MASTERARBEIT / MASTER'S THESIS

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"Pointwise Estimates of the Weighted Bergman Kernel in Finite Dimension"

verfasst von / submitted by

Damir Ferizović, BSc.

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to my mother -
who has lifted the weight
of the world for me

Preface

The goal of this thesis is to earn its author the degree “Master of Science”, in short “MSc.”, in the field of Mathematics under the supervision of Prof. Friedrich Haslinger. This thesis has been written in Latex via the GUI “Kile” for Linux based OS’s; the diagrams have been produced by use of Libre Office’s Draw.

In the present summary we will discuss research papers dealing with pointwise bounds of the Bergman kernel in the weighted case.

We start with J. Ortega-Cerdá’s and J. Marzo’s [MOC09], where we try to fill in the details as often as possible and abandon the crude notion of $a \sim b$.

The second paper that we are going to deal with is H. Delin’s [Del98], where a little issue arises regarding a constant and which we mostly clerify in the Appendix.

We also are going to spend a few pages on the work of [Dal15], which is a very recent paper and somewhat a synthesis of the works [MOC09] and [Del98]. All of them use a trick played by Kerzman, see [Ker72].

The first paper deals with complex dimension one and applies the well known L^2-estimates of L. Hörmander - which, due to its widespread impact, is added in the Appendix; while the other presents bounds in several complex dimensions, but uses Differential Geometry which makes it very hard to access due to this fields almost esoteric appearence. This author warmly suggests Jeffrey M. Lee’s masterfully written [Lee09] for a mathematical rigorous and detailed treatment of this subject.

Hence we will need a good background in manifold theory and Complex Analysis in Several Variables - something that cannot be summerized within a few pages, but for whatever reason, we exactly tried this.

Most of the material presented here has been stolen shamelessly from a few textbooks - we will give the source at the beginning of every chapter, section or near the statement itself, if it is just that part, that is from somewhere else.

The reader should not at all expect a detailed presentation of all the preliminaries, but rather “ε-neighbourhoods” of the main theorems - things that need to be stated, will be stated in most cases, depending on the balast that comes with it, which may well be beyond the scope of this thesis.

One of the certainly many shortcomings of the present work is the missing of a proper introduction to Bergman kernels - but this is actually well known to people working in Complex Analysis as it is presented in perhaps every book on this subject, even newspapers use it sometimes as page fillers; we thus take it for granted; see [Dal15] for a quick summary.

A word of caution here, although the author put great effort and care to do things right, errors might occur and if the reader wishes to use actual parts of this thesis, it might be wise to consult the original sources or ask someone who is familiar with the issues here before doing so; any confusion or worse, caused by the explanations provided here are contrary to this author's intentions and he apologiezes where an apology is to be made.

Finally, this author has the chance to thank his supporter and the helping hands he received.

Thanks to Prof. Frank Vallentin, from the Universität Köln, who told this author about the estimates of Kabatianskii and Levenshtein concerning kissing numbers.

Many thanks to Prof. Joaquim Ortega-Cerdá from Barcelona, who clearified first a severe error this author made concerning doubling measures and second, explained a proof which this author got very wrong and thus closed a big gap in this author's understanding.

With the last lines, the author wishes to express his gratitude to Prof. Friedrich Haslinger - not only for proposing the subject, but also for his vast support, help, kindness and most for his patience.

Damir Ferizović
Vienna

Erläuterung

Thema dieser Arbeit ist die punktweise Abschätzung des gewichteten Bergman Kernes in beliebiger aber endlicher Dimension.

Genaugenommen werden drei Arbeiten präsentiert die sich mit diesem Problem befassen. Diese Arbeiten sind der Reihe nach [MOC09], [Del98] und [Dal15] - wobei die ersten Beiden detailiert beschrieben werden, wenn auch nicht alle Details geklärt werden und die letzte Arbeit lediglich kurz Erwähnung findet, da sie recht neu ist und für eine genaue Untersuchung kaum Zeit blieb, aber genug um ihre Wichtigkeit zu erkennen.

Die Arbeit [MOC09] von Ortega-Cerdá und Marzo liefert Abschätzung in einer Dimension unter relativ milden Voraussetzungen, jene von Delin, [Del98], bietet Abschätzungen in mehreren Veränderlichen und nutzt dafür das nicht gerade leicht nachvollziehbare Gebiet der Differential Geometrie.

Die vorliegende Arbeit ist so aufgeteilt, dass, mit Ausnahme der Grundkenntnisse, so gut wie alle nötigen Begriffe und Resultate in eigenen Kapiteln, den "Preliminieries" zusammengeführt sind die nötig sind um den nachfolgenden Text im Großteil nachvollziehen zu können. Ein Sonderkapitel, das gewisse Abschätzungen der D-quer Gleichung behandelt die kurz Anwendung finden, wurde auf Grund ihrer Wichtigkeit zusammengefasst.

Notation

We state here some symbols that will be used throughout the text, some of them represent a general pattern without being defined explicitly.

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural, real and complex numbers respectively; we identify $\mathbb{C}$ and $\mathbb{R}^2$ and write $z = x + iy$ for $z \in \mathbb{C}$, $i = \sqrt{-1}$ and some $x, y \in \mathbb{R}$. The letter n will exclusively denote the dimension of an underlying vectorspace and is understood to be a natural number.

We will denote by C, C', $\widetilde{C}\ldots$ some positive constants, which may vary from line to line; they actually might be some functions, bounded from above and below by positive constants - whatever makes more sense.

The Lebesgue measure will be denoted by λ and we will not distinguish any cases, whether we integrate over $\mathbb{R}^n$ or $\mathbb{C}^n$ - we simply write $\int \ldots \, \mathrm{d}\lambda$.

$(a,b]$	$=$	$\{x \in \mathbb{R} \ : \ a < x \leq b\}$ for $a, b \in \mathbb{R}$
$\mathbb{R}_{>0}$	$=$	$\{x \in \mathbb{R} \ : \ x > 0\}$
$B(x_0, r)$	$=$	$\{x \in \mathbb{R}^n \ : \ \\|x_0 - x\\| < r\}$ for $r \in \mathbb{R}_{>0}$ and $x_0 \in \mathbb{R}^n$
$B(x_0, r)^c$	$=$	$\{x \in \mathbb{R}^n \ : \ \\|x_0 - x\\| \geq r\}\ldots$ the complement
$C\bar{B}(x_0, r)$	$=$	$\{x \in \mathbb{R}^n \ : \ \\|x_0 - x\\| \leq Cr\}$ for $C > 0$; same for $\mathbb{C}^n$
$B_{h_z}(z, 1)$	$=$	$\{\zeta \in \mathbb{C}^n \ : \ h_z(\zeta, z) < 1\}$ for a metric tensor h_z
δ_j^k	$\ldots$	the Kronecker delta; equals 1 iff $j = k$ and 0 otherwise
$\{u_j\}_j$	$\ldots$	a sequence of u_j's
$(x, 0)$	$=$	$(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n$, given $x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$
$\mathcal{C}^k$	$=$	the set of k-times continuosly differentiable functions
$\mathcal{C}^\infty(X)$	$=$	$\bigcap_k \mathcal{C}^k(X)\ldots$ the set of smooth functions from X to $\mathbb{R}$
$\mathcal{C}^1([0,1], Y)$	$=$	the set of $\mathcal{C}^1$-functions from $[0,1]$ to Y
$\mathcal{C}_0^\infty$	$=$	the set of compactly supported smooth functions
$\bar{z}$	$=$	$x - iy\ldots$ given that $z = x + iy \in \mathbb{C}$
$\bar{z}$ (vector)	$=$	$(\bar{z}_1, \ldots, \bar{z}_n) \in \mathbb{C}^n$, given that $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$
$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$	$=$	$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ respectively; $z \in \mathbb{C}$
∂_j	$=$	$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ for $z \in \mathbb{C}^n$, similarly for $\bar{\partial}_j$
Δ	$=$	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\ldots$ the Laplace operator
pr_1	$\ldots$	the projection on the first slot, i.e. $\mathrm{pr}_1(x_1, x_2) = x_1$
$\partial\varphi \wedge \psi$	$=$	$(\partial\varphi) \wedge \psi$
bX	$\ldots$	the boundary of the set X
$\mathrm{d}S$	$\ldots$	the by $\mathrm{d}\lambda$ induced volume form on a boundary
$\mathcal{J}_{\mathbb{R}}F$	$\ldots$	the Jacobian matrix of F with respect to real variables
σ_n	$\ldots$	will stand for the surface area and the volume of the unit ball
$o(\\|x\\|^2)$	$\ldots$	$F = o(\\|x\\|^2)$ means $F\\|x\\|^{-2} \to 0$ as $\\|x\\| \to 0$
$Y \subset\subset X$	$\ldots$	means that the closure of Y in X is compact in X

Contents

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Chapter 1

Prelimineries I

See the first pages of [Ste93] for a more general treatment or [Sak99] for a few results, mentioning the existence theorem for doubling measures on compact metric spaces due to [VK88] and also its extension in [LS95]. The results concerning estimates of doubling constants are this author's original contribution.

1.1 Doubling Measures

On $\mathbb{R}^n$, the Lebesgue measure λ of a ball B is precisely $2^{-n}\lambda(2B)$ and in what follows we will replace the Lebesgue measure by a more general one with a similar scaling property. Though the metric content of our first definition is subtle, we will come back to it regularly in the next subsection to get a feeling for its strong influence on the measure; meanwhile we will work in $\mathbb{R}^n$ with its standard metric.

1.1.1 Definition and Basic Properties

By the upcoming Example 1.1.3 and Lemma 1.1.4 it is reasonable to demand the following measure μ to be locally finite and non-trivial on $\mathbb{R}^n$, or equivalently: $0 < \mu(B(0,1)) < \infty$. A measure on a complete metric linear space with this property implies the space to be finite dimensional, see [Ruz01].

Definition 1.1.1. *A positive Borel measure*[1] *μ is called doubling measure if there is a constant $C > 0$, such that μ satisfies for any ball B the doubling property:*

$$\mu(2B) \leq C\mu(B). \tag{1.1}$$

The smallest constant $C_d = C_d(\mu)$ with the above property, if existent, is referred to as the doubling constant of μ.

[1] By this we mean a measure which is defined on all open sets of a locally compact Hausdorff space, see for example [Rud09]. Note that this definition is not standard, see [Els05, page 313].

Often, the first thing after a definition is the announcement of an abuse of the given definition - which generally is denoted as notation; so, in the sequel, we will mostly talk about a doubling constant without mentioning its doubling measure or vica versa.

Example 1.1.2. $\mu(B) := \int_B |x| \, \mathrm{d}\lambda$ *is doubling on* $\mathbb{R}$ *with* $C_d = 2$.

Example 1.1.3. *The trivial measures* $\mu \equiv 0$ *or* $\nu \equiv \infty$ *have the doubling property for every positive constant, yet* $C_d(\mu) = 0$ *and* $C_d(\nu)$ *doesn't exist.*

Lemma 1.1.4. *Let* μ *be a doubling measure, then* $\mu(B) = \infty$ *or* $\mu(B) = 0$ *for a ball* B *is equivalent to* $\mu(B) = \infty$ *or* $\mu(B) = 0$ *for every ball* B.

Proof. Let B and B' be given balls, then there exists an $m \in \mathbb{N}$ with $B' \subset 2^m B$ as well as $B \subset 2^m B'$ and thus

$$\mu(B') \leq \mu(2^m B) \leq C^m \mu(B) \quad \text{and} \quad \mu(B) \leq \mu(2^m B') \leq C^m \mu(B').$$

Now B having infinite or zero mass implies this for every other B' too. ■

As a consequence of our metric and for a locally finite and non-trivial doubling measure μ, we have that $C_d > 1$ and by continuity from below, spheres have no mass, i.e. for every ball B we have $\mu(\bar{B} \setminus B) = 0$ and μ has in particular no atoms.

Corollary 1.1.5. *A doubling constant* C_d *exists for a given doubling measure* μ *if and only if* μ *is locally finite.*

Proof. Existence of C_d implies via Example 1.1.3 and Lemma 1.1.4 local finiteness. For the other direction we ignore the case $\mu \equiv 0$ and let C be as in the definition of doubling measures and C' be such that $\mu(B(0,2)) = C'\mu(B(0,1))$, then $C_d \in [C', C]$ is the infimum of all positive constants c, which fulfil (1.1). ■

Remark 1.1.6. *Clearly,* $\mu(3B) \leq \mu(4B) \leq C^2\mu(B)$. *Thus if a doubling constant exists, then also a smallest constant* C_t *with*

$$\mu(3B) \leq C_t \, \mu(B) \quad \textit{and thus} \quad C_d \leq C_t \leq C_d^2.$$

In Proposition 1.1.15 we will see that C_d never equals C_t nor any other C_p for $p \neq 2$, but its proof makes heavy use of the Euclidean metric and its properties. Similar results can be obtained for a general metric space with extra assumptions on its metric for any pair C_d and C_t of doubling constants, for arbitrary but fixed scaling factors, that are to be compared.

1.1.2 Estimates of Doubling Constants

Kissing numbers, once introduced, are a property of the Euclidean metric and we will see how they influence doubling constants. Their rôle might not be unanticipated, but the appearence of the Mandelbrot set however is. Before we go on, let us quickly set an abbreviation: B_n is the unit ball $B_n(0,1)$ of $\mathbb{R}^n$.

Definition 1.1.7. *The maximal amount of disjoint copies of B_n, which fit between $3B_n$ and B_n is called kissing number of n - denoted by k_n.*

We define $k_0 = 0$ for convenience; the odd name originates from billards.

Example 1.1.8. *In $\mathbb{R}^2$, the kissing number is 6 - since the balls with unit radii and centers $2,\ 2e^{\imath\pi/3}, \ldots,\ 2e^{5\imath\pi/3}$ do the job.*

Kissing numbers are only known for a handful of dimensions; the first three are for example $k_1 = 2$, $k_2 = 6$ and $k_3 = 12$. A lower bound for k_n is given by the Minkowski-Hlawka theorem which states that

$$\frac{\zeta(n)}{2^{n-1}} \ \leq\ k_n,$$

where ζ is the Riemann Zeta function. See [CS99] pp. 13-24 and 337-339.

Lemma 1.1.9. *Kissing numbers tend to infinity, more precisely*

$$2n \ \leq\ k_{n-1} + 2 \ \leq\ k_n.$$

Proof. Let $o := (0, \ldots, 0) \in \mathbb{R}^{n-1}$ and $x_{j,0} := (x_j, 0) \in \mathbb{R}^n$. Disjoint balls $B_{n-1}(x_j, 1)$ in $3B_{n-1} \setminus B_{n-1}$ for $1 \leq j \leq k_{n-1}$ and $x_j \in \mathbb{R}^{n-1}$, yield disjoint balls $B_n(x_{j,0}, 1)$, $B_n((o, 2), 1)$ and $B_n((o, -2), 1)$ in $3B_n \setminus B_n$. ∎

Using the notation of Remark 1.1.6 and the conclusion of Corollary 1.1.5, we obtain a growth[2] estimate for the doubling constant C_d which, besides the metric, depends on the dimension of the underlying space only.

Lemma 1.1.10. *For a locally finite, non-trivial doubling measure μ,*

$$\sqrt{k_n + \frac{1}{4}} + \frac{1}{2} \ \leq\ C_t.$$

[2] An estimate from above is not possible in general, take for instance $\mathbb{R}$ and the doubling measures $\{|x|^j \,\mathrm{d}\lambda\}_j$ (see [Ste93, page 40 (8.7)]) whose doubling constants form an unbounded sequence: otherwise use that $-([x_0 - 1] - 1)^{j+1} \leq -[x_0 - 1]^{j+1}$, hence dropping these terms and as $C[x_0 + 1]^{j+1} = C[x_0 + 1][x_0 + 1]^j < j[x_0 + 1]^j$ for j big enough and $[x_0 + 1]^{j+1} + j[x_0 + 1]^j \leq ([x_0 + 1] + 1)^{j+1}$ we have $C[x_0 + 1]^{j+1} < ([x_0 + 1] + 1)^{j+1}$ for j big enough; contradiction.

Proof. Given a ball B, let $k = k_n$ and $B_1, \dots, B_k$ denote the translated copies of B which fit between B and $3B$. Now, since μ is a doubling measure and $B \subset 3B_j$ for every $j \in \{1, \dots, k\}$,

$$\mu(3B) \geq \ \mu(B) + \sum_{j=1}^{k} \mu(B_j) \ \geq \ \mu(B) + \frac{1}{C_t} \sum_{j=1}^{k} \mu(3B_j) \ \geq \left(1 + \frac{k}{C_t}\right) \mu(B). \quad (1.2)$$

Using that $C_t \mu(B) \geq \mu(3B)$, we divide by $\mu(B)$ and perform some elementary manipulations to deduce:

$$0 \ \leq \ C_t^2 - C_t - k. \quad (1.3)$$

As $k \geq 2$, calculating the greater root gives the result. ■

By Lemma 1.1.10 and Remark 1.1.6, we have a lower bound for doubling constants which, by Lemma 1.1.9, tends to infinity as the dimension increases.

Lemma 1.1.11. *For a locally finite doubling measure μ on $\mathbb{R}^n$, we have*

$$C_t \ \leq \ \frac{C_d^2}{1 + \frac{k_n}{C_d^6}} \quad \text{and} \quad C_d \ \leq \ \frac{C_t}{1 + \frac{k_n}{C_t^2}}.$$

Proof. The case $\mu \equiv 0$ is clear; we assume now non-triviality. Since C_t is the best possible constant to fulfil $\mu(3B) \leq C_t \mu(B)$, it implies that there is a sequence of balls $\{B_j\}_j$ such that $\lim_{j \to \infty} \big(\mu(3B_j) - C_t \mu(B_j)\big) = 0$. Thus all we have to show, is that

$$\frac{C_d^2 \mu(B)}{\mu(3B)} \ \geq \ \frac{\mu(4B)}{\mu(3B)} \ \geq \ (1 + \varepsilon) \ \text{ for some } \varepsilon > 0 \text{ and all balls } B.$$

As[3] $\mu(4B \setminus 3B) \ \geq \ \sum_{j=1}^{k_n} \mu(B_j)$ for some disjoint B_j's with radius half the one of B, we obtain by $B \subset 9B_j$ and the same reasoning as in the proof of Lemma 1.1.10:

$$\sum_{j=1}^{k_n} \mu(B_j) \ \geq \ \frac{k_n}{C_t^2} \mu(B) \ \geq \ \frac{k_n}{C_t^2 C_d^2} \mu(4B).$$

Finally we apply Remark 1.1.6 to estimate C_t and see that $\mu(4B) - \mu(3B) = \mu(4B \setminus 3B)$ where we divide by $\mu(3B)$ on the left hand side and by $\mu(4B)$ on the right hand side to find $\varepsilon = k_n C_d^{-6}$.
The second inequality, where one has $2B \subset 9B_j$, is proven analogously. ■

[3] Recall the definition of k_n and note that $4B \backslash 3B$ surely can contain that many and more translates of $\frac{1}{2}B$. Naturally, it would be more useful to have a maximal number of non-overlapping copies of B_n in $8B_n \setminus 6B_n$, denoted by K_n, to be used in the estimates above.

Substituing $k_n \geq 2$ in Lemma 1.1.10, we have as a little application that

$$C_d > \sqrt{2}. \tag{1.4}$$

The conclusions here work by Corollary 1.1.5 perfectly fine for a general metric space S for which we have an $N \in \mathbb{N}$ such that for any balls $B = B(x_0, r)$ in S we have N many disjoint balls $B(x_j, r) \subset (3B \setminus B)$ with $B \subset B(x_j, 3r)$ as well as N many disjoint balls in $8B \setminus 6B$ with similar properties. Also in [VK88], the authors work with metrics which allow only finitely many disjoint balls $B_j \subset 2B$, thus such conditions seem to be natural; see [KW95] for a summary.

Definition 1.1.12. *For $c, k \in \mathbb{C}$ we define $p_0(c,k) = 0$ and then by recursion*

$$p_{j+1}(c,k) = c + kp_j^2(c,k). \tag{1.5}$$

The **Mandelbrot set** is defined to be $\mathcal{M} = \{c \in \mathbb{C} \,:\, |\lim_{j\to\infty} p_j(c,1)| < \infty\}$; see for instance [Dev88], page 315.

Lemma 1.1.13. *For $ck \neq 0$, we have $p_j(c,k) = k^{-1}p_j(ck,1) = cp_j(1,ck)$.*

Proof. The proof is by induction. Surpressing the dependence on c and k, we obtain $p_1 = c$, $p_2 = c + kc^2$, $p_3 = c + k(c + kc^2)^2 = c + kc^2 + 2k^2c^3 + k^3c^4$, etc. and thus can define polynomials $P_j(z)$ and $Q_j(z)$ such that $P_j(z) = zQ_j(z)$ with $p_j(c,k) = \frac{1}{k}P_j(ck) = cQ_j(ck)$; but $p_j(ck,1) = P_j(ck)$ and $p_j(1,ck) = Q_j(ck)$. ∎

Corollary 1.1.14. *$|\lim_{j\to\infty} p_j(c,k)|$ is bounded if and only if $ck \in \mathcal{M}$.*

For $c, k > 0$, the p_j's are strictly increasing[4] and thus converge for $ck \in \mathcal{M}$.

Proposition 1.1.15. *For a locally finite doubling measure μ with doubling constant C_d and constant C_t as in Remark 1.1.6, we have the following relations*

$$C_t \leq \frac{C_d^2}{\lim_{j\to\infty} p_j(1, k_n C_d^{-6})} \text{ and } C_d \leq \frac{C_t}{\lim_{j\to\infty} p_j(1, k_n C_t^{-2})}. \tag{1.6}$$

Proof. The proof is analogous to the one of Lemma 1.1.11, but in the crucial part we do not use Remark 1.1.6 to estimate C_t and C_d, but the improved estimates of Lemma 1.1.11 itself to obtain

$$C_t \leq \frac{C_d^2}{1 + \frac{k_n}{C_d^6}\left(1 + \frac{k_n}{C_d^6}\right)^2}$$

and similar for C_d; iterating this procedure, always using the improved estimates obtained in the step before, yields exactly the definition (1.5) with $c = 1$ and $k = k_n C_d^{-6}$ or $k = k_n C_t^{-2}$ respectively. ∎

[4]By induction we find that $p_{j-1} < p_j$ implies $p_j = c + kp_{j-1}^2 < c + kp_j^2 = p_{j+1}$.

If $C_t > 0$, then (1.6) forces convergence[5] of $\lim_{j\to\infty} p_j(1,\tau)$ for $0 \le \tau \le k_n C_t^{-2}$ and the largest real number in the Mandelbrot set seems to be $\frac{1}{4}$ as numerical evidence suggests, thus we obtain an improvement of Lemma 1.1.10 to $k_n \le \frac{1}{4} C_t^2$.

Using this for $\mu = \lambda$ and $C_t = 3^n$ gives a general upper bound for k_n:

$$k_n \;\le\; \frac{1}{4} 3^{2n}. \tag{1.7}$$

Far better results are obtained in [CZ14], where the authors are using an asymptotic bound given by Kabatianskii and Levenshtein of about $(1.321)^n$. The bound of (1.7) could be improved by finding more suitable measures, or, by fixing a B in (1.2) where one has $\mu(3B) \le C\mu(B)$ for $C < C_t$.

If a doubling measure μ is absolutely continuous with respect to Lebesgue measure, we obtain a sharp result concerning growth. Let f be in $L^1(\mathbb{R}^n, \lambda)$ such that for arbitrary $x_0 \in \mathbb{R}^n$ and $r > 0$ with $B = B(x_0, r)$ we have

$$\mu(2B) = \int_{2B} f \,\mathrm{d}\lambda \;\le\; C_d \int_B f \,\mathrm{d}\lambda = C_d\, \mu(B)$$

or equivalently

$$\int_{B(0,r)} \big(2^n f(2y + x_0) - C_d f(y + x_0)\big)\,\mathrm{d}\lambda \;\le\; 0; \tag{1.8}$$

where we used the transformation formula for $\phi_j(y) = jy + x_0$, $j \in \{1,2\}$.

Proposition 1.1.16. *Let $\mu = f\,\mathrm{d}\lambda$ be a locally finite and non-trivial doubling measure with doubling constant C_d and $f \in L^1(\mathbb{R}^n)$, it follows that*

$$2^n \;\le\; C_d.$$

Proof. Suppose f is smooth for now; then, f being a density of a measure implies non-negativity and we use equation (1.8) to exclude the existence of an x_0 with $f(x_0) > C_d/2^n f(x_0)$.

For arbitrary f, let χ be a compactly supported, smooth and positive function on $\mathbb{R}^n$, then, by Fubini and the doubling property, the convolution $f * \chi$ is also a smooth density for a doubling measure[6]:

$$\begin{aligned}\int_{2B} f * \chi \,\mathrm{d}\lambda &= \int_{\mathbb{R}^n} \chi(x) \int_{2B} f(y - x)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x)\\ &\le \int_{\mathbb{R}^n} \chi(x) C_d \int_B f(y-x)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x) = C_d \int_B f * \chi\,\mathrm{d}\lambda.\end{aligned}$$

Now a standard argument finishes the proof. ■

[5]Note that by (1.6): $\lim_{j\to\infty} p_j(1, k_n C_d^{-6}) < C_d$ and $\lim_{j\to\infty} p_j(1, k_n C_t^{-2}) < C_t 2^{-\frac{1}{2}}$ by (1.4).

[6]This trick has already been used in [Chr91], first lines on page 206. By this, the doubling constant of the newly obtained doubling measure can only decrease compared to the old one.

1.1.3 The Lemma of M. Christ

The next lemma is [Chr91, Lemma 2.1]. A similar result is [KW95, Lemma 4] on page 539, referring to [Wu93, Lemma 1] on page 78.

The following is trivially true when $\mu(B) \in \{0, \infty\}$ by Lemma 1.1.4, hence we regard the case where $0 < \mu(B) < \infty$ and thus a doubling constant exists by Corollary 1.1.5. A dual statement will be given in Lemma 1.2.4.

Lemma 1.1.17. *For a locally finite and non-trivial doubling measure μ, there exist positive constants $\varepsilon = \varepsilon(C_d, n)$ and $C = C(C_d, \varepsilon)$, such that for any balls B', B with $B' \subset B$ the following holds true:*

$$\mu(B') \ \leq \ C \left(\frac{\lambda(B')}{\lambda(B)} \right)^{\varepsilon} \mu(B).$$

Proof. **Step 1:** We regard balls B' and B with the same center, thus $B' = B(x, \rho')$ and $B = B(x, \rho)$ with $0 < \rho' < \rho$. By Remark 1.1.6 and iterating (1.2) we find $m, \bar{m} \in \mathbb{N}$, where $\bar{m} = \bar{m}(n)$ and $m = m(\bar{m})$, such that

$$\mu(2^m B) \geq \mu(3^{\bar{m}} B) \geq \left(1 + \frac{k_n}{C_d^2}\right)^{\bar{m}} \mu(B) \geq 2\mu(B). \tag{1.9}$$

Next we choose an $r \in \mathbb{N}$ so that $2^{-rm} B \subseteq B' \subseteq 2^{-(r-1)m} B$, thus

$$\left(\frac{\lambda(B')}{\lambda(B)} \right)^{\frac{1}{nm}} \geq \frac{1}{2^r}; \tag{1.10}$$

iterate (1.9) to obtain

$$\mu(B) \geq 2^r \mu(2^{-rm} B) \tag{1.11}$$

and conclude with (1.10), (1.11), m-times the doubling property and the choice of r:

$$\left(\frac{\lambda(B')}{\lambda(B)} \right)^{\frac{1}{nm}} \mu(B) \ \geq \mu(2^{-rm} B) \geq \ \frac{1}{C_d^m} \mu(2^{-(r-1)m} B) \geq \frac{1}{C_d^m} \mu(B'), \tag{1.12}$$

which proves the claim for the special case of equal centers.

Step 2: Let $B' = B(x', \rho')$ and $B = B(x, \rho)$ be given with $x, x' \in \mathbb{R}^n$ and $\rho', \rho \in \mathbb{R}_{>0}$ with $B' \subset B$. Obviously

$$B', B \subset B(x', 2\rho) \subset 4B,$$

and we apply (1.12) to B' and $B(x', 2\rho)$, use $\mu(B(x', 2\rho)) \leq \mu(4B) \leq C_d^2 \mu(B)$ and replace $\lambda(B(x', 2\rho)) = \lambda(B(x, 2\rho)) = 2^n \lambda(B)$ to finally obtain

$$C_d^{m+2} \, 2^{\frac{-1}{m}} \ \left(\frac{\lambda(B')}{\lambda(B)} \right)^{\frac{1}{nm}} \mu(B) \geq \mu(B').$$

■

A generalization to the metrics that we have mentioned after Lemma 1.1.11 is possible too provided (1.10) holds for our r and the volume of balls with different centers is comparable.

Remark 1.1.18. *In Lemma 1.1.17 we have $\varepsilon n < 1$ since $m \geq 2$. The integer m can in fact be chosen arbitrary as long as big enough, which clearly will affect C.*

Some immediate consequences for a locally finite and non-trivial doubling measure μ are the following.

Corollary 1.1.19. *$\mu(\mathbb{R}^n) = \infty$ and $\mu(\{x\}) = 0$ for all elements $x \in \mathbb{R}^n$.*

Proof. To obtain contradictions, set for the first assertion $B' = B(0,1)$ and $B = 2^m B'$ in Lemma 1.1.17 and let $m \to \infty$; and for the second one, set for an arbitrary $x \in \mathbb{R}^n$ $B = B(x,1)$ and $B' = 2^{-m}B$ and let $m \to \infty$. ∎

E. Stein, while concerned with more general metrics, stated on page 9 in [Ste93] some further assumptions in order to "avoid certain technical complications" - one of them was $x \mapsto \mu\big(B(x,\delta) \cap U\big)$ to be continuous for all open U and each $\delta > 0$. For the standard metric this is equivalent to local finiteness as can be seen using continuity from above and below.

1.2 The Radius Function

We henceforth assume to deal with a locally finite, non-trivial doubling measure μ on $\mathbb{R}^n$, until stated otherwise. As μ has no atoms and $\mathbb{R}^n$ has infinite mass, we can always find by continuity from below a positive radius ρ for a given center x, such that we have $\mu\big(B(x,\rho)\big) = 1$.

1.2.1 Definition and Basic Properties

Definition 1.2.1. *We define the radius function $\rho = \rho_\mu : \mathbb{R}^n \to \mathbb{R}_{>0}$ to satisfy $\mu\big(B(x,\rho(x))\big) = 1$ for all $x \in \mathbb{R}^n$.*

From time to time we will talk about the radius function ρ without mentioning the underlying doubling measure. It follows [MMOC03, page 7, equation (6)].

Lemma 1.2.2. *The radius function is Lipschitz, i.e. $|\rho(x) - \rho(y)| \leq \|x - y\|$.*

Proof. Otherwise there are $x, y \in \mathbb{R}^n$ such that $\rho(x) \geq \rho(y)$ and $\|x - y\| < \rho(x) - \rho(y)$, hence a contradiction to Lemma 1.1.4, as

$$B\big(y,\rho(y)\big) \subsetneq B\big(x, \|x-y\| + \rho(y)\big) \subsetneq B\big(x,\rho(x)\big)$$

implies the existence of a ball with zero mass. ∎

Lemma 1.2.3. *Given the radius function ρ for μ, it exists an $r = r(\mu) \in \mathbb{R}_{>0}$, such that $\rho(x) \leq \|x\|$ for $x \in \mathbb{R}^n$ with $\|x\| \geq r$.*

Proof. Let's assume the contrary. Let $\{x_j\}_j$ be a sequence in $\mathbb{R}^n$ which tends in norm to infinity and with the property: $\rho(x_j) > \|x_j\|$. Thus $B(x_j, \|x_j\|) \subset B(x_j, \rho(x_j))$, but also $B(0, \|x_j\|) \subset B(x_j, 2\|x_j\|)$, hence, by definition of ρ,

$$\mu\big(B(0, \|x_j\|)\big) \leq \mu\big(B(x_j, 2\|x_j\|)\big) \leq C_d\, \mu\big(B(x_j, \|x_j\|)\big) \leq C_d$$

which contradicts, by lower continuity, $\mu(\mathbb{R}^n) = \infty$; see Corollary 1.1.19. ∎

We will now give a statement which is dual to Lemma 1.1.17 and was communicated to this author by prof. Joaquim Ortega-Cerdá.

Lemma 1.2.4. *Let μ be an arbitrary doubling measure and set $M \geq \frac{\log(C_d)}{n \log(2)}$, then for any balls B', B with $B' \subset B$, the following holds:*

$$\mu(B) \;\leq\; 2^{nM} \left(\frac{\lambda(B)}{\lambda(B')}\right)^M \mu(B').$$

Proof. Let $r \in \mathbb{N}$ such that $2^{r-1}B' \subset B \subset 2^r B'$ and thus $\mu(B) \leq C_d^r \mu(B')$, hence

$$2^{n(r-1)} \;\leq\; \frac{\lambda(B)}{\lambda(B')} \;\leq\; 2^{nr} \;\leq\; 2^n \frac{\lambda(B)}{\lambda(B')}$$

and taking the M-th power shows $C_d^r \leq 2^{nrM} \leq 2^{nM}\big(\lambda(B)/\lambda(B')\big)^M$. ∎

For the next we follow the proof of [Chr91, Lemma 3.1].

Lemma 1.2.5. *Given the radius function ρ for μ, there is a $\delta_0 \in (0, 1)$ such that for every $\delta \in (0, \delta_0]$, we find a $C = C(\delta, C_d)$ to satisfy for any elements $x, y \in \mathbb{R}^n$ with $\|x - y\| \geq \rho(y)$ and $v \in B(x, \|x - y\|/2)$, the following inequality*

$$\frac{\rho(v)}{\|x-y\|} \;\leq\; C \left(\frac{\rho(y)}{\|x-y\|}\right)^{\delta}.$$

Proof. By the doubling property, Lemma 1.1.17 on $B(y, \rho(y)) \subset B(y, \|x - y\|)$ with $\varepsilon = (nm)^{-1}$, note Remark 1.1.18, and the definition of ρ we have

$$C_d\, \mu\big(B(x, \|x-y\|)\big) \geq \mu\big(B(y, \|x-y\|)\big) \geq C' \left(\frac{\|x-y\|}{\rho(y)}\right)^{\frac{1}{m}}.$$

Let $v \in B(x, \|x - y\|/2)$, then, if $\|x - y\| \geq \rho(v)$, we use Lemma 1.2.4:

$$\mu\big(B(x, \|x-y\|/2)\big) \leq \mu\big(B(v, \|x-y\|)\big) \leq 2^{nM} \left(\frac{\|x-y\|}{\rho(v)}\right)^{nM}$$

and otherwise we use Lemma 1.1.17 again[7]. Combining inequalities, we get

$$\left(\frac{\|x-y\|}{\rho(y)}\right)^{\varepsilon M^{-1}} \leq \tilde{C}\frac{\|x-y\|}{\rho(v)}$$

and since $C_d > \sqrt{2}$, see (1.4), we have $\varepsilon M^{-1} \leq \log(2)[2\log(C_d)]^{-1} < 1$. ■

This result will help us to gain [Chr91, Lemma 3.3]:

Lemma 1.2.6. *Given the radius function ρ for μ and $x, y \in \mathbb{R}^n$ with $\|x-y\| \geq \rho(y)$, then there is an $\epsilon > 1$ and a constant $C = C(\epsilon, C_d)$ such that*

$$\frac{\rho(x)}{\rho(y)} + \frac{\rho(y)}{\rho(x)} \leq C\left(\frac{\|x-y\|}{\rho(y)}\right)^{\epsilon}.$$

Proof. We use Lemma 1.2.5 for $x = v$ and multiply by $\|x-y\|/\rho(y)$ to obtain

$$\frac{\rho(x)}{\rho(y)} \leq \tilde{C}\left(\frac{\|x-y\|}{\rho(y)}\right)^{1-\delta}.$$

By the assumption on $\rho(y)$ it follows that $B(y, \rho(y)) \subset B(x, 2\|x-y\|)$ and thus $2\|x-y\| > \rho(x)$. Hence we apply Lemma 1.2.5 where we are allowed to interchange the roles of x and y and set $y = v$ to obtain

$$\frac{\|x-y\|}{\rho(y)} \geq C'\left(\frac{\|x-y\|}{\rho(x)}\right)^{\delta'}. \tag{1.13}$$

Finally, using the assumption $\|x-y\| \geq \rho(y)$, we get with (1.13)

$$C'^{\frac{-1}{\delta'}}\ \rho(x)\left(\frac{\|x-y\|}{\rho(y)}\right)^{\frac{1}{\delta'}} \geq \|x-y\| \geq \rho(y).$$

C can now be chosen properly and as $1-\delta < \frac{1}{\delta'}$, we choose $\epsilon := \frac{1}{\delta'}$. ■

The following is another corollary of Lemma 1.1.17, [MOC09, Lemma 2.2].

Lemma 1.2.7. *There exists a $\gamma_0 \in (0,1)$ such that for every $\gamma \in (0, \gamma_0]$ and any balls B', B with respective radii $r' \leq r$ and $B' \cap B \neq \emptyset$, we find $C = C(C_d, \gamma)$ and $C' = C'(C_d, \gamma)$, such that*

$$\left(\frac{\mu(B)}{\mu(B')}\right)^{\gamma} C \leq \frac{r}{r'} \leq \left(\frac{\mu(B)}{\mu(B')}\right)^{\frac{1}{\gamma}} C'.$$

[7] Where we use Remark 1.1.18 with different m and m' to force their ratio to be less than 1.

Proof. Apply Lemma 1.2.4 and Lemma 1.1.17 to $B' \subset 2B$. ■

Set $r = \rho(x)$ and $r' = \rho(x')$, then this conclusion is often stated as the almost constant character of ρ on intersecting balls or as the comparability of the values of ρ. Let $B' = B = B(x_0, \rho(x_0))$ in Lemma 1.2.7, it then follows[8] that

$$C' r^{\gamma} \leq \mu(rB) \leq C'' r^{\frac{1}{\gamma}}. \tag{1.14}$$

Using Lemma 1.2.7, we will now give a finer growth estimate on ρ, see the lines after [MMOC03, Lemma 1], page 7.

Lemma 1.2.8. *For x with $\|x\| > 1$, there exists $\gamma \in (0,1)$ and $C = C(\gamma, \mu)$ as follows*

$$\frac{1}{C} \|x\|^{1-\frac{1}{\gamma^2}} \leq \rho(x) \leq C \|x\|^{1-\gamma^2}. \tag{1.15}$$

Proof. Let x be given such that $\rho(x) \leq \|x\|$, then, we claim:

$$\left(\left(\frac{1}{r_x} \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}} C' \leq \left(\frac{C''}{\mu(B)} \right)^{\frac{1}{\gamma}} \leq \frac{\rho(x)}{\|x\|} \leq \left(\frac{\tilde{C}'}{\mu(B)} \right)^{\gamma} \leq \left(\left(\frac{1}{r_x} \right)^{\gamma} \right)^{\gamma} \tilde{C}''. \tag{1.16}$$

To show this, we regard the inverses and therefore the reversed chain of inequalities in (1.14), with $x_0 = 0$ and $r_x := \|x\|/\rho(0)$ to confirm the left- and rightmost estimates in (1.16); Lemma 1.2.7 applied to $B = B(0, \|x\|)$ and $B' = B(x, \rho(x))$ yield the inner ones.

Noting that $\gamma < 1$ and multiplying (1.16) by $\|x\|$, we finish the proof by choosing C appropriately. By Lemma 1.2.3, the case $\rho(x) > \|x\|$ does not happen for $\|x\|$ large enough, thus can be controlled by adjusting C if necessary. ■

1.2.2 The Induced Metric

We continue to use the assumptions and notation of the previous section.

Definition 1.2.9. *Let $\mathcal{P}_{x,y}$ be the set of piecewise $C^1([0,1], \mathbb{R}^n)$-paths from x to y; we define the by μ induced metric on $\mathbb{R}^n$ by*

$$d(x,y) = d_\mu(x,y) = \inf_{\gamma \in \mathcal{P}_{x,y}} \int_0^1 \frac{\|\nabla \gamma(t)\|}{\rho(\gamma(t))} dt.$$

We say induced metric if the measure doesn't fall into weight; the metric properties are evident, for $\rho \circ \gamma$ is locally bounded from below and above. We arrived at [MMOC03, Lemma 4].

[8] This is basically equation (5) of [MOC09], page 893.

Lemma 1.2.10. *We can find $\epsilon \in (0,1)$, such that for every $r > 0$, there exists $C_r = C(\epsilon, \mu, r) > 0$ with*

$$(a)\ \frac{1}{C_r}\frac{\|x-y\|}{\rho(y)} \le d(x,y) \le C_r\frac{\|x-y\|}{\rho(y)}, \qquad \textit{for } \|x-y\| \le r\rho(y).$$

$$(b)\ \frac{1}{C_r}\left(\frac{\|x-y\|}{\rho(y)}\right)^{\epsilon} \le d(x,y) \le C_r\left(\frac{\|x-y\|}{\rho(y)}\right)^{1+\frac{1}{\epsilon}}, \qquad \textit{for } \|x-y\| > r\rho(y).$$

Proof. For fixed r, we have $B(x, r\rho(x)) = CB(x,\rho(x))$ for a function C with $0 < a \le C \le b < \infty$; thus $r\rho$ is another radius function with analogous properties and hence without loss of generality we assume $r = 1$.

For (a) we use that ρ is almost constant on balls and its minimal and maximal values deviate from $\rho(y)$ by constant factors, see the lines after Lemma 1.2.7. For the right hand side of (b) we set $\gamma(t) = y + t(x-y)$, note that $\|y-\gamma(t)\| > \rho(y)$ for t close to 1, and substitute $\gamma(t)$ for x in Lemma 1.2.6, then

$$\begin{aligned} d(x,y) &\le \int_0^1 \frac{\|x-y\|\mathrm{d}t}{\rho(\gamma(t))} = \|x-y\|\int_0^1 \frac{\rho(y)\mathrm{d}t}{\rho(\gamma(t))\rho(y)} \\ &\le C\int_0^1 \left(\frac{t\|x-y\|}{\rho(y)}\right)^{\frac{1}{\epsilon}}\frac{\|x-y\|}{\rho(y)}\,\mathrm{d}t \le C\left(\frac{\|x-y\|}{\rho(y)}\right)^{1+\frac{1}{\epsilon}}\int_0^1 t^{\frac{1}{\epsilon}}\,\mathrm{d}t. \end{aligned}$$

The left hand side is Lemma 1.2.5 with $v = \gamma(t)$ for arbitrary γ plus the fact that $\int \|\gamma'(t)\|\mathrm{d}t\|x-y\|^{-1} \ge 1$. ■

The next is [MMOC03, Lemma 5], where the proof of (b) works in arbitrary dimension.

Lemma 1.2.11. *If μ is a doubling measure on $\mathbb{C}$, then there exists $C = C(C_d) > 0$ and an $m = m(C_d) \in \mathbb{N}$ such that for any $r > 0$ we have*

$$(a)\ \int_{B(\zeta,r)} \log\left(\frac{2r}{|z-\zeta|}\right) d\mu(z) \le C\mu\big(B(\zeta,r)\big), \qquad \zeta \in \mathbb{C}.$$

$$(b)\ \sup_{\zeta\in\mathbb{C}} \int_{\mathbb{C}} \frac{d\mu(z)}{1+d^m(z,\zeta)} < \infty.$$

Proof. For (a) see [Chr91, Lemma 2.3] with the partition in [MMOC03, Theorem 8]. In (b) we do as in Lemma 2.1.1 for $n = 2$, where details can be found, to obtain

$$\int_{\mathbb{C}} \frac{d\mu(z)}{1+d^m(z,\zeta)} \le 2^{2M}(C_d-1)\sum_{j=1}^{\infty} \frac{\rho(\zeta)^{\epsilon m} j^{2M}}{\rho(\zeta)^{\epsilon m} + j^{\epsilon m}} < \infty \text{ for } m \text{ large enough.}$$

To achieve this we chopped the integral domain into annuli with boundary distances $\rho(\zeta)$ and center ζ, applied Lemma 1.2.10 - where the ϵ is from, performed obvious estimates and used finally Lemma 1.2.4 - where the M is from, to finish the argument. ■

1.2.3 Local Behaviour and Regularization

The next are Lemma 13 and Theorem 14 of [MMOC03] page 14, respectively.

Lemma 1.2.12. *For every $r > 0$ there exists a constant $C = C(r) > 0$ such that for all $z \in \mathbb{C}$*

$$\sup_{w \in B(z, r\rho(z))} |\phi(w) - \phi(z) - \mathscr{H}_z(w)| \leq C, \tag{1.17}$$

where $\mathscr{H}_z$ is a harmonic function in $B(z, r\rho(z))$ with $\mathscr{H}_z(z) = 0$.

Theorem 1.2.13. *Let ϕ be subharmonic with $\Delta\phi \, \mathrm{d}\lambda$ being a doubling measure. There exist a subharmonic $\psi \in C^\infty(\mathbb{C})$, a constant $C > 0$ and functions C_1, C_2 and C_3 with values in a compact intervall $[a, b]$ with $0 < a$ and $b < \infty$ to satisfy*

(*a*) $|\phi - \psi| \leq C$,
(*b*) $\Delta\psi \, \mathrm{d}\lambda$ *is a doubling measure and*
(*c*) $\Delta\psi = C_1 \rho_\psi^{-2} = C_2 \rho_\phi^{-2}$.

Moreover, it will be true that

$$|\nabla(\Delta\psi)| = C_3 \rho_\phi^{-3}.$$

Chapter 2

Estimates of the Weighted Bergman Kernel in One Complex Dimension

Here we will present the work [MOC09] and deal with a subharmonic function ϕ, defined on $\mathbb{C}$, such that $\Delta\phi \ \mathrm{d}\lambda$ is a locally finite, non-trivial doubling measure with associated radius function ρ and induced metric $d(\cdot,\cdot)$.

But first we define the weighted Fock spaces as

$$\mathcal{F}^2_\phi = \{f \in L^2(\mathbb{C}, e^{-2\phi} \ \mathrm{d}\lambda) \ : \ f \text{ is holomorphic}\}$$

and

$$\mathcal{F}^\infty_\phi = \{f \in L^\infty(\mathbb{C}, e^{-\phi} \ \mathrm{d}\lambda) \ : \ f \text{ is holomorphic}\};$$

where the Bergman kernel $\mathcal{B}_\phi$ is the integral operator to the Bergman projection from the L^2-space to the Fock space, also denoted by $\mathcal{B}_\phi$, i.e. for $f \in L^2(\mathbb{C}, e^{-2\phi} \ \mathrm{d}\lambda)$ we have

$$\mathcal{B}_\phi(f)(z) = \int_{\mathbb{C}} f(\zeta)\mathcal{B}_\phi(z,\zeta)e^{-2\phi(\zeta)} \ \mathrm{d}\lambda(\zeta) \in \mathcal{F}^2_\phi.$$

The Bergman kernel, holomorphic in one variable and antiholomorphic in the other, arises naturally by the Riesz representation theorem on the point evaluation functional, which is bounded, in the Hilbert space $\mathcal{F}^2_\phi$ with Hermitean inner product given by the integral with respect to the measure $e^{-2\phi} \ \mathrm{d}\lambda$ - as an illustration set $\mathcal{B}_z := \overline{\mathcal{B}_\phi(z,\cdot)}$ and choose an arbitrary $f \in \mathcal{F}^2_\phi$, it then follows the **reproducing property** of the Bergman kernel:

$$f(z) = \langle f, \mathcal{B}_z \rangle_{\mathcal{F}^2_\phi} = \int_{\mathbb{C}} f(\zeta)\mathcal{B}_\phi(z,\zeta)e^{-2\phi(\zeta)} \ \mathrm{d}\lambda(\zeta).$$

2.1 Two Lemmata and a Peak Function

We will present in the following order [MOC09, Lemma 2.7] and partially [MMOC03, Lemma 18]. The next proof works in arbitrary dimension, we present it for $n = 2$.

Lemma 2.1.1. *Let ϕ be a subharmonic function whose Laplacian is doubling, i.e. $\mu = \Delta\phi \,\mathrm{d}\lambda$. For every $\varepsilon > 0$ and $k \geq 0$ we can find a $C = C(\varepsilon, k, C_d) > 0$ with*

$$\int_{\mathbb{C}} \frac{|z-\zeta|^k}{\exp\big(d(z,\zeta)^\varepsilon\big)} d\mu(z) \leq C\rho^k(\zeta).$$

Proof. We multiply the integrand by $\rho(\zeta)^k\rho(\zeta)^{-k}$ and apply Lemma 1.2.10 where we assume without loss of generality $C_1 = 1$ and abbreviate $\delta = \epsilon\varepsilon$ to obtain

$$\begin{aligned}\int_{\mathbb{C}} \frac{|z-\zeta|^k}{\exp\big(d(z,\zeta)^\varepsilon\big)} d\mu(z) \leq \rho^k(\zeta) &\int_{B(\zeta,\rho(\zeta))} e^{-\left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^\delta} \left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^k \mathrm{d}\mu(z)\\ &+ \rho^k(\zeta) \int_{B(\zeta,\rho(\zeta))^c} e^{-\left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^\delta} \left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^k \mathrm{d}\mu(z)\end{aligned}$$

where the first term on the right is less then $\rho^k(\zeta)$ due to the integral domain.

For the second one we write $A_j(\zeta) = B(\zeta,(j+1)\rho(\zeta)) \setminus B(\zeta, j\rho(\zeta))$ and apply the doubling property and Lemma 1.2.4 to $B(\zeta, j\rho(\zeta))$ and $B(\zeta,\rho(\zeta))$ to see that

$$\mu(A_j(\zeta)) \leq \mu\big(B(\zeta, 2j\rho(\zeta))\big) - \mu\big(B(\zeta, j\rho(\zeta))\big) \leq (C_d - 1)2^{2M}j^{2M};$$

hence

$$\begin{aligned}\int_{B(\zeta,\rho(\zeta))^c} e^{-\left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^\delta} \left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^k \mathrm{d}\mu(z) &= \sum_{j=1}^\infty \int_{A_j(\zeta)} e^{-\left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^\delta} \left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^k \mathrm{d}\mu(z)\\ &\leq \sum_{j=1}^\infty e^{-j^\delta}(j+1)^k \int_{A_j(\zeta)} \mathrm{d}\mu(z)\\ &\leq (C_d-1)2^{2M}\sum_{j=1}^\infty e^{-j^\delta}(j+1)^k j^{2M}.\end{aligned}$$

To continue the estimate above, we use that $j+1 \leq 2j$ and choose $r \in \mathbb{N}$ such that $r\delta - 1 < k + 2M \leq (r+1)\delta - 1$, hence for $\widetilde{C} = (C_d - 1)2^{2(M+k)}$

$$\begin{aligned}(C_d-1)2^{2M}\sum_{j=1}^\infty e^{-j^\delta}(j+1)^k j^{2M} &\leq \widetilde{C}\sum_{j=1}^\infty e^{-j^\delta} j^{k+2M}\\ &\leq \frac{\widetilde{C}}{\delta}\sum_{j=1}^\infty e^{-j^\delta}\ j^{r\delta}\ \delta\ j^{\delta-1}.\end{aligned}$$

By the transformation formula, all we need to do is to show that

$$\int_1^\infty e^{-t}t^r \,\mathrm{d}t < \infty,$$

but this is clear and the proof complete. ∎

Lemma 2.1.2. *For any $r>0$ there exists $C = C(r) > 0$ such that for any entire function f and $z \in \mathbb{C}$*

$$(a)\ |f(z)|^2 e^{-2\phi(z)} \le C \int_{B(z,r\rho(z))} |f(\zeta)|^2 e^{-2\phi(\zeta)}\rho^{-2}(\zeta)d\lambda(\zeta),$$

$$(b)\ \left|\nabla\big(|f|e^{-\phi}\big)(z)\right|^2 \le C\rho^{-2}(z) \int_{B(z,r\rho(z))} |f(\zeta)|^2 e^{-2\phi(\zeta)}\rho^{-2}(\zeta)d\lambda(\zeta).$$

The following is a construction of a "peak" function as described in the appendix of [MMOC03].

For any ϕ, subharmonic with $\Delta\phi \,\mathrm{d}\lambda$ doubling, we will point out how to show the existence of positive C and δ, such that for all $\eta \in \mathbb{C}$ there is a holomorphic P_η with $P_\eta(\eta) = 1$, and

$$|P_\eta(z)| \le Ce^{\phi(z)-\phi(\eta)} \min\left\{1, \left(\frac{\rho(\eta)}{|z-\eta|}\right)^\delta\right\}. \tag{2.1}$$

Suppose there exists a holomorphic h_η with $h_\eta(\eta) = 0$, $h'_\eta(\eta) = 1$ and

$$|h_\eta(z)| \le Ce^{\phi(z)-\phi(\eta)}\frac{\rho^2(\eta)}{\rho(z)} \text{ for } z \in \mathbb{C},$$

then we set

$$P_\eta(z) = \frac{h_\eta(z)}{|z-\eta|}$$

and use Lemma 1.2.6 - the first half of its proof, to gain for $z \notin B(\eta, \rho(\eta))$:

$$|P_\eta(z)| \le Ce^{\phi(z)-\phi(\eta)}\frac{\rho(\eta)}{|z-\eta|}\left(\frac{|z-\eta|}{\rho(\eta)}\right)^\epsilon = Ce^{\phi(z)-\phi(\eta)}\left(\frac{\rho(\eta)}{|z-\eta|}\right)^{1-\epsilon}.$$

So one focuses on obtaining such a holomorphic function by defining

$$F(z) = (z-\eta)\chi\left(\frac{|z-\eta|^2}{\rho^2(\eta)}\right)e^{H_\eta(z)},$$

where H_η is holomorphic and $\mathrm{Re}(H_\eta) = \mathscr{H}_\eta$, with the notation of Lemma 1.2.12, and χ is a smooth cut-off function with $\chi \equiv 1$ on $B(0,1)$, $\chi \equiv 0$ on $\mathbb{C} \setminus B(0,2)$ and $|\chi'|$ bounded.

Hence by construction, Lemma 1.2.12 and Lemma 1.2.6, we have

$$\rho(z)|F(z)|e^{-\phi(z)} \leq C\rho^2(\eta)e^{-\phi(\eta)}.$$

Thus h_η will simply be $F - u$, where u is as in the next lemma.

Lemma 2.1.3. *There exists a solution to $\bar{\partial}u = \bar{\partial}F$ such that $u(\eta) = 0$ and $\|u\|_{\mathcal{F}^\infty_\phi} \leq C\rho^2(\eta)e^{-\phi(\eta)}$.*

Proof. By Hörmanders L^2 exitence theory, see equation (A.10), there exists for every well behaved subharmonic ψ in $\mathbb{C}$ a solution to $\bar{\partial}u = \bar{\partial}F$ such that

$$\int_{\mathbb{C}} |u|^2 e^{-2\psi} \leq C \int_{\mathbb{C}} |\bar{\partial}F|^2 \frac{e^{-2\psi}}{\Delta\psi}.$$

We set $\psi = \phi + 2v$, where

$$v(z) = \log(|z-\eta|) - \frac{1}{8\pi}\int_{B_s} \log(|z-\eta|)\Delta\phi(\zeta)\ \mathrm{d}\lambda(\zeta)$$

and $s \in \mathbb{R}_{>0}$ is such that $\mu(B_s) := \mu(B(\eta, s\rho(\eta))) = 8\pi$ for $\mu = \Delta\phi\ \mathrm{d}\lambda$. This s depends on the doubling constant only, as one could define the radius function $\rho_s(\eta) = s(\eta)\rho(\eta)$ and obtain the same results as for ρ, especially Lemma 1.2.8, thus ρ_s/ρ is bounded. v is bounded from above and on the annulus where $\bar{\partial}F$ is supported, it is also bounded from below.

By calculation and Theorem 1.2.13, it follows that

$$\Delta\psi \geq \Delta\phi - \frac{4\pi}{\mu(B_s)}\Delta\phi = \frac{1}{2}\Delta\phi = \frac{C}{\rho^2}.$$

We have $|\bar{\partial}F| \leq Ce^{\mathscr{H}_\eta}$ and deduce from Hörmander's estimate and Lemma 1.2.12 that

$$\|u\|_{\mathcal{F}^2_\phi} \leq \int_{\mathbb{C}} |u|^2 e^{-2\psi} \leq C\int e^{\mathscr{H}_\eta} e^{-2\psi}\rho^2 \leq C'\rho^4(\eta)e^{-2\phi(\eta)},$$

but also

$$e^{-2\psi} = \frac{C}{|z-\eta|^4} \text{ for } |z-\eta| \leq \epsilon\rho(\eta),$$

thus $u(\eta) = \partial u(\eta) = 0$. To see that u is in $\mathcal{F}^\infty_\phi$, it takes a few steps further, but we will only note that in the end it is an application of Lemma 4.5.1. ■

2.2 Pointwise Estimates on the Diagonal

We prove these estimates also near the diagonal, i.e. in a neighbourhood of the set $\{(z,z) \in \mathbb{C}^2\}$; also we drop the ϕ in $\mathcal{B}_\phi$ and write simply $\mathcal{B}$.

Proposition 2.2.1. *There exists a $C > 0$, such that*

$$\frac{1}{C}\frac{e^{2\phi(z)}}{\rho^2(z)} \leq \mathcal{B}(z,z) \leq C\frac{e^{2\phi(z)}}{\rho^2(z)} \qquad \textit{for } z \in \mathbb{C}. \tag{2.2}$$

Proof. By the previous subsection we can construct a holomorphic peak function P'_η with $P'_\eta(\eta) = 1$ for a fixed $\eta \in \mathbb{C}$ satisfying (2.1) for any suitable subharmonic function. We will apply this to $m^{-1}\phi$ for $m \in \mathbb{N}_{>1}$ and take the m-th power of (2.1) to obtain for $P_\eta := \left(P'_\eta\right)^m$ the relation[1]

$$|P_\eta(z)| \leq Ce^{\phi(z)-\phi(\eta)} \min\left\{1, \left(\frac{\rho(\eta)}{|z-\eta|}\right)^m\right\}.$$

We define an entire function by

$$f_\eta(z) = C_0 \frac{e^{\phi(\eta)}}{\rho(\eta)} P_\eta(z),$$

where C_0 will be chosen soon. Then by the inequality for P_η and integration in polar coordinates we get for some constants C, C'

$$\begin{aligned} \int_{\mathbb{C}} |f_\eta(z)|^2 e^{-2\phi(z)} \,\mathrm{d}\lambda(z) \leq&\ CC_0^2 + \int_{\mathbb{C}\setminus B(\eta,\rho(\eta))} \left(\frac{\rho(\eta)}{|z-\eta|}\right)^{2m} \frac{1}{\rho^2(\eta)} \,\mathrm{d}\lambda(z) \\ =&\ C'C_0^2 \leq 1 \end{aligned}$$

for C_0 small enough. Now $f_\eta(\eta) = C_0 e^{\phi(\eta)}\rho(\eta)^{-1}$, thus for some constant and the norm being expressable in terms of the inner product, we have by the reproducing property

$$\mathcal{B}(z,z) = \sup\left\{|f(z)|^2 \;:\; f \in \mathcal{F}^2_\phi,\ \|f\|^2_{\mathcal{F}^2_\phi} \leq 1\right\} \geq C\frac{e^{2\phi(z)}}{\rho^2(z)}.$$

Here one also sees that this expression is never zero. For the other direction, use Lemma 2.1.2 (a) on[2] $\mathcal{B}_z$ as well as the comparability of ρ on intersecting balls,

[1] Note that $\rho(\Delta\phi \,\mathrm{d}\lambda) < \rho(m^{-1}\Delta\phi \,\mathrm{d}\lambda)$ and that they are comparable by Lemma 1.2.4.

[2] The definition of $\mathcal{B}_z$ can be found at the very beginning of this chapter.

adjust this Lemma's C to C' and increase the domain of integration respectively, to obtain

$$\begin{aligned} |\mathcal{B}_z(\zeta)|^2 e^{-2\phi(\zeta)} &\leq C \int_{B(\zeta,\rho(\zeta))} |\mathcal{B}_z(w)|^2 e^{-2\phi(w)} \rho^{-2}(w) \, \mathrm{d}\lambda(w) \\ &\leq C' \int_{\mathbb{C}} |\mathcal{B}_z(w)|^2 e^{-2\phi(w)} \rho^{-2}(\zeta) \, \mathrm{d}\lambda(w) \\ &= C' \mathcal{B}(z,z) \rho^{-2}(\zeta). \end{aligned} \tag{2.3}$$

Finally we set $\zeta = z$, divide and are done. ∎

Proposition 2.2.2. *Let $\mathcal{B}(z,\zeta)$ be the Bergman kernel for $\mathcal{F}^2_\phi$, then there exists a $C = C(C_d) > 0$ and a $C' = C'(C_d) > 0$, such that for any $z, \zeta \in \mathbb{C}$*

$$|\mathcal{B}(z,\zeta)| \leq C \frac{e^{\phi(z)+\phi(\zeta)}}{\rho(z)\rho(\zeta)}. \tag{2.4}$$

Moreover, there is an $r = r(z) > 0$, such that for all $\zeta \in B(z, r\rho(z))$,

$$|\mathcal{B}(z,\zeta)| C' \geq \frac{e^{\phi(z)+\phi(\zeta)}}{\rho(z)\rho(\zeta)}. \tag{2.5}$$

Proof. We fix $z \in \mathbb{C}$ and use (2.3) and Proposition 2.2.1 to see (2.4). Also, by Proposition 2.2.1 we have (2.5) for $z = \zeta$. Otherwise the mean value property for functions on $\mathbb{R}$, using precomposition with $t \mapsto (1-t)z + t\zeta$ with $\zeta \in B(z, r\rho(z))$ and applying Lemma 2.1.2 (b) as well as the reproducing property of the Bergman kernel on itself, comparability of $\rho(z)$ and $\rho(\zeta)$ and (2.2), gives

$$\left| |\mathcal{B}_z(\zeta)|^2 e^{-\phi(\zeta)} - |\mathcal{B}_z(z)|^2 e^{-\phi(z)} \right| \leq C(z) e^{\phi(z)} |z-\zeta| \rho^{-3}(z) \leq C(z) r e^{\phi(z)} \rho^{-2}(z)$$

after taking square roots; for r small enough, (2.5) still holds. ∎

Though ρ can go to zero, there is a limit on how fast this can happen and we still obtain a nice neighbourhood of the diagonal where (2.5) holds.

2.3 Pointwise Estimates off the Diagonal

The following is [MOC09, Theorem 1.1].

Theorem 2.3.1. *Let ϕ be a subharmonic function whose Laplacian is doubling, i.e. $\mu = \Delta\phi \, \mathrm{d}\lambda$, and $\mathcal{B}(z,\zeta)$ be the Bergman kernel for $\mathcal{F}^2_\phi$. There exist positive constants $C = C(C_d)$ and $\varepsilon = \varepsilon(C_d)$ such that for any $z, \zeta \in \mathbb{C}$*

$$|\mathcal{B}(z,\zeta)| \leq C \frac{1}{\rho(z)\rho(\zeta)} \frac{e^{\phi(z)+\phi(\zeta)}}{\exp\left(\frac{|z-\zeta|}{\rho(z)}\right)^{\varepsilon}}. \tag{2.6}$$

A symmetric version of Theorem 2.3.1 can be obtained using Lemma 1.2.10 or Lemma 1.2.7 depending on which of the following is true

$$|z-\zeta| \geq C \max\{\rho(z), \rho(\zeta)\} \quad \text{or} \quad |z-\zeta| < C \max\{\rho(z), \rho(\zeta)\};$$

thus, with a possibly different ε but still only depending on C_d, we get

$$|\mathcal{B}(z,\zeta)| \leq C \frac{1}{\rho(z)\rho(\zeta)} \frac{e^{\phi(z)+\phi(\zeta)}}{\exp\big(d(z,\zeta)^{\varepsilon}\big)}. \tag{2.7}$$

The proof is seperated into several steps and we will have to establish a few lemmata until we reach our goal. First, we introduce a regularized version of ϕ with an exponent which will be chosen later on, hence, we fix $\zeta \in \mathbb{C}$ and let

$$\varphi_\epsilon(z) = \varphi_{\epsilon,\zeta}(z) = \left(\frac{(z-\zeta)(\bar{z}-\bar{\zeta})}{\rho^2(\zeta)}\right)^{\epsilon/2} = \left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^{\epsilon}.$$

This function is subharmonic as can be read off of its Laplacian, see Proposition A.1.9, and satisfies

$$\frac{\partial \varphi_\epsilon}{\partial z}(z) = \frac{\epsilon |z-\zeta|^{\epsilon-2}(\bar{z}-\bar{\zeta})}{2\rho^{\epsilon}(\zeta)} \quad \text{and} \quad \Delta\varphi_\epsilon(z) = 4\frac{\partial^2 \varphi_\epsilon}{\partial \bar{z}\partial z}(z) = \frac{\epsilon^2 |z-\zeta|^{\epsilon-2}}{\rho^{\epsilon}(\zeta)},$$

thus

$$\Delta\varphi_{2\epsilon}(z) = 4\left|\frac{\partial \varphi_\epsilon}{\partial z}(z)\right|^2.$$

The Laplacian of φ is not bounded above, so we convolute with the characteristic function of $B(0,\rho(\zeta))$, denoted by $\chi_{\rho(\zeta)} = \chi_{B(0,\rho(\zeta))}$ where we put emphasize on the fact that the center is zero, and define

$$\psi_\epsilon = \frac{1}{\pi\rho^2(\zeta)} \chi_{\rho(\zeta)} * \varphi_\epsilon.$$

With the measure $(\pi\rho^2(\zeta))^{-1}\, \mathrm{d}\lambda$, we can apply Jensens's inequality to see

$$\left|\frac{\partial \psi_\epsilon}{\partial z}\right|^2 \leq \frac{1}{\pi\rho^2(\zeta)} \chi_{\rho(\zeta)} * \left|\frac{\partial \varphi_\epsilon}{\partial z}\right|^2,$$

and plain calculation gives

$$\Delta\psi_{2\epsilon}(z) = \left(\frac{1}{\pi\rho^2(\zeta)} \chi_{\rho(\zeta)} * \Delta\varphi_{2\epsilon}\right)(z) = \left(\frac{1}{\pi\rho^2(\zeta)} \chi_{\rho(\zeta)} * 4\left|\frac{\partial \varphi_\epsilon}{\partial z}\right|^2\right)(z).$$

The next is [MOC09, Lemma 3.2].

Lemma 2.3.2. *For any $C > 0$, there exists an $\epsilon_0 \in (0,1)$ which depends on C and the doubling constant of $\Delta\phi$ $\mathrm{d}\lambda$ only, such that for $\epsilon \in (0,\epsilon_0]$ and $\rho = \rho_{\Delta\phi}$, we have the estimate:*

$$\begin{aligned}\Phi_\epsilon(w) := \frac{\Delta\psi_{2\epsilon}}{4}(w) &= \frac{\epsilon^2}{\pi\rho^2(\zeta)}\int_{B(0,\rho(\zeta))}\left|\frac{|w-z-\zeta|^{\epsilon-2}(\bar{w}-\bar{z}-\bar{\zeta})}{2\rho^\epsilon(\zeta)}\right|^2 \mathrm{d}\lambda(z)\\ &= \frac{\epsilon^2}{\pi\rho^2(\zeta)}\int_{B(\zeta,\rho(\zeta))}\frac{|w-z|^{2\epsilon-2}}{4\rho^{2\epsilon}(\zeta)}\,\mathrm{d}\lambda(z) \le \frac{C}{\rho^2(w)}.\end{aligned}$$

Proof. **Case 1.** Suppose that $B(w,\rho(w)) \cap B(\zeta, 2\rho(\zeta)) \neq \emptyset$. The function Φ_ϵ has a maximum in $w = \zeta$ as $2\epsilon - 2 \le 0$, hence it suffices to show that $\Phi_\epsilon(\zeta) \le C\rho^{-2}(w)$. By polar coordinates,

$$\begin{aligned}\Phi_\epsilon(\zeta) &= \frac{1}{\pi\rho^2(\zeta)}\int_{B(0,\rho(\zeta))}\frac{\epsilon^2|\zeta-z-\zeta|^{2\epsilon-2}}{4\rho^{2\epsilon}(\zeta)}\,\mathrm{d}\lambda(z)\\ &= \frac{\epsilon^2}{4\pi\rho^{2\epsilon+2}(\zeta)}\int_{B(0,\rho(\zeta))}|z|^{2\epsilon-2}\,\mathrm{d}\lambda(z)\\ &= \frac{\epsilon^2}{4\pi\rho^{2\epsilon+2}(\zeta)}\int_0^{\rho(\zeta)}\int_0^{2\pi} t^{2\epsilon-1}\,\mathrm{d}\theta\mathrm{d}t = \frac{\epsilon}{4\rho^2(\zeta)},\end{aligned}$$

so we need

$$\frac{\epsilon}{4} \le C\left(\frac{\rho(\zeta)}{\rho(w)}\right)^2,$$

and this property holds for $0 < \epsilon \le \epsilon_0$ by comparability, see the lines after Lemma 1.2.7.

Case 2. Suppose that $B(w,\rho(w)) \cap B(\zeta, 2\rho(\zeta)) = \emptyset$. Then by translation invariance of Lebesgue measure it follows that

$$\begin{aligned}\Phi_\epsilon(w) &= \frac{1}{\pi\rho^2(\zeta)}\int_{B(0,\rho(\zeta))}\frac{\epsilon^2|w-z-\zeta|^{2\epsilon-2}}{4\rho^{2\epsilon}(\zeta)}\,\mathrm{d}\lambda(z)\\ &= \frac{\epsilon^2}{4\pi\rho^{2\epsilon+2}(\zeta)}\int_{B(\zeta,\rho(\zeta))}|w-u|^{2\epsilon-2}\,\mathrm{d}\lambda(u)\\ &\le \frac{C'\epsilon^2}{\rho^{2\epsilon}(\zeta)|w-\zeta|^{2-2\epsilon}}.\end{aligned}$$

So we need for some positive constants C'' and $\widetilde{C}$

$$\frac{\epsilon^2|w-\zeta|^{2\epsilon-2}}{\rho^{2\epsilon}(\zeta)} \le C''\frac{1}{\rho^2(w)},$$

or equivalently

$$\frac{1}{\epsilon}\left(\frac{|w-\zeta|}{\rho(\zeta)}\right)^{1-\epsilon} \geq \widetilde{C}\frac{\rho(w)}{\rho(\zeta)};$$

which follows by the first half of the proof of Lemma 1.2.6. Finally, by the fact that $\epsilon^{-1} \to \infty$ as $\epsilon \to 0$, the last inequality is satisfied for all ϵ small enough. ■

Let $\widetilde{\phi} \in C^\infty(\mathbb{C})$ be the, by Theorem 1.2.13, regularized version of ϕ. The next is [MOC09, Lemma 3.1].

Lemma 2.3.3. *There exist constants $\epsilon_0, C_1, C_2 \in (0,1)$ depending on the doubling constant of $\Delta\phi \,\mathrm{d}\lambda$ only, such that for every $\epsilon \in (0, \epsilon_0]$*

$$\left|\frac{\psi_\epsilon}{\partial z}\right|^2(z) \leq C_1 \Delta\widetilde{\phi}(z) \quad \textit{and} \quad \Delta\psi_\epsilon(z) \leq C_2\Delta\widetilde{\phi}(z).$$

Proof. By Theorem 1.2.13 there exists a $C' > 0$ such that

$$\frac{1}{\rho^2(z)} \leq C'\Delta\widetilde{\phi}(z).$$

Let $\epsilon_0 > 0$ be the one given by Lemma 2.3.2 for $C \in (0, 1/(4C'))$. For $\epsilon \in (0, \epsilon_0]$ we have

$$\left|\frac{\psi_\epsilon}{\partial z}\right|^2(z) \leq \Phi_\epsilon(z) \leq \frac{C}{\rho^2(z)} \leq CC'\Delta\widetilde{\phi}(z)$$

and

$$\Delta\psi_\epsilon(z) \leq 4\Phi_{\epsilon/2}(z) \leq \frac{4C}{\rho^2(z)} \leq 4CC'\Delta\widetilde{\phi}(z).$$

An obvious choice of C_1 and C_2 finishes the proof. ■

From now on we will fix an $\epsilon > 0$ as proposed by Lemma 2.3.3 for some suitable constants.

Lemma 2.3.4. *Set $\varrho = \widetilde{\phi} - \psi$, then for positive, bounded functions C', $\widetilde{C}$:*

$$\Delta\varrho = C'\Delta\widetilde{\phi} \quad \textit{and} \quad \rho_\varrho^2 = \widetilde{C}\rho_{\widetilde{\phi}}^2.$$

Proof. As ψ is subharmonic: $\Delta\widetilde{\phi} \geq \Delta\widetilde{\phi} - \Delta\psi = \Delta\varrho$. The other inequality follows from Lemma 2.3.3 since $\Delta\varrho \geq (1 - C_2)\Delta\widetilde{\phi}$, for C_2 as in the Lemma. The relation of the radius functions is Theorem 1.2.13. ■

Proof of Theorem 2.3.1. Let $z, \zeta \in \mathbb{C}$ be fixed points such that $B(z,\rho(z)) \cap B(\zeta,\rho(\zeta)) = \emptyset$. Let χ be a smooth cut-off function with values in $[0,1]$ and support in $B(\zeta,\rho(\zeta))$ such that $\chi \equiv 1$ on $B(\zeta, 1/2\rho(\zeta))$ and

$$|\bar{\partial}\chi|^2 \leq C\frac{\chi}{\rho^2(\zeta)} \quad \text{for } z \in B(\zeta,\rho(\zeta)).$$

We have that

$$\begin{aligned}
|\mathcal{B}_z(\zeta)|^2 e^{-2\phi(\zeta)} &\leq \frac{C'}{\rho^2(\zeta)} \int_{B(\zeta,1/2\rho(\zeta))} |\mathcal{B}_z(w)|^2 e^{-2\phi(w)} \, \mathrm{d}\lambda(w) \\
&= \frac{C'}{\rho^2(\zeta)} \int_{B(\zeta,1/2\rho(\zeta))} \chi(w)|\mathcal{B}_z(w)|^2 e^{-2\phi(w)} \, \mathrm{d}\lambda(w) \\
&\leq \frac{\widetilde{C}}{\rho^2(\zeta)} \|\mathcal{B}_z\|^2_{L^2(\chi e^{-2\phi})}.
\end{aligned}$$

But $\|\mathcal{B}_z\|^2_{L^2(\chi e^{-2\phi})} = \sup_f |\langle f, \mathcal{B}_z\rangle_{L^2(\chi e^{-2\phi})}|$ where the supremum runs over all holomorphic f in $B(\zeta,\rho(\zeta))$ with

$$\int |f|^2 e^{-2\phi}\chi \, \mathrm{d}\lambda = 1.$$

As $f\chi \in L^2(e^{-2\phi})$ one has

$$\langle f, \mathcal{B}_z\rangle_{L^2(\chi e^{-2\phi})} = \mathcal{B}(f\chi)(z),$$

where $\mathcal{B} = \mathcal{B}_\phi$ stands for the Bergman projection too, as explained at the beginning of this chapter. Now

$$u = f\chi - \mathcal{B}(f\chi)$$

is the canonical[3] solution of

$$\bar{\partial}u = \bar{\partial}(f\chi) = f\bar{\partial}\chi$$

in $L^2(e^{-2\phi})$ and, since $\chi(z) = 0$, one has

$$|\langle f, \mathcal{B}_z\rangle_{L^2(\chi e^{-2\phi})}| = |\mathcal{B}(f\chi)(z)| = |u(z)|.$$

As $B(z,\rho_\phi(z)) \cap B(\zeta,\rho_\phi(\zeta)) = \emptyset$, $f\chi$ vanishes outside of $B(\zeta,\rho_\phi(\zeta))$ and therefore u is holomorphic in $B(z, r\rho_\varrho(z))$ for some $r > 0$, where we used $\rho_\varrho = \widetilde{C}\rho_{\tilde{\phi}} = C\rho_\phi$

[3]Thus the solution of minimal distance to the Fock space $\mathcal{F}^2_\phi$.

by Lemma 2.3.4. By Lemma 2.1.2 (a)

$$
\begin{aligned}
|u(z)|^2 e^{-2\phi(z)+2\psi(z)} &\leq C|u(z)|^2 e^{-2\widetilde{\phi}(z)+2\psi(z)} \;=\; C|u(z)|^2 e^{-2\varrho(z)} \\
&= C' \int_{B(z,r\rho_\varrho(z))} \frac{|u(w)|^2 e^{-2\varrho(w)}}{\rho_\varrho^2(w)} \, \mathrm{d}\lambda(w) \\
&\leq \frac{\widetilde{C}}{\rho_\varrho^2(z)} \int_{\mathbb{C}} |u(w)|^2 e^{-2\varrho(w)} \, \mathrm{d}\lambda(w) \\
&\leq \frac{C''}{\rho_\phi^2(z)} \int_{\mathbb{C}} |u(w)|^2 e^{-2\varrho(w)} \, \mathrm{d}\lambda(w).
\end{aligned}
\tag{2.8}
$$

We estimate this last integral with the classical Hörmander theorem, see equation (A.10) and note that $\Delta\phi \, \mathrm{d}\lambda$ is supposed to be a non-trivial doubling measure and hence cannot be zero on an open set.

Theorem 2.3.5. *Let $\Omega \subset \mathbb{C}$ be a domain and $\phi \in C^2(\Omega)$ be such that $\Delta\phi \geq 0$. For any $f \in L^2_{loc}(\Omega)$ there exist a solution u to $\bar{\partial} u = f$ such that*

$$\int |u|^2 e^{-2\phi} \, \mathrm{d}\lambda \leq \int \frac{|f|^2}{\Delta\phi} e^{-2\phi} \, \mathrm{d}\lambda.$$

And also a variant due to Berndtsson [Ber99, Lemma 2.2].

Theorem 2.3.6. *If*

$$\left| \frac{\partial \psi}{\partial w} \right|^2 \leq C \Delta\widetilde{\phi}, \quad \textit{with } 0 < C < 1$$

and for any g one can find v such that $\bar{\partial} v = g$ with

$$\int |v|^2 e^{-2\phi-2\psi} \, \mathrm{d}\lambda \leq \int \frac{|g|^2}{\Delta\widetilde{\phi}} e^{-2\phi-2\psi} \, \mathrm{d}\lambda, \tag{2.9}$$

then for the canonical solution v_0 in $L^2(e^{-2\phi})$, one has

$$\int |v_0|^2 e^{-2\phi+2\psi} \, \mathrm{d}\lambda \leq C' \int \frac{|g|^2}{\Delta\widetilde{\phi}} e^{-2\phi+2\psi} \, \mathrm{d}\lambda,$$

where $C' = 6/(1-C)^2$.

We know that $\Delta(\widetilde{\phi} + \psi) \geq 0$, then applying Theorem 2.3.5 to $\bar{\partial}(f\chi)$, one has v such that $\bar{\partial} v = \bar{\partial}(f\chi)$ with

$$
\begin{aligned}
\int |v|^2 e^{-2\widetilde{\phi}-2\psi} \, \mathrm{d}\lambda &\leq \int \frac{|\bar{\partial} v|^2}{\Delta(\widetilde{\phi}+\psi)} e^{-2\widetilde{\phi}-2\psi} \, \mathrm{d}\lambda \\
&\leq \int \frac{|\bar{\partial} v|^2}{\Delta\widetilde{\phi}} e^{-2\widetilde{\phi}-2\psi} \, \mathrm{d}\lambda.
\end{aligned}
$$

As $|\phi - \widetilde{\phi}| \leq C$ by Theorem 1.2.13 (a), we have that (2.9) holds and by Theorem 2.3.6

$$\int |u|^2 e^{-2\phi+2\psi}\ \mathrm{d}\lambda \leq C \int \frac{|\bar{\partial} u|^2}{\Delta\widetilde{\phi}} e^{-2\phi+2\psi}\ \mathrm{d}\lambda.$$

Furthermore, $\Delta\widetilde{\phi} = \widetilde{C}\rho_\phi^{-2}$, so one can estimate (2.8) as

$$\begin{aligned}\frac{C'}{\rho_\phi^2(z)} \int_{\mathbb{C}} |u(w)|^2 e^{-2\varrho(w)}\ \mathrm{d}\lambda(w) &\leq \frac{C''}{\rho_\phi^2(z)} \int_{B(\zeta,\rho_\phi(\zeta))} \frac{|\bar{\partial}(f\chi)(w)|^2}{e^{2\varrho(w)}} \rho_\phi^2(w)\ \mathrm{d}\lambda(w) \\ &\leq \frac{\widetilde{C}'}{\rho_\phi^2(z)} \int_{B(\zeta,\rho_\phi(\zeta))} \frac{|f(w)|^2}{e^{2\varrho(w)}} \rho_\phi^2(w) \frac{\chi(w)}{\rho_\phi^2(\zeta)}\ \mathrm{d}\lambda(w).\end{aligned} \tag{2.10}$$

The function ψ is bounded from above in $B(\zeta, \rho_\phi(\zeta))$ by a constant depending on the doubling constant of $\Delta\phi\ \mathrm{d}\lambda$ only. For $w \in B(\zeta, \rho_\phi(\zeta))$ we have

$$\frac{1}{\pi\rho_\phi^2(\zeta)} \int_{\mathbb{C}} \chi_{\rho_\phi(\zeta))}(w-u)\varphi(u)\ \mathrm{d}\lambda(u) \leq \frac{1}{\pi\rho_\phi^2(\zeta)} \int_{B(\zeta,2\rho_\phi(\zeta))} \varphi(u)\ \mathrm{d}\lambda(u) \leq C''2^\epsilon.$$

Thus (2.10) can be estimated by

$$\int_{B(\zeta,\rho_\phi(\zeta))} \frac{\rho_\phi^2(w)|f(w)|^2}{\rho_\phi^2(\zeta)\rho_\phi^2(z)} \frac{\chi(w)}{e^{2\varrho(w)}}\ \mathrm{d}\lambda(w) \leq \int_{B(\zeta,\rho_\phi(\zeta))} \frac{C|f(w)|^2\chi(w)}{\rho_\phi^2(z)e^{2\phi(w)}}\ \mathrm{d}\lambda(w) = \frac{C'}{\rho_\phi^2(z)}.$$

Finally, we have

$$|\mathcal{B}(\zeta, z)|^2 \leq \frac{C''}{\rho_\phi^2(z)\rho_\phi^2(\zeta)} \frac{e^{2\phi(z)+2\phi(\zeta)}}{e^{2\psi(z)}}. \tag{2.11}$$

All that is left to do for us, is to get rid of the expression involving the ψ in (2.11) in the next Lemma, where we use φ as defined prior to Lemma 2.3.2.

Lemma 2.3.7. *If $B(\zeta, \rho_\phi(\zeta)) \cap B(w, \rho_\phi(w)) = \emptyset$, there exists a $C > 0$ such that*

$$|\psi(w) - \varphi(w)| \leq C.$$

Proof. Using subharmonicity we have

$$\psi(w) - \varphi(w) = \frac{1}{\rho^\epsilon(\zeta)} \left(\frac{1}{\pi\rho_\phi^2(\zeta)} \int_{B(\zeta,\rho_\phi(\zeta))} |w-u|^\epsilon\ \mathrm{d}\lambda(u) - |w-\zeta|^\epsilon \right) \geq 0.$$

On the other hand, if $|w - \zeta| \leq 2\rho(\zeta)$ we see

$$\psi(w) = \frac{1}{\pi\rho^{2+\epsilon}(\zeta)} \int_{B(\zeta,\rho_\phi(\zeta))} |w-u|^\epsilon\ \mathrm{d}\lambda(u) \leq 3^\epsilon$$

and therefore $0 \le \psi(w) - \varphi(w) \le 3^\epsilon$.

For $|w-\zeta| \ge 2\rho(\zeta)$ and writing $v(z) = |w-z|^\epsilon$ and $B = B(\zeta, \rho_\phi(\zeta))$ as abbreviations, we have

$$\begin{aligned}
&\psi(w) - \varphi(w) \\
&\quad = \frac{1}{\rho^\epsilon(\zeta)} \left(\frac{1}{\pi \rho_\phi^2(\zeta)} \int_B v(u) \, \mathrm{d}\lambda(u) - v(\zeta) \right) \\
&\quad = \frac{1}{2\pi\rho^\epsilon(\zeta)} \int_B \left(\log\left[\frac{\rho(\zeta)}{|u-\zeta|}\right] + \frac{1}{2}\left[\left[\frac{|u-\zeta|}{\rho(\zeta)}\right]^2 - 1\right]\right) \Delta v(u) \, \mathrm{d}\lambda(u) \\
&\quad \le \frac{1}{2\pi\rho^\epsilon(\zeta)} \int_B \log\left(\frac{\rho(\zeta)}{|u-\zeta|}\right) \Delta v(u) \, \mathrm{d}\lambda(u),
\end{aligned}$$

for the second equality see [BOC97, Sect. 3.3.]. By Lemma 1.2.11 the last integral is smaller than

$$\frac{1}{2\pi\rho^\epsilon(\zeta)} \int_{B(\zeta,\rho_\phi(\zeta))} \Delta v(u) \, \mathrm{d}\lambda(u)$$

times a constant C, depending on the doubling[4] constant $C_d(v)$ of $\Delta v \, \mathrm{d}\lambda$ only, where $C_d(v)$ depends on ϵ only. For any $u \in B(\zeta, \rho_\phi(\zeta))$ one deduces from $|w-\zeta| \ge 2\rho(\zeta)$ that $|w-u| \ge \rho(\zeta)$, and

$$\int_{B(\zeta,\rho_\phi(\zeta))} \Delta v(u) \, \mathrm{d}\lambda(u) \le \left(\frac{\epsilon}{2}\right)^2 \frac{1}{\rho^{2-\epsilon}(\zeta)} \lambda\big(B(\zeta, \rho_\phi(\zeta))\big);$$

hence

$$\psi(w) - \varphi(w) \le C\epsilon^2.$$

■

This finishes the exposition of the proof of Theorem 2.3.1. ■

[4] $\Delta v \, \mathrm{d}\lambda$ is a doubling measure as it is a polynom of degree one, see [Ste93], page 40 (8.7).

Chapter 3

Prelimineries II

It would be utopic to collect all of manifold theory needed for the presentation of the papers under consideration, so we give a few details which this author regards very helpful and that are not found in standard books on Complex Analysis. The reader who wants to learn or refresh his knowledge about Differential Geometry may follow this author's suggestion and consult the truly masterfully written book [Lee09] - if the author has had this book from the beginning on, he would had saved a lot of time, nerves and tears.

3.1 Tensor Products

This section is an important one. Tensor products crop up often in higher mathematics but are banned in undergraduate courses; one rather uses multilinear algebra to get things settled. We will follow the first chapter of [Fed96], which is an extraordinary book too, though quite hard to tackle.

This author claims that Cartesian products $V_1 \times \ldots \times V_k$ of some finite dimensional vector spaces V_j are peculiar to the study of linear maps whereas tensor products $V_1 \otimes \ldots \otimes V_k$ are peculiar to the study of k-linear maps by means of linear maps. One thus can also understand higher order terms of Taylor expansions by linear maps. We assume throughout $v_j \in V_j$.

A function f which maps a Cartesian product of k vectorspaces V_j into the vectorspace W, in symbols

$$f : V_1 \times \ldots \times V_k \to W$$

is said to be **k-linear** if and only if for any k-tuple $(v_1, \ldots, v_k)$ in $V_1 \times \ldots \times V_k$ and any j the function given by

$$\begin{array}{rcl} V_j & \to & W \\ x & \mapsto & f(v_1, \ldots, v_{j-1}, x, v_{j+1}, \ldots, v_k) \end{array}$$

is linear.

Definition 3.1.1. *The tensor product of the vectorspaces* $V_1, \ldots, V_k$ *is a vectorspace denoted as*

$$V_1 \otimes \ldots \otimes V_k$$

with an, up to isomorphisms, unique k-linear map

$$\begin{array}{rcl} \mathscr{M} : V_1 \times \ldots \times V_k & \to & V_1 \otimes \ldots \otimes V_k \\ (v_1, \ldots, v_k) & \mapsto & v_1 \otimes \ldots \otimes v_k \end{array}$$

which are both jointly characterized by the property that for any k-linear map $\mathscr{L} : V_1 \times \ldots \times V_k \to W$*, with* W *an arbitrary vectorspace, there exists a unique linear map, say* $\mathscr{G}_{\mathscr{L}}$*, of* $V_1 \otimes \ldots \otimes V_k$ *into* W *such that* $\mathscr{L} = \mathscr{G}_{\mathscr{L}} \circ \mathscr{M}$*. Thus the following diagram commutes:*

$$\begin{tikzcd} & V_1 \otimes \ldots \otimes V_k \arrow[dr, "\mathscr{G}_{\mathscr{L}}"] & \\ V_1 \times \ldots \times V_k \arrow[ur, "\mathscr{M}"] \arrow[rr, "\mathscr{L}"] & & W \end{tikzcd}$$

This diagram is the so called **universal property** for tensor products; uniqueness, up to linear isomorphisms, can be read off; existence is the hard part. To **construct a tensor product**, let us consider the vectorspace F, consisting of those real valued functions on $V_1 \times \ldots \times V_k$ with finite supports.

We define the injective map

$$\phi : V_1 \times \ldots \times V_k \to F,$$

where we write $\phi_{v_1,\ldots,v_k}$ for the function $\phi(v_1, \ldots, v_k)$, by

$$\phi_{v_1,\ldots,v_k}(w_1, \ldots, w_k) = \begin{cases} 1 & \text{if } w_j = v_j, \\ 0 & \text{else.} \end{cases}$$

Let G be the vectorsubspace of the functionspace F, generated by all elements of the two types:

$$\phi(v_1, \ldots, v_{j-1}, x, v_{j+1}, \ldots, v_k) + \phi(v_1, \ldots, v_{j-1}, y, v_{j+1}, \ldots, v_k) -$$

$$-\phi(v_1, \ldots, v_{j-1}, x + y, v_{j+1}, \ldots, v_k)$$

and

$$\phi(v_1, \ldots, v_{j-1}, cv_j, v_{j+1}, \ldots, v_k) - c\phi(v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_k)$$

with $c \in \mathbb{R}$. It is noteworthy that $\mathbb{R}$ could be replaced by any other finitely generated free module, more on this will follow in the next section.

Let $\psi_1(j, x, y)$ and $\psi_2(j, c, x)$ denote the two expressions above respectively, with the dependence on the v's surpressed. By plugging in and a bit arithmetics, one obtains for $a \in \mathbb{R}$ the identities

$$a\psi_2(j, c, x) = \psi_2(j, ac, x) - \psi_2(j, a, cx),$$
$$a\psi_1(j, x, y) = \psi_1(j, ax, ay) - \psi_2(j, a, x) - \psi_2(j, a, y) + \psi_2(j, a, x + y).$$

Lemma 3.1.2. *The quotient space F/G is a model for $V_1 \otimes \ldots \otimes V_k$ with $\mathscr{M}$ being the composition of the canonical embedding $\pi : F \to F/G$ and ϕ. The vectors $e^1_{j_1} \otimes \ldots \otimes e^k_{j_k}$ form a basis of F/G.*

Proof. For the case $k = 2$ only, as it works exactly the same for larger products. Let[1] $a = \sum_{s=1}^{m} \phi_{\mathfrak{v}_s} \in F$ with $\mathfrak{v}_s \in V_1 \times V_2$, then

$$[a] = \left\{ \sum_{s=1}^{m} \phi_{\mathfrak{v}_s} + g \ : \ g \in G \right\} \in F/G.$$

We claim that $\pi \circ \phi$ is n-linear, thus

$$\pi(\phi_{ax,v+w}) = [\phi_{ax,v+w}] = a[\phi_{x,v}] + a[\phi_{x,w}]$$

for $a \in \mathbb{R}$, $x \in V_1$ and $v, w \in V_2$; but this is the very way equivalence is defined, for, with $z = v + w$ and $(-\phi_{ax,z} + a\phi_{x,z}) \in G$ we have:

$$[\phi_{ax,z}] = [\phi_{ax,z} - \phi_{ax,z} + a\phi_{x,z}] = a[\phi_{x,z}]$$

and analogously[2] for the sum. A basis is obviously given by elements of the form

$$\left[\phi_{e^1_j, e^2_s}\right] =: e^1_j \otimes e^2_s.$$

Given a bilinear map $\mathscr{L} : V_1 \times V_2 \to W$, with W a vectorspace and let e^1_j and e^2_s denote basisvectors of V_1 and V_2 respectively. If $v = \sum_{j=1}^{n_1} a_j e^1_j \in V_1$ and $w = \sum_{s=1}^{n_2} b_s e^2_s \in V_2$, then $\mathscr{L}(v, w) = \sum_{j=1}^{n_1} \sum_{s=1}^{n_2} a_j b_s \mathscr{L}(e^1_j, e^2_s)$. Hence one sets $\mathscr{G}_{\mathscr{L}}(e^1_j \otimes e^2_s) = \mathscr{L}(e^1_j, e^2_s)$ and sees uniqueness. ■

Corollary 3.1.3. *We have* $\dim(V_1 \otimes \ldots \otimes V_k) = \prod_{j=1}^{k} \dim(V_j)$.

[1] For simplicity we assume the element to have range $\{0, 1\}$ only, otherwise we use the identities on ψ_1 and ψ_2 in addition.

[2] Since G is a vector space and by the disjointness of equivalence classes, we may pick out a particular representative to see that the equivalence class of a sum is the sum of the equivalence classes.

The dimension of a tensor product actually might be small, regard for instance[3] $V_1, \ldots, V_k = \mathbb{R}$. A basis argument proves the next, where we use as usual product before sum.

Corollary 3.1.4. *If $V = P \oplus Q$, then $V \otimes W = P \otimes W \oplus Q \otimes W$.*

Since scalar multiplication is a bilinear map from $\mathbb{R} \times V$ into V, we can and will interpret $\mathbb{R} \otimes V$ as V through the isomorphism $c \otimes v \mapsto cv$.

3.1.1 Graded Algebra

A **graded Algebra** is formaly a pair (A, ν) where A is a vectorspace with a specified direct sum decomposition $A = \oplus_{j=0}^{\infty} A_j$ and a bilinear function (associative mulitiplication) $\nu : A \times A \to A$ such that $\nu(A_m \times A_n) \subset A_{m+n}$ and an isomorphism $\mathbb{R} \simeq A_0$ which maps 1 onto an unit element of the ring A.

Instead of $\nu(x, y)$ we will simply write xy, for $x \in A_m$ and $y \in A_n$; which will be anti-commutative in most cases, hence $xy = (-1)^{mn} yx$.

If A and B are graded algebras, then the graded tensor product

$$A \otimes B = \bigoplus_{j=0}^{\infty} \bigoplus_{m+n=j} A_m \otimes B_n$$

can be made a graded algebra with either of the following two standard definitions of multiplication:

1. $(a \otimes b_m) \cdot (c_k \otimes d) = (ac_k) \otimes (b_m d) \ldots$ commutative product,
2. $(a \otimes b_m) \cdot (c_k \otimes d) = (-1)^{mk} (ac_k) \otimes (b_m d) \ldots$ anti-commutative product,

where $a, c \in A$ and $b, d \in B$, with $b = \sum b_j$ and $c = \sum c_j$ such that $b_j \in B_j$ and $c_j \in A_j$, $m, k \in \mathbb{N}$ - thus the product is a sum of expressions as above.

The **tensor algebra** of a vectorspace V is

$$\otimes_* V = \bigoplus_{j=0}^{\infty} \bigotimes_{n=1}^{j} V$$

with the bilinear composition $(x, y) \mapsto x \otimes y$ as multiplication and the understanding that $\otimes^0 V = \mathbb{R}$.

Let us consider the **two sided ideal** $\mathfrak{I}$ of $\otimes_* V$ generated by elements $x \otimes x$ of $V \otimes V$ for $x \in V$.

[3] In that case, the dimension formula for linear maps implies that there are only multiples of one n-linear map from $\mathbb{R}^n$ to $\mathbb{R}$; buzzword: determinant.

Definition 3.1.5. *The quotient algebra*

$$\wedge_* V = \otimes_* V \Big/ \mathfrak{I}$$

is called the exterior algebra of the vectorspace V. We set

$$\wedge_m V = \otimes_{j=1}^m V \Big/ \left(\otimes_{j=1}^m V \cap \mathfrak{I}\right).$$

Clearly, $\wedge_* V = \oplus_{m=0}^\infty \wedge_m V$ with the multiplication denoted by the wedge symbol $\wedge$, induced by the tensor algebra, turns this quotient algebra into a graded one. We have in particular $\wedge_0 V = \mathbb{R}$ and $\wedge_1 V = V$.

The well known anti-commutativity of the **wedge product** follows by multilinearity

$$\begin{aligned} (v_1 + w_1) \wedge (v_1 + w_1) \wedge v_3 \wedge \ldots \wedge v_n = 0 \text{ and thus} \\ v_1 \wedge w_1 \wedge v_3 \wedge \ldots \wedge v_n = -w_1 \wedge v_1 \wedge v_3 \wedge \ldots \wedge v_n. \end{aligned}$$

3.2 More on Tensors

This is an excerpt of [Lee09], Appendix D and chapter 7; chapter 6 is the source regarding fibre (fiber) bundles.

3.2.1 Generalization

Remember, that a ring is a set R with pairings defined on it, usually denoted by $+$ and $\cdot$, such that $(R, +)$ is an Abelian group, $(R, \cdot)$ is a halfgroup, and that both pairings are related to each other by the usual rules for parantheses.

Definition 3.2.1. *Let R be a ring. A left R-module V is an Abelian group $(V, +)$ together with an operation $R \times V \to V$, written as $(a, v) \mapsto av$, such that*

1. *$(a + b)v = av + bv$ for all $a, b \in R$ and all $v \in V$;*
2. *$a(v + w) = av + aw$ for all $a \in R$ and all $v, w \in V$;*
3. *$(ab)v = a(bv)$ for all $a, b \in R$ and all $v \in V$.*

If R has an identity 1 (for V), i.e. $1v = v$ for all $v \in V$, we say that V is a unitary R-module. If R is commutative, we also want $av = va$ to hold.

It is necessary to demand that $av = va$ as the following example shows.

Example 3.2.2. *Let $V = \mathbb{R}^2$ and R be the algebra of commuting upper triangular matrices; i.e. R is the subset of the upper triangular matrices in $\mathbb{R}^{2\times 2}$, such that for any two matrices of R, written as $(a, b; 0, c)$ and $(\alpha, \beta; 0, \gamma)$, we have $b(\alpha - \gamma) = \beta(a - c)$ - then $av \neq (v^t a)^t$ in general.*

We will henceforth **assume** V to be a **unitary** R**-module**, where R will be **commutative** and possesing an **identity**.

Definition 3.2.3. *A map $L : V \to W$ between modules over the same ring R is called a module homomorphism or a linear map if*

$$L(av + bw) = aL(v) + bL(w) \text{ for all } v, w \in V \text{ and } a, b \in R,$$

and thus call it linear over R.

The **set** of **all module homomorphisms** between modules V and W over a commutative ring R **is itself** a **left** R**-module** and is denoted by $\mathrm{Hom}_R(V, W)$; operations are defined pointwise.

A **submodule** is defined to be a subset of V, that is closed under the operations inherited from V; the intersection of all submodules containing a given set $A \subset V$ is called the **submodule generated by** A and is denoted by $\langle A \rangle$. A is called **generating set**. If $\langle A \rangle = V$ for a finite set A, then we say that V is **finitely generated**.

The usual notions for kernel and image of a linear map $L : V \to W$, being submodules themselves, are given by

$$\begin{aligned} &\mathrm{kern}\, L = \{v \in V \ : \ L(v) = 0\} \subset V, \\ &\mathrm{range}\, L = \{w \in W \ : \ w = L(v) \text{ for some } v \in V\} = L(V) \subset W. \end{aligned}$$

A set of elements $\{e_1, \ldots, e_k\}$ of a module is said to be **linearly dependent** over R, if there exist ring elements $r_1, \ldots, r_k \in R$, not all zero, such that $r_1 e_1 + \ldots + r_k e_k = 0$. Otherwise, they are said to be **linearly independent** over R; $\{e_1, \ldots, e_k\}$ is called a linearly independent set over R - and to save some ink, we stop mentioning the dependence on R from now on.

One word of caution, it might happen that there is a $v \in V$ such that $rv = 0$ for a non-zero[4] $r \in R$. Thus linear dependent elements are by no means always expressable as a linear combination of a proper subset of them.

If a linearly independent set $\{e_1, \ldots, e_k\}$ is maximal in size, then we say that the **module** has **rank** k. Thus given any $w \in V$, $\{e_1, \ldots, e_k, w\}$ is linearly dependent, but there is no reason to assume that an equation like $w = r_1 e_1 + \ldots + r_k e_k$ holds true, since R might contain non-invertible elements.

[4] By the Example above, let v be an eigenvector to the eigenvalue zero.

If B is a generating set for a module V, such that every element of V has a unique expression as a finite R-linear combination of elements of B, then we say that B is a **basis** for V.

If an R-module has a basis, then it is referred to as a **free module**. If this basis is finite, we indicate this by referring to the module as a **finitely generated free module**.

Definition 3.2.4. *For modules under consideration, there are the following, potentially different, notions of dimension*

1. *The cardinality*[5] *c of the largest index set J, that induces a linear ordering on the set of submodules of V with*

$$0 = V_c \text{ and } V_j \subsetneq V_k \text{ if } j < k \text{ for } j, k \in J.$$

2. *The cardinality of the largest linearly independent set.*

3. *The cardinality of a basis.*

For a finitely generated free module V, all these notions are equivalent and the rank is called dimension, V a finite dimensional free module.

Definition 3.2.5. *For $j \in \{1, \ldots, k\}$, let the V_j's and W be modules over a ring R. A map $\nu : V_1 \times \ldots \times V_k \to W$ is called (k-)multilinear if for each fixed j, the map*

$$\begin{aligned} \nu_j : V_j &\to W \\ v &\mapsto \nu(v_1, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_k) \end{aligned}$$

is linear, where the v_s for $s \neq j$ are arbitrary, but fixed elements from their respective modules each. The R-module of all multilinear maps between this two sets will be denoted by $L_R(V_1, \ldots, V_k; W)$, if $V_j = V_1$ for all j, then we abbreviate this to $L^k_R(V_1; W)$.

The **dual** of an R-module V is the module $V^* = L_R(V; R)$. With this, one is immediately tempted to identify elements of V with ones in $V^{**} = L_R(V^*; R)$ as evaluation maps, i.e. act with $v \in V$ on $\alpha \in V^*$ via $v(\alpha) = \alpha(v)$. This provides naturally a map between V and V^{**}; in case that this is an isomorphism - which is true for a finitely generated free module, we say that V is **reflexive**.

[5] Since we are talking of cardinalities, we may assume that there is a largest element in J.

Proposition 3.2.6. *If $f : V_1 \to W_1$ and $g : V_2 \to W_2$ are module homomorphisms, then there is a unique homomorphism*

$$f \otimes g : V_1 \otimes V_2 \to W_1 \otimes W_2,$$

the tensor product of f and g, with characterizing properties that $f \otimes g$ is linear and that $f \otimes g(v_1 \otimes v_2) = f(v_1) \otimes g(v_2)$ for all $v_j \in V_j$ with $j \in \{1, 2\}$.

This result can easily be augmented to the case of k-many homomorphisms.

Elements of $\otimes_{j=1}^{k} V_j$, that may be written as $v_1 \otimes \ldots \otimes v_k$ for some v_j's, are called simple or **decomposable**; $(v_1 + w_1) \otimes v_2$ is a decomposable element.

Lemma 3.2.7. *There are the following natural isomorphisms:*

$$\begin{gathered} (V \otimes W) \otimes U \cong V \otimes (W \otimes U) \cong V \otimes W \otimes U \\ (v \otimes w) \otimes u \longleftrightarrow v \otimes (w \otimes u) \longleftrightarrow v \otimes w \otimes u \end{gathered}$$

and

$$\begin{gathered} V \otimes W \cong W \otimes V \\ v \otimes w \longleftrightarrow w \otimes v. \end{gathered}$$

In particular, these are the reasons why people do not distinguish the order of how tensor products are obtained - they are identified; and thus, strictly speaking, we regard equivalence classes of tensor products.

Proposition 3.2.8. *We have, by commutativity of R, $V \otimes R \cong V \cong R \otimes V$.*

Proposition 3.2.9. *For R-modules V_j, W with $j \in \{1, \ldots, k\}$, we have*

$$L_R(V_1 \otimes \ldots \otimes V_k; W) \cong L_R(V_1, \ldots, V_k; W).$$

Theorem 3.2.10. *Given free R-modules V_j, with $j \in \{1, \ldots, k\}$, and corresponding bases $\{e_1^j, \ldots, e_{n_j}^j\}$ with $n_j = \dim V_j$; the set of all decomposable elements of the form $e_{j_1}^1 \otimes \ldots \otimes e_{j_k}^k$ is a basis for $V_1 \otimes \ldots \otimes V_k$.*

Proposition 3.2.11. *There is a unique R-module homomorphism $\iota : L_R(V_1; W_1) \otimes \ldots \otimes L_R(V_k; W_k) \to L(V_1 \otimes \ldots \otimes V_k; W_1 \otimes \ldots \otimes W_k)$ such that if $f_1 \otimes \ldots \otimes f_k$ is a decomposable element of $L_R(V_1; W_1) \otimes \ldots \otimes L_R(V_k; W_k)$, then*

$$\iota(f_1 \otimes \ldots \otimes f_k)(v_1 \otimes \ldots \otimes v_k) = f_1(v_1) \otimes \ldots \otimes f_k(v_k).$$

If the modules are all finitely generated and free, then this is an isomorphism.

Clearly $R \otimes R = R$.

Corollary 3.2.12. *There is a unique R-module homomorphism $\iota : V_1^* \otimes \ldots \otimes V_k^* \to (V_1 \otimes \ldots \otimes V_k)^*$ such that if $\alpha_1 \otimes \ldots \otimes \alpha_k$ is a decomposable element of $V_1^* \otimes \ldots \otimes V_k^*$, then*

$$\iota(\alpha_1 \otimes \ldots \otimes \alpha_k)(v_1 \otimes \ldots \otimes v_k) = \alpha_1(v_1) \otimes \ldots \otimes \alpha_k(v_k).$$

If the modules are all finitely generated and free, then this is an isomorphism.

Corollary 3.2.13. *There is a unique R-module homomorphism $\iota_0 : W \otimes V^* \to L_R(V; W)$ such that if $w \otimes \beta$ is a decomposable element of $W \otimes V^*$, then*

$$\iota_0(w \otimes \beta)(v) = \beta(v)w.$$

If V and W are finitely generated and free, then this is an isomorphism.

Theorem 3.2.14. *If $\varphi_j : V_j \times W_j \to U_j$ are bilinear maps for $j \in \{1, \ldots, k\}$, then there is a unique bilinear map*

$$\varphi : \otimes_{j=1}^k V_j \times \otimes_{j=1}^k W_j \to \otimes_{j=1}^k U_j$$

such that for $v_j \in V_j$ and $w_j \in W_j$,

$$\varphi(v_1 \otimes \ldots \otimes v_k, w_1 \otimes \ldots \otimes w_k) = \varphi_1(v_1, w_1) \otimes \ldots \otimes \varphi_k(v_k, w_k).$$

The special case $U_j = R$ for all j turns φ into a scalar product.

3.2.2 Some Multilinear Algebra

We continue with the understanding that the ring under consideration is commutative with identity and that all modules are finitely generated and free. We keep the notation, i.e. the meanings of R, V and W are clear.

Definition 3.2.15. *An algebraic W-valued tensor on V is a multilinear mapping of the form*

$$\tau : V_1 \times \ldots \times V_m \to W,$$

where each factor V_j is either V or V^*. Let the amount of V^*'s occuring be r, the amount of V's be s, then we say that τ is r-contravariant and s-covariant and that the tensor is of total type $\binom{r}{s}$.
A tensor who's total type is $\binom{0}{0}$, is simply an element of the ring R; we allow r or s to be negative, in this case we mean the constant map 0.

An example is given by $\tau : V \times V^* \times V \to R$ and also by $\tau' : V^* \times V \times V \to R$, which both are of **total type** $\binom{1}{2}$ but are indeed different. To indicate this, we will use a more specific notation and say that the first tensor is of **type** $\binom{\ 1\ }{1 \ \ 1}$ and the second one of type $\binom{1}{\ 2}$. The set of all W-valued tensors on V of type $\binom{\ 1}{1\ 1}$ is denoted by $T_1{}^1{}_1(V;W)$. The general pattern should be clear.

If all V^*'s are on the left hand side, then we say that the tensor is in **consolidated form**, as was the second one in the example above.

We always have the **module homomorphism**

$$(\otimes^r V) \otimes (\otimes^s V^*) \to T^r{}_s(V;R),$$

where we identify an element $u_1 \otimes \ldots \otimes u_r \otimes \beta^1 \otimes \ldots \otimes \beta^s$ as the multilinear map which acts as follows:

$$(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_s) \mapsto \alpha^1(u_1) \cdots \alpha^r(u_r)\beta^1(v_1) \cdots \beta^s(v_s).$$

The reader should note how we define the action, i.e. by reflexivity, thus an element of V is identified with one of V^{**}. This is naturally generalized to a map

$$\otimes : T^r{}_s(V) \times T^t{}_u(V) \to T^r{}_s{}^t{}_u(V)$$

where we dropped the R in our notation.

Proposition 3.2.16. *Given a basis for V by $\{e_1, \ldots, e_n\}$ and the dual basis for V^* by $\{e^1, \ldots, e^n\}$, then elements of the form*

$$e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{j_1} \otimes \ldots \otimes e^{j_s}$$

are a basis for $T^r{}_s(V)$ and every tensor τ can be written as

$$\sum \tau^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{j_1} \otimes \ldots \otimes e^{j_s}$$

where $\tau^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} = \tau(e_{i_1}, \ldots, e_{i_r}, e^{j_1}, \ldots, e^{j_s})$.

If a tensor is not in consolidated form, a rearrangement of the above will do.

A covariant tensor $\tau \in T^0{}_k(V;W)$ is said to be **alternating** if

$$\tau(v_1, \ldots, v_k) = \operatorname{sgn}(\sigma)\tau(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for all v_j's in V and all permutations σ of the letters $\{1, \ldots, k\}$. If $l : U \to V$ is a linear map, then the map $l^* : T^0{}_k(V;W) \to T^0{}_k(U;W)$ is defined by

$$(l^*\tau)(u_1, \ldots, u_k) = \tau(l(u_1), \ldots, l(u_k)).$$

This is the **pull-back** of τ by l.

Proposition 3.2.17. *Given $l \in L(V;V)$, $\alpha \in T^0{}_k(V)$ and $\beta \in T^0{}_s(V)$, then*

$$l^*(\alpha \otimes \beta) = l^*\alpha \otimes l^*\beta.$$

Definition 3.2.18. *Given a basis for V by $\{e_1, \ldots, e_n\}$ and the dual basis for V^* by $\{e^1, \ldots, e^n\}$; let $\tau \in T^r{}_s(V)$, then for $k \leq r$ and $j \leq s$, we denote the contraction of the k-th contravariant by the j-th covariant slot of τ by $C^k_j \tau \in T^{r-1}{}_{s-1}(V)$, which is defined*[6] *by*

$$\begin{aligned} &C^k_j \tau(\alpha^1, \ldots, \alpha^{r-1}, v_1, \ldots, v_{s-1}) \\ &\quad := \sum_{i=1}^{n} \tau(\alpha^1, \ldots, \underbrace{e^i}_{k\text{-th position}}, \ldots, \alpha^{r-1}, v_1, \ldots, \underbrace{e_i}_{j\text{-th position}}, \ldots, v_{s-1}). \end{aligned}$$

This can be regarded as the swallowing of the j-th covariant component by the k-th contravariant component as can be easily seen on decomposable elements.

Example 3.2.19. *Regard $\tau = v \otimes \alpha \in T^1{}_1(V)$, then*

$$v(\alpha) = \alpha(v) = \sum_j \alpha_j e^j \Big(\sum_i v_i e_i \Big) = \sum_j \alpha_j v_j = C^1_1 \tau.$$

This example illustrates a very common trick, to obtain a desired manipulation of an element, one takes tensor products and uses contraction - in the example above, the inner product of v and α is obtained by contraction of $v \otimes \alpha$.

3.3 Vector- and Tensorfields

We start with definitions of fibre bundles and move on to the construction of tensor bundles.

3.3.1 Fibre Bundles

Definition 3.3.1. *Let F, M and E be smooth manifolds and let $\pi : E \to M$ be a smooth map. The quadruple*[7] *$\xi := (E, \pi, M, F)$ is called a (locally trivial) fibre bundle if for each $p \in M$, there is an open set U containing p and a smooth diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

[6] Where we tacitly assume that $k, j > 1$.

[7] One also writes $\pi : E \to M$ or $E \to M$ if F is clear from the context.

The space E is called **total space**, π the **bundle projection**, M the **base space** and F is called the **typical fibre**. For each $p \in M$, the set $E_p := \pi^{-1}(p)$ is called the **fibre over** p.

Example 3.3.2. $\mathrm{pr}_1 : M \times F \to M$ *is called product or trivial bundle.*

The maps ϕ are said to be **local trivializations** of the bundle; they must look like $(\pi, \Phi) := (\pi|_{\pi^{-1}(U)}, \Phi)$, where $\Phi : \pi^{-1}(U) \to F$ is a smooth map with the property that $\Phi|_{E_p} : E_p \to F$ is a diffeomorphism. Φ is called the **principal part** of the local trivialization. A pair (U, ϕ), where ϕ is a local trivialization over $U \subset M$, is called **bundle chart**.

Definition 3.3.3. *Given $\xi_1 = (E_1, \pi_1, M, F_1)$ and $\xi_2 = (E_2, \pi_2, M, F_2)$ smooth fibre bundles over the same base space M. A bundle morphism over M from ξ_1 to ξ_2 is a smooth map $h : E_1 \to E_2$, such that the following diagram commutes:*

$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

If h is a diffeomorphism, the notions of bundle isomorphism and automorphism crop up naturally. A smooth **section**[8] of a fibre bundle $\xi = (E, \pi, M, F)$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$, i.e. $\sigma(p) \in E_p$ for $p \in M$.

3.3.2 Vector- and Tensor Bundles

Definition 3.3.4. *Let V be a finite dimensional vector space over $\mathbb{R}$. A smooth $\mathbb{R}$-vector bundle with typical fibre V is a fibre bundle $\xi = (E, \pi, M, V)$, such that*

1. *for every $p \in M$, the set $E_p = \pi^{-1}(p)$ has the structure of a vector space over $\mathbb{R}$, isomorphic to V;*

2. *every $p \in M$ is in the domain of some vectorbundle chart (U, ϕ) with $\phi = (\pi, \Phi)$, such that for each $x \in U$, the map $\Phi|_{E_x} : E_x \to V$ is a vector space isomorphism.*

Instead of $\mathbb{R}$, one could perfectly well regard $\mathbb{C}$; we will also use the terms **real** or **complex vector bundle** for ξ and the dimension of V is called **rank** of ξ.

The most important example is that of the tangent bundle TM; more can be found in [Lee09], page 274 - "Vector bundle construction theorem".

[8]We mention that this is the notion of a global section, which may not exist in general.

Example 3.3.5. *We set* $TM = \bigcup_{p \in M} T_pM$*, the disjoint union of tangent spaces, all isomorphic to some* $\mathbb{R}^n$*.*

To construct a tensor bundle, we start out with a real k-vector bundle $\xi = (E, \pi, M, \mathbb{R}^k)$. The disjoint union ${T^r}_s(E) = \bigcup_{p \in M} {T^r}_s(E_p)$ will be given a smooth structure, such that it becomes a manifold and total space of a bundle with ${T^r}_s(E_p)$ as fibre over p and base space M, i.e. ${T^r}_s(\xi) = ({T^r}_s(E), \pi, M, \mathbb{R}^m)$[9].

Given a vectorbundle chart (U, ϕ) for ξ, we construct one for ${T^r}_s(\xi)$ as follows; since $\phi = (\pi, \Phi)$, where $\Phi : \pi^{-1}(U) \to \mathbb{R}^k$ is a diffeomorphism and $\Phi_p := \Phi|_{E_p} : E_p \to \mathbb{R}^k$ is a linear isomorphism for each p, we define $\Phi_p^{r,s} : {T^r}_s(E_p) \to {T^r}_s(\mathbb{R}^k)$ by

$$
\begin{aligned}
&(\Phi_p^{r,s}\tau_p)(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_s) \\
&\qquad := \tau_p\Big((\Phi_p)^*\alpha^1, \ldots, (\Phi_p)^*\alpha^r, \Phi_p^{-1}v_1, \ldots, \Phi_p^{-1}v_s\Big),
\end{aligned}
$$

where $(\Phi_p)^*$ is the pull-back of Φ_p, introduced prior to Proposition 3.2.17. Naturally, $\pi^{-1}(U)$ will be $\bigcup_{p \in U} {T^r}_s(E_p)$ and we use this to obtain a topology on ${T^r}_s(\xi)$ and further combine the $\Phi_p^{r,s}$ to form a vectorbundle chart

$$
\phi^{r,s} = (\pi, \Phi^{r,s}) : \pi^{-1}(U) \to U \times {T^r}_s(\mathbb{R}^k).
$$

Since ${T^r}_s(\mathbb{R}^k)$ can be interpreted as a set of tensors and further as a vectorspace and U is diffeomorphic[10] to some set in $\mathbb{R}^k$; the vectorbundle chart above can be interpreted as a chart on a manifold, giving a smooth structure on ${T^r}_s(\xi)$, which finishes the construction.

The bundle ${T^r}_s(TM) \to M$ is called the (r, s)-**tensor bundle on** M and sections of this bundle are called **tensor fields**.

The above construction can also be applied to W-valued tensors, simply by changing ${T^r}_s(E_p)$ to ${T^r}_s(E_p; W)$.

3.4 Metric Tensors

Next we want to introduce the important notion of musical isomorphisms for manifolds with a metric.

In order to do so, let us collect a few definitions. A **symmetric bilinear form** g on a finite dimensional vectorspace V is **nondegenerate** if and only if $g(v, w) = 0$ for all $w \in V$ implies that $v = 0$; in this case g is called **scalar product**. g is said to be **positive definite** if $g(v, v) \geq 0$ for all $v \in V$ and

[9] Here π might have changed, though we won't put any emphasize on it; m is chosen suitably.
[10] At least every point in U has a neighbourhood in U with this property.

$g(v,v)=0$ implies $v=0$; the notion of negative definite is defined analogously. A positive definite scalar product is referred to as an **inner product**. The orthogonal complement to $\text{span}\{v\}$ is the vectorspace of all $w \in V$ such that $g(v,w)=0$. (V,g) is denoted **scalar product space** or **inner product space** with the obvious dependence on g.

The **length** or **norm** of a vector $v \in V$ is the number $|g(v,v)|^{\frac{1}{2}}$.

Given an inner product space (V,g), then by Riesz representation, there is a unique $g_\flat(v) := v^\flat := \flat v$ in V^*, such that $v^\flat(w)$ equals $g(v,w)$. The inverse is written as $g^\sharp := \sharp(\cdot) := (\cdot)_\sharp$, i.e. $(v^\flat)_\sharp = v$ - these are the **musical isomorphisms** and they are related through inversion, see [Lee09].

We force their name to make sense by defining an inner product g^* on V^* via

$$g^*(\alpha,\beta) = g(\alpha_\sharp,\beta_\sharp).$$

It follows that the dual basis to an orthonormal basis will be orthonormal too. Given scalar product spaces (V_j,g_j), then by Theorem 3.2.14, there is a **unique bilinear form** that turns $\otimes_j V_j$ into a scalar product space. We use this to endow the ${T^r}_s(V)$ with an inner product.

To see how wedge products and metrics go together and what the reason for the standard inner product on the Grassman algebra is, one sets

$$\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_r} \right\rangle = \delta_j^r$$

and extends this to all forms via the musical isomorphisms and Theorem 3.2.14.

Example 3.4.1. *Take* $v = \sum_{r,j=1}^n v_{jr}\ \mathrm{d}x_j \wedge \mathrm{d}x_r$, $w = \sum_{r,j=1}^n w_{jr}\ \mathrm{d}x_j \wedge \mathrm{d}x_r$ *to obtain*

$$\langle v,w\rangle = \sum_{r,j=1}^n v_{jr}w_{jr}\langle \mathrm{d}x_j \wedge \mathrm{d}x_r, \mathrm{d}x_j \wedge \mathrm{d}x_r\rangle = \sum_{r,j=1}^n v_{jr}w_{jr}$$

as by definition $\|\mathrm{d}x_j \wedge \mathrm{d}x_r\|^2 = \langle \mathrm{d}x_j \wedge \mathrm{d}x_r, \mathrm{d}x_j \wedge \mathrm{d}x_r\rangle = 1$.

Definition 3.4.2. *A Riemannian metric g on the tangent bundle TM is a map $p \mapsto g_p(\cdot,\cdot)$ that assigns to each $p \in M$ an inner product $g_p(\cdot,\cdot)$ on T_pM, which is smooth in the sense that $p \mapsto g_p(s_1(p), s_2(p))$ is smooth for all smooth sections s_1 and s_2 of TM. We define the Riemannian distance or the by g induced distance function $d_g(\cdot,\cdot)$ to be*

$$d_g(z,\zeta) = \inf_{c \in \mathcal{P}_{z,\zeta}} \int_I \sqrt{g_{c(t)}(\dot{c}(t),\dot{c}(t))}\mathrm{d}t$$

where $\mathcal{P}_{z,\zeta}$ is the set of rectifiable paths from $I=[0,1]$ to M with $c(0)=z$ and $c(1)=\zeta$.

3.5 Integration

We suppose that M is a smooth manifold of dimension n and denote its boundary by bM, which might as well be empty.

The **support** of a differential form α on M is the closure of the set where it is not the zero-functional. Let $\alpha = a\mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n$ be a differential form with the function "a" being zero outside a compact set K and local coordinates u_j in K. The element $\mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n$ is referred to as a local **top form** or **volume form** for M, written as $\mathrm{d}V$, and M is said to be **orientable** if there exists a nowhere vanishing volume form, also called the orientation - in this case, there is also a natural orientation for the submanifold bM. We define the integral of α to be

$$\int_M \alpha := \int_M a(u)\mathrm{d}u_1 \ldots \mathrm{d}u_n$$

where the last integral is a welldefined integral in $\mathbb{R}^n$ with respect to Lebesgue measure due to the diffeomorphism property and orientation. The above integral is always zero if we do not integrate a volume form.

Theorem 3.5.1. ***(Stokes)***[11] *For compactly supported differential forms α of degree $n-1$, we have*

$$\int_M \mathrm{d}\alpha = \int_{bM} \alpha.$$

We are now in the position to define an inner product for tensor fields via Example 3.4.1 and integrals. Let $\mathrm{d}V$ be a volume form and v and w be arbitrary differential forms, then we define

$$\langle v, w\rangle := \int_{p\in M} \langle v_p, w_p\rangle \mathrm{d}V \tag{3.1}$$

which is always zero unless v and w are of same degree and in this case we recover the standard definition of inner products for tensor fields.

3.6 A bit Complex Differential Geometry

Unfortunately, this author is not able to add any insight in this section and has no time to invest in its improvement; we will find a mere collection of notions as anywhere else and hope that it is of some sense and use. We follow [Ran98], pages 122-136.

Two promising sources might have been [Wel08] and [Nar73].

[11] It has little to do with Stokes, see [Pen07, page 245-6].

3.6.1 Basic Notions

A **complex manifold** M of complex dimension n is defined just like a smooth manifold, with the exception that **charts**[12] **map into** $\mathbb{C}^n$ and the **transition maps** are supposed to be **biholomorphic**. An atlas with these properties is called **atlas for the complex structure of** M.

Such an atlas induces a smooth structure on M with real dimension $2n$, thus one has the notion of **tangent space** and **-vector** as well as the theory of **multilinear maps** at hand to **extend** these notions $\mathbb{C}$**-linearly**, such that a tangent vector v of the **complex vector space** $T_p^{\mathbb{C}}M$ with $v = x + iy$ for $x, y \in T_pM$, acts on a complex valued function $f = g + ih$ as $v(f) = x(f) + iy(f)$ via $x(f) = x(g) + ix(h)$ and $y(f) = y(g) + iy(h)$ for $g, h \in \mathcal{C}^\infty(M, \mathbb{R})$ - note that contrary to the real case, the complex dimension of $T_p^{\mathbb{C}}M$ is $2n$.

A **complex valued multilinear map** is analogously to functions simply a multilinear map plus i-times another such map.

If $\varphi = (z_1, \ldots, z_n)$ is a complex chart, hence complex coordinates, near some point $p \in M$ and $z_j = x_j + iy_j$, for $j \in \{1, \ldots, n\}$, are the decompositions into real- and imaginary part of the coefficient functions, then $(x_1, y_1, \ldots, x_n, y_n)$ form a real chart or coordinates near p with

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \right\}$$

as a basis for T_pM and

$$\{\mathrm{d}x_1, \mathrm{d}y_1, \ldots, \mathrm{d}x_n, \mathrm{d}y_n\}$$

a basis for the cotangent space T_p^*M, which will be rewritten[13] to

$$\left\{ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_n} \right\}$$

and

$$\{\mathrm{d}z_1, \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}z_n, \mathrm{d}\bar{z}_n\},$$

where this expressions are $\mathbb{C}$-linearly independent and span $T_p^{\mathbb{C}}M$ as well as $T_p^{*\mathbb{C}}M$ and that one has[14]

$$\mathrm{d}z_j \left(\frac{\partial}{\partial \bar{z}_k} \right) = 0 \quad \text{and} \quad \mathrm{d}\bar{z}_j \left(\frac{\partial}{\partial \bar{z}_k} \right) = \delta_{jk}.$$

[12] (U, φ) with $U \subset M$ and $\varphi : U \to \mathbb{C}^n$.

[13] Well, not exactly rewritten, but re-stated such that they fit their complex purpose.

[14] With similar formulae for $\frac{\partial}{\partial z_j}$.

The differential of a smooth function f at p is given by

$$\mathrm{d}f_p = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p)\mathrm{d}x_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}(p)\mathrm{d}y_j$$

or by

$$\mathrm{d}f_p = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(p)\mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}(p)\mathrm{d}\bar{z}_j.$$

The complexification of the above basis can be obtained in terms of an endomorphism J which mimics multiplication by i on $\mathbb{R}^{2n}$, actually on T_pM, is called an **almost complex structure** on T_pM if it satisfies $J^2 = -\operatorname{id}$.

The scalar product of $a + ib$ and $v \in T_p^{\mathbb{C}}M$, $a, b \in \mathbb{R}$, is defined as $(a + ib)v = av + bJ(v)$. We can decompose the complex vector space $T_p^{\mathbb{C}}M$ into spaces $T_p^{1,0}M$ and $T_p^{0,1}M$ - the eigenspaces to the eigenvalues i and $-i$ of J respectively. **Forms of bidegree** or of **type** $(1, 0)$ are elements of the vector space

$$\Lambda^{1,0}(T_p^*M) := \{\omega \in T_p^{*\mathbb{C}}M \ : \ \omega(Jv) = i\omega(v) \text{ for all } v \in T_p^{\mathbb{C}}M\}$$

which is simply the span of $\{\mathrm{d}z_1, \dots, \mathrm{d}z_n\}$.

$\Lambda^{p,0}(T_p^*M)$ is spanned by the wedge product of p-many different $\mathrm{d}z_j$'s and if we write $\overline{\mathrm{d}z_1 \wedge \mathrm{d}z_2}$ for $\mathrm{d}\bar{z}_1 \wedge \mathrm{d}\bar{z}_2$, then we define analogously $\Lambda^{0,p}(T_p^*M) := \overline{\Lambda^{p,0}(T_p^*M)}$ to be the vector space of forms of bidegree $(0, p)$. An element of $\Lambda^{p,q}(T_p^*M)$, hence a form of bidegree (p, q), is just a linear combination of elements like

$$\mathrm{d}z_{j_1} \wedge \mathrm{d}z_{j_2} \wedge \dots \mathrm{d}z_{j_p} \wedge \ \mathrm{d}\bar{z}_{k_1} \wedge \mathrm{d}\bar{z}_{k_2} \wedge \dots \mathrm{d}\bar{z}_{k_q} =: \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K$$

for $I = (j_1, \dots, j_p)$ and $K = (k_1, \dots, k_q)$ and is of degree $p + q$. In general, the set[15] of forms of degree r is a direct sum of the various $\Lambda^{p,q}$ with[16] $p + q = r$, i.e.

$$\Lambda^r = \oplus_{p+q=r}\Lambda^{p,q}.$$

Now we will augment Example 3.4.1 to the complex case, hence one obtains

$$\left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_r} \right\rangle = \frac{1}{2}\delta_j^r$$

and

$$\langle \mathrm{d}z_j, \mathrm{d}z_r \rangle = 2\delta_j^r$$

by the definition of the real case.

The **interior product** of differential forms γ and α is defined to satisfy

$$\langle \gamma \lrcorner \alpha, \beta \rangle = \langle \alpha, \bar{\gamma} \wedge \beta \rangle \tag{3.2}$$

for every β such that α and $\bar{\gamma} \wedge \beta$ have matching bidegrees.

[15] Here we adapt our notion.

[16] We really want p and q to be non-negative integers.

3.6.2 The $\bar{\partial}$-Operator

We now define a central object in the study of Complex Analysis in Several Variables, the **d-bar operator**, as

$$\begin{aligned}\bar{\partial} : \Lambda^{p,q} &\to \Lambda^{p,q+1}\\ \sum_{IK} a_{IK} \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K &\mapsto \sum_{j=1}^{n} \sum_{IK} \frac{\partial a_{IK}}{\partial \bar{z}_j} \mathrm{d}\bar{z}_j \wedge \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K\end{aligned}$$

where we tacitly used summation of increasing indices, i.e. for $I = \{1, \ldots, n\}$ we write $\sum_I \mathrm{d}z^I = \sum_I \mathrm{d}z_{s_1} \wedge \ldots \wedge \mathrm{d}z_{s_p}$ for all $\{s_1, \ldots, s_p\} \subset \{1, \ldots, n\}$ with $s_1 < \ldots < s_p$.

Similarly for the operator $\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}$, which sums up to $\mathrm{d} = \partial + \bar{\partial}$ and since $\mathrm{d}^2 = 0$, this implies[17] $\bar{\partial}^2 = \partial^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

For the next we need to recall that for a bounded operator $A : B_1 \to B_2$ between Banach spaces B_1 and B_2 we say that A has closed graph if the space $\mathrm{Gra}(A) := \{(x, Ax) \ : \ x \in B_1\}$ is closed in $B_1 \times B_2$ with norm $\| \cdot \|_1 + \| \cdot \|_2$. A similar concept is true for unbounded, densely defined operators like $\bar{\partial}$, see [Has14].

Remark 3.6.1. *The $\bar{\partial}$-operator is densely defined with closed graph.*

3.6.3 The $*$-Operator

The **Hodge star operator** is closely connected to the volume[18] form

$$\mathrm{d}V = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$$

and variants thereof.

Theorem 3.6.2. *Let $\mathrm{d}V$ be a volume form on an n-dimensional complex manifold M. There exists a unique $\mathbb{C}$-linear map $*$, defined on the whole Grassman algebra of M, acting as follows on $\varphi \in \Lambda^r$:*

$$\begin{aligned}&* \varphi \in \Lambda^{n-r},\\ &* * \varphi = (-1)^{(n-r)r} \varphi \text{ and}\\ &\psi \wedge * \bar{\varphi} = \langle \psi, \varphi \rangle \mathrm{d}V \text{ for all } \psi;\end{aligned}$$

and in addition, satisfies $*1 = \mathrm{d}V$ and is real, i.e. $\overline{*\varphi} = *\bar{\varphi}$.

[17] One says it follows by bidegree reasons - this has a similar meaning as for ordinary wedge products; forms of different degrees sum up to zero if they are zero, simply because one is dealing with multilinear maps and they can be applied to something that annihilates all the other terms, hence showing that the remaining form must be zero.

[18] There is only one such volume form except for multiplication by a smooth function, which is always positive or negative, and contributes an orientation on the manifold.

For our case $M = \mathbb{C}^n$ will suffice and the star operator has the additional properties

$$
\begin{aligned}
&* (\Lambda^{p,q}) \subset \Lambda^{n-q,n-p} \text{ for } p,q \in \{1,\dots,n\},\\
&* * \varphi = (-1)^{p+q}\varphi \text{ for } \varphi \in \Lambda^{p,q} \text{ and}\\
&* \, \mathrm{d}z^S = \frac{(-1)^{q(q-1)/2}}{2^{n-q} i^n} \mathrm{d}z^S \wedge \left(\wedge_{\nu \in S'} \mathrm{d}\bar{z}_\nu \wedge \mathrm{d}z_\nu\right),
\end{aligned}
$$

given that $|S| = q$, where $S \subset \{1,\dots,n\}$ and S' is the complement of S in $\{1,\dots,n\}$.

We will need to adapt our notion of star operator to the metric at hand to

$$\langle \alpha, \xi \rangle_\omega dV = \alpha \wedge (*\bar{\xi}), \tag{3.3}$$

which changes $*$ to $*_\omega = w \cdot *$ for a bounded, positive weight function w.

3.6.4 Kähler Metrics

We now copy pages 81-83 and 101-102 from [Mor07] and continue to use J as the almost complex structure from earlier.

A **Hermitean metric** on an n-dimensional complex manifold M is a Riemannian metric h, such that

$$h(\alpha,\beta) = h(J\alpha, J\beta) \text{ for all } \alpha,\beta \in TM.$$

The **fundamental 2-form** of h is defined by $\Omega(\alpha,\beta) = h(J\alpha,\beta)$.

The extension by $\mathbb{C}$-linearity of h, also denoted by h, to the complexified tangent space $T^{\mathbb{C}}M = \bigcup_{p\in M} T_p^{\mathbb{C}}M$ satisfies

$$
\begin{aligned}
&h(\bar{\zeta},\bar{\eta}) = \overline{h(\zeta,\eta)} \text{ for all } \zeta,\eta \in T^{\mathbb{C}}M;\\
&h(\zeta,\bar{\zeta}) \ > 0 \text{ outside the zero set of } \zeta;\\
&h(\zeta,\eta) = 0 \text{ if } \zeta,\eta \text{ are both of bidegree (0,1) or (1,0).}
\end{aligned}
$$

Set

$$h_{j,\bar{k}} := h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right),$$

then

$$\Omega = i \sum_{j,k=1}^{n} h_{j,\bar{k}} \, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k.$$

The local $i\partial\bar{\partial}$-lemma, which can be found on page 68 of [Mor07], ensures locally the existence of a real valued function u with $\Omega = i\partial\bar{\partial}u$ and consequently

$$h_{j,\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}.$$

A Hermitean metric h is called **Kähler metric**[19] if Ω is closed, i.e. $\mathrm{d}\Omega = 0$. The function u from above is a local **Kähler potential** of the metric h.

Theorem 3.6.3. *A Hermitean metric h on a complex manifold M is Kähler if and only if around each point of M, there exist holomorphic coordinates in which h osculates to the standard[20] Hermitean metric to the order of two.*

We define the following linear operators with use of the interior product, see (3.2):

$$\begin{array}{rcl|rcl} \delta : \Omega^{k+1}M & \to & \Omega^k M & \Lambda : \Lambda^{k+1}M & \to & \Lambda^k M \\ \alpha & \mapsto & -(-1)^{nk} * \mathrm{d} * \alpha & \alpha & \mapsto & \frac{1}{2}\sum_{j=1}^{2m} Je_j \lrcorner e_j \lrcorner \alpha \end{array}$$

and with $[P, Q] = P \circ Q - Q \circ P$, one of the famous **Kähler identities** reads as follows

$$[\Lambda, \delta] = 0.$$

In the paper of Delin, however, things are stated in the following way

$$\Lambda := \omega\lrcorner : \Lambda^{(p,q)} \to \Lambda^{(p-1,q-1)} \tag{3.4}$$

and we take for granted that:

$$\partial^* = -i[\bar{\partial}, \Lambda] \tag{3.5}$$

and

$$\Lambda\xi = -i^{n-1}(-1)^{\frac{n(n+1)}{2}} * \xi \tag{3.6}$$

for ξ being an $(n, 1)$-form.

[19] Moroianu also demands a certain tensor to vanish for this definition, but we only deal with the prototypical case and do not have to worry.

[20] By this we mean $\frac{i}{2}\sum \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$.

Chapter 4

Estimates of the Weighted Bergman Kernel in Several Complex Dimensions

We will work in the Hilbert space setting of $L^2(\mathbb{C}^n, \varphi)$, thus regard functions $f : \mathbb{C}^n \to \mathbb{C}$ such that

$$\int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \, \mathrm{d}\lambda < \infty,$$

which is basically L^2 with measure $e^{-\varphi} \, \mathrm{d}\lambda$ and inner product as known, just the measures changed. The Bergman space $A^2(\mathbb{C}^n, \varphi)$, i.e. the set of entire functions in $L^2(\mathbb{C}^n, \varphi)$, is a closed subspace of it, see [HKZ00] for more content.

The "orthogonal projection" from $L^2(\mathbb{C}^n, \varphi)$ to $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$, i.e. the set of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$, and kernel[1] $A^2(\mathbb{C}^n, \varphi)$, is denoted $\bar{\partial}$ and is well behaved on holomorphic bundles, see [Mor07]. This is not the case for its brother in law, ∂, which depends on φ and where we have

$$\partial_\varphi = e^{\varphi} \partial e^{-\varphi} = \partial - \partial\varphi \wedge . \tag{4.1}$$

Its formal adjoint is defined through the equality $\langle \bar{\partial} f, g \rangle = \langle f, \bar{\partial}^*_\varphi g \rangle$, where f is a function and g a $(0,1)$-form and involves an integration by parts which is given, due to the presence of $e^{-\varphi}$, by

$$\bar{\partial}^*_\varphi = e^{\varphi} \bar{\partial}^* e^{-\varphi} = \bar{\partial}^* + \partial\varphi \lrcorner. \tag{4.2}$$

The wedge and interior products are scetched in the preliminaries to this chapter; the main properties of plurisubharmonic functions and alike is collected in the Appendix.

[1] This author has luckily been reminded that the kernel of $\bar{\partial}_{(0,1)} : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \to L^2_{(0,2)}(\mathbb{C}^n, \varphi)$ is bigger than just $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$, as $\bar{z}_1 \mathrm{d}\bar{z}_2 + \bar{z}_2 \mathrm{d}\bar{z}_1 \in \mathrm{kern}(\bar{\partial}_{(0,1)})$.

4.1 Overview and Statements

The plan is as follows. First, technical lemmata and theorems are developped which provide one with integral estimates; then the Bergman projection is applied to a radial cut-off function with support in a ball, centered at a generic element, which will lead, in combination with the estimates obtained so far, to an estimate of the Bergman kernel in this ball - where the second variable is fixed as the center of this ball.

Second, the link presented here between integral estimates and pointwise estimates is a lemma due to Bo Berndtsson, Lemma 4.5.1 - which comes with a rather unpleasant unknown a_φ; to get control of this term, Lemma 4.4.1 is applied, and here it is essential that $\partial\bar{\partial}\varphi$ is bounded on $\mathbb{C}^n$.

H. Delin's theorems will now be summerized.

Theorem 4.1.1. *For f a closed $(0,1)$-form on a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and φ a strictly plurisubharmonic $\mathcal{C}^2$-function defined there with a given weight w and constant $\varepsilon \in (0, \sqrt{2})$ such that $|\partial w|_{i\partial\bar{\partial}\varphi} \leq \varepsilon w$, we have that the $L^2_\varphi(\Omega)$-minimal solution to $\bar{\partial} u = f$ satisfies*

$$\int_\Omega |u|^2 e^{-\varphi} w \; \mathrm{d}\lambda \leq \frac{2}{(\varepsilon - \sqrt{2})^2} \int_\Omega |f|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} w \; \mathrm{d}\lambda.$$

We will use $|\cdot|_{i\partial\bar{\partial}\varphi}$ to denote the norm obtained by the Kähler metric with Kähler form $i\partial\bar{\partial}\varphi$, but in general, we will speak of a Kähler form ω and denote by $dV = \frac{\omega^n}{n!}$ its volume form; see this chapter's preliminaries for a less crude scetch.

We will have to assume the Kähler form locally approximable by a constant Hermitian $(1,1)$-form h, such that

$$h_z \geq \omega_\zeta \text{ for } \zeta \in B_{h_z}(z, 1), \tag{4.3}$$

as metrics on the tangent spaces, i.e. the Hermitian form majorises ω in the unit ball with respect to the Kähler metric with form given by h_z.

If the inequality holds in the larger set $B_\omega(z, 1)$, then we surely have (4.3), hence there are no problems in finding such a form when ω is continuous. A linear change of coordinates, denoted by η, that turns h into the Euclidean metric, $\eta(h) = \beta$, changes (4.3) to $\eta(\omega) \leq \beta$, where

$$\beta = \frac{i}{2}\partial\bar{\partial}|z|^2 = \frac{i}{2}\sum \; \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

Using normal coordinates of ω at z and an abuse of notation (where we write ω_z for the diagonal matrix in the given coordinates), we can calculate eigenvalues of ω_z with respect to h_z as solutions to the equation

$$\lambda_j(\omega_z | h_z) := \det(\lambda h_z - \omega_z) = 0$$

for $1 \leq j \leq n$. By linear algebra, this definition is independent of choice of coordinate representations for ω and h as both are invertible everywhere.

Also the notation

$$\det(\omega_z|h) = \prod_j^n \lambda_j(\omega_z|h_z)$$

and

$$\det(\omega_z) = \prod_j^n \lambda_j(\omega_z|\beta) := \prod_j^n \lambda_j(\omega_z)$$

as well as

$$c_{\omega,h}(z) = \inf_{\zeta \in B_{h_z}(z,1)} \lambda_{\min}(\omega_\zeta|h_z)$$

which is smaller than 1 by (4.3), where

$$\lambda_{\min}(\omega_\zeta|h_z) = \min_{\{v \,:\, h_z(v,\bar{v})=1\}} \omega_\zeta(v,\bar{v})$$

will be used. In addition, let $d_\omega(z,\zeta)$ denote the distance of z and ζ in the metric induced by ω.

Theorem 4.1.2. *Let φ be a strictly plurisubharmonic $\mathcal{C}^2$-function on $\mathbb{C}^n$ and h satisfy* (4.3) *with $\omega = i\partial\bar{\partial}\varphi$, and assume that $\varepsilon \in (0, \sqrt{2})$. Then the weighted Bergman kernel satisfies*

$$|\mathcal{B}_\varphi(z,\zeta)|^2 \leq \frac{C}{(\sqrt{2}-\varepsilon)^2 c_{\omega,h}(\zeta)} \det(h_z)\det(h_\zeta) e^{\varphi(z)} e^{\varphi(\zeta)} e^{-\varepsilon d_\omega(z,\zeta)}, \tag{4.4}$$

where C is a constant that depends on the dimension only.

4.2 Weighted L^2-Identities

A weighted L^2-identity for $(n,1)$-forms will be derived, satisfying the boundary conditions of the $\bar{\partial}$-Neumann problem.

For the next, it should be noted that the boundary term does not really depend on the metric, though H. Delin has included it for ease of sign determination.

Lemma 4.2.1. *Let φ and w be sufficiently smooth real valued functions. Let γ be an $(n-1,0)$-form defined in a neighbourhood of a smooth bounded set $G = \{z : \nu(z) < 0\}$, where ν is the defining function for G with $\mathrm{d}\nu \neq 0$ on bG. If γ satisfies the boundary condition $\partial\nu \wedge \gamma = 0$ on bG, then*

$$
\begin{aligned}
&2\operatorname{Re}\int_G i^n\bar\partial\partial_\varphi\gamma\wedge\bar\gamma e^{-\varphi}w = \\
&\qquad\int_G i^n\partial\bar\partial\varphi\wedge\gamma\wedge\bar\gamma e^{-\varphi}w+(-1)^n\int_G i^n\bar\partial\gamma\wedge\overline{\bar\partial\gamma}e^{-\varphi}w \\
&\qquad+(-1)^{n-1}\int_G i^n\partial_\varphi\gamma\wedge\overline{\partial_\varphi\gamma}e^{-\varphi}w+\int_G i^n\partial w\wedge\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi}) \\
&\qquad+\int_{bG} i^n*(\partial\bar\partial\nu\wedge\gamma\wedge\bar\gamma)e^{-\varphi}w\frac{\mathrm{d}S}{|\mathrm{d}\nu|}.
\end{aligned}
\tag{4.5}
$$

Proof. By equation (4.1), we write $\bar\partial(\bar\gamma e^{-\varphi})=\bar\partial(\overline{\gamma e^{-\varphi}})=\overline{\partial(\gamma e^{-\varphi})}=\overline{\partial_\varphi\gamma}e^{-\varphi}$ and since $\bar\partial\gamma\wedge\bar\gamma e^{-\varphi}=(\bar\partial\gamma)e^{-\varphi}\wedge\bar\gamma$, we repeat this calculation without bars and γ replaced by $\gamma'=\bar\partial\gamma$, (thus $\partial(\gamma' e^{-\varphi})=\partial_\varphi\gamma' e^{-\varphi}=\partial_\varphi\bar\partial\gamma e^{-\varphi}$) to obtain

$$
\begin{aligned}
\partial\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi})&=\partial\left(\bar\partial\gamma\wedge\bar\gamma e^{-\varphi}+(-1)^{n-1}\gamma\wedge\overline{\partial_\varphi\gamma}e^{-\varphi}\right)\\
&=\Big(\partial_\varphi\bar\partial\gamma\wedge\bar\gamma+(-1)^n\bar\partial\gamma\wedge\overline{\bar\partial\gamma}+(-1)^{n-1}\partial_\varphi\gamma\wedge\overline{\partial_\varphi\gamma}+\gamma\wedge\overline{\bar\partial\partial_\varphi\gamma}\Big)e^{-\varphi}.
\end{aligned}
\tag{4.6}
$$

The powers of (-1) stem from the fact that $\gamma\wedge\bar\gamma$ is an $(n-1,n-1)$-form, $\bar\partial\gamma\wedge\bar\gamma$ is an $(n-1,n)$-form and the Leibniz rule for ∂ and $\bar\partial$. Then, by equation (4.1), the equality $\partial\bar\partial=-\bar\partial\partial$ and the fact that $\partial\varphi\wedge\gamma$ is an $(n,0)$-form, we have

$$
\partial_\varphi\bar\partial\gamma=-\bar\partial\partial\gamma-\partial\varphi\wedge\bar\partial\gamma=-\bar\partial\partial_\varphi\gamma+\bar\partial(\partial\varphi\wedge\gamma)-\partial\varphi\wedge\bar\partial\gamma=-\bar\partial\partial_\varphi\gamma+\bar\partial\partial\varphi\wedge\gamma
$$

and hence the first term on the right hand side of (4.6) equals

$$
\partial_\varphi\bar\partial\gamma\wedge\bar\gamma=-\bar\partial\partial_\varphi\gamma\wedge\bar\gamma+\partial\bar\partial\varphi\wedge\gamma\wedge\bar\gamma. \tag{4.7}
$$

Writing $\gamma\wedge\overline{\bar\partial\partial_\varphi\gamma}=(-1)^{n-1}\overline{\bar\partial\partial_\varphi\gamma\wedge\bar\gamma}$, we see that the first term on the left hand side of (4.7) is, besides the factor $(-1)^n$, the conjugate of the last term of (4.6). Multiplying by i^n makes it exactly the conjugate, thus substituing (4.7) into (4.6) and using that $-2\operatorname{Re}z=-\bar z-z$, yields

$$
\begin{aligned}
i^n\partial\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi})=\Big(&-2\operatorname{Re}\ i^n\bar\partial\partial_\varphi\gamma\wedge\bar\gamma i^n\partial\bar\partial\varphi\wedge\gamma\wedge\bar\gamma+\\
&+(-i)^n\bar\partial\gamma\wedge\overline{\bar\partial\gamma}+(-1)^{n-1}i^n\partial_\varphi\gamma\wedge\overline{\partial_\varphi\gamma}\Big)e^{-\varphi}.
\end{aligned}
\tag{4.8}
$$

We multiply (4.8) with w and integrate. For bidegree reasons and by $\bar\partial^2=0$, we have that $\partial(w\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi}))=\mathrm{d}(w\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi}))$, hence Stokes theorem yields

$$
\int_G i^n w\partial\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi})=\int_{bG} i^n w\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi})-\int_G i^n\partial w\wedge\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi}). \tag{4.9}
$$

The boundary condition

$$\partial\nu \wedge \gamma = 0 \text{ for points on } bG \tag{4.10}$$

implies that for any $(0,n)$-form α, the wedge product $\gamma\wedge\alpha$ vanishes as a differential form on the submanifold bG, otherwise the topform $\mathrm{d}\nu\wedge\gamma\wedge\alpha$ wouldn't be zero on all of the boundary, contradicting (4.10). Using similar arguments in the boundary integral of (4.9) gives

$$\begin{aligned}\int_{bG} i^n w\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi}) &= \int_{bG} i^n\left(\bar\partial\gamma\wedge\bar\gamma + (-1)^{n-1}\gamma\wedge\overline{\partial_\varphi\gamma}\right)we^{-\varphi}\\ &= \int_{bG} i^n\bar\partial\gamma\wedge\bar\gamma we^{-\varphi}\end{aligned} \tag{4.11}$$

since $\overline{\partial_\varphi\gamma}$ is a $(0,n)$-form.

We define $\partial\nu\wedge\gamma = \nu\left(\frac{1}{\nu}\partial\nu\wedge\gamma\right) =: \nu A$ and want A to be bounded on $\overline{G}$; thus we need the extra assumption that $\partial\nu\wedge\gamma$ vanishes at least of same order as ν does. Then

$$\bar\partial\left(\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi}\right) = \bar\partial\left(\nu A\wedge\bar\gamma e^{-\varphi}\right) = \bar\partial\nu\wedge A\wedge\bar\gamma e^{-\varphi} + \nu\bar\partial\left(A\wedge\bar\gamma e^{-\varphi}\right).$$

This vanishes for points on the boundary of G - since ν does so on the boundary and by the conjugate of (4.10) (after re-arangement). Thus on the boundary, again by (4.10) and $\partial\bar\partial = -\bar\partial\partial$

$$\begin{aligned}0 &= \bar\partial\left(\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi}\right)\\ &= \bar\partial\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi} - \partial\nu\wedge\bar\partial\gamma\wedge\bar\gamma e^{-\varphi} + (-1)^n\partial\nu\wedge\gamma\wedge\overline{\partial_\varphi\gamma}e^{-\varphi}\\ &= -\partial\bar\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi} - \partial\nu\wedge\bar\partial\gamma\wedge\bar\gamma e^{-\varphi},\end{aligned}$$

and in conclusio

$$\partial\bar\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi} = -\partial\nu\wedge\bar\partial\gamma\wedge\bar\gamma e^{-\varphi} \text{ on } bG. \tag{4.12}$$

We use the representation[2]

$$\int_{bG}\alpha = \int_{bG} *(\mathrm{d}\nu\wedge\alpha)\frac{\mathrm{d}S}{|\mathrm{d}\nu|},$$

[2] We can think of this alpha as a volume form on the boundary, thus a multiple of the by $\bigwedge_j \mathrm{d}z_j\wedge\mathrm{d}\bar z_j$ induced volume form $\mathrm{d}S$ - the star operator applied to the top form will give us exactly this unknown multiple.

where α is a form of degree $2n-1$, and obtain with the conjugate of (4.10) the extension of (4.11) to its final form:

$$\begin{aligned}\int_{bG} i^n \bar{\partial}\gamma \wedge \bar{\gamma} e^{-\varphi} w &= i^n \int_{bG} * \left(\partial\nu \wedge \bar{\partial}\gamma \wedge \bar{\gamma}\right) e^{-\varphi} w \frac{\mathrm{d}S}{|\mathrm{d}\nu|} \\ &= -i^n \int_{bG} * \left(\partial\bar{\partial}\nu \wedge \gamma \wedge \bar{\gamma}\right) e^{-\varphi} w \frac{\mathrm{d}S}{|\mathrm{d}\nu|}.\end{aligned}$$

We thus have collected all the terms, tracing back ends the proof. ■

As already mentioned, this author thinks that an extra assumption has to be made; we want $\partial\nu \wedge \gamma$ to vanish at least of same order as ν at the boundary such that $\nu^{-1}\partial\nu \wedge \gamma = \big(\partial \log(|\nu|)\big) \wedge \gamma$ to be bounded on bG. In the case of balls - which we are mainly interessted in, this assumption is satisfied.

Let $\nu = (|z|-1)(|z|+1)$ and $\gamma = \sum g_k (-1)^{k+1} \wedge_{j \neq k} \mathrm{d}\bar{z}_j$ be a generic $(n-1,0)$-form. Now, $\bar{\partial}\nu \wedge \gamma = (\sum z_k g_k) \wedge_j \mathrm{d}\bar{z}_j$ is supposed to be zero on the boundary and by this explicit representation, we see that $\bar{\partial}\nu \wedge \gamma$ must vanish linearly for $\nu^{-1}\bar{\partial}\nu \wedge \gamma$ to be bounded. This will be true if the g_k's can be linearly approximated, which will be the case.

The next task is to translate (4.5) into an identity for $(n,1)$-forms using the star operator defined by a Kähler metric, i.e. for an orthonormal basis θ_j of $(1,0)$-forms in the metric ω, let $\theta = \theta_1 \wedge \ldots \wedge \theta_n$, then

$$i^n (-1)^{\frac{n(n+1)}{2}} \theta \wedge \bar{\theta} = \mathrm{d}V = \langle \theta, \theta \rangle_\omega \mathrm{d}V = \theta \wedge \overline{*\theta}$$

thus

$$* \, \theta = i^n (-1)^{\frac{n(n+1)}{2}} \theta. \tag{4.13}$$

Proposition 4.2.2. *Let ξ be an $(n,1)$-form on a smooth, bounded, open set $G = \{z \,:\, \nu(z) < 0\}$ in a Kähler manifold. Here, ν is a global defining function of G with $\mathrm{d}\nu \neq 0$ on bG. Assume ξ satisfies $\partial\nu \lrcorner \xi = 0$ on bG, and that φ and w are sufficiently smooth real valued functions, then*

$$\begin{aligned} 2\,\mathrm{Re} \int_G \langle \bar{\partial}\bar{\partial}^*_\varphi \xi, \xi \rangle_\omega e^{-\varphi} w \mathrm{d}V = \\ & \int_G i \langle \partial\bar{\partial}\varphi \wedge \Lambda\xi, \xi \rangle_\omega e^{-\varphi} w \mathrm{d}V + \int_G \left|\bar{\partial}^*_\varphi \xi\right|^2_\omega e^{-\varphi} w \mathrm{d}V \\ & + \int_G |\partial^* \xi|^2_\omega e^{-\varphi} w \mathrm{d}V - \int_G \left|\bar{\partial}\xi\right|^2_\omega e^{-\varphi} w \mathrm{d}V \\ & + \int_G i \left(\langle \partial w \wedge \Lambda\xi, \bar{\partial}^*_\varphi \xi \rangle_\omega + \langle \partial w \wedge \bar{\partial}\Lambda\xi, \xi \rangle_\omega \right) e^{-\varphi} \mathrm{d}V \\ & + \int_{bG} \langle i\partial\bar{\partial}\nu \wedge \Lambda\xi, \xi \rangle_\omega e^{-\varphi} w \frac{\mathrm{d}S}{|\mathrm{d}\nu|}. \end{aligned} \tag{4.14}$$

Proof. This will be similar to Lemma 4.2.1 in terms of ξ, where $\gamma = *\xi$. The major difference is the introduction of a Kähler metric, which helped H. Delin to get control of the term $\bar{\partial}\gamma \wedge \overline{\bar{\partial}\gamma}$. Multiplying the entire identity (4.5) with the factor $(-1)^{n-1}(-1)^{\frac{n(n-1)}{2}} = -(-1)^{\frac{n(n+1)}{2}}$, the sign of each term will be the same as the corresponding one in (4.14).

As $\partial\nu \lrcorner \xi = 0$ on bG, we see that for any $(n,0)$-form α

$$0 = \langle \alpha, \partial\nu \lrcorner \xi\rangle_\omega \mathrm{d}V = \langle \bar{\partial}\nu \wedge \alpha, \xi\rangle_\omega \mathrm{d}V = \bar{\partial}\nu \wedge \alpha \wedge \overline{*\xi} = \overline{\partial\nu \wedge \bar{\alpha} \wedge \gamma},$$

which satisfies the boundary condition of Lemma 4.2.1.

Now, the integrand on the left hand side in (4.14) yields by (4.13)

$$\begin{aligned}\langle \bar{\partial}\bar{\partial}^*_\varphi \xi, \xi\rangle_\omega \mathrm{d}V &= \langle \bar{\partial}(- * \partial_\varphi * \xi), \xi\rangle_\omega \mathrm{d}V = -\bar{\partial}(*\partial_\varphi * \xi) \wedge \overline{*\xi} \\ &= -(-1)^{\frac{n(n+1)}{2}} i^n \bar{\partial}\partial_\varphi \gamma \wedge \bar{\gamma}.\end{aligned}$$

For the right hand side, we have the following for each term. The first integrand is by (3.6),

$$\begin{aligned}i\langle \partial\bar{\partial}\varphi \wedge \Lambda\xi, \xi\rangle_\omega \mathrm{d}V &= i\partial\bar{\partial}\varphi \wedge (-i^{n-1}(-1)^{\frac{n(n+1)}{2}} * \xi) \wedge \overline{*\xi} \\ &= -i^n(-1)^{\frac{n(n+1)}{2}} \partial\bar{\partial}\varphi \wedge \gamma \wedge \bar{\gamma}.\end{aligned}$$

Using $** = (-1)^n$ and (4.13), we have for the second term

$$\langle \bar{\partial}^*_\varphi \xi, \bar{\partial}^*_\varphi \xi\rangle_\omega \mathrm{d}V = *\partial_\varphi * \xi \wedge \overline{* * \partial_\varphi * \xi} = i^n(-1)^{\frac{n(n+1)}{2}}(-1)^n \partial_\varphi\gamma \wedge \overline{\partial_\varphi\gamma}.$$

By writing the third and fourth term of (4.14) in normal coordinates, we will obtain that they together correspond to the term $\bar{\partial}\gamma \wedge \overline{\bar{\partial}\gamma}$ of (4.5). Note that all the terms involved consist only of first order derivatives of the forms, hence the following approach is valid when the metric is Kähler.

We may switch to normal coordinates at a point z_0 in G (thus $\omega_{z_0} = \beta = i\partial\bar{\partial}|z|^2$ - without factor $\frac{1}{2}$ since the $\mathrm{d}z_k$ constitute an orthonormal basis at that point, i.e. $|\mathrm{d}z_k|^2 = |\mathrm{d}z_k|^2_\omega = 1$), see Theorem 3.6.3. Let

$$\xi = \sum \xi_j \ \wedge_j^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k.$$

One obtains that, at z_0

$$*\xi = \gamma = i^n(-1)^{\frac{n(n+1)}{2}} \sum \xi_k \mathrm{d}\widehat{z}_k,$$

where $\mathrm{d}\widehat{z}_k = (-1)^{k+1} \bigwedge_{j\neq k} \mathrm{d}z_j$; differentiating, which is possible in normal coordinates, we have, still at z_0,

$$\begin{aligned}
\bar\partial\gamma \wedge \overline{\bar\partial\gamma} &= \Big(\sum_{j,k} \bar\partial_j \xi_k \mathrm{d}\bar z_j \wedge \mathrm{d}\widehat{z}_k\Big) \wedge \Big(\overline{\sum_{j,k} \bar\partial_j \xi_k \mathrm{d}\bar z_j \wedge \mathrm{d}\widehat{z}_k}\Big) \\
&= (-1)^{2n-1}\Big(\sum_{j,k} \bar\partial_j \xi_k \overline{\bar\partial_k \xi_j}\Big) \bigwedge_{j=1}^{n} \mathrm{d}z_j \wedge \bigwedge_{j=1}^{n} \mathrm{d}\bar z_j \\
&= \Big(\frac{1}{2}\sum_{j,k} |\bar\partial_j\xi_k - \bar\partial_k\xi_j|^2 - \sum_{j,k} |\bar\partial_j\xi_k|^2\Big) \bigwedge_{j=1}^{n} \mathrm{d}z_j \wedge \bigwedge_{j=1}^{n} \mathrm{d}\bar z_j \\
&= \Big(\big|\bar\partial\xi\big|_\omega^2 - \big|\bar\partial\gamma\big|_\omega^2\Big) \bigwedge_{j=1}^{n} \mathrm{d}z_j \wedge \bigwedge_{j=1}^{n} \mathrm{d}\bar z_j.
\end{aligned}$$

Where we have used that in normal coordinates $|f|^2$ coincides with $|f|_\omega^2$ at z_0. We have that $|\bar\partial\gamma| = |*\bar\partial\gamma|$ by definitions of the star operator for Euclidean norm and the absolute value.

For the equality $|*\bar\partial\gamma| = |*\bar\partial*\xi| = |\partial^*\xi|$ we use that $\bar\partial^* = -*\partial*$ as used earlier in this proof (but now for $\varphi = 0$) and that $\Big|\overline{*\bar\partial*\xi}\Big| = |*\bar\partial*\xi|$.

We use that

$$\bigwedge_{j=1}^{n} \mathrm{d}z_j \wedge \bigwedge_{j=1}^{n} \mathrm{d}\bar z_j = (-1)^{\frac{n(n-1)}{2}} \bigwedge_{j=1}^{n} \ \mathrm{d}z_j \wedge \mathrm{d}\bar z_j = \frac{1}{i^n n!}(-1)^{\frac{n(n-1)}{2}}\beta^n$$

and conclude

$$-i^n(-1)^{\frac{n(n-1)}{2}}\partial\gamma \wedge \overline{\bar\partial\gamma} = \Big(\big|\bar\partial\gamma\big|_\omega^2 - \big|\bar\partial\xi\big|_\omega^2\Big)\,\mathrm{d}V = \Big(\big|\partial^*\xi\big|_\omega^2 - \big|\bar\partial\xi\big|_\omega^2\Big)\,\mathrm{d}V,$$

which gives the identity for the third and fourth term, at an arbitrary point z_0.

The two terms arising from differentiation of the weight function w are by (3.6)

$$\begin{aligned}
i\langle \partial w \wedge \Lambda\xi, \bar\partial_\varphi^*\xi\rangle_\omega \mathrm{d}V &= (-1)^{\frac{n(n+1)}{2}} i^n \langle \partial w \wedge *\xi, *\partial_\varphi * \xi\rangle_\omega \mathrm{d}V \\
&= (-1)^{\frac{n(n+1)}{2}} i^n \partial w \wedge \gamma \wedge \overline{**\partial_\varphi\gamma} \\
&= (-1)^n(-1)^{\frac{n(n+1)}{2}} i^n \partial w \wedge \gamma \wedge \overline{\partial_\varphi\gamma}
\end{aligned}$$

as well as

$$\begin{aligned}
&i\langle \partial w \wedge \bar\partial\Lambda\xi, \xi\rangle_\omega \mathrm{d}V = \\
&\qquad - i^n(-1)^{\frac{n(n+1)}{2}} \langle \partial w \wedge \bar\partial * \xi, \xi\rangle_\omega \mathrm{d}V = -i^n(-1)^{\frac{n(n+1)}{2}} \partial w \wedge \bar\partial\gamma \wedge \bar\gamma.
\end{aligned}$$

The boundary integral is treated similarly as the first term:

$$i\langle \partial\bar\partial\nu \wedge \Lambda\xi, \xi\rangle_\omega = *\big(i\langle \partial\bar\partial\nu \wedge \Lambda\xi, \xi\rangle_\omega \mathrm{d}V\big) = -i^n(-1)^{\frac{n(n+1)}{2}} * \big(\partial\bar\partial\nu \wedge \gamma \wedge \bar\gamma\big)$$

which finishes the proof. ■

4.3 A weighted L^2-Estimate

Given the following lemma, we will prove Theorem 4.1.1 on the spot.

It is appropriate to mention that this author claims that we actually demand $\bar{\partial}(u\alpha) = \hat{f}$ in the upcoming, where $\alpha = \| \bigwedge_j^n \mathrm{d}z_j \|_\omega$ and $\hat{f} = \sum_{j=1}^n f_j \beta_j \bigwedge_{r=1}^n \mathrm{d}z_r \wedge \mathrm{d}\bar{z}_j$ with $\beta_j = \| \bigwedge_{r=1}^n \mathrm{d}z_r \wedge \mathrm{d}\bar{z}_j \|_\omega$. The β_j's and α are continuous, bounded weights on the coefficients of f and u, if ω is continuous and bounded, such that $|u|_\omega^2 = |u\alpha|^2$ and $|f|_\omega^2 = |\hat{f}|^2$ - a subtle change, without side-effects nor echo, no harm done.

Lemma 4.3.1. *Given a closed $(n,1)$-form f on a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and a weight function w on Ω satisfying $|\partial w|_\omega \leq \varepsilon w$ for some $\varepsilon \in (0, \sqrt{2})$. Let φ be a strictly plurisubharmonic $\mathcal{C}^2$ function on Ω and u be the $L_\varphi^2(\Omega, \omega, \mathrm{d}V)$-minimal solution to $\bar{\partial} u = f$, then*

$$\int |u|_\omega^2 \, e^{-\varphi} w \mathrm{d}V \leq \frac{2}{(\sqrt{2} - \varepsilon)^2} \int |f|_\omega^2 \, e^{-\varphi} w \mathrm{d}V, \tag{4.15}$$

where the metric is given by $\omega = i\partial\bar{\partial}\varphi$.

The existence of such a φ is guaranteeded, see the Appendix.

Proof of Theorem 4.1.1. We apply Lemma 4.3.1 to $\tilde{f} = f \wedge_j^n \mathrm{d}z_j$, where f is a closed $(0,1)$-form, and get the $L_\varphi^2(\Omega, \omega, \mathrm{d}V)$-minimal solution $\tilde{u} = u \wedge_j^n \mathrm{d}z_j$ to $\bar{\partial}\tilde{u} = \tilde{f}$. Then $\bar{\partial} u = f$ and by (4.13) as well as (3.3)

$$\begin{aligned} |\tilde{u}|_\omega^2 \, \mathrm{d}V &= i^n (-1)^{\frac{n(n-1)}{2}} \tilde{u} \wedge \bar{\tilde{u}} = \alpha^2 i^n (-1)^{\frac{n(n-1)}{2}} |u|^2 \bigwedge_{j=1}^n \mathrm{d}z_j \wedge \bigwedge_{j=1}^n \mathrm{d}\bar{z}_j \\ &= 2^n |u\alpha|^2 \bigwedge_{j=1}^n \ \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j \end{aligned}$$

and similarly that

$$\left|\tilde{f}\right|_\omega^2 \mathrm{d}V = \left|f\right|_\omega^2 \langle \wedge_j^n \mathrm{d}z_j, \wedge_j^n \mathrm{d}z_j \rangle_\omega \mathrm{d}V = 2^n |\hat{f}|^2 \bigwedge_{j=1}^n \ \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

Hence the function $u\alpha$ is the $L_\varphi^2(\Omega, \ \mathrm{d}\lambda)$-minimal solution to $\bar{\partial}(u\alpha) = \hat{f}$ for $(0,1)$-forms; by Lemma 4.3.1, the desired inequality for u and f follows. ■

Proof of Lemma 4.3.1. First we assume the Ω is smooth, bounded and strictly pseudoconvex domain and that φ is smooth. Let f be a $(n,1)$-form and ξ the $\bar{\partial}$-Neumann solution to

$$\Box'' \xi = (\bar{\partial}\bar{\partial}_\varphi^* + \bar{\partial}_\varphi^* \bar{\partial})\xi = f \text{ on } \Omega. \tag{4.16}$$

If $\bar{\partial} f = 0$, we have

$$\bar{\partial}\xi = 0$$

by a commutativity property of the $\bar{\partial}$-Neumann solution operator, see for instance page 83 of [Has14], and hence the second term in (4.16) vanishes, thus

$$\Box''\xi = \bar{\partial}\bar{\partial}^*_\varphi \xi = f.$$

Let

$$u = \bar{\partial}^*_\varphi \xi.$$

This is the $L^2_\varphi(\Omega, \omega, \mathrm{d}V)$-minimal solution to $\bar{\partial} u = f$, since u is orthogonal to all holomorphic $(n, 0)$-forms H by

$$\int \langle u, H\rangle_\omega e^{-\varphi}\mathrm{d}V = \int \langle \bar{\partial}^*_\varphi \xi, H\rangle_\omega e^{-\varphi}\mathrm{d}V = \int \langle \xi, \bar{\partial} H\rangle_\omega e^{-\varphi}\mathrm{d}V = 0;$$

minimality follows by substituing $(u' - u)$ in H, for another solution u' since $\bar{\partial}(u'-u) = 0$, which is holomorphicity, plus Cauchy- Schwarz for the inner-product $\langle\cdot,\cdot\rangle_\omega$. In addition, from $\bar{\partial}\xi = 0$ it follows by (3.5) that $\bar{\partial}\Lambda\xi = [\bar{\partial}, \Lambda]\xi = i\partial^*\xi$.

We apply Proposition 4.2.2 to this ξ and by the strict pseudoconvexity of Ω, the boundary integral will be positive and we drop it, obtaining

$$\begin{aligned} 2\,\mathrm{Re}\int \langle f, \xi\rangle_\omega e^{-\varphi} w\mathrm{d}V \geq & \int i\langle \partial\bar{\partial}\varphi \wedge \Lambda\xi, \xi\rangle_\omega e^{-\varphi} w\mathrm{d}V \\ & + \int |u|^2_\omega e^{-\varphi} w\mathrm{d}V + \int |\partial^*\xi|^2_\omega e^{-\varphi} w\mathrm{d}V \\ & + \int i\langle \partial w \wedge \Lambda\xi, u\rangle_\omega e^{-\varphi}\mathrm{d}V \\ & - \int \langle \partial w \wedge \partial^*\xi, \xi\rangle_\omega e^{-\varphi}\mathrm{d}V. \end{aligned} \tag{4.17}$$

We use the bound on ∂w and the standard estimate for integrals and that

$$|\partial w \wedge \Lambda\xi|^2_\omega = |\partial w|^2_\omega |\Lambda\xi|^2_\omega \leq \varepsilon^2 w^2 |\xi|^2_\omega \tag{4.18}$$

with $|\Lambda\xi|^2_\omega = |{*\xi}|^2_\omega$, to estimate the third line on the right hand side of (4.17) by

$$\begin{aligned} \int & \left|\langle \partial w \wedge \Lambda\xi, u\rangle_\omega e^{-\varphi}\right|\mathrm{d}V \\ & \leq \frac{\delta}{2w}\int |\partial w \wedge \Lambda\xi|^2_\omega e^{-\varphi}\mathrm{d}V + \frac{w}{2\delta}\int |u|^2_\omega e^{-\varphi}\mathrm{d}V \\ & \leq \frac{\delta\varepsilon^2}{2}\int |\xi|^2_\omega e^{-\varphi} w\mathrm{d}V + \frac{1}{2\delta}\int |u|^2_\omega e^{-\varphi} w\mathrm{d}V, \end{aligned} \tag{4.19}$$

where we used repeatedly the large constant - small constant trick:

$$0 \le \left(\frac{1}{\sqrt{\delta}}a + \sqrt{\delta}b\right)^2 \Rightarrow ab \le \frac{1}{2\delta}a^2 + \frac{\delta}{2}b^2$$

for a $\delta > 0$ which will be determined later on. For the last term of (4.17), we do the same

$$\begin{aligned}
&\int \left|\langle \partial w \wedge \partial^* \xi, \xi\rangle_\omega e^{-\varphi}\right| \mathrm{d}V \\
&\qquad \le \frac{1}{\varepsilon^2 w}\int |\partial w \wedge \partial^* \xi|_\omega^2 \, e^{-\varphi}\mathrm{d}V + \frac{\varepsilon^2 w}{4}\int |\xi|_\omega^2 \, e^{-\varphi}\mathrm{d}V \qquad (4.20)\\
&\qquad \le \int |\partial^* \xi|_\omega^2 \, e^{-\varphi} w\mathrm{d}V + \frac{\varepsilon^2}{4}\int |\xi|_\omega^2 \, e^{-\varphi} w\mathrm{d}V.
\end{aligned}$$

To get control of the term $\int |\xi|_\omega^2 \, e^{-\varphi} w\mathrm{d}V$, H. Delin used an appropriate metric in order that $\partial\bar{\partial}\varphi$ became a majorant.

So we use on Ω the metric induced by the form $\omega = i\partial\bar{\partial}\varphi$, then

$$i\langle \partial\bar{\partial}\varphi \wedge \Lambda\xi, \xi\rangle_\omega = \langle \omega \wedge \Lambda\xi, \xi\rangle_\omega = |\Lambda\xi|_\omega^2 = |\xi|_\omega^2 .$$

Multiplying (4.19) and (4.20) by (-1) and augmenting inequality (4.17) to the right, we see that the term with $|\partial^*\xi|_\omega^2$ cancels out and we are left with

$$\begin{aligned}
&2\,\mathrm{Re}\int \langle f, \xi\rangle_\omega w\mathrm{d}V \\
&\qquad \ge \left(1 - \frac{\varepsilon^2\delta}{2} - \frac{\varepsilon^2}{4}\right)\int |\xi|_\omega^2 \, e^{-\varphi} w\mathrm{d}V + \left(1 - \frac{1}{2\delta}\right)\int |u|_\omega^2 \, e^{-\varphi} w\mathrm{d}V,
\end{aligned} \qquad (4.21)$$

where δ is positive but arbitrary for now. Further, we estimate

$$\left|2\,\mathrm{Re}\int \langle f, \xi\rangle_\omega w\mathrm{d}V\right| \le \frac{1}{\delta'}\int |f|_\omega^2 \, e^{-\varphi} w\mathrm{d}V + \delta'\int |\xi|_\omega^2 \, e^{-\varphi} w\mathrm{d}V \qquad (4.22)$$

and choose $\delta > \frac{1}{2}$ such that

$$\left(1 - \frac{\varepsilon^2\delta}{2} - \frac{\varepsilon^2}{4}\right) = \delta' > 0,$$

which is possible if $\varepsilon^2 < 4 < 2(2\delta+1)$. Since $\varepsilon \in (0, \sqrt{2})$, we chose $\delta = \sqrt{\varepsilon^{-2} - 4^{-1}}$ and combine (4.21) and (4.22) to obtain

$$\begin{aligned}
\int |u|_\omega^2 \, e^{-\varphi} w\mathrm{d}V &\le \frac{1}{\left(1 - \frac{\varepsilon^2\delta}{2} - \frac{\varepsilon^2}{4}\right)\left(1 - \frac{1}{2\delta}\right)}\int |f|_\omega^2 \, e^{-\varphi} w\mathrm{d}V \\
&= 2\big(2 - \varepsilon\sqrt{4 - \varepsilon^2}\big)^{-1}\int |f|_\omega^2 \, e^{-\varphi} w\mathrm{d}V \\
&\le 2\big(\sqrt{2} - \varepsilon\big)^{-2}\int |f|_\omega^2 \, e^{-\varphi} w\mathrm{d}V.
\end{aligned}$$

This proves the result for very nice domains.

For an arbitrary pseudoconvex set Ω, we take an increasing sequence[3] of smooth, bounded and strictly pseudoconvex domains Ω_j with $\bigcup \Omega_j = \Omega$. Let u_j be the L^2-minimal solution to $\bar{\partial} u_j = f$ on Ω_j and thus u_j satisfies there the result obtained so far. For $j \geq j_0$

$$\begin{aligned}\int_{\Omega_{j_0}} |u_j|^2_\omega \, e^{-\varphi} w \mathrm{d}V &\leq \int_{\Omega_j} |u_j|^2_\omega \, e^{-\varphi} w \mathrm{d}V \\ &\leq 2\big(\sqrt{2} - \varepsilon\big)^{-2} \int_{\Omega_j} |f|^2_\omega \, e^{-\varphi} w \mathrm{d}V \\ &\leq 2\big(\sqrt{2} - \varepsilon\big)^{-2} \int_{\Omega} |f|^2_\omega \, e^{-\varphi} w \mathrm{d}V.\end{aligned}$$

It follows that $\{u_j\}_j$ is a bounded sequence in $L^2(\Omega_{j_0}, \omega, e^{-\varphi} w \mathrm{d}V)$ as well as in $L^2(\Omega_{j_0}, \omega, e^{-\varphi} \mathrm{d}V)$, where one sets $w \equiv 1$. By a diagonal argument, the following sequence

$$A_j = \int_{\Omega_j} |u_j|^2_\omega \, e^{-\varphi} \mathrm{d}V \leq \int_{\Omega} |f|^2_\omega \, e^{-\varphi} \mathrm{d}V$$

is bounded. We thus can find a weak limit u of a subsequence of $\{u_j\}_j$ in $L^2(\Omega_{j_0}, \omega, e^{-\varphi} \mathrm{d}V)$ such that $\{A_j\}_j$ converges too. Since w is locally bounded, u is also a weak limit of a subsequence of $\{u_j\}_j$ in $L^2(\Omega_{j_0}, \omega,\ e^{-\varphi} w \mathrm{d}V)$, and as such it solves $\bar{\partial} u = f$ on Ω_{j_0}.

Now, by increasing j_0 and taking further subsequences, we may assume without loss of generality $u_j \to u$ weakly in $L^2(\Omega_{j_0}, \omega, e^{-\varphi} w \mathrm{d}V)$ as well as in $L^2(\Omega_{j_0}, \omega, e^{-\varphi} \mathrm{d}V)$ for every j_0.

Since weak convergence decreases norms, we have

$$\begin{aligned}\int_{\Omega_{j_0}} |u|^2_\omega \, e^{-\varphi} w \mathrm{d}V &\leq \lim_{j \to \infty} \int_{\Omega_{j_0}} |u_j|^2_\omega \, e^{-\varphi} w \mathrm{d}V \\ &\leq 2\big(\sqrt{2} - \varepsilon\big)^{-2} \int_{\Omega} |f|^2_\omega \, e^{-\varphi} w \mathrm{d}V,\end{aligned} \tag{4.23}$$

which, by monotone convergence[4], is valid on whole Ω.

To prove that u is the $L^2_\varphi(\Omega, \omega, \mathrm{d}V)$-minimal solution we regard similarly to (4.23)

$$\int_{\Omega_{j_0}} |u|^2_\omega \, e^{-\varphi} \mathrm{d}V \leq \lim_{j \to \infty} \int_{\Omega_{j_0}} |u_j|^2_\omega \, e^{-\varphi} \mathrm{d}V \leq \lim_{j \to \infty} A_j,$$

[3] This is always possible, see the Appendix.

[4] Regard the sequence $\{\chi_j u\}_j$, where χ_j is the characteristic function of Ω_j.

and extend[4] to Ω. Suppose u_0 to be the minimal solution, then, since u_j is so in Ω_j,

$$A_j = \int_{\Omega_j} |u_j|^2_\omega \, e^{-\varphi} \mathrm{d}V \leq \int_{\Omega_j} |u_0|^2_\omega \, e^{-\varphi} \mathrm{d}V \leq \int_{\Omega} |u_0|^2_\omega \, e^{-\varphi} \mathrm{d}V$$
$$\leq \int_{\Omega} |u|^2_\omega \, e^{-\varphi} \mathrm{d}V \leq \lim_{j\to\infty} A_j.$$

As j approaches infinity, one obtains equal norms of u_0 and u, and thus, by uniqueness, $u_0 = u$.

If φ is not smooth, but still in $\mathcal{C}^2$, we may use convolution with an approximate identity on Ω_j and hence get a sequence of smooth $\{\varphi_j\}_j \downarrow \varphi$ with uniform convergence of $i\partial\bar{\partial}\varphi_j \to i\partial\bar{\partial}\varphi$ in Ω_j by continuity.

We proceed as above where we take this detail into account when choosing subsequences $\{u_j\}_j$ and finish the proof. ■

4.4 The L^2-Estimate of the Kernel

The next lemma is crucial. In our case, where we exclusively regard balls with constant radii, it can be shown that the appearing constant is universal, i.e. independent of the domain, see the Appendix for an almost complete argument. H. Delin seemingly also refers to [LG86, Theorem 2.28], page 42; where another proof for balls can be found and which is much easier to understand.

Lemma 4.4.1. *Assume ω is a positive, bounded, continuous,* d*-closed $(1,1)$-form on a neighbourhood of a smooth, strictly pseudoconvex, star shaped domain. Then there exists a plurisubharmonic function ψ on the domain such that $i\partial\bar{\partial}\psi = \omega$ and $\|\psi\|_{L^\infty} \leq C\|\omega\|_{L^\infty}$ where C depends on the dimension as well as on the domain itself.*

Proof. Since ω is closed, the Poincaré lemma says there exists w such that $\mathrm{d}w = \omega$. Decomposing w we may write $\mathrm{d}(w_{0,1} + w_{1,0}) = \omega$ where

$$w_{0,1} = \sum_{j,k} \int_0^1 t\omega_{kj}(tz)\mathrm{d}t \; z_j \mathrm{d}\bar{z}_k$$

and

$$w_{1,0} = \overline{w_{0,1}} = \sum_{j,k} \int_0^1 t\omega_{kj}(tz)\mathrm{d}t \; \bar{z}_j \mathrm{d}z_k. \tag{4.24}$$

So $w_{0,1}$ and $w_{1,0}$ are bounded by $\|\omega\|_{L^\infty}$ and by bidegree reasons we have that $\bar{\partial}w_{0,1} = \partial w_{1,0} = 0$. By (4.24) it will be enough to solve $i\bar{\partial}v = w_{0,1}$ and to set $\psi = 2\operatorname{Re} v$; that way we obtain $i\partial\bar{\partial}\psi = i\partial\bar{\partial}(v+\bar{v}) = \partial w_{0,1} + \bar{\partial}w_{1,0} = \mathrm{d}(w_{0,1}+w_{1,0}) = \omega$.

The proof will be finished if we can find a v which is bounded in supremum norm by $w_{0,1}$, a result[5] that can be found in the book of Henkin and Leiterer [HL84], Theorem 2.6.1 on page 82. It says, there exists a constant C, such that if w is a continuous $(0,q)$-form on $\overline{D}$, where D is supposed to be a smooth and strongly pseudoconvex set, with $\bar{\partial} w = 0$ in D, then one can find a solution v to $\bar{\partial} v = w$ which is of bounded Hölder norm, i.e.

$$\|v\|_{\mathcal{C}^{1/2}(D)} \leq C\|w\|_{L^\infty}.$$

Especially, $\|v\|_{L^\infty} \leq C\|w\|_{L^\infty}$. ■

The plan is as follows; by the formula for the Bergman kernel and how it transforms under holomorphic mappings, we map the Hermitian form h_z to the Euclidean metric β near z; such that our transformed Kähler metric is majorized by the Euclidean one (near z).

Let this map be η and let $\mathcal{L}$ be the linear operator representing h in the Euclidean metric, thus

$$\langle v, v\rangle_h = \langle v, \mathcal{L}v\rangle_\beta = \langle \eta v, \eta v\rangle_\beta$$

and such that

$$\eta\big(B_h(\zeta_0, 1)\big) = B_\beta(\eta(\zeta_0), 1). \tag{4.25}$$

We see that we can take η as the square root of the positive operator $\mathcal{L}$, then η is in fact a linear operator for which $(\eta^{-1})^* h = \beta$, the pull-back of h. Also $\mathcal{J}_{\mathbb{C}}\eta$ is constant, and

$$\mathcal{B}_\varphi(z,\zeta) = |\mathcal{J}_{\mathbb{C}}\eta|^2 \mathcal{B}_{\varphi\circ\eta^{-1}}\big(\eta(z), \eta(\zeta)\big) = \det(h)\mathcal{B}_{\varphi\circ\eta^{-1}}\big(\eta(z), \eta(\zeta)\big). \tag{4.26}$$

By $i\partial\bar{\partial}\varphi \leq h$ on $B_h(0,1)$ it follows $i\partial\bar{\partial}(\varphi\circ\eta^{-1}) = (\eta^{-1})^* i\partial\bar{\partial}\varphi \leq \beta$ on $\eta\big(B_h(0,1)\big) = B_\beta(0,1)$ and also

$$d_\varphi(z,\zeta) = d_{\varphi\circ\eta^{-1}}(\eta(z), \eta(\zeta)).$$

Finally we remind ourselves of the invariance under change of coordinates of the eigenvalues with respect to a given form.

Proposition 4.4.2. *Let $\mathcal{B}_\varphi(z,\zeta)$ be the Bergman kernel in $L^2_\varphi(\mathbb{C}^n)$ and $d(z,\zeta)$ be the distance function of the metric with metric form given by $\omega = i\partial\bar{\partial}\varphi$ and let h_{ζ_0} satisfy* (4.3)*. Then for some $\varepsilon \in (0, \sqrt{2})$ and any fixed ζ_0, we have for a constant $C > 0$, depending on the dimension only, that*

$$\int |\mathcal{B}_\varphi(z,\zeta_0)|^2 e^{-\varphi} e^{\varepsilon d(z,\zeta_0)}\, \mathrm{d}\lambda \leq \frac{C\det(h_{\zeta_0}) e^{\varphi(\zeta_0)}}{(\sqrt{2}-\varepsilon)^2 \inf_{\zeta\in B_{h_{\zeta_0}}(\zeta_0,1)} \lambda_{min}(\omega(\zeta)|h_{\zeta_0})}. \tag{4.27}$$

[5]We will use this for the case of balls only and hence [HL84, Theorem 2.2.2] will suffice and give us a universal constant as has been explained in the Appendix.

Proof. First we assume that $\omega \leq \beta$ in $B_\beta(\zeta_0, 1)$ and hence we assume h_{ζ_0} to be the Euclidean metric. Let $\chi : \mathbb{C}^n \to \mathbb{R}$ be a non-negative radial function compactly supported in $B_\beta(\zeta_0, 1)$ with $\int \chi \, \mathrm{d}\lambda = 1$. Then for any harmonic function g,

$$g(\zeta_0) = \int_{B_\beta(\zeta_0,1)} \chi \cdot g \, \mathrm{d}\lambda. \tag{4.28}$$

Let in addition $H(\zeta)$ be a holomorphic function in $B_\beta(\zeta_0, 1)$ with $H(\zeta_0) = 0$ and define

$$v(\zeta) = \chi(\zeta) e^{\varphi(\zeta) + \overline{H(\zeta)}}.$$

Since $\mathcal{B}_\varphi(z, \zeta)$ is antiholomorphic in ζ, we have by (4.28) that for any z

$$\mathcal{B}_\varphi v(z) = \int_{B_\beta(\zeta_0,1)} \mathcal{B}_\varphi(z,\zeta)\chi(\zeta) e^{\overline{H(\zeta)}} \, \mathrm{d}\lambda(\zeta) = \mathcal{B}_\varphi(z,\zeta_0) e^{\overline{H(\zeta_0)}} = \mathcal{B}_\varphi(z,\zeta_0).$$

Now we set $f = \bar{\partial} v$, hence $\bar{\partial} f = 0$, and let u denote the $L^2_\varphi(\mathbb{C}^n, \mathrm{d}\lambda)$-minimal solution to $\bar{\partial} u = f$. We thus can decompose v orthogonally as

$$v = u + \mathcal{B}_\varphi v,$$

or

$$\mathcal{B}_\varphi(z, \zeta_0) = v(z) - u(z). \tag{4.29}$$

By compact support of v, it vanishes unless z and ζ_0 are close to each other and, since χ is bounded,

$$|v(z)|^2 e^{-\varphi(z)} \leq C e^{\varphi(z) + 2\operatorname{Re} H(z)}; \tag{4.30}$$

which will become dominated by a part coming from the estimate on u.

We will use Theorem 4.1.1 to estimate the $L^2_\varphi(\mathbb{C}^n,\ \mathrm{d}\lambda)$-minimal solution; but first note that

$$f = \bar{\partial} v = \big(\bar{\partial}\chi + \chi \bar{\partial}(\varphi + \overline{H})\big) e^{\varphi + \overline{H}}.$$

The norm $|\cdot|_\omega$ on 1-forms is given by the inverse[6] of ω and can be estimated from above by the Euclidean norm and $\lambda^{-1}_{\min}(\omega|\beta)$, see its definition prior to Theorem 4.1.2. This yields

$$|f|^2_\omega \, e^{-\varphi} \leq \frac{2}{\lambda_{\min}(\omega|\beta)} \left(\left|\bar{\partial}\chi\right|^2_\beta e^{\varphi + 2\operatorname{Re} H} + \chi^2 \left|\bar{\partial}(\varphi + \overline{H})\right|^2_\beta e^{\varphi + 2\operatorname{Re} H} \right). \tag{4.31}$$

In the Euclidean metric, $\bar{\partial}\chi$ is bounded by a constant and since H is holomorphic ($\bar{\partial} H = 0$) and φ is plurisubharmonic, one can see that

$$\left|\bar{\partial}(\varphi + \overline{H})\right|^2_\beta e^{\varphi + 2\operatorname{Re} H} = \left|\bar{\partial}(\varphi + 2\operatorname{Re} H)\right|^2_\beta e^{\varphi + 2\operatorname{Re} H} \leq \Delta_\beta e^{\varphi + 2\operatorname{Re} H}.$$

[6] This stems from a fact concerning musical isomorphisms.

Since χ^2 has compact support, Green's formula:

$$\int_D (u\Delta v + v\Delta u)\,\mathrm{d}\lambda = \int_{bD}\left(u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N}\right)\mathrm{d}S$$

allows us to apply the Δ_β on χ^2 instead the exponential factor. This yields

$$\int |f|^2_\omega\, e^{-\varphi}\,\mathrm{d}\lambda \le \frac{2}{\inf_{\zeta\in B_\beta(\zeta_0,1)}\lambda_{\min}(\omega(\zeta)|\beta)}\int_{B_\beta(\zeta_0,1)}\left(C+\Delta_\beta\chi^2\right)e^{\varphi+2\operatorname{Re}H}\,\mathrm{d}\lambda.$$

Now, $\Delta_\beta\chi^2$ is bounded, and can be absorbed into C. Also, $d_\omega(z,\zeta_0)$ is bounded by 1 on $B_\beta(\zeta_0,1)$, since $\omega\le\beta$ there and hence $w=e^{\varepsilon d(z,\zeta_0)}$ is bounded from above and below by positive constants. It therefore can be inserted into the integral, and we obtain by Theorem 4.1.1,

$$\begin{aligned}\int |u(z)|^2 e^{-\varphi(z)}e^{\varepsilon d(z,\zeta_0)}\,\mathrm{d}\lambda&\\ &\le \frac{2}{(\sqrt{2}-\varepsilon)^2}\int |f(z)|^2_\omega\, e^{-\varphi(z)}e^{\varepsilon d(z,\zeta_0)}\,\mathrm{d}\lambda \qquad (4.32)\\ &\le \frac{C}{(\sqrt{2}-\varepsilon)^2\inf_{\zeta\in B_\beta(\zeta_0,1)}\lambda_{\min}(\omega(\zeta)|\beta)}\int_{B_\beta(\zeta_0,1)} e^{\varphi+2\operatorname{Re}H}\,\mathrm{d}\lambda.\end{aligned}$$

By (4.30), the same estimate holds with $u(\cdot)$ replaced by $\mathcal{B}_\varphi(\cdot,\zeta_0)$. The only thing that remains to complete the proof for the case $\omega\le\beta$ is to estimate

$$\int_{B_\beta(\zeta_0,1)} e^{\varphi+2\operatorname{Re}H}\,\mathrm{d}\lambda.$$

By Lemma 4.4.1, there exists a plurisubharmonic function ψ with $i\partial\bar\partial\psi=\omega=i\partial\bar\partial\varphi$, which is bounded by a constant on $B_\beta(\zeta_0,1)$. Then, since $\psi-\varphi$ is pluriharmonic, there exists a holomorphic function $\widetilde{H}$ such that $2\operatorname{Re}\widetilde{H}=\psi-\varphi$. We define now $H=\widetilde{H}-\widetilde{H}(\zeta_0)=\psi-\varphi-\psi(\zeta_0)-\varphi(\zeta_0)$, then we may write

$$\int_{B_\beta(\zeta_0,1)} e^{\varphi+2\operatorname{Re}H}\,\mathrm{d}\lambda = \int_{B_\beta(\zeta_0,1)} e^{\psi-\psi(\zeta_0)+\varphi(\zeta_0)}\,\mathrm{d}\lambda \le Ce^{\varphi(\zeta_0)}$$

since ψ is bounded. This finishes the proof for the case $\omega\le\beta$ around ζ_0.

The general case is substitution and variable transform[7] of the function η, as mentioned earlier with ζ_0 as origin, into the case discussed above. In detail, we

[7]Note that $\det\mathcal{J}_\mathbb{R}F=|\det\mathcal{J}_\mathbb{C}F|^2$ as can be found in [Ran98, Lemma 2.1], page 19.

apply (4.26) for the first equality, use variable transform by η and (4.25) to use the result obtained in the special case:

$$\begin{aligned}
\int_{B_h(\zeta_0,1)} & |\mathcal{B}_\varphi(z,\zeta_0)|^2 e^{\varepsilon d_\varphi(z,\zeta_0)} e^{-\varphi(z)} \, \mathrm{d}\lambda(z) \\
&= \int_{B_h(\zeta_0,1)} |\mathcal{B}_{\varphi\circ\eta^{-1}}(\eta(z),\eta(\zeta_0))|^2 |\mathcal{J}_{\mathbb{C}}\eta|^4 e^{\varepsilon d_\varphi(z,\zeta_0)} e^{-\varphi(z)} \, \mathrm{d}\lambda(z) \\
&= \int_{\eta(B_h(\zeta_0,1))} |\mathcal{B}_{\varphi\circ\eta^{-1}}(z,\eta(\zeta_0))|^2 |\mathcal{J}_{\mathbb{C}}\eta|^2 e^{\varepsilon d_{\varphi\circ\eta^{-1}}(z,\eta(\zeta_0))} e^{-\varphi\circ\eta^{-1}(z)} \, \mathrm{d}\lambda(z) \\
&\leq \frac{C \det(h_{\zeta_0}) e^{-\varphi\circ\eta^{-1}(\eta(\zeta_0))}}{(\sqrt{2}-\varepsilon)^2 \inf_{\xi\in B_\beta(\eta(\zeta_0),1)} \lambda_{\min}(\partial\bar\partial\varphi\circ\eta^{-1}(\xi)|\beta)} \\
&\leq \frac{C \det(h_{\zeta_0}) e^{-\varphi(\zeta_0)}}{(\sqrt{2}-\varepsilon)^2 \inf_{\zeta\in B_{h_{\zeta_0}}(\zeta_0,1)} \lambda_{\min}(\omega|h_{\zeta_0})},
\end{aligned}$$

which completes the proof. ■

4.5 Pointwise Estimates

We turn now to the key to pointwise estimates, [Ber97, Lemma 3.1]:

Lemma 4.5.1. *Let φ be plurisubharmonic on $B = B_\beta(\zeta_0,1)$. Set*

$$M_\varphi = \{v \leq 0 \ : \ \partial\bar\partial v = \partial\bar\partial\varphi \ \text{on } B\}$$

and put $a_\varphi = \sup_{M_\varphi} v(0)$. Assume that $u \in L^2_{loc}(B)$ satisfies

$$\int_B |u|^2 e^{-\varphi} \leq 1$$

and

$$\sup_B |\bar\partial u|^2 e^{-\varphi} \leq 1.$$

Then

$$|u(0)|^2 e^{-\varphi(0)+a_\varphi} \leq C,$$

where C is a universal constant independent of φ.

Remark 4.5.2. *In the case we are interessted in, $i\partial\bar\partial\varphi$ is uniformly bounded, and Lemma 4.4.1 then says that a_φ is bounded by a constant. The conclusion is that*

$$|u(0)|^2 \leq C e^{\varphi(0)}.$$

Proof of Lemma 4.5.1. When $\varphi \equiv 0$, we take a cut-off function χ on B which equals 1 when $|z| < 1/2$. Let

$$K(z, \zeta) = \sigma_n \partial |\zeta - z|^{2-2n}$$

be the Bochner-Mortinelli kernel, see [Ran98, page 148]. Then, with $K_z = K(z, 0)$,

$$u(0) = \int_B \bar{\partial}(\chi u) K_z \, \mathrm{d}\lambda = \int_B \chi \bar{\partial} u K_z \, \mathrm{d}\lambda + \int u \bar{\partial} \chi K_z \, \mathrm{d}\lambda.$$

The first term on the right hand side can be estimated by $\|\bar{\partial} u\|_{L^\infty}$, since $K_z \in L^1(B)$; the second by $\|u\|_{L^2}$, since $\bar{\partial}\chi = 0$ for $|z| < 1/2$, proving the lemma for the case $\varphi \equiv 0$.

For other plurisubharmonic φ, we let $v \in M_\varphi$. Then $v - \varphi$ is pluriharmonic, and hence there is a holomorphic H such that $v = \varphi + 2 \operatorname{Re} H$. Thus

$$\int_B |ue^H|^2 e^{-v} \, \mathrm{d}\lambda = \int_B |u|^2 e^{-\varphi} \leq 1,$$

and

$$|\bar{\partial}(ue^H)|^2 e^{-v} = |e^H \bar{\partial} u|^2 e^{-v} = |\bar{\partial} u|^2 e^{-\varphi} \leq 1.$$

Since $e^{-v} > 1$, the same inequalities hold with the factor removed and the case $\varphi \equiv 0$ applies to ue^H; we find that

$$|u(0)|^2 e^{v(0) - \varphi(0)} = |u(0) e^{H(0)}|^2 \leq C.$$

Taking supremum over all v finishes the proof. ■

Proof of Theorem 4.1.2. We are going to use the left hand side of (4.27), Lemma 4.5.1 and work along the same lines as in the proof of Proposition 4.4.2.

For a given point z_0, let h_{z_0} be an Hermitian form on the tangent space, satisfying (4.3). Let η be the complex linear mapping for h_{z_0} as defined on page 69. Furthermore, $d_\omega(\cdot, z_0) \leq 1$ on $B_\omega(z_0, 1) \supseteq B_{h_{z_0}}(z_0, 1)$, so by the triangle inequality $d_\omega(z_0, z) + d_\omega(z, \zeta_0) \geq d_\omega(z_0, \zeta_0)$ and by choice of ε,

$$e^{\varepsilon d_\omega(z, \zeta_0)} e^{-\varepsilon d_\omega(z_0, \zeta_0)} \geq e^{-\varepsilon d_\omega(z_0, z)} \geq e^{-\sqrt{2}} \text{ for } z \in B_\omega(z_0, 1).$$

Now we start by integrating the Bergman kernel around an arbitrary point z_0, perform variable transformation by η^{-1} (note (4.25)), use the inequality of exponential factors above and finally Proposition 4.4.2. In formulae:

$$\begin{aligned}
&\int_{B_\beta(\eta(z_0),1)} |\mathcal{B}_\varphi(\eta^{-1}(z),\zeta_0)|^2 e^{-\varphi(\eta^{-1}(z))}\,\mathrm{d}\lambda(z)\\
&\qquad\qquad = \det(h_{z_0}) \int_{B_{h_{z_0}}(z_0,1)} |\mathcal{B}_\varphi(z,\zeta_0)|^2 e^{-\varphi(z)}\,\mathrm{d}\lambda(z)\\
&\qquad\qquad \le e^{\sqrt{2}} \det(h_{z_0}) \int |\mathcal{B}_\varphi(z,\zeta_0)|^2 e^{\varepsilon d_\omega(z,\zeta_0)} e^{-\varphi(z)}\,\mathrm{d}\lambda(z) e^{-\varepsilon d_\omega(z_0,\zeta_0)}\\
&\qquad\qquad \le \frac{C\det(h_{z_0})\det(h_{\zeta_0}) e^{\varphi(\zeta_0)} e^{-\varepsilon d_\omega(z_0,\zeta_0)}}{(\sqrt{2}-\varepsilon)^2 \inf_{\zeta\in B_{h_{\zeta_0}}(\zeta_0,1)} \lambda_{\min}(\omega(\zeta)|h_{\zeta_0})}.
\end{aligned}$$

Further, $\bar\partial\mathcal{B}(\cdot,\zeta_0)=0$, so Lemma 4.5.1 applies for $\mathcal{B}(\eta^{-1}(\cdot),\zeta_0)$ with the plurisubharmonic function $\varphi\circ\eta^{-1}$, yielding

$$\begin{aligned}
&|\mathcal{B}_\varphi(z_0,\zeta_0)|^2 e^{-\varphi\circ\eta^{-1}(\eta(z_0))+a_{\varphi\circ\eta^{-1}}}\\
&\qquad\qquad \le \frac{C\det(h_{z_0})\det(h_{\zeta_0}) e^{\varphi(\zeta_0)} e^{-\varepsilon d_\omega(z_0,\zeta_0)}}{(\sqrt{2}-\varepsilon)^2 \inf_{\zeta\in B_{h_{\zeta_0}}(\zeta_0,1)} \lambda_{\min}(\omega(\zeta)|h_{\zeta_0})}.
\end{aligned}$$

But $a_{\varphi\circ\eta^{-1}}$ was bounded, independently of φ, by Lemma 4.4.1, so Theorem 4.1.2 is proved. ∎

Chapter 5

Estimates in Several Complex Dimensions using Supremum Norm

Here we present [Dal15] and as noted in the foreword, we will skim through the paper only. It is interesting for us, as it combines tools from the already presented papers and makes only slight use of Differential Geometry.

The plan of Dall'Ara might be summerized as follows: first find a compactly supported function f_z such that the Bergman projection applied to it gives the Bergman kernel with fixed second variable z;

second, use the canonical solution operator N to write the Bergman projection as the difference of the identity and $\bar{\partial}^* N \bar{\partial}$;

third, estimate the norms of f_z and $\bar{\partial}^* N \bar{\partial} f_z$ through some value these functions assume within their support;

fourth, obtain point-to-integral[1] estimates via Lemma 5.0.7 of the Bergman kernel, combine the first and second step and use step three.

That way one can relax the condition of strict plurisubharmonicity of the weight φ, but adds conditions on the measures in use, similar to [Chr91], but also on the **weighted Kohn Laplacian** $\Box_\varphi$, where we use the definitions made at the beginning of the previous chapter (see (4.2)):

$$\Box_\varphi := \bar{\partial}^*_\varphi \bar{\partial} + \bar{\partial} \bar{\partial}^*_\varphi$$

and abbreviates[2]

$$\mathcal{E}_\varphi(u, v) := (\bar{\partial} u, \bar{\partial} v)_\varphi + (\bar{\partial}^*_\varphi u, \bar{\partial}^*_\varphi v)_\varphi.$$

[1]By this we mean that the point evaluation is dominated by the norm, i.e. the integral of the function.

[2]The subindex φ indicates the metric tensor obtained by this function.

We follow Dall'Ara and say that $\mathcal{E}_\varphi$ is **κ-coercive** if

$$\mathcal{E}_\varphi(u,u) \geq \kappa^2 \|u\|_\varphi$$

for u in the domain of the weighted Kohn Laplacian and κ being some measurable function $\mathbb{C}^n \to [0,\infty)$.

Overview

After an introduction to the problem and stating some inequalities concerning integrals and the $\bar{\partial}_\varphi^*$ operator, we encounter a radius function ρ just like in the chapter on doubling measures.

Definition 5.0.3. *A Borel function $\rho : \mathbb{R}^n \to (0,\infty)$ is said to be a radius function if there exists a constant $0 < C < \infty$ with*

$$C^{-1}\rho(x) \leq \rho(y) \leq C\rho(x)$$

for all $y \in B(x,\rho(x))$, where $x \in \mathbb{R}^n$ is arbitrary.

This is exactly the important statement of comparability. It follows immediatley the notion of distance function $d(\cdot,\cdot)$, introduced as earlier.

The next steps are motivated by physics. It seems that one needs the energy of order r^{-2} to localize a free particle in a given ball B with radius r. If there is a potential V acting on the particle in B, then this affects the energy needed and the idea is now to associate to V a radius function, such that the additional energy needed in B' with radius ρ_V, is comparable to the one when $V \equiv 0$ on B' - thus simplifying calculations to a mere adjustment of constants.

Our potential is going to be a measurable function

$$V : \mathbb{R}^n \to (0,\infty),$$

which is locally bounded, not almost everywhere zero and satisfies the following L^∞-doubling condition:

$$\|V\|_{L^\infty(B(x,2r))} \leq C_d \|V\|_{L^\infty(B(x,r))}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. This is quiet familiar to us and we obtain quiet familiar properties, just like $\|V\|_{L^\infty(B)} > 0$ for every ball B. The difference on the other hand is that we are dealing with pointwise terms.

Let us define $f : \mathbb{R}^n \times \mathbb{R}_+ \to (0,\infty)$ by

$$f(x,r) := r^2 \|V\|_{L^\infty(B(x,r))}$$

to set

$$\rho(x) = \rho_V(x) := \sup\{r > 0 \ : \ f(x,r) \leq 1\}$$

which exists as f is small if r is, and f is strictly increasing to ∞ if r does. It might not be a great surprise that ρ turns out to be a radius function and that it obeys inequalities that we have already encountered - but again, the nature is different and one is tempted to see what can be saved to this setting.

Proposition 5.0.4. *For $x \in \mathbb{R}^n$, we have*

$$\frac{1}{4C_d\rho_V^2(x)} \leq \|V\|_{L^\infty(B(x,\rho_V(x)))} \leq \frac{1}{\rho_V^2(x)}.$$

Admissible Weights

Now some additional structure is imposed on V, see [Dal15] pp. 13-14.

Definition 5.0.5. *A $\mathcal{C}^2$-plurisubharmonic weight $\varphi : \mathbb{C}^n \to \mathbb{R}$ is said to be admissible if:*

1. *the following L^∞-doubling condition holds:*

$$\sup_{B(z,2r)} \Delta\varphi \leq C_d \sup_{B(z,r)} \Delta\varphi$$

   for all $z \in \mathbb{C}^n$, $r > 0$ and a finite, non-zero C_d,

2. *there exists $c > 0$ such that*

$$\inf_{z\in\mathbb{C}^n} \sup_{w\in B(z,c)} \Delta\varphi(w) > 0.$$

For an admissible weight φ, we set $V = V_\varphi \equiv \Delta\varphi$.

Since $\Delta\varphi$ is four times the trace of the complex Hessian of φ, ρ_V is also called the maximal eigenvalue radius function as it becomes comparable to the maximal eigenvalue.

Proposition 5.0.6. *The radius function associated to an admissible weight is bounded.*

This is an important statement, as Lemma 4.4.1 will be applied and an analysis of the result of Henkin and Leiterer, see the Appendix, shows that a universal constant can be obtained in combination with the radius[3].

The next lemma is [Dal15, Lemma 13] and a substitute for Lemma 4.5.1. It will be applied to obtain a point-to-integral estimate of the Bergman kernel.

[3]For an unbounded radius function, already Propositions 1 and 3 of the Appendix of [HL84] might be a problem, which are an important ingredient of the proof of [HL84, Theorem 2.2.2].

Lemma 5.0.7. *There exists a constant $C > 0$, depending on φ only, such that if $h : B(z, r) \to \mathbb{C}$ is holomorphic and $r \leq \rho(z)$, then*

$$|h(z)|^2 e^{-2\varphi(z)} \leq \frac{C}{\lambda(B(z,r))} \int_{B(z,r)} |h|^2 e^{-2\varphi} \, \mathrm{d}\lambda.$$

Proof. Using Lemma 4.4.1 on balls $B(x, r)$, Dall'Ara obtains a real valued, bounded and continuous function ψ in $B(x, r)$ with $\partial\bar{\partial}\psi = \partial\bar{\partial}\varphi$ and $\|\psi\|_\infty \leq Cr^2 \|\partial\bar{\partial}\varphi\|_\infty$, where C is universal; with Proposition 5.0.4, this gives $\|\psi\|_\infty \leq C$, where ψ changes with the domain.

Applying $\partial\bar{\partial}$ on $\psi - \varphi$, one sees that this term is pluriharmonic and thus there is a holomorphic H with $\operatorname{Re} H = \psi - \varphi$ in $B(x, r)$.

One has, for some constant $C > 0$,

$$|h(z)|^2 e^{-2\varphi(z)} \leq C|h(z)|^2 e^{2\psi - 2\varphi(z)} = |h(z)e^{H(z)}|^2.$$

By the mean value property, the standard estimate of integrals, Jensen's inequality and the bound on ψ, all applied to the holomorphic function he^H, we find

$$\begin{aligned}
|h(z)e^{H(z)}|^2 &\leq \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |he^H|^2 \, \mathrm{d}\lambda \\
&= \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |h|^2 e^{2\psi - 2\varphi} \, \mathrm{d}\lambda \\
&\leq \frac{C}{\lambda(B(z,r))} \int_{B(z,r)} |h|^2 e^{-2\varphi} \, \mathrm{d}\lambda.
\end{aligned}$$

■

What happens next are derivations of estimates to the canonical solution operator N_φ, see for instance [Has14], the construction of a peak function f_z, compactly supported in $B(z, \rho(z))$ for which upper bounds of its norm are derived. This and more is the content of Lemma 16 in [Dal15].

The Bergman projection will then soon be applied to f_z, which equals the Bergman kernel with z in the second variable. Now, it is used that the Bergman kernel is holomorphic in the first variable - Lemma 5.0.7 is applied, yielding an integral estimate.

Since the Bergman projection of f_z can be written as $f_z - \bar{\partial}^*_\varphi N_\varphi \bar{\partial} f_z$, for which already integral estimates have been derived in [Dal15, Lemma 16], an orthogonality argument finishes the proof.

Appendix A

Solutions and Estimates to the $\bar{\partial}$-Equation

A.1 Hörmander's L^2-Estimates

Almost all of this section is an incomplete copy of [Hö94], chapters three and four.

A.1.1 Subharmonic Functions

The plan is to go quickly through the material needed to understand the terms in use. Interesting things will be proved if it takes not too long nor needs tools which need to be introduced exclusively for this purpose, otherwise they are stated only. X will denote an open, non-empty set in $\mathbb{R}^n$.

First, we will define harmonic functions...

Definition A.1.1. *Let $u \in \mathcal{C}^2(X)$. One says u is harmonic, if it satisfies the Laplace equation*

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0.$$

...second, we recall the definition of upper semicontinuity...

Definition A.1.2. *Let T_S be a topological space and $f : T_S \to [-\infty, \infty]$ a function. If $\{y \in T_S \ : \ f(y) < \alpha\}$ is open for every $\alpha \in \mathbb{R}$, we say f is upper semicontinuous.*

... and third, the definition of subharmonic functions.

Definition A.1.3. *A function $u : X \to [-\infty, \infty)$ is called subharmonic if*

(a) *u is upper semicontinuous;*

(b) *for every compact set K of X and every continuous function h on K which is harmonic in the interior of K, the inequality $u \le h$ is valid in K if it holds in bK.*

$u \equiv -\infty$ is subharmonic, but we are going to ignore this case. We list now some properties of subharmonic functions, see [Hö94] pp. 141-148, 156.

Theorem A.1.4. *If u is subharmonic in X and $c > 0$, then cu is subharmonic in X. The maximum of finitely many subharmonic functions u_j in X is subharmonic in X. If for a family of functions $\{u_j\}_{j\in I}$ the supremum $u(x) = \sup_{j\in I} u_j(x)$ is upper semicontinuous with values in $[-\infty, \infty)$, then u is subharmonic. If $\{u_j\}_j$ is a decreasing sequence of subharmonic functions, then $u(x) = \lim u_j(x)$ is subharmonic.*

Theorem A.1.5. *Let u be upper semicontinuous with values in $[-\infty, \infty)$. Each of the following conditions is necessary and sufficient for u to be subharmonic in X:*

1. *Condition (b) in Definition A.1.3 is fulfilled for closed balls $\bar{B} \subset X$.*

2. *If $X_\delta = \{x \in X \ : \ \bar{B}_\delta(x) \subset X\}$ for $\delta > 0$, the mean value function*

$$M(x, r) = M_u(x, r) = \int_{|y|=1} \frac{u(x + ry)}{\sigma_n} \mathrm{d}S, \quad \text{for } x \in X_r, \tag{A.1}$$

is an increasing function of $r \in [0, \delta]$ for $x \in X_\delta$.

3. *For every positive measure ν in $[0, \delta]$ with $\delta > 0$, we have for $x \in X_\delta$*

$$u(x) \int_{r\in[0,\delta]} \int_{|y|=1} \mathrm{d}S(y)\mathrm{d}\nu(r) \le \int_{r\in[0,\delta]} \int_{|y|=1} u(x + ry)\mathrm{d}S(y)\mathrm{d}\nu(r). \tag{A.2}$$

4. *For every $\delta > 0$ and every $x \in X_\delta$, there is a positive measure supported by $[0, \delta]$, but not by $\{0\}$ such that* (A.2) *is valid.*

[Hö94, Exercise 3.2.2]. If u is upper semicontinuous, then u is subharmonic if and only if $\overline{\lim}_{r\to 0}(M(x, r) - u(x))r^{-2} \ge 0$ for $x \in X$ with $u(x) > -\infty$.

Remark A.1.6. *By condition 3. of the above theorem and by upper semicontinuity, it follows that for every positive measure μ with compact support on $\mathbb{R}$ and every $x \in X$ we have*

$$u(x) \int_{\mathbb{R}} \int_{|y|=1} \mathrm{d}S(y)\mathrm{d}\mu(r) = \lim_{\delta\to 0} \int_{\mathbb{R}} \int_{|y|=1} u(x + \delta ry)\mathrm{d}S(y)\mathrm{d}\mu(r).$$

Corollary A.1.7. *The finite sum of subharmonic functions is subharmonic.*

Corollary A.1.8. *If u is subharmonic in an open, connected set X, then $u \in L^1_{loc}(X)$ and hence $u(x) > -\infty$ almost everywhere.*

Proposition A.1.9. *If $u \in \mathcal{C}^2(X)$ and M is as in Theorem A.1.5, then*

$$\lim_{r\to 0} \frac{M(x,r) - u(x)}{r^2} = \frac{\Delta u(x)}{2n}, \quad \text{for } x \in X, \tag{A.3}$$

and u is subharmonic in X if and only if $\Delta u \geq 0$.

Definition A.1.10. *$\Delta u \geq 0$ in the sense of distribution theory means that*

$$\int u\Delta v \,\mathrm{d}\lambda \geq 0, \quad \text{if } v \in \mathcal{C}_0^\infty(X) \text{ with } v \geq 0.$$

Other properties in the sense of distribution theory or in the sense of distributions are defined analogously.

Theorem A.1.11. *If u is subharmonic in X and $u \in L^1_{loc}(X)$, then $\Delta u \geq 0$ in the sense of distribution theory. Conversely, if U is a distribution in X and $\Delta U \geq 0$, then U is defined by a unique subharmonic u in X.*

Theorem A.1.12. *If φ is convex and increasing on $\mathbb{R}$, $\varphi(-\infty) = \lim_{t^-} \varphi(t)$, and if u is subharmonic in X, then $\varphi \circ u$ is subharmonic in X.*

Proof. If the distance from $x \in X$ to bX is greater than $r > 0$, then

$$\begin{aligned}\varphi(u(x)) \leq \varphi\big(M_u(x,r)\big) &= \varphi\left(\int_{|y|=1} \frac{u(x+ry)}{\sigma_n} \mathrm{d}S(y)\right) \\ &\leq \int_{|y|=1} \frac{\varphi\big(u(x+ry)\big)}{\sigma_n} \mathrm{d}S(y) = M_{\varphi\circ u}(x,r),\end{aligned}$$

where the first inequality follows from Theorem A.1.5 and the fact that φ is increasing; the second one follows from Jensen's inequality for φ is convex and $\sigma_n^{-1}\mathrm{d}S$ is a probability measure.

Convex functions are continuous in the interior of their domains, thus $\varphi \circ u$ is upper semicontinuous. The result follows now from Theorem A.1.5, since M_u is increasing in r and thus $\varphi\big(M_u\big)$ is so too. ■

A.1.2 Plurisubharmonic Functions

Again, we are going to present the essence without going much into details. Note, that this section ends with the important definition of pseudoconvex sets and that X will denote an open, non-empty set in $\mathbb{C}^n$; see [Hö94], pages 225-228.

Definition A.1.13. *We call $u : X \to [-\infty, \infty)$ plurisubharmonic (psh.) if*

> (a) *u is an upper semicontinuous function;*
> (b) *for arbitrary z and w in $\mathbb{C}^n$ the function $\tau \mapsto u(z + \tau w)$ is subharmonic in the open subset of $\mathbb{C}$ where it is defined.*

Analogous statements to the subharmonic case hold almost by definition.

Theorem A.1.14. *If u is plurisubharmonic in X and $c > 0$, then cu is plurisubharmonic in X. The maximum of finitely many plurisubharmonic functions u_j in X and their sum are plurisubharmonic in X. If for a family of functions $\{u_j\}_{j\in I}$ the supremum $u(x) = \sup_{j\in I} u_j(x)$ is upper semicontinuous with values in $[-\infty, \infty)$, then u is plurisubharmonic. If $\{u_j\}_j$ is a decreasing sequence of plurisubharmonic functions, then $u(x) = \lim u_j(x)$ is plurisubharmonic.*

We would like to recall the definition of a polydisc $P_R(z)$ for a polyradius $R = (R_1, \dots, R_n) \in \mathbb{R}^n_{>0}$ and center $z = (z_1, \dots, z_n) \in \mathbb{C}^n$: $P_R(z) = B_{R_1}(z_1) \times \dots \times B_{R_n}(z_n)$.

Theorem A.1.15. *If u is plurisubharmonic in X and, for some polyradius R, $z \in X_R = \{\zeta \in X \ : \ \bar{P}_R(\zeta) \subset X\}$, then*

$$M_u(z, r_1, \dots, r_n) = \frac{1}{2\pi} \underset{\theta_j\in[0,2\pi)}{\int \dots \int} u\big(z_1 + r_1 e^{i\theta_1}, \dots, z_n + r_n e^{i\theta_n}\big) \ \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$

is an increasing function of r_j for $j \in \{1, \dots, n\}$ with $0 \le r_j < R_j$. In particular, if Φ is a continuous, non-negative function with support in $[-1, 1]$ and $x \in X_\delta = \{y \in X \ : \ \bar{B}_\delta(y) \subset X\}$ for $\delta > 0$, then

$$\int_{|w|<1} u(z + rw)\Phi(|w|) \ \mathrm{d}\lambda(w)$$

is an increasing function of $r \in [0, \delta)$, hence u is subharmonic.

Theorem A.1.16. *Let $0 \le \varphi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ with support in the unit ball and assume that $\varphi(z)$ is radial, i.e. it only depends on $|z_1|, \dots, |z_n|$. If $u \in L^1_{loc}$ is plurisubharmonic in X and*

$$X_\varepsilon = \Big(\bigcup_{w\in X^c} \bar{B}_\varepsilon(w)\Big)^c,$$

i.e. the subset of X which has more than ε distance to bX, then the convolution

$$u_\varepsilon(z) = \int u(z - \varepsilon\zeta)\varphi(\zeta)\, \mathrm{d}\lambda(\zeta)$$

is a plurisubharmonic function in $\mathcal{C}^\infty(X_\varepsilon)$; in addition, if $\int \varphi(\zeta)\, \mathrm{d}\lambda(\zeta) = 1$ then $u_\varepsilon \searrow u$ as $\varepsilon \searrow 0$.

Proof. It is well known that $u_\varepsilon \in \mathcal{C}^\infty$; that it decreases to u follows from the monotonicity of the means M_{u_ε} in Theorem A.1.15 and the upper semicontinuity of u, thus we plug in the definition of M_u and change the order of integration. For $z \in X_\varepsilon$, $w \in \mathbb{C}^n$ and sufficiently small $r > 0$ we have

$$\int_0^{2\pi} u_\varepsilon(z + rwe^{i\theta})\frac{\mathrm{d}\theta}{2\pi} = \int \varphi(\zeta)\, \mathrm{d}\lambda(\zeta) \int_0^{2\pi} u(z + rwe^{i\theta} - \zeta\varepsilon)\frac{\mathrm{d}\theta}{2\pi} \geq u_\varepsilon(z),$$

proving plurisubharmonicity by *3.* of Theorem A.1.5. ■

Corollary A.1.17. *A function $u \in \mathcal{C}^2(X)$ is plurisubharmonic if and only if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0, \quad \textit{for } z \in X \textit{ and } w \in \mathbb{C}^n. \tag{A.4}$$

If u is any plurisubharmonic function in $L^1_{loc}(X)$, then $\partial^2 u/\partial z_j \partial \bar{z}_k$ is a measure and (A.4) *is valid in the sense of measure theory, i.e.*

$$\int v(z) \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k\, \mathrm{d}\lambda \geq 0, \quad \textit{for } v \in \mathcal{C}_0^\infty(\mathbb{C}^n) \textit{ with } v \geq 0.$$

Conversely, a distribution for which this is true is defined by a unique plurisubharmonic function.

The corollary above contained a very important form in the study of Complex Analysis in Several Variables, let us note and use it here.

Definition A.1.18. *The form* (A.4) *in Corollary A.1.17 is called the Levi form of u. A function u is strictly plurisubharmonic if and only if its Levi form is positive for all $w \in \mathbb{C}^n \setminus \{0\}$.*

Corollary A.1.19. *If Y is an open set in $\mathbb{C}^m$ and $f : X \to Y$ is a holomorphic function, then $u \circ f$ is a plurisubharmonic function in X for every plurisubharmonic function u in Y.*

We arrived at page 236 of [Hö94].

Definition A.1.20. *Let δ be a continuous, homogeneous distance function in $\mathbb{C}^n$, i.e.*

$$\delta(tz) = |t|\delta(z), \quad \text{for } t \in \mathbb{C},\ z \in \mathbb{C}^n \quad \text{and } \delta(z) > 0,\ \text{if } z \neq 0.$$

We define the boundary distance for X by

$$d_X(z) = \inf_{w \in X^c} \delta(z - w) \quad \text{for } z \in X,$$

if $X \neq \mathbb{C}^n$; otherwise we define d_X to be identical zero.

Theorem A.1.21. *The following conditions are equivalent:*

1. *There is a plurisubharmonic function u in X, such that for every $t \in \mathbb{R}$ $\{z \in X \ : \ u(z) \leq t\} \subset\subset X$.*
2. *If $K \subset X$ is compact, then*

$$\widehat{K} = \bigcap_{u \text{ psh.}} \{z \in X \ : \ u(z) \leq \sup_K u\} \subset\subset X.$$

3. *For every continuous, homogeneous distance function d_X in $\mathbb{C}^n$, the function $z \mapsto -\log d_X(z)$ is plurisubharmonic in X.*
4. *For some continuous, homogeneous distance function d_X in $\mathbb{C}^n$, the function $z \mapsto -\log d_X(z)$ is plurisubharmonic in X.*

Definition A.1.22. *An open set $X \subset \mathbb{C}^n$ is called pseudoconvex if the equivalent conditions in Theorem A.1.21 are fulfilled.*

Theorem A.1.23. *Let $X \subset \mathbb{C}^n$ be a pseudoconvex, open set and K a compact subset of X. Let $\widehat{K}$ be as in Theorem A.1.21. If Y is an open set with $\widehat{K} \subset\subset Y \subset X$, then one can find $u \in \mathcal{C}^\infty(X)$ satisfying condition 1. in Theorem A.1.21 so that the Levi form is strictly positive definite and $u < 0$ in K but $u \geq 1$ in $X \setminus Y$.*

A.1.3 Pluriharmonic Functions

X will denote a domain, thus an open, non-empty and simply connected set in $\mathbb{C}^n$; our source is [PfESo15].

Definition A.1.24. *A $\mathcal{C}^2$-function u in X such that u and $-u$ are plurisubharmonic in X is called pluriharmonic, i.e. it satisfies*

$$\frac{\partial^2 u}{\partial \bar{z}_j \partial z_r} = 0, \ \text{for } j, r \in \{1, \ldots, n\}. \tag{A.5}$$

Pluriharmonic functions are locally the real and imaginary parts of holomorphic functions, and to every pluriharmonic function u on a domain X, we can find a holomorphic function H in X with $\mathrm{Re}(H) = u$.

Remark A.1.25. *Given a pluriharmonic function u in X, then for $z_0 \in X$ and writing $z_j = x_j + iy_j$ for $(z_1, \dots, z_n) \in X$, we construct a holomorphic function $H = u + iv$ in X via*

$$v(z) = \int_{z_0}^{z} \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} \mathrm{d}y_j - \frac{\partial u}{\partial y_j} \mathrm{d}x_j \right) + C.$$

Proof. One confirms that $\frac{\partial H}{\partial \bar{z}_j} = 0$ using (A.5) and applies Hartogs Theorem[1]. ■

A.1.4 Hilbert Space Methods and the Estimates

Since these estimates are used repeatedly in several areas of mathematics, it surely is a good idea to get familiar with them; see [Hö94] pp. 248-258.

As often, H_1, H_2 and H_3 will denote Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_j$ and norms distinguished by suitable subindices $j \in \{1, 2, 3\}$. The domain of an operator T will be denoted by $\mathrm{dom}\, T$; $\mathrm{range}\, T$ and $\mathrm{kern}\, T$ are self-explanatory. $(0,1)$-forms with coefficients in a function space E will generally be denoted by $E_{(0,1)}$. We will, following Hörmander, handle the one-dimensional case first and conclude with the case in several dimensions. X is an open set in $\mathbb{C}$ for now.

Hence the main topic is the inhomogeneous Cauchy-Riemann equation:

$$\frac{\partial u}{\partial \bar{z}} = f. \tag{A.6}$$

Suppose that for every $f \in L^2(X, e^{-\psi})$ there is a solution $u \in L^2(X, e^{-\varphi})$ to (A.6), where φ and ψ are assumed to be smooth functions for now, thus we define

$$\begin{aligned} N : L^2(X, e^{-\psi}) &\to L^2(X, e^{-\varphi}) \\ f &\mapsto u \end{aligned}$$

for the unique u that satisfies (A.6) with minimal distance[2] to $A^2(X, \varphi)$. Note that $\partial/\partial\bar{z}$ has closed graph, see Remark 3.6.1, and is surjective by assumption, this is also true for the restriction of $\partial/\partial\bar{z}$ to range(N), i.e. range$(N) \times L^2(X, e^{-\psi})$ is closed in $L^2(X, e^{-\varphi}) \times L^2(X, e^{-\psi})$. Now

$$\begin{aligned} \mathrm{pr}_2 : \mathrm{range}(N) \times L^2(X, e^{-\psi}) &\to L^2(X, e^{-\psi}) \\ (u, f) &\mapsto f \end{aligned}$$

[1] Hartogs Theorem states that analycity in every dimension seperately, implies analycity - i.e. the function is holomorphic.

[2] $A^2(X, \varphi)$ is the Bergman space, see the beginning of the fourth chapter.

is a bounded, bijective operator between Banach spaces and thus has a bounded inverse ${\mathrm{pr}_2}^{-1}$ by the inverse mapping theorem; it then follows that $N = \mathrm{pr}_1 \circ ({\mathrm{pr}_2}^{-1})$ is the composition of bounded operators. To sum up we have[3]

$$\|Nf\|_\varphi = \int |u|^2 e^{-\varphi}\, \mathrm{d}\lambda \;\le\; C \int |f|^2 e^{-\psi}\, \mathrm{d}\lambda = C\|f\|_\psi. \tag{A.7}$$

We will regard $\partial/\partial\bar{z}$ as an unbounded operator T in $L^2(X, e^{-\varphi})$, defined in the sense of distributions. Thus

$$\int f\bar{v}e^{-\varphi}\, \mathrm{d}\lambda = -\int u \frac{\partial(\bar{v}e^{-\varphi})}{\partial\bar{z}}\, \mathrm{d}\lambda = -\int u\overline{\partial_\varphi v}e^{-\varphi}\, \mathrm{d}\lambda, \quad \text{for } v \in \mathcal{C}_0^\infty(X)$$

where ∂_φ is defined as

$$\partial_\varphi v = \frac{e^\varphi \partial\left(ve^{-\varphi}\right)}{\partial z} = \frac{\partial v}{\partial z} - v\frac{\partial\varphi}{\partial z}.$$

By the above equality, Cauchy-Schwarz[4] and (A.7), we obtain

$$\left|\int f\bar{v}e^{-\varphi}\, \mathrm{d}\lambda\right|^2 \le C\int |f|^2 e^{-\psi}\, \mathrm{d}\lambda \int |\partial_\varphi v|^2 e^{-\varphi}\, \mathrm{d}\lambda, \quad \text{for } v \in \mathcal{C}_0^\infty(X)$$

and since f was arbitrary, set $f = ve^{\psi-\varphi}$ and divide to see that

$$\int |v|^2 e^{\psi-2\varphi}\, \mathrm{d}\lambda \le C\int |\partial_\varphi v|^2 e^{-\varphi}\, \mathrm{d}\lambda, \quad \text{for } v \in \mathcal{C}_0^\infty(X). \tag{A.8}$$

Conversely, reversing the argument and extending the map

$$L^2(X, e^{-\varphi}) \ni \partial_\varphi v \mapsto \int f\bar{v}e^{-\varphi}\, \mathrm{d}\lambda, \quad \text{for } v \in \mathcal{C}_0^\infty(X),$$

by Hahn-Banach's theorem we conclude that (A.8) guarantees that (A.6) always has a solution satisfying (A.7) when $f \in L^2(X, e^{-\psi})$. To study the estimate (A.8) we integrate by parts

$$\begin{aligned}
&\int |\partial_\varphi v|^2 e^{-\varphi}\, \mathrm{d}\lambda \\
&\qquad = -\int \frac{\partial(\partial_\varphi v)}{\partial\bar{z}}\bar{v}e^{-\varphi}\, \mathrm{d}\lambda = \int \frac{\partial^2\varphi}{\partial z\partial\bar{z}}|v|^2 e^{-\varphi}\, \mathrm{d}\lambda - \int \partial_\varphi \frac{\partial v}{\partial\bar{z}}\bar{v}e^{-\varphi}\, \mathrm{d}\lambda \\
&\qquad = \int \left|\frac{\partial v}{\partial\bar{z}}\right|^2 e^{-\varphi}\, \mathrm{d}\lambda + \frac{1}{4}\int |v|^2 \Delta\varphi e^{-\varphi}\, \mathrm{d}\lambda.
\end{aligned}$$

[3]This result also follows by a more general setting, see iv) in [Has14, Lemma 4.28].
[4]Note that our measure is $e^{-\varphi}\, \mathrm{d}\lambda$.

Hence

$$4\int |\partial_\varphi v|^2 e^{-\varphi}\, \mathrm{d}\lambda \geq \int |v|^2 \Delta\varphi e^{-\varphi}\, \mathrm{d}\lambda. \tag{A.9}$$

This gives the estimate (A.8) if $e^\psi = Ce^\phi\Delta\varphi/4$. Thus we have proved that if $\varphi \in \mathcal{C}^2$ and $\Delta\varphi > 0$, then equation (A.6) has a solution u with

$$\int_X |u|^2 e^{-\varphi}\, \mathrm{d}\lambda \leq 4\int_X |f|^2 e^{-\varphi} \frac{\mathrm{d}\lambda}{\Delta\varphi}. \tag{A.10}$$

Here and in what follows it is tacitly understood that u and f are in L^2_{loc}, and the result states that when the right hand side is also finite, then there is a solution u of (A.6) satisfying (A.10).

If φ is just subharmonic, we can apply the preceding result on

$$\psi(z) = \varphi(z) + a\log(1+|z|^2),$$

where $a > 0$. We then have

$$\Delta\psi \geq a\Delta\log(1+|z|^2) = 4a\partial\frac{z}{1+|z|^2} = 4a\frac{(1+|z|^2) - z\bar{z}}{(1+|z|^2)^2} = \frac{4a}{(1+|z|^2)^2}.$$

Since $e^{-\psi} = e^{-\varphi}(1+|z|^2)^{-a}$,

$$\frac{4e^{-\psi}}{\Delta\psi} \leq \frac{e^{-\varphi}}{a}(1+|z|^2)^{2-a},$$

it follows that (A.6) has a solution with

$$\begin{aligned} a\int_X |u(z)|^2 e^{-\varphi(z)}(1+|z|^2)^{-a}\, \mathrm{d}\lambda(z) \\ \leq \int_X |f(z)|^2 e^{-\varphi(z)}(1+|z|^2)^{2-a}\, \mathrm{d}\lambda(z), \end{aligned} \tag{A.11}$$

provided that φ is smooth and subharmonic, and the right hand side is finite. The smoothness assumption on φ can and will be removed. Now.

Theorem A.1.26. *Let X be a connected, open set in $\mathbb{C}$ and φ a subharmonic function in X; let $a > 0$. If $f \in L^2_{loc}(X)$ and the right-hand side of* (A.11) *is finite, then the Cauchy-Riemann equation* (A.6) *has a solution $u \in L^2_{loc}(X)$ such that* (A.11) *holds.*

Proof. Let Y_j be open sets increasing to X with $Y_j \subset\subset X$, very much like the interiors of a compact exhaustion. By a standard regularization we can find a sequence of subharmonic functions $\varphi_j \in \mathcal{C}^\infty(Y_j)$ with $\varphi_j \geq \varphi_{j+1}$ in X. If the right-hand side of (A.11) is finite, it follows that there exists a solution u_j of (A.6) in Y_j such that

$$a \int_{Y_j} |u_j(z)|^2 e^{-\varphi_j(z)}(1+|z|^2)^{-a} \, \mathrm{d}\lambda(z) \leq \int_X |f(z)|^2 e^{-\varphi(z)}(1+|z|^2)^{2-a} \, \mathrm{d}\lambda(z);$$

where we have extended the domain of integration on the right hand side. Hence we can choose a subsequence u_{j_k} which is weakly convergent in $L^2(Y_j, e^{-\varphi_j})$ for every fixed j. The limit u satisfies (A.6) in X since differential operators are continuous in the distributional topology and the above inequality holds for every j, with u_j replaced by u. Letting $j \to \infty$, we obtain (A.11) by monotone convergence. ∎

We now turn our attention to the case in several dimensions. Note first that the domain of the adjoint of an operator $T : H_1 \to H_2$, consists of these $y \in H_2$ such that $f(x) := \langle Tx, y\rangle_2$ is a bounded linear functional on $\mathrm{dom}(T)$. As T is densely defined, there is a unique extension of f to all of H_1 and thus by the Riesz representation theorem a unique $T^*y \in H_1$ such that $f(x) = \langle x, T^*y\rangle_1$; see [Has14].

Lemma A.1.27. *Let T be a linear, closed and densely defined operator from H_1 to H_2. Let F be a closed subspace of H_2 containing the range of T. Then* $\mathrm{range}\, T = F$ *if and only if*

$$\|f\|_2 \leq C\|T^*f\|_1, \quad \text{for } f \in F \cap \mathrm{dom}\, T^*. \tag{A.12}$$

Here, $\bar\partial$ will play the role of T, defined on scalar functions and F will be a space of $(0,1)$-forms annihilated by the $\bar\partial$ operator. In addition, we will have a closed and densely defined operator $S : H_2 \to H_3$ with $\mathrm{range}\, T \subset \mathrm{kern}\, S = F$. Using the graph-norm, the inequality follows if

$$\|f\|_2^2 \leq C^2\big(\|T^*f\|_1^2 + \|Sf\|_3^2\big), \quad \text{for } f \in \mathrm{dom}\, S \cap \mathrm{dom}\, T^*. \tag{A.13}$$

Let X be open in $\mathbb{C}^n$. The L^2-spaces with respect ot smooth densities $e^{-\varphi_j}$ for $j \in \{1,2,3\}$ consisting of functions, $(0,1)$-forms and $(0,2)$-forms respectively will play the role of the Hilbert spaces mentioned in the Lemma above. $T : H_1 \to H_2$ and $S : H_2 \to H_3$ will both denote $\bar\partial$ with different bidegrees. These are closed and densely defined operators since $\bar\partial$ is closed in the distribution topology. We want to prove (A.13); to this end we will choose weights so that the set of $(0,1)$-forms with coefficients in $\mathcal{C}_0^\infty(X)$, denoted by $\mathcal{D}_{(0,1)}(X)$, is dense in $\mathrm{dom}\, S \cap \mathrm{dom}\, T^*$ in the graph norm: $\|f\|_2 + \|T^*f\|_1 + \|Sf\|_3$. Note that

$$T^*f = -\sum_{j=1}^n e^{\varphi_1} \frac{\partial(e^{-\varphi_2} f_j)}{\partial z_j}, \quad \text{if} \quad f = \sum_{j=1}^n f_j \mathrm{d}\bar z_j.$$

Lemma A.1.28. *Suppose there exists a sequence $\chi_j \in \mathcal{C}_0^\infty(X)$ such that $0 \le \chi_j \le 1$ and on a given compact subset K of X, all $\chi_j \equiv 1$ for $j \ge N(K) \in \mathbb{N}$ as well as*

$$e^{-\varphi_{k+1}}|\bar\partial\chi_j|^2 \le Ce^{-\varphi_k} \quad \textit{for} \quad k \in \{1,2\} \quad \textit{and } j \ge 1. \tag{A.14}$$

Then $\mathcal{D}_{(0,1)}(X)$ is dense in $\operatorname{dom} S \cap \operatorname{dom} T^*$ *in the graph norm.*

For $f \in \mathcal{D}_{(0,1)}(X)$ we shall now prove an estimate of the form (A.13) by satisfying (A.14) with

$$\varphi_k = \varphi + (k-3)\psi, \quad \text{for} \quad k \in \{1,2,3\}, \tag{A.15}$$

provided that

$$|\bar\partial\chi_j|^2 \le Ce^{\psi} \quad \text{for } j \ge 1. \tag{A.16}$$

For the following we will need φ to be strictly plurisubharmonic and X to be pseudoconvex. With the choice of (A.15) we have

$$\begin{aligned} T^*f &= -e^{\varphi-2\psi}\sum_{j=1}^n \frac{\partial(e^{\psi-\varphi}f_j)}{\partial z_j}, \quad \text{or} \\ e^{\psi}T^*f &= -\sum_{j=1}^n \left(\frac{\partial}{\partial z_j} - \frac{\partial\varphi}{\partial z_j}\right) f_j - \langle\partial\psi, f\rangle. \end{aligned}$$

Since $\|T^*f\|_1^2 = \int |e^{\psi}T^*f|^2 e^{-\varphi}\,\mathrm{d}\lambda$, it follows that for $\varepsilon > 0$

$$\begin{aligned} &\int \Big|\sum_{j=1}^n \left(\frac{\partial}{\partial z_j} - \frac{\partial\varphi}{\partial z_j}\right) f_j\Big|^2 \frac{\mathrm{d}\lambda}{e^{\varphi}} \\ &\qquad\qquad \le (1+\varepsilon)\|T^*f\|_1^2 + \left(1 + \frac{1}{\varepsilon}\right)\int |f|^2|\partial\psi|^2\frac{\mathrm{d}\lambda}{e^{\varphi}}. \end{aligned}$$

We have

$$|\bar\partial f|^2 = \frac{1}{2}\sum_{j,k=1}^n \left|\frac{\partial f_j}{\partial\bar z_k} - \frac{\partial f_k}{\partial\bar z_j}\right|^2 = \sum_{j,k=1}^n \left|\frac{\partial f_j}{\partial\bar z_k}\right|^2 - \sum_{j,k=1}^n \frac{\partial f_j}{\partial\bar z_k}\frac{\partial\bar f_k}{\partial z_j},$$

and since[5]

$$\begin{aligned} &\int \left(\Big(\frac{\partial}{\partial z_j} - \frac{\partial\varphi}{\partial z_j}\Big) f_j \overline{\Big(\frac{\partial}{\partial z_k} - \frac{\partial\varphi}{\partial z_k}\Big) f_k} - \frac{\partial f_j}{\partial\bar z_k}\frac{\partial\bar f_k}{\partial z_j}\right) e^{-\varphi}\,\mathrm{d}\lambda \\ &\quad = \int \left(\left[\Big(\frac{\partial}{\partial z_j} - \frac{\partial\varphi}{\partial z_j}\Big), \frac{\partial}{\partial\bar z_k}\right] f_j\right)\bar f_k e^{-\varphi}\,\mathrm{d}\lambda = \int \frac{\partial^2\varphi}{\partial z_j\partial\bar z_k} f_j\bar f_k e^{-\varphi}\,\mathrm{d}\lambda, \end{aligned}$$

[5]The brackets in the second line denote the commutator.

we conclude that for $f \in \mathcal{D}_{(0,1)}(X)$:

$$\int \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} f_j \bar{f}_k e^{-\varphi} \,\mathrm{d}\lambda$$

$$\leq \int \Big| \sum_{j=1}^{n} \left(\frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} \right) f_j \Big|^2 e^{-\varphi} \,\mathrm{d}\lambda + \int |\bar{\partial} f|^2 e^{-\varphi} \,\mathrm{d}\lambda$$

$$\leq (1+\varepsilon) \|T^* f\|_1^2 + \|Sf\|_3^2 + \left(1 + \frac{1}{\varepsilon}\right) \int |f|^2 |\partial \psi|^2 e^{-\varphi} \,\mathrm{d}\lambda.$$

By strict plurisubharmonicity, we have for a positive, continuous function c

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq c(z) \sum_{j=1}^{n} |w_j|^2, \text{ for } z \in X,\ w \in \mathbb{C}^n.$$

Summing up, when $f \in \mathcal{D}_{(0,1)}(X)$, we proved

$$\int \left(c - \left(1 + \frac{1}{\varepsilon}\right) |\partial \psi|^2 \right) |f|^2 e^{-\varphi} \,\mathrm{d}\lambda \leq (1+\varepsilon) \|T^* f\|_1^2 + \|Sf\|_3^2. \tag{A.17}$$

By Theorem A.1.23, we can choose for every $a \in \mathbb{R}$ a strictly plurisubharmonic function $s \in \mathcal{C}^\infty(X)$ such that

$$X_a = \{ z \in X \ : \ s(z) < a \} \subset\subset X.$$

So let us drop the cutoff functions χ_j in (A.16) which are not identically 1 in X_{a+2}; we can choose ψ, satisfying (A.16) and being identically 0 in X_{a+1}. We replace φ by $\widetilde{\varphi} = \varphi + \chi(s)$ where the non-negative and convex function χ vanishes on $(-\infty, a)$, but then grows so fast that

(*a*) $\widetilde{\varphi} - 2\psi = \varphi + \chi(s) - 2\psi \geq \varphi$ and

(*b*) $\chi'(s) \sum_{j,k=1}^{n} \frac{\partial^2 s}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq \left(1 + \frac{1}{\varepsilon}\right) |\partial \psi|^2 \sum_{j=1}^{n} |w_j|^2$, for $z \in X,\ w \in \mathbb{C}^n$.

If we apply (A.17) with $\varphi_j = \widetilde{\varphi} + (j-3)\psi$, and adjust T, S, H_j for that φ, we get

$$\int c |f|^2 e^{-\widetilde{\varphi}} \,\mathrm{d}\lambda \leq (1+\varepsilon) \|T^* f\|_1^2 + \|Sf\|_3^2 \tag{A.18}$$

for $f \in \mathcal{D}_{(0,1)}(X)$ and hence when $f \in \operatorname{dom} S \cap \operatorname{dom} T^*$.

Let the coefficients of a $(0,1)$-form g be in $L^2_{loc}(X)$, assume that $\bar{\partial} g = 0$ and that

$$M = \int |g|^2 \frac{e^{-\varphi}}{c} \,\mathrm{d}\lambda \leq \infty.$$

If ψ is chosen so that $e^{-\psi}c$ is bounded, then $g \in H_2$ since $\varphi_2 - \varphi \geq \widetilde{\varphi} - \psi - (\widetilde{\varphi} - 2\psi) = \psi$, so $e^{-\varphi_2} \leq (e^{-\psi}c)(e^{-\varphi}/c)$. Since $2\varphi_2 - \varphi - \widetilde{\varphi} = \widetilde{\varphi} - 2\psi - \varphi \geq 0$, we obtain by Cauchy-Schwarz and (A.18)

$$|\langle g, f\rangle_2|^2 \leq M \int c|f|^2 e^{-\widetilde{\varphi}}\, \mathrm{d}\lambda \leq M(1+\varepsilon)\|T^*f\|_1^2 + M\|Sf\|_3^2,$$

when $f \in \operatorname{dom} S \cap \operatorname{dom} T^*$, and we claim that

$$|\langle g, f\rangle_2|^2 \leq M(1+\varepsilon)\|T^*f\|_1^2, \quad \text{for } f \in \operatorname{dom} T^*. \tag{A.19}$$

This is clear if $Sf = 0$. If f is orthogonal to the kernel of S, which contains the range of T, then $T^*f = 0$ and $\langle g, f\rangle_2 = 0$ for g is in the kernel of S, since $\bar{\partial} g = 0$.

Using a slight modification of the theorems of Riesz and Hahn-Banach for the bounded anti-linear map

$$T^*f \mapsto \langle g, f\rangle_2, \quad \text{for } f \in \operatorname{dom} T^*$$

one gets a $u_a \in L^2(X, e^{-\varphi_1})$ such that

$$\int |u_a|^2 e^{-\varphi_1}\, \mathrm{d}\lambda \leq M(1+\varepsilon), \quad \text{and } \langle g, f\rangle_2 = \langle u_a, T^*f\rangle_1. \tag{A.20}$$

But this means that $\bar{\partial} u_a = g$. Recalling that $\varphi_1 = \varphi$ in X_a, we can choose sequences $a_j \to \infty$ and $\varepsilon_j \to 0$ such that u_{a_j} converges weakly in $L^2(X_a)$ for every a, to a limit u. We get $\bar{\partial} u = g$ in X since $\bar{\partial}$ is continuous in the distribution topology and from (A.20) we obtain for every a

$$\int_{X_a} |u|^2 e^{-\varphi_1}\, \mathrm{d}\lambda \leq M.$$

We proved

Proposition A.1.29. *Let X be a pseudoconvex open set in $\mathbb{C}^n$ and let $\varphi \in \mathcal{C}^2(X)$ be strictly plurisubharmonic, hence*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq c(z) \sum_{j=1}^{n} |w_j|^2, \ \textit{for } z \in X, \ w \in \mathbb{C}^n,$$

where c is a positive, continuous function in X. If the coefficients of a $(0,1)$-form g are in $L^2_{loc}(X)$ and $\bar{\partial} g = 0$, it follows that one can find a $u \in L^2(X, e^{-\varphi})$ with $\bar{\partial} u = g$ and

$$\int_{X_a} |u|^2 e^{-\varphi}\, \mathrm{d}\lambda \leq \int |g|^2 \frac{e^{-\varphi}}{c}\, \mathrm{d}\lambda,$$

provided that the right hand side is finite.

If $\varphi \in \mathcal{C}^2(X)$ is just plurisubharmonic, we set

$$\psi(z) = \varphi(z) + a \log(1 + |z|^2)$$

and it follows

$$\sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq \frac{a}{(1+|z|^2)^2} \sum_{j=1}^{n} |w_j|^2, \text{ for } z \in X,\ w \in \mathbb{C}^n.$$

If $f \in L^2_{(0,1)}(X, e^{-\varphi}(1 + |\cdot|^2)^{2-a})$ and $\bar{\partial} f = 0$, we conclude that the equation $\bar{\partial} u = f$ has a solution in $L^2_{(0,1)}(X, e^{-\varphi}(1+|\cdot|^2)^{-a})$ satisfying (A.11). By repeating the proof of Theorem A.1.26 on connected components of X, we can remove the smoothness assumptions on φ and end this section with Hörmander's L^2-existence theorem and L^2-estimates on that solution:

Theorem A.1.30. *Let X be a pseudoconvex open set in $\mathbb{C}^n$, φ a plurisubharmonic function in X, and $a > 0$. If the coefficients of a $(0,1)$-form f are in $L^2_{loc}(X)$ and $\bar{\partial} f = 0$, then the eqation $\bar{\partial} u = f$ has a solution $u \in L^2_{loc}(X)$ such that* (A.11) *holds, provided its right hand side is finite.*

A.2 Estimates by Supremum Norm

This section has a rather optional character where this author tries to convince that Lemma 4.4.1 yields constants independent of the domain in our case. See [LG86, Theorem 2.28] on page 42 where this is done in a simpler way. When we mention page numbers, we mean the ones of [HL84].

We are going to track down all lemmata, theorems and notation which lead to [HL84, Theorem 2.2.2], which is sufficient to prove Lemma 4.4.1 if we restrict our domains to be balls. We will do a careful analysis of the proofs to point out that our special case, where all domains are balls with equal radii, yields constants independent of the center. Unfortunately the most crucial part - the proof of Theorem A.2.17, has been omitted due to its level of sophistication. No proofs will be presented, instead it is assumed that the reader has a copy of the book [HL84] at hand and we just point out how and where the statements in it are affected when dealing with our case and why all constants under consideration are thus universal.

A.2.1 Introduction

Regarding a bounded domain with sufficiently smooth boundary $D \subset \mathbb{C}$, which we can and will think of as a ball in the sequel, we have the formula of Cauchy-Green

$$f(z) = \frac{1}{2\pi i}\int_{bD}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{D}\frac{\bar{\partial}f(\zeta)\wedge \mathrm{d}\zeta}{\zeta - z} \tag{A.21}$$

for every complex-valued $\mathcal{C}^1$-function f in $\bar{D}$; and a solution u to $\frac{\partial u}{\partial \bar{z}} = f$ in D is given by

$$u(z) := \frac{-1}{2\pi i}\int_{bD}\frac{f(\zeta)\mathrm{d}\bar{\zeta}\wedge \mathrm{d}\zeta}{\zeta - z}. \tag{A.22}$$

Though there is a generalization to higher dimensions as can be seen on page 17, these two equations will serve us as a prototype of integral operators which decompose any nice f.

Let us first collect some notation for differential forms as stated on page 46. A $\mathcal{C}^k_{(p,q)}$-form will denote a differential form of bidegree (p,q) who's coefficients are $\mathcal{C}^k$-functions. Given two $\mathcal{C}^1$-maps $u = (u_1, \ldots, u_n) : X \to \mathbb{C}^n$ and $v = (v_1, \ldots, v_n) : X \to \mathbb{C}^n$, we define the differential forms $\omega(u) := \mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n$ and

$$\omega'(v) := \sum_{j=1}^{n}(-1)^{j+1}v_j \bigwedge_{k\neq j}^{n} \mathrm{d}v_k.$$

Also we use the notation

$$\langle v, u\rangle := \sum_{j=1}^{n} v_j u_j.$$

If there is a second manifold Y and the maps u and v are defined on $X \times Y$ and $\mathcal{C}^1$ with respect to the x-coordinate, we denote the exterior derivative with respect to x with d_x and the differential forms introduced earlier as ω_x and ω'_x to emphasize the use of d_x[6]. We will think of D as a ball in $\mathbb{C}^n$.

Proposition A.2.1. *For a smooth manifold X and u, v as above, the differential form*

$$\frac{\omega'(v)\wedge\omega(u)}{\langle v, u\rangle^n}$$

is closed, in the sense of distributions, in $\{x \in X \ : \ \langle v(x), u(x)\rangle \neq 0\}$.

[6]The reader might want to recall that the exterior derivative d is defined on a manifold, this time it is a product manifold $X \times Y$ and it is easy to see that $d = d_x \times d_y$, where d_x is the exterior derivative defined on X; similar on Y.

One of the central definitions will now be given, see pages 47 and 49. We set for $(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n$

$$\frac{\omega'_{z,\zeta}(\bar{\zeta}-\bar{z}) \wedge \omega(\zeta)}{|\zeta - z|^{2n}} := \sum_{j=1}^{n} (-1)^{j+1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} \bigwedge_{k \neq j}^{n} (\mathrm{d}\bar{\zeta}_k - \mathrm{d}\bar{z}_k) \wedge \omega(\zeta). \tag{A.23}$$

Definition A.2.2. *In analogy to the integral operators of the Cauchy-Green formula* (A.21)*, we set for bounded forms f and g on D and bD respectively and $z \in D$,*

$$(B_D f)(z) := \frac{(n-1)!}{(2\pi i)^n} \int_{\zeta \in D} f(\zeta) \wedge \frac{\omega'_{z,\zeta}(\bar{\zeta}-\bar{z}) \wedge \omega(\zeta)}{|\zeta - z|^{2n}}$$

and

$$(B_{bD} g)(z) := \frac{(n-1)!}{(2\pi i)^n} \int_{\zeta \in bD} g(\zeta) \wedge \frac{\omega'_{z,\zeta}(\bar{\zeta}-\bar{z}) \wedge \omega(\zeta)}{|\zeta - z|^{2n}}.$$

Definition A.2.3. *For a measurable set $Y \subset \mathbb{C}^n$ and every measurable function g and continuous function f on Y, both complex-valued, $\alpha \in (0,1)$, we define*

1. *the supremum norm by*
$$\|g\|_{0,Y} := \sup_{y \in Y} |g(y)|,$$
and denote by $L^\infty(Y)$ the Banach space $\{g \ : \ \|g\|_{0,Y} < \infty\}$;

2. *by*
$$\|f\|_{\alpha,Y} := \|f\|_{0,Y} + \sup_{z,\zeta \in Y} \frac{|f(z) - f(\zeta)|}{|z-\zeta|^\alpha},$$
the α-Hölder norm and by $\mathcal{H}^\alpha(Y)$ we denote the space of all continuous functions f in Y with $\|f\|_{\alpha,Y} < \infty$ - the α-Hölder continuous functions;

3. *by $\mathcal{C}^\alpha(Y)$, these continuous f with $\|f\|_{\alpha,K} < \infty$ for every $K \subset\subset Y$;*

4. *by $\mathcal{C}^{k+\alpha}$, the k-times continuous differentiable functions who's k-th derivative belongs to $\mathcal{C}^\alpha$ and*

5. *by $L^\infty_{(p,q)}$, we denote the differential forms of bidegree (p,q) with coefficients in the space L^∞; same for $\mathcal{H}^\alpha_{(p,q)}$, etc.*

$\|\cdot\|$ will denote the norm on differential forms obtained by the standard Euclidean inner product when extended to forms via the procedure using musical isomorphisms as described in the section dealing with metric tensors of the Prelimineries II. The next is Lemma 1.8.5 on page 51.

Lemma A.2.4. *Let $\alpha \in (0,1)$.*

1. *For every bounded differential form f on D, $B_D f$ is a $\mathcal{C}^\alpha$-form in D. Moreover, there is a constant $C_\alpha < \infty$ such that for every bounded differential form f on D:*
$$\|B_D f\|_{\alpha,D} \leq C_\alpha \|f\|_{0,D}.$$

2. *If f is $\mathcal{C}^k$, then $B_D f$ is $\mathcal{C}^{k+\alpha}$.*

Proof. The first thing one obtains is for $z, \xi \in D$

$$\|B_D f(z) - B_D f(\xi)\| \leq C_1 \|f\|_{0,D} \sum_{j=1}^{n} \int_{\zeta \in D} \left| \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} - \frac{\bar{\zeta}_j - \bar{\xi}_j}{|\zeta - \xi|^{2n}} \right| \mathrm{d}\lambda,$$

where C_1 is a constant depending on the dimension only. Obviously, the integral is translation invariant, hence Proposition 1 of Appendix 1 in [HL84], page 202 - which is used for the next step to estimate the integral, yields indeed a universal constant. ■

A.2.2 Leray Maps and Other Operators

We arrived at pages 52 and 53. For the upcoming we will need to integrate with respect to a subset of the variables at hand, this is done analogously to the case of functions; i.e. integrating out a variable, very much like $F(y) = \int f(x, y)\mathrm{d}x$.

Definition A.2.5. *A $\mathbb{C}^n$-valued $\mathcal{C}^1$-map $w(z, \zeta) = (w_1(z, \zeta), \ldots, w_n(z, \zeta))$ defined for $(z, \zeta) \in D \times U_{bD}$, where U_{bD} is some neighbourhood of bD, is called a Leray map for D if*
$$\langle w(z, \zeta), \zeta - z\rangle \neq 0 \text{ for all } (z, \zeta) \in D \times bD.$$
We also define

$$\eta^w(z, \zeta, \lambda) := (1 - \lambda)\frac{w(z, \zeta)}{\langle w(z, \zeta), \zeta - z\rangle} + \lambda \frac{\bar{\zeta} - \bar{z}}{\langle \bar{\zeta} - \bar{z}, \zeta - z\rangle}$$

for all $z \in D$, $\lambda \in [0, 1]$ and ζ in U_{bD}. As well as

$$\omega'(w(z, \zeta)) := \sum_{j=1}^{n} (-1)^{j+1} w_j(z, \zeta) \bigwedge_{k\neq j}^{n} \bar{\partial}_{z,\zeta} w_k(z, \zeta)$$

and
$$\bar{\omega}'(\eta^w(z, \zeta, \lambda)) := \sum_{j=1}^{n} (-1)^{j+1} \eta^w_j(z, \zeta, \lambda) \bigwedge_{k\neq j}^{n} (\bar{\partial}_{z,\zeta} + \mathrm{d}_\lambda)\eta^w_k(z, \zeta, \lambda).$$

Definition A.2.6. *Let f be a bounded differential form on bD, then we define for $z \in D$*
$$L^w_{bD} f(z) := \frac{(n-1)!}{(2\pi i)^n} \int_{\zeta \in bD} f(\zeta) \wedge \frac{\omega'(w(z,\zeta)) \wedge \omega(\zeta)}{\langle w(z,\zeta), \zeta - z \rangle^n}$$
and
$$R^w_{bD} f(z) := \frac{(n-1)!}{(2\pi i)^n} \int_{\zeta \in bD, 0 \le \lambda \le 1} f(\zeta) \wedge \bar{\omega}'(\eta^w(z,\zeta,\lambda)) \wedge \omega(\zeta)$$
where we first integrate ζ out, and in the second integral the couple (ζ, λ).

Now we have collected a few operators, let us see of what use they are. A first glimpse is the Martinelli-Bochner formula.

Theorem A.2.7. *If D is as in our standing assumption and f as well as $\bar{\partial} f$ are continuous in $\bar{D}$, then*
$$f = B_{bD} f - B_D \bar{\partial} f \text{ in } D.$$

Which can be extended to the Leray formula, see page 56:

Theorem A.2.8. *Let $w(z,\zeta)$ be the Leray map for D. Then for every continuous function f in $\bar{D}$ with $\bar{\partial} f$ also continuous on $\bar{D}$, we have, in D,*
$$f = L^w_{bD} f - R^w_{bD} f - B_D \bar{\partial} f.$$

Corollary A.2.9. *In case f is continuous in $\bar{D}$ and holomorphic in D, we have*
$$f = L^w_{bD} f.$$

Now we present the Koppelman formula, which is a generalization of the Martinelli-Bochner formula to differential forms, see page 57.

Theorem A.2.10. *Let f be a continuous $(0,q)$-form on $\bar{D}$ such that $\bar{\partial} f$ is also continuous on $\bar{D}$. Then we have, in D,*
$$(-1)^q f = B_{bD} f - B_D \bar{\partial} f + \bar{\partial} B_D f,$$
where all the terms on the right are continuous too.

The theorem is stated for $(0,q)$-forms rather then for (p,q)-forms since the latter can be interpreted as $(0,q)$-forms with values in $(p,0)$-forms - a holomorphic vector bundle...

On page 59 we find the Koppelman-Leray formula, which will be essential for us since it enables us to solve the inhomogeneous Cauchy-Riemann equation if there exists a Leray map $w(z,\zeta)$ which is holomorphic in z.

Theorem A.2.11. *Let us assume that w has continuous derivatives of order up to two in z and up to one in ζ for $z \in D$ and $\zeta \in U_{bD}$. Then for every continuous $(0,q)$-form f on $\bar{D}$ with $\bar{\partial} f$ also continuous on $\bar{D}$ we have, in D,*

$$(-1)^q f = L^w_{bD} f - \big(R^w_{bD} + B_D\big)\bar{\partial} f + \bar{\partial}\big(R^w_{bD} + B_D\big) f,$$

where all the forms $L^w_{bD} f$, $\bar{\partial} B_D f$, etc. are continuous too.

Now we see how this gives a continuous solution to the equation $\bar{\partial} u = f$; see page 60.

Corollary A.2.12. *Let $w(\cdot, \zeta)$ be holomorphic in D. We set for $1 \le q \le n$*

$$T_q := (-1)^q \big(R^w_{bD} + B_D\big)$$

and for f, as in the theorem above, we conclude

$$f = \bar{\partial} T_q f + T_{q+1} \bar{\partial} f,$$

which in case f is holomorphic reduces to

$$f = \bar{\partial} T_q f =: \bar{\partial} u.$$

In fact, u is $\mathcal{C}^\alpha_{(0,q-1)}(D)$. If f is $\mathcal{C}^k_{(0,q)}(D)$, then u is in addition $\mathcal{C}^{k+\alpha}_{(0,q-1)}(D)$.

Proof. This is just the Koppelman-Leray formula, where the term $L^w_{bD} f$ is zero if $q \ge 1$, due to the fact that $w(z, \zeta)$ is holomorphic in $z \in D$... ■

A.2.3 Hölder Estimates for the $\bar{\partial}$-Equation

For the upcoming it is assumed that $D = \{z \in \mathbb{C}^n \ : \ \nu(z) < 0\}$ where ν is a real valued $\mathcal{C}^2$-function in $\mathbb{C}^n$ (we always think of $\nu(z) = |z|^2 - 1$), such that for some $\alpha > 0$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \nu(z)}{\partial x_j \partial x_k} t_j t_k \ge \alpha |t|^2 \text{ for all } z \in bD \text{ and } t \in \mathbb{R}^{2n},$$

where $x_j = x_j(z)$ are the real coordinates of $z \in \mathbb{C}^n$, such that $z_j = x_j(z) + i x_{j+n}(z)$, see page 68.

In our case[7] $\nu(z)$ will simply be $|z|^2 - 1 = \sum_{j=1}^n (x_j^2 + x_{j+n}^2) - 1$ and it fulfils the above inequality with $\alpha \le 2$.

Definition A.2.13. *We define a central object by*

$$w_\nu(\zeta) := 2 \left(\frac{\partial \nu(\zeta)}{\partial \zeta_1}, \dots, \frac{\partial \nu(\zeta)}{\partial \zeta_n} \right).$$

[7]For simplicity we set the center of the ball under consideration to be zero.

Lemma A.2.14. *There exists a neighbourhood U_{bD} of bD and numbers $\varepsilon, \beta > 0$, such that for all $\zeta \in U_{bD}$ and $z \in \mathbb{C}^n$ with $|\zeta - z| \leq \varepsilon$*

$$\operatorname{Re}\langle w_\nu(\zeta), \zeta - z\rangle \geq \nu(\zeta) - \nu(z) + \beta|\zeta - z|^2.$$

Proof. Again, in our case

$$\begin{aligned}\operatorname{Re}\langle w_\nu(\zeta), \zeta - z\rangle &= \sum_{j=1}^{2n} \frac{\partial \nu(\zeta)}{\partial x_j} x_j(\zeta - z) = \sum_{j=1}^{2n} 2x_j(\zeta)x_j(\zeta - z) \\ &= |\zeta|^2 - 1 - (|z|^2 - 1) + |\zeta - z|^2;\end{aligned}$$

which actually shows equality for arbitrary ε and $\beta = 1$. ■

Corollary A.2.15. *w_ν is a Leray map for D.*

Our goul is to give estimates of the solution to the equation $\bar{\partial} u = f$ in terms of the supremum norm of f; we already have estimated $B_D f$ in Lemma A.2.4, all we need is thus to estimate R^w_{bD}, see Corollary A.2.12.

The most important lemma for us will be the following one; page 69.

Lemma A.2.16. *Suppose w is $\mathcal{C}^1$ in $U_{\bar{D}} \times U_{bD}$ with the following properties:*

1. *$w(z, \zeta)$ and $\mathrm{d}_\zeta w(z, \zeta)$ depend holomorphically on $z \in U_{\bar{D}}$.*
2. *w is a Leray map for D.*
3. *For every point $\xi \in bD$, there is a neighbourhood U_ξ of ξ and real $\mathcal{C}^1$-functions $t_j(z, \zeta)$ for $1 \leq j \leq 2n - 1$, defined for $z, \zeta \in U_\xi$, such that the following conditions are fulfilled:*
   - *for fixed $z \in U_\xi$, $\{t_j(z, \cdot)\}_{1 \leq j \leq 2n-1}$ are real coordinates on $bD \cap U_\xi$,*
   - *for some $\delta > 0$ and all $z \in D \cap U_\xi$ and $\zeta \in bD \cap U_\xi$ we have that*

$$|\langle w(z, \zeta), \zeta - z\rangle| \geq \delta\Big(|t_1(z, \zeta)|^2 + \sum_{j=1}^{2n-1} |t_j(z, \zeta)|^2 + \operatorname{dist}(z, bD)\Big). \tag{A.24}$$

Then for some constant C and every continuous differential form f on $\bar{D}$

$$\|R^w_{bD} f\|_{\frac{1}{2}, D} \leq C\|f\|_{0,D}.$$

Proof. In our case $w(z,\zeta)$ will be $w_\nu(\zeta)$ for $\nu(\zeta)=|\zeta-c|^2-1$, where $c=(c_1,\dots,c_n)$ is the center of the ball under consideration, and since it is independent of z, it is holomorphic in z. It is a Leray map by Corollary A.2.15.

Let $\Phi(z,\zeta) := \langle w(z,\zeta), \zeta - z\rangle$; in the proof it is deduced that the coefficients of $R^w_{bD}f$ are linear combinations of integrals of the following type

$$E(z) := \int_{bD} \frac{f_I\psi}{\Phi(z,\zeta)^{n-s-1}|\zeta-z|^{2s+2}} \bigwedge_{j\neq m} \mathrm{d}\bar{\zeta}_j \wedge \omega(\zeta),$$

(equation (2.2.7)) where $0 \le s \le n-2$, $m \in \{1,\dots,n\}$ and f_I is some coefficient of the differential form f and ψ is the product of some of the functions w_j, $\bar{\zeta}_j - \bar{z}_j$ and $\frac{\partial w_j}{\partial \bar{\zeta}_k}$ for $1 \le k \le n$.

The factors $\frac{\partial w_j}{\partial \bar{\zeta}_k}$ will be constants; the factors w_j will be $|\zeta_j - c_j|^2$ and are thus bounded by one. One concludes that ψ will contain at least one of the factors $\bar{\zeta}_j - \bar{z}_j$ and thus, with a universal constant in our case, will obey

$$|\psi| \le C_1|\zeta - z|.$$

The estimates of the integrals E are obtained via [HL84, Proposition 2] in Appendix 1 on page 203, which also comes with a constant depending on a path γ; but in our case, this path can easily be translated, hence the constant is universal. The last constant we have to take care of is the one in estimating

$$\int_{bD\cap U_\xi} \frac{\mathrm{d}\lambda}{|\Phi(z,\zeta)|^{n-s-j}|\zeta-z|^{2s+1+j}} \le C_\xi \operatorname{dist}(z,bD)^{-\frac{1}{2}} \text{ for } j \in \{0,1\}$$

where $\xi \in bD$; this represents the equations (2.2.10) and (2.2.11).

The estimates are obtained by writing the $(2n-1)$-dimensional Lebesgue measure $\mathrm{d}\lambda$ as $\pi(z,\zeta)\wedge \mathrm{d}_\zeta t_1(z,\zeta)\wedge\ldots\wedge \mathrm{d}_\zeta t_{2n-1}(z,\zeta)$ on $bD\cap U_\xi$, where one also chooses $R,\gamma,\Gamma > 0$, such that $|\zeta - z| \ge \gamma|t(z,\zeta)|$ and $|\pi(z,\zeta)| \le \Gamma$ for $(z,\zeta) \in U_\xi \times (bD\cap U_\xi)$ to perform appropriate substitution and use these estimates to see that $C_\xi = \frac{\Gamma}{\delta\gamma}$.

Hence we will have a universal constant in this lemma if we know this for δ,γ and Γ - which is done in the proof of [HL84, Theorem 2.2.2]. The last step is to use Proposition 3 of Appendix 1, which comes with a universal constant, since $R=1$ for us. ■

Theorem A.2.17. *(Theorem 2.2.2) There exists a constant $C < \infty$, independent of the center of D, such that for every continuous differential form f on $\bar{D}$,*

$$\|R^{w_\nu}_{bD}f + B_D f\|_{\frac{1}{2},D} \le C\|f\|_{0,D}.$$

In particular, if f is a continuous $(0,q)$-form on $\bar{D}$, such that $\bar{\partial}f = 0$ on D, then the solution of $\bar{\partial}u = f$ given by Corollary A.2.12 admits the estimate

$$\|u\|_{\frac{1}{2},D} \le C\|f\|_{0,D}.$$

Bibliography

[Ber97] Bo Berndtsson. Uniform estimates with weights for the $\bar{\partial}$-equation. *Journal of Geometric Analysis*, 7(2):195–215, 1997.

[Ber99] Bo Berndtsson. Weighted estimates for the $\bar{\partial}$-equation. *In: Complex Analysis and Geometry, Columbus, OH*, 9:43–57, 1999.

[BOC97] J. Bruna and J. Ortega-Cerdà. On L^p-solutions of the Laplace equation and zeros of holomorphic functions. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 24(3):571–591, 1997.

[Chr91] Michael Christ. On the $\bar{\partial}$ equation in weighted L^2 norms in $\mathbb{C}^1$. *The Journal of Geometric Analysis*, 1(3):193–230, 1991.

[CS99] John H. Conway and Neil J. A. Sloane. *Sphere Packings, Lattices and Groups*. Springer-Verlag, 1999.

[CZ14] Henry Cohn and Yufei Zhao. Sphere packing bounds via spherical codes. *Duke Math. J.*, 163(10):1965–2002, 2014. arXiv:1212.5966v2.

[Dal15] Gian Maria Dall'Ara. Pointwise estimates of weighted Bergman kernels in several complex variables. 2015. arXiv:1502.00865v1.

[Del98] Henrik Delin. Pointwise estimates for the weighted Bergman projection kernel in $\mathbb{C}^n$, using a weighted L^2 estimate for the $\bar{\partial}$ equation. *Annales de l'institut Fourier*, 48(4):967–997, 1998.

[Dev88] Robert L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley Publishing Company, Inc., 1988. Second Edition.

[Els05] Jürgen Elstrodt. *Maß- und Integrationstheorie*. Springer Berlin Heidelberg New York, 2005. Vierte Auflage.

[Fed96] Herbert Federer. *Geometric Measure Theory*. Springer Berlin Heidelberg, 1996.

[Has14] Friedrich Haslinger. *The $\bar{\partial}$ Neumann Problem and Schrödinger Operators*. Walter De Gruyter Inc, 2014.

[HKZ00] Haakan Hedenmalm, Boris Korenblum, and Kehe Zhu. *Theory of Bergman Spaces*. Springer-Verlag New York Berlin Heidelberg, 2000.

[HL84] Gennadi Henkin and Jürgen Leiterer. *Theory of Functions on Complex Manifolds*. Birkhäuser Verlag, 1984.

[Hö94] Lars Hörmander. *Notions of Convexity*. Birkhäuser, 1994.

[Ker72] Norbert Kerzman. The Bergman kernel function. Differentiability at the boundary. *Math. Ann.*, 195:149–158, 1972. MR 0294694 (45 3762).

[KW95] Robert Kaufman and Jang-Mei Wu. Two problems on doubling measures. *Revista Matemática Iberoamericana*, 11(3):527–545, 1995.

[Lee09] Jeffrey M. Lee. *Manifolds and Differential Geometry*. American Mathematical Society, 2009.

[LG86] Pierre Lelong and Lawrence Gruman. *Entire Functions of Several Complex Variables*. Springer Berlin Heidelberg New York Tokyo, 1986.

[LS95] J. Luukkainen and E. Saksman. Every complete doubling metric space carries a doubling measure. *Proceedings of the American Mathematical Society*, 126:531–534, 1995.

[MMOC03] N. Marco, X. Massaneda, and J. Ortega-Cerdà. Interpolating and sampling sequences for entire functions. *Geom. Funct. Anal.*, 182(4):862–914, 2003.

[MOC09] Jordi Marzo and Joaquim Ortega-Cerdà. Pointwise estimates for the Bergman kernel of the weighted Fock space. *The Journal of Geometric Analysis*, 19:890–910, 2009.

[Mor07] Andrei Moroianu. *Lectures on Kähler Geometry*. Cambridge University Press, 2007.

[Nar73] Raghavan Narasimhan. *Analysis on Real and Complex Manifolds*. North-Holland Publishing Company, 1973. Second Edition.

[Pen07] Roger Penrose. *The Road to Reality*. Vintage Books, 2007.

[PfESo15] Encyclopedia of Mathematics. Pluriharmonic function. E.D. Solomentsev (originator). `http://www.encyclopediaofmath.org/index.php?title=Pluriharmonic_function&oldid=16481`, October 2015.

[Ran98] R. Michael Range. *Holomorphic Functions and Integral Representations in Several Complex Variables*. Springer-Verlag, 1998.

[Rud09] Walter Rudin. *Reelle und Komplexe Analysis*. Oldenburg, 2009.

[Ruz01] Michael Ruzhansky. On uniform properties of doubling measures. *Proceedings of the American Mathematical Society*, 129(11):3413–3416, 2001.

[Sak99] Eero Saksman. Remarks on the nonexistence of doubling measures. *Annales Academiæ Scientiarum Fennicæ Mathematica*, 24:155–163, 1999.

[Ste93] Elias M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oszillatory Integrals.* Princton University Press, 1993.

[VK88] A. L. Vol'berg and S. V. Konyagin. On measures with the doubling condition. *Math. USSR Izvestiya*, 30:629–638, 1988.

[Wel08] Raymond O. Wells. *Differential Analysis on Complex Manifolds.* Springer Science + Business Media, LLC, 2008. Third Edition.

[Wu93] Jang-Mei Wu. Null sets for doubling and dyadic doubling measures. *Annales Academiæ Scientiarum Fennicæ Series A. I. Mathematica*, 18:77–91, 1993.