MASTER'S THESIS

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Scaling limits of random one-dimensional diffusions

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Introduction

"Aging" occurs in out-of-equilibrium physical systems. It is becoming increasingly important in physics and mathematics.
Bouchaud's trap model on a graph $G = (V, \mathcal{E})$ has been introduced to analyze aging. It is defined as follows:
Let $E = (E_x)_{x \in V}$ be a collection of i.i.d. random variables exponentially distributed with mean one, we introduce the continuous time Markov chain $X(t)$ with state space V such that

$$\mathbb{P}(X(t+dt) = y | X(t) = x, E) = \begin{cases} w_{xy}\, dt & \text{if } x, y \text{ are connected in } G \\ 0 & \text{otherwise,} \end{cases}$$

where

$$w_{xy} = \nu \exp(-\beta((1-a)E_x - aE_y)).$$

Here β stands for the inverse temperature, $a \in [0,1]$ expresses the balance of power between x and y. The parameter ν is introduced for technical reasons.
The Markov chain $X(t)$ resides at site x for a certain amount of time, which depends on E_x. This time increases, if E_x does. Thus, E_x can be viewed as the depth of the trap at site x.
Bouchaud's trap model illustrates the motion of the physical system between the states with energies E_x. It can be viewed as a simplification of spin-glass dynamics. The states of $X(t)$ represent those of a spin-glass system with particularly low energy. The spin-glass dynamics reside mostly in the deepest traps and run through the others especially fast. Thus, all smaller traps can be neglected, when it comes to analyzing long time behaviour of dynamics.

For the sake of this thesis, the case $G = (\mathbb{Z}, \mathcal{E})$, where $\mathcal{E} = \{(x,y) : |x-y| = 1)\}$ are the nearest neighbor edges is most important:

Let $(\tau_x)_{x \in \mathbb{Z}}$ be i.i.d. random variables with unbounded expectation, which

fulfill the following requirement:
Let $\alpha > 0$. There exists a slowly varying function L, such that

$$\mathbb{P}[\tau_x \geqslant u] = u^{-\alpha} L(u).$$

Suppose $\tau_0 > c > 0$, for $a \in [0,1]$, the one-dimensional Bouchaud trap model, $\mathbf{BTM}(\mathbb{Z}, \tau, a)$, is a continuous time Markov chain $X(t)$ with state space $\mathbb{Z}$ starting in $X(0) = 0$ and random jump rates

$$\omega_{xy} = \begin{cases} \frac{1}{2}\mathbb{E}[\tau_0^{-a}]\tau_x^{-(1-a)}\tau_y^{-a} & \text{if } (x,y) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

This model was analyzed in terms of "aging" for $a = 0$ by Fontes, Isopi and Newman in [7] and for $a > 0$ by Ben Arous and Černý in [6].

Aging occurs if there exists a limit of a two-point function $F(t_\omega, t_\omega + t)$, which represents the change of the $\mathbf{BTM}(\mathbb{Z}, \tau, a)$ at time $t + t_\omega$ after time t_ω, if t goes to infinity.
Introduced by Fontes, Isopi and Newman a good estimate for the change of the system and therefore a good choice for F is

$$R^q(t, t + \theta t) = \mathbb{E} \sum_{i \in \mathbb{Z}} \mathbb{P}[X((1 + \theta)t) = i | \tau, X(t)]^2.$$

R^q represents the average (here the average with respect to the starting point) probability that two independent realizations of $X(t)$ with equal site at time t_ω will arrive at the same site after time $t + t_\omega$.
Fontes, Isopi and Newman managed to prove existence of the following limit for $a = 0$:

$$\lim_{t \to \infty} R^q(t, t + \theta t) = \lim_{t \to \infty} \mathbb{E} \sum_{i \in \mathbb{Z}} \mathbb{P}[X((1 + \theta)t) = i | \tau, X(t)]^2 = R^q(\theta). \quad (1)$$

This has been extended by Ben Arous and Černý for the case $a > 0$. Thus, existence of "aging" has been proven for $\mathbf{BTM}(\mathbb{Z}, \tau, a)$ for all $a \in [0,1]$. The main tool to prove (1) consists of computing the scaling limit of $X(t)$:
Let $\varepsilon > 0$ we define the rescaled process:

$$X^\varepsilon(t) := \varepsilon X(t/\varepsilon c_\varepsilon),$$

where

$$c_\varepsilon = (\inf\{t \geqslant 0 : \mathbb{P}(\tau_0 > t) \leqslant \varepsilon\})^{-1}.$$

It has been proven in [7] that $X^\varepsilon(t)$ converges almost surely in distribution to a diffusion $M(t)$ as $\varepsilon \to 0$, where $M(t)$ is defined by its speed measure $\rho := \sum_{i\in\mathbb{Z}} v_i \delta_{x_i}$ with $(x_i, v_i)_{i\in\mathbb{Z}}$ being an inhomogeneous Poisson point process on $\mathbb{R} \times (0,\infty)$ with density measure $dx\, \alpha v^{-1-\alpha}\, dv$. We will refer to $M(t)$ as the FIN-diffusion, since this diffusion was first introduced by Fontes, Isopi, and Newman.
For the proof that the scaling limit of $X(t)$ is the FIN-diffusion they applied a theorem (see chapter 1 theorem 9) proved by C. Stone in [8]. This theorem states that vague convergence of a sequence of speed measures implies convergence in distribution of the corresponding diffusions.
Consequently, they showed vague convergence of the speed measures to get convergence in distribution of the corresponding rescaled diffusions to the FIN-diffusion.

In this thesis we focus on diffusions, which can be interpreted as continuous versions of the $\mathbf{BTM}(\mathbb{Z}, \tau, a)$. Analogously to the $\mathbf{BTM}(\mathbb{Z}, \tau, a)$ we want to compute their scaling limits.
To be concrete: Let $(\Omega_1, \mathcal{A}_1, \mathcal{P}_1)$ and $(\Omega_2, \mathcal{A}_2, \mathcal{P}_2)$ be probability spaces. The diffusions will be defined on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P})$, with $\mathbb{P} := \mathcal{P}_1 \otimes \mathcal{P}_2$.

Let ξ be a Poisson Point Process on $\mathbb{R}$ with intensity 1, defined on $(\Omega_1, \mathcal{A}_1, \mathcal{P}_1)$. Consequently, $\xi([a,b])$ has a Poisson distribution with parameter $b-a$. For $\omega_1 \in \Omega_1$ let $(x_i(\omega_1))_{i\in\mathbb{Z}}$ be the points of the Poisson point process corresponding to the realization ω_1.
Let $(Y_i)_{i\in\mathbb{Z}}$ be a sequence of positive $i.i.d$ random variables on $(\Omega_2, \mathcal{A}_2, \mathcal{P}_2)$.
Let $h \in C^\infty(\mathbb{R})$ with $supp(h) = [-1,1]$ and let $c > 0$.

We study scaling limits of two diffusion models with random coefficients.
The first case corresponds to $a = 0$ in the discrete case:
We consider a random one-dimensional diffusion solving for each $\omega \in \Omega_1 \times \Omega_2$ a SDE (stochastic differential equation) without drift

$$dX_t = (f_\omega(X_t))^{-1/2} dW_t, \qquad X_0 = 0 \tag{2}$$

with

$$f_\omega(x) = \widehat{c} \sum_{i=-\infty}^{\infty} h(x - x_i(\omega_1))Y_i(\omega_2) + c$$

where $\widehat{c}$ is to be defined dependent on the requirement for Y_i:
We study the asymptotic behavior of the solution $X(t)$ to (2) under two different assumptions:

1. $(Y_i)_{i\in\mathbb{Z}}$ satisfy
$$\mathbb{E}[Y_i] < \infty \tag{3}$$
and we set
$$\widehat{c} := \frac{1}{\left(\int_{-1}^1 h(x)\,dx\right) + \frac{c}{\mathbb{E}[Y_1]}}.$$

2. $(Y_i)_{i\in\mathbb{Z}}$ have unbounded expectation and the following condition holds: There exists a slowly varying function L, such that
$$\mathcal{P}_2[Y_i \geqslant u] = u^{-\alpha}L(u) \tag{4}$$
and we set
$$\widehat{c} := \left(\int_{-1}^1 h(x)\,dx\right)^{-1}.$$

In both cases we want to determine the scaling limit of X_t.
Provided $\mathbb{E}[Y_i] < \infty$, we define the rescaled process $X^\varepsilon(t) := \varepsilon X(\varepsilon^{-2}c_2^{-1}t)$, where $c_2 := (\mathbb{E}[Y_1])^{-1}$.
Analogously to $\mathbf{BTM}(\mathbb{Z}, \tau, a)$ we exploit Stone's theorem to compute the limit of $X^\varepsilon(t)$, as $\varepsilon \to 0$, which will be the Brownian motion.
In the other case, provided (4), for $\varepsilon > 0$ and

$$c_\varepsilon = (\inf\{t \geqslant 0 : \mathbb{P}(Y_1 > t) \leqslant \varepsilon\})^{-1}, \tag{5}$$

we define the rescaled process $X^\varepsilon(t) := \varepsilon X\left(\frac{t}{c_\varepsilon \varepsilon}\right)$. Since for Y_i the same conditions are fulfilled as for the rescaled $\mathbf{BTM}(\mathbb{Z}, \tau, a)$, which converges to the FIN-diffusion $M(t)$, we expect $X^\varepsilon(t)$ to converge to $M(t)$ as well. We proceed equally to the first case by using Stone's theorem and obtain almost sure convergence in distribution to the FIN-diffusion.

The second diffusion, which we will examine, is derived from a SDE with non-zero drift:
Assume x_i, Y_i and h are equally defined as before. Suppose (4) holds, we consider

$$dX_t = V'_\omega(X_t)\ dt + \ dW_t \qquad X_0 = x, \tag{6}$$

where for each $\omega \in \Omega_1 \times \Omega_2$

$$V_\omega(x) := -\frac{1}{2}\log\Big(c_1 \sum_{i=-\infty}^{\infty} h(x - x_i(\omega_1))Y_i(\omega_2) + c\Big).$$

Again we show that for c_ε as introduced in (5) and some constant C the rescaled process $X^\varepsilon_\omega(t) := \varepsilon X_\omega\left(\frac{tC}{c_\varepsilon \varepsilon}\right)$ converges to the FIN-diffusion $M(t)$.

This thesis is organized as follows: In the first chapter we will establish a general theory concerning one-dimensional diffusions, derived from stochastic differential equations. One dimensional diffusions coming from SDEs without drift, can be expressed as a time changed Brownian motion. A time changed Brownian motion is characterized by the speed measure of a diffusion. The speed measure can be obtained by the SDE.
The second chapter then concentrates on proving the convergence results of the diffusions as illustrated above.

1. One-dimensional diffusions

Let W_t be a one-dimensional Brownian motion. We focus on the stochastic differential equation (SDE)

$$dX_t = \sigma(X_t)\, dW_t + b(X_t)\, dt, \quad X_0 = x. \tag{1.1}$$

A process X_t satisfies (1.1) if

$$X_t = x + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b(X_s)\, ds.$$

We assume that σ and b are both locally Lipschitz and that σ is bounded from below by some positive constant such that a solution to (1.1) exists.

Let $C := C([0,\infty))$. The canonical process Z_t is defined on C by $Z_t(\omega') = \omega'(t)$.

Let $X(x,t,\omega)$ be the solution to (1.1). For each x we introduce a probability measure $\mathbb{P}^x$ on C, determined by

$$\mathbb{P}^x(Z_{t_1}(\omega') \in E_1, ..., Z_{t_n}(\omega') \in E_n) = \mathbb{P}(X(x,t_1,\omega) \in E_1, ..., X(x,t_n,\omega) \in E_n) \tag{1.2}$$

whenever $t_1, ..., t_n \in [0,\infty)$ and $E_1, ..., E_n$ are Borel sets in $\mathbb{R}$. Hence, x indicates the starting point of the canonical process Z_t with probability measure $\mathbb{P}^x$. Defined on the smallest σ-field containing these cylindrical sets, $\mathbb{P}^x$ can be extended to $\mathscr{F}'_\infty := \bigcup_{t\in\mathbb{R}^+} \mathscr{F}'_t$, where $\mathscr{F}'_t$ is a right continuous filtration. We now state our first important theorem:

Definition 1. *The shift operators $\theta_t : C \to C$ are defined by $\theta_t(\omega)(s) = \omega(t+s)$. A stochastic process Z_t is strong Markov if*

$$\mathbb{E}^x[f(Y \circ \theta_T)|\mathscr{F}'_T] = \mathbb{E}^{Z_T}[Y] \tag{1.3}$$

holds $\mathbb{P}^x$-a.s., whenever $x \in \mathbb{R}$, Y is bounded and $\mathscr{F}'_\infty$-measurable and T is a finite stopping time.

Theorem 1. *The canonical process $(\mathbb{P}^x, Z_t)$ is a strong Markov process.*

The proof of this theorem can be found in [1] (Chapter 1 Theorem 5.1).

1.1 Properties of diffusions

Definition 2. *For each $x \in \mathbb{R}$ let $\mathbb{P}^x$ be a probability measure defined on $\mathscr{F}'_\infty$. If X_t is a a continuous strong Markov process with respect to $\mathbb{P}^x$ ((1.3) holds), then we say that $(\mathbb{P}^x, X_t)$ is a diffusion.*

From Theorem 1 we obtain: $(\mathbb{P}^x, Z_t)$, the canonical process of the solution of the SDE (1.1), is a diffusion. Let W_t be a Brownian motion defined on some $(\Omega, \{\mathscr{F}\}_t, \mathscr{F}, \mathbb{P})$. Then the canonical Brownian motion $(\mathbb{P}^x, Z_t)$, analogously defined to (1.2), is also a diffusion. We will now prove a certain property of the canonical Brownian motion:

Proposition 1. *Let $(\mathbb{P}^x, X_t)$ be the canonical Brownian motion and let $\tau_{[a,b]} := \inf\{t > 0 | X_t \notin [a,b]\}$. If $a < x < b$, then $\tau_{[a,b]} < \infty$ a.s., and*

$$\mathbb{P}^x(X(\tau_{[a,b]}) = a) = \frac{b-x}{b-a}, \quad \mathbb{P}^x(X(\tau_{[a,b]}) = b) = \frac{x-a}{b-a}. \tag{1.4}$$

Proof. We write τ instead of $\tau_{[a,b]}$. $X_t^2 - t$ is a martingale, so is the stopped process $X^2_{\tau\wedge t} - \tau \wedge t$. Hence,

$$\mathbb{E}^x[X^2_{\tau\wedge t}] = \mathbb{E}^x[\tau \wedge t] + x.$$

For $t \leqslant \tau, |X_t| \leqslant |a| + |b|$, so Fatou's lemma yields: $\mathbb{E}^x\tau \leqslant (|a| + |b|)^2 + |x|$ and thus $\tau < \infty$ a.s. Optional stopping theorem applied on the martingale X_t gives:

$$x = \mathbb{E}^x X_\tau = a\mathbb{P}^x(X_\tau = a) + b\mathbb{P}^x(X_\tau = b). \tag{1.5}$$

As $\tau < \infty$ a.s.,

$$1 = \mathbb{P}^x(X_\tau = a) + \mathbb{P}^x(X_\tau = b). \tag{1.6}$$

Solving the linear system (1.5) and (1.6) for the two unknowns $\mathbb{P}^x(X_\tau = a)$ and $\mathbb{P}^x(X_\tau = b)$ completes the proof. $\square$

Taking $b \to \infty$ and $x > a$, continuity of the probability measure gives:

$$\mathbb{P}^x(T_a < \infty) = 1. \tag{1.7}$$

Definition 3. *Let $(\mathbb{P}^x, X_t)$ be a diffusion, which takes values in an interval I, and $T_y := \inf\{t : X_t = y\}$ if*

$$\textit{for all } x, y \textit{ in the interior of } I : \mathbb{P}^x(T_y < \infty) = 1 \tag{1.8}$$

*we say the diffusion is **regular**.*

Definition 4. *We say that a regular diffusion $(\mathbb{P}^x, X_t)$ is on **natural scale** if* (1.4) *holds for every interval $[a, b]$.*

1.2 Speed measures

Assume we are given a regular diffusion $(\mathbb{P}^x, X_t)$ on an open interval I on natural scale. For $(a, b) \subseteq I$, we define

$$G_{a,b}(x, y) = \begin{cases} 2(x-a)(b-y)/(b-a) & \text{for } a < x \leqslant y < b \\ 2(y-a)(b-x)/(b-a) & \text{for } a < y \leqslant x < b \end{cases} \tag{1.9}$$

and set $G_{a,b}(x, y) = 0$ if x or y is not in (a, b).

We call the measure $m(dx)$ for which

$$\mathbb{E}^x(\tau_{(a,b)}) = \int G_{a,b}(x, y)\, m(dy), \quad \text{whenever } (a, b) \subseteq I \text{ and } x \in I \tag{1.10}$$

holds, speed measure of the diffusion $(\mathbb{P}^x, X_t)$. From (1.10) we notice that the speed measure computes how fast the diffusion moves in average through intervals.

Proposition 2. *The speed measure for the canonical Brownian motion is the Lebesgue measure.*

Proof. Since $X_t^2 - t$ is a martingale, so is the stopped process $(X_{t\wedge\tau_{(a,b)}})^2 - t \wedge \tau_{(a,b)}$. Thus,

$$\mathbb{E}^x(\tau_{(a,b)} \wedge t) = \mathbb{E}^x(X_{t\wedge\tau_{(a,b)}} - x)^2. \tag{1.11}$$

We have $|X_t| \leqslant |a| + |b|$ for $t \leqslant \tau$. Hence, for $t \to \infty$ monotone convergence on the left and the dominated convergence on the right yield

$$\begin{aligned}\mathbb{E}^x(\tau_{(a,b)}) &= \mathbb{E}^x(X_{\tau_{(a,b)}} - x)^2 = \\ &= (b-x)^2\mathbb{P}^x(X_{\tau_{(a,b)}} = b) + (x-a)^2\mathbb{P}^x(X_{\tau_{(a,b)}} = a).\end{aligned}$$

By Proposition 1 this equals

$$(x-a)(b-x),$$

which is the same as the integral with respect to the Lebesgue measure of $G_{a,b}(x,y)$. Hence,

$$\mathbb{E}^x(\tau_{(a,b)}) = \int G_{a,b}(x,y)\,dy.$$

□

We now state without proof a very important theorem, which we will apply later in order to determine the speed measure of a diffusion corresponding to a solution of a stochastic differential equation.

Theorem 2 (Trotter)**.** *Let W_t be a Brownian motion. There exists a family of non-decreasing processes $l(t,x)$ that are jointly continuous in x and t a.s. such that*

1. *if f is nonnegative Borel function, then*

$$\int_0^t f(W_s)\,ds = \int_{\mathbb{R}} f(x)l(t,x)\,dx, \quad a.s.,$$

   where the null set can be taken independent of f;

2. *$l(t,x) \to \infty$ a.s., as $t \to \infty$;*

3. *The set of t on which $l(t,x)$ increases is precisely the set $\{t : W_t = x\}$;*

4. *$l(t,x)$ may be defined by the formula*

$$|W_t - x| - |W_0 - x| = \int_0^t \operatorname{sgn}(W_s - x)\,dW_s + l(t,x).$$

This theorem is verified in [2] (pages 68-71).

Definition 5. *The process $l(t,x)$ is called local time of the Brownian motion W_t.*

For the next theorem we introduce some notations:
Let m be a measure on an open interval I such that $0 < m(a,b) < \infty$ for every interval $(a,b) \subseteq I$. Let $(\mathbb{P}^x, W_t)$ be the canonical Brownian motion, we set

$$\phi_t := \int l(t,x)\, m(dx), \qquad \psi_t := \inf\{u : \phi_u > t\}, \qquad X_t := W_{\psi_t}. \tag{1.12}$$

Theorem 3. *Under $\mathbb{P}^x$, X_t as defined by (2.39), is a regular diffusion on natural scale with speed measure m.*

Proof. X_t ***is continuous:***

$l(t,x)$ is continuous in t and for every $t \in [0,\infty)$ Lebesgue integrable with respect to $m < \infty$. By dominated convergence theorem, we obtain the continuity of the process ϕ_t. Let $s < u$, if $t \in (s,u)$, then $W_t = x$ for some $x \in \mathbb{R}$. Since $l(t,x)$ grows at t by (3) of Theorem 2, it follows that $l(u,x) - l(s,x) > 0$. Continuity of the local times gives $l(u,y) - l(s,y) > 0$ for all y in a neighborhood $(x-\delta, x+\delta)$ of x. By assumption $m(x-\delta, x+\delta) > 0$, consequently, $\phi_u - \phi_s > 0$. As a result ϕ_t is strictly increasing and continuous. Thus, the inverse ψ_t exists and is continuous. Since the Brownian motion is continuous as well, X_t is continuous.

ψ_t ***is*** $\mathscr{F}_t'$***-stopping time:***

Let $\omega \in \{\psi_t < t\}$, then there exists $u \in [0,t)$ such that $\phi_u(\omega) > t$. ϕ_t is continuous process and there exists a rational s with $\phi_s(\omega) > t$. Consequently, $\{\psi_t < t\} = \bigcup_{0 \leqslant s < t, s \in \mathbb{Q}} \{\phi(s) > t\} \in \mathscr{F}_s' \subseteq \mathscr{F}_t'$. As a result ψ_t is $\mathscr{F}_t'$-stopping time.

$(\mathbb{P}^x, X_t)$ ***is a regular diffusion on natural scale:***

Applying monotone convergence and from (2) of Theorem 2, we get $\phi_t \nearrow \infty$, thus $\psi_t \nearrow \infty$ and $\tau^X_{(a,b)} < \infty$ $\mathbb{P}^x$-a.s., where $\tau^X_{(a,b)}$ denotes the exit time of (a,b)

by X_t and $\tau^W_{(a,b)}$ denotes the corresponding exit time of W_t. In addition,

$$\mathbb{P}^x(X(\tau^X_{(a,b)}) = b) = \frac{x-a}{b-a}.$$

Since X_t is a time changed Brownian motion and therefore a martingale according to optional stopping theorem, this can be proven exactly in the same way as in the proof of Proposition 1.

$(\mathbb{P}^x, X_t)$ ***is a strong Markov process:***

We set $\mathscr{F}^*_t := \mathscr{F}'_{\psi_t}$. Let T be a bounded stopping time for $\mathscr{F}'_t$. Thus,

$$\mathbb{E}^x[f(X_{T+t})|\mathscr{F}^*_T] = \mathbb{E}^x[f(W(\psi_{T+t}))|\mathscr{F}_{\psi_T}]. \tag{1.13}$$

ψ_T is a stopping time for $\mathscr{F}'_t$. ($\{T \leqslant t\} \in \mathscr{F}'_{\psi_t} \Rightarrow \{\psi_T \leqslant t\} = \{T \leqslant \phi_t\} \in \mathscr{F}'_t$) Moreover, $\psi_{T+t} = \psi_t \circ \theta_{\psi_T}$. Thus, by the strong Markov property of the canonical Brownian motion (1.13) equals

$$\mathbb{E}^{W(\psi_T)}[f(W_{\psi_t})] = \mathbb{E}^{X_T}[f(X_t)].$$

As shown in [2], this is equivalent to the strong Markov property.

The speed measure of $(\mathbb{P}^x, X_t)$ ***is m:***

$$\begin{aligned}\mathbb{E}^x\tau^X_{(a,b)} &= \mathbb{E}^x\int_0^\infty \mathbb{1}_{(a,b)}(X(s \wedge \tau^X_{(a,b)}))\,ds \\ &= \mathbb{E}^x\int_0^\infty \mathbb{1}_{(a,b)}(W(\psi(s \wedge \tau^{W(\psi)}_{(a,b)})))\,ds.\end{aligned} \tag{1.14}$$

Since ϕ_t is strictly increasing, $\psi_0 = 0$. Moreover, we have already shown that $\psi_t \nearrow \infty$. As a result substitution w.r.t. Lebesgue-Stieltjes integrals yields that (1.14) equals

$$\begin{aligned}
&= \mathbb{E}^x \int_0^\infty \mathbb{1}_{(a,b)}(W(t \wedge \tau^W_{(a,b)}))\, d\phi_t \\
&= \mathbb{E}^x \int \int_0^\infty \mathbb{1}_{(a,b)}(W(t \wedge \tau^W_{(a,b)}))\, dl(t,y)\, m(dy) \\
&= \mathbb{E}^x \int \int_0^{\tau^W_{(a,b)}} dl(t,y)\, m(dy) \\
&= \int \mathbb{E}^x l(\tau^W_{(a,b)}, y)\, m(dy). \qquad (1.15)
\end{aligned}$$

(4) of Theorem 2 gives

$$\begin{aligned}
\mathbb{E}^x l(\tau^W_{(a,b)}, y) &= \mathbb{E}^x \left[|W_{\tau^W_{(a,b)}} - y| - |W_0 - y| - \int_0^{\tau^W_{(a,b)}} \operatorname{sgn}(W_s - y)\, dW_s \right] \\
&= \mathbb{E}^x |W(\tau^W_{(a,b)}) - y| - |x - y|, \qquad (1.16)
\end{aligned}$$

since $M(\tau^W_{(a,b)} \wedge t) = \int_0^{\tau^W_{(a,b)} \wedge t} \operatorname{sgn}(W_s - x)\, dW_s$ is martingale according to optional stopping theorem. By Proposition 1, this equals:

$$|a - y| \frac{b - x}{b - a} + |b - y| \frac{x - a}{b - a} - |x - y| = G_{a,b}(x, y).$$

which together with (1.15) implies

$$\mathbb{E}^x \tau^X_{(a,b)} = \int G_{a,b}(x, y)\, m(dy)$$

and completes the proof. □

1.3 Diffusions as solutions of SDEs

Assume X_t is given as the solution to

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \qquad (1.17)$$

where we require σ and b to be Lipschitz and bounded above and σ to be bounded from below by a positive constant.

We introduce the following operator

$$\mathscr{L}f(x) := \frac{1}{2}a(x)f''(x) + b(x)f'(x), \tag{1.18}$$

where $a(x) := \sigma^2(x)$.

Proposition 3. *Suppose M_t is a continuous martingale with $[M]_t$ strictly increasing and $[M]_\infty = \infty$. Then M is a time change of Brownian motion, that is there exists a one-dimensional Brownian motion X such that $M_t = X_{[M]_t}$.*

Proof. We set $\tau(u) := \inf\{t : [M]_t > u\}$ and $X_u := M_{\tau(u)}$. $[M]_t$ is the limit of sums of continuous functions so it is continuous too. According to our assumption $[M]_u$ is strictly increasing, so $\tau(u)$ coincides with $(M_u)^{-1}$ on the image of M_u. Thus, $\tau(u)$ is the inverse of a continuous, strictly increasing function and therefore continuous too. Consequently so is X. Let $u_1 < u_2$. $\tau(u_1) < \tau(u_2)$ are stopping times. Now optional stopping theorem yields

$$\mathbb{E}[X_{u_2}|\mathscr{F}_{\tau(u_1)}] = \mathbb{E}[M_{\tau(u_2)}|\mathscr{F}_{\tau(u_1)}] = M_{\tau(u_1)} = X_{u_1}.$$

Thus, X_u is a martingale with respect to the filtration $\mathscr{F}_{\tau(u)}$. From the definition of quadratic variation we get that $M_t^2 - [M]_t$ is a martingale. Again according to the optional stopping theorem we get

$$\begin{aligned}\mathbb{E}[X_{u_2}^2 - u_2|\mathscr{F}_{\tau(u_1)}] = \mathbb{E}[M_{\tau(u_2)}^2 - [M]_{\tau(u_2)}|\mathscr{F}_{\tau(u_1)}] = M_{\tau(u_1)}^2 - [M]_{\tau(u_1)} = \\ = X_{u_1}^2 - u_1.\end{aligned}$$

Thus, $X_t^2 - t$ is a martingale, hence according to the definition of quadratic variation, we obtain $[X]_t = t$. By Levy's Theorem it follows that X_t is a Brownian motion and we have $M_t = X_{[M]_t}$.

□

Theorem 4. *Given a solution X_t to (1.17). Let $s(x)$ be the solution to $\mathscr{L}s(x) = 0$, then there exist constants c_1, c_2 and x_0 such that*

$$s(x) = c_1 + c_2 \int_{x_0}^{x} e^{-\int_{x_0}^{y} 2b(\omega)/a(\omega)\, d\omega}\, dy. \tag{1.19}$$

Moreover, $s(X_t)$, is a regular diffusion on natural scale.

Proof. Since $a(x) > c^2 > 0$

$$0 = \mathscr{L}s(x) = \frac{1}{2}a(x)s''(x) + b(x)s'(x)$$

is equivalent to

$$\frac{s''(x)}{s'(x)} = -2\frac{b(x)}{a(x)},$$

which is equivalent to

$$(\log s'(x))' = -2b(x)/a(x).$$

This ordinary differential equation can be solved by integrating. Thus, for $x_0 \in \mathbb{R}$

$$s'(x) = \exp\left\{-\int_{x_0}^{x} 2b(\omega)/a(\omega)d\omega + c\right\}. \tag{1.20}$$

Integrating again yields (1.19).

As according to (1.17) both σ and b are continuous, $s(x)$ given by (1.19) is C^2. Now let X_t be a solution to (1.17), we use Itô's formula:

$$\begin{aligned} ds(X_t) &= s'(X_t)\,dX_t + \frac{1}{2}s''(X_t)\,(dX_t)^2 \\ &= \sigma(X_t)s'(X_t)\,dW_t + b(X_t)s'(X_t)\,dt + \frac{1}{2}\sigma(X_t)^2 s''(X_t)\,dt \\ &= (\sigma s')(X_t)\,dW_t + \underbrace{\frac{1}{2}(bs' + \sigma^2 s'')(X_t)\,dt}_{=0}. \end{aligned}$$

Consequently, $s(X_t)$ is a martingale. The quadratic variation of $Y_t := s(X_t)$ is $\int_0^t (\sigma s')^2(X_s)\,ds$. As $\sigma s'$ is positive and bounded from below, the quadratic variation is strictly increasing and converges to infinity. Thus, according to Theorem 3, Y_t can be written as a time change of a Brownian motion: $Y_t = \widehat{W}(\psi_t)$. Analogously to the proof of Theorem 3 it follows from monotone convergence and (2) of Theorem 2 that $\tau^Y_{(a,b)} < \infty$ $\mathbb{P}^x$-a.s. Let $(\mathbb{P}^x, Z_t)$ be the corresponding canonical martingale process to Y_t. Equal steps as used in the proof of Proposition 1 show that $(\mathbb{P}^x, Z_t)$ is on natural scale. Finally, by Theorem 1 it follows that $(\mathbb{P}^x, Z_t)$ is a diffusion, which completes the proof.

□

Definition 6. *Let X_t be a solution to* (1.17). *Then $s(x)$, defined as in* (1.19), *is called scale function of the diffusion X_t.*

In the proof of Theorem 4 we have seen that if $Y_t := s(X_t)$, then

$$dY_t = (s'\sigma)(s^{-1}(Y_t))\,dW_t. \tag{1.21}$$

We continue with the case, when b in (1.17) is zero, that is

$$dX_t = \sigma(X_t)dW_t \tag{1.22}$$

Theorem 5. *Let X_t be a solution to* (1.22), *then its speed measure is given by*

$$m(dx) = \frac{1}{a(x)}\,dx \tag{1.23}$$

and $X_t = \widehat{W}(\psi_t)$, where $\widehat{W}$ is a Brownian motion and

$$\psi_t := \inf\{u : \phi_u > t\}, \qquad \phi_t := \int l(t,x)\,m(dx). \tag{1.24}$$

Proof. Since X_t satisfies (1.22), X_t is a martingale. $[X]_t = \int_0^t a(X_s)\,ds$ is strictly increasing and $[X]_\infty = \infty$ as $a(x) \geqslant c$. According to Theorem 3 X_t is a time changed Brownian motion: $X_t = \widehat{W}([X]_t)$. We set:

$$\psi_t := [X]_t = \int_0^t a(X_s)\,ds.$$

The inverse of ψ_t, namely, ϕ_t, exists and it holds that

$$\frac{d\phi_t}{dt} = \frac{1}{a(X_{\phi_t})} = \frac{1}{a(\widehat{W}_t)},$$

integrating gives

$$\phi_t = \int_0^t \frac{1}{a(\widehat{W}_s)}\,ds = \int l(t,y)\frac{1}{a_y}\,dy$$

for all t, where the second equality follows from Theorem 2.2. Now we set $m(dx) := \dfrac{1}{a(x)}\,dx$. Since $a(x) \geqslant c^2$, for any $a, b \in \mathbb{R}$

$$0 < m(a,b) < \infty$$

holds. Theorem 3 yields that the corresponding canonical process $(\mathbb{P}^x, Z_t)$ to X_t, is a regular diffusion with speed measure m. $\square$

We state without proof (see [1] page 83):

Theorem 6. *If* $(\mathbb{P}_i^x, X_t)$, $i = 1, 2$, *are two diffusions on natural scale on an open interval* I *with the same speed measure* m, *then* $\mathbb{P}_1^x = \mathbb{P}_2^x$ *on* $\mathscr{F}_{\tau_I}$.

1.4 Invariant measures

Occasionally it is important, to determine a measure μ, which remains equal under the "flow" of a diffusion X_t. In other words, X_t is distributed according to μ for all $t \geqslant 0$.
Let

$$dX_t = \sigma(X_t)\, dW_t + b(X_t)\, dt, \qquad X_0 = x. \tag{1.25}$$

Let $f \in C^2(\mathbb{R})$ and $\mathscr{L}$ defined as in (1.18), the *adjoint* operator $\mathscr{L}^*$ is given by

$$\mathscr{L}^* f(x) := \frac{\partial^2}{\partial^2 x}\left(\sigma^2(x) f(x)\right) - \frac{\partial}{\partial x}\left(b(x) f(x)\right), \tag{1.26}$$

that is for f and g satisfying suitable regularity conditions, by using integration by parts,

$$\int f(x) \mathscr{L} g(x)\, dx = \int g(x) \mathscr{L}^* f(x)\, dx. \tag{1.27}$$

Definition 7. *A measure* μ *is invariant for a strong Markov family if* $\mathbb{E}^\mu f(X_t) = \int f(x)\, \mu(dx)$ *for all* t *and all bounded continuous* f, *where* $\mathbb{E}^\mu f(X_t) = \int \mathbb{E}^y f(X_t)\, \mu(dy)$.

Theorem 7. *Let*

$$\partial_t u(x,t) = \mathscr{L} u(x,t), \quad t > 0, x \in \mathbb{R}, \tag{1.28}$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}. \tag{1.29}$$

The solution to (1.28) *is*

$$u(x,t) = \mathbb{E}^x f(X_t).$$

Proof. Let t_0 be fixed and let $M_t = u(X_t, t_0 - t)$. One can show that the solution u to (1.28) is C^2 in x and C^1 for $t > 0$ (see [9]). Itô's formula on $\mathbb{R} \times [0, t_0)$, gives

$$u(X_t, t_0 - t) = \text{martingale} + \int_0^t \mathscr{L} u(X_s, t_0 - s)\, ds + \int_0^t (-\partial_t u)(X_s, t_0 - s))\, ds.$$

Due to $\partial_t u = \mathscr{L} u$, M_t is a martingale, and $\mathbb{E}^x M_0 = \mathbb{E}^x M_{t_0}$. Since

$$\mathbb{E}^x M_{t_0} = \mathbb{E}^x u(X_{t_0}, 0) = \mathbb{E}^x f(X_{t_0}),$$

and

$$\mathbb{E}^x M_0 = \mathbb{E}^x u(X_0, t_0) = u(x, t_0).$$

the result follows for t_0. As t_0 can be chosen arbitrarily, the proof is complete. $\square$

For the next theorem we assume the coefficients of $\mathscr{L}$ to be smooth:

Theorem 8. *Assume there exists a nonnegative solution v to $\mathscr{L}^* v = 0$, which satisfies suitable regularity conditions. Let $\mu(dx) = v(x)\, dx$. Then μ is invariant for the process associated to $\mathscr{L}$.*

Proof. Let f be continuous and let $u(x,t) = \mathbb{E}^x f(X_t)$. Then

$$0 = \int u(x,t) \mathscr{L}^* v(x)\, dx = \int \mathscr{L} u(x,t) v(x)\, dx = \int \partial_t u(x,t) v(x)\, dx.$$

Thus,

$$\mathbb{E}^\mu f(X_t) = \int \mathbb{E}^x f(X_t)\, \mu(dx) = \int u(x,t) v(x)\, dx \tag{1.30}$$

is a constant function of t. For $t \to 0$,

$$\mathbb{E}^\mu f(X_t) = \int \mathbb{E}^x f(X_t)\, \mu(dx) \to \int f(x)\, \mu(dx).$$

Hence, $\mathbb{E}^\mu f(X_t) = \int f(x)\, \mu(dx)$ for all t. $\square$

1.5 Convergence of diffusions in distribution

A diffusion on natural scale can be expressed as a time-changed Brownian motion, defined by its speed measure. Stone's theorem, which we will state in the paragraph below, represents our main tool to prove the convergence results about diffusions coming from SDEs in the second chapter of this thesis. We notice that Stone's theorem originally not only involves statements about convergence for time-changed Brownian motions, but also for some additive functionals on diffusions. For more details see [8].

Theorem 9 (C. Stone)**.** *Let $\widehat{W}_t$ be a Brownian motion and μ^ε, μ be a collection of non-random locally finite measures, let*

$$Y^\varepsilon(t) := \widehat{W}(\psi(\mu^\varepsilon)(t))$$

with

$$\phi(\mu^\varepsilon)(t) := \int l(t,x)\,\mu^\varepsilon(dx), \quad \psi(\mu^\varepsilon)(t) := \inf\{u : \phi(\mu^\varepsilon)(t) > t\}$$

and

$$Y(t) := \widehat{W}(\psi(\mu)(t))$$

with

$$\phi(\mu)(t) := \int l(t,x)\,\mu(dx), \quad \psi(\mu)(t) := \inf\{u : \phi(\mu)(t) > t\}. \tag{1.31}$$

For any deterministic $t_0 > 0$, let $v^\varepsilon(t_0)$ denote the distribution of $Y^\varepsilon(t_0)$ and v denote the distribution of $Y(t_0)$. Suppose that

$$\mu^\varepsilon \xrightarrow{v} \mu,$$

then, as $\varepsilon \to 0$,

$$v^\varepsilon \xrightarrow{w} v.$$

2. Scaling limits of random one-dimensional diffusions

In this chapter we consider diffusion processes, derived from two stochastic differential equations related to the $\mathbf{BTM}(\mathbb{Z}, \tau, a)$.
We will study the asymptotic behavior of these random diffusions. In other words, we will determine their scaling limits.
We proceed as follows:
We will prove convergence of an equally distributed sequence of diffusion processes, which we introduce as time changed Brownian motions defined by their speed measures.
Then we use Stone's theorem, which implies, it suffices to verify vague convergence of the corresponding speed measures to prove convergence in distribution of the respective diffusion processes.
Thus, the main work consists of proving vague convergence of the speed measures.

The first SDE is given as follows:
Let $(\Omega_1, \mathcal{A}_1, \mathcal{P}_1)$ and $(\Omega_2, \mathcal{A}_2, \mathcal{P}_2)$ be probability spaces. Let $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P})$, with $\mathbb{P} := \mathcal{P}_1 \otimes \mathcal{P}_2$.
Let ξ be a Poisson Point Process on $\mathbb{R}$ with intensity 1, defined on $(\Omega_1, \mathcal{A}_1, \mathcal{P}_1)$, so $\xi([a, b]) \sim P((b - a))$, which means that $\xi([a, b])$ has a Poisson distribution with parameter $(b - a)$. For $\omega_1 \in \Omega_1$ let $(x_i(\omega_1))_{i \in \mathbb{Z}}$ be the points of the Poisson point process taken for this ω_1.

Let $(Y_i)_{i \in \mathbb{Z}} \geqslant 0$ be a sequence of $i.i.d.$ random variables on $(\Omega_2, \mathcal{A}_2, \mathcal{P}_2)$. Let $h \in C^\infty(\mathbb{R})$ with $\sup(h) = [-1, 1]$ and let $c > 0$.

For each $\omega \in \Omega_1 \times \Omega_2$ we consider the following 1-dimensional SDE:

$$dX_t = (f_\omega(X_t))^{-1/2} dW_t, \qquad X_0 = x, \tag{2.1}$$

with

$$f_\omega(x) = \hat{c} \sum_{i=-\infty}^{\infty} h(x - x_i(\omega_1)) Y_i(\omega_2) + c,$$

where $\hat{c}$ is to be defined dependent on the requirement for Y_i.
In order to compute the scaling limit of a solution X_t to (2.1) two assumptions must be differentiated:

1. $(Y_i)_{i \in \mathbb{Z}}$ defined as above satisfy

$$\mathbb{E}[Y_i] < \infty \tag{2.2}$$

   and we set

$$\hat{c} := \frac{1}{\left(\int_{-1}^{1} h(x)\,dx\right) + \frac{c}{\mathbb{E}[Y_1]}}.$$

2. $(Y_i)_{i \in \mathbb{Z}}$ defined as above have unbounded expectation and the following condition holds: There exists a slowly varying function L, such that

$$\mathcal{P}_2[Y_i \geqslant u] = u^{-\alpha} L(u), \quad \text{with} \ \ \alpha \in (0,1) \tag{2.3}$$

   and we set

$$\hat{c} := \left(\int_{-1}^{1} h(x)\,dx\right)^{-1}.$$

2.1 Weak convergence to Brownian motion

Suppose (2.2) holds:

Since $(Y_i)_{i \in \mathbb{Z}}$ have finite expectation, we can apply the strong law of large numbers to prove that the speed measures, which will be defined in the proof of the following theorem below, converge to the Lebesgue measure. This implies that the rescaled process converges to the Brownian motion:

Theorem 10. *Let $\omega \in \Omega_1 \times \Omega_2$. Let $X_\omega(t)$ be a solution to* (2.1), *where $(Y_i)_{i\in\mathbb{Z}}$ satisfy* (2.2), *then as ε converges to zero:*

$$\varepsilon X(\varepsilon^{-2} c_2^{-1} t) \xrightarrow{w} B(t) \quad \mathbb{P} - a.s.,$$

where $c_2 := (\mathbb{E}[Y_1])^{-1}$ and $B(t)$ is a Brownian motion.

Proof. According to Theorem 5 there exists a Brownian motion $\widehat{W}_t$ such that $X_\omega(t) = \widehat{W}(\psi(\mu_\omega)(t))$, where $\psi(\mu_\omega)$ is defined as above in (1.31) with

$$\mu_\omega(dx) = f_\omega(x)dx,$$

the speed measure of the diffusion $X_\omega(t)$.

Let $\widehat{W}^\varepsilon_t := \varepsilon \widehat{W}(\varepsilon^{-2}t)$, which is also a Brownian motion according to Lévy's Theorem.

Let $\varepsilon > 0$, we introduce the scaled speed measure

$$\mu^\varepsilon_\omega(dx) := c_2 \cdot f_\omega(\varepsilon^{-1}x)dx = c_2(\hat{c}\sum_{i\in\mathbb{Z}} h(\varepsilon^{-1}x - x_i(\omega_1))Y_i(\omega) + c)\, dx.$$

Let λ be the Lebesgue-measure. If

$$\mathbb{P}(\omega \in \Omega_1 \times \Omega_2 : \mu^\varepsilon_\omega \xrightarrow[\varepsilon\to 0]{v} \lambda) = 1 \tag{2.4}$$

holds, by Theorem 9

$$\widehat{W}^\varepsilon(\psi(\mu^\varepsilon)(t)) \overset{d}{=} \widehat{W}(\psi(\mu^\varepsilon)(t)) \xrightarrow{w} \widehat{W}(\psi(\lambda)(t)) \;\; \mathbb{P} - a.s.$$

and we obtain

$$\widehat{W}^\varepsilon(\psi(\mu^\varepsilon)(t)) \xrightarrow{w} \widehat{W}(\psi(\lambda)(t)) \;\; \mathbb{P} - a.s.$$

We have shown in Theorem 2 that the speed measure of the canonical Brownian motion is the Lebesgue measure. Let $(\mathbb{P}^x, Z_t)$ be the corresponding canonical process to $\widehat{W}(\psi(\lambda)(t))$ and $(\widehat{\mathbb{P}}^x, W_t)$ the canonical Brownian motion, then Theorem 6 yields

$$(\mathbb{P}^x, Z_t) = (\widehat{\mathbb{P}}^x, W_t).$$

Consequently $B(t) := \widehat{W}(\psi(\lambda)(t))$ is a Brownian motion.

It remains to prove that

$$\widehat{W}^{\varepsilon}(\psi(\mu^{\varepsilon})(t)) = \varepsilon X(\varepsilon^{-2}c_2^{-1}t).$$

Let $l^{\varepsilon}(t,x)$ be the local time of $\widehat{W}^{\varepsilon}$. An easy computation gives

$$l^{\varepsilon}(t,x) = \varepsilon l(\varepsilon^{-2}t, \varepsilon^{-1}x).$$

We write $\mu^{\varepsilon}(dx)$ instead of $\mu^{\varepsilon}_{\omega}(dx)$, $\phi^{\varepsilon}(t)$ instead of $\phi(\mu^{\varepsilon})(t)$ and $\psi^{\varepsilon}(t)$ instead of $\psi(\mu^{\varepsilon})(t)$. Thus,

$$\phi^{\varepsilon}(t) = \int l^{\varepsilon}(t,x)\,\mu^{\varepsilon}(dx) = \int \varepsilon l(\varepsilon^{-2}t, \varepsilon^{-1}x)(c_2 f(\varepsilon^{-1}x))\,dx.$$

By using substitution $y = \varepsilon^{-1}x$ we get

$$= \int \varepsilon^2 l(\varepsilon^2 t, y)(c_2 f(y))\,dy = c_2\varepsilon^2\phi(\varepsilon^2 t).$$

Thus,

$$\psi^{\varepsilon}(t) = \varepsilon^{-2}\psi(c_2^{-1}\varepsilon^{-2}t).$$

As a result

$$\widehat{W}^{\varepsilon}(\psi(\mu^{\varepsilon})(t) = \varepsilon\widehat{W}(\varepsilon^{-2}\psi^{\varepsilon}(t)) = \varepsilon\widehat{W}(\psi(\varepsilon^{-2}c_2^{-1}t)) = \varepsilon X(\varepsilon^{-2}c_2^{-1}t).$$

Hence,

$$\varepsilon X(\varepsilon^{-2}c_2^{-1}t) \xrightarrow{d} B(t) \quad \mathbb{P} - a.s.$$

It remains to show (2.4):

We need some preparatory lemmas:
The following property immediately can be derived from the definition of vague convergence:

Lemma 1. *Let h_ε and h be nonnegative functions in $C(\mathbb{R})$. If*

$$\mu_\varepsilon(dx) := h_\varepsilon(x)dx \text{ and } \mu(dx) := h(x)dx$$

and

$$\mu_\varepsilon([a,b]) \xrightarrow{\varepsilon\to 0} \mu([a,b]) \quad \forall a < b \in \mathbb{R},$$

then

$$\mu_\varepsilon \xrightarrow{v} \mu.$$

Proof. Let f be a bounded continuous function with compact support K and $I = [a,b]$ with $K \subseteq I$. Let $\varepsilon > 0$. As I is compact, the function $|f|$ is bounded from above by some $L > 0$ and

$$\left|\int_K f(x)h_\varepsilon(x)\,dx - \int_K f(x)h(x)\,dx\right| = \left|\int_a^b f(x)h_\varepsilon(x) - f(x)h(x)\,dx\right|$$

$$\leqslant \sup_x |f(x)| \left|\int_a^b h_\varepsilon(x) - h(x)\,dx\right| \leqslant L \left|\int_a^b h_\varepsilon(x) - h(x)\,dx\right| \xrightarrow{\varepsilon\to 0} 0$$

□

We set $\varepsilon := \dfrac{1}{n}$.

Lemma 2. *Let q_n be measures on the Borel σ-Algebra, defined by*

$$q_n([a,b]) := \frac{1}{n}\int_{na}^{nb} \sum_{i=-\infty}^{\infty} \left(Y_i\ h(y - x_i)\right) + c\,dy, \tag{2.5}$$

$$\forall a < b \in \mathbb{R} \tag{2.6}$$

then

$$q_n([a,b]) \xrightarrow{n\to\infty} (c_h\,\mathbb{E}[Y_1] + c)(b-a) \quad \mathbb{P} - a.s., \tag{2.7}$$

$$\forall a < b \in \mathbb{R} \tag{2.8}$$

where

$$c_h := \int_{-1}^{1} h(x)\,dx.$$

Proof. $\xi([a,b])$ has Poisson distribution with parameter $(b-a)$. Thus, for $\eta(n) := [na, nb]$, it follows that $\xi(\eta(n))$ has Poisson distribution with parameter $n(b-a)$.

Let $A_1 := \{\omega_1 \in \Omega_1 : (b-a) = \lim_{n\to\infty} \frac{\xi(\eta(n))(\omega_1)}{n}\}$. We will show now that $\mathcal{P}_1(A_1) = 1$ holds: To see this, write

$$\xi(\eta(n)) = \sum_{1\leqslant i\leqslant n} (\xi([(n-i+1)a + (i-1)b, (n-i)a + ib]) \tag{2.9}$$

as the sum of n i.i.d. random variables, which have Poisson $(b-a)$-distribution and therefore are integrable. The strong law of large numbers yields

$$\frac{\xi(\eta(n))}{n} \to (b-a) \quad \mathcal{P}_1 - a.s. \text{ as } n \to \infty. \tag{2.10}$$

We set $\eta'(n) := [na-1, na]$, $\eta''(n) := [nb, nb+1]$, $\eta^{(3)} := [na, na+1]$ and $\eta^{(4)} := [nb-1, nb]$. Let

$A_2^1(n) := \{\omega_1 \in \Omega_1 : \xi(\eta'(n)) \leqslant \lceil\sqrt{n}\rceil\}$, $A_2^1 := \{\omega_1 \in \Omega : \omega_1 \in \liminf_{n\to\infty} A_2^1(n)\}$,
$A_2^2(n) := \{\omega_1 \in \Omega_1 : \xi(\eta''(n)) \leqslant \lceil\sqrt{n}\rceil\}$, $A_2^2 := \{\omega_1 \in \Omega : \omega_1 \in \liminf_{n\to\infty} A_2^2(n)\}$,
$A_2^3(n) := \{\omega_1 \in \Omega_1 : \xi(\eta^{(3)}(n)) \leqslant \lceil\sqrt{n}\rceil\}$, $A_2^3 := \{\omega_1 \in \Omega : \omega_1 \in \liminf_{n\to\infty} A_2^3(n)\}$,
$A_2^4(n) := \{\omega_1 \in \Omega_1 : \xi(\eta^{(4)}(n)) \leqslant \lceil\sqrt{n}\rceil\}$, $A_2^4 := \{\omega_1 \in \Omega : \omega_1 \in \liminf_{n\to\infty} A_2^4(n)\}$.

We will show that $\mathcal{P}_1(A_2^1) = 1$ by proving $\mathcal{P}_1((A_2^1)^c) = 0$. For any $t \in \mathbb{R}$ we have

$$\mathcal{P}_1(\xi(\eta'(n)) \geqslant \lceil\sqrt{n}\rceil)) \leqslant \mathcal{P}_1(e^{(t\xi(\eta'(n))} \geqslant e^{t\lceil\sqrt{n}\rceil}) \leqslant \frac{\mathbb{E}[e^{t\xi(\eta'(n))}]}{e^{t\lceil\sqrt{n}\rceil}} =: (1).$$

Let $t := \dfrac{1}{\sqrt{(\lceil\sqrt{n}\rceil)}}$. As $\mathbb{E}[e^{t\xi(\eta'(n))}] = e^{c'(e^t-1)}$ we get for n large:

$$(1) \leqslant c'' \cdot e^{-\sqrt{\lceil\sqrt{n}\rceil}}.$$

Since

$$\sum_{i=1}^{\infty} \mathcal{P}_1(\xi(\eta'(n)) \leqslant \sum_{i=1}^{\infty} e^{-\sqrt{\lceil\sqrt{n}\rceil}} \leqslant \sum_{i=1}^{\infty} e^{-\sqrt[4]{n}} < \infty,$$

Borel-Cantelli lemma gives $\mathcal{P}_1((A_2^1)^c) = 0$. Analogously we get $\mathcal{P}_1((A_2^2)^c) = \mathcal{P}_1((A_2^3)^c) = \mathcal{P}_1((A_2^4)^c) = 0$. Hence, $\mathcal{P}_1(A_2) = \mathcal{P}_1(A_2') = \mathcal{P}_1(A_2^3) = \mathcal{P}_1(A_2^4) = 1$.
Then for $A := A_1 \cap A_2 \cap A_2' \cap A_2^3 \cap A_2^4$ and
$B := \{\omega_2 \in \Omega_2 : \lim_{n\to\infty} \dfrac{1}{n} \sum_{i=1}^n Y_i(\omega_2) < \infty\}$ it holds that $\mathbb{P}(A \times B) = \mathcal{P}_1(A)\mathcal{P}_2(B) = 1$.

For $\omega_1 \in A$ and $\omega_2 \in B$ we have

$$
\begin{aligned}
&q_n([a,b])(\omega_1,\omega_2) = \\
&= \frac{1}{n}\int_{na}^{nb} \sum_{i=-\infty}^{\infty} Y_i(\omega_2)\ h(y - x_i(\omega_1)) + c\,dy \\
&= \frac{1}{n}\sum_{i=-\infty}^{\infty} Y_i(\omega_2) \int_{na}^{nb} h(y - x_i(\omega_1))\,dy + \frac{1}{n}\int_{na}^{nb} c\,dy \\
&= \frac{1}{n}\sum_{i:x_i\in[na,nb]} Y_i(\omega_2) \int_{na}^{nb} h(y - x_i(\omega_1))\,dy \\
&+ \underbrace{\frac{1}{n}\sum_{i:x_i\in[na-1,na]} Y_i(\omega_2) \int_{na}^{na+1} h(y - \lambda x_i(\omega_1))\,dy}_{=:c_1(n)} \\
&+ \underbrace{\frac{1}{n}\sum_{i:x_i\in[nb,nb+1]} Y_i(\omega_2) \int_{nb-1}^{nb} h(y - x_i(\omega_1))\,dy}_{=:c_2(n)} \\
&- \underbrace{\frac{1}{n}\sum_{i:x_i\in[na,na+1]} Y_i(\omega_2) \int_{x_i-1}^{na} h(y - x_i(\omega_1))\,dy}_{=:s_1(n)} \\
&- \underbrace{\frac{1}{n}\sum_{i:x_i\in[nb-1,nb]} Y_i(\omega_2) \int_{nb}^{x_i+1} h(y - x_i(\omega_1))\,dy}_{=:s_2(n)} + c(b-a)
\end{aligned}
$$

$$= \frac{1}{n} \sum_{i: x_i \in [na, nb]} Y_i(\omega_2) \underbrace{\int_{-1}^{1} h(y)\, dy}_{=: c_h} + c(b-a) + c_1(n) + c_2(n) + s_1(n) + s_2(n)$$

$$= \underbrace{c_h \cdot \frac{1}{n} \sum_{i=1}^{\xi(\eta(n))(\omega_1)} Y_i(\omega_2)}_{(\bigstar)} + c(b-a) + c_1(n) + c_2(n) + s_1(n) + s_2(n).$$

We have $(b-a) = \lim_{n\to\infty} \frac{\xi(\eta(n))(\omega_1)}{n}$. Thus, for all $\varepsilon > 0$ there exits $n_0 : \forall n \geqslant n_0 : \xi(\eta(n))(\omega_1) \leqslant \lceil n(b-a) \rceil + n\varepsilon$. Hence,

$$(\bigstar) \leqslant c_h \cdot \frac{1}{n} \sum_{i=1}^{\lceil n(b-a) \rceil + \lceil n\varepsilon \rceil} Y_i(\omega_2) = c_h \cdot \frac{1}{n} \sum_{i=1}^{n\lceil (b-a) \rceil} Y_i(\omega_2) + c_h \cdot \frac{1}{n} \sum_{i=\lceil n(b-a) \rceil + 1}^{\lceil n(b-a) \rceil + \lceil n\varepsilon \rceil} Y_i(\omega_2)$$

$$\leqslant c_h \cdot (b-a) \cdot \frac{1}{\lceil n(b-a) \rceil - 1} \sum_{i=1}^{\lceil n(b-a) \rceil} Y_i(\omega_2) + c_h \varepsilon \cdot \frac{1}{\lceil n\varepsilon \rceil - 1} \sum_{i=\lceil n(b-a) \rceil + 1}^{\lceil n(b-a) \rceil + \lceil n\varepsilon \rceil} Y_i(\omega_2).$$

By the strong law of large numbers this converges to

$$c_h \cdot (b-a) \cdot \mathbb{E}[Y_1] + c\varepsilon \cdot \mathbb{E}[Y_1]$$

for n drawing closer to infinity. Since ε was arbitrary, we obtain

$$\limsup_{n\to\infty} (\bigstar) \leqslant c_h \cdot \mathbb{E}[Y_1].$$

Analogously we get the lower bound.
Finally we need to show that $c_1(n)$ and $c_2(n)$ as well as $s_1(n)$ and $s_2(n)$ converge to 0 for $n \to \infty$: First it holds that $\exists N \in \mathbb{N} : \forall n \geqslant N :$

$$|\{i : x_i \in [\frac{na-1}{(b-a)}, \frac{na}{(b-a)}]\} \leqslant \lceil \sqrt{n} \rceil .$$

h is continuous and has compact support and consequently is bounded. Thus, $\forall n : \exists \;\; C'' > 0$

$$\int_{na}^{na+1} h(y - x_i(\omega_1))\, dy \leqslant C''.$$

Consequently,

$$\lim_{n\to\infty} c_1(n) \leqslant \lim_{n\to\infty} \Big(\frac{1}{\lceil\sqrt{n}\rceil}\frac{1}{\lceil\sqrt{n}\rceil}\sum_{i=1}^{\lceil\sqrt{n}\rceil} Y_i(\omega_2)C''\Big)$$
$$= \lim_{n\to\infty} \Big(\frac{1}{\lceil\sqrt{n}\rceil}\Big)C'' \lim_{n\to\infty} \Big(\frac{1}{\lceil\sqrt{n}\rceil}\sum_{i=1}^{\lceil\sqrt{n}\rceil} Y_i(\omega_2)\Big) = 0\cdot C''\mathbb{E}[Y_1] = 0,$$

again by the strong law of large numbers. $c_2(n) \to 0$, $s_1(n) \to 0$ and $s_2(n) \to 0$ follow equally.
As a result we get $\rho_{\frac{1}{n}}([a,b]) \xrightarrow{n\to\infty} c_h \cdot (b-a)\mathbb{E}[Y_1] + c(b-a) \quad \mathbb{P}-a.s.$ □

Hence, by Lemma 1 and Lemma 2 we get (2.4). □

2.2 Weak convergence to FIN-diffusion

Provided that Y_i have finite expectation, we proved convergence of the rescaled diffusion to the Brownian motion. It remains to show convergence for the second case, when Y_i have unbounded expectation:

$$dX_t = (f_\omega(X_t))^{-1/2} dW_t, \qquad X_0 = x, \text{ with}$$

$$f_\omega(x) = \hat{c}\sum_{i=-\infty}^{\infty} h(x - x_i(\omega_1))Y_i(\omega_2) + c,$$

where

$$\hat{c} := \left(\int_{-1}^{1} h(x)\,dx\right)^{-1}.$$

Suppose (2.3) holds: There exists a slowly varying function L, such that

$$\mathcal{P}_2[Y_i \geqslant u] = u^{-\alpha}L(u), \quad \text{with} \quad \alpha \in (0,1).$$

Under this assumption, we will prove very similar to [7] that the speed measures, which are defined in the proof of the theorem below, converge to a measure ρ. This implies that the rescaled process converges to the FIN-diffusion, which is defined as follows:

Definition 8. *Let $(x_i, v_i)_{i\in\mathbb{Z}}$ be an inhomogeneous Poisson point process on $\mathbb{R}\times(0,\infty)$ with density measure $dx\alpha v^{-1-\alpha}dv$ and let $\rho := \sum_i v_i\delta_{x_i}$ be a discrete measure on the Borel-σ-algebra $\mathscr{B}(\mathbb{R})$. Let*

$$\phi(t) := \int l(t,x)\,\rho(dx) \quad \text{and} \quad \psi(t) := \inf\{u : \phi(u) > t\}$$

and let $\widehat{W}(t)$ be a Brownian motion, we denote $M(t) := \widehat{W}(\psi(t))$ and call $M(t)$ FIN-diffusion.

Theorem 11. *Let $\omega \in \Omega_1 \times \Omega_2$. Let $X_\omega(t)$ be a solution to (2.1), where $(Y_i)_{i\in\mathbb{Z}}$ satisfy (2.3). Let*

$$c_\varepsilon := (\inf[t \geqslant 0 : \mathbb{P}(Y_1 > t) \leqslant \varepsilon])^{-1}. \tag{2.11}$$

Let M_t be the FIN-diffusion. Then there exist processes $\bar{X}_t$ and $\bar{M}_t$, defined on the same probability space $(\Omega_1 \times \overline{\Omega}_2, \mathcal{A}_1 \otimes \overline{\mathcal{A}_2}, \overline{\mathbb{P}})$ such that for all $t > 0$:

$$\bar{X}_t \overset{d}{=} X_t, \quad \bar{M}_t \overset{d}{=} M_t$$

and as ε converges to zero:

$$\varepsilon\bar{X}((c_\varepsilon\varepsilon)^{-1}t) \xrightarrow{w} \bar{M}(t) \quad \overline{\mathbb{P}} - a.s.$$

Proof. According to Theorem 5 there exists a Brownian motion $\widehat{W}_t$ such that $X_\omega(t) = \widehat{W}(\psi(\mu_\omega)(t))$, where $\psi(\mu_\omega)$ is defined as above in (1.31) with

$$\mu_\omega(dx) = f_\omega(x)dx,$$

the speed measure of the diffusion $X_\omega(t)$.

Let $\widehat{W}_t^\varepsilon := \varepsilon\widehat{W}(\varepsilon^{-2}t)$, which is a Brownian motion too according to Lévy's Theorem.
Let $\varepsilon > 0$, we introduce the scaled speed measure

$$\mu_\omega^\varepsilon(dx) := \frac{c_\varepsilon}{\varepsilon} f_\omega(\varepsilon^{-1}x)dx = \frac{c_\varepsilon}{\varepsilon}(c_1 \sum_{i\in\mathbb{Z}} h(\varepsilon^{-1}x - x_i(\omega_1))Y_i(\omega)) + c\,dx.$$

We have to show that for every $\varepsilon > 0$ there exist i.i.d. random variables $(Y_i^\varepsilon)_{i\in\mathbb{Z}}$ defined on a probability space $(\Omega_1 \times \overline{\Omega}_2, \mathcal{A}_1 \otimes \overline{\mathcal{A}_2}, \overline{\mathbb{P}})$ such that

$$Y_i^\varepsilon \overset{d}{=} Y_1,$$

and such that for couplings $\overline{\mu}^{\varepsilon}$ of the speed measures μ^{ε} defined by

$$\overline{\mu}^{\varepsilon}_{\omega}(dx) := \frac{c_{\varepsilon}}{\varepsilon}\left(b_{\varepsilon}(\varepsilon^{-1}x)\right)\, dx, \tag{2.12}$$

where

$$b_{\varepsilon}(x) := \widehat{c}\sum_{i\in\mathbb{Z}} h(x - x_i(\omega_1))Y_i^{\varepsilon}(\omega) + c$$

there exists a coupling $\overline{\rho}$ of the speed measure ρ such that all couplings are defined on an equal probability space $(\Omega_1 \times \overline{\Omega}_2, \mathcal{A}_1 \otimes \overline{\mathcal{A}_2}, \overline{\mathbb{P}})$ and such that

$$\overline{\mu}^{\varepsilon}(dx) \xrightarrow{v} \overline{\rho}(dx) \quad \overline{\mathbb{P}} - a.s. \tag{2.13}$$

holds. Provided (2.13), then for $\bar{M}(t) := \widehat{W}(\overline{\rho})(t) \stackrel{d}{=} M(t)$ Theorem 9 gives

$$\widehat{W}^{\varepsilon}(\overline{\mu}^{\varepsilon})(t) \xrightarrow{w} \bar{M}(t) \qquad \overline{\mathbb{P}} - a.s.$$

Let $\bar{\phi}_{\omega}(t) := \int l(t,x)\,\overline{\mu}_{\omega}(dx)$, where $\overline{\mu} := \overline{\mu}^1$ and $\bar{\psi}_{\omega}(t) := \inf\{u : \phi_{\omega}(u) > t\}$ and let

$$\begin{aligned}\bar{\phi}^{\varepsilon}_{\omega}(t) &:= \int l^{\varepsilon}(t,x)\,\overline{\mu}^{\varepsilon}_{\omega}(dx) = \int \varepsilon l(\varepsilon^{-2}t, \varepsilon^{-1}x)\frac{c_{\varepsilon}}{\varepsilon}\left(b_{\varepsilon}(\varepsilon^{-1}x)\right)\, dx = \\ &= \int l(\varepsilon^{-2}t, \varepsilon^{-1}x)c_{\varepsilon}\left(b_{\varepsilon}(\varepsilon^{-1}x)\right)\, dx.\end{aligned}$$

By using substitution $[y = \varepsilon^{-1}x \Rightarrow \varepsilon^{-1}dy = dx]$ we obtain

$$\bar{\phi}^{\varepsilon}_{\omega}(t) = \int \varepsilon l(\varepsilon^{-2}t, y)c_{\varepsilon}\left(b_{\varepsilon}(y)\right)\, dy = c_{\varepsilon}\varepsilon\bar{\phi}_{\omega}(\varepsilon^{-2}t).$$

Therefore,

$$\bar{\psi}^{\varepsilon}_{\omega}(t) = \varepsilon^2\bar{\psi}_{\omega}\left(\frac{t}{c_{\varepsilon}\varepsilon}\right).$$

Now let $\bar{X}(t) := \widehat{W}^{\varepsilon}(\overline{\mu}(t)) \stackrel{d}{=} X(t)$, then

$$\begin{aligned}\varepsilon\bar{X}_{\omega}(t(c_{\varepsilon}\varepsilon)^{-1}) &= \varepsilon\widehat{W}(\bar{\psi}_{\omega}(t(c_{\varepsilon}\varepsilon)^{-1})) = \varepsilon\widehat{W}(\varepsilon^{-2}\bar{\psi}^{\varepsilon}_{\omega}(t)) = \\ &= \widehat{W}^{\varepsilon}(\bar{\psi}^{\varepsilon}_{\omega}(t))).\end{aligned}$$

□

It remains to prove (2.13):

Vague convergence of speed measures to ρ

In order to show the almost sure vague convergence of the rescaled random speed measures μ^ε to ρ, we need to construct a coupling of μ^ε as well as ρ, such that both μ^ε and ρ are defined on the same probability space.
We introduce a Levy process $V(x), x \in \mathbb{R}$ with $V(0) = 0$ that has stationary and independent increments with càdlàg paths defined on $(\overline{\Omega}_2, \overline{\mathscr{A}}_2, \overline{\mathcal{P}}_2)$ through its characteristic functions:

$$\mathbb{E}[e^{ir(V(x+x_0)-V(x_0))}] = \exp\left[x\alpha \int_0^\infty (e^{ir\omega} - 1)\omega^{-1-\alpha}\ d\omega\right].$$

Let $\overline{\rho}$ be the Lebesgue-Stieltjes measure on $\mathbb{R}$ associated to V, i.e. $\overline{\rho}((a,b]) = V(b) - V(a)$ for all $a, b \in \mathbb{R}, a < b$. We state without proof:

Lemma 3. *1. $\overline{\rho}(dx) = \sum_j v_j \delta_{x_j}(dx)$, where (x_j, v_j) is an inhomogeneous Poisson point process with density $dx\alpha v^{-1-\alpha}dv$.*

2. $V(\varepsilon) \overset{d}{=} \varepsilon^{1/\alpha} V(1)$

3. $\lim_{y\to\infty} y^\alpha \overline{\mathbb{P}}(V_1 > y) = 1$

Now we define the function $G : [0,\infty) \to [0,\infty)$ such that

$$\overline{\mathbb{P}}(V(1) > G(x)) = \mathbb{P}(Y_1 > x). \tag{2.14}$$

The function is well-defined since $V(1)$ has continuous distribution, it is non-decreasing and right-continuous. Hence, it has non-decreasing right-continuous generalized inverse G^{-1}.

Let $g_\varepsilon : [0,\infty) \to [0,\infty)$ be defined as

$$g_\varepsilon := c_\varepsilon G^{-1}(\varepsilon^{-1/\alpha} x) \ \ \forall\ x \geqslant 0. \tag{2.15}$$

Lemma 4. *Let*

$$\tau_i^\varepsilon = \frac{1}{c_\varepsilon} g_\varepsilon(V(\varepsilon(i+1)) - V(\varepsilon i)), \tag{2.16}$$

then for any $\varepsilon > 0$: τ_i^ε are i.i.d. with the same distribution as Y_1.

Proof. We have to show that for any $\varepsilon > 0, t \in \mathbb{R}$ and for all $i \in \mathbb{Z}$:

$$\overline{\mathbb{P}}(\tau_i^\varepsilon \leqslant t) = \overline{\mathbb{P}}(Y_1 \leqslant t) \tag{2.17}$$

As the increments of V are independent, so are τ_i^ε. Moreover, we get from the stationarity of the increments that

$$\tau_i^\varepsilon = \frac{1}{c_\varepsilon} g_\varepsilon(V(\varepsilon(i+1)) - V(\varepsilon i)) \stackrel{d}{=} \frac{1}{c_\varepsilon} g_\varepsilon(V(\varepsilon)) = \tau_0^\varepsilon.$$

Since $g_\varepsilon^{-1}(\cdot) = \varepsilon^{1/\alpha} G(\cdot / c_\varepsilon)$ it holds that

$$\begin{aligned} \overline{\mathbb{P}}(\tau_i^\varepsilon > t) = \overline{\mathbb{P}}(\tau_0^\varepsilon > t) = \overline{\mathbb{P}}(g_\varepsilon(V_\varepsilon) > c_\varepsilon t) = \overline{\mathbb{P}}(V_\varepsilon > g_\varepsilon^{-1}(c_\varepsilon t)) = & \quad (2.18) \\ = \overline{\mathbb{P}}(V(\varepsilon) > \varepsilon^{1/\alpha} G(t)) \quad \forall i \in \mathbb{Z}. & \quad (2.19) \end{aligned}$$

According to (1) of Lemma 3: $V(\varepsilon) \stackrel{d}{=} \varepsilon^{1/\alpha} V(1)$. Hence,

$$\overline{\mathbb{P}}(V(\varepsilon) > \varepsilon^{1/\alpha} G(t)) = \overline{\mathbb{P}}(V(1) > G(t)) = \overline{\mathbb{P}}(Y_1 > t) \quad \forall \varepsilon > 0, t \in \mathbb{R},$$

completing the proof. □

Construction of the coupling:

Let $(\Omega_1 \times \overline{\Omega_2}, \mathcal{A}_1 \otimes \overline{\mathcal{A}_2}, \overline{\mathbb{P}})$, where $\overline{\mathbb{P}} := \mathcal{P}_1 \otimes \overline{\mathcal{P}}_2$ and for each $\omega_1 \in \Omega_1$ and $a < b$, with $a, b \in \mathbb{R}$ let

$$\overline{\rho}_{\omega_1}((a,b]) := V(b) - V(a). \tag{2.20}$$

Moreover, if $Y_i^\varepsilon := \tau_i^\varepsilon$, then

$$\overline{\mu}_\omega^\varepsilon(dx) := \frac{c_\varepsilon}{\varepsilon} \left(c_1 \sum_{i \in \mathbb{Z}} h(\varepsilon^{-1} x - x_i(\omega_1)) \tau_i^\varepsilon(\omega) + c \right) dx \tag{2.21}$$

coincides with (2.12) and according to Lemma 3 and Lemma 4 the following equations hold:

$$\rho \stackrel{d}{=} \bar{\rho} \quad \overline{\mu}^\varepsilon \stackrel{d}{=} \mu^\varepsilon.$$

Consequently, $\bar{\rho}$ and $\overline{\mu}^\varepsilon$ are defined on the same probability space and it is possible to show (2.13).

In order to do this, we have to prove some properties:
The subsequent lemmas (Lemmas 5-10) as well as their proofs are taken from [7] (see page 598 ff)

Lemma 5. *For any fixed $y > 0, g_\varepsilon(y) \to y$ as $\varepsilon \to 0$.*

Lemma 6. *For any $\delta' > 0$, there exist constants C' and C'' in $(0, \infty)$ such that*

$$g_\varepsilon(x) \leqslant C' x^{1-\delta'} \quad \text{for } \varepsilon^{1/\alpha} \leqslant x \leqslant 1 \text{ and } \varepsilon \leqslant C''. \tag{2.22}$$

In order to proof these two lemmas we use the following four subsidiary lemmas, which we proof later.

Lemma 7. $\frac{1}{\varepsilon}\overline{\mathbb{P}}(Y_1 > \frac{1}{c_\varepsilon}) \to 1$ *as* $\varepsilon \to 0$.

Lemma 8. *For* $y > 0 : \frac{1}{\varepsilon}\overline{\mathbb{P}}(Y_1 > \frac{y}{c_\varepsilon}) \to \frac{1}{y^\alpha}$ *as* $\varepsilon \to 0$.

Lemma 9. *For any* $\lambda > 0$*:* $\frac{c_\varepsilon}{c_{\lambda\varepsilon}} \to \lambda^{-1/\alpha}$ *as* $\varepsilon \to 0$ *and thus* $c_\varepsilon = \varepsilon^{1/\alpha}\tilde{L}(\varepsilon^{-1})$*, where* $\tilde{L}$ *is a positive slowly varying function at infinity.*

Lemma 10. *There exists* $\lambda > 0$ *sufficiently small such that* $G^{-1}(y) \leqslant \frac{1}{c_{\lambda/y^\alpha}}$ *for* $y \geqslant 1$ *or, equivalently* $g_\varepsilon(x) \leqslant \frac{c_\varepsilon}{c_{\lambda\varepsilon/x^\alpha}}$ *for* $x \geqslant \varepsilon^{1/\alpha}$.

Proof of Lemma 5. Let g_ε^{-1} be the right-continuous generalized inverse of g_ε. To prove $g_\varepsilon(y) \to y$, it suffices to prove that $g_\varepsilon^{-1}(y) \to y$. Now $G^{-1}(V_1) \stackrel{d}{=} Y_1$, so $g_\varepsilon(\varepsilon^{1/\alpha}V_1) = c_\varepsilon G^{-1}(\varepsilon^{-1/\alpha}\varepsilon^{1/\alpha}V_1) \stackrel{d}{=} c_\varepsilon Y_1$ and thus $\overline{\mathbb{P}}(Y_1 > y/c_\varepsilon)$ equals

$$\overline{\mathbb{P}}(c_\varepsilon Y_1 > y) = \overline{\mathbb{P}}(g_\varepsilon(\varepsilon^{1/\alpha}V_1) > y) = \overline{\mathbb{P}}(\varepsilon^{1/\alpha}V_1 > g_\varepsilon^{-1}(y)) = \overline{\mathbb{P}}(V_1 > \varepsilon^{-1/\alpha}g_\varepsilon^{-1}(y)). \tag{2.23}$$

By Lemma 3.(3.):

$$\varepsilon^{-1}\overline{\mathbb{P}}(V_1 > \varepsilon^{-1/\alpha}y) \to \frac{1}{y^\alpha} \tag{2.24}$$

as $\varepsilon \to 0$. By (2.23) and Lemma 8:

$$\varepsilon^{-1}\overline{\mathbb{P}}(V_1 > \varepsilon^{-1/\alpha}g_\varepsilon^{-1}(y)) = \varepsilon^{-1}\overline{\mathbb{P}}(Y_1 > \frac{y}{c_\varepsilon}) \to \frac{1}{y^\alpha} \tag{2.25}$$

as $\varepsilon \to 0$. This implies that $\varepsilon^{-1}\overline{\mathbb{P}}(V_1 > \varepsilon^{-1/\alpha}g_\varepsilon^{-1}(y))/\varepsilon^{-1}\overline{\mathbb{P}}(V_1 > \varepsilon^{-1/\alpha}y) \to 1$ as $\varepsilon \to 0$ and this plus (2.24) implies that $\limsup_{\varepsilon\to 0} g_\varepsilon^{-1}(y) \leqslant y$ and $\liminf_{\varepsilon\to 0} g_\varepsilon^{-1}(y) \geqslant y$.

Since indirect assume: $\limsup_{\varepsilon\to 0} g_\varepsilon^{-1}(y) = c > y \Rightarrow$ Let $\delta > 0$ with $c - \delta > y$ for $\varepsilon > 0 : \exists \varepsilon_0 \leqslant \varepsilon : g_\varepsilon^{-1} \geqslant c - \delta$

Therefore:

$$\varepsilon_0^{-1}\mathbb{P}(V_1 > \varepsilon_0^{-1/\alpha} g_\varepsilon^{-1}(y)) \leqslant \varepsilon_0^{-1}\mathbb{P}(V_1 > \varepsilon_0^{-1/\alpha}(c-\delta)$$

As this inequality holds for any $\varepsilon > 0$, we get:

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{P}(V_1 > \varepsilon^{-1/\alpha} g_\varepsilon^{-1}(y)) \leqslant \limsup_{\varepsilon\to 0}(\varepsilon_0^{-1}\mathbb{P}(V_1 > \varepsilon_0^{-1/\alpha}(c-\delta)) \leqslant \frac{1}{(c-\delta)^\alpha} \tag{2.26}$$

$$\Rightarrow 1 = \frac{\limsup_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{P}(V_1 > \varepsilon^{-1/\alpha} g_\varepsilon^{-1}(y))}{\limsup_{\varepsilon\to 0} \varepsilon^{-1}\overline{\mathbb{P}}(V_1 > \varepsilon^{-1/\alpha} y)} \leqslant \frac{1/(c-\delta)^\alpha}{1/y^\alpha} < 1,$$

which is a contradiction. Analogously we get $\liminf_{\varepsilon\to 0} g_\varepsilon^{-1}(y) \geqslant y$, which completes the proof of Lemma 5. ☐

Proof of Lemma 6. By Lemmas 9 and 10 for $x \geqslant \varepsilon^{1/\alpha}$:

$$g_\varepsilon(x) \leqslant \lambda^{-1/\alpha} x \frac{\tilde{L}(\varepsilon)^{-1}}{\tilde{L}((x^\varepsilon/\lambda)\varepsilon^{-1})} \tag{2.27}$$

for $\lambda > 0$ small enough; the value of λ will be chosen later. We now use a result about slowly varying functions, stating that $\tilde{L}(x) = a(x)\exp(\int_1^x \frac{\triangle(y)}{y}\, dy)$, where $a(x) \to c \in (0,\infty)$ as $x \to \infty$ and $\triangle(y) \to 0$ as $y \to \infty$. The quotient in the right-hand side of (2.27) then becomes

$$\frac{a(\varepsilon^{-1})}{a((x^\alpha/\lambda)\varepsilon^{-1})} \exp\left\{\int_1^{\varepsilon^{-1}} \frac{\triangle(y)}{y}\, dy\right\} \exp\left\{-\int_1^{(x^\alpha/\lambda)\varepsilon^{-1}} \frac{\triangle(y)}{y}\, dy\right\} = \tag{2.28}$$

$$= \frac{a(\varepsilon^{-1})}{a((x^\alpha/\lambda)\varepsilon^{-1})} \exp\left\{\int_{(x^\alpha/\lambda)\varepsilon^{-1}}^{\varepsilon^{-1}} \frac{\triangle(y)}{y}\, dy\right\}. \tag{2.29}$$

If $\varepsilon \leqslant \lambda$ such that $(x^\alpha/\lambda)\varepsilon^{-1} \geqslant 1/\lambda \geqslant \varepsilon^{-1}$, then the absolute value of the latter integral is bounded above by

$$\delta\left|\int_{(x^\alpha/\lambda)\varepsilon^{-1}}^{\varepsilon^{-1}} \frac{1}{y}\, dy\right| \leqslant \delta|\log(x^\alpha/\lambda|, \tag{2.30}$$

where $\delta = \delta(\lambda) = \sup\{|\triangle(y)|, y > 1/\lambda\}$ and thus the exponential in (2.29) is bounded above (for $\lambda \leqslant 1, x \leqslant 1$) by

$$\lambda^{-\delta} x^{-\alpha\delta}. \tag{2.31}$$

Thus, given $\delta' > 0$, we choose $\lambda \in (0,1)$ such that $\alpha\delta(\lambda) \leqslant \delta'$ and such that $a(y) \in [c/2, 2c]$ for $y \geqslant \lambda^{-1}$. The lemma now follows from (2.27) - (2.31) with $C' = 4\lambda^{-(1+\delta')/\alpha}$ and $C'' = \lambda$. □

To complete the proof of our two main lemmas, it remains to prove the subsidiary Lemmas 7, 8, 9 and 10.

Proof of Lemma 7. By the definition of 2.11 of c_ε, $\overline{\mathbb{P}}(Y_1 > c_\varepsilon^{-1}) \leqslant \varepsilon$ and $\overline{\mathbb{P}}(Y_1 > x) > \varepsilon$ for all $x < c_\varepsilon^{-1}$. Thus, if the statement of the lemma is not true, then there must exist $\delta \in (0,1)$ and a sequence (ε_i) with $\varepsilon_i > 0$ for all i and $\varepsilon_i \to 0$ as $i \to \infty$ such that $\overline{\mathbb{P}}(Y_1 > \delta' c_{\varepsilon i}^{-1}) > \varepsilon_i$ and so

$$\frac{\overline{\mathbb{P}}(Y_1 > \delta' c_{\varepsilon i}^{-1})}{\overline{\mathbb{P}}(Y_1 > c_{\varepsilon i}^{-1})} \geqslant \delta^{-1} \tag{2.32}$$

for all i. Since $c_{\varepsilon i}^{-1} \to \infty$ and $\overline{\mathbb{P}}(Y_1 > \cdot)$ is regularly varying at infinity (with exponent $-\alpha$), it follows that for any $\lambda > 0$:

$$\lim_{t\to\infty} \frac{\overline{\mathbb{P}}(Y_1 > \lambda t}{\overline{\mathbb{P}}(Y_1 > t)} = \lambda^{-\alpha}, \tag{2.33}$$

which contradicts (2.32) since $(\delta')^\alpha > \delta$. □

Proof of Lemma 8. This is a consequence of Lemma 7, the fact that $c_\varepsilon^{-1} \to \infty$ as $\varepsilon \to 0$, and (2.2), from which it follows that

$$\frac{\overline{\mathbb{P}}(Y_1 > y/c_\varepsilon}{\overline{\mathbb{P}}(Y_1 > 1/c_\varepsilon)} \to \frac{1}{y^\alpha}. \tag{2.34}$$

□

Proof of Lemma 9. By Lemma 7: $(\lambda\varepsilon)^{-1}\overline{\mathbb{P}}(Y_1 > 1/c_{\lambda\varepsilon}) \to 1$ or equivalently $\varepsilon^{-1}\overline{\mathbb{P}}(Y_1 > 1/c_{\lambda\varepsilon}) \to \lambda$ as $\varepsilon \to 0$ while, by Lemma 8, $\varepsilon^{-1}\overline{\mathbb{P}}(Y_1 > y/c_\varepsilon) \to 1/y^\alpha$. Now by taking $y^\alpha = \lambda^{-1}$ we get, that $c_\varepsilon \lambda^{1/\alpha}/c_{\lambda\varepsilon} \to 1$, (this can be proved analogously to the last part of the proof of Lemma 5), which is equivalent to $c_\varepsilon/c_{\lambda\varepsilon} \to \lambda^{-1/\alpha}$ as $\varepsilon \to 0$. □

Proof of Lemma 10. We need to show that $G^{-1}(y) \leqslant z$,. It is enough to show that $G(z) > y$. So we want to prove that $G(1/c_{\lambda/y^\alpha}) > y$ for $y \geqslant 1$ and some $\lambda > 0$. By the definition (2.14) of G, $G(x) > y$ would be a consequence of $\overline{\mathbb{P}}(V_1 > y) > \overline{\mathbb{P}}(Y_1 > x)$, where we take $x = 1/c_{\lambda/y^\alpha}$. Now there exists $K > 0$ such that $\overline{\mathbb{P}}(V_1 > y) > K/y^\alpha$ for $y \geqslant 1$ [by Lemma 3.(3.)], so it suffices to show that $\overline{\mathbb{P}}(Y_1 > 1/c_{\lambda/y^\alpha} \leqslant K/y^\alpha$ for $y \geqslant 1$ and some $\lambda > 0$; or, equivalently, taking $\varepsilon = \lambda/y^\alpha$, it suffices to show that for some $\lambda > 0$ and all $\varepsilon \leqslant \lambda, \overline{\mathbb{P}}(Y_1 > 1/c_\varepsilon) \leqslant K\varepsilon/\lambda$, or $\overline{\mathbb{P}}(Y_1 > 1/c_\varepsilon)/\varepsilon \leqslant K\lambda$. By Lemma 7, we may choose λ small enough so that for $\varepsilon \leqslant \lambda, \overline{\mathbb{P}}(Y_1 > 1/c_\varepsilon)/\varepsilon \leqslant 2$ and also small enough that $K/\lambda \geqslant 2$. □

Lemma 11. *Let for all* $\varepsilon > 0$ f_ε *be a non-decreasing right-continuous function with* $\lim_{\varepsilon\to 0} f_\varepsilon(x) = x$ $\forall x \in \mathbb{R}$. *Let* $y : \mathbb{R} \to \mathbb{R}$ *be a non-decreasing function that is right-continuous in* 0 *with* $\lim_{\varepsilon\to 0} y(\varepsilon) = z$. *Then*

$$\lim_{\varepsilon\to 0} f_\varepsilon(y(\varepsilon)) = z.$$

Proof. Let $\delta > 0$. Since f_ε is right-continuous and non-decreasing it holds that

$$\begin{aligned}\exists \mu_1 > 0 : \forall x < y \in \mathbb{R} : 0 \leqslant x - z < y - z < \mu_1 : 0 &\leqslant f_\varepsilon(y) - f_\varepsilon(x) \\ &\leqslant f_\varepsilon(y) - f_\varepsilon(z) < \frac{\delta}{3}.\end{aligned}$$

y is right-continuous in 0. Thus,

$$\eta : \forall \varepsilon \leqslant \eta : 0 \leqslant y(\eta) - y(\varepsilon) < \mu_1.$$

Since $\lim_{\varepsilon\to 0} y(\varepsilon) = z$, there exists

$$\gamma > 0 : \forall \varepsilon \leqslant \gamma : y(\varepsilon) - z < \frac{\delta}{3}.$$

Let $\varepsilon_1 := \min(\eta, \gamma)$. Due to $\lim_{\varepsilon\to 0} f_\varepsilon(x) = x$, there exits

$$\varepsilon_2 > 0 : \forall \varepsilon \leqslant \varepsilon_2 : 0 \leqslant f_\varepsilon(y(\varepsilon_1)) - y(\varepsilon_1) < \frac{\delta}{3}.$$

For $\varepsilon \leqslant \min(\varepsilon_1, \varepsilon_2)$ we have

$$\begin{aligned}|f_\varepsilon(y(\varepsilon)) - z| &= |f_\varepsilon(y(\varepsilon)) - f_\varepsilon(y(\varepsilon_1)) + f_\varepsilon(y(\varepsilon_1)) - y(\varepsilon_1) + y(\varepsilon_1) - z| \\ &\leqslant |f_\varepsilon(y(\varepsilon)) - f_\varepsilon(y(\varepsilon_1))| + |f_\varepsilon(y(\varepsilon_1)) - y(\varepsilon_1)| + |y(\varepsilon_1) - z| \leqslant \delta.\end{aligned}$$

□

Having collected all required properties, we continue with the proof of (2.13). We have to show that for every $f \in C_c(\mathbb{R})$:

$$\int_{\mathbb{R}} f \, d\overline{\mu}^{(\varepsilon)} \to \int_{\mathbb{R}} f \, d\overline{\rho} \quad \bar{\mathbb{P}} - a.s. \tag{2.35}$$

Before we get into the proof, we illustrate its main concept: Examining the summands of the sum within the speed measure of $\overline{\mu}^{(\varepsilon)}$ we notice that the process V only has jumps in y_i and is constant elsewhere. Applying the above lemmas we will see that for all indices i of the sum within the speed measure $\overline{\mu}^{(\varepsilon)}$, for which the process $V_{\varepsilon(i+1)} - V_{\varepsilon(i)}$ does not have a jump, the summand converges to zero as ε draws closer to zero. Consequently, taking the sum over the jumps of V yields the same sum. Moreover, the number of jumps of V larger some $\delta > 0$ is finite within a bounded interval, which enables us to exchange limit and sum for those indices. Taking δ to zero ensures that eventually all jumps are covered. In detail:

Proof of (2.13). We verify (2.35): Let $f \in C_c(\mathbb{R})$ with $I := supp(f)$ and w.l.o.g. $I = [a, b]$, where $a < b \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} f \, d\overline{\mu}^{(\varepsilon)} = \int_{\mathbb{R}} f(x) \cdot \frac{c_\varepsilon}{\varepsilon} \cdot \left(\hat{c} \sum_{i=-\infty}^{\infty} h(\varepsilon^{-1}x) - x_i) \frac{1}{c_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) + c \right) dx$$

$$= \hat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} f(x) \cdot \left(\sum_{i=-\infty}^{\infty} h(\varepsilon^{-1}v - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) + c \right) \, dx =: (1).$$

Let $(y_i, v_i)_{i \in \mathbb{N}}$ be the jump process of V. For $\delta > 0$ we define $J^\delta := \{y_i : v_i > \delta\}$ and $J^\delta_\varepsilon = \{i \in \mathbb{Z} : i \in I, V_{\varepsilon(i+1)} - V_{\varepsilon i} > \delta\}$.

Hence,

$$
\begin{aligned}
(1) = {} & \underbrace{\hat{c}\int_{\mathbb{R}} \frac{1}{\varepsilon} f(x) \cdot \left(\sum_{i \in J_\varepsilon^\delta} h(\varepsilon^{-1}x - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \right) dx}_{:=K_1^\delta(\varepsilon)} \\
& + \underbrace{\hat{c}\int_{\mathbb{R}} \frac{1}{\varepsilon} f(x) \cdot \left(\sum_{i \in J^\varepsilon_{\varepsilon^{1/\alpha}} \setminus J_\delta^\varepsilon} h(\varepsilon^{-1}x - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \right) dx}_{:=K_2^\delta(\varepsilon)} \\
& + \underbrace{\hat{c}\int_{\mathbb{R}} \frac{1}{\varepsilon} f(x) \cdot \left(\sum_{i \in J_0^\varepsilon \setminus J^\varepsilon_{\varepsilon^{1/\alpha}}} h(\varepsilon^{-1}x - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \right) dx}_{:=K_3(\varepsilon)} \\
& + \underbrace{\hat{c}\int_{\mathbb{R}} \frac{c_\varepsilon}{\varepsilon} f(x) \cdot c \, dx}_{:=K_4(\varepsilon)} .
\end{aligned}
$$

The process V only has finitely many jumps larger than δ within I. Let $|J^\delta \cap I| =: n < \infty$.

For any $\varepsilon > 0$ and $y \in J^\delta$ we have: $\varepsilon\lfloor \frac{y}{\varepsilon} \rfloor \leqslant y \leqslant \varepsilon(\lfloor \frac{y}{\varepsilon} \rfloor + 1) \Rightarrow$ for ε small: $\{\left\lfloor \frac{y}{\varepsilon} \right\rfloor | y \in J^\delta \wedge \left\lfloor \frac{y}{\varepsilon} \right\rfloor \in I\} \subseteq J_\varepsilon^\delta$. For ε small there is at most only one jump in each $\left[\varepsilon\left\lfloor \frac{y}{\varepsilon} \right\rfloor, \varepsilon(\left\lfloor \frac{y}{\varepsilon} \right\rfloor + 1)\right]$. So $J_\varepsilon^\delta = \{\left\lfloor \frac{y}{\varepsilon} \right\rfloor | y \in J^\delta \wedge \left\lfloor \frac{y}{\varepsilon} \right\rfloor \in I\}$ and $|J_\varepsilon^\delta \cap I| = |J^\delta \cap I| < \infty$. We consider the first sum and compute its limit for $\varepsilon \to 0$:
W.l.o.g. we assume that f is a nonnegative function. All functions within the integral converge uniformly and we can move the limit inside:

$$\begin{aligned}
\lim_{\varepsilon\to 0} K_1^\delta(\varepsilon) &= \lim_{\varepsilon\to 0} \widehat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} f(x) \cdot g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \left(\sum_{i \in J_\varepsilon^\delta} h\left(\frac{x - \varepsilon x_i}{\varepsilon}\right)\right) dx \\
&= \lim_{\varepsilon\to 0} \widehat{c} \sum_{i \in J_\varepsilon^\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_I \frac{1}{\varepsilon} f(x) \cdot h\left(\frac{x - \varepsilon x_i}{\varepsilon}\right)\ dx \\
&= \lim_{\varepsilon\to 0} \widehat{c} \sum_{i \in J_\varepsilon^\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{\varepsilon x_i - \varepsilon}^{\varepsilon x_i + \varepsilon} \frac{1}{\varepsilon} f(x) \cdot h\left(\frac{x - \varepsilon x_i}{\varepsilon}\right)\ dx.
\end{aligned}$$

Applying Substitution $\left[z = \dfrac{x - \varepsilon x_i}{\varepsilon} \Rightarrow \varepsilon^{-1} dx = dz\right]$ gives

$$\lim_{\varepsilon\to 0} K_1^\delta(\varepsilon) = \lim_{\varepsilon\to 0} \widehat{c} \sum_{i \in J_\varepsilon^\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{-1}^{1} f(\varepsilon z - \varepsilon x_i) \cdot h(z)\ dz.$$

Let i_j $(1 \leqslant j \leqslant n)$ be the indices of the jumps larger δ. Let $z_j := y_{i_j}$. It follows that

$$\lim_{\varepsilon\to 0} K_1^\delta(\varepsilon) = \widehat{c} \cdot \sum_{j=1}^{n} \underbrace{\lim_{\varepsilon\to 0} g_\varepsilon(V_{\varepsilon(\lfloor \frac{z_j}{\varepsilon} \rfloor + 1)} - V_{\varepsilon \lfloor \frac{z_j}{\varepsilon} \rfloor})}_{=:A_j(\varepsilon)} \underbrace{\lim_{\varepsilon\to 0} \int_{-1}^{1} f(\varepsilon z - \varepsilon x_{\lfloor \frac{z_j}{\varepsilon} \rfloor}) h(z)\, dz}_{=:B_j(\varepsilon)}.$$

Since V is increasing, we get: $V_{\varepsilon(\lfloor \frac{z_j}{\varepsilon} \rfloor + 1)} - V_{\varepsilon \lfloor \frac{z_j}{\varepsilon} \rfloor} \geqslant \lim_{\varepsilon\to 0} V_{\varepsilon(\lfloor \frac{z_j}{\varepsilon} \rfloor + 1)} - V_{\varepsilon \lfloor \frac{z_j}{\varepsilon} \rfloor} = V_{y_{i_j}+} - V_{y_{i_j}-} = v_{i_j}$. Let $y_j(\varepsilon) := V_{\varepsilon(\lfloor \frac{z_j}{\varepsilon} \rfloor + 1)} - V_{\varepsilon \lfloor \frac{z_j}{\varepsilon} \rfloor}$. Now according to Lemma 5: $\lim_{\varepsilon\to 0} g_\varepsilon(x) = x$. Moreover, g_ε is right-continuous and $y_j(\varepsilon)$ is non decreasing and we get from Lemma 11:

$$A_j(\varepsilon) = \lim_{\varepsilon\to 0} g_\varepsilon(V_{\varepsilon(\lfloor \frac{z_j}{\varepsilon} \rfloor + 1)} - V_{\varepsilon \lfloor \frac{z_j}{\varepsilon} \rfloor}) = \lim_{\varepsilon\to 0} g_\varepsilon(y_j(\varepsilon)) = v_{i_j}.$$

As $(x_i)_{i\in\mathbb{N}}$ is a Poisson Point Process, the waiting times e_i between two consecutive points x_i and x_{i+1} are i.i.d. with exponential distribution. Therefore according to the strong law of large numbers we get that

$$\lim_{\varepsilon\to 0} \varepsilon x_{\lfloor \frac{z_j}{\varepsilon} \rfloor} = \lim_{\varepsilon\to 0} \frac{\varepsilon}{z_j} z_j x_{\lfloor \frac{z_j}{\varepsilon} \rfloor} \leqslant z_j \lim_{\varepsilon\to 0} \frac{1}{\lfloor \frac{z_j}{\varepsilon} \rfloor} \sum_{i=0}^{\lfloor \frac{z_j}{\varepsilon} \rfloor} e_j = z_j \mathbb{E}[e_1] = z_j$$

and

$$\lim_{\varepsilon\to 0}\varepsilon x_{\lfloor \frac{z_j}{\varepsilon}\rfloor} = \lim_{\varepsilon\to 0}\frac{\varepsilon}{z_j} z_j x_{\lfloor \frac{z_j}{\varepsilon}\rfloor} \overset{e_1 \text{ bounded}}{\geqslant} z_j \lim_{\varepsilon\to 0}\frac{1}{\lfloor \frac{z_j}{\varepsilon}\rfloor+1}\sum_{i=0}^{\lfloor \frac{z_j}{\varepsilon}\rfloor+1} e_j - z_j\lim_{\varepsilon\to 0}\frac{c}{\lfloor \frac{z_j}{\varepsilon}\rfloor+1} =$$
$$= z_j\mathbb{E}[e_1] = z_j.$$

Hence, we have $\varepsilon x_{\lfloor \frac{z_j}{\varepsilon}\rfloor} \to z_j = y_{i_j}$.
Now we get for $B_j(\varepsilon)$ by dominated convergence that

$$B_j(\varepsilon) = \lim_{\varepsilon\to 0}\int_{-1}^{1} f(\varepsilon z-\varepsilon x_{\lfloor \frac{z_j}{\varepsilon}\rfloor})h(z)\,dz = \int_{-1}^{1}\lim_{\varepsilon\to 0} f(\varepsilon z-\varepsilon x_{\lfloor \frac{z_j}{\varepsilon}\rfloor})h(z)\,dz$$
$$= \int_{-1}^{1} f(y_{i_j})h(z)\,dz = f(y_{i_j})\underbrace{\int_{-1}^{1} h(z)\,dz}_{=:\hat{c}^{-1}}.$$

As a result we obtain that

$$\lim_{\varepsilon\to 0}K_1^{\delta}(\varepsilon) = \widehat{c}\cdot\sum_{j=1}^{n}\lim_{\varepsilon\to 0} g_\varepsilon(V_{\varepsilon(\lfloor \frac{z_j}{\varepsilon}\rfloor+1)} - V_{\varepsilon\lfloor \frac{z_j}{\varepsilon}\rfloor})\lim_{\varepsilon\to 0}\int_{-1}^{1} f(\varepsilon z-\varepsilon x_{\lfloor \frac{z_j}{\varepsilon}\rfloor})h(z)\,dz$$
$$= \widehat{c}\cdot\sum_{j=1}^{n} v_{i_j} f(y_{i_j})(\widehat{c})^{-1} = \sum_{j=1}^{n} v_{i_j} f(y_{i_j}).$$

We continue with the second sum. For ε small the support of $f(\varepsilon s(\varepsilon^{-1}v))$ is subset of the compact interval I'. Thus,

$$K_2^{\delta}(\varepsilon) = \widehat{c}\int_{\mathbb{R}}\frac{1}{\varepsilon}f(x)\cdot\left(\sum_{i\in J^{\varepsilon}_{\varepsilon^{1/\alpha}}\setminus J^{\varepsilon}_{\delta}} h(\varepsilon^{-1}x-x_i)g_\varepsilon(V_{\varepsilon(i+1)}-V_{\varepsilon i})\right)dx =$$
$$\leqslant \widehat{c}\int_I\sum_{i\in J^{\varepsilon}_{\varepsilon^{1/\alpha}}\setminus J^{\varepsilon}_{\delta}} g_\varepsilon(V_{\varepsilon(i+1)}-V_{\varepsilon i})f(x)h\Big(\frac{x-\varepsilon x_i}{\varepsilon}\Big)\frac{1}{\varepsilon}\,dx.$$

Since all functions are positive applying Fubini's Theorem yields that this is

$$\leqslant \widehat{c}\sum_{i\in J^{\varepsilon}_{\varepsilon^{1/\alpha}}\setminus J^{\varepsilon}_{\delta}} g_\varepsilon(V_{\varepsilon(i+1)}-V_{\varepsilon i})\int_I f(x)h\Big(\frac{x-\varepsilon x_i}{\varepsilon}\Big)\frac{1}{\varepsilon}\,dx.$$

Using substitution $\left[z = \dfrac{x - \varepsilon x_i}{\varepsilon} \Rightarrow \varepsilon^{-1} dx = dz\right]$ gives that the above term is

$$\leqslant \hat{c} \sum_{i \in J^{\varepsilon}_{\varepsilon^{1/\alpha}} \setminus J^{\varepsilon}_{\delta}} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{-1}^{1} f(\varepsilon z - \varepsilon x_i) h(z)\, dz,$$

which is

$$\leqslant C_1 \cdot \sum_{i \in J^{\varepsilon}_{\varepsilon^{1/\alpha}} \setminus J^{\varepsilon}_{\delta}} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}).$$

By Lemma 6 we see that for δ' such that $\delta' + \alpha \leqslant 1$, this is

$$\leqslant C \cdot \sum_{i \in J^{\varepsilon}_{\varepsilon^{1/\alpha}} \setminus J^{\varepsilon}_{\delta}} (V_{\varepsilon(i+1)} - V_{\varepsilon i})^{1-\delta'} \leqslant$$

$$\leqslant C \cdot \sum_{j : v_j \leqslant \delta \ \textit{with}\ y_j \in I} v_j^{1-\delta'} =: H_\delta.$$

The density of the inhomogeneous Poisson Point Process (y_j, v_j) is $dx\alpha v^{-1-\alpha} dv$. Hence:

$$\mathbb{E}(H_\delta) \leqslant \alpha |I| \int_0^\delta \omega^{1-\delta'} \omega^{-1-\alpha} d\omega \to 0 \ \ as\ \delta \to 0.$$

H_δ is decreasing and positive. Therefore the limit $\lim_{\delta \to 0} H_\delta$ exists $\bar{\mathbb{P}}$-a.s. Now according to the dominated convergence theorem $\lim_{\delta \to 0} H_\delta = 0$ $\bar{\mathbb{P}}$-a.s.

For next sum we notice that as g_ε is non-decreasing and by using Lemma 10 we get $g_\varepsilon(x) \leqslant g_\varepsilon(\varepsilon^{1/\alpha}) \leqslant C c_\varepsilon$ for all $x \leqslant \varepsilon^{1/\alpha}$. Thus, after completing equal steps as for the second sum we see that

$$K_3(\varepsilon) = \hat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} f(x)) \cdot \left(\sum_{i \in J^{\varepsilon}_{0} \setminus J^{\varepsilon}_{\varepsilon^{1/\alpha}}} h(\varepsilon^{-1} x - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \right) dx$$

is

$$\leqslant C \cdot \sum_{i \in J^{\varepsilon}_{0} \setminus J^{\varepsilon}_{\varepsilon^{1/\alpha}}} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \leqslant C' C c_\varepsilon \cdot \sum_{i \in J^{\varepsilon}_{0} \setminus J^{\varepsilon}_{\varepsilon^{1/\alpha}}} 1 \leqslant C' C c_\varepsilon \cdot \sum_{i \in \varepsilon^{-1} I \cap \mathbb{Z}} 1 \leqslant$$

$$\leqslant C'Cc_\varepsilon \cdot |I| \cdot \varepsilon^{-1} \to 0 \quad \text{as } \varepsilon \to 0.$$

This convergence holds, because there exists $k > 0$ with $-k + \dfrac{1}{\alpha} > 1$ such that $c_\varepsilon \leqslant C\varepsilon^{-k+1/\alpha}$(Lemma 9).

We study the last term:

$$K_4(\varepsilon) = \hat{c} \int_{\mathbb{R}} \frac{c_\varepsilon}{\varepsilon} f(x) \cdot c \, dx \leqslant C \cdot \frac{c_\varepsilon}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

The convergence again follows from the fact that there exists $k > 0$ with $-k + \dfrac{1}{\alpha} > 1$ such that $c_\varepsilon \leqslant C\varepsilon^{-k+1/\alpha}$(Lemma 9).

As a result we obtain

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} f \, d\tilde{p}^{(\varepsilon)} = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \left(K_1^\delta(\varepsilon) + K_2^\delta(\varepsilon) + K_3(\varepsilon) + K_4(\varepsilon) \right)$$

$$= \lim_{\delta \to 0} \sum_{j: v_j \geqslant \delta} f(x_j) v_j = \int_{\mathbb{R}} f(x) \delta_{y_i}(dx) = \int_{\mathbb{R}} f \, d\bar{\rho}.$$

□

2.3 Weak convergence to FIN-diffusion with drift

The second SDE we are going to consider has nonzero drift. Thus, the solution of this SDE has a scale function, which doesn't equal the identity function.

The speed measure of a solution to this SDE is very similar to the speed measure of a solution to the first SDE. The main difference being that a scale function is involved. This scale function, however, converges to the identity function.

Hence, provided the same assumptions as in section 2, we can proceed analogously to section 2, when it comes to determining the scaling limit of this diffusion.

Thus, proceeding in the same way as in section 2, we introduce suitably rescaled speed measures, which define our diffusion processes with equal distributions as the rescaled processes of the solution to the SDE, and prove that these speed measures converge to ρ. Applying Stone's theorem again yields that its scaling limit is the FIN-diffusion.

Let $(\Omega_1, \mathcal{A}_1, \mathcal{P}_1)$ and $(\Omega_2, \mathcal{A}_2, \mathcal{P}_2)$ be probability spaces. Let $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P})$, with $\mathbb{P} := \mathcal{P}_1 \otimes \mathcal{P}_2$.

Let ξ be a Poisson Point Process with intensity 1, defined on $(\Omega_1, \mathcal{A}_1, \mathcal{P}_1)$, so $\xi([a, b])$ has Poisson distribution with parameter $(b - a)$. For $\omega_1 \in \Omega_1$ let $(x_i(\omega_1))_{i\in\mathbb{Z}}$ be the points of the Poisson point process taken for this ω_1.

Let $(Y_i)_{i\in\mathbb{Z}} \geqslant 0$ be a sequence of *i.i.d.* random variables on $(\Omega_2, \mathcal{A}_2, \mathcal{P}_2)$, which satisfy the following condition: There exists a slowly varying function L, such that

$$\mathbb{P}[Y_i \geqslant u] = u^{-\alpha} L(u)$$

holds.
Let $h \in C^\infty(\mathbb{R})$ with $\sup(h) = [-1, 1]$ and let $c > 0$. Moreover, let c_1 be a constant, which will be defined later in the proof of Lemma 12.

For each $\omega \in \Omega_1 \times \Omega_2$, we introduce the function:

$$V_\omega(x) = \frac{1}{2} \log \left(c_1 \sum_{i=-\infty}^{\infty} h(x - x_i(\omega_1)) Y_i(\omega_2) + c \right)$$

We consider the following SDE

$$dX_t = V'_\omega(X_t)\ dt + \ dW_t. \tag{2.36}$$

Theorem 12. *Let for each $\omega \in \Omega_1 \times \Omega_2$ $X_\omega(t)$ be a solution to (2.36). Let $M(t)$ be the FIN-diffusion. Then there exist processes $\bar{X}(t)$ and $\bar{M}(t)$, defined on the same probability space, such that*

$$\bar{X}(t) \stackrel{d}{=} X(t) \quad \bar{M}(t) \stackrel{d}{=} M(t)$$

and for $\varepsilon \to 0$

$$\varepsilon \bar{X}\left(\frac{tC}{c_\varepsilon \varepsilon^2}\right) \xrightarrow{w} \bar{M}(t) \qquad \bar{\mathbb{P}} - a.s., \tag{2.37}$$

where

$$c_\varepsilon := (\inf[t \geqslant 0 : \mathbb{P}(Y_1 > t) \leqslant \varepsilon])^{-1} \tag{2.38}$$

and C is a constant which will be introduced in the proof.

Before we prove this theorem we have to verify some properties:

Lemma 12. *Let $X_\omega(t)$ be a solution to* (2.36). *Let $s_\omega(x)$ be a scale function to $X_\omega(t)$, then for $|x| \to \infty$:*

$$\frac{s_\omega(x)}{x} \to 1 \quad \mathbb{P} - a.s.$$

Proof. We show convergence for $x \to \infty$. Convergence for $x \to -\infty$ follows analogously. Let $x \in \mathbb{N}$. We know from (1.20) that

$$\begin{aligned} s'_\omega(x) &= \exp\Big(-\int_0^x 2V'_\omega(u)du + \tilde{c}\Big) = \exp\Big(-2V_\omega(x) + c'\Big) \\ &= \exp\Big(\log\Big(c_1 \sum_{i=-\infty}^{\infty} h(x - x_i(\omega_1))Y_i(\omega_2) + c\Big)^{-1} + c'\Big) \\ &= \Big(c_1 \sum_{i=-\infty}^{\infty} h(x - x_i(\omega_1))Y_i(\omega_2) + c\Big)^{-1} \cdot \overline{c} \end{aligned}$$

Consequently, we choose our scale function as follows:

$$\begin{aligned} s_\omega(x) :&= \int_0^x s'_\omega(y)dy = \sum_{j=1}^{x} \int_{j-1}^{j} s'_\omega(y)dy = \\ &= (c_1)^{-1} \sum_{j=1}^{x} \underbrace{\int_{j-1}^{j} \Big(\sum_{i=-\infty}^{\infty} h(y - x_i(\omega_1))Y_i(\omega_2) + c\Big)^{-1} \cdot \overline{c}\, dy}_{:=Z_j(\omega_2)} \end{aligned}$$

This presentation of the scale function will allow us to apply the strong law of large numbers. For each $\omega_1 \in \Omega_1$, we have Z_j is independent from Z_{j+k} for $k \geqslant 3$.

We proceed with the crucial definition of the still not defined constant c_1 by setting $c_1 := \mathbb{E}[Z_1]$. Now

$$\frac{s_\omega(x)}{x} = (c_1)^{-1} \frac{1}{x} \sum_{j=1}^{x} Z_j(\omega_2) = (c_1)^{-1} \sum_{k=0}^{2} \frac{1}{x} \sum_{\substack{1 \leqslant j \leqslant x \\ \text{with } j mod 3 = k}} Z_j(\omega_2).$$

Let $D_0 := \{j | 0 < 3j \leqslant x\}$, $D_1 := \{j | 0 < 3j+1 \leqslant x\}$, $D_2 := \{j | 0 < 3j+2 \leqslant x\}$

We have $\left\lfloor \frac{x}{3} \right\rfloor - 1 \leqslant |D_i| \leqslant \left\lfloor \frac{x}{3} \right\rfloor + 1$ for $i \in \{0, 1, 2\}$ and

$$\frac{1}{\left\lfloor \frac{x}{3} \right\rfloor + 1} \sum_{j \in D_i} Z_j(\omega_2) \leqslant \frac{1}{\frac{x}{3}} \sum_{\substack{1 \leqslant j \leqslant x \\ \text{with } jmod3=i}} Z_j(\omega_2) \leqslant \frac{1}{\left\lfloor \frac{x}{3} \right\rfloor - 1} \sum_{j \in D_i} Z_j(\omega_2)$$

for $i \in \{0, 1, 2\}$.
Since Z_j are of finite expectation, according to the strong law of large numbers we get

$$\frac{1}{\left\lfloor \frac{x}{3} \right\rfloor + 1} \sum_{j \in D_i} Z_i \xrightarrow{x \to \infty} \mathbb{E}[Z_1] \quad \mathcal{P}_2 - \text{a.s.}$$

and

$$\frac{1}{\left\lfloor \frac{x}{3} \right\rfloor - 1} \sum_{j \in D_i} Z_i \xrightarrow{x \to \infty} \mathbb{E}[Z_1] \quad \mathcal{P}_2 - \text{a.s.}$$

for $i \in \{0, 1, 2\}$. As a result we get for $0 \leqslant i \leqslant 2$::

$$\frac{1}{\frac{x}{3}} \sum_{\substack{1 \leqslant j \leqslant x \\ \text{with } jmod3=i}} Z_j \xrightarrow{x \to \infty} \mathbb{E}[Z_1] \quad \mathcal{P}_2 - \text{a.s.}$$

Finally, by definition of c_1

$$\begin{aligned} \frac{s(x)}{x} = (c_1)^{-1} \sum_{k=0}^{2} \frac{1}{x} \sum_{\substack{1 \leqslant j \leqslant x \\ \text{with } j \mod 3=k}} Z_j &\xrightarrow{x \to \infty} (c_1)^{-1} \left(\frac{1}{3}\mathbb{E}[Z_1] + \frac{1}{3}\mathbb{E}[Z_1] + \frac{1}{3}\mathbb{E}[Z_1] \right) \\ &= 1 \quad \mathcal{P}_2 - \text{a.s.} \end{aligned}$$

For $x \in \mathbb{R}^+$, by using the monotonicity of the scale function we obtain

$$\frac{s(x)}{x} \leqslant \frac{s(\lceil x \rceil)}{x} = (c_1)^{-1} \frac{1}{x} \sum_{j=1}^{\lceil x \rceil} Z_j = \frac{\lceil x \rceil}{x} (c_1)^{-1} \frac{1}{\lceil x \rceil} \sum_{j=1}^{\lceil x \rceil} Z_j \xrightarrow{x \to \infty} 1 \quad \mathcal{P}_2 - \text{a.s.}$$

and

$$\frac{s(x)}{x} \geqslant \frac{s(\lfloor x \rfloor)}{x} = (c_1)^{-1} \frac{1}{x} \sum_{j=1}^{\lfloor x \rfloor} Z_j = \frac{\lfloor x \rfloor}{x} (c_1)^{-1} \frac{1}{\lfloor x \rfloor} \sum_{j=1}^{\lfloor x \rfloor} Z_j \xrightarrow{x \to \infty} 1 \quad \mathcal{P}_2 - \text{a.s.}$$

$\square$

Lemma 13. *Let* $(m_\varepsilon)_{(0<\varepsilon<1)}$ *be positive finite measures and*

$$\phi_t^\varepsilon := \int l(t,x)\, m_\varepsilon(dx), \qquad \psi_t^\varepsilon := \inf\{u : \phi_u > t\}. \tag{2.39}$$

If

$$\widehat{W}^\varepsilon(\psi^\varepsilon(t)) \xrightarrow{w} M(t) \ \ \mathbb{P}-a.s. \quad and \quad \frac{s_\omega(x)}{x} \to 1 \ \ \mathbb{P}-a.s.,$$

then

$$\varepsilon s_\omega^{-1}(\varepsilon^{-1}\widehat{W}^\varepsilon(\psi^\varepsilon(t))) \xrightarrow{w} M(t) \qquad \mathbb{P}-a.s.$$

Proof. We will omit ω in the following proof.(Assume $\omega \in \Omega_1 \times \Omega_2$ such that both convergence conditions hold.) The set $\{\widehat{W}^\varepsilon(\psi^\varepsilon(t)) : 0 < \varepsilon < 1, s \in \mathbb{R}\}$ is sequentially weakly compact, as for any sequence within this set one can always find a weakly converging subsequence. Hence, according to Prokhorov's theorem, it is tight. Now

$$\begin{aligned}
&\varepsilon s^{-1}(\varepsilon^{-1}\widehat{W}^\varepsilon(\psi^\varepsilon(t))) - \widehat{W}^\varepsilon(\psi^\rho(t)) = \\
&= \varepsilon s^{-1}(\varepsilon^{-1}\widehat{W}^\varepsilon(\psi^\varepsilon(t))) - \widehat{W}^\varepsilon(\psi^\varepsilon(t)) + \widehat{W}^\varepsilon(\psi^\varepsilon(t)) - \widehat{W}^\varepsilon(\psi^\rho(t)) \\
&= \mathbb{1}_{\{|\widehat{W}^\varepsilon(\psi^\varepsilon(t))| \leqslant M\}}(\varepsilon s^{-1}(\varepsilon^{-1}\widehat{W}^\varepsilon(\psi^\varepsilon(t))) - \widehat{W}^\varepsilon(\psi^\varepsilon(t))) \\
&+ \mathbb{1}_{\{|\widehat{W}^\varepsilon(\psi^\varepsilon(t))| > M\}}(\varepsilon s^{-1}(\varepsilon^{-1}\widehat{W}^\varepsilon(\psi^\varepsilon(t))) - \widehat{W}^\varepsilon(\psi^\varepsilon(t))) \\
&+ \widehat{W}^\varepsilon(\psi^\varepsilon(t)) - \widehat{W}^\varepsilon(\psi^\rho(t)).
\end{aligned} \tag{2.40}$$

s^{-1} is an increasing function. It follows that $\Sigma_\varepsilon : [0, M] \to \mathbb{R}$, defined by $\Sigma_\varepsilon(x) := \varepsilon s^{-1}(\varepsilon^{-1}x)$, are a monotonically decreasing sequence of functions for $\varepsilon \to 0$, defined on a compact topological space. In addition, according to Lemma 12 Σ_ε converges pointwise to the identity function. Hence, we can apply Ulisse Dini's theorem (see [10] p. 238) and obtain

$$\sup_{x \in [0,M]} |\Sigma_\varepsilon(x) - x| \xrightarrow{\varepsilon \to 0} 0.$$

Thus, the first term on the right-hand side of (2.40) converges almost surely to zero. Consequently, it also converges weakly to zero.

We continue with the second term of the right-hand side: We notice that the set $\{\widehat{W}^\varepsilon(\psi^\varepsilon(t)) : 0 < \varepsilon < 1, s \in \mathbb{R}\}$ is sequentially weakly compact, so Prokhorov's theorem yields:

$$\mathbb{P}((\mathbb{1}_{\{|\widehat{W}^\varepsilon(\psi^\varepsilon(t))|>M\}}(\varepsilon s^{-1}(\varepsilon^{-1}\widehat{W}^\varepsilon(\psi^\varepsilon(t))) - \widehat{W}^\varepsilon(\psi^\varepsilon(t))) - 0) > \gamma) = \\ = \mathbb{P}(|\widehat{W}^\varepsilon(\psi^\varepsilon(t))| > M) \xrightarrow{M\to\infty} 0,$$

which implies weak convergence to 0. The last term on the right-hand side converges weakly to 0 according to assumption. □

Proof of Theorem 12. Let $X_\omega(t)$ be a solution to (2.36). We have already shown in the proof of Lemma 12 that

$$s'_\omega(x) = \left(c_1 \sum_{i=-\infty}^{\infty} h(x - x_i(\omega_1))Y_i(\omega_2) + c\right)^{-1} \cdot \bar{c}.$$

If $Y_\omega(t) := s(X_\omega(t))$, then according to (1.21) Y_t satisfies

$$dY_t = g(Y_t)dW_t \tag{2.41}$$

with

$$g(y) = s'_\omega \circ s_\omega^{-1}(y).$$

Let $(\widetilde{\mathbb{P}}^x, \widetilde{Z}_\omega(t))$ be the corresponding canonical process of $Y_\omega(t)$. $(\widetilde{\mathbb{P}}^x, \widetilde{Z}_\omega(t))$ is a regular diffusion on natural scale. According to (1.23) its speed measure satisfies

$$m_\omega(dx) = g_\omega(x)^{-2}dx = (s'_\omega \circ s_\omega^{-1})^{-2}(x)dx = \\ = \left(c_1 \sum_{i=0}^{\infty} h(s_\omega^{-1}(x) - x_i(\omega_1))Y_i(\omega_2) + c\right)^2 \cdot \bar{c}\, dx.$$

Let $\phi_\omega(t) := \int l(t,x)\, m_\omega(dx)$ and $\psi_\omega(t) := \inf\{u : \phi_\omega(u) > t\}$. According to Theorem 5: $Y_\omega(t) = s_\omega(X_\omega(t)) = \widehat{W}(\psi_\omega(t))$, consequently $X_\omega(t) = s^{-1}(Y_\omega(t)) = s^{-1}(\widehat{W}(\psi_\omega(t))$.

Let

$$m^{\varepsilon}_{\omega}(dx) := \frac{c_{\varepsilon}}{\varepsilon C}\left(c_1 \sum_{i=0}^{\infty} h(s_{\omega}^{-1}(\varepsilon^{-1}x) - x_i(\omega_1))Y_i(\omega_2) + c\right)^2 \cdot \bar{c}\, dx.$$

where c_{ε} is defined according to (2.38) and C is a constant, which will be defined later.

According to Lemma 3 and Lemma 4 for $\varepsilon > 0$ and for all $i \in \mathbb{Z} : \tau_i^{\varepsilon} \stackrel{d}{=} Y_1$ and $\overline{m}^{\varepsilon}$ defined by

$$\overline{m}^{\varepsilon}_{\omega}(dx) := \frac{c_{\varepsilon}}{\varepsilon C}\left(c_1 \sum_{i=0}^{\infty} h(s_{\omega}^{-1}(\varepsilon^{-1}x) - x_i(\omega_1))\tau_i^{\varepsilon}(\omega_2) + c\right)^2 \cdot \bar{c}\, dx$$

on $(\Omega_1 \times \overline{\Omega}_2, \mathcal{A}_1 \otimes \overline{\mathcal{A}_2}, \overline{\mathbb{P}})$ are couplings of m^{ε}.

We again define $\bar{\rho} := V(b) - V(a)$ for $a < b$ with $a, b \in \mathbb{R}$. $\bar{\rho}$ as shown in Lemma 3 equals $\sum_j v_j \delta_{x_j}(dx)$. $\overline{m}^{\varepsilon}$ and $\bar{\rho}$ are defined on the same probability space $(\Omega_1 \times \overline{\Omega}_2, \mathcal{A}_1 \otimes \overline{\mathcal{A}_2}, \overline{\mathbb{P}})$. Consequently, it is possible to show that

$$\overline{m}^{\varepsilon}(dx) \xrightarrow{v} \overline{\rho}(dx) \quad \overline{\mathbb{P}} - a.s. \tag{2.42}$$

holds. Theorem 9 then gives

$$\widehat{W}^{\varepsilon}(\overline{m}^{\varepsilon})(t) \xrightarrow{w} \widehat{W}(\overline{\rho})(t) =: \bar{M}(t) \qquad \overline{\mathbb{P}} - a.s.$$

Let $\bar{\phi}_{\omega}(t) := \int l(t,x)\, \overline{m}_{\omega}(dx)$ with $\overline{m} := \overline{m}^1$ and $\bar{\psi}_{\omega}(t) := \inf\{u : \phi_{\omega}(u) > t\}$ and let

$$\begin{aligned}
\bar{\phi}^{\varepsilon}_{\omega}(t) = & \int l^{\varepsilon}(t,x)\, \overline{m}^{\varepsilon}_{\omega}(dx) = \\
& = \int \varepsilon l(\varepsilon^{-2}t, \varepsilon^{-1}x) \frac{c_{\varepsilon}}{\varepsilon C}\left(c_1 \sum_{i=0}^{\infty} h(s_{\omega}^{-1}(\varepsilon^{-1}x) - x_i(\omega_1))\tau_i^{\varepsilon}(\omega_2) + c\right)^2 \cdot \bar{c}\, dx = \\
& = \int l(\varepsilon^{-2}t, \varepsilon^{-1}x) c_{\varepsilon} C^{-1}\left(c_1 \sum_{i=0}^{\infty} h(s_{\omega}^{-1}(\varepsilon^{-1}x) - x_i(\omega_1))\tau_i^{\varepsilon}(\omega_2) + c\right)^2 \cdot \bar{c}\, dx.
\end{aligned}$$

By applying substitution $[y = \varepsilon^{-1}x \Rightarrow \varepsilon^{-1}dy = dx]$ we obtain

$$\bar{\phi}^{\varepsilon}_{\omega}(t) = \int \varepsilon l(\varepsilon^{-2}t, y) c_{\varepsilon} C^{-1} \left(c_1 \sum_{i=0}^{\infty} h(s_{\omega}^{-1}(y) - x_i(\omega_1)) \tau_i^{\varepsilon}(\omega_2) + c \right)^2 \cdot \bar{c}\, dx = \\ = \varepsilon c_{\varepsilon} C^{-1} \cdot \bar{\phi}_{\omega}(\varepsilon^{-2}t).$$

Therefore

$$\bar{\psi}^{\varepsilon}_{\omega}(t) = \varepsilon^2 \bar{\psi}_{\omega} \left(\frac{Ct}{c_{\varepsilon}\varepsilon} \right).$$

Finally, let $\bar{X}(t) := s^{-1}(\widehat{W}^{\varepsilon}(\overline{m})(t))$ then

$$\varepsilon \bar{X}_{\omega}(tC(c_{\varepsilon}\varepsilon)^{-1}) = \varepsilon s^{-1}(\widehat{W}(\bar{\psi}_{\omega}(tC(c_{\varepsilon}\varepsilon)^{-1}))) = \varepsilon s^{-1}(\widehat{W}(\varepsilon^{-2}\bar{\psi}^{\varepsilon}_{\omega}(t))) = \\ = \varepsilon s^{-1}(\varepsilon^{-1}\widehat{W}^{\varepsilon}(\bar{\psi}^{\varepsilon}_{\omega}(t))).$$

From Lemma 12 we have $\dfrac{s_{\omega}(x)}{x} \xrightarrow{x\to\infty} 1$ $\bar{\mathbb{P}} - a.s.$ and since $\widehat{W}^{\varepsilon}(\bar{\psi}^{\varepsilon}(t)) \xrightarrow{w}$ $\bar{M}(t)$ $\bar{\mathbb{P}} - a.s.$

Lemma 13 yields that $\bar{\mathbb{P}} - a.s. : \varepsilon \bar{X}(t(c_{\varepsilon}\varepsilon)^{-1}) = \varepsilon s^{-1}(\varepsilon^{-1}\widehat{W}^{\varepsilon}(\bar{\psi}^{\varepsilon}(t))) \xrightarrow{w} \bar{M}(t)$.
□

It remains to prove (2.42):

Proving (2.42) involves the same procedure, which we already used in the proof of (2.13). Additional technical work needs to be done due to the scale function s. We will add some substitutions and exploit the fact that $s(x)$ converges to the identity function for $x \to \infty$. This will eventually transform the integral, such that analogously to (2.13) all lemmas can be applied.

Proof of (2.42). Let $f \in C_c(\mathbb{R})$ with $I := supp(f)$ and $I = [a, b]$.

$$\int_{\mathbb{R}} f\, d\overline{m}^{(\varepsilon)} = \int_{\mathbb{R}} f(x) \frac{c_{\varepsilon}}{\varepsilon} \left(c_1 \sum_{i=-\infty}^{\infty} h(s^{-1}(\varepsilon^{-1}x) - x_i) \frac{1}{c_{\varepsilon}} g_{\varepsilon}(V_{\varepsilon(i+1)} - V_{\varepsilon i}) + c \right)^2 \bar{c}\, dx.$$

Applying substitution $[x = \varepsilon s(y) \Rightarrow \varepsilon s'(y)dy = \varepsilon \left(c_1 \sum_{i=-\infty}^{\infty} h(y-x_i)\tau_i + c\right)^{-1} dy = dx]$ yields

$$\int_{\mathbb{R}} f\, d\overline{m}^{(\varepsilon)} = \int_{\mathbb{R}} f(\varepsilon s(y)) \cdot \frac{c_\varepsilon}{\varepsilon} \cdot \varepsilon \cdot \left(c_1 \sum_{i=-\infty}^{\infty} h(y-x_i) \frac{1}{c_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) + c \right) \bar{c}\, dy.$$

Again by substitution $[y = \varepsilon^{-1} v \Rightarrow \varepsilon^{-1} dv = dy]$ we obtain

$$\begin{aligned} \int_{\mathbb{R}} f\, d\overline{m}^{(\varepsilon)} &= \widehat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} f(\varepsilon s(\varepsilon^{-1} v)) \cdot \left(\sum_{i=-\infty}^{\infty} h(\varepsilon^{-1} v - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \right) dv \\ &\quad + \underbrace{\widehat{c} \int_{\mathbb{R}} \frac{c_\varepsilon}{\varepsilon} f(\varepsilon s(\varepsilon^{-1} v)) c\, dv}_{=:K_4(\varepsilon)}. \end{aligned} \tag{2.43}$$

Let $(y_i, v_i)_{i \in \mathbb{N}}$ be the jump process of V. For $\delta > 0$ we define $J^\delta := \{y_i : v_i > \delta\}$ and $J^\delta_\varepsilon = \{i \in \mathbb{Z} : \varepsilon S(\varepsilon^{-1} i) \in I, V_{\varepsilon(i+1)} - V_{\varepsilon i} > \delta\}$. Therefore we can express the first term on the right hand side of (2.43) as

$$\begin{aligned} &\underbrace{\widehat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} f(\varepsilon s(\varepsilon^{-1} v)) \cdot \left(\sum_{i \in J^\delta_\varepsilon} h(\varepsilon^{-1} v - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \right) dv}_{=:K_1^\delta(\varepsilon)} \\ &+ \underbrace{\widehat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} f(\varepsilon s(\varepsilon^{-1} v)) \cdot \left(\sum_{i \in J^\varepsilon_{\varepsilon^{1/\alpha}} \setminus J^\varepsilon_\delta} h(\varepsilon^{-1} v - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \right) dv}_{=:K_2^\delta(\varepsilon)} \\ &+ \underbrace{\widehat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} f(\varepsilon s(\varepsilon^{-1} v)) \cdot \left(\sum_{i \in J^\varepsilon_0 \setminus J^\varepsilon_{\varepsilon^{1/\alpha}}} h(\varepsilon^{-1} v - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \right) dv}_{=:K_3(\varepsilon)} \end{aligned}$$

Since according to Lemma 12 $\varepsilon s(\varepsilon^{-1} v) \to v$, for ε small, there exists small neighborhood I' of I such that $J^\delta_\varepsilon \subseteq I'$.The process V only has finitely many jumps $> \delta$ within I'. Let $|J^\delta \cap I'| =: n < \infty$.
For any $\varepsilon > 0$ and $y \in J^\delta$ we have: $\varepsilon \left\lfloor \frac{y}{\varepsilon} \right\rfloor \leqslant y \leqslant \varepsilon(\left\lfloor \frac{y}{\varepsilon} \right\rfloor + 1) \Rightarrow$ for ε small:

$\{\left\lfloor \frac{y}{\varepsilon} \right\rfloor | y \in J^\delta \wedge \left\lfloor \frac{y}{\varepsilon} \right\rfloor \in I'\} \subseteq J^\delta_\varepsilon$. For ε small there is at most only one jump in each $\left[\varepsilon\left\lfloor \frac{y}{\varepsilon} \right\rfloor, \varepsilon(\left\lfloor \frac{y}{\varepsilon} \right\rfloor + 1)\right]$. So $J^\delta_\varepsilon = \{\left\lfloor \frac{y}{\varepsilon} \right\rfloor | y \in J^\delta \wedge \left\lfloor \frac{y}{\varepsilon} \right\rfloor \in I'\}$ and $|J^\delta_\varepsilon \cap I'| = |J^\delta \cap I'| < \infty$.
We consider the first sum and compute its limit for $\varepsilon \to 0$:

Let $\gamma > 0$. For ε small it holds that $f(\varepsilon s^{-1}(\varepsilon^{-1}v))$ has compact support I' and $f_+(\varepsilon s^{-1}(\varepsilon^{-1}v)) \leqslant f_+(v) + \gamma \mathbb{1}_{I'}(v)$ as well as $f_-(\varepsilon s^{-1}(\varepsilon^{-1}v)) \leqslant f_-(v) + \gamma \mathbb{1}_{I'}(v)$ $\forall v \in \mathbb{R}$. W.l.o.g. we assume that f is a nonnegative function. All functions within the integral converge uniformly and we can move the limit inside:

$$\begin{aligned}
\lim_{\varepsilon\to 0} K_1^\delta(\varepsilon) &= \lim_{\varepsilon\to 0} \hat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} f(\varepsilon s^{-1}(\varepsilon^{-1}v)) \cdot g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \left(\sum_{i \in J^\delta_\varepsilon} h\left(\frac{v - \varepsilon x_i}{\varepsilon}\right) \right) dv \\
&\leqslant \lim_{\varepsilon\to 0} \hat{c} \int_{\mathbb{R}} \frac{1}{\varepsilon} (f(v) + \gamma \mathbb{1}_{I'}) \cdot g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \left(\sum_{i \in J^\delta_\varepsilon} h\left(\frac{v - \varepsilon x_i}{\varepsilon}\right) \right) dv \\
&= \lim_{\varepsilon\to 0} \hat{c} \sum_{i \in J^\delta_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_I \frac{1}{\varepsilon} f(v) \cdot h\left(\frac{v - \varepsilon x_i}{\varepsilon}\right) dv \\
&+ \lim_{\varepsilon\to 0} \hat{c} \sum_{i \in J^\delta_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{I'} \frac{1}{\varepsilon} \gamma \cdot h\left(\frac{v - \varepsilon x_i}{\varepsilon}\right) dv \\
&= \lim_{\varepsilon\to 0} \hat{c} \sum_{i \in J^\delta_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{\varepsilon x_i - \varepsilon}^{\varepsilon x_i + \varepsilon} \frac{1}{\varepsilon} f(v) \cdot h\left(\frac{v - \varepsilon x_i}{\varepsilon}\right) dv \\
&+ \lim_{\varepsilon\to 0} \hat{c} \sum_{i \in J^\delta_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{\varepsilon x_i - \varepsilon}^{\varepsilon x_i + \varepsilon} \frac{1}{\varepsilon} \gamma \cdot h\left(\frac{v - \varepsilon x_i}{\varepsilon}\right) dv.
\end{aligned}$$

Using substitution $[z = \frac{v - \varepsilon x_i}{\varepsilon} \Rightarrow \varepsilon^{-1} dv = dz]$ gives

$$\begin{aligned}
\lim_{\varepsilon\to 0} K_1^\delta(\varepsilon) &= \lim_{\varepsilon\to 0} \hat{c} \sum_{i \in J^\delta_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{-1}^{1} f(\varepsilon z - \varepsilon x_i) \cdot h(z)\, dz \\
&+ \lim_{\varepsilon\to 0} \hat{c} \sum_{i \in J^\delta_\varepsilon} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{-1}^{1} \gamma \cdot h(z)\, dz.
\end{aligned}$$

Let i_j $(1 \leqslant j \leqslant n)$ be the indices of the jumps larger δ. Let $z_j := y_{i_j}$. It follows that

$$\lim_{\varepsilon\to 0} K_1^\delta(\varepsilon) = \widehat{c}\cdot\sum_{j=1}^n \underbrace{\lim_{\varepsilon\to 0} g_\varepsilon(V_{\varepsilon(\lfloor\frac{z_j}{\varepsilon}\rfloor+1)} - V_{\varepsilon\lfloor\frac{z_j}{\varepsilon}\rfloor})}_{=:A_j(\varepsilon)} \underbrace{\lim_{\varepsilon\to 0}\int_{-1}^1 f(\varepsilon z - \varepsilon x_{\lfloor\frac{z_j}{\varepsilon}\rfloor})h(z)\,dz}_{=:B_j(\varepsilon)}$$
$$+\,\widehat{c}\cdot\sum_{j=1}^n \lim_{\varepsilon\to 0} g_\varepsilon(V_{\varepsilon(\lfloor\frac{z_j}{\varepsilon}\rfloor+1)} - V_{\varepsilon\lfloor\frac{z_j}{\varepsilon}\rfloor}) \lim_{\varepsilon\to 0}\int_{-1}^1 \gamma h(z)\,dz.$$

We have already shown in the proof of (2.13) that

$$A_j(\varepsilon) = v_{i_j} \quad\text{and}\quad B_j(\varepsilon) = f(y_{i_j})\underbrace{\int_{-1}^1 h(z)\,dz}_{=:c_h}.$$

Consequently,

$$\lim_{\varepsilon\to 0} K_1^\delta(\varepsilon) = \widehat{c}\cdot\sum_{j=1}^n \lim_{\varepsilon\to 0} g_\varepsilon(V_{\varepsilon(\lfloor\frac{z_j}{\varepsilon}\rfloor+1)} - V_{\varepsilon\lfloor\frac{z_j}{\varepsilon}\rfloor}) \lim_{\varepsilon\to 0}\int_{-1}^1 f(\varepsilon z - \varepsilon x_{\lfloor\frac{z_j}{\varepsilon}\rfloor})h(z)\,dz$$
$$+\,\widehat{c}\cdot\sum_{j=1}^n \lim_{\varepsilon\to 0} g_\varepsilon(V_{\varepsilon(\lfloor\frac{z_j}{\varepsilon}\rfloor+1)} - V_{\varepsilon\lfloor\frac{z_j}{\varepsilon}\rfloor}) \lim_{\varepsilon\to 0}\int_{-1}^1 \gamma h(z)\,dz$$
$$= \widehat{c}\cdot\sum_{j=1}^n v_{i_j} f(y_{i_j})c_h + \widehat{c}\cdot\sum_{j=1}^n v_{i_j}\gamma c_h = C\sum_{j=1}^n v_{i_j} f(y_{i_j}) + \gamma\left(\widehat{c}\cdot\sum_{j=1}^n v_{i_j} c_h\right).$$

Since γ can be chosen arbitrary small

$$\lim_{\varepsilon\to 0} K_1^\delta(\varepsilon) = C\sum_{j:v_j\geqslant\delta} f(x_j)v_j.$$

We continue with the second sum. For ε small the support of $f(\varepsilon s(\varepsilon^{-1}v))$ is subset of the compact interval I'. Thus,

$$K_2^\delta(\varepsilon) = \widehat{c}\int_{\mathbb{R}} \frac{1}{\varepsilon} f(\varepsilon s(\varepsilon^{-1}v))\cdot\left(\sum_{i\in J^\varepsilon_{\varepsilon^{1/\alpha}}\setminus J^\varepsilon_\delta} h(\varepsilon^{-1}v - x_i) g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i})\right) dv =$$
$$= \widehat{c}\int_{I'} \sum_{i\in J^\varepsilon_{\varepsilon^{1/\alpha}}\setminus J^\varepsilon_\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) f(\varepsilon s(\varepsilon^{-1}v)) h\Big(\frac{v-\varepsilon x_i}{\varepsilon}\Big)\frac{1}{\varepsilon}\,dv.$$

All functions are positive and by Fubini's Theorem we get

$$K_2^\delta(\varepsilon) = \hat{c} \sum_{i \in J^\varepsilon_{\varepsilon^{1/\alpha}} \setminus J^\varepsilon_\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{I'} f(\varepsilon s(\varepsilon^{-1} v)) h\Big(\frac{v - \varepsilon x_i}{\varepsilon}\Big) \frac{1}{\varepsilon}\, dv.$$

For ε small, we find $\gamma > 0$ such that

$$\begin{aligned} K_2^\delta(\varepsilon) &\leqslant \hat{c} \sum_{i \in J^\varepsilon_{\varepsilon^{1/\alpha}} \setminus J^\varepsilon_\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{I'} (f(v)) + \gamma) h\Big(\frac{v - \varepsilon x_i}{\varepsilon}\Big) \frac{1}{\varepsilon}\, dv \\ &\leqslant \hat{c} \sum_{i \in J^\varepsilon_{\varepsilon^{1/\alpha}} \setminus J^\varepsilon_\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{\varepsilon x_i - \varepsilon}^{\varepsilon x_i + \varepsilon} (f(v)) + \gamma) h\Big(\frac{v - \varepsilon x_i}{\varepsilon}\Big) \frac{1}{\varepsilon}\, dv. \end{aligned}$$

Then substitution $[z = \dfrac{v - \varepsilon x_i}{\varepsilon} \Rightarrow \varepsilon^{-1} dv = dz]$ yields

$$\begin{aligned} K_2^\delta(\varepsilon) &\leqslant \hat{c} \sum_{i \in J^\varepsilon_{\varepsilon^{1/\alpha}} \setminus J^\varepsilon_\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}) \int_{-1}^{1} (f(\varepsilon z - \varepsilon x_i) + \gamma) h(z)\, dz \\ &\leqslant C_1 \cdot \sum_{i \in J^\varepsilon_{\varepsilon^{1/\alpha}} \setminus J^\varepsilon_\delta} g_\varepsilon(V_{\varepsilon(i+1)} - V_{\varepsilon i}). \end{aligned}$$

By Lemma 6 we get

$$\begin{aligned} K_2^\delta(\varepsilon) &\leqslant C \cdot \sum_{i \in J^\varepsilon_{\varepsilon^{1/\alpha}} \setminus J^\varepsilon_\delta} (V_{\varepsilon(i+1)} - V_{\varepsilon i})^{1-\delta'} \\ &\leqslant C \cdot \sum_{j : v_j \leqslant \delta \text{ with } y_j \in I} v_j^{1-\delta'} =: H_\delta. \end{aligned}$$

In the proof of (2.13) we have already verified that $\lim_{\delta \to 0} H_\delta = 0 \quad \bar{\mathbb{P}}$-a.s. $\lim_{\varepsilon \to 0} K_3(\varepsilon) = 0$ and $\lim_{\varepsilon \to 0} K_4(\varepsilon) = 0$ can be shown by analogous steps as used in the proof of (2.13).

We obtain

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\mathbb{R}} f\, d\tilde{p}^{(\varepsilon)} &= \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \big(K_1^\delta(\varepsilon) + K_2^\delta(\varepsilon) + K_3(\varepsilon) + K_4(\varepsilon)\big) = \\ &= \lim_{\delta \to 0} C \sum_{j : v_j \geqslant \delta} f(x_j) v_j = C \int_{\mathbb{R}} f(x) \delta_{y_i}(dx) = C \int_{\mathbb{R}} f\, d\bar{\rho}. \end{aligned}$$

□

Appendix

Abstract

We establish a general theory concerning one-dimensional diffusions, derived from stochastic differential equations. One dimensional diffusions coming from SDEs without drift, can be expressed as time changed Brownian motions. A time changed Brownian motion is characterized by the speed measure of a diffusion. The speed measure can be obtained by the SDE.

We devote ourselves to determining scaling limits of random diffusions derived from stochastic differential equations.
At first we consider a diffusion defined by a stochastic differential equation without drift. Suitably rescaling this diffusion we obtain sequences of diffusions which converge weakly. Depending on the assumptions we show that one suitably rescaled sequence converges weakly to the Brownian motion and another suitably rescaled sequence converges weakly to the FIN-diffusion. Applying Stone's theorem it is sufficient to show vague convergence of the respective speed measures in order to obtain weak convergence of the rescaled diffusion sequences. Analogously we proceed for the second diffusion coming form a stochastic differential equation which has non-zero drift. Using Stone's theorem we show weak convergence to the FIN-diffusion of a suitably rescaled sequence of diffusions.

Zusammenfassung

Wir entwickeln eine allgemeine Theorie über eindimensionale Diffusionen, welche durch stochastische Differentialgleichungen definiert sind. Eindimensionale Diffusionen können als zeitlich geänderte Brownsche Bewegungen dargestellt werden. Eine zeitlich geänderte Brownsche Bewegung wird durch das Geschwindigkeitsmaß der Diffusion bestimmt.

Wir beschäftigen uns mit den Grenzwerten skalierter eindimensionaler Diffusionen, die durch stochastische Differentialgleichungen definiert sind.
Zuerst studieren wir eine Diffusion, welche von einer stochastischen Differentialgleichung ohne Drift bestimmt wird. Indem wir diese Diffusion passend skalieren, erhalten wir schwach konvergierende Folgen von Diffusionen. Abhängig von den Voraussetzungen konvergiert eine passend skalierte Folge von Diffusionen gegen die Brownsche Bewegung und eine andere passend skalierte Folge gegen die FIN-Diffusion.
Um diese Konvergenzaussagen zu beweisen, verwenden wir Stone's Theorem. Stone's Theorem impliziert, dass es ausreicht, die vage Konvergenz der Geschwindigkeitsmaße der entsprechenden skalierten Diffusionen zu zeigen, um die schwache Konvergenz der skalierten Diffusionen zu erhalten. Die zweite Diffusion, mit der wir uns auseinandersetzen, wird durch eine stochastische Differentialgleichung mit Drift bestimmt. Wir verfahren analog wie zuvor und zeigen, dass eine passend skalierte Folge von Diffusionen gegen die FIN-Diffusion konvergiert, indem wir Stone's Theorem anwenden.

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