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Abstract

Gabor frames provide stable, discrete time-frequency representations in $L^2(\mathbb{R}^d)$. It is therefore of great interest to determine pairs (g, Λ), consisting of an L^2-function g and a time-frequency lattice Λ, that generate a frame.

The advances of Gabor analysis in the last thirty years led to numerous characterizations of the frame inequality for Gabor systems. We give a comprehensive account of known results. The novelty lies in the systematic approach: The central result is the duality theory for Gabor frames over arbitrary lattices. We therefore develop the duality theory first and then obtain all characterizations for separable lattices as easy consequences. The well-known criterion of Ron and Shen follows by a simple Fourier series argument. In the case of a rational lattice, further periodization yields the results of Zeevi and Zibulski, where the frame property is linked to the spectral properties of a family of finite dimensional matrices.

For M^1-windows, the combination of the duality theory with Wiener's lemma yields a dozen additional characterizations without inequalities. All of them are expressed as properties of the canonical operators associated to the Gabor system.

Contents

Introduction

The goal of time-frequency analysis is to represent any function $f \in L^2(\mathbb{R}^d)$ as a superposition of translated and modulated versions of a fixed window function $g \in L^2(\mathbb{R}^d)$. While integral representations provide good tools for theoretical purposes, for applications it is more convenient to have a discrete representation, i.e., a series expansion of the form

$$f = \sum_{\lambda \in \Lambda} c_\lambda \, \pi(\lambda) g, \tag{0.1}$$

where Λ is a discrete subset of the time-frequency plane $\mathbb{R}^{2d}$, $c_\lambda \in \mathbb{C}$ and

$$\pi(\lambda)g(x) := e^{2\pi i \lambda_2 \cdot x} g(x - \lambda_1)$$

denotes the time-frequency shift with respect to $\lambda = (\lambda_1, \lambda_2) \in \Lambda \subseteq \mathbb{R}^{2d}$.

This type of expansion was proposed by Gabor [19] in 1946 with respect to the one-dimensional Gaussian $\varphi(x) = e^{-\pi x^2}$ and the integer lattice $\Lambda = \mathbb{Z}^2$. Only much later, in the 1980s, Bastiaans [5] and Janssen [31] could indeed prove that every $f \in L^2(\mathbb{R})$ possesses such a representation. Unfortunately, the series is numerically unstable and converges only in a distributional sense [31].

In order to obtain stable expansions, Daubechies, Grossmann and Meyer [12] combined Gabor analysis with frame theory in 1986. A family $\mathcal{G}(g, \Lambda) := \{\pi(\lambda)g : \lambda \in \Lambda\}$ is called a *Gabor frame* if there exist positive constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|^2 \qquad \forall f \in L^2(\mathbb{R}^d). \tag{0.2}$$

In this case, every $f \in L^2(\mathbb{R}^d)$ has a frame expansion of the form (0.1) with unconditional L^2-convergence. Since frames are overcomplete, the frame expansion is more robust with respect to noise and measuring errors than basis expansions with unique coefficients.

In general, it is difficult to determine whether the Gabor system $\mathcal{G}(g, \Lambda)$ constitutes a frame. Especially the existence of a lower bound is hard to show as it amounts to proving the invertibility of an operator. As a result, numerous characterizations of the frame inequality (0.2) have appeared in the literature over the last thirty years.

Historically, Gabor analysis was studied first for rectangular lattices and only later extended to more general subsets of the time-frequency plane. The early investigations of Janssen [32, 33] and Daubechies et al. [13] covered a series of duality results for Gabor frames over rectangular lattices. Their proofs employ the Poisson Summation Formula and some basic properties of the short-time Fourier transform, a technique introduced to signal analysis by Tolimieri and Orr [46, 47]. The duality theory was extended to various levels of generality, most notably by Feichtinger and Kozek [15] by means of the Kohn-Nirenberg correspondence. Meanwhile, Ron-Shen [42] and Zeevi-Zibulski [51, 50] discovered additional characterizations for Gabor frames over separable, respectively rectangular lattices with rational density.

In this thesis, we adopt the point of view that the duality theory is the central result of Gabor analysis. Its streamlined proof for general lattices uses the same methods as Janssen and Daubechies. As the general result does not require more effort than the early criteria for rectangular lattices, we state and prove the duality theory for arbitrary time-frequency lattices first. All known characterizations for rectangular lattices will then follow as easy consequences. We hope that this approach and the resulting proofs are more accessible than the original literature.

This thesis is organized as follows: Chapter 1 serves as a collection of preliminaries. We establish the basics of frame theory in general Hilbert spaces as well as Gabor frames in $L^2(\mathbb{R}^d)$. The last section is devoted to the short-time Fourier transform and modulation spaces, which form an important class of function spaces for time-frequency analysis.

Chapter 2 is devoted to the duality theory of Gabor analysis, which covers all characterizations for Gabor frames with respect to general time-frequency lattices. The duality theory relates the frame property of $\mathcal{G}(g,\Lambda)$ to the adjoint system $\mathcal{G}(g,\Lambda^\circ)$ being a Riesz sequence, or equivalently, the existence of a dual window. The key argument in its proof is the Poisson Summation Formula applied to a product of short-time Fourier transform.

In Chapter 3, we consider the case where g is in the modulation space $M^1(\mathbb{R}^d)$. After investigating the boundedness of the frame operator on modulation spaces, we derive twelve additional characterizations for Gabor frames. All of them are expressed without inequalities, i.e., as properties of the four canonical operators associated to $\mathcal{G}(g,\Lambda)$, respectively $\mathcal{G}(g,\Lambda^\circ)$. Most of those characterizations follow directly from the duality theory or basic functional analysis, but in two arguments we require two versions of Wiener's Lemma.

In Chapter 4, we consider Gabor systems $\mathcal{G}(g,\Lambda)$ with L^2-functions g and separable lattices Λ. The well-known criterion of Ron and Shen follows from the duality theory by a simple application of Plancherel's theorem. In the case of a rational lattice, i.e., a rectangular lattice with rational density, further periodization yields the characterizations of Zeevi and Zibulski by means of a family of finite dimensional matrices. Finally, we clarify the relation of various Zeevi-Zibulski matrices appearing in the literature.

1. Basic Concepts of Time-Frequency Analysis

This chapter serves as a collection of results which are needed in the main part of this thesis. By and large, we follow the textbooks of Christensen [7] and Gröchenig [21] with some tweaks.

We establish the basics of frame theory in Hilbert spaces and cover some characterizations for Riesz bases and sequences. The section about Gabor frames includes the first trivial reformulations of the frame inequality by means of the canonical operators corresponding to the Gabor system.

The last section is devoted to the short-time Fourier transform and modulation spaces. We collect a series of elementary results for the short-time Fourier transform, which will be used extensively throughout the thesis.

A brief look at Wiener amalgam spaces $W(L^\infty, L^p)$ concludes this chapter. We state an important sampling property and the key ingredient for the duality theory of Gabor frames: Poisson's Summation Formula.

1.1. Frame Theory

A sequence $\{f_j : j \in J\}$ in a Hilbert space $\mathcal{H}$ is a *frame* if there exist positive constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The constants satisfying the frame condition (1.1) are called *frame bounds*. If the frame bounds can be chosen such that $A = B$, we say $\{f_j : j \in J\}$ is a *tight frame*. Finally, let A_{opt} and B_{opt} denote the *optimal frame bounds*, i.e., the maximal lower frame bound respectively the minimal upper frame bound.

As a consequence of the lower frame inequality, a frame is complete, i.e., finite linear combinations of its elements are dense in $\mathcal{H}$. In general, completeness does not guarantee a series expansion, but we will see that every frame admits a series representation

$$f = \sum_{j\in J} c_j f_j \quad \forall f \in \mathcal{H}.$$

Note that the coefficients in such an expansion need not be unique: The union of two orthonormal bases constitutes a tight frame with frame bounds $A = B = 2$, but there are uncountably many choices for the coefficients to represent any given $f \in \mathcal{H}$.

To every sequence $\{f_j : j \in J\}$, we associate the following four canonical operators:

Definition 1.1.1. Let $\{f_j : j \in J\}$ be a sequence in a Hilbert space $\mathcal{H}$. Then the *analysis operator* (or *coefficient operator*) C is defined for all $f \in \mathcal{H}$ by

$$(Cf)(j) := \langle f, f_j \rangle \quad \forall j \in J$$

and maps elements of the Hilbert space to sequences.

The *synthesis operator* (or *reconstruction operator*) D maps sequences to elements of $\mathcal{H}$ via

$$Dc := \sum_{j \in J} c_j f_j,$$

whenever the series is well-defined. Initially, we will restrict the domain of D to the subspace of all finite sequences, to avoid convergence issues.

The *frame operator* is defined as the composition $S := DC$ and given explicitly by

$$Sf = \sum_{j \in J} \langle f, f_j \rangle f_j \quad \forall f \in \mathcal{H},$$

when well-defined.

Finally, the *Gramian operator* $G := CD$ maps sequences to sequences. Interpreted as a matrix, its entries are given by $G_{j,k} = \langle f_k, f_j \rangle$ for all $j, k \in J$.

Definition 1.1.2. A sequence $\{f_j : j \in J\}$ is called a *Bessel sequence* for $\mathcal{H}$ if there exists a $B > 0$ such that

$$\sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B \|f\|^2 \tag{1.2}$$

holds for all $f \in \mathcal{H}$.

The Bessel condition (1.2) states that the corresponding analysis operator C is a bounded linear operator from $\mathcal{H}$ to $\ell^2(J)$. In that case, all canonical operators of Definition 1.1.1 are well-defined and bounded as the following proposition shows.

Proposition 1.1.3. *Suppose $\{f_j : j \in J\}$ is a Bessel sequence. Then the analysis operator C is a bounded operator from $\mathcal{H}$ to $\ell^2(J)$. Furthermore, the operators C and D are adjoint on the subspace of all finite sequences.*

Consequently, the synthesis operator D can be extended to a bounded linear operator from $\ell^2(J)$ to $\mathcal{H}$.

Proof. The boundedness of C is already established for a Bessel sequence. For any finite sequence c, we obtain

$$\langle C^*c, f\rangle = \langle c, Cf\rangle = \sum_{j\in J} c_j \overline{\langle f, f_j\rangle} = \Big\langle \sum_{j\in J} c_j f_j, f\Big\rangle = \langle Dc, f\rangle \quad \forall f \in \mathcal{H}.$$

Thus $D = C^*$ on all finite sequences with bounded extension $C^* : \ell^2(J) \to \mathcal{H}$. □

Corollary 1.1.4. *Suppose $\{f_j : j \in J\}$ is a Bessel sequence, then $Dc = \sum_{j\in J} c_j f_j$ with unconditional convergence for all $c \in \ell^2(J)$.*

Proof. This is an immediate consequence of the boundedness of the synthesis operator and the unconditional convergence of square-summable sequences. □

Now that we have established that the canonical operators are well-defined and bounded whenever $\{f_j : j \in J\}$ is a Bessel sequence, we return to frames.

Proposition 1.1.5. *Let $\{f_j : j \in J\}$ be a sequence in $\mathcal{H}$, then the following are equivalent:*

(i) $\{f_j : j \in J\}$ is a frame for $\mathcal{H}$ with frame bounds $A, B > 0$.

(ii) The frame operator $S : \mathcal{H} \to \mathcal{H}$ is a positive invertible operator satisfying

$$AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}, \tag{1.3}$$

where $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$.

In particular $\{f_j : j \in J\}$ is a tight frame if and only if $S = AI_{\mathcal{H}}$.

Proof. Since

$$\langle Sf, f\rangle = \langle Cf, Cf\rangle = \sum_{j\in J} |\langle f, f_j\rangle|^2 \quad \forall f \in \mathcal{H},$$

the frame inequality (1.1) is equivalent to the operator inequality (1.3). □

Proposition 1.1.5 states that a sequence is a frame if and only if its corresponding frame operator is a well-defined bounded positive invertible operator on $\mathcal{H}$. The invertibility will be crucial in reversing the action of the frame operator to obtain a series expansion.

Proposition 1.1.6 (Frame Expansion)**.** *Let $\{f_j : j \in J\}$ be a frame for $\mathcal{H}$ with frame bounds $A, B > 0$. Then $\{S^{-1}f_j : j \in J\}$ is a frame, the* canonical dual frame*, with frame bounds $B^{-1}, A^{-1} > 0$.*

Furthermore, every $f \in \mathcal{H}$ has the frame expansion

$$f = \sum_{j\in J} \langle f, S^{-1}f_j\rangle f_j = \sum_{j\in J} \langle f, f_j\rangle S^{-1} f_j, \tag{1.4}$$

which converges unconditionally in $\mathcal{H}$.

Proof. By Proposition 1.1.5, the frame operator S is a positive invertible operator satisfying $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$. Since the spectrum of S is contained in $[A, B]$, the spectrum of its inverse is contained in $[B^{-1}, A^{-1}]$ by the spectral mapping theorem. Hence, S^{-1} is positive and satisfies

$$B^{-1}I_{\mathcal{H}} \leq S^{-1} \leq A^{-1}I_{\mathcal{H}}. \tag{1.5}$$

In particular, S^{-1} is self-adjoint and we obtain

$$\begin{aligned}\sum_{j\in J} |\langle f, S^{-1}f_j\rangle|^2 &= \sum_{j\in J} |\langle S^{-1}f, f_j\rangle|^2 = \langle C(S^{-1}f), C(S^{-1}f)\rangle \\ &= \langle S(S^{-1}f), S^{-1}f\rangle = \langle S^{-1}f, f\rangle\end{aligned}$$

for all $f \in \mathcal{H}$. Consequently, inequality (1.5) becomes

$$B^{-1}\|f\|^2 \leq \langle S^{-1}f, f\rangle = \sum_{j\in J} |\langle f, S^{-1}f_j\rangle|^2 \leq A^{-1}\|f\|^2$$

for all $f \in \mathcal{H}$. Hence, $\{S^{-1}f_j : j \in J\}$ is a frame with frame bounds $B^{-1}, A^{-1} > 0$.
Finally, the frame expansions

$$f = S(S^{-1}f) = \sum_{j\in J}\langle S^{-1}f, f_j\rangle f_j = \sum_{j\in J}\langle f, S^{-1}f_j\rangle f_j$$

and

$$f = S^{-1}(Sf) = \sum_{j\in J}\langle f, f_j\rangle S^{-1}f_j$$

converge unconditionally by Corollary 1.1.4, since the coefficients are in $\ell^2(J)$. □

Applying S^{-1} to the frame expansion (1.4), yields the series expansion of the inverse frame operator S^{-1}.

Corollary 1.1.7. *If $\{f_j : j \in J\}$ is a frame for $\mathcal{H}$, then the inverse fame operator S^{-1} is given by*

$$S^{-1}f = \sum_{j\in J}\langle f, S^{-1}f_j\rangle S^{-1}f_j.$$

Thus S^{-1} is the frame operator with respect to the dual frame $\{S^{-1}f_j : j \in J\}$.

We end this section with two lemmata about tight frames.

Lemma 1.1.8. *If $\{f_j : j \in J\}$ is a frame for $\mathcal{H}$, then $\{S^{-\frac{1}{2}}f_j : j \in J\}$ is a tight frame with frame bounds $A = B = 1$.*

Proof. Since the frame operator S is a positive invertible operator, $S^{-\frac{1}{2}}$ is a well-defined positive operator. For all $f \in L^2(\mathbb{R}^d)$, we have

$$f = S^{-\frac{1}{2}} S(S^{-\frac{1}{2}} f) = \sum_{j\in J} \langle S^{-\frac{1}{2}} f, f_j \rangle S^{-\frac{1}{2}} f_j = \sum_{j\in J} \langle f, S^{-\frac{1}{2}} f_j \rangle S^{-\frac{1}{2}} f_j.$$

Using this, we obtain

$$\langle f, f \rangle = \left\langle f, \sum_{j\in J} \langle f, S^{-\frac{1}{2}} f_j \rangle S^{-\frac{1}{2}} f_j \right\rangle = \sum_{j\in J} |\langle f, S^{-\frac{1}{2}} f_j \rangle|^2.$$

Hence $\{S^{-\frac{1}{2}} f_j : j \in J\}$ is a tight frame with frame bounds $A = B = 1$. □

Lemma 1.1.9. *A sequence $\{f_j : j \in J\}$ is a tight frame for $\mathcal{H}$ with frame bounds $A = B = 1$ and $\|f_j\| = 1$ for all $j \in J$ if and only if it is an orthonormal basis.*

Proof. Suppose $\{f_j : j \in J\}$ is a tight frame for $\mathcal{H}$ with frame bounds $A = B = 1$ and $\|f_j\| = 1$ for all $j \in J$. Then, the frame inequality (1.1) implies

$$1 = \|f_k\|^2 = \sum_{j\in J} |\langle f_k, f_j \rangle|^2 = 1 + \sum_{j\neq k} |\langle f_k, f_j \rangle|^2$$

and therefore $\langle f_k, f_j \rangle = \delta_{k,j}$. □

1.2. Riesz Bases and Riesz Sequences

A sequence $\{f_j : j \in J\}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence* if there exist positive constants $A, B > 0$ such that

$$A\|c\|_{\ell^2}^2 \le \Big\| \sum_{j\in J} c_j f_j \Big\|^2 \le B\|c\|_{\ell^2}^2 \tag{1.6}$$

for all $c \in \ell^2(J)$.

A Riesz sequence is called *Riesz basis* if it is complete in $\mathcal{H}$, i.e., its linear span is dense in $\mathcal{H}$.

The completeness of a Riesz basis is an additional assumption and does not follow from the norm equivalence (1.6): Take an orthonormal basis for $\mathcal{H}$ and remove one element. This family still satisfies (1.6), but its linear span is clearly not dense in $\mathcal{H}$. In particular, a Riesz sequence need not be a frame since the latter is always complete.

Conversely, a frame need not be a Riesz sequence: The union of two orthonormal bases is a frame, but there exists a non-trivial linear combination of zero since this system is linearly dependent. This contradicts the lower bound in (1.6).

Obviously, this is also a counterexample of a frame that is not a Riesz basis. But as it turns out, the converse is true: Every Riesz basis constitutes a frame. To prove this, we need to recall a few well known facts from functional analysis. Proofs not given can be found for example in [43].

Proposition 1.2.1. *A bounded operator $T : \mathcal{B}_1 \to \mathcal{B}_2$ between two Banach spaces is injective if and only if its adjoint has dense range.*

In combination with the closed range theorem, Proposition 1.2.1 implies the following corollary.

Corollary 1.2.2. *Let $T : \mathcal{B}_1 \to \mathcal{B}_2$ be a bounded operator between two Banach spaces, then T is invertible if and only if T^* is invertible.*

Lemma 1.2.3. *A bounded linear operator $T : \mathcal{B}_1 \to \mathcal{B}_2$ between Banach spaces satisfies*

$$A\|f\|^2_{\mathcal{B}_1} \leq \|Tf\|^2_{\mathcal{B}_2} \leq B\|f\|^2_{\mathcal{B}_1} \quad \forall f \in \mathcal{B}_1$$

if and only if it is injective and has closed range.

Now, we can prove the following.

Proposition 1.2.4. *Every Riesz basis is a frame.*

Proof. By Lemma 1.2.3, the Riesz conditions (1.6) imply that $D : \ell^2(J) \to \mathcal{H}$ is injective and has closed range. Since the range of D contains $\mathrm{span}\{f_j : j \in J\}$ and the latter is dense in $\mathcal{H}$ by assumption, the synthesis operator is bijective. Consequently, $C = D^*$ is also a bijection and $S = DC$ is a well-defined bounded positive invertible operator which implies that $\{f_j : j \in J\}$ is a frame. □

As seen in the proof of Proposition 1.2.4, for Riesz bases, the analysis and synthesis operators are bijections. As a consequence, the coefficients in its frame expansion (1.4) are unique. Thus every Riesz basis is a Schauder basis.

We exploit Lemma 1.2.3 further to arrive at a few more trivial characterizations for frames and Riesz sequences.

Corollary 1.2.5. *Let $\{f_j : j \in J\}$ be a sequence in $\mathcal{H}$. Then the following are equivalent:*

(i) *$\{f_j : j \in J\}$ is a frame for $\mathcal{H}$.*

(ii) *The corresponding analysis operator $C : \mathcal{H} \to \ell^2(J)$ is bounded, injective and has closed range.*

(iii) *The corresponding synthesis operator $D : \ell^2(J) \to \mathcal{H}$ is a well-defined, bounded and surjective operator.*

An analogous characterization in terms of the corresponding analysis and synthesis operator can be obtained for Riesz sequences. In the following, we will see more sophisticated characterizations of Riesz sequences and Riesz bases, which will be crucial in the duality theory of Gabor frames in Chapter 2.

Definition 1.2.6. Two sequences $\{f_j : j \in J\}$ and $\{g_k : k \in J\}$ in $\mathcal{H}$ are called *biorthogonal* if they satisfy

$$\langle f_j, g_k \rangle = \delta_{j,k} \quad \forall j, k \in J,$$

where $\delta_{j,k}$ denotes the Kronecker delta.

Proposition 1.2.7. *Let $\{f_j : j \in J\}$ be a sequence in a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) $\{f_j : j \in J\}$ *is a Riesz basis.*

(ii) $\{f_j : j \in J\}$ *is complete and its corresponding Gramian operator is a bounded positive invertible operator on $\ell^2(J)$.*

(iii) $\{f_j : j \in J\}$ *is the image of an orthonormal basis under a bounded invertible operator on $\mathcal{H}$.*

(iv) $\{f_j : j \in J\}$ *is a complete Bessel sequence and it has a complete biorthogonal sequence $\{g_j : j \in J\}$, which is also a Bessel sequence.*

Proof. *(i)*$\Leftrightarrow$*(ii)*: Since

$$\langle Gc, c \rangle = \langle Dc, Dc \rangle = \Big\| \sum_{j \in J} c_j f_j \Big\|^2 \quad \forall c \in \ell^2(J),$$

the Riesz condition (1.6) is equivalent to the operator inequality

$$A I_{\ell^2} \leq G \leq B I_{\ell^2}$$

and hence equivalent to G being a bounded positive invertible operator on $\ell^2(J)$.

(i)$\Rightarrow$*(iii)*: Let $\{e_j : j \in J\}$ be an orthonormal basis for $\mathcal{H}$. Then every $f \in \mathcal{H}$ has an orthonormal expansion $f = \sum_{j \in J} c_j e_j$ for a unique $c \in \ell^2(J)$ with norm equality $\|f\|_{\mathcal{H}} = \|c\|_{\ell^2}$.

For $f = \sum_{j \in J} c_j e_j \in \mathcal{H}$ define $Tf := \sum_{j \in J} c_j f_j$. Note that the linear operator $T : \mathcal{H} \to \mathcal{H}$ is well defined: The coefficients are unique and the series converges as the coefficients are square-summable and $\{f_j : j \in J\}$ is a frame by Proposition 1.2.4.

Interpreting the Riesz inequality (1.6) with respect to T becomes

$$A\|f\|_{\mathcal{H}}^2 = A\|c\|_{\ell^2}^2 \leq \|Tf\|_{\mathcal{H}}^2 \leq B\|c\|_{\ell^2}^2 = B\|f\|_{\mathcal{H}}^2$$

for all $f \in \mathcal{H}$. Therefore, T is bounded, injective and has closed range by Lemma 1.2.3. Since $\{f_j : j \in J\}$ is complete, T is even surjective. Hence $\{f_j : j \in J\}$ is the image of the orthonormal basis $\{e_j : j \in J\}$ under the bounded and invertible operator T.

(iii)⇒(iv): By assumption, there exists an orthonormal basis $\{e_j : j \in J\}$ and a bounded invertible operator $T : \mathcal{H} \to \mathcal{H}$ with $Te_j = f_j$. Then $\{f_j : j \in J\}$ is a complete Bessel sequence, since

$$\sum_{j\in J} |\langle f, f_j\rangle|^2 = \sum_{j\in J} |\langle f, Te_j\rangle|^2 = \|T^* f\|^2 \leq \|T^*\|^2 \|f\|^2.$$

Set $g_j := (T^*)^{-1} e_j$ for $j \in J$, then $\{g_j : j \in J\}$ is also a complete Bessel sequence and

$$\langle f_j, g_k\rangle = \langle Te_j, (T^*)^{-1} e_k\rangle = \delta_{j,k}.$$

(iv)⇒(i): Suppose $\{f_j : j \in J\}$ is a Bessel sequence with biorthogonal Bessel sequence $\{g_j : j \in J\}$. Since $\{f_j : j \in J\}$ is a Bessel sequence, the corresponding synthesis operator D is bounded from $\ell^2(J)$ to $\mathcal{H}$ by Proposition 1.1.3. Thus the upper inequality of the Riesz condition (1.6) is satisfied.

For the lower bound consider an arbitrary sequence $c \in \ell^2(J)$ and define $f := \sum_{j\in J} c_j f_j$. Since $\{f_j : j \in J\}$ and $\{g_j : j \in J\}$ are biorthogonal, the coefficients are precisely $c_j = \langle f, g_j\rangle$. Hence

$$\|c\|_{\ell^2}^2 = \sum_{j\in J} |\langle f, g_j\rangle|^2 \leq B\|f\|^2$$

where we used the fact that $\{g_j : j \in J\}$ is a Bessel sequence with Bessel bound $B > 0$. Dividing by B yields the lower inequality of the Riesz condition (1.6). □

We isolate an important characterization for Riesz sequences:

Lemma 1.2.8. *Let $\{f_j : j \in J\}$ be a sequence in a Hilbert space $\mathcal{H}$. Then $\{f_j : j \in J\}$ is a Riesz sequence if and only if $\{f_j : j \in J\}$ is a Bessel sequence and there exists a biorthogonal Bessel sequence $\{g_j : j \in J\}$.*

Proof. Any Riesz sequence is a Riesz basis for its closed linear span, hence there exists a biorthogonal Bessel sequence by Proposition 1.2.7.

The converse is direction *(iv)⇒(i)* of Proposition 1.2.7 ad verbatim, since the completeness of the Bessel sequences was never used to show that the Riesz conditions (1.6) are satisfied. □

1.3. Gabor Frames

The *Fourier transform* of an integrable function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x,$$

where $x \cdot \xi = \sum_{n=1}^{d} x_n\xi_n$ denotes the inner product on $\mathbb{R}^d$. With this normalization, *Plancherel's theorem* is of the form

$$\|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2$$

and the Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R}^d)$ satisfying *Parseval's formula*

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle.$$

We define the *translation* respectively *modulation operator* by $T_x f(t) = f(t-x)$ and $M_\xi f(t) = e^{2\pi i\xi\cdot t} f(t)$. They are bijective isometries on all L^p-spaces and for $1 \le p < \infty$ the operator groups $\{T_x\}$ and $\{M_\xi\}$ are continuous with respect to the strong operator topology on the space of all bounded linear operators on $L^p(\mathbb{R}^d)$, this means for all $f \in L^p(\mathbb{R}^d)$ we have

$$\begin{aligned} \lim_{x\to 0} \|T_x f - f\|_{L^p} &= 0, \\ \lim_{\xi\to 0} \|M_\xi f - f\|_{L^p} &= 0. \end{aligned} \tag{1.7}$$

With regard to the Fourier transform, they satisfy

$$\begin{aligned} (T_x f)\widehat{\ } &= M_{-x}\hat{f}, \\ (M_\xi f)\widehat{\ } &= T_\xi \hat{f}. \end{aligned}$$

Since a modulation in time corresponds to a shift in frequency, operators of the form $\pi(z) := T_x M_\xi$ are called *time-frequency shifts* for $z = (x, \xi) \in \mathbb{R}^{2d}$. Note that translation and modulation do not commute, but they satisfy the following *commutation relation*

$$T_x M_\xi = e^{-2\pi i\xi\cdot x} M_\xi T_x. \tag{1.8}$$

Let $g \in L^2(\mathbb{R}^d)$ be a non-zero window function and $\Lambda \subseteq \mathbb{R}^{2d}$ an index set. Then the collection

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$$

of time-frequency shifts of g is called a *Gabor system*.

Since we are interested in discrete time-frequency representations, the index set needs to be a discrete. Moreover, introducing a group structure yields much desired symmetry, e.g., the dual frame is again a Gabor frame, hence we consider only full-rank lattices $\Lambda \subseteq \mathbb{R}^{2d}$. Whenever we speak of a lattice in the following, we implicitly mean that it is of full rank.

Every *lattice* $\Lambda \subseteq \mathbb{R}^{2d}$ is determined by $\Lambda = A\mathbb{Z}^{2d}$ where $A \in GL(\mathbb{R}^{2d})$ is an invertible $2d \times 2d$ matrix with real coefficients. The matrix A is unique up to a multiplication by a unimodular matrix, i.e., an integer matrix whose inverse is also an integer matrix.

Definition 1.3.1. Let $g \in L^2(\mathbb{R}^d)$ be a non-zero window function and $\Lambda \subseteq \mathbb{R}^{2d}$ a lattice. The Gabor system $\mathcal{G}(g, \Lambda)$ is called a *Gabor frame*, if there exist positive constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|^2 \quad \forall f \in L^2(\mathbb{R}^d).$$

We will state some invariance properties of the frame inequality.

Proposition 1.3.2. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then the following are equivalent:*

(i) $\mathcal{G}(g, \Lambda)$ *is a frame with frame bounds* $A, B > 0$.

(ii) $\mathcal{G}(\pi(z)g, \Lambda)$ *is a frame with frame bounds* $A, B > 0$ *for all* $z \in \mathbb{R}^{2d}$.

(iii) $\mathcal{G}(D_r g, D\Lambda)$ *is a frame with frame bounds* $A, B > 0$ *for all* $r > 0$, *where* $D_r g(t) := r^{d/2} g(rt)$ *denotes the* dilation operator *and*

$$D := \begin{pmatrix} \frac{1}{r} I_d & 0 \\ 0 & r I_d \end{pmatrix}$$

with I_d *the* d*-dimensional identity matrix.*

(iv) $\mathcal{G}(\hat{g}, \mathcal{I}\Lambda)$ *is a frame with frame bounds* $A, B > 0$, *where*

$$\mathcal{I} = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$$

denotes the standard symplectic matrix.

Proof. The equivalence *(i)*⇔*(ii)* follows from the commutation relation (1.8). Equivalence *(i)*⇔*(iii)* is nothing else than a change of variables in the integral and *(i)*⇔*(iv)* follows by applying Plancherel's theorem. □

In Definition 1.1.1, we defined four canonical operators corresponding to a given sequence in a Hilbert space. For a Gabor system $\mathcal{G}(g,\Lambda)$, the *analysis operator* $C_{g,\Lambda}$ is given by

$$C_{g,\Lambda}f(\lambda) = \langle f, \pi(\lambda)g\rangle \quad \forall \lambda \in \Lambda.$$

The *synthesis operator* $D_{g,\Lambda}$ of a sequence $c = (c_\lambda)_{\lambda\in\Lambda}$ is

$$D_{g,\Lambda}c = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g.$$

For the *frame operator* $S_{g,\Lambda} = D_{g,\Lambda}C_{g,\Lambda}$ we obtain

$$S_{g,\Lambda}f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g.$$

The *Gramian operator* $G := G_{g,\Lambda} = C_{g,\Lambda}D_{g,\Lambda}$ has the entries

$$G_{\lambda,\mu} = \langle \pi(\mu)g, \pi(\lambda)g\rangle.$$

Again, the synthesis operator is well-defined for finite sequences. If the Gabor system $\mathcal{G}(g,\Lambda)$ is a Bessel sequence, all of the above operators are well-defined and bounded on $L^2(\mathbb{R}^d)$ respectively $\ell^2(\Lambda)$.

Corollary 1.3.3. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then the following are equivalent:*

(i) $\mathcal{G}(g,\Lambda)$ is a frame.

(ii) The frame operator $S_{g,\Lambda}$ is a bounded positive invertible operator on $L^2(\mathbb{R}^d)$.

(iii) The analysis operator $C_{g,\Lambda} : L^2(\mathbb{R}^d) \to \ell^2(\Lambda)$ is bounded, injective and has closed range.

(iv) The synthesis operator $D_{g,\Lambda} : \ell^2(\Lambda) \to L^2(\mathbb{R}^d)$ is well-defined, bounded and surjective.

Proof. This is Proposition 1.1.5 and Corollary 1.2.5. □

By Proposition 1.1.6, every Gabor frame $\mathcal{G}(g,\Lambda)$ has a dual frame. But a priori, we do not know if this dual frame is again a Gabor frame. Fortunately, this is the case if the index set Λ has enough structure, i.e., a group structure, as the following results show.

Lemma 1.3.4. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Suppose $\mathcal{G}(g,\Lambda)$ is a Bessel sequence, then the frame operator $S_{g,\Lambda}$ commutes with time frequency shifts $\pi(\lambda)$ for all $\lambda \in \Lambda$.*

Proof. We need to show $\pi(\lambda)^{-1}S_{g,\Lambda}\pi(\lambda)f = S_{g,\Lambda}f$. The left-hand side is

$$\begin{aligned}\pi(\lambda)^{-1}S_{g,\Lambda}\pi(\lambda)f &= \sum_{\mu\in\Lambda}\langle\pi(\lambda)f, \pi(\mu)g\rangle\pi(\lambda)^{-1}\pi(\mu)g \\ &= \sum_{\mu\in\Lambda}\langle f, \pi(\lambda)^{-1}\pi(\mu)g\rangle\pi(\lambda)^{-1}\pi(\mu)g.\end{aligned}$$

Note that interchanging $\pi(\lambda)^{-1}$ with the sum is justified since $\mathcal{G}(g, \Lambda)$ is a Bessel sequence and time-frequency shifts are unitary, hence bounded, operators on $L^2(\mathbb{R}^d)$.

The commutation relation of translation and modulation (1.8) implies

$$\pi(\lambda)^{-1}\pi(\mu) = e^{2\pi i(\mu_2-\lambda_2)\cdot\lambda_1}\pi(\mu-\lambda).$$

The resulting phase factors cancel each other and since $\Lambda - \lambda = \Lambda$, we obtain

$$\pi(\lambda)^{-1}S_{g,\Lambda}\pi(\lambda)f = \sum_{\mu\in\Lambda}\langle f, \pi(\mu)g\rangle\pi(\mu)g = S_{g,\Lambda}f$$

after renaming the summation parameter. □

Proposition 1.3.5. *Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame. Then there exists a* dual window *$\gamma \in L^2(\mathbb{R}^d)$, the* canonical dual window *$\gamma = S_{g,\Lambda}^{-1}g$, such that $\mathcal{G}(\gamma, \Lambda)$ is a dual frame for $\mathcal{G}(g, \Lambda)$.*

Furthermore, every $f \in L^2(\mathbb{R}^d)$ has the frame expansion

$$f = \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)\gamma\rangle\pi(\lambda)g = \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)g\rangle\pi(\lambda)\gamma,$$

with unconditional L^2-convergence.

Proof. Combine Proposition 1.1.6 and Lemma 1.3.4. □

1.4. Modulation Spaces and the Short-Time Fourier Transform

From the perspective of time-frequency analysis, the Fourier transform has a major drawback: The resulting frequency representation lacks any information about when those frequencies actually occur.

A more suitable approach for a simultaneous time-frequency representation is the following: First, cut-out a time-segment and then take the Fourier transform. This leads to the following definition.

Definition 1.4.1. Let $g \in L^2(\mathbb{R}^d)$ be a non-zero window function. Then the *short-time Fourier transform* of $f \in L^2(\mathbb{R}^d)$ is defined as

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\xi}\,\mathrm{d}t$$

for all $x, \xi \in \mathbb{R}^d$.

Recall that the translation $T_x f(t) = f(t-x)$ and modulation $M_\xi f(t) = e^{2\pi i\xi\cdot t}f(t)$ satisfy

$$T_x M_\xi = e^{-2\pi i\xi\cdot x}M_\xi T_x.$$

With this in mind, the short-time Fourier transform can be rewritten as

$$\begin{aligned} V_g f(x,\xi) &= (f\cdot T_x\overline{g})\widehat{\ }(\xi) \\ &= \langle f, M_\xi T_x g\rangle \\ &= \langle \hat{f}, T_\xi M_{-x}\hat{g}\rangle \\ &= e^{-2\pi i x\cdot\xi}V_{\hat{g}}\hat{f}(\xi,-x). \end{aligned} \tag{1.9}$$

The representation as an inner product makes it possible to generalize the short-time Fourier transform beyond $L^2(\mathbb{R}^d)$ via duality. Let $\mathcal{S}(\mathbb{R}^d)$ denote the *Schwartz space*, i.e., all smooth functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that the seminorms

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d}|x^\alpha D^\beta f(x)|$$

are finite for all multi-indices $\alpha, \beta \in \mathbb{N}^d$, and $\mathcal{S}'(\mathbb{R}^d)$ its topological dual.

Then the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined as

$$V_g f(x,\xi) := \langle f, M_\xi T_x g\rangle.$$

Definition 1.4.2. Let $1 \le p, q \le \infty$ and $g \in \mathcal{S}(\mathbb{R}^d)$ be a fixed non-zero window function. Then the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ whose *modulation space norm*

$$\|f\|_{M^{p,q}} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\xi)|^p\,\mathrm{d}x\right)^{q/p}\mathrm{d}\xi\right)^{1/q}$$

is finite with the usual modification for $p, q = \infty$.

Furthermore, we write $M^p(\mathbb{R}^d)$ instead of $M^{p,p}(\mathbb{R}^d)$ when $p = q$.

We quickly state a few important properties of modulation spaces. For proofs and an in-depth introduction to the theory, we refer to the Chapters 11–12 in Gröchenig's textbook [21].

Most importantly, the definition is independent of the window function; in fact, different non-zero windows $g \in \mathcal{S}(\mathbb{R}^d)$ yield equivalent norms [21, Prop. 11.3.2]. If $1 \leq p, q < \infty$, then the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a dense in $M^{p,q}(\mathbb{R}^d)$ [21, Prop. 11.3.4].

Modulation spaces are Banach spaces [21, Thm. 11.3.5], which are invariant under time-frequency shifts (cf. Lemma 1.4.4). For $p = q$, they are also invariant under the Fourier transform by (1.9).

A straight forward computation using Plancherel and Fubini yields $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Furthermore, one can show the inclusion $M^1(\mathbb{R}^d) \subseteq M^p(\mathbb{R}^d) \subseteq M^\infty(\mathbb{R}^d)$ [21, Cor. 12.1.10].

Since modulation spaces are defined via L^p-norms, the duality is exactly as one would expect: For $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have $(M^p(\mathbb{R}^d))^* = M^q(\mathbb{R}^d)$ [21, Thm. 11.3.6].

With the duality and inclusion property for $M^p(\mathbb{R}^d)$, the short-time Fourier transform can be further extended to windows and functions in dual modulation spaces. Again, one can show that non-zero windows $g \in M^1(\mathbb{R}^d)$ yield equivalent norms on $M^p(\mathbb{R}^d)$ [21, Thm. 11.3.7]. Furthermore, the short-time Fourier transform satisfies the following inversion formula.

Theorem 1.4.3 (Inversion Formula [21, Thm. 11.3.7]). *Let $g, \gamma \in M^1(\mathbb{R}^d)$, then we have for all $f \in M^p(\mathbb{R}^d)$*

$$f = \frac{1}{\langle \gamma, g \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x, \xi) M_\xi T_x \gamma \, \mathrm{d}\xi \, \mathrm{d}x, \tag{1.10}$$

where the integral has to be understood in the weak sense, i.e.,

$$\langle f, h \rangle = \frac{1}{\langle \gamma, g \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x, \xi) \overline{V_\gamma h(x, \xi)} \, \mathrm{d}\xi \, \mathrm{d}x \tag{1.11}$$

for all $h \in (M^p(\mathbb{R}^d))^*$.

In the following, we collect a few technical properties of the short-time Fourier transform, which we will need in the subsequent chapters. Recall that $\pi(z) = M_{z_2} T_{z_1}$ denotes the time-frequency shift with respect to $z = (z_1, z_2) \in \mathbb{R}^{2d}$.

Lemma 1.4.4 (Covariance Property). *Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $g \in M^p(\mathbb{R}^d)$ and $f \in M^q(\mathbb{R}^d)$, then the following holds:*

(a) For all $x, \xi, y, \eta \in \mathbb{R}^d$ we have

$$V_g(M_\eta T_y f)(x, \xi) = e^{-2\pi i (\xi - \eta) \cdot y} V_g f(x - y, \xi - \eta).$$

(b) For all $z, w \in \mathbb{R}^{2d}$ we have

$$V_{\pi(w)g}(\pi(w)f)(z) = e^{2\pi i z\cdot \mathcal{I}w} V_g f(z), \tag{1.12}$$

with $\mathcal{I} = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ where I_d is the d-dimensional identity matrix.

Proof. (a) We use the definition of the short-time Fourier transform by means of the (M^p, M^q)-duality and apply the commutation relation (1.8):

$$\begin{aligned} V_g(M_\eta T_y f)(x,\xi) &= \langle M_\eta T_y f, M_\xi T_x g\rangle \\ &= \langle f, T_{-y} M_{\xi-\eta} T_x g\rangle \\ &= e^{-2\pi i(\xi-\eta)\cdot y}\langle f, M_{\xi-\eta} T_{x-y} g\rangle. \end{aligned}$$

(b) Using (1.8), we obtain after a straight forward computation

$$\pi(w)^*\pi(z)\pi(w) = e^{-2\pi i z\cdot \mathcal{I}w}\pi(z).$$

Now, identity (1.12) is just an exercise in rewriting:

$$V_{\pi(w)g}(\pi(w)f)(z) = \langle f, \pi(w)^*\pi(z)\pi(w)g\rangle = e^{2\pi i z\cdot \mathcal{I}w}\langle f, \pi(z)g\rangle.$$

□

Lemma 1.4.5. *Let $1 \le p,q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $g, \gamma \in M^1(\mathbb{R}^d)$, $f \in M^p(\mathbb{R}^d)$ and $h \in M^q(\mathbb{R}^d)$, then we have for all $z \in \mathbb{R}^{2d}$*

$$\left(V_g f \cdot \overline{V_\gamma h}\right)^\wedge(z) = \left(V_h f \cdot \overline{V_\gamma g}\right)(-\mathcal{I}z).$$

Proof. Observe that the product $V_g f \cdot \overline{V_\gamma h}$ is in $L^1(\mathbb{R}^{2d})$ by Hölder's inequality for L^p-spaces. Hence, the Fourier transform is defined as an integral and we obtain

$$\begin{aligned} \left(V_g f \cdot \overline{V_\gamma h}\right)^\wedge(z) &= \int_{\mathbb{R}^{2d}} V_g f(w) \cdot \overline{V_\gamma h(w)} e^{-2\pi i(w\cdot z)}\,\mathrm{d}w \\ &= \int_{\mathbb{R}^{2d}} V_g f(w) \cdot \overline{V_{\pi(-\mathcal{I}z)\gamma}\pi(-\mathcal{I}z)h(w)}\,\mathrm{d}w \\ &= \langle f, \pi(-\mathcal{I}z)h\rangle\overline{\langle g, \pi(-\mathcal{I}z)\gamma\rangle}, \end{aligned}$$

where we first used (1.12) in combination with the fact $\mathcal{I}^2 = -I_{2d}$ and then the inversion formula (1.11) to separate the integral into two dual pairings. □

Proposition 1.4.6. *If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $V_g f$ is a Schwartz function on $\mathbb{R}^{2d}$.*

Proof. For this proof it is advantageous to view the short-time Fourier transform in the following way:

Let $(f \otimes g)(x,t) := f(x)g(t)$ denote the tensor product and define the asymmetric coordinate transform $\mathcal{T}_a$ for functions $F : \mathbb{R}^{2d} \to \mathbb{C}$ by

$$\mathcal{T}_a F(x,t) = F(t, t-x)$$

and the partial Fourier transform $\mathcal{F}_2$ by

$$\mathcal{F}_2 F(x,\xi) = \int_{\mathbb{R}^d} F(x,t) e^{-2\pi i t\cdot\xi}\,\mathrm{d}t.$$

Then the short-time Fourier transform can be written as

$$V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \overline{g}).$$

Now observe that the tensor product $f \otimes \overline{g}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ whenever $f, g \in \mathcal{S}(\mathbb{R}^d)$. Since the operators $\mathcal{T}_a$ and $\mathcal{F}_2$ are isomorphisms on $\mathcal{S}(\mathbb{R}^{2d})$, we have $V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \overline{g}) \in \mathcal{S}(\mathbb{R}^{2d})$. □

In the following chapters, we need to check if $V_g f$ belongs to certain modulation spaces. This involves computing its short-time Fourier transform. To indicate the different dimensions, we will denote the short-time Fourier transform of functions $F, G : \mathbb{R}^{2d} \to \mathbb{C}$ by $\mathcal{V}_G F$.

Lemma 1.4.7. *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then for all $g \in M^1(\mathbb{R}^d)$ and $f \in M^\infty(\mathbb{R}^d)$ we have*

$$\mathcal{V}_{V_\varphi \varphi}(V_g f)(z,\zeta) = e^{-2\pi i z_2\cdot\zeta_2}\overline{V_\varphi g(-z_1-\zeta_2,\zeta_1)}V_\varphi f(-\zeta_2, z_2+\zeta_1)$$

for all $z = (z_1, z_2) \in \mathbb{R}^{2d}$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

We follow the proof of Cordero and Gröchenig [8].

Proof. First observe that Lemma 1.4.4 implies

$$T_z(V_\varphi \varphi)(x,\xi) = V_\varphi \varphi(x - z_1, \xi - z_2) = e^{2\pi i(\xi - z_2)\cdot z_1} V_\varphi(M_{z_2} T_{z_1}\varphi)(x,\xi). \tag{1.13}$$

Since $V_\varphi \varphi \in \mathcal{S}(\mathbb{R}^{2d})$ by Proposition 1.4.6 and $V_g f \in L^\infty(\mathbb{R}^{2d})$ by definition, their short-time Fourier transform is well-defined and can be expressed as an integral. Using (1.13) and Lemma 1.4.5 yields

$$\mathcal{V}_{V_\varphi \varphi}(V_g f)(z,\zeta) = \iint_{\mathbb{R}^{2d}} V_g f(x,\xi)\overline{T_z(V_\varphi\varphi)(x,\xi)} e^{-2\pi i(x\cdot\zeta_1 + \xi\cdot\zeta_2)}\,\mathrm{d}x\,\mathrm{d}\xi$$

$$\begin{aligned}
&= \iint_{\mathbb{R}^{2d}} V_g f(x,\xi)\overline{V_\varphi(M_{z_2}T_{z_1}\varphi)(x,\xi)} e^{-2\pi i(\xi - z_2)\cdot z_1} e^{-2\pi i(x\cdot\zeta_1 + \xi\cdot\zeta_2)}\,\mathrm{d}x\,\mathrm{d}\xi \\
&= e^{2\pi i z_2\cdot z_1}\big(V_g f\cdot \overline{V_\varphi(M_{z_2}T_{z_1}\varphi)}\big)\widehat{\ }(\zeta_1, \zeta_2 + z_1) \\
&= e^{2\pi i z_2\cdot z_1}\big(V_{(M_{z_2}T_{z_1}\varphi)} f\cdot \overline{V_\varphi g}\big)(-z_1 - \zeta_2, \zeta_1) \\
&= e^{2\pi i z_2\cdot z_1}\langle f, M_{\zeta_1}T_{(-z_1-\zeta_2)}M_{z_2}T_{z_1}\varphi\rangle\overline{V_\varphi g(-z_1-\zeta_2,\zeta_1)} \\
&= e^{-2\pi i z_2\cdot\zeta_2} V_\varphi f(-\zeta_2, z_2 + \zeta_1)\overline{V_\varphi g(-z_1 - \zeta_2, \zeta_1)},
\end{aligned}$$

where we used the commutation relation $T_{(-z_1-\zeta_2)}M_{z_2} = e^{2\pi i z_2\cdot(z_1+\zeta_2)}M_{z_2}T_{(-z_1-\zeta_2)}$ to obtain the last line. □

Lemma 1.4.8. *Suppose $g \in M^1(\mathbb{R}^d)$ and $f \in M^p(\mathbb{R}^d)$ for $1 \le p \le \infty$, then $V_g f : \mathbb{R}^{2d} \to \mathbb{C}$ is continuous.*

Proof. Fix an arbitrary $w \in \mathbb{R}^{2d}$. By assumption, $f \in M^p(\mathbb{R}^d) \subseteq M^\infty(\mathbb{R}^d)$ and therefore

$$\begin{aligned}
|V_g f(w+z) - V_g f(w)| &= |\langle f, \pi(w+z)g - \pi(w)g\rangle| \\
&\le \|f\|_{M^\infty}\|e^{2\pi i w_2\cdot z_1}\pi(z)\pi(w)g - \pi(w)g\|_{M^1} \\
&\le \|f\|_{M^\infty}\big(|e^{2\pi i w_2\cdot z_1} - 1|\cdot\|\tilde{g}\|_{M^1} + \|\pi(z)\tilde{g} - \tilde{g}\|_{M^1}\big)
\end{aligned}$$

for $\tilde{g} := \pi(w)g$. Since $w = (w_1, w_2) \in \mathbb{R}^{2d}$ is fixed, the first term can be made sufficiently small as $z \to 0$.

For the second term, fix a window $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\|\pi(z)\tilde{g} - \tilde{g}\|_{M^1} = \|V_\varphi(\pi(z)\tilde{g}) - V_\varphi\tilde{g}\|_{L^1} = \|T_z M_{(0,-z_1)}V_\varphi\tilde{g} - V_\varphi\tilde{g}\|_{L^1}$$

by Lemma 1.4.4. Now the continuity follows from (1.7), i.e., the fact that $(x,\xi) \mapsto M_\xi T_x$ is continuous with respect to the strong operator topology on the space of bounded operators on L^1. □

We conclude this section with the two most fundamental principles upon which the subsequent chapters are based: Poisson's Summation Formula and a convenient sampling property in Wiener amalgam spaces $W(L^\infty, L^p)$.

The *Wiener amalgam space* $W(L^\infty, L^p)$ consists of all functions $f \in L^\infty(\mathbb{R}^d)$, such that the Wiener amalgam norm

$$\|f\|_{W(L^\infty,L^p)} := \Big(\sum_{n\in\mathbb{Z}^d}\|f\cdot T_n\chi_{[0,1]^d}\|_{L^\infty}^p\Big)^{1/p} \tag{1.14}$$

is finite, with $\chi_{[0,1]^d}$ the characteristic function of the d-dimensional unit cube. Note that $W(L^\infty, L^\infty) = L^\infty(\mathbb{R}^d)$.

Furthermore, we denote by $W(\mathcal{C}, L^p)$ the space of all continuous functions in $W(L^\infty, L^p)$. The Wiener amalgam space $W(\mathbb{R}^d) := W(L^\infty, L^1)$ is important enough to warrant a distinct notation.

The amalgam norm in (1.14) is taken with respect to the integer lattice $\mathbb{Z}^d$, but different lattices $\Lambda \subseteq \mathbb{R}^d$ yield equivalent norms. This is easy to see as $[0,1]^d$ can be covered by finitely many lattice cells of Λ and conversely, the fundamental domain of Λ can be covered by finitely many integer translates of the unit cube.

For a more comprehensive introduction to Wiener amalgam spaces we refer to Heil's excellent exposition [28].

Proposition 1.4.9 (Gröchenig [21, Prop. 12.1.4]). *The modulation space $M^1(\mathbb{R}^d)$ is continuously embedded into $W \cap \mathcal{F}W(\mathbb{R}^d)$. In particular, all $f \in M^1(\mathbb{R}^d)$ are continuous and there exists a constant $C > 0$ such that*

$$\|f\|_W \leq C\|f\|_{M^1}.$$

Since the coefficient operator $C_{g,\Lambda}$ is nothing else than the short-time Fourier transform V_g sampled along the lattice $\Lambda \subseteq \mathbb{R}^{2d}$, the following sampling property will be frequently applied later on.

Proposition 1.4.10 (Sampling Property). *Suppose $f \in W(L^\infty, L^p)$ is continuous, then for all lattices $\Lambda \subseteq \mathbb{R}^d$ the restriction $f|_\Lambda$ is in $\ell^p(\Lambda)$ and*

$$\|f|_\Lambda\|_{\ell^p} \leq C_\Lambda \|f\|_{W(L^\infty,L^p)}.$$

Proof. For all $\lambda \in n + [0,1]^d$ and $n \in \mathbb{Z}^d$, we have

$$|f(\lambda)| \leq \|f \cdot T_n\chi_{[0,1]^d}\|_\infty.$$

Let $C_\Lambda := \max_{k\in\mathbb{Z}^d} \#\{\lambda \in \Lambda \cap n + [0,1]^d\}$ denote the maximal number of lattice points over all integer translates of $[0,1]^d$. Clearly, this bounds the number of lattice points in every translate of the unite cube, hence

$$\Big(\sum_{\lambda\in\Lambda} |f(\lambda)|^p\Big)^{1/p} \leq \Big(\sum_{n\in\mathbb{Z}^d} C_\Lambda \|f \cdot T_n\chi_{[0,1]^d}\|_\infty^p\Big)^{1/p} = C_\Lambda^{1/p}\|f\|_{W(L^\infty,L^p)}.$$

□

Theorem 1.4.11 (Poisson Summation Formula). *Suppose $f \in W \cap \mathcal{F}W(\mathbb{R}^d)$, then*

$$\sum_{n\in\mathbb{Z}^d} f(x+n) = \sum_{n\in\mathbb{Z}^d} \hat{f}(n)e^{2\pi i n\cdot x} \tag{1.15}$$

holds pointwise and both sides converge absolutely for all $x \in \mathbb{R}^d$.

Proof. Since $f \in W \cap \mathcal{F}W(\mathbb{R}^d)$, both sides converge absolutely for every $x \in \mathbb{R}^d$. Furthermore $f \in W(\mathbb{R}^d) \subseteq L^1(\mathbb{R}^d)$, hence its periodization

$$\varphi(x) := \sum_{n\in\mathbb{Z}^d} f(x+n)$$

is in $L^1(\mathbb{T}^d)$ with Fourier coefficients

$$\hat{\varphi}(k) = \int_{\mathbb{T}^d} \sum_{n\in\mathbb{Z}^d} f(x+n)e^{-2\pi i k\cdot x}\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)e^{-2\pi i k\cdot x}\,\mathrm{d}x = \hat{f}(k).$$

Note that f is continuous by Riemann-Lebesgue, hence φ is also continuous with an absolutely convergent Fourier series. Consequently, the Poisson Summation Formula (1.15) holds pointwise. □

Since $M^1(\mathbb{R}^d) \subseteq W \cap \mathcal{F}W(\mathbb{R}^d)$ by Proposition 1.4.9, Theorem 1.4.11 and the following generalization to arbitrary lattices also holds for all $f \in M^1(\mathbb{R}^d)$.

Recall that any lattice $\Lambda \subseteq \mathbb{R}^d$ is determined by $\Lambda = A\mathbb{Z}^d$ where $A \in GL(\mathbb{R}^d)$ is an invertible matrix. Let $\mathrm{vol}(\Lambda) := |\det A|$ denote the *volume of the lattice* and $\Lambda^\perp = (A^T)^{-1}\mathbb{Z}^d$ the *dual lattice.*

Corollary 1.4.12. *Suppose $f \in W \cap \mathcal{F}W(\mathbb{R}^d)$ and $\Lambda = A\mathbb{Z}^d$ is a lattice, then*

$$\sum_{\lambda\in\Lambda} f(x+\lambda) = \frac{1}{\mathrm{vol}(\Lambda)} \sum_{\mu\in\Lambda^\perp} \hat{f}(\mu)e^{2\pi i\mu\cdot x}$$

holds pointwise and both sides converge absolutely for all $x \in \mathbb{R}^d$.

Proof. Note that $f(x+\lambda) = f\big(A(A^{-1}x+n)\big)$ for $\lambda = An$. We compute the Fourier transform

$$\begin{aligned}(f\circ A)\hat{}\,(k) &= \int_{\mathbb{R}^d} f(Ax)e^{-2\pi i x\cdot k} \\ &= \int_{\mathbb{R}^d} f(Ax)e^{-2\pi i Ax\cdot (A^{-1})^T k} \\ &= |\det(A)|^{-1}\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot (A^{-1})^T k} \\ &= \mathrm{vol}(\Lambda)^{-1}\hat{f}\big((A^T)^{-1}k\big).\end{aligned}$$

Since amalgam norms with respect to different lattices yield equivalent norms, the composition $f\circ A$ is also in $W \cap \mathcal{F}W(\mathbb{R}^d)$. Hence Theorem 1.4.11 applies and we obtain

$$\begin{aligned}\sum_{\lambda\in\Lambda} f(x+\lambda) &= \sum_{n\in\mathbb{Z}^d} f\big(A(A^{-1}x+n)\big) \\ &= \frac{1}{\mathrm{vol}(\Lambda)}\sum_{n\in\mathbb{Z}^d} \hat{f}\big((A^T)^{-1}n\big)e^{2\pi i n\cdot A^{-1}x} \\ &= \frac{1}{\mathrm{vol}(\Lambda)}\sum_{\mu\in\Lambda^\perp} \hat{f}(\mu)e^{2\pi i\mu\cdot x}.\end{aligned}$$

□

2. Characterizations for Arbitrary Lattices

The duality theory for Gabor frames includes all characterizations of the frame property for Gabor systems generated by an L^2-window and an arbitrary time-frequency lattice. It links the frame property of $\mathcal{G}(g, \Lambda)$ to the adjoint system $\mathcal{G}(g, \Lambda^\circ)$ being a Riesz sequence, or equivalently, to the existence of a dual window.

The Fundamental Identity of Gabor Analysis will be our main tool for relating $\mathcal{G}(g, \Lambda)$ to its adjoint system $\mathcal{G}(g, \Lambda^\circ)$. Another milestone is the Bessel duality, which states that $\mathcal{G}(g, \Lambda)$ is a Bessel sequence if and only if $\mathcal{G}(g, \Lambda^\circ)$ is one. We will need both to prove the duality theory in Section 2.4.

2.1. The Fundamental Identity of Gabor Analysis

The duality theory of Gabor analysis relates properties of the Gabor system $\mathcal{G}(g, \Lambda)$ to those of its dual system $\mathcal{G}(g, \Lambda^\circ)$ along the adjoint lattice. At the core of its arguments always lies the Poisson Summation Formula

$$\sum_{\lambda \in \Lambda} F(\lambda) = \operatorname{vol}(\Lambda)^{-1} \sum_{\mu \in \Lambda^\perp} \hat{F}(\mu) \tag{2.1}$$

applied to a product of short-time Fourier transforms.

For the lattice $\Lambda = A\mathbb{Z}^{2d}$ determined by the invertible matrix $A \in GL(\mathbb{R}^{2d})$, the *dual lattice* is given by

$$\Lambda^\perp := (A^T)^{-1}\mathbb{Z}^{2d}$$

and the *volume* of the lattice is denoted by $\operatorname{vol}\Lambda := |\det(A)|$. The reciprocal value $\delta(\Lambda) := \operatorname{vol}(\Lambda)^{-1}$ is called the *density* or *redundancy* of the lattice.

We already computed the Fourier transform of a product of short-time Fourier transforms in Lemma 1.4.5 and obtained

$$\big(V_g f \cdot \overline{V_\gamma h}\big)\hat{}\,(z) = \big(V_h f \cdot \overline{V_\gamma g}\big)(-\mathcal{I}z), \tag{2.2}$$

where $\mathcal{I} = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ denotes the standard symplectic matrix.

The dual lattice on the right-hand side of (2.1) together with the resulting symplectic twist of (2.2) motivates the definition of the *adjoint lattice* as

$$\Lambda^\circ := \mathcal{I}(A^T)^{-1}\mathbb{Z}^{2d}.$$

A straight forward computation reveals that the adjoint lattice is characterized by the commutation relation

$$\Lambda^\circ = \{\mu \in \mathbb{R}^{2d} : \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda) \quad \forall \lambda \in \Lambda\}.$$

Theorem 2.1.1 (Fundamental Identity of Gabor Analysis)**.** *Let $f, h \in M^1(\mathbb{R}^d)$, $g, \gamma \in L^2(\mathbb{R}^d)$, and $\Lambda = A\mathbb{Z}^{2d}$ be a lattice. Then*

$$\sum_{\lambda\in\Lambda} V_g f(\lambda)\overline{V_\gamma h(\lambda)} = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} V_g \gamma(\mu)\overline{V_f h(\mu)}, \tag{2.3}$$

with absolute convergence on both sides.

Note that due to $V_g f(x,\xi) = e^{-2\pi i x\cdot\xi}\,\overline{V_f g(-x,-\xi)}$, the Fundamental Identity of Gabor Analysis (2.3) also holds for $f, h \in L^2(\mathbb{R}^d)$ and $g, \gamma \in M^1(\mathbb{R}^d)$.

We already outlined the idea of the proof. The technical difficulty lies in showing that the product of short-time Fourier transforms $V_g f \cdot \overline{V_\gamma h}$ is in the modulation space $M^1(\mathbb{R}^{2d})$, such that the Poisson Summation Formula in the form of Corollary 1.4.12 is applicable.

We follow Feichtinger and Luef [16] and resolve this technicality with three lemmata, involving Hölder inequalities and estimates for the short-time Fourier transform in Wiener amalgam spaces.

The *Wiener amalgam space* $W(\mathcal{F}L^1, L^p)$ consists of all complex-valued functions $f : \mathbb{R}^d \to \mathbb{C}$, such that for a fixed window function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ the amalgam norm

$$\|f\|_{W(\mathcal{F}L^1,L^p)} := \left(\int_{\mathbb{R}^d} \|(f \cdot T_x\overline{g})\widehat{\ }\|_{L^1}^p \,\mathrm{d}x\right)^{1/p}$$

is finite with the usual modification for $p = \infty$.

Wiener amalgam spaces of this type are Fourier images of modulation spaces, as the following short computation shows:

$$\begin{aligned}
\|f\|_{W(\mathcal{F}L^1,L^p)} &= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |\,(f \cdot T_x\overline{g})\widehat{\ }(\xi)|\,\mathrm{d}\xi\right)^p \mathrm{d}x\right)^{1/p} \\
&= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|\,\mathrm{d}\xi\right)^p \mathrm{d}x\right)^{1/p} \\
&= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_{\hat{g}} \hat{f}(\xi,-x)|\,\mathrm{d}\xi\right)^p \mathrm{d}x\right)^{1/p} \\
&= \|\hat{f}\|_{M^{1,p}_{\hat{g}}}
\end{aligned} \tag{2.4}$$

Consequently, the Wiener amalgam space norm of f is equivalent to a modulation space norm of $\hat{f}$. Since modulation space norms with respect to different windows yield equivalent norms [21, Prop. 11.3.2], we have the same property for Wiener amalgam spaces.

Furthermore, we have $W(\mathcal{F}L^1, L^1)(\mathbb{R}^d) = M^1(\mathbb{R}^d)$ with equivalent norms, which is evident from the first line of (2.4).

The first lemma is Hölder's inequality for Wiener amalgam spaces. For convenience, we will use $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$ and $a \asymp b$ if $C^{-1}a \leq b \leq Ca$ for some $C > 0$.

Lemma 2.1.2. *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $f \in W(\mathcal{F}L^1, L^p)$ and $h \in W(\mathcal{F}L^1, L^q)$, then $f \cdot h \in W(\mathcal{F}L^1, L^1)$ with norm estimate*

$$\|f \cdot h\|_{W(\mathcal{F}L^1, L^1)} \lesssim \|f\|_{W(\mathcal{F}L^1, L^p)} \|h\|_{W(\mathcal{F}L^1, L^q)}.$$

Proof. As different windows yield equivalent norms, we may assume that the window $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is real-valued. The trick is to express the Wiener amalgam norm on the left-hand side with respect to the window g^2. Then the claim follows from a straight-forward computation using Young's and Hölder's inequality for L^p-spaces:

$$\begin{aligned}
\|f \cdot h\|_{W(\mathcal{F}L^1, L^1)} &\asymp \int_{\mathbb{R}^d} \| \big(f \cdot h \cdot T_x g^2\big)^\wedge \|_{L^1} \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \|(f \cdot T_x g)^\wedge * (h \cdot T_x g)^\wedge\|_{L^1} \,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \|(f \cdot T_x g)^\wedge\|_{L^1} \|(h \cdot T_x g)^\wedge\|_{L^1} \,\mathrm{d}x \\
&\leq \|f\|_{W(\mathcal{F}L^1, L^p)} \|h\|_{W(\mathcal{F}L^1, L^q)}
\end{aligned}$$

□

Lemma 2.1.3 (Cordero-Gröchenig [8])**.** *Let $1 \leq p \leq \infty$. Suppose $f \in M^p(\mathbb{R}^d)$ and $g \in M^1(\mathbb{R}^d)$, then $V_g f \in W(\mathcal{F}L^1, L^p)(\mathbb{R}^{2d})$ with norm estimate*

$$\|V_g f\|_{W(\mathcal{F}L^1, L^p)} \lesssim \|f\|_{M^p} \|g\|_{M^1}.$$

In particular $V_g f \in M^1(\mathbb{R}^{2d})$ for $f, g \in M^1(\mathbb{R}^d)$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $V_\varphi \varphi \in \mathcal{S}(\mathbb{R}^{2d})$ by Proposition 1.4.6. As $g \in M^1(\mathbb{R}^d)$ and $f \in M^p(\mathbb{R}^d) \subseteq M^\infty(\mathbb{R}^d)$, the short-time Fourier transform $V_g f$ is in $L^\infty(\mathbb{R}^{2d})$. Hence $\mathcal{V}_{V_\varphi \varphi}(V_g f)$ is well-defined and Lemma 1.4.7 implies

$$|\mathcal{V}_{V_\varphi \varphi}(V_g f)(x, \xi)| = |V_\varphi g(-x_1 - \xi_2, \xi_1)| |V_\varphi f(-\xi_2, x_2 + \xi_1)| \tag{2.5}$$

for all $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^{2d}$.

For $1 \leq p < \infty$, we take an equivalent modulation space norm with respect to the window $V_\varphi \varphi \in \mathcal{S}(\mathbb{R}^{2d})$ and apply (2.5) to get

$$\begin{aligned}
&\|V_g f\|_{W(\mathcal{F}L^1,L^p)} \asymp \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |\mathcal{V}_{V_\varphi \varphi}(V_g f)(x,\xi)| \,\mathrm{d}\xi \right)^p \mathrm{d}x \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_\varphi g(-x_1-\xi_2,\xi_1)||V_\varphi f(-\xi_2,x_2+\xi_1)| \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2 \right)^p \mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_\varphi g(-\xi_2,-\xi_1)||V_\varphi f(x_1-\xi_2,x_2-\xi_1)| \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2 \right)^p \mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_\varphi g(-w_1,-w_2)||V_\varphi f(x_1-w_1,x_2-w_2)| \,\mathrm{d}w_1 \,\mathrm{d}w_2 \right)^p \mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^{2d}} \Big(|V_\varphi g|^* * |V_\varphi f|(x_1,x_2) \Big)^p \mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/p} = \|\, |V_\varphi g|^* * |V_\varphi f| \,\|_{L^p}
\end{aligned}$$

where $f^*(x) := \overline{f(-x)}$ denotes the involution. Applying Young's inequality yields the desired estimate

$$\|\, |V_\varphi g|^* * |V_\varphi f| \,\|_{L^p} \leq \|V_\varphi f\|_{L^p} \|V_\varphi g\|_{L^1} \asymp \|f\|_{M^p} \|g\|_{M^1}.$$

For $p = \infty$ the argument is analogous with the essential supremum replacing the outer L^p-norm. $\square$

Lemma 2.1.4. *Let $g, \gamma \in M^1(\mathbb{R}^d)$, $f \in M^p(\mathbb{R}^d)$ and $h \in M^q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $\overline{V_g f} \cdot V_\gamma h \in M^1(\mathbb{R}^{2d})$ with norm estimate*

$$\|\overline{V_g f} \cdot V_\gamma h\|_{M^1} \lesssim \|g\|_{M^1} \|\gamma\|_{M^1} \|f\|_{M^p} \|h\|_{M^q}.$$

Proof. It is easy to see that $W(\mathcal{F}L^1, L^p)$ is closed under complex conjugation. Now the claim follows immediately from Lemma 2.1.2 and Lemma 2.1.3. $\square$

Now that we have dealt with the technical difficulties, we are ready to prove the Fundamental Identity of Gabor Analysis. We restate and prove Theorem 2.1.1 in a slightly more general formulation.

With the appropriate modifications, the Fundamental Identity also holds for weighted mixed-norm modulation spaces $M^{p,q}_m(\mathbb{R}^d)$. The proof is analogous and can be found in the original article by Feichtinger and Luef [16].

Theorem 2.1.5 (Fundamental Identity of Gabor Analysis)**.** *Let $g, \gamma \in M^1(\mathbb{R}^d)$, $f \in M^p(\mathbb{R}^d)$, $h \in M^q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then*

$$\sum_{\lambda \in \Lambda} V_g f(\lambda) \overline{V_\gamma h(\lambda)} = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu \in \Lambda^\circ} V_g \gamma(\mu) \overline{V_f h(\mu)},$$

with absolute convergence on both sides.

Proof. By Lemma 2.1.4, the product of short-time Fourier transforms $V_g f \cdot \overline{V_\gamma h}$ is in $M^1(\mathbb{R}^{2d})$. Hence, the Poisson Summation Formula in the form of Corollary 1.4.12 applies and we obtain

$$\begin{aligned}\sum_{\lambda\in\Lambda} V_g f(\lambda)\overline{V_\gamma h(\lambda)} &= \mathrm{vol}(\Lambda)^{-1} \sum_{\nu\in\Lambda^\perp} \big(V_g f \cdot \overline{V_\gamma h}\big)\hat{}\,(\nu) \\ &= \mathrm{vol}(\Lambda)^{-1} \sum_{\nu\in\Lambda^\perp} \big(V_h f \cdot \overline{V_\gamma g}\big)\,(-\mathcal{I}\nu) \\ &= \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} V_h f(\mu)\overline{V_\gamma g(\mu)}\end{aligned}$$

where we used Lemma 1.4.5 to rewrite the Fourier transform in the first line.

After using $V_g f(x,\xi) = e^{-2\pi i x\cdot\xi}\,\overline{V_f g(-x,-\xi)}$ on both factors on the right-hand side, we obtain the desired result. □

2.2. Janssen's Representation of the Frame Operator

As we have seen in Chapter 1, many properties of the Gabor system $\mathcal{G}(g,\Lambda)$ can be rephrased in terms of the corresponding frame operator and the reconstruction formula is given by the mixed frame operator

$$S_{g,\gamma} = \sum_{\lambda\in\Lambda} \langle\,.\,,\pi(\lambda)g\rangle\pi(\lambda)\gamma \tag{2.6}$$

for a suitable $\gamma \in L^2(\mathbb{R}^d)$. Evidently, the frame operator plays an important role in time-frequency analysis and we devote this section to its further study.

The first point we want to clarify is unspectacular but important: When is the frame operator well-defined and bounded? This will be answered in the first two statements of this section.

The second and main point of investigation concerns Janssen's representation of the frame operator. This representation reformulates (2.6) as an infinite linear combination of time-frequency shifts along the adjoint lattice.

Lemma 2.2.1. *Let $g \in M^1(\mathbb{R}^d)$, $f \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then*

$$\sum_{\lambda\in\Lambda} |\langle f,\pi(\lambda)g\rangle|^2 \le C_\Lambda ||g||^2_{M^1} ||f||^2_{L^2} \tag{2.7}$$

where the constant $C_\Lambda > 0$ depends only on the lattice Λ.

In particular, the frame operator $S_{g,\gamma}$ is bounded on $L^2(\mathbb{R}^d)$ for all g,γ in $M^1(\mathbb{R}^d)$.

Proof. Lemma 2.1.4 implies that $|V_g f|^2$ is in $M^1(\mathbb{R}^{2d})$ for all $g \in M^1(\mathbb{R}^d)$ and all $f \in L^2(\mathbb{R}^d)$. Recall that $M^1(\mathbb{R}^{2d})$ is continuously embedded into $W \cap \mathcal{F}W(\mathbb{R}^{2d})$ by Proposition 1.4.9. Hence, Proposition 1.4.10 applies and we obtain

$$\sum_{\lambda \in \Lambda} |V_g f(\lambda)|^2 \leq C_\Lambda || \, |V_g f|^2 \, ||_W \leq C_\Lambda || \, |V_g f|^2 \, ||_{M^1} \leq C_\Lambda ||g||_{M^1}^2 ||f||_{L^2}^2$$

where $C_\Lambda > 0$ depends only on the lattice Λ. □

A different interpretation of Lemma 2.2.1 yields the following corollary.

Corollary 2.2.2. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then the coefficient operator $C_{g,\Lambda}$ is bounded from $M^1(\mathbb{R}^d)$ to $\ell^2(\Lambda)$.*

Consequently, the frame operator $S_{g,\gamma}$ is bounded from $M^1(\mathbb{R}^d)$ into $M^\infty(\mathbb{R}^d)$ for all $g, \gamma \in L^2(\mathbb{R}^d)$.

Proof. If $g \in L^2(\mathbb{R}^d)$, then (2.7) implies that the coefficient operator $C_{g,\Lambda}$ is bounded from $M^1(\mathbb{R}^d)$ to $\ell^2(\Lambda)$ since $|V_g f(\lambda)| = |V_f g(-\lambda)|$. Consequently, its adjoint $D_{g,\Lambda}$ is bounded from $\ell^2(\Lambda)$ to $M^\infty(\mathbb{R}^d)$.

Therefore, $S_{g,\gamma} = D_{\gamma,\Lambda} C_{g,\Lambda} : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ is well-defined and bounded for all $g, \gamma \in L^2(\mathbb{R}^d)$. □

Corollary 2.2.2 also implies that the frame operator $S_{g,\gamma} : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ has the expected series representation

$$S_{g,\gamma} = \sum_{\lambda \in \Lambda} \langle \, . \, , \pi(\lambda) g \rangle \pi(\lambda) \gamma, \tag{2.8}$$

with unconditional convergence in the weak operator topology, i.e., the locally convex topology induced by the seminorms

$$T \mapsto |\langle T f, h \rangle| \quad \forall f, h \in M^1(\mathbb{R}^d).$$

The series representation (2.8) is a superposition of the time-frequency shifted window γ along the lattice Λ. Applying the Fundamental Identity of Gabor Analysis yields a series of time-frequency shifts along the adjoint lattice called *Janssen's representation* of the frame operator.

Theorem 2.2.3. *Let $g, \gamma \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then the Gabor frame operator $S_{g,\gamma} : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ can be written as*

$$S_{g,\gamma} = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu \in \Lambda^\circ} \langle \gamma, \pi(\mu) g \rangle \pi(\mu) \tag{2.9}$$

with unconditional convergence in the weak operator topology.

Proof. Using the Fundamental Identity of Gabor Analysis (2.3), we have for all $f, h \in M^1(\mathbb{R}^d)$

$$\begin{aligned}\langle S_{g,\gamma} f, h\rangle &= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\langle \pi(\lambda)\gamma, h\rangle \\ &= \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \langle \gamma, \pi(\mu)g\rangle\langle \pi(\mu)f, h\rangle\end{aligned} \tag{2.10}$$

with absolute convergence of both sums. Consequently, the right-hand side of (2.9) converges in the weak operator topology and its limit is precisely $S_{g,\gamma}$. □

If both $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(\gamma,\Lambda)$ are Bessel sequences, the frame operator $S_{g,\gamma}$ is bounded on $L^2(\mathbb{R}^d)$ and the series representation (2.8) converges in the L^2-norm. In this case, we would like Janssen's representation to converge in a stronger sense. But for this, the pair (g,γ) has to satisfy an additional property.

Definition 2.2.4. A pair of functions (g,γ) in $L^2(\mathbb{R}^d)$ satisfies *condition (A')* with respect to the lattice $\Lambda \subseteq \mathbb{R}^{2d}$ if

$$\sum_{\mu\in\Lambda^\circ} |\langle \gamma, \pi(\mu)g\rangle| < \infty.$$

If $\gamma = g$, then g is said to satisfy *condition (A)*.

Theorem 2.2.5. *Let $g,\gamma \in L^2(\mathbb{R}^d)$ satisfy condition (A') for the lattice $\Lambda \subseteq \mathbb{R}^{2d}$ and $\mathcal{G}(g,\Lambda)$, $\mathcal{G}(\gamma,\Lambda)$ be Bessel sequences for $L^2(\mathbb{R}^d)$. Then the Gabor frame operator $S_{g,\gamma} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ can be written as*

$$S_{g,\gamma} = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \langle \gamma, \pi(\mu)g\rangle \pi(\mu) \tag{2.11}$$

with absolute convergence in the operator norm.

Proof. As a consequence of condition (A'), the right-hand side of (2.11) converges absolutely with respect to the operator norm and

$$\tilde{S} := \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \langle \gamma, \pi(\mu)g\rangle \pi(\mu)$$

defines a bounded linear operator on $L^2(\mathbb{R}^d)$.

Since $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(\gamma,\Lambda)$ are Bessel sequences, the Gabor frame operator $S_{g,\gamma}$ is also bounded on $L^2(\mathbb{R}^d)$.

By equation (2.10), we have

$$\langle S_{g,\gamma} f, h\rangle = \langle \tilde{S} f, h\rangle$$

for all $f, h \in M^1(\mathbb{R}^d)$. Therefore, the bounded linear operators $\tilde{S}$ and $S_{g,\gamma}$ agree on the dense subset $M^1(\mathbb{R}^d)$ of $L^2(\mathbb{R}^d)$, hence they agree on $L^2(\mathbb{R}^d)$. □

The following two statements concern the frame operator for $\gamma = g$. Anticipating the Bessel duality of Theorem 2.3.1 in the next section, we find that condition (A) already implies that $\mathcal{G}(g, \Lambda)$ constitutes a Bessel sequence. Consequently, condition (A) alone is sufficient for Janssen's representation to hold in the strong sense.

Lemma 2.2.6. *If $g \in L^2(\mathbb{R}^d)$ satisfies condition (A) with respect to the lattice $\Lambda \subseteq \mathbb{R}^{2d}$, then $\mathcal{G}(g, \Lambda^\circ)$ is a Bessel sequence.*

Proof. This is an application of Schur's test. If the Gramian operator G, defined by the entries $G_{\mu,\nu} = \langle \pi(\nu)g, \pi(\mu)g \rangle$ for $\mu, \nu \in \Lambda^\circ$, satisfies

$$\sup_{\mu \in \Lambda^\circ} \sum_{\nu \in \Lambda^\circ} |G_{\mu,\nu}| \leq K_1,$$
$$\sup_{\nu \in \Lambda^\circ} \sum_{\mu \in \Lambda^\circ} |G_{\mu,\nu}| \leq K_2$$

for some constants $K_1, K_2 > 0$, then Schur's test implies that G is bounded on $\ell^2(\Lambda^\circ)$.

By taking absolute values in each entry of the Gramian, we gain a Toeplitz structure since

$$|G_{\lambda,\mu}| = |\langle \pi(\mu)g, \pi(\lambda)g \rangle| = |\langle g, \pi(\lambda - \mu)g \rangle|.$$

Consequently, all rows respective columns have the same sum, which is bounded by condition (A). Therefore, the assumptions of Schur's test are satisfied and G is bounded on $\ell^2(\Lambda^\circ)$. Hence $\mathcal{G}(g, \Lambda^\circ)$ is a Bessel sequence. □

Corollary 2.2.7. *Assume that $g \in L^2(\mathbb{R}^d)$ satisfies condition (A) with respect to the lattice $\Lambda \subseteq \mathbb{R}^{2d}$. Then the Gabor frame operator $S_{g,\Lambda}$ is a bounded linear operator on $L^2(\mathbb{R}^d)$ and can be written as*

$$S_{g,\Lambda} = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu \in \Lambda^\circ} \langle g, \pi(\mu)g \rangle \pi(\mu)$$

with absolute convergence in the operator norm.

Proof. Combine Lemma 2.2.6, Theorem 2.2.5, and Theorem 2.3.1. □

We conclude this section with Janssen's representation for windows in the modulation space $M^1(\mathbb{R}^d)$. We will see that M^1-functions are nice enough that no additional assumptions are needed to obtain absolute convergence in the operator norm.

Lemma 2.2.8. *Let $g, \gamma \in M^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then*

$$\sum_{\mu \in \Lambda^\circ} |\langle \gamma, \pi(\mu) g \rangle| \leq C_\Lambda ||g||_{M^1} ||\gamma||_{M^1} < \infty.$$

In particular, condition (A') is always satisfied for $g, \gamma \in M^1(\mathbb{R}^d)$.

Proof. By Lemma 2.1.3, the short-time Fourier transform $V_g \gamma$ is in $M^1(\mathbb{R}^{2d})$. Therefore, Proposition 1.4.10 applies and we obtain

$$\sum_{\mu \in \Lambda^\circ} |\langle \gamma, \pi(\mu) g \rangle| \leq C_\Lambda ||V_g \gamma||_W \leq C_\Lambda ||g||_{M^1} ||\gamma||_{M^1}.$$

□

Since the Gabor system for M^1-functions always constitutes a Bessel sequence, we obtain the following corollary.

Corollary 2.2.9. *Let $g, \gamma \in M^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then Janssen's representation of the Gabor frame operator* (2.11) *converges absolutely in the operator norm.*

Proof. Combine Lemma 2.2.1 and Lemma 2.2.8. □

2.3. Bessel Duality

Theorem 2.3.1 (Bessel Duality). *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then $\mathcal{G}(g, \Lambda)$ is a Bessel sequence if and only if $\mathcal{G}(g, \Lambda^\circ)$ is a Bessel sequence.*

For separable lattices, the Bessel duality principle was discovered independently by Daubechies et al. [13], Janssen [33], and Ron-Shen [42]. Feichtinger and Zimmermann [18] generalized this result to arbitrary lattices and recently, Jakobsen and Lemvig [30] proved that the Bessel duality also holds for Gabor systems over a closed subgroup of the time-frequency plane.

We follow the proof of Jakobsen and Lemvig [30], which modifies the idea of Janssen in [33].

The idea is to transport the problem to the adjoint lattice, where we can use the Bessel estimate for $\mathcal{G}(g, \Lambda^\circ)$. Unsurprisingly, Janssen's representation of the frame operator will play a central role in the proof.

Proof. Since $(\Lambda^\circ)^\circ = \Lambda$, it suffices to prove only one direction. We assume that $\mathcal{G}(g, \Lambda^\circ)$ is a Bessel sequence and show that all functions in $M^1(\mathbb{R}^d)$ satisfy the Bessel inequality with respect to $\mathcal{G}(g, \Lambda)$. As $M^1(\mathbb{R}^d)$ is a dense subspace of $L^2(\mathbb{R}^d)$, this implies that $\mathcal{G}(g, \Lambda)$ is a Bessel sequence for $L^2(\mathbb{R}^d)$ by a standard density argument.

By Corollary 2.2.2, the operator $S_{g,\Lambda}$ is bounded from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$. Taking $h \in M^1(\mathbb{R}^d)$ with normalization $\|h\|_{L^2} = 1$, the inversion formula for the short-time Fourier transform (1.10) implies for all $f \in M^1(\mathbb{R}^d)$

$$\begin{aligned}\langle S_{g,\Lambda} f, f\rangle &= \int_{\mathbb{R}^{2d}} \overline{V_h f(z)} V_h(S_{g,\Lambda} f)(z)\,\mathrm{d}z \\ &= \int\limits_{\mathbb{R}^{2d}/\Lambda^\circ} \sum_{\mu\in\Lambda^\circ} \overline{V_h f(z-\mu)} V_h(S_{g,\Lambda} f)(z-\mu)\,\mathrm{d}z \\ &= \int\limits_{\mathbb{R}^{2d}/\Lambda^\circ} \sum_{\mu\in\Lambda^\circ} \langle \pi(\mu)^*\pi(z)h, f\rangle \langle S_{g,\Lambda} f, \pi(\mu)^*\pi(z)h\rangle\,\mathrm{d}z.\end{aligned} \tag{2.12}$$

Observe that we used $\pi(z-\mu) = e^{2\pi i(\mu_2 - z_2)\cdot\mu_1}\pi(\mu)^*\pi(z)$ twice and the resulting phase factors cancel each other.

Set $\varphi_z(\mu) := \langle \pi(\mu)^*\pi(z)h, f\rangle = V_f(\pi(z)h)(\mu)$. This yields a sequence indexed by Λ° for every $z \in \mathbb{R}^{2d}$ and every f in $M^1(\mathbb{R}^d)$.

Using Theorem 2.2.3, i.e., the weak version of Janssen's representation, the integrand becomes

$$\begin{aligned}&\sum_{\mu\in\Lambda^\circ} \varphi_z(\mu)\langle S_{g,\Lambda} f, \pi(\mu)^*\pi(z)h\rangle \\ &\qquad = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \varphi_z(\mu) \sum_{\nu\in\Lambda^\circ} \langle g, \pi(\nu)g\rangle \langle \pi(\nu)f, \pi(\mu)^*\pi(z)h\rangle.\end{aligned}$$

Finally, we replace ν by $\nu - \mu$ in the inner sum and obtain

$$\begin{aligned}&\sum_{\mu\in\Lambda^\circ} \varphi_z(\mu)\langle S_{g,\Lambda} f, \pi(\mu)^*\pi(z)h\rangle \\ &\qquad = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \varphi_z(\mu) \sum_{\nu\in\Lambda^\circ} \langle g, \pi(\mu)^*\pi(\nu)g\rangle \langle \pi(\mu)^*\pi(\nu)f, \pi(\mu)^*\pi(z)h\rangle \\ &\qquad = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \varphi_z(\mu) \sum_{\nu\in\Lambda^\circ} \langle \pi(\mu)g, \pi(\nu)g\rangle \overline{\varphi_z(\nu)} \\ &\qquad = \mathrm{vol}(\Lambda)^{-1} \left\langle \sum_{\mu\in\Lambda^\circ} \varphi_z(\mu)\pi(\mu)g, \sum_{\nu\in\Lambda^\circ} \varphi_z(\nu)\pi(\nu)g \right\rangle.\end{aligned}$$

Note that interchanging the sums with the inner product is justified, as the sums converge unconditionally in $L^2(\mathbb{R}^d)$ since $\varphi_z \in \ell^2(\Lambda^\circ)$ by Lemma 2.2.1 and $\mathcal{G}(g, \Lambda^\circ)$ is a Bessel sequence.

Recall that the synthesis operator for a Bessel sequence is bounded. Therefore, we obtain the following estimate for all $z \in \mathbb{R}^{2d}$

$$\sum_{\mu\in\Lambda^\circ} \varphi_z(\mu)\langle S_{g,\Lambda} f, \pi(\mu)^*\pi(z)h\rangle = \mathrm{vol}(\Lambda)^{-1}\|D_{g,\Lambda^\circ}\varphi_z\|^2 \le B\|\varphi_z\|_{\ell^2}^2,$$

where $B := \mathrm{vol}(\Lambda)^{-1} B_{\Lambda^\circ}$ and B_{Λ° is the Bessel bound of $\mathcal{G}(g, \Lambda^\circ)$.

Using this estimate in (2.12) yields for all $f \in M^1(\mathbb{R}^d)$

$$\begin{aligned}
\langle S_{g,\Lambda} f, f\rangle &\leq B \int\limits_{\mathbb{R}^{2d}/\Lambda^\circ} \sum_{\mu \in \Lambda^\circ} |\langle \pi(\mu - z)h, f\rangle|^2 \,\mathrm{d}z \\
&= B \int_{\mathbb{R}^{2d}} |V_f h(z)|^2 \,\mathrm{d}z \\
&= B\langle f, f\rangle,
\end{aligned}$$

where we used the orthogonality relations (1.11) in the last line.

We have shown, that the Bessel inequality for $\mathcal{G}(g, \Lambda)$ holds on the dense subspace $M^1(\mathbb{R}^d)$, hence $\mathcal{G}(g, \Lambda)$ is a Bessel sequence for $L^2(\mathbb{R}^d)$. □

2.4. Duality Theory

While the characterizations for Gabor frames in Corollary 1.3.3 were just functional analytic reinterpretations of the frame inequality, the duality theory goes beyond such a superficial reformulation. The structures of $\mathcal{G}(g, \Lambda)$ and its dual system $\mathcal{G}(g, \Lambda^\circ)$ are intimately related: A Gabor frame on one side corresponds to a Riesz sequence on the other. Furthermore, dual windows of $\mathcal{G}(g, \Lambda)$ are characterized by a biorthogonality condition along the adjoint lattice.

Theorem 2.4.1 (Duality Theory)**.** *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then the following are equivalent:*

(i) *$\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.*

(ii) *$\mathcal{G}(g, \Lambda^\circ)$ is a Bessel sequence and there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma, \Lambda^\circ)$ is a Bessel sequence and*

$$\langle \gamma, \pi(\mu)g\rangle = \mathrm{vol}(\Lambda)\delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ, \tag{2.13}$$

where $\delta_{\mu,0}$ denotes the Kronecker delta.

(iii) *$\mathcal{G}(g, \Lambda^\circ)$ is a Riesz sequence for $L^2(\mathbb{R}^d)$.*

The biorthogonality relation (2.13) were discovered by the engineers Wexler and Raz [49] and characterize all possible dual windows (see Corollary 2.4.4).

Janssen [32, 33], Daubechies et al. [13] and Ron-Shen [42] made the results of Wexler-Raz rigorous and further expanded upon them which became the duality theory for separable lattices. The theory for general lattices is due to Feichtinger and Kozek [15]. Recently, Jakobsen and Lemvig [30] formulated density and duality theorems for Gabor frames along a closed subgroup of the time-frequency plane.

We follow the proof sketch given in the survey article by Gröchenig [23].

Proof. *(i)$\Rightarrow$(ii)*: Since $\mathcal{G}(g,\Lambda)$ is a Gabor frame, there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma,\Lambda)$ is a frame by Proposition 1.3.5. Furthermore, the reconstruction formula

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma$$

holds for all $f \in L^2(\mathbb{R}^d)$ with unconditional L^2-convergence. This justifies interchanging summation and inner product and we have

$$\langle f,h\rangle = \sum_{\lambda\in\Lambda} \langle f,\pi(\lambda)g\rangle\langle \pi(\lambda)\gamma, h\rangle = \sum_{\lambda\in\Lambda} V_g f(\lambda)\overline{V_\gamma h(\lambda)}$$

for all $f, h \in L^2(\mathbb{R}^d)$.

To apply the Fundamental Identity of Gabor Analysis (2.3) on the right-hand side, we restrict ourselves to $f, h \in M^1(\mathbb{R}^d)$ and obtain

$$\langle f,h\rangle = \sum_{\lambda\in\Lambda} V_g f(\lambda)\overline{V_\gamma h(\lambda)} = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} V_g\gamma(\mu)\overline{V_f h(\mu)}. \tag{2.14}$$

Replacing f, h by time-frequency shifted versions $\pi(z)f$, $\pi(z)g$ in $M^1(\mathbb{R}^d)$, equation (2.14) becomes

$$\begin{aligned}
\langle f,h\rangle &= \langle \pi(z)f, \pi(z)h\rangle \\
&= \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} V_g\gamma(\mu)\overline{V_{\pi(z)f}(\pi(z)h)(\mu)} \\
&= \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} V_g\gamma(\mu)\overline{V_f h(\mu)} e^{-2\pi i\mu\cdot\mathcal{I}z} \qquad (2.15)
\end{aligned}$$

for all $z \in \mathbb{R}^{2d}$, where we used Lemma 1.4.4 in the last step. Observe that (2.15) is a constant Fourier series, hence all its Fourier coefficients vanish except the one corresponding to $\mu = 0$. Explicitly, this means

$$\mathrm{vol}(\Lambda)^{-1} V_g\gamma(\mu)\overline{V_f h(\mu)} = \langle f,h\rangle \delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ \tag{2.16}$$

for all $f, h \in M^1(\mathbb{R}^d)$.

If $\mu = 0$, we have $\overline{V_f h(0)} = \langle f,h\rangle$ and (2.16) implies

$$\mathrm{vol}(\Lambda)^{-1} V_g\gamma(0) = \delta_{0,0}. \tag{2.17}$$

If $\mu \neq 0$, then the right-hand side of (2.16) vanishes. Therefore, one of the factors on the left must be zero. Since $\overline{V_f h(\mu)} = \langle \pi(\mu)f, h\rangle$ cannot be zero for all $f, h \in M^1(\mathbb{R}^d)$, we have

$$\mathrm{vol}(\Lambda)^{-1} V_g\gamma(\mu) = 0 = \delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ \setminus \{0\}. \tag{2.18}$$

Combining (2.17) and (2.18) yields the biorthogonality relations (2.13). The remaining assertions follow from Theorem 2.3.1.

(ii)⇒(i): By Theorem 2.3.1, the Gabor systems $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(\gamma,\Lambda)$ are Bessel sequences. Therefore, it remains to show the lower frame inequality for $\mathcal{G}(g,\Lambda)$.

The biorthogonality relation (2.13) implies condition (A'), hence Janssen's representation of the frame operator, i.e., Theorem 2.2.5, yields

$$S_{g,\gamma}f = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \langle\gamma,\pi(\mu)g\rangle\pi(\mu)f = f \tag{2.19}$$

for all $f \in L^2(\mathbb{R}^d)$.

Using (2.19) and the fact that $D_{\gamma,\Lambda}$ is bounded, we obtain

$$\|f\|^2 = \|S_{g,\gamma}f\|^2 = \|D_{\gamma,\Lambda}(C_{g,\Lambda}f)\|^2 \le B \sum_{\lambda\in\Lambda} |V_g f(\lambda)|^2$$

for all $f \in L^2(\mathbb{R}^d)$, which implies the lower frame inequality.

(ii)⇒(iii): By assumption, the Bessel sequences $\mathcal{G}(g,\Lambda^\circ)$ and $\mathcal{G}(\gamma,\Lambda^\circ)$ satisfy the biorthogonal condition (2.13), thus

$$\langle\pi(\nu)\gamma,\pi(\mu)g\rangle = c\langle\gamma,\pi(\mu-\nu)g\rangle = c\,\mathrm{vol}(\Lambda)\delta_{\mu-\nu,0} \quad \forall\mu,\nu\in\Lambda^\circ \tag{2.20}$$

with phase factor $c = e^{-2\pi i(\mu_2-\nu_2)\cdot\nu_1}$. Define $\tilde{\gamma} := \mathrm{vol}(\Lambda)^{-1}\gamma$, then (2.20) implies that $\mathcal{G}(\tilde{\gamma},\Lambda^\circ)$ is a biorthogonal Bessel sequence for $\mathcal{G}(g,\Lambda^\circ)$. Therefore, $\mathcal{G}(g,\Lambda^\circ)$ is a Riesz sequence by Lemma 1.2.8.

(iii)⇒(ii): By assumption $\mathcal{G}(g,\Lambda^\circ)$ is a Riesz sequence, i.e., a Riesz basis for its closed linear span. Theorem 1.2.7 implies that there exists a Bessel sequence $\{e_\nu : \nu\in\Lambda^\circ\}$ in $\overline{\mathrm{span}}\{\mathcal{G}(g,\Lambda^\circ)\}$ such that

$$\langle e_\nu,\pi(\mu)g\rangle = \delta_{\nu,\mu} \quad \forall\mu,\nu\in\Lambda^\circ.$$

This yields

$$\begin{aligned} C_{g,\Lambda^\circ}(\pi(\nu)e_0)(\mu) &= \langle\pi(\nu)e_0,\pi(\mu)g\rangle \\ &= e^{-2\pi i(\mu_2-\nu_2)\cdot\nu_1}\langle e_0,\pi(\mu-\nu)g\rangle \\ &= \delta_{0,\mu-\nu} \\ &= \langle e_\nu,\pi(\mu)g\rangle = C_{g,\Lambda^\circ}(e_\nu)(\mu) \end{aligned} \tag{2.21}$$

for all $\mu,\nu\in\Lambda^\circ$.

Recall that the coefficient operator of a Riesz basis is injective. Therefore, the restriction of C_{g,Λ° to $\overline{\mathrm{span}}\{\mathcal{G}(g,\Lambda^\circ)\}$ is injective. Since $\overline{\mathrm{span}}\{\mathcal{G}(g,\Lambda^\circ)\}$ is closed under time-frequency shifts with respect to Λ°, equation (2.21) implies $e_\nu = \pi(\nu)e_0$ for all $\nu\in\Lambda^\circ$.

After the normalization $\gamma := \mathrm{vol}(\Lambda)^{-1}e_0$, the Bessel sequence $\mathcal{G}(\gamma,\Lambda^\circ) = \{\mathrm{vol}(\Lambda)^{-1}e_\nu : \nu\in\Lambda^\circ\}$ satisfies the biorthogonality relations (2.13). □

Remark 2.4.2 (Frame Bounds and an Alternative Proof)**.** An alternative way to prove the equivalence *(i)*$\Leftrightarrow$*(iii)* can be seen in the proof of the Bessel duality, i.e., Theorem 2.3.1, where we related the frame operator $S_{g,\Lambda}$ to the Gramian $G_{g,\Lambda^\circ} = C_{g,\Lambda^\circ} D_{g,\Lambda^\circ}$ of the dual system.

Modifying the ideas of Janssen in [33] to suit general lattices, one can show that

$$AI_{L^2} \le S_{g,\Lambda} \le BI_{L^2} \quad \Longleftrightarrow \quad AI_{\ell^2} \le \mathrm{vol}(\Lambda)^{-1} G_{g,\Lambda^\circ} \le BI_{\ell^2}.$$

Hence, the family $\mathcal{G}(g,\Lambda)$ is a frame with frame bounds $A, B > 0$ if and only if $\mathcal{G}(g,\Lambda^\circ)$ is a Riesz sequence with bounds $\mathrm{vol}(\Lambda)A, \mathrm{vol}(\Lambda)B > 0$ respectively.

As we have seen in Section 1.3, every Gabor frame $\mathcal{G}(g,\Lambda)$ has a canonical dual window $\gamma^\circ = S_{g,\Lambda}^{-1} g$ that gives rise to the frame expansion

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma^\circ\rangle \pi(\lambda) g = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda)\gamma^\circ \tag{2.22}$$

for all $f \in L^2(\mathbb{R}^d)$.

In general, i.e., if $\mathcal{G}(g,\Lambda)$ is not a Riesz sequence, the coefficients in (2.22) are not unique. Hence, there may be other windows $\gamma \in L^2(\mathbb{R}^d)$ that satisfy the reconstruction property $S_{g,\gamma} = S_{\gamma,g} = I_{L^2}$. This motivates the following definition.

Definition 2.4.3. Let $g \in L^2(\mathbb{R}^d)$ and $\mathcal{G}(g,\Lambda)$ be a Bessel sequence. We call $\gamma \in L^2(\mathbb{R}^d)$ a *dual window* for $\mathcal{G}(g,\Lambda)$, if $\mathcal{G}(\gamma,\Lambda)$ is a Bessel sequence and the reconstruction property $S_{g,\gamma} = S_{\gamma,g} = I_{L^2}$ is satisfied.

We isolate the following characterization for dual windows from the proof of the equivalence *(i)*$\Leftrightarrow$*(ii)* in Theorem 2.4.1.

Corollary 2.4.4. *Suppose $g, \gamma \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ such that $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(\gamma,\Lambda)$ are Bessel sequences. Then γ is a dual window for $\mathcal{G}(g,\Lambda)$ if and only if the Wexler-Raz biorthogonality relations* (2.13) *are satisfied.*

Proof. There is no need to prove anything new, we just put a different focus on equivalence *(i)*$\Leftrightarrow$*(ii)* of Theorem 2.4.1.

Assume that $\gamma \in L^2(\mathbb{R}^d)$ is a dual window for $\mathcal{G}(g,\Lambda)$, hence the reconstruction property (2.22) is satisfied with L^2-convergence since both $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(\gamma,\Lambda)$ are Bessel sequences. If we continue as in *(i)*$\Rightarrow$*(ii)*, we find that γ and g satisfy the biorthogonality relations.

Conversely, let g and γ satisfy the biorthogonality relations. As in *(ii)*$\Rightarrow$*(i)*, we arrive at equation (2.19), which is equivalent to $S_{g,\gamma} = I_{L^2}$. Since the assumptions are symmetric, we also have $S_{\gamma,g} = I_{L^2}$ after interchanging the roles of g and γ. $\square$

We conclude this section with a characterization for tight Gabor frames.

Corollary 2.4.5. *A Gabor system* $\mathcal{G}(g,\Lambda)$ *is a tight frame if and only if* $\mathcal{G}(g,\Lambda^\circ)$ *is an orthogonal system. In this case the frame bound satisfies* $A = \mathrm{vol}(\Lambda)^{-1}\|g\|^2$.

Proof. Assume $\mathcal{G}(g,\Lambda)$ is a tight frame, then $S_{g,\Lambda} = AI_{L^2}$ by Proposition 1.1.5. Hence the canonical dual window is of the form $\gamma = S_{g,\Lambda}^{-1}g = \frac{1}{A}g$ and satisfies the Wexler-Raz biorthogonality relations (2.13) by Corollary 2.4.4. Thus $\mathcal{G}(g,\Lambda^\circ)$ is an orthogonal system since

$$\langle g, \pi(\mu)g\rangle = A\langle \gamma, \pi(\mu)g\rangle = A\,\mathrm{vol}(\Lambda)\delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ. \tag{2.23}$$

Conversely, let $\mathcal{G}(g,\Lambda^\circ)$ be an orthogonal system, i.e.,

$$\langle g, \pi(\mu)g\rangle = \|g\|^2\delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ.$$

Then g satisfies condition (A) and Janssen's representation of the frame operator, i.e., Theorem 2.2.6, yields

$$S_{g,\Lambda}f = \mathrm{vol}(\Lambda)^{-1}\sum_{\mu\in\Lambda^\circ}\langle g, \pi(\mu)g\rangle\pi(\mu)f = \mathrm{vol}(\Lambda)^{-1}\|g\|^2 f.$$

Consequently, the frame operator is a multiple of the identity operator on $L^2(\mathbb{R}^d)$ and thus $\mathcal{G}(g,\Lambda)$ a tight frame.

The claim regarding the frame bound follows immediately from (2.23). □

2.5. Density Theorem

The duality theory has an insightful consequence for the density of the lattice: If we sample too sparsely in the time-frequency plane, reconstruction will be impossible regardless of the window function.

Theorem 2.5.1 (Density Theorem)**.** *Let* $g \in L^2(\mathbb{R}^d)$ *and* $\Lambda \subseteq \mathbb{R}^{2d}$ *be a lattice. Then the following holds:*

(a) If $\mathcal{G}(g,\Lambda)$ *is a frame, then* $0 < \mathrm{vol}(\Lambda) \leq 1$.

(b) If $\mathcal{G}(g,\Lambda)$ *is a Riesz sequence, then* $\mathrm{vol}(\Lambda) \geq 1$.

(c) $\mathcal{G}(g,\Lambda)$ *is a Riesz basis if and only if it is a frame and* $\mathrm{vol}(\Lambda) = 1$.

For a comprehensive historic account on the density theorem, we refer the interested reader to the excellent survey article by Heil [29].

Proof. *(a)* Let $\mathcal{G}(g,\Lambda)$ be a frame and $S := S_{g,\Lambda}$ be the corresponding frame operator. Recall that the frame operator is a positive invertible operator, hence $S^{-1/2}$ is a well-defined positive operator on $L^2(\mathbb{R}^d)$.

By Corollary 2.4.4, the canonical dual window $\gamma := S^{-1}g$ satisfies the Wexler-Raz biorthogonality relation (2.13). This yields

$$\mathrm{vol}(\Lambda) = \langle S^{-1}g, g\rangle = \langle S^{-1/2}g, S^{-1/2}g\rangle = \|S^{-1/2}g\|_{L^2}^2. \tag{2.24}$$

Since $\mathcal{G}(S^{-1/2}g,\Lambda)$ is a tight frame with frame bound $A = B = 1$ by Lemma 1.1.8, we obtain

$$\begin{aligned}\|S^{-1/2}g\|_{L^2}^4 &= |\langle S^{-1/2}g, S^{-1/2}g\rangle|^2\\ &\leq \sum_{\lambda\in\Lambda} |\langle S^{-1/2}g, \pi(\lambda)S^{-1/2}g\rangle|^2 = \|S^{-1/2}g\|_{L^2}^2.\end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25) yields

$$0 < \mathrm{vol}(\Lambda) = \|S^{-1/2}g\|_{L^2}^2 \leq 1.$$

(b) Observe that the volume of the adjoint lattice is $\mathrm{vol}(\Lambda^\circ) = \mathrm{vol}(\Lambda)^{-1}$. Therefore, the claim is equivalent to *(a)* by Theorem 2.4.1.

(c) Suppose $\mathcal{G}(g,\Lambda)$ is a frame with $\mathrm{vol}(\Lambda) = 1$. Then $\mathcal{G}(S^{-1/2}g,\Lambda)$ is again a tight frame with frame bound $A = B = 1$ and equation (2.24) implies that the window function has norm $\|S^{-1/2}g\|_{L^2}^2 = \mathrm{vol}(\Lambda) = 1$.

Thus, Lemma 1.1.9 implies that $\mathcal{G}(S^{-1/2}g,\Lambda)$ is an orthonormal basis. Consequently, $\mathcal{G}(g,\Lambda)$ is the image of an orthonormal basis under the bounded invertible operator $S^{1/2}$, hence a Riesz basis by Proposition 1.2.7.

The converse follows from combining *(a)* and *(b)*. □

While the last 30 years has seen numerous results on sufficient conditions for Gabor frames [6, 10, 14, 17, 42, 48], the gap between necessary and sufficient conditions is still massive. For an arbitrary window function, we hardly know which lattices generate a frame.

The only exception is the one-dimensional Gaussian $\varphi(x) = e^{-\pi x^2}$, where a full characterization was obtained independently by Seip-Wallstén [44, 45] and Lyubarskiĭ [41], confirming a conjecture of Daubechies and Grossmann [11].

Theorem 2.5.2 (Seip-Wallstén [44, 45], Lyubarskiĭ [41])**.** *Consider the one-dimensional Gaussian $\varphi(x) = e^{-\pi x^2}$ and let $\Lambda \subseteq \mathbb{R}^2$ be a lattice. Then $\mathcal{G}(\varphi,\Lambda)$ is a frame for $L^2(\mathbb{R})$ if and only if* $\mathrm{vol}(\Lambda) < 1$.

Theorem 2.5.2 reveals yet again the interdisciplinary nature of time-frequency analysis. The proof relies on the fact that the short-time Fourier transform with respect to the Gaussian is closely related to the Bargmann transform. This transports the frame inequality in $L^2(\mathbb{R}^d)$ to a sampling problem in the Bargmann-Fock space of holomorphic functions, where the sophisticated methods of complex analysis apply.

Unfortunately, only the Gaussian possesses an analytic short-time Fourier transform [1]. Hence the methods of complex analysis do not carry further, which explains why this result has remained singular for 25 years.

Remark 2.5.3. Seip-Wallstén and Lyubarskiĭ's results are even stronger. They considered Gabor systems $\mathcal{G}(\varphi,\Lambda)$ along uniformly discrete sets $\Lambda \subseteq \mathbb{R}^2$ and proved that $\mathcal{G}(\varphi,\Lambda)$ is a frame if and only if the *Beurling density* $D^-(\Lambda)$ satisfies

$$D^-(\Lambda) := \liminf_{r\to\infty} \frac{1}{r^2} \min_{(x,\xi)\in\mathbb{R}^2} \operatorname{card}\{\lambda \in \Lambda : \lambda \in (x,\xi) + [0,r]^2\} > 1.$$

Remark 2.5.4. The extraordinary result of Seip-Wallstén and Lyubarskiĭ holds only for dimension $d = 1$. For the d-dimensional Gaussian $\varphi_d(x) = e^{-\pi|x|^2}$, the system $\mathcal{G}(\varphi_d(x),\Lambda)$ need not be a frame if $\operatorname{vol}(\Lambda) < 1$. Even for general rectangular lattices of the form

$$\Lambda := \prod_{j=1}^{d} \alpha_j\mathbb{Z} \times \prod_{j=1}^{d} \beta_j\mathbb{Z} \quad \text{with } \operatorname{vol}(\Lambda) = \alpha_1\beta_1\cdots\alpha_d\beta_d < 1,$$

the Gabor system $\mathcal{G}(\varphi_d,\Lambda)$ might fail to be a frame:

Due to the tensor product structure of both the window and the lattice, $\mathcal{G}(\varphi_d,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if $\mathcal{G}(\varphi,\alpha_j\mathbb{Z}\times\beta_j\mathbb{Z})$ is a frame in $L^2(\mathbb{R})$ for all $j = 1,\ldots,d$. This is precisely the case if all $\alpha_j\beta_j < 1$, but in dimension $d \geq 2$ the condition on the volume leaves enough freedom for at least one $\alpha_j\beta_j \geq 1$.

3. Characterizations for M^1-Window Functions

Since the upper frame inequality is always satisfied by Lemma 2.2.1, only the existence of a lower frame bound remains to be checked for Gabor families generated by an M^1-function.

Gröchenig [22] listed fourteen equivalent conditions for the invertibility of the frame operator on $L^2(\mathbb{R}^d)$. All of them are expressed without any inequalities, but as properties of the four canonical operators, i.e., the analysis, synthesis, frame and Gramian operators. The key element is a version of Wiener's Lemma [25] which implies that for $g \in M^1(\mathbb{R}^d)$, the invertibility of $S_{g,\Lambda}$ on $L^2(\mathbb{R}^d)$ is equivalent to the invertibility on all modulation spaces simultaneously.

But first, we need to clarify that the frame operator is well-defined and bounded on modulation spaces.

3.1. Boundedness of the Frame Operator

In this section, we will show that frame operator $S_{g,\Lambda}$ with respect to $g \in M^1(\mathbb{R}^d)$ is bounded on all modulation spaces $M^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$. This requires some technical preparation.

Lemma 3.1.1. *Let $1 \leq p \leq \infty$. Suppose $g, \varphi \in M^1(\mathbb{R}^d)$ and $f \in M^p(\mathbb{R}^d)$, then*

$$|V_g f| \leq \frac{1}{|\langle \varphi, \varphi \rangle|} |V_\varphi f| * |V_g \varphi|.$$

Proof. We use the inversion formula (1.10) and obtain for all $z \in \mathbb{R}^{2d}$

$$\begin{aligned}
|\langle f, \pi(z)g \rangle| &\leq \frac{1}{|\langle \varphi, \varphi \rangle|} \int_{\mathbb{R}^{2d}} |\langle f, \pi(w)\varphi \rangle| |\langle \pi(w)\varphi, \pi(z)g \rangle| \, \mathrm{d}w \\
&= \frac{1}{|\langle \varphi, \varphi \rangle|} \int_{\mathbb{R}^{2d}} |\langle f, \pi(w)\varphi \rangle| |\langle \varphi, \pi(z-w)g \rangle| \, \mathrm{d}w \\
&= \frac{1}{|\langle \varphi, \varphi \rangle|} \left(|V_\varphi f| * |V_g \varphi| \right)(z).
\end{aligned}$$

□

The following proposition is Young's inequality for Wiener amalgam spaces.

Proposition 3.1.2 (Gröchenig [21, Thm. 11.1.5])**.** *Let $1 \leq p \leq \infty$. Suppose $g \in W(L^\infty, L^1)$ and $f \in L^p(\mathbb{R}^d)$, then*

$$\|f * g\|_{W(L^\infty,L^p)} \leq \|f\|_{L^p}\|g\|_{W(L^\infty,L^1)}.$$

The proof is rather lengthy but uneventful and employs the usual estimations for functions in Wiener amalgam spaces. We therefore omit the proof and refer to Gröchenig's textbook [21].

Lemma 3.1.3. *Let $1 \leq p \leq \infty$. Suppose $g \in M^1(\mathbb{R}^d)$ and $f \in M^p(\mathbb{R}^d)$, then $V_g f \in W(L^\infty, L^p)$ with norm estimate*

$$\|V_g f\|_{W(L^\infty,L^p)} \lesssim \|g\|_{M^1}\|f\|_{M^p}.$$

Proof. Fix $\varphi \in M^1(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$. We apply Young's inequality for amalgam spaces, Proposition 3.1.2, to the pointwise estimate of Lemma 3.1.1. Furthermore, the embedding of $M^1(\mathbb{R}^d)$ into $W(L^\infty, L^1)$ is continuous by Proposition 1.4.9, hence

$$\begin{aligned}\|V_g f\|_{W(L^\infty,L^p)} &\leq \| \, |V_\varphi f| * |V_g\varphi| \, \|_{W(L^\infty,L^p)} \leq \|V_\varphi f\|_{L^p}\|V_g\varphi\|_{W(L^\infty,L^1)} \\ &\lesssim \|f\|_{M^p}\|V_g\varphi\|_{M^1} \lesssim \|f\|_{M^p}\|g\|_{M^1}\|\varphi\|_{M^1},\end{aligned}$$

where we used Lemma 2.1.3 in the last step. □

Finally, we can prove the main result of this section, the boundedness of the frame operator on modulation spaces.

Proposition 3.1.4. *Let $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then there exits a positive constant $C_\Lambda > 0$ depending only on lattice Λ but not on $1 \leq p \leq \infty$ or $g \in M^1(\mathbb{R}^d)$, such that*

(a) $C_{g,\Lambda} : M^p(\mathbb{R}^d) \to \ell^p(\Lambda)$ is bounded with $\|C_{g,\Lambda}f\|_{\ell^p} \leq C_\Lambda\|g\|_{M^1}\|f\|_{M^p}$;

(b) $D_{g,\Lambda} : \ell^p(\Lambda) \to M^p(\mathbb{R}^d)$ is bounded with $\|D_{g,\Lambda}c\|_{M^p} \leq C_\Lambda\|g\|_{M^1}\|c\|_{\ell^p}$;

(c) $S_{g,\Lambda} : M^p(\mathbb{R}^d) \to M^p(\mathbb{R}^d)$ is bounded with $\|S_{g,\Lambda}f\|_{M^p} \leq C_\Lambda^2\|g\|_{M^1}^2\|f\|_{M^p}$;

(d) $G_{g,\Lambda} : \ell^p(\Lambda) \to \ell^p(\Lambda)$ is bounded with $\|G_{g,\Lambda}c\|_{\ell^p} \leq C_\Lambda^2\|g\|_{M^1}^2\|c\|_{\ell^p}$.

Proof. It suffices to prove claim *(a)*, then the rest follows by dualization and composition.

(a) Let $1 \leq p \leq \infty$, then $V_g f$ is in $W(L^\infty, L^p)$ by Lemma 3.1.3. Furthermore, recall that $V_g f$ is continuous by Lemma 1.4.8. Now, the sampling property of Proposition 1.4.10 applies which yields

$$\|C_{g,\Lambda}f\|_{\ell^p} = \|V_g f|_\Lambda\|_{\ell^p} \leq C_\Lambda\|V_g f\|_{W(L^\infty,L^p)} \leq C_\Lambda\|g\|_{M^1}\|f\|_{M^p}.$$

□

3.2. Fourteen Characterizations without Inequalities

The previous section concerned the boundedness of the canonical operators on (respectively between) modulation spaces $M^p(\mathbb{R}^d)$ and ℓ^p-spaces for $1 \le p \le \infty$. With this technicality out of the way, we are now able to state the fourteen equivalent formulations for the existence of a lower bound in the frame inequality.

We will closely follow Gröchenig's original work [22] in this section.

Theorem 3.2.1 (Gröchenig [22]). *Let $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then the following are equivalent:*

(i) $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, i.e., $S_{g,\Lambda}$ is invertible on $L^2(\mathbb{R}^d)$.

(ii) $S_{g,\Lambda}$ is invertible on $M^1(\mathbb{R}^d)$.

(iii) $S_{g,\Lambda}$ is invertible on $M^\infty(\mathbb{R}^d)$.

(iv) $S_{g,\Lambda}$ is injective on $M^\infty(\mathbb{R}^d)$.

(v) $C_{g,\Lambda}$ is injective from $M^\infty(\mathbb{R}^d)$ to $\ell^\infty(\Lambda)$.

(vi) $D_{g,\Lambda}$ defined on $\ell^1(\Lambda)$ has dense range in $M^1(\mathbb{R}^d)$.

(vii) $D_{g,\Lambda}$ is surjective from $\ell^1(\Lambda)$ onto $M^1(\mathbb{R}^d)$.

(viii) D_{g,Λ° is injective from $\ell^\infty(\Lambda^\circ)$ to $M^\infty(\mathbb{R}^d)$.

(ix) C_{g,Λ° defined on $M^1(\mathbb{R}^d)$ has dense range in $\ell^1(\Lambda^\circ)$.

(x) C_{g,Λ° is surjective from $M^1(\mathbb{R}^d)$ onto $\ell^1(\Lambda^\circ)$.

(xi) G_{g,Λ° is invertible on $\ell^1(\Lambda^\circ)$.

(xii) G_{g,Λ° is invertible on $\ell^\infty(\Lambda^\circ)$.

(xiii) G_{g,Λ° is injective on $\ell^\infty(\Lambda^\circ)$.

(xiv) $\mathcal{G}(g,\Lambda^\circ)$ is a Riesz sequence in $L^2(\mathbb{R}^d)$, i.e., G_{g,Λ° is invertible on $\ell^2(\Lambda^\circ)$.

In most parts, the proof requires only some basic facts of functional analysis. But the key element is Wiener's Lemma in two versions. One links the invertibility of the frame operator on $L^2(\mathbb{R}^d)$ to its invertibility on modulation spaces. The other, Wiener's Lemma for the twisted convolution, links the invertibility of the Gramian on $\ell^2(\Lambda^\circ)$ to the its invertibility on all $\ell^p(\Lambda^\circ)$ for $1 \le p \le \infty$.

The *twisted convolution* of two finite sequences a, c indexed with respect to the lattice $\Lambda \subseteq \mathbb{R}^{2d}$ is defined as

$$(c \natural a)(\lambda) := \sum_{\mu \in \Lambda} c_\mu a_{\lambda-\mu} e^{-2\pi i \sigma(\mu, \lambda-\mu)},$$

where $\sigma(\mu, \lambda) := \mu_1 \cdot \lambda_2$ for $\mu = (\mu_1, \mu_2), \lambda = (\lambda_1, \lambda_2) \in \Lambda \subseteq \mathbb{R}^{2d}$. By Young's inequality

$$\|c \natural a\|_{\ell^p} \leq \| \, |c| * |a| \, \|_{\ell^p} \leq \|a\|_{\ell^1} \|c\|_{\ell^p},$$

with $|a|_\lambda := |a_\lambda|$, the twisted convolution extends to ℓ^p-spaces.

In particular, the *twisted convolution operator* C_a with respect to a fixed sequence $a \in \ell^1(\Lambda)$, defined by

$$C_a c := c \natural a,$$

is well-defined and bounded on all $\ell^p(\Lambda)$ for $1 \leq p \leq \infty$.

The Gramian operator $G := G_{g,\Lambda}$ of a Gabor system $\mathcal{G}(g, \Lambda)$ is precisely such a twisted convolution operator. Recall that its entries are given by

$$G_{\lambda,\mu} = \langle \pi(\mu) g, \pi(\lambda) g \rangle = e^{-2\pi i \sigma(\mu, \lambda-\mu)} \langle g, \pi(\lambda - \mu) g \rangle. \tag{3.1}$$

Let us denote $a_\lambda := \langle g, \pi(\lambda) g \rangle$, then the action of the Gramian is

$$(Gc)(\lambda) = \sum_{\mu \in \Lambda} G_{\lambda,\mu} c_\mu = \sum_{\mu \in \Lambda} c_\mu a_{\lambda-\mu} e^{-2\pi i \sigma(\mu, \lambda-\mu)} = (c \natural a)(\lambda) = (C_a c)(\lambda).$$

Since $g \in M^1(\mathbb{R}^d)$, the sequence a is absolutely summable by Proposition 3.1.4, hence C_a is well-defined and bounded on all $\ell^p(\Lambda)$ for $1 \leq p \leq \infty$.

Now, we can state *Wiener's Lemma for the twisted convolution*.

Theorem 3.2.2 (Gröchenig-Leinert [25])**.** *Suppose that $a \in \ell^1(\Lambda)$ and that the twisted convolution operator $C_a c = c \natural a$ is invertible on $\ell^2(\Lambda)$. Then the inverse is $C_a^{-1} = C_b$ for a unique $b \in \ell^1(\Lambda)$. Consequently, C_a is invertible simultaneously on all $\ell^p(\Lambda)$ for $1 \leq p \leq \infty$.*

For a proof, we refer to Gröchenig and Leinert's original work [25]. Alternative proofs can be found in [3, 26].

We have already seen in (3.1), that the phase factor in the twisted convolution relates to combining two time-frequency shifts. It is therefore unsurprising, although it requires some work, that Wiener's Lemma for the twisted convolution can be transferred to operators of the form $\sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda)$ with $a \in \ell^1(\Lambda)$, i.e., absolutely convergent sums of time-frequency shifts.

Theorem 3.2.3 (Gröchenig-Leinert [25])**.** *Let $S = \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda)$. If $a \in \ell^1(\Lambda)$ and S is invertible on $L^2(\mathbb{R}^d)$, then S^{-1} is again an absolutely convergent series of time-frequency shifts $S^{-1} = \sum_{\lambda \in \Lambda} b_\lambda \pi(\lambda)$ for a unique $b \in \ell^1(\Lambda)$. Consequently, S is invertible simultaneously on all modulation spaces M^p for $1 \leq p \leq \infty$.*

The following two lemmata settle some small technical details in the proof of Theorem 3.2.1. The first concerns the convergence of infinite linear combinations of time-frequency shifts with coefficients in $\ell^\infty(\Lambda)$, while the second states that time-frequency shifts are linearly independent.

Lemma 3.2.4. *If $c \in \ell^\infty(\Lambda)$, then the operator $\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)$ is bounded from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$ and the sum converges unconditionally in the weak operator topology.*

Proof. By Lemma 3.1.4, we have $\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) f = D_{f,\Lambda} c \in M^\infty(\mathbb{R}^d)$ for all $f \in M^1(\mathbb{R}^d)$. Furthermore,

$$\Big|\sum_{\lambda\in\Lambda} c_\lambda \langle \pi(\lambda) f, g\rangle\Big| \le \|c\|_{\ell^\infty} \sum_{\lambda\in\Lambda} |\langle \pi(\lambda) f, g\rangle| \le C_\Lambda \|c\|_{\ell^\infty} \|f\|_{M^1} \|g\|_{M^1}, \qquad (3.2)$$

which implies that the sum $\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)$ converges unconditionally in the weak operator topology. Since $M^\infty(\mathbb{R}^d)$ is the dual space of $M^1(\mathbb{R}^d)$, inequality (3.2) also implies

$$\Big\|\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) f\Big\|_{M^\infty} \le C_\Lambda \|c\|_{\ell^\infty} \|f\|_{M^1}.$$

Hence, $\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)$ is a bounded operator from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$. $\square$

Lemma 3.2.5. *If $\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) = 0$ for some $c \in \ell^\infty(\Lambda)$, then $c = 0$.*

Proof. By Lemma 3.2.4, the sum $\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)$ converges with respect to the weak operator topology. Hence, we have for all $g, h \in M^1(\mathbb{R}^d)$

$$\Big\langle \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)\pi(z) g, \pi(z) h\Big\rangle = \sum_{\lambda\in\Lambda} c_\lambda \langle \pi(\lambda)\pi(z) g, \pi(z) h\rangle = 0$$

by assumption.

Since $\pi(z)^*\pi(\lambda)\pi(z) = e^{2\pi i(z_1\lambda_2 - z_2\lambda_1)}\pi(\lambda)$, we obtain

$$\sum_{\lambda\in\Lambda} c_\lambda \langle \pi(\lambda) g, h\rangle e^{2\pi i(z_1\lambda_2 - z_2\lambda_1)} = 0,$$

an absolutely convergent Fourier series which vanishes everywhere.

Consequently,

$$c_\lambda \langle \pi(\lambda) g, h\rangle = 0 \qquad \forall \lambda \in \Lambda$$

for all $g, h \in M^1(\mathbb{R}^d)$, which is only possible if $c = 0$. $\square$

Proof of Theorem 3.2.1. The structure or the proof is the following. We will show three chains of implications starting at *(i)* and ending with *(viii)*:

1. *(i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(iv)* $\Rightarrow$ *(v)* $\Rightarrow$ *(vi)* $\Rightarrow$ *(viii)*

2. *(i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(vii)* $\Rightarrow$ *(viii)*

3. *(i)* $\Rightarrow$ *(xiv)* $\Rightarrow$ *(xi)* $\Rightarrow$ *(xii)* $\Rightarrow$ *(xiii)* $\Rightarrow$ *(viii)*

To complete the circle, we will then prove

4. *(viii)* $\Rightarrow$ *(ix)* $\Rightarrow$ *(x)* $\Rightarrow$ *(i)* .

(i)$\Rightarrow$*(ii)*: Janssen's representation of the frame operator, Theorem 2.2.5, allows us to write $S_{g,\Lambda}$ as

$$S_{g,\Lambda} = \mathrm{vol}(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \langle g, \pi(\mu)g\rangle \pi(\mu),$$

where $a_\mu := \mathrm{vol}(\Lambda)^{-1}\langle g, \pi(\mu)g\rangle$ is absolutely summable by Lemma 2.2.8.

By assumption, the frame operator is invertible on $L^2(\mathbb{R}^d)$. Hence, Wiener's Lemma, Theorem 3.2.3, implies that $S_{g,\Lambda}$ is invertible on all modulation spaces $M^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$.

(ii)$\Leftrightarrow$*(iii)*: A bounded linear operator between Banach spaces is invertible if and only if its adjoint is invertible. Since the adjoint of $S_{g,\Lambda} : M^1(\mathbb{R}^d) \to M^1(\mathbb{R}^d)$ is $S_{g,\Lambda} : M^\infty(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$, equivalence *(ii)*$\Leftrightarrow$*(iii)* follows immediately.

(iii)$\Rightarrow$*(iv)*: Obviously, an invertible operator must be injective.

(iv)$\Rightarrow$*(v)*: If $S_{g,\Lambda} = D_{g,\Lambda}C_{g,\Lambda}$ is injective on $M^\infty(\mathbb{R}^d)$, then in particular $C_{g,\Lambda}$ must be injective from $M^\infty(\mathbb{R}^d)$ to $\ell^\infty(\Lambda)$.

(v)$\Rightarrow$*(vi)*: A bounded linear operator between Banach spaces has dense range if and only if its adjoint is injective. Since $C_{g,\Lambda} : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda)$ is the adjoint operator of $D_{g,\Lambda} : \ell^1(\Lambda) \to M^1(\mathbb{R}^d)$, this implies the claim.

(vi),*(vii)*$\Rightarrow$*(viii)*: By assumption, $D_{g,\Lambda}(\ell^1(\Lambda))$ is dense in $M^1(\mathbb{R}^d)$ (or equal to $M^1(\mathbb{R}^d)$). Since finite sequences are dense in $\ell^1(\Lambda)$ and $D_{g,\Lambda} : \ell^1(\Lambda) \to M^1(\mathbb{R}^d)$ is bounded by Proposition 3.1.4, finite linear combinations of the form

$$f = \sum_{\lambda\in F} a_\lambda \pi(\lambda)g, \tag{3.3}$$

where F is a finite subset of Λ, are also dense in $M^1(\mathbb{R}^d)$.

Now assume

$$D_{g,\Lambda^\circ}c = \sum_{\mu\in\Lambda^\circ} c_\mu \pi(\mu)g = 0$$

for some $c \in \ell^\infty(\Lambda^\circ)$.

The goal is to show that the operator $\sum_{\mu\in\Lambda^\circ} c_\mu\pi(\mu)$ vanishes on $M^1(\mathbb{R}^d)$, then $c = 0$ since time-frequency shifts are independent by Lemma 3.2.5. Note that the sum converges unconditionally with respect to the weak operator topology and maps M^1-functions to M^∞-distributions by Lemma 3.2.4.

Take an arbitrary $f \in M^1(\mathbb{R}^d)$ of the form (3.3), then for all $h \in M^1(\mathbb{R}^d)$ we have

$$\begin{aligned}\Big\langle \sum_{\mu\in\Lambda^\circ} c_\mu \pi(\mu) f, h \Big\rangle &= \sum_{\mu\in\Lambda^\circ}\sum_{\lambda\in F} \langle a_\lambda c_\mu \pi(\mu)\pi(\lambda) g, h\rangle \\ &= \sum_{\lambda\in F} a_\lambda \sum_{\mu\in\Lambda^\circ} \langle c_\mu \pi(\mu) g, \pi(\lambda)^* h\rangle \\ &= \sum_{\lambda\in F} a_\lambda \Big\langle \sum_{\mu\in\Lambda^\circ} c_\mu \pi(\mu) g, \pi(\lambda)^* h \Big\rangle = 0,\end{aligned}$$

where we used the fact that time-frequency shifts with respect to Λ commute with time-frequency shifts of the adjoint lattice Λ°.

Consequently,

$$\sum_{\mu\in\Lambda^\circ} c_\mu \pi(\mu) = 0$$

on a dense subset of $M^1(\mathbb{R}^d)$. As $\sum_{\mu\in\Lambda^\circ} c_\mu\pi(\mu)$ is bounded from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$, the operator vanishes on all of $M^1(\mathbb{R}^d)$.

Now, Lemma 3.2.5 implies that $c = 0$ and therefore D_{g,Λ° is injective.

This concludes the first chain of implications. For the second, we only need to prove the following.

(ii)$\Rightarrow$(vii): If $S_{g,\Lambda} = D_{g,\Lambda} C_{g,\Lambda}$ is bijective on $M^1(\mathbb{R}^d)$, then $D_{g,\Lambda}$ must be surjective from $\ell^1(\Lambda)$ onto $M^1(\mathbb{R}^d)$.

We continue with the third chain:

(i)$\Leftrightarrow$(xiv): This is Theorem 2.4.1.

(xiv)$\Rightarrow$(xi),(xii): Recall that the Gramian operator G_{g,Λ° can be interpreted as a twisted convolution operator C_a for $a_\mu := \langle g, \pi(\mu) g\rangle \in \ell^1(\Lambda^\circ)$. Then $G_{g,\Lambda^\circ} = C_a$ is invertible on $\ell^2(\Lambda^\circ)$ by assumption, hence invertible on all $\ell^p(\Lambda^\circ)$ for $1 \le p \le \infty$ by Theorem 3.2.2.

(xi)$\Leftrightarrow$(xii): Again, the operator $G_{g,\Lambda^\circ} : \ell^1(\Lambda^\circ) \to \ell^1(\Lambda^\circ)$ is invertible if and only if its adjoint $G_{g,\Lambda^\circ} : \ell^\infty(\Lambda^\circ) \to \ell^\infty(\Lambda^\circ)$ is invertible.

(xii)$\Rightarrow$(xiii): Obviously, an invertible operator must be injective.

(xiii)$\Rightarrow$(viii): If the Gramian $G_{g,\Lambda^\circ} = C_{g,\Lambda^\circ} D_{g,\Lambda^\circ}$ is bijective on $\ell^\infty(\Lambda^\circ)$, then D_{g,Λ° must be injective from $\ell^\infty(\Lambda^\circ)$ to $M^\infty(\mathbb{R}^d)$

This concludes the third chain of implications. It remains to show the last chain from *(viii)* to *(i)*.

(viii)$\Leftrightarrow$(ix): Again, the operator $C_{g,\Lambda^\circ} : M^1(\mathbb{R}^d) \to \ell^1(\Lambda^\circ)$ has dense range, if and only if its adjoint $D_{g,\Lambda^\circ} : \ell^\infty(\Lambda^\circ) \to M^\infty(\mathbb{R}^d)$ is injective.

(ix)$\Rightarrow$(x): By assumption $C_{g,\Lambda^\circ}(M^1(\mathbb{R}^d))$ is dense in $\ell^1(\Lambda^\circ)$. Hence for every $1 > \varepsilon > 0$, there exists a $\varphi \in M^1(\mathbb{R}^d)$ such that

$$\|C_{g,\Lambda^\circ}\varphi - e_0\|_{\ell^1} < \varepsilon$$

where $e_0(\mu) := \delta_{\mu,0}$. Explicitly, this means

$$\sum_{\mu \in \Lambda^\circ} |\langle \varphi, \pi(\mu)g \rangle - \delta_{\mu,0}| = |\langle \varphi, g \rangle - 1| + \sum_{\mu \in \Lambda^\circ \setminus \{0\}} |\langle \varphi, \pi(\mu)g \rangle| < \varepsilon.$$

We define the *cross-Gramian* Φ by

$$\Phi_{\mu,\nu} := \langle \pi(\nu)\varphi, \pi(\mu)g \rangle$$

for $\mu, \nu \in \Lambda^\circ$, then

$$\begin{aligned}
\|\Phi - I_{\ell^1}\|_{\ell^1 \to \ell^1} &\leq \sup_{\nu \in \Lambda^\circ} \sum_{\mu \in \Lambda^\circ} |\Phi_{\mu,\nu} - \delta_{\mu,\nu}| \\
&= |\langle \varphi, g \rangle - 1| + \sup_{\nu \in \Lambda^\circ} \sum_{\mu \in \Lambda^\circ \setminus \{\nu\}} |\langle \pi(\nu)\varphi, \pi(\mu)g \rangle| \\
&= |\langle \varphi, g \rangle - 1| + \sum_{\mu \in \Lambda^\circ \setminus \{0\}} |\langle \varphi, \pi(\mu)g \rangle| < \varepsilon < 1.
\end{aligned}$$

Consequently, the cross-Gramian is invertible on $\ell^1(\Lambda^\circ)$. (Its inverse is given by the Neumann series $\Phi^{-1} = \sum_{k=0}^{\infty} (I_{\ell^1} - \Phi)^k$.)

Now, let $a \in \ell^1(\Lambda^\circ)$ be arbitrary. Since Φ is invertible on $\ell^1(\Lambda^\circ)$, there exists a sequence $c \in \ell^1(\Lambda^\circ)$ such that $\Phi c = a$. Set

$$f := \sum_{\nu \in \Lambda^\circ} c_\nu \pi(\nu)\varphi = D_{\varphi,\Lambda^\circ} c,$$

then $f \in M^1(\mathbb{R}^d)$ by Proposition 3.1.4 since $\varphi \in M^1(\mathbb{R}^d)$ and $c \in \ell^1(\Lambda^\circ)$. Moreover, we have

$$\begin{aligned}
(C_{g,\Lambda^\circ} f)(\mu) &= \langle f, \pi(\mu)g \rangle = \langle D_{\varphi,\Lambda^\circ} c, \pi(\mu)g \rangle \\
&= \langle c, C_{\varphi,\Lambda^\circ}(\pi(\mu)g) \rangle = \sum_{\nu \in \Lambda^\circ} c_\nu \overline{\langle \pi(\mu)g, \pi(\nu)\varphi \rangle} = (\Phi c)(\nu) = a_\nu.
\end{aligned}$$

Hence, C_{g,Λ° is surjective from $M^1(\mathbb{R}^d)$ to $\ell^1(\Lambda^\circ)$.

(x) $\Rightarrow$ *(i)*: Since $C_{g,\Lambda}$ is surjective from $M^1(\mathbb{R}^d)$ to $\ell^1(\Lambda^\circ)$, there exists a function $\gamma \in M^1(\mathbb{R}^d)$ such that $C_{g,\Lambda^\circ}\gamma = e_0$ where e_0 is defined as $e_0(\mu) := \mathrm{vol}(\Lambda)\delta_{\mu,0}$ for all $\mu \in \Lambda^\circ$. In other words, the biorthogonality relations

$$C_{g,\Lambda^\circ}\gamma(\mu) = \langle \gamma, \pi(\mu)g \rangle = \mathrm{vol}(\Lambda)\delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ$$

are satisfied.

Since both g and γ are in $M^1(\mathbb{R}^d)$, their respective Gabor systems are Bessel sequences by Lemma 2.2.1. Now, the duality theory (Theorem 2.4.1) implies that $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. $\square$

4. Characterizations for Separable Lattices

The duality theory of Chapter 2 is a fascinating result that reveals the deep structural connection between the Gabor system $\mathcal{G}(g,\Lambda)$ and its adjoint system $\mathcal{G}(g,\Lambda^\circ)$. In combination with the additional structure of a separable lattice, this yields numerous characterizations, which have seen several applications.

Among them is the well-known criterion of Ron and Shen [42], which follows from the duality theory by an application of Plancherel's theorem. A modified version of Ron-Shen featuring the Zak transform is used extensively in Janssen's work [34, 35].

For rational lattices, i.e., a rectangular lattices with rational redundancy, the criterion of Ron and Shen can be further periodized. After an application of Plancherel's theorem, this yields the characterizations for Gabor frames of Zeevi and Zibulski [51, 50] via a family of finite-dimensional matrices.

4.1. The Criterion of Ron and Shen

In this and the following section, we consider separable lattices of the form $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ for $\alpha,\beta > 0$. Observe that the adjoint of such a *rectangular lattice* is

$$\Lambda^\circ = \mathcal{I}\begin{pmatrix} \frac{1}{\alpha}I_d & 0 \\ 0 & \frac{1}{\beta}I_d \end{pmatrix}\mathbb{Z}^{2d} = \frac{1}{\beta}\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d,$$

i.e., again a rectangular lattice.

For convenience, we denote $\mathcal{G}(g,\alpha,\beta) := \mathcal{G}(g,\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ and analogously the corresponding frame operators by $S_{g,\alpha,\beta}, C_{g,\alpha,\beta}, D_{g,\alpha,\beta}$.

Definition 4.1.1. Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha,\beta > 0$ be a separable lattice. The *pre-Gramian matrix* $P(x)$ is defined by

$$P(x)_{j,k} = \overline{g}\Big(x + \alpha j - \frac{k}{\beta}\Big) \quad \forall j,k \in \mathbb{Z}^d$$

and the *Ron-Shen matrix* $R(x) := P(x)^* P(x)$ has the entries

$$R(x)_{k,l} = \sum_{j\in\mathbb{Z}^d} g\Big(x + \alpha j - \frac{k}{\beta}\Big)\overline{g}\Big(x + \alpha j - \frac{l}{\beta}\Big) \quad \forall k,l \in \mathbb{Z}^d.$$

Theorem 4.1.2. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha, \beta > 0$ be a separable lattice. Then the following are equivalent:*

(i) $\mathcal{G}(g, \alpha, \beta)$ *is a frame for* $L^2(\mathbb{R}^d)$.

(ii) $\mathcal{G}(g, \alpha, \beta)$ *is a Bessel sequence and there exists a dual window* $\gamma \in L^2(\mathbb{R}^d)$ *such that* $\mathcal{G}(\gamma, \alpha, \beta)$ *is a Bessel sequence and*

$$\langle \gamma, M_{\frac{l}{\alpha}} T_{\frac{k}{\beta}} g \rangle = (\alpha\beta)^d \delta_{k,0}\delta_{l,0} \quad \forall k, l \in \mathbb{Z}^d, \tag{4.1}$$

where $\delta_{k,0}$ *denotes the Kronecker delta.*

(iii) $\mathcal{G}(g, \frac{1}{\beta}, \frac{1}{\alpha})$ *is a Riesz sequence for* $L^2(\mathbb{R}^d)$.

(iv) There exist positive constants $A, B > 0$, *such that for all* $c \in \ell^2(\mathbb{Z}^d)$ *and almost all* $x \in \mathbb{R}^d$

$$A\|c\|_{\ell^2}^2 \leq \sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k \overline{g}\Big(x + \alpha j - \frac{k}{\beta}\Big)\Big|^2 \leq B\|c\|_{\ell^2}^2. \tag{4.2}$$

(v) There exist positive constants $A, B > 0$, *such that the spectrum of almost every Ron-Shen matrix is contained in the interval* $[A, B]$. *This means*

$$\sigma(R(x)) \subseteq [A, B] \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

(vi) The set of pre-Gramians $\{P(x)\}$ *is uniformly bounded on* $\ell^2(\mathbb{Z}^d)$ *and has a set of uniformly bounded left-inverses. This means that there exist* $\Gamma(x)$ *for almost all* $x \in \mathbb{R}^d$, *such that*

$$\begin{aligned} \Gamma(x)P(x) &= I_{\ell^2(\mathbb{Z}^d)} \quad &&\text{for a.e. } x \in \mathbb{R}^d, \\ \|\Gamma(x)\| &\leq C &&\text{for a.e. } x \in \mathbb{R}^d, \end{aligned}$$

where $I_{\ell^2(\mathbb{Z}^d)}$ *denotes the identity on* $\ell^2(\mathbb{Z}^d)$.

The first three characterizations is just a special case of Theorem 2.4.1. The conditions *(iv)* and *(v)* were discovered by Ron and Shen in their fundamental work [42]. The last criterion is due to Gröchenig and Stöckler [27].

Remark 4.1.3. The results of Ron and Shen in [42] are more general. They considered *separable lattices* of the form $P\mathbb{Z}^d \times Q\mathbb{Z}^d$ with invertible matrices $P, Q \in GL(\mathbb{R}^d)$. In this setting, the results of Theorem 4.1 hold with the appropriate modifications. This means replacing the scalar-multiplication with $\alpha, \beta, 1/\alpha, 1/\beta$ by the matrix-vector multiplication with P, Q, P^{-1}, Q^{-1} respectively.

The proofs remain identical, but the resulting Fourier series are now periodic on the fundamental domain of the lattice $P\mathbb{Z}^d$ instead of $Q_\alpha = [0, \alpha]^d$.

The proof of the equivalences *(iii)* ⇔ *(iv)* ⇔ *(v)* follows Gröchenig's survey article [23]. The proof of the last criterion is the original by Gröchenig and Stöckler [27].

Proof of Theorem 4.1.2. *(i)* ⇔ *(ii)* ⇔ *(iii)*: This is Theorem 2.4.1.

(iv)⇔*(v)*: For all sequences $c \in \ell^2(\mathbb{Z}^d)$, we have

$$\sum_{j\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d} c_k\overline{g}\Big(x+\alpha j-\frac{k}{\beta}\Big)\Big|^2 = \langle P(x)c, P(x)c\rangle = \langle R(x)c, c\rangle.$$

Hence, inequality (4.2) becomes

$$A\|c\|_{\ell^2}^2 \le \langle R(x)c, c\rangle \le B\|c\|_{\ell^2}^2 \quad \forall c \in \ell^2(\mathbb{Z}^d),$$

for almost all $x \in \mathbb{R}^d$, which is equivalent to $\sigma(R(x)) \subseteq [A, B]$ for almost all $x \in \mathbb{R}^d$.

(iv)⇒*(iii)*: We need to show that $D_{g,\frac{1}{\beta},\frac{1}{\alpha}} : \ell^2(\mathbb{Z}^{2d}) \to L^2(\mathbb{R}^d)$ is bounded from above and below.

Due to the separable structure of the lattice, we can rewrite the synthesis operator as

$$(D_{g,\frac{1}{\beta},\frac{1}{\alpha}}c)(x) = \sum_{k,l\in\mathbb{Z}^d} c_{k,l}M_{\frac{l}{\alpha}}T_{\frac{k}{\beta}}g(x) = \sum_{k\in\mathbb{Z}^d}\Big(\sum_{l\in\mathbb{Z}^d} c_{k,l}e^{2\pi i\frac{l}{\alpha}\cdot x}\Big)T_{\frac{k}{\beta}}g(x),$$

where $p_k(x) := \sum_{l\in\mathbb{Z}^d} c_{k,l}e^{2\pi i\frac{l}{\alpha}\cdot x}$ is a periodic Fourier series on $Q_\alpha := [0, \alpha]^d$ with square-summable coefficients $(c_{k,l})_{l\in\mathbb{Z}^d}$. Consequently, the Fourier series p_k is in $L^2(Q_\alpha)$ and we have

$$\int_{Q_\alpha} |p_k(x)|^2\,\mathrm{d}x = \alpha^d\sum_{l\in\mathbb{Z}^d}|c_{k,l}|^2$$

by Plancherel's theorem.

We periodize the L^2-norm of the synthesis operator with respect to Q_α as follows:

$$\begin{aligned}
\|D_{g,\frac{1}{\beta},\frac{1}{\alpha}}c\|_{L^2}^2 &= \Big\|\sum_{k\in\mathbb{Z}^d} p_k\cdot T_{\frac{k}{\beta}}g\Big\|_{L^2}^2\\
&= \int_{\mathbb{R}^d}\Big|\sum_{k\in\mathbb{Z}^d} p_k(x)g\Big(x-\frac{k}{\beta}\Big)\Big|^2\,\mathrm{d}x\\
&= \int_{Q_\alpha}\sum_{j\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d} p_k(x)g\Big(x+\alpha j-\frac{k}{\beta}\Big)\Big|^2\,\mathrm{d}x.
\end{aligned}$$

Observe that $\big(p_k(x)\big)_{k\in\mathbb{Z}^d}$ is in $\ell^2(\mathbb{Z}^d)$ for almost all $x\in\mathbb{R}^d$ since

$$\int_{Q_\alpha}\sum_{k\in\mathbb{Z}^d}|p_k(x)|^2\,\mathrm{d}x=\sum_{k\in\mathbb{Z}^d}\int_{Q_\alpha}|p_k(x)|^2\,\mathrm{d}x=\alpha^d\sum_{k,l\in\mathbb{Z}^d}|c_{k,l}|^2<\infty.$$

Hence, we can use assumption (4.2) with respect to the sequence $\big(p_k(x)\big)$ and obtain for the lower bound

$$\begin{aligned}\|D_{g,\frac{1}{\beta},\frac{1}{\alpha}}c\|_{L^2}^2&\geq\int_{Q_\alpha}A\sum_{k\in\mathbb{Z}^d}|p_k(x)|^2\,\mathrm{d}x\\&=A\sum_{k\in\mathbb{Z}^d}\int_{Q_\alpha}|p_k(x)|^2\,\mathrm{d}x\\&=\alpha^dA\sum_{k,l\in\mathbb{Z}^d}|c_{k,l}|^2=\alpha^dA\|c\|_{\ell^2}^2\end{aligned}$$

for all $c\in\ell^2(\mathbb{Z}^{2d})$. The upper bound follows analogously.

(iii)⇒(iv): By assumption, the synthesis operator $D_{g,\frac{1}{\beta},\frac{1}{\alpha}}$ is bounded from above and below. This means we have

$$A\|c\|_{\ell^2}^2\leq\|D_{g,\frac{1}{\beta},\frac{1}{\alpha}}c\|_{L^2}^2\leq B\|c\|_{\ell^2}^2.$$

for all $c\in\ell^2(\mathbb{Z}^{2d})$.

For $a,b\in\ell^2(\mathbb{Z}^d)$, define the sequence c by $c_{k,l}:=a_kb_l$. Then $c\in\ell^2(\mathbb{Z}^{2d})$ since

$$\|c\|_{\ell^2(\mathbb{Z}^{2d})}^2=\sum_{k,l\in Z^d}|c_{k,l}|^2=\sum_{k,l\in\mathbb{Z}^d}|a_k|^2|b_l|^2=\|a\|_{\ell^2(\mathbb{Z}^d)}^2\|b\|_{\ell^2(\mathbb{Z}^d)}^2.$$

Every $p\in L^2(Q_\alpha)$ can be written as Fourier series $p(x)=\sum_{l\in\mathbb{Z}^d}b_le^{2\pi il\cdot\frac{x}{\alpha}}$ with coefficients $b\in\ell^2(\mathbb{Z}^d)$. Hence we obtain for arbitrary $a\in\ell^2(\mathbb{Z}^d)$ and $p\in L^2(Q_\alpha)$

$$\begin{aligned}\frac{A}{\alpha^d}\|a\|_{\ell^2(\mathbb{Z}^d)}^2\int_{Q_\alpha}|p(x)|^2\,\mathrm{d}x&=A\|a\|_{\ell^2(\mathbb{Z}^d)}^2\|b\|_{\ell^2(\mathbb{Z}^d)}^2=A\|c\|_{\ell^2(\mathbb{Z}^{2d})}^2\\&\leq\|D_{g,\frac{1}{\beta},\frac{1}{\alpha}}c\|_{L^2}^2=\Big\|\sum_{k,l\in\mathbb{Z}^d}a_kb_lM_{\frac{l}{\alpha}}T_{\frac{k}{\beta}}g\Big\|_{L^2}^2\\&=\int_{\mathbb{R}^d}\Big|\sum_{k\in\mathbb{Z}^d}a_k\sum_{l\in\mathbb{Z}^d}b_le^{2\pi il\cdot\frac{x}{\alpha}}g\Big(x-\frac{k}{\beta}\Big)\Big|^2\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}|p(x)|^2\Big|\sum_{k\in\mathbb{Z}^d}a_kg\Big(x-\frac{k}{\beta}\Big)\Big|^2\,\mathrm{d}x\qquad(4.3)\\&=\int_{Q_\alpha}\sum_{j\in\mathbb{Z}^d}|p(x+\alpha j)|^2\Big|\sum_{k\in\mathbb{Z}^d}a_kg\Big(x+\alpha j-\frac{k}{\beta}\Big)\Big|^2\,\mathrm{d}x\end{aligned}$$

$$= \int_{Q_\alpha} |p(x)|^2 \sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} a_k g\Big(x+\alpha j - \frac{k}{\beta}\Big)\Big|^2 \,\mathrm{d}x.$$

Since $L^2(Q_\alpha)$ contains all characteristic functions of measurable subsets in Q_α, (4.3) implies

$$\frac{A}{\alpha^d}\|a\|_{\ell^2}^2 \le \sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} a_k g\Big(x+\alpha j - \frac{k}{\beta}\Big)\Big|^2 \quad \text{for a.e. } x\in\mathbb{R}^d$$

for all $a \in \ell^2(\mathbb{Z}^d)$. The upper bound follows analogously.

(v)⇒(vi): Suppose that the spectrum of almost all $R(x)$ is contained in the interval $[A,B]$ for some positive constants $A, B > 0$. Then the set of pre-Gramians is uniformly bounded by $B^{\frac{1}{2}}$ since $R(x) = P(x)^*P(x)$.

As $R(x)$ is invertible, we may define $\Gamma(x) := R(x)^{-1}P(x)^*$. Then

$$\Gamma(x)P(x) = I_{\ell^2(\mathbb{Z}^d)}$$

and

$$\|\Gamma(x)\| \le \|R(x)^{-1}\|\|P(x)\| \le A^{-1}B^{\frac{1}{2}}.$$

(vi)⇒(iv): By assumption, $P(x)$ possesses a uniformly bounded left inverse for almost all $x\in\mathbb{R}^d$. This implies

$$\begin{aligned}\|c\|_{\ell^2}^2 &= \|\Gamma(x)P(x)c\|_{\ell^2}^2 \le \|\Gamma(x)\|^2\|P(x)c\|_{\ell^2}^2 \\ &\le C^2\langle R(x)c,c\rangle \le C^2\|P(x)\|^2\|c\|_{\ell^2}^2\end{aligned} \tag{4.4}$$

for all $c\in\ell^2(\mathbb{Z}^{2d})$ and almost all $x\in\mathbb{R}^d$. Since $P(x)$ is also uniformly bounded for almost all $x\in\mathbb{R}^d$, inequality (4.4) is equivalent to the Ron-Shen criterion (4.2). □

Remark 4.1.4 (Frame Bounds)**.** Remark 2.4.2 indicates, how the frame bounds of $\mathcal{G}(g,\alpha,\beta)$ are related to the Riesz bounds of the dual system $\mathcal{G}(g,\frac{1}{\beta},\frac{1}{\alpha})$. Furthermore, the proof of Theorem 4.1.2 showed how those are connected to the bounds of the Ron-Shen criterion (4.2).

In summary, the following statements concerning the frame, Riesz, and Ron-Shen bounds are equivalent:

(i) $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds $A, B > 0$.

(ii) $\mathcal{G}(g,\frac{1}{\beta},\frac{1}{\alpha})$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ with bounds $(\alpha\beta)^dA, (\alpha\beta)^dB > 0$.

(iii) The Ron-Shen criterion (4.2) is satisfied with respect to $\beta^dA, \beta^dB > 0$.

We isolate the following Ron-Shen characterization for Bessel sequences from Theorem 4.1.2.

Lemma 4.1.5. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha, \beta > 0$ be a separable lattice. Then $\mathcal{G}(g, \alpha, \beta)$ is a Bessel sequence if and only if the upper inequality of* (4.2) *holds.*

Proof. The proof of this claim is implicit in the proof of equivalence *(iii)*$\Leftrightarrow$*(iv)* in Theorem 4.1.2. $\square$

At first glance, the Ron-Shen criterion (4.2) does not look like a big improvement to the frame inequality. In both cases, the difficulty lies with the lower inequality, which corresponds the invertibility of the related operator.

However, there are instances where the Ron-Shen criterion is fairly easy to check. If the window has compact support, the Ron-Shen matrices $R(x)$ are sparse, which improves inequality (4.2) considerably.

In the setting of Daubechies, Landau and Landau's "painless non-orthonormal expansions" [12], the Ron-Shen matrix is even of diagonal form and yields the following characterization for Gabor frames.

Theorem 4.1.6 (Painless Non-Orthonormal Expansions)**.** *Suppose $g \in L^\infty(\mathbb{R}^d)$ with $\operatorname{supp} g \subseteq [0, L]^d$. If $\alpha \leq L$ and $\beta \leq \frac{1}{L}$, then $\mathcal{G}(g, \alpha, \beta)$ is a frame if and only if*

$$0 < \operatorname*{ess\,inf}_{x \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} |g(x - \alpha k)|^2.$$

We follow the proof given in Gröchenig's survey article [23].

Proof. By assumption, we have $\frac{1}{\beta} \leq L$ and therefore, the common support of $T_{\frac{k}{\beta}} g$ and $T_{\frac{l}{\beta}} \overline{g}$ is at most a null set for $k \neq l$. Since the values on a set of measure zero is negligible for L^2-functions, we may assume without loss of generality that the supports are disjoint. Consequently,

$$\begin{aligned} R(x)_{k,l} &= \sum_{j \in \mathbb{Z}^d} g\Big(x + j\alpha - \frac{k}{\beta}\Big) \overline{g}\Big(x + j\alpha - \frac{l}{\beta}\Big) \\ &= \sum_{j \in \mathbb{Z}^d} \Big| g\Big(x + j\alpha - \frac{k}{\beta}\Big)\Big|^2 \delta_{k,l}. \end{aligned}$$

This diagonal form implies that the operator norm of $R(x)$ is uniformly bounded for almost all $x \in \mathbb{R}^d$ by

$$B := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \sum_{j \in \mathbb{Z}^d} |g(x - \alpha j)|^2 \leq \left\lceil \frac{L}{\alpha} \right\rceil^d \|g\|_{L^\infty},$$

where $\lceil \frac{L}{\alpha} \rceil$ denotes the closest integer greater or equal to $\frac{L}{\alpha}$.

If $A := \operatorname{ess\,inf}_{x\in\mathbb{R}^d} \sum_{j\in\mathbb{Z}^d} |g(x-\alpha j)|^2 > 0$, then

$$\langle R(x)c, c\rangle \geq A\|c\|_{\ell^2}^2$$

for all $c \in \ell^2(\mathbb{Z}^d)$ and almost all $x \in \mathbb{R}^d$. Hence, the spectrum of almost all $R(x)$ is contained in $[A, B]$ and therefore, $\mathcal{G}(g, \alpha, \beta)$ is a frame by Theorem 4.1.2 *(v)*.

Conversely, if $\mathcal{G}(g, \alpha, \beta)$ is a frame, then the spectrum of almost all $R(x)$ is contained in $[A, B]$ for some $A, B > 0$. Due to the diagonal form of $R(x)$, we find that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^d} \sum_{j\in\mathbb{Z}^d} |g(x-\alpha j)|^2 \geq A > 0.$$

□

Remark 4.1.7. Under the assumptions of Theorem 4.1.6, one can show that the frame operator is the multiplication operator

$$S_{g,\alpha,\beta}f(x) = \Big(\beta^{-d} \sum_{k\in\mathbb{Z}^d} |g(x-\alpha k)|^2\Big) f(x)$$

via Janssen's representation of the frame operator (2.9).

Evidently, the canonical dual window is therefore

$$\gamma(x) = \beta^d \Big(\sum_{k\in\mathbb{Z}^d} |g(x-\alpha k)|^2\Big)^{-1} g(x)$$

with the same smoothness and support as g.

The following result by Gröchenig and Janssen [24] states that for "nice" windows and irrational lattices it suffices to check the invertibility of the Ron-Shen matrix at just one point. Recall that $Q_\alpha := [0, \alpha]^d$ denotes the d-dimensional cube of side length $\alpha > 0$.

Theorem 4.1.8 (Gröchenig-Janssen [24])**.** *Assume that $g \in W(\mathcal{C}, L^1)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ is a separable lattice with $\alpha\beta \notin \mathbb{Q}$. Then the following are equivalent:*

(i) *$\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$.*

(ii) *There exists a single $x_0 \in Q_\alpha$ such that the Ron-Shen matrix $R(x_0)$ is invertible on $\ell^2(\mathbb{Z}^d)$.*

We only give a sketch of the proof. For the full proof, we refer to the original article by Gröchenig and Janssen [24].

Proof (Sketch). Since $g \in W(\mathcal{C}, L^1)$, the upper bound of inequality (4.2) is always satisfied by Lemma 4.1.5 and [21, Proposition 6.2.2].

The invertibility at a fixed point $x_0 \in Q_\alpha$ implies the operator inequality

$$AI_{\ell^2} \leq R(x_0) \leq BI_{\ell^2}$$

for suitable constants $A, B > 0$.

From the definition of the Ron-Shen matrix it is evident that a time-shift of the Ron-Shen matrix by $\frac{n}{\beta}$ results in an index shift by n:

$$R\Big(x_0 - \frac{n}{\beta}\Big)_{k,l} = \sum_{j \in \mathbb{Z}^d} g\Big(x_0 - \frac{n}{\beta} + \alpha j - \frac{k}{\beta}\Big)\overline{g}\Big(x_0 - \frac{n}{\beta} + \alpha j - \frac{l}{\beta}\Big) = R(x_0)_{k+n,l+n}.$$

Clearly, this symmetric index shift does not change the spectrum. Hence,

$$AI_{\ell^2} \leq R\Big(x - \frac{n}{\beta}\Big) \leq BI_{\ell^2} \qquad \forall n \in \mathbb{Z}^d. \tag{4.5}$$

Since $\alpha\beta$ is irrational, the set $\{x = x_0 - \frac{n}{\beta} \pmod{\alpha} : n \in \mathbb{Z}^d\}$ is dense in $Q_\alpha = [0, \alpha]^d$ by Kronecker's theorem [38]. Consequently, the Ron-Shen inequality holds on a dense subset of Q_α.

By a continuity argument, inequality (4.5) can be extended all of Q_α. For this, one needs to show that the map $x \mapsto R(x)$ is uniformly continuous with respect to the operator norm on $\ell^2(\mathbb{Z}^d)$. (see Gröchenig and Janssen [24, Lemma 2]) $\square$

We return to general rectangular lattices $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha, \beta > 0$. The goal of the remainder of this section is to expand Theorem 4.1.2 by an additional characterization involving the Zak transform.

The *Zak transform* with respect to the parameter $\alpha > 0$ is defined by

$$Z_\alpha f(x, \xi) := \sum_{k \in \mathbb{Z}^d} f(x - \alpha k) e^{2\pi i \alpha k \cdot \xi}.$$

We start with proving a few basic properties of the Zak transform.

Lemma 4.1.9 (Quasiperiodicity of the Zak transform)**.** *Let $\alpha > 0$, then for $n \in \mathbb{Z}^d$ we have for all $x, \xi \in \mathbb{R}^d$*

$$Z_\alpha f\Big(x, \xi + \frac{n}{\alpha}\Big) = Z_\alpha f(x, \xi)$$

and

$$Z_\alpha f(x + \alpha n, \xi) = e^{2\pi i \alpha n \cdot \xi} Z_\alpha f(x, \xi).$$

In particular, the Zak transform $Z_\alpha f$ is completely determined by its values on $Q_\alpha \times Q_{\frac{1}{\alpha}} \subseteq \mathbb{R}^{2d}$.

Proof. Both claims follow from a straight forward index shift in the summation of the Zak transform. $\square$

Lemma 4.1.10. *Let $\alpha > 0$, then the following holds:*

(a) If $f \in L^2(\mathbb{R}^d)$, then $Z_\alpha f$ is defined almost everywhere on $\mathbb{R}^{2d}$. Furthermore, $Z_\alpha f(x, \,.\,) \in L^2(Q_{\frac{1}{\alpha}}, \mathrm{d}\xi)$ for almost every $x \in \mathbb{R}^d$.

(b) If $f \in W(L^\infty, L^1)$, then $Z_\alpha f \in L^\infty(\mathbb{R}^{2d})$.

(c) If $f \in W(\mathcal{C}, L^1)$, then $Z_\alpha f$ is continuous on $\mathbb{R}^{2d}$.

Proof. *(a)* The periodization of the L^2-norm of f yields

$$\int_{Q_\alpha} \sum_{k \in \mathbb{Z}^d} |f(x - \alpha k)|^2 \, \mathrm{d}x = \int_{\mathbb{R}^d} |f(x)|^2 < \infty.$$

This implies that the sequences $(f(x-\alpha k))_{k\in\mathbb{Z}^d}$ are in $\ell^2(\mathbb{Z}^d)$ for almost all $x \in \mathbb{R}^d$. Consequently, $Z_\alpha f(x, \,.\,)$ is a Fourier series with square-summable coefficients and therefore in $L^2(Q_{\frac{1}{\alpha}}, \mathrm{d}\xi)$. In particular, $Z_\alpha f$ is defined almost everywhere on $\mathbb{R}^{2d}$.

(b) This claim follows from a straight forward estimate:

$$\begin{aligned}
\|Z_\alpha f\|_{L^\infty} &\leq \operatorname*{ess\,sup}_{x \in Q_\alpha} \sum_{k \in \mathbb{Z}^d} |f(x - \alpha k)| \leq \sum_{k \in \mathbb{Z}^d} \operatorname*{ess\,sup}_{x \in Q_\alpha} |f(x - \alpha k)| \\
&= \sum_{k \in \mathbb{Z}^d} \|f \cdot T_{\alpha k} \chi_{Q_\alpha}\|_{L^\infty} \lesssim \|f\|_{W(L^\infty, L^1)}.
\end{aligned}$$

(c) Let $(x, \xi) \in \mathbb{R}^{2d}$ and $(x_n, \xi_n)_{n\in\mathbb{N}} \in \mathbb{R}^{2d}$ be a sequence converging to (x, ξ), where w.l.o.g. the x-coordinates are already sufficiently close, i.e., $x_n \in x + \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right]^d$ for all $n \in \mathbb{N}$.

Then for all $k \in \mathbb{Z}^d$ and for all $n \in \mathbb{N}$, we have

$$|f(x_n - \alpha k) e^{2\pi i \alpha k \cdot \xi_n}| \leq \|f \cdot T_{\alpha k} \chi_{x + [-\frac{\alpha}{2}, \frac{\alpha}{2}]^d}\|_{L^\infty} =: g(k).$$

Since $f \in W(L^\infty, L^1)$, the sequence $(g(k))_{k\in\mathbb{Z}^d}$ is an absolute summable majorizing sequence. Now, the continuity in (x, ξ) follows from Lebesgue's dominated convergence theorem for the point measure the continuity of f. $\square$

With the help of the Zak transform, we expand Theorem 4.1.2 by an additional characterization. We denote the d-dimensional torus by $\mathbb{T}^d := [0, 1]^d$.

Theorem 4.1.11. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha, \beta > 0$ be a separable lattice. Then the following are equivalent:*

(i) $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$.

(vii) $\{Z_{\frac{1}{\beta}} g(x + \alpha j, \beta\,.\,) : j \in \mathbb{Z}^d\}$ is a frame for $L^2(\mathbb{T}^d)$ for almost all $x \in \mathbb{R}^d$ with frame bounds independent of x.

Proof. By Theorem 4.1.2, $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame if and only if there exist positive constants $A, B > 0$, such that

$$A\|c\|_{\ell^2}^2 \leq \sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k \overline{g}\Big(x + \alpha j - \frac{k}{\beta}\Big)\Big|^2 \leq B\|c\|_{\ell^2}^2$$

for all $c \in \ell^2(\mathbb{Z}^d)$ and almost all $x \in \mathbb{R}^d$.

We interpret the sum over $k \in \mathbb{Z}^d$ as an inner product of sequences. By periodization of the L^2-norm, it is easy to see that the sequences

$$g_j := \left(g\Big(x + \alpha j - \frac{k}{\beta}\Big)\right)_{k\in\mathbb{Z}^d}$$

are in $\ell^2(\mathbb{Z}^d)$ for all $j \in \mathbb{Z}^d$ and for almost all $x \in \mathbb{R}^d$. Hence, the corresponding Fourier series

$$f(\xi) := \sum_{k\in\mathbb{Z}^d} c_k e^{2\pi i k\cdot\xi},$$

$$Z_{\frac{1}{\beta}} g(x + \alpha j, \beta\xi) = \sum_{k\in\mathbb{Z}^d} g\Big(x + \alpha j - \frac{k}{\beta}\Big)e^{2\pi i k\cdot\xi}$$

are in $L^2(\mathbb{T}^d, \mathrm{d}\xi)$ and Parseval's formula yields

$$\begin{aligned}\sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k \overline{g}\Big(x + \alpha j - \frac{k}{\beta}\Big)\Big|^2 &= \sum_{j\in\mathbb{Z}^d} \Big| \int_{\mathbb{T}^d} f(\xi)\overline{Z_{\frac{1}{\beta}} g(x + \alpha j, \beta\xi)}\,\mathrm{d}\xi\Big|^2 \\ &= \sum_{j\in\mathbb{Z}^d} \Big|\langle f, Z_{\frac{1}{\beta}} g(x + \alpha j, \beta\,.\,)\rangle_{L^2(\mathbb{T}^d)}\Big|^2.\end{aligned}$$

Consequently, $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame if and only if the frame inequality

$$A\|f\|_{L^2(\mathbb{T}^d)}^2 \leq \sum_{j\in\mathbb{Z}^d} \Big|\langle f, Z_{\frac{1}{\beta}} g(x + \alpha j, \beta\,.\,)\rangle_{L^2(\mathbb{T}^d)}\Big|^2 \leq B\|f\|_{L^2(\mathbb{T}^d)}^2$$

holds for all $f \in L^2(\mathbb{T}^d)$ and almost all $x \in \mathbb{R}^d$.

□

4.2. Applications

One major goal in time-frequency analysis is to determine whether $\mathcal{G}(g, \alpha, \beta)$ constitutes a frame. This may be done by applying one of the characterizations of the previous section. Depending on the function and lattice parameters, some characterizations may be more advantageous then others.

Ideally, one would like to determine the *(reduced) frame set* to a given window function $g \in L^2(\mathbb{R}^d)$

$$\mathcal{F}(g) := \{(\alpha, \beta) \in \mathbb{R}_+^2 : \mathcal{G}(g, \alpha, \beta) \text{ is a frame}\},$$

i.e., all pairs of lattice parameters that generate a frame with respect to the window g. The last thirty years have shown that this is a hard problem. Even for the characteristic function of an interval, a rather simple function, this proved to be extremely difficult. The frame set of $\chi_{[0,1]}$ is highly complicated and Janssen's investigation [36] left open several blank spots. Only recently, Dai and Sun [9] devised an algorithm to determine when $\mathcal{G}(\chi_{[0,1]}, \alpha, \beta)$ is a frame.

Previous to the result of Gröchenig and Stöckler [27] for totally positive functions, we knew the frame set for only six functions (not counting trivial modifications such as scaling, dilation and time-frequency shifts). Those functions were the Gaussian $\varphi(x) = e^{-\pi x^2}$, the one-side exponential $g(x) = e^{-x}\chi_{\mathbb{R}_+}(x)$, its Fourier transform $g(x) = (1 + 2\pi i x)^{-1}$, the symmetric exponential $g(x) = e^{-|x|}$, its Fourier transform $g(x) = (1 + 4\pi^2 x^2)^{-1}$, and the hyperbolic secant $g(x) = 2(e^x + e^{-x})^{-1} = (\cosh x)^{-1}$.

The results of Seip-Wallstén [44, 45] and Lyubarskiĭ [41] characterize all time-frequency lattices which generate a Gabor frame for the one-dimensional Gaussian (cf. Theorem 2.5.2). The Gaussian marks the only function, where the *full frame set* is known, i.e., the set of all time-frequency lattices that generate a frame.

The results of the remaining five functions are due to Janssen [34, 35] respectively Janssen and Strohmer [37] and employ the methods of section 4.1.

Proposition 4.2.1 (Janssen [34])**.** *Let $g(x) = e^{-x}\chi_{\mathbb{R}_+}(x)$ or $g(x) = (1+2\pi i x)^{-1}$. Then $\mathcal{G}(g, \alpha, \beta)$ is a frame if and only if $\alpha\beta \leq 1$.*

Proof. If $g(x) = e^{-x}\chi_{\mathbb{R}_+}(x)$, then its Fourier transform is $\hat{g}(\xi) = (1 + 2\pi i \xi)^{-1}$. Therefore, it suffices to prove the claim for the one-sided exponential, since $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame if and only if $\mathcal{G}(\hat{g}, \beta, \alpha)$ is a Gabor frame by Proposition 1.3.2.

By the Theorem 2.5.1, the condition $\alpha\beta \leq 1$ is necessary for $\mathcal{G}(g, \alpha, \beta)$ to be a frame. On the other hand, it is easy to see that

$$\gamma(x) := \beta\, e^x \big(\chi_{[0,\alpha)}(x) - \chi_{[-\alpha,0)}(x)\big)$$

and $g(x) = e^{-x}\chi_{\mathbb{R}_+}(x)$ satisfy the biorthogonality relations (4.1) for $\alpha\beta \leq 1$.

Note that $g, \gamma \in W(L^\infty, L^1)$, hence both $\mathcal{G}(g, \alpha, \beta)$ and $\mathcal{G}(\gamma, \alpha, \beta)$ are Bessel sequences for all $\alpha, \beta > 0$ by [21, Proposition 6.2.2]. Consequently, $\mathcal{G}(g, \alpha, \beta)$ is a frame for all $\alpha\beta \leq 1$ by Theorem 4.1.2 *(ii)*. $\square$

Theorem 4.2.2 (Janssen [35], Janssen-Strohmer [37])**.** *Let $g(x) = e^{-|x|}$, $g(x) = (1 + 4\pi^2x^2)^{-1}$ or $g(x) = 2(e^x + e^{-x})^{-1} = (\cosh x)^{-1}$. Then $\mathcal{G}(g, \alpha, \beta)$ is a frame if and only if $\alpha\beta < 1$.*

Observe that all three functions are in the Wiener amalgam space $W(\mathcal{C}, L^1)$. Consequently, the upper frame inequality is again always satisfied by [21, Proposition 6.2.2]. On the other hand, the amalgam Balian-Low theorem [4, 10, 39] restricts the necessary density condition even further to $\alpha\beta < 1$. In this light, Theorem 4.2.2 states that the necessary conditions on the lattice are already sufficient.

The result for the symmetric exponential $g(x) = e^{-|x|}$ is due to Janssen [35]. Janssen's proof uses the Ron-Shen criterion reformulated by means of the Zak transform (cf. Theorem 4.1.11) and shows the lower inequality directly by long winded computation and case distinctions. This also covers the result for the Poisson kernel $g(x) = (1 + 4\pi^2x^2)^{-1}$ as the Fourier transform of the symmetric exponential.

Janssen and Strohmer [37] proved the result for the hyperbolic secant $g(x) = 2(e^x + e^{-x})^{-1} = (\cosh x)^{-1}$ by relating its Zak transform to that of the Gaussian $\varphi(x) = e^{-\pi x^2}$.

We turn to Gröchenig and Stöckler's result [27] for totally positive functions of finite type. The showed that that the density condition $\alpha\beta < 1$ is necessary and sufficient for finite type $M \geq 2$ by constructing an explicit dual window.

Definition 4.2.3. A real-valued function $g : \mathbb{R} \to \mathbb{R}$ in $L^1(\mathbb{R})$ is called *totally positive of finite type M*, if its Fourier transform admits a factorization of the form

$$\hat{g}(\xi) = Ce^{2\pi i\delta\xi} \prod_{j=1}^{M} (1 + 2\pi i\delta_j\xi)^{-1}, \tag{4.6}$$

where $C > 0$, $M \in \mathbb{N}$, $\delta \in \mathbb{R}$ and $\delta_j \in \mathbb{R} \setminus \{0\}$.

Lemma 4.2.4. *Let g be a totally positive function of finite type $M \geq 2$, then $g \in M^1(\mathbb{R})$.*

Proof. If g is a totally positive function of finite type $M \geq 2$, then its Fourier transform satisfies at least $|\hat{g}(\xi)| \lesssim (1 + |\xi|)^{-2}$. Clearly, $\hat{g}$ is smooth and all its derivatives are in $L^1(\mathbb{R})$ since they have similar decay. Consequently, g decays faster than any polynomial.

The decay of g and $\hat{g}$ is enough such that the embedding estimates of [20] are satisfied, which guarantees $g \in M^1(\mathbb{R})$. $\square$

There are two important consequences of Lemma 4.2.4: First, the upper frame bound is always satisfied for totally positive functions of finite type $M \geq 2$ by Lemma 2.2.1.

Second, M^1-windows cannot generate a frame at critical density by the Amalgam Balian-Low theorem [4, 10, 39]. This implies the necessary density condition $\alpha\beta < 1$ for totally positive functions of type $M \geq 2$.

Gröchenig and Stöckler's [27] result states that this necessary condition is already sufficient.

Theorem 4.2.5 (Gröchenig-Stöckler [27]). *Let $g \in L^2(\mathbb{R}^d)$ be a totally positive function of finite type $M \geq 2$. Then $\mathcal{G}(g, \alpha, \beta)$ is a frame if and only if $\alpha\beta < 1$.*

Note that Theorem 4.2.5 does not hold for totally positive functions of type $M = 1$. Clearly, the one-sided exponential is of this type but generates a frame at critical density by Proposition 4.2.1.

The proof of Theorem 4.2.5 is highly technical and involves constructing a dual window by means of the following lemma.

Lemma 4.2.6 (Gröchenig-Stöckler [27]). *Assume that there exists a (Lebesgue measurable) vector-valued function $\sigma(x)$ from $\mathbb{R}^d$ to $\ell^2(\mathbb{Z}^d)$ with period α, such that*

$$\sum_{j \in \mathbb{Z}^d} \sigma_j(x) \overline{g}\Big(x + \alpha j - \frac{k}{\beta}\Big) = \delta_{k,0} \quad \text{for a.e. } x \in \mathbb{R}^d.$$

If $\mathcal{G}(g, \alpha, \beta)$ is a Bessel sequence and $\sum_{j \in \mathbb{Z}^d} \|\sigma_j\|_{L^\infty(Q_\alpha)} < \infty$, then $\mathcal{G}(g, \alpha, \beta)$ is a frame. Moreover, a dual window is given by

$$\gamma(x) = \beta^d \sum_{j \in \mathbb{Z}^d} \sigma_j(x) \chi_{Q_\alpha}(x - \alpha j) \quad \forall x \in \mathbb{R}^d,$$

where χ_{Q_α} denotes the characteristic function of the set $Q_\alpha := [0, \alpha]^d$.

Proof. Since $\sum_{j \in \mathbb{Z}^d} \|\sigma_j\|_{L^\infty(Q_\alpha)} < \infty$, the dual window γ is in the Wiener space $W(L^\infty, L^1)$. Hence, $\mathcal{G}(\gamma, \alpha, \beta)$ is a Bessel sequence by [21, Proposition 6.2.2].

We verify that g and γ satisfy the Wexler-Raz biorthogonality conditions (4.1):

$$\begin{aligned}
\langle \gamma, M_{\frac{l}{\alpha}} T_{\frac{k}{\beta}} g \rangle &= \int_{\mathbb{R}^d} \gamma(x) \overline{g}\Big(x - \frac{k}{\beta}\Big) e^{-2\pi i \frac{l}{\alpha} \cdot x} \, \mathrm{d}x \\
&= \int_{Q_\alpha} \sum_{j \in \mathbb{Z}^d} \gamma(x + \alpha j) \overline{g}\Big(x + \alpha j - \frac{k}{\beta}\Big) e^{-2\pi i \frac{l}{\alpha} \cdot x} \, \mathrm{d}x \\
&= \beta^d \int_{Q_\alpha} \sum_{j \in \mathbb{Z}^d} \sigma_j(x) \overline{g}\Big(x + \alpha j - \frac{k}{\beta}\Big) e^{-2\pi i \frac{l}{\alpha} \cdot x} \, \mathrm{d}x \\
&= \beta^d \int_{Q_\alpha} \delta_{k,0} \, e^{-2\pi i \frac{l}{\alpha} \cdot x} \, \mathrm{d}x
\end{aligned}$$

$$= (\alpha\beta)^d \delta_{k,0}\delta_{l,0}.$$

Hence $\mathcal{G}(g,\alpha,\beta)$ is a Bessel sequence with biorthogonal Bessel sequence $\mathcal{G}(\gamma,\alpha,\beta)$ and therefore a frame by Theorem 4.1.2. □

Since totally positive functions of finite type are defined via their Fourier transform, we obtain the following corollary of Theorem 4.2.5.

Corollary 4.2.7. *Let $g \in L^2(\mathbb{R}^d)$ be a finite product of the form* (4.6) *with $M \geq 2$. Then $\mathcal{G}(g,\alpha,\beta)$ is a frame if and only if $\alpha\beta < 1$.*

Proof. Combine Theorem 4.2.5 and the fact that $\mathcal{G}(g,\alpha,\beta)$ is a frame if and only if $\mathcal{G}(\hat{g},\beta,\alpha)$ is a frame by Proposition 1.3.2. □

4.3. The Criteria of Zeevi and Zibulski

The Ron-Shen criterion of Theorem 4.1.2 states that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame if and only if the inequality

$$A\|c\|_{\ell^2}^2 \leq \sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k \overline{g}\Big(x+\alpha j - \frac{k}{\beta}\Big)\Big|^2 \leq B\|c\|_{\ell^2}^2$$

is satisfied for all $c \in \ell^2(\mathbb{Z}^d)$ and almost all $x \in \mathbb{R}^d$. In Section 4.1, we interpreted the sum over $k \in \mathbb{Z}^d$ as an inner product of sequences and used Parseval's formula to derive a characterization of Gabor frames involving the Zak transform.

With the additional structure of a rational lattice, i.e., a rectangular lattice where $\alpha\beta \in \mathbb{Q}$, we may interpret the sum over $k \in \mathbb{Z}^d$ as a convolution after a suitable periodization and apply Plancherel's theorem on the outer ℓ^2-norm. This yields characterizations of Zeevi and Zibulski [51, 50], where only the spectral properties of a family of finite dimensional matrices need to be checked.

Recall that the *Zak transform* for the parameter $\alpha > 0$ is defined as

$$Z_\alpha f(x,\xi) = \sum_{k\in\mathbb{Z}^d} f(x-\alpha k)e^{2\pi i\alpha k\cdot\xi}$$

and quasi-periodic on $Q_\alpha \times Q_{\frac{1}{\alpha}}$ by Lemma 4.1.9. Furthermore for $f \in L^2(\mathbb{R}^d)$, the Zak transform is defined almost everywhere on $\mathbb{R}^{2d}$ by Lemma 4.1.10.

Definition 4.3.1. We define the matrix $\mathcal{P}(x,\xi)$ by its entries

$$\mathcal{P}(x,\xi)_{r,s} := \overline{Z_{\alpha q} g}\Big(x+\alpha r - \frac{s}{\beta}, \xi\Big),$$

where $r \in E_q := \{0,1,\ldots,q-1\}^d$ and $s \in E_p := \{0,1,\ldots,p-1\}^d$.

The matrix $\mathcal{R}(x,\xi) := \mathcal{P}(x,\xi)^*\mathcal{P}(x,\xi)$ has the entries

$$\mathcal{R}(x,\xi)_{s,t} = \sum_{r\in E_q} Z_{\alpha q}g\Big(x+\alpha r-\frac{s}{\beta},\xi\Big)\overline{Z_{\alpha q}g}\Big(x+\alpha r-\frac{t}{\beta},\xi\Big),$$

where $s,t\in E_p := \{0,1,\dots,p-1\}^d$.

In the following, let $(L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$ denote all vector-valued functions $f := (f_s)_{s\in E_p}$, where all components f_s are in $L^2(Q_{\frac{1}{\alpha q}})$.

Lemma 4.3.2. *Let $g\in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$ with $\alpha\beta = \frac{p}{q}\in\mathbb{Q}$. Then for all $c\in\ell^2(\mathbb{Z}^d)$ and all $x\in\mathbb{R}^d$, we have*

$$\sum_{j\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d} c_k\overline{g}\Big(x+\alpha j-\frac{k}{\beta}\Big)\Big|^2 \tag{4.7}$$

$$= (\alpha q)^d\int_{Q_{\frac{1}{\alpha q}}}\sum_{r\in E_q}\Big|\sum_{s\in E_p} f_s(\xi)\cdot\overline{Z_{\alpha q}g}\Big(x+\alpha r-\frac{s}{\beta},\xi\Big)\Big|^2\,\mathrm{d}\xi \tag{4.8}$$

$$= (\alpha q)^d\int_{Q_{\frac{1}{\alpha q}}}\langle\mathcal{R}(x,\xi)f(\xi),f(\xi)\rangle_{\ell^2(E_p)}\,\mathrm{d}\xi$$

where $f_s(\xi) := \sum_{k\in\mathbb{Z}^d} c_{pk+s}e^{2\pi i\alpha qk\cdot\xi}$ and $f := (f_s)_{s\in E_p}\in(L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$.

Proof. We start with (4.7) and periodize both sums. In the outer sum we replace j by $qj+r$ with $j\in\mathbb{Z}^d$ and $r\in E_q$. We periodize the inner sum modulo E_p, hence k becomes $pk+s$ where k runs through $\mathbb{Z}^d$ and s through E_p.

Since $\frac{1}{\beta} = \frac{\alpha p}{q}$, this yields

$$\begin{aligned}\sum_{j\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d} c_k\overline{g}\Big(x+\alpha j-\frac{k}{\beta}\Big)\Big|^2 &= \sum_{r\in E_q}\sum_{j\in\mathbb{Z}^d}\Big|\sum_{s\in E_p}\sum_{k\in\mathbb{Z}^d} c_{pk+s}\,\underbrace{\overline{g}\Big(x+\alpha r-\frac{s}{\beta}+\alpha q(j-k)\Big)}_{=:g_{r,s}(j-k)}\Big|^2\\ &= \sum_{r\in E_q}\sum_{j\in\mathbb{Z}^d}\Big|\sum_{s\in E_p}(c_{p\,\cdot\,+s}*g_{r,s})(j)\Big|^2.\end{aligned}$$

We use Plancherel's theorem with respect to the domain $Q_{\frac{1}{\alpha q}}$ and obtain

$$\sum_{r\in E_q}\sum_{j\in\mathbb{Z}^d}\Big|\sum_{s\in E_p}(c_{p\,\cdot\,+s}*g_{r,s})(j)\Big|^2$$

$$= (\alpha q)^d \sum_{r\in E_q} \int_{Q_{\frac{1}{\alpha q}}} \Big| \sum_{s\in E_p} (c_{p\,\cdot\,+s} * g_{r,s})^{\smile}(\xi)\Big|^2 \,\mathrm{d}\xi$$
$$= (\alpha q)^d \sum_{r\in E_q} \int_{Q_{\frac{1}{\alpha q}}} \Big| \sum_{s\in E_p} (c_{p\,\cdot\,+s})^{\smile}(\xi)\cdot (g_{r,s})^{\smile}(\xi)\Big|^2 \,\mathrm{d}\xi.$$

Now write

$$f_s(\xi) := (c_{p\,\cdot\,+s})^{\smile}(\xi) = \sum_{k\in\mathbb{Z}^d} c_{pk+s} e^{2\pi i\alpha q k\cdot\xi},$$
$$\begin{aligned}(g_{r,s})^{\smile}(\xi) &= \sum_{k\in\mathbb{Z}^d} \overline{g}\Big(x+\alpha r - \frac{s}{\beta} + \alpha q k\Big) e^{2\pi i\alpha q k\cdot\xi} \\ &= \overline{\sum_{k\in\mathbb{Z}^d} g\Big(x+\alpha r - \frac{s}{\beta} + \alpha q k\Big) e^{-2\pi i\alpha q k\cdot\xi}} \\ &= \overline{Z_{\alpha q} g}\Big(x+\alpha r - \frac{s}{\beta}, \xi\Big),\end{aligned}$$

then

$$\begin{aligned}&\sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k \overline{g}\Big(x+\alpha j - \frac{k}{\beta}\Big)\Big|^2 \\ &\quad= (\alpha q)^d \int\limits_{Q_{\frac{1}{\alpha q}}} \sum_{r\in E_q} \Big| \sum_{s\in E_p} f_s(\xi)\overline{Z_{\alpha q} g}\Big(x+\alpha r - \frac{s}{\beta}, \xi\Big)\Big|^2 \,\mathrm{d}\xi \\ &\quad= (\alpha q)^d \int\limits_{Q_{\frac{1}{\alpha q}}} \sum_{r\in E_q} \Big| \big[\mathcal{P}(x,\xi) f(\xi)\big]_r\Big|^2 \,\mathrm{d}\xi \\ &\quad= (\alpha q)^d \int\limits_{Q_{\frac{1}{\alpha q}}} \langle \mathcal{P}(x,\xi) f(\xi), \mathcal{P}(x,\xi) f(\xi)\rangle_{\ell^2(E_q)} \,\mathrm{d}\xi \\ &\quad= (\alpha q)^d \int\limits_{Q_{\frac{1}{\alpha q}}} \langle \mathcal{R}(x,\xi) f(\xi), f(\xi)\rangle_{\ell^2(E_p)} \,\mathrm{d}\xi,\end{aligned}$$

where $[f]_r := f_r$ denotes the r-th component of the vector $(f_t)_{t\in E_q}$. □

Corollary 4.3.3. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$. Then the following are equivalent:*

(i) $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$.

(ii) There exist positive constants $A, B > 0$, such that for all $c \in \ell^2(\mathbb{Z}^d)$ and almost all $x \in \mathbb{R}^d$

$$A\|c\|_{\ell^2}^2 \le \sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k \overline{g}\Big(x+\alpha j - \frac{k}{\beta}\Big)\Big|^2 \le B\|c\|_{\ell^2}^2. \tag{4.9}$$

(iii) There exist positive constants $A, B > 0$, such that for all $f := (f_s)_{s\in E_p} \in (L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$ and almost all $x \in Q_\alpha$

$$\int\limits_{Q_{\frac{1}{\alpha q}}} A\|f(\xi)\|^2_{\ell^2(E_p)}\,\mathrm{d}\xi \le \int\limits_{Q_{\frac{1}{\alpha q}}} \langle \mathcal{R}(x,\xi)f(\xi), f(\xi)\rangle_{\ell^2(E_p)}\,\mathrm{d}\xi \le \int\limits_{Q_{\frac{1}{\alpha q}}} B\|f(\xi)\|^2_{\ell^2(E_p)}\,\mathrm{d}\xi \quad (4.10)$$

Furthermore, the constants $A, B > 0$ in (4.9) *and* (4.10) *are equal.*

Proof. Since

$$\begin{aligned}
\|c\|^2_{\ell^2(\mathbb{Z}^d)} &= \sum_{s\in E_p}\sum_{k\in\mathbb{Z}^d} |c_{pk+s}|^2 \\
&= \sum_{s\in E_p} (\alpha q)^d \int\limits_{Q_{\frac{1}{\alpha q}}} |f_s(\xi)|^2\,\mathrm{d}\xi \\
&= (\alpha q)^d \int\limits_{Q_{\frac{1}{\alpha q}}} \|f(\xi)\|^2_{\ell^2(E_p)}\,\mathrm{d}\xi,
\end{aligned}$$

the equivalence follows from Lemma 4.3.2. □

It is easy to see that $(L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{(L^2)^{E_p}} := \int\limits_{Q_{\frac{1}{\alpha q}}} \langle f(\xi), g(\xi)\rangle_{\ell^2(E_p)}\,\mathrm{d}\xi.$$

In this light, condition *(iii)* states that the spectra of $\mathcal{R}(x, .) : (L^2(Q_{\frac{1}{\alpha q}}))^{E_p} \to (L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$ are contained in the interval $[A, B]$ for almost all $x \in Q_\alpha$.

In the following, we relate the spectral bounds of $\mathcal{R}(x, .)$ to those of the individual matrices $\mathcal{R}(x,\xi)$. This will yield the classical results of Zeevi and Zibulski [51, 50] for the matrix $\mathcal{R}(x,\xi)$.

Definition 4.3.4. Let $\lambda^{\mathcal{R}}_{\max}(x,\xi)$ and $\lambda^{\mathcal{R}}_{\min}(x,\xi)$ denote the maximal respective minimal eigenvalue of $\mathcal{R}(x,\xi)$ and define

$$\begin{aligned}
\lambda^{\mathcal{R}}_{\max} &:= \operatorname{ess\,sup}\{\lambda^{\mathcal{R}}_{\max}(x,\xi) : (x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}\}, \\
\lambda^{\mathcal{R}}_{\min} &:= \operatorname{ess\,inf}\{\lambda^{\mathcal{R}}_{\min}(x,\xi) : (x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}\}.
\end{aligned}$$

Theorem 4.3.5. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$. Then the following are equivalent:*

(i) $\mathcal{G}(g, \alpha, \beta)$ *is a frame for* $L^2(\mathbb{R}^d)$.

(ii) $0 < \lambda_{\min}^{\mathcal{R}} \leq \lambda_{\max}^{\mathcal{R}} < \infty$.

The proof requires some elementary results of measure theory in linear algebra which can be found in Appendix A.1.

Proof. *(ii)*$\Rightarrow$*(i)*: By the definition of $\lambda_{\max}^{\mathcal{R}}$ and $\lambda_{\min}^{\mathcal{R}}$, we have

$$0 < \lambda_{\min}^{\mathcal{R}} \leq \lambda_{\min}^{\mathcal{R}}(x,\xi) \leq \lambda_{\max}^{\mathcal{R}}(x,\xi) \leq \lambda_{\max}^{\mathcal{R}} < \infty$$

for almost all $(x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}$. Since

$$\lambda_{\min}^{\mathcal{R}}\langle f(\xi), f(\xi)\rangle \leq \langle \mathcal{R}(x,\xi) f(\xi), f(\xi)\rangle \leq \lambda_{\max}^{\mathcal{R}}\langle f(\xi), f(\xi)\rangle$$

holds for all $f \in (L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$ almost everywhere, inequality (4.10) follows by integration over $Q_{\frac{1}{\alpha q}}$.

(i)$\Rightarrow$*(ii)*: By assumption, the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is a frame. Hence, inequality (4.10) holds by Corollary 4.3.3.

Diagonalization of $\mathcal{R}(x,\xi)$ as in Corollary A.1.4 yields measurable functions $\psi_U, \psi_D : \mathbb{R}^{2d} \to M_{E_p}(\mathbb{C})$ such that

$$\mathcal{R}(x,\xi) = \psi_U(x,\xi)\psi_D(x,\xi)\psi_U^*(x,\xi) \qquad \forall (x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}},$$

where $\psi_U(x,\xi)$ is a unitary $E_p \times E_p$-matrix and $\psi_D(x,\xi)$ is of diagonal form at every point.

It is easy to see that $\psi_U(x, .\,)$ is a unitary operator on $(L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$, hence inequality (4.10) becomes

$$\int_{Q_{\frac{1}{\alpha q}}} A\|f(\xi)\|^2_{\ell^2(E_p)}\,\mathrm{d}\xi \leq \int_{Q_{\frac{1}{\alpha q}}} \langle \psi_D(x,\xi) f(\xi), f(\xi)\rangle_{\ell^2(E_p)}\,\mathrm{d}\xi \leq \int_{Q_{\frac{1}{\alpha q}}} B\|f(\xi)\|^2_{\ell^2(E_p)}\,\mathrm{d}\xi \tag{4.11}$$

for all $f \in (L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$.

Now suppose $\lambda_{\min}^{\mathcal{R}} = 0$, then for every $N \in \mathbb{N}$ there exists an $\Omega_N \subseteq Q_\alpha \times Q_{\frac{1}{\alpha q}}$ with $|\Omega_N| > 0$, i.e., of positive (Lebesgue-)measure, such that

$$\lambda_{\min}^{\mathcal{R}}(x,\xi) < \frac{1}{N} \qquad \forall (x,\xi) \in \Omega_N.$$

Define $\Omega_{N,x} := \{\xi \in Q_{\frac{1}{\alpha q}} : (x,\xi) \in \Omega_N\}$ for $x \in Q_\alpha$. Observe that the sections $\Omega_{N,x}$ are measurable for all $x \in Q_\alpha$ since Ω_N is measurable. Furthermore, the collection $\Omega_N^+ := \{x \in Q_\alpha : |\Omega_{N,x}| > 0\}$ cannot be a null set as this would imply that Ω_N is a null set by Fubini.

Now fix an arbitrary $x \in \Omega_N^+$. Clearly, the diagonal entries of $\psi_D(x,\xi)$ consist of all eigenvalues of $\mathcal{R}(x,\xi)$ and by assumption, they fall below $1/N$ on $\Omega_{N,x}$. Since E_p is finite, there must already exist a coordinate $s_0 \in E_p$ and a set $\tilde{\Omega}_{N,x} \subseteq \Omega_{N,x}$ of positive measure such that the s_0-th diagonal entry of $\psi_D(x,\xi)$ is less than $1/N$ on $\tilde{\Omega}_{N,x}$.

Now define $f^x := (f_s^x)_{s\in E_p}$ by

$$f_s^x(\xi) := \begin{cases} \chi_{\tilde{\Omega}_{N,x}}(\xi) & s = s_0 \\ 0 & s \neq s_0 \end{cases}.$$

Then all f_s^x are measurable and $f^x \in (L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$. Furthermore, we have

$$\int\limits_{Q_{\frac{1}{\alpha q}}} \langle \psi_D(x,\xi) f^x(\xi), f^x(\xi)\rangle_{\ell^2(E_p)} \,\mathrm{d}\xi < \frac{1}{N}\|f^x(\xi)\|^2_{\ell^2(E_p)}. \tag{4.12}$$

We have shown that for every $N \in \mathbb{N}$, we find a set of positive measure Ω_N^+ and corresponding $f^x \in (L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$ for all $x \in \Omega_N^+$ such that (4.12) holds. But this contradicts the lower inequality of (4.11).

If the upper bound for the eigenvalues is violated, an analogous argument yields a contradiction to the upper inequality of (4.11).

□

We immediately obtain the following results for integer redundancy $\alpha\beta = \frac{1}{N}$ respectively at critical density $\alpha\beta = 1$.

Corollary 4.3.6. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta = \frac{1}{N}$ for $N \in \mathbb{N}$. Then $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ if and only if there exist positive constants $A, B > 0$ such that*

$$A \leq \sum_{r\in E_N} |Z_{\frac{1}{\beta}} g(x+\alpha r,\xi)|^2 \leq B \quad \textit{for a.e. } (x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha N}}$$

where $E_N := \{1,\dots,N-1\}^d$.

Proof. Since $\alpha\beta = \frac{1}{N}$, the operation of $\mathcal{R}(x,\xi)$ is just a scalar-multiplication by

$$\mathcal{R}(x,\xi) = \sum_{r\in E_q} Z_{\alpha N} g\Big(x+\alpha r-\frac{1}{\beta},\xi\Big)\overline{Z_{\alpha N} g}\Big(x+\alpha r-\frac{1}{\beta},\xi\Big) = \sum_{r\in E_N} |Z_{\frac{1}{\beta}} g(x+\alpha r,\xi)|^2,$$

where we used the quasi-periodicity of the Zak transform (Lemma 4.1.9). Now, the claim follows from Theorem 4.3.5. □

Corollary 4.3.7. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta = 1$. Then $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ if and only if there exist positive constants $A, B > 0$ such that*

$$A \leq |Z_\alpha g(x, \xi)|^2 \leq B \qquad \text{for a.e. } (x, \xi) \in Q_\alpha \times Q_{\frac{1}{\alpha}}.$$

Proof. This follows immediately from Corollary 4.3.6 by setting $N = 1$. □

We return again to rational lattices for $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$. Unfortunately, computing the eigenvalues of a matrix is costly. In the following, we want to replace the condition on the eigenvalues by a condition on the determinants $\det(\mathcal{R}(x, \xi))$.

The first result is a characterization for Bessel sequences.

Lemma 4.3.8. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$. Then the following are equivalent:*

(i) $\mathcal{G}(g, \alpha, \beta)$ *is a Bessel sequence in* $L^2(\mathbb{R}^d)$.

(ii) $\lambda^{\mathcal{R}}_{\max} < \infty$.

(iii) $Z_{\alpha q} g \in L^\infty(Q_\alpha \times Q_{\frac{1}{\alpha q}})$.

Proof. Recall Lemma 4.1.5, which states that $\mathcal{G}(g, \alpha, \beta)$ is a Bessel sequence if and only if the upper Ron-Shen criterion holds. By Lemma 4.3.2, this is equivalent to the upper inequality of (4.10). Hence, equivalence *(i)*$\Leftrightarrow$*(ii)* is implicit in the proof of Theorem 4.3.5.

For the remaining implications, we consider the upper inequality of (4.10) in a slightly different version: Due to identity (4.8), $\mathcal{G}(g, \alpha, \beta)$ is a Bessel sequence if and only if

$$\int_{Q_{\frac{1}{\alpha q}}} \sum_{r \in E_q} \Big| \sum_{s \in E_p} f_s(\xi) \cdot \overline{Z_{\alpha q} g}\Big(x + \alpha r - \frac{s}{\beta}, \xi\Big) \Big|^2 \,\mathrm{d}\xi \leq B \int_{Q_{\frac{1}{\alpha q}}} \|f(\xi)\|^2_{\ell^2(E_p)} \,\mathrm{d}\xi \quad (4.13)$$

holds for all $f := (f_s)_{s \in E_p} \in (L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$ and almost all $x \in Q_\alpha$.

(iii)$\Rightarrow$*(i)*: If $Z_{\alpha q} g \in L^\infty(Q_\alpha \times Q_{\frac{1}{\alpha q}})$, then

$$\begin{aligned} &\sum_{r \in E_q} \int_{Q_{\frac{1}{\alpha q}}} \Big| \sum_{s \in E_p} f_s(\xi) \cdot \overline{Z_{\alpha q} g}\Big(x + \alpha r - \frac{s}{\beta}, \xi\Big) \Big|^2 \,\mathrm{d}\xi \\ &\qquad \leq \sum_{r \in E_q} \|Z_{\alpha q} g\|^2_{L^\infty} \int_{Q_{\frac{1}{\alpha q}}} \Big(\sum_{s \in E_p} |f_s(\xi)| \Big)^2 \,\mathrm{d}\xi \end{aligned}$$

$$\leq q^d p^{2d} \|Z_{\alpha q} g\|_{L^\infty}^2 \int\limits_{Q_{\frac{1}{\alpha q}}} \sum_{s\in E_p} |f_s(\xi)|^2 \, \mathrm{d}\xi$$

$$\leq B \int\limits_{Q_{\frac{1}{\alpha q}}} \|f(\xi)\|_{\ell^2(E_p)}^2 \, \mathrm{d}\xi,$$

where we used the standard trick

$$\Big(\sum_{s\in E_p} |f_s(\xi)|\Big)^2 \leq \Big(\sum_{s\in E_p} \max_{s\in E_p}\{|f_s(\xi)|\}\Big)^2 = p^{2d} \max_{s\in E_p}\{|f_s(\xi)|^2\} \leq p^{2d} \sum_{s\in E_p} |f_s(\xi)|^2.$$

(i)⇒(iii): Suppose that $Z_{\alpha q} g \notin L^\infty(Q_\alpha \times Q_{\frac{1}{\alpha q}})$. This means that for every $N \in \mathbb{N}$ there exists sets with positive measure $\Omega_N \subseteq Q_\alpha \times Q_{\frac{1}{\alpha q}}$, such that

$$\Big|Z_{\alpha q} g\Big(x + \alpha r - \frac{s}{\beta}\Big)\Big| > N \quad \forall (x, \xi) \in \Omega_N.$$

As in the proof of Theorem 4.3.5, let $\Omega_{N,x} := \{\xi \in Q_{\frac{1}{\alpha q}} : (x, \xi) \in \Omega_N\}$ denote the measurable sections of Ω_N and $\Omega_N^+ := \{x \in Q_\alpha : |\Omega_{N,x}| > 0\}$. Recall that Ω_N^+ must be of positive measure by Fubini.

For $x \in \Omega_N^+$, define for all $s \in E_p$

$$f_s^x(\xi) := \begin{cases} \overline{\operatorname{sgn}\Big(Z_{\alpha q} g\Big(x + \alpha r - \frac{s}{\beta}\Big)\Big)} & \xi \in \Omega_{N,x} \\ 0 & \xi \notin \Omega_{N,x} \end{cases},$$

where

$$\operatorname{sgn}(z) := \begin{cases} \frac{z}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

denotes the signum function.

Clearly, all f_s^x are measurable and we obtain

$$\int\limits_{Q_{\frac{1}{\alpha q}}} \|f^x(\xi)\|_{\ell^2(E_p)}^2 \, \mathrm{d}\xi = \int\limits_{Q_{\frac{1}{\alpha q}}} \sum_{s\in E_p} |f_s^x(\xi)|^2 \, \mathrm{d}\xi = \sum_{s\in E_p} \|f_s^x\|_{L^2(Q_{\frac{1}{\alpha q}})}^2 \leq p^d \, |\Omega_{N,x}|.$$

Consequently, $f^x := (f_s^x)_{s\in E_p}$ is in $(L^2(Q_{\frac{1}{\alpha q}}))^{E_p}$ but

$$\int\limits_{Q_{\frac{1}{\alpha q}}} \sum_{r\in E_q} \Big| \sum_{s\in E_p} f_s^x(\xi) \cdot \overline{Z_{\alpha q} g}\Big(x + \alpha r - \frac{s}{\beta}, \xi\Big)\Big|^2 \, \mathrm{d}\xi$$

$$
\begin{aligned}
&\geq \int\limits_{\Omega_{N,x}} \sum_{r\in E_q} \Big| \sum_{s\in E_p} \underbrace{f_s^x(\xi)\cdot \overline{Z_{\alpha q} g}\Big(x+\alpha r-\frac{s}{\beta},\xi\Big)}_{=|Z_{\alpha q} g(x+\alpha r-\frac{s}{\beta},\xi)|>N} \Big|^2 \,\mathrm{d}\xi \\
&> \int\limits_{\Omega_{N,x}} \sum_{r\in E_q} \Big| \sum_{s\in E_p} N \Big|^2 \,\mathrm{d}\xi \\
&= q^d p^{2d} N^2 |\Omega_{N,x}| \\
&\geq (qp)^d N^2 \int\limits_{Q_{\frac{1}{\alpha q}}} \|f^x(\xi)\|^2_{\ell^2(E_p)} \,\mathrm{d}\xi
\end{aligned}
$$

for all $x \in \Omega_N^+$. Clearly, this yields a contradiction to (4.13). □

Theorem 4.3.9. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$. Suppose $\mathcal{G}(g,\alpha,\beta)$ is a Bessel sequence, then the following are equivalent:*

(i) $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$.

(ii) The determinant is uniformly bounded away from zero for almost all matrices $\mathcal{R}(x,\xi)$. This means that there exists a constant $C > 0$, such that

$$\det(\mathcal{R}(x,\xi)) \geq C > 0 \quad \text{for a.e. } (x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}.$$

Proof. In the following, let $\lambda_j(x,\xi)$ for $j \in E_p$ denote all eigenvalues (including multiples) of $\mathcal{R}(x,\xi)$. Recall that all $\mathcal{R}(x,\xi)$ are positive semi-definite, hence their eigenvalues are non-negative.

(i)⇒(ii): By Theorem 4.3.5, the eigenvalues of almost all $\mathcal{R}(x,\xi)$ satisfy

$$0 < \lambda^{\mathcal{R}}_{\min} \leq \lambda^{\mathcal{R}}_{\min}(x,\xi) \leq \lambda^{\mathcal{R}}_{\max}(x,\xi) \leq \lambda^{\mathcal{R}}_{\max} < \infty.$$

Since the determinant of a matrix is equal to the product of its eigenvalues, we have

$$\det(\mathcal{R}(x,\xi)) = \prod_{j\in E_p} \lambda_j(x,\xi) \geq (\lambda^{\mathcal{R}}_{\min})^{q^d} > 0$$

for almost all $(x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}$.

(ii)⇒(i): By assumption, we have

$$0 < C \leq \det(\mathcal{R}(x,\xi)) = \prod_{j\in E_p} \lambda_j(x,\xi)$$

for almost all $(x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}$.

Since $\mathcal{G}(g,\alpha,\beta)$ is a Bessel sequence, Lemma 4.3.8 implies that the eigenvalues of almost all $\mathcal{R}(x,\xi)$ are uniformly bounded by $\lambda^{\mathcal{R}}_{\max} < \infty$.

This implies

$$0 < \frac{C}{(\lambda_{\max}^{\mathcal{R}})^{q^d-1}} \leq \frac{C}{\prod_{j\neq\min}\lambda_j(x,\xi)} \leq \lambda_{\min}^{\mathcal{R}}(x,\xi)$$

almost everywhere on $Q_\alpha \times Q_{\frac{1}{\alpha q}}$ and therefore $\lambda_{\min}^{\mathcal{R}} > 0$. Consequently, $\mathcal{G}(g,\alpha,\beta)$ is a frame by Theorem 4.3.5. □

We obtain the following corollary for window functions whose Zak transform is continuous. Recall that by Lemma 4.1.10, all functions in $W(\mathcal{C},L^1)$ have a continuous Zak transform. In particular, Corollary 4.3.10 holds for all windows $g \in M^1(\mathbb{R}^d)$.

Corollary 4.3.10. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$. Suppose that $Z_{\alpha q}g$ is continuous on $Q_\alpha \times Q_{\frac{1}{\alpha q}}$, then the following are equivalent:*

(i) $\mathcal{G}(g,\alpha,\beta)$ *is a frame for* $L^2(\mathbb{R}^d)$.

(ii) $\det(\mathcal{R}(x,\xi)) \neq 0$ *for all* $(x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}$.

(iii) $\operatorname{rank}(\mathcal{P}(x,\xi)) = p^d$ *for all* $(x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}$.

Proof. Since $Z_{\alpha q}g$ is continuous, it is in particular bounded on $Q_\alpha \times Q_{\frac{1}{\alpha q}}$. Therefore, $\mathcal{G}(g,\alpha,\beta)$ is a Bessel sequence by Lemma 4.3.8.

(i)⇔*(ii)*: By Theorem 4.3.9, $\mathcal{G}(g,\alpha,\beta)$ is a frame if and only if

$$\det(\mathcal{R}(x,\xi)) \geq C > 0 \qquad \text{for a.e. } (x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}. \tag{4.14}$$

If $Z_{\alpha q}g$ is continuous, then $\det(\mathcal{R}(x,\xi))$ is continuous as a polynomial of continuous functions. Consequently, (4.14) is equivalent to

$$\det(\mathcal{R}(x,\xi)) \neq 0 \qquad \forall (x,\xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}},$$

since by definition, $\mathcal{R}(x,\xi)$ is a positive semi-definite matrix.

(ii)⇔*(iii)*: Since $\mathcal{R}(x,\xi) = \mathcal{P}(x,\xi)^*\mathcal{P}(x,\xi)$, this equivalence follows from well-known facts from linear algebra. □

While $\det(\mathcal{R}(x,\xi)) \neq 0$, respectively $\operatorname{rank}(\mathcal{P}(x,\xi)) = p^d$, almost everywhere is not sufficient for a Gabor frame, Zeevi and Zibulski [51, 50] proved that it characterizes completeness of Gabor systems in $L^2(\mathbb{R}^d)$.

Theorem 4.3.11 (Zeevi-Zibulski [51, 50])**.** *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$. Then the following are equivalent:*

(i) $\mathcal{G}(g, \alpha, \beta)$ *is complete in* $L^2(\mathbb{R}^d)$*.*

(ii) $\det(\mathcal{R}(x, \xi)) \neq 0$ *for almost all* $(x, \xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}$*.*

(iii) $\operatorname{rank}(\mathcal{P}(x, \xi)) = p^d$ *for almost all* $(x, \xi) \in Q_\alpha \times Q_{\frac{1}{\alpha q}}$*.*

For a proof, we refer to original work of Zeevi and Zibulski [51, 50]. Even though they considered slightly different matrices, Theorem 4.3.11 still holds for $\mathcal{R}(x, \xi), \mathcal{P}(x, \xi)$ due to the results of Section 4.4.

We conclude this section with the important example of the one-dimensional Gaussian. It is now a well-known fact, that the time-frequency shifts of the standard Gaussian $\varphi(x) = e^{-\pi x^2}$ along the integer lattice $\Lambda = \mathbb{Z}^2$ are complete but do not generate a frame. We verify this in the following examples via the characterizations of Zeevi and Zibulski.

Example 4.3.12. Consider the one-dimensional Gaussian $\varphi(x) = e^{-\pi x^2}$. At critical density $\alpha = \beta = 1$, the operator $\mathcal{R}(x, \,.\,)$ is just pointwise multiplication with $|Z_1\varphi(x, \,.\,)|^2$ (cf. Corollary 4.3.7). Since

$$Z_1\varphi(\tfrac{1}{2}, \tfrac{1}{2}) = \sum_{k\in\mathbb{Z}} e^{-\pi(\frac{1}{2}-k)^2}(-1)^k = 0,$$

Corollary 4.3.10 implies that $\mathcal{G}(\varphi, 1, 1)$ is not a frame.

Note that $|Z_1\varphi|^2 > 0$ holds almost everywhere. In fact, $Z_1\varphi$ has only finitely many zeroes in $[0, 1]^2$: The Zak transform of the Gaussian is

$$\begin{aligned} Z_1\varphi(x, \xi) &= \sum_{k\in\mathbb{Z}} e^{-\pi(x-k)^2} e^{2\pi i\alpha k\xi} \\ &= e^{-\pi x^2} \sum_{k\in\mathbb{Z}} e^{-\pi k^2} e^{2\pi i\alpha k(\xi - ix)} \\ &= e^{-\pi x^2} \Theta(\alpha(\xi - ix), i) \end{aligned}$$

where

$$\Theta(z, \tau) := \sum_{k\in\mathbb{Z}} e^{\pi i k^2 \tau} e^{2\pi i k z}$$

denotes the *Jacobi-Theta function* for $z \in \mathbb{C}$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. For fixed $\tau \in \mathbb{H}$, the series converges uniform on compact subsets of $\mathbb{C}$. Therefore, the Jacobi-Theta function is an entire function in the first variable.

Suppose $Z_1\varphi(x, \xi) = 0$ on a countably infinite set Ω in $[0, 1]^2$, then $\Theta(z, i)$ vanishes on a set with accumulation point. Since $\Theta(z, i)$ is holomorphic, it must therefore be identically zero on the whole complex plane. Clearly, this is not the case (take $z = 0$), hence $Z_1\varphi$ has only finitely many zeroes.

Consequently, the Gabor system $\mathcal{G}(\varphi, 1, 1)$ is complete by Theorem 4.3.11.

4.4. The Classical Zeevi-Zibulski Matrices

In the previous section, we derived the well-known results of Zeevi and Zibulski for the matrices $\mathcal{R}(x,\xi)$ defined as

$$\mathcal{R}(x,\xi)_{s,t} := \sum_{r\in E_q} Z_{\alpha q}g\Big(x+\alpha r-\frac{s}{\beta},\xi\Big)\overline{Z_{\alpha q}g}\Big(x+\alpha r-\frac{t}{\beta},\xi\Big)$$

for $(x,\xi)\in Q_{\frac{1}{\alpha}}\times Q_{\frac{1}{\alpha q}}$ and $s,t\in E_p := \{0,1,\dots,p-1\}^d$.

These matrices are different from the matrices appearing in Zeevi and Zibulski's original work [51, 50] or Gröchenig's textbook [21]. In this section, we will derive classical Zeevi-Zibulski matrices from the modified Ron-Shen criterion of Theorem 4.1.11 and show how the matrices in [21] are related to $\mathcal{R}(x,\xi)$.

Definition 4.4.1. Let $g\in L^2(\mathbb{R}^d)$ and define the matrix $\mathbf{G}(x,\xi)$ by its entries

$$\mathbf{G}(x,\xi)_{r,s} := \overline{Z_{\frac{1}{\beta}}g}\Big(x+\alpha r,\beta\xi-\frac{\beta s}{p}\Big),$$

for $r\in E_q := \{0,1,\dots,q-1\}^d$ and $s\in E_p := \{0,1,\dots,p-1\}^d$.

Furthermore, the matrix $\mathbf{S}(x,\xi) := \mathbf{G}(x,\xi)^*\mathbf{G}(x,\xi)$ has the entries

$$\mathbf{S}(x,\xi)_{s,t} := \sum_{r\in E_q} Z_{\frac{1}{\beta}}g\Big(x+\alpha r,\beta\xi-\frac{\beta s}{p}\Big)\overline{Z_{\frac{1}{\beta}}g}\Big(x+\alpha r,\beta\xi-\frac{\beta t}{p}\Big)$$

for $s,t\in E_p = \{0,1,\dots,p-1\}^d$.

Proposition 4.4.2. *Let $g\in L^2(\mathbb{R}^d)$ and $\Lambda=\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$ with $\alpha\beta=\frac{p}{q}\in\mathbb{Q}$. Then the following are equivalent:*

(i) $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds $A,B>0$.

(ii) There exist positive constants $A,B>0$, such that for all $f:=(f_s)_{s\in E_p}\in (L^2(Q_{\frac{1}{p}}))^{E_p}$ and almost all $x\in Q_\alpha$

$$\int_{Q_{\frac{1}{p}}}(p\beta)^d A\|f(\xi)\|^2_{\ell^2(E_p)}\,\mathrm{d}\xi \le \int_{Q_{\frac{1}{p}}}\langle\mathbf{S}(x,\xi)f(\xi),f(\xi)\rangle_{\ell^2(E_p)}\,\mathrm{d}\xi \le \int_{Q_{\frac{1}{p}}}(p\beta)^d B\|f(\xi)\|^2_{\ell^2(E_p)}\,\mathrm{d}\xi.$$

Furthermore, the constants $A,B>0$ in (i) and (ii) are the same.

The following proof is based on notes by Karlheinz Gröchenig.

Proof. Recall Theorem 4.1.11 which states that the Ron-Shen criterion is equivalent to

$$\beta^d A\|f\|^2_{L^2(\mathbb{T}^d)} \le \sum_{j\in\mathbb{Z}^d} \Big|\langle f, Z_{\frac{1}{\beta}} g(x+\alpha j, \beta\,.\,)\rangle_{L^2(\mathbb{T}^d)}\Big|^2 \le \beta^d B\|f\|^2_{L^2(\mathbb{T}^d)} \tag{4.15}$$

where $A, B > 0$ are the frame bounds of $\mathcal{G}(g,\alpha,\beta)$ (see Remark 4.1.4) and $\mathbb{T}^d := [0,1]^d$ denotes the d-dimensional torus.

Now, we periodize the sum over $j \in \mathbb{Z}^d$ and write $j = ql + r$ with $r \in E_q$ and $l \in \mathbb{Z}^d$. Since $\alpha = \frac{p}{\beta q}$, we have

$$\alpha j = \frac{p}{\beta} l + \alpha r$$

and thus

$$Z_{\frac{1}{\beta}} g(x+\alpha j, \beta\xi) = Z_{\frac{1}{\beta}} g\Big(x + \frac{p}{\beta} l + \alpha r, \beta\xi\Big) = e^{2\pi i \frac{p}{\beta} l\cdot\xi} Z_{\frac{1}{\beta}} g(x+\alpha r, \beta\xi)$$

by the quasi-periodicity of the Zak transform (see Lemma 4.1.9).

This yields

$$\begin{aligned}\sum_{j\in\mathbb{Z}^d} \Big|\langle f, Z_{\frac{1}{\beta}} g(x+\alpha j, \beta\,.\,)\rangle_{L^2(\mathbb{T}^d)}\Big|^2 &= \sum_{r\in E_q}\sum_{l\in\mathbb{Z}^d} \Big|\langle f, e^{2\pi i \frac{p}{\beta} l\cdot}\, Z_{\frac{1}{\beta}} g(x+\alpha r, \beta\,.\,)\rangle_{L^2(\mathbb{T}^d)}\Big|^2 \\ &= \sum_{r\in E_q}\sum_{l\in\mathbb{Z}^d} \Big|(f\cdot \overline{Z_{\frac{1}{\beta}} g}(x+\alpha r, \beta\,.\,))\widehat{\ }(pl)\Big|^2. \end{aligned} \tag{4.16}$$

Suppose $f \in L^2(\mathbb{T}^d)$ is periodically extended to $\mathbb{R}^d$, then

$$\sum_{s\in E_p} f\Big(x - \frac{s}{p}\Big)$$

is periodic on $Q_{\frac{1}{p}}$ and its Fourier coefficients are

$$\begin{aligned}\Big(\sum_{s\in E_p} f\Big(\,.\, - \frac{s}{p}\Big)\Big)\widehat{\ }(l) &= \int_{Q_{\frac{1}{p}}} \sum_{s\in E_p} f\Big(x - \frac{s}{p}\Big) e^{-2\pi i p l\cdot x}\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d} f(x) e^{-2\pi i p l\cdot x}\,\mathrm{d}x = \hat{f}(pl).\end{aligned}$$

Hence, its Fourier series is

$$\sum_{s\in E_p} f\Big(x - \frac{s}{p}\Big) = p^d \sum_{l\in\mathbb{Z}^d} \Big(\sum_{s\in E_p} f\Big(\,.\, - \frac{s}{p}\Big)\Big)\widehat{\ }(l) e^{2\pi i p l\cdot x}$$

$$= p^d \sum_{l\in\mathbb{Z}^d} \hat{f}(pl)e^{2\pi i pl\cdot x}.$$

By Plancherel's theorem, we have

$$\sum_{l\in\mathbb{Z}^d} |\hat{f}(pl)|^2 = \sum_{l\in\mathbb{Z}^d} \Big|\Big(\sum_{s\in E_p} f\Big(\,.\, - \frac{\beta s}{p}\Big)\Big)\hat{}\,(l)\Big|^2 = \frac{1}{p^d}\int_{Q_{\frac{1}{p}}} \Big|\sum_{s\in E_p} f\Big(x - \frac{s}{p}\Big)\Big|^2 \,\mathrm{d}x.$$

We continue with (4.16) and obtain

$$\sum_{r\in E_q}\sum_{l\in\mathbb{Z}^d} \Big|(f\cdot \overline{Z_{\frac{1}{\beta}} g}(x+\alpha r, \beta\,.\,))\hat{}\,(pl)\Big|^2$$

$$= \frac{1}{p^d}\sum_{r\in E_q}\int_{Q_{\frac{1}{p}}} \Big|\sum_{s\in E_p} f\Big(\xi - \frac{s}{p}\Big)\cdot \overline{Z_{\frac{1}{\beta}} g}\Big(x+\alpha r, \beta\xi - \frac{\beta s}{p}\Big)\Big|^2 \,\mathrm{d}\xi.$$

Now write

$$\tilde{f} := (T_{\frac{s}{p}} f)_{s\in E_p} \in (L^2(Q_{\frac{1}{p}}))^{E_p}$$

and let $[\tilde{f}]_r$ denote the r-th component of the vector $\tilde{f} \in (L^2(Q_{\frac{1}{p}}))^{E_p}$. Then

$$\sum_{j\in\mathbb{Z}^d} \Big|\langle f, Z_{\frac{1}{\beta}} g(x+\alpha j, \beta\,.\,)\rangle_{L^2(\mathbb{T}^d)}\Big|^2$$

$$\begin{aligned}
&= \frac{1}{p^d}\sum_{r\in E_q}\int_{Q_{\frac{1}{p}}} \Big|\sum_{s\in E_p} f\Big(\xi - \frac{s}{p}\Big)\cdot \overline{Z_{\frac{1}{\beta}} g}\Big(x+\alpha r, \beta\xi - \frac{\beta s}{p}\Big)\Big|^2 \,\mathrm{d}\xi \\
&= \frac{1}{p^d}\int_{Q_{\frac{1}{p}}} \sum_{r\in E_q} \Big|\big[\mathbf{G}(x,\xi)\tilde{f}(\xi)\big]_r\Big|^2 \,\mathrm{d}\xi \\
&= \frac{1}{p^d}\int_{Q_{\frac{1}{p}}} \langle \mathbf{G}(x,\xi)\tilde{f}(\xi), \mathbf{G}(x,\xi)\tilde{f}(\xi)\rangle_{\ell^2(E_q)} \,\mathrm{d}\xi \\
&= \frac{1}{p^d}\int_{Q_{\frac{1}{p}}} \langle \mathbf{S}(x,\xi)\tilde{f}(\xi), \tilde{f}(\xi)\rangle_{\ell^2(E_p)} \,\mathrm{d}\xi.
\end{aligned}$$

Consequently, inequality (4.15) becomes

$$\int_{Q_{\frac{1}{p}}} (p\beta)^d A\|f(\xi)\|^2_{\ell^2(E_p)} \,\mathrm{d}\xi \le \int_{Q_{\frac{1}{p}}} \langle \mathbf{S}(x,\xi)f(\xi), f(\xi)\rangle_{\ell^2(E_p)} \,\mathrm{d}\xi \le \int_{Q_{\frac{1}{p}}} (p\beta)^d B\|f(\xi)\|^2_{\ell^2(E_p)} \,\mathrm{d}\xi$$

where $f \in (L^2(Q_{\frac{1}{p}}))^{E_p}$ and $A, B > 0$ are the frame bounds of $\mathcal{G}(g,\alpha,\beta)$. □

Proceeding as in section 4.3 shows that all results for $\mathcal{R}(x,\xi), \mathcal{P}(x,\xi)$ also hold for $\mathbf{S}(x,\xi), \mathbf{G}(x,\xi)$ respectively. By Theorem 4.3.5, the spectral bounds

$$\lambda^{\mathbf{S}}_{\max} := \operatorname{ess\,sup}\{\lambda^{\mathbf{S}}_{\max}(x,\xi) : (x,\xi) \in Q_{\frac{1}{\beta}} \times Q_{\frac{1}{p}}\},$$

$$\lambda^{\mathbf{S}}_{\min} := \operatorname{ess\,inf}\{\lambda^{\mathbf{S}}_{\min}(x,\xi) \,:\, (x,\xi) \in Q_{\frac{1}{\beta}} \times Q_{\frac{1}{p}}\}$$

satisfy

$$\begin{aligned} \lambda^{\mathbf{S}}_{\max} &= (p\beta)^d B_{\mathrm{opt}} \\ \lambda^{\mathbf{S}}_{\min} &= (p\beta)^d A_{\mathrm{opt}} \end{aligned} \tag{4.17}$$

where $A_{\mathrm{opt}}, B_{\mathrm{opt}} > 0$ are the optimal frame bounds.

Remark 4.4.3. Despite appearing slightly different, the matrix $(p\beta)^{-d}\mathbf{S}(x,\xi)$ is precisely the matrix Zeevi and Zibulski considered in their original work [51, 50]. The apparent difference stems from the fact that they defined the Zak transform as

$$\tilde{Z}_\alpha f(x,\xi) := \alpha^{d/2} \sum_{k\in\mathbb{Z}^d} f(\alpha(x-k)) e^{2\pi i k\cdot\xi},$$

which is quasi-periodic on $\mathbb{T}^d \times \mathbb{T}^d$.

The original Zeevi-Zibulski matrix is defined for $\tilde{x} \in \mathbb{T}^d$ and $\xi \in Q_{\frac{1}{p}}$ by

$$\tilde{\mathbf{S}}(\tilde{x},\xi)_{s,t} := \frac{1}{p^d} \sum_{r\in E_q} \tilde{Z}_{\frac{1}{\beta}} g\Big(\tilde{x} + \frac{p}{q} r, \xi - \frac{s}{p}\Big) \overline{\tilde{Z}_{\frac{1}{\beta}} g\Big(\tilde{x} + \frac{p}{q} r, \xi - \frac{t}{p}\Big)}$$

where $s,t \in E_p := \{0,1,\ldots,p-1\}$.

Since

$$\begin{aligned} \overline{\tilde{Z}_{\frac{1}{\beta}} g\Big(\tilde{x} + \frac{p}{q} r, \xi - \frac{s}{p}\Big)} &= \beta^{-d/2} \overline{\sum_{k\in\mathbb{Z}^d} f\Big(\frac{1}{\beta}\Big(\tilde{x} + \frac{p}{q} r - k\Big)\Big) e^{2\pi i k\cdot(\xi - \frac{s}{p})}} \\ &= \beta^{-d/2}\, \overline{Z_{\frac{1}{\beta}} g}\Big(\frac{\tilde{x}}{\beta} + \alpha r, \beta\xi - \frac{\beta s}{p}\Big) \\ &= \beta^{-d/2}\, \mathbf{G}(x,\xi)_{r,s} \end{aligned}$$

for $x := \frac{\tilde{x}}{\beta} \in Q_{\frac{1}{\beta}}$, we have $\tilde{\mathbf{S}}(\tilde{x},\xi) = (p\beta)^{-d}\mathbf{S}(x,\xi)$. In particular, the spectral bounds of $\tilde{\mathbf{S}}$ are precisely the optimal frame bounds by (4.17).

Now, we turn to the Zeevi-Zibulski matrices appearing in Gröchenig's textbook [21].

Definition 4.4.4. Let $g \in L^2(\mathbb{R}^d)$ and define the matrix $\mathcal{Q}(x,\xi)$ by

$$\mathcal{Q}(x,\xi)_{r,s} := \overline{Z_\alpha g}\Big(x + \frac{\alpha s}{p}, \xi - \beta r\Big) e^{-2\pi i r\cdot s/q}$$

for $r \in E_q := \{0,1,\ldots,q-1\}^d$ and $s \in E_p := \{0,1,\ldots,p-1\}^d$.

Furthermore, the matrix $\mathcal{A}(x,\xi) := \frac{\alpha^d}{p^d}\mathcal{Q}(x,\xi)^*\mathcal{Q}(x,\xi)$ has the entries

$$\mathcal{A}(x,\xi)_{s,t} := \frac{\alpha^d}{p^d} \sum_{r\in E_q} Z_\alpha g\Big(x + \frac{\alpha s}{p}, \xi - \beta r\Big) \overline{Z_\alpha g}\Big(x + \frac{\alpha t}{p}, \xi - \beta r\Big) e^{2\pi i r\cdot(s-t)/q}$$

for $s,t \in E_p = \{0,1,\ldots,p-1\}^d$.

Note the additional factor $1/p^d$ in the definition of the matrix $\mathcal{A}(x,\xi)$ which is missing in Gröchenig's textbook [21].

The following connection between the matrices $\mathcal{Q}(x,\xi)$ and $\tilde{\mathcal{P}}(x,\xi)$ is due to Lyubarskiĭ and Nes [40].

Proposition 4.4.5. *Let $g \in L^2(\mathbb{R}^d)$ and $\alpha\beta = \frac{p}{q}$ with $p, q \in \mathbb{N}$ relative prime. Then*

$$\mathcal{Q}(x,\xi) = W \operatorname{diag}\left(e^{2\pi i\alpha t\cdot\xi}\right)_{t\in E_q} \tilde{\mathcal{P}}(x,\xi) \operatorname{diag}\left(e^{-2\pi i\alpha\xi\cdot\frac{s+qm_s}{p}}\right)_{s\in E_p},$$

where $m_s \in E_p$ denotes the unique element in E_p such that $s \equiv -qm_s (\mathrm{mod}\ p)$ and the matrices $W, \tilde{\mathcal{P}}(x,\xi)$ are defined by its entries

$$\tilde{\mathcal{P}}(x,\xi)_{t,s} := \overline{Z_{\alpha q} g}\Big(x + \alpha t - \frac{m_s}{\beta}, \xi\Big)$$

and

$$W_{r,t} := e^{-2\pi i\frac{p}{q} r\cdot t}$$

for $r, t \in E_q := \{0, 1, \dots, q-1\}^d$ and $s \in E_p := \{0, 1, \dots, p-1\}^d$.

Proof. Since $p, q \in \mathbb{N}$ are relatively prime, there exist $m_s \in E_p$ and $k_s \in \mathbb{Z}^d$ such that

$$s = pk_s - qm_s \tag{4.18}$$

by the Euclidean algorithm. It is easy to check that there is a one-to-one correspondence between $s \in E_p$ and $m_s \in E_p$.

Substituting (4.18) into the definition of $\mathcal{Q}(x,\xi)$ yields

$$\begin{aligned}
\mathcal{Q}(x,\xi)_{r,s} &= \sum_{k\in\mathbb{Z}^d} \overline{g}\Big(x + \frac{\alpha s}{p} - \alpha k\Big) e^{-2\pi i\alpha k\cdot(\xi-\beta r)} e^{-2\pi i r\cdot s/q} \\
&= \sum_{k\in\mathbb{Z}^d} \overline{g}\Big(x - \frac{m_s}{\beta} - \alpha(k - k_s)\Big) e^{-2\pi i\alpha k\cdot\xi} e^{2\pi i\frac{p}{q}k\cdot r} e^{-2\pi i\frac{p}{q} r\cdot k_s} \\
&= e^{-2\pi i\alpha k_s\cdot\xi} \sum_{k'\in\mathbb{Z}^d} \overline{g}\Big(x - \frac{m_s}{\beta} - \alpha k'\Big) e^{-2\pi i\alpha k'\cdot\xi} e^{2\pi i\frac{p}{q} r\cdot k'}
\end{aligned}$$

after the index shift $k' := k - k_s$.

Now, we periodize the sum over $k' \in \mathbb{Z}^d$ with respect to E_q. This means, we write $k' = qm - t$ with $t \in E_q$ and $m \in Z^d$ and obtain

$$\begin{aligned}
\mathcal{Q}(x,\xi)_{r,s} &= e^{-2\pi i\alpha k_s\cdot\xi} \sum_{t\in E_q}\sum_{m\in\mathbb{Z}^d} \overline{g}\Big(x - \frac{m_s}{\beta} + \alpha t - \alpha q m\Big) e^{-2\pi i\alpha qm\cdot\xi} e^{2\pi i\alpha t\cdot\xi} e^{-2\pi i\frac{p}{q} r\cdot t} \\
&= e^{-2\pi i\alpha\xi\cdot\frac{s+qm_s}{p}} \sum_{t\in E_q} \overline{Z_{\alpha q} g}\Big(x - \frac{m_s}{\beta} + \alpha t, \xi\Big) e^{2\pi i\alpha t\cdot\xi} e^{-2\pi i\frac{p}{q} r\cdot t},
\end{aligned}$$

since $k_s = (s + qm_s)/p$. As a matrix multiplication, this is precisely

$$\mathcal{Q}(x,\xi) = W \operatorname{diag}\left(e^{2\pi i\alpha t\cdot\xi}\right)_{t\in E_q} \tilde{\mathcal{P}}(x,\xi) \operatorname{diag}\left(e^{-2\pi i\alpha\xi\cdot\frac{s+qm_s}{p}}\right)_{s\in E_p}.$$

$\square$

Due to the one-to-one correspondence between $s \in E_p$ and $m_s \in E_p$, the matrix $\tilde{\mathcal{P}}(x,\xi)$ differs only by a permutation of columns from $\mathcal{P}(x,\xi)$. Furthermore, $W^*W = q^d I_{E_q}$ and the remaining diagonal matrices are unitary. Consequently, all results of section 4.3 also hold for the matrices $\mathcal{A}(x,\xi)$ and $\mathcal{Q}(x,\xi)$ instead of $\mathcal{R}(x,\xi)$ and $\mathcal{P}(x,\xi)$ respectively.

Recall that by Theorem 4.3.5, the spectral bounds of $\mathcal{R}(x,\xi) = \mathcal{P}(x,\xi)^*\mathcal{P}(x,\xi)$ satisfy

$$\begin{aligned}\lambda^{\mathcal{R}}_{\max} &= \beta^d B_{\text{opt}}\\ \lambda^{\mathcal{R}}_{\min} &= \beta^d A_{\text{opt}}\end{aligned}$$

where $A_{\text{opt}}, B_{\text{opt}} > 0$ denote the optimal frame bounds of $\mathcal{G}(g,\alpha,\beta)$.

Let $U := \operatorname{diag}\left(e^{-2\pi i\alpha\xi\cdot\frac{s+qm_s}{p}}\right)_{s\in E_p}$, then

$$\mathcal{A}(x,\xi) = \frac{\alpha^d}{p^d}\mathcal{Q}(x,\xi)^*\mathcal{Q}(x,\xi) = \frac{(\alpha q)^d}{p^d}U^*\tilde{\mathcal{P}}(x,\xi)^*\tilde{\mathcal{P}}(x,\xi)U,$$

where $U^*\tilde{\mathcal{P}}(x,\xi)^*\tilde{\mathcal{P}}(x,\xi)U$ has the same spectral bounds as $\mathcal{R}(x,\xi)$. This implies that the spectral bounds of $\mathcal{A}(x,\xi)$ are precisely the optimal frame bounds since $(\alpha q/p)^d = \beta^d$.

A. Appendix

A.1. Measurability in Linear Algebra

Let M_n denote all complex-valued $n \times n$-matrices and H_n all $n \times n$ Hermitian matrices. We equip M_n with the usual (product) topology along with the induced Borel σ-algebra.

Furthermore, let $\pi_j : \prod_{k=1}^{N} X_k \to X_j$ denote the natural projection upon the j-th coordinate.

Theorem A.1.1 (Azoff [2]). *Let X and Y be complete separable metric spaces and E a closed σ-compact subset of $X \times Y$. Then $\pi_1(E)$ is a Borel set in X and there exists a Borel-measurable function $\varphi : \pi_1(E) \to Y$ whose graph is contained in E.*

Proof. An elementary proof can be found in Azoff's original paper [2]. $\square$

Corollary A.1.2. *There exist Borel-measurable functions $\psi_U, \psi_D : H_n \to M_n$ associating with each Hermitian matrix A a unitary matrix $\psi_U(A)$ and a diagonal matrix $\psi_D(A)$ such that $\psi_U(A)^* A \psi_U(A) = \psi_D(A)$.*

Proof. This follows immediately from the proof of Corollary 2 in [2]. $\square$

Lemma A.1.3. *Let $A(x) = \big(f_{kl}(x)\big)_{k,l=1}^{n}$ for all $x \in \mathbb{R}^d$ denote a matrix-valued function on $\mathbb{R}^d$. If all $f_{kl} : \mathbb{R}^d \to \mathbb{C}$ are (Lebesgue-Borel) measurable, then $A : \mathbb{R}^d \to M_n$ is (Lebesgue-Borel) measurable.*

Proof. Since every open set in M_n can be written as a countable union of sets of the form

$$B := \prod_{k,l=1}^{n} (a_{kl}, b_{kl}),$$

it suffices to show that the pre-image of those sets under A are measurable.

Let B be such a product of arbitrary open intervals. Then clearly, the pre-image is

$$\begin{aligned} A^{-1}(B) &= \{x \in \mathbb{R}^d : a_{kl} < f_{kl}(x) < b_{kl} \quad \forall k, l = 1, \dots, n\} \\ &= \bigcap_{k,l=1}^{n} f_{kl}^{-1}\big((a_{kl}, b_{kl})\big) \end{aligned}$$

and hence measurable. $\square$

Corollary A.1.4. *Let* $A(x) = \big(f_{kl}(x)\big)_{k,l=1}^{n}$ *for all* $x \in \mathbb{R}^d$ *denote a matrix-valued function on* $\mathbb{R}^d$ *where all* $f_{kl} : \mathbb{R}^d \to \mathbb{C}$ *are (Lebesgue-Borel) measurable. Furthermore, suppose* $A(x)$ *is Hermitian for all* $x \in \mathbb{R}^d$*. Then there exit exist (Lebesgue-Borel) measurable functions* $\psi_U, \psi_D : \mathbb{R}^d \to M_n$ *associating to each* $x \in \mathbb{R}^d$ *a unitary matrix* $\psi_U(x)$ *and a diagonal matrix* $\psi_D(x)$ *such that* $\psi_U(x)^* A(x) \psi_U(x) = \psi_D(x)$*.*

Proof. Combine Lemma A.1.3 with Corollary A.1.2. $\square$

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Zusammenfassung

Gaborframes liefern stabile, diskrete Zeit-Frequenz-Darstellungen in $L^2(\mathbb{R}^d)$. Es ist daher von großem Interesse, Paare (g, Λ), bestehend aus einer L^2-Funktion g und einem Zeit-Frequenz-Gitter Λ, zu bestimmen, die einen Frame erzeugen.

Die Fortschritte der Gaboranalysis der letzten dreißig Jahre lieferten zahlreiche Charakterisierungen der Frameungleichung für Gaborsysteme. Wir geben eine vollständige Darstellung bekannter Resultate. Die Neuheit liegt in der systematischen Herangehensweise: Das zentrale Resultat ist die Dualitätstheorie für Gaborframes über beliebige Gitter. Wir entwickeln daher die Dualitätstheorie zuerst und erhalten dann daraus alle Charakterisierungen für separable Gitter als einfache Folgerungen. Das wohlbekannte Kriterium von Ron und Shen folgt durch ein einfaches Fourierreihenargument. Im Fall eines rationalen Gitters liefert eine weitere Periodisierung die Resultate von Zeevi und Zibulski, in denen die Frameeigenschaft mit den Spektraleigenschaften einer Familie von endlichdimensionalen Matrizen verknüpft wird.

Für M^1-Fenster liefert die Kombination der Dualitätstheorie mit Wieners Lemma ein Dutzend zusätzliche Charakterisierungen ohne Ungleichungen, das heißt als Eigenschaften der zum Gaborsystem gehörenden kanonischen Operatoren.