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Introduction

The main goal of this thesis is to study the set $\mathcal{Y}(f)$ of solutions to an implicit equation

$$f(x, \boldsymbol{y}(x)) = 0, \quad \boldsymbol{y}(0) = 0 \tag{0.1}$$

given by a vector $f = (f_1(x, y), \ldots, f_m(x, y))$ of either algebraic, convergent or formal power series in two sets of variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_N)$. In 1968, Micheal Artin ([Art68]) showed that if $f(x, y) = (f_1(x, y), \ldots, f_m(x, y))$ is a vector of convergent power series, then every formal solution $\widehat{\boldsymbol{y}}(x) \in \mathbb{C}\,[|x|]^N$ to Equation 0.1 can be approximated up to any order by solutions to Equation 0.1 inside $\mathbb{C}\{x\}^N$, where $\mathbb{C}\{x\}$ denotes the ring of convergent power series. Phrased differently, the set of convergent solutions $\mathcal{Y}(f)$ is dense in the set of formal solutions $\widehat{\mathcal{Y}}(f)$ in the $\mathfrak{m}$-adic topology. Shortly after Artin, Arkadiusz Płoski([Pło74], [Pło15]) proved upon inspection of Artin's proof that it is even possible to find a parametrized family of convergent solutions $\Psi(x, t) \in \mathbb{C}\{x, t\}^N$, where $t = (t_1, \ldots, t_k)$ is a set of new variables, which passes through $\widehat{\boldsymbol{y}}(x)$ - in the sense that $\Psi(x, \boldsymbol{t}(x)) \in \mathcal{Y}(f)$ for every $\boldsymbol{t}(x) \in \mathbb{C}\{x\}^k$ with $\boldsymbol{t}(0) = 0$ and that there exists a $\widehat{\boldsymbol{t}}(x) \in \mathbb{C}\,[|x|]^k$ such that $\Psi(x, \widehat{\boldsymbol{t}}(x)) = \widehat{\boldsymbol{y}}(x)$. These results where then generalized in various directions, and Dorin Popescu succeded in 1985 ([Pop85]) in proofing the General Néron Desingularization Theorem. This theorem has numerous applications and yields as a corollary Artin's conjecture that every excellent Henselian local ring A has the (now-called) *Artin Approximation Property* (AAT), which means that every polynomial $f(y)$ with coefficients in A that admits a solution $\widehat{\boldsymbol{y}}$ to $f(\boldsymbol{y}) = 0$ in the completion $\widehat{A}$, has a solution $\boldsymbol{y}$ in A. Another consequence of General Néron Desingularization is that the Nested Approximation problem admits a positive answer when the equation are given by algebraic power series. For further information, we refer the reader to [Ron15].

While big efforts were undertaken to understand the approximation properties of rings, there was few work at that time in which $\mathcal{Y}(f)$ was studied as a geometric object - with the notable exception of John Nash, whose unpublished paper in 1965 (which appeared only in 1995 [Nas65]), laid down the foundation to the study of arc spaces X_∞, the set of all (parametrized) curves $\boldsymbol{y}(t)$ on an algebraic variety X_∞. Arc spaces became popular in the 90ies and have been studied in connection with the Nash-problem, which establishes a connection between components of certain families of arcs on the variety and the essential divisors that appear in a resolution of the variety.

In the first chapter we investigate various aspects of the geometry of an *arquile variety* $\mathcal{Y}(f)$ defined either over the space of algebraic, convergent or formal power series. The respective ring over which we work will be denoted by $\mathbb{C}\langle\langle x\rangle\rangle$ and its maximal ideal of series that vanish at 0 by $\mathbb{C}_0\langle\langle x\rangle\rangle$. The equation $f(x,\boldsymbol{y}(x)) = 0$ corresponds to an infinite system of polynomial equations $F_\beta(y_{\alpha,j}) = 0$ in the coefficients $(y_{\alpha,j})_{\alpha\in\mathbb{N}^n,1\leq j\leq N}$. One approach, which is for example carried out in the theory of arc spaces, is to identify the solutions to $f(x,\boldsymbol{y}(x)) = 0$ as the closed points of $\mathrm{spec}(\mathbb{C}[(y_{\alpha,j})_{\alpha\in\mathbb{N}^n,1\leq j\leq N}]/\langle(F_\beta)_{\beta\in\mathbb{N}^n}\rangle)$, by assigning to each series $\boldsymbol{y}(x)$ the kernel $\mathcal{K}_{\boldsymbol{y}}$ of the evaluation map $\mathbb{C}[(y_{\alpha,j})_{\alpha\in\mathbb{N}^n,1\leq j\leq N}] \to \mathbb{C}$, which evaluates to coordinates at the coefficients of $\boldsymbol{y}$. The infinite-dimensional version of Hilbert's Nullstellensatz given by Lang ([Lan52]) shows that each closed point (maximal ideal) stems from such an evaluation map, i.e., corresponds to a formal power series $\boldsymbol{y}$. A disadvantage of this viewpoint is that one has to work in a non-Noetherian setting, in which geometric concepts such as regularity are no longer at one's disposal. Also, this approach works only in the formal category. Instead, we embed $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ into $\mathrm{spec}(\mathbb{C}_0\langle\langle x,y\rangle\rangle)$, by identifying a series $\boldsymbol{y}(x) \in \mathbb{C}_0\langle\langle x\rangle\rangle^N$ with the ideal $\mathcal{P}_{\boldsymbol{y}}$ formed by all $f(x,y) \in \mathbb{C}_0\langle\langle x,y\rangle\rangle$ which solve $f(x,\boldsymbol{y}(x)) = 0$. In the same way we can identify an arquile variety $\mathcal{Y}$ with its ideal of relations $\mathcal{I}_{\mathcal{Y}}$. An arquile variety is called *regular* at a point $\boldsymbol{y}_0 \in \mathcal{Y}$, if it can be locally defined by a system $f(x,\boldsymbol{y}(x)) = (f_1(x,\boldsymbol{y}(x)) = \cdots = f_m(x,\boldsymbol{y}(x))) = 0$ so that a full minor Δ of $\partial_y f(x,\boldsymbol{y}(x))$ does not vanish (as a series in $\mathbb{C}\,[|x|]$) in a neighborhood of $\boldsymbol{y}_0$. The tuple (f,Δ) is then called a regular pair. The Jacobian criterion in our setting goes then reads as follows: The regularity of $\mathcal{Y}$ at a point $\boldsymbol{y}_0 \in \mathcal{Y}$ is equivalent to the regularity of the local ring $(\mathbb{C}\langle\langle x,y\rangle\rangle/\mathcal{I}_{\mathcal{Y}})_{\mathcal{P}_{\boldsymbol{y}_0}}$. We show with the Płoski- Tougeron Theorem (Theorem 1.2.4), that at a regular point a Zariski-dense subset of $\mathcal{Y}$ can be parametrized by a map $\boldsymbol{t}(x) \mapsto \Psi(x,\boldsymbol{t}(x))$ and is isomorphic to a power series module $\mathfrak{m}\mathbb{C}\langle\langle x\rangle\rangle^{N-m}$.

A guiding question for the research during this thesis was to find suitable assumptions on f which would guarantee that the equation $f(x,\boldsymbol{y}(x)) = 0$ becomes (in a vague sense) smooth if a sufficiently high jet of $\boldsymbol{y}$ is fixed. In many examples one can observe that implicit equations start to behave nicely for high-order terms. If

$f(x,y) \in \mathfrak{m}\mathbb{C}\,[|x,y|]^m$, then the *arquile map* $f_\infty \colon \mathfrak{m}\mathbb{C}\,[|x|]^N \to \mathfrak{m}\mathbb{C}\,[|x|]^m$ can be interpreted as a formally analytic map (see [Bru09]). The Fréchet derivative of f_∞ at a point $\boldsymbol{y}_0(x) \in \mathfrak{m}\mathbb{C}\,[|x|]^N$ is the $\mathbb{C}\,[|x|]$-module morphism $\boldsymbol{y}(x) \mapsto (\partial_y f(x, \boldsymbol{y}_0(x))) \cdot \boldsymbol{y}(x)$. If the initial terms of the tangential modules $\mathrm{Im}(\partial_y f(x, \boldsymbol{y}(x))$ are stable in neighborhood of a point $\boldsymbol{y}_0$, then f_∞ is said to have *constant rank* and f_∞ can be linearized by formally analytic automorphisms of source and target. In the case of a single parameter x, if (f, Δ) is a regular pair at $\boldsymbol{y}_0$, then it can be shown that f_∞ has constant rank around $\boldsymbol{y}_0$. However, it turns out that this statement is false in several variables.

In the case of a single variable, every regular point is *smooth* in the sense that an $\mathfrak{m}$-adic neighorhood of $\boldsymbol{y}_0$ in $\mathcal{Y}$ is isomorphic to a free module $\mathbb{C}_0\langle\langle x\rangle\rangle^{N-m}$. This result is illustrated with Example 1.7.3. In the single parameter case, $\mathcal{Y}$ can be decomposed into arquile locally closed subsets $\mathcal{Y} = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_s$, so that each $\mathcal{W}_i$ is smooth in the sense that every point $\boldsymbol{y} \in \mathcal{W}_i$ admits a neighborhood which is isomorphic to a free module $\mathbb{C}_0\langle\langle x\rangle\rangle^k$.

The regular part $\mathcal{Y}(f) \setminus \mathcal{Y}(\Delta)$ of a regular pair (f, Δ) can be decomposed into disjoint strata $\mathcal{Y}_d$ on which the x_n-order of Δ is constant to d. With the aid of a linearization theorem for arquile maps $f_\infty \colon \boldsymbol{y}(x) \mapsto f(x, \boldsymbol{y}(x))$ along certain submodules of $\mathcal{Y}_d$, we will show that each stratum $\mathcal{Y}_d$ is isomorphic to a product $\mathcal{Z}_d^* \times \mathfrak{m}\mathbb{C}\langle\langle x\rangle\rangle^{N-m}$, where $\mathcal{Z}_d^*$ is a set which is given locally by arquile equations in a power series space $\mathbb{C}\langle\langle x'\rangle\rangle$ in one variable less. This construction is based on Płoski's proof and gives a geometric interpretation of his techniques. Iterating this construction, we are able to show that the smooth points are dense in an arquile variety. In the case of several variables, we do not know whether every regular point is already smooth, but we suspect that this is not the case. If however the columns of $\partial_y f(x, \boldsymbol{y}_0(x))$ span an open submodule of $\mathbb{C}\langle\langle x\rangle\rangle^m$, then the arquile map $f_\infty \colon \boldsymbol{y}(x) \mapsto f(x, \boldsymbol{y}(x))$ can be linearized and $\boldsymbol{y}_0$ is a smooth point of $\mathcal{Y}(f)$. Most parts of the first chapter are joint work with H. Hauser and will appear in a joint paper.

In the second chapter we develop the analytic toolbox that we need in order to describe the isomorphisms between $\mathcal{Y}_d$ and $\mathcal{Z}_d^* \times \mathfrak{m}\mathbb{C}\langle\langle x\rangle\rangle^{N-m}$. These maps are (rational) textile maps ([Bru09]), which are maps between power series spaces, whose coefficients are polynomial (rational) in the coefficients of the input series. The space of convergent power series has a natural locally convex inductive topology, in which every arquile map $f_\infty \colon \mathbb{C}_0\{x\}^N \to \mathbb{C}_0\{x\}^m$ defines a holomorphic function. This point of view has been introduced by H.Hauser and G. Müller in [HM94], to establish a linearization theorem for arquile maps of constant rank. We review the rank theorem by Hauser and Müller and show a version of it for arquile maps whose tangential map has a cofinite image. In the second section we discuss analytic textile maps. The principal statement is that a textile map F is analytic whenever F preserves convergence, so if $F(\boldsymbol{y})$ is convergent whenever $\boldsymbol{y}$ is convergent. This statement under a mild assumption on the growth of the support of the coefficient functions. With the aid of majorization techniques as in the Cauchy-Kovalevskaya Theorem, we show an Implicit and Inverse Function Theorem for textile maps which are small perturbations of arquile maps.

In the third chapter a new class of spaces which admit filtrations both by Banach spaces and finite-dimensional Hilbert spaces is introduced. The main example, from which these spaces are axiomatized, is the space of convergent power series with its filtrations by homogeneous polynomials and Banach spaces of power series of fixed radius of convergence. These spaces behave stable with respect to direct compliments and provide a framework for the study of normal forms of power series equations. The homomorphisms of these spaces are the homogeneous maps, for which a formal Penrose inverse will be constructed. In the final section we discuss composition operators and extend a theorem by Eakin & Harris ([EH77]) to homogeneous composition operators between spaces of entire functions.

Chapter 1

Analytic varieties in power series spaces

1.1 Arquile Varieties

Let $f(x, y)$ be a vector of either algebraic, convergent or formal power series in two sets of variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_N)$. The aim of this chapter is to study the set $\mathcal{Y}(f)$ of implicit solutions $f(x, \boldsymbol{y}(x)) = 0$ with the same quality (algebraic, convergent, formal) as f. In order to avoid unnecessary technical complications and to achieve a local Noetherian setting, we consider only solutions $\boldsymbol{y}$ without constant term, so which satisfy $\boldsymbol{y}(0) = 0$. We will call these solution sets *arquile varieties*, as they can be seen as generalizations of pointed *arc* spaces (see e.g. [Ish07]). In order to distinguish variables from power series, we will denote power series by bold letters, whenever the same letter is used as a variable. To keep notation short, we will at several places write $f(\boldsymbol{y})$ instead of $f(x, \boldsymbol{y}(x))$. The power series space $\mathbb{C}\langle\langle x\rangle\rangle$ stands either for the space $\mathbb{C}\langle x\rangle$ of algebraic series, $\mathbb{C}\{x\}$ of convergent series or $\mathbb{C}\,[|x|]$ of formal power series. A consequence of Artin's Approximation Theorem is that every algebraic power series is convergent, which yields the hieararchy of rings

$$\mathbb{C}\langle x\rangle \subset \mathbb{C}\{x\} \subset \mathbb{C}\,[|x|]\,.$$

Each of these rings is a regular local ring whose dimension equals the number of variables and which is closed under solving implicit equations $f(x, \boldsymbol{y}(x)) = 0$ of systems $f(x, y) \in \mathbb{C}_0\langle\langle x, y\rangle\rangle^N$ with $\det(\partial_y f(0, 0)) \neq 0$ (see [Rui93]). By $\mathfrak{m}$ and $\mathbb{C}_0\langle\langle x\rangle\rangle$ we will denote the maximal ideal of all series in $\mathbb{C}\langle\langle x\rangle\rangle$ which vanish at 0. The Taylor expansion of f shows that the equation $f(x, \boldsymbol{y}(x)) = 0$ corresponds to an infinite system of polynomial equations in the coefficients $(y_{\alpha,i})_{\alpha\in\mathbb{N}^n, 1\le i\le N}$ of $\boldsymbol{y}$, so $\mathcal{Y}(f)$ can be seen as an infinite-dimensional algebraic variety and Zariski-closed subset of the spectrum $\operatorname{spec}(\mathbb{C}[(y_{\alpha,i})_{\alpha\in\mathbb{N}^n, 1\le i\le N}])$ of a polynomial ring in infinitely many variables. The closed points of $\operatorname{spec}(\mathbb{C}[(y_{\alpha,i})_{\alpha\in\mathbb{N}^n,1\leq i\leq N}])$ correspond to solutions of $f(\boldsymbol{y})=0$. This approach is used in the theory of arc spaces, which studies the set of parametrized formal curves on an algebraic variety. The disadvantage of this viewpoint is that the ring $\mathbb{C}[(y_{\alpha,i})_{\alpha\in\mathbb{N}^n,1\leq i\leq N}]$ is not Noetherian, which means that algebraic-geometric concepts such as regularity are not at one's disposal.

Instead, we will embed $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ in $\operatorname{spec}(\mathbb{C}\langle\langle x,y\rangle\rangle)$, identifying series $\boldsymbol{y}\in\mathbb{C}_0\langle\langle x\rangle\rangle^N$ with points $\mathcal{P}_{\boldsymbol{y}}$ in $\operatorname{spec}(\mathbb{C}\langle\langle x,y\rangle\rangle)$: Every $\boldsymbol{y}\in\mathbb{C}_0\langle\langle x\rangle\rangle^N$ defines a $\mathbb{C}$-algebra substitution homomorphism $\varepsilon_{\boldsymbol{y}}\colon\mathbb{C}\langle\langle x,y\rangle\rangle\to\mathbb{C}\langle\langle x\rangle\rangle, f(x,y)\mapsto f(x,\boldsymbol{y}(x))$. The assignments $\boldsymbol{y}\mapsto\mathcal{P}_{\boldsymbol{y}}:=\ker(\varepsilon_{\boldsymbol{y}})$ is an injection $\mathbb{C}_0\langle\langle x\rangle\rangle^N\to\operatorname{spec}(\mathbb{C}\langle\langle x,y\rangle\rangle)$, since $\mathcal{P}_{\boldsymbol{y}}$ contains $y_i-\boldsymbol{y}_i(x)$. The relation between between arquile varieties $\mathcal{Y}$ and their defining equations is expressed via a Galois connection between the power set of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ and the set of ideals of $\mathbb{C}\langle\langle x,y\rangle\rangle$, assigning on the one hand to a subset W of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ the set of implicit relations I_W on W in $\mathbb{C}\langle\langle x,y\rangle\rangle$, and on the other hand to an ideal I the set of common solutions $f(\boldsymbol{y})=0$ of all members f of I. In this way we can talk about properties of an arquile variety at a point $\boldsymbol{y}_0\in\mathcal{Y}$ in terms of the ring properties of $\mathbb{C}\langle\langle x,y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}_0}}\big/\mathcal{I}_{\mathcal{Y}}\cdot\mathbb{C}\langle\langle x,y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}_0}}$.

Definition 1.1.1. Let U be a subset of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ which is open in the $\mathfrak{m}$-adic topology, and denote by $\widehat{U}$ the closure in the $\mathfrak{m}$-adic topology of U in $\mathbb{C}_0\,[|x|]^N$. For a subset I of $\mathbb{C}\langle\langle x,y\rangle\rangle$, we define

$$\mathcal{Y}_U(I):=\{\,\boldsymbol{y}\in U\mid f(x,\boldsymbol{y}(x))=0\ \forall f\in I\,\}$$
$$\mathcal{Y}_{\widehat{U}}(I):=\{\,\boldsymbol{y}\in\widehat{U}\mid f(x,\boldsymbol{y}(x))=0\ \forall f\in I\,\}$$

as the set of common solutions $f(\boldsymbol{y})=0$ of all members $f\in I$ in $\mathbb{C}_0\langle\langle x\rangle\rangle^N$, $\mathbb{C}_0[[x]]^N$, respectively. Sets of the form $\mathcal{Y}_U(I)$ will be called *arquile varieties* in U. If I is defined by either convergent or algebraic power series, then $\mathcal{Y}_U(I)$ is dense in $\mathcal{Y}_{\widehat{U}}(I)$ in the $\mathfrak{m}$-adic topology, because of Artin's Approximation Theorem ([Art68]). Dually, if W is an arbitrary subset of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$, then let

$$\mathcal{I}_W=\{\ f\in\mathbb{C}\langle\langle x,y\rangle\rangle\ \mid\ f(\boldsymbol{y})=0\text{ for all }\boldsymbol{y}\in W\ \}$$
$$\widehat{\mathcal{I}}_W=\{\ f\in\mathbb{C}\,[|x,y|]\ \mid\ f(\boldsymbol{y})=0\text{ for all }\boldsymbol{y}\in W\ \}$$

be the ideals of relations on W inside $\mathbb{C}\langle\langle x,y\rangle\rangle$ and ideal of formal relations on W. A natural question is, whether the formal relations $\widehat{\mathcal{I}}_W$ are generated by the relations $\mathcal{I}_W$ inside $\mathbb{C}\langle\langle x,y\rangle\rangle$. We will show in subsection 1.3.1 that this is the case when W is an arquile variety, which can be seen as a dual version to Artin's Theorem. For a fixed open set U, we obtain the following Galois connection between the power set $\mathfrak{P}(U)$ of U and the set $\mathfrak{I}(\mathbb{C}\langle\langle x, y\rangle\rangle)$ of ideals of $\mathbb{C}\langle\langle x, y\rangle\rangle$, given by the assignments

$$\mathfrak{I}(\mathbb{C}\langle\langle x, y\rangle\rangle) \to \mathfrak{P}(U), I \mapsto \mathcal{Y}_U(I)$$
$$\mathfrak{P}(\mathbb{C}_0\langle\langle x\rangle\rangle^N) \to \mathfrak{I}, W \mapsto \mathcal{I}_W.$$

An ideal I will be called *U-saturated*, if $I = \mathcal{I}_{\mathcal{Y}_U(I)}$, so if it coincides with the ideal of relations on $\mathcal{Y}_U(I)$. If Y is a subset of U, then $\mathcal{Y}_U(\mathcal{I}_Y)$ is called the *Zariski*-closure of Y in U. The restriction of the Galois connection yields a bijection between the arquile varieties in U and the set of U-saturated ideals. In the next lemma we will see that the arquile varieties form the closed sets of a topology, which we call the *arquile topology*. If we identify $U \subset \mathbb{C}_0\langle\langle x\rangle\rangle^N$ with $\mathbb{U} \subset \operatorname{spec}(\mathbb{C}\langle\langle x, y\rangle\rangle)$, the image of the map $\boldsymbol{y} \mapsto \mathcal{P}_{\boldsymbol{y}}$, then the topology given by the arquile varieties in $\mathbb{U}$ is the relative Zariski topology induced from $\operatorname{spec}(\mathbb{C}\langle\langle x, y\rangle\rangle)$ on $\mathbb{U}$ ([Cla13, p.23]).

Lemma 1.1.2. *Let U be an open subset of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$.*

(1) *If $W \subset W' \subset U$ then $\mathcal{I}_W \supset \mathcal{I}_{W'}$.*
(2) *If $I \subset I' \subset \mathbb{C}\langle\langle x, y\rangle\rangle$ then $\mathcal{Y}_U(I) \supset \mathcal{Y}_U(I')$.*
(3) *$\mathcal{Y}_U(I_1 \cap \cdots \cap I_l) = \mathcal{Y}_U(I_1) \cup \cdots \cup \mathcal{Y}_U(I_l)$ for all $I_1, \ldots, I_l \subset \mathbb{C}\langle\langle x, y\rangle\rangle$.*
(4) *$\mathcal{Y}_U(I) = \mathcal{Y}_U(\mathcal{I}_{\mathcal{Y}_U(I)})$ for every $I \subset \mathbb{C}\langle\langle x, y\rangle\rangle^N$.*
(5) *If $W \subset U$ then $\mathcal{I}_W = \mathcal{I}_{\mathcal{Y}_U(\mathcal{I}_W)}$.*
(6) *If $U \subset U'$, U' open, then every U-saturated ideal is also U'-saturated.*

Proof: (1) By definition $\mathcal{I}_W = \cap_{\boldsymbol{y}\in W}\mathcal{P}_{\boldsymbol{y}} \supset \cap_{\boldsymbol{y}\in W'}\mathcal{P}_{\boldsymbol{y}} = \mathcal{I}_{W'}$. (2) If $\boldsymbol{y} \in \mathcal{Y}_U(I')$ then $f(\boldsymbol{y}) = 0$ for all $f \in I'$ and hence for all $f \in I$, consequently $\boldsymbol{y}$ is contained in $\mathcal{Y}_U(I)$. (3) "$\supset$" follows from (2). "$\subset$" Since $\prod_{k=1}^n I_k \subset \cap_{k=1}^n I_k$ we have that $\mathcal{Y}_U(\cap_{k=1}^l I_k) \subset \mathcal{Y}_U(\prod_{k=1}^l I_k)$. The product ideal is generated by the set $\{ f_1 \cdots f_l \mid f_i \in I_i \}$. If $\boldsymbol{y} \in \mathcal{Y}_U(\prod_{k=1}^l I_k)$ then $f_1(\boldsymbol{y}) \cdots f_l(\boldsymbol{y}) = 0$ (for all $f_i \in I_i$). Suppose that $\boldsymbol{y} \notin \mathcal{Y}_U(I_1) \cup \cdots \cup \mathcal{Y}_U(I_{l-1})$. Then for every $i \in \{1, \ldots, l-1\}$ there exist $f_i \in I_i$ so that $f_i(\boldsymbol{y}) \neq 0$. Let now $g \in I_l$ be arbitrary. Then $0 = (f_1 \cdots f_{l-1} \cdot g)(\boldsymbol{y}) = (f_1 \cdots f_{l-1})(\boldsymbol{y})g(\boldsymbol{y})$, thus $g(\boldsymbol{y}) = 0$, and therefore $\boldsymbol{y} \in \mathcal{Y}_U(I_l)$. This yields $\mathcal{Y}_U(\cap_{k=1}^l I_k) \subset \mathcal{Y}_U(I_1) \cup \cdots \cup \mathcal{Y}_U(I_l)$ and hence both sets are equal. (4) By definition, $I \subset \mathcal{I}_{\mathcal{Y}_U(I)}$. So (2) implies that $\mathcal{Y}_U(I) \supset \mathcal{Y}_U(\mathcal{I}_{\mathcal{Y}_U(I)}) \supset \mathcal{Y}_U(I)$. (5) By definition $W \subset \mathcal{Y}_U(\mathcal{I}_W)$, hence $\mathcal{I}_W \supset \mathcal{I}_{\mathcal{Y}_U(\mathcal{I}_W)}$ and again by definition $\mathcal{I}_W \subset \mathcal{I}_{\mathcal{Y}_U(\mathcal{I}_W)}$. (6) If $f \in \mathcal{I}_{\mathcal{Y}_{U'}(I)}$, then $f \in \mathcal{I}_{\mathcal{Y}_U(I)} = I$. $\square$

Note that a prime ideal I need not to be saturated, since $\mathcal{Y}(I)$ might be empty. But if an ideal is saturated, then all its associated primes are saturated.

Theorem 1.1.3. *Let U be an open subset of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$. Let $I = \mathcal{I}_{\mathcal{Y}_U(I)}$ be a U-saturated ideal with $\mathcal{Y}_U(I) \neq \emptyset$ and let $\operatorname{Ass}(I) = \{I_1, \dots, I_s\}$ be the associated primes of I. Then:*

(1) $\mathcal{Y}_U(I) = \mathcal{Y}_U(I_1) \cup \cdots \cup \mathcal{Y}_U(I_s)$

(2) $\mathcal{Y}_U(I_k) \neq \emptyset$ *for all* $k \in \{1, \dots, s\}$

(3) *The associated primes of I are again U-saturated, so $I_k = \mathcal{I}_{\mathcal{Y}_U(I_k)}$.*

(4) *An arquile variety $\mathcal{Y}$ in U is irreducible if and only if $\mathcal{I}_{\mathcal{Y}}$ is prime.*

Proof: Since I is a radical ideal, it is the intersection $I = I_1 \cap \cdots \cap I_k$ of its associated primes, and this decomposition is irredundant. (1) follows now directly from Lemma 1.1.2.
(2) Suppose that $\mathcal{Y}_U(I_{n_0}) = \emptyset$ (wlog $n_0 = 1$). Consequently $\mathcal{Y}_U(I) = \bigcup_{k=2}^s \mathcal{Y}_U(I_k) = \mathcal{Y}_U(\bigcap_{k=2}^s I_k)$. Since $\mathcal{I}_{\mathcal{Y}_U(I)} \subset \bigcap_{k=2}^s I_k$ and as $\mathcal{I}_{\mathcal{Y}_U(I)}$ is saturated, we both ideals have to coincide because $\mathcal{Y}_U(\bigcap_{k=2}^s I_k) = \mathcal{Y}_U(I)$. But as the the prime decomposition of $\mathcal{Y}_U(I)$ is irredundant, $\mathcal{I}_{\mathcal{Y}_U(I)}$ has to be properly contained in $\bigcap_{k=2}^s I_k$ - a contradiction. Hence $\mathcal{Y}_U(I_1) \neq \emptyset$.
(3) Suppose that $I_k \subsetneq \mathcal{I}_{\mathcal{Y}_U(I_k)}$. Then let $I_{1,1}, \dots, I_{1,r}$ be an irredundant prime decomposition of $\mathcal{I}_{\mathcal{Y}_U(I_k)} = \bigcap_{j=1}^r I_{1,j}$. We obtain

$$I \subset \bigcap_{j=1}^r I_{1,j} \cap I_2 \cap \cdots \cap I_s.$$

Since the solution set of both sets coincides and as I is by definition the largest ideal vanishing on $\mathcal{Y}_U(I)$, we obtain in fact that

$$I = \bigcap_{j=1}^r I_{1,j} \cap I_2 \cap \cdots \cap I_s.$$

The above prime decomposition can be reduced to an irredundant one. Since a radical ideal has a unique irredundant prime decomposition, it follows that I_1 has to appear in this decomposition, and since $I_i \neq I_j$ for $i \neq j$ it follows that $I_1 = I_{1,1}$ (and that $r = 1$), which leads to the contradiction $\mathcal{I}_{\mathcal{Y}_U(I_1)} \subset I_1$. (4) Assume that $\mathcal{Y}$ is irreducible. The ideal $\mathcal{I}_{\mathcal{Y}}$ is U-saturated and radical. So if $\mathcal{I}_{\mathcal{Y}} = I_1 \cap \cdots \cap I_s$ is the decomposition of $\mathcal{I}_{\mathcal{Y}}$ into its associated primes, then $\mathcal{Y}$ decomposes into $\mathcal{Y}_U(I_1) \cup \cdots \cup \mathcal{Y}_U(I_s)$, with $\mathcal{Y}_U(I_l) \neq \emptyset$. Since $\mathcal{Y}$ is irreducible, it follows that $s = 1$ and that $\mathcal{I}_{\mathcal{Y}}$ is prime. Conversely, assume that I is prime. If $\mathcal{Y}_U(I)$ splits into arquile varieties $\mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_s$, then $I = \mathcal{I}_{\mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_s} = \mathcal{I}_{\mathcal{Y}_1} \cap \cdots \cap \mathcal{I}_{\mathcal{Y}_s}$. Hence $I = \mathcal{I}_{\mathcal{Y}_i}$ for one $i \in \{1, \dots, s\}$, which implies that $\mathcal{Y}_i = \mathcal{Y}_U(I)$ and therefore $\mathcal{Y}_U(I)$ is irreducible. $\square$

Corollary 1.1.4. *A radical ideal I is U-saturated if and only if all of its associated primes are U-saturated.*

Proof: Let $\mathrm{Ass}(I) = \{I_1, \dots, I_s\}$ and suppose that each I_l is U-saturated. Then $I = \bigcap_{l=1}^s I_l = \bigcap_{l=1}^s \mathcal{I}_{\mathcal{Y}_U(I_l)} = \mathcal{I}_{\bigcup_{l=1}^s \mathcal{Y}_U(I_l)} = \mathcal{I}_{\mathcal{Y}_U(I)}$. The converse follows from the previous theorem. $\square$

A theorem by Kolchin states that if $X \subset \mathbb{C}^N$ is an irreducible algebraic variety, then the arc space X_∞ consisting of all arc $\boldsymbol{y}(t)$ on X with arbitrary starting point $y(0) \in X$ is irreducible in the Zariski topology induced from $\mathbb{C}[(y_{k,i})_{k\in\mathbb{N},1\le i\le N}]$ (see [Kol73, Chap IV, Prop 10], or [Ish07, Lemma 3.10].) Nevertheless, the space of arcs centered at 0, $X_{\infty,0} = \{\boldsymbol{y} \in X_\infty \mid \boldsymbol{y}(0) = 0\}$ may be reducible. For example, the space of centered arcs $X_{\infty,0}$ defined by $y_1y_2 - y_3^m = 0$ has $m-1$ irreducible components (see [JK13]).

Likewise, it is not true that an arquile variety defined by a prime ideal is irreducible in the arquile topology.

Example 1.1.5. Let I be the prime ideal of $\mathbb{C}\,[|x, y_1, y_2, y_3|]$ generated by

$$y_1^2(y_1^2 - t^2) - y_2^7 \quad \text{and} \quad y_1^2(y_1^4 - t^4) - y_3^7$$

Then the arquile variety $\mathcal{Y} = \mathcal{Y}(I) = \{(0,0,0), (t,0,0), (-t,0,0)\}$ consists of three isolated points and is thus reducible. The saturation $\mathcal{I}_{\mathcal{Y}(I)}$ of I is the intersection

$$\mathcal{I}_{\mathcal{Y}(I)} = \langle y_1, y_2, y_3\rangle \cap \langle y_1 - t, y_2, y_3\rangle \cap \langle y_1 + t, y_2, y_3\rangle.$$

This pathology does not occur in the case that the ideal can be generated by series which do not depend on the parameter t. In this case, I is saturated whenever it is a radical ideal and $\mathcal{Y}(I)$ is irreducible whenever I is prime (see Theorem 1.3.18).

Definition 1.1.6. An *arquile map* is a map $F\colon \mathbb{C}_0\langle\langle x\rangle\rangle^N \to \mathbb{C}_0\langle\langle x\rangle\rangle^m$ given by substitution $\boldsymbol{y} \mapsto f(x, \boldsymbol{y}(x))$ for a vector of series in $\mathbb{C}_0\langle\langle x, y\rangle\rangle^m$. Given a series $f(x,y) \in \mathbb{C}_0\langle\langle x, y\rangle\rangle^m$, we denote by f_∞ the arquile map $f_\infty\colon \mathbb{C}_0\langle\langle x\rangle\rangle^N \to \mathbb{C}_0\langle\langle x\rangle\rangle^m$ induced by f. Arquile maps are uniformly continuous with respect to the $\mathfrak{m}$-adic topology, because $f_\infty(\boldsymbol{y} + \mathfrak{m}^d\mathbb{C}\langle\langle x\rangle\rangle^N) - f_\infty(\boldsymbol{y}) \subset \mathfrak{m}^d\mathbb{C}\langle\langle x\rangle\rangle^N$ for all $\boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N$.

Theorem 1.1.7. *Let $f(x,y) \in \mathbb{C}_0\langle\langle x, y\rangle\rangle^N$ and assume the $\det(\partial_y f(0,0)) \neq 0$. Then the arquile map $f_\infty\colon \mathbb{C}_0\langle\langle x\rangle\rangle^N \to \mathbb{C}_0\langle\langle x\rangle\rangle^N$ is an isomorphism with arquile inverse.*

Proof: The map $H(x,y) = (x, f(x,y)) \colon (\mathbb{C}^{n+N}, 0) \to (\mathbb{C}^{n+N}, 0)$ satisfies the condition of the Inverse Function Theorem, so there exists a $G(x,y) = (\varphi(x,y), \psi(x,y)) \in \mathbb{C}_0\langle\langle x,y\rangle\rangle^{n+N}$, so that $H(\varphi(x,y), \psi(x,y)) = (x,y)$ and $G(x, f(x,y)) = (x,y)$. It follows that $\varphi(x,y) = x$, so $G(x,y) = (x, \psi(x,y))$. In particular, $f_\infty \circ \psi_\infty = \psi_\infty \circ f_\infty$. $\square$

Definition 1.1.8. A map $F \colon \mathbb{C}_0\,[|x|]^N \to \mathbb{C}_0\,[|x|]^m\,, \boldsymbol{y} \mapsto F(\boldsymbol{y}) = \sum_{\beta\in\mathbb{N}^n} F_\beta(\boldsymbol{y})x^\beta$ is called *textile* ([Bru09]), if each coefficient function F_β is a polynomial in finitely many of the of the coefficients $(y_{\alpha,i})_{\alpha\in\mathbb{N}^n, 1\le i\le N}$ of $\boldsymbol{y} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_N)$, where $\boldsymbol{y}_i = \sum_{\alpha\in\mathbb{N}^n} y_{\alpha,i}x^\alpha$. Likewise, we say that $F \colon \mathbb{C}_0\langle\langle x\rangle\rangle^N \to \mathbb{C}_0\langle\langle x\rangle\rangle^m$ is textile if it is the restriction of a textile map $\widehat{F} \colon \mathbb{C}_0\,[|x|]^N \to \mathbb{C}_0\,[|x|]^m$ which maps $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ to $\mathbb{C}_0\langle\langle x\rangle\rangle^m$. Note that textile maps are continuous with respect to the $\mathfrak{m}$-adic topology and that every arquile map is textile. In section 2.2 we will discuss textile maps in more detail. These maps where introduced and studied by C. Bruschek in his thesis ([Bru09]). A map $F \colon \widehat{U} \to \mathbb{C}_0\,[|x|]^m$ defined on a subset $\widehat{U}$ of $\mathbb{C}_0\,[|x|]^N$ is called *rationally textile*, if each coefficient function $F_\beta(\boldsymbol{y})$ is a rational function in the coefficients of $\boldsymbol{y}$ and if each of the denominators of F_β vanishes nowhere in $\widehat{U}$. Again, a function $F \colon U \to \mathbb{C}_0\langle\langle x\rangle\rangle^m$ defined on a subset U of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ will be called rationally textile, if it is the restriction of a rationally textile function $\widehat{F} \colon \widehat{U} \to \mathbb{C}_0\,[|x|]^m$. An example of a rationally textile map is given by the solution operator $S \colon U \to \mathbb{C}_0\,[|x|]^N$ defined on the textile open subset U of $\mathbb{C}_0\langle\langle x,y\rangle\rangle^N$ formed by those series which satisfy the condition of the implicit function theorem and which assigns to a given $f(x,y)$ the unique solution $\boldsymbol{y}$ to $f(x, \boldsymbol{y}(x)) = 0$. Instances of rationally textile maps which will appear in this thesis are all compositions of arquile maps with division maps that assign the quotients in the Weierstrass Division (Theorem 2.3.11). These maps will serve as parametrizations $F \colon U \to \mathcal{Y}$ of parts of arquile varieties $\mathcal{Y}$. A point $\boldsymbol{y} \in \mathcal{Y}$ will be called *smooth* (in the sense of Differential Geometry), if there exists a rationally textile map defined on an open subset U of a product of power series spaces which maps U bijectively onto a neighborhood of $\boldsymbol{y} \in \mathcal{Y}$. We will show in section 1.5 that these points are dense in the $\mathfrak{m}$-adic topology. The collection of the zero sets $F^{-1}(0)$ of textile maps $F \colon \mathbb{C}_0\langle\langle x\rangle\rangle^N \to \mathbb{C}_0\langle\langle x\rangle\rangle$ form the closed sets of a topology on $\mathbb{C}_0\langle\langle x\rangle\rangle^N$, which we call the *textile topology*. The closed sets are given by the solutions of usually infinite systems of polynomials in the coefficients $y_{\alpha,i}$ of $\boldsymbol{y}$. If a set V can be defined by a finite number of polynomials, then V will be called a *cofinite textile closed set*. If $\boldsymbol{y}_0 \in \mathbb{C}\,[|x|]^N$, then the kernel $\mathfrak{M}_{\boldsymbol{y}_0}$ of the substitution homomorphism $\mathbb{C}[(y_{\alpha,i})_{\alpha\in\mathbb{N}^n, i\in\mathbb{N}}] \to \mathbb{C}$ given by evaluation of the variables $y_{\alpha,i}$ at the coefficients of $\boldsymbol{y}_0$ is a maximal ideal of $\mathbb{C}[(y_{\alpha,i})_{\alpha\in\mathbb{N}^n, 1\le i\le N}]$. The assignment $\iota \colon \mathbb{C}_0\,[|x|]^N \to \operatorname{specmax}(\mathbb{C}[(y_{\alpha,i})_{\alpha\in\mathbb{N}^n, 1\le i\le N}]), \boldsymbol{y} \mapsto \mathfrak{M}_{\boldsymbol{y}}$ is an isomorphism.

If specmax($\mathbb{C}[(y_{\alpha,i})_{\alpha\in\mathbb{N}^n,1\le i\le N}]$) is equipped with the Zariski topology, then the initial topology induced by ι on $\mathbb{C}_0\left[|x|\right]^N$ coincides with the textile topology. The textile topology is strictly stronger then the arquile topology and the $\mathfrak{m}$-adic topology. A subset W of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ is called *textile locally closed*, if it is the compliment $V_1 \setminus V_2$ of a textile closed set in another. We say that W is a *textile cofinitely locally closed set*, if it is given as the difference of two cofinite textile closed sets, so if it can be defined by a finite number of polynomial equalities $p(y_{\alpha,i}) = 0$ and the non-vanishing $q(y_{\alpha,i}) \neq 0$ of another finite system.

1.2 The Płoski-Tougeron Theorem

We will show in this section that if a maximal minor of $\partial_{\boldsymbol{y}} f$ at a solution to $f(\boldsymbol{y}) = (f_1(\boldsymbol{y}), \ldots, f_m(\boldsymbol{y})) = 0$ does not vanish, then Zariski dense subset of $\mathcal{Y}(f)$ can be parametrized by an arquile morphism $\Psi_\infty \mathbb{C}_0\langle\langle x\rangle\rangle^{N-m} \to \mathcal{Y}(f)$. In the case that x is a single variable, Ψ_∞ maps $\mathbb{C}_0\langle\langle x\rangle\rangle^{N-m}$ onto a neighborhood of $\boldsymbol{y}$ in $\mathcal{Y}$ (see also Theorem 1.5.2).

Definition 1.2.1. Let $f \in \mathbb{C}_0\langle\langle x, y\rangle\rangle^m$. A solution $\boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N$ to $f(\boldsymbol{y}) = 0$ is called *simple* if there exists an m-minor Δ of $\partial_y(f)$, so that $\Delta(x, \boldsymbol{y}(x)) \neq 0$. Conversely, we say that $f_1, \ldots, f_m$ form a *regular system* at $\boldsymbol{y}$, if $\boldsymbol{y}$ is a simple solution to $f(\boldsymbol{y}) = 0$.

Theorem 1.2.2 (Identity Theorem)**.** *Let $f(x, y) \in \mathbb{C}\left[|x, y|\right]$ and suppose that $\mathcal{Y}(f) \cap \mathbb{C}\langle\langle x\rangle\rangle^N$ is $\mathfrak{m}$-open in $\mathbb{C}\langle\langle x\rangle\rangle^N$. Then $f = 0$.*

Proof: By assumption there exists a $\boldsymbol{y}_0 \in \mathcal{Y}(f) \cap \mathbb{C}\langle\langle x\rangle\rangle^N$ and a $d \in \mathbb{N}$ so that $f(\boldsymbol{y}) = 0$ for all $\boldsymbol{y} \in \boldsymbol{y}_0 + \mathfrak{m}^d\mathbb{C}\langle\langle x\rangle\rangle^N$. After a linear change of coordinates we can assume that f_∞ vanishes on $\mathfrak{m}^d\mathbb{C}\langle\langle x\rangle\rangle^N$. The facts that $f_\infty \colon \mathbb{C}_0\left[|x|\right]^N \to \mathbb{C}\left[|x|\right]$ is continuous in the $\mathfrak{m}$-adic topology and that $\widehat{\mathbb{C}\langle\langle x\rangle\rangle} = \mathbb{C}\left[|x|\right]$ imply that f_∞ vanishes on $\mathfrak{m}^d\mathbb{C}\left[|x|\right]^N$. Let $\alpha_1, \ldots, \alpha_{N'}$ be an enumeration of all monomials of $\mathbb{C}^n$ of degree d. We introduce new variables $(z_{i,\alpha})_{1\le i\le N, 1\le j\le N'}$ and

$$\varphi(x, z) = \left(\sum_{1\le j\le N'} z_{1,j} x^{\alpha_j}, \ldots, \sum_{1\le j\le N'} z_{N,j} x^{\alpha_j} \right).$$

Then the image of $\mathbb{C}\left[|x|\right]^{N\cdot N'}$ under $z(x) \mapsto \varphi(x, z(x))$ is $\mathfrak{m}^d\mathbb{C}\left[|x|\right]^N$. Therefore the map $(f(x, \varphi(x, z)))_\infty \colon z(x) \mapsto f(x, \varphi(x, z(x)))$ is identically zero, which implies that

$f(x, \varphi(x,z)) = 0$ (compare with [Wob12, Lemma 5.2.3]). Observe that

$$\det\left(\frac{\partial(\varphi_1(x,z),\ldots,\varphi_N(x,z))}{\partial(z_{1,1},\ldots,z_{N,1})}\right) = x^{\alpha_1\cdot N}.$$

Let $\mathbb{K}$ be the quotient field of $\mathbb{C}\,[|x|]$. Consider φ as an element of $\mathbb{K}[|z|]^N$ and f as an element of $\mathbb{K}[|y|]$. The partial Jacobian $\frac{\partial(\varphi_1(x,z),\ldots,\varphi_N(x,z))}{\partial(z_{1,1},\ldots,z_{N,1})} = x^{\alpha_1\cdot N}$ is invertible in $\mathbb{K}$, so by the inverse function theorem there exists a series $\psi(z) \in \mathbb{K}\,[|z|]^N$such that

$$\varphi(x, \psi_1(z), z_{1,2}, \ldots, z_{1,N'}, \ldots, \psi_N(z), z_{N,2}, \ldots, z_{N,N'}) = (z_{1,1}, \ldots, z_{N,1}).$$

Together with the fact $f(x, \varphi(x,z)) = 0$ this implies that $f(x, z_{1,1}, \ldots, z_{N,1}) = 0$ and thus $f = 0$. □

Definition 1.2.3. Let $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_k)$ and let $R = \mathbb{C}\langle\langle z\rangle\rangle,\ S = \mathbb{C}\langle\langle w\rangle\rangle$ be rings of convergent power series. Let $\Phi\colon R \to S$ be a $\mathbb{C}$-algebra homomorphism, i.e. $\Phi(f(z)) = f(\phi(w))$ for some $\phi \in \mathbb{C}_0\langle\langle w\rangle\rangle^n$. Then the generic rank $\mathrm{grk}(\Phi)$ of Φ is defined as the rank of $\frac{\partial(\phi_1,\ldots,\phi_n)}{\partial(w_1,\ldots,w_k)}$ over S.

Theorem 1.2.4 (Płoski-Tougeron, [Pło15])**.** *Let $f(x,y) \in \mathbb{C}_0\langle\langle x,y\rangle\rangle^m$, let $\boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N$ be a simple solution to $f(\boldsymbol{y}) = 0$ and let Δ be an $m-$minor of $\partial_y(f)$ with $\Delta(\boldsymbol{y}) \neq 0$. Let $z = (z_1, \ldots, z_{N-m})$ be a set of new variables. Then there exists a series $\Psi(x,z) \in \mathbb{C}_0\langle\langle x,z\rangle\rangle^N$ with the following properties:*

1. *$f(x, \Psi(x,z)) = 0$.*
2. *$\Psi(x, 0) = \boldsymbol{y}(x)$.*
3. *Let $\varepsilon_\Psi\colon \mathbb{C}\langle\langle x,y\rangle\rangle \to \mathbb{C}\langle\langle x,z\rangle\rangle,\ g(x,y) \mapsto g(x, \Psi(x,z))$. Then $\mathrm{grk}(\varepsilon_\Psi) = n+N-m$.*
4. *The image of the arquile map $\Psi_\infty\colon \mathbb{C}_0\langle\langle x\rangle\rangle^{N-m} \to \mathbb{C}_0\langle\langle x\rangle\rangle^N,\ \boldsymbol{z} \mapsto \Psi(\boldsymbol{z})$ is contained in $\mathbb{C}_0\langle\langle x\rangle\rangle^N \setminus \mathcal{Y}(\Delta)$.*
5. *The height of $\mathcal{K} = \ker(\varepsilon_\Psi)$ is m, and $\langle f_1, \ldots, f_m\rangle_{\mathcal{P}_{\boldsymbol{y}}} = \mathcal{K}_{\mathcal{P}_{\boldsymbol{y}}}$.*
6. *$\mathcal{K}$ is U-saturated for every $\mathfrak{m}$-adic neighborhood U of $\boldsymbol{y}$.*
7. *There exists an $h \in \mathbb{C}\langle\langle x,y\rangle\rangle$ so that $\mathcal{Y}_U(\mathcal{K}) \setminus \mathcal{Y}(h) = \mathcal{Y}_U(f_1, \ldots, f_m) \setminus \mathcal{Y}(h)$.*
8. *For every sufficiently small $\mathfrak{m}$-adic neighborhood $U_0 \subset U$ of $\boldsymbol{y}$, the image $\Psi_\infty(\Psi_\infty^{-1}(U_0))$ is dense in $\mathcal{Y}_{U_0}(f)$.*

Proof: After a permutation of the y-variables, we may suppose that

$$\Delta(x,y) = \det\left(\frac{\partial(f_1(x,y),\ldots,f_m(x,y))}{\partial(y_{N+m+1},\ldots,y_N)}\right).$$

Accordingly we divide y into two blocks of variables: Let $y^{(1)} = (y_1, \ldots, y_{N-m})$ and let

$y^{(2)} = (y_{N-m+1}, \dots, y_N)$. Let $v = (v_1, \dots, v_N), z = (z_1, \dots, z_{N-m}), a = (a_1, \dots, a_m)$ be three sets of new variables. Let $L_1(x, y) = \partial_{y^{(1)}} f(x, y)$, $L_2(x, y) = \partial_{y^{(2)}} f(x, y)$. We expand

$$f(x, y + v) = f(x, y) + \partial_y f(x, y)v + q(x, y, v),$$

where

$$q(x, y, v) = \sum_{1 \le i,j \le N} q_{i,j}(x, y, v) v_i v_j$$

is at least quadratic in v. We set

$$F(x, z, a) := f\left(x, \boldsymbol{y}(x) + \begin{pmatrix} \Delta^2(x, \boldsymbol{y}(x)) \cdot z \\ \Delta(x, \boldsymbol{y}(x)) \cdot a \end{pmatrix}\right) =$$

$$= \underbrace{f(x, \boldsymbol{y}(x))}_{=0} + \Delta^2(x, \boldsymbol{y}(x)) L_1(x, \boldsymbol{y}(x)) \cdot z + \Delta(x, \boldsymbol{y}(x)) L_2(x, \boldsymbol{y}(x)) \cdot a + \Delta^2(x, \boldsymbol{y}(x)) Q(x, z, a),$$

where $Q(x, z, a)$ is quadratic in (z, a). Let L_2^* denote the classical adjoint of L_2. Since $\det L_2^*(x, \boldsymbol{y}) \neq 0$, we can multiply the previous equation with L_2^* from the left without changing its solutions, and $L_2^*(x, \boldsymbol{y}(x)) F(x, \boldsymbol{z}(x), \boldsymbol{a}(x)) = 0$ expands to

$$\Delta^2(x, \boldsymbol{y}) L_2^*(x, \boldsymbol{y}(x)) L_1(x, \boldsymbol{y}(x)) \cdot z + \Delta^2(x, \boldsymbol{y}(x)) \cdot a + \Delta^2(x, \boldsymbol{y}(x)) L_2^*(x, \boldsymbol{y}(x)) Q(x, z, a) = 0$$

$$\Leftrightarrow \Delta^2(x, \boldsymbol{y}) \underbrace{\left(L_2^*(x, \boldsymbol{y}(x)) L_1(x, \boldsymbol{y}(x)) \cdot z + \mathrm{id}_{\mathbb{C}^m} \cdot a + L_2^*(x, \boldsymbol{y}(x)) Q(x, z, a)\right)}_{=:H(x,z,a)} = 0$$

Since $H(0) = 0$, $\partial_a H(0) = \mathrm{id}_{\mathbb{C}^m}$, we can apply the classical Implicit Function Theorem to conclude the existence of a series $\boldsymbol{a}(x, z) \in \mathfrak{m}\mathbb{C}\langle\!\langle x, z\rangle\!\rangle^m$ solving $H(x, z, \boldsymbol{a}(x, z)) = 0$. Then

$$\Psi(x, z) = \boldsymbol{y}(x) + \begin{pmatrix} \Delta^2(x, \boldsymbol{y}(x)) \cdot z \\ \Delta(x, \boldsymbol{y}(x)) \cdot \boldsymbol{a}(x, z) \end{pmatrix}$$

solves $f(x, \Psi(x, z)) = 0$. Since $Q(x, 0, 0) = 0$, it follows that $a(x, 0) = 0$, whence $\Psi(x, 0) = \boldsymbol{y}(x)$, which concludes the proof of (1) and (2).

(3) The generic rank of ε_Ψ is the rank of

$$\frac{\partial(x_1, \dots, x_n, \Psi_1(x, z), \dots, \Psi_N(x, z))}{\partial(x_1, \dots, x_n, z_1, \dots, z_{N-m})} = \left(\begin{array}{c|c} \mathrm{id}_{\mathbb{C}^n} & 0 \\ \hline \partial_x(\Psi(x, z)) & \mathrm{id}_{\mathbb{C}^{N-m}} \cdot \Delta(x, \boldsymbol{y}(x)) \end{array}\right),$$

which is $n + N - m$.

(4) We claim that $\Delta(x, \Psi(x, \boldsymbol{z}(x))) \neq 0$ for all $\boldsymbol{z}(x) \in \mathbb{C}_0\langle\!\langle x\rangle\!\rangle^{N-m}$, which follows from the following fact: If $g(x, y)$ is an arbitrary power series without constant term, and

$\boldsymbol{y}(x), \boldsymbol{h}(x) \in \mathfrak{m}\mathbb{C}\langle\langle x\rangle\rangle^N$, then $g(x, \boldsymbol{y} + g(x, \boldsymbol{y}(x)) \cdot \boldsymbol{h}(x))$ coincides up to multiplication with a unit with $g(x, \boldsymbol{y}(x))$. Indeed, expand

$$g(x, y + v) = g(x, y) + \partial_y g(x, y) \cdot v + q(x, y, v),$$

where q is quadratic in h. Substituting $v = g(x)\boldsymbol{h}(x)$ and $y = \boldsymbol{y}(x)$ yields

$$g(x, \boldsymbol{y}(x)) = g(x, \boldsymbol{y}(x)) + \partial_y g(x, \boldsymbol{y}(x)) \cdot g(x, \boldsymbol{y}(x))\boldsymbol{h}(x) + g^2(x, \boldsymbol{y}(x))\tilde{q}(x, \boldsymbol{h}(x)),$$

where $\tilde{q}(x, h)$ is quadratic in h, yielding

$$g(x, \boldsymbol{y}(x)) = g(x, \boldsymbol{y}(x)) \underbrace{(1 + \partial_y g(x, \boldsymbol{y}(x))\boldsymbol{h}(x) + g(x, \boldsymbol{y}(x))\tilde{q}(x, \boldsymbol{h}(x)))}_{u(x)},$$

and since $h(x), g(x, \boldsymbol{y}(x))$ vanish at zero, $u(x)$ is a unit. Now it follows at once that $\Delta(x, \Psi(x, \boldsymbol{z}(x)))$ and $\Delta(x, \boldsymbol{y}(x))$ coincide up to multiplication with a unit.
(5) After a linear change of coordinates we can suppose that $\boldsymbol{y} = 0$. Then $\Psi(x, z) = (\Delta^2(x)z, \Delta(x)\boldsymbol{a}(x, z))$. We factorize ε_Ψ into $\Phi_1 \circ \Phi_2$, where

$$\begin{aligned}
&\Phi_1 \colon \mathbb{C}\langle\langle x, y\rangle\rangle \to \mathbb{C}\langle\langle x, z\rangle\rangle,\ f(x, y) \mapsto f(x, z, \boldsymbol{a}(x, z))\\
&\Phi_2 \colon \mathbb{C}\langle\langle x, y\rangle\rangle \to \mathbb{C}\langle\langle x, y\rangle\rangle,\ f(x, y) \mapsto f(x, \Delta^2(x)y^{(1)}, \Delta(x)y^{(2)}).
\end{aligned}$$

and set $\varphi_1 = (z, \boldsymbol{a}(x, z)),\ \varphi_2 = (\Delta^2(x) \cdot y^{(1)}, \Delta(x) \cdot y^{(2)})$, so that $\Phi_2 = \varepsilon_{\varphi_2}$, $\Phi_1 = \varepsilon_{(\boldsymbol{z},\boldsymbol{a})}$. Let us show at first that Φ_2 is injective. Denote by $\mathbb{K}$ the quotient field of $\mathbb{C}\langle\langle x\rangle\rangle$. The Jacobian matrix

$$\partial_y(\varphi_2)(0) = \operatorname{diag}(\Delta^2(x), \dots, \Delta^2(x), \Delta(x), \dots, \Delta(x))$$

is invertible in $\mathbb{K}$. By the Inverse Function theorem there exists hence a series vector $\varphi_2^{-1}(y) \in \mathbb{K}[[y]]^N$ with $\varphi_2^{-1} \circ \varphi_2 = \mathrm{id}$. Hence the extension $\overline{\Phi_2} \colon \mathbb{K}[[y]] \to \mathbb{K}[[y]]$ of Φ_2 is an isomorphism with inverse $\varepsilon_{\varphi_2^{-1}}$. As $\mathbb{C}\,[|x, y|] = \mathbb{C}\,[|x|]\,[[y]] \subset \mathbb{K}[[y]]$, it follows in particular that Φ_2 is injective. The homomorphism $\Phi_1 \colon \mathbb{C}\langle\langle x, y\rangle\rangle \to \mathbb{C}\langle\langle x, z\rangle\rangle$ is onto, and induces therefore and isomorphism $\overline{\Phi_1} \colon \mathbb{C}\langle\langle x, y\rangle\rangle / \ker(\Phi_1) \to \mathbb{C}\langle\langle x, z\rangle\rangle$. Therefore $\operatorname{ht}(\ker(\Phi_1)) = \dim(\mathbb{C}\langle\langle x, y\rangle\rangle) - \dim(\mathbb{C}\langle\langle x, z\rangle\rangle) = m$. Since Φ_2 is injective, it follows that $\ker(\Phi_1 \circ \Phi_2) \subset \ker(\Phi_1)$ and therefore $\operatorname{ht}(\ker(\varepsilon_\Psi)) \leq m$. Using Theorem 1.3.5 we see that $\langle f_1, \dots, f_m\rangle_{\mathcal{P}_{\boldsymbol{y}}}$ is a prime ideal (since regular rings are integral domains) of $\mathbb{C}\langle\langle x, y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}}$ and that its height is m. Hence $\ker(\varepsilon_\Psi)_{\mathcal{P}_{\boldsymbol{y}}} = \langle f_1, \dots, f_m\rangle_{\mathcal{P}_{\boldsymbol{y}}}$, because both ideals are prime, $\langle f_1, \dots, f_m\rangle_{\mathcal{P}_{\boldsymbol{y}}} \subset \ker(\varepsilon_Y)_{\mathcal{P}_{\boldsymbol{y}}}$ and since $\operatorname{ht}(\ker(\varepsilon_\Psi)_{\mathcal{P}_{\boldsymbol{y}}}) \leq m$. (6)

Let U be an open neighborhood of $\boldsymbol{y}$ and assume that $g(x,y)$ vanishes on $\mathcal{Y}(\mathcal{K}) \cap U$. We need to show that $g \in \mathcal{K}$. Since arquile maps are continuous with respect to the $\mathfrak{m}$-adic topology and as Ψ_∞ maps 0 to $\boldsymbol{y}$, the preimage $\Psi_\infty^{-1}(U)$ is a neighborhood of 0. By assumption, $g(x, \Psi(x, \boldsymbol{z}(x))) = 0$ for all $\boldsymbol{z}(x) \in \Psi_\infty^{-1}(U)$. The identity theorem for arquile maps (Theorem 1.2.2) implies that $g(x, \Psi(x, z)) = 0$, which means that $g \in \ker \varepsilon_\Psi = \mathcal{K}$.
(7) Since $\langle f_1, \dots, f_m \rangle_{\mathcal{P}_{\boldsymbol{y}}}$ is prime and coincides with $\mathcal{K}_{\mathcal{P}_{\boldsymbol{y}}}$, $\mathcal{K}$ is an associated prime of $\langle f_1, \dots, f_m \rangle$ and hence there exists an $h \in \mathbb{C}\langle\langle x, y \rangle\rangle \setminus \mathcal{P}_{\boldsymbol{y}}$ so that $(\langle f_1, \dots, f_m \rangle \colon h) = \mathcal{K}$. As $\langle f_1, \dots, f_m \rangle \subset \mathcal{K}$, we have $\mathcal{Y}(f_1, \dots, f_m) \supset \mathcal{Y}(\mathcal{K})$. Let $\tilde{\boldsymbol{y}} \in \mathcal{Y}(f_1, \dots, f_m) \setminus \mathcal{Y}(h)$ and $g \in \mathcal{K}$. Then $h \cdot g \in \langle f_1, \dots, f_m \rangle$, whence $(h \cdot g)(\tilde{\boldsymbol{y}}) = 0$, and as $h(\tilde{\boldsymbol{y}}) \neq 0$, it follows that $g(\tilde{\boldsymbol{y}}) = 0$. Therefore $\mathcal{Y}_U(f_1, \dots, f_m) \setminus \mathcal{Y}(h) = \mathcal{Y}_U(\mathcal{K}) \setminus \mathcal{Y}(h)$.
(8) Let $U_0 \subset U \setminus \mathcal{Y}(h)$ be an $\mathfrak{m}$-adic neighborhood of $\boldsymbol{y}$. Then $\mathcal{Y}_{U_0}(f) = \mathcal{Y}_{U_0}(\mathcal{K})$. The proof of (6) shows that $g \in \mathbb{C}\langle\langle x, y \rangle\rangle$ vanishes on $\mathcal{Y}_{U_0}(\mathcal{K})$ iff g vanishes on $\mathrm{Im}(\Psi_\infty) \cap U_0$, which shows that the Zariski-closure of both sets coincides. $\square$

Example 1.2.5. Let $n = N = 2$, and let $f(x,y) = y_1 x_1 + y_2 x_2$. Then f_∞ is linear, $f(x,0) = 0$, $f_{y_2}(x,y) = x_2$, and $f_{y_2}^*(x,y) = 1$. Let $\Psi_1(x,t) = (x_2^2 t, -x_2 x_1 t)$ be the curve through the solution 0, as constructed in the Ploski-Bourbaki-Tougeron theorem, with respect to $f_{y_2}(x,0)$. Analogously the construction can be repeated with respect to $f_{y_1}(x,0)$, yielding the solution curve $\Psi_2(x,t) := (-x_1 x_2 t, x_1^2 t)$. In particular, the map Ψ_∞ need not to be onto, even if f is linear in y.

1.3 Regularity

Definition 1.3.1. Let U be an $\mathfrak{m}$-open subset of $\mathbb{C}_0\langle\langle x \rangle\rangle^N$ and let $\mathcal{Y}$ be an arquile variety in U. We say that $\mathcal{Y}$ is regular at $\boldsymbol{y} \in \mathcal{Y}$ if

$$\mathbb{C}\langle\langle x, y \rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}} \Big/ (I_{\mathcal{Y}})_{\mathcal{P}_{\boldsymbol{y}}}$$

is a regular local ring. We will see that geometrically, this means that a Zariski neighborhood N of $\boldsymbol{y}$ is defined by a regular system of equations (Proposition 1.3.11[4]). Conversely, if $\boldsymbol{y}$ is a simple solution to $f(\boldsymbol{y}) = 0$, then $\mathcal{Y}(f)$ is regular at $\boldsymbol{y}$. At a regular point, a dense subset of the irreducible component of $\boldsymbol{y}$ in $\mathcal{Y}$ can be parametrized by an arquile morphism $\Psi_\infty \colon \mathbb{C}_0\langle\langle x \rangle\rangle^k \to \mathcal{Y}$.

Remark. A pathology which comes with this definition is that singular points may be $\mathfrak{m}$-adically isolated from the regular points - which does not correspond to the expectation formed by the study of finite-dimensional analytic varieties, in which the regular points are dense in the Euclidean (as well as in the Zariski) topology. We will introduce in the next section a weaker, more local concept, to deal with this situation.

Example 1.3.2 ([Hic05])**.** Let X be the Whitney Umbrella $f(y) = y_1^2 - y_2 y_3^2 = 0$ and consider the curve $\boldsymbol{y}_0(t) = (0, t, 0)$, which is contained in the singular locus of X. The ideal $I = \langle f \rangle$ is saturated (see Theorem 1.3.18) and hence $\boldsymbol{y}_0 \in \mathcal{Y}^{\text{sing}}(f)$. Let $U_n = \boldsymbol{y}_0 + t^n \cdot \mathbb{C}\langle\langle t \rangle\rangle^3$. Then $U_n \cap \mathcal{Y}(f) \subset \mathcal{Y}^{\text{sing}}(f)$ for all $n \geq 2$: Write $\boldsymbol{y} \in U_n$ as $\boldsymbol{y}(t) = (at^n, t + bt^n, ct^n)$. Then $\boldsymbol{y} \in \mathcal{Y}(f)$ if $a^2 t^{2n} = (t + bt^n) c^2 t^{2n}$, and a simple degree comparison shows that this is only the case when $a = c = 0$, which means that $\boldsymbol{y} \in \mathcal{Y}^{\text{sing}}(f)$.

In the following we will formulate an adopted version of the Jacobian criterion, which will show that an arquile variety can be defined by a regular system at a regular point.

Definition 1.3.3. Let R be a local integral domain and let $A \in \mathrm{M}^{k \times l}(R)$. Then the rank $\mathrm{rank}(A)$ of A is the rank of A over the quotient field of R. It is easy to see that the rank of A coincides with the determinant rank of A over R, that is, $\mathrm{rank}(A)$ equals the size of the largest non-vanishing minor of A. If $\mathfrak{q}$ is a prime ideal, then by $\mathrm{rank}(A) \mod \mathfrak{q}$ we denote the rank of A over $R/\mathfrak{q}$.

Lemma 1.3.4. *Let $\varphi\colon R \to S$ be a homomorphism of local integral domains and let $\mathfrak{p}$ be the kernel of φ. Let $A = (a_{i,j})_{i,j} \in \mathrm{M}^{k \times l}(R)$ and let $\varphi(A)$ be the matrix $(\varphi(a_{i,j}))_{i,j} \in \mathrm{M}^{k \times l}(S)$. Then $\mathrm{rank}(A) \mod \mathfrak{p}$ equals the rank of $\varphi(A)$ over S.*

Proof: Observe that if B is a square matrix over R, then $\det(\varphi(B)) = \varphi(\det(B))$, because of Leibniz' determinant formula. So the minors of A which are not contained in $\mathfrak{p}$ correspond to the non-vanishing minors of $\varphi(A)$, which proves the claim. □

We can now formulate the usual Jacobian criterion for power series spaces (see for example [Rui93, p.32], [dJP00, p.154]).

Theorem 1.3.5 (Jacobian Criterion)**.** *Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal in $\mathbb{C}\langle\langle x \rangle\rangle$ and let $\mathfrak{q}$ be a prime ideal in $\mathbb{C}\langle\langle x \rangle\rangle$ containing I.*

(1) *The following are equivalent:*
  (a) $\mathbb{C}\langle\langle x \rangle\rangle_{\mathfrak{q}} \big/ I_{\mathfrak{q}}$ *is a regular local ring.*
  (b) $\mathrm{rank}\left(\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots,x_n)}\right) \mod \mathfrak{q} = \mathrm{ht}(I_{\mathfrak{q}})$.
  (c) $\mathrm{rank}\left(\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots,x_n)}\right) \mod \mathfrak{q} \geq \mathrm{ht}(I_{\mathfrak{q}})$.

(2) *Let* $m = \mathrm{ht}(I_q)$*, and let* $g_1, \dots, g_m \subset I$*. If* $\mathrm{rank}\left(\frac{\partial(g_1,\dots,g_m)}{\partial(x_1,\dots,x_n)}\right) = m \mod \mathfrak{q}$*, then* $g_1, \dots, g_m$ *generate* $I_{\mathfrak{q}}$*.*

Corollary 1.3.6 ([Rui93, p.31])**.** *Let* $\mathfrak{p}$ *be a prime ideal in* $\mathbb{C}\langle\langle x\rangle\rangle$ *of height* m*. Then there exist* $f_1, \dots, f_m \in \mathfrak{p}$ *so that an* m*-minor of* $\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_n)}$ *is not contained in* $\mathfrak{p}$*.*

The following basic facts from Commutative Algebra will be used repeatedly.

Theorem 1.3.7. *Let* R *be a Noetherian local integral domain. Let* I *be an ideal in* R *and let* $I \subset \mathfrak{p} \subset \mathfrak{q}$*, where* $\mathfrak{p}, \mathfrak{q} \in \mathrm{spec}(R)$*.*

(1) *Let* $I = I_1 \cap I_2 \cap \cdots \cap I_s \cap I_{s+1} \cap \cdots \cap I_t$ *be an irredundant primary decomposition of* I*, arranged so that* $I_1, \dots, I_s \in \mathfrak{q}$ *and and* $I_{s+1}, \dots, I_t \notin \mathfrak{q}$*. Then* $I_{\mathfrak{q}} = I_1 R_{\mathfrak{q}} \cap \cdots \cap I_s R_{\mathfrak{q}}$ *is an irredundant primary decomposition of* I *and in particular, the associated primes of* $I_{\mathfrak{q}}$ *are the localized associated primes of* I *which are contained in* $\mathfrak{q}$*.*

(2) *If* $I_{\mathfrak{q}} = \mathfrak{p}_{\mathfrak{q}}$*, then there exists an* $h \in R\backslash\mathfrak{q}$*, so that* $(I\colon h) = \mathfrak{p}$ *and* $\mathfrak{p}$ *is an associated prime of* I*.*

Proof: For (1), we refer to [AM69, p.54]. Let us show (2): Let $\{f_1, \dots, f_s\}$ be a set of generators of I, and choose $f_{s+1}, \dots, f_t \in \mathfrak{p}$ so that $\{f_1, \dots, f_t\}$ generates $\mathfrak{p}$. By assumption $I_{\mathfrak{q}} = \mathfrak{p}_{\mathfrak{q}}$ and hence for every $j \in \{s+1, \dots, t\}$ there exist $r_{i,j} \in R$ and $h_{i,j} \in R \setminus \mathfrak{q}$ so that

$$f_j = \sum_{k=1}^{s} \frac{r_{i,j}}{h_{i,j}} f_i.$$

Set $h = \Pi_{i,j} h_{i,j}$. Then $h \in R \setminus \mathfrak{q}$ and $h \cdot f_j \subset I$ for all j, which yields $h \cdot \mathfrak{p} \subset I$, so $\mathfrak{p} \subset (I : h)$. Since $h \in R \setminus \mathfrak{q} \subset R \setminus \mathfrak{p} \subset R_{\mathfrak{p}}^*$ it follows that $I_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$. Suppose that $\mathfrak{p}$ were a proper subset of $(I : h)$. Then $(I : h)R_{\mathfrak{p}} = R_{\mathfrak{p}}$, since $(I : h)$ contains an element of $R \setminus \mathfrak{p} \subset R_{\mathfrak{p}}^*$. But

$$(I : h)R_{\mathfrak{p}} = (I : h)hR_{\mathfrak{p}} = IR_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}.$$

Thus $\mathfrak{p}R_{\mathfrak{p}} = R_{\mathfrak{p}}$, which implies by Nakayammas lemma that $R_{\mathfrak{p}} = 0$ - a contradiction and therefore $(I : h) = \mathfrak{p}$. This also implies that $\mathfrak{p} \in \mathrm{Ass}(I)$. □

Definition 1.3.8. Let $M(x, y) \in \mathrm{M}^{n\times m}(\mathbb{C}\langle\langle x, y\rangle\rangle)$ and let $\boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle$. Then $\mathrm{rank}_{\boldsymbol{y}}(M)$ is the rank of $M(x, \boldsymbol{y}(x))$ in $\mathrm{M}^{n\times m}(\mathbb{C}\langle\langle x\rangle\rangle)$, so by definition the rank of $M(x, \boldsymbol{y}(x))$ over the quotient field of $\mathbb{C}\langle\langle x\rangle\rangle$. By Lemma 1.3.4 it coincides with $\mathrm{rank}(M) \mod \mathcal{P}_{\boldsymbol{y}}$ and equals the size of the largest minor of M which does not vanish at $\boldsymbol{y}$.

The following is a standard argument used in all proofs of Artin's approximation theorem (see for example [Pło15]).

Lemma 1.3.9. *Let $f = (f_1, \dots, f_m) \in \mathbb{C}_0\langle\langle x, y\rangle\rangle^m$ and let $\boldsymbol{y}$ in $\mathcal{Y}(f)$. Then*

$$\operatorname{rank}_{\boldsymbol{y}}\left(\frac{\partial(f_1, \dots, f_m)}{\partial(x_1, \dots, x_n, y_1, \dots, y_N)}\right) = \operatorname{rank}_{\boldsymbol{y}}\left(\frac{\partial(f_1, \dots, f_m)}{\partial(y_1, \dots, y_N)}\right).$$

Proof: Differentiating $f(x, \boldsymbol{y}(x)) = 0$ with respect to x_j yields the following relation between the x-derivatives and the $y-$derivatives:

$$-\left.\frac{\partial f(x,y)}{\partial x_j}\right|_{(x,\boldsymbol{y}(x))} = \sum_{i=1}^{N} \left.\frac{\partial f(x,y)}{\partial y_i}\right|_{(x,\boldsymbol{y}(x))} \frac{\partial \boldsymbol{y}_i(x)}{\partial x_j}. \tag{1.1}$$

The linear relations Equation 1.1 between the columns of $\partial_x(f)(\boldsymbol{y})$ and $\partial_y(f)(\boldsymbol{y})$ over $\mathbb{C}\langle\langle x\rangle\rangle$ imply that the rank of $(\partial_x(f)(\boldsymbol{y}), \partial_y(f)(\boldsymbol{y}))$ over the quotient field of $\mathbb{C}\langle\langle x\rangle\rangle$ is the same as the rank of $(0, \partial_y(f)(\boldsymbol{y}))$, and therefore

$$\operatorname{rank}_{\boldsymbol{y}}\left(\partial_x(f), \partial_y(f)\right) = \operatorname{rank}_{\boldsymbol{y}}\left(0, \partial_y(f)\right).$$

□

The preceding lemma immediately yields the following reformulation of the Jacobian criterion (Theorem 1.3.5, Theorem 1.3.5).

Corollary 1.3.10 (Arquile Jacobian Criterion)**.** *Let U be an open subset of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$, let $I = \langle f_1, \dots, f_s\rangle \subset \mathbb{C}_0\langle\langle x, y\rangle\rangle$ and let $\boldsymbol{y} \in \mathcal{Y}_U(I)$.*

(1) *The following are equivalent:*
   (a) $\mathbb{C}\langle\langle x, y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}} \big/ I_{\mathcal{P}_{\boldsymbol{y}}}$ *is a regular local ring.*
   (b) $\operatorname{rank}_{\boldsymbol{y}}\left(\frac{\partial(f_1,\dots,f_s)}{\partial(y_1,\dots,y_N)}\right) = \operatorname{ht}(I_{\mathcal{P}_{\boldsymbol{y}}})$.

(2) *Let* $\operatorname{ht}(I_{\mathcal{P}_{\boldsymbol{y}}}) = m$*. If* $\operatorname{rank}_{\boldsymbol{y}}\left(\frac{\partial(f_1,\dots,f_m)}{\partial(y_1,\dots,y_N)}\right) = m$*, then $f_1, \dots, f_m$ generate $I_{\mathcal{P}_{\boldsymbol{y}}}$.*

Proposition 1.3.11. *Let U be an $\mathfrak{m}$-open subset of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ and let $\mathcal{Y}$ be a non-empty arquile variety in U. Let $\mathcal{S}$ be a system of generators for $\mathcal{I}_{\mathcal{Y}}$, let $\boldsymbol{y}_0 \in \mathcal{Y}$ and let m be the height of the localized ideal $\mathcal{I}_{\mathcal{Y}} \cdot \mathbb{C}\langle\langle x, y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}_0}}$ at $\boldsymbol{y}_0$. TFAE:*

(1) *The point $\boldsymbol{y}_0$ is a regular point of $\mathcal{Y}$.*
(2) *There exist $f_1, \dots, f_m \in \mathcal{S}$ so that an $(m \times m)$-minor Δ of the partial Jacobian $\partial_y f$ of $f = (f_1, \dots, f_m)$ does not vanish at $\boldsymbol{y}_0$.*

(3) *There exist* $f_1, \dots, f_m \in \mathcal{S}$ *with* $f_i(\boldsymbol{y}_0) = 0$*, an* $(m \times m)$*-minor* Δ *of the partial Jacobian* $\partial_y f$ *of* $f = (f_1, \dots, f_m)$ *which does not vanish at* $\boldsymbol{y}_0$*, so that*

$$\mathcal{Y}_U(f_1, \dots, f_m) \setminus \mathcal{Y}(\Delta) = \mathcal{Y} \setminus \mathcal{Y}(\Delta).$$

(4) *There exist* $f_1, \dots, f_r \in \mathbb{C}\langle\langle x, y\rangle\rangle$ *with* $f_i(\boldsymbol{y}_0) = 0$*, an* $(m \times m)$*-minor* Δ *of the partial Jacobian* $\partial_y f$ *of* $f = (f_1, \dots, f_m)$ *which does not vanish at* $\boldsymbol{y}_0$*, so that*

$$\mathcal{Y}_U(f_1, \dots, f_m) \setminus \mathcal{Y}(\Delta) = \mathcal{Y} \setminus \mathcal{Y}(\Delta).$$

Proof: The equivalence of (1) and (2) is the preceding Jacobian Criterion and (3) $\Rightarrow$ (4) is trivial. Set $I = \mathcal{I}_\mathcal{Y}$. WLOG we may assume that $\Delta = \det\left(\frac{\partial(f_1,\dots,f_m)}{\partial(y_1,\dots,y_m)}\right)$.
(2) $\Rightarrow$ (3) Because of the Jacobian Criterion, the ring $\mathbb{C}\langle\langle x, y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}_0}}/I_{\mathcal{P}_{\boldsymbol{y}_0}}$ is a regular local ring and hence an integral domain ([dJP00, p. 167]), whence $I_{\mathcal{P}_{\boldsymbol{y}_0}}$ is prime and coincides with the localization of a prime $\mathfrak{p} \in \operatorname{spec}(\mathbb{C}\langle\langle x, y\rangle\rangle)$. Let $I_0 = \langle f_1, \dots, f_m\rangle$. The localization $(I_0)_{\mathcal{P}_{\boldsymbol{y}_0}}$ coincides with $I_{\mathcal{P}_{\boldsymbol{y}_0}}$ (Theorem 1.3.10(2)), and is hence prime. By Theorem 1.3.7, there exists an $h \in \mathbb{C}\langle\langle x, y\rangle\rangle \setminus \mathcal{P}_{\boldsymbol{y}_0}$, so that $(I_0 \colon h) = \mathfrak{p}$. Then set $\tilde{f}_i = h^2 \cdot f_i$ and let I_1 be the ideal in $\mathbb{C}\langle\langle x, y\rangle\rangle$ generated by the $\tilde{f}_i$. Observe that h divides $\widetilde{\Delta} = \det\left(\frac{\partial(\tilde{f}_1,\dots,\tilde{f}_m)}{\partial(y_1,\dots,y_m)}\right)$ and that $\widetilde{\Delta}(\boldsymbol{y}) = h^2(\boldsymbol{y}) \cdot \Delta(\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathcal{Y}(f_1, \dots, f_m)$, in particular

$$\mathcal{Y}(f_1, \dots, f_m) \setminus (\mathcal{Y}(\Delta) \cup \mathcal{Y}(h)) = \mathcal{Y}(\tilde{f}_1, \dots, \tilde{f}_m) \setminus \mathcal{Y}(\widetilde{\Delta}) \tag{1.2}$$

It is easily seen that $(I_1 \colon h^3) = (h^2 I_0 \colon h^3) = (I_0 \colon h)$. The inclusion of ideals $I_1 \subset I \subset \mathfrak{p}$ yields $\mathcal{Y}_U(I_1) \supset \mathcal{Y}_U(I) \supset \mathcal{Y}_U(\mathfrak{p})$. Let $\boldsymbol{y} \in \mathcal{Y}_U(I_1) \setminus \mathcal{Y}(\widetilde{\Delta})$. As h divides $\widetilde{\Delta}$, $h(\boldsymbol{y}) \neq 0$. If $g \in \mathfrak{p}$, then $h^3 \cdot g \in I_1$, whence $(h^3 \cdot g)(\boldsymbol{y}) = 0$, and hence $g(\boldsymbol{y}) = 0$. This shows $\mathcal{Y}_U(\tilde{f}_1, \dots, \tilde{f}_m) \setminus \mathcal{Y}(\Delta) \subset \mathcal{Y}_U(\mathfrak{p})$, and thus $\mathcal{Y}_U(\tilde{f}_1, \dots, \tilde{f}_m) \setminus \mathcal{Y}(\widetilde{\Delta}) = \mathcal{Y}_U(I) \setminus \mathcal{Y}(\widetilde{\Delta}) = \mathcal{Y} \setminus \mathcal{Y}(\widetilde{\Delta})$.
(4) $\Rightarrow$ (1) By assumption, $\boldsymbol{y}_0$ is a simple solution to $f(\boldsymbol{y}_0) = 0$, so we can apply Theorem 1.2.4. Let $z = (z_1, \dots, z_{N-r})$ be a set of new variables, let $\Psi(x, z) \in \mathbb{C}_0\langle\langle x, z\rangle\rangle^N$ and ε_Ψ as in Theorem 1.2.4. The image of $\Psi_\infty \colon \mathbb{C}_0\langle\langle x\rangle\rangle^{N-r} \to \mathbb{C}_0\langle\langle x\rangle\rangle^N$ is contained in $\mathcal{Y}(f) \setminus \mathcal{Y}(\Delta)$, so $\Psi_\infty(\boldsymbol{z}) \in \mathcal{Y}$ for every $\boldsymbol{z} \in \Psi_\infty^{-1}(U)$. Let $g \in I$. Then $g_\infty \circ \Psi_\infty = 0$ on $\Psi_\infty^{-1}(U)$ and thus $g(x, \Psi(x, z)) = 0$ by Theorem 1.2.2. So $g \in \mathcal{K} := \ker(\varepsilon_\Psi)$ and $I \subset \mathcal{K}$. We claim that $I_{\mathcal{P}_{\boldsymbol{y}_0}}$ is prime, i.e. that $\boldsymbol{y}_0$ is contained in exactly one irreducible component of $\mathcal{Y}$. Let $\mathcal{Y}_1, \dots, \mathcal{Y}_l$ be the irreducible components of $\mathcal{Y}$ that contain $\boldsymbol{y}_0$, let $I_i = \mathcal{I}_{\mathcal{Y}_i}$. By Theorem 1.2.4[6], $\mathcal{K}$ is U-saturated and therefore $\mathcal{Y}_U(\mathcal{K})$ is irreducible. The localization of $\langle f_1, \dots, f_r\rangle_{\mathcal{P}_{\boldsymbol{y}_0}}$ coincides with $\mathcal{K}_{\mathcal{P}_{\boldsymbol{y}_0}}$ and $f_i \in \mathcal{K}$, hence there exists

an $h \in \mathbb{C}\langle\langle x, y\rangle\rangle \setminus \mathcal{P}_{\boldsymbol{y}_0}$ so that $(\langle f_1, \dots, f_r\rangle \colon h) = \mathcal{K}$. Equation 1.2 and the proof of (2) $\Rightarrow$ (3) shows that $\mathcal{Y}(f_1, \dots, f_r) \setminus \mathcal{Y}(\Delta \cdot h) = \mathcal{Y}_U(\mathcal{K}) \setminus \mathcal{Y}(\Delta \cdot h) = \mathcal{Y} \setminus \mathcal{Y}(\Delta \cdot h)$. The compliment $\mathcal{Y} \setminus \mathcal{Z}$ of a proper closed subset $\mathcal{Z}$ of an irreducible set $\mathcal{Y}$ is dense in $\mathcal{Y}$. Therefore

$$\mathcal{Y}_U(\mathcal{K}) = \overline{\mathcal{Y}_U(\mathcal{K}) \setminus \mathcal{Y}_U(\Delta \cdot h)} = \bigcup_{i=1}^{l} \overline{\mathcal{Y}_i \setminus \mathcal{Y}(\Delta \cdot h)}.$$

Since $\mathcal{Y}_U(\mathcal{K})$ is irreducible, it follows that $\mathcal{Y}_U(\mathcal{K}) = \overline{\mathcal{Y}_{i_0} \setminus \mathcal{Y}(\Delta \cdot h)} = \mathcal{Y}_{i_0}$ for one $i_0 \in \{1, \dots l\}$ and that $\mathcal{Y}_i \setminus \mathcal{Y}(\Delta \cdot h) = \emptyset$ for $i \neq i_0$. As both $\mathcal{K}$ and I_{i_0} are U-saturated, we obtain that $\mathcal{K} = I_{i_0} = I_1$, so $\mathcal{K}$ is associated to I. Since $\Delta \cdot h \neq 0$, it follows in particular that $\boldsymbol{y}_0$ is contained only in the component $\mathcal{Y}_U(\mathcal{K})$. Let $I = I_1 \cap \dots \cap I_k$ be the decomposition of I into its associated prime. The associated primes of $I_{\mathcal{P}_{\boldsymbol{y}_0}}$ are the localizations of those associated primes which are contained in $\mathcal{P}_{\boldsymbol{y}_0}$, and since I is radical and only one associated prime is contained in $\mathcal{P}_{\boldsymbol{y}_0}$, we have obtain that $I_{\mathcal{P}_{\boldsymbol{y}_0}} = \mathcal{K}_{\mathcal{P}_{\boldsymbol{y}_0}}$. But $\mathcal{K}_{\mathcal{P}_{\boldsymbol{y}_0}} = \langle f_1, \dots, f_m\rangle_{\mathcal{P}_{\boldsymbol{y}_0}}$ (Theorem 1.2.4(5)), and once more the Jacobian Criterion implies the regularity of $\mathbb{C}\langle\langle x, y\rangle\rangle / \mathcal{K}_{\mathcal{P}_{\boldsymbol{y}_0}} = \mathbb{C}\langle\langle x, y\rangle\rangle / I_{\mathcal{P}_{\boldsymbol{y}_0}}$. □

Definition 1.3.12. A pair (f, Δ) as in (3) of the previous theorem is called a *regular pair.*

Theorem 1.3.13. *Let U be an $\mathfrak{m}$-open subset of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$, let $\mathcal{Y}$ be a non-empty arquile subset of U and let $\mathcal{Y} = \mathcal{Y}_1 \cup \dots \cup \mathcal{Y}_l$ be the irreducible components of $\mathcal{Y}$.*

1. *The regular locus is covered by a finite number of arquile open subsets of the form $\mathcal{Y}_U(f_1, \dots, f_m) \setminus \mathcal{Y}(\Delta)$, where (f, Δ) is a regular pair for $\mathcal{Y}$.*
2. *The pairwise intersections $\mathcal{Y}_i \cap \mathcal{Y}_j$ $(i \neq j)$ of the irreducible components are contained in the singular locus of $\mathcal{Y}$.*
3. *A point $\boldsymbol{y}$ lies in the regular locus* $\mathrm{Reg}(\mathcal{Y})$ *if and only $\boldsymbol{y}$ is contained in exactly one component $\mathcal{Y}_i$ and in the regular locus of $\mathcal{Y}_i$.*
4. *The singular locus is an arquile closed proper subset of $\mathcal{Y}$. The height of the defining ideal of* $\mathrm{Sing}(\mathcal{Y})$ *is strictly larger than the height of $I_{\mathcal{Y}}$.*
5. *There exists a finite filtration of $\mathcal{Y}$ by arquile closed subsets $Y = \mathcal{W}_0 \supset \mathcal{W}_1 \supset \mathcal{W}_2 \supset \dots \supset \mathcal{W}_k = \emptyset$ so that each locally closed subset $\mathcal{W}_i \setminus \mathcal{W}_{i+1}$ admits a finite covering as in* (1).

Proof: Let $I_1 = \mathcal{I}_{\mathcal{Y}_1}, \dots, I_l = \mathcal{I}_{\mathcal{Y}_l}$ be the associated primes of $I = \mathcal{I}_{\mathcal{Y}}$, and let $\boldsymbol{y} \in \mathcal{Y}$. Since $I = I_1 \cap \dots \cap I_l$ and hence $I_{\mathcal{P}_{\boldsymbol{y}}}$ is radical, $I_{\mathcal{P}_{\boldsymbol{y}}}$ is the intersection $(I_{i_1})_{\mathcal{P}_{\boldsymbol{y}}} \cap \dots \cap (I_{i_k})_{\mathcal{P}_{\boldsymbol{y}}}$ of the localization of those associated primes I_{i_j} which are contained in $\mathcal{P}_{\boldsymbol{y}}$ ([dJP00, p.33]).

(1) The arquile Zariski topology is Noetherian and therefore every open subset is compact (see [Har77b, p.80]). By the previous theorem every regular point has a neighborhood $\mathcal{Y}(f) \setminus \mathcal{Y}(\Delta)$ induced by a regular pair, so these sets form an open covering of $\mathrm{Reg}(\mathcal{Y})$, which can be reduced to a finite covering because $\mathrm{Reg}(\mathcal{Y})$ is compact.
(2) If $\boldsymbol{y} \in \mathcal{Y}_i \cap \mathcal{Y}_j$ for $i \neq j$, then the localization $I_{\mathcal{P}_{\boldsymbol{y}}}$ is not a prime ideal and hence $\mathcal{Y}$ cannot be regular at $\boldsymbol{y}$.
(3) If $\boldsymbol{y} \in \mathcal{Y}_i$ and $\boldsymbol{y}$ is not contained in any other component $\mathcal{Y}_j$, then $I_{\mathcal{P}_{\boldsymbol{y}}} = (I_i)_{\mathcal{P}_{\boldsymbol{y}}}$. Then it follows directly from the definition that $\boldsymbol{y} \in \mathrm{Reg}(\mathcal{Y})$ if and only if $\boldsymbol{y} \in \mathrm{Reg}(\mathcal{Y}_i)$. Conversely, if $\boldsymbol{y} \in \mathrm{Reg}(\mathcal{Y})$, then $I_{\mathcal{P}_{\boldsymbol{y}}}$ is prime. Since $I_{\mathcal{P}_{\boldsymbol{y}}}$ is prime, there exists hence exactly one I_j which is contained in $\mathcal{P}_{\boldsymbol{y}}$ - which means that $\boldsymbol{y} \in \mathcal{Y}_j$ and $\boldsymbol{y} \notin \mathcal{Y}_i$ for $i \neq j$.
(4) Let $I = I_{\mathcal{Y}}$. Assume at first that $\mathcal{Y}$ is irreducible, i.e. that $\mathcal{I}_{\mathcal{Y}}$ is prime and let $m = \mathrm{ht}(I)$. Then by Theorem 1.3.5 there exist $f_1, \ldots, f_m \in I$ so that an $m-$minor Δ of $\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n,y_1,\ldots,y_N)}$ is not contained in I. Since I is U-saturated, this means that there exists a point $\boldsymbol{y} \in \mathcal{Y}(I)$ so that $\Delta(\boldsymbol{y}) \neq 0$. By Lemma 1.3.9, we can assume that Δ is a minor of the partial Jacobian $\partial_y f$. Then the arquile version of the Jacobian criterion (Theorem 1.3.10) implies that $\boldsymbol{y} \in \mathrm{Reg}(\mathcal{Y})$, so $\mathrm{Reg}(\mathcal{Y})$ is non-empty. Denote by $\mathcal{J}_y(I)$ the ideal generated by I and all m-minors of $\partial_y f$, of all $f = (f_1, \ldots, f_m)$ whose components f_i are contained in I. The singular locus $\mathrm{Sing}(\mathcal{Y})$ is then given by $\mathcal{Y}_U(\mathcal{J}(I))$ (Theorem 1.3.10) and is hence closed. Since $I \subsetneq \mathcal{J}_y(I) \subset \mathcal{I}_{\mathrm{Sing}(\mathcal{Y})}$, the height of $\mathcal{I}_{\mathrm{Sing}(\mathcal{Y})}$ is strictly larger than the height of I, which shows (2) if $I = \mathcal{I}_{\mathcal{Y}}$ is irreducible. Now let I be arbitrary. The singular locus is composed of the union of the singular loci $\mathrm{Sing}(\mathcal{Y}_i)$ of the irreducible components and the pairwise intersection $\mathcal{Y}_i \cap \mathcal{Y}_j$ $(i \neq j)$ (because of (2) and (3)), so

$$\begin{aligned}\mathrm{Sing}(\mathcal{Y}) &= \bigcup_i \mathrm{Sing}(\mathcal{Y}_i) \cup \bigcup_{i \neq j} \mathcal{Y}_i \cap \mathcal{Y}_j = \bigcup_i \mathcal{Y}_U(\mathcal{J}_y(I_i)) \cup \bigcup_{i \neq j} \mathcal{Y}_U(I_i + I_j) = \\ &= \mathcal{Y}_U\left(\bigcap_i \bigcap_{i \neq j} (I_i + I_j) \cap \mathcal{J}_y(I_i)\right).\end{aligned}$$

As $I_1 \cap \cdots \cap I_l$ is irredundant, the height of $I_i + I_j$ must be strictly larger than $\max\{\mathrm{ht}(I_i), \mathrm{ht}(I_j)\}$. We have already seen that $\mathrm{ht}(\mathcal{J}_y(I_i)) > \mathrm{ht}(I_i)$. Therefore the height of $(I_i + I_j) \cap \mathcal{J}_y(I_i)$ is strictly larger than $\mathrm{ht}(I) = \min \mathrm{ht}(I_i)$, and the same holds for the intersection of all $(I_i + I_j) \cap \mathcal{J}_y(I_i)$. The height of the saturated ideal can be only larger, whence $\mathrm{ht}(I_{\mathrm{Sing}(\mathcal{Y})}) > \mathrm{ht}(I_{\mathcal{Y}})$.

(5) Set $\mathcal{W}_1 = \operatorname{Sing}(\mathcal{Y})$, $\mathcal{W}_2 = \operatorname{Sing}(\mathcal{W}_2) \subset \mathcal{W}_1$ and so on. This descending chain of closed subsets becomes eventually stationary. Since $\mathcal{W}_i \setminus \mathcal{W}_{i+1} = \mathcal{W}_i \setminus \operatorname{Sing}(\mathcal{W}_i)$, the finite covering by open sets induced by regular system follows directly from (1). $\square$

Corollary 1.3.14. *Let $\mathcal{Y}$ be an arquile variety, let $\boldsymbol{y}$ be a regular point of $\mathcal{Y}$, let $\mathcal{Y}_i$ be the irreducible component which contains $\mathcal{Y}$ and let $m = \operatorname{ht}(\mathcal{Y}_i)$. Then there exists an injective arquile map $\Psi_\infty\colon W \to \mathcal{Y}_i$, defined on an open subset W of $\mathbb{C}_0\langle\langle x\rangle\rangle^{N-m}$, which has dense image in $\mathcal{Y}_i$.*

Proof: Since $\mathcal{Y}$ is regular at $\boldsymbol{y}$, it follows from the previous theorem that $\boldsymbol{y} \in \operatorname{Reg}(\mathcal{Y}_i)$. Let (f, Δ), $f = (f_1, \dots, f_m)$ be a regular pair for $\mathcal{Y}_i$ at $\boldsymbol{y}$. Then by Theorem 1.2.4 there exists an injective arquile map $\Psi_\infty\colon \mathbb{C}_0\langle\langle x\rangle\rangle^{N-m} \to \mathcal{Y}(f) \setminus \mathcal{Y}(\Delta)$. The restriction of Ψ_∞ to $W = \Psi_\infty^{-1}(U)$ maps W to $\mathcal{Y}_U(f) \setminus \mathcal{Y}(\Delta) = \mathcal{Y}_i \setminus \mathcal{Y}(\Delta)$. Let ε_Ψ be defined as in Theorem 1.2.4, $\mathcal{K} = \ker(\varepsilon_\Psi)$. Then $\langle f_1, \dots, f_m\rangle_{\mathcal{P}_{\boldsymbol{y}}} = \mathcal{K}_{\mathcal{P}_{\boldsymbol{y}}} = (I_i)_{\mathcal{P}_{\boldsymbol{y}}}$ by Theorem 1.2.4(5) and Theorem 1.3.10(2), and therefore $I_i = \mathcal{K}$. Suppose now that $g \in \mathbb{C}\langle\langle x, y\rangle\rangle$ vanishes on $\Psi_\infty(W)$. Then $g(x, \Psi(x, \boldsymbol{z}(x))) = 0$ for all $\boldsymbol{z}(x) \in W$, hence $g(x, \Psi(x, z)) = 0$ because of the Identity Theorem Theorem 1.2.2, hence $g \in \mathcal{K}$ and thus $\mathcal{K} = \mathcal{I}_{I_{\Psi_\infty(W)}}$. Therefore $\mathcal{Y}_i = \mathcal{Y}_U(\mathcal{K}) = \mathcal{Y}_U(\mathcal{I}_{\Psi_\infty(W}) = \overline{\Psi_\infty(W)}$, as claimed. $\square$

Corollary 1.3.15. *The regular locus of an arquile variety $\mathcal{Y} \subset U$ is Zariski-dense in $\mathcal{Y}$.*

Proof: Assume at first that $\mathcal{Y}$ is irreducible in U, so that $\mathcal{I}_\mathcal{Y}$ is prime. Because of the previous theorem we know that $\operatorname{Sing}(\mathcal{Y})$ is a proper arquile subset of $\mathcal{Y}$. Since $\mathcal{Y}$ is irreducible, it follows that $\operatorname{Reg}(\mathcal{Y}) = \mathcal{Y} \setminus \operatorname{Sing}(\mathcal{Y})$ is dense in $\mathcal{Y}$. Let now $\mathcal{Y}$ be arbitrary, $\operatorname{Ass}(I) = \{ I_1, \dots, I_l \}$, and $\mathcal{Y}_i = \mathcal{Y}_U(I_i)$ the irreducible components. We claim that $\operatorname{Sing}(\mathcal{Y}) \cap \mathcal{Y}_i$ is a proper subset of $\mathcal{Y}_i$ for every $1 \leq i \leq l$. Assume contrarily that there exists a component $\mathcal{Y}_{i_0}$ such that $\mathcal{Y}_{i_0} \subset \operatorname{Sing}(\mathcal{Y})$. WLOG $i_0 = 1$. In particular, this means that the regular locus of $\mathcal{Y}_1$ is completely covered by the other components (Theorem 1.3.13(4)), so we can write $\mathcal{Y}$ as $\operatorname{Sing}(\mathcal{Y}_1) \cup \mathcal{Y}_2 \cup \cdots \cup \mathcal{Y}_l$ and further as $\mathcal{Y}_{1,1} \cup \cdots \cup \mathcal{Y}_{1,s} \cup \mathcal{Y}_2 \cup \cdots \cup \mathcal{Y}_l$, where $\mathcal{Y}_{1,1}, \dots, \mathcal{Y}_{1,s}$ are the irreducible components of $\operatorname{Sing}(\mathcal{Y}_1)$. This decomposition can be reduced to an irredundant decomposition, and the uniqueness implies that $\mathcal{Y}_1 = \mathcal{Y}_{1,j}$ for some $1 \leq j \leq s$. But then we obtain the contradiction $\mathcal{Y}_1 \supsetneq \operatorname{Sing}(\mathcal{Y}_1) \supset \mathcal{Y}_1$, which shows that $\mathcal{Y}_i \setminus \operatorname{Sing}(\mathcal{Y}) \neq \emptyset$. So

$$\overline{\operatorname{Reg}(\mathcal{Y})} = \overline{\mathcal{Y} \setminus \operatorname{Sing}(\mathcal{Y})} = \bigcup_{i=1}^{l} \overline{\mathcal{Y}_i \setminus \operatorname{Sing}(\mathcal{Y})} = \bigcup_{i=1}^{l} \mathcal{Y}_i = \mathcal{Y}.$$

$\square$

1.3.1 Characterization of saturated ideals and a dual version of AAT

Theorem 1.3.16. *A prime ideal $I \subset \mathbb{C}\langle\langle x, y\rangle\rangle$ is U-saturated if and only if there exists a point $\boldsymbol{y} \in \mathcal{Y}_U(I)$ so that $(\mathbb{C}\langle\langle x, y\rangle\rangle/I)_{\mathcal{P}_{\boldsymbol{y}}}$ is regular.*

Proof: Assume at first that I is saturated, so that $I = \mathcal{I}_{\mathcal{Y}_U(I)}$. By Proposition 1.3.11, the regular locus of $\mathcal{Y}_U(I)$ is non-empty, so there exists a $\boldsymbol{y} \in \mathcal{Y}_U(I)$ so that $(\mathbb{C}\langle\langle x, y\rangle\rangle/I)_{\mathcal{P}_{\boldsymbol{y}}}$ is regular. Conversely, suppose that there exists a point $\boldsymbol{y} \in \mathcal{Y}_U(I)$ so that $(\mathbb{C}\langle\langle x, y\rangle\rangle/I)_{\mathcal{P}_{\boldsymbol{y}}}$ is regular. By Theorem 1.3.10 $I_{\mathcal{P}_y}$ is generated by $f_1, \ldots, f_m \in I$, which satisfy that an m-minor of $\partial_y(f)$, $f = (f_1, \ldots, f_m)$, does not vanish at $\boldsymbol{y}$. By Theorem 1.2.4, the local ideal $\langle f_1, \ldots, f_m\rangle_{\mathcal{P}_{\boldsymbol{y}}}$ is the localization $\mathcal{K}_{\mathcal{P}_{\boldsymbol{y}}}$ of a U-saturated prime $\mathcal{K} \subset \mathbb{C}\langle\langle x, y\rangle\rangle$ of height m. But since the prime ideals of $\mathbb{C}\langle\langle x, y\rangle\rangle$ included in $\mathcal{P}_{\boldsymbol{y}}$ are in a one-to-one correspondence with the primes of $\mathbb{C}\langle\langle x, y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}}$, $\mathcal{K} = I$ and in particular, I is U-saturated. □

Corollary 1.3.17. *A radical ideal $I \subset \mathbb{C}\langle\langle x, y\rangle\rangle$ is U-saturated if and only there exists for every associated prime I_i of I a point $\boldsymbol{y} \in \mathcal{Y}_U(I_i)$ such that $(\mathbb{C}\langle\langle x, y\rangle\rangle/I)_{\mathcal{P}_{\boldsymbol{y}}}$ is regular.*

Proof: Let $I = I_1 \cap \cdots \cap I_s$ be the decomposition of I into its associated primes. By Theorem 1.1.3, I is saturated if and only if all I_j are saturated. Then the characterization follows immediately from the previous theorem. □

If the generators of the ideal I do not depend on the parameter x, then the curve selection lemma can be applied to conclude that not all solutions to $f(\boldsymbol{y}) = 0$ are contained in $\mathrm{Sing}(\mathcal{Y}(I))$. This shows that I is saturated. However, by Example 1.3.2, I might not be U-saturated if U is an $\mathfrak{m}$-adic neighborhood U of a point $\boldsymbol{y} \in \mathrm{Sing}(\mathcal{Y})$.

Corollary 1.3.18. *Let $I \subset \mathbb{C}\langle\langle x, y\rangle\rangle$ be an ideal which is generated by series that do not depend on x, let $J = I \cap \mathbb{C}\langle\langle y\rangle\rangle$.*

(1) *If I is a radical ideal, then I is saturated.*
(2) *If I is prime, then $\mathcal{Y}(I)$ is irreducible.*
(3) *If $J \subset \mathbb{C}[y]$ is radical, then I is saturated and $\mathcal{Y}(I)$ is irreducible if and only if J is analytically irreducible.*

Proof: (1), (2) Let $J = I \cap \mathbb{C}\langle\langle y\rangle\rangle$, let $J = J_1 \cap \cdots \cap J_s$ the decomposition of J into its associated primes and let X_i be the irreducible analytic space germ defined by J_i. Because of the curve selection lemma ([Mil68]), we can choose for every component X_i an arc $\boldsymbol{y}(x_1) \in \mathbb{C}_0\langle\langle x_1\rangle\rangle^N$, which does not lie completely in the singular locus of X_i.

The Jacobian Criterion together with the fact that $\mathrm{ht}(J_i) = \mathrm{ht}(J_i \cdot \mathbb{C}\langle\langle x,y\rangle\rangle)$ implies that $(\mathbb{C}\langle\langle x,y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}}/I_i\mathbb{C}\langle\langle x,y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}})$ is regular, which by the previous theorem implies that $J_i \cdot \mathbb{C}\langle\langle x,y\rangle\rangle$ is saturated. By assumption, $I = J \cdot \mathbb{C}\langle\langle x,y\rangle\rangle$. The associated primes of I are $I_i = J_i\mathbb{C}\langle\langle x,y\rangle\rangle$ of $I = J \cdot \mathbb{C}\langle\langle x,y\rangle\rangle$ are therefore all saturated, and thus I is saturated by Theorem 1.1.3.
(3) $I = J \cdot \mathbb{C}\langle\langle x,y\rangle\rangle$ is a radical ideal whenever J is a radical ideal. The irreducible components of $\mathcal{Y}(I)$ stem from the components of the analytic space germ defined by J in $(\mathbb{C}^N, 0)$, which shows the second claim. $\square$

Let $\Phi\colon \mathbb{C}\{z\} \to \mathbb{C}\{w\}$ be a $\mathbb{C}$-algebra homomorphism between two rings of convergent power series, given by substitution of z by a vector of convergent series $\varphi(w)$. The morphism naturally extends to $\widehat{\Phi}\colon \mathbb{C}[|z|] \to \mathbb{C}[|w|]$. Grothendieck ([Gro62]) asked whether it is true that the kernel of $\widehat{\Phi}$ is generated by the convergent relations, the kernel of Φ. A. Gabrielov gave a counter-example in ([Gab71]), and later ([Gab73]) gave conditions in terms of the generic rank of φ, under which the problem admits a positive answer.

Theorem 1.3.19 (Gabrielov, [Gab73], [BZ79, p.47])**.** *Let $R = \mathbb{C}\{z\},\ S = \mathbb{C}\{w\}$ be rings of convergent power series, $z = (z_1,\dots,z_n), w = (w_1,\dots,w_k)$ and let $\Phi\colon R \to S$, be the $\mathbb{C}$-algebra homomorphism $f(z) \mapsto f(\varphi(w))$. Denote by $\widehat{\Phi}\colon \mathbb{C}[|z|] \to \mathbb{C}[|w|]$ the extension of Φ to the completions of R and S. Set*

$$r_1 = \mathrm{grk}(\Phi),\ r_2 = \dim\left(\mathbb{C}[|z|]\big/_{\ker(\widehat{\Phi})}\right),\ r_3 = \dim\left(\mathbb{C}\{z\}\big/_{\ker(\Phi)}\right)$$

Then $r_1 \le r_2 \le r_3$ and if $r_1 = r_2$, then $r_2 = r_3$ and $\widehat{\ker(\Phi)} = \ker\widehat{\Phi}$.

We note that if Φ is induced by algebraic series, and r_4 denotes the dimension of the quotient of $\mathbb{C}\langle z\rangle$ by the kernel of the restriction of Φ to $\mathbb{C}\langle z\rangle$, then $r_3 \le r_4$. Describing the kernel of a morphism Φ between analytic rings can be reformulated as a Nested Artin Approximation problem involving the coordinates of φ. Since the Nested Artin Approximation Problem admits a positive answer in the case of algebraic series (as a consequence of Popescu's General Néron Desingularization Theorem), Grothendieck's question can be answered in the affirmative in the algebraic case. In [Ron13] G. Rond shows that if Φ is induced by algebraic series, then $r_1 = r_4$ and as a consequence we obtain that the convergent, formal relations, respectively, are generated by the algebraic relations.

We now turn to a similar problem. Let $W \subset \mathbb{C}_0\{x\}^N$ be a set of convergent power series, then is it true that the analytic relations $\mathcal{I}_W$ on W generate the formal relations $\widehat{\mathcal{I}}_W$? If $W = \{\boldsymbol{y}\}$ consists only of one point, then the question is easily answered in the affirmative: Consider $\Phi = \varepsilon_{\boldsymbol{y}} \colon \mathbb{C}\{x, y\} \to \mathbb{C}\{x\}, f(x, y) \mapsto f(x, \boldsymbol{y}(x))$. Then $\ker \widehat{\Phi} = \widehat{\ker \Phi}$ follows already from the fact that $\mathbb{C}\{x\} \to \mathbb{C}\,[|x|]$ is a flat extension. If $\mathcal{Y}$ is an irreducible arquile variety, then a Zariski-dense subset of $\mathcal{Y}$ can be parametrized by an arquile map Ψ_∞ (Theorem 1.2.4), and dually $\mathcal{I}_\mathcal{Y}$ is given by the kernel of the associated morphism ε_Ψ. In this manner we can relate our question to Gabrielov's Theorem.

Theorem 1.3.20. *Let $\mathbb{C}\langle\langle x\rangle\rangle \in \{\mathbb{C}\{x\}, \mathbb{C}\langle x\rangle\}$ be either the ring of algebraic or convergent power series. Let $\mathcal{Y}$ be an arquile variety in an $\mathfrak{m}$-adically open subset U of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$. Then the ideal of relations on $\mathcal{Y}$ within $\mathbb{C}\langle\langle x, y\rangle\rangle$ generates the ideal of formal relations on $\mathcal{Y}$, that is,*

$$\widehat{\mathcal{I}}_\mathcal{Y} = \mathcal{I}_\mathcal{Y} \cdot \mathbb{C}\,[|x, y|]\,.$$

Proof: Assume at first that $\mathcal{I}_\mathcal{Y}$ is prime. Let $\boldsymbol{y} \in \operatorname{Reg}(\mathcal{Y})$ (Theorem 1.3.13) and let (f, Δ) be a regular system for $\mathcal{Y}$ at $\boldsymbol{y}$. Let $\Psi(x, z)$ be as in Theorem 1.2.4 through $\boldsymbol{y}$. Then as in Theorem 1.3.16 it follows that $\mathcal{I}_\mathcal{Y} = \ker(\varepsilon_\Psi)$, where $\varepsilon_\Psi \colon \mathbb{C}\langle\langle x, y\rangle\rangle \to \mathbb{C}\langle\langle x, z\rangle\rangle, f(x, y) \mapsto f(x, \Psi(x, z))$. Let $g(x, y) \in \widehat{\mathcal{I}}_\mathcal{Y}$. Then $g(x, \Psi(x, \boldsymbol{z}(x))) = 0$ for all $\boldsymbol{z} \in \Psi_\infty^{-1}(U)$, since $\Psi(x, \boldsymbol{z}(x)) \in \mathcal{Y}$. The Identity Theorem (Theorem 1.2.2) implies then that $g(x, \Psi(x, z)) = 0$ and therefore $\widehat{\mathcal{I}}_\mathcal{Y} \subset \ker(\widehat{\varepsilon_\Psi})$. By Theorem 1.2.4

$$\operatorname{grk}(\varepsilon_\Psi) = n + N - m = \dim\left(\mathbb{C}\langle\langle x, y\rangle\rangle \big/ \ker(\varepsilon_\Psi)\right),$$

and thus ε_Ψ satisfies Gabrielov's rank condition. So we can apply Gabrielov's theorem (Theorem 1.3.19) to conclude that $\widehat{\ker(\varepsilon_\Psi)} = \ker(\widehat{\varepsilon_\Psi})$ and therefore $\widehat{\mathcal{I}}_\mathcal{Y} \subset \mathcal{I}_\mathcal{Y} \cdot \mathbb{C}\,[|x, y|]$. The converse inclusion is trivially satisfied and hence $\widehat{\mathcal{I}}_\mathcal{Y} = \mathcal{I}_\mathcal{Y} \cdot \mathbb{C}\,[|x, y|]$. Now the general case. Let $\mathcal{I}_\mathcal{Y} = \mathcal{I}_{\mathcal{Y}_1} \cap \cdots \cap \mathcal{I}_{\mathcal{Y}_s}$ be the decomposition of $\mathcal{I}_\mathcal{Y}$ into its associated primes and let $\mathcal{Y}_U(I_i) = \mathcal{Y}_i$ be the irreducible components of $\mathcal{Y}$. Then

$$\widehat{\mathcal{I}}_\mathcal{Y} = \widehat{\mathcal{I}}_{\mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_s} = \bigcap_{i=1}^{s} \widehat{\mathcal{I}}_{\mathcal{Y}_i} = \bigcap_{i=1}^{s} \left(\mathcal{I}_{\mathcal{Y}_i} \cdot \mathbb{C}\,[|x, y|]\right) \overset{(\star)}{=} \left(\bigcap_{i=1}^{s} \mathcal{I}_{\mathcal{Y}_i}\right) \cdot \mathbb{C}\,[|x, y|] = \mathcal{I}_\mathcal{Y} \cdot \mathbb{C}\,[|x, y|]$$

where $(\star)$ follows from the fact that $\left(\bigcap_{i=1}^{k} J_i\right) \cdot \mathbb{C}\,[|x, y|] = J_1 \cdot \mathbb{C}\,[|x, y|] \cap \cdots \cap J_s \cdot \mathbb{C}\,[|x, y|]$ for arbitrary ideals $J_i \subset \mathbb{C}\langle\langle x, y\rangle\rangle$ (see [Rui93, p.100]). $\square$

1.3.2 Weak Regularity

Example 1.3.2 shows that singular points may be isolated from regular points in the $\mathfrak{m}$-adic topology. To remedy this problem, we introduce a weaker notion of regularity, which is a local concept with respect to the $\mathfrak{m}$-adic topology.

Definition 1.3.21. Let U be an $\mathfrak{m}$-open subset of $\mathfrak{m}\mathbb{C}\langle\langle x\rangle\rangle^N$ and let $\mathcal{Y} \subset U$ be an arquile subset. We say that $\mathcal{Y}$ is *weakly regular* at $\boldsymbol{y} \in W$, if there exists an $\mathfrak{m}$-adic neighborhood W of $\boldsymbol{y}$, $f_1, \ldots, f_m \in \mathbb{C}\langle\langle x, y\rangle\rangle$, so that

$$\mathcal{Y}_W(f_1, \ldots, f_m) = \mathcal{Y} \cap W$$

and such that an $m-$minor Δ of $\partial_y(f)$ vanishes nowhere on W. So $\mathcal{Y}$ is weakly regular at $\boldsymbol{y} \in \mathcal{Y}$, if $\mathcal{Y}$ can be defined in an $\mathfrak{m}$-adic neighborhood by a regular system. As expected, regularity implies weak regularity, but the converse is not true (Example 1.3.23).

Lemma 1.3.22. *If $\boldsymbol{y}$ is a regular point of the arquile subset $\mathcal{Y}$ in U, then $\mathcal{Y}$ is weakly regular at $\boldsymbol{y}$.*

Proof: By Proposition 1.3.11, there exists a regular pair (f, Δ) so that $\mathcal{Y}_U(f)\backslash\mathcal{Y}(\Delta) = \mathcal{Y} \setminus \mathcal{Y}(\Delta)$. As Δ does not vanish at $\boldsymbol{y}$, there exists an $\mathfrak{m}$-adic neighborhood W of $\boldsymbol{y}$ in U on which Δ stays non-zero. But then

$$W \cap \mathcal{Y}(f) = W \cap (\mathcal{Y}_U(f) \setminus \mathcal{Y}(\Delta)) = W \cap \mathcal{Y}_i,$$

and therefore $\mathcal{Y}$ is weakly regular at $\boldsymbol{y}$. $\square$

Example 1.3.23. The Whitney umbrella (Example 1.3.2) $\mathcal{Y} = \mathcal{Y}(y_1^2 - y_2 y_3^2)$ provides an example of a weakly regular but non-regular point. We have seen that $(0, t, 0) \in \mathrm{Sing}(\mathcal{Y})$. But $\mathcal{Y} \cap U_n$ is given by the equations $y_1 = 0, y_3 = 0$, where $U_n = (0, t, 0) + t^n \cdot \mathbb{C}\langle\langle t\rangle\rangle^3,\ n \geq 2$. So $\mathcal{Y}$ is weakly regular at $\boldsymbol{y}$.

Definition 1.3.24. The local ideal $\mathcal{I}_{(\mathcal{Y},\boldsymbol{y})}$ of an arquile variety $\mathcal{Y}$ at $\boldsymbol{y} \in \mathcal{Y}$ is given by all $g \in \mathbb{C}\langle\langle x, y\rangle\rangle$ which vanish some $\mathfrak{m}$-adic neighborhood of $\boldsymbol{y}$ in $\mathcal{Y}$. It corresponds to the defining ideal of the germ (with respect to $\mathfrak{m}$-adic local equivalence) of $\mathcal{Y}$ at $\boldsymbol{y}$.

The Noethernity of $\mathbb{C}\langle\langle x, y\rangle\rangle$ implies that $\mathcal{I}_{(\mathcal{Y},\boldsymbol{y})} = \mathcal{I}_{W\cap\mathcal{Y}}$ for every sufficiently small neighborhood W of $\boldsymbol{y}$. Indeed, the set of ideals $\{\mathcal{I}_{\mathcal{Y}\cap W} \mid W \text{ is an } \mathfrak{m} - \text{neighborhood of } \boldsymbol{y}\}$ has a maximal element $\mathcal{I}_{\mathcal{Y}\cap W_0}$. If now $W \subset W_0$, then $\mathcal{I}_{W\cap\mathcal{Y}} \supset \mathcal{I}_{W_0\cap\mathcal{Y}}$, and therefore

$\mathcal{I}_{W\cap\mathcal{Y}} = \mathcal{I}_{W_0\cap\mathcal{Y}}$ because $\mathcal{I}_{W_0\cap\mathcal{Y}}$ is maximal. It follows that $\mathcal{I}_{(\mathcal{Y},\boldsymbol{y})} = \mathcal{I}_{\mathcal{Y}\cap W}$ for every neighborhood W contained in W_0. Also weak regularity can be characterized by an adopted Jacobian criterion.

Theorem 1.3.25. *Let U be an $\mathfrak{m}$-open subset of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ and let $\mathcal{Y}$ be an arquile subset of U. The following are equivalent:*

(1) *$\mathcal{Y}$ is weakly regular at $\boldsymbol{y}$.*

(2) $\mathbb{C}\langle\langle x,y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}}\Big/\mathcal{I}_{(\mathcal{Y},\boldsymbol{y})}\cdot\mathbb{C}\langle\langle x,y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}}$ *is a regular local ring.*

Proof: Pick a sufficiently small neighborhood W_0 of $\boldsymbol{y}$ such that $\mathcal{I}_{W\cap\mathcal{Y}} = \mathcal{I}_{(\mathcal{Y},\boldsymbol{y})}$ for every neighborhood $W \subset W_0$ and let $\mathcal{Z} = W_0 \cap \mathcal{Y}$ be the corresponding arquile subset of W_0. (2)$\Rightarrow$(1) Then $\boldsymbol{y}$ is a regular point of $\mathcal{Z}$ (Theorem 1.3.10) and hence a weakly regular point of $\mathcal{Z}$ (Lemma 1.3.22). So $\mathcal{Z}$ and therefore $\mathcal{Y}$ is defined in an $\mathfrak{m}$-adic neighborhood of $\boldsymbol{y}$ by a regular system, which means that $\mathcal{Y}$ is weakly regular at $\boldsymbol{y}$.
(1) $\Rightarrow$ (2) Let $W \subset W_0$ be a neighborhood of $\boldsymbol{y}$ so that $\mathcal{Y}\cap W = \mathcal{Y}(f_1,\ldots,f_m)\cap W$ and an m-minor Δ of $\partial_y(f)$, $f = (f_1,\ldots,f_m)$, vanishes nowhere on W. In particular, $\mathcal{Y}\cap W = \mathcal{Y}\cap W\setminus\mathcal{Y}(\Delta) = (\mathcal{Y}(f_1,\ldots,f_m)\cap W)\setminus\mathcal{Y}(\Delta)$. It follows now from Proposition 1.3.11(4) that $\boldsymbol{y}$ is a regular point of the arquile subset $\mathcal{Y}\cap W$ of W, which means that $\mathbb{C}\langle\langle x,y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}}/\mathcal{I}_{\mathcal{Y}\cap W}\mathbb{C}\langle\langle x,y\rangle\rangle_{\mathcal{P}_{\boldsymbol{y}}}$ is regular. The fact $\mathcal{I}_{\mathcal{Y}\cap W} = \mathcal{I}_{(\mathcal{Y},\boldsymbol{y})}$ concludes the proof. $\square$

Theorem 1.3.26. *The weakly regular points of an arquile variety $\mathcal{Y} \subset U$ are dense in $\mathcal{Y}$.*

Proof: Let $\boldsymbol{y} \in \mathcal{Y}$ and let W be an arbitrary $\mathfrak{m}$-adic neighborhood of $\boldsymbol{y}$. By Proposition 1.3.11, the regular locus of $(W\cap\mathcal{Y})$ is non-empty. Let $\boldsymbol{y}_1 \in \mathrm{Reg}(W\cap\mathcal{Y})$. Then $(W\cap\mathcal{Y})$ and thus $\mathcal{Y}$ is weakly regular at $\boldsymbol{y}_1$. Since W was arbitrary, the claim follows. $\square$

1.4 Factorization of the regular locus

In this section, we deal with the parametrization of the solution set of a regular system $\mathcal{Y}(f)\setminus\mathcal{Y}(\Delta)$, where Δ is a maximal minor of $\partial_y f$. The open subset $\mathbb{C}_0\langle\langle x\rangle\rangle^N\setminus\mathcal{Y}(\Delta)$ is partitioned into the strata $\mathcal{S}_d$, on which the x_n-order of Δ is d. Each stratum $\mathcal{S}_d$ is then partitioned into a family $(\mathcal{V}_z)_{z\in\mathcal{R}_d}$ of $\mathbb{C}\langle\langle x\rangle\rangle$-submodules, along which f_∞ can be linearized. The family of submodules can be trivialized to a product $\mathcal{Z}_d\times\mathbb{C}\langle\langle x\rangle\rangle^N$ of a textile cofinitely locally closed subset $\mathcal{Z}_d$ of a module $\mathbb{C}\langle\langle x'\rangle\rangle$ over the power series space $\mathbb{C}\langle\langle x'\rangle\rangle$ in one variable less and a free module $\mathbb{C}\langle\langle x\rangle\rangle^N$. We will show then that $\mathcal{Y}(f)\cap\mathcal{S}_d$

is isomorphic to a product $\mathcal{Z}_d^* \times \mathbb{C}\langle\langle x\rangle\rangle^{N-m}$ of a local arquile subvariety $\mathcal{Z}_d^*$ of $\mathbb{C}\langle\langle x'\rangle\rangle^{N'}$ with $\mathbb{C}\langle\langle x\rangle\rangle^{N-m}$, where m is the co-dimension of $\mathcal{Y}(f)$. This construction follows the proof of Płoski's desingularization theorem ([Pło74], [Pło15]). The variety $\mathcal{Z}_d^*$ may be singular, but since the weakly regular points of an arquile variety are dense in the arquile topology, this construction can be generically iterated to locally factorize $\mathcal{Z}_d^*$ into a product of a variety of a power series space in two variables less and so forth. This will show that at a generic point an arquile variety is locally isomorphic to a nested product $\mathfrak{m}_1^{\mu_1}\mathbb{C}\langle\langle x_1\rangle\rangle^{N_1} \times \mathfrak{m}_2^{\mu_2}\mathbb{C}\langle\langle x_1, x_2\rangle\rangle^{N_2} \times \cdots \times \mathfrak{m}_n^{\mu_n}\mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle^{N_n}$, where $\mathfrak{m}_k = \langle x_1, \ldots, x_k\rangle$ denotes the maximal ideal of $\mathbb{C}\langle\langle x_1, \ldots, x_k\rangle\rangle$.

Definition 1.4.1. Let $f = (f_1, \ldots, f_m) \in \mathfrak{m}\mathbb{C}\langle\langle x, y\rangle\rangle^m$ and assume that $m \leq N$. Fix a minor Δ of the partial Jacobian $\partial_y(f)$. After reordering the y-coordinates we may assume that

$$\Delta = \det\left(\frac{\partial(f_1, \ldots, f_m)}{\partial(y_{N-m+1}, \ldots, y_N)}\right).$$

We split y accordingly into two variable blocks: Set $y^{(1)} = (y_1, \ldots, y_{N-m})$ and $y^{(2)} = (y_{N-m+1}, \ldots, y_N)$, so that $y = (y^{(1)}, y^{(2)})$. The partial y-Jacobians are then

$$\partial_{y^{(1)}}(f) := \frac{\partial(f_1, \ldots, f_m)}{\partial(y_1, \ldots, y_{N-m})} = \frac{\partial f}{\partial y^{(1)}}$$
$$\partial_{y^{(2)}}(f) := \frac{\partial(f_1, \ldots, f_m)}{\partial(y_{N-m+1}, \ldots, y_N)} = \frac{\partial f}{\partial y^{(2)}}$$

For $\boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N$ let $D(\boldsymbol{y})$ be the diagonal matrix

$$D(\boldsymbol{y}) := \operatorname{diag}(\underbrace{\Delta^2(\boldsymbol{y}), \ldots, \Delta^2(\boldsymbol{y})}_{N-m \text{ times}}, \underbrace{g(\boldsymbol{y}), \ldots, g(\boldsymbol{y})}_{m \text{ times}})$$

and let $\mathcal{V}_{\boldsymbol{y}}$ be the affine submodule of $\mathbb{C}\langle\langle x\rangle\rangle^N$ (as in Theorem 1.2.4)

$$\mathcal{V}_{\boldsymbol{y}} := \boldsymbol{y} + \begin{pmatrix} \Delta^2(\boldsymbol{y}) \cdot \mathbb{C}_0\langle\langle x\rangle\rangle^{N-m} \\ \Delta(\boldsymbol{y}) \cdot \mathbb{C}_0\langle\langle x\rangle\rangle^m \end{pmatrix} = \boldsymbol{y} + D(\boldsymbol{y}) \cdot \mathbb{C}_0\langle\langle x\rangle\rangle^N.$$

We will write elements of $\mathcal{V}_{\boldsymbol{y}}$ as $\boldsymbol{y} + \boldsymbol{v}$, and $\boldsymbol{v} = D(\boldsymbol{y}) \cdot \boldsymbol{a}$. A key observation is that even though the modules $\mathcal{V}_{\boldsymbol{y}}$ depend in a nonlinear way on $\boldsymbol{y}$, the module $\mathcal{V}_{\boldsymbol{y}}$ does not change along itself, that is, $\mathcal{V}_{\boldsymbol{y}+\boldsymbol{v}} = \mathcal{V}_{\boldsymbol{y}}$ for all $\boldsymbol{y}$ and $\boldsymbol{v} = D(\boldsymbol{y}) \cdot \boldsymbol{a}$. Let $v = (v_1, \ldots, v_N)$ and $a = (a_1, \ldots, a_N)$ be two pairs of new variables.

Theorem 1.4.2. *Let Δ and $\mathcal{V}_{\boldsymbol{y}}$ be defined as above and assume that $\Delta \in \mathbb{C}_0\langle\langle x, y\rangle\rangle$.*

(1) *There exists a unit $u(x, z, a)$ so that $\Delta(x, z + D(x, z)a) = u(x, z, a) \cdot \Delta(x, z)$.*

(2) *For any $\boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N, \boldsymbol{v} = D(\boldsymbol{y}) \cdot \boldsymbol{a} \in \mathbb{C}\langle\langle x\rangle\rangle^N$,*

$$\mathcal{V}_{\boldsymbol{y}+\boldsymbol{v}} = \mathcal{V}_{\boldsymbol{y}}. \tag{1.3}$$

Proof: We expand $\Delta(x, z + v) = \Delta(x, z) + L(x, z) \cdot v + q(x, z, v)$. Let us substitute $v = D(x, z)a$. Since $q(x, z, v)$ is quadratic in v, we can factor it into $\tilde{q}(x, z, a)\Delta(x, z)^2$, where $\tilde{q}$ is quadratic in a. We factor $D(x, z)$ into $\Delta(x, z) \cdot \tilde{D}(x, z)$, where $\tilde{D}(x, z) = \operatorname{diag}(\Delta(x, z), \ldots, \Delta(x, z), 1, \ldots, 1)$. Hence

$$\begin{aligned}\Delta(x, z + v) &= \Delta(x, z) + L(x, z) \cdot D(x, z) \cdot a + \Delta(x, z)^2\tilde{q}(x, z, a) = \\ &= \Delta(x, z) \underbrace{\left(1 + L(x, z)\tilde{D}(x, z)a + \Delta(x, z)\tilde{q}(x, z, a)\right)}_{=:u(x,z,a)}.\end{aligned}$$

The series $u(x, z, a)$ is a power series with non-vanishing constant term and hence a unit. In particular, $D(\boldsymbol{y})$ and $D(\boldsymbol{y} + \boldsymbol{v})$ coincide up multiplication with an invertible diagonal matrix and consequently the modules generated by the columns of both matrices coincide. Since $\boldsymbol{v} = D(\boldsymbol{y})\boldsymbol{a}$,

$$\begin{aligned}\mathcal{V}_{\boldsymbol{y}+\boldsymbol{v}} = \boldsymbol{y} + D(\boldsymbol{y})\boldsymbol{a} + D(\boldsymbol{y} + \boldsymbol{v})\mathbb{C}_0\langle\langle x\rangle\rangle^N &= \boldsymbol{y} + D(\boldsymbol{y})\boldsymbol{a} + D(\boldsymbol{y}) \cdot \mathbb{C}_0\langle\langle x\rangle\rangle^N = \\ &= \boldsymbol{y} + D(\boldsymbol{y}) \cdot \mathbb{C}_0\langle\langle x\rangle\rangle^N = \mathcal{V}_{\boldsymbol{y}},\end{aligned}$$

and the claim follows. $\square$

Remark. The statement of the previous theorem is true for any module of the form $\boldsymbol{y} + G(\boldsymbol{y}) \cdot \mathbb{C}_0\langle\langle x\rangle\rangle^N$, where $G(\boldsymbol{y})$ is a diagonal module $\langle g(\boldsymbol{y})^{p_1}\rangle \times \cdots \times \langle g(\boldsymbol{y})^{p_N}\rangle$.

Theorem 1.4.3. *There exists a series $h(x, z, a) \in \mathbb{C}_0\langle\langle x, z, a\rangle\rangle^N$ with $\partial_a h(0) = \mathrm{id}_{\mathbb{C}^N}$, so that*

$$f(\boldsymbol{y} + D(\boldsymbol{y})\boldsymbol{a}) = f(\boldsymbol{y}) + \partial_y f(\boldsymbol{y})D(\boldsymbol{y})h(\boldsymbol{y}, \boldsymbol{a}). \tag{1.4}$$

Proof: As usually we expand $f(x, z + v) = f(x, z) + \partial_y f(x, z)v + q(x, z, v)$, where

$$q(z, v) = \sum_{1 \le i,j \le N} q_{i,j}(z, v)v_i v_j.$$

Substituting v by $\boldsymbol{v} = D(\boldsymbol{y})\boldsymbol{a}$ yields

$$q(\boldsymbol{y}, \boldsymbol{v}) = \sum_{1\le i,j\le N} q_{i,j}(\boldsymbol{y}, D(\boldsymbol{y})\boldsymbol{a})\Delta(\boldsymbol{y})^{l_i+l_j}\boldsymbol{a}_i\boldsymbol{a}_j,$$

where

$$l_i = \begin{cases} 2 & \text{if } i \in \{1, \dots, N-m\} \\ 1 & \text{if } i \in \{N-m+1, \dots, N\} \end{cases}$$

Since $l_i + l_j \ge 2$, we can factor out $\Delta(\boldsymbol{y})$ so that $q(\boldsymbol{y}, D(\boldsymbol{y})\boldsymbol{a}) = \Delta^2(\boldsymbol{y})\tilde{q}(\boldsymbol{y}, \boldsymbol{a})$, where $\tilde{q}(x, z, a) \in \mathbb{C}_0\langle\langle x, z, a\rangle\rangle^m$ is at least quadratic in a. Let $\partial^*_{y^{(2)}} f$ denote the classical adjoint of $\partial_{y^{(2)}} f$. Then

$$\begin{aligned} \Delta^2(\boldsymbol{y})\tilde{q}(\boldsymbol{y}, \boldsymbol{a}) &= (\partial_{y^{(2)}} f(\boldsymbol{y}))(\partial^*_{y^{(2)}} f(\boldsymbol{y}))\Delta(\boldsymbol{y}) \cdot \tilde{q}(\boldsymbol{y}, \boldsymbol{a}) \\ &= (\partial_{y^{(2)}} f(\boldsymbol{y}))\Delta(\boldsymbol{y})(\partial^*_{y^{(2)}} f(\boldsymbol{y})) \cdot \tilde{q}(\boldsymbol{y}, \boldsymbol{a}) \\ &= \partial_y f(\boldsymbol{y}, \boldsymbol{a}) \cdot D(\boldsymbol{y}) \underbrace{\begin{pmatrix} 0 \\ (\partial^*_{y^{(2)}} f(\boldsymbol{y})) \cdot \tilde{q}(\boldsymbol{y}, \boldsymbol{a}) \end{pmatrix}}_{p(\boldsymbol{y}, \boldsymbol{a})}, \end{aligned}$$

where $p(x, z, a) = (0, \dots, 0, \partial^*_{y^{(2)}} f(x, z, a) \cdot \tilde{q}(x, z, a)) \in \mathbb{C}_0\langle\langle x, z, a\rangle\rangle^N$. If we set now $h(x, z, a) = a + p(x, z, a)$, then

$$\begin{aligned} f(\boldsymbol{y} + D(\boldsymbol{y})\boldsymbol{a}) = f(\boldsymbol{y}) + \partial_y f(\boldsymbol{y}) \cdot D(\boldsymbol{y})\boldsymbol{a} + \partial_y f(\boldsymbol{y}, \boldsymbol{a}) \cdot D(\boldsymbol{y})p(\boldsymbol{y}, \boldsymbol{a}) = \\ = f(\boldsymbol{y}) + \partial_y f(\boldsymbol{y}) \cdot D(\boldsymbol{y})h(\boldsymbol{y}, \boldsymbol{a}) \end{aligned}$$

as claimed. □

Remark. For every $\boldsymbol{y}$ the map $\boldsymbol{a} \mapsto h(x, \boldsymbol{y}, \boldsymbol{a})$ is an arquile automorphism of $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ (by Theorem 1.1.7). The previous theorem shows that up to a coordinate change in $\boldsymbol{a}$, the map $f_\infty(\boldsymbol{y} + D(\boldsymbol{y}) \cdot \boldsymbol{a})$ is linear in $\boldsymbol{a}$. In [HW16], modules with properties Equation 1.3 and Equation 1.4 were dubbed *Division Modules*, providing an abstract setting in which the linearization works. However, the only known examples of such modules are diagonal modules of the form $\boldsymbol{y} + \langle\Delta(\boldsymbol{y})^{l_1}\rangle \times \dots \times \langle\Delta(\boldsymbol{y})^{l_N}\rangle$, where the exponents satisfy $l_i + l_j \ge l_k + 1$ for all $i, j \in \{1, \dots, N\}$ and $k \in \{N-m+1, \dots, N\}$.

Remark. A consequence of Theorem 1.4.2 is that the relation $\boldsymbol{y} \sim \boldsymbol{y}' :\Leftrightarrow \boldsymbol{y} \in \mathcal{V}_{\boldsymbol{y}'}$ defines an equivalence relation (if Δ does not have a constant term) on $\mathbb{C}_0\langle\langle x\rangle\rangle^N$. If $\mathcal{N}$ is now a canonical normal form (that is, there exists for each $\mathcal{V}_{\boldsymbol{y}}$ a unique $\boldsymbol{z} \in \mathcal{N}$ so that $\mathcal{V}_{\boldsymbol{z}} = \mathcal{V}_{\boldsymbol{y}}$), then $\mathcal{N} \times \mathbb{C}_0\langle\langle x\rangle\rangle^N \to \mathbb{C}_0\langle\langle x\rangle\rangle^N, (\boldsymbol{z}, \boldsymbol{a}) \mapsto \boldsymbol{z} + D(\boldsymbol{y})\boldsymbol{a}$ is an isomorphism, which trivializes the module bundle $(\mathcal{V}_{\boldsymbol{y}})_{\boldsymbol{y}\in\mathcal{N}}$. Using a variation of the Weierstrass Division, we will show that locally the normal form can be chosen as a space of x_n-polynomials. This corresponds to the induction step in Płoski's proof. In order to be able to apply the Weierstrass-Grauert Division Theorem with respect to $\Delta(\boldsymbol{y})$, we have to restrict to strata on which $\Delta(\boldsymbol{y})$ is x_n-regular of a fixed order d.

Definition 1.4.4. If $h(x)$ is any power series in $x = (x_1, \dots, x_n)$, we define the x_n-order $\operatorname{ord}_{x_n} h(x)$ as the order of $h(0, x_n)$. We set

$$\mathcal{S}_d = \left\{ \ \boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N \ \mid \ \operatorname{ord}_{x_n} \Delta(\boldsymbol{y}) = d \ \right\}$$

and $\mathcal{S} = \bigcup_{d\geq 1} \mathcal{S}_d$. Note that if $\Delta(\boldsymbol{y}) \neq 0$, then one can apply a linear coordinate change in the x-variables so that $\Delta(\boldsymbol{y})$ becomes x_n regular. The condition $\operatorname{ord}_{x_n} \Delta(\boldsymbol{y}) = d$ means $\operatorname{ord} \Delta(0, x_n, \boldsymbol{y}(0, x_n)) = d$ and depends only on a finite number of polynomial equalities and inequalities in the d-jet of $\boldsymbol{y}(0, x_n)$. Indeed, let

$$\boldsymbol{y}(0, x_n) = \boldsymbol{p}(x_n) = (\sum_{j=1}^{\infty} p_{1,j} x_n^j, \dots, \sum_{j=1}^{\infty} p_{N,j} x_n^j)$$

and expand

$$\Delta(0, x_n, \boldsymbol{p}(x_n)) = \sum G_k(p_{i,j}) x_n^k,$$

where the G_k are polynomials in the coefficients of $p(x_n)$. Each G_k only depends on $(p_{i,j})_{1\leq i\leq N, 1\leq j\leq k}$, so on the $k-$jet of $\boldsymbol{p}(x_n)$. Hence the set $\mathcal{S}_d$ is given by

$$\begin{aligned} G_1(p_{i,j}) &= 0 \\ G_2(p_{i,j}) &= 0 \\ &\vdots \\ G_{d-1}(p_{i,j}) &= 0 \\ G_d(p_{i,j}) &\neq 0 \end{aligned}$$

which means that $\mathcal{S}_d$ is a cofinitely textile locally closed set. Since the conditions only depend on finitely many coefficients, $\mathcal{S}_d$ is open in the $\mathfrak{m}$-adic topology. We will now turn to the division by $D(\boldsymbol{y})$. In principle, we want to apply on each component the ordinary Weierstraß division, so that we can decompose $\boldsymbol{y}$ into $D(\boldsymbol{y})\boldsymbol{a}+\boldsymbol{z}$, where $\boldsymbol{z}$ is a vector of x_n-polynomials of fixed maximal degrees. But since we require the quotient $\boldsymbol{a}$ to vanish at zero, this means dividing by $(x_1\Delta(\boldsymbol{y}), \ldots, x_n\Delta(\boldsymbol{y}))$, so we have to apply the more general Grauert-Hironaka Division. For $d \in \mathbb{N}$, set

$$\mathcal{P}_d := \{\, p \in \mathbb{C}\langle\langle x' \rangle\rangle[x_n] \mid \deg_{x_n}(p) \le d \,\} \quad \widehat{\mathcal{P}}_d := \{\, p \in \mathbb{C}\,[|x'|]\,[x_n] \mid \deg_{x_n}(p) \le d \,\}$$
$$\mathcal{R}_d := \mathbb{C}x_n^d \oplus \mathcal{P}_{d-1} \subset \mathcal{P}_d \qquad \widehat{\mathcal{R}}_d := \mathbb{C}x_n^d \oplus \widehat{\mathcal{P}}_{d-1} \subset \widehat{\mathcal{P}}_d$$

Lemma 1.4.5. *Let $g(x) \in \mathbb{C}\langle\langle x \rangle\rangle$ be x_n-regular of order d. Then for every $h(x) \in \mathbb{C}\langle\langle x \rangle\rangle$ there exist unique $a(x) \in \mathbb{C}_0\langle\langle x \rangle\rangle$ and $z(x) \in \mathcal{R}_d$ such that*

$$h(x) = g(x)a(x) + z(x).$$

Proof: Fix $d \in \mathbb{N}$ and let $<$ be a monomial order on $\mathbb{C}\langle\langle x \rangle\rangle$ so that $\mathrm{It}_<(p) = p_{0,d}x_n^d$, whenever $p(x)$ is x_n-regular of order d. The set $\{x_1 g(x), \ldots, x_n g(x)\}$ is a standard basis of the ideal $g \cdot \mathbb{C}_0\langle\langle x \rangle\rangle$. The initial module of $g(x) \cdot \mathbb{C}_0\langle\langle x \rangle\rangle$ is hence generated by $\{x_n^{d+1} x_n^d x_{n-1}, \ldots, x_n^d, x_1\}$. The remainder space of in the Grauert-Hironaka division (Theorem 2.3.6) by $g \cdot \mathbb{C}_0\langle\langle x \rangle\rangle$ is therefore $\mathcal{R}_d$ and thus there exist a unique vector $q = (q_n, \ldots, q_1) \in \mathbb{C}\langle\langle x \rangle\rangle_\Gamma$ and $\boldsymbol{z}(x) \in \mathcal{R}_d$ so that $h = q_n x_n g + \ldots q_1 x_1 q + z = g\boldsymbol{a} + \boldsymbol{z}$, where $\boldsymbol{a} = q_n x_n + \cdots + q_1 x_1$, which shows the existence. In the case that we are working with algebraic power series, so if $\mathbb{C}\langle\langle x \rangle\rangle = \mathbb{C}\langle x \rangle$, we observe that Hironaka's box condition is satisfied, which guarantees that that the remainder $\boldsymbol{z}$ is again algebraic. Hence $g(x)a(x) = h(x) - z(x)$ is algebraic, and as the Weierstrass Division theorem is valid for the ring of algebraic power series, also $a(x)$ is algebraic. Concerning the uniqueness: Let $\tilde{\boldsymbol{a}} \in \mathbb{C}_0\langle\langle x \rangle\rangle, \tilde{\boldsymbol{z}} \in \mathcal{R}_d$ with $h(x) = g(x)\tilde{\boldsymbol{a}}(x) + \tilde{\boldsymbol{z}}(x)$. Since $\{x_1 g, \ldots, x_n g\}$ is a standard basis, the remainder is unique, hence $\boldsymbol{z} = \tilde{\boldsymbol{z}}$, and consequently $\boldsymbol{a} = \tilde{\boldsymbol{a}}$. $\square$

Theorem 1.4.6. *For $d \in \mathbb{N}$ let*

$$\mathcal{Z}_d = \left\{ \boldsymbol{z} = (\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}) \in \mathcal{R}_{2d}^{N-m} \times \mathcal{R}_d^m \;\middle|\; \operatorname{ord}_{x_n} \Delta(\boldsymbol{z}) = d \right\} \tag{1.5}$$

(1) *Then the map*

$$\Psi_d \colon \mathcal{Z}_d \times \mathbb{C}_0\langle\!\langle x \rangle\!\rangle^N \to \mathcal{S}_d, (\boldsymbol{z}, \boldsymbol{a}) \mapsto \boldsymbol{z} + D(\boldsymbol{z}) \cdot \boldsymbol{a} = \boldsymbol{z} + \boldsymbol{v}$$

is an arquile isomorphism with rational-textile inverse given by componentwise Weierstraß Division as in the previous lemma. By π_d we denote the remainder map $\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{v} \mapsto \boldsymbol{z}$.

(2) *The restriction of Ψ_d to $\{\boldsymbol{z}\} \times \mathbb{C}_0\langle\!\langle x \rangle\!\rangle^N$ is an isomorphism onto $\mathcal{V}_{\boldsymbol{z}}$ and the family of submodules $\{\mathcal{V}_{\boldsymbol{z}}\}_{\boldsymbol{z} \in \mathcal{Z}_d}$ partitions $\mathcal{S}_d$ into disjoint submodules. In particular, $\pi_d(\boldsymbol{y}) = \boldsymbol{z}$ for all $\boldsymbol{y} \in \mathcal{V}_{\boldsymbol{z}}$ and $\mathcal{V}_{\boldsymbol{y}} = \mathcal{V}_{\pi_d(\boldsymbol{y})}$.*

Proof: Let $\boldsymbol{y} \in \mathcal{S}_d$. Dividing the components $\boldsymbol{y}_i$ by $g(\boldsymbol{y})^2$ for $i \in \{1, \ldots, N-m\}$ and $\boldsymbol{y}_i$ by $g(\boldsymbol{y})$ for $i \in \{N-m+1, \ldots, N\}$ yields a unique representation $\boldsymbol{y} = D(\boldsymbol{y})\tilde{\boldsymbol{a}} + \boldsymbol{z}$ with $\tilde{\boldsymbol{a}} \in \mathbb{C}_0\langle\!\langle x \rangle\!\rangle^N$ and $\boldsymbol{z} \in \mathcal{R}_{2d}^{N-m} \times \mathcal{R}_d^m$. By Theorem 1.4.2, $\Delta(\boldsymbol{y})$ and $\Delta(\boldsymbol{z})$ coincide up to multiplication with a unit and hence $\operatorname{ord}_{x_n} \Delta(\boldsymbol{z}) = d$, which means that $\boldsymbol{z} \in \mathcal{Z}_d$ as claimed. Since $D(\boldsymbol{z}) = u(\boldsymbol{z}, \boldsymbol{a})D(\boldsymbol{y})$, and as $\boldsymbol{z}$ is in the direct monomial compliment of $D(\boldsymbol{y})$ and thus of $D(\boldsymbol{z})$, the remainder in the division of $\boldsymbol{y}$ by $D(\boldsymbol{z})$ is the same as with respect to $D(\boldsymbol{y})$, which shows that $\pi_d(\mathcal{V}_{\boldsymbol{z}}) = \{\boldsymbol{z}\}$. The fact $\mathcal{V}_{\boldsymbol{y}} = \mathcal{V}_{\boldsymbol{z}}$ shows that Ψ_d is onto. Since the initial terms of the generators $\{x_1\Delta(\boldsymbol{y}), \ldots, x_n\Delta(\boldsymbol{y})\}$ are constant for $\boldsymbol{y} \in \mathcal{S}_d$, the assignment of $\boldsymbol{y} = D(\boldsymbol{y})\tilde{\boldsymbol{a}} + \boldsymbol{z} \mapsto \tilde{\boldsymbol{a}}$, $\boldsymbol{y} \mapsto \boldsymbol{z}$ is rationally textile in $\boldsymbol{y}$ by Theorem 2.3.11. So π_d is rationally textile, and the inverse of Ψ_d is given by division (assignment of quotient and remainder) of $\boldsymbol{y} = \Psi_d(\boldsymbol{z}, \boldsymbol{v})$ through $\pi_d(\boldsymbol{y}) = \boldsymbol{z}$, which is a rationally textile map on $\mathcal{S}_d$. □

Proposition 1.4.7 (Linearization of arquile maps)**.** *There exists a rational-textile isomorphism*

$$\chi_d \colon \mathcal{Z}_d \times \mathbb{C}_0\langle\!\langle x \rangle\!\rangle^N \to \mathcal{S}_d$$

so that the composition

$$f_\infty \circ \chi_d \colon \mathcal{Z}_d \times \mathbb{C}_0\langle\!\langle x \rangle\!\rangle^N \to \mathbb{C}_0\langle\!\langle x \rangle\!\rangle^m$$

is linear in the second coordinate $\boldsymbol{a}$,

$$f_\infty \circ \chi_d(\boldsymbol{z}, \boldsymbol{a}) = f_\infty(\boldsymbol{z}) + \partial_y f(\boldsymbol{z}) \cdot D(\boldsymbol{z}) \cdot \boldsymbol{a}.$$

Proof: Let $\Psi_d^{-1}\colon \mathcal{S}_d \to \mathcal{Z}_d \times \mathbb{C}_0\langle\langle x\rangle\rangle^N, \boldsymbol{y} \mapsto (\boldsymbol{z}, \boldsymbol{a})$ be the divsion map as in the previous theorem. By Equation 1.4 there exists a vector $h(x,z,a) \in \mathbb{C}\langle\langle x,z,a\rangle\rangle, \partial_a h(0) = \mathrm{id}_{\mathbb{C}^N}$ so that

$$f(\boldsymbol{z} + D(\boldsymbol{z})\boldsymbol{a}) = f(\boldsymbol{z}) + \partial_y f(\boldsymbol{z}) \cdot D(\boldsymbol{z}) \cdot h(\boldsymbol{z}, \boldsymbol{a}).$$

By Theorem 1.1.7, the map

$$\varphi_d\colon \mathcal{Z}_d \times \mathbb{C}_0\langle\langle x\rangle\rangle^N \to \mathcal{Z}_d \times \mathbb{C}_0\langle\langle x\rangle\rangle^N, (\boldsymbol{z}, \boldsymbol{a}) \mapsto (\boldsymbol{z}, h(\boldsymbol{z}, \boldsymbol{a}))$$

is an isomorphism with arquile inverse $\varphi_d^{-1}(\boldsymbol{z}, \boldsymbol{a}) = (\boldsymbol{z}, g(\boldsymbol{z}, \boldsymbol{a}))$. Set $\chi_d = \Psi_d \circ \varphi_d^{-1}$. Then it follows immediately that $f_\infty \circ \chi_d(\boldsymbol{z}, \boldsymbol{a}) = f(\boldsymbol{z}) + \partial_y f(\boldsymbol{z}) \cdot D(\boldsymbol{z}) \cdot \boldsymbol{a}$. □

Definition 1.4.8. A subset $\mathcal{Z}$ of a free power series module $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ is called *locally arquile*, if $\mathcal{Z}$ is locally defined by arquile equations, so if every point $\boldsymbol{z} \in \mathcal{Z}$ has an $\mathfrak{m}$-adic neighborhood W so that $W \cap \mathcal{Z} = \mathcal{Y}_W(f_1, \dots, f_k)$ for some $f_i \in \mathbb{C}_0\langle\langle x,y\rangle\rangle$.

Theorem 1.4.9. *Let $\mathcal{Y}_d = \mathcal{Y}(f_1, \dots, f_m) \cap \mathcal{S}_d$, and let $\mathcal{Z}_d^* = \pi_d(\mathcal{Y}_d)$ be the image of $\mathcal{Y}_d$ under the projection π_d.*

(1) $\mathcal{Z}_d^* = \left\{ \boldsymbol{z} \in \mathcal{Z}_d \;\middle|\; (\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot f(\boldsymbol{z}) \equiv 0 \mod \Delta(\boldsymbol{z})^2 \mathbb{C}_0\langle\langle x\rangle\rangle^m \right\}$.

(2) *There exists an isomorphism*

$$\Phi_d\colon \mathcal{Z}_d^* \times \mathbb{C}_0\langle\langle x\rangle\rangle^{N-m} \to \mathcal{Y}_d$$

which is the composition of an arquile map with a division mapping, with rational-textile inverse. The map extends to an isomorphism

$$\widehat{\Phi}_d\colon \widehat{\mathcal{Z}}_d^* \times \mathbb{C}_0\,[|x|]^{N-m} \to \widehat{\mathcal{Y}}_d,$$

where

$$\widehat{\mathcal{Z}}_d^* = \left\{ \boldsymbol{z} \in \widehat{\mathcal{Z}}_d \;\middle|\; \mathrm{ord}_{x_n}(\Delta(\boldsymbol{z})) = d, (\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot f(\boldsymbol{z}) \equiv 0 \mod \Delta(\boldsymbol{z})^2 \mathbb{C}_0\,[|x|]^m \right\}$$
$$\widehat{\mathcal{Y}}_d = \left\{ \widehat{\boldsymbol{y}} \in \mathbb{C}_0\,[|x|]^N \mid f(\boldsymbol{y}) = 0,\ \mathrm{ord}_{x_n} \Delta(\boldsymbol{y}) = d \right\}.$$

(3) *Each point $\boldsymbol{z}_0 \in \mathcal{Z}_d^*$ has an $\mathfrak{m}-$adic neighborhood U so that $U \cap \mathcal{Z}_d^*$ is given by arquile equations in the x_n-coefficients of $\boldsymbol{z}$, so $\mathcal{Z}_d^*$ is a locally arquile subset of a free module $\mathbb{C}\langle\langle x'\rangle\rangle^{N'}$ over the power series space $\mathbb{C}\langle\langle x'\rangle\rangle$ in one variable less.*

(4) *For every $\boldsymbol{z} \in \mathcal{Z}_d^*$ fixed, the map $\Phi_d(\boldsymbol{z}, \_)\colon \mathbb{C}_0\langle\langle x\rangle\rangle^{N-m} \to \mathcal{V}_d$ is an isomorphism onto $\mathcal{Y} \cap \mathcal{V}_{\boldsymbol{z}}$.*

Proof: Let χ_d be as in the previous theorem so that $f_\infty \circ \chi_d(\boldsymbol{z}, \boldsymbol{a}) = f(\boldsymbol{z}) + \partial_y f(\boldsymbol{z}) D(\boldsymbol{z})\boldsymbol{a}$. Then $\boldsymbol{z} \in \mathcal{Z}_d$ is contained in $\mathcal{Z}_d^*$ if and only if there exists a $\boldsymbol{y} \in \mathcal{V}_{\boldsymbol{z}}$ so that $f(\boldsymbol{y}) = 0$, which means that

$$f(\boldsymbol{z}) + \partial_y f(\boldsymbol{z}) \cdot D(\boldsymbol{z})\boldsymbol{a} = 0 \tag{1.6}$$

is solvable for some $\boldsymbol{a} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N$. As the determinant of $\partial_{y^{(2)}} f(\boldsymbol{z})$ and hence of the adjoint $\partial^*_{y^{(2)}} f(\boldsymbol{z})$ is non-zero, Equation 1.6 is equivalent to

$$(\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot f(\boldsymbol{z}) + (\partial^*_{y^{(2)}} f(\boldsymbol{z}))(\partial_{y^{(1)}} f(\boldsymbol{z}), \partial_{y^{(2)}} f(\boldsymbol{z})) D(\boldsymbol{z}) \cdot \boldsymbol{a} = 0 \tag{1.7}$$

$$\Leftrightarrow (\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot f(\boldsymbol{z}) + \left((\partial^*_{y^{(2)}} f(\boldsymbol{z}))\partial_{y^{(1)}} f(\boldsymbol{z}), \mathrm{id}_{\mathbb{C}^m}\right) g(\boldsymbol{z})^2 \cdot \boldsymbol{a} = 0, \tag{1.8}$$

which is solvable if and only if

$$(\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot f(\boldsymbol{z}) \equiv 0 \mod \Delta(\boldsymbol{z})^2 \mathbb{C}_0\langle\langle x\rangle\rangle. \tag{1.9}$$

Denote by $Q(\_, \Delta(\boldsymbol{z}))\colon \mathbb{C}_0\langle\langle x\rangle\rangle^m \to \mathbb{C}_0\langle\langle x\rangle\rangle^m$ the map which assigns the quotient in $\mathbb{C}_0\langle\langle x\rangle\rangle^m$ in the Weierstrass Division with respect by $\Delta(\boldsymbol{z})$. If $\boldsymbol{z} \in \mathcal{Z}_d^*$, then the solutions of $f(\boldsymbol{y}) = 0$ inside $\mathcal{V}_{\boldsymbol{z}}$ are given by

$$\Phi_d(\boldsymbol{z}, \boldsymbol{a}^{(1)}) = \tag{1.10}$$

$$= \boldsymbol{z} + D(\boldsymbol{z}) \cdot \left[\begin{pmatrix} 0 \\ Q\left((\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot f(\boldsymbol{z}), \Delta(\boldsymbol{z})\right) \end{pmatrix} + \begin{pmatrix} a^{(1)} \\ -(\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot (\partial^*_{y^{(1)}} f(\boldsymbol{z})) a^{(1)} \end{pmatrix} \right] \tag{1.11}$$

The inverse of Φ_d is given by the division maps

$$\Phi_d^{-1}(\boldsymbol{y}) = \begin{pmatrix} \pi_d(\boldsymbol{z}) \\ Q(\boldsymbol{y}^{(1)} - \pi_d^{(1)}(\boldsymbol{z}), \Delta^2(\pi_d(\boldsymbol{z}))) \end{pmatrix}$$

and is rationally textile, which concludes the proof of (1) and (2).
(3) We will formulate Equation 1.9 as an arquile equation in the coefficients of the remainder in the division. Let

$$\mathrm{proj}_d\colon \mathbb{C}\langle\langle x\rangle\rangle^N \to \mathbb{C}[x_n]^N, \mathrm{proj}_d(\boldsymbol{y}) = \left(j^{2d}(\boldsymbol{y}^{(1)}(0, x_n)), j^d(\boldsymbol{y}^{(2)}(0, x_n))\right) \in \mathbb{C}[x_n]^N.$$

Let $\boldsymbol{z}_0 \in \mathcal{Z}_d$ and set $\boldsymbol{p}_0 = \mathrm{proj}_d(\boldsymbol{z}_0)$. Then let U be the $\mathfrak{m}$-adically open neighborhood $U = \{\ \boldsymbol{z} \in \mathcal{R}_d \mid \mathrm{proj}_d(\boldsymbol{z}) = \boldsymbol{p}_0\ \}$ of $\boldsymbol{z}_0$. Since $\mathrm{ord}_{x_n} \Delta(\boldsymbol{z}) = d$ only depends on $\mathrm{proj}_d(\boldsymbol{z})$, the neighborhood U is in fact contained in $\mathcal{Z}_d$. If $\boldsymbol{z} \in U$, then we can write

$\boldsymbol{z} = (\boldsymbol{z}_1, \dots, \boldsymbol{z}_N)$ as

$$\boldsymbol{z}_i(x', x_n) = \boldsymbol{p}_i(x_n) + \sum_{j=0}^{2d-1} \boldsymbol{v}_{i,j}(x')x_n^j \quad \text{for } i \in \{1, \dots, N-m\},$$
$$\boldsymbol{z}_i(x', x_n) = \boldsymbol{p}_i(x_n) + \sum_{j=0}^{d-1} \boldsymbol{v}_{i,j}(x')x_n^j \quad \text{for } i \in \{N-m+1, \dots, N\},$$

where $\boldsymbol{v}_{i,j} \in \mathbb{C}_0\langle\langle x' \rangle\rangle$. Accordingly, let $v = (v_{i,j})_{i,j}$ be a set of new variables, and let $P(x_n, v) = (P_1(x_n, v), \dots, P_N(x_n, v))$ be the polynomial

$$P_i(x_n, v) = \boldsymbol{p}_i(x_n) + \sum_{j=0}^{2d-1} v_{i,j}x_n^j \quad \text{for } i \in \{1, \dots, N-m\},$$
$$P_i(x_n, v) = \boldsymbol{p}_i(x_n) + \sum_{j=0}^{d-1} v_{i,j}x_n^j \quad \text{for } i \in \{N-m+1, \dots, N\}.$$

In particular, for every $\boldsymbol{z} \in U$ there exists a unique $\boldsymbol{v} = (\boldsymbol{v}_{i,j})_{i,j} \in \mathbb{C}_0\langle\langle x' \rangle\rangle^{N'}$ so that $\boldsymbol{z} = P(x_n, \boldsymbol{v})$. Let now $h(x, v) = (\partial^*_{y^{(2)}} f(x, P(x_n, v))) \cdot f(x_n, P(x_n, v))$ and $g(x, v) = \Delta^2(x, P(x_n, v))$. So Equation 1.9 on U can be expressed as

$$h(\boldsymbol{v}) \equiv 0 \mod g(\boldsymbol{v}) \cdot \mathbb{C}_0\langle\langle x \rangle\rangle. \tag{1.12}$$

Observe that $g(x, v)$ is x_n-regular of order $2d$, since

$$g(0, x_n, 0) = \Delta^2(0, P(x_n, 0)) = \Delta^2(0, \boldsymbol{p}(x_n)).$$

We apply Lemma 1.4.5 to rewrite Equation 1.12 as

$$h_i(x, v) = g(x, v)q_i(x, v) + \sum_{j=0}^{2d-1} r_{i,j}(x', v)x_n^j + c_i x_n^{2d}$$

for some $q_i(x, v) \in \mathbb{C}_0\langle\langle x', v \rangle\rangle$, $r_{i,j}(x', v) \in \mathbb{C}\langle\langle x', v \rangle\rangle$ and $c_i \in \mathbb{C}$. The uniqueness of quotient and remainder, and the fact that $\mathrm{ord}_{x_n}(g(\boldsymbol{v})) = 2d$ for every $\boldsymbol{v}$ without constant term, implies that $q_i(\boldsymbol{v})$ and $\sum_{j=0} r_{i,j}(\boldsymbol{v})x_n^j + c_i x_n^{2d}$ are the quotient and remainder, respectively, in the division of $h_i(\boldsymbol{v})$ by $g(\boldsymbol{v})$. Let $\boldsymbol{v}_0 \in \mathbb{C}_0\langle\langle x' \rangle\rangle^{N'}$ so that $P(x_n, \boldsymbol{v}_0) = \boldsymbol{z}_0$. Then Equation 1.12 implies that the remainder vanishes, $\sum_{j=0}^{2d-1} r_{i,j}(\boldsymbol{v})x_n^j + c_i x_n^{2d} = 0$, and a comparison of x_n degrees yields $c_i = 0$. Therefore, $\boldsymbol{z} = P(x_n, \boldsymbol{v}) \in U$ is contained in $\mathcal{Z}_d^*$ if and only if

$$r_{i,j}(\boldsymbol{v}) = 0 \quad \text{for all } 1 \leq i \leq m, 0 \leq j \leq 2d-1,$$

which shows that $U \cap \mathcal{Z}_d^*$ can be identified with an arquile variety in $\mathbb{C}\langle\langle x' \rangle\rangle^N$.

$\square$

Remark. In the terminology of Płoski([Pło15]), solutions of Equation 1.9 are called *approximative solutions.*

1.5 Smooth Points

Definition 1.5.1. Let $\mathcal{Y}$ be an arquile variety. We say that $\mathcal{Y}$ is smooth at $\boldsymbol{y} \in \mathcal{Y}$, if there exists a neighborhood W of $\boldsymbol{y}$, so that $\mathcal{Y} \cap W$ is textile isomorphic to a product of the form

$$\mathfrak{m}_1^{\mu_1}\mathbb{C}\langle\langle x_1 \rangle\rangle^{l_1} \times \cdots \times \mathfrak{m}_{n-1}^{\mu_{n-1}}\mathbb{C}\langle\langle x_1, \ldots, x_{n-1} \rangle\rangle^{l_{n-1}} \times \mathfrak{m}_n^{\mu_n}\mathbb{C}\langle\langle x_1, \ldots, x_n \rangle\rangle^{l_n},$$

where $\mu_i, l_i \in \mathbb{N}$.

Theorem 1.5.2. *If x is a single variable, and $\mathcal{Y}$ an arquile variety in $\mathbb{C}_0\langle\langle x \rangle\rangle^N$, then every weakly regular point is already smooth. If $\mathcal{Y}$ is weakly regular at $\boldsymbol{y}_0 \in \mathcal{Y}$, then there exists a neighborhood W of $\boldsymbol{y}$ so that $W \cap \mathcal{Y}$ is textile isomorphic to $\mathbb{C}_0\langle\langle x \rangle\rangle^{N-m}$, where m is the height of $\mathcal{I}_{(\mathcal{Y},\boldsymbol{y}_0)}$.*

Proof: Since $\mathcal{Y}$ is weakly regular at $\boldsymbol{y}_0$, there exists a neighborhood V of $\boldsymbol{y}_0$, a regular pair (f, Δ) such that Δ vanishes nowhere on V and such that $\mathcal{Y}(f) \cap V = \mathcal{Y} \cap V$. After shrinking V, we may assume that $\operatorname{ord}(\Delta(\boldsymbol{y})) = \operatorname{ord}(\Delta(\boldsymbol{y}_0)) = d$ on V. By Theorem 1.4.9 there exists a textile isomorphism $\Phi_d\colon \mathcal{Z}_d^* \times \mathbb{C}_0\langle\langle x \rangle\rangle^{N-m} \to \mathcal{Y}_d(f)$. Let $(\boldsymbol{z}_0, \boldsymbol{a}_0) \in \mathcal{Z}_d^* \times \mathbb{C}_0\langle\langle x \rangle\rangle^{N-m}$ so that $\Phi(\boldsymbol{z}_0, \boldsymbol{a}_0) = \boldsymbol{y}_0$. By Theorem 1.4.9(4), the restriction

$$\Phi := \Phi(\boldsymbol{z}_0, \_)\colon \mathbb{C}_0\langle\langle x \rangle\rangle^{N-m} \to \mathcal{V}_{\boldsymbol{z}}$$

is an isomorphism onto $\mathcal{V}_{\boldsymbol{z}} \cap \mathcal{Y}(f)$. Recall that $\mathcal{V}_{\boldsymbol{z}} = \mathcal{V}_{\boldsymbol{y}_0}$ (Theorem 1.4.6) and that

$$\mathcal{V}_{\boldsymbol{y}_0} = \boldsymbol{y}_0 + \begin{pmatrix} \Delta^2(\boldsymbol{y}_0) \cdot \mathbb{C}_0\langle\langle x \rangle\rangle^{N-m} \\ \Delta(\boldsymbol{y}_0) \cdot \mathbb{C}_0\langle\langle x \rangle\rangle^m \end{pmatrix} = \boldsymbol{y}_0 + \begin{pmatrix} x^{2d} \cdot \mathbb{C}_0\langle\langle x \rangle\rangle^{N-m} \\ x^d \cdot \mathbb{C}_0\langle\langle x \rangle\rangle^m \end{pmatrix},$$

because $\Delta(\boldsymbol{y}_0)$ factors into the product of x^d with a unit in $\mathbb{C}\langle\langle x \rangle\rangle$, as x is a single variable. In particular, $\mathcal{V}_{\boldsymbol{y}_0}$ is an $\mathfrak{m}$-open neighborhood of $\boldsymbol{y}_0$. Choose now μ_1 so that $x^{\mu_1} \cdot \mathbb{C}\langle\langle x\rangle\rangle^{N-m} \subset \Phi^{-1}(V)$. Set $W = V \cap \mathcal{V}_{\boldsymbol{y}}$. Then the restriction of Φ to $x^{\mu_1} \cdot \mathbb{C}\langle\langle x\rangle\rangle^{N-m}$ is an isomorphism onto $W \cap \mathcal{Y}$, which shows that $\mathcal{Y}$ is smooth at $\boldsymbol{y}_0$. $\square$

Proposition 1.5.3. *The smooth points of an arquile variety $\mathcal{Y}$ are $\mathfrak{m}$-adically dense in $\mathcal{Y}$.*

Proof: By Theorem 1.3.26 the weakly regular points are $\mathfrak{m}$-adically dense, so it suffices to show that the smooth points are dense in the set of weakly regular points of $\mathcal{Y}$. So let $\boldsymbol{y}_0 \in \mathcal{Y}$ be a weakly regular point and let W_0 be a neighborhood so that $\mathcal{Y} \cap W_0$ is given by a system $f_1(\boldsymbol{y}) = \cdots = f_m(\boldsymbol{y}) = 0$, so that an m-minor Δ of $\partial_y(f)$ vanishes nowhere on W_0. Let $W \subset W_0$ be an arbitrary neighborhood of W_0. We need to show that $\mathcal{Y}$ has a smooth point in W, i.e. that $\mathcal{Y}_W(f_1, \ldots, f_m)$ has a smooth point. The proof is conducted by induction on n (the length of x). In the case of a single variable $n = 1$, every weakly regular point is already smooth by the last theorem. Now we show the induction step $n-1 \to n$. After a linear change of coordinates and a permuation of the y-coordinates, $\Delta(\boldsymbol{y}_0)$ is x_n-regular of order $d \geq 0$ and is the determinant of $\frac{\partial(f_1,\ldots,f_m)}{\partial(y_1,\ldots,y_{N-m})}$. By Theorem 1.4.9 there exists a textile isomorphism $\Phi_d \colon \mathcal{Z}_d^* \times \mathbb{C}_0\langle\langle x\rangle\rangle^{N-m} \to \mathcal{Y}_d$, where $\mathcal{Y}_d = \mathcal{Y}(f_1, \ldots, f_m) \cap \{\boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N \mid \operatorname{ord}_{x_n}(\Delta(\boldsymbol{y})) = d\}$ and $\mathcal{Z}_d^*$ is a locally arquile subset of $\mathbb{C}_0\langle\langle x_1, \ldots, x_{n-1}\rangle\rangle^{N'}$. Let $(\boldsymbol{z}_0, \boldsymbol{a}_0) \in \mathcal{Z}_d^* \times \mathbb{C}_0\langle\langle x\rangle\rangle^{N-m}$ so that $\Phi(\boldsymbol{z}_0, \boldsymbol{a}_0) = \boldsymbol{y}_0$. Let

$$V = V_1 \times \left(\boldsymbol{a}_0 + \mathfrak{m}_n^{\mu_n}\mathbb{C}\langle\langle x\rangle\rangle^{N-m}\right)$$

be an open neighborhood of $(\boldsymbol{z}_0, \boldsymbol{a}_0)$ such that $V_1 \cap \mathcal{Z}_d^*$ is an arquile variety in V_1 (in one variables less) and so that the image of V under Φ is contained in W. Because of the induction hypothesis, $V_1 \cap \mathcal{Z}_d^*$ has a smooth point $\boldsymbol{z}_1$, so there exists a textile isomorphism

$$\widetilde{\Phi} \colon \mathfrak{m}_1^{\mu_1}\mathbb{C}\langle\langle x_1\rangle\rangle^{l_1} \times \cdots \times \mathfrak{m}_{n-1}^{\mu_{n-1}}\mathbb{C}\langle\langle x_1, \ldots, x_{n-1}\rangle\rangle^{l_{n-1}} \to \mathcal{Z}_d^* \cap V_2$$

onto a neighborhood $V_2 \cap \mathcal{Z}_d^*$ of $\boldsymbol{z}_1$ in $\mathcal{Z}_d^*$. Let $\tau_{\boldsymbol{a}_0}$ be the translation $\mathbb{C}\langle\langle x\rangle\rangle^{N-m} \to \mathbb{C}\langle\langle x\rangle\rangle^{N-m}$, $\boldsymbol{a} \mapsto \boldsymbol{a} + \boldsymbol{a}_0$. Then the textile map $\Phi = \Phi_d \circ (\widetilde{\Phi} \times \tau_{\boldsymbol{a}_0})$,

$$\Phi \colon \mathfrak{m}_1^{\mu_1}\mathbb{C}\langle\langle x_1\rangle\rangle^{l_1} \times \cdots \times \mathfrak{m}_{n-1}^{\mu_{n-1}}\mathbb{C}\langle\langle x_1, \ldots, x_{n-1}\rangle\rangle^{l_{n-1}} \times \mathfrak{m}_n^{\mu_n}\mathbb{C}\langle\langle x\rangle\rangle^{N-m} \to \mathcal{Y}_d$$

is an isomorphism onto a neighborhood of $\Phi(\boldsymbol{z}_1, 0)$ in $\mathcal{Y}(f)$, since $\mathcal{Y}_d(f)$ is $\mathfrak{m}$-open in $\mathcal{Y}(f)$. In fact, the construction shows that the image of Φ lies in $\mathcal{Y}(f) \cap W = \mathcal{Y} \cap W$, so Φ maps onto an open neighborhood of $\Phi(\boldsymbol{z}_1, 0)$ in $\mathcal{Y} \cap W$, and there $\Phi(\boldsymbol{z}_1, 0)$ is a smooth point of $\mathcal{Y} \cap W$. $\square$

1.6 Finitely regular points and isolated singularities

Definition 1.6.1. We say that an arquile variety $\mathcal{Y}$ is *finitely regular* at $\boldsymbol{y}_0 \in \mathcal{Y}$, if there exists an $\mathfrak{m}$-neighborhood W of $\boldsymbol{y}_0$ and $f(x, y) \in \mathbb{C}\langle\langle x, y\rangle\rangle^m$ so that $\mathcal{Y} \cap W = \mathcal{Y}(f) \cap W$ and

$$\partial_y f(x, \boldsymbol{y}_0(x)) \cdot \mathbb{C}_0\langle\langle x\rangle\rangle^N$$

is open in the $\mathfrak{m}$-adic topology. In the next chapter we will see that the arquile map f_∞ can under this hypothesis be linearized by a textile isomorphism in a neighborhood of $\boldsymbol{y}_0$, if we are are working with either convergent or formal power series. In particular, finitely regular points are smooth. Unfortunately, we do not know whether this isomorphism also preserves algebraic series, which is why the case of algebraic series is excluded from our results below.

Like regular points, we can also characterize finitely regular points in terms of the minors of $\partial_y f(x, \boldsymbol{y}_0(x))$ of a defining system of $\mathcal{Y}$ at $\boldsymbol{y}_0$.

Theorem 1.6.2. *Let $\mathcal{Y}$ be an arquile variety over $\mathbb{C}\langle\langle x\rangle\rangle \in \{\mathbb{C}\{x\}, \mathbb{C}[|x|]\}$ and let $\boldsymbol{y}_0$ be a point in $\mathcal{Y}$.*

(1) *$\boldsymbol{y}_0$ is a finitely regular point of $\mathcal{Y}$ if and only if there exists a local defining system $f(x, y) \in \mathbb{C}\langle\langle x, y\rangle\rangle^m$ for $\mathcal{Y}$ at $\boldsymbol{y}_0$ such that the ideal of m-minors of $\partial_y f(x, \boldsymbol{y}_0(x))$ is $\mathfrak{m}$-open in $\mathbb{C}\{x\}$.*
(2) *Every finitely regular point of $\mathcal{Y}$ is a weakly regular point.*
(3) *Every finitely regular point of $\mathcal{Y}$ is a smooth point of $\mathcal{Y}$.*

Proof: (1) will be proven in the remainder of this subsection, see in particular Theorem 1.6.7 and (2) is a direct consequence of (1). (3) Let $f(x, y)$ be a local defining system at $\boldsymbol{y}_0$ of $\mathcal{Y}$ as in (1). By Proposition 2.1.11, there exists a $\delta > 0$ and an analytic automorphism Ψ of $U := \boldsymbol{y}_0(x) + \mathfrak{m}^\delta \mathbb{C}\{x\}^N$, so that $f_\infty \circ \Psi = L$, where L is the tangential map $b(x) \mapsto \partial_y f(x, \boldsymbol{y}_0(x)) \cdot b(x)$ of f_∞. So $\mathcal{Y} \cap U$ is textile isomorphic to the kernel of L, and since linear arquile varieties are smooth (see Theorem 1.7.1), the claim follows. $\square$

The concept of finitely regular points is closely related to that of finite determinacy and to isolated singularities of subvarieties of $\mathbb{C}^n$. In the following, we are going to review the basic concepts of isolated singularities of varieties in $\mathbb{C}^n$. For simplicity, we restrict the discussion at first to the hypersurface case.

Definition 1.6.3. Let Ω be an open zero neighborhood in $\mathbb{C}^n$, and let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function. The *Milnor number* $\mu(f) \in \mathbb{N} \cup \{\infty\}$ of is defined as

$$\mu(f) = \dim_{\mathbb{C}} \mathbb{C}\{x\} \Big/ _{\mathbb{C}\{x\}} \langle \partial_{x_1} f(x), \ldots, \partial_{x_n} f(x) \rangle .$$

The function f has an *isolated singularity* at 0, if there exists a neighborhood V of 0 so that the 0 is the only singular point of the analytic variety $\Omega \cap V(f)$ defined by f in Ω.

Theorem 1.6.4. *Let $U \subset \mathbb{C}^n$ be an open zero-neighborhood, and let $f\colon U \to (\mathbb{C}, 0)$ be an analytic map.*

(1) *The following are equivalent:*
   (a) $\mu(f) < \infty$.
   (b) *f has an isolated singularity at* 0.
   (c) *There exists an $l \in \mathbb{N}$ so that $\mathfrak{m}^l \subset \langle \partial_{x_1} f(x), \ldots, \partial_{x_n} f(x) \rangle$.*

(2) *If any of the conditions of* (1) *is satisfied, then $\mathfrak{m}^{\mu(f)} \subset \langle \partial_{x_1} f(x), \ldots, \partial_{x_n} f(x) \rangle$.*

Proof: See [GLS07, p.113] and [Rui93, p.92] □

Remark. If f is a polynomial, it is possible to calculate the Milnor number of f for example with the Computer-Algebra Software SINGULAR (see [GLS07, p.111]). For example, let $f(z, w) = z^3 + z^2 w + w z^2 + z^3$. Then $\mu(f) = 4$ can be computed as follows:

```
ring r = 0,(z,w),ds; \\
poly f = z3 + z2w + wz2 + z3;\\
ideal j = jacob(f);\\
vdim(std(j));  // Calculates the Milnor number of f
//-> 4
```

Remark. In the case of two variables, a holomorphic function $f(x_1, x_2)$ has an isolated singularity at 0 if and only if f is square-free (see [Rui93, p. 72] and [Loo84, p.7]).

Definition 1.6.5. Let $(R, \mathfrak{m})$ be a local ring. A regular sequence in R is a tuple $r_1, \ldots, r_s \in \mathfrak{m}$ so that each r_i is a non-zero divisor in $R/\langle r_1, \ldots r_{i-1} \rangle$. Geometrically, a regular sequence corresponds to a (set-theoretic) complete intersection. Let X be a germ of an analytic variety in $\mathbb{C}^n$ of co-dimension k. Then X is called a set-theoretic complete intersection if there exist $f_1, \ldots, f_k \in \mathbb{C}\{x\}$ so that X is locally given by $f_1 = \cdots = f_k = 0$, so if X can be defined by $\operatorname{codim} X$-many equations. An analytic variety is a set-theoretic complete intersection if and only if it can be defined by a regular sequence $f_1, \ldots, f_k \in \mathfrak{m}\mathbb{C}\{x\}$, and in this case the ideal $I = \langle f_1, \ldots, f_k \rangle$ is unmixed, i.e. all associated primes of I have the same height as I (see [Loo84],[Mat89]). An analytic variety X is said to have an isolated singularity at x_0, if x_0 is a singular point of X and if $X \setminus \{x_0\}$ is locally regular around x_0.

Theorem 1.6.6 ([Loo84, p.7])**.** *Let $f_1, \ldots, f_k \in \mathfrak{m}\mathbb{C}\{x\}$ be a regular sequence in $\mathfrak{m}\mathbb{C}\{x\}$, let X be the analytic variety defined by $f_1, \ldots, f_k$ at 0 and let J_k be the ideal generated by all $k \times k$-minors of $\partial_x(f)$. The following are equivalent:*

(1) *X has an isolated singularity at* 0*.*

(2) *J_k contains a power of the maximal ideal $\mathfrak{m}$.*

In particular, if the components of $f = (f_1, \ldots, f_k)$ define an isolated complete intersection singularity at 0, then then the image $\partial_x(f(x)) \cdot \mathbb{C}\{x\}^n$ of the Jacobian of f is $\mathfrak{m}$-adically open in $\mathbb{C}\{x\}^k$. In fact, also the converse is true.

Theorem 1.6.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let $L\colon R^N \to R^m$ be a module homomorphism and let $A \in R^{N \times m}$ be the matrix representation of L. Denote by $J_m(A)$ the ideal generated by the $m \times m$-minors of A. The image of L is $\mathfrak{m}$-open if and only $J_m(A)$ is $\mathfrak{m}$-open.*

Proof: Denote by M the cokernel $R^m / \operatorname{Im}(L)$ of L. The sequence

$$R^N \xrightarrow{L} R^m \to M \to 0$$

is a free presentation of M. The ideal $J_m(A)$ is called the first Fitting ideal of M (usually denoted by $F_1(M)$). Recall that the annihilator $\operatorname{ann}(M)$ of a module M is the ideal which consists of those elements $s \in R$ which satisfy $s \cdot m = 0$ for all $m \in M$. One has $F_1(M) \subset \operatorname{ann}(M)$, and the radicals of both ideals coincide (see [BE77]). In particular, $J_m(A)$ is $\mathfrak{m}-$primary if and only if $\operatorname{ann}(M)$ is $\mathfrak{m}$-primary. Since R is Noetherian, there exists for every ideal I in R a natural number k_0 such that $(\sqrt{I})^{k_0} \subset I$. The image of L is open if and only if there exists a $k \in \mathbb{N}$ so that $\mathfrak{m}^k \cdot R^m \subset \operatorname{Im}(L)$, which is equivalent to saying that $\mathfrak{m}^k$ annihilates $M = \operatorname{coker}(L)$. Since $\sqrt{\operatorname{ann}(M)} = \sqrt{F_1(M)}$, this is the case if and only if $F_1(M) = J_m(A)$ contains $\mathfrak{m}^{k_0}$ (for some k_0), which means that $J_m(A)$ is open. $\square$

Corollary 1.6.8. *Assume that $f_1, \dots, f_k \in \mathfrak{m}\mathbb{C}\{x, y\}$ define an isolated complete intersection singularity at 0 and that $f(x, 0) = 0$. Then $\operatorname{Im} \partial_y f(x, 0)$ is $\mathfrak{m}$-open and f_∞ can be factorized as in Proposition 2.1.11.*

Proof: By the two previous theorems, $M(x, y) = (\partial_x f, \partial_y f) \cdot \mathbb{C}\{x, y\}^N$ is open, so it contains $\langle x, y\rangle^{k_0}$ for some k_0. Differentiating $f(x, 0) = 0$ with respect to x shows that the columns of $\partial_x f(x, 0)$ are $\mathbb{C}\{x\}$-linear combinations of $\partial_y f(x, 0)$ (Lemma 1.3.9). Hence $M(x, 0) = \partial_y f(x, 0) \cdot \mathbb{C}\{x\}^N$, and thus $\partial_y f(x, 0) \cdot \mathbb{C}\{x\}^N$ contains $\langle x\rangle^{k_0}$. $\square$

1.7 Special Cases

1.7.1 The linear case

If $f(x, y)$ is linear in y, the requirement $\boldsymbol{y}(0) = 0$ is superfluous, since $f(x, \boldsymbol{y}(x))$ is well-defined for all $\boldsymbol{y} \in \mathbb{C}\langle\langle x\rangle\rangle^N$. So it is more natural to study the module $\ker(f)$ and not just $\mathcal{Y}(f)$, which is $\ker(f) \cap \mathbb{C}_0\langle\langle x\rangle\rangle^N$. So let $f(x, y) = A(x) \cdot y$, where $A(x) \in \mathrm{M}^{m \times N}(\mathbb{C}\langle\langle x\rangle\rangle)$. By the rank of a module M defined over an integral domain R we understand the rank of $\operatorname{Quot}(R) \otimes_R M$ over $\operatorname{Quot}(R)$. The $\operatorname{Quot}(\mathbb{C}\langle\langle x\rangle\rangle)$-module $\operatorname{Quot}(\mathbb{C}\langle\langle x\rangle\rangle) \otimes_{\mathbb{C}\langle\langle x\rangle\rangle} \ker(f)$ is the kernel of the extension $\overline{f}$ of f to $\operatorname{Quot}(\mathbb{C}\langle\langle x\rangle\rangle)^N$, because localizing is flat. Choose a basis $b_1, \dots, b_{N-m}$ for $\ker(\overline{f})$, where $m = \operatorname{rank}(\overline{f}) = \operatorname{rank}(f)$, by the rank-nullity theorem. Then $\ker(f) = \langle b_1, \dots, b_{N-m}\rangle \cap \mathbb{C}\langle\langle x\rangle\rangle^N$. However, this description is not very satisfactory as one does not obtain a parametrization of the kernel. If $n > 1$, then $\mathbb{C}\langle\langle x\rangle\rangle$ is not a principal ideal domain, and finitely generated modules over $\mathbb{C}\langle\langle x\rangle\rangle$ will in general not be free. An alternative is offered by the Grauert-Hironaka division theorem. The kernel $\ker(f)$ is as $\mathbb{C}$-vector space isomorphic to a subspace E_Γ of $\mathbb{C}\langle\langle x\rangle\rangle^l$. However, this presentation has two flaws: Neither is E_Γ a $\mathbb{C}\langle\langle x\rangle\rangle$-module, nor is the rank of f reflected in any way. With the parametrization theorem developed section 1.4, we obtain the following description.

Theorem 1.7.1. *Let $f(x, y)$ be linear in y and let m be the rank of f. Then there exists $k_1, \dots, k_{n-1} \geq 0$ and a $\mathbb{C}\langle\langle x_1\rangle\rangle$-linear isomorphism*

$$\Phi\colon \mathbb{C}\langle\langle x_1\rangle\rangle^{k_1} \times \mathbb{C}\langle\langle x_1, x_2\rangle\rangle^{k_2} \times \cdots \times \mathbb{C}\langle\langle x_1, \dots, x_{n-1}\rangle\rangle^{k_{n-1}} \times \mathbb{C}\langle\langle x_1, \dots, x_n\rangle\rangle^{N-m} \to \ker(f)$$

with the property that for any $l \leq n$ fixed and every,

$$\boldsymbol{a} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_l) \in \mathbb{C}\langle\langle x_1\rangle\rangle^{k_1} \times \mathbb{C}\langle\langle x_1, x_2\rangle\rangle^{k_2} \times \cdots \times \mathbb{C}\langle\langle x_1, \dots, x_l\rangle\rangle^{k_l},$$

the map $\Phi(\boldsymbol{a}, \_)$ is $\mathbb{C}\langle\langle x_1, \dots, x_{l+1}\rangle\rangle$-linear.

Proof: We conduct the proof by induction on n. If $n = 1$, then $\mathbb{C}\langle\langle x\rangle\rangle$ is a principal ideal domain and hence the torsion-free module $\ker(f)$ is free (see [dJP00, Theorem 1.2.7]). $n-1 \to n$. Write $f(x,y)$ as $A(x)\cdot y$. WLOG we may assume that A has full rank: Consider the equation over $\mathrm{Quot}(\mathbb{C}\langle\langle x\rangle\rangle)$. If a row is linearly dependent on others, we may delete it without changing the solutions to the equation. So there exists a full minor Δ of A which is not zero. After a linear change of coordinates it is x_n-regular of some order d. The minor is independent of $\boldsymbol{y}$. So $\ker(f) = \mathcal{Y}_d \cong \mathcal{Z}_d^* \times \mathbb{C}\langle\langle x\rangle\rangle^{N-m}$. It is easy to see that the Weierstrass division is $\mathbb{C}\langle\langle x'\rangle\rangle$-linear. $\mathcal{Z}_d^*$ is a (global) arquile set which can be defined by $\mathbb{C}\langle\langle x'\rangle\rangle-$linear equations. So we can apply the induction hypothesis, which yields a factorization as stated. $\square$

1.7.2 The single parameter case

In this section we assume that x is a single variable. In this case, every element $g(x) \in \mathbb{C}\langle\langle x\rangle\rangle$ factors into a product of $x^{\mathrm{ord}(g)}$ with a unit in $\mathbb{C}\langle\langle x\rangle\rangle$ and every submodule of $\mathbb{C}\langle\langle x\rangle\rangle^N$ is open in the $\mathfrak{m}$-adic topology. A direct consequence is that every weakly regular point of an arquile variety $\mathcal{Y}$ in a power series space of one variable is already smooth (see Theorem 1.5.2). The filtration by the iterated (Theorem 1.3.13(5)) singular loci induces then a partition of $\mathcal{Y}$ into smooth subsets.

Theorem 1.7.2. *Let $\mathcal{Y}$ be an arquile variety in an $\mathfrak{m}$-open subset of a free power series module $\mathbb{C}_0\langle\langle x\rangle\rangle^N$ in one variable. Then there exists a partition of $\mathcal{Y}$ into arquile locally closed subsets $\mathcal{Y} = V_1 \cup \cdots \cup V_s$, so that every V_k is smooth in the sense that every $\boldsymbol{y} \in V_k$ admits an $\mathfrak{m}$-adic neighborhood in V_k, which is textile-isomorphic to an open subset $\mathfrak{m}^k\mathbb{C}\langle\langle x\rangle\rangle^l$, where $l = N - \mathrm{ht}(\mathcal{I}_{(V_k,\boldsymbol{y})})$.*

Proof: By Theorem 1.3.13(5) we can filter $\mathcal{Y}$ by the iterated singular loci: $W_0 = \mathcal{Y} \supsetneq W_1 = \mathrm{Sing}(\mathcal{Y}) \supsetneq \cdots \supsetneq W_s = \emptyset$. Then the differences $V_k = W_k \setminus W_{k+1} = \mathrm{Reg}(W_k)$ are the regular loci of varieties W_k and form a partition of $\mathcal{Y}$. Every regular point is weakly regular and hence smooth, and so every point $\boldsymbol{y} \in V_k = \mathrm{Reg}(W_k)$ admits a neighborhood which is isomorphic to $\mathfrak{m}^l\mathbb{C}\langle\langle x\rangle\rangle^k$, where $k = N - \mathrm{ht}(\mathcal{I}_{(V_k,\boldsymbol{y})})$ (see Theorem 1.5.2 and Theorem 1.3.25). $\square$

Let (f,Δ) be a regular pair and consider $\mathcal{Y}_d = \{\boldsymbol{y} \in \mathbb{C}_0\langle\langle x\rangle\rangle^N \mid \mathrm{ord}(\Delta(\boldsymbol{y})) = d\}$. The remainder space in the division by $D(\boldsymbol{y})$ is the space $\mathcal{R}_{2d}^{N-m} \times \mathcal{R}_d^m$ of polynomial vectors of degree $\leq 2d$ in the first $N-m$ components and of degree $\leq d$ in the last m components. Let $\boldsymbol{z} = (\boldsymbol{z}_1,\ldots,\boldsymbol{z}_N) \in \mathcal{R}_{2d}^{N-m} \times \mathcal{R}_d^m$ and write $\boldsymbol{z}_i$ as $\boldsymbol{z}_i = \sum_{k=0}^{2d} z_{i,j}x^j$ (if $1 \leq i \leq N-m$) and $\boldsymbol{z}_i = \sum_{k=0}^{d} z_{i,j}x^j$ (if $N-m+1 \leq i \leq N$). The coefficients of $\Delta(\boldsymbol{z})$ are polynomials in the coefficients of $\boldsymbol{z}$, so there exists polynomials $G_k((z_{i,j})_{1\leq i\leq N, 1\leq j\leq k})$ such that

$$\Delta(\boldsymbol{z}) = \sum_{k=0}^{\infty} G_k((z_{i,j})_{1\leq i\leq N,1\leq j\leq k})x^k.$$

The order condition $\operatorname{ord}(\Delta(\boldsymbol{z})) = d$ is then given by

$$\begin{aligned}
G_1(z_{1,1}, \dots, z_{N,1}) &= 0\\
G_2(z_{1,1}, z_{1,2}, \dots, z_{N,1}, z_{N,2}) &= 0\\
&\vdots\\
G_{d-1}(z_{1,1}, \dots, z_{1,d-1}, \dots, z_{N,1}, \dots, z_{N,d-1}) &= 0\\
G_d(z_{1,1}, \dots, z_{1,d}, \dots, z_{N,1}, \dots, z_{N,d}) &\neq 0.
\end{aligned}$$

If $\boldsymbol{z} \in \mathcal{Z}_d$, then $\boldsymbol{z} \in \mathcal{Z}_d^*$ if

$$(\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot f(\boldsymbol{z}) \equiv 0 \mod \Delta^2(\boldsymbol{z}) \cdot \mathbb{C}_0\langle\langle x\rangle\rangle^m.$$

Since $\Delta^2(\boldsymbol{z})$ is conjugated to x^{2d}, the module $\Delta(\boldsymbol{z})\mathbb{C}_0\langle\langle x\rangle\rangle^m$ equals $x^{2d+1}\mathbb{C}_0\langle\langle x\rangle\rangle^m$ and the equation simplifies to

$$\operatorname{ord}((\partial^*_{y^{(2)}} f(\boldsymbol{z})) \cdot f(\boldsymbol{z}) \geq 2d+1 \quad .(*)$$

Again we expand $(\partial^*_{y^{(2)}} f(\boldsymbol{z}))$ as $\sum F_k(z_{i,j})x^k$, where each F_k is a polynomial in the coefficients of $\boldsymbol{z}$. The order condition $(*)$ can then be written as

$$F_1(z_{i,1}) = \cdots = F_{2d}((z_{i,j})_{1\leq j\leq 2d}) = 0,$$

and therefore the set of approximative solutions $\mathcal{Z}_d^*$ is a finite-dimensional locally closed variety in some finite-dimensional affine space $\mathbb{C}^k$.

Example 1.7.3. Let $\mathcal{Y}(f)$ be the arquile variety given by $f(y) = y_1^2 - y_2y_3$. Since the parameter x does not appear in the equation and since f is prime, $\operatorname{Sing}(\mathcal{Y}) = \mathcal{Y}(f, \partial_{y_1} f, \partial_{y_2} f, \partial_{y_3} f) = \{(0,0,0)\}$ (see Theorem 1.3.18) and the regular part is covered by $\mathcal{Y}(f)\setminus\mathcal{Y}(\partial_{y_i} f)$. We fix $i = 1$, so we consider the set $\mathcal{Y}(f)\setminus\mathcal{Y}(2y_1)$, which we stratify into $\mathcal{Y}_d = \{\boldsymbol{y} \in \mathcal{Y}(f) \mid \operatorname{ord} \boldsymbol{y}_1 = d\}$. It is easily seen that

$$\mathcal{Z}_d = \left\{ \boldsymbol{z} = \left(z_{1,d}x^d, \sum_{k=1}^{2d} z_{2,k}x^k, \sum_{k=1}^{2d} z_{3,k}x^k \right) \middle| \, z_{1,d} \neq 0 \right\}$$

In the hypersurface case, the defining minor for $\mathcal{Z}_d$ stems from the 1×1 matrix $\partial_{y_1} f$, whence $\partial_{y_1}^* f = 1$ and thus the equation for $\mathcal{Z}_d^*$ simplifies to $f(\boldsymbol{z}) \equiv 0 \mod x^{2d+1}$. Therefore $\mathcal{Z}_d^*$ is given by the equations

$$\begin{aligned} z_{1,d} &\neq 0 \\ z_{2,1}z_{3,1} &= 0 \\ &\vdots \\ \sum_{j=1}^{2d-1} z_{2,2d-1-j}z_{3,j} &= 0 \\ z_{1,d}^2 - \sum_{j=1}^{2d} z_{2,2d-j}z_{3,j} &= 0 \end{aligned}$$

inside $\mathcal{Z}_d$. By the factorization theorem, $\mathcal{Y}_d$ is isomorphic to $\mathcal{Z}_d^* \times \mathbb{C}_0\langle\langle x \rangle\rangle^2$. An isomorphism $\Phi_d = (\Phi_{d,1}, \Phi_{d,2}, \Phi_{d,3})\colon \mathcal{Z}_d^* \times \mathbb{C}_0\langle\langle x \rangle\rangle^2 \to \mathcal{Y}_d$ can be explicitly computed and is given by

$$\begin{aligned} \Phi_{d,1}(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{a}_2, \boldsymbol{a}_3) &= \boldsymbol{z}_1 + x^d \left(-\boldsymbol{z}_{1,d} + \sqrt{z_{1,d}^2 - \frac{\boldsymbol{z}_1 - \boldsymbol{z}_2\boldsymbol{z}_3}{x^{2d}} + \boldsymbol{z}_2\boldsymbol{a}_3 + \boldsymbol{z}_3\boldsymbol{a}_2 + x^{2d}\boldsymbol{a}_2\boldsymbol{a}_3} \right) \\ \Phi_{d,2}(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{a}_2, \boldsymbol{a}_3) &= \boldsymbol{z}_2 + x^{2d}\boldsymbol{a}_2 \\ \Phi_{d,3}(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{a}_2, \boldsymbol{a}_3) &= \boldsymbol{z}_3 + x^{2d}\boldsymbol{a}_3 \end{aligned}$$

Let us quickly explain how the isomorphism is constructed. We expand $f(\boldsymbol{z} + \boldsymbol{v})$ into $f(\boldsymbol{z}) + \partial_y(f)(\boldsymbol{z}) \cdot \boldsymbol{v} + q(\boldsymbol{z}, \boldsymbol{v})$

$$f(\boldsymbol{z} + \boldsymbol{v}) = (\boldsymbol{z}_1^2 - \boldsymbol{z}_2\boldsymbol{z}_3) + 2\boldsymbol{z}_1\boldsymbol{v}_1 - \boldsymbol{z}_2\boldsymbol{v}_3 - \boldsymbol{z}_3\boldsymbol{v}_2 + (\boldsymbol{v}_1^2 - \boldsymbol{v}_2\boldsymbol{v}_3).$$

Substitute $\boldsymbol{v}_1 = x^d\boldsymbol{a}_1$, $\boldsymbol{v}_2 = x^{2d}\boldsymbol{a}_2$, $\boldsymbol{v}_3 = x^{2d}\boldsymbol{a}_3$. We assume that $\boldsymbol{z} \in \mathcal{Z}_d^*$, so there exists an $\boldsymbol{h}_{\boldsymbol{z}}(x) \in \mathbb{C}_0\langle\langle x \rangle\rangle$ so that $\boldsymbol{z}_1^2 - \boldsymbol{z}_2\boldsymbol{z}_3 = x^{2d} \cdot \boldsymbol{h}_{\boldsymbol{z}}(x)$. Observe that $\boldsymbol{z}_1 = x^{2d}z_{1,d}$. We want to solve $f(\boldsymbol{z} + \boldsymbol{v}) = 0$, which in $\boldsymbol{a}$-coordinates takes the form

$$(\boldsymbol{z}_1^2 - \boldsymbol{z}_2\boldsymbol{z}_3) + 2\boldsymbol{z}_1x^d\boldsymbol{a}_1 - \boldsymbol{z}_2x^{2d}\boldsymbol{a}_3 - \boldsymbol{z}_3x^{2d}\boldsymbol{a}_2 + (x^{2d}\boldsymbol{a}_1^2 - x^{2d}\boldsymbol{a}_2x^{2d}\boldsymbol{a}_3) = 0$$

$$\Leftrightarrow x^{2d}\left(\boldsymbol{h}_{\boldsymbol{z}}(x) + 2z_{1,d}\boldsymbol{a}_1 - \boldsymbol{z}_2\boldsymbol{a}_3 - \boldsymbol{z}_3\boldsymbol{a}_2 + \boldsymbol{a}_1^2 - x^{2d}\boldsymbol{a}_2\boldsymbol{a}_3\right) = 0$$

and therefore

$$\boldsymbol{a}_1 = -z_{1,d} + \sqrt{z_{1,d}^2 - \frac{\boldsymbol{z}_1 - \boldsymbol{z}_2\boldsymbol{z}_3}{x^{2d}} + \boldsymbol{z}_2\boldsymbol{a}_3 + \boldsymbol{z}_3\boldsymbol{a}_2 + x^{2d}\boldsymbol{a}_2\boldsymbol{a}_3}$$

□

If an arquile variety $\mathcal{Y}$ can be defined by equations which do not depend on the parameter x, i.e. $\mathcal{Y} = \mathcal{Y}(f_1, \ldots, f_k)$, $f_i \in \mathbb{C}_0\langle\langle y_1, \ldots, y_N\rangle\rangle^k$, the situation becomes simpler and some pathologies cannot occur. For example, we have already seen in Theorem 1.3.18 that any radical ideal which can be generated by series not depending on x is saturated. Geometrically, $\mathcal{Y}$ correspond to the set of arcs centered at 0 on the complex space germ X defined by f at 0. The curve selection lemma states that when X is an irreducible analytic variety and x_0 is a (closed) point contained in a proper subvariety Z of X, then there exists a curve $\gamma(x)\colon (\mathbb{C}, 0) \to X, \gamma(0) = 0$, so that the image of γ is not contained in Z. One can ask whether an analogous statement holds for arquile varieties: Given $\mathcal{Y}$ an irreducible arquile variety, $\mathcal{W} \subset \mathcal{Y}$ a proper arquile subvariety and a point $\boldsymbol{w} \in \mathcal{W}$, is it true that there exists an analytic map $\Phi\colon \mathbb{C}_0\langle\langle x\rangle\rangle^N \to \mathcal{Y}, \Phi(0) = 0$, so that the image of Φ is not contained in $\mathcal{W}$? However, as Example 1.3.2 shows, it can occur that $\mathcal{Y}$ and $\mathcal{W}$ coincide in an $\mathfrak{m}$-adic neighborhood of a point in $\mathcal{W}$, so that in general the curve-selection theorem for arquile varieties cannot be true. If $\mathcal{Y} = \mathcal{Y}(f)$, and f does not depending on x, then it Nevertheless, it is possible to show a similar statement. We can use a resolution of singularities to lift curves up to a reparametrization to the resolved smooth spaces $\widehat{X}$. In [Izu90], S.Izumi proves the following theorem:

Theorem 1.7.4 ([Izu90, Lemma 2.1])**.** *Let X be a pure $r-$dimensional germ of a complex space at* 0*, and let $\boldsymbol{y}(x)$ be an analytic arc through zero. Then there exist a series $\Psi(z_1, \ldots, z_r) \in \mathbb{C}\{z_1, \ldots, z_r\}^N$ with*

(1) $\operatorname{grk}(\Psi) = r$
(2) *There exists a $q \in \mathbb{N}$ so that $\Psi(x, 0, \ldots, 0) = \boldsymbol{y}(x^q)$*
(3) $\Psi\colon (\mathbb{C}^r, 0) \to (X, 0)$*.*

The theorem shows that a given arc $\boldsymbol{y}$ on X can be reparametrized so that the new arc $\nu(x) = \boldsymbol{y}(x^q)$ is part of a full-rank family.

Theorem 1.7.5. *Let $\mathcal{Y}$ be an irreducible arquile variety defined by an ideal $I \subset \mathbb{C}\{y\}$, and let $\mathcal{Z}$ be a proper subvariety defined by an ideal $J \subset \mathbb{C}\{y\}$. Let $\boldsymbol{y} \in \mathcal{Z}$. Then there exists a $\Gamma(x, s) \in \mathbb{C}\{x, s\}$, where s is an additional variable, so that*

(1) $\Psi \in \mathcal{Y} \setminus \mathcal{Z}$.

(2) $\Psi(x, 0) = \boldsymbol{y}(x^q)$ *for some* $q \in \mathbb{N}$.

1.7.3 The case of single independent variable

Theorem 1.7.6. *Let $N = 1$, i.e. y is a single variable, and let $f(x, y) \in \mathbb{C}_0\,[|x, y|]^m$, $m \geq 1$. Then the number of solutions to $f(\boldsymbol{y}) = 0$ is bounded by* $\operatorname{ord}(f)$. *Especially, if f is convergent, then every solution is already convergent.*

Proof: Let $\boldsymbol{y}_1$ be a solution to $f(\boldsymbol{y}) = 0$. Since $(y - \boldsymbol{y}_1(x)) \in \mathbb{C}_0\,[|x, y|]$ is $y-$regular we can divide f by $(y - \boldsymbol{y}_1(x))$ and obtain a representation $f = (y - \boldsymbol{y}_1(x))q + R$, where $R(x) \in \mathbb{C}\,[|x|]$. Evaluating both sides at $\boldsymbol{y}_1(x)$ yields $0 = R(x)$, whence $f = (y - \boldsymbol{y}_1(x))q$. If $\boldsymbol{y}_2(x) \neq \boldsymbol{y}_1$ is another implicit solution of $f = 0$, then it must also solve $q(\boldsymbol{y}_2) = 0$. Dividing q by $(y - \boldsymbol{y}_2(x))$ yields a representation $f = (y - \boldsymbol{y}_1(x))(y - \boldsymbol{y}_2(x))q'(x, y)$. Since $\operatorname{ord}(y - \boldsymbol{y}_1) = 1 > 0$ there can exist at most $\operatorname{ord}(f)$-many solutions. By Artin's theorem, every formal solution is the limit of a sequence of convergent solutions, and since a such sequence is finely constant, every solution must be already convergent. $\square$

Chapter 2

Analytic and textile maps between power series spaces

If the space of convergent power series is equipped with its natural inductive locally convex topology, then every arquile map $f_\infty \colon \mathbb{C}_0\{x\}^N \to \mathbb{C}_0\{x\}^m$ defines a holomorphic function. This point of view has been introduced by H.Hauser and G. Müller in [HM94], to establish a linearization theorem for arquile maps of constant rank. We review the basic notions of analytic functions between locally convex spaces and proof an inversion theorem for holomorphic maps between (DFS)-spaces. Then we review the rank theorem by Hauser and Müller and show a version of it for arquile maps whose tangential map has a cofinite image. The second section then turns to analytic maps. We discuss the analytic solvability of textile equations $F(\boldsymbol{y}) = 0$ and give a characterization of analytic textile maps. In the third section we review the division theorem of Grauert-Hironaka, which is a tool needed in the factorization theorem for arquile varieties as well as in the linearization theorem for arquile maps.

2.0.4 Holomorphic maps between locally convex spaces

Definition 2.0.7. Let E, F be locally convex vector spaces, let $U \subset E$ be an open subset of E and denote by F' the continuous dual space of F. A function $f \colon U \to F$ is called Gâteaux- or G-holomorphic, if for every $\xi \in U, w \in E, \varphi \in F'$, the mapping of one complex variable $z \mapsto \varphi \circ f(\xi + z \cdot w)$ is holomorphic at 0. A function f is Gâteaux-holomorphic if and only the limit

$$\lim_{z \to 0} \frac{\varphi \circ f(\xi + z \cdot w) - \varphi \circ f(\xi)}{z}$$

exists for all $\xi \in U, w \in E, \varphi \in F'$ ([Din99, p.149]). We call a map $f \colon U \to F$ holomorphic, if it is G-holomorphic and continuous with respect to the locally convex topologies on E and F.

Definition 2.0.8. A function $f\colon U \to F$ is called *curve-holomorphic*, if the composition $\varphi \circ f \circ \gamma$ is holomorphic for all $\varphi \in F'$ and every analytic curve $\gamma\colon \mathbb{D} \to U$. While every holomorphic function is curve-holomorphic, the converse is in general not true.

Theorem 2.0.9 ([KM97])**.** *Assume that each of the spaces E and F is either a Fréchet or* (DFS)*-space and let $f\colon U \to F$ be a function defined on an open subset U of E. Then f is holomorphic if and only if it is curve-holomorphic.*

2.0.5 The Inverse Function Theorem in (DFS)-spaces

In the following we will proof a version of the inverse function theorem for analytic maps between (DFS)-spaces. Similar versions were obtained by D.Pisanelli ([Pis86]) and Hauser & Müller ([HM94]). A (DFS)-space is an inductive limit of sequences of Banach spaces $(E_n)_{n\in\mathbb{N}}$ with compact linking maps. The theorem presented below is tailored to maps between spaces of convergent power series which are high-order perturbations of the identity and which permit norm estimates that guarantee that a ball of E_n is mapped into one of the same step E_n. In this situation one can apply the inverse function theorem for Banach spaces and glue together the local inverses along the different Banach steps E_n. A theorem by L. Harris gives a precise estimate on the size of the domain of the local inverse. Under suitable assumptions on the ratio of the size of the image to the size of the domain of the local operators, the domains of the inverse fill out a whole neighborhood in E.

Definition 2.0.10. An inductive sequence of Banach spaces $(E_n)_{n\in\mathbb{N}}$ is a sequence of Banach space such that $E_n \hookrightarrow E_{n+1}$ and so that $\|x\|_{n+1} \leq \|x\|_n$ if $x \in E_n$. We say that the sequence is compact, if all inclusion maps $\iota_n\colon E_n \to E_{n+1}$ are compact. The inductive limit of a compact inductive sequence of Banach spaces is called a (DFS) space or Silva space. For the convenience of the reader, we list some of the properties of (DFS)-spaces which we are going to use.

Theorem 2.0.11 ([FW68], [Din99], [Wob12])**.** *Let $(E_n)_{n\in\mathbb{N}}$ be a compact inductive sequence of Banach spaces and let $E = \varinjlim E_n$.*

(1) *E carries the final topology with respect to the inclusions $\iota_n\colon E_n \to E$. A function $f\colon E \to Y$ into an arbitrary topological space is continuous if and only if all its restrictions $f\colon E_n \to Y$ are continuous.*
(2) *$(E_n)_{n\in\mathbb{N}}$ is sequentially retractive, that is, if a sequence $(x_n)_{n\in\mathbb{N}}$ converges in E to $x \in E$, then there exists a step E_{n_0}. which contains the sequence $(x_n)_{n\in\mathbb{N}}$ and $\{x\}$, and x_n converges to x in the norm topology of E_{n_0}.*

(3) $(E_n)_{n\in\mathbb{N}}$ *is regular, that is, every bounded subset is already contained and bounded in some step* E_{n_0}.

(4) E *is a Montel space, that is, every bounded set is relative compact.*

(5) *Every compact subset of* E *is metrizable.*

(6) *A function* $f\colon U \to F$ *defined on an open subset* U *of* E *with values in a locally convex space is analytic if and only if all restrictions* $f\colon U_n := U \cap E_n \to F$ *are holomorphic.*

Theorem 2.0.12 ([Har77a, Proposition2])**.** *Let* $(X, \| \ \|)$ *be a complex Banach space, and let* $B_M := \{x \in X \mid \|x\| < M\}$. *Let* $h\colon B_K \to B_M$ *be a holomorphic function and suppose that the inverse* $T_0 h^{-1}$ *of the tangential map exists with* $\left\|T_0 h^{-1}\right\|^{-1} \geq a$. *Set*

$$\nu = \frac{K^2 a}{4M}, \ \rho = \frac{K^2 a^2}{8M}$$

Then h *maps* B_ν *biholomorphically onto a domain covering* $B_\rho(h(0))$. *Moreover,* $h(x) \neq h(0)$ *whenever* $0 < \|x\| < \frac{K^2 a}{M}$.

Proposition 2.0.13. *Let* E *be a* (DFS)*-space and let* $(E_n)_{n\in\mathbb{N}}$ *be a compact inductive sequence of Banach spaces with* $E = \varinjlim E_n$. *By* $B_{n,M}$ *we denote the open ball of size* M *in* E_n. *Let* U *be an open subset of* E *and let* $f\colon U \to E$ *be a holomorphic mapping with* $f(0) = 0$ *and assume that the tangential map* $T_0 f$ *of* f *at is* 0 *is the identity* id_E. *Suppose that there exist sequences* $(K_k)_{k\in\mathbb{N}}, (M_k)_{k\in N}$ *of positive real numbers with the following properties:*

(1) $B_{k,K_k} \subset B_{k+1,K_{k+1}} \subset U$,

(2) $f(B_{k,K_k}) \subset B_{k,M_k}$,

(3) $f\colon B_{k,K_k} \to B_{k,M_k}$ *is analytic as a local map on the Banach space* E_k,

(4) $B_{k,\nu_k} \subset B_{k+1,\nu_{k+1}}$, $B_{k,\rho_k} \subset B_{k+1,\rho_{k+1}}$ *where* $\nu_k = \frac{K_k^2}{4M_k}$ *and* $\rho_k = \frac{K_k^2}{8M_k}$

for all $k \in \mathbb{N}$. *Set*

$$V = \bigcup_{k\in\mathbb{N}} B_{k,\nu_k}, \ W = \bigcup_{k\in\mathbb{N}} B_{k,\rho_k}.$$

Then f maps V biholomorphically onto a domain Ω which contains W.

Remark. Condition 4) is for example satisfied if $\frac{K_k^2}{M_k} \leq \frac{K_{k+1}^2}{M_{k+1}}$.

Proof: Let f_k denote the restriction of f to B_{k,ν_k}, considered as a local analytic map on the Banach space E_k. By Theorem 2.0.12 f_k maps B_{k,ν_k} onto an open domain Ω_k, that contains B_{k,ρ_k}. Let $f_k^{-1}\colon \Omega_k \to B_{k,\nu_k}$ denote the analytic inverse of f_k. Since $B_{k,\nu_k} \subset B_{k+1,\nu_{k+1}}$ and $f_{k+1}|_{E_k} = f_k$ it follows that $\Omega_k \subset \Omega_{k+1}$ for all $k \in \mathbb{N}$. Since, of course, the inverse is unique, we can glue together the local inverses f_k^{-1} together to an inverse $f^{-1}\colon \Omega \to V$ of $f\big|_V$. We now show that Ω is open in E. Since E carries the final topology with respect to the inclusions $\iota_k\colon E_k \to E$, this is equivalent to $\Omega \cap E_k$ being open for all $k \in \mathbb{N}$. Since $\Omega_k \subset \Omega_n$ for $n \geq k$, we have that $\Omega \cap E_k = \bigcup_{n\geq k} \Omega_n \cap E_k$. As the inclusion $E_k \to E_n$ $(n \geq k)$ is continuous, all $\Omega_n \cap E_k$ are open and therefore Ω is open. So it remains to show that $f^{-1}\colon \Omega \to E$ is holomorphic, for which it suffices to show that $f^{-1}\big|_{\Omega\cap E_k}$ is holomorphic for all $k \in \mathbb{N}$. If $x_0 \in \Omega \cap E_k$, then there exists a $n \geq k$ so that $x_0 \subset \Omega_n$. Then $f^{-1}\big|_{\Omega\cap E_k}$ is locally at x_0 just the composition of f_n^{-1} with the inclusion $E_k \to E_n$, and therefore an analytic map to E_n and hence to E. $\square$

In the application of the previous theorem in Proposition 2.1.11 we also need the following fact about holomorphic maps between Banach spaces. Recall that a function $f\colon E \to F$ between locally convex spaces is called locally bounded, if there exists for every $x_0 \in E$ a neighborhood W of x_0 such that $f(W)$ is bounded in F.

Theorem 2.0.14. *Let E, F be Banach spaces and let $f\colon U \to F$ be a map defined on an open subset U of E. Assume that F has a Schauder basis $(y_k)_{k\in\mathbb{N}}$ and let* $\mathrm{proj}_k\colon F \to \mathbb{C}$ *be the continuous coordinate projections $\sum_{l=0}^{\infty} c_l y_l \mapsto c_k$. Then f is holomorphic if and only if it is locally bounded and if all coordinate functions* $\mathrm{proj}_k \circ f$ *are holomorphic.*

Proof: If f is holomorphic, all is coordinate functions are holomorphic as compositions of holomorphic functions. The local boundedness follows from a theorem by S.Dineen ([Din99, p.153]), which concludes the first part of the proof. Now assume conversely that f is locally bounded and that all its coordinates $f_k := \mathrm{proj}_k \circ f$ are holomorphic. The analyticity of maps between Banach spaces can be tested along analytic curves ([KM97]), so it suffices to show that $\phi \circ f \circ \gamma\colon \mathbb{D} \to \mathbb{C}$ is holomorphic for every analytic curve $\gamma\colon \mathbb{D} \to U$ and every continuous linear functional $\phi \in F'$. For every $\varepsilon \in (0,1)$, the image of $\overline{\mathbb{D}}_\varepsilon$ under γ is compact in E, and admits hence a finite covering by open sets $W_1, \dots, W_k$ such that $f(W_k)$ is bounded (because of the local boundedness of f), thus $f \circ \gamma$ is bounded on $\overline{\mathbb{D}}_\varepsilon$. Therefore $\phi \circ f \circ \gamma$ is bounded on every compact subset of $\mathbb{D}$. Write ϕ as $\sum_k \phi_k \mathrm{proj}_k$. Then $\phi \circ f \circ \gamma(t) = \sum_k \phi_k f_k(\gamma(t))$. All $f_k \circ \gamma$ are holomorphic, therefore $\phi \circ f \circ \gamma$ is the pointwise limit of the locally bounded family $\{\sum_{k=0}^{N} \phi_k f_k \circ \gamma\}_{N\in\mathbb{N}}$ and hence itself holomorphic because of Montel's theorem. $\square$

2.1 Constant rank and linearization of analytic maps between power series spaces

In this section we shortly review the constant rank theorem for analytic maps between power series spaces and especially for arquile maps. The rank theorem was proven by H.Hauser and G.Müller in [HM94] in the analytic and by C. Bruschek [Bru09] in the formal setting and states that an arquile map of constant rank can be linearized by analytic automorphisms. In the single variable case, an arquile map $f_\infty \colon \boldsymbol{y}(x) \mapsto f(x, \boldsymbol{y}(x))$ which is induced by a vector of convergent power series whose Jacobian $\partial_y f(x,y)$ has maximal rank at $\boldsymbol{y}_0(x)$, has constant rank as a map between power series spaces $\mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$. In several variables, regularity no longer implies constant rank, since, unlike in the one-variable case, arbitrary submodules of $\mathbb{C}\,[|x|]^N$ do not need to be open in the $\mathfrak{m}$-adic topology. We will show that if the image of the tangential map of an arquile map is open in the $\mathfrak{m}$-adic topology, then the arquile map can be linearized in an $\mathfrak{m}$-adic neighborhood.

Definition 2.1.1. For $r \in \mathbb{R}_+$ let $\|\ \|_r$ be the pseudo-norm

$$\|\ \|_r : \mathbb{C}\,[|x|] \to [0,\infty], f = \sum_{\alpha\in\mathbb{N}^n} f_\alpha x^\alpha \mapsto \sum_{\alpha\in\mathbb{N}^n} |f_\alpha| r^{|\alpha|}.$$

Let ℓ_r^1 be the weighted ℓ^1-space $\ell_r^1 = \{\ f \in \mathbb{C}\,[|x|] \ |\ \|f\|_r < \infty\ \}$ equipped with $\|\ \|_r$, which defines a norm on ℓ_r^1. Each ℓ_r^1 is a Banach-Algebra, so $\|f \cdot g\|_r \leq \|f\|_r \cdot \|g\|_r$. For every $0 < r' < r$ we have that $\|f\|_{r'} < \|f\|_r$ and therefore the injection $\ell_r^1 \hookrightarrow \ell_{r'}^1$ is continuous. The family $(\ell_r^1)_{r\in\mathbb{R}_+}$ forms a *Banach scale*. The space of convergent power series is the union $\mathbb{C}\{x\} = \bigcup_{r\in\mathbb{R}_+} \ell_r^1$ and is equipped with inductive topology, so that it is the inductive limit $\varinjlim_{r\in\mathbb{R}_+} \ell_r^1$ in the category of locally convex spaces. The inclusion $\ell_r^1 \to \ell_{r'}^1$ (for $0 < r' < r$) is compact, and since $\varinjlim_{n\in\mathbb{N}} \ell_{r_n}^1 = \varinjlim_{r\in\mathbb{R}_+} \ell_r^1$ for every sequence $(r_n) \searrow 0$ of positive real numbers, $\mathbb{C}\{x\}$ with the inductive topology is a (DFS)-space (see Theorem 2.0.11). The topological dual space of $\mathbb{C}\{x\}$, equipped with the strong topology, can be identified with the space $\mathcal{H}(\mathbb{C}^N)$ of entire functions on $\mathbb{C}^n$ with the topology of uniform convergence on compact subset, via the duality

$$\langle\ ,\ \rangle \colon \mathbb{C}\{x\} \times \mathcal{H}(\mathbb{C}^n) \to \mathbb{C},\ \left(\sum f_\alpha x^\alpha, \sum h_\alpha x^\alpha\right) \mapsto \sum f_\alpha \phi_\alpha.$$

As a consequence of the Grauert-Hironaka Division Theorem, every finitely generated submodule of $\mathbb{C}\{x\}^N$ is closed (Theorem 2.3.6). For details we refer the reader to [GR13], [HM94], and [Wob12].

Theorem 2.1.2 ([HM94]). *Let $f_\infty \colon \mathbb{C}_0\{x\}^N \to \mathbb{C}_0\{x\}^m$ be an arquile map associated to a convergent power series $f \in \mathbb{C}_0\{x,y\}^m$. Then*

(1) *f_∞ is a holomorphic map.*

(2) *The Fréchet derivative of F at a point $\boldsymbol{y}_0 \in \mathbb{C}\{x\}$ is the $\mathbb{C}\{x\}$-linear map*

$$T_{\boldsymbol{y}_0} f_\infty \colon \mathbb{C}_0\{x\}^N \to \mathbb{C}_0\{x\}^m, \boldsymbol{y}(x) \mapsto (\partial_y f(x, \boldsymbol{y}_0(x))) \cdot \boldsymbol{y}(x)$$

Remark. Since $\mathbb{C}_0\{x\}$ is a (DFS)-space, a map $\mathbb{C}_0\{x\}^N \to \mathbb{C}_0\{x\}^m$ is holomorphic if and only if it is curve-holomorphic ([KM97]). This implies that every solution operator which "comes from a theorem which allows a version with parameters" is holomorphic. For example, let U be the open subset of $\mathbb{C}\{x,y\}^N$ of all those $f \in \mathbb{C}_0\{x,y\}^N$ which satisfy the condition $\det(\partial_y f(0,0)) \neq 0$ and let $\Sigma \colon U \to \mathbb{C}_0\{x\}^N$ be the solution operator which assigns to a given f the unique solution $\boldsymbol{y}$ to $f(x, \boldsymbol{y}(x)) = 0$. A holomorphic curve $\gamma(t) \colon (\mathbb{C}, 0) \to U$ is just a series $F(t,x,y)$ with $F(t_0, x, y) \in U$ for all t_0 close to zero. In particular, $F(t,x,y)$ satisfies the condition of the implicit function theorem, so there exists a unique $\sigma(t,x)$ solving $F(t,x,\sigma(t,x)) = 0$. The uniqueness implies that $\Sigma(F(t,x,y)) = \sigma(t,x)$, and hence Σ maps analytic curves to analytic curves, i.e. it is curve-holomorphic.

Definition 2.1.3. Let E, F be locally convex spaces, let U be an open subset of E and let $f \colon U \to F$ be an analytic map. By $T_a f$ we denote the tangential map of f at $a \in U$. An analytic curve in E is a germ of an analytic map $\gamma \colon (\mathbb{C}, 0) \to E$, the set of all such curves will be denoted by $\mathcal{C}(E)$. We say that f has *constant rank* at $a_0 \in E$, if the following conditions are satisfied:

(1) There exists a closed subspace J of F so that $\operatorname{Im} T_{a_0} \oplus J = F$.

(2) For every analytic curve in E with $\gamma(0) = a_0$, every $\sigma \in \mathcal{C}(F)$, there exist unique $b \in \mathcal{C}(E)$ and $\tau \in \mathcal{C}(J)$ so that

$$\sigma(t) = (T_{\gamma(t)} f) b(t) + \tau(t)$$

This definition is per se weaker then the usual one, to which we will refer as *pointwise constant rank*: f has pointwise constant rank if F splits into $\operatorname{Im} T_a \oplus J$ for all $a \in U$. However, in applications of the rank theorem in this context, constant rank implies pointwise constant rank ([HM94, p.100]). A concept related to constant rank is that of flatness. An analytic map $f \colon U \to F$ is called *flat* at $a_0 \in U$ if for every curve γ through a_0, every $v \in \ker T_{a_0}$ can be lifted an analytic curve $V(t)$ in E with $V(0) = v$, satisfying $(T_{\gamma(t)} f) V(t) = 0$. Constant rank implies flatness, and the converse is true for arquile maps.

Proposition 2.1.4 ([HM94, p.99])**.** *Let $f_\infty\colon \mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ be an arquile map. Then f_∞ is flat at $\boldsymbol{y}_0 \in \mathbb{C}\{x\}^N$ if and only f_∞ has constant rank at $\boldsymbol{y}_0$.*

Using the Division Theorem by Grauert/Hironaka, it is possible to construct a type of pseudo-inverse σ (which is called a *scission* in [HM94]) to the tangential map $T_{a_0} f_\infty$ of an arquile map. Restricted to a sufficiently neighborhood $\mathfrak{m}^c\mathbb{C}\{x\}^N$, the higher-order terms of $\sigma \circ f_\infty$ become in some sense contractive, and f_∞ can be linearized if it has constant rank on $\mathfrak{m}^c\mathbb{C}\{x\}^N$.

Proposition 2.1.5 (Rank Theorem ([HM94, p.96]))**.** *Let $f_\infty\colon \mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ be an arquile map satisfying $f_\infty(0) = 0$. Then there exists a $\delta \in \mathbb{N}$ so that for every $c \geq \delta$ for which the restricted map $f_\infty\colon \mathfrak{m}^c\mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ has constant rank at 0 there exist local analytic isomorphisms Φ of $\mathfrak{m}^c\mathbb{C}\{x\}^N$ and Ψ of $\mathbb{C}\{x\}^m$ at 0 which linearize f, i.e. such that $\Phi \circ f_\infty \circ \Psi = T_0 f_\infty$.*

In order to calculate a direct compliment of a finitely generated power series module, one usually has to calculate a standard basis for the module, which is a laborious task. Flatness, however, can sometimes be easily checked. The tangential map $T_{\boldsymbol{y}_0} f_\infty$ of an an arquile map f_∞ at a point $\boldsymbol{y}_0$ is the $\mathbb{C}\{x\}$-module homomorphism $b(x) \mapsto \partial_y f(x, \boldsymbol{y}_0(x)) \cdot b(x)$. An analytic curve $\gamma\colon (\mathbb{C}, 0) \to \mathbb{C}\{x\}^N$ is just a vector of power series $\gamma(t,x) \in \mathbb{C}\{t,x\}^N$, so f_∞ is flat at $\boldsymbol{y}_0$ if for every $\gamma(t,x) \in \mathbb{C}\{t,x\}^N$ with $\gamma(0,x) = \boldsymbol{y}_0(x)$, every $v(x) \in \ker \partial_y f(x, \boldsymbol{y}_0(x))$ can be lifted to a $V(t,x) \in \mathbb{C}\{t,x\}^N$ solving $\partial_y f(x, \gamma(t,x)) \cdot V(t,x) = 0$ and $V(0,x) = \boldsymbol{y}_0(x)$.

Theorem 2.1.6. *Let $x = x_1$ be a single variable. Let $f(x,y) = (f_1, \ldots, f_m) \in \mathbb{C}\{x,y\}$ and let f_∞ be the arquile map associated to f. Assume that $\boldsymbol{y}_0$ is a simple solution to $f(x, \boldsymbol{y}(x)) = 0$. Then there exists an $\mathfrak{m}$-open module in $\mathbb{C}\{x\}^N$ so that the restriction of f_∞ to U has constant rank.*

Proof: After reordering the variables and conducting a transformation $y \mapsto y - \boldsymbol{y}_0(x)$ we may assume that $\boldsymbol{y}_0 = 0$ and that $\Delta(x) = \det(\partial_{y^{(2)}} f(x,0)) \neq 0$, as in section 1.4. Let d be the order of $\Delta(x)$ and factor $\Delta(x)$ into $x^d \cdot u(x)$, where $u(x)$ is a unit in $\mathbb{C}\{x\}$. Set $U = \mathcal{V}_0 = \langle x \rangle^{2d+1}\mathbb{C}\{x\}^{N-m} \times \langle x \rangle^{d+1}\mathbb{C}\{x\}^m$ as in Theorem 1.4.2. We will show that f_∞ restricted to U is flat. Let $\boldsymbol{v}_0 \in \ker(\partial_y f(x,0)) \cap U$ and let $\gamma\colon (\mathbb{C}, 0) \to \mathbb{C}\{x\}$ be a curve through zero, with values in U. We have to show that v can be lifted to $V(t,x) \in \mathbb{C}\{t,x\}$ solving $(\partial_y f(x, \gamma(t,x))) \cdot V(t,x) = 0$ with $V(0,x) = \boldsymbol{v}_0$. Write $V(t,x)$

as $\boldsymbol{v}_0 + t \cdot w(t,x)$, $\boldsymbol{v}_0 = (x^{2d+1}\boldsymbol{a}_0^{(1)}, x^d \boldsymbol{a}_0^{(2)})$, $w = (x^{2d+1}\boldsymbol{a}^{(1)}(x,t), x^{d+1}\boldsymbol{a}^{(2)}(x,t))$. Then

$$\begin{aligned}
&\partial_y f(x,\gamma(t,x))V(t,x) = 0 \\
\Leftrightarrow\ &\partial^*_{y^{(2)}} f(x,\gamma(t,x)) \cdot \partial_y f(x,\gamma(t,x))V(t,x) = 0 \\
\Leftrightarrow\ &\partial^*_{y^{(2)}} f(x,\gamma(t,x))\partial_{y^{(1)}} f(x,\gamma(t,x))x^{2d+1}(\boldsymbol{a}_0^{(1)} + t\cdot \boldsymbol{a}^{(1)}) + u(x)x^{2d+1}(\boldsymbol{a}_0^{(2)} + t \cdot \boldsymbol{a}^{(2)}) = 0 \\
\Leftrightarrow\ &u(x)^{-1}\left(\partial^*_{y^{(2)}} f(x,\gamma(t,x))\partial_{y^{(1)}} f(x,\gamma(t,x))(\boldsymbol{a}_0^{(1)} + t\cdot \boldsymbol{a}^{(1)}\right) + \boldsymbol{a}_0^{(2)} = -t\cdot \boldsymbol{a}^{(2)}
\end{aligned}$$

The left side of the equation vanishes for $t = 0$, and can therefore be divided by t. Choosing $\boldsymbol{a}^{(1)}$ arbitrary and $\boldsymbol{a}^{(2)}$ as above yields then the desired lifting $V(t,x)$ of $v(x)$. □

Remark. The previous theorem follows of course also from Proposition 1.4.7

Example 2.1.7. In several variables, the previous statement is false. Let $f(x_1,x_2,y_1,y_2) = x_1^3 y_1 - x_2 y_2^3$. Then $\boldsymbol{y}_0(x) := 0$ is a regular point of $\mathcal{Y}(f)$, since $\partial_{y_1} f(x,0) = x_1^3$. We claim that f does not have constant rank at $\boldsymbol{y}_0$. Since constant rank is equivalent to flatness, it is enough to show that there exists a curve $\gamma(t,x) \in \mathbb{C}\{t,x\}^2$ with $\gamma(0) = 0$ and an element $b_0(x)$ of $\ker(\partial_y f(x,0))$ which cannot be lifted along $\gamma(t)$ to a relation $b(t,x)$ with $v(0,x) = v_0$ among the columns of $\partial_y f(x,\gamma(t,x))$, i.e. which solves $\partial_y f(x,\gamma(t,x)) \cdot b(t,x) = 0$. The curve can be chosen in fact in an arbitrary $\mathfrak{m}$-adic neighborhood of 0: We consider the curve $\gamma(t) = (tx_1^n, tx_2^n)$, where n is an arbitrary positive integer. Let $b_0 = (0,(x_1+x_2)) \in \ker \partial_y f(x,0)$ and let $b(t,x) = b_0 + t\cdot(h_1(t,x),h_2(t,x)) \in \mathbb{C}\{t,x\}^2$ be a deformation of b_0. Suppose that $b(t,x)$ solves

$$(\partial_y f)(x,\gamma(t,x)) \cdot b(t,x) = (x_1^3, -3x_2(tx_2^n)^2) \cdot (th_1(t,x),(x_1+x_2) + th_2(t,x)) = 0$$

Then

$$x_1^3 \cdot th_1(t,x) - (3x_2 t^2 x_2^{2n})(x_1 + x_2 + th_2(t,x)) = 0$$

and hence

$$-(3x_2t^2x_2^{2n})(x_1 + x_2 + th_2(t,x)) \equiv 0 \mod x_1^3$$

which leads to the contradiction

$$(x_1 + x_2 + th_2(t,x)) \equiv 0 \mod x_1^3.$$

In particular, this shows that the module of initial terms of $\partial_y f(x,0)$ is not stable under $\mathfrak{m}$-adic small pertubations of $\boldsymbol{y}_0(x)$: For every $n \in \mathbb{N}$ there exists an $h \in \mathfrak{m}^n\mathbb{C}\{x\}^2$ so that $\mathrm{In}\langle \partial_y f(x, \boldsymbol{y}_0(x) + h(x))\rangle \neq \mathrm{In}\langle \partial_y f(x, \boldsymbol{y}_0(x))\rangle$.

Nevertheless it is rather simple to directly solve the equation $x_1^3 y_1 - x_2 y_2^3 = 0$. Since $\mathbb{C}\{x\}$ is a unique factorization domain, the above equation is equivalent to $x_1^3 x_2 w_1 - x_2 x_1^3 w_2^3 = 0$, under the transformation $y_1 = x_2 w_1$, $y_2 = x_1 w_2$, which simplifies to $w_1 = w_2^3$. So we obtain an isomorphism $\varphi\colon \mathbb{C}\{x\} \to \mathcal{Y}(f)$, $\varphi(w_2(x)) = (x_2 w_2(x)^3, x_1 w_2(x))$. Phrased in geometric terms, $\mathcal{Y}(f)$ is contained in the proper submodule $M = x_2 \cdot \mathbb{C}\{x\} \times x_1 \cdot \mathbb{C}\{x\}$ of the affine space $\mathfrak{m}\mathbb{C}\{x\}^2$. On the algebraic side we observe that even though f generates the saturated ideal $\mathcal{I}_{\mathcal{Y}}$, i.e. every arquile function that vanishes on $\mathcal{Y} = \mathcal{Y}(f)$ is a multiple of f, there exist additional relations which are not part of $\mathcal{I}_{\mathcal{Y}}$ that can be added to f without changing the solution set $\mathcal{Y}$. Note that the membership condition $\boldsymbol{y}(x) \in M$ is not arquile (it is textile), this is why the fact that $Y(f) \subset M$ is not reflected by $I_{\mathcal{Y}}$. It is not difficult to show that f_∞ restricted to M has constant rank.

2.1.1 Finite Codimension

In the case of a single variable x, a matrix $A(x) \in \mathbb{C}\,[|x|]^{m \times N}$ of generic full rank induces an $\mathfrak{m}$-adically open $\mathbb{C}\,[|x|]$-module morphism $\mathbb{C}\,[|x|]^N \to \mathbb{C}\,[|x|]^m$. This is no longer true in the case of several variables and is the reason why an arquile map f_∞ at a regular point may not have constant rank. If, however, $T_{\boldsymbol{y}_0} f_\infty$ is $\mathfrak{m}$-adically open, then f_∞ has constant rank at $\boldsymbol{y}_0$. If the tangential map $T_{\boldsymbol{y}_0} f_\infty$ is $\mathfrak{m}$-open, then there exists an $\mathfrak{m}$-adic neighborhood U of $\boldsymbol{y}_0$, so that the image of the non-linear terms of f_∞ are included in the image of the tangential part. In [Bru09] such maps were called *quasi-submersions* and the author observed that these maps have constant rank.

Theorem 2.1.8. *Let $f_\infty\colon \mathfrak{m}\mathbb{C}\{x\}^N \to \mathfrak{m}\mathbb{C}\{x\}^m$ be an arquile map with $f_\infty(0) = 0$ so that the module generated by the columns of $\partial_y f(x, 0)$ is $\mathfrak{m}$-adically open in $\mathbb{C}\{x\}^m$. Then there exists a $\delta \in \mathbb{N}$ so that for all $d \geq \delta$ the restriction of $f_\infty\colon \mathfrak{m}^d\mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ can be linearized by local automorphisms.*

Proof: Let $A(x, y) = \partial_y f(x, y)$. By assumption, there exists a $\delta \in \mathbb{N}$ so that $\mathfrak{m}^\delta\mathbb{C}\{x\}^m \subset A(x, 0) \cdot \mathfrak{m}\mathbb{C}\{x\}^N$. Let $d \in \mathbb{N}, d \geq \delta$ and let $\boldsymbol{y} \in \mathfrak{m}^d\mathbb{C}\{x\}^N$. Using Taylor expansion, we see that $A(x, \boldsymbol{y}(x)) = A(x, 0) + H(x)$, where $H(x)$ is a matrix with entries of order $\geq d$. Hence the columns of $H(x)$ are included in the image of $A(x, 0)$ and consequently $H(x)$ factors into $A(x, 0) \cdot B(x)$ for some matrix $B(x) \in \mathfrak{m}\mathbb{C}\{x\}^{N \times N}$. So $A(x, \boldsymbol{y}(x)) = A(x, 0)(\mathrm{id} + B(x))$ and since all entries of $B(x)$ vanish at zero, $(\mathrm{id} + B(x))$ is invertible and therefore $\operatorname{Im} A(x, \boldsymbol{y}(x)) = \operatorname{Im} A(x, 0)$ for all $\boldsymbol{y} \in \mathfrak{m}^c\mathbb{C}\{x\}^N$. In particular, $f_\infty\colon \mathfrak{m}^c\mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ has pointwise constant rank and may therefore be linearized by Proposition 2.1.5. $\square$

Under the strong hypothesis of the previous theorem, we are able to prove a more precise statement on the linearization in this situation. Our arguments and techniques follow closely the ones of [HM94]).

Definition 2.1.9. In the following, we consider $\mathbb{C}\{x\}$ with the homogeneous ℓ^1-norm structure: For a power series $f = \sum f_\alpha x^\alpha$, we set $\|f\|_r = \sum |f_\alpha| r^{|\alpha|} = \sum_{k=0} \sum_{|\alpha|=k} |f_\alpha| r^k$. Let $E \subset \mathbb{C}\{x\}^m$, and $F \subset \mathbb{C}\{x\}^N$ be linear subspaces and let $T\colon E \to F$ be a linear map. Then let $\|T\|_{r,\mathrm{OP}} := \sup_{b\in E, \|b\|_r=1} \|T(b)\|_r$. Note that $\|T\|_{r,\mathrm{OP}}$ might not be finite, but if T is induced from a $\mathbb{C}\{x\}$-module homomorphism, then $\|T(b)\|_r$ is finite for all sufficiently small r (see [HM94]).

Lemma 2.1.10. *Let $L\colon \mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ be a $\mathbb{C}\{x\}$-linear map and suppose that $L(\mathfrak{m}^c\mathbb{C}\{x\}^N)$ contains $\mathfrak{m}^\delta\mathbb{C}\{x\}^m$. Let $E = L^{-1}(\mathfrak{m}^\delta\mathbb{C}\{x\}^m)$. Then there exists a right-inverse*
$\sigma\colon \mathfrak{m}^\delta\mathbb{C}\{x\}^m \to E$ of $L\big|_E$ with the following properties:

(1) *σ satisfies the estimate*
$$\|\sigma\|_{r,\mathrm{OP}} < \frac{C}{r^{\delta-c}}$$
for all $0 < r < r_0$, and some constant $C > 0$, which is independent of r.

(2) *If $\operatorname{ord} v(x) \geq \delta + d$, then $\operatorname{ord} \sigma(v(x)) \geq c + d$.*

Proof: We choose the set of monomials $\{x^{\alpha,i} \mid |\alpha| = \delta, \alpha \in \mathbb{N}^n, 1 \leq i \leq m\}$ as a standard basis for $\mathfrak{m}^\delta\mathbb{C}\{x\}^m$, where $x^{\alpha,i} = x^\alpha e_i$ and e_i denotes the i^{th} standard unit vector of $\mathbb{C}^m$. By assumption, there exist $b_{\alpha,i}(x) \in \mathfrak{m}^c\mathbb{C}\{x\}^N$ so that $x^{\alpha,i} = L(b_{\alpha,i})$ for all $|\alpha| = \delta$ and $1 \leq i \leq m$. We enumerate the monomials $\{(\alpha, i) \mid |\alpha| = \delta, 1 \leq i \leq m\}$ as $\gamma_1, \ldots, \gamma_{N'}$ (where $N' = m \cdot \binom{d+n-1}{d}$)). Set $B(x) := (b_{\gamma_1}(x), \ldots, b_{\gamma_{N'}}(x)) \in \mathfrak{m}^c\mathbb{C}\{x\}^{N\times N'}$. Let $\tilde{L}\colon \mathbb{C}\{x\}^{N'} \to \mathfrak{m}^\delta\mathbb{C}\{x\}^m$, $(q_i)_{1\leq i\leq N'} \mapsto \sum q_i x^{\gamma_i}$. Let Δ be the Newton diagram generated by the x^{γ_i}, and let $\mathcal{O}_\Gamma \subset \mathbb{C}\{x\}^{N'}$ thee vector space of the canonical normal forms of the quotients (Theorem 2.3.6), so that each $f \in \mathbb{C}\{x\}^m$ admits a unique representation $f = \sum q_i x^{\gamma_i} + p$ such that $q = (q_1, \ldots, q_m) \in \mathcal{O}_\Gamma$ and no term of p is divisible by any of the x^{γ_i}. Every $f \in \mathfrak{m}^\delta\mathbb{C}\{x\}^m$ admits a unique representation as $f = q_1 x^{\gamma_1} + \cdots + q_{N'} x^{\gamma_{N'}}$, with $(q_1, \ldots, q_{N'}) \in \mathcal{O}_\Gamma$. By the construction of Γ, the Newton diagram of f is partitioned according to the divisibility of the terms by $q_i x^{\gamma_i}$, whence $\|f\|_r = \sum_{i=1}^{N'} \|x^{\gamma_i} q_i\|_r$ and thus $\sum_{i=1}^{N'} \|q_i\|_r \leq \frac{1}{r^\delta} \|f\|_r$. The map $\tilde{\sigma}\colon \mathfrak{m}^\delta \mathbb{C}\{x\}^m \to \mathcal{O}_\Gamma, f \mapsto q(f)$ is a right inverse of $\tilde{L}$ and satisfies the estimate $\|\tilde{\sigma}\|_{r,\mathrm{OP}} \leq \frac{1}{r^\delta}$. Since $\operatorname{ord} b_{\gamma_i} = c$, there exist a $C > 0$ and $r_0 > 0$, so that for all $r \in (0, r_0)$ the estimate $N' \cdot \max_{1\leq i \leq N'} \|b_{\gamma_i}\|_r < C \cdot r^c$ holds. Let now finally be $T_B\colon \mathbb{C}\{x\}^{N'} \to \mathfrak{m}^c\mathbb{C}\{x\}^N, q \mapsto B \cdot q$. Then $\tilde{L} = L \circ T_B$ and $\sigma := T_B \circ \tilde{\sigma}$ is a right-inverse of the restriction of L to $E = L^{-1}(\mathfrak{m}^\delta\mathbb{C}\{x\}^m)$. Indeed, if $w \in \mathfrak{m}^\delta \mathbb{C}\{x\}^m$, we have $(L \circ T_B \circ \tilde{\sigma})(w) = (\tilde{L} \circ \tilde{\sigma})(w) = w$ and σ satisfies the estimate $\|\sigma\|_{r,\mathrm{OP}} = \|T_B \circ \tilde{\sigma}\|_{r,\mathrm{OP}} \leq C \cdot r^c \cdot \frac{1}{r^\delta} = \frac{C}{r^{\delta-c}}$. For the second property, observe that $\tilde{\sigma}$ decreases the order by δ and that T_B increases the order by c, since $\operatorname{ord} B \geq c$. □

Remark. Condition (2) of the preceeding theorem, formulated in the terminology of [Bru09], signifies that the *degree of contraction* of σ is greater or equal to $c - \delta$.

Proposition 2.1.11. *Let $f_\infty\colon \mathfrak{m}\mathbb{C}\{x\}^N \to \mathfrak{m}\mathbb{C}\{x\}^m$ be an arquile map with $f_\infty(0) = 0$ so that the module generated by the columns of $\partial_y f(x, 0))$ is $\mathfrak{m}$-adically open in $\mathbb{C}\{x\}^m$. Let $L = T_0 f_\infty\colon \mathfrak{m}\mathbb{C}\{x\}^N \to \mathfrak{m}\mathbb{C}\{x\}^m$ be the tangential map $b(x) \mapsto \partial_y f(x, 0) \cdot b(x)$. Then there exists an $\mathfrak{m}$-adic zero neighborhood U and a global textile isomorphism $\Psi\colon U \to U$ so that f_∞ factors into*

$$f_\infty = L \circ \Psi.$$

If $\mathfrak{m}^\delta\mathbb{C}\{x\}^m \subset \partial_y f(x, 0) \cdot \mathfrak{m}\mathbb{C}\{x\}^N$, then U can be chosen as $\mathfrak{m}^\delta\mathbb{C}\{x\}^N$.

Proof: By assumption, there exists a $\delta \in \mathbb{N}$ so that $\mathfrak{m}^\delta \cdot \mathbb{C}\{x\}^m \subset \partial_y f(x, 0) \cdot \mathfrak{m}\mathbb{C}\{x\}^N$. Set $U = \mathfrak{m}^\delta\mathbb{C}\{x\}^N$ and let $y(x) \in U$. We expand $f(x, \boldsymbol{y}(x))$ as

$$\underbrace{f(x, 0)}_{=0} + \partial_y f(x, 0) \cdot \boldsymbol{y}(x) + q(x, \boldsymbol{y}(x)),$$

where $q(x, y) = \sum_{1\leq i,j\leq N} q_{i,j}(x, y) y_i y_j$ is the non-linear part of f in y. Let $L\colon \mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$, $\boldsymbol{y}(x) \mapsto \partial_y f(x, 0) \cdot \boldsymbol{y}(x)$ and let $\sigma\colon \mathfrak{m}^\delta\mathbb{C}\{x\}^m \to \mathfrak{m}\mathbb{C}\{x\}^N$ be a right-inverse of L as in the previous lemma. If $\boldsymbol{y}(x) \in U$, then $\operatorname{ord} q(x, \boldsymbol{y}(x)) \geq 2\delta$ and $\operatorname{ord} \sigma(q(x, \boldsymbol{y}(x))) \geq \delta + 1$, thus $\sigma(q(x, \boldsymbol{y}(x))) \in U$. For $\boldsymbol{y}(x) \in \mathfrak{m}^\delta\mathbb{C}\{x\}^N$, we have

$$q(x, \boldsymbol{y}(x)) = L \circ \sigma(q(x, \boldsymbol{y}(x))).$$

and hence

$$f(x, \boldsymbol{y}(x)) = L(\boldsymbol{y}(x)) + L(\sigma(q(x, \boldsymbol{y}(x)))) = L\,\underbrace{(\boldsymbol{y}(x) + \sigma(q_\infty(\boldsymbol{y}(x)))}_{=:\Psi(\boldsymbol{y}(x))}$$

It remains to show that $\Psi\colon U \to U$ is an analytic isomorphism, for which we will apply Proposition 2.0.13. Set $H = \sigma \circ q_\infty$, so that $\Psi = \mathrm{id} + H$. As q consists only of quadratic or higher terms, $T_0 q_\infty = 0$, and hence the tangential map of Ψ is the identity at $0 \in U$. Let $B_{R,K}$ be the open unit ball in $\ell^1(R)^N \subset \mathbb{C}\{x\}^N$. We claim that for every $\varepsilon > 0$ we have

$$\mathfrak{m}^{\delta+1}\mathbb{C}\{x\}^N = \bigcup_{R>0} B_{R,\varepsilon R^{\delta-1}}.$$

Let $g(x) \in \mathfrak{m}^\delta \mathbb{C}\{x\}^N$ and let $\sum_{k\geq\delta} g_k(x)$ be the decomposition of $g(x)$ into its homogeneous terms. Let $R > 0$ so that $\|g(x)\|_R < \infty$. If $0 < S < R$, then

$$\begin{aligned}\|g(x)\|_S = \sum_{k\geq\delta} \|g_k\|_S = \sum_{k\geq\delta} \|g_k\|_R \left(\frac{S}{R}\right)^k \leq \left(\frac{S}{R}\right)^\delta \|g(x)\|_R = \\ = S^{\delta-1} \underbrace{\frac{S}{R^\delta} \|g(x)\|_R}_{\xrightarrow{S\to 0} 0}.\end{aligned}$$

So for R fixed and S small enough, we obtain $\|g(x)\|_S < \varepsilon S^{\delta-1}$. The same estimate shows that

$$B_{R,\varepsilon R^{\delta-1}} \subset B_{S,\varepsilon S^{\delta-1}},$$

for every $0 < S < R$. Now we show that there exist constants $M, \varepsilon, R_0 \in \mathbb{R}_+$ so that

$$H\colon B_{R,\varepsilon R^{\delta-1}} \to B_{R,M\varepsilon^2 R^{\delta-1}}$$

for all $0 < R < R_0$. By the previous lemma, there exists a constant $C > 0$ and $R_0 \in (0,1)$ so that $\|\sigma\|_{R,\mathrm{OP}} < \frac{C}{R^{\delta-1}}$ for all $R \in (0, R_0)$, where we choose R_0 so small that $\|q\|_{R_0} < \infty$. Set $\varepsilon = R_0$, and $\tilde{M} := \max_{i,j} \|q_{i,j}(x,y)\|_\varepsilon < \infty$. Then for $0 < R \leq \varepsilon$ and $\|\boldsymbol{y}(x)\|_R < \varepsilon$ we obtain $\|q_{i,j}(x, \boldsymbol{y}(x))\|_R < \tilde{M}$. Now the crucial estimate is that

$$\|q(x, \boldsymbol{y}(x))\|_R \leq \sum_{i,j} \|q_{i,j}(x, \boldsymbol{y}(x))\|_R \|y_i(x)\|_R \|y_j(x)\|_R \leq \tilde{M} \cdot \|\boldsymbol{y}(x)\|_R^2$$

for all $\|\boldsymbol{y}(x)\|_R \leq \varepsilon$ and $0 < R \leq \varepsilon$. Set $M := \tilde{M} \cdot C$. Then for $\boldsymbol{y}(x) \in B_{R,\varepsilon R^{\delta-1}}$, $0 < R < R_0$, we obtain the estimates $\|H(\boldsymbol{y}(x))\|_R = \|q_\infty(\boldsymbol{y}(x))\|_R < \tilde{M}\varepsilon^2 R^{2\delta-2}$,

$$\|\sigma \circ q_\infty(\boldsymbol{y}(x))\|_R < \varepsilon^2 M R^{\delta-1}$$

and

$$\|\Psi(\boldsymbol{y}(x))\|_R = \|(\mathrm{id} + H)(\boldsymbol{y}(x))\|_R < \varepsilon R^{\delta-1}(1 + M\varepsilon)$$

Therefore,

$$\Psi\colon B_{R,\varepsilon R^{\delta-1}} \to B_{R,\varepsilon R^{\delta-1}(1+M\varepsilon)}.$$

Since Ψ has polynomial and hence analytic coefficient functions and as it is bornological (in the sense the it maps bounded to bounded sets), it is holomorphic as a local map $B_{R,\varepsilon R^{\delta-1}} \to B_{R,\varepsilon R^{\delta-1}(1+M\varepsilon)}$ of the Banach space ℓ^1_R by Theorem 2.0.14. We check that condition (4) of Proposition 2.0.13 is satisfied. Let

$$\nu_R = \frac{\varepsilon^2 R^{2(\delta-1)}}{4\varepsilon R^{\delta-1}(1+M\varepsilon)} = \frac{\varepsilon}{4(1+M\varepsilon)} R^{\delta-1}$$
$$\rho_R = \frac{\varepsilon^2 R^{2(\delta-1)}}{8\varepsilon R^{\delta-1}(1+M\varepsilon)} = \frac{\varepsilon}{8(1+M\varepsilon)} R^{\delta-1}$$

Then $B_{R,\nu_R} \subset B_{S,\nu_S}$ and $B_{R,\rho_R} \subset B_{S,\rho_S}$. By Proposition 2.0.13 Ψ is therefore a biholomorphism between $\bigcup_{0\leq R<R_0} B_{R,\nu_R} = \mathfrak{m}^\delta\mathbb{C}\{x\}^N$ and $\bigcup_{0\leq R<R_0} B_{R,\rho_R} = \mathfrak{m}^\delta\mathbb{C}\{x\}^N$ and the claim follows.

$\square$

Remark. The previous result holds also in the formal setting. It is easily seen that $\Psi - \mathrm{id}$ has a positive degree of contraction in the sense of [Bru09] and hence Ψ is a formal textile isomorphism by [Bru09, Theorem 3]. In the terminology of the cited author, the map f_∞ is a *quasi-submersion* if the conditions of the theorem are satisfied.

Remark. The condition that $T_0 f_\infty$ is open in the $\mathfrak{m}-$adic topology is of course quite restrictive. However, it seems to be the only known sufficient condition for the linearization of an arquile map, with is stable with respect to arbitrary pertubations at a sufficient high degree and which is determined by a sufficiently high $k-$jet of f. In general, constant rank is destroyed by $\mathfrak{m}$-adic pertubations: For example, take $g = 0$ and f as in Example 2.1.7. Then g_∞ has constant rank, but $(x_1^n \cdot f)_\infty$ does not have constant rank for any n.

Definition 2.1.12. Let $f\colon X \to Y$ be an arbitrary function. Then a pseudo-inverse of f is a map $g\colon Y \to X$ so that $f \circ g \circ f = f$, so any map which assigns to a point $f(x)$ in the image of f a point z in the pre-image $f^{-1}(f(x))$. Note that the terminology is not unified, depending on the author pseudo-inverses are also called generalized inverses or scissions.

Corollary 2.1.13. *Let $f_\infty\colon \mathfrak{m}\mathbb{C}\{x\}^N \to \mathfrak{m}\mathbb{C}\{x\}^m$ be an arquile map with $f_\infty(0) = 0$ and assume that $\partial_y f(x,0) \cdot \mathfrak{m}\mathbb{C}\{x\}^N$ contains $\mathfrak{m}^\delta\mathbb{C}\{x\}^N$. Then the restriction of f_∞ to $U = \mathfrak{m}^{\delta+1}\mathbb{C}\{x\}^N$ admits an analytic textile pseudo-inverse.*

Proof: By [HM94] every linear map $L\colon \mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ admits a continuous linear scission σ. So let σ be the scission to the tangential map L of f_∞. The restriction of f_∞ to U factorizes into $L \circ \Psi$ as in the previous theorem. It is easily checked that $\Psi^{-1} \circ \sigma$ is an analytic pseudo-inverse. $\square$

Proposition 2.1.14 (Homogenization of arquile maps). *Let $f(x,y) \in \mathfrak{m}\mathbb{C}\{x,y\}^N$ with $f(x,0) = 0$ and let $\partial_y f(x,0) = H_d(x) + \sum_{k\geq d+1} H_k(x)$ be the decomposition of $\partial_y f(x,0)$ into its homogeneous parts, and assume that $H_d(x)\cdot \mathfrak{m}\mathbb{C}\{x\}^N$ is $\mathfrak{m}$-open. Denote by L_d the linear operator associated to H_d. Then there exists an $\mathfrak{m}-$adic zero neighborhood U and a textile isomorphism $\varphi\colon U \to U$ so that the restriction of f_∞ factors into*

$$f_\infty\big|_U = L_d \circ \varphi.$$

If $\mathfrak{m}^\delta\mathbb{C}\{x\}^m \subset H_d(x)\cdot\mathfrak{m}\mathbb{C}\{x\}^N$, then U can be chosen as $\mathfrak{m}^\delta\mathbb{C}\{x\}^N$.

Proof: Let $L = T_0(f) = L_d + S$, where S denotes the multiplication operator of the higher terms $b(x) \mapsto (\sum_{k\geq d+1} H_k(x))\cdot b(x)$. Assume that $\mathfrak{m}^\delta\mathbb{C}\{x\}^m \subset H_d \cdot \mathfrak{m}\mathbb{C}\{x\}^N$. Since H_d is homogeneous, the terms of order δ in the image of H_d can be generated by homogeneous terms of order $c = \delta - d$, so $H_d \cdot \mathfrak{m}^c\mathbb{C}\{x\}^N = \mathfrak{m}^\delta\mathbb{C}\{x\}^m$. By Proposition 2.1.11, there exists a textile automorphism $\Psi\colon U \to U$, where $U = \mathfrak{m}^\delta\mathbb{C}\{x\}^N$, so that $f_\infty\big|_U$ factors into $L \circ \Psi$. Now we are going to factorize L. Let $\sigma\colon \mathfrak{m}^\delta\mathbb{C}\{x\}^m \to \mathfrak{m}^c\mathbb{C}\{x\}^N$ be a right-inverse of $L_d\big|_{\mathfrak{m}^c\mathbb{C}\{x\}^N}$ as in Lemma 2.1.10, satisfying $\|\sigma\|_{r,\mathrm{OP}} < \frac{C}{r^{\delta-c}}$ and for all $0 < r < r_0$. Then

$$L = (L_d + S) = L_d \circ (\mathrm{id} + \sigma \circ S).$$

Since $\operatorname{ord} S \geq d+1$, there exists an $M > 0$ so that $\|S\|_r < Mr^{d+1}$ for all $0 < r < r_1$ (for some $M > 0, r_1 < r_0$. Note that $\sigma \circ S$ maps $\mathfrak{m}^\delta\mathbb{C}\{x\}^N$ to $\mathfrak{m}^{\delta+1}\mathbb{C}\{x\}^m$ and satisfies the estimate

$$\|\sigma \circ S\|_r \leq \frac{C}{r^{\delta-c}} M r^{d+1} = CMr$$

for all $r < r_1$. In particular, $\|\sigma \circ S\|_r < 1$ for all sufficiently small r, and for such r the map $\mathrm{id} + \sigma \circ S$ is an isomorphism $(\mathfrak{m}^\delta \ell_r^1)^N \to (\mathfrak{m}^\delta \ell_r^1)^N$, and thus a linear automorphism of $\mathfrak{m}^\delta\mathbb{C}\{x\}^N$. $\square$

Example 2.1.15. Let $f(x,y) = (f_1(x,y), f_2(x,y))$, where

$$f_1(x,y) = x_1y_1 + x_2y_1 - x_2y_2 + x_2y_3 + g_1(x,y)$$
$$f_2(x,y) = x_2y_1 + x_1y_2 - x_2y_2 - x_1y_3 + g_2(x,y)$$

with $\operatorname{ord} g_i \geq 3$ and $g_i(x,0) = 0$. The lowest order terms of $\partial_y f(x,0)$ are

$$H_1(x) = \begin{pmatrix} x_1 + x_2 & -x_2 & x_2 \\ x_2 & x_1 - x_2 & -x_1 \end{pmatrix}$$

The set of minors of $H_1(x)$ is $\{x_1^2, x_2^2, -x_1^2 - x_1x_2 - x_2^2\}$ and generates $\mathfrak{m}^2\mathbb{C}\{x\}$. Using the adjoints of the 2×2 submatrices, we see that $\operatorname{Im}(L_1) = \mathfrak{m}^2\mathbb{C}\{x\}^2$. Therefore the linear map associated to the lowest order terms $L_1 \colon \mathfrak{m}\mathbb{C}\{x\}^3 \to \mathfrak{m}^2\mathbb{C}\{x\}^2$ is onto and the restriction of f_∞ to $U = \mathfrak{m}^2\mathbb{C}\{x\}^3$ factors into $L_1 \circ \Psi$, where Ψ is a textile automorphism of U.

Definition 2.1.16. For $d \in \mathbb{N}$, let $\mathcal{J}_d(\mathbb{C}^n, 0)$ be the set of polynomial local biholomorphisms of $(\mathbb{C}^n, 0)$, which fix the origin and whose degree is less or equal to d. Note that if $f \in \operatorname{Aut}(\mathbb{C}^n, 0)$, then $j^d(f) \in \mathcal{J}_d(\mathbb{C}^n, 0)$, so $\operatorname{Aut}(\mathbb{C}^n, 0) = \mathcal{J}_d(\mathbb{C}^n, 0) \bigoplus \mathfrak{m}^{d+1}\mathbb{C}\{x\}^n$.

Definition 2.1.17. If $w(x) \in \mathfrak{m}\mathbb{C}\{x\}^n$, then let $C_w \colon \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ be the associated composition operator $f(x) \mapsto f(w(x))$. For any $k \in \mathbb{N}$, let $C_{w,k} \colon \mathbb{C}\{x\}^k \to \mathbb{C}\{x\}^k, (f_1, \ldots, f_k) \mapsto (C_w(f_1), \ldots, C_w(f_k))$ be the extension of C_w to $\mathbb{C}\{x\}^k$.

Theorem 2.1.18. *Let $f(x) = (f_1, \ldots, f_m) \in \mathbb{C}\{x\}^m$ be a vector of convergent power series, let $L \colon \mathbb{C}\{x\}^n \to \mathbb{C}\{x\}^m$ be the tangential map $b(x) \mapsto \partial_x f(x) \cdot b(x)$. We decompose* $\operatorname{Aut}(\mathbb{C}^n, 0)$ *into the direct sum $\mathcal{J}_d(\mathbb{C}^n) \bigoplus \mathfrak{m}^{d+1}\mathbb{C}\{x\}^n$, and write $\varphi(x)$ as $j^d(\varphi(x)) + h^d(\varphi(x))$, where $h^d(\varphi(x)) = \varphi(x) - j^d(\varphi(x)) \in \mathfrak{m}^{d+1}\mathbb{C}\{x\}^n$. Set $U_d = \mathfrak{m}^{d+1}\mathbb{C}\{x\}^n$. Assume that the image of L is $\mathfrak{m}$-open. Then there exists a $\delta > 0$, textile analytic maps $\Lambda \colon \mathcal{J}_\delta(\mathbb{C}^n, 0) \times \mathfrak{m}\mathbb{C}\{x\}^m \to \mathfrak{m}\mathbb{C}\{x\}^m$ and $\Upsilon \colon \mathcal{J}_\delta(\mathbb{C}^n, 0) \times U_\delta \to U_\delta$, so that for every $p(x) \in \mathcal{J}_\delta(\mathbb{C}^n, 0)$,*

(1) *$\Lambda(p(x), \_)$ is a linear automorphism of $\mathbb{C}\{x\}^m$,*
(2) *$\Upsilon(p(x), \_)$ is an analytic automorphism of U_δ,*

so that

$$f_\infty \colon \operatorname{Aut}(\mathbb{C}^n, 0) \to \mathbb{C}\{x\}^m$$

factorizes into

$$f_\infty(\varphi) = f_\infty(j^\delta(\varphi)) + \Lambda(j^\delta(\varphi), \_) \circ L \circ \Upsilon(j^\delta(\varphi), h^\delta(\varphi)).$$

In other words, f_∞ can be linearized up to the finite-dimensional part $j^\delta(\varphi)$, in the sense that $f_\infty(j^\delta(\varphi)+h^\delta(\varphi))$ is conjugated to an affine linear map in $h^\delta(\varphi)$ for every fixed $j^\delta(\varphi)$. If $m=1$, then δ can be chosen as the Milnor number of f, and for general m as any number δ such that $\mathfrak{m}^\delta\mathbb{C}\{x\}^m \subset L(\mathbb{C}\{x\}^n)$.

Proof: By assumption there exists a $\delta \in \mathbb{N}$ such that $\mathfrak{m}^\delta\mathbb{C}\{x\}^m \subset L(\mathbb{C}\{x\}^N)$. Let $\varphi(x) \in \mathrm{Aut}(\mathbb{C}^n, 0)$, and set $p(x) = j^\delta(\varphi)$, $h(x) = h^\delta(\varphi)$. Then

$$f(\varphi(x)) = f(p(x)+h(x)) = C_{p,m}C_{p,m}^{-1}f(p(x)+h(x)) = C_{p,m} \circ f(x+h(p^{-1}(x))),$$

with $h(p^{-1}(x)) \in U_d$, where p^{-1} denotes the composite inverse of p. Set $F(x,y) = f(x+y)$ and consider $F_\infty\colon \mathfrak{m}^{\delta+1}\mathbb{C}\{x\}^n \to \mathbb{C}\{x\}^m$. Since $\partial_y(F(x,0)) = \partial_x(f(x))$, $\mathfrak{m}^{\delta+1}\mathbb{C}\{x\}^m \subset \partial_y(F(x,0)) \cdot \mathfrak{m}\mathbb{C}\{x\}^N$, and so we can apply Proposition 2.1.11 to conclude the existence of a textile biholomorphism $\Psi\colon \mathfrak{m}^{\delta+1}\mathbb{C}\{x\}^n \to \mathfrak{m}^{\delta+1}\mathbb{C}\{x\}^n$ so that $F_\infty = F(x,0) + L \circ \Psi$. Since $f(p(x)+h(x)) = C_{p,m}F(x, h(p^{-1}(x)))$ we obtain that

$$f(\varphi(x)) = f(p(x)) + C_{p,m} \circ L \circ \Psi(h(p^{-1}(x))). \tag{2.1}$$

Then the maps

$$\begin{aligned}\Lambda(p(x), g(x)) &:= C_{p,m}(g(x))\\ \Upsilon(p(x), h(x)) &:= \Psi \circ C_{p,n}^{-1}(h(x))\end{aligned}$$

linearize f_∞ as in the statement of the theorem. In the hypersurface case $m=1$, we have that $\mathfrak{m}^{\mu(f)} \subset \partial_x(f)$ by Theorem 1.6.4, and in this case we can thus choose δ as $\mu(f)$. $\square$

In [LM07] the authors showed that if a holomorphic map $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ has generic full rank, then the induced map $H\colon \mathrm{Aut}(\mathbb{C}^n,0) \to \mathrm{Gl}_n(\mathbb{C}) \times \mathfrak{m}\mathbb{C}\{x\}^n, \varphi \mapsto (j^1(\varphi), f_\infty(\varphi))$ admits an analytic left inverse $\Phi\colon \mathrm{Gl}_n(\mathbb{C}) \times \mathfrak{m}\mathbb{C}\{x\}^n \to \mathrm{Aut}(\mathbb{C}^n,0)$. The following theorem is in a similar spirit:

Corollary 2.1.19. *Keep the notation and the assumptions on f from the previous theorem. Then there exists an analytic pseudo-inverse Φ of the map*

$$H\colon \mathrm{Aut}(\mathbb{C}^n,0) \to \mathcal{J}_\delta(\mathbb{C}^n,0) \times \mathfrak{m}\mathbb{C}\{x\}^m, H(\varphi) = (j^\delta(\varphi), f_\infty(\varphi)).$$

In other words, $\Phi(p, \_)$ is an analytic pseudo-inverse of the map $U_\delta \ni h \mapsto f_\infty(p+h)$ for every fixed $p \in \mathcal{J}_\delta(\mathbb{C}^n,0)$ and $\Phi(p,\_)$ varies analytically with p.

Proof: Let σ be a scission of $L\colon U_\delta = \mathfrak{m}^{\delta+1}\mathbb{C}\{x\}^n \to \mathbb{C}\{x\}^m$, and let $\Psi\colon U_\delta \to U_\delta$ be as in the proof of the previous theorem. For $(p,g) \in \mathcal{J}_\delta(\mathbb{C}^n,0)\times\mathbb{C}\{x\}^m$ we define (see Equation 2.1)

$$\Phi(p,g) := p + C_{p,n}\circ\Psi^{-1}\circ\sigma\circ C_{p,m}^{-1}(h^\delta(g - f_\infty(p))),$$

where $h^\delta(g)$ are the terms of order $\geq \delta+1$ of g . We check that Φ is indeed a pseudo-inverse of H:

$$\begin{aligned} H\circ\Phi\circ H(\varphi) &= H\circ\Phi(p, f_\infty(p) + \underbrace{C_{p,m}\circ L\circ\Psi\circ C_{p,n}^{-1}(h^\delta(\varphi)))}_{\mathrm{ord}\geq\delta+1} = \\ &= H(p + C_{p,n}\circ\Psi^{-1}\circ\sigma\circ L\circ\Psi\circ C_{p,n}^{-1}(h^\delta(\varphi))) \\ &= (p, f_\infty(p) + C_{p,m}\circ L\circ\Psi\circ C_{p,n}^{-1}\circ C_{p,n}\circ\Psi^{-1}\circ\sigma\circ L\circ\Psi\circ C_{p,n}^{-1}(h^\delta(\varphi))) = \\ &= (p, f_\infty(p) + C_{p,m}\circ L\circ\Psi\circ C_{p,n}^{-1}(h^\delta(\varphi))) = (p, f_\infty(\varphi)) = H(\varphi) \end{aligned}$$

□

2.2 Textile maps

Definition 2.2.1. A map $F = (F_1,\ldots,F_{N'})\colon \mathbb{C}\left[|x|\right]^N \to \mathbb{C}\left[|x|\right]^{N'}$, $F_i(\boldsymbol{y}) = \sum_{\beta\in\mathbb{N}^n} F_{\beta,i}(\boldsymbol{y})x^\beta$ between two free modules of formal power series is called *textile*, if each coefficient function $F_{\beta,i}$ depends polynomially on a finite number of the coefficients $(y_{\alpha,i})_{\alpha\in\mathbb{N}^n,1\leq i\leq N}$ of the input $\boldsymbol{y} = (\boldsymbol{y}_1,\ldots,\boldsymbol{y}_N), \boldsymbol{y}_i = \sum_{\alpha\in\mathbb{N}^n}\boldsymbol{y}_{\alpha,i}x^\alpha$. We will restrict the discussion to textile maps $F\colon \mathfrak{m}\mathbb{C}\left[|x|\right]^N \to \mathfrak{m}\mathbb{C}\left[|x|\right]^{N'}$, so it is convenient to define the set of all monomial exponents $\mathcal{M}_n$ as

$$\mathcal{M}_n = \{\ \gamma\colon \mathbb{N}^n \to \mathbb{N}\ \mid\ \gamma(0) = 0,\ \gamma(\alpha)\neq 0 \text{ for only finitely many } \alpha\in\mathbb{N}^n\ \}.$$

Then, for $\boldsymbol{y} = (\boldsymbol{y}_1,\ldots,\boldsymbol{y}_N)\in\mathbb{C}\left[|x|\right]^N$, $\boldsymbol{y}_i = \sum y_{\alpha,i}x^\alpha$ and $\gamma = (\gamma_1,\ldots,\gamma_N)\in\mathcal{M}_n^N$ we define

$$\boldsymbol{y}^\gamma := \boldsymbol{y}_1^{\gamma_1}\cdots\boldsymbol{y}_N^{\gamma_N},\ \boldsymbol{y}_i^{\gamma_i} = \prod_{\alpha\in\mathbb{N}^n} y_{\alpha,i}^{\gamma_i(\alpha)}.$$

For notational simplicity we assume that $N' = 1$. Then the polynomial coefficient functions F_β can be written as

$$F_\beta(\boldsymbol{y}) = \sum_{\gamma\in\mathcal{M}^N} F_{\beta,\gamma}\boldsymbol{y}^\gamma,$$

where only finitely many of the $F_{\beta,\gamma}$ are not zero. In applications, the support

$$\operatorname{supp}(F) = \left\{ \ (\beta,\gamma) \in \mathbb{N}^n \times \mathcal{M}_n^N \ \mid \ F_{\beta,\gamma} \neq 0 \ \right\}$$

underlies certain growth conditions. We define the weight of a monomial $\gamma \in \mathcal{M}_n^N$ as the vector

$$\operatorname{wt}(\gamma) = \sum_{i=1}^{N} \sum_{\alpha \in \mathbb{N}^n} \gamma_i(\alpha) \cdot \alpha \in \mathbb{N}^n,$$

the *shape* as

$$\operatorname{sh}(\gamma) = (\operatorname{sh}(\gamma_1), \dots, \operatorname{sh}(\gamma_N)),\ \operatorname{sh}(\gamma_i) = \sum_{\alpha \in \mathbb{N}^n} \gamma(\alpha).$$

and the *multiplicity*

$$\mu(\gamma) := \prod_{i=1}^{N} \mu(\gamma_i),\ \mu(\gamma_i) := \frac{\operatorname{sh}(\gamma_i)!}{\prod_{\alpha \in \mathbb{N}^n} \gamma_i(\alpha)!}$$

We will say that the support is of *linear growth*, if there exists an affine-linear function $L\colon \mathbb{N}^n \to \mathbb{N}^n,\ L(\alpha) = \lambda \cdot \alpha + v$, where $\lambda \in \mathbb{N},\ v \in \mathbb{N}^n$, such that

$$\operatorname{supp}(F) \subset \left\{ \ (\beta,\gamma) \in \mathbb{N}^n \times \mathcal{M}_n^N \ \mid \ \operatorname{wt}(\gamma) \leq L(\beta) \ \right\},$$

and hence

$$F_\beta(\boldsymbol{y}) = \sum_{\operatorname{wt}(\gamma) \leq L(\beta)} F_{\beta,\gamma} \boldsymbol{y}^\gamma.$$

If L can be chosen as the identity, then we will call F of *strict linear growth.* A textile map $F\colon \mathfrak{m}\mathbb{C}\,[|x|]^N \to \mathfrak{m}\mathbb{C}\,[|x|]^{N'}$ is called *analytic*, if $F(\mathfrak{m}\mathbb{C}\{x\}^N) \subset \mathfrak{m}\mathbb{C}\{z\}^{N'}$ and if the restriction on the convergent series is a holomorphic map $F\colon \mathfrak{m}\mathbb{C}\{x\}^N \to \mathfrak{m}\mathbb{C}\{z\}^{N'}$. The analyticity of textile maps whose support is of linear growth admits the following characterization: Such a map is analytic if and only if it maps convergent series to convergent series.

Proposition 2.2.2. *Let $F\colon \mathfrak{m}\mathbb{C}\,[|x|]^N \to \mathfrak{m}\mathbb{C}\,[|x|]$ be a textile map whose support is of linear growth. For $R \in \mathbb{R}_+$ let $\|\boldsymbol{y}\|_{R,\infty} := \sup_{\alpha\in\mathbb{N}^n} |y_\alpha| R^{|\alpha|}$ be the weighted ∞-pseudo-norm of a power series $\boldsymbol{y} = \sum_{\alpha\in\mathbb{N}^n} y_\alpha x^\alpha$ and let $B_{R,M}$ be the closed ball $\{\boldsymbol{y} \in \mathfrak{m}\mathbb{C}\,[|x|] \mid \|\boldsymbol{y}\|_{R,\infty} \leq M\}$. The following are equivalent:*

(1) *F is analytic.*
(2) *F preserves convergence, that is, $F(\mathbb{C}\{x\}^N) \subset \mathbb{C}\{x\}^{N'}$.*
(3) *There exist $R, M, K, S \in \mathbb{R}_+$ so that $F(B_{R,M}^N) \subset B_{S,K}$.*

(4) *For all* $R, M \in \mathbb{R}_+$, *there exist* $S, K \in \mathbb{R}_+$ *so that* $F(B^N_{R,M}) \subset B_{S,K}$.

(5) *There exist* $S, K, R \in \mathbb{R}_+$ *so that*

$$\sup_{\beta\in\mathbb{N}^k} R^{|\beta|} \sup_{\gamma\in\mathcal{M}_n^N} |F_{\beta,\gamma}| \frac{K^{|\mathrm{sh}(\gamma)|}}{S^{|\,\mathrm{wt}(\gamma)|}} < \infty.$$

Proof: The equivalence of (1), (2), (3), (4) and (5) were proven in [Wob12] in Proposition 5.3.2. and Theorem 4.3.2. By definition, (1) $\Longrightarrow$ (2). We will show that (2) $\Longrightarrow$ (1). We consider the restriction $G := F\big| : (\ell_R^\infty)^N \to \mathfrak{m}\mathbb{C}\{x\}$ of F to $(\ell_R^\infty)^N$ for some $R \in R_+$. Since $\mathfrak{m}\mathbb{C}\{x\} = \bigcup_{n,k\in\mathbb{N}_+} B_{1/n,1/k}$, we have that $\ell_R^\infty = \bigcup_{n,k\in\mathbb{N}_+} G^{-1}(B_{1/n,1/k})$. We claim that each $G^{-1}(B_{1/n,1/k})$ is closed. First observe that $B_{1/n,1/k}$ is compact in $\mathfrak{m}\mathbb{C}\{x\}^N$, since it is bounded and closed and as $\mathfrak{m}\mathbb{C}\{x\}^N$ is a Montel space ([Wob12]). Since $\mathfrak{m}\mathbb{C}\{x\}^N$ is an (LF)-space, every compact subset is metrizable and therefore sequentially compact by a result of B.Cascales and J.Orihuela ([CO86]). Let now $(\boldsymbol{y}_k)_{k\in\mathbb{N}}$ be a sequence in $G^{-1}(B_{1/n,1/k})$ which converges to some $\boldsymbol{y} \in \ell_R^\infty$. Then $(G(\boldsymbol{y}_k))_{k\in\mathbb{N}}$ is a sequence in the compactum $B_{1/n,1/k}$. Let $(G(\boldsymbol{y}_{n_k}))_{k\in\mathbb{N}}$ be an arbitrary convergent subsequence. The continuity of the polynomial coordinate function F_β implies that its limit coincides with $G(a)$. This holds for every convergent subsequence, thus $\lim_{k\to\infty} G(\boldsymbol{y}_k) = G(\boldsymbol{y})$. and therefore $\boldsymbol{y} \in G^{-1}(B_{1/n,1/k})$. As the countable family of closed sets $G^{-1}(B_{1/n,1/k})$ covers $(\ell_R^\infty)^N$ there exist n_0, k_0 such that $G^{-1}(B_{1/n_0,1/k_0})$ has non-empty interior, by Baire's Theorem. Therefore F is bounded on a ball $\boldsymbol{y}_0 + B^N_{R,M}$. Let τ be the translation $\boldsymbol{y} \mapsto \boldsymbol{y}_0 + \boldsymbol{y}$. Then $F \circ \tau$ is bounded on $B^N_{R,M}$. Therefore $F \circ \tau$ satisfies the condition of (3), is therefore analytic and hence also F is analytic. □

Definition 2.2.3. A useful class of textile maps is given by those which are dominated arquile maps. Originally (in [Bru09]), arquile maps were called tactile maps, whence we refer to those maps as *tactilly bounded*. A textile function whose support is of strict linear growth is called tactilly bounded, if there exists $M, R > 0$ so that

$$|F_{\beta,\gamma}| \leq \frac{M\mu(\gamma)}{R^{|\beta|-|\,\mathrm{wt}(\gamma)|+|\mathrm{sh}(\gamma)|}}.$$

The estimates on the right-handed side correspond to the β-coefficient of the arquile (and hence textile) map $\boldsymbol{y} \mapsto g(x, \boldsymbol{y}(x))$, where $g(x,y) = M \cdot \sum_{\alpha\in\mathbb{N}^n,\beta\in\mathbb{N}^N} \frac{x^\alpha y^\beta}{R^{|\alpha|+|\beta|}}$. The modulus of the coefficient $F_\beta(\boldsymbol{y})$ is bounded by the one of $g(x, |\boldsymbol{y}|(x))$, that is,

$$|F_\beta(\boldsymbol{y})| \leq |g(x, |\boldsymbol{y}|(x))[\beta]|,$$

where $[\beta]$ denotes the extraction of the β^{th} coefficient and $|\boldsymbol{y}|(x) := \sum_{\alpha\in\mathbb{N}^n} |y_\alpha| x^\alpha$

. We then say that $g(x, y)$ (respectively g_∞) *dominates* F. For further details see [Wob12], where this class of maps was introduced. The majorization by arquile maps makes it possible to generalize theorems such as the Inverse Function Theorem (see Theorem 2.2.6) of the Cauchy-Kovalevskaya Theorem ([Wob12]) to this class. If F is any textile map and, then the tangential map $T_{\boldsymbol{a}}(F)\colon \mathbb{C}_0\left[|x|\right]^N \to \mathbb{C}_0\left[|x|\right]$ of F at $\boldsymbol{a} \in \mathbb{C}_0\left[|x|\right]^N$ is given by the tangential maps of the coefficient functions, so $T_{\boldsymbol{a}}F\colon \boldsymbol{y} \mapsto \sum_{\beta\in\mathbb{N}^n}((T_{\boldsymbol{a}}F_\beta)(\boldsymbol{y}))x^\beta$. If F is analytic, then the tangential map is the Fréchet derivative of F. If $F = f_\infty$ is an arquile map $\boldsymbol{y}(x) \mapsto f(x, \boldsymbol{y}(x))$, then $T_{\boldsymbol{a}}f_\infty$ is the $\mathbb{C}\left[|x|\right]$-module morphism ([HM94])

$$\boldsymbol{y}(x) \mapsto \partial_y f(x, \boldsymbol{a}(x)) \cdot \boldsymbol{y}(x).$$

We now turn to the Implicit Function Theorem for textile maps. If $f(x, y) \in \mathfrak{m}\mathbb{C}\{x, y\}^N$, $y = (y_1, \dots, y_N)$, with $\partial_y f(0, 0)$ invertible, then the matrix $\partial_y f(x, 0)$ is invertible over $\mathbb{C}\{x\}$. So after multiplying with $(\partial_y f(x, 0))^{-1}$, we can assume that $\partial_y f(x, 0)$ is the identity on $\mathfrak{m}\mathbb{C}\{x\}^N$. Then $f(x, \boldsymbol{y}(x)) = f(x, 0) + \boldsymbol{y}(x) + q(x, \boldsymbol{y}(x))$, where $q(x, y)$ denote the terms of f which are at least quadratic in y. Under the assumption that $T_0(F) = \mathrm{id}$, we can generalize the Implicit Function Theorem to textile maps. Let us introduce one more piece of terminology: We say that a textile map F is *quadratic*, if F is of strict linear growth and if $F(0) = T_0F = 0$.

Theorem 2.2.4. *Let $F\colon \mathbb{C}_0\left[|x|\right]^N \to \mathbb{C}_0\left[|x|\right]^N$ be a textile map of strict linear growth. Assume that $T_0F = \mathrm{id}$.*

(1) *There exists a unique formal series $\boldsymbol{y}_0 \in \mathfrak{m}\mathbb{C}\left[|x|\right]^N$ solving $F(\boldsymbol{y}_0) = 0$.*
(2) *If F is tactilly bounded, then $\boldsymbol{y}_0$ is convergent.*

Proof: In order to use a majorization argument for the second part, we consider $G(\boldsymbol{y}) = F(-\boldsymbol{y})$ instead. Write $G(\boldsymbol{y})$ as $F(0) - \boldsymbol{y} + Q(\boldsymbol{y})$, where Q is quadratic. We want to solve

$$\boldsymbol{y} = F(0) + Q(\boldsymbol{y}).$$

For notational simplicity we assume that $n = N = 1$, so that x and y are single variables. Write $\boldsymbol{y}(x)$ as $\sum_{k=1}^\infty y_k x^k$, let $F(0) = \sum_{k=1}^\infty c_k x^k$ and

$$Q(\boldsymbol{y}) = \sum Q_k(\boldsymbol{y})x^k$$
$$Q_k(\boldsymbol{y}) = \sum_{\mathrm{wt}(\gamma)\le k} q_{k,\gamma}\boldsymbol{y}^\gamma$$

Therefore

$$y_k = c_k + \sum_{\mathrm{wt}(\gamma) \le k} q_{k,\gamma} \boldsymbol{y}^\gamma.$$

Note that the condition $\mathrm{wt}(\gamma) \le k$ implies that Q_k can only linearly depend on y_k, and since Q is quadratic, this term must be zero, whence Q is polynomial in $y_1, \ldots, y_{k-1}$. We observe that y_k is the evaluation of a polynomial (independent of F) with positive coefficients, evaluated at input data $c_k, (q_{k,\gamma})_\gamma, (y_1, \ldots, y_{k-1})$. Recursively we express $y_1, \ldots, y_{k-1}$ in the same manner and obtain

$$y_k = R_k \left((c_i)_{1 \le i \le k}, (q_{l,\gamma})_{1 \le l \le k, \gamma} \right)$$

where R_k is a universal polynomial (independent of F), with positive coefficients. In particular, this system is always solvable, which concludes the proof of (1). The fact that R_k has positive coefficients can be exploited to majorize the modulus of the coefficients of $\boldsymbol{y}$ by the coefficients from another system, which dominates the coefficients of F:

$$|y_k| \le R_k \left((|c_i|)_{1 \le i \le k}, (|q_{l,\gamma}|)_{1 \le l \le k, \gamma} \right).$$

If F is tactilly bounded (which means that Q is tactilly bounded and $F(0)$ is convergent), then the right-handed side of the preceding equation is the k-th coefficient of the solution of a non-degenerate analytic equation of the form

$$y = \sum_{k \ge 1} \frac{M}{R^k} x^k + \sum_{\beta \ge 2} \frac{M}{R^{k+\beta}} x^k y^\beta.$$

The solution of this equation is analytic and thus geometrically bounded by the Cauchy-estimates, therefore $|y_k| \le \frac{K}{S^k}$ for some $K, S \in \mathbb{R}_+$, and hence the solution $\boldsymbol{y}$ to $F(\boldsymbol{y}) = 0$ is a convergent power series. $\square$

Remark. The condition that $T_0 F = \mathrm{id}$ is vital to the analytic case, and cannot simply be replaced by merely requiring that $T_0 F$ is invertible: The tactile boundedness of F may not be carried over to $(T_0 F)^{-1} F$. The only problem, however, which can occur here, is that the terms of the diagonal $\frac{\partial F_k}{\partial y_k}(0)$ of $T_0 F$ are too small in comparison to other ones. In particular, if F is tactilly bounded and if $|\frac{\partial F_k}{\partial y_k}(0)| \ge 1$ for all $k \in \mathbb{N}$, then $(T_0 F)^{-1} F$ is again tactilly bounded.

Example 2.2.5. Let x be a single variable and let us consider the textile map $F \colon \mathfrak{m}\mathbb{C}\{x\} \to \mathfrak{m}\mathbb{C}\{x\}, \boldsymbol{y} = \sum y_n x^n \mapsto \sum F_n(\boldsymbol{y}) x^n$, where $F_n(\boldsymbol{y}) = \frac{y_n}{2^n} - y_{n-1}$ for $n > 1$ and $F_1(\boldsymbol{y}) = \frac{y_1}{2} - 1$. Then all coefficients of F are bounded by 1, in particular F

is tactilly bounded. The unique solution to $F(\boldsymbol{y}) = 0$ is the series $\sum_{n\geq 1} 2^{n(n-1)/2} x^n$, which is not a convergent power series.

Theorem 2.2.6 (Textile Inversion Theorem)**.** *Let $F\colon \mathfrak{m}\mathbb{C}\{x\}^N \to \mathfrak{m}\mathbb{C}\{x\}^N$ be a tactilly bounded map and assume that $T_0F = \mathrm{id}$. Then F is a global textile isomorphism.*

Proof: By the preceding theorem, there exists an $\boldsymbol{y} \in \mathfrak{m}\mathbb{C}\{x\}^N$ so that $F(\boldsymbol{y}) = 0$, so after an affine-linear coordinate change we may assume that $F(0) = 0$. Then $F = \mathrm{id} + Q$, where Q is quadratic. Let $g(x, y) \in \mathfrak{m}\mathbb{C}\{x, y\}^N$ so that g_∞ dominates F. Let $B_{R,M}$ be the open weighted ℓ^1-ball $\left\{ \boldsymbol{y} \in \mathfrak{m}\mathbb{C}\{x\}^N \mid \|\boldsymbol{y}\|_R < M \right\}$, where $\|\boldsymbol{y}\|_R = \sum_{\alpha\in\mathbb{N}^n, 1\leq k\leq N} |y_{\alpha,k}| R^{|\alpha|}$. Let $K > 0$ so that $\|g(x, y)\|_K \leq M < \infty$. Then if $0 < R < K$ and $\|\boldsymbol{y}\|_R < K$, we have the estimate $\|g(x, \boldsymbol{y}(x))\|_R < M$, which implies that $\|F(\boldsymbol{y})\|_R \leq M$. Therefore F maps $B_{R,K}$ to $B_{R,M}$ and with Theorem 2.0.14 it is easily seen that F is analytic as a local map $B_{R,K} \to B_{R,M}$ of the Banach space ℓ^1_R. We apply Proposition 2.0.13. Let

$$\nu_R = \frac{K^2}{4M}$$
$$\rho_R = \frac{K^2}{8M}$$

Then, by Proposition 2.0.13, F is a biholomorphism between $\bigcup_{0<R<K} B_{R,\nu_R} = \mathbb{C}_0\{x\}^N$ and $\bigcup_{0<R<K} B_{R,\rho_R} = \mathbb{C}_0\{x\}^N$. $\square$

We will shortly discuss the inversion of the tangential map of a textile function. For notational simplicity, we restrict the discussion to the case of a textile map $F\colon \mathfrak{m}\mathbb{C}\,[|x|] \to \mathfrak{m}\mathbb{C}\,[|x|]$ and where x is a single variable. Let e_k denote the standard unit vector and let $e^*_k \in \mathcal{M}_1$ be the monomial vector

$$e^*_k(n) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{else} \end{cases}.$$

The tangential map T_0F is again textile, and the coordinates functions $(T_0F)_n(\boldsymbol{y})$ are the maps

$$(T_0F)_n(\boldsymbol{y}) = F_{n,e^*_1} \cdot y_1 + \cdots + F_{n,e^*_n} \cdot y_n.$$

Hence the tangential map can be identified with an infinite lower-triangular matrix.

Definition 2.2.7. Let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ and $B = (b_{i,j})_{i,j\in\mathbb{N}}$ be infinite matrices. We formally define the product as

$$A \cdot B := C = (c_{i,j})_{i,j \in \mathbb{N}}, \text{ where } c_{i,j} := \sum_{k \in \mathbb{N}} a_{i,k} b_{k,j}.$$

We say that an infinite matrix D is *lower triangular* if $d_{i,j} = 0$ whenever $j > i$ and call D *unipotent* if in addition all diagonal elements $d_{i,i}$ of D are equal to one. The product of two lower triangular matrices is well-defined and again a lower triangular matrix. For $n \in \mathbb{N}$ we set $D_n := (d_{i,j})_{1 \leq i,j \leq n} \in \mathbb{C}^{n \times n}$. A lower triangular matrix D defines a textile map of strict linear growth on $\mathfrak{m}\mathbb{C}[|x|]$ via $\boldsymbol{y} \mapsto \sum_{n=1}^{\infty} (d_{n,1} y_1 + \cdots + d_{n,n} y_n) x^n$. By Proposition 2.2.2, D is analytic if and only if there exist $C, S, T \in \mathbb{R}_+$ so that $|d_{i,j}| \leq \frac{CS^j}{T^i}$, and D is tactilly bounded if there exist $M, R > 0$ so that $|d_{i,j}| \leq \frac{MR^j}{R^i}$. Note that $T_0 F$ is tactilly bounded if F is of this type.

Lemma 2.2.8. *Let $A = (a_{i,j})_{i,j \in \mathbb{N}}$ be a lower triangular matrix. Then:*

(1) $(A^k)_n = (A_n)^k$

(2) *Let $A = \mathrm{id} - B$ be unipotent. Then A is invertible and*

$$A^{-1} = \sum_{k=0}^{\infty} B^k$$

Theorem 2.2.9. *Let $C, S, T \in \mathbb{R}_+$, let $B = (b_{i,j})_{i,j \in \mathbb{N}_+}$ where*

$$b_{i,j} = \begin{cases} 0 & \text{if } j \geq i \\ \frac{CS^j}{T^i} & \text{else} \end{cases}$$

and set $U = (u_{i,j})_{i,j \in \mathbb{N}_+} = \sum_{k=0}^{\infty} B^k$. Then $U = (1 - B)^{-1}$ is an unipotent lower triangular matrix and its sub-diagonal coefficients are

$$u_{i,j} = \frac{CS^j}{T^i} \prod_{k=1+j}^{i-1} (1 + C\frac{S^k}{T^k}) \text{ for } j < i, \tag{2.2}$$

where we use the convention that $\prod_{k \in \emptyset} a_k = 1$.

Proof: Obviously U is an unipotent lower triangular matrix. Because of Lemma 2.2.8 and the fact that all B_n are nilpotent of order n we have that $U_n = \sum_{k=0}^{n} B_n^k$ and that

$$U_{n+1} = \left(\begin{array}{c|c} U_n & 0 \\ \hline u_{n+1} & 0 \end{array} \right),$$

where $u_{n+1} = (u_{n+1,1}, \dots, u_{n+1,n})$. B_{n+1} has the structure

$$\left(\begin{array}{c|c} B_n & 0 \\ \hline b_{n+1} & 0 \end{array}\right)$$

and

$$B_{n+1}^k = \left(\begin{array}{c|c} B_n^k & 0 \\ \hline b_{n+1} \cdot B_n^{k-1} & 0 \end{array}\right),$$

which yields that $u_{n+1} = b_{n+1} \cdot \sum_{k=0}^n B_n^k = b_{n+1} \cdot U_n$. We claim that Equation 2.2 holds for all U_n. For $n = 1$ we have $U_1 = 1$, so there is nothing to show. Now suppose that the estimates hold for U_n. Let $1 \leq j \leq n$. Then

$$\begin{aligned} u_{n+1,j} &= b_{n+1,j} \cdot 1 + \sum_{k=1}^n b_{n+1,k} u_{k,j} = \frac{CS^j}{T^{n+1}} + \sum_{k=j+1}^n \frac{CS^k}{T^{n+1}} \frac{CS^j}{T^k} \prod_{l=1+j}^{k-1} (1 + \frac{S^l}{T^l}) \\ &= \frac{CS^j}{T^{n+1}} \left(1 + \sum_{k=j+1}^n \frac{CS^k}{T^k} \prod_{l=1+j}^{k-1} (1 + C\frac{S^l}{T^l}) \right) = \\ &= \frac{CS^j}{T^{n+1}} \prod_{k=1+j}^{n+1-1} (1 + C\frac{S^k}{T^k}) \end{aligned}$$

□

Theorem 2.2.10. *Let x be a single variable and let $F \colon \mathfrak{m}\mathbb{C}\{x\} \to \mathfrak{m}\mathbb{C}\{x\}$ be a tactilly bounded map. Assume that the diagonal elements of $T_0 F$ are bounded from below by 1, that is, assume that*

$$|F_{n,e_n^*}| \geq 1.$$

Then $T_0 F$ has a tactilly bounded inverse and F is a global textile analytic isomorphism $\mathfrak{m}\mathbb{C}\{x\} \to \mathfrak{m}\mathbb{C}\{x\}$.

Proof: Let D be the infinite diagonal matrix whose diagonal elements $d_{n,n}$ are F_{n,e_n^*}. Let $H = D^{-1} \circ F$. Since all diagonal elements of D^{-1} are bounded by 1, H is again tactilly bounded. Let $M, R > 0$ so that

$$|H_{n,\gamma}| \leq \frac{M\mu(\gamma)}{R^{n - \mathrm{wt}(\gamma) + \mathrm{sh}(\gamma)}}.$$

Then

$$|H_{n,e_k^*}| \leq \frac{MR^k}{RR^n}$$

for $n \neq k$ and $G_{n,e_n^*} = 1$. Let $\mathrm{id} - B$ be the matrix representation of T_0H and let $L = (\mathrm{id} - B)^{-1} = (l_{i,j})_{i\geq 1, j\geq 1}$. Let us denote by $|B|$ the matrix $(|b_{i,j}|)_{i\geq 1, j\geq 1}$ The modulus of the entries of $L = (\mathrm{id} - B)^{-1}$ is bounded by the entries of $(\mathrm{id} - |B|)^{-1}$. Set $C = M/R$. By the previous theorem, the sub-diagonal entries of L can be estimated by

$$\begin{aligned}
|l_{i,j}| &\leq \frac{CR^j}{R^i} \prod_{k=1+j}^{i-1} (1 + C) = \frac{C}{1+C} \frac{R^j}{R^i} (1 + C)^{i-j} = \\
&\leq \frac{C}{1+C} \frac{(R/(1+C))^j}{(R/(1+C)^i} \leq \\
&\leq KS^{i-j}
\end{aligned}$$

where $S = (1+C)/R$ and $K = C/(1+C)$. Then $|\sum_{j=1}^{i} l_{i,j} y_j| \leq |y_i| + \sum_{i \neq j} KS^{i-j} |y_j|$, and hence L is dominated by the linear arquille map $\varphi(x, \boldsymbol{y}(x)) = \varphi(x) \cdot \boldsymbol{y}(x)$, where $g(x) = 1 + \sum_{k\geq 1} KS^k x^k$. Let G_∞ be an arquile map which dominates H. Then $L \circ H$ is dominated by $\varphi_\infty \circ G_\infty$, thus $L \circ H$ is tactilly bounded with $T_0 L \circ H = \mathrm{id}$. Therefore $L \circ H$ and hence G is an analytic isomorphism by Theorem 2.2.6. $\square$

Definition 2.2.11. Let U be a subset of $\mathbb{C}[|x|]^N$. A map $F\colon U \to \mathbb{C}[|x|]$ is called *rationally textile* if each coefficient function F_β is a quotient of polynomials in the coefficients of the input $\boldsymbol{y}$ and if non of the denominators vanish on U. A typical example is the map which assigns to an invertible power series its multiplicative inverse. The denominators of the inversion map only depends on the constant term $\boldsymbol{y}(0)$. Likewise, the denominators of rational textile maps which appear in practice usually only depend on a finite number of the coefficients of $\boldsymbol{y}$. In Theorem 2.3.11 we will show that the quotient and remainder in the Grauert-Hironaka division are analytic rationally textile functions in the dividend and the divisors if the initial monomials of the dividends are fixed.

2.3 Division Theorems for Power Series

Definition 2.3.1. A monomial ordering on $\mathbb{N}^n$ is a well ordering $<$ (i.e. a total ordering s.t. every non-empty set has a minimal element) which is compatible with addition, that is, $\alpha < \beta$ implies that $\alpha+\gamma < \beta+\gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}^n$. Accordingly, monomials are ordered with respect to their exponents, and $x^\alpha < x^\beta$ implies that $x^\alpha x^\gamma < x^\beta x^\gamma$. We also need orderings on free modules over $\mathbb{C}[|x|], \mathbb{C}\{x\}$ respectively. Let $f = (f_1, \dots, f_N) \in \mathbb{C}[|x|]^N$, $f_k = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha,k} x^\alpha$. We can identify f with $f_1 x_{n+1}^0 + \dots + f_N x_{n+1}^{N-1}$ in $\bigoplus_{k=1}^N \mathbb{C}[|x|]\, x_{n+1}^{k-1}$. Instead of $x^\alpha x_n^k$ we will shortly write $x^{\alpha,k}$, so that we can write f as $\sum_{\alpha \in \mathbb{N}^n, 0 \le k \le N-1} c_{\alpha,k} x^{\alpha,k}$. So the set of monomials of $\mathbb{C}\{x\}^N$ is

$$M_{n,N} := \{(\alpha, k) \in \mathbb{N}^{n+1} \mid 0 \le k \le N-1\}.$$

Let $<$ be a given monomial ordering on $\mathbb{N}^n$. Then we say that a total ordering $<_M$ on $M_{n,N}$ is a *module ordering* (compatible with $<$), if for all $\alpha, \beta \in \mathbb{N}^n$, $\nu, \mu \in \mathbb{N}^{n+1}$

(1) $\alpha < \beta$ implies $(\alpha, 0) + \nu < (\beta, 0) + \nu$
(2) $\nu <_M \mu$ implies $(\alpha, 0) + \nu < (\alpha, 0) + \mu$.

If $\mu \in \mathbb{N}^{n+1}$, then let $\underline{\mu}$ denote the first n components $(\mu_1, \dots, \mu_n)$ of $\mu = (\underline{\mu}, \mu_{n+1})$. Given two monomials x^μ, x^ν, say that x^μ divides x^ν, which we denote by $x^\mu | x^\nu$, if $\mu_{n+1} = \nu_{n+1}$ and if $\underline{x^{\mu}}$ divides $\underline{x^{\nu}}$. The most important monomial orderings in this context are the ones given by weight vectors $\omega = (\underline{\omega}, \omega_{n+1}) \in \mathbb{R}_+^{n+1}$: Set $\omega\mu := \omega \cdot \mu = \sum \mu_k \omega_k$. If the entries of ω are linearly independent over $\mathbb{Z}$, then ω induces a module-ordering on $M_{n,N}$ (which respects the monomial ordering induced by $\underline{\omega}$ on $\mathbb{N}^n$) by the rule $\mu <_\omega \nu$ iff $\omega\mu < \omega\nu$. For example, if $q_1, \dots, q_{n+1}$ are pairwise coprime positive integers, then the components of $\omega = (\log(q_1), \dots, \log(q_{n+1}))$ are linearly independent over $\mathbb{Z}$. Fix a monomial ordering $<$ on $M_{n,N}$ induced by some weight vector ω. For $f = \sum_{k=0}^{N-1} \sum_\alpha f_{\alpha,k} x^{\alpha,k} \in \mathbb{C}[|x|]^N$ we set

$$\begin{aligned}
\mathrm{In}(f) &= \min\{(\alpha, k) : f_{\alpha,k} \neq 0\} \\
\mathrm{Ic}(f) &= f_{\mathrm{In}(f)} \\
\mathrm{It}(f) &= f_{\mathrm{In}(f)} x^{\mathrm{In}(f)} \\
\mathrm{tail}(f) &= f - \mathrm{It}(f).
\end{aligned}$$

2.3.1 The Grauert-Hironaka Division Theorem for convergent power series

A weight vector also induces a Banach scale structure consisting of weighted ℓ^p-spaces on $\mathbb{C}\{x\}^N$. In division theorems usually the ℓ^1-structure is used, since ℓ^1-spaces are Banach algebras which thus enable good norm estimates for multiplication operators.

Definition 2.3.2. For a fixed weight vector ω, we introduce the following weighted sequence spaces. For $r \in \mathbb{R}_+, f = (f_1, \dots, f_N) \in \mathbb{C}\{x\}^N$, $f = \sum_{\alpha,k} c_{\alpha,k} x^{\alpha,k}$ we set

$$\|f\|_{\omega,r} = \sum_{(\alpha,k)\in M_{n,N}} |c_{\alpha,k}| r^{\omega\cdot(\alpha,k)} = \sum_{k=1}^{N} \|f_k\|_{\omega,r} \, r^{\omega_{n+1}\cdot(k-1)}$$
$$(\ell^1_{\omega,r})^N = \left\{ f \in \mathbb{C}\,[|x|]^N \;\middle|\; \|f\|_{\omega,r} < \infty \right\}$$

So the free $\ell^1_{\overline{\omega},r}$-module $(\ell^1_{\overline{\omega},r})^N = (\ell^1_{\omega,r})^N$ is considered with the weighted sum norm induced by $\ell^1_{\overline{\omega},r}$. The space of convergent power series $\mathbb{C}\{x\}$ is the union of all $\ell^1_{\omega,r}$ (independent of $\omega \in \mathbb{R}^n_+$) and will be equipped with the inductive topology, which turns it into the inductive limit (as locally convex space) of the inductive sequence $(\ell^1_{\omega,r})_{r\in\mathbb{R}_+}$. The inductive topology has many good properties and $\mathbb{C}\{x\}$ is a (DFN)-space (the strong dual of a nuclear Fréchet - Schwartz space (see for example [GR13], [HM94] or [Wob12])). Let $\omega = (\underline{\omega}, \omega_{n+1})$. Then $\mathbb{C}\{x\} = \varinjlim_{r\in\mathbb{R}_+} \ell^1_{\underline{\omega},r}$ and the inductive topology on $\mathbb{C}\{x\}^N$ coincides with the product topology induced by $\mathbb{C}\{x\}$ on $\mathbb{C}\{x\}^N$. From now, on we will fix a weight vector ω and omit ω in the notation of the norm, so we will write $\|f\|_r$ instead of $\|f\|_{\omega,r}$.

The key estimate in the division theorems of Grauert-Hironaka and its extension to entire functions is that the norm of the leading terms dominates the norm of the tail.

Theorem 2.3.3. *Let $f \in \mathbb{C}\{x\}^N, g \in \mathbb{C}\{x\}$. Then:*
(1) $\lim_{r\to 0} \|\mathrm{tail}(f)\|_r \cdot (\left\|x^{\mathrm{In}(f)}\right\|_r)^{-1} = 0$
(2) $\|f \cdot g\|_r \leq \|g\|_r \|f\|_r$

Proof: 1) We order the terms $\mathrm{supp}(f)$ of f with respect to $<$ in increasing order $c_{\mu_1} \leq c_{\mu_2} \leq \dots$. Then $f = \sum_{j\geq 1} c_{\mu_j} x^{\mu_j}$ and $\mathrm{tail}(f) = \sum_{j\geq 2} c_{\mu_j} x^{\mu_j}$. Let

$$h(r) := \frac{\|\mathrm{tail}(f)\|_r}{\|x^{\mathrm{In}(f)}\|_r} = \sum_{j=2}^{\infty} |c_{\mu_j}| r^{\omega\mu_j - \omega\mu_1}.$$

Since f is convergent, $h(r) < \infty$ for all r smaller than a certain r_0. As all exponents of the monomials $r^{\omega\mu_j-\omega\mu_1}$ are strictly positive, the sum tends uniformly to 0 for $r\to 0$. (2) is Mertens inequality for absolutely convergent series (see also [GR13]). $\square$

Definition 2.3.4. Let $\{\mu_1,\dots,\mu_p\}\in M_{n,N}$. Set $\langle\mu_1,\dots,\mu_k\rangle:=\cup_{i=1}^k(\mu_i+\mathbb{N}^n\times\{0\})$ and $\Delta:=\langle\mu_1,\dots,\mu_p\rangle$. Then the set of monomials which can be divided by some x^{μ_i} is $x^\Delta:=\{x^\delta\mid\delta\in\Delta\}$. We partition Δ by setting $\Delta_1:=\langle\mu_1\rangle$ and iteratively defining $\Delta_i:=\langle\mu_i\rangle\setminus\langle\mu_1,\dots,\mu_{i-1}\rangle$. So this means that we partition x^Δ into sets of monomials which can be divided by the respective monomial generators x^{μ_i}. Note that if $\gamma\in\Delta_i$, then $\gamma-\mu_i\in\mathbb{N}^n\times\{0\}$. Let $\pi\colon\mathbb{N}^{n+1}\to\mathbb{N}^n$ be the projection $(\alpha_1,\dots,\alpha_{n+1})\mapsto(\alpha_1,\dots,\alpha_n)$. We define

$$\Gamma_i=\begin{cases}\pi(\Delta_i-\mu_i) & \text{if } \Delta_i\neq\emptyset\\ \{0\} & \text{else}\end{cases}$$

We set

$$\begin{aligned}\mathcal{O}_{\Gamma_i}&=\{q\in\mathbb{C}\{x\}\mid \text{all terms of } q \text{ are contained in } x^{\Gamma_i}\}\\ \mathcal{O}_\Gamma&=\mathcal{O}_{\Gamma_1}\times\cdots\times\mathcal{O}_{\Gamma_p}\\ \mathcal{O}_{\Delta^c}&=\{g=\sum g_\nu x^\nu\in\mathbb{C}\{x\}^N\mid g_\nu=0 \text{ if } \nu\in\Delta\}\\ \mathcal{O}_\Gamma(r)&=\mathcal{O}_\Gamma\cap(\ell_r^1)^m,\mathcal{O}_{\Delta^c}(r)=\mathcal{O}_{\Delta^c}\cap(\ell_r^1)^N\end{aligned}$$

Note that $\mathcal{O}_{\Delta^c}$ is the set of all $f\in\mathbb{C}\{x\}^N$ which consist only of terms which are not divisible by any x^{μ_i}. The space $\mathcal{O}_\Gamma$ corresponds to a canonical normal form of the quotients in the Grauert division presented below.

Remark. The notation is not unified - depending on the author Δ can also refer to Δ^c.

Definition 2.3.5. A standard basis of a finitely generated submodule M of $\mathbb{C}\{x\}^N$ is a finite set $\{f_1,\dots,f_m\}\subset M$ of generators of M, such that $\{\mathrm{In}(f_1),\dots,\mathrm{In}(f_m)\}$ generates $\mathrm{In}(M)$. Every finitely generated module of $\mathbb{C}\{x\}^N$ possesses a standard basis, which can be shown for example by using the fact that $\mathbb{C}\{x\}^N$ is a Noetherian module ([dJP00, p.271]).

Theorem 2.3.6 ([dJP00]). *Let $f_1, \ldots, f_m, g \in \mathbb{C}\{x\}^N$ and let $<$ be a fixed monomial ordering induced by some weight vector ω.*

(1) *Then there exist $q_1, \ldots, q_m \in \mathbb{C}\{x\}$, $R \in \mathbb{C}\{x\}^N$ so that*

$$g = q_1 f_1 + \ldots q_p f_p + R$$

and no term of R is divisible by any $\mathrm{It}(f_1), \ldots, \mathrm{It}(f_p)$.

(2) *There exist* unique $(q_1, \ldots, q_m) \in \mathcal{O}_\Gamma$, $R \in \mathcal{O}_{\Delta^c}$ *so that* $g = q_1 f_1 + \cdots + q_m f_m + R$.

(3) *There exists an r_0 so that for all $0 < r \leq r_0$ the module ${}_{\ell^1_r}\langle f_1, \ldots, f_m\rangle$ is closed in $(\ell^1_r)^N$.*

Proof: Assume at first that $f_k = x^{\mu_k}$. Let $\Delta = \dot\cup_i \Delta_i$ as above, $g = \sum c_{\alpha,k} x^{\alpha,k}$. Set

$$q_i = \sum_{\alpha \in \Gamma_i} c_{\mu_i + \alpha} x^\alpha = \frac{1}{x^{\mu_k}} \sum_{\mu \in \Delta_i} c_\mu x^\mu$$
$$R = \sum_{\mu \in \Delta^c} c_\mu x^\mu$$

Then $\Psi \colon \mathcal{O}_\Gamma(r) \times \mathcal{O}_{\Delta^c}(r) \to (\ell^1_r)^N, (q_1, \ldots, q_m, R) \mapsto \sum q_k x^{\mu_k} + R$ is a continuous linear isomorphism. The construction shows that it is surjective, and injectivity follows from the decomposition of Δ. Let $q_i \in \ell^1_r, R \in (\ell^1_r)^N$. Then

$$\|\Psi(q, R)\|_r \leq \sum_{i=1}^N r^{\omega\mu_i} \|q_i\|_r + \|R\|_r$$

Note that

$$\|q_i(g)\|_r \leq \frac{1}{r^{\omega\mu_i}} \|g\|_r \tag{2.3}$$

Now the general case. WLOG we may suppose that the coefficients of the initial terms $\mathrm{It}(f_i)$ are normalized for all $i = 1, \ldots, m$. Let $\mu_i = \mathrm{In}(f_i)$ and let Ψ as above with respect to $x^{\mu_1}, \ldots, x^{\mu_p}$. Let $\Phi \colon \mathcal{O}_\Gamma(r) \times \mathcal{O}_{\Delta^c}(r) \to (\ell^1_r)^N$, $(q_1, \ldots, q_m) \mapsto \sum q_i f_i + R$ and consider $\Phi \circ \Psi^{-1}$. Let id_r be the identity map on $(\ell^1_r)^N$.

$$\left\|\Phi \circ \Psi^{-1} - \mathrm{id}_r\right\|_r = \left\|\sum q_i(g)(x^{\mu_i} - f_i)\right\|_r = \left\|\sum q_i(g)\mathrm{tail}(f_i)\right\|_r \leq \tag{2.4}$$
$$\leq \sum \|q_i(g)\mathrm{tail}(f_i)\|_r \leq \sum \|q_i(g)\|_r \|\mathrm{tail}(f_i)\|_r \leq \tag{2.5}$$
$$\leq \sum \|g\|_r \frac{1}{r^{\omega \cdot \mathrm{In}(f_i)}} \|\mathrm{tail}(f_i)\|_r \tag{2.6}$$

By Theorem 2.3.3 there is an $r_0 \in \mathbb{R}_+$ so that we have $\frac{1}{r^{\omega \cdot \mathrm{In}(f_i)}} \|\mathrm{tail}(f_i)\|_r < \frac{1}{2m}$ for all

$r \leq r_0$, which yields that

$$\left\| \Phi \circ \Psi^{-1} - \mathrm{id}_r \right\|_r \leq \frac{1}{2} \|g\|_r$$

for $r \leq r_0$. So $\Phi \circ \Psi^{-1}$ and thus Φ is invertible and the claim follows. As the corresponding maps are isomorphisms, a representation as in (2) is necessarily unique.
(3) Assume at first that $\{f_1, \ldots, f_m\}$ forms a standard basis for ${}_{\mathbb{C}\{x\}}\langle f_1, \ldots, f_m\rangle =: M$. Then $g \in M$ if and only if $R(g) = 0$. Let $r < r_0$ and let $g \in (\ell_r^1)^N$. Then there exist unique $q \in \mathcal{O}_\Delta(r)$ so that $g = \sum q_i f_i + R$. Hence $R(g) = 0$ if and only if $g \in {}_{\ell_r^1}\langle f_1, \ldots, f_m\rangle$, and $M \cap (\ell_r^1)^N = {}_{\ell_r^1}\langle f_1, \ldots, f_m\rangle$. The map $g \mapsto R(g)$ equals $\mathrm{proj}_{\mathcal{O}_{\Delta^c}} \circ \Phi^{-1} \colon (\ell_r^1)^N \to (\ell_r^1)^N$ and is thus a continuous endomorphism of $(\ell_r^1)^N$ for $r < r_0$. Hence we obtain that $\ker(\mathrm{proj}_{\mathcal{O}_{\Delta^c}} \circ \Phi^{-1})) = {}_{\ell_r^1}\langle f_1, \ldots, f_m\rangle$ is closed in $(\ell_r^1)^N$. In the case that $f_1, \ldots, f_m$ do not form a standard basis, let $g_1, \ldots, g_l$ be a standard basis of M and let r_0 be chosen as above for $g_1, \ldots, g_l$. There exists an $r_1 > 0$ so that all $g_j, f_j \in (\ell_r^1)^N$ and such that there exists $q_{i,j} \in \ell_{r_1}^1$ so that $\sum_j f_j q_{i,j} = g_i$. Let $r < \min\{r_0, r_1\}$, and $\lambda_i \in \ell_r^1$. Then $\sum \lambda_i g_i = \sum_j (\sum_i \lambda_i q_{i,j}) f_j$ and as $\lambda_i q_{i,j} \in \ell_r^1$, we obtain that ${}_{\ell_r^1}\langle g_1, \ldots, g_l\rangle \subset {}_{\ell_r^1}\langle f_1, \ldots, f_m\rangle$. The same argument shows the converse inclusion, and therefore ${}_{\ell_r^1}\langle g_1, \ldots, g_l\rangle = {}_{\ell_r^1}\langle f_1, \ldots, f_m\rangle$ for $r < \min\{r_0, r_1\}$, which in particular implies that ${}_{\ell_r^1}\langle f_1, \ldots, f_m\rangle$ is closed. $\square$

Definition 2.3.7. Let $\mu_1, \ldots, \mu_m \in M_{n,N}$ be a set of monomial vectors, let $<$ be a fixed monomial ordering on $\mathbb{C}\{x\}^N$, let Γ, Δ^c be defined as before. Let $z = (z_1, \ldots, z_k)$ be a set of new variables. We set

$$\mathcal{O}_{z,\Gamma_i} = \{q(x,z) = \sum_{\alpha \in \mathbb{N}^n} q_\alpha(z) x^\alpha \in \mathbb{C}\{x,z\} \mid q_\alpha = 0 \text{ whenever } \alpha \notin \Gamma_i\}$$
$$\mathcal{O}_{z,\Gamma} = \mathcal{O}_{z,\Gamma_1} \times \cdots \times \mathcal{O}_{z,\Gamma_p}$$
$$\mathcal{O}_{z,\Delta^c} = \{g = \sum g_\nu(z) x^\nu \in \mathbb{C}\{z\}^N \mid g_\nu = 0 \text{ if } \nu \in \Delta\}$$

One of the applications of the Grauert division theorem is that it provides a method for finding direct compliments to finitely generated $\mathbb{C}\{x\}$-modules.

Theorem 2.3.8 ([HM94])**.** *Let $f_1, \ldots, f_m \in \mathbb{C}\{x\}^N$, let M be the $\mathbb{C}\{x\}$-module generate by $f_1, \ldots, f_m$ and let $L \colon \mathbb{C}\{x\}^m \to \mathbb{C}\{x\}^N$ be the module homomorphism $(q_1, \ldots, q_m) \mapsto q_1 f_1 + \cdots + q_m f_m$ induced by the f_i.*

(1) *Assume that $f_1, \ldots, f_m$ form a standard basis. Then*

$$\mathbb{C}\{x\}^m = E_\Gamma \bigoplus \ker(L), \ \mathbb{C}\{x\}^N = M \bigoplus E_{\Delta^c}$$

(2) *There exists a continuous linear scission* $\sigma\colon \mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ *of* L *which is compatible with the* ℓ_r^1*-structure of* $\mathbb{C}\{x\}^N$ *and* $\mathbb{C}\{x\}^m$*.*

Theorem 2.3.9 (Parametric Grauert-Hironaka Division Theorem)**.** *Let* $\mu_1, \dots, \mu_m \in M_{n,N}$ *be a set of monomial vectors, let* $<$ *be a fixed monomial ordering on* $\mathbb{C}\{x\}^N$*, let* $z = (z_1, \dots, z_k)$ *be an additional set of variables, and let* $\mathcal{O}_{z,\Gamma}, \mathcal{O}_{z,\Delta^c}$ *be defined with respect to* $\{\mu_1, \dots, \mu_m\}$ *as above. Let* $f_1, \dots, f_m \in \mathbb{C}\{z\}^N$ *be a set of convergent power series. Assume that* $f_1, \dots, f_m$ *converge on the polycylinder* $\mathbb{D}_{r_0}^{n+k}$ *and assume that there exist* $u_1(z), \dots, u_m(z)$ *which converge on* $\mathbb{D}_{r_0}^k$ *such that* $u_i(z) \neq 0$ *for all* $z \in \mathbb{D}_{r_0}$*, so that*

$$\mathrm{It}_< f_i(x, z_0) = u_i(z_0) x^{\mu_i}$$

for every fixed $z_0 \in \mathbb{D}_r^k$*. Then for every* $g \in \mathbb{C}\{x, z\}$ *there exist unique* $q \in \mathcal{O}_{z,\Gamma}$ *and* $R \in \mathcal{O}_{z\Delta^c}$ *so that*

$$f = q_1 f_1 + \cdots + q_m f_m + R.$$

Proof: Since $u_1, \dots, u_m$ do not depend on x, it is clearly enough to proof the theorem under the assumption that all $u_i = 1$. Then $f_i(x, z) = x^{\mu_i} + \sum_{\beta > \mu_i} f_{\beta,i}(z) x^\beta$. Let $\widetilde{<}$ be any extension of $<$ on $\mathbb{C}\{x, z\}^N$ induced by some weight vector, so that the restriction of $\widetilde{<}$ to $\mathbb{C}\{x\}^N$ coincides with $<$. Then the initial term of each f_i with respect to $\widetilde{<}$ is x^{μ_i}, and the parametric version follows immediately from the standard Grauert-Hironaka Division Theorem. $\square$

2.3.2 The Grauert-Hironaka Division Theorem for formal power series

We will now prove the Grauert-Hironaka Division Theorem for formal power series. Let us introduce again some notation:

$$\begin{aligned}
\widehat{\mathcal{O}}_{\Gamma_i} &= \{q \in \mathbb{C}\,[|x|] \mid \text{all terms of } q \text{ are contained in } x^{\Gamma_i}\} \\
\widehat{\mathcal{O}}_{\Gamma} &= \widehat{\mathcal{O}}_{\Gamma_1} \times \cdots \times \widehat{\mathcal{O}}_{\Gamma_p} \\
\widehat{\mathcal{O}}_{\Delta^c} &= \{g = \sum g_\nu x^\nu \in \mathbb{C}\,[|x|]^N \mid g_\nu = 0 \text{ if } \nu \in \Delta\}
\end{aligned}$$

Theorem 2.3.10. *Let $f_1, \dots, f_m, f \in \mathbb{C}\,[|x|]^N$ and let $<$ be a fixed monomial ordering induced by some weight vector ω.*

(1) *There exist* unique $(q_1, \dots, q_m) \in \widehat{\mathcal{O}}_\Gamma,\ R \in \widehat{\mathcal{O}}_{\Delta^c}$ *so that* $g = q_1 f_1 + \cdots + q_m f_m + R$

(2) *Assume that* $\mathrm{Ic}(q_i) = 1$ *for all* $1 \leq i \leq m$*. Then the map*

$$\Phi_f^{-1} \colon \mathbb{C}\,[|x|]^N \to \widehat{\mathcal{O}}_\Gamma \times \widehat{\mathcal{O}}_{\Delta^c},\ \ g \mapsto (q, R)$$

is a textile isomorphism.

(3) *Let* $\widehat{\mathcal{U}} = \left\{ (f_1, \dots, f_m) \in (\mathbb{C}\,[|x|]^N)^m \;\middle|\; f_i = x^{\mu_i} + \sum_{\mu > \mu_i} f_\mu(z) x^\mu \right\}$*. The map*

$$\widehat{\mathcal{U}} \times \mathbb{C}\,[|x|]^N \to \widehat{\mathcal{O}}_\Gamma \times \widehat{\mathcal{O}}_{\Delta^c}, (f, g) \mapsto \Phi_f^{-1}(g)$$

is a textile map. That is, quotient and remainder in the Hironaka Division depend in a textile way on the dividend and the divisors.

Proof: Let $\Phi \colon \widehat{\mathcal{O}}_\Gamma \times \widehat{\mathcal{O}}_{\Delta^c} \to \mathbb{C}\,[|x|]^N, (q_1, \dots, q_m, R) \mapsto q_1 f_1 + \cdots + q_m f_m + R$ and let $\Psi \colon (q_1, \dots, q_m, R) \mapsto q_1 x^{\mu_1} + \cdots + q_m x^{\mu_m} + R$. Then Ψ is clearly in isomorphism. We show that $\Phi \circ \Psi^{-1}$ is invertible by interpreting $H := \mathrm{id} - \Phi \circ \Psi^{-1}$ as a small perturbation of the identity on $\mathbb{C}\,[|x|]^N$. Let ω be the weight vector which induces $<$ and define the weighted order of an element $f \in \mathbb{C}\,[|x|]^N$ as $\mathrm{ord}_< f = \omega(\mathrm{In}(f))$. Let $g = x^{\mu_1} q_{\Psi,1}(g) + \cdots + x^{\mu_m} q_{\Psi,m}(g) + R$, with $q_\Psi \in \widehat{\mathcal{O}}_\Gamma, R \in \widehat{\mathcal{O}}_{\Delta^c}$. It is easily checked that

$$\mathrm{ord}_< q_{\Psi,i}(g) \geq \mathrm{ord}(g) - \mathrm{ord}(x^{\mu_i}).$$

Set $\varepsilon_i = \mathrm{ord}(\mathrm{tail}(f_i)) - \mathrm{ord}(\mu_i)$ and $\varepsilon = \min \varepsilon_i > 0$. Then

$$\begin{aligned} \mathrm{ord}_< H(g) = \mathrm{ord}_<(\mathrm{id} - \Phi \circ \Psi^{-1})(g) = \mathrm{ord}_< \left(\sum (x^{\mu_i} - f_i) q_{\Psi,i}(g) \right) &\geq \\ \geq \min_{1 \leq i \leq m} \mathrm{ord}_<(\mathrm{tail}(f_i)) + \mathrm{ord}_< q_{\Psi,i}(g) &\geq \\ \geq \min_{1 \leq i \leq m} \varepsilon_i + \mathrm{ord}(g) \geq \varepsilon + \mathrm{ord}(g) & \end{aligned}$$

Since $\mathrm{ord}_< H(g)$ is strictly larger than $\mathrm{ord}_< g$, the sum $\sum_{k=0}^\infty H^k(g) = (\Phi \circ \Psi)^{-1}$ is well-defined. The inverse of Φ is hence given by $\Psi^{-1} \circ (\Phi \circ \Psi^{-1})^{-1}$. We claim that the coefficients of $\Phi_f^{-1}(g)$ depend polynomially on a finite number of the coefficients of f and g.Let $\alpha \in M_{n,N}$ and choose $K_\alpha \in \mathbb{N}$ so that $K_\alpha \cdot \varepsilon \geq \alpha$. If $k > K_\alpha(g)$, then

$$\mathrm{ord}_< H^k(g) > k\varepsilon + \mathrm{ord}(g) > \mathrm{ord}_< x^\alpha$$

and in particular, $H^k(g)$ does not contain any x^α-terms. Therefore

$$\Phi \circ \Psi^{-1}(g)[\alpha] = \sum_{k=0}^{K_\alpha} H^k(g)[\alpha].$$

The operator H is a shift-operator followed by a multiplication operator, hence the coefficients of $H^k(g)$ depend polynomially on a finite number of the coefficients of f and g, therefore $\Phi \circ \Psi^{-1}$ and hence $\Phi_f^{-1}(g)$ is a textile map in (f, g). $\square$

Theorem 2.3.11 (Variational Grauert-Hironaka Division)**.** *Let $\mu_1, \ldots, \mu_m$ be a fixed set of monomial vectors $\mu_i \in M_{n,N} = \mathbb{N}^n \times \{0, \ldots, N-1\}$ and let*

$$\widehat{\Omega} = \{f = (f_1, \ldots, f_m) \in (\mathbb{C}\,[|x|]^N)^m \mid \mathrm{In}(f_i) = \mu_i\}$$
$$\Omega = \{f = (f_1, \ldots, f_m) \in (\mathbb{C}\{x\}^N)^m \mid \mathrm{In}(f_i) = \mu_i\}$$

We denote by $Q(f,g) = (Q_1(f,g), \ldots, Q_m(f,g)), R(f,g)$ the unique quotient $Q(f,g) \in \widehat{\mathcal{O}}_\Gamma$ and remainder $R(f,g) \in \widehat{\mathcal{O}}_{\Delta^c}$ of a power series vector $g \in \mathbb{C}\,[|x|]^N$, so that

$$g = Q_1(f,g) \cdot f_1 + \cdots + Q_m(f,g) \cdot f_m + R$$

as in Theorem 2.3.6. Then the map

$$\widehat{\Omega} \times \mathbb{C}\,[|x|]^N \to \widehat{\mathcal{O}}_\Gamma \times \widehat{\mathcal{O}}_{\Delta^c}, (f,g) \mapsto (Q(f,g), R(f,g))$$

is a rational-textile map and its restriction

$$\Omega \times \mathbb{C}\{x\}^N \to \mathcal{O}_\Gamma \times \mathcal{O}_{\Delta^c}, (f,g) \mapsto (Q(f,g), R(f,g))$$

is holomorphic.

Proof: As in Theorem 2.3.10, let

$$\widehat{\mathcal{U}} = \{f \in (\mathbb{C}\,[|x|]^N)^m \mid \mathrm{It}(f_i) = x^{\mu_i}\}$$
$$\mathcal{U} = \{f \in (\mathbb{C}\{x\}^N)^m \mid \mathrm{It}(f_i) = x^{\mu_i}\}$$

be the set normalized dividends with initial monomial μ_i. In the cited theorem it was shown that the division $(f,g) \mapsto (Q(f,g), R(f,g))$ is textile on $\widehat{\mathcal{U}} \times \mathbb{C}\,[|x|]^N$. The normalization $f_i \mapsto f_i/\mathrm{Ic}(f_i))$ is a rational-textile map $\widehat{\Omega} \to \widehat{\mathcal{U}}$. Since

$$Q\left(\frac{f_1}{\mathrm{Ic}(f_1)}, \ldots, \frac{f_m}{\mathrm{Ic}(f_m)}, g\right) = \left(\frac{1}{\mathrm{Ic}(f_1)} Q_1(f,g), \ldots, \frac{1}{\mathrm{Ic}(f_m)} Q_m(f,g)\right),$$
$$R\left(\frac{f_1}{\mathrm{Ic}(f_1)}, \ldots, \frac{f_m}{\mathrm{Ic}(f_m)}, g\right) = R(f,g),$$

the divsion is rationally textile on $\widehat{\Omega} \times \mathbb{C}\,[|x|]^N$ and it remains to show that it is analytic. The set $\mathcal{U}$ is $\mathfrak{m}$-adic open and can be identified with a subvector space, which is closed in the inductive topology, of $(\mathbb{C}\{x\}^N)^m$. It is hence a (DFS)-space, so analyticity can be tested along holomorphic curves (see Theorem 2.0.9). A holomorphic curve

$$(F,G)\colon (\mathbb{C},0) \to (\mathbb{C}\{x\}^N)^m \times \mathbb{C}\{x\}^N$$

is a germ of a convergent power series $(F(x,t), G(x,t)) \in (\mathbb{C}\{x,t\}^N)^m \times \mathbb{C}\{x\}^N$, and the image of $F(x,t)$ lies in $\mathcal{U}$ if $F_i(x,t) = x^{\mu_i} + \sum_{\mu > \mu_i} F_\mu(t) x^\mu$. So the analyticity reduces to the parametric version of the Grauert Division Theorem (Theorem 2.3.9), by which $Q(F(x,t), G(x,t))$ and $R(F(x,t), G(x,t))$ are analytic. $\square$

2.3.3 The Grauert-Hironaka Division Theorem for algebraic power series

In the case of algebraic power series, the Division Theorem is without further assumptions on the divisors no longer true. If $f_1, \ldots, f_m, g$ are algebraic power series, then the remainder of g in the division by $f_1, \ldots, f_m$ need not to be any longer an algebraic power series (see [ACJH14]). If, however, the module generated by $f_1, \ldots, f_m$ satisfies Hironaka's Box condition, then the remainder will be algebraic.

Theorem 2.3.12 ([Hir77], [ACJH14]). *Let M be a submodule $\mathbb{C}\,[|x|]^N$ which is generated by vectors of algebraic power series, and let $\widehat{\mathcal{O}}_{\Delta^c}$ be the direct monomial compliment of M in $\mathbb{C}\,[|x|]^N$. Suppose M satisfies* Hironaka' Box Condition*, that is, that $\widehat{\mathcal{O}}_{\Delta^c}$ is of the form*

$$\prod_{l=1}^{N} \bigoplus_{j=0}^{n} \bigoplus_{\gamma \in \Gamma_{l,j}} \mathbb{C}[[x_1, \ldots, x_j]] x^\gamma$$

where $\Gamma_{l,j}$ are finite subsets of $\mathbb{N}^n$. If $g \in \mathbb{C}\langle x \rangle^N$ is a vector of algebraic series, then the unique remainder $r \in \widehat{\mathcal{O}}_{\Delta^c}$ in the division of g by a standard basis of M is again algebraic.

We do not know whether the canonical quotients in the division by an algebraic standard basis (under the assumptions of the previous theorem) are again algebraic. However, the Weierstrass Division Theorem remains true for algebraic power series.

Theorem 2.3.13 (Weierstrass Division Theorem [Laf65], [Rui93])**.** *Let $f \in \mathbb{C}\langle x\rangle$ be an algebraic power series which is x_n-regular of order d. Then for every $g \in \mathbb{C}\langle x\rangle$ there exists a unique quotient $q \in \mathbb{C}\langle x\rangle$ and a unique remainder $r(x) \in \mathbb{C}\langle x\rangle \cap \bigoplus_{k=0}^{d-1} \mathbb{C}[[x_1, \ldots, x_{n-1}]]x_n^k$ so that $g = q \cdot f + r$.*

2.3.4 Division of entire functions by polynomials

This section is inspired by a paper by Apel et al ([ASTW96]) and Mitiagin/Djakov ([DM80a]). In this section we will establish a division for vectors of entire functions by vectors of polynomials. The division is very similar to Grauert's division theorem, and yields a normal form of $f \in \mathcal{H}(\mathbb{C}^n)^N$ with respect to a finite set of polynomials $p_1, \ldots, p_k$. The difference in comparison to the local version is that the division is carried out through the maximal terms $\exp(p_i)$ and not through the initial terms $\mathrm{It}(p_i)$.

Definition 2.3.14. Let $<$ be a term ordering on $\mathbb{C}[[x]]^m$. For a polynomial vector $p(x) = \sum p_\alpha x^\alpha \in \mathbb{C}[x]^N$, let $\exp(p) = \max\{\alpha \mid p_\alpha \neq 0\}$ and let

$$\mathrm{lt}(p) = p_{\exp(p)} x^{\exp(p)}$$

denote the *leading term* of p. By lack of better options we call $\mathrm{stail}(p) = p - \mathrm{lt}(p)$ the *swallowtail* of p.

Lemma 2.3.15. *Let $p \in \mathbb{C}[x]^N$ be a polynomial vector. Then*

$$\lim_{r\to\infty} \frac{\|p\|_r}{r^{\omega(\exp(p))}} = 0$$

Proof: As all powers in powers of r in

$$\frac{\|p\|_{\omega,r}}{r^{\omega(\exp(p))}} = \sum_{\alpha\in M_{n,m}} |p_\alpha| r^{\omega(\alpha)-\omega(\exp(p))}$$

are negative, the value tends to zero for r going to infinity. $\square$

Definition 2.3.16. Let $\{\mu_1, \ldots, \mu_p\} \in M_{n,N}$, let $\Delta_1 \cup \cdots \cup \Delta_p$ be the according partition of $\Delta = \langle \mu_1, \ldots, \mu_p \rangle$. We set

$$\begin{aligned} E_{\Gamma_i} &:= \{q \in \mathcal{H}(\mathbb{C}^n) \mid \text{all terms of } q \text{ are contained in } x^{\Gamma_i}\} \\ E_\Gamma &:= E_{\Gamma_1} \times \cdots \times E_{\Gamma_m} \\ E_{\Delta^c} &= \{g = \sum g_\nu x^\nu \in \mathcal{H}(\mathbb{C}^n)^N \mid g_\nu = 0 \text{ if } \nu \in \Delta\} \end{aligned}$$

Theorem 2.3.17. *Let $f_1, \ldots, f_m \in \mathbb{C}[x]^N$ be a finite set of polynomial vectors.*

(1) *For every $g \in \mathcal{H}(\mathbb{C}^n)$, there exist $q_1, \ldots, q_m \in \mathcal{H}(\mathbb{C}^n)$ and $R \in \mathcal{H}(\mathbb{C}^n)^N$, such that no term of R is divisible by any $\mathrm{lt}(f_i)$ so that*

$$g = f_1 q_1 + \cdots + f_m q_m + R$$

(2) *For every $g \in \mathcal{H}(\mathbb{C}^n)$, there exist unique $q = (q_1, \ldots, q_m) \in F_\Gamma, R \in F_{\Delta^c}$ so that $g = f_1 q_1 + \ldots f_m q_m + R$. If g is a polynomial, then both q and R are polynomials.*

(3) *There exists an $r_0 \in \mathbb{R}_+$ so that for all $r \geq r_0$ the following holds: For every $g \in (\ell^1_{\omega,r})^N$ there exist $q_1, \ldots, q_m \in \ell^1_{\omega,r}$ and $R \in (\ell^1_{\omega,r})^N$.*

Proof: The proof runs analogous to the the local case. Assume at first that $f_k = x^{\mu_k}$. Let $\Delta = \dot\cup_i \Delta_i$ as above, $f = \sum c_{\alpha,k} x^{\alpha,k}$. Set

$$\begin{aligned} q_i &= \sum_{\alpha \in \Gamma_i} c_{\mu_i + \alpha} x^\alpha = \frac{1}{x^{\mu_k}} \sum_{\mu \in \Delta_i} c_\mu x^\mu \\ R &= \sum_{\mu \in \Delta^c} c_\mu x^\mu \end{aligned}$$

Then $\Psi \colon E_\Gamma(r) \times E_{\Delta^c}(r) \to (\ell^1_r)^N, (q_1, \ldots, q_m, R) \mapsto \sum q_k x^{\mu_k} + R$ is a continuous linear isomorphism. The construction shows that it is surjective, and injectivity follows from the decomposition of Δ. Let $q_i \in \ell^1_{\omega,r}, R \in \ell^1_r$. Then

$$\|\Psi(q,R)\|_r \leq \sum_{i=0}^{N-1} s^{\omega\mu_i} \|q_i\|_{\omega,r} + \|R\|_r$$

Note that

$$\|q_i(g)\|_{\omega,r} \leq \frac{1}{r^{\omega\mu_i}} \|f\|_r$$

Now the general case. WLOG we may suppose that the coefficients of the leading terms $\mathrm{lt}(f_i)$ are normalized for all $i = 1, \ldots, m$. Let $\mu_i = \mathrm{It}(f_i)$ and let Ψ as above with respect to $x^{\mu_1}, \dots, x^{\mu_p}$. Let $\Phi \colon E_\Gamma(r) \times E_{\Delta^c}(r) \to (\ell_r^1)^N$, $(q_1, \dots, q_m) \mapsto \sum q_i f_i + R$ and consider $\Phi \circ \Psi^{-1}$. Let id_r be the identity map on $(\ell_r^1)^N$.

$$\begin{aligned}\left\| \Phi \circ \Psi^{-1} - \mathrm{id}_r \right\|_r = \left\| \sum q_i(g)(x^{\mu_i} - f_i) \right\|_r = \left\| \sum q_i(g) \operatorname{stail}(f_i) \right\|_r \leq \\ \leq \sum \left\| q_i(g) \operatorname{stail}(f_i) \right\|_r \leq \sum \left\| q_i(g) \right\|_{\omega,r} \left\| \operatorname{stail}(f_i) \right\|_r \leq \\ \leq \sum \|f\|_r \frac{1}{r^{\omega \cdot \exp(f_i)}} \left\| \operatorname{stail}(f_i) \right\|_r\end{aligned}$$

By Lemma 2.3.15 there is an $r_0 \in \mathbb{R}_+$ so that for all $r \geq r_0$ we have $\frac{1}{r^{\omega \cdot \exp(f_i)}} \left\| \operatorname{stail}(f_i) \right\|_r < \frac{1}{2m}$, which yields that

$$\left\| \Phi \circ \Psi^{-1} - \mathrm{id}_r \right\|_r \leq \frac{1}{2} \|f\|_r$$

for $r \geq r_0$. So $\Phi \circ \Psi^{-1}$ is invertible, which gives that Φ is invertible and the claim follows. As the corresponding maps are isomorphisms, the representation is necessarily unique. If f is polynomial, we can apply the classical multivariate polynomial division algorithm to obtain a representation as stated (see [AL94]).The third part follows from the proof of (1).

□

Remark. Suppose that $g, f_1, \dots, f_p$ are polynomials. Then the series $q_1, \dots, q_m, R$ as in the previous theorem coincide with the output of the multinomial polynomial division algorithm applied to g with respect to the ordered set $\{f_1, \dots, f_p\}$. (see [AL94]).

Definition 2.3.18. Let $f, f_1, \dots, f_p$ as above. Then $R = R(f) = \mathrm{NF}(f \mid f_1, \dots, f_l)$ is called the normal form of f with respect to $f_1, \dots, f_p$.

Corollary 2.3.19 ([ASTW96],[DM80b])**.** *Let $f_1, \dots, f_p \in \mathbb{C}[x]^N$, let M be the $\mathcal{H}(\mathbb{C}^n)-$module generated by $f_1, \dots, f_p$ and let $f \in \mathcal{H}(\mathbb{C}^n)^N$.*

(1) *Assume that $\{f_1, \dots, f_p\}$ forms a Gröbner basis for ${}_{\mathbb{C}[x]}\langle f_1, \dots, f_p \rangle$. Then $f \in M$ if and only if* $\mathrm{NF}(f \mid f_1, \dots, f_p) = 0$.
(2) *Submodules of $\mathcal{H}(\mathbb{C}^n)^N$ generated by polynomials are closed in the compact-open topology.*
(3) *There exists an $r_0 > 0$ so that for all $r \geq r_0$ the module ${}_{\ell_r^1}\langle f_1, \dots, f_p \rangle$ is closed in $(\ell_r^1)^N$.*

Proof: (1) We need to show that if $f \in M$ then $R := \mathrm{NF}(f \mid f_1, \dots, f_p) = 0$. Since $f \in M$, also $R \in M$ and we can write R as $f_1 q_1 + \dots + f_m q_m$. Set $R_k =$

$f_1 j^k(q_1) + \dots f_m j^k(q_m)$. The sequence R_k converges to R in $\mathcal{H}(\mathbb{C}^n)^N$ in the compact-open topology. The map $f \mapsto \mathrm{NF}(f \mid f_1, \dots, f_p)$ is continuous and hence $\mathrm{NF}(R_k)$ converges to $\mathrm{NF}(R) = R$. As $\{q_1, \dots, q_m\}$ forms a Gröbner Basis and $R_k \in {}_{\mathbb{C}[x]}\langle f_1, \dots, f_m \rangle$ we obtain $\mathrm{NF}(R_k \mid q_1, \dots, q_m) = 0$ (by Theorem 2.3.17) and hence $\mathrm{NF}(R) = 0$. (2) Without loss of generality we can assume that the f_i form a Gröbner basis and hence the module membership $f \in M$ can be expressed as $\mathrm{NF}(f \mid f_1, \dots, f_p) = 0$, and since $f \mapsto \mathrm{NF}(f \mid f_1, \dots, f_p)$ is continuous it follows that M is closed. (3) By Theorem 2.3.17 there exists an r_0 so that for every $r \geq r_0$, every $f \in (\ell^1_r)^N$ can be written as $f_1 q_1 + \dots + f_p q_p + \mathrm{NF}(f \mid f_1, \dots, f_p)$, with all q_i and $\mathrm{NF} \in (\ell^1_r)^N$. Again, we can assume that the f_i form a Gröbner basis, and the membership $f \in {}_{\ell^1_r}\langle f_1, \dots, f_p \rangle$ is equivalent to $\mathrm{NF} = 0$. $\square$

Chapter 3

Flat analytic spaces and Homogeneous Operators

The $\mathbb{C}$-algebra $\mathbb{C}\{x\}$ of convergent power series has two useful filtrations:

– $\mathbb{C}\{x\}$ is the inductive limit $\varinjlim_r \ell^1_r$ of weighted ℓ^1-Banach spaces. (3.1)

– $\mathbb{C}\{x\}$ is filtered by spaces of (weighted) homogeneous polynomials $H_k(\omega)$. (3.2)

The space of homogeneous polynomials of a fixed degree carries a well-known inner product, the Bombieri inner product (and its normalized cousin, the Fischer product). One of its applications is that it allows to formulate the properties of a canonical solution of system of polynomial or analytic equations in terms of orthogonality relations between the coefficients. For example, Fischer ([Fis18], [NS66]) showed that if P is a homogeneous polynomial, then every polynomial F can be uniquely written as $F = PQ + R$ with $\overline{P}(\frac{\partial}{\partial x})(R) = 0$. This last relation can be interpreted (with the suitable product) as a decomposition of the space of polynomials into the ideal generated by P and its orthogonal complement $\{R \mid \overline{P}(\frac{\partial}{\partial x})(R) = 0\}$. The Bombieri inner product is also compatible with the inductive locally convex structure of $\mathbb{C}\{x\}$ as $\varinjlim_r \ell^1_r$ in the sense that the duality on the homogeneous levels extends to a duality between power series, which is well-defined on $\mathbb{C}\{x\} \times \mathcal{H}(\mathbb{C}^n)$ and recovers the inductive topology as the strong topology on $\mathbb{C}\{x\}$ with respect to this duality. We introduce a new class of spaces called *flat analytic spaces*, which have similar filtrations as in Equation 3.1. These spaces appear naturally if one wants to construct normal forms of solutions to power series equations and as direct complements of modules generated by homogeneous power series. The structure morphisms of flat analytic spaces are the so-called homogeneous operators. We will show that for such operators a generalized Penrose inverse can be constructed. In the last section we shortly discuss composition operators and show a strong injectivity theorem for homogeneous composition operators acting on spaces of entire functions, generalizing a theorem by Eakin and Harris ([EH77]) to this setting.

3.1 Filtrations by homogeneous polynomials and flat analytic spaces

Definition 3.1.1. Let $H_k(\mathbb{C}^n)$ denote the space $\{\sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha \mid |\alpha| = k \; \forall c_\alpha \neq 0\}$ of homogeneous polynomials of degree k in n variables. There are two commonly used inner products on these spaces, which have been studied by various authors ([Fis18],[Rez93], [Pin12], [BBEM90]). Let $P = \sum p_\alpha x^\alpha, Q = \sum q_\alpha x^\alpha \in H_k(\mathbb{C}^n)$. Then we introduce the Bombieri inner product

$$\langle P, Q\rangle = \sum \binom{\alpha}{|\alpha|} p_\alpha \overline{q_\alpha}.$$

The Bombieri product has a couple of useful properties, for example the norm associated to it is sup-multiplicative and adjoints of multiplication operator are differential operators. If $F = \sum f_\alpha x^\alpha$ is any polynomial, let $F(D)$ be the differential operator $\sum f_\alpha D^\alpha$, where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}}$, and let $\overline{F}$ be the conjugate $\sum \overline{c_\alpha} x^\alpha$.

Theorem 3.1.2 ([BBEM90], [Rez93])**.** *Let $P \in H_d(\mathbb{C}^n)$ and $Q \in H_k(\mathbb{C}^n)$, let $T_k \colon H_k(\mathbb{C}^n) \to H_{k+d}(\mathbb{C}^n)$ be the multiplication operator $Q \mapsto P \cdot Q$ and let $\|\ \ \|$ be the norm induced by the Bombieri inner product $\langle\ ,\ \rangle$ on $H_k(\mathbb{C}^n)$.*

(1) $\|P \cdot Q\| \geq (\frac{k!d!}{(k+d)!})^{1/2} \|P\| \cdot \|Q\|$

(2) $\|P\|^2 = \overline{P}(D)(P)$

(3) *The adjoint operator of T_k is the differential operator $\frac{k!}{(k+d)!}\overline{P}(D)$.*

Proof: For the proofs of (1) and (2) we refer to [Rez93].
(3) If $\varepsilon, \delta \in \mathbb{N}^n$, then

$$\langle x^\varepsilon, D^\delta x^{\varepsilon+\delta}\rangle = \binom{\varepsilon}{|\varepsilon|}\binom{\varepsilon+\delta}{\varepsilon} = \frac{|\varepsilon+\delta|!}{|\varepsilon|!}\binom{\varepsilon+\delta}{|\varepsilon+\delta|} = \frac{|\varepsilon+\delta|!}{|\varepsilon|!}\langle x^{\varepsilon+\delta}, x^{\varepsilon+\delta}\rangle.$$

Let $F \in H_{k+d}(\mathbb{C}^n)$. Then

$$\begin{aligned}\langle Q\cdot P, F\rangle &= \sum_{\varepsilon\in\mathbb{N}^n} Q_\epsilon \sum_{\delta\in\mathbb{N}^n} P_\delta \overline{F}_{\varepsilon+\delta}\langle x^{\varepsilon+\delta}, x^{\varepsilon+\delta}\rangle = \\ &= \sum_{\varepsilon\in\mathbb{N}^n} Q_\epsilon \sum_{\delta\in\mathbb{N}^n} \overline{\overline{P}}_\delta \overline{F}_{\varepsilon+\delta}\frac{(d+k)!}{k!}\langle x^\varepsilon, D^\delta(x^{\varepsilon+\delta})\rangle = \langle Q, \frac{(d+k)!}{k!}\overline{P}(D)F\rangle \quad \square\end{aligned}$$

Let P be a homogeneous polynomial of degree d and let $T_k\colon H_k(\mathbb{C}^n) \to H_{k+d}(\mathbb{C}^n)$ be the associated multiplication operator $Q \mapsto P \cdot Q$. Then a natural question ([Rez93]) is to find a bound for the operator norm of $\|T_k\|_{\mathrm{OP}}$ independent of k. For $P \in \mathcal{P}(\mathbb{C}^n)$, let $\|P\|_{\mathbb{B}} = \sup_{|z|=1} |P(z)|$ be the supremum norm of P on the Euclidean ball in $\mathbb{C}^n$.

Theorem 3.1.3 ([Pin12])**.** *Let P as above be homogeneous polynomial of degree d and let $T_k\colon H_k(\mathbb{C}^n) \to H_{k+d}(\mathbb{C}^n)$ be the multiplication operator $Q \mapsto P \cdot Q$. Then*

$$\limsup_{k\to\infty} \|T_k\|_{\mathrm{OP}} = \|P\|_{\mathbb{B}}\,.$$

The Bombieri product induces via application on the homogeneous parts a duality between the convergent power series and the space of entire functions: Let $f = \sum_{k\in\mathbb{N}} F_k(x) \in \mathbb{C}\{x\}$, and let $g = \sum_{k\in\mathbb{N}} G_k(x) \in \mathcal{H}(\mathbb{C}^n)$, $F_k, G_k \in H_k(\mathbb{C}^n)$. Then $\langle F, G\rangle := \sum \langle H_k, G_k\rangle$. It is easily checked that $\langle\ ,\ \rangle$ is well-defined, i.e. that $\langle F, G\rangle < \infty$. The duality recovers the usual topologies of $\mathbb{C}\{x\}$ (the inductive topology) and $\mathcal{H}(\mathbb{C}^n)$ (locally uniform convergence) as the strong topologies of this duality.

Definition 3.1.4. We now introduce the Fischer inner product ([Fis18]), which is a weighted variant of the Bombieri inner product. For $P, Q \in H_d(\mathbb{C}^n)$, let

$$[P, Q]_d = d!\langle P, Q\rangle$$

We equip the space $\mathcal{P}(\mathbb{C}^n) = \bigoplus_{d\in\mathbb{N}} H_d(\mathbb{C}^n)$ of all polynomials with the direct sum inner product $[\ ,\ ]$ and the associated Fischer norm $\|\ \|_2$. A nice feature of $[\ ,\ ]$ is that the adjoint of a polynomial multiplication operator is the corresponding differential operator.

Theorem 3.1.5 ([Ren08])**.** *Let P be a polynomial and let $T_P\colon \mathcal{P}(\mathbb{C}^n) \to \mathcal{P}(\mathbb{C}^n)$, $F \mapsto P \cdot F$ be the associated multiplication operator. Then the adjoint of T_P with respect to $[\ ,\ ]$ is the differential operator $\overline{P}(D)$.*

Proof: Let $P = P_0 + \cdots + P_d$ be the decomposition of P into its homogeneous parts. Accordingly we decompose T_P into $T_{P_0} + \cdots + T_{P_d}$. The operator $T \mapsto T^*$ is linear, so it suffices to show that $T_P^* = \overline{P}(D)$ holds for homogeneous polynomial P_d of degree d. Let $F = F_0 + \cdots + F_e, Q = Q_0 + \cdots + Q_e,\ F_l, Q_l \in H_l(\mathbb{C}^n)$. Then

$$[P_d \cdot F, Q] = \sum_{l=0}^{e} (k+d)!\langle P_d F_k, Q_{k+d}\rangle = \sum_{l=0}^{e} (k+d)!\frac{k!}{(k+d)!}\langle F_k, \overline{P}(D)Q\rangle = [F, \overline{P}(D)Q]$$

□

Restated in terms of the Fischer norm, Theorem 3.1.2 takes the following form:

Theorem 3.1.6 ([Ren08], [BBEM90], [Rez93]). *Let $P \in H_d(\mathbb{C}^n)$ and $Q \in H_k(\mathbb{C}^n)$, let $T_k\colon H_k(\mathbb{C}^n) \to H_{k+d}(\mathbb{C}^n)$ be the multiplication operator $Q \mapsto P \cdot Q$. Then*
(1) $\|P \cdot Q\|_2 \geq \|P\|_2 \cdot \|Q\|_2$
(2) $\|P\|_2^2 = \overline{P}(D)(P)$

An interesting aspect of the Fischer inner product is that it can be extended to the Bargmann-Fock

$$\mathcal{F}(\mathbb{C}^n) = \left\{ f \in \mathcal{H}(\mathbb{C}^n) \;\middle|\; \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} d\lambda < \infty| \right\}$$

space of entire functions and represented as a weighted integral. The inner product $(\ ,\ )$ on $\mathcal{F}(\mathbb{C}^n)$ is given by

$$(f, g) := \frac{1}{\pi^d} \int_{\mathbb{C}^n} f(z)\overline{g}(z)e^{-|z|^2} d\lambda.$$

If f, g are polynomials, then

$$(f, g) = [f, g].$$

3.1.1 Flat Analytic Spaces

We will now define a class of weighted sequence spaces which are filtered finite-dimensional Hilbert spaces, and offer a framework to use the properties Equation 3.1. These spaces have been introduced by B.Lamel.

Definition 3.1.7. Let $(F_\alpha)_{\alpha\in\mathbb{N}^d}$ be a sequence of finite-dimensional normed spaces over $\mathbb{C}$, let $\|\ \|_\alpha$ be the norm of F_α. Then the flat analytic space $\mathcal{F}$ over the *floors* $(F_\alpha)_{\alpha\in\mathbb{N}^d}$ is the vector space

$$\mathcal{F} = \left\{ (f_\alpha)_{\alpha\in\mathbb{N}^d} \subset \prod F_\alpha \;\middle|\; \exists r > 0 \text{ such that } \sum \|f_\alpha\|_\alpha r^{|\alpha|} < \infty \right\}.$$

We call $\widehat{\mathcal{F}} := \prod F_\alpha$ the formal analytic space over the floors $(F_\alpha)_{\alpha\in\mathbb{N}^d}$. The projection $\widehat{\mathcal{F}} \to F_\alpha$ will be denoted by π_α. For $r > 0$ let $\mathcal{F}_r$ be the weighted ℓ^1 Banach space

$$\mathcal{F}_r = \left\{ (f_\alpha)_{\alpha\in\mathbb{N}^d} \subset \prod F_\alpha \;\middle|\; \sum \|f_\alpha\|_\alpha r^{|\alpha|} < \infty \right\}.$$

We equip $\mathcal{F}$ with its natural topology as the inductive limit of the *steps* $\mathcal{F}_r$. For every other choice of $p \in [1, \infty]$,

$$\mathcal{F} = \varinjlim_{r} \mathcal{F}_{r,p}$$

where $\mathcal{F}_{r,p} = \left\{ (f_\alpha)_{\alpha \in \mathbb{N}^d} \subset \prod F_\alpha \;\middle|\; \sum \|f_\alpha\|_\alpha^p r^{p|\alpha|} < \infty \right\}$ (for $p \neq \infty$), equipped with the corresponding $p-$norm, and the obvious definition in the case that $p = \infty$. If $s < r$ then $\mathcal{F}_r \hookrightarrow \mathcal{F}_s$ and it is not difficult to show that the inclusion is compact (the proof works analogously as the one for weighted ℓ^1-spaces of power series, see for example [Wob12])), and hence a flat analytic space is a (DFS)-space. In particular, the open mapping theorem holds for continuous linear maps between flat analytic spaces. From now on, we will always assume that each floor F_α is equipped with an inner product $\langle \_, \_ \rangle_\alpha$ and the induced norm. The topological dual space, equipped with the strong topology, is isomorphic to

$$\mathscr{H}(\mathcal{F}) = \varprojlim_{r>0} \mathcal{F}_{1/r,q} \ ,$$

for every $q \in (1, \infty]$, via the duality

$$\langle \ , \ \rangle \colon \mathcal{F} \times \mathscr{H}(\mathcal{F}) \to \mathbb{C}, \ \left((f_\alpha)_{\alpha \in \mathbb{N}^d}, (g_\alpha)_{\alpha \in \mathbb{N}^d}\right) \mapsto \sum \langle f_\alpha, g_\alpha \rangle_\alpha.$$

The structural morphisms of flat analytic spaces $\mathcal{E} \to \mathcal{F}$ are the homogeneous maps (see below), which, roughly speaking, map one floor to another, and are nothing else then an (ascending) sequence of linear operators between the finite dimensional floors $E_\alpha \to F_\alpha$. We can carry over several constructions from the floorwise level, in which we work with Hilbert spaces, to the whole flat analytic space and reproduce results for homogeneous maps which are usually only available for linear maps between Hilbert spaces. For example, we will see that the image of a homogeneous map has a closed complement, provided that the image is closed.

Example 3.1.8. For every $\alpha \in \mathbb{N}^d$, let $F_\alpha = \mathbb{C}$. Then $\mathcal{F} = \mathbb{C}\{x_1, \ldots, x_d\}$, $\mathscr{H}_{\mathcal{F}} = \mathcal{H}(\mathbb{C}^d)$ and the duality is given by $\langle \sum f_\alpha x^\alpha, \sum g_\alpha x^\alpha \rangle = \sum f_\alpha \overline{g}_\alpha$.

Example 3.1.9. Let $F_k = H_k(\mathbb{C}^d)$ be the space of homogeneous polynomials of degree k, equipped with the Bombieri inner product. Then again $\mathcal{F} = \mathbb{C}\{x_1, \ldots, x_d\}$, $\mathscr{H}_{\mathcal{F}} = \mathcal{H}(\mathbb{C}^d)$. The duality is given by

$$\langle \sum_{k \in \mathbb{N}} F_k, \sum G_k \rangle = \sum \langle F_k, G_k \rangle_k = \sum_{k \in \mathbb{N}} \sum_{|\alpha| = k} \binom{\alpha}{k} f_\alpha \overline{g_\alpha},$$

where H_k, G_k are the homogeneous parts of $F = \sum_{\alpha \in \mathbb{N}^d} f_\alpha x^\alpha, G = \sum_{\alpha \in \mathbb{N}^d} g_\alpha x^\alpha$.

3.1.2 Complemented subspaces

Definition 3.1.10. Let $\mathcal{F}$ be a flat analytic space over the floors $(F_\alpha)_{\alpha\in\mathbb{N}^d}$. A flat analytic subspace of $\mathcal{F}$ is a flat analytic space $\mathcal{G}$ over the floors $(G_\alpha)_{\alpha\in\mathbb{N}^d}$, such that G_α is a subspace of F_α for every $\alpha \in \mathbb{N}^d$. We will show that every flat analytic subspace is complemented (Theorem 3.1.14).

Lemma 3.1.11. *Let $\mathcal{G}$ be flat analytic subspace of $\mathcal{F}$. Then $\mathcal{G}$ is a closed subspace of $\mathcal{F}$, and its natural inductive topology coincides with the topology induced by $\mathcal{F}$.*

Proof: The projections $\pi_\alpha \colon \mathcal{F} \to F_\alpha$ are continuous, and so $\mathcal{G} = \bigcap_\alpha \pi^{-1}(G_\alpha)$ is closed in $\mathcal{F}$. The inclusion $\mathcal{G}_r \to \mathcal{G}$ is continuous if $\mathcal{G}$ is considered with the subspace topology induced by $\mathcal{F}$, and hence $\mathrm{id}\colon \varinjlim \mathcal{G}_r \to \mathcal{G}$ is continuous. As a closed subspace of a (DFS)-space $\mathcal{G}$ is again of such nature, we can apply the open mapping theorem to conclude that $\varinjlim \mathcal{G}_r \cong \mathcal{G}$. □

Remark. Let $(G_\alpha)_{\alpha\in\mathbb{N}^d}$ and $(H_\alpha)_{\alpha\in\mathbb{N}^d}$ be two sequences of complemented subspaces of the floors F_α, i.e. $F_\alpha = G_\alpha \bigoplus H_\alpha$ for every $\alpha \in \mathbb{N}^d$. Let $\mathcal{G}$ and $\mathcal{H}$ be the associated flat analytic spaces. Then it is in general *not* true, that $\mathcal{F} = \mathcal{G} \bigoplus \mathcal{H}$. While both subspaces are closed and the floorwise complementation induces a decomposition on the formal level, i.e. $\widehat{\mathcal{F}} = \widehat{\mathcal{G}} \bigoplus \widehat{\mathcal{H}}$, the decomposition does not work on an analytic level, as seen in the following example given by B. Lamel.

Example 3.1.12. Let $F_k := \mathbb{C}^2$ for every $k \in \mathbb{N}$ and let $\mathcal{F}$ be the associated analytic space over $\mathbb{N}^2$. Let e_1, e_2 be the standard unit vectors of $\mathbb{C}^2$, and set $v_k = e_2 + (k!)^{-1}e_1$. Let $H_k = \langle e_2\rangle$ and $G_k = \langle v_k\rangle$. Let $x = (x_k)_{k\in\mathbb{N}^2}$, where $x_k = e_1$ for all $k \in \mathbb{N}$. Suppose that $x = g + h$ for some $g \in \mathcal{G}, h \in \mathcal{H}$. Then $x_k = g_k + h_k$, and hence $g_k = k!v_k,\ h_k = -k!e_2$. But $g \in \widehat{\mathcal{G}} \setminus \mathcal{G},\ h \in \widehat{\mathcal{H}} \setminus \mathcal{H}$. This shows that $\mathcal{G} + \mathcal{H} \subsetneq \mathcal{F}$. Let proj_k be the canonical extension to $\prod F_j$ of the projection of F_k onto H_k along G_k and set $P = (\mathrm{proj}_k)_{k\in\mathbb{N}}$. Then $P\colon \mathcal{F} \to \hat{\mathcal{F}}$ and $P(\mathcal{F})$ is not contained in $\mathcal{F}$. If this failure is excluded, then the decomposition by the induced by the complemented subspaces G_k, H_k works also analytically.

Theorem 3.1.13. *Let $(G_\alpha)_{\alpha\in\mathbb{N}^d}$ and $(H_\alpha)_{\alpha\in\mathbb{N}^d}$ be two sequences of complemented subspaces in the F_α and let $\mathcal{G}, \mathcal{H}$ be the associated analytic spaces. Let* $\mathrm{proj}_\alpha \colon \widehat{\mathcal{F}} \to H_\alpha$ *be the canonical extension of the projection of F_α onto H_α along G_α, and let* $\mathrm{proj}_{\mathcal{H}} \colon \mathcal{F} \to \widehat{\mathcal{H}}$, $\mathrm{proj}_{\mathcal{H}}((f_\alpha)_{\alpha\in\mathbb{N}^d}) := (\mathrm{proj}_\alpha(f))_{\alpha\in\mathbb{N}^d} = (\mathrm{proj}_\alpha(f_\alpha))_{\alpha\in\mathbb{N}^d}$ *be the floorwise projection.*

The following are equivalent:

(1) $\mathrm{proj}_{\mathcal{H}}(\mathcal{F}) \subset \mathcal{F}$.
(2) $\mathcal{F} = \mathcal{G} \bigoplus \mathcal{H}$.
(3) $\mathcal{F} = \mathcal{G} \bigoplus \mathcal{H}$ *and the projection on* $\mathcal{H}$ *along* $\mathcal{G}$ *is* $\mathrm{proj}_{\mathcal{H}}$.
(4) *There exist* $S, M > 0$ *so that* $\sup_{\alpha\in\mathbb{N}^d} \|\mathrm{proj}_\alpha\|_{\mathrm{OP},F_\alpha} S^\alpha \leq M < \infty$, *where* $\|\mathrm{proj}_\alpha\|_{\mathrm{OP},F_\alpha}$ *is the operator norm of the projection as an operator* $F_\alpha \to H_\alpha$.

Proof: (1)$\Rightarrow$(3) By definition, $\mathrm{proj}_{\mathcal{H}}|_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$, $\mathrm{proj}_{\mathcal{H}}^2 = \mathrm{proj}_{\mathcal{H}}$, $\mathrm{proj}_{\mathcal{H}}(\mathcal{F}) \subset \hat{\mathcal{H}} \cap \mathcal{F} = \mathcal{F}$ and hence $\mathrm{proj}_{\mathcal{H}} \colon \mathcal{F} \to \mathcal{H}$ is a projection on $\mathcal{H}$. The projection of F_α onto G_α along H_α is $\mathrm{id}_{F_\alpha} - \mathrm{proj}_\alpha\big|_{\mathcal{F}_\alpha}$. Therefore $\mathrm{id} - \mathrm{proj}_{\mathcal{H}} \colon \mathcal{F} \to \mathcal{G}$ is the projection of $\mathcal{F}$ onto $\mathcal{G}$ along $\mathcal{H}$. This shows that $\mathcal{F} = \mathcal{G} \bigoplus \mathcal{H}$ on an algebraic level, and since both subspaces are closed and therefore (DFS)-spaces, the open mapping theorem implies that $\mathcal{F} = \mathcal{G} \bigoplus \mathcal{H}$ as locally convex spaces. In particular, $\mathrm{proj}_{\mathcal{H}}$ is continuous (as the projection on $\mathcal{H}$ along $\mathcal{G}$ is unique).
(2) $\Rightarrow$ (1) Let $\pi_{\mathcal{H}}, \pi_{\mathcal{G}}$ be the projections onto $\mathcal{H}, \mathcal{G}$, respectively. Then $\mathrm{proj}_{\mathcal{H}} = \mathrm{proj}_{\mathcal{H}}(\pi_{\mathcal{H}} + \pi_{\mathcal{G}}) = \mathrm{proj}_{\mathcal{H}}(\pi_{\mathcal{H}}) + 0 = \pi_{\mathcal{H}}$. (3) $\Rightarrow$ (4) $\mathrm{proj}_{\mathcal{H}}$ is continuous, and therefore preserves bounded sets. Let $\mathcal{B}_{R,M} = \{(x_\alpha)_{\alpha\in\mathbb{N}^d} \mid \sup_\alpha \|x_\alpha\|_\alpha R^\alpha \leq M\}$. Observe that $\mathcal{B}_{1,1} = \Pi_{\alpha\in\mathbb{N}^d}\mathcal{B}_\alpha$, where $\mathcal{B}_\alpha$ is the unit ball in F_α. Since $\mathrm{proj}_{\mathcal{H}}(\mathcal{B})$ is bounded, there exists an $S, M > 0$ so that $\mathrm{proj}_{\mathcal{H}}(\mathcal{B}_{1,1}) \subset \mathcal{B}_{S,M}$. Choose $x_\alpha \in \mathcal{B}_\alpha$ so that $\|\mathrm{proj}_\alpha(x_\alpha)\|_\alpha = \|\mathrm{proj}_\alpha\|_{\mathrm{OP},F_\alpha}$. Since $x = (x_\alpha)_{\alpha\in\mathbb{N}^d} \in \mathcal{B}_{1,1}$, we obtain $\sup_\alpha \|\mathrm{proj}_\alpha\|_\alpha S^\alpha \leq M < \infty$.
(4) $\Rightarrow$ (1). If $x \in \mathcal{F}$, then $\exists R, K$ so that $\sup_\alpha \|x_\alpha\|_\alpha R^\alpha \leq K$. Hence

$$\sup_\alpha \|\mathrm{proj}_\alpha(x_\alpha)\| (SR)^\alpha \leq \sup_\alpha \|\mathrm{proj}_\alpha\|_{\mathrm{OP}} R^\alpha \|x_\alpha\|_\alpha S^\alpha \leq M \cdot K,$$

and so $\mathrm{proj}_{\mathcal{H}}(x) \in \mathcal{F}$. □

Corollary 3.1.14 (and Definition)**.** *Assume that every* F_α *is equipped with an inner product and the corresponding Hilbert-norm. Let* $(H_\alpha)_{\alpha\in\mathbb{N}^d}$ *be an arbitrary sequence of subspaces with each* H_α *contained in* F_α*. Let* $H_\alpha^\perp$ *be the orthogonal complement of* H_α *in* F_α *and let* $\mathcal{H}^\perp$ *be the flat analytic space over the floors* $\left(H_\alpha^\perp\right)_{\alpha\in\mathbb{N}^d}$*. Further, let* $\pi_{H_\alpha} \colon \widehat{\mathcal{F}} \to H_\alpha$ *be the canonical extension of the orthogonal projection of* F_α *onto* H_α *and let* $\pi_H \colon \widehat{\mathcal{F}} \to \widehat{\mathcal{H}}$ *be the* orthogonal projection *on* $\widehat{\mathcal{H}}$, $\pi_H((f_\alpha)_{\alpha\in\mathbb{N}^d}) := (\pi_\alpha(f_\alpha))_{\alpha\in\mathbb{N}^d}$*. Then*

$$\mathcal{F} = \mathcal{H} \bigoplus \mathcal{H}^\perp$$

and the orthogonal projection $\pi \colon \mathcal{H} \to \mathcal{H}$ *is analytic. In particular, every flat analytic subspace is complemented.*

Proof: We apply the previous theorem with $G_\alpha = F_\alpha^\perp$. The operator norm of each of orthogonal projection is of course bounded by 1, thus the fourth part of the previous theorem yields that the projections are continuous and that $\mathcal{F} = \mathcal{H} \bigoplus \mathcal{H}^\perp$. $\square$

3.2 Homogeneous operators

Let $\mathcal{E}, \mathcal{F}$ be flat analytic spaces over the floors $(E_n)_{n\in\mathbb{N}}, (F_n)_{n\in\mathbb{N}}$. The canonical embedding of E_n into $\mathcal{E}$ will be denoted by ι_n, the coordinate projections $\mathcal{E} \to E_n$ by π_n and by abuse of notation the analogous maps for $\mathcal{F}$ will be denoted by the same letters.

Definition 3.2.1. A linear operator $T\colon \mathcal{E} \to \mathcal{F}$ between flat analytic spaces is called *homogeneous*, if there exists a strictly increasing function $\varphi\colon \mathbb{N} \to \mathbb{N}$ so that

$$T \circ \iota_k = \iota_{\varphi(k)} \circ \pi_{\varphi(k)} \circ T \circ \iota_k,$$

which will be shortly expressed by $T(E_k) \subset F_{\varphi(k)}$. The map φ will be called the *shift* of T. This means that each coordinate function $\pi_n \circ T$ depends only on the k^{th} entry e_k (if $n = \varphi(k)$) or is identically zero (if $n \notin \mathrm{Im}(\varphi)$). For $k \in \mathbb{N}$, let then

$$T_k\colon E_k \to F_{\varphi(k)},\ T_k = \pi_{\varphi(k)} \circ T \circ \iota_k$$

denote the floorwise operators of T. A linear operator $T\colon \mathcal{E}^1 \times \cdots \times \mathcal{E}^N \to \mathcal{F}$ will be called homogeneous, if T is homogeneous with respect to the natural filtration $(E_k^1 \times \cdots \times E_k^N)_{k\in\mathbb{N}}$ of $\mathcal{E}^1 \times \cdots \times \mathcal{E}^N$. An arbitrary map $T\colon \mathcal{E} \to \mathcal{F}$ will be called *quasi-homogeneous* if there exist linear topological automorphisms Ψ of $\mathcal{F}$, φ of $\mathcal{E}$, so that the induced map $\Psi \circ T \circ \varphi^{-1}$ is homogeneous.

Example 3.2.2. Let $T\colon \mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ be the linear map associated to a vector of homogeneous polynomials of degree d. Then T is homogeneous since $T_k\colon H_k(\mathbb{C}^n)^N \to H_{k+d}(\mathbb{C}^n)^m$.

Example 3.2.3. Let $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots y_N), z = (z_1, \ldots, z_k)$ be three sets of variables, and let $\varphi(x, z) \in H_d(\mathbb{C}^{n+k})^N$ be a vector of homogeneous polynomials of degree d. Let $\omega \in \mathbb{N}^{n+N}$ be the weight vector which assigns to each x_i the weight 1 and each to y_i the weight d. Then the composition operator $C_\varphi\colon \mathbb{C}\{x, y\} \to \mathbb{C}\{x, z\},\ f(x, y) \mapsto f(x, \varphi(x, z))$ is a homogeneous operator with respect to the filtration of $\mathbb{C}\{x, y\}$ by $H_k(\omega)$ and $\mathbb{C}\{x, z\}$ by $H_k(\mathbb{C}^{n+k})$.

Example 3.2.4. More generally, assume that all $\varphi_i(x,z)$ are homogeneous with respect to the same weight $\upsilon = (\mu,\nu) \in \mathbb{N}^{n+k}$. Let d_i be the weighted degree of φ_i, let $\delta = (d_1,\dots,d_N)$ and let $\omega = (\mu,\delta)$. Then we claim that $C_\varphi \colon \mathbb{C}\{x,y\} \to \mathbb{C}\{x,z\}$ is a homogeneous operator, if we filtrate $\mathbb{C}\{x,y\}$ with by the weighted homogeneous polynomials $H_k(\omega)$ and $\mathbb{C}\{x,z\}$ by $H_k(\upsilon)$. Recall that a polynomial $P(x_1,\dots,x_n)$ with complex coefficients is weighted homogeneous of degree d with respect to some weight μ, if and only if $P(t^{\mu_1}x_1,\dots,t^{\mu_n}x_n) = t^d P(x_1,\dots,x_n)$ holds for all $t \in \mathbb{C}$. Let us shortly write $t^\mu x$ instead of $(t^{\mu_1}x_1,\dots,t^{\mu_n}x_n)$. If now $H(x,y) \in H_k(\omega)$, then

$$\begin{aligned}(C_\varphi(H))(t^\mu x, t^\nu z) &= \sum_{\mu\cdot\alpha+\delta\cdot\beta=k} H_{\alpha,\beta}(t^\mu x)^\alpha(\varphi_1(t^\mu x,t^\nu z),\dots,\varphi_N(t^\mu x,t^\nu z))^\beta = \\ &= \sum_{\mu\cdot\alpha+\delta\cdot\beta=k} H_{\alpha,\beta}x^\alpha t^{\mu\cdot\alpha}(t^{d_1}\varphi_1(x,z),\dots,t^{d_N}\varphi(x,z))^\beta = \\ &= \sum_{\mu\cdot\alpha+\delta\cdot\beta=k} H_{\alpha,\beta}t^{\mu\cdot\alpha+\delta\cdot\beta}x^\alpha(\varphi_1(x,z),\dots,\varphi(x,z))^\beta = t^k(C_\varphi(H))(x,z)\end{aligned}$$

Definition 3.2.5. Let $T\colon \mathcal{E} \to \mathcal{F}$ be a homogeneous operator with shift φ. Then let $T_k^*\colon F_{\varphi(k)} \to E_k$ be the adjoint of $T_k\colon E_k \to F_{\varphi(k)}$. The *floorwise adjoint* T' to T is the map $\widehat{\mathcal{F}} \to \widehat{\mathcal{E}}$, $T'(f) = (T_k^*(f_{\varphi(k)}))_{k\in\mathbb{N}}$.

We recall that an adjoint operator to a linear operator $T\colon E \to F$ between dualities $\langle E,G\rangle$, $\langle F,H\rangle$ is an operator $T^*\colon H \to G$, which satisfies $\langle T(e),g\rangle = \langle e,T^*(g)\rangle$ ([Sch66, p.128]). The adjoint may not always exist, as the dual map of T might fail to map H into G. The adjoint exists if and only if T is weakly continuous (i.e. if E, F are equipped with the weak topologies $\sigma(E,G), \sigma(F,H)$, respectively [Sch66, p.128]). If E and F are equipped with topologies which are consistent with the respective dualities, then every continuous operator $T\colon E \to F$ is also weakly continuous and has therefore an adjoint map, which is continuous if both H and G are equipped with their strong topologies (uniform convergence on bounded sets) ([Sch66, p.130]).

Theorem 3.2.6. *Let $T\colon \mathcal{E} \to \mathcal{F}$ be a homogeneous operator and let T' be its floorwise adjoint, let $T^*\colon \mathscr{H}(\mathcal{F}) \to \mathscr{H}(\mathcal{E})$ be the adjoint operator of T. Then*

(1) $T^* = T'$

(2) *Assume that $T(\mathscr{H}(\mathcal{E})) \subset \mathscr{H}(\mathcal{F})$. Then T' is analytic, i.e. $T'(\mathcal{F}) \subset \mathcal{E}$ and $T'\colon \mathcal{F} \to \mathcal{E}$ is continuous.*

Proof: (1) Let $e = (e_k)_{k\in\mathbb{N}} \in \mathcal{E}, h = (h_n)_{n\in\mathbb{N}} \in \mathscr{H}(\mathcal{F})$. Then

$$\begin{aligned}\langle T(e), h\rangle &= \sum_{n\in\mathbb{N}} \langle T_n(e), h_n\rangle_n = \sum_{k\in\mathbb{N}} \langle T_{\varphi(k)}(\iota(e_k)), h_{\varphi(k)}\rangle_{\varphi(k)} = \\ &= \sum_{k\in\mathbb{N}} \langle T_k(e_k), h_{\varphi(k)}\rangle_{\varphi(k)} = \sum_{k\in\mathbb{N}} \langle e_k, T_k^*(h_{\varphi(k)})\rangle_k = \langle e, T'(h)\rangle\end{aligned}$$

Set $g_k = T_k^*(h_{\varphi(k)}), g = (g_k)_{k\in\mathbb{N}} = T'(h)$. It remains to show that $g \in \mathcal{H}_{\mathcal{E}}$. For every $k \in \mathbb{N}$ choose $\delta_k \in E_k$ so that $\langle \delta_k, g_k\rangle = \|g_k\|_k$ and such that $\|\delta_k\| = 1$. Then $\delta(R) := (\delta_k R^k)_{k\in\mathbb{N}} \in \mathcal{E}$ for every $R \in \mathbb{R}_+$ and since $\sum_k \|g_k\|_k R^k = \langle \delta(R), g\rangle = \langle T(\delta(R)), h\rangle \in \mathbb{C}$, the sequence $\|g_k\|_k R^k$ is bounded, and hence $T'(\mathscr{H}(\mathcal{F})) \subset \mathscr{H}(\mathcal{E})$. Therefore T' is the adjoint operator of T. (2) The same arguments as before show that T' is the adjoint of $T_| \colon \mathscr{H}(\mathcal{E}) \to \mathscr{H}(\mathcal{F})$. □

Example 3.2.7. The conditions of the previous theorem are for example met by multiplication operators associated to matrices of homogeneous polynomials. If A is an $m \times n$ matrix of homogeneous polynomials, then the multiplication $f \mapsto A \cdot f$ is continuous both as an operator $\mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ and $\mathcal{H}(\mathbb{C}^n)^N \to \mathcal{H}(\mathbb{C}^n)^m$.

3.2.1 The Penrose inverse of a homogeneous operator

Definition 3.2.8. If $L\colon V \to W$ is a linear map between two finite-dimensional Hilbert spaces V and W, then the Penrose inverse L^+ of L is the linear map $L^+\colon W \to V$ which assigns to an element $w \in W$ the unique $T^+(w) \in V$, which is of least norm among all vectors in V whose image approximates w in the best way. That is, $T^+(w)$ solves the two-stage optimization problem ([BIG03, p.109])

(1) $\min\{\|w - T(v)\| \mid v \in V\}$
(2) $\min\{\|\tilde{v}\| \mid \|w - T(\tilde{v})\| = \min_{v\in V} \|w - T(v)\|\}$

So the Penrose inverse is the unique map $L\colon W \to V$ which solves

(1) $w - LL^+(w) \in \operatorname{Im}(T)^\perp$
(2) $L^+(w) \in \ker(T)^\perp$

for all $w \in W$.

Lemma 3.2.9 ([BIG03])**.** *Let $L\colon V \to W$ be a linear map between two finite-dimensional Hilbert spaces, and let $\pi_I\colon W \to \operatorname{Im}(L)$, $\pi_{K^\perp}\colon V \to \ker(L)^\perp$ be the orthogonal projections on $\operatorname{Im}(L)$ and $\ker(L)^\perp$, the orthogonal complement of the kernel of L, respectively. The Penrose inverse of L has the following representations:*

(1) $L^+ = (\pi_I \circ L|_{K^\perp})^{-1} \circ \pi_I$

(2) $L^+ = L^* \circ (\pi_I \circ LL^*|_{\mathrm{Im}(L)})^{-1} \circ \pi_I$
(3) *If L is surjective, then $L^+ = L^*(LL^*)^{-1}$*
(4) *If L is injective, then $L^+ = (L^*L)^{-1}L^*$*

Definition 3.2.10. Let $T\colon \mathcal{E} \to \mathcal{F}$ be a homogeneous operator between flat analytic spaces, with shift φ. We denote by T_k^+ the Penrose inverse of $T_k\colon E_k \to F_{\varphi(k)}$. Then the *Penrose inverse* of T^+ is the map

$$T^+\colon \widehat{\mathcal{F}} \to \widehat{\mathcal{E}},\ f = (f_n)_{n\in\mathbb{N}} \mapsto (T_k^+(f_{\varphi(k)}))_{k\in\mathbb{N}}.$$

T^+ maps $\mathcal{F}$ into $\mathcal{E}$ if and only if the image of T is closed (see below). But even if the image is not closed, $\mathrm{Im}(T)$ is mapped to $\mathcal{E}$. In particular, if T is injective, then T^+ is a left-inverse of T. By $\ker(T)^\perp$ we denote the closed subspace in $\mathcal{E}$ which is the flat analytic space over the floor sequence $(\ker T_k)^\perp$. This gives a decomposition of $\mathcal{E}$ into $\ker T \oplus (\ker T)^\perp$.

Theorem 3.2.11. *Let $T\colon \mathcal{E} \to \mathcal{F}$ be a homogeneous operator. Then*
(1) $T^+(\mathrm{Im}(T)) \subset \mathcal{E}$.
(2) *The following are equivalent:*
  (a) *The Penrose inverse of T is analytic.*
  (b) *T has closed image in $\mathcal{F}$.*
  (c) $T(\widehat{\mathcal{E}}) \cap \mathcal{F} = T(\mathcal{E})$.

Proof: (1) Let $f = T(e) \in \mathrm{Im}(T)$, and choose $r > 0$ so that $\sum \|e_k\|_k r^k < \infty$. Since $T_k^+T_k(e_k)$ is the orthogonal projection along the kernel of T, $\left\|T_k^+T_k(e)\right\|_k \leq \|e_k\|$, and hence $\|T^+(f)\|_r = \sum \left\|T_k^+T_k(e_k)\right\|_k r^k < \infty$, which shows that $T^+(\mathrm{Im}(T)) \subset \mathcal{E}$.
(i) $\Leftrightarrow$ (ii) Let $\pi_{\mathrm{Im}(T)} = (\pi_{\mathrm{Im}(T_k)})_{k\in\mathbb{N}}$ be the orthogonal projection on the image of T and let $\pi_{\ker(T)^\perp} = (\pi_{\ker(T_k)^\perp})_{k\in\mathbb{N}}$ be the orthogonal projection onto the orthogonal complement of $\ker(T)$. Suppose at first that T has closed range. Then by Theorem 3.1.14 $\pi_{\mathrm{Im}(T)}$ is analytic and $S := \pi_{\mathrm{Im}(T)} \circ T\Big|_{\ker(T)^\perp}$ is a continuous isomorphism onto the image of T, as the open mapping theorem holds in the realm of (DFS)-spaces and since a closed subspace of a (DFS)-space is again a (DFS)-space. Since $T_k^+ = (\pi_{\mathrm{Im}(T_k)} \circ T\Big|_{\ker(T_k)^\perp})^{-1} \circ \pi_{\mathrm{Im}(T_k)}$ and as T^+ is floorwisely defined as T_k^+, we have $T^+ = (\pi_{\mathrm{Im}(T)} \circ T\Big|_{\ker(T)^\perp})^{-1} \circ \pi_{\mathrm{Im}(T)}$, which is an analytic map $\mathcal{F} \to \mathcal{E}$. Conversely, suppose that T^+ is an analytic map $\mathcal{F} \to \mathcal{E}$. Then $\mathrm{Im}(T)^\perp$, the kernel of T^+, is closed and is therefore a flat analytic space over the floor sequence $(\mathrm{Im}(T_k)^\perp)_{k\in\mathbb{N}}$. By Theorem 3.1.14 we obtain a direct sum decomposition of $\mathcal{F}$ into $\overline{\mathrm{Im}(T)} \bigoplus \mathrm{Im}(T)^\perp$, where

$\overline{\mathrm{Im}(T)}$ denotes the topological closure of $\mathrm{Im}(T)$ in E. The fact that $T_k T_k^+ = \mathrm{id}$ on $\mathrm{Im}(T_k)$ carries over to $TT^+ = \mathrm{id}$ on $\overline{\mathrm{Im}(T)}$, whence $\mathrm{Im}(T) = \overline{\mathrm{Im}(T)}$.
(ii) $\Leftrightarrow$ (iii) Assume that $\mathrm{Im}(T)$ is closed and that $f \in T(\widehat{\mathcal{E}}) \cap \mathcal{F}$. Then $T^+(f) \subset \mathcal{E}$ and $T(T^+(f)) = f$, so $f \in T(\mathcal{E})$. Conversely, assume that $T(\widehat{\mathcal{E}}) \cap \mathcal{F} = T(\mathcal{E})$. Let $(\mathbf{e}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ so that the images $T(\mathbf{e}_n)$ converge to $f \in \widehat{\mathcal{F}}$, write $\mathbf{e}_n$ as $(e_{k,n})_{k\in\mathbb{N}}$. The projections onto the F_α are continuous, whence the coordinates $T_k(e_{k,n})$ converge to $f_{\phi(k)}$ (for $n \to \infty$). The images of linear maps between finite-dimensional spaces are closed, hence there exist $e_k \in E_k$ such that $T(e_k) = f_{\phi(k)}$. We can therefore solve $T(\widehat{e}) = f$ formally with $\widehat{e} \in \widehat{\mathcal{E}}$ and thus also inside $\mathcal{E}$, which shows that $f \in \mathrm{Im}(T)$. $\square$

Let us shortly note that the construction of an analytic Pseudo-inverse is not limited to taking orthogonal complements to the image and kernel. Formally, one can choose arbitrary complements. If the projections along the respective spaces are analytic and if the homogeneous operator has closed image, then it is possible to construct an analytic inverse with respect to the chosen complements.

Definition 3.2.12. Let $T\colon \mathcal{E} \to \mathcal{F}$ be a homogeneous operator with shift φ, $T_k\colon E_k \to F_{\varphi(k)}$. For every $k \in \mathbb{N}$ let $N_k \bigoplus \ker(T_k) = E_k$ and $M_k \bigoplus \mathrm{Im}(T_k) = F_{\varphi(k)}$ be complemented subspaces to $\ker(T_k), \mathrm{Im}(T_k)$, respectively. Denote by $\mathrm{proj}_{N_k}\colon E_k \to N_k$ the projections along $\ker(T_k)$ onto $\mathcal{N}_k$ and by $\mathrm{proj}_{\mathrm{Im}(T_k)}$ the projection of F_k onto $\mathrm{Im}(T_k)$ along M_k. Let $\mathcal{N}$ be the flat analytic space over the floor sequence $(N_k)_{k\in\mathbb{N}}$, respectively, and let

$$\mathrm{proj}_{\mathcal{N}} = (\mathrm{proj}_{N_k})_{k\in\mathbb{N}}\colon \mathcal{E} \to \widehat{\mathcal{N}}, \qquad \mathrm{proj}_{\mathrm{Im}(T)} = (\mathrm{proj}_{\mathrm{Im}(T_k)})_{k\in\mathbb{N}}\colon \mathcal{F} \to \widehat{\mathrm{Im}(T)}$$

be the projections on $\widehat{\mathcal{N}}$ and $\widehat{\mathrm{Im}(T)}$. Then the pseudo-inverse S with respect to the complements $(N_k)_{k\in\mathbb{N}}, (M_k)_{k\in\mathbb{N}}$ is the linear map

$$S\colon \widehat{\mathcal{F}} \to \widehat{\mathcal{E}}, S((f_k)_{k\in\mathbb{N}}) = \Big(S_k(f_{\varphi(k)})\Big)_{k\in\mathbb{N}} \text{ where } S_k = (\mathrm{proj}_{\mathrm{Im}(T_k)} \circ T_k|_{N_k})^{-1}(\mathrm{proj}_{\mathrm{Im}(T_k)})$$

Theorem 3.2.13. *Let T be a homogeneous map with shift φ, and let $N_k \bigoplus \ker(T_k) = E_k$ and $M_k \bigoplus \mathrm{Im}(T_k) = F_{\varphi(k)}$ be complemented subspaces to $\ker(T_k), \mathrm{Im}(T_k)$, respectively. Let $\mathrm{proj}_{\mathcal{N}}$ and $\mathrm{proj}_{\mathrm{Im}(T)}$ be defined as above. Suppose that T has closed image and that both $\mathrm{proj}_{\mathcal{N}}$ and $\mathrm{proj}_{\mathrm{Im}(T)}$ are analytic. Then the pseudoinverse S with respect to the complements $(N_k)_{k\in\mathbb{N}}$ and $(M_k)_{k\in\mathbb{N}}$ is analytic and $S\colon \mathcal{F} \to \mathcal{E}$ equals*

$$\mathrm{proj}_{\mathcal{N}} \circ (\mathrm{proj}_{\mathrm{Im}(T)} \circ T|_{\mathcal{N}})^{-1} \circ \mathrm{proj}_{\mathrm{Im}(T)}.$$

Proof: The analyticity of the projections yields direct sum decompositions

$$\mathcal{E} = \ker(T) \bigoplus \mathcal{N} \qquad \mathcal{F} = \mathrm{Im}(T) \bigoplus \mathcal{M}.$$

Again the open mapping theorem for (DFS)-spaces implies that $\mathrm{proj}_{\mathrm{Im}(T)} \circ T\big|_{\mathcal{N}}$ has a continuous inverse. It is easy to check that the k^{th} floorwise operator of $(\mathrm{proj}_{\mathrm{Im}(T)} \circ T\big|_{\mathcal{N}})^{-1} \circ \mathrm{proj}_{\mathrm{Im}(T)}$ coincides with $(\mathrm{proj}_{\mathrm{Im}(T_k)} \circ T_k\big|_{N_k})^{-1}(\mathrm{proj}_{\mathrm{Im}(T_k)})$. □

Since every submodule of $\mathbb{C}\{x\}^m$ is closed by Theorem 2.3.6(3), the next theorem is a direct consequence of Theorem 3.2.11.

Corollary 3.2.14. *Let $T\colon \mathbb{C}\{x\}^N \to \mathbb{C}\{x\}^m$ be the homogeneous $\mathbb{C}\{x\}$-module homomorphism $T\colon \boldsymbol{y} \mapsto H(x) \cdot \boldsymbol{y}$, where $H(x)$ is a matrix of homogeneous polynomials. Then the Penrose inverse T^+ of T is analytic.*

Another approach besides showing that the image of the homogeneous operator T is closed, is to directly estimate the norms of the floorwise Penrose inverses, in order to show the analyticity of T^+. This means that one has to find bounds (from below) for the smallest non-zero eigenvalues of $T_k T_k^*$. The following example illustrates this approach, but it seems difficult to use it in more complicated situations.

Example 3.2.15. Let $p = \gamma x_1^2 + x_1 x_2 + \gamma x_2^2$, and let $T\colon (a,b) \mapsto ((\partial_{x_1} p)a, (\partial_{x_2} p)b) = (2\gamma x_1 + x_2) \cdot a + (2\gamma x_2 + x_1) \cdot b$ be the homogeneous linear operator $T\colon \mathbb{C}\{x\}^2 \to \mathfrak{m}\mathbb{C}\{x\}$ associated to the differential of p. We filter $\mathbb{C}\{x\}$ by $H_k(\mathbb{C}^2)$, equipped with the Bombieri inner product. Let $T_k\colon H_k(\mathbb{C}^2)^2 \to H_{k+1}(\mathbb{C}^2)$ be the restriction of T to the k^{th} floor of homogeneous polynomials of degree k. We assume that $\gamma \in \mathbb{R}_+, \gamma \neq 1/2$. Then all T and all T_k are surjective, which implies that the pseudoinverse of T_k can be calculated as

$$T_k^+ = T_k^*(T_k T_k^*)^{-1}.$$

The adjoint of T_k is the differential operator

$$H_k \ni f_k \mapsto \frac{1}{k+1}\begin{pmatrix} 2\gamma f_{x_1} + f_{x_2} \\ 2\gamma f_{x_2} + f_{x_1} \end{pmatrix}$$

The operator $T_k T_k^*\colon H_{k+1} \to H_{k+1}$ maps $f \in H_{k+1}$ to

$$\begin{aligned} &\frac{1}{k+1}(2\gamma x_1 + x_2)(2\gamma f_{x_1} + f_{x_2}) + \frac{1}{k+1}(2\gamma x_2 + x_1)(2\gamma f_{x_2} + f_{x_1}) = \\ &= \frac{1}{k+1}((4\gamma^2+1)x_1 + 4\gamma x_2) f_{x_1} + \frac{1}{k+1}((4\gamma^2+1)x_2 + 4\gamma x_1) f_{x_2}. \end{aligned}$$

We calculate the matrix representation of $T_k T_k^*$ with respect to the canonical orthonormal basis

$$\left\{ x^\alpha \binom{k+1}{\alpha}^{1/2} \;\middle|\; |\alpha| = k+1 \right\}.$$

The monomial x^α is mapped to

$$T_k T_k^*(x^\alpha) = \frac{1}{k+1}\left((4\gamma^2+1)(\alpha_1+\alpha_2)x^\alpha + 4\gamma\alpha_1 x^{\alpha-e_1+e_2} + (4\gamma)\alpha_2 x^{\alpha-e_2+e_1}\right),$$

and so the coefficients of $T_k T_k^*(x^\alpha \binom{k+1}{\alpha}^{1/2})$ with respect to the orthonormal basis are

$$\left\langle T_k T_k^*\left(x^\alpha \binom{\alpha}{k+1}^{1/2}\right), x^\beta \binom{\beta}{k+1}^{1/2} \right\rangle = \begin{cases} \frac{(4\gamma^2+1)}{k+1} & \beta = \alpha \\ \frac{4\gamma}{k+1}(\alpha_1)^{(1/2)}(\alpha_2+1)^{(1/2)} & \beta = \alpha - e_1 + e_2 \\ \frac{4\gamma}{k+1}(\alpha_2)^{(1/2)}(\alpha_1+1)^{(1/2)} & \beta = \alpha - e_2 + e_1 \\ 0 & \text{else} \end{cases}$$

In lexicographical order, the matrix representation of $T_k T_k^*$ is the tridiagonal symmetric matrix

$$S_k = \frac{1}{k+1}\begin{pmatrix} (4\gamma^2+1)(k+1) & 4\gamma(k+1)^{1/2} & 0 & & \\ 4\gamma(k+1)^{1/2} & (4\gamma^2+1)(k+1) & 4\gamma k^{1/2}2^{1/2} & & \\ 0 & 4\gamma k^{1/2}2^{1/2} & (4\gamma^2+1)(k+1) & & \\ & & \ddots & \ddots & \ddots \\ & & & & \\ & & & & \end{pmatrix}$$

By Gershgorin's circle theorem, the eigenvalues of $S_k = (s_{i,j})_{i,j}$ are contained in the union of the closed Euclidean balls $B(s_{j,j}, \sum_{i\neq j}|a_{i,j}|)$. Hence the smallest Eigenvalue λ_k of S_k is larger or equal to $\min_{0\leq j\leq k+1} s_{j,j} - \sum_{i\neq j} a_{i,j}$, so

$$\begin{aligned} \lambda_k &\geq \min_{0\leq j\leq k+1}(4\gamma^2+1) - 4\gamma\frac{1}{k+1}\left((k+1-j)^{1/2}(j+1)^{1/2} + (k+2-j)^{1/2}j^{1/2}\right) \\ &\geq (4\gamma^2+1) - 4\gamma\frac{k+2}{k+1} \end{aligned}$$

The quadratic function $q(\gamma) = 4\gamma^2 + 1 - 4\gamma$ reaches its minimum at $\gamma = 1/2$ and is strictly positive for all real values different form zero. In particular, $4\gamma^2 + 1 > 4\gamma$ for $\gamma \neq 0$ and therefore λ_k is bounded from below by a constant $\tilde{C} > 0$ for all $k \geq k_0$, and thus there exists a constant $C > 0$ so that $\lambda_k \geq C$ for all $k \in \mathbb{N}$. The operator norm of T_k is bounded by the maximum of the supremum norms of the coefficient polynomials on the unit sphere, so

$$\|T_k\|_{\mathrm{OP}} \leq \max\{\sup_{|x_1|^2+|x_2|^2=1} |2\gamma x_1 + x_2|, \sup_{|x_1|^2+|x_2|^2=1} |2\gamma x_2 + x_1|\} \leq \sqrt{4\gamma^2+1}.$$

Therefore

$$\|T_k\|^+_{\mathrm{OP}} \leq \|T_k^*\|_{\mathrm{OP}} \left\|(T_k T_k^*)^{-1}\right\|_{\mathrm{OP}} \leq \|T_k\|_{\mathrm{OP}} \lambda_k^{-1} \leq \sqrt{4\gamma^2+1} C^{-1} =: M.$$

Now let $f = \sum_k F_k \in \ell^1(R)$, where F_k is the homogeneous part of order k of f and $\sum_k \|F_k\| R^k < \infty$. Then

$$\left\|T^+(F)\right\|^1_R = \sum \left\|T_k^+ F_{k+1}\right\|_k R^k \leq M/R \sum \|F_{k+1}\|_{k+1} R^{k+1} \leq M/R \|f\|^1_R.$$

These estimates show that T^+ is a bounded operator $\ell^1(R) \to \ell^1(R)^2$. The quality of the estimate is the same as via the approach the Grauert Division algorithm, however, the estimate holds for all $R > 0$ and not only for all R smaller then a certain R_0.

3.2.2 Composition operators

In the following, we will review some results on the question when a composition operator between power series algebras has closed image. This question is intimately related to the solvability of analytic equations and the Nested Artin approximation theorem. Composition operators induced by algebraic maps always have closed image. In the analytic case the closedness of the image of C_φ is equivalent to the a rank condition on the Jacobian of φ.

Definition 3.2.16. An *analytic algebra* is a quotient of a ring of convergent power series $\mathbb{C}\{y_1, \dots, y_n\}/I$ by an ideal $I \subset \mathbb{C}\{y_1, \dots, y_N\}$. An analytic homomorphism $\Phi\colon R \to S$ between two analytic algebras $R = \mathbb{C}\{y_1, \dots, y_N\}/I$, $S = \mathbb{C}\{x_1, \dots, x_n\}/J$, is a local $\mathbb{C}$-algebra homomorphism. It turns out that every such homomorphism is induced from a composition operator $C_\varphi\colon \mathbb{C}\{y_1, \dots, y_N\} \to \mathbb{C}\{x_1, \dots, x_n\}, f(y) \mapsto f(\varphi(x))$ ([GR13, p.79]). Analytic algebras carry a natural structure as a locally convex space in the form of a Banach scale and (DFS)-space. We will refer to the quotient topology induced by $\mathbb{C}\{x\} = \lim_{r\to\infty} \ell^1_r$ on S as the *inductive* topology. In [BZ79], Becker and Zane studied the functional analytic properties of analytic algebras and showed that many properties of analytic algebra morphisms such as the closedness can be read off the algebraic properties of the rings and the power series vector φ which induces the composition operator. Let S be a domain and let $\Phi\colon R \to S$ be an analytic algebra homomorphism induced by the composition operator $C_\varphi\colon \mathbb{C}\{y_1,\dots,y_N\} \to \mathbb{C}\{x_1,\dots,x_n\}$. The the *generic rank* $\operatorname{grk}(\Phi)$ of φ is the rank of the Jacobi matrix $\frac{\partial(\varphi_1,\dots,\varphi_N)}{\partial(x_1,\dots,x_n)}$ over the quotient field of S. In other terms, it is the size of the largest minor of $\frac{\partial(\varphi_1,\dots,\varphi_N)}{\partial(x_1,\dots,x_n)}$ which is not contained in J.

Remark. As a local ring, an analytic algebra can be also equipped with the Krull (= $\mathfrak{m}$-adic) topology. A third natural topology is given by the coefficientwise convergence of power series, which is the initial topology with respect to the coordinate projections $\operatorname{proj}_\beta\colon \sum c_\alpha x^\alpha \mapsto c_\beta$.

Theorem 3.2.17 ([BZ79, Cor 2.12, Thm 2.8])**.** *Let $\Phi\colon R \to S$ be a local homomorphism of analytic algebras, with S a domain. TFAE:*

(1) *$\Phi(R)$ is closed in S in the Krull topology.*
(2) *$\Phi(R)$ is closed in S in the inductive topology.*
(3) $\operatorname{grk}(\Phi) = \dim R/\ker(\Phi) = \dim \widehat{R}/\widehat{\ker(\Phi)}$.

Corollary 3.2.18. *Let $\Phi\colon \mathbb{C}\{y_1,\dots,y_N\} \to \mathbb{C}\{x_1,\dots,x_n\}$ be an analytic homomorphism. If* $\operatorname{grk}(\Phi) = N$*, then Φ is a closed map and an isomorphism onto its image.*

Proof: Let $\varphi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^N,0)$ be the holomorphic map which induces $C_\varphi = \Phi$. We show at first that C_φ is injective. Suppose that $C_\varphi(f(y)) = f(\varphi(x)) = 0$. By assumption, the rank of $\frac{\partial(\varphi_1,\dots,\varphi_N)}{\partial(x_1,\dots,x_n)}$ is N. The rank theorem implies that the image of every open zero neighborhood has non-empty interior. Let V be a connected open set around $0 \in \mathbb{C}^N$ so that f is defined on V. Then $\varphi(\varphi^{-1}(V)) \subset V$ has non-empty interior. Since $f \circ \varphi = 0$, it follows that f vanishes on $\varphi(\varphi^{-1}(V))$, thus $f = 0$ and C_φ is injective. Since $\operatorname{grk}(\Phi) = N = \dim \mathbb{C}\{y_1,\dots,y_N\}/\ker(\Phi)$, the image of Φ is closed. Hence Φ is an isomorphism onto its image (as the open mapping theorem holds for maps between (DFS)-spaces). Therefore the image of a closed set under Φ is closed in $\Phi(R)$ and thus in S, since $\Phi(R)$ is closed in S. □

If R and S are rings of germs of analytic maps on reduced algebraic varieties, and if Φ is an algebraic map between the corresponding algebraic varieties, then the image of Φ is closed without further assumptions.

Theorem 3.2.19 ([Tou76], [Bec77, Thm B])**.** *Let R, S be the local rings of analytic functions of germs of reduced algebraic varieties and let $\Phi\colon R \to S$ be a local homomorphism. If Φ is induced by an algebraic map, then the image of Φ is closed in the Krull topology.*

This theorem has been generalized by G. Rond ([Ron06], [Ron13]) to the case where R, S are quotients of convergent power series algebras by ideals generated by algebraic series, if Φ is an algebraic power series vector and if S is an integral domain.

Theorem 3.2.20. *Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_N)$, $(z_1, \dots, z_k)$ and let $\varphi(x, z) \in \mathbb{C}[x, z]^N$ be a vector of homogeneous polynomials with respect to the same weight $\upsilon \in \mathbb{N}^{n+k}$. Let d_i be the weighted degree with respect to υ of $\varphi_i(x, z)$, let $\delta = (d_1, \dots, d_N)$ and let $\omega = (\mu, \delta) \in \mathbb{N}^{n+N}$. We consider $\mathbb{C}\{x, y\}$ as a flat analytic space over the spaces of weighted homogeneous polynomials $(H_l(\omega))_{l\in\mathbb{N}}$ and $\mathbb{C}\{x, z\}$ as a flat analytic space over $(H_k(\upsilon))_{k\in\mathbb{N}}$. Then $C_\varphi \colon \mathbb{C}\{x, y\} \to \mathbb{C}\{x, z\}$ is a homogeneous operator and its Penrose inverse φ^+ is analytic.*

Proof: The operator is homogeneous (see Example 3.2.4) and by Becker's Theorem B the image of φ is closed, which is equivalent to the analyticity of the Penrose inverse of φ (Theorem 3.2.11). $\square$

Definition 3.2.21. We say that two vectors of convergent power series $f(x) \in \mathbb{C}\{x\}^N$ and $g(x) \in \mathbb{C}\{x\}^N$ are *right-equivalent*, if there exists a local biholomorphism of $(\mathbb{C}^n, 0)$ so that $f(x) = g(\psi(x))$. A germ V of an analytic space is called quasi-homogeneous, if there exist local coordinates in which V can be defined by weighted homogeneous polynomials (which are weighted homogeneous with respect to the same weight ω). Put differently, this mean that there exist $f_1, \dots, f_k$ which define V, so that $f = (f_1, \dots, f_k)$ is right-equivalent to a vector of quasi-homogeneous polynomials.

Lemma 3.2.22. *If $\varphi(x) \in \mathbb{C}\{x\}^N$ is right-equivalent to a weighted homogeneous polynomial, then C_φ is a quasi-homogeneous operator.*

Proof: By assumption, there exists a local isomorphism h of $(\mathbb{C}^n, 0)$ and vector of weighted homogeneous polynomials p so that $\varphi(x) = p(h(x))$. Hence C_φ factors into $C_h \circ C_p$, where C_h is an automorphism of $\mathbb{C}\{x\}$ and C_p is a homogeneous map. $\square$

The problem when a germ V of an analytic variety is quasi-homogeneous has been studied classically under the assumption that V is an isolated hypersurface singularity. Saito ([Sai71]) gave a full characterization and showed that an isolated hypersurface singularity $V(f)$ is quasi-homogeneous if and only if the Milnor number $\mu(f)$ coincides with the Tjurina number $\tau(f)$. For isolated complete intersection singularities of higher co-dimension, the same statement holds true, which has been proven by [Vos02] 40 years after Saito's result. In the hypersurface case, $\tau(f) = \mu(f)$ can be equivalently stated as $f \in \langle \partial_{x_1} f, \ldots, \partial_{x_n} f \rangle$. The more restrictive question, when an isolated singularity can be defined after a change of coordinates by homogeneous polynomials, has only been recently answered by Yau and Zuo ([YZ15]): $V(f) \subset \mathbb{C}^n$ is homogeneous if and only if $\mu(f) = \tau(f) = (v-1)^n$, where v is the order of f (called the multiplicity).

3.2.3 Composition operators on spaces of entire functions

If we study composition operators on the local level, that is, between rings of germs of analytic functions, then the closedness of the image is characterized by the rank criterion (Theorem 3.2.17). On a global level, this problem has been classically studied for composition operators $C_\varphi \colon C^\infty(M) \to C^\infty(N)$ between spaces of smooth functions on smooth manifolds, usually under the assumption the φ is semi-proper and real-analytic ([BMP96]). In the real-analytic setting, Domanski et al ([DG12]) gave criteria for the closedness of the image of a composition operator $C_\varphi \colon \mathcal{A}(M) \to \mathcal{A}(N)$ induced by a real-analytic map and which acts on the real-analytic functions on real-analytic manifolds M, N.

Theorem 3.2.23 ([DL05, Theorem 3.1])**.** *Let $\varphi \colon M \to N$ be a real-analytic map between two real-analytic manifolds and let $C_\varphi \colon \mathcal{A}(M) \to \mathcal{A}(N)$ be the associated composition operator. Then*

(1) *If C_φ has closed range, then $\varphi(M)$ is analytic with global equations.*
(2) *If C_φ is open onto its closed range, then φ is semi-proper.*

A map $\varphi \colon X \to Y$ between two topological spaces is called *semi-proper*, if there exists for every compact subset K of Y a compact subset L of X so that $\operatorname{Im}(\varphi) \cap K = \varphi(L)$. A subset S of N is called analytic with global equations, if S is the zero set of a real analytic function f on N.
For composition operators $C_\varphi \colon \mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(\mathbb{C}^k)$ between spaces of entire functions, it seems that there does not exist any characterization of the closedness of $\operatorname{Im}(C_\varphi)$. The analogously stated necessary conditions of the real analytic setting are not necessarily met by an operator with closed image: In $\mathbb{C}^2$, we can proof the following result:

Theorem 3.2.24. *Let $q(z,w) = q_1(z)w^1 + \cdots + q_d(z)w^d$ be any polynomial in $\mathbb{C}^2$ without constant w-term and let $\varphi(z,w) = (z, q(z,w))$. Then the associated composition operator $C_\varphi \colon \mathcal{H}(\mathbb{C}^2) \to \mathcal{H}(\mathbb{C}^2)$ has closed range.*

Proof: If $q \equiv 0$, then the image of C_φ is $\mathcal{H}(\mathbb{C})$ and thus closed. Now assume that $q \not\equiv 0$, and let $z_1, \ldots, z_s$ be the common zeros of the w-coefficients $q_1(z), \ldots, q_d(z)$.

Then

$$\mathrm{Im}(\varphi) = \left(\mathbb{C}^2 \setminus \bigcup_{i=1}^{s} \{z = z_i\}\right) \cup \{(z_1, 0), \ldots, (z_s, 0)\}\,.$$

Let h be in the closure of the image and choose a sequence $(f_k)_{k\in\mathbb{N}}$ of entire functions so that $\lim_{k\to\infty} C_\varphi(f_k) = h$. Set $U = \mathbb{C}^2 \setminus \{g_w = 0\}$. For $x \in U$, φ is a local isomorphism, so $C_\varphi\colon \mathcal{O}_{\varphi(x)} \to \mathcal{O}_x$ is a a topological isomorphism. Hence $(f_k)_{k\in\mathbb{N}}$ converges locally uniformly on $\varphi(U)$ to some $f \in \mathcal{H}(\varphi(U))$. If $x \in \{g_w = 0\}$, still $C_\varphi\colon \mathcal{O}_{\varphi(x)} \to \mathcal{O}_x$ has generic full rank, is therefore injective and has closed image. So there exists an $F \in \mathcal{O}_{\varphi(x)}$ solving $C_\varphi(F) = h$. Let V be a connected open neighborhood of $\varphi(x)$ so that F is analytic on V. Observe that $\varphi(U)$ is dense (and open) in $\mathrm{Im}(\varphi)$. Let $x_1 \in U \cap \varphi^{-1}(V)$. Since $C_\varphi\colon \mathcal{O}_{\varphi(x_1)} \to \mathcal{O}_{x_1}$ is injective, and as $C_\varphi(F) = h = C_\varphi(f)$, we obtain that F has to coincide with f around $\varphi(x_1)$. So F coincides with f on $\varphi(U \cap \varphi^{-1}(V))$ and is thus an analytic extension of f around $\varphi(x)$. Since x was arbitrary, we can thus extend f to $\mathrm{Im}(\varphi)$ so that f still solves $C_\varphi(f) = h$. It remains to show that f can be extended to an entire function, i.e. that it can be extended across the hyperplanes $\{z = z_i\}$. But f is holomorphic on $\mathbb{C}^2 \setminus \bigcup_{i=1}^{s}\{z = z_i\}$ and extends locally at $(z_i, 0) \in \{z = z_i\}$ since $(z_i, 0) = (z_i, q(z, 0)) = \varphi(z_i, 0)$. So we can apply Hartogs removable singularity theorem (of functions across graphs) and extend f to an entire function. □

Remark. If $C_\varphi\colon \mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(\mathbb{C}^d)$ has closed range, then it is automatically open onto its image (by the open mapping theorem for Fréchet spaces).

Example 3.2.25. Let $\varphi(z, w) = (z, zw)$. Then by the previous theorem $C_\varphi(\mathcal{H}(\mathbb{C}^2))$ is closed. But φ is neither semi-proper, nor is the image of φ closed in $\mathbb{C}^2$. The image of φ is $\mathbb{C} \setminus \{z = 0\} \cup (0, 0)$. Let $K = \{(z, 1) \mid |z| \leq 1\}$. Then the only set which is mapped onto $K \cap \mathrm{Im}(\varphi)$ is the hyperbola $\{(z, 1/z) \mid |z| \leq 1, z \neq 0\}$, which is obviously not compact, thus φ is not semi-proper.

For weighted homogeneous polynomials, we can easily extend the local results.

Definition 3.2.26. Let W be an arbitrary subset of $\mathbb{C}^n$. We say that a function $f\colon W \to \mathbb{C}$ is holomorphic, if it is locally at each point $w_0 \in W$ the restriction of a holomorphic function defined in a neighborhood of w_0 in $\mathbb{C}^n$.

Theorem 3.2.27. *Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}^N$ be a vector of weighted homogeneous polynomials with respect to the same weight ω. Assume that φ has generic full rank. Then the following holds:*

(1) *Let f be a formal power series. If $f \circ \varphi$ is an entire function, then f is already entire.*

(2) *Every function which is holomorphic on the image of φ extends to an entire function.*

(3) *The image of $C_\varphi \colon \mathcal{H}(\mathbb{C}^N) \to \mathcal{H}(\mathbb{C}^n)$ is closed.*

(4) *The image of C_φ has a closed complement, and C_φ^+ is a continuous linear left inverse of C_φ.*

(5) *The extension operator $\mathcal{H}(\mathrm{Im}(\varphi)) \to \mathcal{H}(\mathbb{C}^N)$ is continuous.*

Proof: (1) Let $h = f \circ \varphi$ be entire. The image of the local operator $L := C_\varphi \colon \mathbb{C}\{y\} \to \mathbb{C}\{x\}$ is closed (by Theorem 3.2.17), so by Theorem 3.2.20 L admits an analytic Penrose inverse $L^+ \colon \mathbb{C}\{x\} \to \mathbb{C}\{y\}$. As L is injective, L^+ is in fact a continuous left-inverse . The continuity of L^+ implies that the radius of convergence is preserved uniformly by L^+ : For every $r > 0$ there exists an $s > 0$ so that $L^+(f) \in \ell^1_s(y)$, whenever $f \in \ell^1_r(x)$, where $\ell^1_s(x) \subset \mathbb{C}\{x\}$ and $\ell^1_s(y) \subset \mathbb{C}\{y\}$ are the usual weighted ℓ^1 spaces. Indeed, recall that the ball $B_{s,M}(x)$ of radius M in $\ell^1_s(x)$ is bounded in $\mathbb{C}\{x\}$ and that the family $\{B_{s,M}(x) \mid s, M \in \mathbb{R}_+\}$ forms a fundamental system of the bounded sets in $\mathbb{C}\{x\}$. So for every r, M there exist s, K so that $T^+(B_{r,M}(x)) \subset B_{s,K}(y)$, since T^+ preserves bounded sets. Now if $f \neq 0 \in \ell^1_r(x)$ is arbitrary, then $L^+(f \cdot \|f\|_r^{-1} M) \subset B_{s,K}(y)$, and hence $L^+ \colon \ell^1_r(x) \to \ell^1_s(y)$. Let now $r = 1$ for notational simplicity and let s be chosen so that $L^+ \colon \ell^1_1(x) \to \ell^1_s(y)$. We will now show that f is an entire function. Let d_i be the weighted degree of φ_i. For $t > 0$, set $f_t(y_1, \ldots, y_N) = f(t^{d_1} y_1, \ldots, t^{d_N} y_N)$ and let $t^\omega x := (t^{\omega_1} x_1, \ldots, t^{\omega_n} x_n)$. Then

$$h(t^\omega x) = f(\varphi(t^\omega x)) = f(t^{d_1}\varphi_1(x), \ldots, t^{d_N}\varphi_N(x)) = f_t(\varphi(x))$$

By assumption, $h(x)$ is an entire function, hence $f_t(\varphi(x)) = h(t^\omega x) \in \ell^1_1(x)$ and by the choice of s we can conclude that $f_t(y) \in \ell^1_s(y)$ for every $t > 0$. Let $d_0 = \min_{1 \le i \le N} d_i$. Then $f(y) \in \ell^1_{st^{d_0}}(y)$ for all $t > 1$ and therefore $f \in \mathcal{H}(\mathbb{C}^N)$. (2) Let $f(y)$ be holomorphic on $\mathrm{Im}(\varphi)$. Then $h(x) = f(\varphi(x)) \in \mathcal{H}(\mathbb{C}^n)$, and we can apply (1) to conclude that f is an entire function. (3) Let h be an entire function in the closure of $\mathrm{Im}(C_\varphi)$ and choose a sequence $f_k \in \mathcal{H}(\mathbb{C}^N)$ so that $C_\varphi(f_k)$ converges to h in the compact-open topology. In particular, $C_\varphi(f_k)$ converges to h locally at $0 = \varphi(0)$. The local operator $L \colon \mathbb{C}\{y\} \to \mathbb{C}\{x\}$ has closed image, hence there exists an $f \in \mathbb{C}\{y\}$ such that $C_\varphi(f) = h$. We can use the same arguments as in 1) to conclude that f is an entire function, whence the image of C_φ is closed. (4) We choose a flat analytic structure in which C_φ is a homogeneous operator: Let $\delta = (d_1, \ldots, d_N)$ be the vector of the weighted degrees of the φ_i, let $H_k(\delta) \subset \mathbb{C}[y]$ and $H_k(\omega) \subset \mathbb{C}[x]$ be the weighted homogeneous steps corresponding to the weights δ and ω, respectively. Then C_φ maps $H_k(\delta)$ into $H_k(\omega)$ and we denote the restriction of C_φ to $H_k(\delta)$ by T_k. We claim that $\mathcal{H}(\mathbb{C}^n) = \mathrm{Im}(C_\varphi) \bigoplus \mathrm{Im}(C_\varphi)^\perp$, where $\mathrm{Im}(C_\varphi)^\perp = \mathcal{H}_{\mathrm{Im}(C_\varphi)^\perp}$. Let $\pi = (\pi_k)_{k\in\mathbb{N}}$ be the (floorwise) orthogonal projection onto $\widehat{\mathrm{Im}(C_\varphi)}$. The operator norm of each $\pi_k \colon H_k(\omega) \to H_k(\omega)$ is 1, so the norm of $\pi \colon \ell^1_r(x) \to \ell^1_r(x)$ is 1 and π is a continuous projection on $\mathcal{H}(\mathbb{C}^n)$ (see also Theorem 3.1.14). So π maps $\mathcal{H}(\mathbb{C}^n)$ onto $\mathrm{Im}(C_\varphi)$ and has $\mathrm{Im}(C_\phi)^\perp$ as kernel, thus $\mathcal{H}(\mathbb{C}^n) = \mathrm{Im}(C_\varphi) \bigoplus \mathrm{Im}(C_\varphi)^\perp$. The image of C_φ is closed and C_φ is injective, thus $S = (C_\varphi|_{\ker(C_\varphi)^\perp})^{-1}$ is a continuous inverse of the restriction. The construction of C_φ^+ (which was conducted via the homogeneous floors) shows that $C_\varphi^+ = S \circ \pi$, and is thus analytic. (5) If f is holomorphic on $\mathrm{Im}(\varphi)$, then the extension of f is given by $C_\varphi^+(f(\varphi(x))$. So the extension operator is the composition of C_φ^+ with the extension of the composition operator to $\mathcal{H}(\mathrm{Im}(\varphi))$.

□

Remark. If the requirement that φ is weighted homogeneous is dropped, then the first statement of the previous theorem is of course false: Let $f(y) = \sqrt{1+y}$ and let $\varphi(x) = 2x + x^2$. Then $f(\varphi(x)) = (1 + x)$, but f is only locally analytic. The local statement

$$f(\varphi(x)) \text{ convergent implies that } f(x) \text{ is convergent}$$

is true if and only $\varphi(x) \in \mathbb{C}\{x\}^N, \varphi(0) = 0$ has generic full rank N and was proven by Eakin and Harris [EH77]. Juhlin and Lamel ([JL13]) have proven a concrete version of this result, and have also shown that an analytic left-inverse of C_φ can be constructed which depends analytically on φ inside a textile cofinitely open subsets of $\mathbb{C}\{x\}^N$. There exist also concrete estimates for the radius of convergence of f in terms by the one of $f \circ \varphi$ for certain classes of quasi-analytic function ([Mou02], [CC01], [Bie04]).

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Appendix

Abstract

The main objective of this thesis is to understand certain aspects of the geometry of the set of solutions $\mathcal{Y}(f)$ to an implicit power series equation $f(x, \boldsymbol{y}(x)) = 0$. This equation corresponds to an infinite system of polynomial equations $F_i(y_\alpha) = 0$ in the coefficients y_α of $\boldsymbol{y}$. The usual approach to understand these systems is to consider $\mathcal{Y}(f)$ as the closed points of the affine scheme $\operatorname{spec}(\mathbb{C}[y_\alpha]/\langle F_i \rangle)$ in an infinite-dimensional polynomial ring, which has the disadvantage that finite-dimensional geometrical concepts such as regularity are not available. Instead, we identify power series $\boldsymbol{y}(x)$ with (non-closed) points $\mathcal{P}_{\boldsymbol{y}}$ of $\operatorname{spec}(\mathbb{C}[|x, y|])$, assigning to a series $\boldsymbol{y}(x)$ the ideal of relations formed by all $f \in \mathbb{C}[|x, y|]$ solving $f(x, \boldsymbol{y}(x)) = 0$. A point $\boldsymbol{y}_0 \in \mathcal{Y}$ is said to be regular, if $\mathcal{Y}$ can be locally defined by a system of $f(x, \boldsymbol{y}(x)) = 0$ whose partial Jacobian $\partial_y f(x, \boldsymbol{y}(x))$ has full rank over $\mathbb{C}[|x|]$ at $\boldsymbol{y}_0$. We show a Jacobian Criterion adopted to this setting. The regular part $\mathcal{Y}(f) \setminus \mathcal{Y}(\Delta)$, where a maximal minor Δ of $\partial_y(f(x, y))$ does not vanish, can be decomposed into strata $\mathcal{Y}_d$ according to the order d of $\Delta(\boldsymbol{y})$. With the aid of a linearization theorem for arquile maps $f_\infty \colon \boldsymbol{y}(x) \mapsto f(x, \boldsymbol{y}(x))$ we show that the subsets $\mathcal{Y}_d$ are isomorphic to a product of a local arquile variety $\mathcal{Z}_d$ in one variable less and a free module $\mathbb{C}[|x|]^{N'}$. These constructions provide a geometrical interpretation of Artin's ([Art68]) and Płoski's ([Pło74]) approximation theorems. If x is a single variable, then our results show that every regular point is smooth in the sense that a neighborhood is isomorphic to free power series module $\mathbb{C}[|x|]^N$. The case of several variables is more complicated, but we are able to show that the points at which $\mathcal{Y}$ is smooth are dense. The results are shown over the ring of either algebraic, convergent or formal power series. The second chapter forms the analytic toolbox which is needed for the linearization techniques and the constructions of the trivializing morphisms $\mathcal{Y}_d \to \mathcal{Z}_d^* \times \mathbb{C}[|x|]^{N'}$. These maps are (rational) textile maps ([Bru09]), which are functions between power series spaces whose coefficients depend polynomially (rationally) on the coefficients of the input. We show an inversion and implicit function theorem for textile maps which are mild perturbations of arquile

maps $\boldsymbol{y}(x) \mapsto f(x, \boldsymbol{y}(x))$. The last part of this chapter reviews the division theorem by Grauert and Hironaka in various settings. In the final chapter we introduce a class of spaces which admit filtrations by Banach spaces and finite dimensional Hilbert spaces. The homomorphisms of these spaces are homogeneous operators, for which a formal Penrose inverse is constructed. In the last section we generalize a result by Eakin & Harris [EH77] to homogeneous composition operators between spaces of entire functions.

Zusammenfassung

In dieser Arbeit werden verschiedene Aspekte der Geometrie von so genannten arquilen Varietäten, welche die Lösungsmengen $\mathcal{Y}(f)$ von impliziten Potenzreihengleichungen $f(x, \boldsymbol{y}(x)) = 0$ sind, untersucht. Diese Gleichungen enstsprechen einem unendlichen System an polynomialen Gleichungen $F_i(y_\alpha) = 0$ in den Koeffizienten y_α von $\boldsymbol{y}(x)$. Der übliche Zugang, der beispielsweise in der Theorie der arc spaces gewählt wird, ist, die Lösungsmenge $\mathcal{Y}(f)$ als die Menge der abgeschloßenen Punkte des affinen Schemas $\operatorname{spec}(\mathbb{C}[y_\alpha]/\langle F_i\rangle)$ zu interpretieren. Der Nachteil dieses Ansatzes ist jedoch, dass geometrische Konzepte wie das der Regularität im Unendlichdimensionalen nicht mehr zu Verfügung stehen. Stattdessen identifizieren wir Potenzreihen $\boldsymbol{y}(x)$ mit (nicht-abgeschloßenen) Punkten $\mathcal{P}_{\boldsymbol{y}}$ in $\operatorname{spec}(\mathbb{C}\,[|x, y|])$, indem wir einer Potenzreihe das Ideal ihrer Relationen in $\mathbb{C}\,[|x, y|]$ zuordnen, bestehend aus denjenigen $f(x, y)$, welche $f(x, \boldsymbol{y}(x)) = 0$ lösen. Eine Lösung $\boldsymbol{y}_0 \in \mathcal{Y}$ heißt regulär, wenn $\mathcal{Y}$ lokal bei $\boldsymbol{y}_0$ durch ein System $f(x, \boldsymbol{y}(x)) = 0$ definiert werden kann, dessen partielle Jacobimatrix $\partial_y f(x, y)$ bei $\boldsymbol{y}_0$ vollen Rang über $\mathbb{C}\,[|x|]$ hat. Wir charakterisieren diese Eigenschaft mit einem adaptiertem Jakobi-Kriterium. Die reguläre Teilmenge $\mathcal{Y}(f) \setminus \mathcal{Y}(\Delta)$, wobei Δ ein maximaler Minor von $\partial_y f(x, y)$ sei, wird entsprechend der Ordnung d von $\Delta(x, \boldsymbol{y}(x))$ in Strata $\mathcal{Y}_d$ zerlegt. Mithilfe eines Linearisierungstheorems für arquile Abbildungen $\boldsymbol{y}(x) \mapsto f(x, \boldsymbol{y}(x))$ wird gezeigt, dass jedes Stratum $\mathcal{Y}_d$ zu einem Produkt $\mathcal{Z}_d^* \times \mathbb{C}\,[|x|]^{N'}$ isomorph ist, wobei $\mathcal{Z}_d^*$ eine lokal-arquile Varietät über einem Potenzreihenring in einer Variablen weniger ist. Diese Konstruktionen ergeben eine geometrische Interpretation der Beweise von Artin ([Art68]) und Płoski([Pło74]). Wenn x ein eindimensionaler Parameter ist, dann zeigt unser Resultat, dass eine arquile Varietät bei einem regulären Punkt glatt ist, d.h. dass sie eine Umgebung besitzt, welche zu einem Produkt von freien Potenzeihenmodulen isomorph ist. Der mehrdimensionale Fall ist etwas komplizierter, aber wir können zeigen, dass die Menge der glatten Punkte dicht ist. Diese Resultate werden über den Ringen der algebraischen, konvergenten und formalen Potenzeihen gezeigt. Im zweiten Kapitel wird der für die Linearisierung und Trivialisierung $\mathcal{Y}_d \to \mathcal{Z}_d^* \times \mathbb{C}\,[|x|]^{N'}$ notwendige analytische Rahmen geschaffen. Die Abbildugen, welche hier auftreten sind die s.g. (rational-) textilen Abbildungen ([Bru09]), welche Abbildungen zwischen Potenzreihenräumen sind, deren Koeffizienten polynomial (rational) von den Koeffizienten des Inputs abhängen. Es wird ein (analytischer) Inverser- und Impliziter Funktionensatz für textile Abbildungen, welche eine geringe Störung von arquilen Abbildungen $\boldsymbol{y}(x) \mapsto f(x, \boldsymbol{y}(x))$ sind, gezeigt. Im letzten Teil dieses Abschnitt werden die Divisionssätze von Grauert-Hironaka besprochen. Im

dritten Kapitel führen wir eine neue Klasse von Räumen ein, welche wie die Ringe der konvergenten Potenzreihen Filtrierungen durch Banachräume und endlichdimensionale Hilberträume besitzen. Die Homomorphismen dieser Räume sind die homogenen Abbildungen, für welche eine formale Penrose-Inverse konstruiert wird. Im letzten Teil wird ein Resultat von Eakin u. Harris auf hom. Kompositionsoperatoren zw. Räumen von ganzen Funktionen verallgemeinert.

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