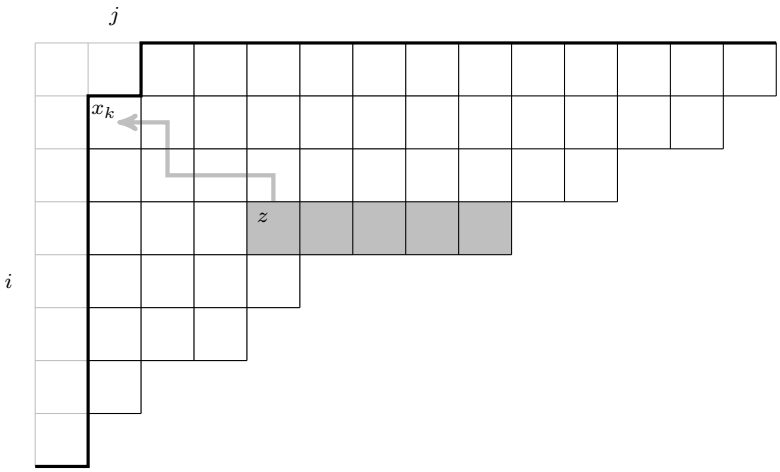


FIGURE 21. possible positions for the cell z

We will resolve problem (P1) by induction. To do so we need an argument connecting the paths of the jeux de taquin starting in x_k and x_{k+1}. Actually we have already met this argument in Figure 17, but we have another look at it on a larger scale. The upper part of Figure 22 uses grey arrows to emphasise the standard property and a black path p. The middle part of Figure 22 has additional arrows in black to emphasise the relations generated by the jeu de taquin along this path.

On the one hand this picture again illustrates that the backward jeu de taquin indeed reverses the jeu de taquin, because the reverse path must always choose the tip of a black arrow over its shaft. On the other hand the next jeu de taquin must choose the shafts of these arrows over their tips (if it runs into the situation to choose between them) and hence the path q must stay weakly north of p. We illustrate this in the lower part of

Figure 22 by displaying only the relations generated by p together with p in grey and a possible path q in a darker grey.

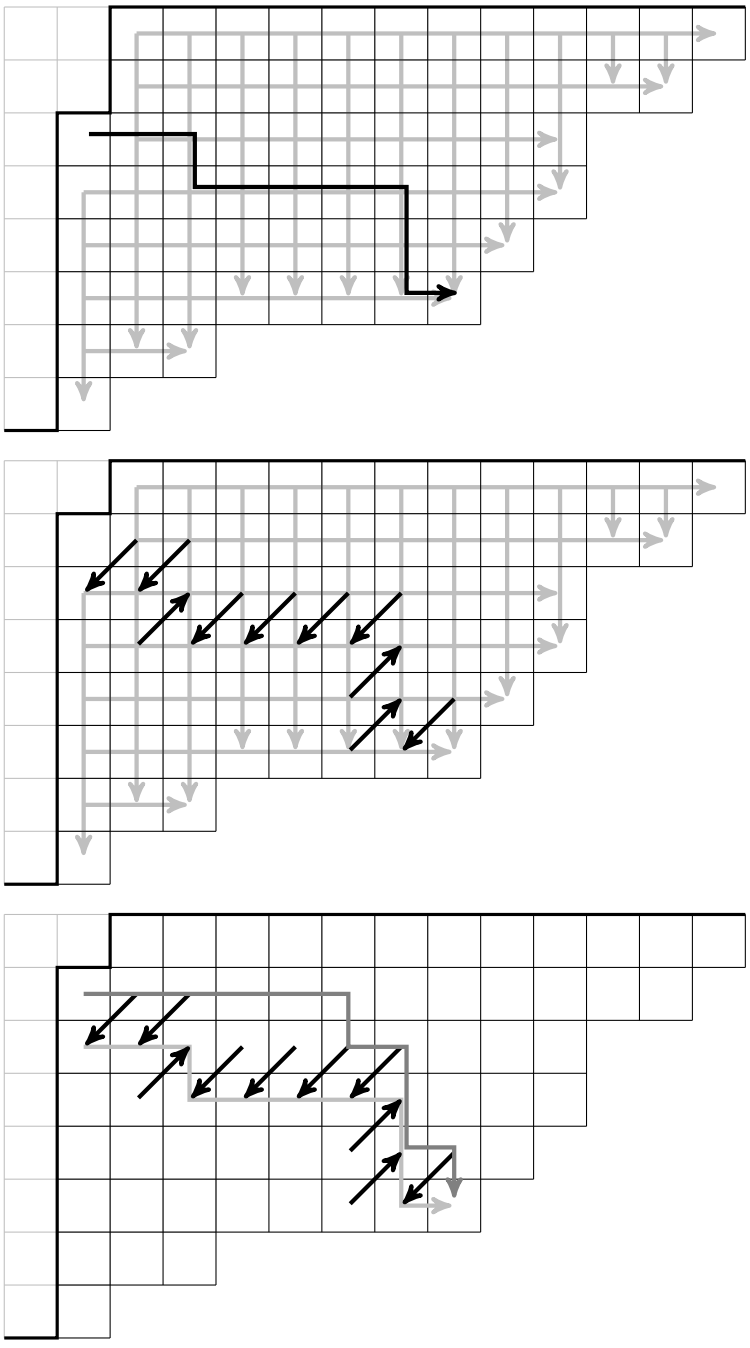


FIGURE 22. Additional relations between entries

We reverse this picture to conclude the following lemma.

LEMMA 2.12. *Let* $\bar{x} = \binom{i}{j}$ *and* $x = \binom{i+1}{j}$. *Suppose* $\bar{x}'$ *is the path-order-maximal cell in* $\mathcal{C}_{\bar{x}}$ *and* x' *is the path-order-maximal cell in* $\mathcal{C}_x$. *Further suppose that the reverse path* q *produced by* $\mathrm{bjdt}_{\bar{x}'}$ *ends in* $\bar{x}$ *and the reverse path* p *produced by* $\mathrm{bjdt}_{x'}$ *contains a cell* $y = \binom{k}{l}$ *south of* q. *Then* p *ends in* x.

PROOF. We show that p has a cell in column $l-1$ (if $l > j$) that is south of q. Iteratively it follows that p contains a cell in column j that is south of q and hence contains x. Suppose q takes a west step from $\bar{y} = \binom{k'}{l}$ to $\bar{z} = \binom{k'}{l-1}$, then we know that after this step the entry of $\bar{y}$ will be smaller than the entry of $z' = \binom{k'+1}{l-1}$ and hence p must take a west step from $y' = \binom{k'+1}{l}$ to z' unless it did so somewhere in the leg of y'. Either way p contains a cell to the south of $\bar{z}$. □

Figure 23 illustrates the argument from the proof of Lemma 2.12. The upper part shows a path q and the relations generated by its slides. The lower part shows a path p, the relations that must have been there before the sliding procedure and again the relations which were generated by q. The emphasised arrows illustrate how a west step of q forces a west step of p in the row below. That the cells x' and $\bar{x}'$ agree is coincidence.

We can now use Lemma 2.12 to resolve problem (P1) as promised before.

PROPOSITION 2.13. *All candidate cells in* $\mathcal{C}_x$ *are vertices of the* x*-tree.*

PROOF. As mentioned before the proposition is true, if x is in the first row. We induce southwards along a column. We can suppose by induction that for the addition of $\bar{x} = \binom{i}{j} = x_{k+1}$ all candidate cells in $\mathcal{C}_{\bar{x}}$ lie in the $\bar{x}$-tree. Let $x = \binom{i+1}{j} = x_k$ be the south neighbour of $\bar{x}$. We have to prove that all candidate cells in $\mathcal{C}_x$ lie in the x-tree. Let $\bar{y} = \binom{i'}{j'}$ be the path-order-maximal element in $\mathcal{C}_{\bar{x}}$. By induction we know that there is a reverse path q from $\bar{y}$ to $\bar{x}$. We will prove that every reverse path p from a candidate cell y in $\mathcal{C}_x$ contains a cell to the south of q and hence by Lemma 2.12 is contained in the x-tree.

We know that the candidate cells in $\mathcal{C}_{\bar{x}}$ other than $\bar{y}$ are $\prec^p_{U_{k+1}}$-smaller than $\bar{y}$ in the path-order. Hence, they are either a cell on $p \setminus \{\bar{y}\}$ such that $\bar{y}$ lies in an east subtree (i.e. cells after a west step of p) or they lie in a south subtree of such a cell. We call the cells which could be candidate cells **candidate-suspects**. By the above argument we find three groups of candidate-suspects for $\bar{x}$, namely the cells

(S1) in the rows from i to $i'-1$ west of q,
(S2.1) in the rows strictly below i' south of q and
(S2.2) in the rows strictly below i' but not south of q.

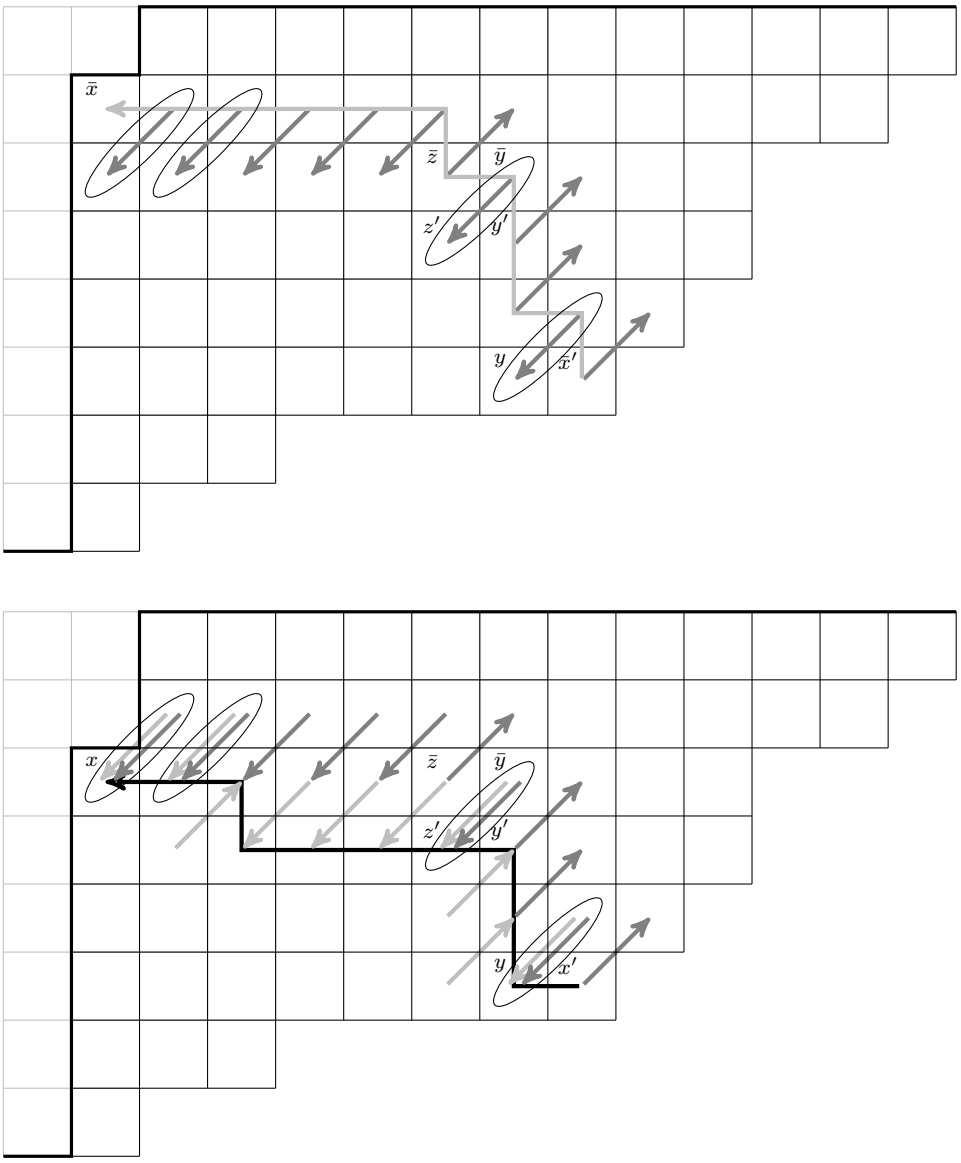


FIGURE 23. Additional relations between entries

Each of these properties defines a convex cell set. We do not call $\bar{y}$ a candidate-suspect, since we know that it is a candidate cell, but therefore we know that there is no other candidate cell in the row i'. We know that the cells described by (S1) and (S2.1) are in the $\bar{x}$-tree. The (S2.2)-cells need not to, but still they could lie in the $\bar{x}$-tree.

Considering $\text{backtrack}_{\bar{x}}^{(U,H)}$, we know that the candidate cells for $\bar{x}$ in the convex cell sets (S2.x) are also candidate cells in $\mathcal{C}_x$ and if we move a candidate cell in $\mathcal{C}_{\bar{x}}$ that lies in the convex cell set (S1) one to the south and one to the east, we get a candidate cell

in $\mathcal{C}_x$. Moreover, there could be -1's in the first row of the hook-tableau which grow to 0's and we could get additional candidate cells. The latter two cases are described by the property

(T1) that a cell lies in the rows from $i+1$ to i' south of q.

Hence, the candidate-suspects for $\mathcal{C}_x$ are described by (T1), (S2.1) and (S2.2). The candidate cells in the convex cell sets (T1) and (S2.1) are to the south of q and it remains to show that the reverse paths starting in the candidate cells in the convex cell set (S2.2) contain a cell south of q. Consider the convex cell set $\varpi = \{z \in \lambda \mid \bar{y} \prec z\}$ containing the convex cell set (S2.2) and the arm and leg of y. The backward jeu de taquin from y does not affect the entries in ϖ at all, hence, the reverse paths starting in the convex cell set (S2.2) do not depend on whether we want to process x or $\bar{x}$. These reverse paths have to leave ϖ and hence, they reach at some point the arm or the leg of $\bar{y}$. The paths reaching the arm cannot join q with a north step, hence, they are $\prec^p_{U_k}$-larger than $\bar{y}$ in the corresponding path-order and, hence, they contain no candidate cells. Thus all reverse paths starting in a candidate cell (for both x and $\bar{x}$) contain a cell in the leg of $\bar{y}$ which is south of q. □

Figure 24 shows a reverse path and the corresponding convex cell sets containing the candidate-suspects described by (T1), (S1), (S2.1) and (S2.2). Figure 25 shows the convex cell set ϖ with some possible parts of reverse paths.

After we resolved (P1) we are left with problem (P3). It is now obvious that NPS step by step reverses SPN, because in this direction it is clear which cell to use in every step. Unfortunately, so far we have neither that NPS is injective, nor that SPN is surjective – both would immediately finish the proof. Hence, we have not much of a choice but to prove that the path-order-maximal cell is indeed the correct candidate cell. In order to do so, we will first refine the argument from Figure 22 to the technical lemma below and then wonder where candidate cells come from.

LEMMA 2.14. *Let $T \in \mathrm{T}(\lambda)$ and $\binom{1}{j} = x_s \prec x_{s-1} \prec \cdots \prec x_r = \binom{\lambda'_j}{j}$ be the cells of the j^{th} column of λ. For $r \le t \le s$ let p_t be the path generated by $\mathrm{add}^T_{x_t}$. Fix $r \le \underline{k} \le s$ and consider the paths $p_{\underline{k}}$ and p_k for $\underline{k} < k \le s$.*

1. *If $p_{\underline{k}}$ contains an east step $e_{\underline{k}} = (\binom{i_{\underline{k}}}{j'}, \binom{i_{\underline{k}}}{j'+1})$ from column j' to column $j'+1$ and some p_k contains an east step $e_k = (\binom{i_k}{j'}, \binom{i_k}{j'+1})$ between the same columns, then e_k is north of $e_{\underline{k}}$ i.e. $i_k < i_{\underline{k}}$.*
2. *p_k enters column j' north of all cells of $p_{\underline{k}}$.*

PROOF. We argued before that these statements are true for $k = \underline{k}-1$ (see Figure 22). We need to extend the argument marginally to see that it holds for $k = \underline{k}+t$ where p_k

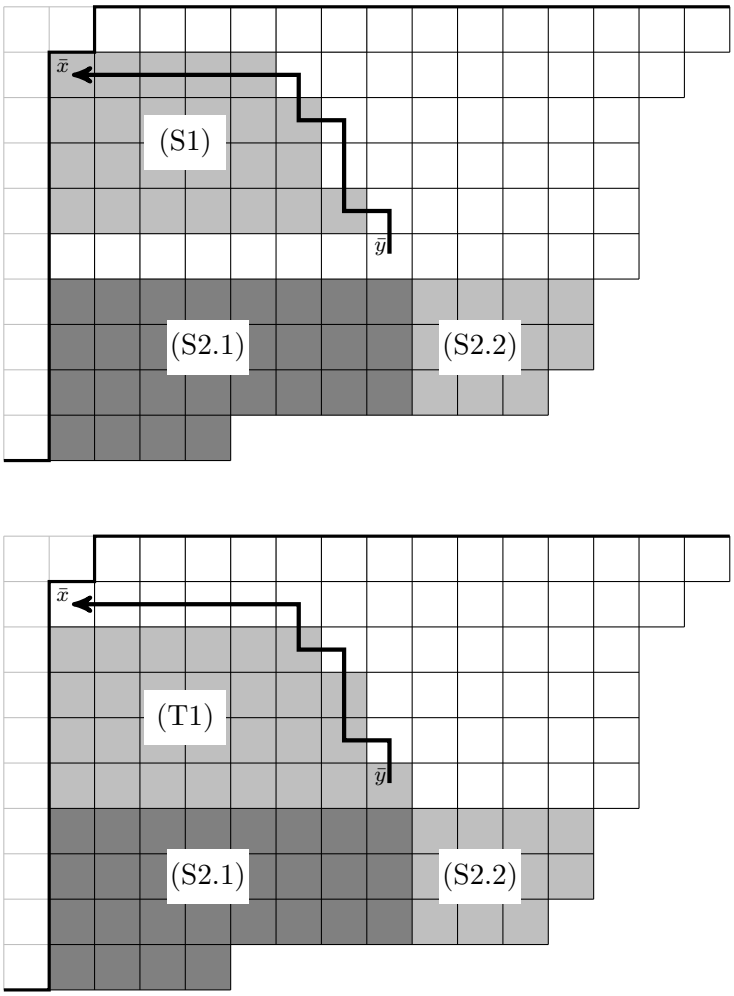


FIGURE 24. Candidate-suspects

is the first path after $p_{\underline{k}}$ (in the NPS-algorithm) that contains an east step between the columns in question.

First, we observe that (2) is fulfilled emptily if $p_{\underline{k}}$ has no cells in column j'. Secondly, we observe that it is true for $j' = j$. Thirdly, we observe that (2) holds for $j' > j$ if (1) holds for $j' - 1$. Hence, we have to show that if (2) holds for column j' so does (1).

Now suppose p_l is the first path after p_k that contains an east step from column j' to column $j' + 1$. We know that neither of the paths p_t for $k > t > \underline{k}$ has reached either of the cells $\binom{i_{\underline{k}}}{j'}$ or $\binom{i_{\underline{k}}-1}{j'+1}$ since, after the jeu de taquin producing the path $p_{\underline{k}}$ was processed the entry of $\binom{i_{\underline{k}}-1}{j'+1}$ was smaller than the entry of $\binom{i_{\underline{k}}}{j'}$, hence these paths would have had to reach $\binom{i_{\underline{k}}-1}{j'+1}$ first – but neither did since neither contains the necessary east step.

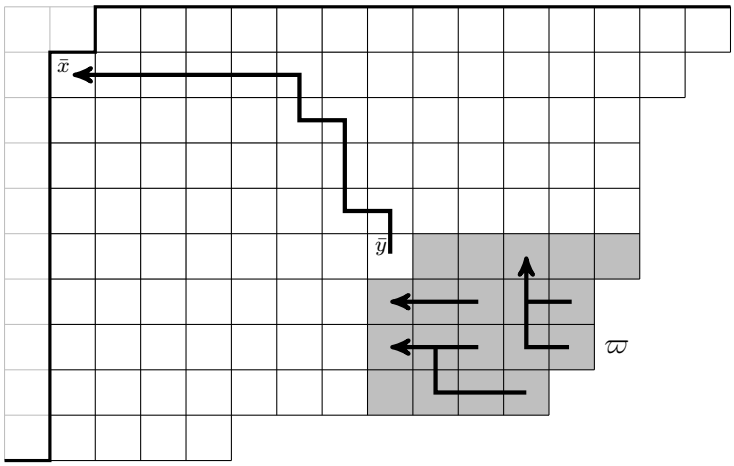


FIGURE 25. The convex cell set ϖ

Hence, the east step of p_k from column j' to column $j'+1$ must either be $(\binom{i_k-1}{j'}, \binom{i_k-1}{j'+1})$ or north of it and the induction is complete. □

We are going to work from south to north trough a column and consider the positions of candidate cells to resolve the final problem (P3).

PROPOSITION 2.15. *Let the cells $x_s, \dots, x_r$ be as in Lemma 2.14. Let $r \le k \le s$ and consider the application of the* NPS*-algorithm to the tabloid T of shape λ and let (U_k, H_k) be the intermediate pair after x_k was processed. Then the entry $T(x_k)$ dropped to the $\prec^p_{U_k}$-maximal candidate cell in $\mathcal{C}_{x_k}$.*

PROOF. Suppose the entry $T(x_k)$ drops from x_k to $x'_k = \binom{i'}{j'}$ along the path p_k. The statement obviously holds for $k = r$ since there will be a unique candidate cell. For $k > r$ there are three types of candidate cells in $\mathcal{C}_{x_k}$, namely

(C1) the cells $\binom{i}{j}$ with $i < i'$ such that there is a candidate cell $\binom{i+1}{j+1} \in \mathcal{C}_{x_{k-1}}$,
(C2) the cells $\binom{i}{j} \in \mathcal{C}_{k-1}$ with $i > i'$ and
(C3) the cell x'_k.

We claim that

(1) the cells described by (C1) are west and weakly south of p_k (and hence $\prec^p_{U_k}$-smaller than x'_k) and
(2) the reverse paths from cells described by (C2) contain a cell south of p_k (and hence also these cells are $\prec^p_{U_k}$-smaller than x'_k).

We first have a look at the (C1)-cells. The corresponding candidate cells in $\mathcal{C}_{x_{k-1}}$ are either west and weakly south of some path p_t with $r \le t < k$ by induction or they are some target cell of one of these paths which had been (C2)-cells since. Either way

the candidate cell in $\mathcal{C}_{x_k}$ (which is by one to the west and north of it) is west and weakly south of p_k by Lemma 2.14 (1).

We split the set of cells described by (C2) in

(C2.1) the cells $\binom{i}{j}$ with $j < j'$ and
(C2.2) the cells $\binom{i}{j}$ with $j \geq j'$.

Obviously, the (C2.1) cells are $\prec^p_{U_k}$-smaller than x'_k since they are south of p_k.

Let p_t with $t > k$ be the last path that reached the convex cell set (C2.2) so far. By induction the cell x'_t where it ended was the $\prec^p_{U_t}$-maximal cell among the cells in $\mathcal{C}_{x_t}$. By induction the (C2) cells of x'_t have a cell to the south of p_t in their reverse path. Also by induction the (C1) cells of x'_t are west and weakly south of p_t. Since p_t was the last path reaching the convex cell set (C2.2) of x'_k the cells and reverse paths there are still the same and x'_t is $\prec^p_{U_t}$- and $\prec^p_{U_k}$-maximal among the (C2.2) cells of x'_k. By Lemma 2.14 (2) the reverse path from x'_k has a cell in column j' that is strictly to the south of the cell where p_k entered this column. This finishes the proof. □

Figure 26 shows a possible development of paths, candidate cells with the known parts of their reverse paths, additional relations and the first column of the intermediate hook-tableaux while processing a column.

2.3. Notes on the algorithm. We used the column-wise order to define the NPS-algorithm. This raises the natural question if we could have used other orders as well. Obviously, we could have used the row-wise order, since the row-wise algorithm corresponds to the column-wise algorithm of the conjugated partition. We observe that one of the important ingredients of the NPS-algorithm is that the cells of a column are processed one after the other. Considering the operation track^T_x, we see that a column of the hook-tableau is calculated in one series of steps and not touched after the column was completed. Before the column was processed the convex cell set that was sorted so far was a Young diagram and afterwards it is as well. We conclude that we can conjugate the sorted convex cell set (and with it the hook-tableau) between treating two columns and the algorithm would still yield a bijection.

Remark 2.16. If we build up a standard Young tableau S by in each step choosing either the topmost non-full row or the leftmost non-full column and fill it with the least possible entries, then the corresponding order $\prec_S$ is suitable for the NPS-bijection. Figure 27 shows an example of a standard Young tableaux defining such an order.

Finally, we want to explain, why the above proof does not work for insets. The main problem is that a reverse path q containing the east step $(\binom{2}{1}, \binom{2}{0})$ cannot force a reverse path to its south to take an east step to the south of this east step. Hence, the conclusion

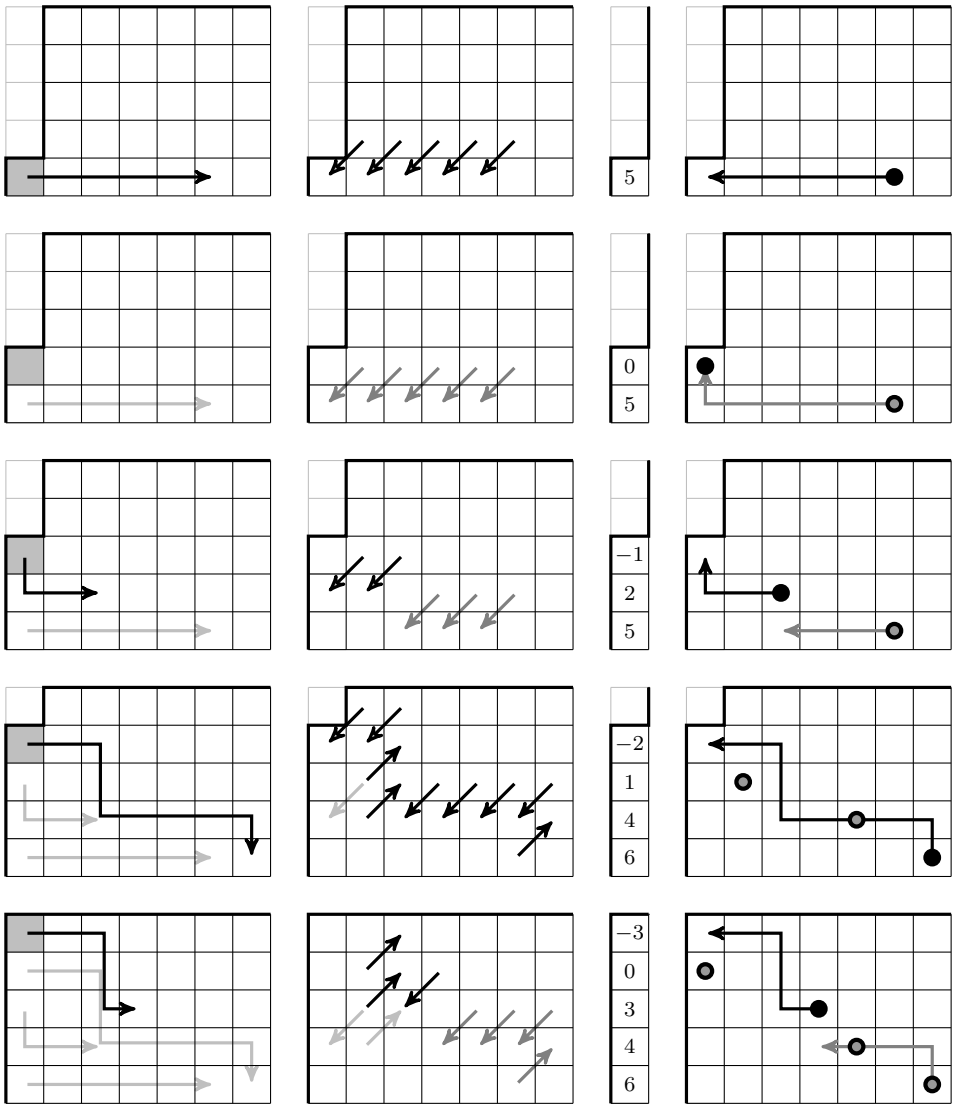


FIGURE 26. Processing a column

that the next reverse path p in the induction contains a cell in the first column that is south of q is wrong, and thus, the proofs of Propostion 2.15 and Lemma 2.12 fail.

Additionally, paths taking the west step $(\binom{1}{0}, \binom{1}{1})$ cannot force later paths to take a west step to the north of this west step. Hence, the proof of Lemma 2.14 does not work. This latter problem baulks the definition of the hook-tableau, but as we will see later, it is no problem for distribution vectors, i.e. it does not hinder the uniform distribution.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
17	26	35	44	45	46	47	48	49	50	51	52	53	54	55	
18	27	36	56	57	58	59	60	61	62	63	64	65	66		
19	28	37	67	74	75	76	77	78	79	80	81	82	83		
20	29	38	68	84	85	86	87	88	89	90	91	92	93		
21	30	39	69	94	95	96	97	98	99	100	101	102	103		
22	31	40	70	104	108	111	114	115	116	117	118	119			
23	32	41	71	105	109	112	120	121	122	123	124	125			
24	33	42	72	106	110	113	126	127	128	129	130				
25	34	43	73	107											

FIGURE 27. Other order for which the NPS-algorithm works

3. Complexity theorems

Considering the uniform distribution of the column-wise (or row-wise or...) algorithm one can understand JDT_S as a method to get a random standard Young tableau from a random permutation. This observation motivated Krattenthaler and Müller [**KM13**] to consider the runtime of the algorithm and they came up with the surprising conjecture that for any fixed λ the column-wise and row-wise algorithm agree on their average runtime.

Remember Definition 1.18 and Definition 1.19 (jeu de taquin and $\prec_S$-algorithm) and observe, that it makes no difference whether we interpret JDT_S as sorting algorithm operating on tabloids of shape λ or as insertion algorithm that adds a cell x_i before the application of jdt_{x_i}. Especially, recall that the exponents $r(x_i, F_i)$ denote the number of jeu de taquin steps a jeu de taquin executes.

Recall further that given a tabloid T of shape λ and a standard Young tableau S and letting $x_n \prec_S x_{n-1} \prec_S \cdots \prec_S x_1$ be the cells of λ the NPS-algorithm creates a sequence of intermediate fillings

$$U_i = \mathrm{add}^T_{x_i} \cdots \mathrm{add}^T_{x_1} \emptyset$$

and along with it the fillings u_i such that $U_i = \mathrm{jdt}_{x_i} u_i$.

DEFINITION 3.1 (Runtime). *Let T be a tabloid of shape $\lambda \vdash n$ and $S \in \mathrm{SYT}(\lambda)$. The* **runtime** *$r_S(T)$ of $\mathrm{JDT}_S(T)$ is the total number of jeu de taquin steps that appear while T is sorted with the $\prec_S$-algorithm, i.e.*

$$r_S(T) := \sum_{i=1}^{n} r(x_i, u_i).$$

We can now give a name to the average runtime.

DEFINITION 3.2 (Complexity). *With the setting of Definition 3.1 we define the* **complexity** *of the $\prec_S$-algorithm as*

$$C(S) := \frac{1}{n!} \sum_{T \in \mathrm{T}(\lambda)} r_S(T).$$

With this nomenclature the conjecture of Krattenthaler and Müller reads as follows.

CONJECTURE 3.3 (Krattenthaler, Müller). *For a given shape λ the row-wise and column-wise algorithm have the same complexity, i.e.*

$$C(C) = C(R).$$

In joint work with Sulzgruber [**NS13**] we could confirm Conjecture 3.3 by proving the following stronger result.

THEOREM 3.4 (Complexity Theorem). *Let $U, V \in \mathrm{SYT}(\lambda)$ then*

$$\mathbf{z}(\lambda, U) = \mathbf{z}(\lambda, V) \text{ implies that } C(U) = C(V).$$

The rest of Section 3 follows [**NS13**] closely.

In order to prove the Complexity Theorem we will need some more notation which we are going to introduce in Section 3.1. We will then present two proofs in Section 3.2 and Section 3.3. In Section 3.4 we give an example, state some notes and refer to further results.

3.1. Additional notation. The following definition can be understood as the second part of Definition 2.1.

DEFINITION 3.5 (height). *Let λ be a Young diagram and $x = \binom{i}{j} \in \lambda$ a cell. We define the* **coleg** *of x as*

$$l'(x) := i - 1,$$

the **coarm** *of x as*

$$a'(x) := j - 1$$

and the **height** *of x as*

$$h'(x) := a'(x) + l'(x).$$

We will need to consider the **height of an entry** a, hence, we introduce the convention

$$h'(a, T) := h'(T^{-1}(a)).$$

Note that we used the coordinates of the cell x in Definition 3.5. This is different from the definition of the hook where we used cardinalities of the arm and leg. This

difference becomes visible if one applies the same definition to skew shapes, where the height does not depend on the northwestern border. Hence, it makes no difference for the height of the entries during the application of the NPS-algorithm if we use diagrams or skew diagrams. This is also the reason why the coarm, coleg and height carry no subscript that indicates the shape.

DEFINITION 3.6 (Dropping zone). *Let P be some diagram and $x \in P$ a cell. The* **dropping zone** *of x is the convex cell set $\mathrm{J}(x)$ of cells where a path starting in x might end, i.e.*

$$\mathrm{J}(x) := \{y \in P \mid x \preceq y\}.$$

We know that U_{i-1} is standard and x_i is the only cell of $\mathrm{J}(x_i)$ that lies not in the domain of U_{i-1}, hence, u_i is standard on $J(x_i) \setminus \{x_i\}$. When jdt_{x_i} is applied, we say that the entry $b = T(x_i)$ **drops** into the dropping zone of x_i, i.e. it takes a (possibly empty) sequence of east and south steps, each of which increases its height by 1. It is clear that b cannot take any east or south steps later on, but it can take north and west steps when it lies on the path of a larger entry that drops later.

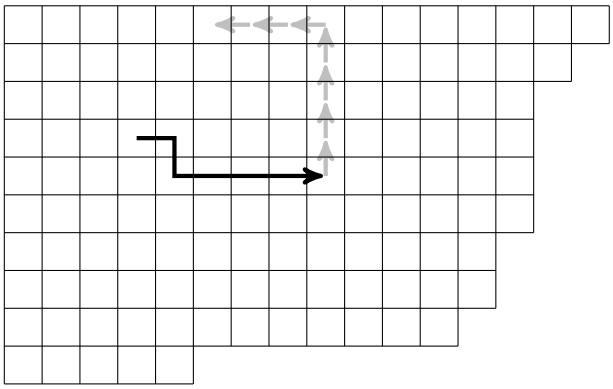


FIGURE 28. Possible movement of an entry during the application of JDT

In order to investigate the complexity, the sequence of intermediate fillings $(U_i)_i$ is not fine enough. We will need a sequence that reflects the status after every jeu de taquin step. For easier notation we will forget about skewing for the time being and record the entire tabloid.

DEFINITION 3.7 (Sequence of intermediate tabloids). *Given a tabloid T of shape λ and standard Young tableau S and let $x_n \prec_S x_{n-1} \prec_S \cdots \prec_S x_1$ be the cells of λ. Let $\sigma_S(1,T), \ldots, \sigma_S(r_S(T),T)$ be the jeu de taquin steps applied during the application of JDT_S to T in the order of their application. The tabloid*

$$T_i := \sigma_S(i,T) \cdots \sigma_S(1,T)\ T$$

is called **intermediate tabloid** *and the sequence*

$$\mathbf{T}_S := (T = T_0, T_1, \ldots, T_{r_S(T)-1}, T_{r_S(T)} = \mathrm{JDT}_S(T))$$

is the **sequence of intermediate tabloids**.

Note, that we added the tabloid to the arguments of the jeu de taquin step, hence, we can really identify the jeu de taquin step with the executed transposition.

With this nomenclature we formulate the above mentioned fact how the heights of entries are changed by a jeu de taquin step as an evident lemma.

LEMMA 3.8. *Suppose there is a jeu de taquin step $\sigma_S(i,T) = (a,b)$ with $a < b$, then*

$$\begin{aligned} h'(a,T_{i+1}) &= h'(a,T_i) - 1 \qquad \text{and} \\ h'(b,T_{i+1}) &= h'(b,T_i) + 1. \end{aligned}$$

We will also need a refinement for the runtime.

DEFINITION 3.9. *Let in the above setting be $x_k \in \lambda$. The* **intermediate runtime** *is the number $r_S(k,T)$ of jeu de taquin steps that occur during the application of* JDT_S *until the end of* jdt_{x_k}*, i.e.*

$$r_S(k,T) := \sum_{i=1}^{k} r(x_i, T_i).$$

As mentioned above we want to consider the heights of entries.

DEFINITION 3.10 (Total initial height). *Before* JDT *is applied, the starting point is some tabloid $T \in \mathrm{T}(\lambda)$, in which an entry b has some height $h'(b,T)$ to which we refer as* **initial height**. *Consequently we call*

$$\alpha_\lambda(b) := \sum_{T \in \mathrm{T}(\lambda)} h'(b,T)$$

the **total initial height**.

One of the keys to the Complexity Theorem is the following remark.

REMARK 3.11. Note that $\alpha_\lambda(b)$ does not depend on b, i.e.

$$\begin{aligned}\alpha_\lambda(b) &= \sum_{T\in \mathrm{T}(\lambda)} h'(b,T)\\ &= \sum_{T\in \mathrm{T}(\lambda)} \left(\sum_{x\in\lambda,\, T(x)=b} h'_\lambda(x) \right)\\ &= \sum_{x\in\lambda} \left(\sum_{T\in \mathrm{T}(\lambda),\, T(x)=b} h'_\lambda(x) \right)\\ &= (n-1)! \sum_{x\in\lambda} h'_\lambda(x).\end{aligned}$$

Therefore, we may write

$$\alpha_\lambda := \alpha_\lambda(b).$$

We want to mention some computational facts of minor importance.

REMARK 3.12. The total initial height could as well be calculated as a sum of hook lengths

$$\alpha_\lambda = (n-1)! \left(-n + \sum_{x\in\lambda} h_\lambda(x) \right)$$

or in terms of λ_i

$$\alpha_\lambda = (n-1)! \sum_{i\in\mathbb{N}} \left(\tbinom{\lambda_i}{2} + (i-1)\lambda_i \right).$$

Here $\binom{\lambda_i}{2}$ denotes of course the binomial coefficient rather than a cell.

3.2. Exchange numbers. The first proof of Theorem 3.4 relies on the two insights, that the total terminal height of an entry depends solely on the distribution vector, and that an entry is exchanged equally often with every larger entry. Before we start the proof we make these notions explicit.

DEFINITION 3.13 (Total terminal height). *Analogously to the total initial height we define the* **total terminal height** *of the entry b with the given order-defining $S \in \mathrm{SYT}(\lambda)$ as*

$$\omega_S(b) := \sum_{T\in \mathrm{T}(\lambda)} h'(b, \mathrm{JDT}_S(T)).$$

The following remark is another key step on the way to the Complexity Theorem.

REMARK 3.14. The total terminal height does not depend on the actual algorithm JDT_S but rather on the distribution vector $\mathbf{z}(\lambda, S)$, i.e.

$$\begin{aligned}\omega_S(b) &= \sum_{T\in \mathrm{T}(\lambda)} h'(b, \mathrm{JDT}_S(T)) \\ &= \sum_{U\in \mathrm{SYT}(\lambda)} \left(\sum_{T\in \mathrm{T}(\lambda),\, \mathrm{JDT}_S(T)=U} h'(b,U) \right) \\ &= \sum_{U\in \mathrm{SYT}(\lambda)} \mathrm{z}_U(\lambda, S) h'(b,U).\end{aligned}$$

Now, we turn our attention to the exchange of entries. We observe that during the application of JDT_S to the tabloid T the transposition (a, b) may occur at most once. Under this light the functions in the following definition not just decide whether a transposition occurs but also counts them.

DEFINITION 3.15 (exchange functions). *Let $T \in \mathrm{T}(\lambda)$ be a tabloid, $x, y \in \lambda$ be cells, a, b be entries and $S \in \mathrm{SYT}(\lambda)$ define the $\prec_S$-algorithm. The* **local exchange function** *decides whether during the application of JDT_S a transposition (a, b) appears when a is the entry of x and b is the entry of y, i.e.*

$$\varepsilon_S(a,b,x,y,T) := \begin{cases} 1 & \exists\, 1 \le i \le r_S(T) :\ T_{i-1}(x) = a, T_{i-1}(y) = b, \\ & \qquad\qquad T_i(x) = b \text{ and } T_i(y) = a \\ 0 & \text{else.} \end{cases}$$

Analogously, the **exchange function** *decides whether (a, b) occurs at all during the application of JDT_S, i.e.*

$$\varepsilon_S(a,b,T) := \begin{cases} 1 & \exists\, 1 \le i \le r_S(T) :\ \sigma_S(i,T) = (a,b) \\ 0 & \text{else.} \end{cases}$$

Note that the exchange function can be understood as a sum of local exchange functions, i.e.

$$\varepsilon_S(a,b,T) = \sum_{x,y\in\lambda} \varepsilon_S(a,b,x,y,T).$$

Moreover, they evidently satisfy

$$\begin{aligned}\varepsilon_S(a,b,x,y,T) &= \varepsilon_S(b,a,y,x,T) \quad \text{and} \\ \varepsilon_S(a,b,T) &= \varepsilon_S(b,a,T).\end{aligned}$$

DEFINITION 3.16 (exchange numbers). *With the setting of Definition 3.15 the* **local exchange numbers** *count the tabloids of shape λ such that a and b are exchanged at x*

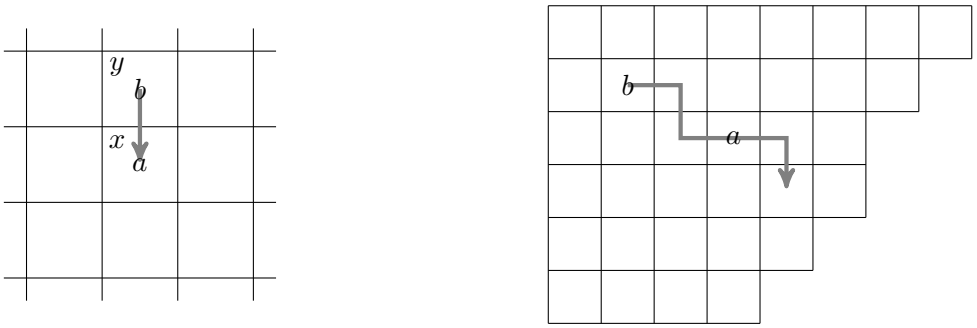


FIGURE 29. Contributions to the local exchange- and exchange function

and y, i.e.

$$\varepsilon_S(a,b,x,y) := \sum_{T\in \mathrm{T}(\lambda)} \varepsilon_S(a,b,x,y,T).$$

Analogously, the **exchange numbers** *count the tabloids of shape λ such that a and b are exchanged, i.e.*

$$\varepsilon_S(a,b) := \sum_{T\in \mathrm{T}(\lambda)} \varepsilon_S(a,b,T).$$

Of course, the exchange numbers are connected with the same relation as the local exchange function

$$\varepsilon_S(a,b) = \sum_{x,y\in\lambda} \varepsilon_S(a,b,x,y).$$

We already depicted the third key fact in Figure 28. Now we have the notation to formulate it.

LEMMA 3.17. *Let $\lambda \vdash n$ be a partiton and $S \in \mathrm{SYT}(\lambda)$ define the $\prec_S$-algorithm. Consider the entry b. We have*

$$\alpha_\lambda + \sum_{a=1}^{b-1} \varepsilon_S(a,b) - \sum_{c=b+1}^{n} \varepsilon_S(b,c) = \omega_S(b).$$

PROOF. The lemma says, that the total initial height plus the number of steps b takes away from the top left corner minus the number of steps b takes towards the top left corner is its total terminal height, i.e. this is a direct consequence of Lemma 3.8. □

The key insight on the way to the Complexity Theorem is that entries are exchanged equally often with larger entries.

PROPOSITION 3.18. *Let $S \in \mathrm{SYT}(\lambda)$ define JDT_S, $x, y \in \lambda$ be cells and a, b, c be entries with $a < b$ and $a < c$. Then*

$$\varepsilon_S(a,b,x,y) = \varepsilon_S(a,c,x,y) \quad \textit{and} \tag{3.1}$$

$$\varepsilon_S(a,b) = \varepsilon_S(a,c). \tag{3.2}$$

Hence, for $a < b$ we may denote the (local) exchange numbers by

$$\begin{aligned} \varepsilon_S(a,x,y) &:= \varepsilon_S(a,b,x,y) \quad \text{and} \\ \varepsilon_S(a) &:= \varepsilon_S(a,b) \quad \text{respectively.} \end{aligned}$$

PROOF. Obviously, (3.2) follows from (3.1) by summation over all pairs of cells $x, y \in \lambda$. It is sufficient to show (3.1) for $c = b + 1$.

Let $x_n \prec_S \cdots \prec_S x_1$ be the cells of λ, consider $T \in \mathrm{SYT}(\lambda)$ and let $1 \le i, j \le n$ be such that $T(x_i) = b$ and $T(x_j) = b+1$. Without loss of generality we may assume $i > j$. Let

$$\begin{aligned} \sigma &= (b, b+1) \qquad \text{and} \\ \mathrm{T}^* &= \sigma\, T. \end{aligned}$$

We will consider the sequences of intermediate tabloids

$$\begin{aligned} \mathbf{T_S} &= (T_0, T_1, \ldots, T_{r_S(T)}) \quad \text{and} \\ \mathbf{T_S^*} &= (T_0^*, T_1^*, \ldots, T_{r_S(T^*)}^*). \end{aligned}$$

We abbreviate

$$\begin{aligned} \sigma_k &:= \sigma_S(k, T) \quad \text{and} \\ \sigma_k^* &:= \sigma_S(k, T^*). \end{aligned}$$

By definition we have $T_0 = \sigma\, T_0^*$.

For $1 \le k \le r_S(j-1, T)$ we have $T_k = \sigma\, T_k^*$ and $\sigma_k = \sigma_k^*$ since none of these jeu de taquin steps involves or is influenced by the entries of x_i or x_j. Moreover, we have $r_S(k, T) = r_S(k, T^*)$.

Now, consider jdt_{x_j}. All entries different from b and $b+1$ are either less than both b and $b+1$ or greater than both b and $b+1$. Hence, the path of the entry of x_j does not depend on whether it is b or $b+1$ and we get $T_k = \sigma\, T_k^*$ for $r_S(j-1, T) < k \le r_S(j, T) = r_S(j, T^*)$.

For the same reason the paths of jdt_{x_l} for $j < l < i$ are the same in T and T^* and we get $T_k = \sigma\, T_k^*$ for $r_S(j, T) < k \le r_S(i-1, T) = r_S(i-1, T^*)$ as well.

The situation for jdt_{x_i} is a little different. Still, b follows exactly the path that $b+1$ would have taken for the above reasons, but if and only if the final step of the path of

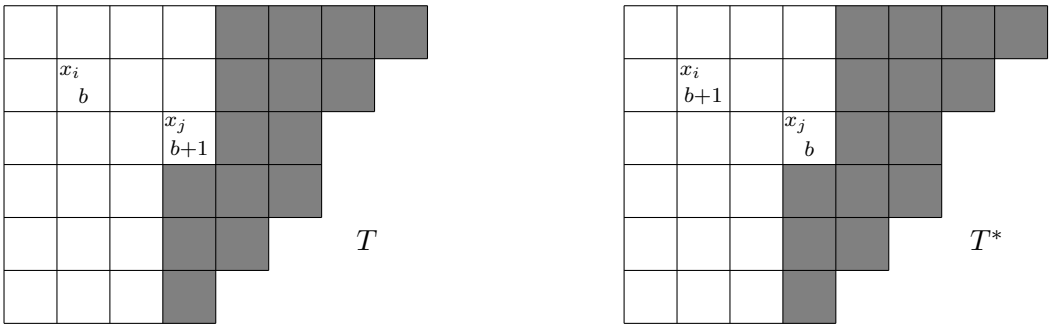


FIGURE 30. Unaffected jeu de taquin

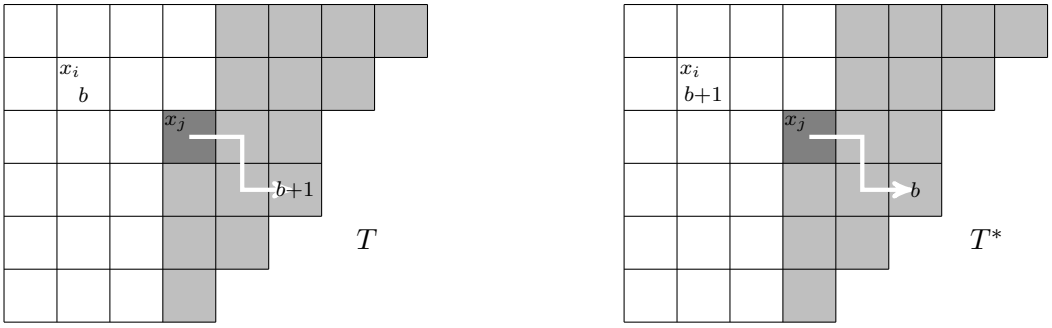


FIGURE 31. Unaffected jeux de taquin path

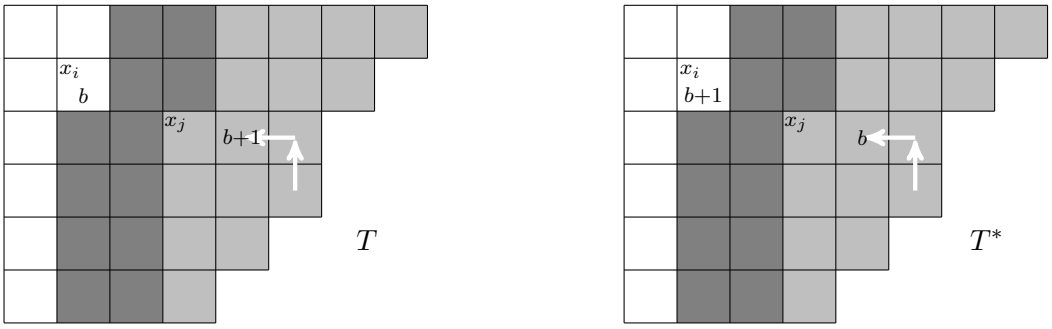


FIGURE 32. More unaffected jeux de taquin

$b+1$ was σ, b cannot take this step. Hence, the path of b may or may not be by one shorter than the path of $b+1$ would have been, and hence, our argument works exactly up to $k = r_S(i, T^*)$ but not necessarily for $k = r_S(i, T)$. Figure 33 shows a situation where b and $b+1$ do not interfere, Figure 34 a situation, where σ occurs.

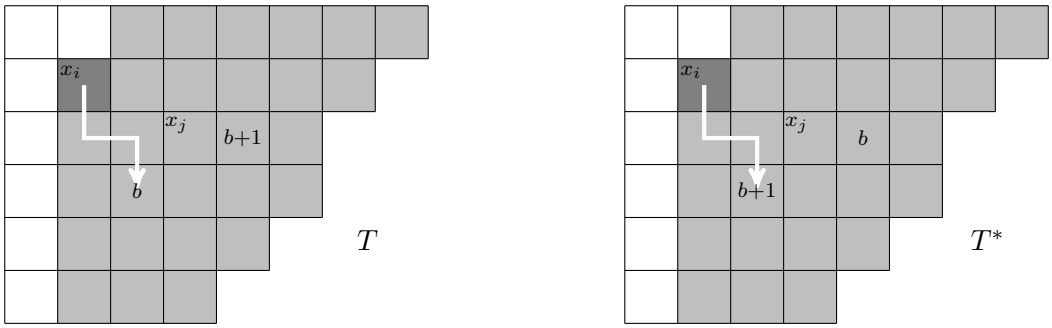


FIGURE 33. A path such that b and $b+1$ do not interfere

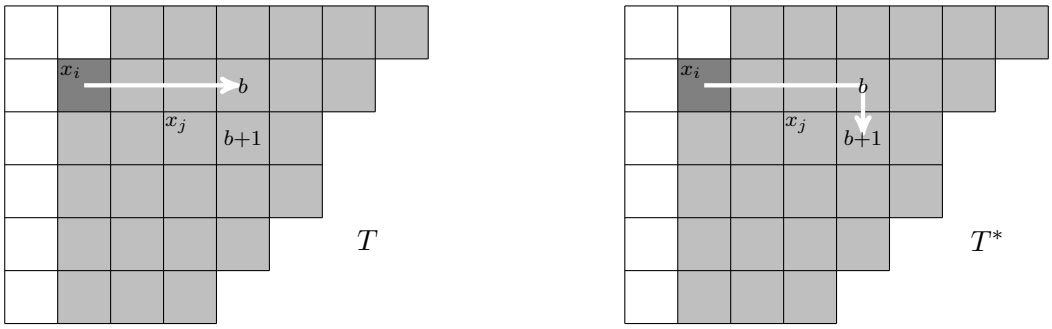


FIGURE 34. A path with interfering b and $b+1$

Summarising, we get

$$\text{for all } 0 \le k \le r_S(i, T^*) : \ T_k = T^*_k$$

and for $k > r_S(i, T^*)$ we know that neither b nor $b+1$ can be exchanged with an entry that is smaller than b. For $a < b$ we conclude that, if there is a transposition $\sigma_k = (a, b)$ then there is a transposition $\sigma^*_k(a, b+1)$ involving the same cells. Analogously, for every transposition $\sigma_k = (a, b+1)$ there is a transposition $\sigma^*_k(a, b)$ involving the same cells. I.e.

$$\begin{aligned} \varepsilon_S(a, b, x, y, T) &= \varepsilon_S(a, b+1, x, y, T^*) \quad \text{and} \\ \varepsilon_S(a, b, x, y, T^*) &= \varepsilon_S(a, b+1, x, y, T). \end{aligned}$$

We see that already the summation over these two tabloids yields equality. Since σ is an involution (i.e. partitions $\mathrm{T}(\lambda)$ in such pairs), summation over T yields (3.1). □

REMARK 3.19. In a situation like in Figure 34 σ is actually reversed during the application and it turns out, that $\mathrm{JDT}_S(T) = \mathrm{JDT}_S(T^*)$. Hence, one might be tempted to believe, that exchanging b and $b+1$ in T either leads to exchanging b and $b+1$ in

$\mathrm{JDT}_S(T)$ or does not have any effect on $\mathrm{JDT}_S(T)$ at all. This, however, is wrong! In a situation like shown in Figure 35 $\mathrm{JDT}_S(T)$ and $\mathrm{JDT}_S(T^*)$ might be totally different.

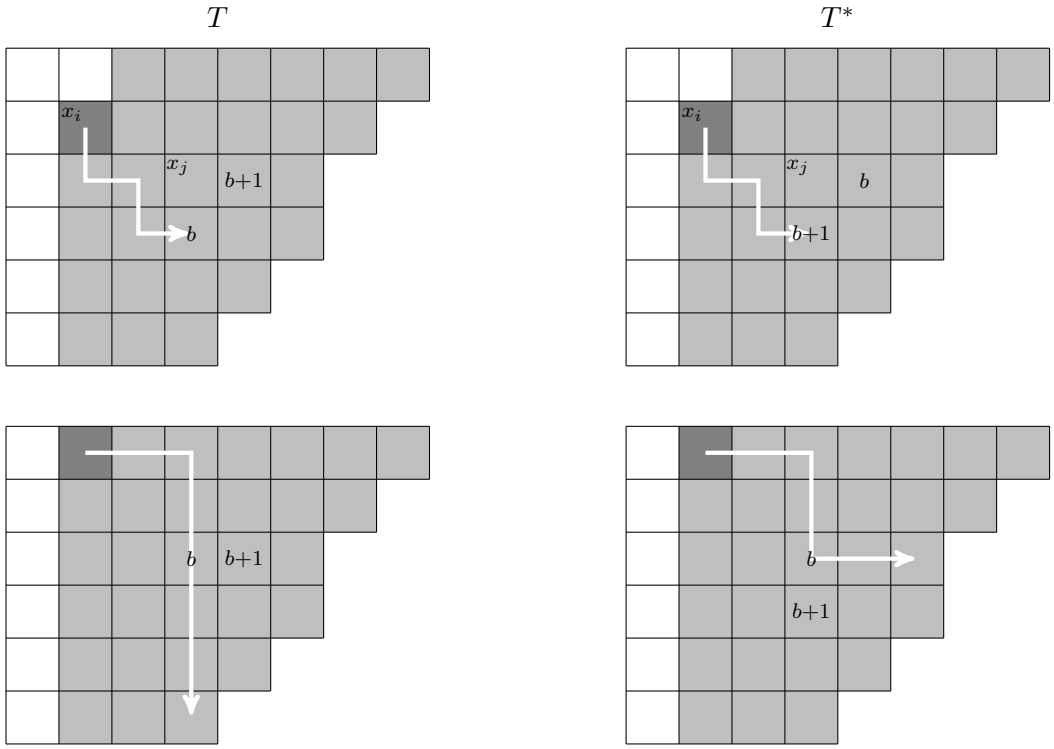


FIGURE 35. A situation leading to essentially different outputs

Equipped with Propostion 3.18 we are well prepared to derive a recursion for the exchange numbers.

THEOREM 3.20 (Exchange Numbers). *Let $\lambda \vdash n$ be a partition and $S \in \mathrm{SYT}(\lambda)$ define the $\prec_S$-order algorithm. For $1 < b < n$ we have the recursion*

$$(n-b)\varepsilon_S(b) = \alpha_\lambda - \omega_S(b) + \sum_{a=1}^{b-1} \varepsilon_S(a).$$

PROOF. Considering Propostion 3.18 we can replace $\varepsilon_S(a,b)$ and $\varepsilon_S(b,c)$ by $\varepsilon_S(a)$ and $\varepsilon_S(b)$ respectively in Lemma 3.17 and we get

$$\alpha_\lambda + \sum_{a=1}^{b-1} \varepsilon_S(a) - \sum_{c=b+1}^{n} \varepsilon_S(b) = \omega_S(b).$$

Again from Propostion 3.18 we know that all the terms in the second sum agree, hence, we get

$$\alpha_\lambda + \sum_{a=1}^{b-1} \varepsilon_S(a) - (n-b)\varepsilon_S(b) = \omega_S(b)$$

and the result follows. □

REMARK 3.21. Theorem 3.20 gives not just a recursion, but the recursion produces its own initial condition. Since, the sum is empty if $b = 1$. Moreover, the entry $b = 1$ will always end up in the top left corner, hence $\omega_S(1) = 0$. We get

$$\varepsilon_S(1) = \frac{\alpha_\lambda}{n-1}.$$

Hence, we can calculate the exchange numbers recursively and with Remark 3.11 and Remark 3.14 for $S, T \in \mathrm{SYT}(\lambda)$ and all entries b it follows that

$$\mathbf{z}(\lambda, S) = \mathbf{z}(\lambda, T) \text{ implies } \varepsilon_S(b) = \varepsilon_T(b).$$

Since every jeu de taquin step exchanges one entry with a smaller one and the other with a larger one, counting how often an entry is exchanged with a larger entry counts the jeu de taquin steps. I.e. we get the Complexity Theorem as corollary.

COROLLARY 3.22 (Complexity Theorem). *Let $\lambda \vdash n$ be a partition and $S, T \in \mathrm{SYT}(\lambda)$. We can calculate the complexity as*

$$C(S) = \frac{1}{n!} \sum_{a=1}^{n} (n-a)\varepsilon_S(a).$$

Moreover,

$$\mathbf{z}(\lambda, S) = \mathbf{z}(\lambda, T) \text{ implies } C(S) = C(T).$$

This solves the questions raised by the conjecture of Krattenthaler and Müller.

COROLLARY 3.23. *Since the column-wise and row-wise algorithms both yield uniform distribution we have*

$$C(R) = C(C).$$

Finally, we want to mention a nice technical aspect.

REMARK 3.24. For every $S \in \mathrm{SYT}(\lambda)$ such that the $\prec_S$-algorithm yields uniform distribution the parameter $\omega_S(b)$ can be expressed as

$$\omega_S(b) = \frac{n!}{f_\lambda} \sum_{T \in \mathrm{SYT}(\lambda)} h'(b, T).$$

This gives a reasonable efficient way to calculate the complexity in the case of uniform distribution.

3.3. Drop-functions and signed exit numbers. While we were searching a promising attempt to prove the conjecture of Krattenthaler and Müller we invented so called drop-functions and conjectured that these agree for the column-wise and row-wise algorithm – this would have implied the conjecture as we will see in a moment. Most surprisingly, it turned out, that also the drop-functions depend solely on the distribution vector – as also the signed exit numbers do, which are derived from the local exchange numbers and will take the place of the exchange numbers on the way to the Drop-Function Theorem. We start this section by introducing a statistic measuring an intermediate height.

DEFINITION 3.25 (maximal height). *Let $T \in \mathrm{T}(\lambda)$, $S \in \mathrm{SYT}(\lambda)$, $x_n \prec_S \cdots \prec_S x_1$ be the cells of λ and $b = T(x_i)$ for some $1 \leq i \leq n$. The* **maximal height** *of b is the maximum of the heights of b during the application of JDT_S to T. Since we know from before that this height is taken at the end of jdt_b we may define*

$$\beta_S(b, T) := h'(b, T_{r(i,T)}).$$

Analogously to the other height statistics we define the **total maximal height** *by summation over $\mathrm{T}(\lambda)$, i.e.*

$$\beta_S(b) := \sum_{T \in \mathrm{T}(\lambda)} \beta(S(b, T)).$$

We mentioned before that we can count jeu de taquin steps by counting south- and east-steps of entries. Obviously, the entry b takes overall $\beta_S(b) - \alpha_\lambda$ steps to the south or east, hence we can calculate the complexity as

$$\begin{aligned} C(S) &= \frac{1}{n!} \sum_{b=1}^{n} (\beta_S(b) - \alpha_\lambda) = \\ &= \frac{1}{n!} \sum_{b=1}^{n} \beta_S(b) - \frac{\alpha_\lambda}{(n-1)!} = \\ &= \frac{1}{n!} \sum_{b=1}^{n} \beta_S(b) - \sum_{x \in \lambda} h'_\lambda(x). \end{aligned}$$

From any of the above right hand sides we see, that the Complexity Theorem might be explained using the parameter $\beta_S(b)$. I.e. if $\mathbf{z}(\lambda, S) = \mathbf{z}(\lambda, T)$ implied $\beta_S(b) = \beta_T(b)$, then the Complexity Theorem would follow. This raises the question of locating the drop target of the entries. The rest of this subsection is devoted to treating this question.

For the definition of the Drop-function we use the Iverson bracket, which is a generalisation of the Kronecker delta. Given an expression E (that evaluates to true or false),

the Iverson bracket is defined as

$$[E] := \begin{cases} 1 & \text{if } E \text{ is true} \\ 0 & \text{if } E \text{ is false.} \end{cases}$$

DEFINITION 3.26 (Drop-function). *Let $T \in \mathrm{T}(\lambda)$, $S \in \mathrm{SYT}(\lambda)$, $x_n \prec_S \cdots \prec_S x_1$ be the cells of λ and $b = T(x_i)$ for some $1 \leq i \leq n$ and let $x \in \lambda$. The* **drop-function** *counts the tabloids of shape λ such that (using the $\prec_S$-algorithm) b drops to x, i.e.*

$$d_S(b, x) := \sum_{T \in \mathrm{T}(\lambda)} [b = T_{r_S(i,T)}(x)].$$

Here the Iverson bracket decides, whether b drops in T to x. We could have used the Kronecker-delta for this definition, but chose the Iverson bracket to enhance the readability (especially of the subscripts).

If an entry takes no east- or south-step, we still say that it dropped – namely to its starting position. We see that

$$\sum_{x \in \lambda} d_S(b, x) = n!$$

because b drops exactly once in every $T \in \mathrm{T}(\lambda)$. Moreover, we have

$$d_S(1, x) = (n-1)!$$

for all $x \in \lambda$, since the entry 1 starts at every cell in $(n-1)!$ tabloids and always drops to its starting position.

As we did for the complexity, we will give a formula for the drop-function that depends on some intermediate quantity for which we can give a recursion. This recursion will again depend solely on $\mathbf{z}(\lambda, S)$. To do so, we also need to adapt our initial and terminal parameters.

DEFINITION 3.27 (entry-count). *An entry-count counts how often an entry b appears as entry of a specified cell x. The* **initial entry-count**

$$\alpha^\lambda := \alpha^\lambda(b, x) := \sum_{T \in \mathrm{T}(\lambda)} [b = T(x)] = (n-1)!$$

was mentioned just above. We will need the more interesting **terminal entry-count**

$$\omega^S(b, x) := \sum_{T \in \mathrm{T}(\lambda)} [b = \mathrm{JDT}_S(T)(x)]$$

where $S \in \mathrm{SYT}(\lambda)$ defines the $\prec_S$-algorithm.

The mentioned intermediate quantity that will take the place of the exchange numbers is defined as follows.

DEFINITION 3.28 (Signed exit number). *With the usual notation we define the* **signed exit number** *as*

$$\Delta_U(b,x) := \sum_{y\in \mathrm{N}_x^-(\lambda)} \varepsilon_S(b,x,y) - \sum_{y\in \mathrm{N}_x^+(\lambda)} \varepsilon_S(b,y,x).$$

The signed exit number counts the tabloids such that during the application of the $\prec_S$-algorithm the entry b leaves the cell x with a north- or west-step (c,b) for a previously fixed entry $c>b$ and subtracts the number of tabloids where the entry b enters the cell x with a north- or west-step (c,b) (for the same a – any other fixed entry greater than b would be fine as well, since we have (3.1)).

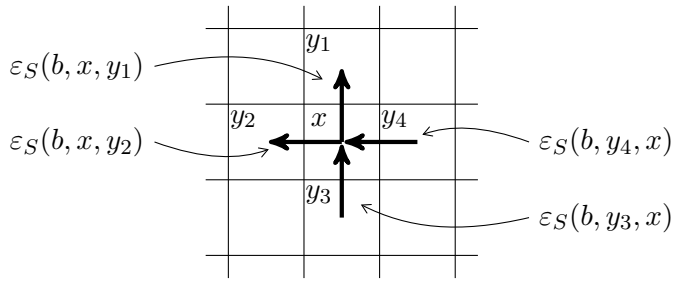


FIGURE 36. Signed exit number

Since we have already done the work on the local exchange numbers, we need nothing more and can directly attack the recursion for the signed exit numbers.

THEOREM 3.29 (Signed exit numbers). *Let $T\in \mathrm{T}(\lambda)$ and $S\in \mathrm{SYT}(\lambda)$ define the $\prec_S$-algorithm. For all cells $x\in\lambda$ and for all entries $1\le b\le n$ we have the recursion*

$$(n-b)\Delta_S(b,x) = \alpha^\lambda - \omega^S(b,x) + \sum_{a=1}^{b-1}\Delta_S(a,x).$$

PROOF. Fix an entry b and a cell x. We can calculate the corresponding terminal entry-count from the initial entry-count by adding the number of times b enters x and subtracting the number of times, b leaves x. That is

$$\omega^S(b,x) = \alpha^\lambda + \sum_{a\neq b}\sum_{y\in \mathrm{N}_\lambda(x)} \Big(\varepsilon(b,a,y,x) - \varepsilon(b,a,x,y)\Big).$$

We use our knowledge about the local exchange numbers to transform the above double sum in the following way.

$$\begin{aligned}
&\sum_{a\neq b}\sum_{y\in \mathrm{N}_\lambda(x)}\Big(\varepsilon(b,a,y,x)-\varepsilon(b,a,x,y)\Big)= \\
&\qquad=\sum_{a=1}^{b-1}\Bigg(\sum_{y\in \mathrm{N}_\lambda^-(x)}\varepsilon_S(a,b,x,y)-\sum_{y\in \mathrm{N}_\lambda^+(x)}\varepsilon_S(a,b,y,x)\Bigg)+ \\
&\qquad\quad+\sum_{c=b+1}^{n}\Bigg(\sum_{y\in \mathrm{N}_\lambda^+(x)}\varepsilon_S(b,c,y,x)-\sum_{y\in \mathrm{N}_\lambda^-(x)}\varepsilon_S(b,c,x,y)\Bigg)= \\
&\qquad=\sum_{a=1}^{b-1}\Bigg(\sum_{y\in \mathrm{N}_\lambda^-(x)}\varepsilon_S(a,x,y)-\sum_{y\in \mathrm{N}_\lambda^+(x)}\varepsilon_S(a,y,x)\Bigg)+ \\
&\qquad\quad+\sum_{c=b+1}^{n}\Bigg(\sum_{y\in \mathrm{N}_\lambda^+(x)}\varepsilon_S(b,y,x)-\sum_{y\in \mathrm{N}_\lambda^-(x)}\varepsilon_S(b,x,y)\Bigg)= \\
&\qquad=\sum_{a=1}^{b-1}\Delta_S(a,x)-(n-b)\Delta_S(b,x).
\end{aligned}$$

Here, the first equality distinguishes the four possible cases

(1) b is the larger of the exchanged entries, then b takes an east- or south step, hence
   (1.1) b can enter x from $\mathrm{N}_\lambda^-(x)$ or
   (1.2) b can leave x to $\mathrm{N}_\lambda^+(x)$.
(2) b is the smaller of the exchanged entries, then b takes a west- or north step, hence
   (2.1) b can enter x from $\mathrm{N}_\lambda^+(x)$ or
   (2.2) b can leave x to $\mathrm{N}_\lambda^-(x)$.

The second equality is Propostion 3.18, and the third equality is the definition of the signed exit number. Basic transformation finishes the proof. □

Remark 3.30. Again, the recursion yields its own initial condition. Let $b=1$, then the sum on the right hand side is empty and we get

$$(n-1)\Delta_S(1,x)=\alpha^\lambda-\omega^S(1,x).$$

Since 1 always ends up in the top left corner we have $\omega^S(1,x)=n!\,\delta_{x,\binom{1}{1}}$ and we have

$$\Delta_S(1,x)=\begin{cases}-(n-1)! & \text{if } x=\binom{1}{1}\\ (n-2)! & \text{else.}\end{cases}$$

Again, we can conclude from the recursion and the fact that $\omega^S(b,x)$ depends solely on the distribution vector rather than on S that

$$\mathbf{z}(\lambda,S)=\mathbf{z}(\lambda,T) \text{ implies } \Delta_S(b,x)=\Delta_T(b,x)$$

for all $S,T\in \mathrm{SYT}(\lambda)$, $x\in\lambda$, and entries b.

Remark 3.31. Intuitively, the signed exit number measures how much stronger or weaker a cell is as source of pushing b towards the top left corner, than it is as sink for b on its way there. More formally we can say that for a fixed entry c summing over all $x\in\lambda$ the signed exit number counts every exchange of b with c once with positive and once with negative sign, hence, we have for all $1\leq b\leq n$ and all $S\in\mathrm{SYT}(\lambda)$

$$\sum_{x\in\lambda}\Delta_S(b,x)=0.$$

We are now going to express the drop-function in terms of the signed exit number and conclude that it also depends solely on the distribution vector.

Theorem 3.32 (Drop-Function). *With the usual notation the drop-function can be derived from the signed exit numbers as*

$$d_S(b,x)=\alpha^\lambda+\sum_{a=1}^{b-1}\Delta_S(a,x).$$

Furthermore,

$$\mathbf{z}(\lambda,S)=\mathbf{z}(\lambda,T) \text{ implies } d_S(b,x)=d_T(b,x).$$

Proof. The second claim follows from Remark 3.30, if the first claim holds. For the first claim we argue as follows.

An entry b drops to a cell x if and only if it enters it from $\mathrm{N}_\lambda^-(x)$ or starts there and does not leave it towards $\mathrm{N}_\lambda^+(x)$. There are α^λ tabloids such that b starts in x. There are

$$\sum_{T\in\mathrm{T}(\lambda)}\sum_{a=1}^{b-1}\sum_{y\in\mathrm{N}_\lambda^-(x)}\varepsilon_S(b,a,y,x,T)$$

tabloids such that b enters x from $\mathrm{N}_\lambda^-(x)$ (since it can enter x from this direction only, if a smaller entry a leaves x in the reverse direction). Finally, there are

$$\sum_{T\in\mathrm{T}(\lambda)}\sum_{a=1}^{b-1}\sum_{y\in\mathrm{N}_\lambda^+(x)}\varepsilon_S(b,a,x,y,T)$$

tabloids such that b leaves x towards $\mathrm{N}_\lambda^+(x)$ (since it can leave x in this direction only, if a smaller entry a enters x in the reverse direction). Summation yields

$$\begin{aligned} d_S(b,x) &= \alpha^\lambda + \sum_{a=1}^{b-1} \Big(\sum_{y \in \mathrm{N}_\lambda^-(x)} \varepsilon_S(a,b,x,y) - \sum_{y \in \mathrm{N}_\lambda^+(x)} \varepsilon_S(a,b,y,x) \Big) = \\ &= \alpha^\lambda + \sum_{a=1}^{b-1} \Big(\sum_{y \in \mathrm{N}_\lambda^-(x)} \varepsilon_S(a,x,y) - \sum_{y \in \mathrm{N}_\lambda^+(x)} \varepsilon_S(a,y,x) \Big) = \\ &= \alpha^\lambda + \sum_{a=1}^{b-1} \Delta_S(a,x). \end{aligned}$$

□

As pointed out before, this gives an alternative proof of the Complexity Theorem that avoids exchange numbers, but still heavily needs local exchange numbers. As we have seen before, it is much easier to prove the Complexity Theorem from the exchange numbers, hence, this is no excuse for considering drop-functions. Our point of view is, that these results on the drop-functions and signed exit numbers are surprising and interesting enough to be granted a section on their own.

3.4. Notes on the complexity theorems. We want to close Section 3 with a couple of remarks.

Remark 3.33. Neither of the arguments in Section 3 used the form of the border of the Young diagrams. Hence the proofs work for arbitrary diagrams, e.g. skew diagrams of shifted diagrams (such are e.g. studied in [**Fis02**]).

Remark 3.34. Although there exists a version of the NPS-algorithm for the shifted case (see [**Fis01**]), the results of Section 3 are not of too much use for diagrams with other shapes, since little or nothing is known about their distribution vectors. Thus, we have no knowledge about the ω-parameters and hence cannot use the formulas to calculate complexities or drop-functions.

Remark 3.35. In the easiest case of a diagram (that consists of a single row, i.e. $\lambda = (n)$) we can explicitly calculate the drop-function using a different approach. We can index the cells with a single coordinate $1 \leq x \leq n$. In this case we also face a single standard tableau, hence we may index the drop-function with n rather than $S \in \mathrm{SYT}(\lambda)$. We define the **partial drop function** counting the number of tabloids in which the entry b drops from the starting position x to y as

$$d_n(b,x,y) = |\{T \in \mathrm{T}((n)) : \ b = T(x) = \mathrm{jdt}_x \cdots \mathrm{jdt}_n T(y)\}|$$

where the indices of the jeux de taquin are cells.

We can directly count these tabloids in the following way: There are $n-b$ entries larger than b. To allow b to drop to y, exactly $n-y$ of these must start to the right of x and consequently $y-b$ of them are left to start to the left of x. Hence, we may choose these positions and permute the entries smaller and the entries larger than b. The reader may find it helpful to compare this arguments with Figure 37.

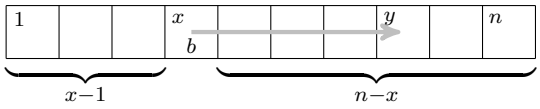


FIGURE 37. A jeu de taquin in a one-lined diagram

We get

$$d_n(b,x,y) = \binom{x-1}{y-b}\binom{n-x}{n-y}(b-1)!(n-b)!.$$

The drop-function could now be evaluated by summation over x and the Chu-Vandermonde identity. Instead we prefer to demonstrate how we can easily calculate the drop-function by deriving a recursion from the partial drop-function. Exchanging b and $b+1$ defines a bijection between the tabloids in which $b+1$ drops from x to y and the tabloids in which either b drops from x to y and $b+1$ starts to the left of x, or b drops from x to $y-1$ and $b+1$ starts to the right of x. The first case contributes $\frac{y-b}{n-b}\,d_n(b,x,y)$ because there are $n-b$ entries greater than b of which $y-b$ must be to the left of x (one of the latter must be $b+1$, hence, choosing one of this cells and permuting the rest yields $(y-b)(n-b-1)!$ permutations of the greater entries, since there are $(n-b)!$ such permutations, cancellation yields the coefficient). The second case contributes $\frac{n-y+1}{n-b}\,d_n(b,x,y-1)$ because of the $n-b$ entries that are greater than b now $n-y+1$ must be to the right of x and we can argue analogously. These situations are depicted in Figure 38 and Figure 39.

FIGURE 38. An exchange fixing the drop-target

We conclude

$$d_n(b+1,x,y) = \frac{y-b}{n-b}\,d_n(b,x,y) + \frac{n-y+1}{n-b}\,d_n(b,x,y-1).$$

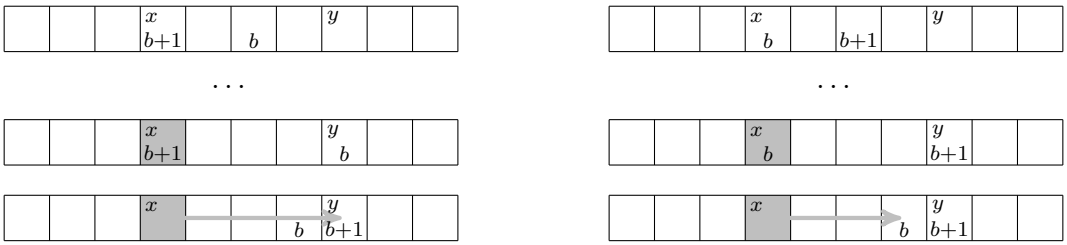


FIGURE 39. An exchange altering the drop-target

To get a recursion for the drop function we may sum over x.

$$d_n(b+1,y) = \frac{y-b}{n-b}\, d_n(b,y) + \frac{n-y+1}{n-b}\, d_n(b,y-1).$$

Since the entry 1 always drops to its starting position, we get the initial condition

$$d_n(1,y) = (n-1)!$$

for all $y \in (n)$. We prove by induction that

$$d_n(b,y) = \begin{cases} \frac{n!}{n-b+1} & \text{if } y \geq b \\ 0 & \text{else.} \end{cases}$$

Obviously this holds for $b = 1$. Suppose it is true for $a < b+1$. We get

$$\begin{aligned} d_n(b+1,y) &= \frac{y-b}{n-b}\, d_n(b,y) + \frac{n-y+1}{n-b}\, d_n(b,y-1) = \\ &= \frac{y-b}{n-b}\frac{n!}{n-b+1}\,[y \geq b] + \frac{n-y+1}{n-b}\frac{n!}{n-b+1}\,[y > b], \end{aligned}$$

where $[p]$ is again the Iverson bracket. If $y < b$ both Iverson-brackets in the above equation are 0, if $y = b$ still the right hand Iverson bracket is 0 and in the left summand $y - b$ vanishes. Hence, we have

$$\text{for all } y < b+1 : \; d_n(b+1,y) = 0$$

as desired. For $y \geq b+1$ both Iverson-brackets take the value 1 and we get

$$\begin{aligned} d_n(b+1,y) &= \frac{y-b}{n-b}\frac{n!}{n-b+1} + \frac{n-y+1}{n-b}\frac{n!}{n-b+1} = \\ &= \frac{n!}{n-b+1}\left(\frac{y-b}{n-b} + \frac{n-y+1}{n-b}\right) = \\ &= \frac{n!}{\cancel{n-b+1}}\frac{\cancel{n-b+1}}{n-b} = \\ &= \frac{n!}{n-(b+1)+1} \end{aligned}$$

and the induction is complete.

REMARK 3.36. The definitions and proofs in Section 3.2 and Section 3.3 might give the impression, that proving the Complexity Theorem is straightforward and that pretty much everything depends solely on the distribution vector. This, however, is not more than an illusion. E.g. on the first glance the partial drop function from Remark 3.35 looks promising to yield another proof for the Complexity Theorem. But a short investigation shows that its generalisation to Young diagrams depends on S rather than $\mathbf{z}(\lambda, S)$.

Moreover, the above calculation uses the fact that b drops to $y-1$ if the exchange with $b+1$ moves $b+1$ to a cell to the right of x. The same statement for the general case appears to be very difficult to handle, since one would have to count the situations such that b drops to a cell above $b+1$ (see Figure 34). Hence, it looks pretty impossible to derive a suitable recursion for the general drop-function from some general partial drop-function.

REMARK 3.37. The numbers $\frac{n!}{n-b+1}$ from Remark 3.35 appear in a well known integer sequence, actually

$$\gcd\left\{\frac{n!}{n-b+1} : 1 \leq b \leq n\right\} = \frac{n!}{\operatorname{lcm}\{1,\ldots,n\}}.$$

Thus we have

$$\gcd\{d_n(b,y) : 1 \leq b \leq y \leq n\} = \frac{n!}{\operatorname{lcm}\{1,\ldots,n\}}.$$

Surprisingly, in a computer experiment for some partitions $\lambda \vdash n$ with $n \leq 10$ using the row-wise algorithm this equation held for most but not all cases. In these other cases we still observed

$$\gcd\{d_S(b,y) : 1 \leq b \leq n, y \in \lambda\} \mid \frac{n!}{\operatorname{lcm}\{1,\ldots,n\}}.$$

In the light of Remark 3.37 we formulate the following conjecture.

CONJECTURE 3.38. *Let $\lambda \vdash n$ be a partition and $S \in \mathrm{SYT}(\lambda)$ such that the $\prec_S$-algorithm is uniformly distributed. Then*

$$\frac{n!}{\operatorname{lcm}\{1,\ldots,n\} \cdot \gcd\{d_S(b,y) : 1 \leq b \leq n, y \in \lambda\}} \in \mathbb{N}.$$

We want to close Section 3 by refering to further results.

REMARK 3.39. Sulzgruber investigated the results of this section for the uniformly distributed case [**Sul14**]. For this case he derived very nice bijections proving the Drop-function Theorem, the Exchange number Theorem and a version of the Complexity Theorem and he was able to give formulae for the drop-function, the exchange numbers and the complexity.

4. The jeu de taquin on d-complete posets

Riegler investigated some aspects of the jeu de taquin on insets and I joined this work with some observations. This resulted in the article [**NR14**] and was the starting point for the investigation of the topics discussed in Section 5. In Section 4.1 we define d-complete posets and explain why they are interesting. In Section 4.2 we have a look at the double-tailed diamond and present the proof of the Δ-Theorem to which we heavily contributed. Finally, we mention the other topics treated in [**NR14**] in Section 4.3.

4.1. d-complete posets. The desire was to generalise hook-lengths such that the hook-length formula applies to more posets than just Young diagrams. Proctor [**Pro99**] introduced d-complete posets and used its intervals to define such hook-lengths. A further generalisation by Ishikawa and Tagawa are **leaf-posets** [**IT07**]. Even more general are the so called **hook-length posets**, which are the posets counted by some hook-length formula (see [**Sta11**]). Neither of these generalisations are of any interest for the present work, hence, we return to a less general case.

Definition 4.1 (d-complete poset). *Let P be a poset and $k \geq 3$ an integer. An interval $[x, z]$ in P is called d_k-**interval** if it is isomorphic to the symmetric double-tailed diamond with $2k$ cells, i.e.*

$$[x, z] \cong (\left(k-1; 1^{k-1}\right), \succ).$$

*An interval $[x, y]$ is called d_k^--**interval** if it isomorphic to a d_k-interval without the top element, i.e.*

$$[x, y] \cong (\left(k-2; 1^{k-1}\right), \succ).$$

Note that in the case $k = 3$ this is not an interval since it has two maximal elements, nevertheless we denote it by $[x, y]$ (the reader may in that case understand the parameter y as the pair of maximal elements, but since it will not matter for our purposes we will not introduce any formalism for it).

*P is d_k-**complete** if it satisfies the three conditions*

1. *$[x, y]$ is a d_k^--interval implies that there exists a $z \in P$ such that $[x, z]$ is a d_k-interval,*
2. *$[x, z]$ is a d_k-interval implies that z covers only elements in $[x, z]$ and*
3. *$[x, z]$ is a d_k-interval implies that there exists no $x' \neq x$ such that $[x', z]$ is a d_k-interval.*

*P is called d-**complete** if P is d_k-complete for all $k \geq 3$.*

Proctors classification of d-complete posets [**Pro99**] uses 15 different classes. Insets $(m; \lambda)$ with $m \geq \lambda'$, the double-tailed diamond $(m; 1^n)$ with $m \geq n$, Young diagrams

and shifted Young diagrams (all under reverse tableau order) are among the d-complete posets, their skew versions, however, are not.

Although it will be of no use for the rest of the present work we want to mention the generalisation of the hook-length, which is of course necessary to have a hook-length formula.

DEFINITION 4.2 (d-complete hook-length). *Let P be a d-complete poset. For every $x \in P$ the hook-length $h_d(x)$ is given by the following algorithm.*

(1) *Assign to all minimal elements of P the hook-length* 1.
(2) *Choose $z \in P$ such that all $x \leq z$ were already assigned a hook-length.*
   (2.1) *If z is not the top element of a d_k-interval, set*
   $$h_d(z) \leftarrow |\{x \in P \mid x \leq z\}|.$$
   (2.2) *If otherwise z is the top element of a d_k-interval $[x, z]$ (note that this is unique) set*
   $$h_d(z) \leftarrow h_d(l) + h_d(r) - h_d(x)$$
   where l and r are the non-comparable elements in $[x, z]$.
(3) *Repeat* (2) *until every $x \in P$ was assigned a hook-length.*

REMARK 4.3. It indeed turns out, that this generalises the earlier defined hook-lengths on Young diagrams, i.e. let P be a Young diagram, then $h_d(x) = h(x)$ for all $x \in P$. Moreover, these hook-lengths qualify for a hook-formula

$$f_P = \frac{|P|!}{\prod_{x \in P} h_d(x)}.$$

The proof of the hook-length formula for d-complete posets is not bijective (see e.g. [**IT07**]) and to the best of our knowledge no such proof is known. The idea of such a proof would be to run some JDT and find a suitable track_x^T such that a bijection to the pairs of a standard filling and a "hook-filling" is established. An indispensable preliminary for such an NPS-like algorithm is a uniformly distributed JDT. The investigation of the jeu de taquin raises remarkable questions and yields results which by themselves are interesting. We turn entirely away from the question of counting standard fillings towards these questions on the distribution.

4.2. The Δ-Theorem. The double-tailed diamond has only two standard fillings (C and R). Considering the column-wise algorithm the distribution vector appears as

$$\mathbf{z}((m; 1^n), C) = \begin{pmatrix} \mathrm{z}_C((m; 1^n), C) \\ \mathrm{z}_R((m; 1^n), C) \end{pmatrix}.$$

Using this distribution vector, we introduce the following statistic.

DEFINITION 4.4. *We call the difference of the entries of the distribution vector* $\mathbf{z}((m;1^n),C)$ *the* **weighted deviation coefficient of the double-tailed diamond**, *i.e.*

$$\Delta_{(m;1^n)} := \mathrm{z}_C((m;1^n),C) - \mathrm{z}_R((m;1^n),C).$$

The nomenclature of this definition might appear somewhat bizarre at first sight, but will become clear in Section 5. Riegler first considered this statistic and proved the following result using "right-left-k–minima" of permutations (for details we refer to [**NR14**]).

THEOREM 4.5 (Δ-Theorem). *For integers* $m,n \geq 2$ *the weighted deviation coefficient of the double-tailed diamond is given by*

$$\Delta_{(m;1^n)} = (-1)^m \binom{n-1}{m} m!n!$$

Note that this expression vanishes for $m \geq n$, hence, as soon as m is large enough such that the double-tailed diamond is d-complete, the jeu de taquin exhibits a uniform distribution. In joint work with Riegler we gave a purely combinatorial proof based on an involution. Before we present this proof, we have to do some preparations. First of all we adapt the sequence of intermediate tabloids as sequence of intermediate fillings

$$\mathrm{T}_C = (F = F_0, \ldots, F_{r_C(F)} = \mathrm{JDT}_C(F))$$

as introduced in Definition 3.7. Secondly, we abbreviate

$$x = x_{n+1} \quad \text{and} \quad y = x_n.$$

We will refer to the set $\{x,y\}$ as the incomparable elements. We also introduce a name for the bottom part of the double-tailed diamond.

DEFINITION 4.6. *Let* $x_{n+m} \prec_C \cdots \prec_C x_1$ *be the cells of the double-tailed diamond* $(m;1^n)$. *The* **tail** *of* $(m;1^n)$ *is the set*

$$\mathrm{TAIL}\,(m;1^n) := \{x_{n-1}, \ldots, x_1\} = \{\tbinom{1}{2}, \ldots, \tbinom{1}{n}\}.$$

DEFINITION 4.7 (Double-tailed diamond types). *Let* $F \in \mathrm{T}\,(m;1^n)$ *be a filling of the double-tailed diamond* $(m;1^n)$. *We define the* **type** *of* F *as*

$$\tau(F) := \begin{cases} 1 & \text{if } \mathrm{JDT}_C(F) = C \\ -1 & \text{if } \mathrm{JDT}_C(F) = R. \end{cases}$$

Analogously, we define the k**-type** *as the "type" after* k *rounds of jeu de taquin, i.e.*

$$\tau_k(F) := \begin{cases} 1 & \text{if } F_k(x) < F_k(y) \\ -1 & \text{if } F_k(x) > F_k(y). \end{cases}$$

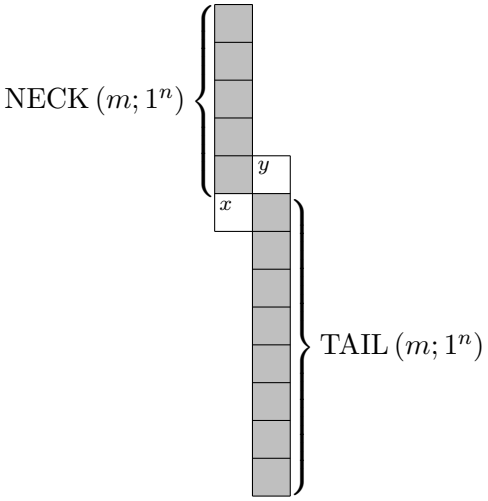


FIGURE 40. Parts of the double-tailed diamond

In the light of this definition it makes sense to use a **modified Iverson bracket** that returns -1 instead of 0 if the condition p is wrong, i.e.

$$[p]^* := 2[p] - 1.$$

With this notation we have

$$\tau(F) = [\mathrm{JDT}_C(F) = C]^* \quad \text{and} \quad \tau_k(F) = [F_k(x) < F_k(y)]^*.$$

The second proof of the Δ-Theorem relies on a type-inversing involution (i.e. a mapping matching fillings of different type). In the case $m \geq n$ the involution is defined on the entire set of fillings $\mathrm{T}(m;1^n)$ such that $\Delta_{(m;1^n)}$ vanishes. For $m < n$ we find a set of exceptional fillings which are all of the same type and the involution will be defined on the remaining fillings. The key fact for the proof is the following observation.

LEMMA 4.8. *Let $x_{n+m} \prec_C \cdots \prec_C x_1$ be the cells of the double-tailed diamond $(m;1^n)$ and $F \in \mathrm{T}(m;1^n)$ be a filling. For $k \geq n+1$ the relative order of the entries of the incomparable elements after k rounds of jeu de taquin depends solely on the relative order of the entries of the cells $x_1, \ldots, x_k$. I.e. let the permutation $p : \{-m+1, \ldots, n\} \to \{-m+1, \ldots, n\}$ be order-preserving on $F(\{x_1, \ldots, x_k\})$ and let $F^* = p \circ F$, then*

$$F_k(x) < F_k(y) \text{ if and only if } F^*_k(x) < F^*_k(y).$$

One may consider Lemma 4.8 as evident, nevertheless we will revise this property of the jeu de taquin in more detail in Section 5.2.

We split the Δ-Theorem in the three cases $m > n$, $m = n$ and $m < n$, we start with the case of equality.

LEMMA 4.9. *Under the column-wise algorithm the double-tailed diamond* $(n;1^n)$ *has the distribution vector*

$$\mathbf{z}((n;1^n),C) = n(2n-1)!\,\mathbf{1}_2.$$

PROOF. The factor $n(2n-1)!$ is obvious if $\mathrm{z}_C((n;1^n),C) = \mathrm{z}_R((n;1^n),C)$ since $|\mathrm{T}(n;1^n)| = (2n)!$. Hence, we only need to prove uniform distribution. We observe that the final round of jeu de taquin reverses the relation between the entries of x and y if and only if the initial entry of the top cell is larger than n, i.e. with the initial filling $F \in \mathrm{T}(n;1^n)$ we have

$$\tau(F) = [F(x_{2n}) \le 0]^* \tau_{2n-1}(F).$$

This is true because

- if $F(x_{2n}) < 0$, it will end up somewhere in the neck without having any effect on the relative order of the entries of the incomparable elements,
- if $F(x_{2n}) = 0$, the entries of the incomparable elements must be -1 and 1, hence, 0 will take the place of -1 preserving the relative order,
- if $F(x_{2n}) = 1$, the entries of the incomparable elements must be -1 and 0, hence, 1 will take the place of -1 reversing the relative order, and
- if $F(x_{2n}) > 1$, the entries of the incomparable elements must again be -1 and 0 and the entry of the top element of the tail must be 1, hence, 1 will be shifted to where -1 has been and the relative order is reversed.

The first two cases are depicted in Figure 41, the latter two in Figure 42.

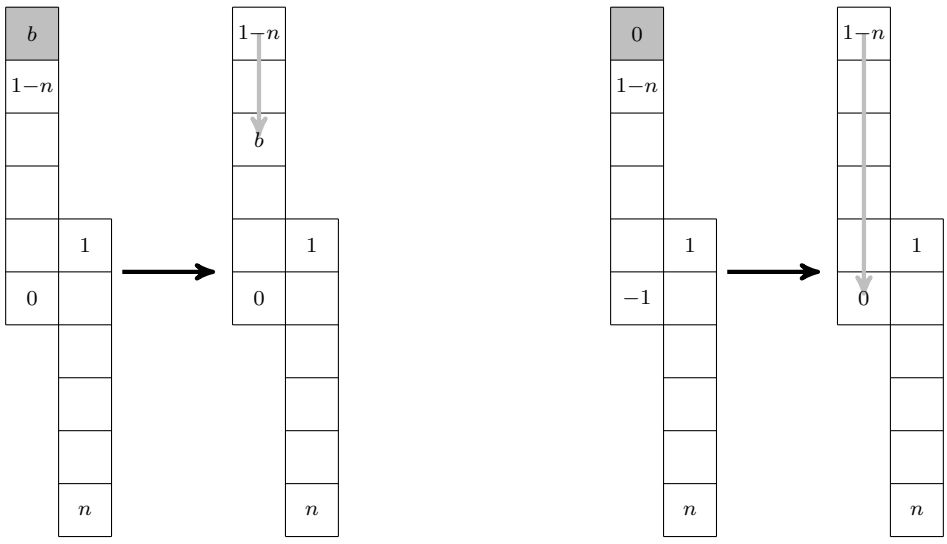


FIGURE 41. Stabilised relative order

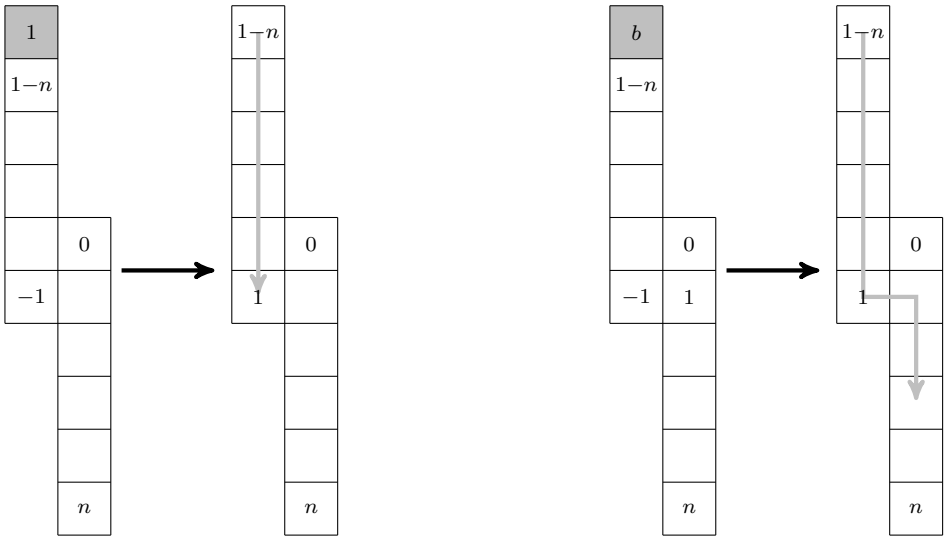


FIGURE 42. Reversed relative order

Hence, we are left with constructing an involution that is order-preserving on $\{x_1, \dots, x_{2n-1}\}$ and flips the entry of the top cell between the lower and upper half of its possible values. Consider the filling F with the top entry $F(x_{2n}) = b$, we cyclically shift the entries $b, \dots, 1-b$ with the following permutations

$$\Phi(F) := \begin{cases} (b, 1-b, -b, -b-1 \dots, b+1)F & \text{if } b \leq 0 \\ (b, 1-b, 2-b, 3-b \dots, b-1)F & \text{if } b \geq n+1. \end{cases}$$

Intuitively this is, we mirror b at $\frac{1}{2}$ and use the unique order preserving shift of the entries "above which b jumped (including the one where it landed)" to reestablish bijectivity. It is clear, that Φ is an involution and has the desired properties. □

EXAMPLE 4.10. Figure 43 shows two fillings in $\mathrm{T}\left(6; 1^6\right)$ that are paired by Φ and have different type. The entries involved in the involution are highlighted, the others greyed.

As second case we treat $m > n$.

COROLLARY 4.11. *For $m > n$ and using the column-wise algorithm the double-tailed diamond $(m; 1^n)$ has the distribution vector*

$$\mathbf{z}((m; 1^n), C) = \frac{(m+n)!}{2}\, \mathbf{1}_2.$$

PROOF. Obviously, if two fillings F and G agree on their upper $m-n$ entries, i.e.

$$F(x_{n+m}) = G(x_{n+m}), F(x_{n+m-1}) = G(x_{n+m-1}), \dots, F(x_{2n+1}) = G(x_{2n+1})$$

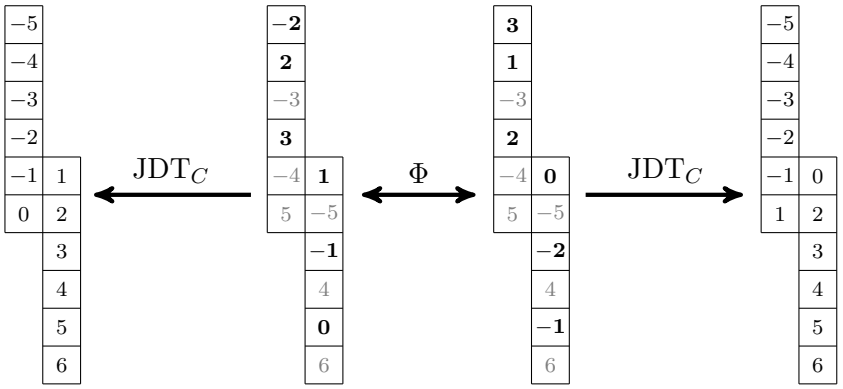


FIGURE 43. Typeinversing involution

and on their $2n$-type, i.e.

$$\tau_{2n}(F) = \tau_{2n}(G),$$

then they also agree on their type

$$\tau(F) = \tau(G).$$

Hence, if we leave the top $m-n$ entries unchanged and apply Φ to the relative order of the bottom $2n$ elements, we get the desired involution. For $F \in \mathrm{T}\,(m;1^n)$ let p_F be the unique order preserving bijection

$$p_F : F((n;1^n)) \to \{1-n,\dots,n\}$$

then we can write this involution as

$$\Phi_{(m;1^n)}(F)(x_i) := \begin{cases} p_F^{-1} \circ \Phi \circ p_F(F)(x_i) & \text{if } i \leq 2n \\ F(x_i) & \text{else.} \end{cases}$$

Since the restriction of $\Phi_{(m;1^n)}$ to $(n;1^n)$ is type-inversing by Lemma 4.9, we have defined a type-inversing involution on $\mathrm{T}\,(m;1^n)$. □

EXAMPLE 4.12. Figure 44 shows two fillings in $\mathrm{T}\left(9;1^6\right)$ that are paired by $\Phi_{(9,6)}$ and have different type. Again the entries involved in the involution are highlighted, the others greyed.

We want to remark here, that this proof of Corollary 4.11 provided the inspiration to consider distribution matrices. Before we turn to them we complete our consideration of the involution-approach. We turn to the case $m < n$.

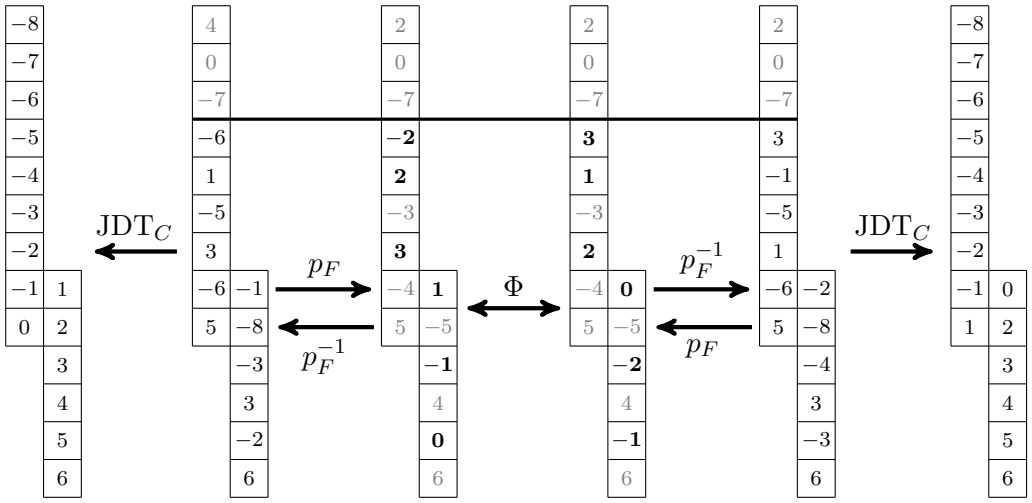


FIGURE 44. Type-inversing involution for $m > n$

REMARK 4.13. The expression $\binom{n-1}{m}n!m!$ is nice but misleading. We rewrite it using the descending factorial

$$\binom{n-1}{m}n!m! = \frac{(n-1)!}{\cancel{m!}(n-m-1)!}n!\cancel{m!} = (n-1)^{\underline{m}}n!$$

PROOF OF THE Δ-THEOREM. Again we have to slightly modify the situation such that we can apply Φ. The problem we are facing is, that the entries in the first column may be too large. E.g. if $F(x_{2n})$ was larger than m it has no counterpart among the entries. We will be looking for the topmost cell, such that we can define an involution. It will turn out, that we can (reasonably) define an involution "based" at x_i if $F(x_i) \leq 2(i-n)-m+1$ (note: there are $i-n$ cells below and including x_i in the first column, the same amount of topmost cells in the second column leads to $2(i-n)$; if we fill these cells with the smallest possible entries, the largest of these entries will be $2(i-n)-m+1$ since the smallest entry overall is $1-m$). We call fillings without such an entry exceptional. We define the set $\mathcal{E}(m;1^n)$ of **exceptional fillings** as

$$\mathcal{E}(m;1^n) := \{F \in \mathrm{T}(m;1^n) \mid \ \forall\, n < i \leq n+m:\ F(x_i) > 2(i-n)-m+1\}.$$

The number of exceptional fillings can easily be calculated – there are $n-m$ values to choose as entry of x_{n+m}, then there are $n-m+2$ possible values for the entry of x_{n+m-1} one of which was already chosen as entry for x_{n+m}; iterating this argument we pick up factors up to $n-1$ for the cell x; the remaining n entries permute. We get

$$|\mathcal{E}(m;1^n)| = (n-m)(n-m+1)\cdots(n-1)n! = (n-1)^{\underline{m}}n!$$

which meets the claimed value of the weighted deviation coefficient. We will first establish our involution on $\mathrm{T}\,(m;1^n)\setminus\mathcal{E}\,(m;1^n)$ and then finish the proof by showing that the exceptional fillings have the correct type.

Let $F\in\mathrm{T}\,(m;1^n)\setminus\mathcal{E}\,(m;1^n)$ and $j\in\{n+1,n+2,\dots,n+m\}$ be maximal such that $F(x_j)\leq 2(j-n)-m+1$. The entries $1-m,2-m,\dots,2(j-n)-m+1$ are among the entries of the cells x_i with $i\leq j$. We call the set of cells carrying these entries

$$\mathrm{small}(F):=F^{-1}(\{1-m,2-m,\dots,2(j-n)-m+1\}).$$

The entries of $\mathrm{small}(F)$ can easily be centred around $\frac{1}{2}$ by adding $n+m-j$, such that Φ applies to them. Therefore we define

$$p_F(a):=a+n+m-j$$

and

$$\Phi_{(m;1^n)}(F)(x_i):=\begin{cases}p_F^{-1}\circ\Phi\circ p_F(F(x_i)) & \text{if } x_i\in\mathrm{small}(F)\\ F(x_i) & \text{else.}\end{cases}$$

Under F_{j-1} the entries $1-m,\dots,2(j-n)-m+1$ are located at the cells $x_{2n-j+1},\dots,x_j$. Therefore, the very same arguments as in the proof of Lemma 4.9 yield that $\Phi_{(m;1^n)}$ is a type-inversing involution. Moreover, $\Phi_{(m;1^n)}$ preserves the property that x_j is the $\prec_C$-minimal cell with entry smaller or equal $2(j-n)-m+1$ and hence is indeed matching elements of $\mathrm{T}\,(m;1^n)\setminus\mathcal{E}\,(m;1^n)$.

Finally, we wonder about the type of the exceptional fillings. From their definition it follows that all entries in the first column are positive, hence $1-m$ and $2-m$ are located in the second column. It follows that $F_n(y)=1-m$ with $2-m$ just below it. Hence, we get

$$F_{n+1}(x)=2-m\quad\text{and}\quad F_{n+1}(y)=1-m$$

and we derived the $(n+1)$-type

$$\tau_{n+1}(F)=-1.$$

We know that the entry $F(x_i)$ of a cell x_i in the neck reverses the x_{i-1}-type if and only if it is larger than both $F_{i-1}(x)$ and $F_{i-1}(y)$ (see the arguments in the proof of Lemma 4.9). The definition of the exceptional fillings ensures for every cell x_i in the neck, that $F(x_i)$ is large enough to reverse the $(i-1)$-type. Hence, every exceptional filling F has type

$$\tau(F)=(-1)^m$$

and the proof is complete. □

Example 4.14. Figure 45 shows two fillings in $\mathrm{T}\left(6;1^9\right)$ that are paired by $\Phi_{(6;1^9)}$ and have different type. The cells containing entries that might be affected by $\Phi_{(m;1^n)}$

are highlighted, the others greyed. Again the entries that are actually involved in the involution are highlighted, the others greyed. Moreover, we marked the topmost cell x_j with an entry smaller than $2(j-n)$.

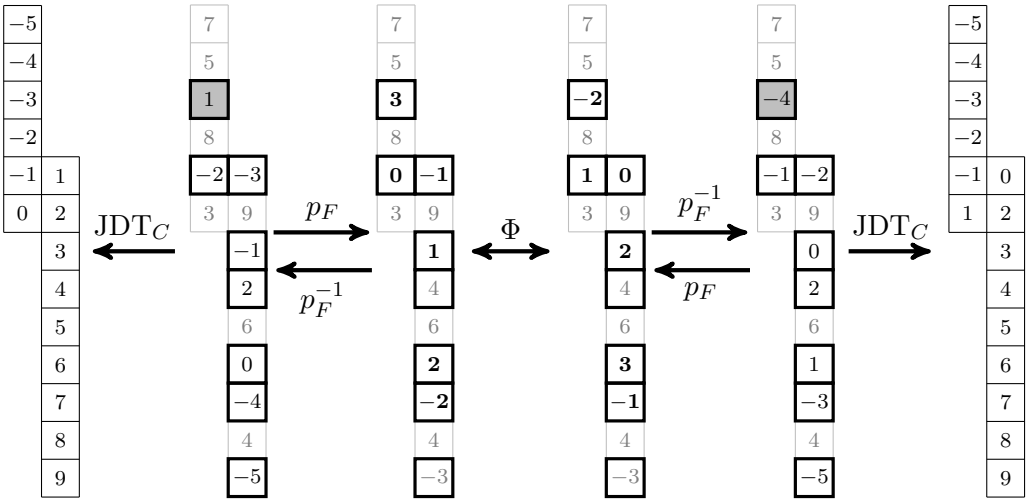


FIGURE 45. Type-inversing involution for $m < n$

REMARK 4.15. In both cases $m < n$ and $m > n$ we named the involution $\Phi_{(m;1^n)}$. This is not just no clash of notation (since the two cases are obviously disjoint) but actually both specialise to Φ for $m = n$. Hence, we can consider all three as one function $\Phi_{(m;1^n)}$.

REMARK 4.16. We claimed that $\Phi_{(m;1^n)}$ is an involution on the set of all non-exceptional fillings. Indeed we proved a finer statement. Namely, define the sets

$$\mathcal{E}_j := \{F \in \mathrm{T}(m;1^n) \mid \ j \text{ is maximal with } F(x_j) \leq 2(n-j) - m + 1\}.$$

The sets $\mathcal{E}_j$ for $n+1 \leq j \leq n+m$ partition the set of non-exceptional fillings. Above we proved that each of these $\mathcal{E}_j$'s contains equally many fillings of both types.

REMARK 4.17. Earlier we described the term $\binom{n-1}{m}n!$ as misleading. The reason for this is, that it canonically counts the set of fillings such that the entries $1-m, \dots 0$ are all located in the tail. With the same argumentation as in the above proof all these fillings have type $(-1)^m$. Hence, we could have chosen them as exceptional fillings. Unfortunately, it appears to be remarkably more difficult to define a suitable involution in this case – although we know that it must exist, we could not write it down or have any hint if it has a nice form.

4.3. Further result. Besides the Δ-Theorem [**NR14**] contains other interesting considerations. We want to point out a formula for the expected value of the entry of the cell $\binom{2}{1}$ in a standard Young tableaux, namely let X be a uniformly distributed random variable taking values from $\mathrm{SYT}(\lambda)$, then

$$\mathbb{E}(X(\tbinom{2}{1})) = \prod_{i=1}^{\lambda'_1} \frac{n+1}{n+1-\lambda_i}.$$

For details we refer to [**NR14**].

5. Distribution matrices of the jeu de taquin

The outline of Section 5 is as follows. We build up the notions chronologically to present the motivations during the research – this is probably not the best way to introduce the topic, but we chose it because the definitions are very young, and at the time being it is unclear which approach is "the best one". Hence, we describe in Section 5.1 our first considerations leading to distribution matrices. The first version of distribution matrices is introduced in Section 5.2. In Section 5.3 we present the conjectures which are the spice making the topic of distribution matrices interesting. In Section 5.4 we revisit the Δ-Theorem and treat a further special case of an inset in Section 5.5. In Section 5.6 we introduce the general versions of distribution matrices. We consider these on Young diagrams in Section 5.7 which leads to further conjectures presented in Section 5.8. Section 5.9 and Section 5.10 are devoted to the structure of distribution matrices on insets. Finally, Section 5.11 treats the structure of the distribution vectors of insets.

5.1. Motivation. As we mentioned before, the idea of considering distribution matrices was inspired from the fact that adding further top cells to a d-complete double-tailed diamond does not influence the uniform distribution. The tiny insight that got the ball rolling was, that the jeu de taquin on an inset can be viewed as an operation on the relative orders of the body of the inset. I.e. the idea was the following algorithm.

Algorithm 5.1. Given an inset $(m;\lambda)$ with $\lambda \vdash n$, a filling $F \in \mathrm{T}(m;\lambda)$ and some order-defining standard filling $S \in \mathrm{SYT}(m;\lambda)$.

(1) There is a unique order-preserving map

$$p_F : F(\mathrm{BODY}(m;\lambda)) \to \{0,\dots,n\}.$$

We consider the inset $(1;\lambda)$ with the filling

$$J \leftarrow p_F \circ F|_{\mathrm{BODY}(m;\lambda)}\,.$$

(2) We play the jeu de taquin on J, i.e.

$$K \leftarrow \mathrm{JDT}_{S|_{(1;\lambda)}}(J).$$

(3) Actually p_F^{-1} is the unique order-preserving map

$$p_F^{-1} : \{0, \dots, n\} \to \{-m, \dots, n\} \setminus F(\mathrm{NECK}(m;\lambda)).$$

We consider the inset $(m;\lambda)$ with the filling L defined by

$$L \leftarrow \begin{cases} F|_{\mathrm{NECK}(m;\lambda)} & \text{on } \mathrm{NECK}(m;\lambda) \\ p_F^{-1} \circ K & \text{on } \mathrm{BODY}(m;\lambda). \end{cases}$$

(4) We play the jeu de taquin on L, i.e.

$$M \leftarrow \mathrm{jdt}_{(3-m,0)} \cdots \mathrm{jdt}_{(1,0)} L.$$

We again adopt the notation x_i for the cells in reverse order to $\prec_S$ and F_i for the corresponding intermediate fillings. Then we can formulate the mentioned tiny insight as a tiny lemma.

LEMMA 5.2. *With the above notation we have* $F_{n+1} = L$ *and* $\mathrm{JDT}(F) = M$.

PROOF. Every jeu de taquin step is determined by the relative order of the entries of the cell and its outneighbours which is preserved by p. □

EXAMPLE 5.3. Given the inset $(4; 3, 2^2, 1)$ with the filling F as in Figure 46 and using the column-wise order, we see that the above algorithm and the column-wise algorithm indeed agree on their outcome.

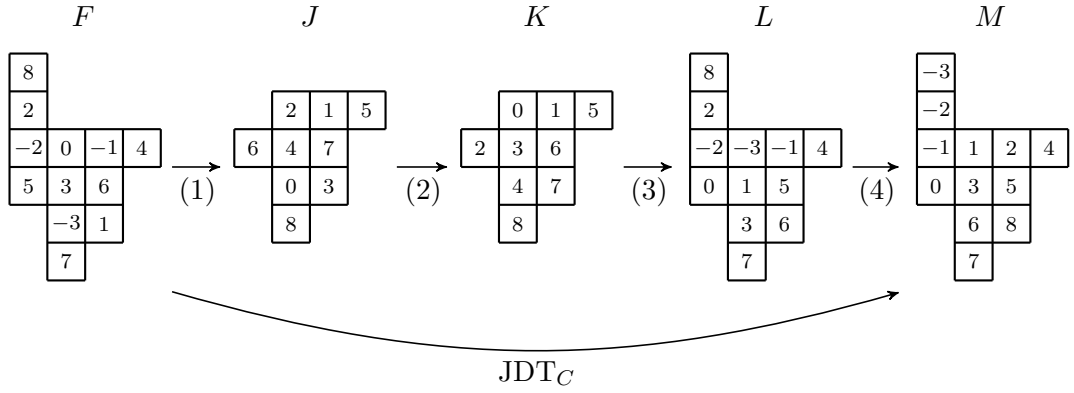


FIGURE 46. The application of the above algorithm to F

Thinking about the situation we recognise $K \in \mathrm{SYT}(1;\lambda)$ and $M|_{\mathrm{BODY}(m;\lambda)} \in \mathrm{SYT}(1;\lambda)$ are both standard fillings of the same inset and that the latter together with

m fully defines M. We can extract a refined simulation from the above algorithm by not adding the neck in one step but by one cell a time. Moreover, we can forget about the part of the neck that is already ordered. We derive the following refined simulation.

SIMULATION 5.4. Given a positive integer t, an inset $(1;\lambda)$, a standard filling ${}_tF \in \mathrm{SYT}(1;\lambda)$ and an integer $b \in \{-t,\dots,n\}$. Consider the inset $(2;\lambda)$ with the filling G that puts b in the top left corner and the rest of the filling is ${}_tF$ with all entries smaller or equal b shifted down by 1, i.e.

$$G = \begin{cases} b & \text{on } \{\binom{1}{0}\} \\ (b, b-1, \dots, -t)\,{}_tF & \text{on } (1;\lambda). \end{cases}$$

Now we apply the jeu de taquin to the top left cell and forget about this cell afterwards, i.e.

$${}_{t+1}F = \mathrm{jdt}_a(G)|_{(1;\lambda)}$$

and we denote $\mathrm{SIM}_b : {}_tF \mapsto {}_{t+1}F$.

This simulation describes what happens if we add a cell at the top of the neck (when everything else is already sorted). Then the body has already the relative order of a standard filling. There are $n+t$ possibilities which entry could be carried by the new top cell. The negative ones get stuck somewhere in the neck, and the non-negative ones will drop to somewhere in the body such that ${}_tF, {}_{t+1}F : (1;\lambda) \to \{0,\dots,n\}$ are both standard fillings and $\mathrm{SIM}_b : \mathrm{SYT}(1;\lambda) \to \mathrm{SYT}(1;\lambda)$ maps standard fillings to standard fillings.

The attentive reader will object that ${}_tF((1;\lambda))$ will never contain negative values and hence the lower bound $-t$ is useless and could as well have been chosen as -1. This, of course, is true. Nevertheless, we want to take the opportunity to point out the intuition again: the new cell is added at the top of a neck of length t. The entries $-t,\dots,-1$ are in this neck and everything smaller than or equal to b including these entries is shifted down by one. We are simulating this operation by placing b at $\binom{1}{0}$. The reader may insist, that $-t$ is still useless, but at the end of the day, we prefer to remark the intuition than to remark that the shift reduces to the identity for negative b.

Moreover, I apologise for putting the index t that indicates the length of the (virtual) neck to the left. This choice was made to avoid a clash of notation with the intermediate fillings.

EXAMPLE 5.5. Continuing the above example with ${}_1F = K$ we find the transitions given in Figure 47. Note that we cannot use $\mathrm{SIM}_{F(1-i,0)}$ in the i-th transition but that we need to use the image under the corresponding order-preserving map. Indeed ${}_4F$ agrees with the body of M.

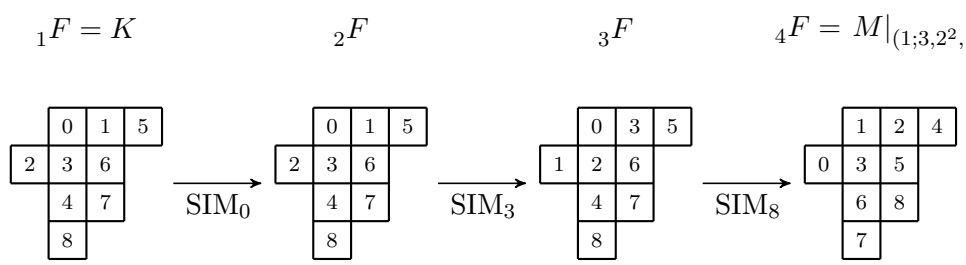


FIGURE 47. Simulating the neck step by step.

We refine our above insight to the following theorem.

THEOREM 5.6. *Given an inset $(m;\lambda)$ and a filling $F \in \mathrm{T}(m;\lambda)$. For $1 \le t \le m$ let p_t be the unique order-preserving maps $p_t : F((t+1;\lambda)) \to \{-t,\dots,n\}$. For $2 \le i \le m$ let $b_i = p_i(F(3-i,0))$ be the entry of the "next top cell of the neck in the corresponding relative order". Let $_1F = \mathrm{JDT}(p_1 \circ F|_{(1;\lambda)})$ be the sorted body under the corresponding relative order. Then*

$$\mathrm{JDT}(F)|_{(1;\lambda)} = \mathrm{SIM}_{b_m} \circ \cdots \circ \mathrm{SIM}_{b_2} \circ {}_1F. \tag{5.1}$$

PROOF. Both sides of (5.1) apply the same sequence of jeu de taquin steps with the same relative orders. □

For the moment it will be enough to consider the simulation of the neck cells and define distribution matrices only for them. Afterwards we are going to work through two interesting examples before we consider distribution matrices also for other cells – which will enable us to explain a little of the structure of the distribution matrices. On the way we will pick up a couple of remarkable conjectures.

5.2. Simulating a neck-cell with a distribution matrix. We are now going to change our point of view from what happens with one filling to what happens with the collection of all fillings.

Remember Definition 1.21 (distribution vector). We defined distribution vectors using multiplicities $\mathrm{z}_U(P,S)$ indexed by standard fillings $U \in \mathrm{SYT}(P)$. Note that the neck of an inset does not contribute to the structure of its standard fillings (the entries of the neck will always be the same for any standard filling). Hence, we may as well use the standard fillings of its body $U \in \mathrm{SYT}(\mathrm{BODY}(P))$ as indices. From here on we will not distinguish if in such a case the index comes from the inset or its body.

Given the distribution vector $\mathbf{z}((m;\lambda),S)$ we can use SIM_b to create a tool to calculate $\mathbf{z}((m+1;\lambda),S)$. To be more precise SIM_b tells us what happens with the relative order of the body when the entry of the (new) top cell is b. I.e. if there are $\mathrm{z}_U((m;\lambda),S)$

fillings of $(m;\lambda)$ sorted to U this will contribute as summand to $\mathrm{z}_{\mathrm{SIM}_b(U)}((m+1;\lambda),S)$ for every $b \in \{1-m,\dots,n\}$. This motivates the definition of the distribution matrices.

DEFINITION 5.7. *Let m be a positive integer and consider the inset $(1;\lambda)$ and standard fillings $U,V \in \mathrm{SYT}(1;\lambda)$. We define*

$$\mathrm{A}_{U,V}(m;\lambda) := |\{b \in \{1-m,\dots,n\} \mid \mathrm{SIM}_b(V) = U\}|$$

and combine these entries to the **distribution matrix**

$$\mathbf{A}(m;\lambda) := (\mathrm{A}_{U,V}(m;\lambda))_{U,V}\,.$$

We can use the distribution matrices to iteratively compute the distribution vectors.

LEMMA 5.8. *In the above setting we have*

$$\mathbf{z}((m+1;\lambda),S) = \mathbf{A}(m;\lambda)\,\mathbf{z}((m;\lambda),S).$$

PROOF. The lemma claims for the components of $\mathbf{z}((m+1;\lambda),S)$ that

$$\mathrm{z}_U((m+1;\lambda),S) = \sum_{V\in\mathrm{SYT}(1;\lambda)} \mathrm{A}_{U,V}(m;\lambda)\,\mathrm{z}_V((m;\lambda),S).$$

But this is nothing else than applying SIM_b for every b that can appear as entry of the top cell. And since Theorem 5.6 tells us, that the simulation is really doing the right thing, the equation holds. □

We right away recognise the simplest and though very important property of the distribution matrices.

LEMMA 5.9. *In the above setting we have*

$$\mathbf{A}(m;\lambda) + I = \mathbf{A}(m+1;\lambda).$$

PROOF. For every negative value of b the simulation SIM_b stabilises every standard filling of the body. The definition of $\mathbf{A}(m+1;\lambda)$ uses exactly one negative value of b more than the definition of $\mathbf{A}(m;\lambda)$. □

Hence, we can consider $\mathbf{A}(1;\lambda)$ as the distribution matrix – all the others differ only by addition of identities. Moreover, we observe a second important property of the distribution matrices.

PROPOSITION 5.10. *The sum of every row and the sum every column of $\mathbf{A}(1;\lambda)$ is $n+2$.*

PROOF. Given a standard filling V. Obviously for every $b \in \{-1,\dots,n\}$ there is a SIM_b that maps V somewhere, and, hence, the entries of the columns sum to $n+2$. For

$U \in \mathrm{SYT}(1;\lambda)$ it is also easy to construct fillings V_b for every b such that $\mathrm{SIM}_b(V_b) = U$. To do so one uses the backward jeu de taquin as described in Definition 2.7. Hence, the entries of the rows sum to $n+2$ as well. □

Thus, we get the following corollaries for free.

COROLLARY 5.11. *Let $(1;\lambda)$ be an inset. The leading eigenvector of $\mathbf{A}(1;\lambda)$ is $\mathbf{1}$ and its eigenvalue is $\epsilon_1 = n+2$.*

Note that we denote eigenvalues with ϵ_i rather than the usual λ_i since we need the latter for the partitions.

COROLLARY 5.12. *$\mathbf{A}(1;\lambda)$ can be written as the sum of $n+2$ permutation matrices.*

Actually it is easy to explicitly find $n+2$ such permutation matrices, since SIM_b for a fixed b is a permutation.

5.3. Conjectures. We know from results of Proctor [**Pro99**] that the number of standard fillings of d-complete[3] insets (among other posets) are given by a hook-length formula. But to the best of our knowledge no bijective proof was found so far. A bijection in the sense of Novelli, Pak and Stoyanovskii would require that there exists a $\prec_S$-algorithm that yields uniform distribution i.e. the $\mathrm{z}_U((m;\lambda),S)$ are equal if $m \geq \lambda'_1$. We formulate this as conjecture for the row-wise and column-wise algorithm meanwhile we suspect that it will hold for many others as well.

CONJECTURE 5.13 (Inset-Uniform-Distribution). *Let $\lambda \vdash n$ and $(m;\lambda)$ be a d-complete inset. Let $S \in \mathrm{SYT}(m;\lambda)$ represent the column-wise or row-wise order. Then the $\prec_S$-algorithm is uniformly distributed, i.e.*

$$\mathbf{z}((m;\lambda),S) = c(m;\lambda)\ \mathbf{1}$$

with a constant $c(m;\lambda) = \frac{(n+m)!}{|\mathrm{SYT}(m;\lambda)|}$.

It is tempting to believe that the existence of a hook-length formula has to include the existence of an $\prec_S$-algorithm yielding uniform distribution – nevertheless, this is unclear. However, we conjecture an enormously stronger statement.

CONJECTURE 5.14 (Inset-Eigenvector-Distribution). *Let $\lambda \vdash n$ and $S \in \mathrm{SYT}(1;\lambda)$ represent the column- or row-wise order. Then the distribution matrix $\mathbf{A}(1;\lambda)$ has an eigenvector v_y to the eigenvalue $\epsilon_y = 2 - \lambda'_1$ such that with $c(1;\lambda) = \frac{(n+1)!}{|\mathrm{SYT}(1;\lambda)|}$ we have*

$$\mathbf{z}((1;\lambda),S) = c(1;\lambda)\ \mathbf{1} + v_y.$$

[3]Remember that an inset $(m;\lambda)$ is d-complete if $m \geq \lambda'_1$ – this is the only property of d-completeness that will matter for our investigations.

The Inset-Eigenvector-Distribution Conjecture straightforwardly implies the Inset-Uniform-Distribution-Conjecture because adding the identity to the distribution matrix stabilises the eigenvectors but adds 1 to the eigenvalues. Therefore, a consequence of the conjecture is

$$v_y \in \ker\left(\mathbf{A}(\lambda'_1 - 1; \lambda)\right).$$

But since Lemma 5.8 implies

$$\mathbf{z}((\lambda'_1; \lambda), S) = \mathbf{A}(\lambda'_1 - 1; \lambda)\mathbf{A}(\lambda'_1 - 2; \lambda) \cdots \mathbf{A}(1; \lambda)\ \mathbf{z}((1; \lambda), S),$$

the coefficient of v_y eventually vanishes and the distribution vector of a d-complete inset under the $\prec_S$-algorithm would be a multiple of $\mathbf{1}$.

The factor $\frac{(n+1)!}{|\mathrm{SYT}(1;\lambda)|}$ is non-ambiguous. The vectors $\mathbf{1}$ and v_y are orthogonal, hence, the components of v_y sum to zero. Therefore $\mathbf{1}$ and its coefficient count permutations.

Now we have a look at an even more surprising conjecture.

Conjecture 5.15 (Inset-Spectrum). *Let $\lambda \vdash n$ be a partition with $\lambda_1 \geq 2$ and $\lambda'_1 \geq 2$. The eigenvalues of $\mathbf{A}(1; \lambda)$ are*

$$\epsilon_1 = n + 2$$

and

$$n, n-1, \ldots, 2 - \lambda'_1.$$

The Inset-Spectrum Conjecture was the initial reason why we decided to have a closer look at the distribution matrices. Note that the Inset-Spectrum Conjecture includes the claim that the above ϵ_y is the least eigenvalue of $\mathbf{A}(1; \lambda)$. The Perron-Frobenius eigenvalue ϵ_1 is easily given by Propostion 5.11, its multiplicity is 1. We make no statement about the multiplicity of the other eigenvalues, but in general their eigenspaces are of higher dimension. Meanwhile, $\mathbf{A}(1; \lambda)$ is in general not diagonalisable.

We excluded the double-tailed diamond from the Inset-Spectrum-Conjecture because it has only two standard fillings hence only two eigenvalues – but these are the essential ones. In the next two sections we have a look at the double-tailed diamond and the inset $(m; 2, 1^{n-2})$, which is interesting, because – as mentioned before – all the eigenvalues are simple.

5.4. The Δ-Theorem revised. We are now able to present an elegant proof of the Δ-Theorem. Remember, we derived a weighted deviation coefficient of

$$\Delta_{(m;1^n)} = (-1)^m \binom{n-1}{m} m!n!.$$

We rewrote its absolute value as $(n-1)^{\underline{m}}n!$. The coefficient itself can be rewritten analogously as

$$\Delta_{(m;1^n)} = (-1)^m(n-1)^{\underline{m}}n! = (1-n)^{\overline{m}}n!.$$

The observation that $\Delta_{(m;1^n)}$ can be expresses as an inner product

$$\Delta_{(m;1^n)} = \mathrm{z}_C((m;1^n),C) - \mathrm{z}_R((m;1^n),C) = \mathbf{z}((m;1^n),C)\cdot\begin{pmatrix}1\\-1\end{pmatrix}$$

enables us to formulate the Δ-Theorem in the form

$$\mathbf{z}((m;1^n),C)\cdot\begin{pmatrix}1\\-1\end{pmatrix} = (1-n)^{\overline{m}}n! \tag{5.2}$$

which is ideal for the following proof.

Alternative proof of the Δ-Theorem. The standard fillings of $(1;1^n)$ are R and C. The simulation maps

$$\mathrm{SIM}_b : R \mapsto \begin{cases} R & b \le 0 \\ C & b > 0 \end{cases} \quad \text{and} \quad \mathrm{SIM}_b : C \mapsto \begin{cases} C & b \le 0 \\ R & b > 0 \end{cases}.$$

Figure 48 depicts the transitions of SIM_b for the values $b \in \{-1,\dots,n\}$.

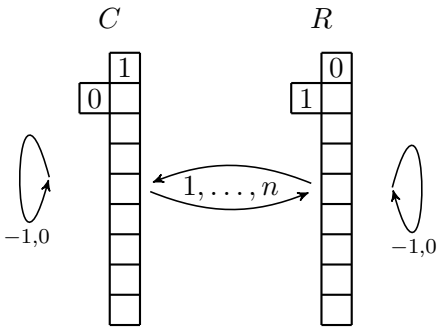


Figure 48. SIM_b on the double-tailed diamond visualised as digraph

Hence, we get the distribution matrix

$$\mathbf{A}\,(1;1^n) = \begin{pmatrix}2 & n\\ n & 2\end{pmatrix}.$$

The characteristic polynomial factors as

$$\det\begin{pmatrix}2-\epsilon & n\\ n & 2-\epsilon\end{pmatrix} = (2-\epsilon)^2 - n^2 = (2-\epsilon-n)(2-\epsilon+n).$$

Hence, we find the eigenvalues

$$\epsilon_1 = 2 + n \quad \text{and} \quad \epsilon_2 = 2 - n.$$

We know from before that $\mathbf{1}$ is the leading eigenvector and spectral theory or half a line of calculation tells us that the second eigenvector is orthogonal, i.e. the eigenvectors are

$$v_1 = \mathbf{1} \qquad \text{and} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We proceed with a straightforward counting argument yielding the initial distribution vector $\mathbf{z}((1; 1^n), C)$. Namely, we observe that every filling which has the 0 somewhere in the right column will be sorted to the standard filling R. Only if the 0 starts at the cell $\binom{2}{0}$ it stays there. Since there is no condition on the positive entries we acquire a factor $n!$, hence

$$\mathbf{z}((1; 1^n), C) = n! \begin{pmatrix} 1 \\ n \end{pmatrix}.$$

Straightforward computation yields coefficients to write the distribution vector as

$$\mathbf{z}((1; 1^n), C) = \frac{(n+1)!}{2}\, \mathbf{1} + \frac{(1-n)n!}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Applying the distribution matrices yields

$$\begin{aligned}
\mathbf{z}((m; 1^n), C) &= \mathbf{A}(m-1; 1^n) \cdots \mathbf{A}(1; 1^n)\ \mathbf{z}((1; 1^m), C) = \\
&= \mathbf{A}(m-1; 1^n) \cdots \mathbf{A}(1; 1^n) \left(\frac{(n+1)!}{2}\, \mathbf{1} + \frac{(1-n)n!}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \\
&= (2+n)^{\overline{m-1}} \frac{(n+1)!}{2}\, \mathbf{1} + (2-n)^{\overline{m-1}} \frac{(1-n)n!}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \\
&= \frac{(n+m)!}{2}\, \mathbf{1} + \frac{(1-n)^{\overline{m}} n!}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{aligned}$$

Plugging this into the lefthand side of (5.2) we get

$$\begin{aligned}
\mathbf{z}((m; 1^n), C) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \left(\frac{(n+m)!}{2}\, \mathbf{1} + \frac{(1-n)^{\overline{m}} n!}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \\
&= \frac{(n+m)!}{2}\, \mathbf{1} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{(1-n)^{\overline{m}} n!}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \\
&= 0 + \frac{(1-n)^{\overline{m}} n!}{2} 2 = \\
&= (1-n)^{\overline{m}} n!. \qquad \square
\end{aligned}$$

Considering this proof the nomenclature weighted deviation coefficient becomes reasonable. $\mathbf{z}((m;1^n),C)$ can be combined from $\mathbf{1}$ (with the **counting coefficient**) and the **deviation vector** v_2 with the weighted deviation coefficient. It is called weighted because v_2 is not normalised but we preferred to write it with integer components. The same applies for the Inset-Eigenvector-Distribution Conjecture, where we call v_y the deviation vector and Δ_S the weighted deviation coefficient (if the conjecture holds, it is obviously possible to write v_y with integer entries, hence, we would prefer also here the weighted version). With normalised v_2 or v_y the Δ-parameters would be a Cartesian measure for the distance from the distribution vector to the line $\mathbf{x} = \xi\,\mathbf{1}$.

5.5. The case $(1;2,1^{n-2})$**.** We can prove the Inset-Eigenvector-Distribution Conjecture and the Inset-Spectrum Conjecture for the case $(1;2,1^{n-2})$ under the column-wise algorithm. Before we attack the case for general n we work through the example $(1;2,1^2)$. There are seven standard fillings (see Figure 49).

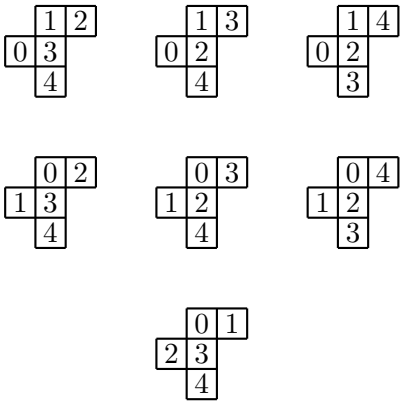


FIGURE 49. Standard fillings of $(1;2,1^2)$

If we order them as they are row-wisely from left to right in Figure 49 we get the distribution vector

$$\mathbf{z}((1;2,1^2),C) = \begin{pmatrix} 8 \\ 8 \\ 8 \\ 24 \\ 24 \\ 24 \\ 24 \end{pmatrix}. \tag{5.3}$$

This can easily be seen by counting arguments which we will present for the case of general n or simply by running the algorithm. The distribution matrix can be calculated

equally fast. It produces the transitions shown in Figure 50 and is given by

$$\mathbf{A}(1;2,1^2) = \begin{pmatrix} 2 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 2 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

where we emphasised the structure by printing the zeros grey.

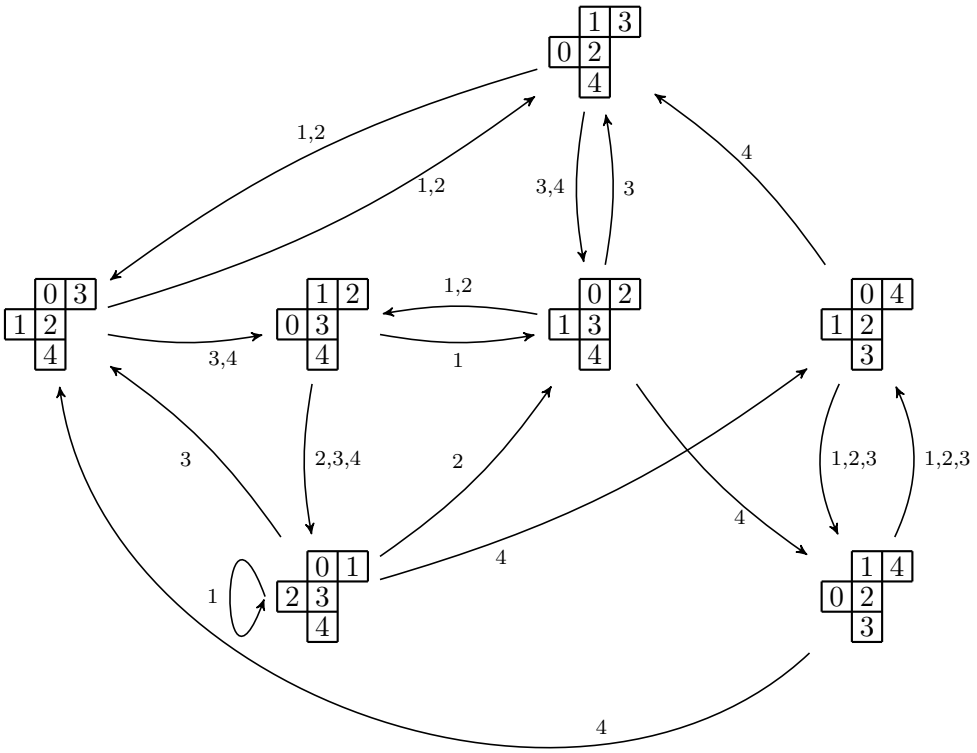


FIGURE 50. SIM_b for $(1;2,1^2)$

Any computer algebra system quickly determines the eigenvalues

$$\epsilon_1 = 6,\ \epsilon_2 = 4,\ \epsilon_3 = 3,\ \epsilon_4 = 2,\ \epsilon_5 = 1,\ \epsilon_6 = 0 \text{ and } \epsilon_7 = -1$$

and the corresponding eigenvectors

$$v_1 = \mathbf{1},\ v_2 = \begin{pmatrix} -3 \\ 2 \\ 7 \\ -4 \\ 1 \\ 6 \\ -9 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 \\ -2 \\ -2 \\ 1 \\ -1 \\ -1 \\ 5 \end{pmatrix},\ v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ -3 \end{pmatrix},\ v_6 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ -2 \\ -1 \\ 0 \\ -3 \end{pmatrix} \text{ and } v_7 = \begin{pmatrix} -4 \\ -4 \\ -4 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}.$$

The eigenpair ϵ_1, v_1 is obvious and the eigenpair ϵ_7, v_7 can as well be easily guessed and checked without the use of a computer. Again, it is straightforward to determine coefficients to express the distribution vector as

$$\mathbf{z}((1;2,1^2),C) = \frac{120}{7}\, v_1 + \frac{16}{7}\, v_7.$$

Hence the Inset-Eigenvector-Distribution Conjecture and the Inset-Spectrum Conjecture both hold for the case $(1;2,1^2)$ and we turn to the case $(1;2,1^{n-2})$.

We start by describing the standard fillings. There are single cells in the first and third column (i.e. in the columns 0 and 2), if we fix their entries $a = S(2,0)$ and $c = S(1,2)$, the entries of the middle column are uniquely determined. Hence, we can denote the fillings by ${}_a[]^c$. If $a = 0$, the top entry in the second column must be 1 and c can take any of the values $2, \ldots, n$. Hence, we collect the possible standard fillings

$${}_2[]^0, \ldots, {}_0[]^n \in \mathrm{SYT}(1;2,1^{n-2}).$$

If $a = 1$, the top entry in the second column must be 0 and we collect analogously the standard fillings

$${}_1[]^2, \ldots, {}_1[]^n \in \mathrm{SYT}(1;2,1^{n-2}).$$

Finally, $a = 2$ is only possible, if the cells in the first row carry the entries 0 and 1, so there is

$${}_2[]^1 \in \mathrm{SYT}(1;2,1^{n-2})$$

and there are no other possible standard fillings. Hence, the number of standard fillings of $(1;2,1^{n-2})$ is

$$\left|\mathrm{SYT}(1;2,1^{n-2})\right| = 2n-1.$$

Note that this equals the number of different eigenvalues claimed by the Inset-Spectrum Conjecture.

Our next task is to determine the distribution vector $\mathbf{z}((1;2,1^{n-2}),C)$. Again we start by recognising that the 0 cannot reach the cell $\binom{2}{0}$ if it starts somewhere else. If the 0 starts in the cell $\binom{2}{0}$ the 1 can start in the second column or at the cell $\binom{1}{2}$. If it starts in the second column the entry c of the cell $\binom{1}{2}$ will not move (for every $c>1$ there are $(n-1)!$ such cases – see Figure 51 left). If 1 starts at the cell $\binom{1}{2}$ it will also reach $\binom{1}{1}$ being exchanged with the entry c that starts there (for every $c>1$ there are $(n-2)!$ such cases – see Figure 51 right). This yields

$$\mathrm{z}_{0[]^c}((1;2,1^{n-2}),C) = (n-1)! + (n-2)! = (n-2)!(1+n-1) = n(n-2)!.$$

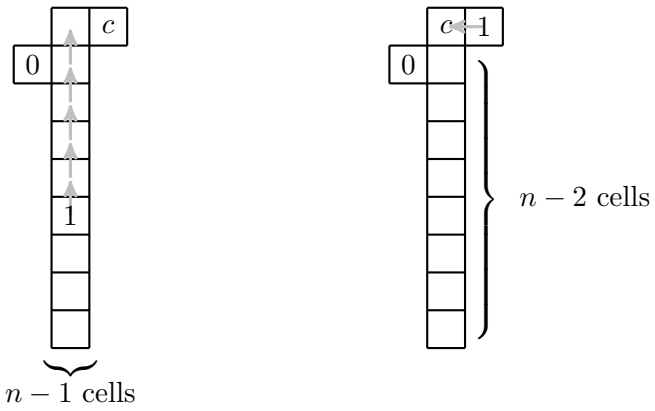


FIGURE 51. Preimages of $_0[]^c$

The outcome $a=1$ will appear if the 0 starts in the second or third column (unless they both start in the first row or the 1 starts at $\binom{1}{2}$). We reach $_1[]^c$ if c starts at $\binom{1}{2}$ and 0 in the second column (these are $(n-1)(n-1)!$ cases for every c – see Figure 52 left) or if the 0 starts at $\binom{1}{2}$ and c completes the first row (these are $(n-1)!$ cases – see Figure 52 right). This yields

$$\mathrm{z}_{1[]^c}((1;2,1^{n-2}),C) = (n-1)(n-1)! + (n-1)! = (n-1)!(n-1+1) = n!.$$

Analogously, $a=2$ requires either the 0 starting in the second row and 1 at $\binom{1}{2}$ or both of them in the first row (see Figure 53) – and the same calculation yields

$$\mathrm{z}_{2[]^1}((1;2,1^{n-2}),C) = n!.$$

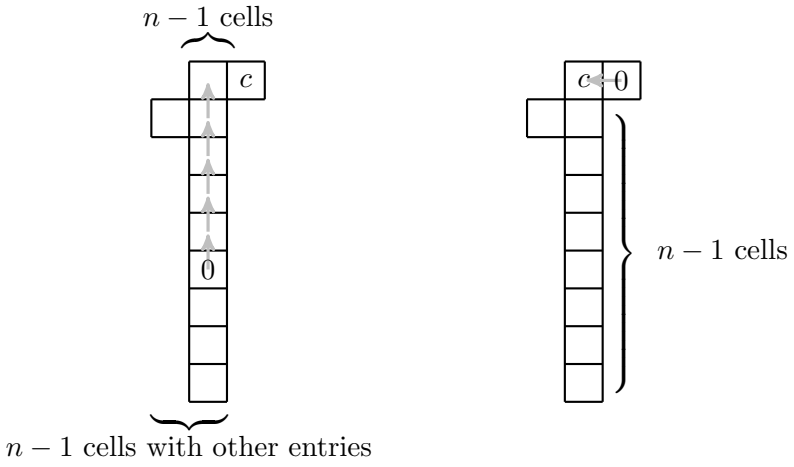


FIGURE 52. Preimages of $_1[]^c$

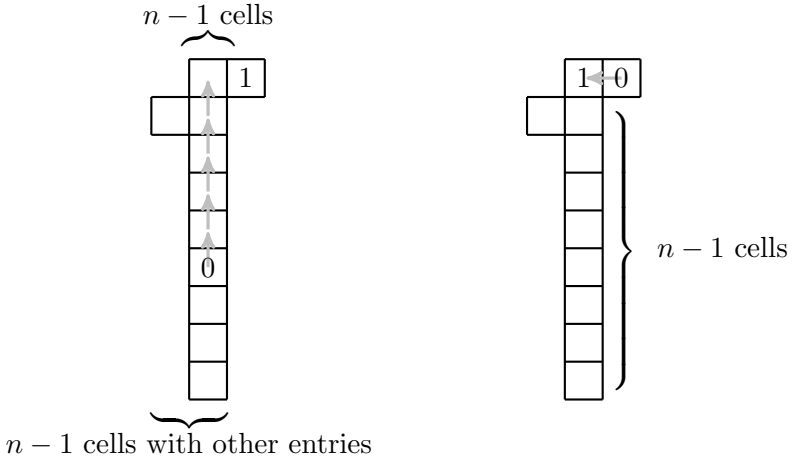


FIGURE 53. Preimages of $_2[]^1$

To ensure that we did not forget any fillings we can recheck by summation, and indeed we find

$$(n-1)n(n-2)! + nn! = n!(1+n) = (n+1)! = \left|\mathrm{T}(1;2,1^{n-2})\right|.$$

If we sort the standard fillings lexicographically we get the beautiful distribution vector

$$\mathbf{z}((1;2,1^{n-2}),C) = \begin{pmatrix} n(n-2)! \\ \vdots \\ n(n-2)! \\ n! \\ \vdots \\ n! \\ n! \end{pmatrix},$$

where the first $n-1$ components are $n(n-2)!$ and $n!$ appears n times (one extra time at the final position which belongs to the standard filling ${}_2[]^1$).

We now turn to calculating the entries of the distribution matrix. The reader may find it helpful to compare the arguments with Figure 50. First we recognise that SIM_b either changes the position of the 0 or stabilises the entire standard filling. Hence the blocks on the diagonal of $\mathbf{A}(1;2,1^{n-2})$ will be diagonal. The standard fillings ${}_0[]^c$ and ${}_1[]^c$ will all be stabilised by SIM_b if and only if $b \in \{-1,0\}$. ${}_2[]^1$ will additionally be stabilised by $b=1$ which will perform two east steps, i.e.

$$\begin{aligned} \mathrm{A}_{{}_0[]^c,{}_0[]^c}(1;2,1^{n-2}) &= 2, \\ \mathrm{A}_{{}_1[]^c,{}_1[]^c}(1;2,1^{n-2}) &= 2 \qquad \text{and} \\ \mathrm{A}_{{}_2[]^1,{}_2[]^1}(1;2,1^{n-2}) &= 3. \end{aligned}$$

Sticking to the argument of the two east steps we see that for every $b \geq 2$ we get $\mathrm{SIM}_b : {}_2[]^1 \mapsto {}_1[]^b$ and, hence,

$$\mathrm{A}_{{}_2[]^1,{}_1[]^c}(1;2,1^{n-2}) = 1.$$

If we consider the standard filling ${}_0[]^c$ and wonder what happens if b is positive, we find, that the jeu de taquin starts with a south step and hence the value of a will be positive. For $c=2$ and $b=1$ the jeu de taquin stops after the first step and we have $\mathrm{SIM}_1 : {}_0[]^2 \mapsto {}_1[]^2$, every larger value of b will shrink c to 1 which means that $\mathrm{SIM}_b : {}_0[]^2 \mapsto {}_2[]^1$ i.e.

$$\begin{aligned} \mathrm{A}_{{}_0[]^2,{}_1[]^2}(1;2,1^{n-2}) &= 1 \qquad \text{and} \\ \mathrm{A}_{{}_0[]^2,{}_2[]^1}(1;2,1^{n-2}) &= n-1. \end{aligned}$$

For every larger value of c the standard filling ${}_0[]^c$ has the 2 as entry of $\binom{2}{1}$, hence, every positive b will produce $a=1$. If $b<c$ the latter will be unchallenged otherwise it will

shrink by 1. We get

$$\begin{aligned} \mathrm{A}_{0[]^c,1[]^c}(1;2,1^{n-2}) &= c-1 \qquad \text{and} \\ \mathrm{A}_{0[]^c,1[]^{c-1}}(1;2,1^{n-2}) &= n+1-c. \end{aligned}$$

For $c \geq 3$ the very same argumentation leads to

$$\begin{aligned} \mathrm{A}_{1[]^c,0[]^c}(1;2,1^{n-2}) &= c-1 \qquad \text{and} \\ \mathrm{A}_{1[]^c,0[]^{c-1}}(1;2,1^{n-2}) &= n+1-c. \end{aligned}$$

We are left with determining the images of $1[]^2$ for positive b. For $b=1$ the jeu de taquin is a single east step, for $b=2$ we get two east steps but the same result, hence

$$\mathrm{A}_{1[]^2,0[]^2}(1;2,1^{n-2}) = 2.$$

For larger b we always find two east steps and $\mathrm{SIM}_b : 1[]^2 \mapsto 0[]^b$, i.e. for $c>2$ we have

$$\mathrm{A}_{1[]^2,0[]^c}(1;2,1^{n-2}) = 1.$$

We can convince ourselves that we did not forget anything by recognising that every row and every column of $\mathbf{A}(1;2,1^{n-2})$ sums up to $n+2$. We use the same lexicographic order as above, suppress the zeros and put lines to separate the values of a and the above arguments to get the beautiful form

$$\mathbf{A}(1;2,1^{n-2}) = \left(\begin{array}{c|c|c|c} \begin{array}{ccccc} 2 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & & 2 \end{array} & \begin{array}{c} 2 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array} & \begin{array}{ccccc} n{-}2 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & n{-}2 & 1 \\ & & & & n{-}1 \end{array} & \\ \hline \begin{array}{ccccc} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & n{-}2 & 1 \\ & & & & n{-}1 \end{array} & \begin{array}{ccccc} n{-}2 & & & & \\ & \ddots & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{array} & \begin{array}{ccccc} 2 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & & 2 \end{array} & \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array} \\ \hline n{-}1 & & & 3 \end{array}\right).$$

Again the eigenpair $\epsilon_1 = n+2, v_1 = \mathbf{1}$ is immediate and with the conditions that the first $n-1$ components shall agree and so shall the last n it is very easy to guess the

eigenvector

$$v_y = \begin{pmatrix} -n \\ \vdots \\ -n \\ n-1 \\ \vdots \\ n-1 \\ n-1 \end{pmatrix}$$

to which we find the eigenvalue $\epsilon_y = 3 - n = 2 - \lambda'_1$. Another short calculation yields coefficients such that

$$\mathbf{z}((1; 2, 1^{n-2}), C) = \frac{(n+1)!}{2n-1}\, \mathbf{1} + \frac{n(n-2)(n-2)!}{2n-1}\, v_y.$$

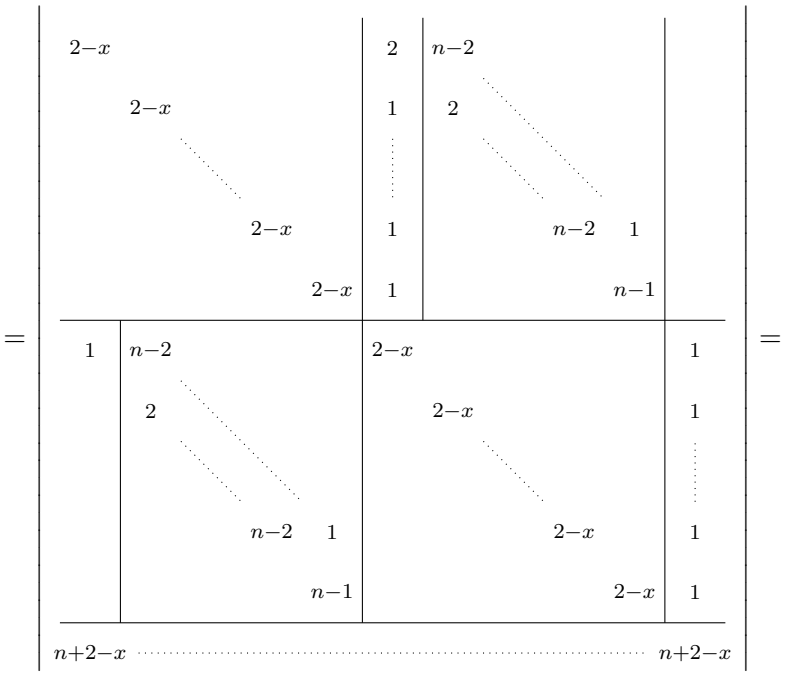
Hence, the Eigenvector-Distribution Conjecture holds for the case $(1; 2, 1^{n-2})$.

We can actually do better and prove the Inset-Spectrum Conjecture for this case as well. We do so by factoring the characteristic polynomial.

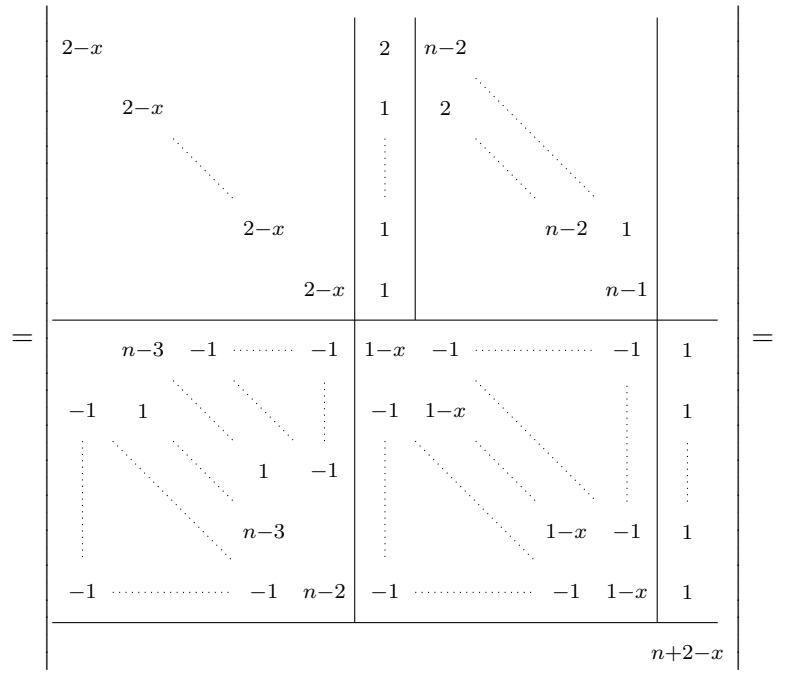
(5.4)

$$\det\left(\mathbf{A}(1; 2, 1^{n-2}) - x\, I\right) = \left|\begin{array}{c|c|c|c} \begin{array}{ccc} 2{-}x & & \\ & \ddots & \\ & & 2{-}x \end{array} & \begin{array}{c} 2 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array} & \begin{array}{ccc} n{-}2 & & \\ & \ddots & \\ & n{-}2 & 1 \end{array} & \\ \hline 1 & \begin{array}{ccc} n{-}2 & & \\ & \ddots & \\ & & 1 \end{array} & \begin{array}{ccc} 2{-}x & & \\ & \ddots & \\ & & 2{-}x \end{array} & \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array} \\ \hline n{-}1 & & & 3{-}x \end{array}\right| =$$

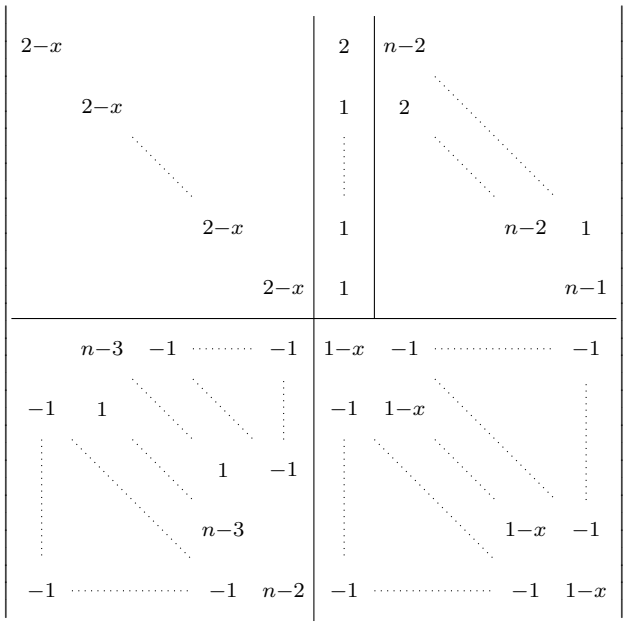
Since we know that all columns have the same sum we start by adding all other rows to the bottom row.



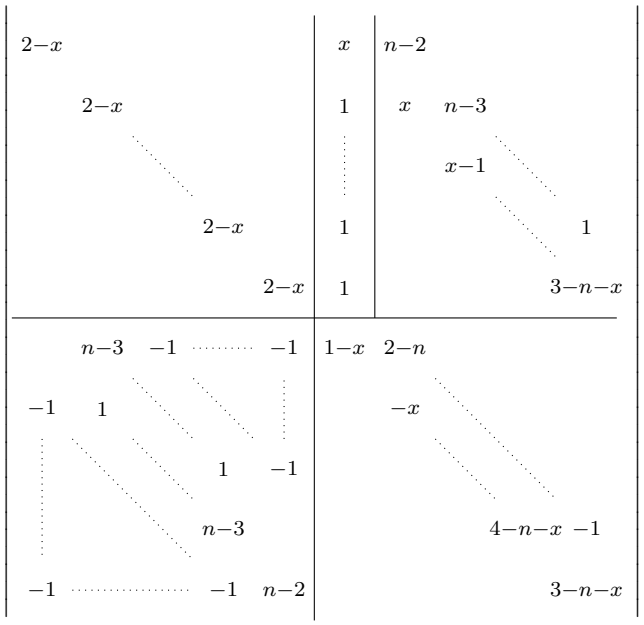
We subtract the last column from every other column to get rid of the entries in the last row. This has no effect on the first $n-1$ rows, but the second $n-1$ rows get 1 subtracted everywhere but in the last column.



We can now expand along the last row to extract the first factor of the polynomial.

$$= (n+2-x)$$

$$=$$

We can get rid of the -1's in the bottom right block by subtracting the i^{th} column from the $(i+n-1)^{\text{st}}$ column where $i = 1, \ldots, n-1$. This also creates some of the diagonal entries that we will need.

$$= (n+2-x)$$

$$=$$

We can now clear a lot in the top right block by adding the i^{th} row to the $(i-n-1)^{\text{st}}$ row where $i = n, \ldots, 2n-2$, and simultaneously modify the diagonal entries in the top left

block as we desire.

$$= (n+2-x)\begin{vmatrix} 2-x & n-3 & -1 & \cdots & -1 & 1 & \\ -1 & 3-x & \ddots & & \vdots & 1 & \\ \vdots & \ddots & \ddots & 1 & -1 & \vdots & \\ & & n-1-x & & & 1 & \\ -1 & \cdots & \cdots & -1 & n-x & 1 & \\ \hline & n-3 & -1 & \cdots & -1 & 1-x & 2-n \\ -1 & 1 & \ddots & & & -x & \ddots \\ \vdots & \ddots & \ddots & 1 & -1 & \ddots & \\ & & n-3 & & & 4-n-x & -1 \\ -1 & \cdots & \cdots & -1 & n-2 & & 3-n-x \end{vmatrix} =$$

We now reached a situation where the sums of the first halfrows vanish in the lower half of the matrix and are $2-x$ in the upper half. We take advantage of this by adding the first $n-2$ columns to the $(n-1)^{\text{st}}$ column.

$$= (n+2-x)\begin{vmatrix} 2-x & n-3 & -1 & \cdots & -1 & 2-x & 1 & \\ -1 & 3-x & \ddots & & -1 & & 1 & \\ \vdots & \ddots & \ddots & 1 & & & \vdots & \\ & & n-1-x & & & & 1 & \\ -1 & \cdots & \cdots & -1 & 2-x & & 1 & \\ \hline & n-3 & -1 & \cdots & -1 & & 1-x & 2-n \\ -1 & 1 & \ddots & & -1 & & -x & \ddots \\ \vdots & \ddots & \ddots & 1 & & & \ddots & \\ & & n-3 & & & & 4-n-x & -1 \\ -1 & \cdots & \cdots & -1 & & & & 3-n-x \end{vmatrix} =$$

We use the $(n-1)^{\text{st}}$ row to clear the 1's, $(2-x)$'s and some -1's by subtracting it from the rows 1 to $n-2$.

$$= (n+2-x)\left|\begin{array}{cccc|cc} 3-x & n-2 & & & & \\ & 4-x & & & & \\ & & 2 & & & \\ & & & n-x & & \\ -1 & \cdots & -1 & 2-x & 1 & \\ \hline & n-3 & -1\cdots & -1 & 1-x & 2-n \\ -1 & 1 & & -1 & & -x \\ & & & 1 & & \\ & & & n-3 & & 4-n-x\ -1 \\ -1 & \cdots & -1 & & & 3-n-x \end{array}\right| =$$

We can now expand along the $(n-1)^{\text{st}}$ column to extract the second factor.

$$= (n+2-x)(2-x)\left|\begin{array}{ccc|cc} 3-x & n-2 & & & \\ & 4-x & & & \\ & & 2 & & \\ & & n-x & & \\ \hline & n-3 & -1\cdots -1 & 1-x & 2-n \\ -1 & 1 & -1 & & -x \\ & & 1 & & \\ & & n-3 & & 4-n-x\ -1 \\ -1 & \cdots & -1 & & 3-n-x \end{array}\right| =$$

Since the top right block is now empty, we can split the determinant in the product of the determinants of the diagonal blocks.

$$= (n+2-x)(2-x) \begin{vmatrix} 3-x & n-2 & & \\ & 4-x & \ddots & \\ & & \ddots & 2 \\ & & & n-x \end{vmatrix} \begin{vmatrix} 1-x & 2-n & & \\ & -x & \ddots & \\ & & \ddots & 4-n-x \; -1 \\ & & & 3-n-x \end{vmatrix} =$$

Since these matrices are upper triangular, we can read off the factors of the characteristic polynomial on the diagonals.

$$= (n+2-x)(n-x)(n-1-x)\dots(3-n-x).$$

Hence, also the Inset-Spectrum Conjecture holds for the case $(1;2,1^{n-2})$. Moreover, all possible eigenvalues exist and are simple. For every larger example we observed that there were some multiplicities of eigenvalues missing i.e. the matrices are not diagonalisable. More importantly, we want to remark that in general the only eigenvalue that is simple is the leading one. Especially note, that the eigenspace of the least eigenvalue is of higher dimension. Hence, it is totally unclear from a spectral point of view which vector in this space should be the deviation vector. This suggests that the most promising approaches to prove the Inset-Eigenvector-Distribution Conjecture are either to guess the deviation vector and derive the correct eigenvalue or to find an induction. At the time being we believe that the latter has better chances to succeed.

5.6. Distribution matrices in a more general setting. If one adds a cell, that is the unique top cell after the addition, it is clear, that it will carry the smallest entry after the jeu de taquin was played. Hence, deleting this cell again makes no difference to the index-set of the distribution vector. The situation is different if we add a top cell which is not unique, since the smallest entry might be stuck in a different top cell such that deleting the added cell shrinks the set of standard fillings.

This yields two different operations treating the addition of an explicit cell x_i. For easier notation we use a shifting function.

Definition 5.16 (Conditional shift). *The* **Conditional shift** $\mathrm{sh}_b : \mathbb{Z} \to \mathbb{Z}$ *stabilises everything larger than* b *and decrements everything else, i.e.*

$$\mathrm{sh}_b(t) = \begin{cases} t & \text{if } t > b \\ t-1 & \text{if } t \leq b. \end{cases}$$

We extend our above notation with the specification of the added cell.

DEFINITION 5.17 (Simulation of a cell). *Let P be a skew diagram and $x \notin P$ a cell such that $P \cup \{x\}$ is a skew diagram. Let $S \in \mathrm{SYT}(P)$ be a standard filling, $n = \max\{S(P)\}$, $N = \min\{S(P)\}$, t a positive integer and $b \in \{N-t, \dots, n\}$. Then the* **simulation of the cell** *x denoted by SIM_b^x is defined by the following algorithm.*

1. *Start with the standard filling S.*
2. *Apply sh_b to the entries of P.*
3. *Add the cell x with entry b.*
4. *Apply jdt_x to this filling of $P \cup \{x\}$ and call the resulting standard filling U.*
5. *Remove the cell x along with its entry.*
6. *Apply $\mathrm{sh}_{U(x)}^{-1}$ to the remaining entries of P to end up with the standard filling $\mathrm{SIM}_b^x(S) \in \mathrm{SYT}(P)$.*

Note that, if x is a unique top cell, $\mathrm{sh}_{U(x)}^{-1}\Big|_P$ is the identity. Intuition tells us that $t = 1$ makes sense, larger values of t simulate some kind of neck attached to x since no value $b < N$ will reach P.

DEFINITION 5.18 (Addition of a cell). *Let P be a skew diagram and $x \notin P$ a cell such that $P \cup \{x\}$ is a skew diagram. Let $S \in \mathrm{SYT}(P)$ be a standard filling, $n = \max\{S(P)\}$, $N = \min\{S(P)\}$ and $b \in \{N-1, \dots, n\}$. Then the* **addition of the cell** *x denoted by ADD_b^x is defined by the following algorithm.*

1. *Start with the standard filling S.*
2. *Apply sh_b to the entries of P.*
3. *Add the cell x with entry b.*
4. *Apply jdt_x to this filling of $P \cup \{x\}$ to end up with the standard filling $\mathrm{ADD}_b^x(S) \in \mathrm{SYT}(P \cup \{x\})$.*

Comparing ADD with the operation add from Section 2 we recognise that they are equal descriptions of the same thing. If we replace the entries of x_k with their image under the unique order-preserving map $p_k : F(\nu_k) \to \{n-k+1, \dots, n\}$ and then apply all $\mathrm{ADD}_{p_k(F(x_k))}^{x_k}$ in the order $k = 1, \dots, k = n$, this is the same as applying $\mathrm{add}_{x_k}^F$. This is again an immediate consequence of the argument that the relative orders are preserved. To illustrate this, we pick up our example from Figure 16 in Figure 54 where we highlighted the entries which are shifted down to create the space for the entry of x_k which is about to drop.

Equipped with these operations we redefine the distribution matrices.

DEFINITION 5.19 (virtual distribution matrix). *Let P be a skew diagram and x_k a cell such that $P \cup \{x_k\}$ is again a skew diagram. Consider the standard fillings $\mathrm{SYT}(P)$*

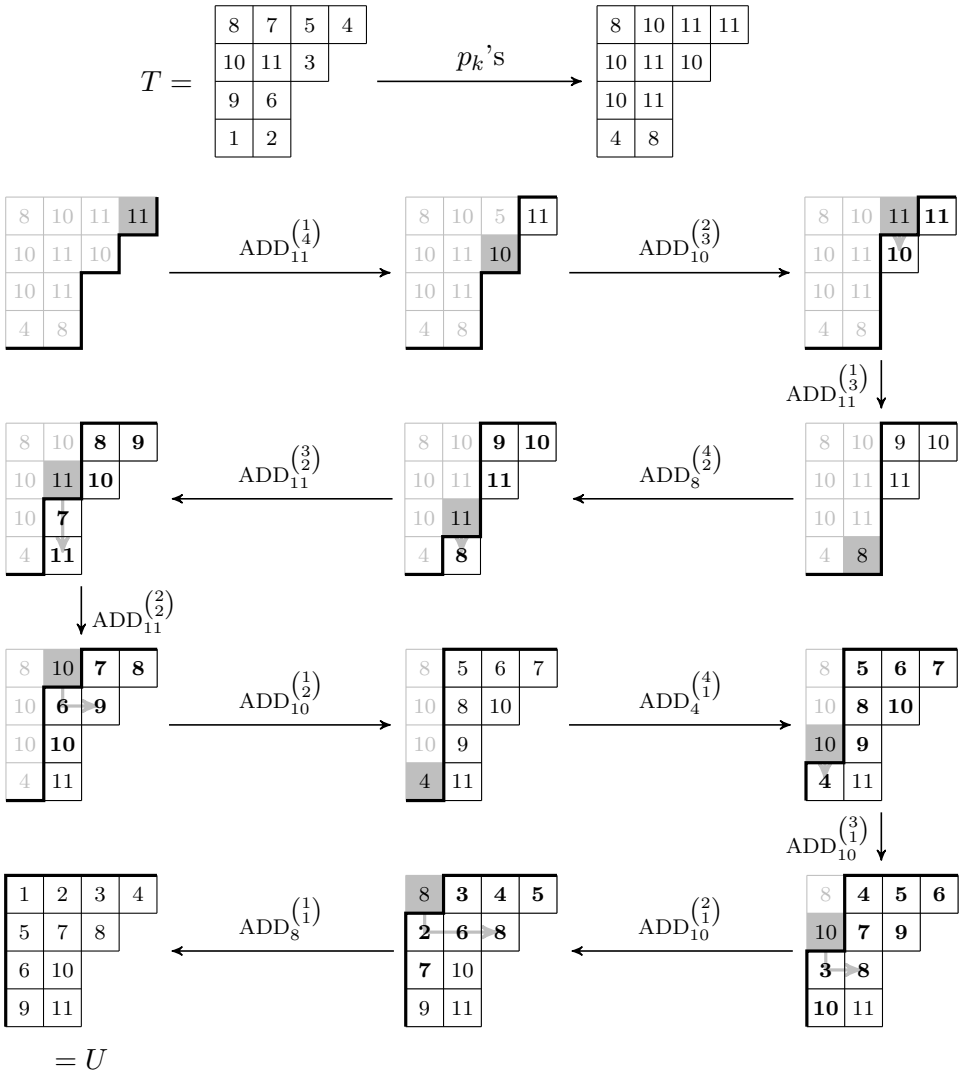


FIGURE 54. Sorting a filling of $(4,3,2,2)$ using ADD

and use for them the letters U *and* V *and let*

$$\begin{aligned} N &= \min V(P), \\ n &= \max V(P) \quad \textit{and} \\ t &\in \mathbb{N}^+. \end{aligned}$$

We define

$$\mathrm{A}_{U,V}(t; x_k \!\times\!\!\mapsto P) := \big|\{b \in \{N-t, \ldots, n\} \mid \mathrm{SIM}_b^{x_k}(V) = U\}\big|$$

and combine these entries to the **virtual distribution matrix**

$$\mathbf{A}(t; x_k \mathrel{\rotatebox[origin=c]{180}{$\rightarrowtail$}} P) := \left(\mathrm{A}_{U,V}(t; x_k \mathrel{\rotatebox[origin=c]{180}{$\rightarrowtail$}} P)\right)_{U,V}.$$

Definition 5.20 (distribution matrix). *Let P be a skew diagram and x_k a cell such that $P \cup \{x_k\}$ is again a skew diagram. Consider the standard fillings* $\mathrm{SYT}(P)$ *and use for them the letter V and let*

$$\begin{aligned} N &= \min V(P) \quad \textit{and} \\ n &= \max V(P). \end{aligned}$$

Further use for standard fillings in $\mathrm{SYT}(P \cup \{x_k\})$ *the letter U. We define*

$$\mathrm{B}_{U,V}(x_k \boxdot\!\!\rightarrow P) := \left|\{b \in \{N-1, \ldots, n\} \mid \mathrm{ADD}_b^{x_k}(V) = U\}\right|$$

and combine these entries to the **distribution matrix**

$$\mathbf{B}(\mathrm{x_k} \boxdot\!\!\rightarrow \mathrm{P}) := \left(\mathrm{B}_{U,V}(x_k \boxdot\!\!\rightarrow P)\right)_{U,V}.$$

If x_k is a unique top cell, and the $t = 1$ agree, the distribution matrix and the virtual distribution matrix are equal. If x_k is in the neck of an inset, both specialise to the distribution matrices defined earlier. The virtual distribution matrices are always square, but they do not reflect the NPS-algorithm. The distribution matrices, however, can be used to follow the sorting algorithm.

Lemma 5.21. *Let P and Q be skew diagrams such that there is a cell x_k with $Q = P \cup \{x_k\}$. Let $S \in \mathrm{SYT}(Q)$ be an algorithm-defining standard filling such that x_k is $\prec_S$-minimal (i.e. is processed last). Then*

$$\mathbf{z}(Q, S) = \mathbf{B}(\mathrm{x_k} \boxdot\!\!\rightarrow \mathrm{P})\ \mathbf{z}(P, S|_P).$$

Proof. The arguments are the same as for Lemma 5.8. □

From Lemma 5.21 the following corollary is immediate.

Corollary 5.22. *Let P be a skew diagram and $S \in \mathrm{SYT}(P)$ define the $\prec_S$-algorithm. Let $x_k \prec_S \cdots \prec_S x_1$ be the cells of P and*

$$\begin{aligned} P_i &:= \bigcup_{j=1}^{i} \{x_i\} \quad \textit{and} \\ S_i &:= S|_{P_i}. \end{aligned}$$

Then for $1 \le l \le k$ we have

$$\mathbf{z}(P_l, S_l) = \mathbf{B}(\mathrm{x_l} \boxdot\!\!\rightarrow \mathrm{P_{l-1}})\mathbf{B}(\mathrm{x_{l-1}} \boxdot\!\!\rightarrow \mathrm{P_{l-2}}) \cdots \mathbf{B}(\mathrm{x_1} \boxdot\!\!\rightarrow \mathrm{P_0})\ \mathbf{1}_1$$

where P_0 is the empty diagram.

Note that the empty diagram has a standard filling (namely the empty filling), hence, $\mathbf{B}(\mathrm{x}_1 \boxdot\!\!\rightarrow \mathrm{P}_0)$ is a (1×1)-matrix and it has the entry 1.

We want to introduce one more matrix. Its definition may seem very unmotivated since like for the virtual distribution matrices its combinatorial use may be questioned. Nevertheless it appeared in our investigations and we do not want to hold it back.

DEFINITION 5.23 (top-cell matrix). *Let P be a skew diagram and let $\{y_1, \ldots, y_k\}$ be the set of top cells of P. Consider the skew diagrams*

$$P^i := P \setminus y_i.$$

We use lines between blocks to denote that a matrix consists of the blocks in between. Then the **top-cell matrix** *is given by*

$$\mathbf{T}(\boxplus\!\!\rightarrow P) := \left(\mathbf{B}(\mathrm{y}_1 \boxdot\!\!\rightarrow \mathrm{P}^1) \middle| \mathbf{B}(\mathrm{y}_2 \boxdot\!\!\rightarrow \mathrm{P}^2) \middle| \ldots \middle| \mathbf{B}(\mathrm{y}_\mathrm{k} \boxdot\!\!\rightarrow \mathrm{P}^\mathrm{k})\right).$$

Since in every standard filling the smallest entry must be located at a top-cell y_i, we can forget about this cell of P to get a standard filling of $P \setminus y_i$. For the same reason $\mathrm{SYT}(P)$ can be partitioned into k subsets corresponding to the sets $\mathrm{SYT}(P^1), \ldots, \mathrm{SYT}(P^k)$. Hence, $\mathbf{T}(\boxplus\!\!\rightarrow P)$ is square.

5.7. Distribution matrices of Young diagrams. We are now going to have a look at distribution matrices in the setting of Young diagrams. First we will be interested in the distribution matrix $\mathbf{B}(\binom{1}{1} \boxdot\!\!\rightarrow \lambda/(1)) = \mathbf{A}(1; \binom{1}{1} \times\!\!\rightarrow \lambda/(1))$ of the top left corner and the distribution vector $\mathbf{z}(\lambda/(1), S)$ before the last cell is added, where S is the restriction of a standard Young tableau that qualifies as order-defining for the Novelli-Pak-Stoyanovskii algorithm (i.e. is given by Remark 2.16). We have

$$\mathbf{z}(\lambda, S) = \mathbf{B}(\tbinom{1}{1} \boxdot\!\!\rightarrow \lambda/(1))\ \mathbf{z}(\lambda/(1), S).$$

From the Novelli-Pak-Stoyanovskii bijection we know that $\mathbf{z}(\lambda, S)$ exhibits uniform distribution. Moreover, we know that the sum of the entries of $\mathbf{z}(\lambda/(1), S)$ counts the fillings of $\lambda/(1)$, hence,

$$(n-1)! = \sum_{T \in \mathrm{SYT}(\lambda/(1))} \mathrm{z}_T(\lambda/(1), S) = \mathbf{z}(\lambda/(1), S) \cdot \mathbf{1}.$$

This means that $\mathbf{z}(\lambda/(1), S)$ lies on a (non-homogeneous) hyperplane orthogonal to $\mathbf{1}$ i.e. there is a vector v_y with $v_y \cdot \mathbf{1} = 0$ such that

$$\mathbf{z}(\lambda/(1), S) = \frac{(n-1)!}{f_\lambda}\ \mathbf{1} + v_y.$$

Since we know that $\mathbf{1}$ is the leading eigenvector of $\mathbf{B}(\binom{1}{1}\boxplus\lambda/(1))$ (having eigenvalue $\varepsilon_1 = n$) we have

$$\mathbf{z}(\lambda, S) = \frac{n!}{f_\lambda}\,\mathbf{1} = \mathbf{B}(\tbinom{1}{1}\boxplus\lambda/(1))\,\frac{(n-1)!}{f_\lambda}\,\mathbf{1}$$

and hence,

$$v_y \in \ker\,(\mathbf{B}(\tbinom{1}{1}\boxplus\lambda/(1))).$$

This can be interpreted as an Eigenvector-Distribution Theorem for Young diagrams.

Let $T \in \mathrm{SYT}(\lambda)$ be such as the above S with the additional preliminary that the top row is processed last, and let $T' = T|_{\lambda/(\lambda_1)}$ be the restriction of T to the Young tableau without the first row. It is clear form the Novelli-Pak-Stoyanovskii algorithm, that the distribution vectors $\mathbf{z}(\lambda, T)$ and $\mathbf{z}(\lambda/(\lambda_1), T')$ both exhibit uniform distribution. We get

$$\underbrace{\mathbf{z}(\lambda/(0), T)}_{=n!\,\mathbf{1}_{f_\lambda}} = \underbrace{\mathbf{B}(\tbinom{1}{1}\boxplus\lambda/(1))\overbrace{\mathbf{B}(\tbinom{1}{2}\boxplus\lambda/(2))\cdots\mathbf{B}(\tbinom{1}{\lambda_1}\boxplus\lambda/(\lambda_1))}^{\mathbf{Q}(\boxplus\!\!\rightarrow\lambda):=}}_{=:\mathbf{P}(\boxminus\!\!\rightarrow\lambda)}\underbrace{\mathbf{z}(\lambda/(\lambda_1), T')}_{=(n-\lambda_1)!\,\mathbf{1}_{f_{\lambda/(\lambda_1)}}} \tag{5.5}$$

Here, we have to discuss the notation for $\mathbf{P}(\boxminus\!\!\rightarrow\lambda)$ and $\mathbf{Q}(\boxplus\!\!\rightarrow\lambda)$. To be consistent with the notations introduced before, the parameters should read $(i)\boxminus\!\!\rightarrow\lambda/(i)$ and $(i)/(1)\boxplus\!\!\rightarrow\lambda/(i)$ but since this is very unhandy and λ carries all information needed, we prefer the notation as introduced.

Note that (5.5) implies that the rows of $\mathbf{P}(\boxminus\!\!\rightarrow\lambda)$ have equal sums. $\mathbf{P}(\boxminus\!\!\rightarrow\lambda)$ is a linear map $\mathbb{R}^{|\mathrm{SYT}(\lambda/(\lambda_1))|} \to \mathbb{R}^{|\mathrm{SYT}(\lambda)|}$ and we can index its columns by $\mathrm{SYT}(\lambda/(\lambda_1))$ and its rows by $\mathrm{SYT}(\lambda)$. Hence, we can write for every $U \in \mathrm{SYT}(\lambda)$

$$\sum_{V\in\mathrm{SYT}(\lambda/(\lambda_1))} \mathrm{P}_{U,V}(\boxminus\!\!\rightarrow\lambda) = \frac{f^{\lambda/(\lambda_1)}}{f^\lambda} n^{\underline{\lambda_1}}. \tag{5.6}$$

Moreover, $\mathrm{ADD}_b^{(1,i)}$ can create from one standard filling of $\lambda/(i)$ as many (not necessarily different) standard fillings of $\lambda/(i-1)$ as there are possible values of b. Hence, the contribution of any 1 in $\mathbf{z}(\lambda/(\lambda_1), R)$ to $\mathbf{z}(\lambda, R)$ is $n^{\underline{\lambda_1}}$. Therefore, also the sums of the columns of $\mathbf{P}(\boxminus\!\!\rightarrow\lambda)$ are equal, namely for all $V \in \mathrm{SYT}(\lambda)$

$$\sum_{U\in\mathrm{SYT}(\lambda/(\lambda_1))} \mathrm{P}_{U,V}(\boxminus\!\!\rightarrow\lambda) = n^{\underline{\lambda_1}}. \tag{5.7}$$

The latter argument also works for $\mathbf{Q}(\boxplus\!\!\rightarrow\lambda)$, i.e. for all $V \in \mathrm{SYT}(\lambda/(1))$

$$\sum_{U\in\mathrm{SYT}(\lambda/(\lambda_1))} \mathrm{Q}_{U,V}(\boxplus\!\!\rightarrow\lambda) = (n-1)^{\underline{\lambda_1-1}}.$$

But equation (5.6) does not apply for $\mathbf{Q}(\sqsupset\!\!\rightarrow\lambda)$ since

$$\frac{(n-1)!}{f^\lambda}\mathbf{1} + v_y = \mathbf{z}(\lambda/(1), R) = \mathbf{Q}(\sqsupset\!\!\rightarrow\lambda)\ \mathbf{z}(\lambda/(\lambda_1), R).$$

We would like to know more about $\mathbf{Q}(\sqsupset\!\!\rightarrow\lambda)$, ideally enough be able to generally guess v_y. We are not sure if this can be achieved, but at least we can say quite a bit about the $\mathbf{B}(\binom{1}{\mathrm{i}}\boxdot\!\!\rightarrow\lambda/(\mathrm{i}))$'s. To do so, we introduce a notation to identify blocks of distribution matrices.

Definition 5.24 ($\mathcal{W} \to \mathcal{U}$ block). *Let $\mathbf{M}(\cdot)$ be a distribution matrix of any kind with index sets $\mathrm{SYT}(P)$ of the rows and $\mathrm{SYT}(Q)$ of the columns (i.e. $\mathbf{M}(\cdot)$ describes how an addition or simulation maps standard fillings of P to standard fillings of Q). Let $\mathcal{W} \subset \mathrm{SYT}(P)$ and $\mathcal{U} \subset \mathrm{SYT}(Q)$, then the $\mathcal{W} \to \mathcal{U}$* **block** *is the matrix*

$$\mathbf{M}^{\mathcal{W}\to\mathcal{U}}(\cdot) := (M_{U,W}(\cdot))_{U\in\mathcal{U},W\in\mathcal{W}}\,.$$

I.e. the $\mathcal{W} \to \mathcal{U}$ block describes how the operation that generates $\mathbf{M}(\cdot)$ transforms standard fillings in $\mathcal{W}$ to standard fillings in $\mathcal{U}$.

Example 5.25. Consider the simulation of $\binom{1}{1}$ with $t = 2$ of a Young diagram λ with $\lambda_1 > 1$ and let $\mathcal{W} = \mathcal{U}$ be the set of standard fillings with the entry λ_1 at $\binom{1}{\lambda_1}$. The first $\lambda_1 - 1$ steps of any path must be east, hence, only values $b \leq \lambda_1$ will contribute to this block (larger b's could only leave λ_1 at $\binom{1}{\lambda_1}$ if $\lambda_1 + 1$ would be shrinked and shifted from $\binom{2}{\lambda_1}$, but $\lambda_1 + 1$ is the entry of $\binom{2}{1}$). All these b's stabilise these standard fillings and hence, this $\mathcal{W} \to \mathcal{U}$ block is $(\lambda_1 + 2)I$.

Proposition 5.26 (1^{st} Addition Property). *Let $\lambda \vdash n$ and $1 < i < \lambda_1$ and consider the skew diagram $\lambda/(i)$ with the first i cells of the first row removed. We are interested in additions of the cell $\binom{1}{i}$ that yield the smallest entry at $\binom{1}{i}$, i.e. we consider the index sets*

$$\begin{aligned}\mathcal{W} &= \mathrm{SYT}(\lambda/(i)) \qquad \text{and}\\ \mathcal{U} &= \{S \in \mathrm{SYT}(\lambda/(i-1)) \mid S(1,i) = i\}.\end{aligned}$$

Then we find the $\mathcal{W} \to \mathcal{U}$ block

$$\mathbf{B}^{\mathcal{W}\to\mathcal{U}}(\tbinom{1}{i}\boxdot\!\!\rightarrow\lambda/(i)) = \left(\mathbf{B}(\tbinom{1}{\mathrm{i}+1}\boxdot\!\!\rightarrow\lambda/(\mathrm{i}+1))\middle|\mathbf{0}\right) + I$$

where $\left(\mathbf{B}(\binom{1}{\mathrm{i}+1}\boxdot\!\!\rightarrow\lambda/(\mathrm{i}+1))\middle|\mathbf{0}\right)$ is the square matrix constructed from the rectangular matrix $\mathbf{B}(\binom{1}{\mathrm{i}+1}\boxdot\!\!\rightarrow\lambda/(\mathrm{i}+1))$ by extending it with zeros.

Proof. Fixing the entry of $\binom{1}{i}$ to i yields no restriction for the other entries (other then that they are not i). Hence we can identify the standard fillings S of $\lambda/(i-1)$ with $S(1,i) = i$ with the standard fillings T of $\lambda/(i)$. Hence, $\mathbf{B}^{\mathcal{W}\to\mathcal{U}}(\binom{1}{i}\boxdot\!\!\rightarrow\lambda/(i))$ is square.

There are two disjoint cases such that the outcome of the addition will satisfy $S(1,i)=i$.

(1) $b=i$: in this case the path of b is trivial and we explained the summand I.
(2) $b>i$ and the original standard filling T satisfies $T(1,i+1)=i+1$: in this case the path will always start with an east step and after this step we are exactly in the starting position of the addition of $\binom{1}{i+1}$ to $\lambda/(i+1)$. This explains the summand $\left(\mathbf{B}(\binom{1}{\mathrm{i+1}})\boxdot\lambda/(\mathrm{i}+1))\middle|\mathbf{0}\right)$.

Note that $T(2,i)=i+1$ is impossible. □

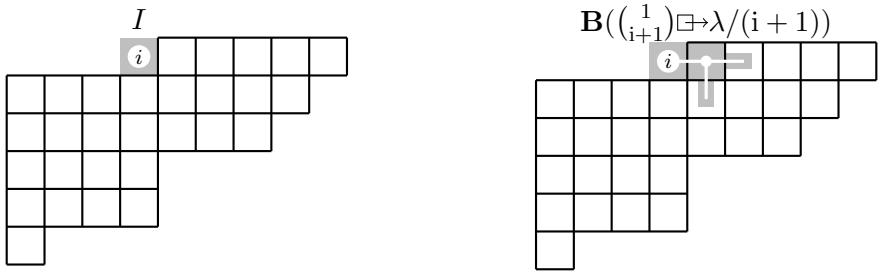


FIGURE 55. Schematic picture of the 1st addition property.

The skew diagram $\lambda/(i)$ has a second top cell, namely $\binom{2}{1}$. We are also interested in the addition of this cell.

PROPOSITION 5.27 (2nd Addition Property). *Let $\lambda\vdash n$ and $1<i<\lambda_1$ and consider the skew diagram $\lambda/(i,1)$. We wonder about the block that keeps the minimal entry at $\binom{1}{i+1}$, this is, we let*

$$\begin{aligned}\mathcal{W} &= \{S\in \mathrm{SYT}(\lambda/(1,i)) \mid S(1,i+1)=i+2\} \qquad \textit{and}\\ \mathcal{U} &= \{S\in \mathrm{SYT}(\lambda/(i)) \mid S(1,i+1)=i+1\}.\end{aligned}$$

Then the corresponding $\mathcal{W}\to\mathcal{U}$ block is given by

$$\mathbf{B}^{\mathcal{W}\to\mathcal{U}}(\tbinom{2}{1}\boxdot\lambda/(i,1)) = \mathbf{B}(\tbinom{2}{1}\boxdot\lambda/(\mathrm{i}+1,1)).$$

PROOF. The only possibility that $i+2$ shrinks to $i+1$ is that $b\geq i+2$. Hence, in $\mathbf{B}(\binom{2}{1}\boxdot\lambda/(\mathrm{i},1))_{\mathcal{U},\mathcal{W}}$ the entry of $(1,i+1)$ is fixed at the minimum and we do not consider $b=i+1$ – but this is exactly the situation of $\mathbf{B}(\binom{2}{1}\boxdot\lambda/(\mathrm{i}+1,1))$. □

Finally, we derive an addition property for the easiest case of a top-cell matrix.

PROPOSITION 5.28 (3rd Addition Property). *Let $\lambda\vdash n$. Then*

$$\mathbf{B}(\tbinom{1}{1}\boxdot\lambda/(1)) = \mathbf{T}(\boxplus\lambda/(1)) + I = \left(\mathbf{B}(\tbinom{1}{2}\boxdot\lambda/(2))\middle|\mathbf{B}(\tbinom{2}{1}\boxdot\lambda/(1^2))\right) + I. \tag{5.8}$$

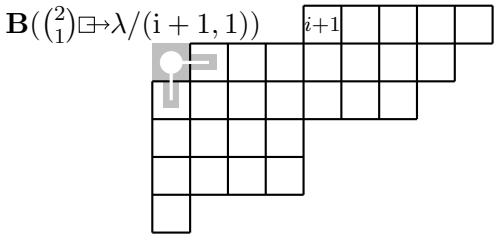


FIGURE 56. Schematic picture of the 2nd addition property.

PROOF. If $b = 1$ every standard filling is fixed and we explained the summand I. If $b \geq 2$ the first step is either east or south. An east step explains the summand $\big(\mathbf{B}(\binom{1}{2}\boxdot\!\!\rightarrow\lambda/(2))\big|\mathbf{0}\big)$ and a south step the summand $\big(\mathbf{0}\big|\mathbf{B}(\binom{2}{1}\boxdot\!\!\rightarrow\lambda/(1^2))\big)$. □

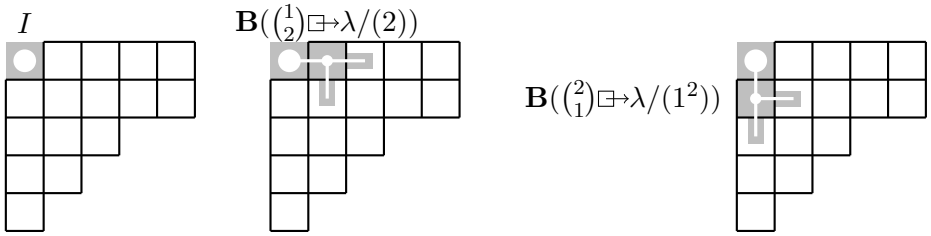


FIGURE 57. Schematic picture of the 3rd addition property.

The above addition properties explain a little of the structure of the distribution matrices but wide areas of them remain in the dark. Of course, a huge amount of further addition properties can be formulated and we wonder if they might be helpful.

QUESTION 5.29. *Is it possible to give general addition properties and use them to derive a general reduction scheme in the sense of* (5.4) *that proves the Spectrum Conjectures?*

5.8. More conjectures. Considering the matrices from the above section, we state the follwing, adventurous conjecture.

CONJECTURE 5.30 (General Young Diagram Spectrum Conjecture). *The eigenvalues of the matrices* $\mathbf{T}(\boxdot\!\!\rightarrow\lambda/\mu) + I$ *and* $\big(\mathbf{B}(\binom{1}{\mathrm{i}}\boxdot\!\!\rightarrow\lambda/(\mathrm{i}))\big|\mathbf{0}\big)$ *are non-negative integers.*

Note that this especially includes a Spectrum Conjecture for $\mathbf{B}(\binom{1}{1}\boxdot\!\!\rightarrow\lambda/(1))$.

We turn our attention back to the distribution vectors. Considering an order defined by $S \in \mathrm{SYT}(\lambda)$ that qualifies for the NPS-algorithm, we know, that either the first row or first column is processed last. Since before this step the intermediate distribution vector exhibits uniform distribution, there are only two possible distribution vectors

$\mathbf{z}(\lambda/(1), S)$ and hence only two possibilities for the deviation vector v_y. We know (by example) that the eigenspace to the least eigenvalue ϵ_y of the distribution matrix is in general of higher dimension. Hence, we should a priori expect these two deviation vectors to be independent. Observations, however, suggest the opposite.

CONJECTURE 5.31. *Let in the above setting v_R and v_C be the deviation vectors of the row-wise and column-wise algorithms. Then the deviation space spanned by v_R and v_C has dimension 1, i.e.*

$$\dim\langle v_R, v_C\rangle = 1.$$

We believe that the following more general version of the same conjecture does not strengthen it.

CONJECTURE 5.32 (Young Diagram Deviation Space Conjecture). *Let $\mathcal{U} \subset \mathrm{SYT}(\lambda)$ be the set of standard fillings defining algorithms that exhibit uniform distribution and denote by v_U the deviation vector corresponding to the $\prec_U$-algorithm. Then the deviation space spanned by the vectors v_U has dimension 1, i.e.*

$$\dim\langle v_U \mid U \in \mathcal{U}\rangle = 1.$$

So far we do not know about an NPS-like bijection for insets, hence, it is much riskier to state the same conjecture for insets. Indeed we are afraid that in this case the search for a counterexample might pay. Nevertheless, we have to state the conjecture.

CONJECTURE 5.33 (Inset Deviation Space Conjecture). *Let $\mathcal{U} \subset \mathrm{SYT}(\lambda_1'; \lambda)$ be as above and let the deviation vectors v_U be analogous to above. Then the deviation space spanned by the vectors v_U has dimension 1, i.e.*

$$\dim\langle v_U \mid U \in \mathcal{U}\rangle = 1.$$

Up to now we stayed very vague with the notions deviation vector and deviation space and trusted in the intuition of the reader. Indeed we denoted the deviation vector as if the Inset Eigenvector Distribution Conjecture was proven. But this is not the only obstacle – we may as well think of a deviation vector of a non-uniformly distributed algorithm.

DEFINITION 5.34 (deviation vector). *Let P be a diagram with a top cell x, use $P' = P \setminus \{x\}$ and let $U \in \mathrm{SYT}(P)$ such that $U(x)$ is the minimal entry. The* **deviation vector** *v_U is defined as*

$$v_U := \mathbf{z}(P, U|_{P'}) - \frac{(|P|-1)!}{|\mathrm{SYT}(P')|}\,\mathbf{1}.$$

Indeed this notation contains the entire information, since P and x are encoded in the standard filling U.

We take a step back to think about the situation. We know that $\mathbf{1}$ is the leading eigenvector of the distribution matrix. We could add a neck of arbitrary length at the position of the top cell. For every cell of the neck we multiply the distribution vector with a distribution matrix that agrees on the leading eigenvector $\mathbf{1}$. We may apply Perron-Frobenius such that the normalised distribution vector converges to $\mathbf{1}$. But the growth of the distribution vector is much too fast such that we will never run into a situation such that $\frac{n!}{|\mathrm{SYT}(P)|}\mathbf{1}$ is the only integer point in the corresponding neighbourhood. This tells us, that the deviation vector will always be small compared to $n!$, but nothing else. A priori the distribution vector could e.g. somehow oscillate around the line $\mathbf{x} = \xi\ \mathbf{1}$. Encouraged by the Eigenvector Distribution Conjectures one might be tempted to conjecture that such a deviation vector is a combination of eigenvectors with positive eigenvalues – or even more surprising that for every order U there is an eigenvalue ϵ_U such that the deviation vector v_U has eigenvalue ϵ_U. Our observations for small cases suggest something far beyond surprising.

Conjecture 5.35 (Young Diagram 2 Conjecture). *Let $U \in \mathrm{SYT}(\lambda)$ define an algorithm that does not exhibit uniform distribution. Then the deviation vector v_U is an eigenvector of $\mathbf{B}(\binom{1}{1}\boxdot\lambda/(1))$ with eigenvalue 2.*

5.9. Distribution matrices on insets. When we started to consider distribution matrices, the intention was to generalise the Δ-Theorem to insets. We soon discovered that they carry a remarkable structure that motivated us to discover the above addition properties. This section is devoted to the investigation of this structure.

We start by partitioning the set of standard fillings of an inset by the entry of the cell $\binom{2}{0}$. From here on we will call this entry g and respectively $\tilde{g}$ after the execution of the jeu de taquin. For the inset $(1;\lambda)$ let

$$\mathcal{U}_i := \{S \in \mathrm{SYT}(1;\lambda) \mid g = i\}.$$

Since the possible values for g are $0,\dots,\lambda_1$, we can write the set of standard fillings as disjoint union

$$\mathrm{SYT}(1;\lambda) = \coprod_{i=0}^{\lambda_1} \mathcal{U}_i.$$

This canonically splits the distribution matrix in blocks $\mathbf{B}^{\mathcal{U}_j\to\mathcal{U}_i}(\binom{1}{0}\boxdot(1;\lambda))$. To avoid additional subscripting, we denote

$$\mathbf{B}^{j\mapsto i}(\tbinom{1}{0}\boxdot(1;\lambda)) := \mathbf{B}^{\mathcal{U}_j\to\mathcal{U}_i}(\tbinom{1}{0}\boxdot(1;\lambda)).$$

The entry of the cell $\binom{2}{1}$ is larger than g and all smaller entries also have to be in the first row (or to be g). We call this entry h and get

$$h \in \{g+1, \dots, \lambda_1 + 1\} \subset \{2, \dots, \lambda_1 + 1\}.$$

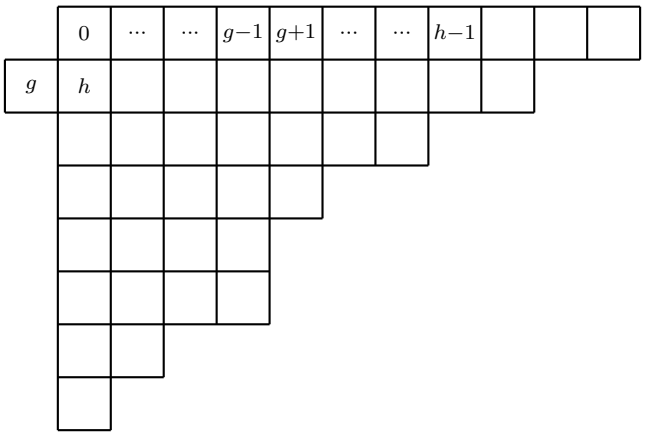


FIGURE 58. The entries of g and h

We can now describe the blocks. We start with the blocks $\mathbf{B}^{g\mapsto g}(\binom{1}{0}\boxdot\!\!\rightarrow(1;\lambda))$. A positive g will be stabilised by $\mathrm{ADD}_b^{\binom{1}{0}}$ if and only if $b < g$. Such a b can only be exchanged with entries smaller than b. But these are all in the first row. Hence, b will take some east steps and stop then without having had any effect on the relative order. Hence, for $b < g$ we have $\mathrm{ADD}_b^{\binom{1}{0}} = \mathrm{id}$ and, since there are $|\{-1, \dots, g-1\}| = g+1$ such values of b, we derived for all $g > 1$

$$\mathbf{B}^{g\mapsto g}(\tbinom{1}{0}\boxdot\!\!\rightarrow(1;\lambda)) = (g+1)\, I.$$

The value $g = 0$ is stabilised by $b \in \{-1, 0\}$ and both values will stabilise the entire tableau. Hence, the block $\mathbf{B}^{0\mapsto 0}(\binom{1}{0}\boxdot\!\!\rightarrow(1;\lambda))$ is given by

$$\mathbf{B}^{0\mapsto 0}(\tbinom{1}{0}\boxdot\!\!\rightarrow(1;\lambda)) = 2\, I.$$

Whenever g is positive the jeu de taquin can start only with an east step. In this situation g will shrink by one if and only if $b \geq g$. In this case we know that actually the first g steps will be east (passing by the entries $0, \dots, g-1$ in the first row). But then we are in a situation we already met, namely

$$\mathbf{B}^{g\mapsto g-1}(\tbinom{1}{0}\boxdot\!\!\rightarrow(1;\lambda)) = \mathbf{B}(\tbinom{1}{\mathrm{g}-1}\boxdot\!\!\rightarrow\lambda/(\mathrm{g}-1)).$$

If $g = 0$ and b is positive we observe jeux de taquin starting with a south step such that the entry of $\binom{2}{0}$ will grow to $\tilde{g}$. There are two possibilities to end up with $\tilde{g}$: either

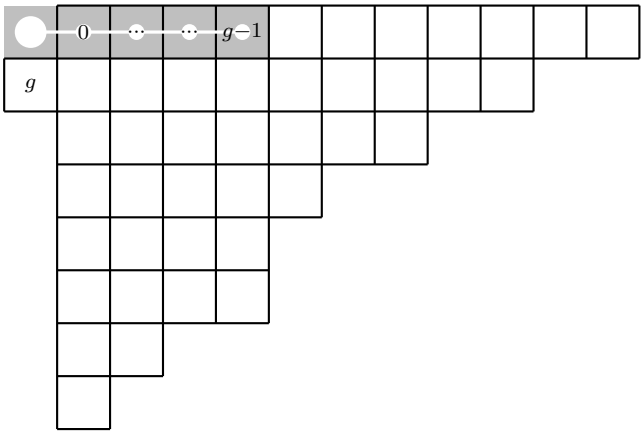


FIGURE 59. The jeux de taquin for the $(g \mapsto g)$-block

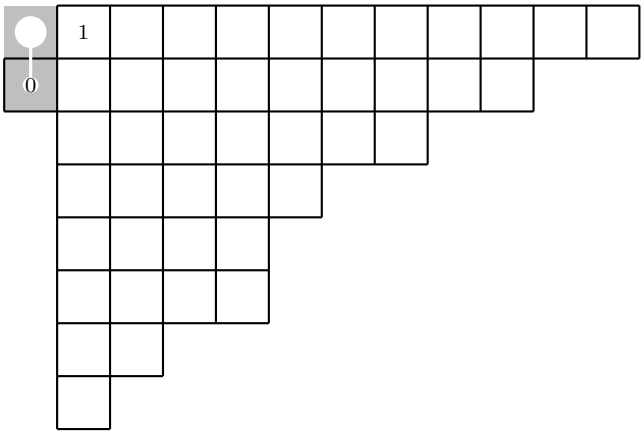


FIGURE 60. The jeux de taquin for the $(0 \mapsto 0)$-block

$b = \tilde{g}$ and $h > b$ (i.e. b stops after the first step), or $h = \tilde{g} + 1$ and $b \geq \tilde{g}$ (i.e. h will shrink by one and slide to $\binom{2}{0}$).

The former includes that $h > \tilde{g}$ and, hence, this case acts as the identity on $\mathrm{SYT}(\lambda/(\tilde{g}))$.

The latter meets the situation of $\mathbf{B}(\binom{2}{1}\boxdot\!\!\rightarrow\lambda/(\tilde{\mathrm{g}}, 1))$.

We observe that $h \leq \tilde{g}$ is impossible. Hence, we can write

$$\mathbf{B}^{0\mapsto\tilde{g}}(\tbinom{1}{0}\boxdot\!\!\rightarrow(1;\lambda)) = \left(I + \left(\mathbf{0}\middle|\mathbf{B}(\tbinom{2}{1}\boxdot\!\!\rightarrow\lambda/(\tilde{\mathrm{g}}, 1))\right)\middle|\mathbf{0}\right).$$

The General-Young-Diagram-Spectrum Conjecture was motivated by observing matrices $I + \left(\mathbf{0}\middle|\mathbf{B}(\binom{2}{1}\boxdot\!\!\rightarrow\lambda/(\tilde{\mathrm{g}}, 1))\right)$ with integer spectrum.

It is easily checked, that all other blocks of $\mathbf{B}(\binom{1}{0}\boxdot\!\!\rightarrow(1;\lambda))$ are $\mathbf{0}$. We obtain the form displayed in Figure 64. The above arguments include something more, not captured

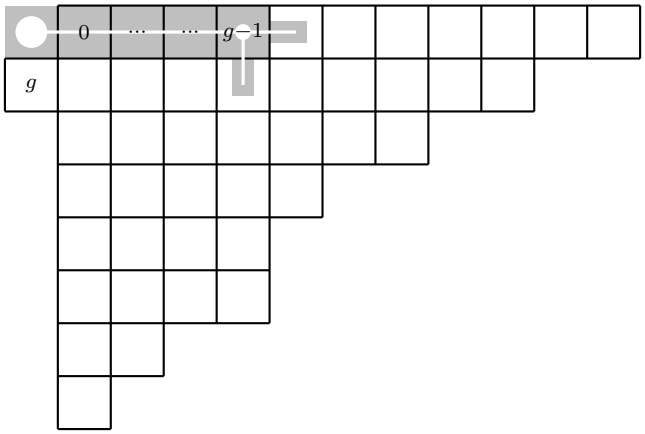


FIGURE 61. The jeux de taquin for the $(g \mapsto g-1)$-blocks

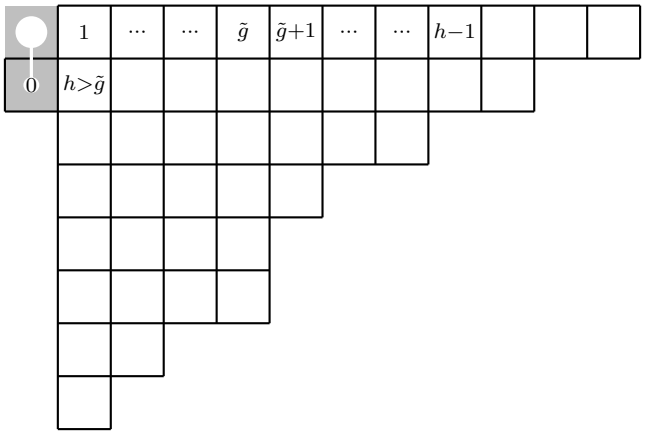


FIGURE 62. The jeu de taquin for the identity part of the $(0 \mapsto \tilde{g})$-blocks

in the picture, namely that $I + \mathbf{B}(\binom{2}{1}\boxdot\!\!\rightarrow\lambda/(\mathrm{i},1))$ lies exactly below the part of $I + \mathbf{B}(\binom{2}{1}\boxdot\!\!\rightarrow\lambda/(\mathrm{i}-1,1))$ that only carries entries from the identity.

Using the row-wise algorithm we know from the Novelli-Pak-Stoyanovskii bijection that by applying the algorithm to the cells up to and including the cell $\binom{1}{2}$ we get the distribution vector

$$\mathbf{z}((1;\lambda)/(1;\lambda_1),R) = \frac{(n-\lambda_1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1}.$$

Analogously to the case of ordinary Young diagrams we derive the distribution vector

$$\begin{aligned}\mathbf{z}((1;\lambda),R) \;=\; & \mathbf{B}(\tbinom{1}{1}\boxdot\!\!\rightarrow(1;\lambda)/(1))\,\mathbf{B}(\tbinom{1}{2}\boxdot\!\!\rightarrow(1;\lambda)/(2))\cdots \\ & \cdots\mathbf{B}(\tbinom{1}{\lambda_1}\boxdot\!\!\rightarrow(1;\lambda)/(\lambda_1))\,\mathbf{B}(\tbinom{2}{0}\boxdot\!\!\rightarrow(1;\lambda)/(1;\lambda_1))\,\mathbf{z}((1;\lambda)/(1;\lambda_1),R).\end{aligned}$$

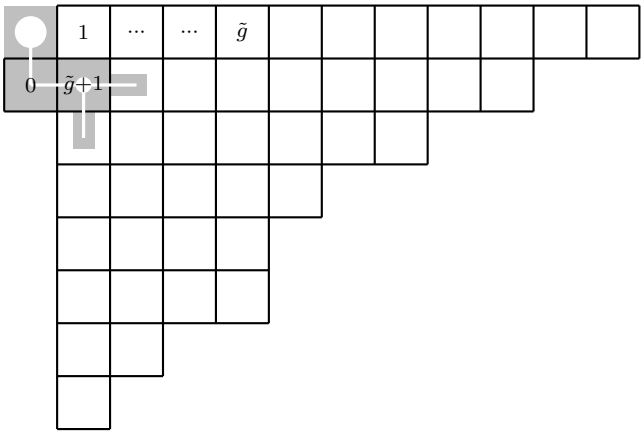


FIGURE 63. The jeux de taquin for the $(0 \mapsto \tilde{g})$-blocks

We already know, that $\mathbf{1}$ is an eigenvector of $\mathbf{B}(\binom{2}{1}\boxdot\!\!\rightarrow(1;\lambda)/(1;\lambda_1,1))$ to the eigenvalue $\epsilon_1 = n - \lambda_1$. Moreover, it is obvious that

$$\mathbf{B}(\tbinom{2}{0}\boxdot\!\!\rightarrow(1;\lambda)/(1;\lambda_1)) = \mathbf{B}(\tbinom{2}{1}\boxdot\!\!\rightarrow(1;\lambda)/(1;\lambda_1,1)) + I.$$

Hence, we get

$$\begin{aligned}\mathbf{z}((1;\lambda),R) &= \mathbf{B}(\tbinom{1}{1}\boxdot\!\!\rightarrow(1;\lambda)/(1))\,\mathbf{B}(\tbinom{1}{2}\boxdot\!\!\rightarrow(1;\lambda)/(2))\cdots \\ &\quad \cdots\mathbf{B}(\tbinom{1}{\lambda_1}\boxdot\!\!\rightarrow(1;\lambda)/(\lambda_1))\,\frac{(n+1-\lambda_1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1}.\end{aligned}$$

This motivates us to have a look at the matrices $\mathbf{B}(\binom{1}{\mathrm{i}}\boxdot\!\!\rightarrow(1;\lambda)/(\mathrm{i}))$. Again, we partition the sets of standard fillings by the values of g and $\tilde{g}$ and consider the blocks $\mathbf{B}^{g\mapsto\tilde{g}}(\binom{1}{i}\boxdot\!\!\rightarrow(1;\lambda)/(i))$. The only non-zero blocks are $\mathbf{B}^{g\mapsto g}(\binom{1}{i}\boxdot\!\!\rightarrow(1;\lambda)/(i))$ and $\mathbf{B}^{g\mapsto g-1}(\binom{1}{i}\boxdot\!\!\rightarrow(1;\lambda)/(i))$. With the same arguments as above we find

$$\mathbf{B}^{g\mapsto g}(\tbinom{1}{i}\boxdot\!\!\rightarrow(1;\lambda)/(i)) = (g-i)\;I$$

and

$$\mathbf{B}^{g\mapsto g-1}(\tbinom{1}{i}\boxdot\!\!\rightarrow(1;\lambda)/(i)) = \mathbf{B}(\tbinom{1}{\mathrm{g}}\boxdot\!\!\rightarrow\lambda/(\mathrm{g})).$$

We summarise these arguments in Figure 65 and display the matrix $\mathbf{B}(\binom{1}{\mathrm{i}}\boxdot\!\!\rightarrow(1;\lambda)/(\mathrm{i}))$ in Figure 66. Note the strong relation between $\mathbf{B}(\binom{1}{\mathrm{i}}\boxdot\!\!\rightarrow(1;\lambda)/(\mathrm{i}))$ and $\mathbf{B}(\binom{1}{1}\boxdot\!\!\rightarrow(1;\lambda))$ by subtracting $(i-1)\;I$ from the latter and finding the former as a right lower block.

Now this matrix looks really inviting, since it would be a square upper-triangular matrix if one could just get rid of the upper left block. We remember that the distribution matrices of simulations are always square. This motivates us to consider the virtual distribution matrix $\mathbf{A}(\binom{1}{i}\!\!\times\!\!\!\rightarrow(1;\lambda)/(i)) := \mathbf{A}(1;\binom{1}{i}\!\!\times\!\!\!\rightarrow(1;\lambda)/(i))$.

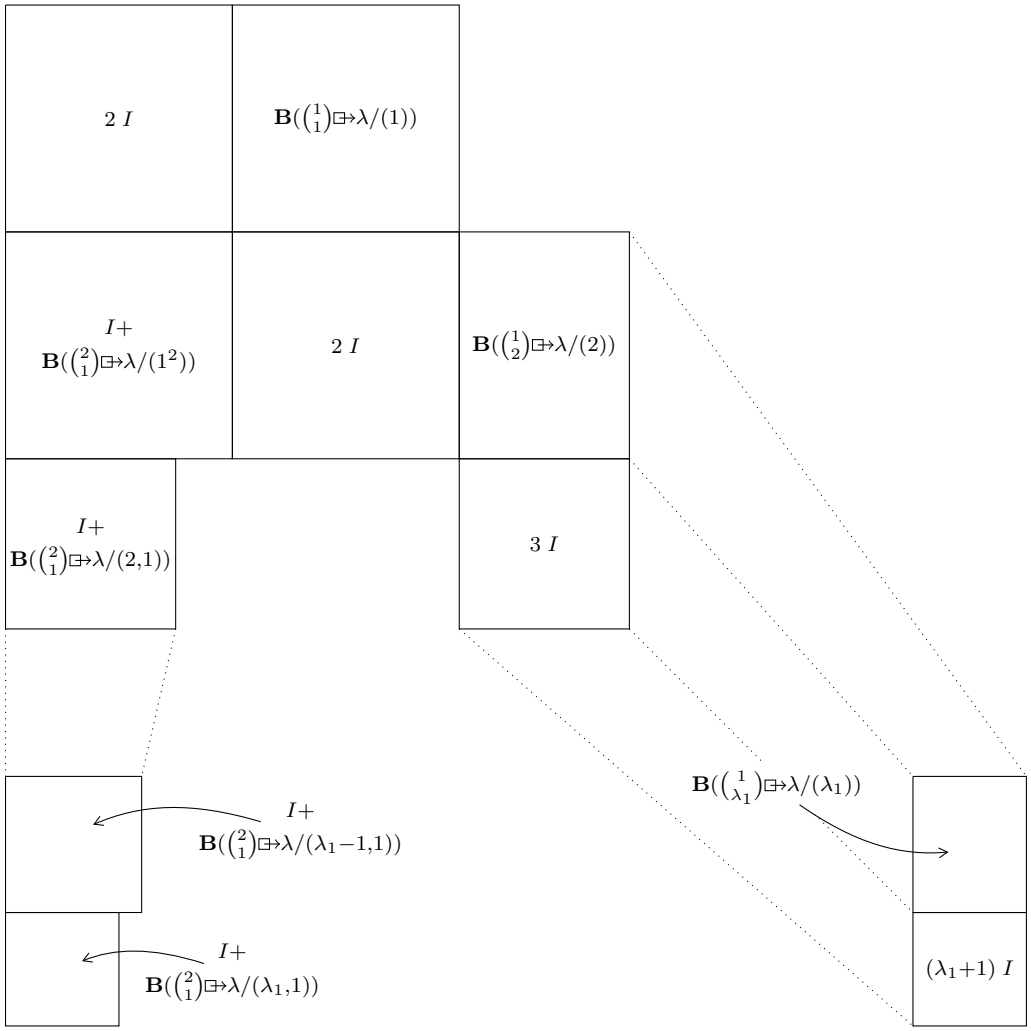


FIGURE 64. The matrix $\mathbf{B}(\binom{1}{0}\boxdot\to(1;\lambda))$

LEMMA 5.36. *In the above setting with* $0 < i < \lambda_1$ *we have for* $i < g \leq \lambda_1$ *and* $i \leq \tilde{g} \leq \lambda_1$

$$\mathbf{A}^{g\mapsto\tilde{g}}(\tbinom{1}{i}\nwarrow\!\!\!\!\!\!\to(1;\lambda)/(i)) = \mathbf{B}^{g\mapsto\tilde{g}}(\tbinom{1}{i}\boxdot\to(1;\lambda)/(i)).$$

PROOF. We know that in this situation we start with the (least) entry i at the cell $\binom{1}{i+1}$. Either the new smallest entry is b or i shrinks and slides to $\binom{1}{i}$. Either way removing the cell $\binom{1}{i}$ again has no effect on the other entries. We identify the standard fillings of $(1;\lambda)/(i)$ with the standard fillings of $(1;\lambda)/(i-1)$ with the smallest entry at $\binom{1}{i}$ if they agree on their values of $(1;\lambda)/(i)$. The statement follows immediately. □

Note that we calculated the blocks $\mathbf{B}^{g\mapsto\tilde{g}}(\binom{1}{i}\boxdot\to(1;\lambda)/(i))$ above and that most of them are zero. Note further that we did not formulate Lemma 5.36 for $i = \lambda_1$. The

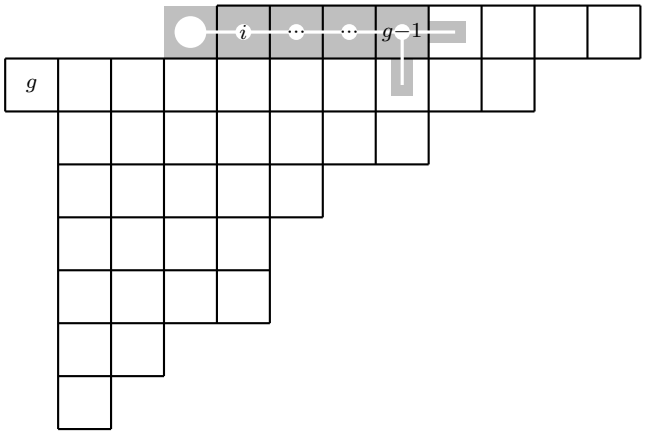


FIGURE 65. The jeux de taquin for $\mathbf{B}(\binom{1}{\mathrm{i}}\boxdot\!\!\rightarrow(1;\lambda)/(\mathrm{i}))$

lemma would be empty, since no g would exist. The following lemma, however, would also apply for $i = \lambda_1$ trivially, but we prefer to formulate it for $i < \lambda_1$ as well.

LEMMA 5.37. *In the above setting with* $0 < i < \lambda_1$ *and* $g = i$ *we get* $\tilde{g} = i$.

PROOF. Either we simulate $b = i - 1$ and g never changes, or we simulate $b \geq i$ and g shrinks to $i - 1$, but then the value that is later deleted at position $\binom{1}{i}$ is greater than $i - 1$ and g will be increased again such that in either case $\tilde{g} = i$. □

Lemma 5.37 tells, that in going from $\mathbf{B}(\binom{1}{\mathrm{i}}\boxdot\!\!\rightarrow(1;\lambda)/(\mathrm{i}))$ to $\mathbf{A}(\binom{1}{i}\times\!\!\rightarrow(1;\lambda)/(i))$ we integrate the block $\mathbf{B}^{i\mapsto i-1}(\binom{1}{i}\boxdot\!\!\rightarrow(1;\lambda)/(i))$ into the block $\mathbf{B}^{i\mapsto i}(\binom{1}{i}\boxdot\!\!\rightarrow(1;\lambda)/(i))$ to get the block $\mathbf{A}^{i\mapsto i}(\binom{1}{i}\times\!\!\rightarrow(1;\lambda)/(i))$. This is almost but not exactly what we wanted. Namely, we got rid of the top block of $\mathbf{B}(\binom{1}{\mathrm{i}}\boxdot\!\!\rightarrow(1;\lambda)/(\mathrm{i}))$, but we did not get an upper triangular matrix, because the block $\mathbf{A}^{i\mapsto i}(\binom{1}{i}\times\!\!\rightarrow(1;\lambda)/(i))$ is not upper triangular. With arguments similar to the ones used for the addition properties one can tell quite a bit about this integration process, but this analysis appears pointless to us, hence we skip it here to baulk neither us nor the reader with it. Nevertheless we present the following theorem to illustrate, that it is maybe not hopeless to attack the Spectrum Conjectures.

THEOREM 5.38 (probably useless Theorem). *For* $1 \leq i < \lambda_1$ *the matrices* $\mathbf{A}(\binom{1}{i}\times\!\!\rightarrow(1;\lambda)/(i))$ *have (maybe among others) eigenvalues* $2, \ldots, \lambda_1 + 1 - i$ *and* $n + 2 - i$.

PROOF. From Lemma 5.36 and Lemma 5.37 follows that the matrix has the form as displayed in Figure 67 and we can read off the eigenvalues $2, \ldots, \lambda_1 + 1 - i$ on the diagonal. The eigenvalue $n + 2 - i$ must be present, since all columns must sum to it. □

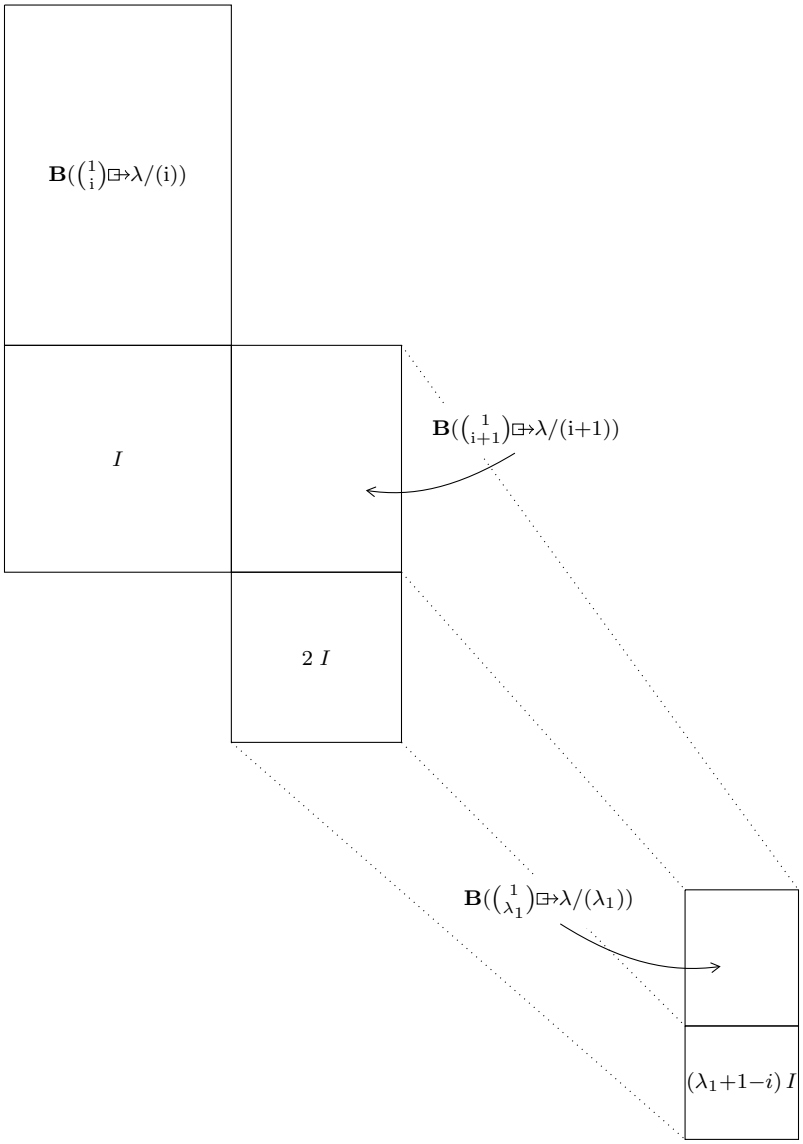


FIGURE 66. The matrix $\mathbf{B}(\binom{1}{\mathrm{i}}\boxplus(1;\lambda)/(\mathrm{i}))$

One might be tempted to think, that the only problem arises with jeux de taquin starting with south steps and that only these generate non-zero entries below the diagonal. This is partly true, but indeed the situation is much worse. The structure we observed so far cannot be sufficient to explain the spectrum. We cannot exclude, that it is possible to derive that the eigenvalues are integers (below $n+2$ which is obviously the leading eigenvalue), but there is no hope to explain the lowest eigenvalue $2-\lambda'_1$ because the number of rows does not appear in the description of the structure of the distribution matrices. If we want to explain this eigenvalue and if and why it is the eigenvalue

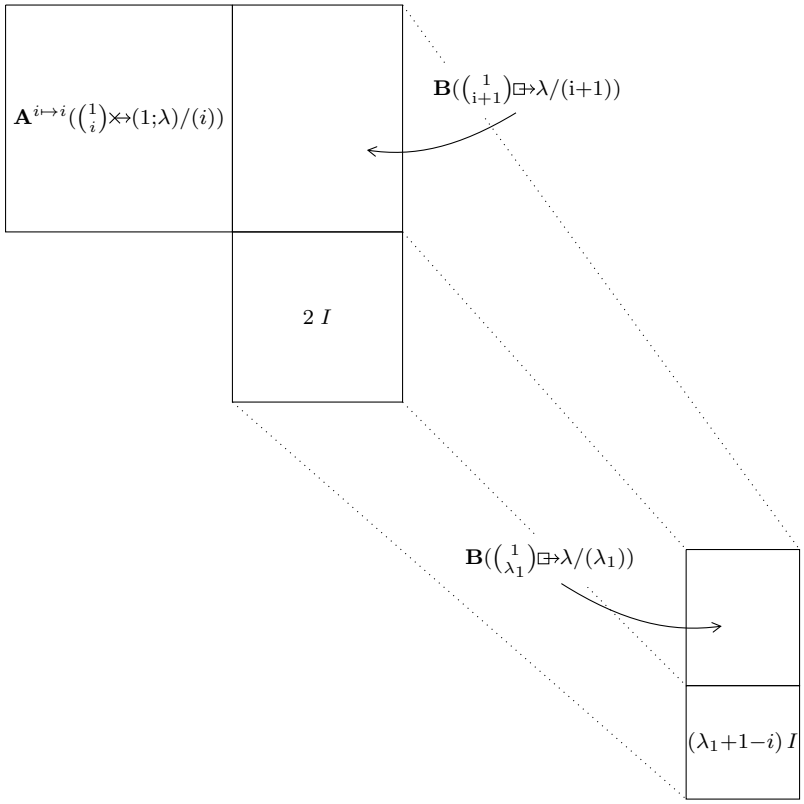


FIGURE 67. The matrix $\mathbf{A}(\binom{1}{i}\nearrow\!\!\!\searrow(1;\lambda)/(i))$

of the deviation vector, we need essentially stronger arguments. We could try some Novelli-Pak-Stoyanovskii-like proof to show the Inset-Uniform-Distribution-Conjecture, but that would basically mean to throw the distribution-matrices-approach away. If we want to really learn more about the distribution matrices, we will probably have to study the matrices $\mathbf{B}(\binom{1}{i}\boxplus\to\lambda/(i))$ and $\mathbf{B}(\binom{2}{1}\boxplus\to\lambda/(i,1))$ and/or to try inductions with the double-tailed diamond and/or the case $(1;2,1^{n-2})$ as starting point.

We close the current section by noting that – if nothing else – the distribution matrices at least provide a way to calculate distribution vectors much more efficiently than by playing the jeu de taquin for $(n+1)!$ tableaux.

5.10. Examples. There is a huge amount of orders that could be chosen to arrange the rows and columns of the distribution matrices. We found no order, that appears to be better than to sort the fillings row-wisely lexicographically starting with the entry g. We have already seen some examples of distribution matrices when discussing the double tailed diamond and the case $(1;2,1^{n-2})$. Here we show $\mathbf{B}(\binom{1}{0}\boxplus\to(1;3,2,1))$ in Figure 68

and $\mathbf{B}(\binom{1}{0}\boxdot\!\!\rightarrow(1;3,2,1,1))$ in Figure 69. We heavily encourage the reader to search the addition properties in these examples.

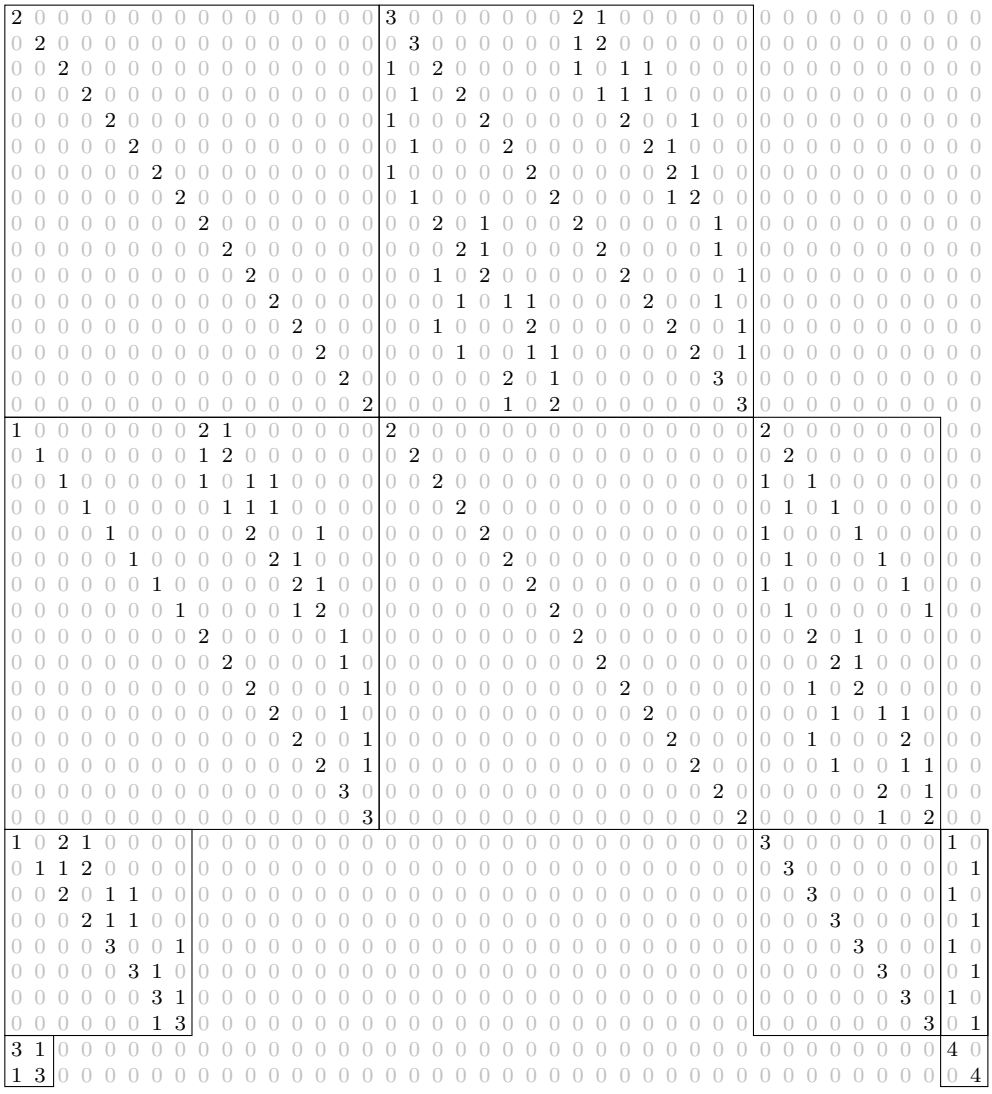


FIGURE 68. $\mathbf{B}(\binom{1}{0}\boxdot\!\!\rightarrow(1;3,2,1))$

5.11. A note on the distribution vectors of insets. We close this chapter with a consideration of the structure of the distribution vectors of insets. After we derived the structure of the corresponding distribution matrices, we would expect to find some kind of similar structure inherent to these vectors.

To do so, we refine our definition of the distribution vector. Namely, we consider the starting and ending position of the minimal entry.

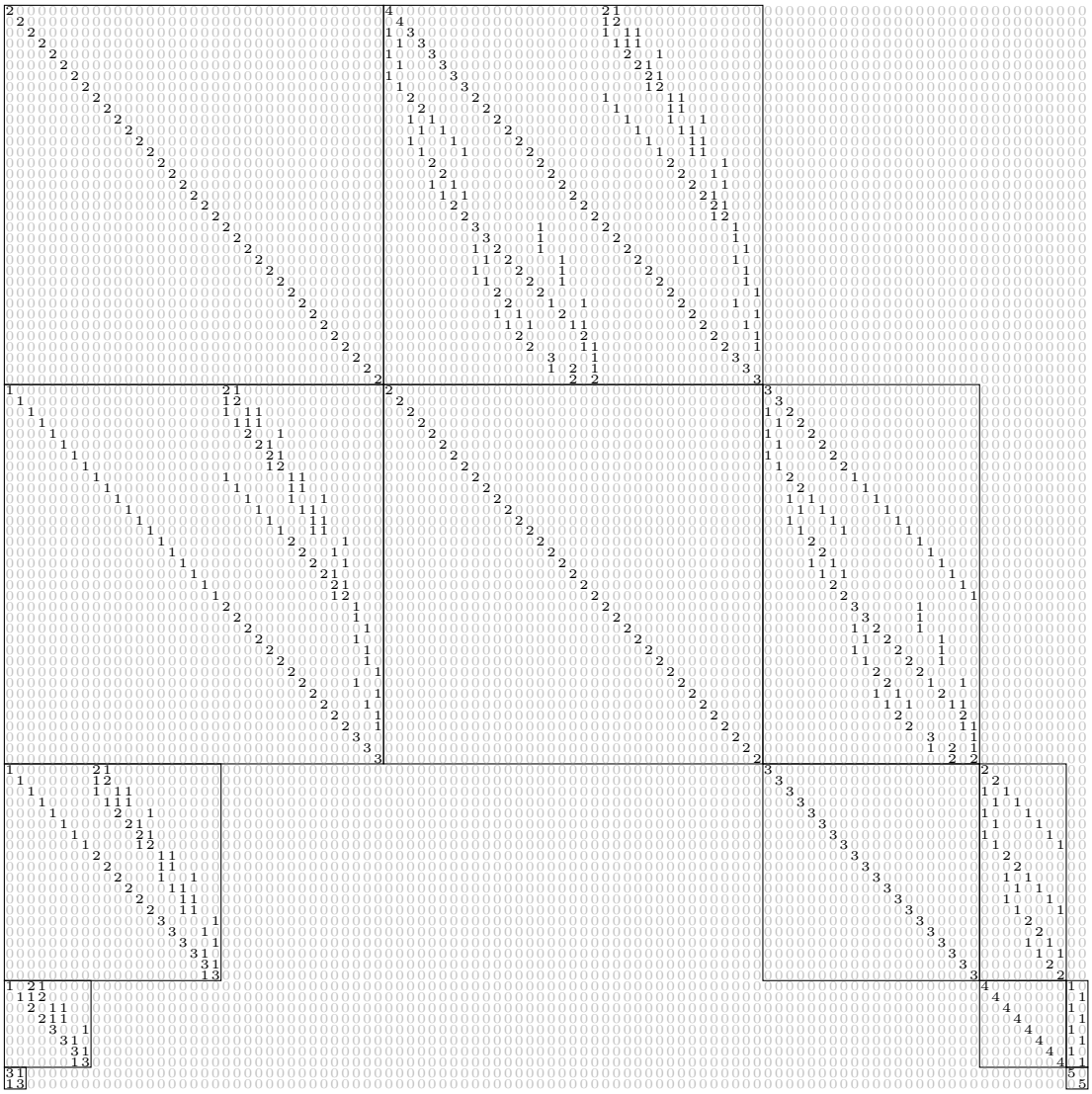


FIGURE 69. $\mathbf{B}(\binom{1}{0}\boxdot\!\!\rightarrow(1;3,2,1,1))$

DEFINITION 5.39. *Let P be some diagram, $S \in \mathrm{SYT}(P)$ define an order and for $w \in P$ let $\mathrm{T}_w(P) \subset \mathrm{T}(P)$ be the set of fillings F such that $F(w)$ is the smallest entry. We define the* w**-start-distribution vector** $\mathbf{z}^{w\rightarrow}(P,S)$ *via its components*

$$\mathbf{z}_S^{w\rightarrow}(P,U) := \left|\mathrm{JDT}^{-1}(U) \cap \mathrm{T}_w(P)\right|$$

for $U \in \mathrm{SYT}$. Analogously, we define the w**-ending-distribution vector** $\mathbf{z}^{\to w}(P,S)$ *via its components*

$$\mathbf{z}^{\to w}_{S}(P,U) := \begin{cases} \mathrm{z}_U(P,S) & \text{if } U(w) \text{ is the minimal entry} \\ 0 & \text{otherwise.} \end{cases}$$

We are going to consider the case of the insets using the row-wise order, i.e. $\mathbf{z}^{\binom{i}{j}\to}((1;\lambda),R)$ and $\mathbf{z}^{\to\binom{i}{j}}((1;\lambda),R)$.

A well known fact which we (due to a remark of Sulzgruber) call the **unstoppability of the** 1, is, that during the application of the $\prec_S$-algorithm the path of the smallest entry from its starting cell w to a top cell depends on S and w and nothing else (especially not on the rest of the filling). The reason for this is of course that whenever a cell is processed that is $\prec$-covered by the cell containing the smallest entry, the smallest entry immediately slides there. We observe a very similar fact.

Lemma 5.40. *Given $S \in \mathrm{SYT}$ and consider cells w and w' with $w' \prec\!\!\cdot_S w$ and $w' \prec\!\!\cdot\, w$ and a filling $F \in \mathrm{T}$ such that $F(w')$ is the minimal entry. Then exchanging the entries of w and w' has no effect on the output of the $\prec_S$-algorithm, i.e.*

$$\mathrm{JDT}_S(F) = \mathrm{JDT}_S((F(w'),F(w))F).$$

Proof. On the lefthand side $\mathrm{jdt}_{w'}$ will be empty and jdt_w will start with $(F(w'),F(w))$, which commutes with all jeu de taquin steps processed before. □

As an immediate consequence we get that the corresponding w- and w'-start-distribution vectors agree.

Corollary 5.41. *Given $S \in \mathrm{SYT}(P)$ and consider cells w and w' with $w' \prec\!\!\cdot_S w$ and $w' \prec\!\!\cdot\, w$, then*

$$\mathbf{z}^{w\to}(P,S) = \mathbf{z}^{w'\to}(P,S).$$

For insets this immediately includes.

Corollary 5.42. *Consider the row-wise algorithm, then the $\binom{i}{j}$-start-distribution vectors of cells in the same row agree, i.e.*

$$\mathbf{z}^{\binom{i}{j}\to}((1;\lambda),R) = \mathbf{z}^{\binom{i}{k}\to}((1;\lambda),R)$$

if $\binom{i}{j}$ and $\binom{i}{k}$ decode cells of the inset.

The same holds obviously for Young diagrams and other appropriate shapes.

Since the distribution vector can be decomposed as

$$\mathbf{z}((1;\lambda),R) = \sum_{w\in(1;\lambda)} \mathbf{z}^{w\to}((1;\lambda),R)$$

we can use this result to break down the distribution vector row-wisely, i.e.

$$\mathbf{z}((1;\lambda),R) = \mathbf{z}^{\binom{2}{1}\rightarrow}((1;\lambda),R) + \sum_{i=1}^{\lambda'_1} \lambda_i \, \mathbf{z}^{\binom{i}{1}\rightarrow}((1;\lambda),R).$$

We further observe that the row-wise algorithm will slide the 0 to $\binom{2}{0}$ if and only if it starts not in the first row. We get

$$\begin{aligned} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda),R) &= \mathbf{z}^{\binom{2}{0}\rightarrow}((1;\lambda),R) + \sum_{i=2}^{\lambda'_1} \lambda_i \, \mathbf{z}^{\binom{i}{1}\rightarrow}((1;\lambda),R) \quad \text{and} \\ \mathbf{z}^{\rightarrow\binom{1}{1}}((1;\lambda),R) &= \lambda_1 \, \mathbf{z}^{\binom{1}{1}\rightarrow}((1;\lambda),R). \end{aligned}$$

The lefthand side of the former can easily be counted directly as

$$\mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda),R) = \frac{(n-\lambda_1+1)n!}{f_\lambda} \begin{pmatrix} \mathbf{1}_{f_\lambda} \\ \hline \mathbf{0} \end{pmatrix}$$

with the following argument: there is uniform distribution after $\binom{2}{1}$ was processed, then the 0 "slides out" giving a factor of $(n-\lambda_1+1)$ and we keep going with the case of an ordinary tabloid where the 1 took the place of the 0. Hence, we get uniform distribution on the standard fillings with the 0 at $\binom{2}{0}$ and the rest of the vector vanishes. Note that we again use a line-notation, this time to separate the parts of vectors.

Looking at the righthand side of the second equation we recognise, that the cell $\binom{1}{1}$ does not play any role i.e.

$$\mathbf{z}^{\binom{1}{1}\rightarrow}((1;\lambda),R) = \mathbf{z}((1;\lambda)/(1),R).$$

This yields

$$\mathbf{z}((1;\lambda),R) = \begin{pmatrix} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda),R) \\ \hline \mathbf{z}^{\rightarrow\binom{1}{1}}((1;\lambda),R) \end{pmatrix} = \begin{pmatrix} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda),R) \\ \hline \lambda_1 \, \mathbf{z}^{\binom{1}{1}\rightarrow}((1;\lambda),R) \end{pmatrix} = \begin{pmatrix} \frac{(n-\lambda_1+1)n!}{f_\lambda} \mathbf{1} \\ \hline \lambda_1 \, \mathbf{z}((1;\lambda)/(1),R) \end{pmatrix}.$$

Looking at the bottom part of this vector, we can repeat the procedure

$$\begin{aligned} \mathbf{z}((1;\lambda)/(1),R) &= \begin{pmatrix} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(1),R) \\ \hline \mathbf{z}^{\rightarrow\binom{1}{2}}((1;\lambda)/(1),R) \end{pmatrix} = \\ &= \begin{pmatrix} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(1),R) \\ \hline (\lambda_1-1) \, \mathbf{z}^{\rightarrow\binom{1}{2}}((1;\lambda)/(1),R) \end{pmatrix} = \\ &= \begin{pmatrix} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(1),R) \\ \hline (\lambda_1-1) \, \mathbf{z}((1;\lambda)/(2),R) \end{pmatrix}. \end{aligned}$$

Note that the upper part of this vector is not a multiple of $\mathbf{1}$ any more, but has still a deviation vector which is in the kernel of $\mathbf{B}((1,1)\boxminus\!\!\rightarrow\lambda/(1))$. For the other cells in the first column we lose this property, i.e. we have

$$\mathbf{z}((1;\lambda)/(i),R) = \begin{pmatrix} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(i),R) \\ \hline (\lambda_1 - i)\,\mathbf{z}((1;\lambda)/(i+1),R) \end{pmatrix}.$$

Finally, we find

$$\mathbf{z}((1;\lambda)/(\lambda_1-1),R) = \begin{pmatrix} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(\lambda_1-1),R) \\ \hline \mathbf{z}((1;\lambda)/(\lambda_1),R) \end{pmatrix} = \begin{pmatrix} \mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(\lambda_1-1),R) \\ \hline \frac{(n-\lambda_1+1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1} \end{pmatrix}.$$

Iteratively plugging in yields

$$\mathbf{z}((1;\lambda),R) = \begin{pmatrix} \frac{(n-\lambda_1+1)n!}{f_\lambda}\,\mathbf{1} \\ \hline \lambda_1{}^{\underline{1}}\,\mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(1),R) \\ \hline \lambda_1{}^{\underline{2}}\,\mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(2),R) \\ \hline \vdots \\ \hline \lambda_1!\,\mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(\lambda_1-1),R) \\ \hline \lambda_1!\,\frac{(n-\lambda_1+1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1} \end{pmatrix} \tag{5.9}$$

This motivates us to wonder about a description of $\mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(i),R)$. Indeed, with the same argument as above we get this distribution vector by applying the corresponding distribution matrices of skew Young diagrams as

$$\mathbf{z}^{\rightarrow\binom{2}{0}}((1;\lambda)/(i),R) = \mathbf{B}(\tbinom{1}{\mathrm{i}+1}\boxminus\!\!\rightarrow\lambda/(\mathrm{i}+1))\cdots\mathbf{B}(\tbinom{1}{\lambda_1}\boxminus\!\!\rightarrow\lambda/(\lambda_1))\;\frac{(n-\lambda_1+1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1}.$$

Thus we get the distribution vector as

$$\mathbf{z}((1;\lambda),R) = \left(\begin{array}{c} \frac{(n-\lambda_1+1)n!}{f_\lambda}\,\mathbf{1} \\ \hline \lambda_1{}^{\underline{1}}\,\mathbf{B}(\binom{1}{2}\square\!\!\to\lambda/(2))\cdots\mathbf{B}(\binom{1}{\lambda_1}\square\!\!\to\lambda/(\lambda_1))\,\frac{(n-\lambda_1+1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1} \\ \hline \lambda_1{}^{\underline{2}}\,\mathbf{B}(\binom{1}{3}\square\!\!\to\lambda/(3))\cdots\mathbf{B}(\binom{1}{\lambda_1}\square\!\!\to\lambda/(\lambda_1))\,\frac{(n-\lambda_1+1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1} \\ \hline \vdots \\ \hline \lambda_1!\,\mathbf{B}(\binom{1}{\lambda_1}\square\!\!\to\lambda/(\lambda_1))\,\frac{(n-\lambda_1+1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1} \\ \hline \lambda_1!\,\frac{(n-\lambda_1+1)!}{f_{\lambda/(\lambda_1)}}\,\mathbf{1} \end{array}\right) =$$

$$= (n-\lambda_1+1)\left(\begin{array}{c} \mathbf{z}(\lambda,R) \\ \hline \lambda_1{}^{\underline{1}}\,\mathbf{z}(\lambda/(1),R) \\ \hline \lambda_1{}^{\underline{2}}\,\mathbf{z}(\lambda/(2),R) \\ \hline \vdots \\ \hline \lambda_1!\,\mathbf{z}(\lambda/(\lambda_1-1),R) \\ \hline \lambda_1!\,\mathbf{z}(\lambda/(\lambda_1),R) \end{array}\right).$$

We see that this distribution vector of the inset $(1;\lambda)$ is built up from the distribution vectors of the skew Young diagrams $\lambda/(i)$, as its distribution matrix is built from the distribution matrices of the top cells of these skew Young diagrams.

CHAPTER 2

Highly arc transitive Digraphs

Before I considered the jeu de taquin I worked on so called highly arc transitive digraphs. I was just about to develop software to visualise parts of highly arc transitive digraphs because with the methods I had at hand I reached a cul-de-sac. In this situation a presentation by Krattenthaler dragged my attention to the complexity of the jeu de taquin. One of my first ideas was to consider the digraph with fillings as vertices and jeu de taquin steps as edges. As mentioned before, this graph neither helped to derive the complexity theorems, nor was it of any use for constructing highly arc transitive digraphs, but half a year later the first version of the involution proving the Δ-Theorem was derived from it.

In this chapter, however, I present my results on highly arc transitive digraphs, that were published before in Discrete Mathematics, i.e. the chapter is almost one to one with my paper [**Neu13**].

1. Introduction

A digraph is highly arc transitive if its automorphism group acts transitively on the set of its n-arcs for every n. We present a way to construct various highly arc transitive digraphs as a fibre product. Doing so, we unify different constructions of highly arc transitive digraphs presented in [**CPW93**], [**DMŠ11**] and [**DL01**] and obtain new highly arc transitive digraphs. Depending on the number and structure of the factors, the produced digraphs can have one or two (or in special cases infinitely many) ends.

Cameron, Praeger and Wormald studied highly arc transitive digraphs in [**CPW93**] and made a conjecture about the subclass of digraphs which have an epimorphism onto the integer line such that the preimages are finite. Möller [**Möl02**] constructed a digraph, that was believed to be a counterexample, but later Šparl [**Špa**] showed that it is not transitive. We give digraphs that are direct fibre products and qualify as counterexamples.

Independently from our work, DeVos, Mohar and Šámal studied highly arc transitive digraphs in [**DMŠ11**]. We must point out that [**DMŠ11**] contains results partially overlapping with ours.[1] They answer a question from [**CPW93**] and clarify the mentioned conjecture. For doing so, they use counterexamples that are isomorphic to ours, but they obtain them with a different approach. Moreover they study the structure of highly arc transitive digraphs with two ends. In Section 3.1 we explain the relation between the present work and [**DMŠ11**].

The outline of Chapter 2 is as follows. In Section 1.1 we recall some notions from [**CPW93**]. In Section 1.2 we introduce the direct fibre product. In Section 2 we use this product to find a counterexample to the mentioned conjecture. In Section 3 we make some remarks on the overlap with [**DMŠ11**], Möller's digraph and the automorphisms we needed for our construction in Section 2. In Section 4 we present further highly arc transitive digraphs that can be constructed using the direct fibre product. This includes representations of some known highly arc transitive digraphs, generalisations of the factors and the thereby obtained new highly arc transitive digraphs.

1.1. Highly arc transitive digraphs. An **n-arc** in a digraph D is a series $(e_i)_{i=0,\dots,n-1}$ of n directed edges $e_i = (x_{e_i}, y_{e_i})$ such that $y_{e_i} = x_{e_{i+1}}$ for all i in the range. A digraph is called **n-arc transitive** if its automorphism group acts transitively on the nonempty set of its n-arcs. A digraph is called **highly arc transitive (HAT)** if it is n-arc transitive for all $n \in \mathbb{N}$. An **alternating walk** is a series $(e_i)_{i=0,\dots,n-1}$ of n directed edges $e_i = (x_{e_i}, y_{e_i})$ such that consecutive edges agree alternatingly on either the initial or terminal vertex, i.e., for every even i in the range $x_{e_i} = x_{e_{i+1}}$ and $y_{e_i} = y_{e_{i+1}}$ for every odd i (or vice verse). Two edges are **reachable** from each other if there exists an alternating walk that contains them both. Being reachable from each other is an equivalence relation on the edge set of a digraph which we call **reachability relation**. Let $D = (V, E)$ be a HAT digraph and $e \in E$. Let $\Delta(e)$ be the subgraph of D that is spanned by the equivalence class of e with respect to the reachability relation. $\Delta(e)$ is independent from e i.e. for all $e_1, e_2 \in E:\ \Delta(e_1) \cong \Delta(e_2)$. Thus, we can speak of it as $\Delta(D)$ and call it the **associated digraph** of D. If $\Delta(D) \neq D$ then $\Delta(D)$ is bipartite. Let $Z = (\mathbb{Z}, \{(i, i+1) \mid i \in \mathbb{Z}\})$ be the digraph representing the integer line. A digraph G has **Property** Z if there is an epimorphism $\varphi : G \to Z$. Note that if G is HAT all edges of $\Delta(e)$ are mapped to the same edge $\varphi(e)$ by φ. The fibre $\varphi^{-1}((i, i+1))$ of every edge of Z induces a subgraph of G which we call **(edge-)layer**. If G is HAT all components of

[1]The results and constructions in this chapter were obtained independently from those of [**DMŠ11**]. As a matter of fact, I learned about [**DMŠ11**] on the day I intended to put [**Neu13**] on the arXiv, when a colleague dragged my attention to the fact that [**DMŠ11**] had appeared on the arXiv just the day before. In view of this, [**Neu13**] was revised so that it took also into account [**DMŠ11**] and it was put on the arXiv with a delay of a few days.

every layer are isomorphic to $\Delta(G)$. We call a fibre $\varphi^{-1}(x)$ a **(vertex-)layer**. Moreover, we will refer to a directed doubleray (i.e. a doubleray such that every finite, connected subsequence is an arc) as line. For more details on these notions we refer to [**CPW93**] or [**Neu10**].

In the paper [**CPW93**] on HAT digraphs, Cameron, Praeger and Wormald stated the following conjecture.

Conjecture 1.1 (Cameron, Praeger, Wormald). *Let D be a connected HAT digraph with Property Z and finite fibres $\varphi^{-1}(x)$. Then the associated digraph $\Delta(D)$ is complete bipartite.*

1.2. The direct fibre product. The direct fibre product appeared earlier e.g. in the construction of the Diestel–Leader digraph (also known as broom-digraph). It was generalised to the horocyclic product which is defined in [**BW08**]. This generalisation is related, but does not agree with the one that we are going to use. Our direct fibre product is a proper subgraph of the direct product. It is induced by a subset of the vertex set that is gained in the following way: given Property Z for both factors, one restricts to vertices agreeing on their image.

Definition 1.2 (Direct fibre product). *Let $G^1 = (V^1, E^1)$ and $G^2 = (V^2, E^2)$ be digraphs with Property Z and let $\varphi^1 : V^1 \to Z$ and $\varphi^2 : V^2 \to Z$ be the arising epimorphisms. The* **direct fibre product** *$G^1 {}_{\varphi^1}\!\times_{\varphi^2} G^2$ is the digraph (V, E) with*

$$\begin{aligned} V &= \{(x, y) \mid x \in V^1,\, y \in V^2,\, \varphi^1(x) = \varphi^2(y)\}, \\ E &= \{((a, b), (x, y)) \mid (a, x) \in E^1,\, (b, y) \in E^2\}. \end{aligned}$$

If the factors are both connected HAT digraphs with Property Z, this direct fibre product gives exactly a connected component of the ordinary direct product – in this case these components are all isomorphic. For general factors, there is no need for the components to be isomorphic. So, in the general case, the direct fibre product picks the "central" component.

It is often convenient to denote the vertices as (n, x, y) where $n = \varphi^1(x) = \varphi^2(y)$. In the situation $x = (i, a)$ and $y = (j, b)$ and $\varphi^{1,2}$ depend only on i and j respectively, it is very convenient to denote the vertices by (n, a, b).

If we need to consider more factors, we generalise our definition of the direct fibre product as follows.

Definition 1.3 (Direct fibre product). *Let I be some set of indices and $G_i = (V^i, E^i)$ for $i \in I$ digraphs with Property Z and let $\Phi := \{\varphi^i : V^i \to Z \mid i \in I\}$ be the set of arising epimorphisms. The* **direct fibre product** *$\prod_I^\Phi := G^i$ is the digraph (V, E)*

with

$$\begin{aligned} V &= \{(x_i)_{i\in I} \mid x_i \in V^i,\ \forall j,k \in I:\ \varphi^j(x_j) = \varphi^k(x_k)\},\\ E &= \{((x_i)_{i\in I},(y_i)_{i\in I}) \mid \forall j \in I:\ (x_j,y_j) \in E^j\}. \end{aligned}$$

Again we denote the vertices by $(n,(x_i)_{i\in I})$ or $(n,(a_i)_{i\in I})$ respectively.

REMARK 1.4 (Further generalisation). We could generalise the direct fibre product by using epimorphisms to digraphs other than Z, as it is done with horocyclic products. This might yield interesting digraphs but we do neither need nor address such constructions here.

REMARK 1.5 (Automorphism groups). Consider the digraph $G = \prod_I^\Phi G^i$ and the automorphism groups $A_i = \mathrm{Aut}(G^i)$ and $A = \mathrm{Aut}(G)$.

(1) Some of the G^i may have shifts and some not. The product is then likely to have no shifts as well. Say A_j is an automorphism group containing a shift. In this case A_j is not a subgroup of A. The same might happen if the shifts of two factors have different step width. Say the step width of the shifts in A_k is 2 and the step width of shifts in A_l is 3. Then the step width of the shifts that may exist in A is likely to not be below 6 and thus neither A_k nor A_l is a subgroup of A. (It will become clear in a moment why we were using the term "is likely" in this paragraph.)

(2) On the other hand, there might appear new shifts. Let $I = \{1,2\}$ and consider the digraphs $G^1 = (V^1,E^1)$ and $G^2 = (V^2,E^2)$ with

$$\begin{aligned} V^1 &= \{(x,y) \mid x \in \mathbb{Z},\ y = 0 \text{ if } x = 0,\ y \in \{0,1\} \text{ if } x \neq 0\},\\ E^1 &= \{((x,y),(a,b)) \mid (x,y),(a,b) \in V^1,\ x+1 = a\},\\ V^2 &= \{(x,y) \mid x \in \mathbb{Z},\ y = 0 \text{ if } x \neq 0,\ y \in \{0,1\} \text{ if } x = 0\} \text{ and}\\ E^2 &= \{((x,y),(a,b)) \mid (x,y),(a,b) \in V^2,\ x+1 = a\}. \end{aligned}$$

The product $G = G^1\,{}_{(x,y)\mapsto x}\!\times_{(x,y)\mapsto x} G^2$ enriches its automorphism group with a shift of arbitrary step width. A_1 and A_2 are canonically embedded in A. $A_1 \times A_2$ does not contain shifts and thus is a proper subgroup of A.

Finally, consider the digraph $G' = G^1\,{}_{(x,y)\mapsto x}\!\times_{(x,y)\mapsto x+1} G^2$ with automorphism group A'. In this case no shifts disappear or arise and we have $A_1 \times A_2 = A'$. (Note that G and G' only differ on the choice of Φ.)

(3) One might think that the shifts are the only problems that prevent us from obtaining a relation like the one immediately above. This, however, is wrong.

Let $I = \{2, 3\}$ and consider the digraph $G^3 = (V^3, E^3)$ to be defined by

$$\begin{aligned} V^3 &= \{(x, y) \mid x \in \mathbb{Z}, y \in \{0, 1\} \text{ if } -1 \leq x \leq 1,\ y = 0 \text{ otherwise}\} \text{ and} \\ E^3 &= \{((x, y), (a, b)) \mid x + 1 = a, y = b\} \cup \{((-2, 0), (-1, 1)), ((1, 1), (2, 0))\} \end{aligned}$$

and the digraph G_2 from above. In the product $G = G_2 \,{}_{(x,y)\mapsto x}\!\times_{(x,y)\mapsto x}\, G_3$ the diamond around the origin of G_2 gets duplicated to the two branches of G_3 around the origin. The automorphism group A of G can act on these branches independently, whereas $A_2 \times A_3$ cannot do so. Hence the latter is again a proper subgroup of the first.

(4) Summarising the above, all we can say, is: let A_i^0 be the groups of automorphisms of G^i that fix the images under φ_i (i.e. do not contain shifts) and let A^0 be analogously defined for the product. Then the direct product of the A_i^0 acts canonically on the fibre product of the G^i but is not necessarily equal to A^0, i.e.

$$\prod_I A_i^0 \leq A^0.$$

2. The counterexample

Our counterexample for Conjecture 1.1 will be a direct fibre product of two factors which we have to define first. Let $L = (V^L, E^L)$ be the digraph with vertex set $V^L = \mathbb{Z} \times \{-1, 0, 1\}$ and $e = ((i, x), (i+1, y))$ be an edge if i is even or if i is odd and $x + y \neq 0$ (the latter is a nice way to encode an alternating 6-cycle). Thus the layers of L are alternating a $K_{3,3}$ and an alternating 6-cycle (see Figure 1).

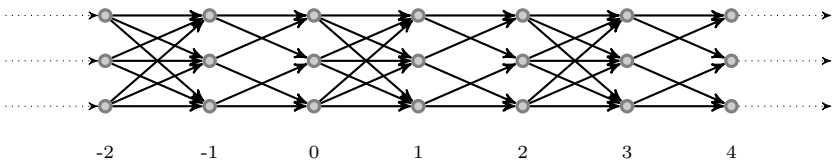


FIGURE 1. The digraph L

L has Property Z with $\varphi^1((i, x)) := i$ but $\varphi^2((i, x)) := i + 1$ is an epimorphism onto Z too. We can now define our counterexample:

$$D := L \,{}_{\varphi^1}\!\times_{\varphi^2}\, L. \tag{2.1}$$

For convenience, we denote $-1/0/1$ by $\texttt{-}/\texttt{o}/\texttt{+}$. As mentioned above, we extract the $\mathbb{Z}$ coordinate and denote the vertices of D with (n, x, y) rather than by $((n, x), (n + 1, y))$ as if D had the vertex set $\mathbb{Z} \times \{\texttt{-},\texttt{o},\texttt{+}\} \times \{\texttt{-},\texttt{o},\texttt{+}\}$.

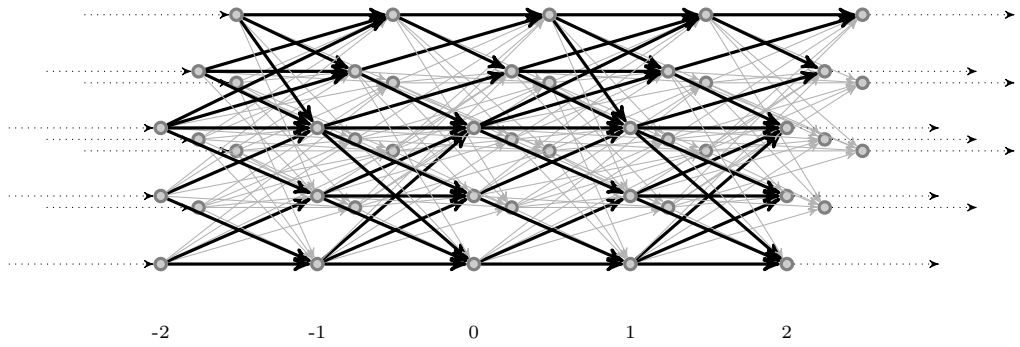


FIGURE 2. The digraph D

LEMMA 2.1. *Let AC_6 be the alternating 6-cycle and $D = (V(D), E(D))$ the above digraph. Let $\psi^1 : V(K_{3,3}) \to \{0,1\}$ and $\psi^2 : AC_6 \to \{0,1\}$ both map the initial vertices to 0 and the terminal vertices to 1. Then $\Delta(e)$ is independent of e and thus, $\Delta(D)$ is welldefined and*

$$\Delta(D) = K_{3,3\ \psi^1 \times_{\psi^2}} AC_6 \tag{2.2}$$

is connected, 1-arc transitive, bipartite but not complete bipartite.

PROOF. For $x = (i, x^1, x^2) \in V(D)$ define $\varphi(x) := i = \varphi^1(x^1) = \varphi^2(x^2)$. Thus, the direct fibre product preserves Property Z. Thus, no alternating walk can leave the layer in which it started. Thus, $\Delta(D)$ (if it exists) must be isomorphic to a subgraph of a layer thus bipartite. By a flip of the coordinates the direct fibre product is commutative. Thus, we do not need to distinguish between even and odd layers in the following. By the construction of D, every layer is of the form of the right side of (2.2). If we consider the subgraphs of a layer that are spanned by the vertices that agree on the second coordinate (note that that means ignoring the $\mathbb{Z}$-coordinate) we get three alternating 6–cycles. If we consider the subgraph spanned by the vertices that have a - in their third coordinate we get a $K_{3,3}$ that connects the three cycles. Thus, the layer is connected and $\Delta(D)$ exists since all layers are isomorphic. Thus, (2.2) holds. The 1-arc transitivity follows from the 1-arc transitivity of the factors in the following way: every edge in one of the factors corresponds to a copy of the other factor in the product. The edge sets of the collection of these copies form a partition of the edge set of the product (compare Figure 3). By the 1-arc transitivity of each one factor its automorphism group acts transitively on these copies and can be embedded in the automorphism group of $\Delta(D)$. Thus, one can adjust in the first step the second coordinates of the endpoints of an edge and in the second step the third coordinates – only restricted by the existence of edges. This is also the main idea of the construction of D and will be used in pretty much every step in the

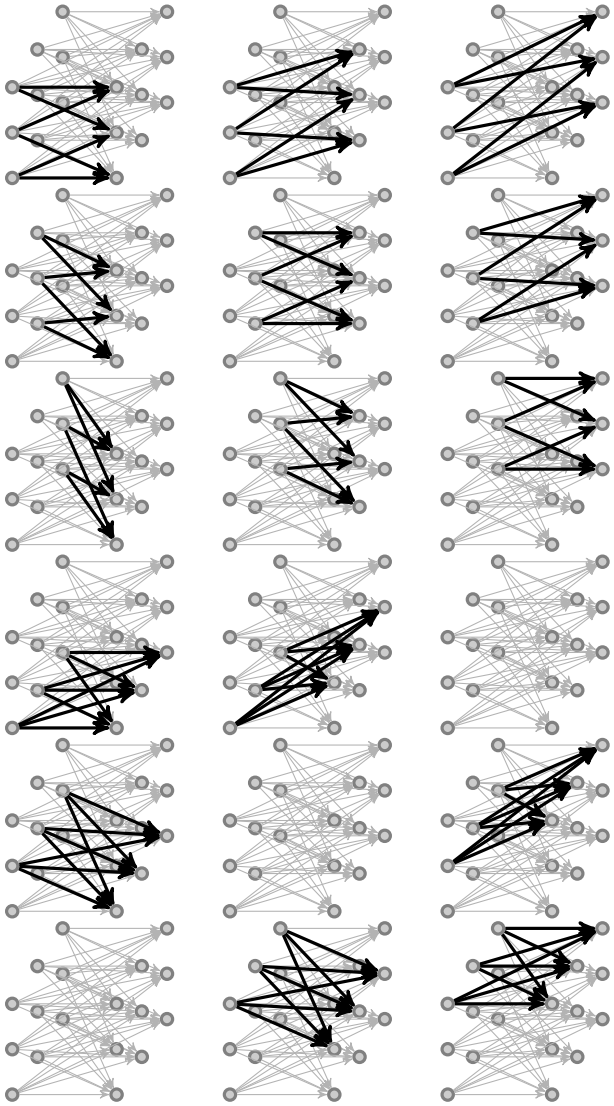


FIGURE 3. $\Delta(D)$

proof of Proposition 2.2. Finally the edge $((0,\mathsf{o},\mathsf{o}),(1,\mathsf{o},\mathsf{o}))$ does not exist in $\Delta(D)$ and thus it is not complete bipartite. □

PROPOSITION 2.2. *The digraph D defined in* (2.1) *is HAT.*

PROOF. We are going to prove very carefully that for every n-arc a (assume initial vertex x with $\varphi(x) = -m$) in D there is an automorphism $\psi \in \mathrm{Aut}(D)$ such that $\psi(a) = z$ where

$$z = (((i,\mathsf{-},\mathsf{-}),(i+1,\mathsf{-},\mathsf{-})))_{i=-n,\ldots,-1}.$$

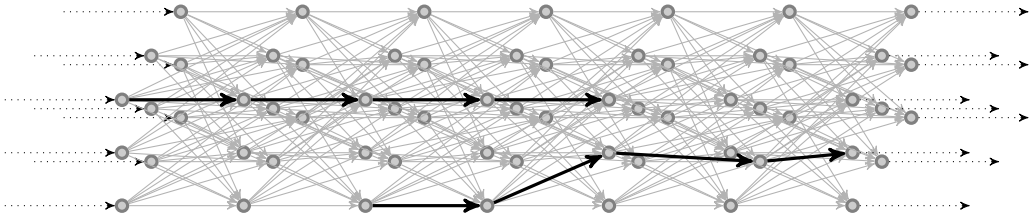


FIGURE 4. Two 4-arcs in D

In order to do so, we use a shift-automorphism σ, transitivity-automorphisms θ_{o}, $\theta_{\mathsf{+}}$, θ^{o} and $\theta^{\mathsf{+}}$ and arranging-automorphisms α_{o}, $\alpha_{\mathsf{+}}$ and α^{o}. For convenience, we denote $\theta_{\mathsf{-}} = \theta^{\mathsf{-}} = \alpha_{\mathsf{-}} = \alpha^{\mathsf{-}} = \alpha^{\mathsf{+}} = \mathrm{id}$. Our automorphism ψ will then have the form

$$\psi := (\sigma^{-1} \circ \alpha_{i_n} \circ \alpha^{i^n}) \circ \cdots \circ (\sigma^{-1} \circ \alpha_{i_1} \circ \alpha^{i^1}) \circ \theta_{i_0} \circ \theta^{i^0} \circ \sigma^m, \tag{2.3}$$

where $m \in \mathbb{Z}$ is of course meant to be a power rather than an index and $i_j, i^j \in \{\mathsf{-},\mathsf{o},\mathsf{+}\}$ where i^j cannot take the value $\mathsf{+}$ for $j > 0$ because there are no edges $((0, x,\mathsf{-}), (1, x,\mathsf{+}))$. We are now going to explain the tasks of the above automorphisms and afterwards check their existence and correctness one by one. In the following, we will refer to the line with the vertices $(i,\mathsf{-},\mathsf{-})$ as **baseline**.

- σ shifts D up by one layer i.e. $\varphi + 1 \equiv \varphi \circ \sigma$. Thus, σ^m shifts the initial vertex of a to the zero-layer i.e. $\varphi \circ \sigma^m(x) = 0$. Moreover, σ stabilises the baseline setwise.
- The transitivity-automorphisms guarantee the transitivity of D. Their task is to map $\sigma^m(x)$ to the baseline. $\theta_{\mathsf{+}}$ and θ_{o} map $(0,\mathsf{+}, y)$ and $(0,\mathsf{o}, y)$ respectively to $(0,\mathsf{-}, y)$, thereby fixing the third coordinate. $\theta^{\mathsf{+}}$ and θ^{o} map $(0, x,\mathsf{o})$ and $(0, x,\mathsf{+})$ respectively to $(0, x,\mathsf{-})$, thereby fixing the second coordinate. Thus, they can be combined in a way to map any given vertex in the zero-layer to $(0,\mathsf{-},\mathsf{-})$. There is no need for the transitivity-automorphisms to stabilise the baseline.
- The arranging-automorphisms guarantee that D is $(s + 1)$-arc transitive if it only is s-arc transitive (where we can understand the transitivity gained from the transitivity-automorphisms as 0-arc transitivity). In order to achieve this, we must be able to map the arcs $((0,\mathsf{-},\mathsf{-}), (1, x, y))$ onto $((0,\mathsf{-},\mathsf{-}), (1,\mathsf{-},\mathsf{-}))$ and at the same time stabilise the negative half of the baseline. α^{o} maps $((0,\mathsf{-},\mathsf{-}), (1,\mathsf{o}, y)) \mapsto ((0,\mathsf{-},\mathsf{-}), (1,\mathsf{-}, y))$ again fixing y. α_{o} and $\alpha_{\mathsf{-}}$ respectively map $((0,\mathsf{-},\mathsf{-}), (1, x,\mathsf{o}))$ and $((0,\mathsf{-},\mathsf{-}), (1, x,\mathsf{+}))$ respectively to $((0,\mathsf{-},\mathsf{-}), (1, x,\mathsf{-}))$ yet again fixing x.
- The brackets in (2.3) therefore map the arc with initial vertex $(0,\mathsf{-},\mathsf{-})$ on the baseline and shift D one step to the left to keep the working-layer the same.

We are now going to have a closer look at every single of these automorphisms:

σ: We start with the shift-automorphism. It will increase the $\mathbb{Z}$ coordinate by one and flip the other coordinates.

$$\sigma : (i, x, y) \mapsto (i + 1, y, x)$$

We recognise that we can describe the edge set of D by joining the conditions of the factors. Thus

$$((i, x_1, y_1), (i + 1, x_2, y_2)) \in E(D) \iff (1) \text{ or } (2)$$

with the conditions

(1) i is even and $y_1 + y_2 \neq 0$

(2) i is odd and $x_1 + x_2 \neq 0$.

We have to prove that the image of every edge is again an edge. We consider

$$\begin{aligned} \sigma(((i, x_1, y_1), (i + 1, x_2, y_2))) &= (\sigma((i, x_1, y_1)), \sigma((i + 1, x_2, y_2))) \\ &= ((i + 1, y_1, x_1), (i + 2, y_2, x_2)) \end{aligned}$$

and the two cases:

(a) i is even. Then we must have $y_1 + y_2 \neq 0$. Obviously, $i + 1$ is odd and by condition (2) from above $((i + 1, y_1, x_1), (i + 2, y_2, x_2))$ is an edge.

(b) i is odd. Analogously, we have $x_1 + x_2 \neq 0$ and $i + 1$ is even and by condition (1) from above $((i + 1, y_1, x_1), (i + 2, y_2, x_2))$ is an edge.

Since $\sigma((i,-,-)) = (i + 1,-,-)$, the baseline is stabilised setwise. Also

$$\varphi(\sigma((i, x, y))) = \varphi((i + 1, y, x)) = i + 1 = \varphi((i, x, y)) + 1$$

holds.

θ_{o}: The automorphism θ_{o} must map

$$\theta_{\mathsf{o}} : (0, \mathsf{o}, y) \mapsto (0, \mathsf{-}, y).$$

We choose the automorphism that exchanges the vertices

$$\begin{aligned} (0, \mathsf{o}, y) &\leftrightarrows (0, \mathsf{-}, y) \quad \text{and} \\ (-1, \mathsf{o}, y) &\leftrightarrows (-1, \mathsf{+}, y). \end{aligned}$$

This is illustrated in Figure 5. Note that this figure shows D projected along the third coordinate. Thus, two more vertices hide behind every vertex and a $K_{3,3}$ (AC_6 respectively – depending on the layer) hides behind every edge. It is enough to find an automorphism in that view. The third coordinate (that is being projected along) cannot cause any problem since inside a layer all the edges represent the same bipartite digraph (either $K_{3,3}$ or AC_6) and the third

coordinate is stabilised. Within a layer, the edges that map onto each other are highlighted with the same style. Note that this automorphism (like all the following ones) involves 12 vertices and 126 edges. These are:

$$\begin{aligned}
((-2,-,y),(-1,\circ,y)) &\leftrightarrows ((-2,-,y),(-1,+,y))\\
((-2,\circ,y),(-1,\circ,y)) &\leftrightarrows ((-2,\circ,y),(-1,+,y))\\
((-2,+,y),(-1,\circ,y)) &\leftrightarrows ((-2,+,y),(-1,+,y))\\
((-1,-,y),(0,-,y)) &\leftrightarrows ((-1,-,y),(0,\circ,y))\\
((-1,\circ,y),(0,+,y)) &\leftrightarrows ((-1,+,y),(0,+,y))\\
((-1,\circ,y),(0,-,y)) &\leftrightarrows ((-1,+,y),(0,\circ,y))\\
((0,-,y),(1,-,y)) &\leftrightarrows ((0,\circ,y),(1,-,y))\\
((0,-,y),(1,\circ,y)) &\leftrightarrows ((0,\circ,y),(1,\circ,y))\\
((0,-,y),(1,+,y)) &\leftrightarrows ((0,\circ,y),(1,+,y))
\end{aligned}$$

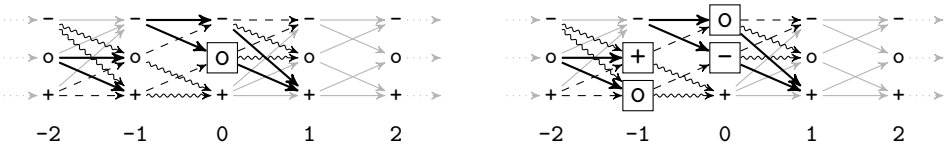


FIGURE 5. $\theta_\circ$

θ_+: The automorphism θ_+ must map

$$\theta_+ : (0,+,y) \mapsto (0,-,y).$$

We choose the automorphism that exchanges the vertices

$$\begin{aligned}
(0,+,y) &\leftrightarrows (0,-,y) \quad \text{and}\\
(-1,+,y) &\leftrightarrows (-1,-,y).
\end{aligned}$$

This is illustrated in Figure 6. For the rest of the proof we will not list the actions on the edges since they are obvious from the figures.

FIGURE 6. θ_+

θ^{o}: The automorphism θ^{o} must map

$$\theta^{\mathsf{o}} : (0, x, \mathsf{o}) \mapsto (0, x, \mathsf{-}).$$

We choose the automorphism that exchanges the vertices

$$\begin{aligned} (0, x, \mathsf{o}) &\leftrightarrows (0, x, \mathsf{-}) \quad \text{and} \\ (1, x, \mathsf{o}) &\leftrightarrows (1, x, \mathsf{+}). \end{aligned}$$

This is illustrated in Figure 7.

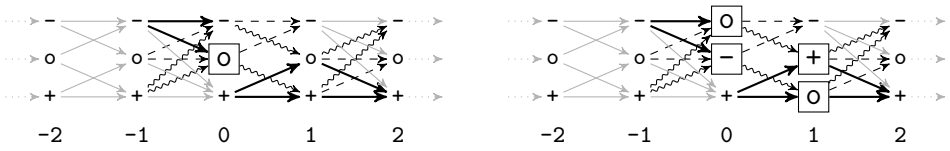


FIGURE 7. θ^{o}

$\theta^{\mathsf{+}}$: The automorphism $\theta^{\mathsf{+}}$ must map

$$\theta^{\mathsf{+}} : (0, x, \mathsf{+}) \mapsto (0, x, \mathsf{-}).$$

We choose the automorphism that exchanges the vertices

$$\begin{aligned} (0, x, \mathsf{+}) &\leftrightarrows (0, x, \mathsf{-}) \quad \text{and} \\ (1, x, \mathsf{+}) &\leftrightarrows (1, x, \mathsf{-}). \end{aligned}$$

This is illustrated in Figure 8.

FIGURE 8. $\theta^{\mathsf{+}}$

α_{o}: The automorphism α_{o} must map the edge

$$((0, \mathsf{-}, \mathsf{-}), (1, \mathsf{o}, y)) \mapsto ((0, \mathsf{-}, \mathsf{-}), (1, \mathsf{-}, y)).$$

We choose the automorphism that exchanges the vertices

$$\begin{aligned} (1, \mathsf{o}, y) &\leftrightarrows (1, \mathsf{-}, y) \quad \text{and} \\ (2, \mathsf{+}, y) &\leftrightarrows (2, \mathsf{o}, y). \end{aligned}$$

This is illustrated in Figure 9.

FIGURE 9. α_{o}

α_{+}: The automorphism α_{+} must map the edge

$$((0,-,-),(1,+,y)) \mapsto ((0,-,-),(1,-,y)).$$

We choose the automorphism that exchanges the vertices

$$\begin{aligned}(1,+,y) &\leftrightarrows (1,-,y) \quad \text{and}\\ (2,+,y) &\leftrightarrows (2,-,y).\end{aligned}$$

This is illustrated in Figure 10.

FIGURE 10. α_{+}

α°: Finally, the automorphism α° must map the edge

$$((0,-,-),(1,x,\mathsf{o})) \mapsto ((0,-,-),(1,x,-)).$$

We choose the automorphism that exchanges the vertices

$$\begin{aligned}(0,x,\mathsf{o}) &\leftrightarrows (0,x,+) \quad \text{and}\\ (1,x,-) &\leftrightarrows (1,x,\mathsf{o}).\end{aligned}$$

This is illustrated in Figure 11.

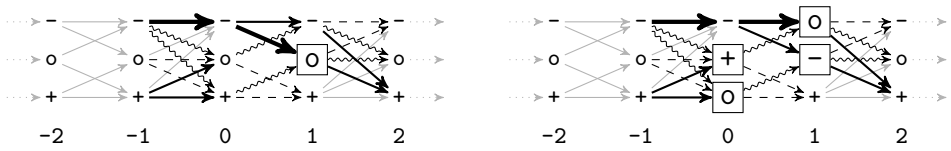


FIGURE 11. α°

□

THEOREM 2.3. *There is a connected HAT digraph with Property Z and finite fibres which has an associated digraph that is* not *complete bipartite, i.e., Conjecture 1.1 does not hold.*

PROOF. The digraph D defined in (2.1) has fibresize 9. By Lemma 2.1 its associated digraph is not complete bipartite. It follows from Lemma 2.1 as well that D is connected. By Proposition 2.2, D is HAT. □

3. Remarks

3.1. On the overlap with the work of DeVos, Mohar and Šámal. Our digraph D was recently constructed by DeVos, Mohar and Šámal [**DMŠ11**, Construction 2] without the use of the direct fibre product. They basically gave the joined condition from the above proof directly and mixed the coordinates in a way such that the shift does not need to flip coordinates. Our constructions from Section 4.2 corresponds in a similar way to [**DMŠ11**, Construction 3].

3.2. Möller's digraph. The digraph Möller constructed in [**Möl02**] looked promising because the arranging-automorphisms work. Unfortunately, he missed to check transitivity. His approach was to use an alternating eight cycle of $K_{2,2}$ as Δ and concatenate them cleverly. The easiest way to see that his digraph is not HAT is probably to see that an edge cannot be mapped to an edge that agrees on the terminal vertex and lies in the same $K_{2,2}$ (the contradiction arises soon in the outward direction).

3.3. On the automorphisms. In the proof of Proposition 2.2 we chose the n-arc

$$z = (((i, -, -), (i+1, -, -)))_{i=-n,\dots,-1}$$

and the automorphism

$$\psi := (\sigma^{-1} \circ \alpha_{i_n} \circ \alpha^{i^n}) \circ \cdots \circ (\sigma^{-1} \circ \alpha_{i_1} \circ \alpha^{i^1}) \circ \theta_{i_0} \circ \theta^{i^0} \circ \sigma^m.$$

We could as well have chosen the n-arc

$$z' = (((i, -, -), (i+1, -, -)))_{i=0,\dots,n-1}$$

in which case we would not need the shifts σ^{-1}. Moreover, we would not need that the shift stabilises the baseline. But we would have needed an argument that the arranging-automorphisms work in every layer (this is not difficult, since all the layers look the same, i.e., just differ by flips of the coordinates). The automorphism would then appear as

$$\psi' := ({}_n\alpha_{i_n} \circ {}_n\alpha^{i^n}) \circ \cdots \circ ({}_1\alpha_{i_1} \circ {}_1\alpha^{i^1}) \circ \theta_{i_0} \circ \theta^{i^0} \circ \sigma^m,$$

where the lower left indices denote the working-layer. More importantly, we notice that the different α- and θ-automorphisms act only on one of the coordinates and thus we need not perform them one after the other but can apply them simultaneously in every layer. We can therefore denote

$$\Theta := \begin{pmatrix} \mathrm{id} \\ \theta_{i_0} \\ \theta^{i^0} \end{pmatrix}, \quad A_j := \begin{pmatrix} \mathrm{id} \\ {}_j\alpha_{i_j} \\ {}_j\alpha^{i^j} \end{pmatrix},$$

and our automorphism reads

$$\psi' = A_n \circ \cdots \circ A_1 \circ \Theta \circ \sigma^m.$$

We chose to use the automorphisms ψ rather than ψ' for the above proof because in that way the number of needed α-automorphisms is finite.

4. More constructions

4.1. The example of McKay and Praeger. In [**CPW93**] a nontrivial HAT digraph is constructed that has Property Z, finite fibres and complete bipartite digraphs as associated digraph. This example can be realised using the direct fibre product. One uses $K_{n,n}$ instead of $K_{3,3}$ and replaces AC_6 with a matching (n horizontal edges). One places m matchings in between the $K_{n,n}$ and multiplies the digraph with its m shifts (see Figure 12).

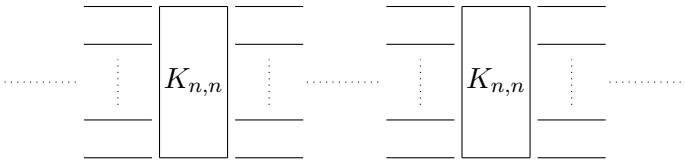


FIGURE 12. Factor of the digraph of Praeger and McKay

4.2. Other factors. Considering the automorphisms in the proof, every 1-arc transitive, connected, noncomplete bipartite digraph (with equal partition sizes) concatenated with a suitable $K_{n,n}$ would have done the job. (Connectedness is not necessary in every situation.) From now on, we will refer to these noncomplete bipartite, 1-arc transitive layers as **involvers** (unless they have outdegree smaller than 2). Figure 13 shows this situation where I stands for the involver.

Like in Section 4.1, we are not restricted to two factors. We are free to place arbitrary many matchings (and some more $K_{n,n}$) between the $K_{n,n}$ and the involver as long as the factor stays periodic (say with the length l of the period). Then the product of the l different shifts is again HAT.

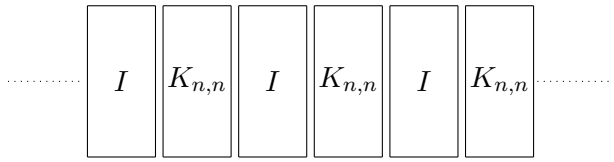


FIGURE 13. Little variation of the factors of D

Moreover, we can use different 1-arc transitive, bipartite digraphs as involvers. We just have to ensure that there is a $K_{n,n}$ between each two of them. Again multiplying the l shifts yield a HAT digraph (see Figure 14).

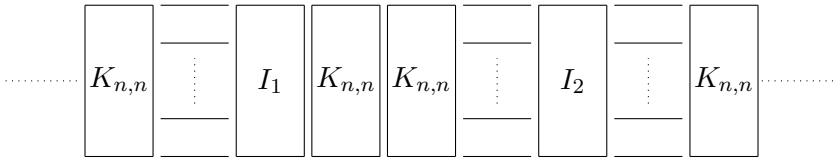


FIGURE 14. More general factors

Indeed we do not even need to keep the fibresizes of the factor constant but can alter them periodically.

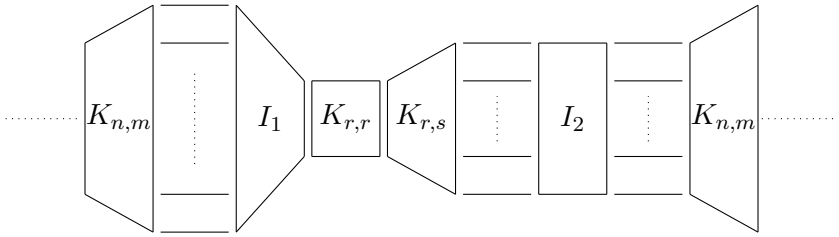


FIGURE 15. Even more general factors

PROPOSITION 4.1. *Let G be a digraph with Property Z and finite fibres such that every layer is either complete bipartite, a matching, or an involver and such that between any two involvers there is a complete bipartite layer. If G is periodic (i.e. there is a shift $\sigma_l \in \mathrm{Aut}(G)$ such that $\varphi \circ \sigma_l(v) = \varphi(v) + l$ for all $v \in V(G)$ and some $l \in \mathbb{Z}$) then the direct fibre product*

$$G^* = \prod_{\{i=0,1,\ldots,l-1\}}^{\{\varphi+i\}} G$$

is highly arc transitive and two-ended.

PROOF. In this more general situation we need to think about the shift globally and the transitivity- and arranging-automorphisms locally in one factor from the $K_{n,m}$ preceding an involver to the $K_{r,s}$ succeeding it. Additionally, we need to take care of the existence of a baseline. We choose some line b in G and label its vertices by zero. The product of b with its shifts induces a line in G^* that contains the vertices $(n, (0)_{i=1,\dots,l})_{n\in\mathbb{Z}}$. We can use this line as baseline.

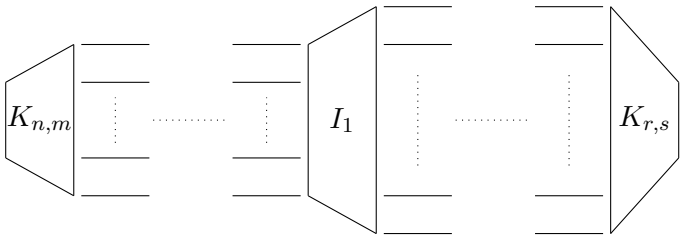


FIGURE 16. Local view of a factor

The shift simply increases the $\mathbb{Z}$ coordinate and circularly shifts the other coordinates:

$$\sigma : (i, x_1, \dots, x_n) \mapsto (i+1, x_n, x_1, \dots, x_{n-1}).$$

We must consider the α- and θ-automorphisms in all the layers of the local view (we can skip the θ for the very first and the θ and α for the very last layer because they actually belong to the preceding or succeeding local view). In contrast to the proof in Section 2 we are not looking for an automorphism that flips a single edge on the baseline, but for an automorphism that maps an arc to the baseline in a single view. That is, we treat one coordinate of the vertices along an arc without influencing the other coordinates.

- The α-automorphisms for the $K_{n,m}$ map an arbitrary edge e_1 onto an other e_0. The corresponding map for the terminal vertices $t_{e_1} \mapsto t_{e_0}$ will be transported by the matchings to the involver. There it can be realised because the 1-arc transitivity of the involver guarantees transitivity on the initial vertices. But it will produce some permutation on its initial vertices (which is transported back by the matchings and stopped at the $K_{n,m}$) and some permutation on its terminal vertices (which is transported by the matchings to the $K_{r,s}$ where it is stopped).
- The θ-automorphisms for the terminal vertices of the $K_{n,m}$ work in exactly the same way.
- In the layers between the $K_{n,m}$ and the involver there are no branchings, thus we have $\alpha = \mathrm{id}$.

- The θ-automorphisms in these layers must again be considered first in the forward direction where they have some effect on the initial and terminal vertices of the involver. These effects are again transported by the matchings to the complete bipartite layers, where they are stopped.
- The same happens for the θ-automorphisms in the initial layer of the involver – only the forward transportation to the involver is now trivial.
- For the α-automorphisms in the involver we need its 1-arc transitivity (which is more than the transitivity on the initial vertices). But then we get the transportation and stoppage as above.
- The rest of the layers can be dealt analogously by first considering the backward direction.

Due to the construction the fibres of G^* are finite. Hence G^* cannot be one-ended. For the same reason it cannot have infinitely many ends. Thus by any of the well know 1-2-infinity theorems it must have two ends if it is only connected. We show the connectedness inductively. G is connected because it contains complete bipartite layers and every vertex of G is contained in some line. Suppose that the product G' of the first $n-1$ factors of G^* is connected. Consider two vertices $v = (v', g)$ and $w = (w', h)$ in $G'' = G' {}_{\varphi'}\times_{\varphi+n-1} G$. Consider lines g_G and h_G in G containing g and h. Choose some complete bipartite layer in G. g_G intersects the initial vertex layer of the complete bipartite layer in g_0 and h_G the corresponding terminal vertex layer in h_0. The lines induce copies of G' in G'' thus (since G is connected) there are paths connecting v with all the vertices $(\cdot, g_0)$ and paths connecting w with all the vertices $(\cdot, h_0)$. Since we chose a complete bipartite layer in G there exists an edge in G'' connecting some vertex $(\cdot, g_0)$ with some vertex $(\cdot, h_0)$. Thus there exists a path connecting v and w. Thus G'' is connected and by induction so is G^*. □

Considering the above proof, we can place more than one involver I_i between two complete bipartite layers if they are compatible in the following sense: there are groups of automorphism $R_i < \mathrm{Aut}(I_i)$ which act 1-arc transitive on the I_i such that the set of permutations that is induced by R_n on the terminal vertices of I_n agrees with the set of permutations that is induced by R_{n+1} on the initial vertices of I_{n+1}.

This description arises from the philosophy of the above proof and can be expressed as follows: consider the subdigraphs T' of G that are spanned by the vertices of the edge-layers between two consecutive complete bipartite layers. The automorphism groups $\mathrm{Aut}(T')$ act transitively on the sets of maximal arcs in T'. We do not need to formulate this condition for various subdigraphs T' but instead use a single subdigraph T. Namely, let σ be a shift of G with step width equal to the period-length and K some complete bipartite layer, then the subdigraph spanned by the edge-layers between K and $\sigma(K)$

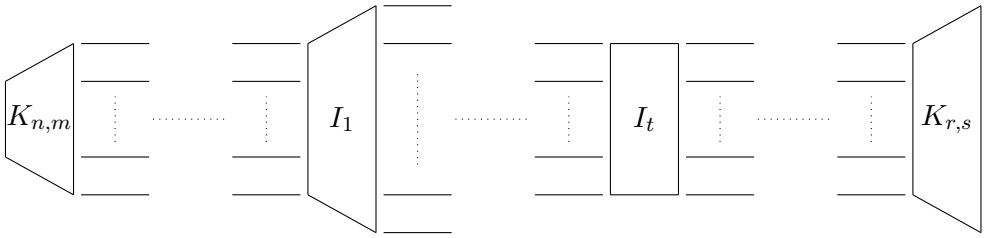


FIGURE 17. Compatible involvers

qualifies as T. Such a digraph T corresponds to the template digraphs used in [**DMŠ11**]. We summarise these considerations in the proposition below.

PROPOSITION 4.2. *Let G be a digraph with Property Z and finite fibres that contains complete bipartite layers and is periodic with period length l. Let the edge layers between a complete bipartite layer and its next shift span a subdigraph T. If $\mathrm{Aut}(T)$ acts transitively on the set of maximal arcs of T then the direct fibre product*

$$G^* = \prod_{\{i=0,1,\dots,l-1\}}^{\{\varphi+i\}} G$$

is HAT and two-ended.

PROOF. The proof runs in analogy to the proof of Proposition 4.1. □

4.3. Factors with infinite fibres. In the above proofs we used the finiteness of the fibres to show the two-endedness. The latter was needed to construct counterexamples to Conjecture 1.1, but we do not need it if we only want to construct a HAT digraph. If we drop finiteness of fibres we can construct one-ended HAT digraphs.

PROPOSITION 4.3. *Let G be a digraph with Property Z that contains complete bipartite layers and is periodic with period length l. Let the edge layers between a complete bipartite layer and its next shift span a subdigraph T. If $\mathrm{Aut}(T)$ acts transitively on the set of maximal arcs of T then the direct fibre product*

$$G^* = \prod_{i\in\{0,1,\dots,l-1\}}^{\{\varphi+i\}} G$$

is HAT. Moreover, if G contains an infinite fibre, G^* is one-ended.

PROOF. It is enough to show the one-endedness since the highly arc transitivity follows from the above proof. Thus suppose that G has an infinite vertex-layer. We first show that G is not locally finite. Either all layers of G are infinite – in this case G contains a $K_{\infty,\infty}$ and is not locally finite – or there is a finite layer – then there is a

finite vertex-layer followed by an infinite one. Since every layer must be 1-arc transitive, all the vertices in a layer must have the same nonvanishing indegree. Thus the total indegree of the infinite layer is infinite. Since it is equal to the total outdegree of the preceding finite layer, G is not locally finite. To see that G^* is one-ended, we recall the proof of connectedness from above. There we chose a complete bipartite layer to connect two lines. We could have chosen it to the right of the two points we intended to connect. Doing this in every step of the above induction we get a path that connects two arbitrary vertices without visiting any vertex that is to the left of the leftmost of these two. Thus, if we cut off the left half of G^* at any layer the remaining right half stays connected. We show that no pair of rays can be finitely separated. In order to do so, we show that the even stronger statement that no pair of vertices v and w can be finitely separated. Since G^* is HAT and not locally finite, every vertex has infinitely many outneighbors. Thus no finite separator S can cut out all outneighbors of a certain vertex. Thus v and w both lie on some right ray. Both these rays reach a region of G^* that is to the right of the rightmost vertex of S. This region is connected and thus v and w are connected. □

4.4. Infinitely many factors. Consider the factor from Figure 15 and let it be non-periodic. We still could use it to construct a HAT digraph if we multiply all its shifts (countably infinitely many). The vertices of such a product can be understood as a $\mathbb{Z}$ coordinate together with a two way infinite sequence with entries at the i-th position from the i-th layer of the factor (which is different for every shift).

PROPOSITION 4.4. *Let G be a digraph with Property Z that contains a complete bipartite layer to the left and the right of any other layer. Let the edge layers between two complete bipartite layers span subdigraphs T_i. If for all i the automorphism group* $\mathrm{Aut}(T_i)$ *acts transitively on the set of maximal arcs of T_i then the direct fibre product*

$$G^* = \prod_{i\in\mathbb{Z}}^{\{\varphi+i\}} G$$

is HAT and not locally finite. Moreover, if the distance between any two neighbouring complete bipartite layers is bounded by some $d \in \mathbb{N}$, then G^* is one-ended.

PROOF. The automorphisms Θ and A_j described in Section 3.3 will have infinitely many entries. But, since the actions on the different coordinates have no effects on one another, the parts in the above proofs concerning highly arc transitivity remain valid. Since any layer of G^* contains a product of infinitely many complete bipartite factors, the vertices have infinite in- and outdegree. The one-endedness will follow from the "right-connectedness" as above. Thus we are left to show the connectedness in the

above sense. Since we have infinitely many factors, we cannot use the above induction, but have to argue as follows. Let v and w be two arbitrary vertices in G^*. Without loss of generality let v be the leftmost of these two vertices and set $w^0 = w$. There is an arc from v to some vertex v^0 that agrees with w^0 on the vertex-layer. We construct two arcs a_v and a_w starting in v^0 and w^0 that meet in $w^d = v^d$. Suppose the a_v and a_w are constructed up to the vertices v^n and w^n. Let I_n be the set of coordinates on which v^n and w^n do not agree and let J_n be the set of coordinates j_n such that $v^n_{j_n}$ and $w^n_{j_n}$ have a common outneighbour x_{j_n} in G (note that $v^n_{j_n}$ and $w^n_{j_n}$ have in particular a common outneighbour if $v^n_{j_n} = w^n_{j_n}$). We set $v^{n+1}_{j_n} = w^{n+1}_{j_n} = x_{j_n}$ for $j_n \in J_n$. For $i_n \in I_n \setminus J_n$ let y_{i_n} be some outneighbour of $v^n_{i_n}$ in G and z_{i_n} some outneighbour of $w^n_{i_n}$ and set $v^{n+1}_{i_n} = y_{i_n}$ and $w^{n+1}_{i_n} = z_{i_n}$. Due to the definition of the direct fibre product, there are edges (v^n, v^{n+1}) and (w^n, w^{n+1}) in G^* which we can attach to a_v and a_w. Since we find a complete bipartite layer in every coordinate after at most d steps, $\bigcup_{i=0,\dots,d-1} J_i$ is an exhaustion of the set of coordinates, and thus I_d is empty. □

Intuitively the key part of the above constructions is the use of the $K_{n,m}$ between the involvers. We used them to stop the movement of the α- and θ-automorphisms into both directions. But in order to construct HAT digraphs it is not necessary to stop this movement if one can guarantee that it does not run into the baseline in backward direction. We can achieve this by using some compatible involvers (finitely many) embedded in a line of matchings. Such a construction yields digraphs with uncountably many components. In the following we denote the cardinality of the set of the real numbers by $\mathfrak{c}$.

PROPOSITION 4.5. *Let G be a digraph with Property Z and let $k < l$ be integers such that for all $j \leq k$ and all $j > l$ the j-th layer is a matching. Let the layers of G be at most countably large. Let the edge-layers between these matchings span a subdigraph T. If $\mathrm{Aut}(T)$ acts transitively on the set of maximal arcs of T, then the direct fibre product*

$$G^* = \prod_{i\in\mathbb{Z}}^{\{\varphi+i\}} G$$

is HAT. If all the involvers are locally finite, so is G^*. If the matchings are not trivial then G^* is not connected and actually the set of its components has cardinality $\mathfrak{c}$.

PROOF. The proof for the highly arc transitivity runs as above. The permutations that appear at the vertex-layers k and l are transported through the rest of the digraph, but that does neither hinder the α- and θ-automorphisms to be automorphisms nor has it any influence on the baseline. If all involvers are locally finite, the in- and outdegrees are products whose factors are either 1 (almost all) or finite, and thus finite. If the

matchings are not trivial, the vertices are defined by their layer and a two way infinite sequence each of whose entries can take more than one value (mutually independently). Thus, a layer contains $\mathfrak{c}$ vertices. Every vertex is contained in some line, thus every component contains a line, thus every component intersects the zero-layer. Hence, there are at most $\mathfrak{c}$ components. Every path starting and ending in the zero-layer passes through a finite number of different layers. In every layer at most $l-k$ coordinates can be altered. Thus, a vertex v in the zero-layer is connected with another vertex of this very layer only if they differ in at most finitely many coordinates. Thus, a component can only contain countably many vertices of the same layer. Hence, G^* has at least $\mathfrak{c}$ components, hence exactly $\mathfrak{c}$ components. In particular, it is not connected. □

A special case of the factors from Proposition 4.5 realises a digraph constructed in [**CPW93**].

Example 4.6. We consider the digraph that was constructed in [**CPW93**, Theorem 4.8] as the direct product (not the direct fibre product) of the so called sequences digraph with Z. The vertices of the sequences digraph are two way infinite sequences $(x_i)_{i\in\mathbb{Z}}$ with entries $x_i \in A$ for $i \leq 0$ and $x_i \in B$ for $i > 0$, where A and B are some sets both containing 0 such that almost all x_i are 0. There is an edge $((x_i)_{i\in\mathbb{Z}}, (y_i)_{i\in\mathbb{Z}})$ if $x_i = y_{i+1}$ for $i \neq 0$ and (x_0, y_1) is an edge of a 1-arc transitive bipartite digraph I with bipartition $A \cup B$ that we chose beforehand. We rename the vertices of the HAT product by the mapping

$$\phi : (n, (x_i)_{i\in\mathbb{Z}}) \mapsto (n, (x_{n-i})_{i\in\mathbb{Z}})$$

and call the resulting digraph D. Thus $D = (V(D), E(D))$ is defined by

$$\begin{aligned} V(D) &= \{(n, (x_i)_{i\in\mathbb{Z}}) \mid n \in \mathbb{Z}, x_{i<n} \in B, x_{i\geq n} \in A\}, \\ E(D) &= \{((n, (x_i)_{i\in\mathbb{Z}}), (n+1, (y_i)_{i\in\mathbb{Z}})) \mid x_i = y_i \text{ for } i \neq n, (x_n, y_n) \in E(I)\}. \end{aligned}$$

Now we define a factor G by extending I with matchings on both sides, i.e. $G = (V(G), E(G))$ with

$$\begin{aligned} V(G) &= \{n \in \mathbb{Z} \mid n \leq 0\} \times A \cup \{n \in \mathbb{Z} \mid n > 0\} \times B, \\ E(G) &= \{((n, a), (n+1, x)) \mid n < 0, a = x\} \cup \\ &= \{((0, a), (1, b)) \mid (a, b) \in E(I)\} \cup \\ &= \{((n, b), (n+1, y)) \mid n > 0, b = y\}. \end{aligned}$$

Building the direct fibre product as in Proposition 4.5 yields a digraph G^* with

$$\begin{aligned} V(G^*) &= \{(n, (x_i)_{i\in\mathbb{Z}}) \mid n \in \mathbb{Z}, x_{i\leq n} \in A, x_{i\geq n} \in B\}, \\ E(G^*) &= \{((n, (x_i)_{i\in\mathbb{Z}}), (n+1, (y_i)_{i\in\mathbb{Z}})) \mid ((n, x_i), (n+1, y_i)) \in E(G)\}, \end{aligned}$$

where the names of the vertices of G are meant to respect the corresponding φ_i. Rewriting the condition on the edges of $E(G^*)$ using the definition of $E(G)$ yields

$$E(G^*) = \{((n, (x_i)_{i\in\mathbb{Z}}), (n+1, (y_i)_{i\in\mathbb{Z}})) \mid x_i = y_i \text{ for } i \neq n, (x_n, y_n) \in E(I)\}.$$

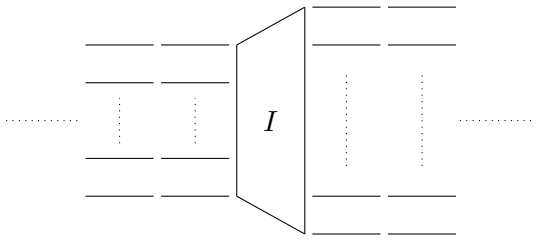


FIGURE 18. A factor realising a construction from [**CPW93**]

Hence, the map $\psi : C \to D$, where C is the component of G^* containing $(0, (0)_{i\in\mathbb{Z}})$ defined by

$$\psi : (n, (x_i)_{i\in\mathbb{Z}}) \mapsto (n, (x_{2n-i})_{i\in\mathbb{Z}})$$

is an isomorphism.

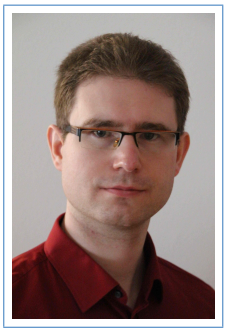
Note that I could be a $K_{1,n}$. In that case, G^* would be a tree and thus had infinitely many ends (unless $n = 1$, in which case $G^* \cong Z$).

4.5. Nonisomorphic factors. Up to now we considered non-HAT factors and created a HAT product. Finally, we want to build new HAT digraphs from existing HAT digraphs. From [**CPW93**, Lemma 4.3, (a)] it follows that the direct fibre product of two HAT digraphs with Property Z is again HAT with Property Z. If the factors have finite fibres so has their product. If one of the factors has a noncomplete bipartite associated digraph so has the product. Thus, we can obtain numerous HAT digraphs with Property Z by multiplying some of the above digraphs, thereby gaining even more counterexamples of Conjecture 1.1.

Building the direct fibre product of the regular tree with indegree 1 and outdegree 2 with the regular tree with indegree 2 and outdegree 1, one gets a Cayley graph of the Lamplighter group. it has $K_{2,2}$ as associated digraph. If one replaces the second tree by the regular tree with indegree 3 and outdegree 1, one gets a digraph that is not quasi-isomorphic to any Cayley graph, is has $K_{3,2}$ as associated digraph. Such digraphs are called Diestel-Leader graphs, and they got to fame for these mentioned properties. Any other choice of regular trees will also yield a HAT product.

We close with a remark on the digraphs we mentioned above to realise the construction from [**CPW93**, Theorem 4.8]. Their direct fibre product with some HAT digraph with Property Z will realise the digraphs from [**CPW93**, Corollary 4.9]. This follows

from the mutual associativity of the direct product and the direct fibre product. This associativity follows immediately from the definitions of the products.



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Peer-Reviewed Journals

[1] **Constructing highly arc transitive digraphs using a direct fibre product**
Discrete Mathematics Volume 313, Issue 23, 6 December 2013, Pages 2816-2829

Preprints

[2] **A complexity theorem for the Novelli-Pak-Stoyanovskii algorithm**
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