Masterarbeit

Titel der Masterarbeit

Affine line bundles and Real affine line bundles

verfasst von

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Contents

1 Introduction

Let X be a compact, complex manifold. The isomorphism classes of holomorphic line bundles on X form an abelian group, called the *Picard group* of X and denoted by $\mathrm{Pic}(X)$. A well known result in the theory of fiber bundles and complex geometry gives a natural identification $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$. It has naturally the structure of a complex Lie group. Generally, one has a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0,$$

with $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/2\pi i H^1(X, \mathbb{Z})$, and $\mathrm{NS}(X)$ is the Neron-Severi group defined as $\ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X))$. If X has a Kähler structure, $2\pi i H^1(X, \mathbb{Z})$ is a lattice in $H^1(X, \mathcal{O}_X)$ and $\mathrm{Pic}^0(X)$, the connected component of the identity,becomes a complex torus.
In the general case, when X is not necessarily Kähler, one can still prove that $2\pi i H^1(X, \mathbb{Z})$ is closed in $H^1(X, \mathcal{O}_X)$, obtaining a complex Lie group structure on $\mathrm{Pic}^0(X)$. This group is not always compact, for example

$$X \text{ is a surface of class VII} \Rightarrow \mathrm{Pic}^0(X) \cong \mathbb{C}^*.$$

Surprisingly, the fact that $2\pi i H^1(X, \mathbb{Z})$ is closed in $H^1(X, \mathcal{O}_X)$ is not presented clearly in the literature, so I give a detailed proof in chapter 2.2.1.
An important theory, developed originally by Atiyah in [2], concerns the concept of Real vector bundles on a Real manifold, i.e. on a manifold endowed with a continuous (or smooth) involution $\iota : X \to X$.

Definition 1.1. *Let (X, ι) be a Real manifold and $E \to X$ a complex vector bundle on X. A ι-Real structure on E is an involutive isomorphism $\tilde{\iota}$: $E \to E$ such that $\tilde{\iota}$ is fiber-wise anti-linear and renders the following diagram commutative.*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\iota}} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota} & X \end{array}$$

In complex geometry, it is often interesting to require compatibility with the complex structures, leading to the next definition:

Definition 1.2. *Let X be a complex manifold. A Real structure, compatible with the complex structure on X, is an anti-holomorphic Real structure on X. A Real structure compatible with ι on a holomorphic vector bundle $E \to X$ is an anti-holomorphic ι-Real structure on E.*

The goals of this article are twofold.

1. Describe explicitly and geometrically the pointed set $\mathrm{Aff}(X) = H^1(X, \mathrm{A}(1))$ of isomorphism classes of holomorphic affine line bundles on X by endowing it with a natural topology using techniques from gauge theory.

2. Generalise the theory of Real structures to affine line bundles.

To achieve the first goal, we used a construction of $\mathrm{Pic}(X)$ based on methods from gauge theory. For a given smooth line bundle L, we identify the set of holomorphic structures on L with the space of integrable semi-connections factorised by a natural action of the complex gauge group of L. We first prove that the fiber over $[L]$ of the natural map $\mathrm{Aff}(X) \to \mathrm{Pic}(X)$ can be identified with the cohomology $H^1(X, L)$ of the sheaf of sections of a representative L. This is proved by first showing that there is a bijective correspondence between $\mathrm{Aff}(X)$ and the set of isomorphism classes of vector bundle epimorphisms $u : E \to X \times \mathbb{C}$, where E is a rank 2 holomorphic vector bundle. We then regard E as extensions of $X \times \mathbb{C}$ by $L = \ker(u)$ to show the above property. One obtains a naive description

$$\mathrm{Aff}(X) = \coprod_{[\mathcal{L}] \in \mathrm{Pic}(X)} H^1(X, \mathcal{L}).$$

There are some problems with this formula. Certainly, for $L_1 \cong L_2$, we know that $H^1(X, L_1) \cong H^1(X, L_2)$ are isomorphic, but there is no canonical choice of an isomorphism.
This problem can be solved using the Poincaré line bundle (also called the universal line bundle) on $\mathrm{Pic}(X) \times X$. This Poincaré line bundle gives us a distinguished choice of a line bundle for each isomorphism class $l \in \mathrm{Pic}(X)$. For every base-point $x_0 \in X$, it is possible to define a holomorphic line bundle $\mathfrak{L}_{x_0} \to \mathrm{Pic}(X) \times X$, called the Poincaré line bundle, on X normalised at x_0. It is characterised by the following two properties:

1. $\mathfrak{L}_{x_0}\restriction_{\{l\} \times X}$ is a holomorphic line bundle belonging to the isomorphism class l for every $l \in \mathrm{Pic}(X)$.

2. $\mathfrak{L}_{x_0}\restriction_{\mathrm{Pic}(X) \times \{x_0\}}$ is trivial.

The pair $(\mathfrak{L}_{x_0}, \mathrm{Pic}(X))$ fulfils the following unique property:

Theorem 1.3. *For any complex manifold Y and holomorphic line bundle $\mathfrak{M} \to Y \times X$ such that $\mathfrak{M}\restriction_{Y \times \{x_0\}}$ is trivial, there exists a unique holomorphic map $\varphi : Y \to \mathrm{Pic}(X)$ such that $\mathfrak{M} \cong (\varphi \times \mathrm{Id})^* \mathfrak{L}_{x_0}$.*

In other words $\mathfrak{L}_{x_0}$ classifies holomorphic families of line bundles on X normalised at x_0. A Poincaré line bundle gives us a choice of a line bundle for each isomorphism class, and the we get

$$\mathrm{Aff}(X) = \coprod_{l \in \mathrm{Pic}(X)} H^1(X, \mathfrak{L}_{x_0}\lceil_{\{l\}\times X}).$$

To prove that there is natural topology on the right hand side, we use again gauge theory, more precisely, a coupled gauge theoretical problem of classifying pairs

$$\{(\delta, \alpha) : \alpha \in A^{0,1}(L), \delta \in \mathfrak{U}^{int}(L), \delta(\alpha) = 0\}.$$

Our disjoint union can then be identified with a quotient of this set of pairs. Several interesting questions naturally develop and will be addressed in a future article.

Q.1: Does $\mathrm{Aff}(X)$ have a natural structure of a complex space? This seems difficult because the dimension of $H^1(X, \mathfrak{L}_{x_0}\lceil_{\{l\}\times X})$ will in general vary as $l \in \mathrm{Pic}^c(X)$ varies.

Q.2: Is it possible to generalise the Poincaré line bundle? In other words, can we define a universal affine line bundle on $\mathrm{Aff}(X) \times X$ satisfying a similar universal property?

Concerning the second goal, we introduce a natural concept of a Real affine line bundle and analogously to the linear case we obtain:

Proposition 1.4. *Let $(A, \tilde{\iota})$ is a Real affine line bundle over the Real manifold X. Then the fixed point locus $A^{\tilde{\iota}}$ is a real affine bundle line on X^{ι}.*

Some interesting problems arise that will be addressed in a future article.

Q.1: Classify the Real affine line bundles over X in relevant cases. For example, the Real *linear* line bundles over Klein surfaces and Real tori are classified in [16]. It seems that so far no results in the affine case are available, in fact, even the concept does not seem to exist in the literature.

Q.2: An interesting family of minimal surfaces of class VII has been introduced and classified by Enoki in [7]. They are constructed as compactifications of affine line bundles over an elliptic curve C. Suppose that (C, ι) is Real elliptic curve and suppose that we have a ι-Real affine line bundle on C, do we get an associated Real structure on the corresponding Enoki surface? Conversely, can we obtain every Real structure on an Enoki surface this way?

2 Bundles

2.1 Classification of fibre bundles on a fixed base

Let X be a topological space, and G a topological group, we will denote by $G_{\mathcal{C}}$ the sheaf associated with the presheaf of continuous G-valued functions defined on open sets of X. Let F be a topological space (called the standard fiber) endowed with an effective continuous action $\alpha : G \times X \to X$ of G. We refer to [11] for the following classification theorem:

Theorem 2.1. *The isomorphism classes of topological fibre bundles X with structure group G, standard fiber F and action α are in a natural one-one correspondence with the elements of the cohomology set $H^1(X, G_{\mathcal{C}})$. The trivial bundle $X \times F$ corresponds to the distinguished element $1 \in H^1(X, G_{\mathcal{C}})$.*

Suppose now that X and F are differentiable (complex) manifolds, G is a (complex) Lie group,on and α is a smooth (respectively holomorphic) action. We define $G_{\mathcal{C}^\infty}$ (respectively G_{hol}) to be the sheaf associated with the presheaf of smooth (respectively holomorphic) G-valued functions on open sets of X. With these conventions one has the following versions of Theorem 2.1:

Theorem 2.2. *The isomorphism classes of differentiable (respectively holomorphic) fibre bundles X with structure group G and standard fiber F are in a natural one-one correspondence with the elements of the cohomology set $H^1(X, G_{\mathcal{C}^\infty})$ (respectively $H^1(X, G_{hol})$). The trivial differentiable (respectively holomorphic) bundle $X \times F$ corresponds to the distinguished element $1 \in H^1(X, G_{\mathcal{C}})$.*

In the statements, for a sheaf of groups $\mathcal{F}$ on X we used the notation $H^1(X, \mathcal{F})$ for the first Cěch cohomology set with values in $\mathcal{F}$. Note that in when $\mathcal{F}$ is a sheaf of abelian groups, this cohomology set (which in general is just a set with distinguished element) becomes an abelian group. Moreover, when X is a (paracompact) manifold, this group coincides with the cohomology group provided by the standard sheaf cohomology theory.

We will be mostly interested in the case when X is a Riemann surface or a compact complex surface, and G is either the multiplicative group $\mathrm{GL}(1) = \mathbb{C}^*$, the complex affine group $\mathrm{A}(1)$ of affine automorphisms of $\mathbb{C}$, or the complex projective linear group $\mathrm{PGL}(2)$ of automorphisms of the complex projective line.

2.2 Line bundles

Let X be compact complex manifold. Taking $F = \mathbb{C}$ and $G = \mathrm{GL}(1) = \mathbb{C}^*$ in Theorem 2.1 we obtain that the isomorphism classes of holomorphic line bundles on X correspond bijectively to $H^1(X, \mathcal{O}_X^*)$. Endowing the set of isomorphism classes of holomorphic line bundles on X with the multiplication structure defined by the tensor product of line bundles, one obtains a group called the *Picard group* of X, and denoted $\mathrm{Pic}(X)$. One can then prove that the bijection $H^1(X, \mathcal{O}_X^*) \to \mathrm{Pic}(X)$ given by Theorem 2.1 is a group isomorphism.

For a connected, compact, complex manifold we have the exact exponential sheaf exact sequence, which reads

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1.$$

From this we get the following cohomology long exact sequence:

$$0 \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \mathbb{Z})$$

$$\longrightarrow H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \to \dots$$

The map $H^0(X, \mathcal{O}_X) \cong \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \cong H^0(X, \mathcal{O}_X^*)$ is surjective, since X is compact, therefore we obtain the exact sequence

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*)$$

$$\longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \to \dots$$

The morphism $H^1(X, \mathcal{O}_X) \to H^2(X, \mathbb{Z})$ sends (the isomorphism class of) a line bundle $\mathcal{L}$ to its Chern class $c_1(\mathcal{L})$. The subgroup

$$\mathrm{NS}(X) := \ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X))$$

of $H^2(X, \mathbb{Z})$ is called the *Neron-Severi group* of X. The kernel $\ker(\mathrm{Pic}(X) \to \mathrm{NS}(X))$ is usually denoted by $\mathrm{Pic}^0(X)$. From the above we get the short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

2.2.1 $\mathrm{Pic}^0(C)$ of a compact, connected, complex surface

As we said above, for a compact, connected, complex manifold,

$$0 \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*)$$

is an exact sequence.
We will show that the image of the first morphism $H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X)$ is closed. We won't indicate the switches between Cěch and deRahm cohomology, it should be clear from the context.
Consider the sequence of morphisms

$$H^1(X,\mathbb{Z}) \longrightarrow H^1(X,i\mathbb{R}) \longrightarrow H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O}_X)$$

induced by the inclusions

$$\mathbb{Z} \xrightarrow{2\pi i} i\mathbb{R} \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_X.$$

By the universal coefficient theorem, $2\pi i H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,i\mathbb{R})$. So what is left to show is the following:

Theorem 2.3. *The morphism $H^1(X,i\mathbb{R}) \to H^1(X,\mathcal{O}_X)$ is injective.*

This result generalises the well known case when X is Kähler. The subtlety of the proof is that even though one does *not* have the direct sum partition $H^1(X,\mathbb{C}) = H^{1,0}(X,\mathbb{C}) \oplus H^{0,1}(X,\mathbb{C})$ in two equal-dimensional subspaces in the general case, it is still possible to give almost the same proof based on the factorisation $A^1(X) = A^{1,0}(X) \oplus A^{0,1}(X)$.
The given proof follows [16], the result and a proof is also available in [9].

Proof. By Dolbeault's theorem we have $H^1(X,\mathcal{O}_X) \cong H^{0,1}(X,\mathbb{C})$. Therefore the morphism has the form

$$H^1(X,i\mathbb{R}) \longrightarrow H^1(X,\mathbb{C}) \longrightarrow H^{0,1}(X,\mathbb{C})$$

$$[\alpha] \longmapsto [\alpha] \longmapsto [\alpha^{0,1}].$$

Let $[\alpha] \in H^1(X,i\mathbb{R})$ and let $\alpha = -\alpha^{1,0} + \alpha^{0,1}$ be a representative of its image, where $\overline{\alpha^{1,0}} = \alpha^{0,1}$. Suppose that $\alpha^{0,1}$ is $\bar\partial$-exact, i.e. $\bar\partial u = \alpha^{0,1}$ for some $u \in A^0(X,\mathbb{C})$. It follows that $\partial\bar u = \alpha^{1,0}$. Since α is closed, we get

$$0 = d\alpha = (\partial+\bar\partial)(-\partial\bar u+\bar\partial u) = -\bar\partial\partial\bar u+\partial\bar\partial u = -\bar\partial\partial\bar u-\bar\partial\partial u = -\bar\partial\partial(2\,\mathrm{Re}(u)),$$

where we used that $\partial\bar\partial = -\bar\partial\partial$ and $\mathrm{Re}(u) = \frac{1}{2}(u+\bar u)$. Therefore $\mathrm{Re}(u)$ is constant by the maximum principle, so replacing u by $u - \mathrm{Re}(u)$, we can suppose that the real part of u is 0, hence $(\partial+\bar\partial)u = \alpha$, since $\bar u = -u$. Hence $[\alpha] = 0$ in $H^1(X,i\mathbb{R})$ and proof is finished. □

For a large class of compact manifolds, this map is more than just injective.

Corollary 2.4. *If X is also Kähler, $H^1(X, i\mathbb{R}) \to H^1(X, \mathcal{O}_X)$ is an isomorphism.*

Proof. If X is Kähler, $H^1(X,\mathbb{C}) \cong H^{1,0}(X,\mathbb{C}) \oplus H^{0,1}(X,\mathbb{C})$ and $H^{1,0}(X,\mathbb{C}) \cong \mathbb{C}^{b_1(X)/2} \cong H^{0,1}(X,\mathbb{C})$. The result readily follows. □

From theorem 2.3 and the discussion before it, we obtain:

Corollary 2.5. *1. The image of $2\pi i H^1(X,\mathbb{Z})$ in $H^1(X,\mathcal{O}_X)$ is a closed subgroup and the quotient $\mathrm{Pic}^0(X) = H^1(X,\mathcal{O}_X)/2\pi i H^1(X,\mathbb{Z})$ is naturally an abelian complex Lie group.*

2. If X is a Kähler surface, $\mathrm{Pic}^0(X)$ is a complex torus of dimension $b_1(X)/2$.

3. If X is Riemann surface, $\mathrm{Pic}^0(X)$ is a complex torus of dimension g, the genus of X.

For a compact Kähler manifold $2\pi i H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,\mathcal{O}_X)$ and a Kähler surface has an even first Betti number. The case of a Riemann surface is clear, for a classical approach to $\mathrm{Pic}^0(X)$ of a Riemann surface X, have a look at Forster's great book on Riemann surfaces [8]. In the case of a surface of class VII, we have $\mathrm{Pic}^0(X) \cong \mathbb{C}^*$ so in general the quotient will not be compact.

2.3 Affine line bundles

In this chapter we will look at the case where G is the affine group $\mathrm{A}(1)$ which acts on $\mathbb{C}$ by affine transformations. A fiber bundle over X with structure group $\mathrm{A}(1)$ and fiber $\mathbb{C}$ is called an affine line bundle. Again, by theorem 2.1, the isomorphism classes of holomorphic affine line bundles are in a one-one correspondence to the pointed set $H^1(X, \mathrm{A}(1)_{hol})$. For the rest of this paper we will denote this set by $\mathrm{Aff}(X) = H^1(X, \mathrm{A}(1)_{hol})$.
Explicitly, the affine group consists of the maps $\mathbb{C} \to \mathbb{C}$, $z \mapsto az + b$ for $a \in \mathbb{C}^*, b \in \mathbb{C}$. It can be regarded as the subgroup of $\mathrm{GL}(2)$ consisting of the matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, with $a \in \mathbb{C}^*, b \in \mathbb{C}$. It fits into an exact sequence of groups

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{A}(1) \longrightarrow \mathbb{C}^* \longrightarrow 1,$$

which gives an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathrm{A}(1)_{hol} \longrightarrow \mathcal{O}_X^* \longrightarrow 1.$$

Passing to the long exact sequence we obtain

$$\ldots \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Aff}(X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathcal{O}_X).$$

For every compact, connected, complex manifold X, the sequence takes the form

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Aff}(X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow 0,$$

since $H^0(X, \mathrm{A}(1)_{hol}) \to H^0(X, \mathcal{O}_X^*)$ and $H^1(X, \mathrm{A}(1)_{hol} \to) \to H^1(X, \mathcal{O}_X^*)$ are surjective.

The morphism $\mathrm{Aff}(X) \to H^1(X, \mathcal{O}_X^*)$ sends the isomorphism class of an affine line bundle to the isomorphism class of its linear part. This is generally defined as follows:

Definition 2.6. *For an affine line bundle $\mathcal{A} \to X$ we can define it's linearisation $\mathcal{A}_{lin}$. Suppose $\mathcal{A}$ is defined by the co-cycles $\{g_{ij} : U_i \cap U_j \to \mathrm{A}(1)\}$. Define the* linearisation *$\mathcal{A}_{lin}$ of $\mathcal{A}$ as the line bundle defined by the co-cycles $\{h_{ij} : U_i \cap U_j \to \mathrm{A}(1) \to \mathbb{C}^*\}$.*

Using principal bundles, we can define the linearisation as follows: Let $P \to X$ be a principal $\mathrm{Aut}_{aff}(\mathbb{C})$-bundle, where $\mathrm{Aut}_{aff}(\mathbb{C})$ denotes the affine automorphisms of $\mathbb{C}$, and $\rho : \mathrm{A}(1) \to \mathrm{Aut}_{aff}(\mathbb{C})$ a holomorphic left action on $\mathbb{C}$ such that $\mathcal{A} = P \times_\rho F$. Then $\mathcal{A}_{lin} = P \times_{\tilde{\rho}} F$, where $\tilde{\rho}$ is defined by the following diagram

$$\begin{tikzcd} A(1) \arrow[r, "\rho"] \arrow[dr, "\tilde{\rho}"'] & \mathrm{Aut}_{aff}(\mathbb{C}) \arrow[d] \\ & \mathrm{Aut}_{lin}(\mathbb{C}). \end{tikzcd}$$

2.4 The geometry of Aff(X)

In this structure we want to take a closer look at the isomorphism class of holomorphic affine line bundles Aff(X). So far we only know that it is a pointed set, where the trivial affine line bundle corresponds to the distinguished element, and that it fits into the exact sequence above. Our first theorem will be the following:

Theorem 2.7. *Let X be a compact, connected, complex manifold. Then*

$$\mathrm{Aff}(X) = \coprod_{[\mathcal{L}]\in\mathrm{Pic}(X)} H^1(X,\mathcal{L}).$$

The technique will be consist of passing to vector bundles of rank 2, where a lot of theory is available. The intermediary result is given by:

Theorem 2.8. *Let X be a compact, connected, complex manifold. There is bijective correspondence between the set* $\mathrm{Aff}(X)$ *and the set of epimorphisms of holomorphic vector bundles $E \to X \times \mathbb{C}$ where E is a rank 2 holomorphic vector bundles.*

Recall that, the group A(1) can be regarded as the subgroup of GL(2) consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \text{ with } a \in \mathbb{C}^*, b \in \mathbb{C}.$$

This injection gives us a map of pointed sets $\mathrm{Aff}(X) \xrightarrow{v} H^1(X, \mathrm{GL}(2)_{hol})$.

Proof part 1: Constructing an epimorphism from an affine bundle.
Let E be a holomorphic rank 2 vector bundle representing $v([\mathcal{A}])$. We will construct an epimorphism onto the trivial bundle $X \times \mathbb{C}$. Let $\{(U_i, \psi_i)\}$ be local trivialisations for E and denote the transition functions by $g_{ij} : U_{ij} \to \mathrm{GL}(2)$. Since the transition functions of E are defined via the transition functions of $\mathcal{A}$, they are of the form $g_{ij}(x) = \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix}$. Therefore the following diagram commutes, where $\psi_{ij} = \psi_i^{-1} \circ \psi_j$.

$$\begin{tikzcd}
U_{ij} \times \mathbb{C}^2 \arrow[r, "\psi_{ij}"] \arrow[d, "\mathrm{Id}\times\mathrm{pr}_2"] & U_{ij} \times \mathbb{C}^2 \arrow[d, "\mathrm{Id}\times\mathrm{pr}_2"] \\
U_{ij} \times \mathbb{C} \arrow[r, "\mathrm{Id}\times\mathrm{Id}"] & U_{ij} \times \mathbb{C}
\end{tikzcd}$$

Let us calculate this in more details:
By definition, we have $\psi_{ij}(x, y) = (x, g_{ij}(x)y)$. Writing $y = (y_1, y_2)$, we get

$$g_{ij}(x)y = \begin{pmatrix} a_{ij}(x) & b_{ij}(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{ij}(x)y_1 + b_{ij}(x)y_2 \\ y_2 \end{pmatrix}$$

This shows that the above diagram commutes and patching together the local functions we obtain an epimorphism $E \xrightarrow{p} X \times \mathbb{C}$. From our explicit calculation, by looking at the preimage of $X \times \{0\}$ and $X \times \{1\}$, we see furthermore:

1. $\ker(p) \cong \mathcal{A}_{lin}$
2. $p^{-1}(X \times \{1\}) \cong \mathcal{A}$
3. Isomorphic affine bundles induce isomorphic maps.

Proof part 2: From an epimorphism to an affine bundle.
Suppose $E \xrightarrow{p} X \times \mathbb{C}$ is an epimorphism onto the trivial bundle. From what we found in the first part of the proof, $p^{-1}(X \times \{1\})$ should be an affine line bundle. Let $\{(U_i, \psi_i)\}$ be local trivialisations for E with transition functions g_{ij}. By definition, p is locally of the form $\mathrm{Id} \times q_i : U_i \times \mathbb{C}^2 \to U_i \times \mathbb{C}$, where q_i is linear map $\mathbb{C}^2 \to \mathbb{C}$. From drawing the same diagram as above, we get that

$$q_i(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) = q_i(\begin{pmatrix} a_{ij}(x) & b_{ij}(x) \\ c_{ij}(x) & d_{ij}(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}),$$

on U_{ij}. As a linear map, q_i is of the form $q_i((y_1, y_2)) = \alpha y_1 + \beta y_2$. Setting $y_1 = 0$, then $y_2 = 0$, comparing coefficients and using the fact that p is an epimorphism shows that there are two possible cases:

1. $c_{ij} = 0$, $d_{ij} = 1$ and q_i is equal to the second projection.
2. $a_{ij} = 1$, $b_{ij} = 0$ and q_i is equal to the first projection.

The same arguments as in part 1 show that $\mathcal{A} = p^{-1}(X \times \{1\})$ defines an affine line bundle, that $\ker(p)$ is the linearisation of $\mathcal{A}$ and that isomorphic epimorphisms define isomorphic bundles. Clearly the two above cases define the same affine line bundles. Just exchange the co-cycle $\left(\begin{smallmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{smallmatrix}\right)$ with the co-cycle $\left(\begin{smallmatrix} 1 & 0 \\ b_{ij} & a_{ij} \end{smallmatrix}\right)$ to switch between second and first projection. The isomorphism between the two epimorphisms is locally defined by $U_i \times \mathbb{C}^2 \to U_i \times \mathbb{C}^2, (x, (y_1, y_2)) \mapsto (x, (y_2, y_1))$, so it suffices to consider the first case which gives us the bijective correspondence. □

Recall the linearisation morphism $\mathrm{Aff}(X) \to H^1(X, \mathcal{O}_X^*)$ from the previous section. We want to know the fiber over a given isomorphism class of line bundles $[\mathcal{L}]$. By the previous theorem 2.8 an affine bundles over $\mathcal{L}$ corresponds, up to isomorphism, to an epimorphism $E \to X \times \mathbb{C}$ with kernel isomorphic to $\mathcal{L}$. So we need to classify these epimorphisms. Instead of looking at vector bundles, we will look at the corresponding sheaves. We denote by $\mathcal{E}$ the sheaf corresponding to a vector bundle E, $X \times \mathbb{C}$ corresponds of course to $\mathcal{O}_X$, and with some abuse of notation, the sheaf corresponding to $\mathcal{L}$ will be denoted by the same symbol. Then an affine line bundle over $\mathcal{L}$ is identified with an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

So given a line bundle $\mathcal{L}$, we need to classify the extensions of $\mathcal{O}_X$ with $\mathcal{L}$. The classes of such extensions are classified by $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{L})$. Each one of those extensions $\mathcal{E}$ will be locally free of rank 2 since $\mathcal{O}_X$ and $\mathcal{L}$ are locally free of rank 1, so defines a vector bundle of rank 2. We cite the following proposition, see [10] for a proof.

Proposition 2.9. *For any sheaf of modules $\mathcal{F}$ and $i \geq 0$, there is a canonical isomorphism of $\mathcal{O}_X(X)$-modules $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong H^i(X, \mathcal{F})$, natural in $\mathcal{F}$.*

In our case, the result follows from the exact sequence obtained from the spectral sequence:

$$0 \longrightarrow H^1(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{F}) \longrightarrow H^0(\mathcal{E}xt^1(\mathcal{O}_X, \mathcal{F})).$$

We assume that $\mathcal{F}$ is locally free so $\mathcal{E}xt^1(\mathcal{O}_X, \mathcal{F})$ vanishes and $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}$, which gives the result. So we obtain that the set of isomorphism classes of holomorphic affine line bundles with linearisation isomorphic to $\mathcal{L}$ is described by $H^1(X, \mathcal{L})$. This proves theorem 2.7, i.e. we obtain

$$\mathrm{Aff}(X) = \coprod_{[\mathcal{L}] \in \mathrm{Pic}(X)} H^1(X, \mathcal{L}),$$

which we can rewrite as

$$\mathrm{Aff}(X) = \coprod_{c \in \mathrm{NS}(X)} \coprod_{[\mathcal{L}] \in \mathrm{Pic}^c(X)} H^1(X, \mathcal{L}).$$

One not very nice thing about this expression, is that we have to make a choice of representative for each isomorphism class $[\mathcal{L}]$ in $\mathrm{Pic}(X)$. So far we have no such canonical choice, and even though for any two $L_1, L_2 \in [\mathcal{L}]$, we have $H^1(X, L_1) \cong H^1(X, L_2)$, there is no canonical isomorphism $L_1 \to L_2$ inducing the isomorphism between the cohomology groups. There is a solution for this problem that we will study in the next chapter.

2.5 A study of $\mathrm{Pic}^c(X)$

Our goal is to give this set additional structure, so we need to study $\mathrm{NS}(X)$, $\mathrm{Pic}^c(X)$ and $H^1(X, \mathcal{L})$. In the case of a compact Riemann surface or surface of class VII, the Neron-Severi group is isomorphic to $\mathbb{Z}$, as we already saw above. Next up, $\mathrm{Pic}^c(X)$ for $c \in \mathrm{NS}(X)$. First we take a look at *smooth* complex line bundles $\mathcal{L}$ with first Chern class $c_1(\mathcal{L}) = c$. Let ξ_X (resp. ξ_X^*) be the sheaf associated to the presheaf of smooth functions with values in $\mathbb{C}$ (resp. $\mathbb{C}^*$) defined on open subsets of X. The part of the long exact sequence corresponding to the $\mathcal{C}^\infty$-exponential sequence,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \xi_X \xrightarrow{\exp} \xi_X^* \longrightarrow 0,$$

that interests us is given by

$$H^1(X, \xi_X) \longrightarrow H^1(X, \xi_X^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \xi_X).$$

Now, ξ_X is a fine sheaf, so all cohomology groups $H^i(X, \xi_X)$ vanish for $i > 0$. Therefore for each $c \in \mathrm{NS}(X)$, there is only one line bundle $\mathcal{L}$, up to diffeomorphism, with $c_1(\mathcal{L}) = c$. So instead of classifying all holomorphic line bundles with Chern class c, we classify the possible holomorphic structures for a given complex smooth line bundle $\mathcal{L}$.

2.5.1 Semi-connections

In this section we will study generalisations of the Dolbeault operator $\bar{\partial} : A^{0,0}(X)^{\oplus n} \to A^{0,1}(X)^{\oplus n}$. As it turns out, instead of trying to find all holomorphic structures on a smooth, complex vector bundle E, we can look for a certain class of operators $A^0(E) \to A^{0,1}(E)$.

Let E be a holomorphic vector bundle of rank n over X. Then, via trivialisations, we define the Dolbeault operator for E, $\bar{\partial}_E : A^0(E) \to A^{0,1}(E)$ satisfying the Leibnitz rule

$$\bar{\partial}_E(f\omega) = \bar{\partial}(f) \otimes \omega + f\bar{\partial}_E(\omega), \forall f \in A^{0,0}(X), \forall \omega \in A^0(E).$$

This extends naturally to an operator $\bar{\partial}_E : A^{p,q}(E) \to A^{p,q+1}(E)$, satisfying $\bar{\partial}_E \circ \bar{\partial}_E = 0$, by setting

$$\bar{\partial}_E(\sigma \otimes \omega) = \bar{\partial}(\sigma) \otimes \omega + (-1)^{p+q}\sigma \wedge \bar{\partial}_E(\omega), \forall \sigma \in A^{p,q}(X), \forall \omega \in A^0(E).$$

We generalise this type of operator in a natural way, by what some people call "the (french) trick of turning a theorem into a definition".

Definition 2.10. *Let E be a smooth, complex vector bundle of rank n over a complex manifold X. A* semi-connection*, or* $(0,1)$-connection *on E is a $\mathbb{C}$-linear operator $\delta : A^0(E) \to A^{0,1}(E)$, satisfying*

$$\delta(f\omega) = \bar{\partial} f \otimes \omega + f\delta\omega, \forall f \in A^0(X, \mathbb{C}), \forall \omega \in A^0(E).$$

Any semi-connection extends to an operator $A^{0,q} \to A^{0,q+1}$, satisfying

$$\delta(\alpha \otimes \omega) = \bar{\partial}\alpha \otimes \omega + (-1)^q \alpha \wedge \delta\omega.$$

If $\delta^2 : A^0 \to A^{0,2} = 0$, the semi-connection is called integrable*. The space of semi-connections is denoted by $\mathfrak{U}(E)$ and the space of integrable semi-connections by $\mathfrak{U}^{int}(E)$.*

Suppose δ_1 and δ_2 are two semi-connections, $f \in A^0(X,\mathbb{C})$ and $\omega \in A^{0,1}(E)$. Then $(\delta_1 - \delta_2)(f\omega) = f(\delta_1 - \delta_2)(\omega)$, so $\delta_1 - \delta_2$ is $A^0(X,\mathbb{C})$ linear, therefore defines an element in $A^{0,1}(\mathrm{End}(E))$. This shows that $\mathfrak{U}(E)$ is an affine space over the vector space $A^{0,1}(\mathrm{End}(E))$. The group $\mathcal{G}_E^{\mathbb{C}} = \mathcal{A}ut(E) = \Gamma(X, \mathrm{GL}(E))$ of complex linear automorphism of E, also called the *complex gauge group* of E, acts on $\mathfrak{U}(E)$ from the left by

$$f \cdot \delta = f \circ \delta \circ f^{-1}.$$

Observing that a semi-connection δ on E induces a semi-connection on $\mathrm{End}(E)$, which we denote by the same symbol, by setting

$$\delta(\alpha)(\omega) = \delta(\alpha(\omega)) - \alpha(\delta(\omega)),$$

enables us to write the action as

$$f \cdot \delta = \delta - (\delta f) f^{-1}.$$

Similarly because the a semi-connection can be extended, we associate to a semi-connection δ an element $\mathfrak{F}_\delta \in A^{0,2}(\mathrm{End}(E))$, corresponding to $\delta \circ \delta$. Integrability is then expressed as $\mathfrak{F}_\delta = 0$.
The result is then the following:

Theorem 2.11. *Let E be a smooth, complex vector bundle of rank n over a complex manifold X. Then there is a bijection between the set of isomorphism classes of holomorphic structure and the set $\mathfrak{U}^{int}(E)/\mathcal{G}_E^{\mathbb{C}}$.*

$$\frac{\{\textit{holomorphic structures on } E\}}{\sim \textit{isomorphism}} \longleftrightarrow \frac{\mathfrak{U}^{int}(E)}{\mathcal{G}_E^{\mathbb{C}}}$$

The integrable semi-connection corresponding to a holomorphic structure on E is given by the Dolbeault operator determined by the holomorphic structure. It is not too hard to see that two isomorphic structures define conjugated semi-connections. The idea of the other direction is first define the holomorphic sections as solutions to $\delta\omega = 0$. Then one shows that this sheaf is a locally free sheaf of rank n over the sheaf $\mathcal{O}_X$. Compared to the relatively easy first direction, this one is harder and essentially utilises a version of the Newlander-Nirenberg theorem. For a proof, utilising slightly different definitions, see [13] or [12].

2.5.2 The Poincaré line bundle

Let us look at how this result helps in our original question. As we discussed in the beginning of this section $\mathrm{Pic}^c(X)$ is the set of holomorphic structures for a smooth line bundle $\mathcal{L}$ with first Chern number $c_1(\mathcal{L}) = c$. So we obtain $\mathrm{Pic}^c(X) = \mathfrak{U}^{int}(\mathcal{L})/\mathcal{G}_{\mathcal{L}}^{\mathbb{C}}$, where $\mathcal{G}_{\mathcal{L}}^{\mathbb{C}} = \mathcal{C}^\infty(X, \mathbb{C}^*)$. If X is a Riemann surface, of course every semi-connection is trivially integrable. The question of additional structure on $\mathrm{Pic}^c(X)$ therefore reduces to studying the space $\mathfrak{U}^{int}(\mathcal{L})/\mathcal{G}_{\mathcal{L}}^{\mathbb{C}}$. Let us recall that $\mathfrak{U}(\mathcal{L})$ is an affine space, in the case of a line bundle over $A^{0,1}(\mathrm{End}(\mathcal{L})) = A^{0,1}(\mathrm{End}(\mathcal{L})) = A^{0,1}(X)$. The space of integrable semi-connections on a line bundle, if non-empty, i.e. $c_1(\mathcal{L}) \in \mathrm{NS}(X)$, is an affine space over the space of closed $(0,1)$-forms $Z^{0,1}(X)$.

Using the space of semi-connections, we will obtain a distinguished choice of a representative for each isomorphism class of holomorphic line bundles on X. Combined, these distinguished representatives give us a universal line bundle on the product $\mathrm{Pic}(X) \times X$. For a given smooth complex line bundle $\mathcal{L}$, this will be achieved by a bundle

$$\begin{array}{c} \mathfrak{L}_{x_0} \\ \downarrow{\scriptstyle p} \\ \mathrm{Pic}^c(X) \times X, \end{array}$$

depending on one, freely choosable point $x_0 \in X$, satisfying the following two properties:

1. For each $l \in \mathrm{Pic}^c(X)$, the restriction $\mathfrak{L}_{x_0}\restriction_{\{l\}\times X}$ is a holomorphic vector bundle belonging to the isomorphism class l.

2. $\mathfrak{L}_{x_0}\restriction_{\mathrm{Pic}^c(X)\times\{x_0\}}$ is trivial, more precisely there is an obvious isomorphism $\mathfrak{L}_{x_0}\restriction_{\mathrm{Pic}^c(X)\times\{x_0\}} = \mathrm{Pic}^c(X) \times \mathcal{L}_{x_0}$.

This is often called the *Poincaré line bundle*, or *universal bundle* of $\mathcal{L}$, but beware that some texts reserve this term for the case where $c = 0$. Now we will show how this bundle looks like if we forget the holomorphic structure for a moment. Fix a point $x_0 \in X$ and denote by $ev_{x_0} : \mathcal{G}_{\mathcal{L}}^{\mathbb{C}} \to \mathbb{C}^*$ the evaluation map $f \mapsto f(x_0)$. The kernel of this map is denoted by $\mathcal{G}_{x_0}^{\mathbb{C}}$ and is isomorphic to $\mathcal{G}_{\mathcal{L}}^{\mathbb{C}}/\mathbb{C}^*$. The action of $\mathcal{G}_{\mathcal{L}}^{\mathbb{C}}$ on $\mathfrak{U}(\mathcal{L})$ induces a free action of $\mathcal{G}_{x_0}^{\mathbb{C}}$ on $\mathfrak{U}(\mathcal{L})$. To see this, recall that the action of $\mathcal{G}_{\mathcal{L}}^{\mathbb{C}}$ on $\mathfrak{U}$ is given by

$$f \cdot \delta = \delta - (\delta f) f^{-1}.$$

For a line bundle, this becomes

$$f \cdot \delta = \delta - (\bar{\partial} f) f^{-1},$$

so if f stabilises δ, $\bar{\partial} f = 0$ and therefore, since X is assumed compact, f is equal to a constant $\alpha \in \mathbb{C}^*$ (in other words, f is an automorphism that is

given by multiplication by a constant in each fiber). The stabiliser of each point is therefore equal to $\mathbb{C}^*$. On also sees that this free action leaves the subspace $\mathfrak{U}^{int}(\mathcal{L})$ invariant. Next, we define an action of $\mathcal{G}^{\mathbb{C}}_{x_0}$ on $\mathfrak{U}^{int}(\mathcal{L}) \times \mathcal{L}$ induced by the action discussed above on $\mathfrak{U}(\mathcal{L})$ and the canonical action on $\mathcal{L}$. The quotient by this action is denoted by $\mathfrak{L}_{x_0}$. The canonical projection to $\mathfrak{U}^{int}(\mathcal{L}) \times X$ gives us a line bundle fulfilling the properties described above. Of course, just like this, so far there is no holomorphic structure on this bundle and there is no good way to see that the restricted bundles $\mathfrak{L}_{x_0}\restriction_{\{l\}\times X}$ are holomorphic line bundles in the isomorphism class of l.
One way to prove it is to use infinite dimensional manifolds. We will outline the construction and give details later. Using the Sobolev completion with respect to the Sobolev norm L^2_k (for $k > n/2$) we obtain a Banach space $A^{0,1}(X)_k$, and the structure of a Banach manifold on the corresponding completion of the affine space $\mathfrak{U}(\mathcal{L})$. The tangent space at each point of this manifold is given by $A^{0,1}(X)_k$. The integrability condition defines a closed Banach submanifold $\mathfrak{U}^{int}(\mathcal{L})$ of $\mathfrak{U}(\mathcal{L})$. Similarly, we complete $\mathcal{G}^{\mathbb{C}}_{\mathcal{L}}$ with respect to the Sobolev norm L^2_{k+1} and we obtain a Banach Lie group $\mathcal{G}^{\mathbb{C}}_{\mathcal{L}_{k+1}}$, which acts smoothly on $\mathfrak{U}(\mathcal{L})$ as before.
The product $\mathfrak{U}^{int}(\mathcal{L})_k \times L$ becomes a holomorphic line bundle over $\mathfrak{U}^{int}(\mathcal{L})_k \times X$. From now on we will omit the Sobolev index to save notations, so we will right simply $A^{0,1}(X), \mathfrak{U}(\mathcal{L}), \mathfrak{U}^{int}(\mathcal{L})$ and $\mathcal{G}^{\mathbb{C}}_{\mathcal{L}}$ instead of $A^{0,1}(X)_k, \mathfrak{U}(\mathcal{L})_k, \mathfrak{U}^{int}(\mathcal{L})_k$ and $\mathcal{G}^{\mathbb{C}}_{\mathcal{L}_{k+1}}$ respectively.
We refer to Appendix A of [17] for the construction of the afore mentioned Sobolov completions.

Therefore we obtain a line bundle

$$\begin{array}{c} \mathfrak{U}^{int}(\mathcal{L}) \times \mathcal{L} \\ \downarrow \\ \mathfrak{U}^{int}(\mathcal{L}) \times X. \end{array}$$

Then using a semi-connection on the space of sections of the resulting bundle, we can prove the properties described before.

Let us return to our bundle $\mathfrak{K} = \mathfrak{U}^{int}(\mathcal{L}) \times \mathcal{L} \to \mathfrak{U}^{int}(\mathcal{L}) \times X$. We define a map $\Theta : A^0(\mathfrak{K}) \to A^{0,1}(\mathfrak{K})$ by

$$\Theta\restriction_{(m,x)} (\sigma)(\xi^{0,1}, v^{0,1}) = \frac{\partial}{\partial \delta_m}(\sigma\restriction_{(m,x)})(\xi^{0,1}) + \delta_m\restriction_{m,x} (\sigma)(v^{0,1}).$$

The two parts need to be explained in detail. For a fixed $x \in X$, a section σ can be regarded as a function $\mathfrak{U}^{int}(\mathcal{L}) \to L_x \cong \mathbb{C}$, therefore the partial derivative $\frac{\partial}{\partial m}$ is well defined. For a function with values in $\mathbb{C}$ we also have a good definition for holomorphy, which would be harder in case of maps between Banach spaces. In the other expression $\delta_m\restriction_{(m,x)} (\sigma_m)(v^{0,1})$, we denote by σ_m the section of $\mathcal{L}$ obtained by fixing m. The δ_m just emphasises that m is a semi-connection on $\mathcal{L}$. We could also write $m\restriction_{m,x} (\sigma)(v^{0,1})$.

Since the action from $\mathcal{G}^{\mathbb{C}}_{x_0}$ on $\mathfrak{U}^{int}(\mathcal{L})$ is defined by $f \cdot \delta = f \circ \delta \circ f^{-1}$,

$$f \circ \delta_m \circ f^{-1}(f \circ \sigma)(v^{0,1}) = f \circ \delta_m(\sigma\restriction_{m,x})(v^{0,1}).$$

Therefore the map Θ is well defined on the quotient. The following proposition holds.

Proposition 2.12. *Θ is an integrable semi-connection on the bundle*

$$\begin{array}{c} \mathfrak{L}_{x_0} = \mathfrak{U}^{int}(\mathcal{L}) \times_{\mathcal{G}^{\mathbb{C}}_{x_0}} \mathcal{L} \\ \downarrow \\ \frac{\mathfrak{U}^{int}(\mathcal{L})}{\mathcal{G}^{\mathbb{C}}_{x_0}} \times X, \end{array}$$

where $\mathfrak{U}^{int}(\mathcal{L}) \times_{\mathcal{G}^{\mathbb{C}}_{x_0}} \mathcal{L}$ denotes the quotient of $\mathfrak{U}^{int}(\mathcal{L}) \times \mathcal{L}$ by the action $(\delta, x) \mapsto (f \cdot \delta, f(x))$.

The result follows essentially from the integrability of the δ's and the Dolbeault operator on the base space, as well as the triviality condition.

Identifying $\mathfrak{U}^{int}(\mathcal{L})/\mathcal{G}^{\mathbb{C}}_{x_0}$ we obtain the following properties of $\mathfrak{L}_{x_0}$ from this proposition:

Corollary 2.13. *(i) $\mathfrak{L}_{x_0}$ is a holomorphic line bundle.*

(ii) $\mathfrak{L}_{x_0}\restriction_{\mathfrak{U}^{int}(\mathcal{L})/\mathcal{G}^{\mathbb{C}}_{x_0} \times \{x_0\}}$ is trivial.

(iii) $\mathfrak{L}_{x_0}\restriction_{\{l\}\times X} \to \{l\} \times X$ is a holomorphic line bundle with isomorphism class l.

Proof. (i) The holomorphic structure is defined by the semi-connection Θ.

(ii) This is clear.

(iii) First of all note that, in general, the restriction of a semi-connection δ to a complex submanifold $Z \subset Y$ of the base manifold of a differentiable bundle $M \to Y$ is well defined, and defines a semi-connection on the restricted bundle $M\restriction_Z$. This restriction is obtained by evaluating the forms $\delta(\alpha)$ on the tangent vectors of Z. This restricted semi-connection will be integrable if the initial semi-connection δ was integrable, and

will define precisely the natural holomorphic structure of $M\restriction_Z$. In other words, the map which associated to an integrable semi-connection δ on M the corresponding holomorphic line bundle $\mathcal{M}_\delta$ commutes with restrictions.
Let us return to our case and fix a representative δ_l of l. Taking into account the construction of the holomorphic structure on $\mathfrak{L}_{x_0}$ (as a quotient) it suffices to show that restricting the holomorphic structure we defined on $\mathfrak{U}^{int}(\mathcal{L}) \times \mathcal{L}$ to $\{\delta_l\} \times X$, one obtains precisely the holomorphic structure defined by δ_l. With this remark in mind, note that in our case, the restriction of Θ to $\{\delta\} \times X$ vanishes precisely on the sections of $\mathcal{L}$ which are holomorphic with respect to δ_l.

□

2.6 The natural topology of $\mathrm{Aff}(X)$

Fixing a $x_0 \in X$, we now have a well defined choice of a representative for each $[\mathcal{L}] \in \mathrm{Pic}(X)$, by using the Poincaré line bundle normalised at x_0.
This allows us to write

$$\mathrm{Aff}(X) = \coprod_{l \in \mathrm{Pic}(X)} H^1(X, \mathfrak{L}_{x_0}\restriction_{\{l\}\times X}).$$

We want to endow this set with a natural topology. One key point is the equality $H^1(X, \mathfrak{L}_{x_0}\restriction_{\{l\}\times X})) = H^{0,1}_{\delta_l}(X, \mathfrak{L}_{x_0}\restriction_{\{l\}\times X})$, between sheaf and Dolbeault cohomology for vector bundles. This leads us to consider the following expression

$$\begin{aligned} T &= \coprod_{\delta \in \mathfrak{U}^{int}(\mathcal{L})} Z^{0,1}_\delta(\mathcal{L})/B^{0,1}_\delta(\mathcal{L}). \\ &= \coprod_{\delta \in \mathfrak{U}^{int}(\mathcal{L})} H^{0,1}_\delta(X, \mathcal{L}) \\ &= \coprod_{\delta \in \mathfrak{U}^{int}(\mathcal{L})} H^1(X, \mathfrak{L}_{x_0}\restriction_{\{\delta\}\times X}) \end{aligned}$$

The gauge group $\mathcal{G}^{\mathbb{C}}_{x_0}$ acts freely on this set, and comparing with the definition of the Poincaré line bundle, we see that

$$\mathrm{Aff}(X) = T/\mathcal{G}^{\mathbb{C}}_{x_0}.$$

Define the set

$$\begin{aligned} P &= \{(\delta, \alpha) \in \mathfrak{U}(\mathcal{L}) \times A^{0,1}(\mathcal{L}) : \delta(\alpha) = 0, \mathfrak{F}_\delta = 0\} \\ &= \{(\delta, \alpha) \in \mathfrak{U}^{int}(\mathcal{L}) \times A^{0,1}(\mathcal{L}) : \delta(\alpha) = 0\}, \end{aligned}$$

which is clearly a topological space (we ignore any additional structure, since it is of no use for here). Next we define an equivalent relation $\sim_P$ on P by

$$\begin{aligned}(\delta_1, \alpha_1) \sim_P (\delta_2, \alpha_2) \Leftrightarrow \exists f \in \mathcal{G}_{x_0}^{\mathbb{C}} \text{ such that } & \\ f \cdot \delta_1 = \delta_2, & \\ \alpha_1 = f\alpha_2 + \delta_2(\lambda), \text{ for some } \lambda \in A^0(\mathcal{L}).&\end{aligned}$$

One checks the equivalence of $P/\sim_P$ and $T/\mathcal{G}_{x_0}^{\mathbb{C}}$. The quotient topology on $P/\sim_P$ then gives us a natural topology for the set $\mathrm{Aff}(X)$. Some remarks on why this topology is natural are in order.

Remark (i) The natural application $\mathrm{Aff}(X) \to \mathrm{Pic}(X)$ is continuous. It is induced by the projection $(\delta, \alpha) \to (\delta)$.

(ii) The restriction of the topology to each $H^1(X, \mathfrak{L}_{x_0}\restriction_l)$ is equivalent to the vector space topology.

(iii) Let Y is a complex manifold and $A \to Y \times X$ a holomorphic affine line bundle. Then the map $Y \to \mathrm{Aff}(X)$, $y \mapsto [A\restriction_{\{y\}\times X}]$ is continuous, where $[A\restriction_{\{y\}\times X}]$ denotes the isomorphism class of the holomorphic affine line bundle $A\restriction_{\{y\}\times X}$ over $\{y\} \times X \cong X$.

The Poincaré line bundles has the following universal property.

Theorem 2.14. *For every complex manifold T and holomorphic line bundle $\mathfrak{M} \to T \times X$ such that the restriction $\mathfrak{M}\restriction_{T\times\{x_0\}}$ is trivial, there exists a unique holomorphic map $f : T \to \mathrm{Pic}(X)$ such that $\mathfrak{M} \cong (f \times \mathrm{Id})^*(\mathfrak{L}_{x_0})$.*

The map f is defined by sending an element $t \in T$ to the isomorphism class of $L\restriction_{\{t\}\times X}$.

2.7 Affine line bundles over an elliptic curve

Let X be an elliptic curve, i.e. a Riemann surface of genus 1. We want to study $\mathrm{Aff}^n(X) \to \mathrm{Pic}^n(X)$ for a fixed $n \in \mathbb{Z}$. The fiber over an element $[\mathcal{L}] \in \mathrm{Pic}^n(X)$ is given by $H^1(X, \mathcal{L})$. In fact, as we will show, the number $\dim(H^1(X, \mathcal{L}))$, if $n \neq 0$, depends only on n and not on the chosen line bundle. We will use the standard notation $h^n(\mathcal{L}) = \dim H^n(X, \mathcal{L})$. Proofs of the following three theorems can be found in [8].

Theorem 2.15. *Let $\mathcal{L}$ be a line bundle of degree $n < 0$ over a Riemann surface X of genus 1. Then $h^0(\mathcal{L}) = 0$.*

Theorem 2.16. *Let $\mathcal{L}$ be a line bundle of degree $n > 0$ over a Riemann surface X of genus 1. Then $h^1(\mathcal{L}) = 0$.*

Theorem 2.17 (Hirzebruch-Riemann-Roch for a Riemann surface)**.**
Let $\mathcal{L}$ be a line bundle over a Riemann surface X of genus g. Then the following formula holds:

$$h^0(\mathcal{L}) - h^1(\mathcal{L}) = \deg \mathcal{L} - g + 1.$$

We return to the case of a line bundle $\mathcal{L}$ over a Riemann surface X of genus 1. Let $n = \deg \mathcal{L}$. Combining the three theorems we obtain the following cases:

(a) $n > 0 : h^0(\mathcal{L}) = n$ and $h^1(\mathcal{L}) = 0$.

(b) $n < 0 : h^0(\mathcal{L}) = 0$ and $h^1(\mathcal{L}) = -n$.

(c) $n = 0$ and $\mathcal{L} \cong \mathcal{O}_X$: $h^0(\mathcal{L}) = h^1(\mathcal{L}) = 1$.

(d) $n = 0$ and $\mathcal{L} \ncong \mathcal{O}_X$: $h^0(\mathcal{L}) = h^1(\mathcal{L}) = 0$.

Note that case (c) follows from

$$H^0(X, \mathcal{O}_X) = \{f : X \to \mathbb{C} \text{ holomorphic}\} \cong \mathbb{C}.$$

Furthermore let us recall that by fixing a base point $x_0 \in X$ (we need such a choice anyway for the Poincaré line bundle), we obtain an isomorphism $x \mapsto (x) - (x_0)$ between X and $\mathrm{Pic}^0(X)$, where (x), respectively (x_0), denote the divisor associated to a point. Also recall that each connected component $\mathrm{Pic}^n(X)$ of the Picard group is isomorphic to $\mathrm{Pic}^0(X)$. We will sketch the proof of the following theorem:

Theorem 2.18. *Let $n \in \mathbb{N} \setminus \{0\}$ and X a Riemann surface of genus 1. Then $\mathrm{Aff}^{-n}(X) \to \mathrm{Pic}^{-n}(X)$ is a holomorphic vector bundle of rank n.*

Sketch of the proof: Denote by $\psi : \mathrm{Pic}^{-n}(X) \times X \to \mathrm{Pic}^{-n}(X)$ the first projection. Regarding the Poincaré line bundle, restricted to $\mathrm{Pic}^{-n}(X) \times X$, as a sheaf denoted by $\mathcal{L}_{x_0}^{-n}$, we see that the function

$$l \mapsto \dim H^1(\{l\} \times X, \mathcal{L}_{x_0}^{-n}\restriction_{\{l\}\times X})$$

is constant and equal to n on $\mathrm{Pic}^{-n}(X)$ by definition of the Poincaré line bundle and the characterisation of the dimension of the first cohomology we gave above. By a theorem of Grauert, related to his direct image theorem, see [5] theorem 2.3 (b), the higher direct image sheaf $R^1\psi_*(\mathcal{L}_{x_0}^{-n})$ is a locally free sheaf, so defines a holomorphic vector bundle. Another corollary of Grauert's theorem implies that the canonical maps

$$R^1\psi_*(\mathcal{L}_{x_0}^{-n})(l) \to H^1(\{l\} \times X, \mathcal{L}_{x_0}^{-n}\upharpoonright_{\{l\}\times X})$$

are isomorphisms for every $l \in \mathrm{Pic}^{-n}(X)$. Using the analytic interpretation of the higher direct image sheaves from [6], one can prove that the topology on $\mathrm{Aff}^{-n}(X)$, introduced using Gauge theoretical methods in section 2.6, coincides with the topology of the holomorphic bundle associated with the sheaf $R^1\psi_*(\mathcal{L}_{x_0}^{-n})$. □

2.8 Projective line bundles

Now we look at fiber bundles with the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ as a fiber and structure group PGL(2).

By definition $\mathrm{PGL}(2) = \mathrm{GL}(2)/\mathbb{C}^* \cdot \mathrm{Id}$, where Id is the identity matrix. We express this as an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{GL}(2) \longrightarrow \mathrm{PGL}(2) \longrightarrow 1,$$

leading to an exact sequence of sheaves

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathrm{GL}(2)_{hol} \longrightarrow \mathrm{PGL}(2)_{hol} \longrightarrow 1.$$

The part of the corresponding long exact sequence that interests us is the following

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \mathrm{GL}(2)_{hol}) \longrightarrow H^1(X, \mathrm{PGL}(2)_{hol}) \longrightarrow H^2(X, \mathcal{O}_X^*).$$

Supposing that X is a Riemann surface, $H^2(X, \mathcal{O}_X^*)$ vanishes and the exact sequence has the form

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \mathrm{GL}(2)_{hol}) \longrightarrow H^1(X, \mathrm{PGL}(2)_{hol}) \longrightarrow 0.$$

To give an interpretation of this exact sequence we define the projective bundle associated to a rank 2 holomorphic vector bundle $E \to X$. The projectivization $\mathbb{P}(E)$ of E is defined as $\mathbb{P}(E) = \coprod_{p \in X} \mathbb{E}_p$, where $\mathbb{E}_p$ denotes the projectivization of the fiber.

The above sequence tells us that every holomorphic projective line bundle over a Riemann surface is the projectivization of a holomorphic vector bundle of rank 2.

Two holomorphic bundles E_1, E_2 of rank 2 define isomorphic projective line bundles if and only if there exists a holomorphic line bundle L on X such that $E_2 \cong E_1 \otimes L$. Therefore, the classification of projective line bundles over a Riemann surface X is equivalent to the classification of rank 2 holomorphic bundles over X (modulo tensor product with line bundles). This shows in particular that the theory of moduli spaces for holomorphic vector bundles over Riemann surfaces plays an important role for the classification of projective line bundles.

Note that the total space of a projective line bundle over a Riemann surface X is (by definition) a ruled surface over X, which is a very important class of surfaces. Indeed, by a classical result in the theory of complex surfaces, see for example [4], one knows that any algebraic surface with Kodaira dimension $-\infty$ is either a rational surface (a surface rationally equivalent to $\mathbb{P}^2_{\mathbb{C}}$) or a ruled surface.

3 Real structures in the sense of Atiyah

3.1 Basic notions

In this chapter we will introduce Real structures in the sense of Atiyah on smooth and on complex manifolds, the classic example being the complex conjugation of the complex plane. The most notable difference to Atiyah's original definition, see [2], is that he introduced the notion in the continuous case, whereas we will only consider the smooth or holomorphic version. For the rest of the chapter, τ will denote complex conjugation in $\mathbb{C}$ or $\mathbb{C}^*$.

Definition 3.1. *First, for a complex manifold X with structure sheaf $\mathcal{O}_X$ of holomorphic function, we define the* complex conjugate manifold $\bar{X}$ *as the complex manifold with the underlying set X and structure sheaf $\overline{\mathcal{O}_X}$, where $\overline{\mathcal{O}_X}(U) := \{\tau \circ f : f \in \mathcal{O}_X(U)\}$.*

Remark (i) Alternatively we could have defined $\bar{X}$ by using charts of X and composing them with complex conjugation in $\mathbb{C}^n$.

(ii) Clearly, we have $\overline{(\overline{X})} = X$.

(iii) For a map $\varphi : X \to Y$ between complex manifolds we have the following equivalences:

$$\begin{aligned} X \xrightarrow{\varphi} Y \text{is holomorphic} \Leftrightarrow & \overline{X} \xrightarrow{\mathrm{Id}} X \xrightarrow{\varphi} Y \text{is anti-holomorphic} \\ \Leftrightarrow & X \xrightarrow{\varphi} Y \xrightarrow{\mathrm{Id}} \overline{Y} \text{is anti-holomorphic} \end{aligned}$$

Definition 3.2. *A* Real structure *on a smooth manifold X is formally defined as a pair (X, ι), where ι is a smooth involution of X. We will usually just say that ι is a Real structure on X, and call (X, ι) a* Real manifold. *If X is a complex manifold and ι an anti-holomorphic involution, we call (X, ι) a* Real complex manifold. *A* Klein surface *is a Real complex manifold such that X is a Riemann surface. The fixed point locus of ι is denote by X^ι.*

We will treat the complex case in more detail, suppose now that (X, ι) is a Real complex manifold. Alternatively, we can think of ι as a holomorphic map $X \to \bar{X}$ (resp. $\bar{X} \to X$), such that $X \xrightarrow{\iota} \bar{X} \xrightarrow{\iota} X$ and $\bar{X} \xrightarrow{\iota} X \xrightarrow{\iota} \bar{X}$ are equal to Id_X, resp. $\mathrm{Id}_{\bar{X}}$.

We want to generalise this to the sheaves $\mathrm{A}(1)_{hol}$ and $\mathrm{PGL}(2)_{hol}$. To achieve this we define a complex conjugation on the groups A(1) and PGL(2). Let us look at the following commutative diagram, derived from the exact sequence in chapter 1:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\tau"] & \mathrm{A}(1) \arrow[r] \arrow[d, dashed] & \mathbb{C}^* \arrow[r] \arrow[d, "\tau"] & 1, \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathrm{A}(1) \arrow[r] & \mathbb{C}^* \arrow[r] & 1.
\end{tikzcd}$$

The morphisms τ on the left and right define a morphism $\mathrm{A}(1) \to \mathrm{A}(1)$, which by abuse of notation, we will also call ν.

An explicit definition of $\nu : \mathrm{A}(1) \to \mathrm{A}(1)$ can be given using the split morphism $s : \mathbb{C}^* \to \mathrm{A}(1)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r, "\phi"] \arrow[d, "\tau"] & \mathrm{A}(1) \arrow[r, "\psi"] \arrow[d, "\nu"] & \mathbb{C}^* \arrow[r] \arrow[d, "\tau"] \arrow[l, bend left, "\tau"] & 1 \\
0 \arrow[r] & \mathbb{C} \arrow[r, "\phi"] & \mathrm{A}(1) \arrow[r, "\psi"] & \mathbb{C}^* \arrow[r] \arrow[l, bend left, "s"] & 1.
\end{tikzcd}$$

Define the map via $\nu(x) = \psi(s(\phi^{-1}(x \cdot \tau(\psi(x))^{-1}))) \cdot s(\tau(\psi(x)))$. Note, that by exactness, $\phi^{-1}(x \cdot \tau(\psi(x))^{-1})$ defines a unique element.

By using an explicit description of A(1) as the maps $\mathbb{C} \to \mathbb{C}$, $z \mapsto az + b$ we get that ν sends such a map to $z \mapsto \bar{a}z + \bar{b}$. Similarly we get a map $\nu : \mathrm{PGL}(2) \to \mathrm{PGL}(2)$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d, "\tau"] & \mathrm{GL}(2) \arrow[r] \arrow[d, "\tau"] & \mathrm{PGL}(2) \arrow[r] \arrow[d, "\nu"] & 1 \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathrm{GL}(2) \arrow[r] & \mathrm{PGL}(2) \arrow[r] & 1.
\end{tikzcd}$$

In the same way we did for the structure sheaf $\mathcal{O}_X$ and the sheaf $\mathcal{O}_X^*$, we can now define for $f \in \mathcal{F}(U)$, $\iota_*(f) \in \mathcal{F}(\iota(U))$ by $\iota_*(f)(x) := \nu(f(\iota(x)))$, where $\mathcal{F}$ is either $\mathrm{A}(1)_{hol}$ or $\mathrm{PGL}(2)_{hol}$.

3.2 Real structures on an elliptic curve

Let now (X, ι) be a Klein surface of genus 1, w.l.o.g $X = \mathbb{C}/ < 1, \alpha >$ with $Re(z) \neq 0$. There are essentially three cases, corresponding to different fixed point sets.

(a) $\alpha = it$, $t \in \mathbb{R}_{>0}$ and $\iota([z]) = [\bar{z}]$.

(b) $\alpha = it$, $t \in \mathbb{R}_{>0}$ and $\iota([z]) = [\frac{1}{2} + \bar{z}]$.

(c) $\alpha = \frac{1}{2} + it$, $t \in \mathbb{R}_{>0}$ and $\iota([z]) = [\frac{1}{2} + \bar{z}]$.

In case (a), the fixed point set is the disjoint union of two circles given by $C_1 = \{[z], Re(z) = 0\}$ and $C_2 = \{[z], z = s + it/2, s \in \mathbb{R}\}$.
In case (b), the fixed point set is empty and in (c) it consists of one circle. For related results and more general notions see [15]. The proof of the above statement can also be found in [1] chapter II, proposition 1.

3.3 Real structures on vector bundles

If $E \to X$ is a complex vector bundle over a smooth manifold and ι a Real structure on X we would like to lift the structure to E. First we need to define what we mean by a lift.

Definition 3.3. *Let $E \to X$ be a smooth, complex vector bundle over a smooth manifold with Real structure ι. A smooth involution $\tilde{\iota} : E \to E$ is called a* Real structure on E, compatible with ι, *if the induced map on the fibres is anti-linear and the following diagram commutes.*

$$\begin{tikzcd} E \arrow[r, "\tilde{\iota}"] \arrow[d, "\pi"] & E \arrow[d, "\pi"] \\ X \arrow[r, "\iota"] & X \end{tikzcd}$$

If $E \to X$ is a holomorphic vector bundle over a Real complex manifold (X, ι), we additionally demand that $\tilde{\iota}$ is anti-holomorphic.

We will look at the special case of a line bundle $\mathcal{L} \xrightarrow{\pi} X$ over a compact, connected complex manifold. Denoting by $[\mathcal{L}]$ the element in $\mathrm{Pic}(X)$ corresponding to $\mathcal{L}$, and denote by $\overline{\mathcal{L}}$ the conjugate bundle of $\mathcal{L}$. Recall the description of the conjugate manifold $\overline{X}$. Pulling back $\overline{X} \xrightarrow{\mathrm{Id}} X$ we get the following commutative diagram:

$$\begin{tikzcd} \mathrm{Id}^*(\mathcal{L}) = \overline{\mathcal{L}} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}^*(\pi)=\pi"] & \mathcal{L} \arrow[d, "\pi"] \\ \overline{X} \arrow[r, "\mathrm{Id}"] & X \end{tikzcd}$$

So $\overline{\mathcal{L}} \to \overline{X}$ is a holomorphic vector bundle.
We define a map $\hat{\iota} : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ by $\hat{\iota}([\mathcal{L}]) := [\iota^*(\overline{\mathcal{L}})]$, illustrated by the following commutative diagram:

$$\begin{tikzcd} \iota^*(\overline{\mathcal{L}}) \arrow[r, "\iota^*"] \arrow[d, "\iota^*(\pi)"] & \overline{\mathcal{L}} \arrow[r, "\mathrm{Id}"] \arrow[d, "\pi"] & \mathcal{L} \arrow[d, "\pi"] \\ X \arrow[r, "\iota"] & \overline{X} \arrow[r, "\mathrm{Id}"] & X \end{tikzcd}$$

Proposition 3.4. *The map $\hat{\iota} : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is an anti-holomorphic involution.*

Remark Note that this does not imply that $\iota^*(\overline{\iota^*(\overline{\mathcal{L}})}) = \mathcal{L}$, it just means that they are isomorphic.

Keeping the remark in mind, we get the following diagram, where the top morphisms are not necessarily induced by the pullback.

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d, "\pi"] & \iota^*(\overline{\mathcal{L}}) \arrow[r] \arrow[d, "\iota^*\pi"] & \mathcal{L} \arrow[d, "\pi"] \\
X \arrow[r, "\iota"] \arrow[rr, bend right, "\mathrm{Id}"'] & X \arrow[r, "\iota"] & X
\end{tikzcd}$$

Suppose next that $[\mathcal{L}] \in \mathrm{Pic}(X)^{\hat{\iota}}$ is a fixed point. Then we obtain *some* anti-holomorphic map $\tilde{\varphi} : \mathcal{L} \to \mathcal{L}$ whose square lifts the identity $\mathrm{Id} : X \to X$. In fact, the following theorem holds (which is stated without proof in [14]). The reader is encouraged to look up further results in the same paper.

Theorem 3.5. *Suppose $X^\iota \neq \emptyset$ and $\mathcal{L}$ a holomorphic line bundle over X. Then*

$$[\mathcal{L}] \in \mathrm{Pic}(X)^{\hat{\iota}} \Leftrightarrow \mathcal{L} \textit{ has a Real structure compatible with } \iota.$$

Proof. Let $\mathcal{L}$ be a holomorphic line bundle over X such that $[\mathcal{L}] \in \mathrm{Pic}(X)^{\hat{\iota}}$. Fixing an isomorphism $\varphi : \mathcal{L} \to \iota^*(\bar{\mathcal{L}})$ we obtain an anti-holomorphic ι-covering isomorphism $\tilde{\varphi} = \tilde{\iota} \circ \varphi$ that covers ι, where $\tilde{\iota} : \iota^*(\bar{\mathcal{L}}) \to \mathcal{L}$ is the map induced by the pullback. Then $\tilde{\varphi} \circ \tilde{\varphi}$ is a holomorphic automorphism of $\mathcal{L}$, hence is equal to $z\,\mathrm{Id}_{\mathcal{L}}$ for some $z \in \mathbb{C}^*$ since X is compact and connected. By assumption there exists an $x \in X^\iota$, so for $v \in \mathcal{L}_x$ we get

$$z\tilde{\varphi}_x(v) = (\tilde{\varphi}_x \circ \tilde{\varphi}_x)(\tilde{\varphi}_x(v)) = (\tilde{\varphi}_x(\tilde{\varphi}_x \circ \tilde{\varphi}_x))(v) = \tilde{\varphi}_x(zv) = \bar{z}\tilde{\varphi}_x(v).$$

So by replacing $\tilde{\varphi}$ with $\frac{1}{\sqrt{|z|}}\tilde{\varphi}$ we can assume $z \in \{-1, 1\}$. We need to show that z cannot be equal to -1. Note first that an anti-linear automorphism of a complex vector space of complex dimension n, regarded as an automorphism of the underlying real vector space, is orientation preserving if n is even and reverses the orientation if n is odd. In our case $n = 1$, so $\tilde{\varphi}_x$ is orientation reversing. On the other hand, the next lemma shows that $\tilde{\varphi}_x$ must be orientation preserving if $z = -1$. This finishes the proof. □

Lemma 3.6. *Let V be a real vector space of dimension 2 and $\psi \in \mathrm{End}(V)$ such that $\psi^2 = -\mathrm{Id}_V$. Then $\mathrm{Tr}(\psi) = 0$ and $\det(\psi) = 1$, in particular ψ is orientation preserving.*

Proof. Recall that in dimension 2, the characteristic polynomial χ_ψ of ψ is given by $T^2 - \mathrm{Tr}(\psi)T + \det(\psi)$. By assumption, the polynomial $T^2 + 1$ annihilates ψ. Since this polynomial has no roots in the real numbers, we conclude that it is the minimal polynomial of ψ. But the minimal polynomial divides the characteristic polynomial and looking at the formula given above we see that $\chi_\psi(T) = T^2 + 1$ and comparing coefficients we obtain the result. □

In the next chapter we will generalise the following result, due to Atiyah, to affine bundles.

Theorem 3.7. *Let $\tilde{\iota} : E \to E$ be a ι-Real structure, compatible with $\iota : X \to X$. Then $\mathcal{L}^{\tilde{\iota}} \to X^{\iota}$ is a real vector bundle.*

This result and others can be found in [3], see also [2].

3.4 Real structures on affine bundles

Now we want to generalise the linear case to affine bundles.

Definition 3.8. *Let $\mathcal{A} \xrightarrow{\pi} X$ be a smooth, complex affine bundle over a Real manifold (X, ι). A smooth involution $\tilde{\iota} : \mathcal{A} \to \mathcal{A}$ is called a* Real structure on $\mathcal{A}$, compatible with ι, *if $\tilde{\iota}$ is a lift of ι and the derivative of the induced maps $\tilde{\iota}_x : \mathcal{A}_x \to \mathcal{A}_{\iota(x)}$ is anti-linear for every $x \in X$. As we had before for vector bundles, if $\mathcal{A} \to X$ is a holomorphic affine bundle over a Real complex manifold (X, ι), we demand that $\tilde{\iota}$ is anti-holomorphic.*

The anti-linearity condition also says that the induced map on the linearisation

$$\tilde{\iota}_{lin} : \mathcal{A}_{lin} \to \mathcal{A}_{lin},$$

is anti-linear on the fibers. In chapter 2.3 we defined a generalisation of complex conjugation for the affine group. Using it we can define the conjugation $\overline{\mathcal{A}}$ of an affine line bundle $\mathcal{A}$. Analogously to the previous chapter we obtain: The result concerning the fixed point locus is similar to the linear case, i.e. theorem 3.7.

Theorem 3.9. *Suppose that $\tilde{\iota} : \mathcal{A} \to \mathcal{A}$ is an ι-Real structure on a smooth, complex affine bundle $\pi : \mathcal{A} \to X$ of rank n over a Real manifold (X, ι) and $X^{\iota} \neq \emptyset$. Then $\mathcal{A}^{\tilde{\iota}}$ is a real affine bundle of rank n over X^{ι}.*

We need some preparatory work before we can give the proof.

Proposition 3.10. *(i) Let A be an affine vector space modelled on a vector space V and $\varphi : A \to A$ an affine involution. Then for any $x \in A$, the barycentre of x and $\varphi(x)$ is a fixed point. Furthermore for any $x \in A^{\varphi}$ the fixed point set A^{φ} is given by $x + V^{T\varphi}$, where $T\varphi$ denotes the tangential map $T\varphi : V \to V$ of φ.*

(ii) Let $\psi : W \to W$ be an anti-linear involution of a complex vector space W of complex dimension n. Then the fixed point set W^{ψ} is a real subspace of real dimension n.

Proof. (i) Let $x \in A$ be arbitrarily chosen. Then barycentre y of x and $\varphi(x)$ is given by $\frac{\varphi(x)-x}{2} + x$. Note that this expression is well defined since $\varphi(x) - x \in V$. A simple calculation shows that y is a fixed point. The rest of the statement is clear.

(ii) We look at ψ as a linear involution of the underlying real vector space. Then, since $\psi^2 = \mathrm{Id}$ we immediately see that $t^2 - 1$ annihilates ψ, so its minimal polynomial splits into factors of degree 1 and hence ψ is diagonalizable. In other words ψ similar to a matrix that has only 1's and -1's at the diagonal. Therefore $W = W_1 \oplus W_{-1}$ is the direct sum of its eigenspaces. Multiplying with i maps W_1 to W_{-1} and W_{-1} to W_1, showing that both W_1 and W_{-1} have real dimension n. Since $W_1 = W^\psi$ we get the result.

□

Lemma 3.11. *Let $\varphi : V \to W$ be linear map of finite dimensional real vector spaces, equivariant with respect to linear involutions $f : V \to V$, $g : W \to W$, i.e. $\varphi(f(x)) = g(\varphi(x))$. If φ is surjective, so is the induced map $\varphi_1 : V^f \to W^g$.*

Proof. As in the proof of proposition (ii) we obtain compositions $V = V_1 \oplus V_{-1}$ and $W = W_1 \oplus W_2$. Of course we cannot say anything about the dimensions of the V_i and W_i. Since φ is equivariant, it maps V_1 into W_1 and V_{-1} into W_{-1}, showing that $\varphi = \varphi_1 \oplus \varphi_{-1}$ and that the induced map $\varphi_1 : V^f = V_1 \to W_1 = W^g$ is well defined. Clearly, if φ_1 is not surjective, φ cannot be surjective. □

Proposition 3.12. *Let $\pi : E \to X$ be a smooth vector bundle of rank n and $F \subseteq E$ a submanifold. Then the following conditions are equivalent:*

(i) F is a subbundle of rank k.

(ii) $F_x = E_x \cap F$ is a linear subspace of dimension k for every $x \in X$.

Proof. Let $0 : X \to E$ be the canonical zero-section, which we can regard as a smooth map $0 : X \to F$ and denote $\pi\restriction_F$ by q. Since $q \circ 0 = \mathrm{Id}_X$, we get that $T_{0_x}q \circ T_x 0 = \mathrm{Id}_{T_xX}$, showing that q is a submersion at 0_x for every $x \in X$. Let us look at the following commutative diagram.

$$\begin{tikzcd}
0^*(TF) \arrow[r] \arrow[d] & TF \arrow[r, "Tq"] \arrow[d] & TX \arrow[d] \\
X \arrow[r, "0"] \arrow[rr, bend right, "\mathrm{Id}"'] & F \arrow[r, "q"] & X
\end{tikzcd}$$

We obtain a fiberwise surjective vector bundle morphism $\varphi : 0^*(TF) \to TX$ from the top row, since we showed that q is a submersion at each 0_x. In other words, we obtain a vector bundle epimorphism and a vector bundle $\ker(\varphi)$, since the kernel of a vector bundle morphism is again a vector bundle. Note that the kernel of $T_{0_x}q : T_{0_x}F \to T_xX$ is exactly the tangent space $T_{0_x}F_x$ which can be identified with F_x since F_x is a vector space. Using this we can identify F with $\ker(\varphi)$, proving the the result. □

There is an analog for affine bundles.

Proposition 3.13. *Let $\pi : A \to X$ be a smooth affine bundle of rank n and $E \subseteq A$ a submanifold of A. Then the following conditions are equivalent:*

(i) E is an affine subbundle of rank k.

(ii) E_x is an affine subspace of dimension k for every $x \in X$ and the restriction $\pi_E : E \to X$ is a submersion.

The method of this proof is essentially the same as the method used in showing that each smooth affine bundle has a global section.

Proof. Obviously (i) implies (ii). For the other direction we can assume that A is a vector bundle, since in the smooth case each affine bundle is isomorphic to a vector bundle. By assumption π_E is a submersion so we find an open cover $\{U_i\}_{i\in I}$ of X and local sections $s_i : U_i \to E$ of $E\restriction_{U_i}$. Since we always work with paracompact manifolds, this cover can be chosen locally finite. Let $\{\rho_i\}_{i\in I}$ be a partition of unity subordinate to the cover $\{U_i\}_{i\in I}$. Then $\sum_{i\in I} \rho_i s_i$ is a smooth global section of A with values in E by (ii), reducing the problem to the case of vector bundles and proposition 3.12. □

The proof makes use of the following theorem, often called the *slice theorem*. We refer the reader to [18] for details.

Theorem 3.14. *Let G be a compact Lie group and M a smooth G-manifold. For $m \in M$ and $H = G_m$, there exists a unique H-representation V and a G-diffeomorphism $\varphi : G \times_H V \to M$ onto an open neighbourhood of Gm such that $\varphi(g, 0) = gm$.*

In the case of $m \in M^G$, $V \cong T_mM$ is called the *tangential representation* at m. We also use the following corollary, again we refer to [18].

Corollary 3.15. *Let G be a compact group acting smoothly on a smooth manifold M. Then the fixed point set M^G is a closed submanifold of M.*

Remark 1. The dimension of M^G may vary on the connected components, so generally M^G is not a pure dimensional submanifold.

2. If (X, ι) is a Real manifold, the corollary implies that X^ι, if non-empty, is a closed submanifold. If (X, ι) is a Real complex manifold of dimension n one can show that X^ι is even a pure dimensional submanifold of real dimension n.

With all the machinery assembled, we can give the proof of our original theorem.

Proof of theorem 3.9: Suppose $x \in X^\iota$. By proposition 3.4 the fixed point set $\mathcal{A}_x^{\tilde{\iota}_x}$ of the fiber map $\tilde{\iota}_x : \mathcal{A}_x \to \mathcal{A}_x$ is a real affine subspace of dimension n. Next, the corollary of the slice theorem tells us that $\mathcal{A}^{\tilde{\iota}}$ is a submanifold of $\mathcal{A}\lceil_{X^\iota}$. Showing that $q = \pi\lceil_{\mathcal{A}^{\tilde{\iota}}}$ is a submersion and applying proposition 3.13 finishes the proof. The slice theorem tells us that the tangent space $T_y\mathcal{A}^{\tilde{\iota}}$ in a point $y \in \mathcal{A}_x^{\tilde{\iota}}$ is given by $(T_y\mathcal{A})^{T_y\tilde{\iota}}$. Now lemma 3.11 implies the surjectivity of $T_yq : T_y\mathcal{A}^{\tilde{\iota}} \to T_xX^\iota$, finishing the proof. $\square$

3.5 A look ahead

Similarly to the linear case, we define

$$\hat{\iota} : \mathrm{Aff}(X) \to \mathrm{Aff}(X),$$

by

$$\hat{\iota}([\mathcal{A}]) = [\iota^*(\overline{\mathcal{A}})].$$

$$\begin{array}{ccccc}
\iota^*(\overline{\mathcal{A}}) & \xrightarrow{\tilde{\iota}} & \overline{\mathcal{A}} & \xrightarrow{\mathrm{Id}} & \mathcal{A} \\
\big\downarrow{\scriptstyle \iota^*(\pi)} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\
X & \xrightarrow{\iota} & \overline{X} & \xrightarrow{\mathrm{Id}} & X
\end{array}$$

Further properties of this map will be addressed in some future article.

Q.1 Is the map $\hat{\iota} : \mathrm{Aff}(X) \to \mathrm{Aff}(X)$ an involution?

Q.2 Is it continuous with respect to the natural topology defined in chapter 2.4?

Q.3 Do we obtain a result similar to theorem 3.5?

Q.4 How does a Real structure on a line bundle lift to an affine line bundle lying above it?

4 Enoki surfaces

Our goal is to study Real structures on Enoki surfaces, a family of complex surfaces introduced and classified by Ichiro Enoki in 1980. See his paper [7]. A general reference for this section is [4]. We will first collect some results and terminology of the classification of complex surfaces. Some generality will be sacrificed to make the exposition more readable and to stay closer to cases that are of interest to us.
For the rest of this chapter X denotes a connected, complex manifold of complex dimension n. The *canonical bundle* of X is defined as $K_X = \bigwedge^n \Omega^1_X$, where Ω^1_X denotes the cotangent bundle of X. It is a line bundle over X.

Definition 4.1. *For $d \in \mathbb{N}$, the d-th* plurigenus *of X is defined as*

$$P^d(X) = h^0(X, K_X) = \dim H^0(X, K_X^{\otimes d}).$$

The Kodaira dimenson $\mathrm{kod}(X)$ *of X is defined as follows:*

$$\mathrm{kod}(X) = \begin{cases} -\infty & \textit{if } P^d(X) = 0 \textit{ for all } d > 0 \\ \min k & \textit{such that } P^d/d^k \textit{ is bounded}\ . \end{cases}$$

Definition 4.2. *A* surface of class VII *is a complex surface S with* $\mathrm{kod}(X) = -\infty$ *and $b_1 = 1$. If X is also minimal, it is a* surface of class VII_0.

Let now S be a complex surface. A curve C on a surface S is a 1-dimensional subspace of S, locally defined by one equation. It naturally corresponds to an effective divisor and vice versa. For a divisor D on S, we denote its *self-intersection number* by $(D)^2$. There are two possible cases for a surface S of class VII_0 that has a curve:

1. There exists a divisor D such that $(D)^2 = 0$.
2. For every divisor D on S, $(D)^2 < 0$.

In his paper, Enoki constructed surfaces $S_{n,\alpha,t}$ for $n > 0$, $0 < |\alpha| < 1$ and $t \in \mathbb{C}^n$ satisfying the following properties:

1. $S_{n,\alpha,t}$ is a surface of class VII_0 with $b_2 = n$.
2. $S_{n,\alpha,t}$ has an effective divisor $D_{n,\alpha,t}$ with $(D_{n,\alpha,t})^2 = 0$.
3. $S_{n,\alpha,t} - D_{n,\alpha,t} = A_{n,\alpha,t}$ is an affine line bundle of degree $-n$ over an elliptic curve.

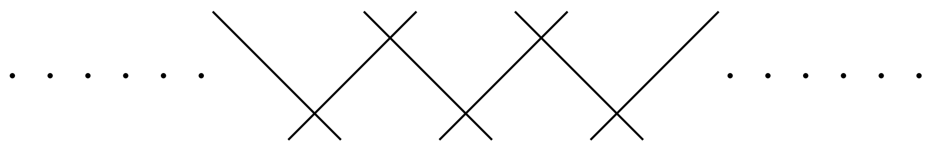
Then, he goes on to prove his main theorem:

Theorem 4.3. *Suppose S is a surface of class* VII_0 *with $b_2 = n$. If there exists an effective divisor D with $(D)^2 = 0$, then S is biholomorphic to $S_{n,\alpha,t}$ and $D = mD_{n,\alpha,t}$ for some n, α, t as above and $m \in \mathbb{Z}$.*

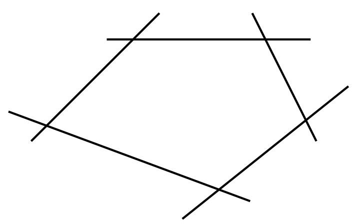
We will provide a very short description of the construction utilised his paper. We mentioned that an Enoki surface is a compactification of an affine line bundle. The important thing to remember is that is *not* the trivial fiber-wise compactification. One would just obtain a ruled surface over an elliptic curve, with fibers $\mathbb{P}^1$, but not a class VII surface. The compactification is along an exceptional divisor D, as stated above. The construction consists of a series of blow-ups of $\mathbb{P}^1 \times \mathbb{C}$, defined by a birational automorphism

$$g_{n,\alpha,t} : (z, w) \to (w^n z + t(w), \alpha w),$$

where n is a natural number, t is a polynomial and $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$. The α corresponds to an elliptic curve $\mathbb{C}^*/ < \alpha >$, that will be the base space of the affine line bundle obtained. What the blow-ups do to construct the exceptional divisor is essentially attaching projective lines at two points, such that two projective lines have transversal intersection or not at all. The following drawing illustrates the idea.



After passing to an inductive limit of certain subspaces for each blow-up, one defines an equivalence relation induces using the $g_{n,\alpha,t}$ and obtain a *cycle of n rational curves*, i.e. a reduced divisor whose irreducible components are rational curves which intersect according to the the drawing below:



More precisely,

1. if $n = 1$, the divisor is singular rational curve C with a simple (nodal) singularity, so that $(C)^2 = 0$.

2. If $n \geq 2$, using a suitable indexation of these curves one obtains $(C_i \cdot C_j) = 1$ when $j = i \pm 1$ modulo n, and $(C_i \cdot C_j) = 0$ otherwise.

This is of course an oversimplification but the idea should be clear. Again, we refer to Enoki's paper [7] for the details.

Some interesting questions arise:

Q.1: If α is a real number, and $t \in \mathbb{R}[x]$ a polynomial with real coefficients, does there exist a natural Real structure on the corresponding Enoki surface?

Q.2: Suppose that the affine line bundle corresponding to an Enoki surface is equipped with a Real structure. Does the structure lift to a Real structure on the Enoki surface?

Q.3: In general, classify all Real structures on an Enoki surface.

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Abstract (deutsche Version)

Die zwei Hauptthemen der Arbeit sind affine Bündel über kompakten, komplexen Mannigfaltigkeiten und Reelle Strukturen im Sinne von Atiyah. Die Beschreibung der Menge der Isomorphieklassen von holomorphen affinen Geradenbündeln mittels der Kohomologie von holomorphen Geradenbündeln (Theorem 2.7), sowie die Einführung einer Topologie auf dieser Menge ist das erste neue Resultat dieser Arbeit. Die Konstruktion der Topologie verwendet Methoden der Eichtheorie, um ein holomorphes Analogon des für abelsche Varietäten bekannten Poincaré Geradenbündels zu definieren. Als Anwendung werden affine Geradenbündel über einer Riemannschen Fläche betrachtet.

Im darauffolgenden Kapitel werden Reelle Strukturen im Sinne von Atiyah behandelt. Zuerst wird in Kapitel 3.3 ein Beweis von [14] Proposition 2.12.(1) gegeben. Anschließend werden Reelle Strukturen auf affinen Bündeln definiert und gezeigt dass die Fixpunktmenge einer solchen Struktur ein affines Bündel ist. Dieses Resultat verallgemeinert das schon bekannte Theorem für Vektorbündel.

Das letze Kapitel 4 behandelt Enokiflächen und stellt weiterführende Fragestellungen vor.

Abstract (english version)

The two main topics of this article are affine bundles over compact, complex manifolds and Real structures in the sense of Atiyah. The first main result describes the set of isomorphism classes of holomorphic affine line bundles using the cohomology of holomorphic line bundles and the description of a topology on this set. The construction of the topology uses gauche theoretical methods as well as a holomorphic analogue of the Poincaré line bundle, well known in the case of abelian varieties. Affine line bundles over a Riemann surface are discussed as an application.

Real structures in the sense of Atiyah are treated in the next chapter. A proof of [14] proposition 2.12.(1) is given in chapter 3.3. Real structures on affine bundles are subsequently defined and it is shown that the fixed point set of such a structure defines an affine bundle. This result generalises the well known case of Real structures on vector bundles.

The last chapter discusses Enoki surfaces and presents problems in the context of the first two chapters.

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2014–2015 **Tutor für Theoretische Informatik**, *Universität Wien*.
Erstellen von Übungsbeispielen für Studenten und Hilfe bei Prüfungen