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Seperable quotients of function spaces and Efimov spaces

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Abstract

This master thesis deals with the separable quotient problem for locally convex spaces of the form $C_p(X)$ which is still open. Its connection with another open problem from topology, namely Efimov's problem, will be addressed. Efimov asked whether there are compact Hausdorff spaces with no convergent sequences and no copies of $\beta\omega$ inside. Nowadays such spaces are called Efimov spaces. A recent result of Kąkol and Śliwa [19] will be presented. They showed that compact Hausdorff spaces X, for which $C_p(X)$ has no separable quotient, are Efimov spaces with few homeomorphisms. Based on de la Vega's ideas in [6], such a space will be constructed using the set-theoretic assumption $\diamondsuit$. Moreover, possible limitations of the construction of such a space are studied. This is the area of research where set theory meets topology and functional analysis.

Zusammenfassung

In der vorliegenden Masterarbeit wird das Problem des separablen Quotienten für lokal-konvexe Räume der Form $C_p(X)$, welches immer noch ungelöst ist, behandelt. Der Zusammenhang mit einem weiteren ungelösten Problem, nämlich dem Efimov-Problem, wird aufgezeigt. Efimov brachte die Frage auf, ob es kompakte Hausdorffräume gibt, die weder konvergente Folgen noch eine Kopie von $\beta\omega$ beinhalten. Heutzutage werden solche Räume Efimovräume genannt. Ein aktuelles Resultat von Kąkol und Śliwa [19] wird präsentiert. Es zeigt, dass kompakte Hausdorffräume X, für die $C_p(X)$ keinen separablen Quotienten besitzt, Efimovräume mit wenigen Homöomorphismen sind. Gestützt auf die Ideen de la Vegas in [6] wird in dieser Arbeit ein solcher Raum unter der mengentheoretischen Annahme $\diamondsuit$ konstruiert. Außerdem werden mögliche Einschränkungen für die Konstruktion eines solchen Raumes untersucht. In diesem Forschungsbereich trifft die Mengenlehre auf die Funktionalanalysis und die Topologie.

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Contents

Introduction

In 1932, Stanislaw Mazur asked a question, which remains one of the most famous problems in functional analysis. It asks whether every Banach space has an infinite-dimensional, separable quotient. This question is partially solved: there are classes of Banach spaces for which we can find such separable quotients. One of the main examples is the class consisting of the Banach spaces of real-valued continuous functions on an infinite compact space with the supremum-norm. A theorem of Rosenthal-Lacey states that every Banach space of this form possesses a quotient isomorphic to c (the space of all convergent sequences of real numbers) or to ℓ_2 (the space of square-summable sequences of real numbers), which are both separable (cf. [19, pp. 104-105]).

Mazur's question makes sense for arbitrary topological vector spaces, in particular for locally convex spaces. One of the most studied examples of locally convex spaces are those of the form $C_p(X)$, the continuous real-valued functions on X with the topology of pointwise convergence. The question, whether all locally convex spaces of this form have an infinite-dimensional, separable quotient is also unsolved until today. However, by a recent result of Kąkol and Śliwa [19, Theorem 4], if X is a compact Hausdorff space such that the locally convex space $C_p(X)$ has no separable quotient, then X cannot contain a copy of $\beta\omega$, the Stone-Čech compactification of the integers. Furthermore, it is not very difficult to show that in this case, X cannot contain convergent sequences either. This means that X must be an Efimov space. These topics will be treated the first chapter.

At that point we encounter another open problem, this time from topology: Are there Efimov spaces? This question was posed by Efimov in 1969, but in fact its origins go back to the time before Mazur formulated his famous question. Alexandroff and Urysohn already asked in 1929, whether there are compact Hausdorff spaces with no convergent sequences (cf. [16, p. 171]). In the next decades, several counter-examples arose. One of them was the space $\beta\omega$, which is a compact Hausdorff space without convergent sequences. This clearly motivates the addition of the supplementary condition (not containing $\beta\omega$) to Alexandroff's and Urysohn's question, in order to obtain the formulation of Efimov's problem. This problem is also partially solved, meaning that under some set-theoretic assumptions (e.g. $\diamondsuit$, CH or $\mathfrak{s} = \omega_1 \wedge \mathfrak{c} = 2^{\omega_1}$) one can construct Efimov spaces (cf. [16, p. 173]). However, there is no known construction without using additional set-theoretic assumptions, i.e. it is not known whether ZFC proves the existence of an Efimov space.

As the first chapter will also show, Kąkol and Śliwa's theorem implies that, if X is a completely regular space with many homeomorphisms between compact subspaces, then $C_p(X)$ has a separable quotient. From this it follows for a compact Hausdorff space X, that if $C_p(X)$ has no separable quotient, then X is not only Efimov, but in addition it cannot have compact disjoint homeomorphic subsets. It is therefore interesting to ask, whether there are Efimov spaces with no infinite closed disjoint homeomorphic subsets. Based on a result of de la Vega ([6, Theorem 5.1] or [7, Theorem 1.5]), who showed under $\diamondsuit$ that there are rigid Efimov spaces, we will construct in Chapter 2 – also under $\diamondsuit$ – an Efimov space with no closed infinite disjoint homeomorphic subsets.

Therefore, it is consistent with ZFC that such Efimov spaces with few homeomorphisms exist, which in turn will mean that the existence of a locally convex space of the form $C_p(X)$ with no separable quotient cannot be excluded this way. At this point it is natural to ask whether this could be achieved by using other set-theoretic assumptions. Therefore, in the third chapter we will study sufficient conditions under which a compact Hausdorff space contains a convergent sequence, and can thus not be Efimov, i.e. we will address the limitations on the constructions of Efimov spaces. This chapter contains two main theorems. One of them is a result of Szentmiklóssy [24, Theorem 3], which implies that the construction from the second chapter cannot be done under MA_{ω_1}. The other one is the Čech-Pospišil-Theorem, which provides a limitation on the size of Efimov spaces. We will deduce that under $\diamondsuit$ there are no Efimov spaces of size less or equal than $\mathfrak{c}$ and can thus determine the size of the space from Chapter 2. To complete the picture, this chapter will be closed with a construction of an Efimov space of size $\mathfrak{c}$, which was done by Fedorčuk using the assumption $\mathfrak{s} = \omega_1 \wedge \mathfrak{c} = 2^{\omega_1}$.

Chapter 1

Separable quotients of function spaces with the topology of pointwise convergence

1.1 The topological vector space of real-valued continuous functions

First we fix our notation and terminology. If V is a vector space over the field $\mathbb{R}$, we say that V is a *topological vector space* if it is endowed with a topology such that the addition $+ : V \times V \to V$ and the scalar multiplication $\cdot : \mathbb{R} \times V \to V$ are continuous maps. If V is a topological vector space and M a linear subspace of V, we can define an equivalence relation on V by calling two points of V equivalent if their difference lies in M. It is natural to define addition and multiplication on the quotient V/M (the set of all equivalence classes induced by the above relation) pointwise, which yields a vector space structure on it. Furthermore one can define a topological structure on this quotient, by declaring a set $U \subseteq V/M$ to be open if and only if the set of all points in V whose equivalence class lies in U is open in the topological space V. It is an exercise to show that this way, the quotient V/M becomes also a topological vector space. The letter ρ will denote the quotient map $V \to V/M$, which maps each x to its equivalence class, which will be written as $x + M$. The map ρ is continuous and open. Usually this construction is considered for closed linear subspaces M, because V/M is Hausdorff if and only if M is closed.

A special class of topological vector spaces are locally convex spaces. Recall that a subspace K of a vector space over $\mathbb{R}$ is called *convex*, if for any two points

$x, y \in K$ and for each $t \in [0,1]$ we have that $tx + (1-t)y \in K$. A *locally convex space* is a topological vector space such that each point has a neighborhood base consising of convex sets, equivalently the point 0 has a neighborhood base consising of convex sets. Such spaces can also be characterized by so-called semi-norms. For more details, see [22, Definition 1.33, Theorems 1.36 and 1.37].

Recall also that a *Banach space* is a topological vector space, whose topology is induced by a norm, such that every Cauchy sequence converges (i.e. the space is complete). It is well-known that quotients of Banach spaces by closed subspaces are again Banach spaces. This result can also be generalized for locally convex spaces as follows. For proofs, we refer to [22, Theorem 1.41] and [8, (12.14.8)].

Theorem 1.1.1. *The quotients of a Banach spaces by a closed linear subspace is again a Banach space. The quotient of a locally convex space by any linear subspace is again a locally convex space.*

For a topological space X the set $C(X)$ of all continuous functions from X to $\mathbb{R}$ can be considered as a linear subspace of $\mathbb{R}^X = \prod_{x \in X} \mathbb{R}$, the topological vector space of all functions from X to $\mathbb{R}$ with pointwise addition and scalar multiplication, endowed with the product topology. The set $C(X)$ together with the linear and topological structures inherited from $\mathbb{R}^X$ forms a topological vector space. Because the convergence of a sequence of functions in this space coincides with pointwise convergence, its topology is called the *topology of pointwise convergence.* Therefore we will denote the set $C(X)$ with its topological vector space structure by $C_p(X)$. The topology on $C_p(X)$ is clearly generated by the sets of the form $\{f \in C(X) : f(x_i) \in (r_i - \varepsilon_i, r_i + \varepsilon_i) \text{ for } i \in n\}$, where $x_0, ..., x_{n-1}$ are distinct points of X, $r_0, ..., r_{n-1} \in \mathbb{R}$ and $\varepsilon_0, ..., \varepsilon_{n-1} > 0$. Sets of this form will be called *basic open sets* in this context. Note also that $C_p(X)$ is a locally convex space: it is not difficult to check that the basic open neighborhoods of 0 are convex.

With all these notions in our pocket we can formulate the separable quotient problem for topological vector spaces of the form $C_p(X)$.

Question 1.1.2. Does every infinite-dimensional topological vector space of the form $C_p(X)$ have an infinite-dimensional separable quotient? In other words: is there a closed linear subspace M of $C_p(X)$ such that the topological vector space $C_p(X)/M$ is infinite-dimensional and separable?

This is a natural analog of the question asked by Mazur in 1932 for Banach spaces, and it can also be considered for several other types of spaces. If for some infinite-dimensional $C_p(X)$ the above question has a positive answer, we say that $C_p(X)$ *has SQ*. If not, we say that $C_p(X)$ *fails to have SQ*.

Note that the question asks only for infinite-dimensional quotients, because one can always find separable quotients of finite dimension. For example one can take M to be the whole $C_p(X)$ and therefore get the quotient to be a trivial vector space, which is clearly separable. It is also not very hard to find one-dimensioinal quotients, which then are isomorphic to the separable space $\mathbb{R}$. This is a consequence of the Hahn-Banach Theorem.

Now that we introduced the topic, we recall some useful facts about (topological) vector spaces. The first fact is a classical isomorphism theorem, which in fact holds generally for vector spaces.

Theorem 1.1.3 (Isomorphism Theorem for vector spaces)**.** *Let V be a topological vector space over $\mathbb{R}$, and $M \subseteq W \subseteq V$ be subspaces of V and $\varphi : V \to \mathbb{R}$ a linear functional. Then the quotient $V/\ker\varphi$ is isomorphic to $\mathrm{im}(\varphi)$. Moreover, we have that $(V/M)/(W/M)$ is isomorphic to V/W. Also, if V is finite-dimensional, V/M is too and we have the dimension formula $\dim V/M = \dim V - \dim M$.*

The next lemma will also be useful, its proof can be found in [4, Corollaire 1, Section I.3] or in [22, Theorem 1.21] for complex vector spaces.

Lemma 1.1.4. *Let M be a finite-dimensional linear subspace of a Hausdorff topological vector space V. Then M is closed.*

We will also use the following result, which is a corollary from the Hahn-Banach Theorem (see [22, Theorem 3.5]).

Theorem 1.1.5. *Let M be a closed linear subspace of a locally convex space V and $x \in V \setminus M$. Then there there is a continuous linear functional $\varphi : V \to \mathbb{R}$ such that $M \subseteq \ker\varphi$ and $\varphi(x) = 1$.*

1.2 Separable algebra quotients

In this section, we will present for infinite completely regular spaces X a characterisation of $C_p(X)$ having a special kind of separable quotient, namely a separable quotient algebra. The source for this is [18].

An *algebra* is a topological vector space V over $\mathbb{R}$, which in addition has an associative, commutative and bilinear vector multiplication, i.e. $(\lambda x)\cdot(\mu y) = \lambda\mu(x\cdot y)$ for all $\lambda, \mu \in \mathbb{R}$ and $x, y \in V$ and the usual distributivity laws should hold, i.e.

$x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in V$. As in the previous section, one can consider quotients of such algebras by closed linear subspaces. It turns out that these quotients may lose their multiplicative algebra structure. However, this structure can be saved, if we consider quotients by ideals. An *ideal* of an algebra V is a linear subspace I of V such that for all $x \in V$ and all $y \in I$, we have $xy \in I$. It is an exercise to check the already mentioned result.

Lemma 1.2.1. *If I is an ideal of an algebra V, then V/I is also an algebra.*

It is clear that the topological vector space $C_p(X)$ forms an algebra with the pointwise multiplication of functions. It makes therefore sense to consider first the special class of algebra quotients of $C_p(X)$ among all quotients. Recall that we are only interested in quotients by closed ideals, because then the quotient is Hausdorff. Therefore, the separable quotient problem for algebras of the form $C_p(X)$ can be rephrased as follows.

Question 1.2.2. Does every infinite-dimensional topological vector space of the form $C_p(X)$ have an infinite-dimensional separable quotient, which is itself again an algebra? In other words: is there a closed ideal I of $C_p(X)$ such that the topological vector space quotient $C_p(X)/I$ is infinite-dimensional and separable?

To shorten our language, we will call such a quotient $C_p(X)/I$ simply a *separable algebra quotient*.

It is natural to ask how the closed ideals of $C_p(X)$ look like. In fact, if X is infinite and completely regular, they can be described very simply, as we will see in Lemma 1.2.4.

In this context, Urysohn's Lemma is very useful. This fact is known for permitting to separate two disjoint closed subsets of a normal space by a continuous function. As in this chapter we will work with completely regular spaces, we formulate a similar fact for this class of topological spaces, see [21] for the proof.

Lemma 1.2.3 (Urysohn's Lemma for completely regular spaces)**.** *Let X be a completely regular space, $K \subseteq X$ compact and $B \subseteq X$ closed such that K and B are disjoint. Then for each $g \in C(K)$ there is some $f \in C(X)$ such that $f \restriction K = g$ and $f \restriction B = 0$.*

Notice that using this lemma, one can see that $C_p(X)$ is a dense subset of the whole product $\mathbb{R}^X$.

Lemma 1.2.4. *Let X be an infinite, completely regular space. The closed ideals of $C_p(X)$ are exactly the sets of the form*

$$I_A := \{f \in C(X) : f(x) = 0 \text{ for all } x \in A\},$$

where A is a closed subspace of X.

Proof. Obviously I_A is a closed ideal of $C_p(X)$. Now let I be a closed ideal of $C_p(X)$ and let $A := \bigcap_{f \in I} f^{-1}(0)$, which is clearly closed by the continuity of all $f \in I$. We show that $I = I_A$. It is clear that $I \subseteq I_A$, because if $f \in I$, then for each $x \in A$ we have $f(x) = 0$.

Conversely, if $f \in I_A$ then $f \restriction A = 0$. Since I is closed, it is enough to show that any basic open neighborhood of f contains a member if I. Let U be a basic open neighborhood of f, i.e. it can be written as $\{g \in C(X) : g(x) \in (f(x) - \varepsilon_x, f(x) + \varepsilon_x) \text{ for } x \in F\}$, where $F \subseteq X$ is finite and ε_x are positive real numbers for $x \in F$. Decompose $F = F_0 \cup F_{\neq 0}$, were $F_0 := \{x \in F : f(x) = 0\}$ and $F_{\neq 0} := F \setminus F_0$. For each $x \in F_{\neq 0}$, we know that $x \notin A$, meaning that there is some $g_x \in I$ such that $g_x(x) \neq 0$. It is clear that the function $g := \sum_{x \in F_{\neq 0}} g_x^2$ is a member of I such that $g(x) \neq 0$ for all $x \in F_{\neq 0}$. For each $x \in F_{\neq 0}$ we may set $h_0(x) := \frac{f(x)}{g(x)}$ and using Lemma 1.2.3 extend h_0 to a continuous function $h \in C(X)$ such that $h \restriction (A \cup F_0) = 0$. We may see that the set $\{x \in X : (g \cdot h)(x) = f(x)\}$ contains F, because if $x \in F_0$, $f(x) = 0 = h(x) = (g \cdot h)(x)$ and if $x \in F_{\neq 0}$ we have that $(g \cdot h)(x) = (g \cdot h_0)(x) = f(x)$. It follows that $g \cdot h$, which is clearly a member of the ideal I, lies in U. This ends the proof. □

The following result is [18, Lemma 15].

Proposition 1.2.5. *Let X be an infinite, completely regular space and A a closed subset of X. Then $C_p(X)/I_A$ is isomorphic to a dense subset of $C_p(A)$.*

Proof. The map $\iota : C_p(X)/I_A \to C_p(A), f + I_A \mapsto f \restriction A$ is a homomorphism. It is well-defined and injective, because $f + I_A = g + I_A$ is equivalent to $f \restriction A = g \restriction A$. So ι is an isomorphism from $C_p(X)/I_A$ onto $\operatorname{ran}(\iota) \subseteq C_p(A)$.

Moreover, $\operatorname{ran}(\iota)$ is dense in $C_p(A)$, because by Lemma 1.2.3 for all distinct $a_0, ..., a_{n-1}$ in A and $r_0, ..., r_{n-1} \in \mathbb{R}$ we can find a continuous function $f \in C(X)$ such that $f(a_i) = r_i$ for all $i \in n$. It is clear that $f \restriction A \in \operatorname{ran}(\iota)$. This way, $\operatorname{ran}(\iota)$ meets every non-empty basic open set, and therefore it is dense in $C_p(A)$. □

Recall that x is a *limit point* of a set A, if for every open neighborhood U of x, we have $U \cap A \setminus \{x\} \neq \emptyset$. For a Hausdorff space X, this is equivalent to saying that for every open neighborhood U of x, we have $U \cap A \setminus \{x\}$ is infinite. There is a useful characterisation of closed sets via limit points, namely a set A is closed if and only if it contains all its limit points (see e.g.[10, Theorem 1.3.4(i)]). This will then be used to prove the following lemma from [18].

Proposition 1.2.6. *Let X be an infinite, completely regular space. If $C_p(X)$ is separable, X has a closed infinite countable subset.*

Proof. Suppose that X had no closed infinite countable subsets. We show that $C_p(X)$ is not separable, i.e. no countable set $\{f_0, f_1, ...\}$ is dense in $C_p(X)$.

Let $f_0, f_1, ... \in C(X)$. If we can find $y_1 \neq y_2$ in X such that $|f_n(y_1) - f_n(y_2)| \leq 1$ for each natural number n, the set $\{f_0, f_1, ...\}$ is not dense in $C_p(X)$. Indeed, by Lemma 1.2.3 we can find some $h \in C(X)$ such that $h(y_1) = 5$ and $h(y_2) = 9$. Let U be the open set $\{h+g : g \in C(X) \text{ with } |g(y_i)| < 1 \text{ for } i \in \{1,2\}\}$. If $f_n \in U$ for some natural number n, then $f_n = h + g$ for some $g \in C(X)$ with $|g(y_i)| < 1$ for $i \in \{1, 2\}$ and furthermore

$$\begin{aligned} |f_n(y_1) - f_n(y_2)| &= |h(y_1) - h(y_2) + g(y_1) - g(y_2)| \\ &\geq ||h(y_1) - h(y_2)| - |g(y_1) - g(y_2)|| \\ &= |4 - |g(y_1) - g(y_2)|| \\ &\geq 2, \end{aligned}$$

which is a contradiction. Hence the set $\{f_0, f_1, ...\}$ does not intersect the open set U, i.e. $\{f_0, f_1, ...\}$ is not dense in $C_p(X)$.

It remains to find distinct y_1, y_2 in X such that $|f_n(y_1) - f_n(y_2)| \leq 1$ for each natural number n. Note first that any countable infinite set S has more than one (in fact uncountably many) limit points, because otherwise the union of S with the set of all its limit points would be a closed countable infinite set, contradicting the assumption. Fix any countable infinite set S and let $y_1 \in X$ be one of its limit points. In particular, y_1 is non-isolated in X. By continuity of f_0 find some open neighborhood U_0 of y_1 such that $f_0[U_0] \subseteq (f_0(y_1) - 1/2, f_0(y_1) + 1/2)$. By setting $V_0 := \overline{U_0}$ and again using continuity of f_0, we get $f_0[V_0] \subseteq [f_0(y_1) - 1/2, f_0(y_1) + 1/2]$.

Furthermore by the continuity of f_1 there is some open neighborhood N_1 of y_1 such that $N_1 \subseteq U_0$ and $f_1[N_1] \subseteq (f_0(y_1) - 1/2, f_0(y_1) + 1/2)$. As y_1 is not isolated, N_1 is not a singleton, so we can find a point in N_1 distinct from y_1. Separate these two points by open subsets of N_1, and denote by U_1 the open subset containing y_1. By letting $V_1 := \overline{U_1}$, it is clear that V_1 is strictly contained in V_0 and again using continuity of f_1 we get $f_1[V_1] \subseteq [f_1(y_1) - 1/2, f_1(y_1) + 1/2]$.

By proceeding this way, we get a strictly decreasing sequence $\langle V_n : n \in \omega \rangle$ of closed neighborhoods of y_1 such that the diameter of $f_n[V_n]$ is at most 1. We may thus pick $x_n \in V_{n+1} \setminus V_n$ for all $n \in \omega$. By the above remark, the countable infinite set $\{x_n : n \in \omega\}$ has more than one limit point. Therefore we can find some $y_2 \neq y_1$, which is a limit point of $\{x_n : n \in \omega\}$. For a fixed n, there are only finitely many x_k's which lie outside of V_n and because for each open neighborhood U of y_2 infinitely many x_k's lie in $U \setminus \{y_2\}$, there is some $x_k \in V_n \cap U \setminus \{y_2\}$. This means that every open neighborhood of y_2 intersects $V_n \setminus \{y_2\}$, i.e. y_2 is also a limit point of V_n. We chose V_n to be closed, hence $y_2 \in V_n$. The desired inequality follows now from the fact that for all $n \in \omega$ the diameter of $f_n[V_n]$ is at most 1. This ends the proof. □

We can conclude with an elegant characterisation for $C_p(X)$ having a separable algebra quotient, proven by Kąkol and Saxon in [18, Theorem 18]. This characterisation will provide an answer for the separable quotient problem for algebras of the form $C_p(X)$, which will be revealed in Example 1.4.3 at the end of the chapter.

Theorem 1.2.7 (Kąkol, Saxon)**.** *Let X be an infinite, completely regular space. Then $C_p(X)$ has a separable algebra quotient if and only if X contains a closed infinite countable subset.*

Proof. If $C_p(X)$ has a separable algebra quotient, there is some closed $A \subseteq X$ such that $C_p(X)/I_A$ is infinite-dimensional and separable. By Proposition 1.2.5, a dense subset of $C_p(A)$ is also separable and therefore so is $C_p(A)$. The set A is infinite, because otherwise $\mathbb{R}^A$ would have finite dimension and thereby also $C_p(X)/I_A$, as it is isomorphic to a subspace of $C_p(A)$, which in turn is a subspace of $\mathbb{R}^A$. Therefore we can use Proposition 1.2.6 to conclude that A contains some closed (in A) infinite countable D. However, A is closed in X and so D is also a closed subset of X.

Conversely, if X contains a closed infinite countable subset D, then by Proposition 1.2.5 the quotient $C_p(X)/I_D$ is isomorphic to a dense subset S of $C_p(D)$. The space S can not be finite-dimensional, because otherwise $C_p(D) = \overline{S} = S$, which would mean that $C_p(D)$ is finite-dimensional. In particular, $C_p(D)$ would then be a closed dense subspace of $\mathbb{R}^D$, which would mean that $\mathbb{R}^D = C_p(D)$ was also of finite dimension. This is a contradiction to the fact that D is infinite. Moreover, also because $C_p(D)$ is a dense subset of $\mathbb{R}^D$, we have that S is also dense in $\mathbb{R}^D$. The latter being a separable and metrizable space, we can conclude that S and hence the quotient algebra $C_p(X)/I_D$ is separable. □

In fact, in [18] it was shown more: the above equivalent conditions are also equivalent to the statement that $C_c(X)$ has a separable quotient, where the latter denotes the set $C(X)$ but this time with the so-called compact-open topology.

1.3 Characterisation of spaces X such that $C_p(X)$ has SQ

In the last section, we saw a characterisation for $C_p(X)$ having a separable algebra quotient for infinite completely regular spaces X. This is clearly a stronger statement than having only an arbitrary separable quotient. Now we are going to focus on general quotients and therefore prove a characterisation for $C_p(X)$ having a separable quotient, where X is again an infinite completely regular space. Unfortunately, this characterisation is not as easy to handle as the one we have already treated. The next statement is [19, Lemma 7], the proof is omitted there.

Proposition 1.3.1. *If X is an infinite completely regular space, the following are equivalent:*

(i) $C_p(X)$ *has SQ.*

(ii) There is a strictly increasing sequence $\langle A_n : n \in \omega \rangle$ *of closed vector subspaces whose union is dense in* $C_p(X)$.

(iii) There is a sequence $\langle \varphi_n : n \in \omega \rangle$ *of non-zero continuous linear functionals on* $C_p(X)$ *such that* $E := \bigcup_{m \in \omega} \bigcap_{n \geq m} \ker \varphi_n$ *is dense in* $C_p(X)$.

(iv) There is a sequence $\langle F_n : n \in \omega \rangle$ *of finite subsets of* X *and non-zero functions* $\xi_n : F_n \to \mathbb{R}$ *such that for any finite* $G \subseteq X$, $g : G \to \mathbb{R}$ *and* $\varepsilon > 0$ *there is a continuous function* $f \in C(X)$ *such that:*

(A) $\sum_{x \in F_n} \xi_n(x) f(x) = 0$ *for all but finitely many* $n \in \omega$,

(B) $|g(x) - f(x)| < \varepsilon$ *for all* $x \in G$.

Proof. (i) $\Rightarrow$ (ii): Let $C_p(X)/M$ be a separable infinite-dimensional quotient and fix a countable dense set $D \subseteq C_p(X)/M$. Notice that the subspace $\operatorname{span} D$ is $\aleph_0$-dimensional. Indeed, it is clear that its dimension cannot be bigger than $\aleph_0$, because it is spanned by a countable set. Moreover if $\operatorname{span} D$ was finite-dimensional and B was a finite base for $\operatorname{span} D$, we would have that

$$C_p(X)/M = \overline{D} = \overline{\operatorname{span} B} = \operatorname{span} B,$$

which implies that the quotient is finite-dimensional.

Hence, we can pick a base $\{x_k : k \in \omega\}$ for $\operatorname{span} D$. We let $A_n := \rho^{-1}[\operatorname{span}\{x_k : k \leq n\}]$, where ρ denotes the quotient map $C_p(X) \to C_p(X)/M$, which is clearly a closed linear subspace of $C_p(X)$. Because $\{x_k : k \in \omega\}$ is a linearly independent set, we have also that A_n is strictly contained in A_{n+1}. To see that $\bigcup_{n\in\omega} A_n$ is dense in $C_p(X)$, we observe that

$$\begin{aligned}\bigcup_{n\in\omega} A_n &= \rho^{-1}[\bigcup_{n\in\omega} \operatorname{span}\{x_k : k \leq n\}] \\ &= \rho^{-1}[\operatorname{span}\{x_k : k \in \omega\}] = \rho^{-1}[\operatorname{span} D] \supseteq \rho^{-1}[D].\end{aligned}$$

Notice that the latter is dense, because every open $U \neq \emptyset$ in $C_p(X)$ meets D. Indeed, $\rho[U] \neq \emptyset$ is open in $C_p(X)/M$ and therefore one can find a point $y \in \rho[U] \cap D$ and hence also some $x \in U$ such that $x + M = y \in D$, which implies that $x \in \rho^{-1}[D]$.

(ii) $\Rightarrow$ (iii): Fix a strictly increasing sequence $\langle A_n : n \in \omega\rangle$ of closed linear subspaces of $C_p(X)$ such that $\bigcup_{n\in\omega} A_n$ is dense and let $x_n \in A_{n+1} \setminus A_n$. Then for each natural number n, we can find by Theorem 1.1.5 a continuous functional $\varphi_n : C_p(X) \to \mathbb{R}$ such that $A_n \subseteq \ker \varphi_n$ and $\varphi_n(x_n) = 1$. Clearly, each φ_n is non-trivial and because $\langle A_n : n \in \omega\rangle$ is increasing, we have $A_n \subseteq \bigcap_{m\geq n} \ker \varphi_m$. Hence $\bigcup_{n\in\omega} \bigcap_{m\geq n} \ker \varphi_m \supseteq \bigcup_{n\in\omega} A_n$ is dense in $C_p(X)$.

(iii) $\Rightarrow$ (i): Let $M := \bigcap_{n\in\omega} \ker \varphi_n$. We show that $C_p(X)/M$ is separable and infinite-dimensional. Let $N_n := \bigcap_{m\geq n} \ker \varphi_m$, which is clearly a closed linear subspace. The sequence $\langle N_n : n \in \omega\rangle$ is clearly increasing but not necessarily strictly increasing. By removing the recurring terms from the sequence, we get an infinite, strictly increasing subsequence $\langle N_{n_k} : k \in \omega\rangle$.

Indeed, if by removing the recurring terms we end up with a finite sequence, then there is some $n \in \omega$ such that for each $m \geq n$ we have $N_m = N_n$. Because $N_n \neq X$ is closed and E is dense in $C_p(X)$, we can find some $f \in E$ such that $f \notin N_n$. However, $f \in E$ implies that $f \in N_k$ for some natural number k. No matter whether $k \geq n$ or $k < n$, we obtain $f \in N_n$, a contradiction.

To simplify our notation, we let $M_k := N_{n_k}$ and $\psi_k := \varphi_{n_k}$. It is a simple task to check that $M_k = \ker \psi_k \cap M_{k+1}$ for each natural number k and hence $\ker \psi_k$ does not contain M_{k+1}. For each k we may pick some $f_k \in M_{k+1} \setminus M_k$. Notice also the kernel of the restriction $\psi_k \restriction M_{k+1}$, is just $\ker \psi_k \cap M_{k+1} = M_k$. By the Isomorphism Theorem for vector spaces (1.1.3), we get that the quotient M_{k+1}/M_k is isomorphic to $\operatorname{im}(\psi_k \restriction M_{k+1})$. The latter, being a non-trivial subspace of the vector space $\mathbb{R}$

(as $\ker \psi_k \not\supseteq M_{k+1}$), coincides with the vector space $\mathbb{R}$. Consequently M_{k+1}/M_k is one-dimensional and therefore generated by $\{f_k + M_k\}$.

We claim that the countable set $D := \{\sum_{j\in k} \alpha_j f_j + M : k \in \omega, \alpha_j \in \mathbb{Q} \text{ for } j \in k\}$ is dense in $C_p(X)/M$. Let $\widetilde{U} \neq \emptyset$ be an open subset of the quotient space. Then $U := \rho^{-1}[\widetilde{U}] \neq \emptyset$ is an open subset of $C_p(X)$ and therefore we can find a function $g \in E \cap U$. This means that there is a natural number k such that $g \in M_k$. If $k = 0$, $g + M$ lies in D and we are done. If $k > 0$, we observe that $g + M_{k-1}$ lies in the one-dimensional space M_k/M_{k-1}, hence we can find some $\beta_{k-1} \in \mathbb{R}$ such that $g + M_{k-1} = \beta_{k-1} f_{k-1} + M_{k-1}$. This implies that $g - \beta_{k-1} f_{k-1} \in M_{k-1}$. Proceeding this way, we can find for each $j \in k$ some real number $\beta_j \in \mathbb{R}$ such that $g - \sum_{j\in k} \beta_j f_j \in M_0 = M$. Hence $g + M = \sum_{j\in k} \beta_j f_j + M \in \widetilde{U}$. For each $j \in k$, let $\langle \alpha_j^i : i \in \omega\rangle$ be a sequence of rationals converging to β_j. Then the sequence $\langle \sum_{j\in k} \alpha_j^i f_j + M : i \in \omega\rangle$ converges to $\sum_{j\in k} \beta_j f_j + M = g + M$ in $C_p(X)/M$. Because $\widetilde{U}$ is an open neighborhood of $g + M$ in $C_p(X)/M$, there is some index $i \in \omega$ such that $\sum_{j\in k} \alpha_j^i f_j + M \in \widetilde{U}$. It is clear that $\sum_{j\in k} \alpha_j^i f_j + M$ also lies in D. As $\widetilde{U}$ was an arbitrary open subset of the quotient, D hits every open subset of $C_p(X)/M$, i.e. D is dense in $C_p(X)/M$. Therefore the quotient $C_p(X)/M$ is separable.

We still need to check that the above quotient is also infinite dimensional. For this, we show by induction that for $k \in \omega$ we have that M_k/M_0 is a linear subspace of $C_p(X)/M$ of dimension k. For $k = 0$, this is trivial. Suppose that the statement holds for some k and observe that $(M_{k+1}/M_0)/(M_k/M_0)$ is isomorphic to M_{k+1}/M_k and hence $\dim M_{k+1}/M_0 - \dim M_k/M_0 = \dim M_{k+1}/M_k$. This yields

$$\dim M_{k+1}/M_0 = \dim M_k/M_0 + \dim M_{k+1}/M_k = k + 1,$$

which ends our induction. If $C_p(X)/M$ had finite dimension m, then M_{m+1}/M_0 would be a subspace of $C_p(X)/M$ of dimension $m + 1$, a contradiction.

(iv) $\Rightarrow$ (iii): Suppose that (iv) holds. We let $\varphi_n(f) := \sum_{x\in F_n} \xi_n(x) f(x)$, which defines clearly a continuous linear functional. Each φ_n is non-zero, because for each $n \in \omega$ we can find some $x_n \in F_n$ such that $\xi_n(x_n) \neq 0$. By complete regularity of X, it is then possible to find some $f_n \in C(X)$ such that $f_n(x_n) = 1$ and $f_n \restriction F_n \setminus \{x_n\} = 0$. This yields $\varphi_n(f_n) \neq 0$, hence the given functional is non-zero.

We show that E as above is dense in $C_p(X)$. Let $U \subseteq C_p(X)$ be a basic open set, i.e. U can be written as $\prod_{x\in X} U_x \cap C(X)$, where $U_x = \mathbb{R}$ for almost all $x \in X$, and for remaining x's, U_x is an open interval. Now apply (iv) to the finite set $G := \{x \in X : U_x \neq \mathbb{R}\}$, the function $g : G \to \mathbb{R}$, where $g(x)$ is the center of U_x, and to $\varepsilon := \min\{\operatorname{diam}(U_x)/2 : x \in G\}$ to get a function $f \in C(X)$ such that (A) and (B) hold. Then (A) clearly implies that $f \in E$ and (B) implies that $f \in U$. This means that U hits E, and because U was an arbitrary basic open set, E is dense in $C_p(X)$.

(iii) $\Rightarrow$ (iv): The converse can we proven similarly: Suppose that (iii) holds, then by a very classical result (see [1, Proposition 0.5.8]) every $\varphi_n(f)$ can be written as $\sum_{x\in F_n} \xi_n(x)f(x)$ for some finite set $F_n \subseteq X$ and some function $\xi_n : F_n \to \mathbb{R}$. For arbitrary G, g and ε as above, fix $U := \prod_{x\in X} U_x \cap C(X)$, where $U_x = \mathbb{R}$ for $x \notin G$ and $U_x := (g(x) - \varepsilon, g(x) + \varepsilon)$ for $x \in G$. The set U is clearly open in $C_p(X)$, and by density of E we have that U hits E. Any $f \in E \cap U$ will satisfy (A) and (B). □

This characterisation is a practical tool for proving the following fact.

A convergent sequence $\langle x_n : n \in \omega\rangle$ is called a *non-trivial convergent sequence* if $x_n \neq x_m$ for $n \neq m$ and $x_n \neq x$ for all n, where x is the limit of this sequence.

Corollary 1.3.2. *Let X be an infinite completely regular space. If $C_p(X)$ fails to have SQ, then X contains no non-trivial convergent sequence.*

Proof. Suppose $\langle x_k : k \in \omega\rangle$ is a non-trivial convergent sequence with limit point x_ω. We use (iv) of the last lemma and put $F_n := \{x_n, x_\omega\}$ with the assigned coefficients $\xi_n(x_\omega) = 1$ and $\xi_n(x_n) = -1$. Let $\varepsilon > 0$ be arbitrary and G any finite subset of X with an assignment $g : G \to \mathbb{R}$.

We define $A := (\{x_k : k < \omega\} \setminus G) \cup \{x_\omega\}$, which is a closed set, and

$$\ell := \begin{cases} g(x_\omega) & \text{if } x_\omega \in G \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 1.2.3 to $K := G \setminus \{x_\omega\}$ and $B := A$ we may find a function $f \in C(X)$ such that $f(x) = g(x)$ for each $x \in G \setminus \{x_\omega\}$ and $f(x) = \ell$ for $x \in A$. Notice that $f(x_\omega) = \ell = g(x_\omega)$ if $x_\omega \in G$ and so $f(x) = g(x)$ for each $x \in G$.

It follows that (B) holds, because $|g(x)-f(x)| = 0 < \varepsilon$ for $x \in G$, and furthermore (A) holds by the computation $\sum_{x\in F_n} \xi_n(x)f(x) = f(x_n) - f(x_\omega) = \ell - \ell = 0$ for all $n \in \omega$ for which $x_n \notin G$. □

1.4 Sufficient conditions for $C_p(X)$ having SQ

Let X be a completely regular and infinite space. Now that we saw a characterisation for $C_p(X)$ having a separable quotient, we will study a sufficient condition for this, which will in turn give us information about how a space X must look like, if $C_p(X)$ would fail to have a separable quotient. The whole section is based on arguments occuring in [19].

The following result was published recently (in 2017, see [19, Theorem 4]). It gives a very elegant sufficient condition for spaces $C_p(X)$ to have SQ, assuming that X is completely regular. We take over the idea of their proof and adapt the notation as well as some technical details in order to simplify it and make it more understandable.

Theorem 1.4.1 (Kąkol, Śliwa)**.** *Let X be a completely regular space with a sequence $\langle K_n : n \in \omega \rangle$ of non-empty compact subsets such that for any $n \in \omega$ the set K_n contains two disjoint subsets homeomorphic to K_{n+1}. Then $C_p(X)$ has SQ.*

Proof. By Proposition 1.3.1, it suffices to check that condition (iii) is satisfied. Without loss of generality, we assume that $K_{n+1} \subseteq K_n$ for each natural number n. Before the proof really starts, we need to fix a bunch of notations.

Let $F[] := K_0$, $F[0] := K_1$ and $F[1]$ be the disjoint copy of K_1 contained in K_0. Inductively let $F[0^{n+1}] := K_{n+1}$ and fix $F[0^n \frown 1]$ to be a homeomorphic copy of K_{n+1}, disjoint from K_{n+1} sitting inside K_n. For each $n \in \omega$ we denote by $M[0^n \frown 1]$ the homeomorphism from $F[0^{n+1}]$ to $F[0^n \frown 1]$. Moreover we let $M[0^n] : F[0^n] \to F[0^n]$ be the identity map.

Inductively we define a map $M[s]$ from $F[0^{|s|}]$ to X for each finite bit string $s \in 2^{<\omega}$ by the following formula:

$$M[s \frown i] := (M[s] \restriction F[0^{|s|} \frown i]) \circ M[0^{|s|} \frown i] \text{ for } i \in \{0, 1\}. \tag{A}$$

We also define $F[s] := \operatorname{ran}(M[s])$. Observe that $F[s \frown i] \subseteq F[s]$, which by induction yields $F[s \frown t] \subseteq F[s]$ for any string t. For $n \in \omega$ we let $F_n := \bigcup_{|s|=n} F[s]$. Note that $F_{n+1} \subseteq F_n$, as well as $F[s]$ and $F[t]$ are disjoint if $s \neq t$ have the same length. Furthermore it is natural to define for $|t| = |s|$ the obvious homeomorphism

$$M[t; s] : F[t] \to F[s], \qquad M[t; s] := M[s] \circ M[t]^{-1}. \tag{B}$$

We show the following equality:

$$M[s_1 \frown t; s_2 \frown t] = M[s_1; s_2] \restriction F[s_1 \frown t], \qquad \text{for } |s_1| = |s_2| \tag{$\star$}$$

by induction over the length of t. If t has length 0, i.e. t is the empty string, then the above statement is trivial. Suppose that the equality holds for strings q of size

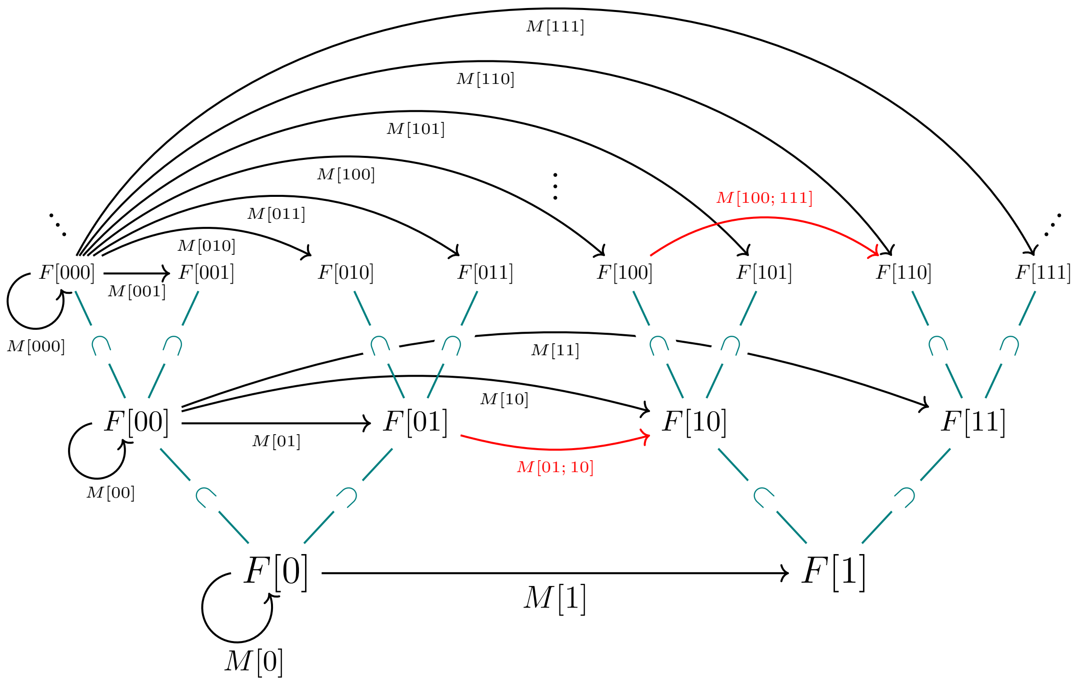

Figure 1.1: Many homeomorphisms in Theorem 1.4.1

k. If t of size $k+1$, we may write $t = i \frown q$ for some bit i and some string q. We let $n := |s_1| = |s_2|$ and compute:

$$\begin{aligned} M[s_1 \frown t, s_2 \frown t] &= M[s_1 \frown i \frown q, s_2 \frown i \frown q] \\ &\overset{\text{IH}}{=} M[s_1 \frown i, s_2 \frown i] \restriction F[s_1 \frown i \frown q] \\ &\overset{\text{(B)}}{=} \left(M[s_2 \frown i] \circ M[s_2 \frown i]^{-1}\right) \restriction F[s_1 \frown t]. \end{aligned}$$

We first focus on the expression $M[s_2 \frown i] \circ M[s_2 \frown i]^{-1}$. By equation (A) we get

$$\begin{aligned} & M[s_2 \frown i] \circ M[s_2 \frown i]^{-1} \\ = \ & (M[s_2] \restriction F[0^n \frown i]) \circ M[0^n \frown i] \circ \left((M[s_1] \restriction F[0^n \frown i]) \circ M[0^n \frown i]\right)^{-1} \\ = \ & (M[s_2] \restriction F[0^n \frown i]) \circ M[0^n \frown i] \circ (M[0^n \frown i])^{-1} \circ (M[s_1] \restriction F[0^n \frown i])^{-1} \\ = \ & (M[s_2] \restriction F[0^n \frown i]) \circ (M[s_1] \restriction F[0^n \frown i])^{-1} \end{aligned}$$

Let us take a look on the map $M[s_1] \restriction F[0^n \frown i]$ going from $F[0^n \frown i]$ to its range, which by (A) is easily seen to be $F[s_1 \frown i]$. We have thus $(M[s_1] \upharpoonright F[0^{|s_1|} \frown i])^{-1} = M[s_1]^{-1} \upharpoonright F[s_1 \frown i]$, which is a map $F[s_1 \frown i] \to F[0^n \frown i]$. We have therefore

$$\begin{aligned} M[s_2 \frown i] \circ M[s_2 \frown i]^{-1} &= (M[s_2] \upharpoonright F[0^n \frown i]) \circ (M[s_1]^{-1} \upharpoonright F[s_1 \frown i]) \\ &= (M[s_2] \circ M[s_1]^{-1}) \upharpoonright F[s_1 \frown i] \\ &= M[s_1; s_2] \upharpoonright F[s_1 \frown i] \end{aligned}$$

This must be restricted to $F[s_1 \frown t] = F[s_1 \frown i \frown q]$, which is a subset of $F[s_1 \frown i]$ and hence we finally get $M[s_1 \frown t, s_2 \frown t] = M[s_1; s_2] \upharpoonright F[s_1 \frown t]$, as desired.

For every natural number n we pick $x[0^n] \in F[0^n]$ and for each string s of size n we put $x[s] := M[s](x[0^n])$. Clearly, we have that $x[s] \in F[s] \subseteq F_{|s|}$ and $M[s;t](x[s]) = M[t](x[0^n]) = x[t]$. Notice that $x[s_1 \frown t] \in F[s_1 \frown t]$, so by $(\star)$ can obtain

$$M[s_1; s_2](x[s_1 \frown t]) = M[s_1 \frown t, s_2 \frown t](x[s_1 \frown t]) = x[s_2 \frown t]. \qquad (\star\star)$$

Now that we fixed all those notations and we checked the above equations, we can go over to the meat of the proof and define the continuous linear functionals φ_n such that item (iii) of Proposition 1.3.1 holds. For $n \in \omega$ we define $\varphi_n : C_p(X) \to \mathbb{R}$ by the formula $\varphi_n(g) := \sum_{|s|=n} g(x[s])$ and by the complete regularity of X we have $\varphi_n \neq 0$ for all n. We need to check that the subspace $E := \bigcup_{n\in\omega} \bigcap_{m\geq n} \ker \varphi_m$ is dense in $C_p(X)$.

For this it suffices to show that for every distinct $z_1, ..., z_n \in X$ there is a function $g \in E$ such that $g(z_1) = 1$ and $g(z_i) = 0$ for all $i \neq 1$. Indeed, if we have this, then for each basic open set is of the form $U = \{f \in C_p(X) : f(z_i) \in (r_i - \varepsilon_i, r_i + \varepsilon_i)$ for $1 \leq i \neq n\}$ for some distinct $z_1, ..., z_n \in X^n$, $(r_1, ..., r_n) \in \mathbb{R}^n$ and $(\varepsilon_1, ..., \varepsilon_n) \in \mathbb{R}^n_+$. For each $1 \leq j \leq n$ we can get a function g_j such that $g_j(z_j) = 1$ and $g_j(z_i) = 0$ for all $i \neq j$. Then we have that the linear combination $f = \sum_i r_i g_i$ lies in E and satisfies $f(z_i) = r_i$ for each i, so $f \in U$. Hence every basic open set U hits E, i.e. E is dense in $C_p(X)$

Let us check that this sufficient condition for E being dense holds. There are two cases. The first case is the one in which z_1 lies outside of F_k for some $k \in \omega$. Because the space X is completely regular, there is a function $g \in C(X)$ such that $g(z_1) = 1$, $g(z_i) = 0$ for each $i \neq 1$ and $g \upharpoonright F_k = 0$. For $m \geq k$ and for each bit string s of length m we obtain $g(x[s]) = 0$, because $x[s] \in F[s] \subseteq F_m \subseteq F_k$. Hence $\varphi_m(g) = 0$ for each $m \geq k$, i.e. $g \in E$.

We consider the second case, the one in which $z_1 \in \bigcap_{k\in\omega} F_k$. Then there is a bit string s_1 of size n such that $z_1 \in F[s_1]$. As we have 2^n bit strings of size n and $2^n > n$ there must be a bit string s_2 of size n such that $z_i \notin F[s_2]$ for all i. We apply the complete regularity of X to get a function $g_1 \in C(F[s_1])$ such that $g_1(z_1) = 1$ and $g_1(z_i) = 0$ for each $i \neq 1$ such that $z_i \in F[s_1]$. This function will be extended onto a continuous function on F_n by defining $g_2 \restriction F[s_1] = g_1$, as well as $g_2 \restriction F[s_2] = -g_1 \circ M[s_2; s_1]$ and $g_2 \restriction F[s] = 0$ for each bit string $s \notin \{s_1, s_2\}$ of size n.

Moreover, by applying Lemma 1.2.3 to the compact sets F_n and $\{z_1, ..., z_n\} \setminus F_n$ it is possible to find a function $g \in C(X)$ with the property that $g \restriction F_n = g_2$ and $g(z_i) = 0$ for all i such that $z_i \notin F_n$. Note that, as $z_1 \in F[s_1] \subseteq F_n$, we have that $g(z_1) = g_2(z_1) = g_1(z_1) = 1$.

For $k \in \omega$ we may compute $\varphi_{n+k}(g)$ by writing each bit string r of length $n + k$ as $r = s ^\frown t$ for bit strings s and t of length n and k respectively and observe that $x[r] \in F[r] \subseteq F[s] \subseteq F_n$. Note that for $s \notin \{s_1, s_2\}$ we have that $g_2(x[s ^\frown t]) = 0$. Consequently,

$$\begin{aligned}
\varphi_{n+k}(g) = \sum_{|r|=n+k} g(x[r]) &= \sum_{|r|=n+k} g_2(x[r]) \\
&= \sum_{|t|=k} g_2(x[s_1 ^\frown t]) + \sum_{|t|=k} g_2(x[s_2 ^\frown t]) \\
&= \sum_{|t|=k} g_1(x[s_1 ^\frown t]) - \sum_{|t|=k} g_1(M[s_1; s_2]^{-1}(x[s_2 ^\frown t])) \\
&\overset{(\star\star)}{=} \sum_{|t|=k} g_1(x[s_1 ^\frown t]) - \sum_{|t|=k} g_1(x[s_1 ^\frown t]) \\
&= 0
\end{aligned}$$

and as $k \in \omega$ was arbitrary, it follows also that $g \in E$. □

This theorem is very powerful and gives us interesting information about X for which $C_p(X)$ fails to have SQ, as we will see in the following corollaries.

For the first application of the above theorem, we shortly introduce the topological space $\beta\omega$, the Stone-Čech compactification of the integers with the discrete topology. One can show that this space is homeomorphic to the space of all ultrafilters on ω endowed with the topology generated by the sets $\{p : p \text{ is an ultrafilter on } \omega \text{ and } A \in p\}$, where $A \subseteq \omega$. Furthermore, $\beta\omega$ is a compact Hausdorff space and the given base consists of clopen (hence also compact) sets. It contains the set of principal ultrafilters (which is obviously homeomorphic to ω) as a dense set, but the remainder

$\beta\omega \setminus \omega$ (i.e. the set of free ultrafilters) is not separable, as we will see in the proof of Lemma 2.4.2.

Corollary 1.4.2. *Let X be an infinite completely regular space. If $C_p(X)$ fails to have SQ, then X contains no copy of $\beta\omega$.*

Proof. For $k, d \in \omega$ we let $k \cdot \mathbb{N} + d := \{km + d : m \in \omega\}$ and $k \cdot \mathbb{N} := k \cdot \mathbb{N} + 0$. Suppose $\beta\omega$ is contained in X, then define $K_n := \{p \in \beta\omega : 2^n \cdot \mathbb{N} \in p\}$. This set is compact by the definition of the topology on $\beta\omega$. Furthermore, $K_{n+1} \subseteq K_n$ is homeomorphic to and disjoint from $\{p \in \beta\omega : (2^{n+1} \cdot \mathbb{N} + 2^n) \in p\} \subseteq K_n$. The claim follows from Theorem 1.4.1. □

Example 1.4.3. The above corollary shows that $C_p(\beta\omega)$ has a separable quotient. However, $\beta\omega$ has no closed infinite countable subset, otherwise it would be a metrizable subspace, thus it would contain a non-trivial convergent sequence, which is impossible in $\beta\omega$. Hence by Theorem 1.2.7 we have that $C_p(\beta\omega)$ has no separable algebra quotient. In particular, this means that there can be separable quotients, which are not algebra quotients and therefore the property of having a separable algebra quotient is strictly stronger than having an arbitrary separable quotient. By the way, we just gave a negative answer for the separable quotient problem for algebras of the form $C_p(X)$ (see Question 1.2.2), because $C_p(\beta\omega)$ has no separable algebra quotient.

Definition 1.4.4. An infinite compact space is called *superrigid* if it contains no two compact disjoint infinite homeomorphic subsets.

Corollary 1.4.5. *Let X be an infinite compact Hausdorff space. If $C_p(X)$ fails to have SQ, then X contains a superrigid subspace Y.*

Proof. Suppose X contains no superrigid subspace. Then every compact subspace contains two homeomorphic disjoint compact infinite sets. Build a sequence $\langle K_n : n \in \omega \rangle$ inductively. Let $K_0 := X$ and if K_n is given pick K_{n+1} one of the two homeomorphic disjoint compact infinite sets contained in K_n. As Theorem 1.4.1 implies that $C_p(X)$ has SQ, we have a contradiction. □

The above corollaries motivate the following definition.

Definition 1.4.6. An infinite compact space is called *Efimov* if it contains neither a subspace homeomorphic to $\beta\omega$, nor a non-trivial convergent sequence.

With this definition, we can rephrase the corollaries as follows.

Corollary 1.4.7. *Let X be an infinite compact Hausdorff space such that $C_p(X)$ fails to have SQ. Then X is Efimov and contains a superrigid subspace. Moreover, this subspace is a superrigid Efimov space.*

Chapter 2

Homeomorphisms of subspaces of Efimov spaces

Let X be an infinite compact Hausdorff space. In the previous chapter, we saw that if $C_p(X)$ fails SQ, then X must be an Efimov space and X has a superrigid subspace, i.e. a closed infinite subspace such that none of its two disjoint infinite subspaces are homeomorphic. This motivates the question whether superrigid Efimov spaces exist. If not, it would mean that the separable quotient problem for $C_p(X)$ is solved, because then all such spaces would have a separable quotient.

Unfortunately, as this section will show, the existence of such spaces is consistent with ZFC. Using very deep results from functional analysis in [11], one can show under $\diamondsuit$ that such spaces exist (see [19, Example 17]). Here we give an alternative construction following the ideas of de la Vega [6], who proved under $\diamondsuit$ that there is a rigid Efimov space. We use his techniques to improve his result and build (also assuming $\diamondsuit$) a superrigid Efimov space.

2.1 The Cantor space

The foundation for our construction will be the Cantor space, which has very convenient topological properties. Those will be recalled first, and then we will move on to more elaborated and technical results, which will play an important role for the construction of our space.

Definition 2.1.1. The *Cantor space* is the set of all bitstrings of length ω, commonly denoted by 2^ω, with the product topology, where $2 := \{0, 1\}$ is endowed with the discrete topology. We will denote the Cantor space by $\mathbb{C}$.

Let us gather some nice properties of this space, which by the well-known Brouwer's Theorem, make the Cantor space unique. This statement is stated for example as a corollary of a theorem in [5, Section 3].

Theorem 2.1.2 (Brouwer)**.** *The Cantor space is the unique (up to homeomorphism) non-empty compact Hausdorff space without isolated points which possesses a countable base consisting of clopen sets. Moreover, the Cantor space is also metrizable.*

The following definitions were introduced by de la Vega in [6, Definition 2.3 and 2.4] and [7, Section 3.1].

Definition 2.1.3. Let S be a countable sequence of closed subsets of $\mathbb{C}$, i.e. $S = \langle S_i : i \in \alpha \rangle$ for some countable ordinal $\alpha \in \omega_1$ and for each $i \in \alpha$, S_i is a closed subset of the Cantor space $\mathbb{C}$. Let also $A_0, A_1 \subseteq \mathbb{C}$ be closed.

We say that $x \in X$ is a *strong limit point* of S if for all neighborhoods U of x there is an $i \in \alpha$ such that $S_i \subseteq U$ and $x \notin S_i$. We say that (A_0, A_1) *preserves* S if whenever $x \in A_0 \cap A_1$ is a strong limit point of S and U is an open neighborhood of x, there are $i_0, i_1 \in \alpha$ with $S_{i_0} \subseteq U \setminus A_1$ and $S_{i_1} \subseteq U \setminus A_0$.

Let $\mathcal{S}$ be a countable set of countable sequences of closed subsets of $\mathbb{C}$ and consider a convergent sequence $B = \langle b_n : n \in \omega \rangle$ in $\mathbb{C}$ with limit point b. We say that B *avoids* $\mathcal{S}$, if whenever b is a strong limit point of $S \in \mathcal{S}$ and U is an open neighborhood of b, there is an $i \in \alpha$ such that $S_i \subseteq U \setminus (B \cup \{b\})$.

A version of the following result was proven in [6, Lemma 2.2] (see also [7, Lemma 3.9]). We adjust the proof to our purpose.

Lemma 2.1.4 (Splitting Lemma)**.** *Suppose $\mathcal{S}$ is a countable set of countable sequences of closed subsets of $\mathbb{C}$. Suppose also that $B = \langle b_n : n \in \omega \rangle \subseteq \mathbb{C}$ is a convergent sequence which avoids $\mathcal{S}$ and let b be its limit point. Suppose furthermore that B is either constant or a non-trivial convergent sequence.*

Then there are closed $A_0, A_1 \subseteq \mathbb{C}$ such that

(i) $A_0 \cup A_1 = \mathbb{C}$,

(ii) $A_0 \cap A_1 = \{b\}$,

(iii) $\{b_{2n} : n \in \omega\} \subseteq A_0$ *and* $\{b_{2n+1} : n \in \omega\} \subseteq A_1$

(iv) (A_0, A_1) *preserves every sequence in* $\mathcal{S}$.

Proof. We will only treat the case in which B is not constant, the proof of the case where B is constant can be found in [9, Proof of Theorem 0.1]. So by assumption,

B is a non-trivial convergent sequence. Without loss of generality we may assume that each sequence in $\mathcal{S}$ has order type ω, because the notion of (A_0, A_1) preserving a given sequence is not related to the order type of the sequence. Because $\mathcal{S}$ is countable, we can therefore enumerate it by $\langle\langle S^i_n : n \in \omega\rangle : i \in \omega\rangle$. It is not difficult to see that there is a decreasing sequence $\langle V_m : m \in \omega\rangle$ consisting of clopen subsets of the Cantor space $\mathbb{C}$ with $V_0 = \mathbb{C}$, $\bigcap_{m\in\omega} V_m = B \cup \{b\}$ and such that every open superset of $B \cup \{b\}$ contains some V_m. The goal will be to build by induction another decreasing sequence $\langle W_m : m \in \omega\rangle$, which also consists of clopen subsets of $\mathbb{C}$ in order to have $W_0 = \mathbb{C}$, $W_m \subseteq V_m$ and $b_m \in W_m \setminus W_{m+1}$ for each natural number m.

Thus we will get that $\bigcap_{m\in\omega} W_m = \{b\}$ and therefore $A_0 := \{b\} \cup \bigcup_{m\in\omega} (W_{2m} \setminus W_{2m+1})$ and $A_1 := \{b\} \cup \bigcup_{m\in\omega} (W_{2m+1} \setminus W_{2m+2})$ will be closed subsets of $\mathbb{C}$, which clearly satisfy the properties (i), (ii) and (iii). To see that A_0 (and A_1 analogously) is closed, consider a sequence in A_0, converging to some point in $\mathbb{C}$. There are two possibilities: if the sequence hits infinitely many $W_{2m} \setminus W_{2m+1}$'s, this means that each W_{2m} contains infinitely many members of the sequence. Hence the limit point of the sequence must lie in each W_{2m}, which means that the sequence must converge to $b \in A_0$. If this does not happen, then the sequence is contained in a clopen subset of A_0, so its limit point lies in A_0.

In remains to build the W_m's such that (iv) holds too. We will guarantee inductively that for all $m \in \omega$ the set $\{b_k : k \geq m\}$ is contained in W_m. For the construction, let $\{O_j : j \in \omega\}$ be a neighborhood base of the point $b \in \mathbb{C}$ and $(\psi, \mu) : \omega \to \omega \times \omega$ such that each $(i, j) \in \omega \times \omega$ is hit by some odd and some even number, i.e. for each (i, j) there is an even number e and an odd number o such that $(\psi, \mu)(e) = (\psi, \mu)(o) = (i, j)$. We clearly need to put $W_0 = \mathbb{C}$ and we are going to build W_{m+1} from a given W_m.

There are two cases: If b is a strong limit point of the sequence $\langle S^{\psi(m)}_n : n \in \omega\rangle$, we can use the assumption that B avoids $\mathcal{S}$ to find some index $n := n(m) \in \omega$ such that $S^{\psi(m)}_n \subseteq W_m \cap O_{\mu(m)}$ and $S^{\psi(m)}_n \cap (B \cup \{b\}) = \emptyset$ hold. Because B is convergent and consists of different points, it is not hard to separate b_m from the rest of the sequence by an open set, and because $S^{\psi(m)}_n$ is closed, we can find some clopen neighborhood U of b_m such that $U \cap B = \{b_m\}$ and U is disjoint from $S^{\psi(m)}_n$. Furthermore, because $S^{\psi(m)}_n$ is closed, we can find some $k \in \omega$ such that $S^{\psi(m)}_n$ is disjoint from V_k and as the sequence $\langle V_k : k \in \omega\rangle$ is decreasing we can assume $k \geq m + 1$. We define $W_{m+1} := (W_m \cap V_k) \setminus U$ in this case.

In case b is not a strong limit point of the sequence $\langle S^{\psi(m)}_n : n \in \omega\rangle$, we pick a clopen neighborhood U of b_m such that $U \cap B = \{b_m\}$ and put $W_{m+1} := (W_m \cap$

$V_{m+1})\setminus U$. In both cases we clearly have that $b_m \notin W_{m+1}$ and, using the assumption that all b_n's are distinct, $\{b_k : k \geq m+1\} \subseteq W_{m+1}$ (hence $b_{m+1} \in W_{m+1}$). Moreover, it is also evident that W_{m+1} is contained in W_m and in V_{m+1} (that is why we chose $k \geq m+1$ in the first case).

We show that (iv) is satisfied: If b is a strong limit point of $\langle S_n^i : n \in \omega \rangle$, and O_j a basic neighborhood of b, then we can find m_0 and m_1 such that $(\psi, \mu)(2m_0) = (\psi, \mu)(2m_1+1) = (i, j)$. Let $n_0 := n(2m_0)$ and $n_1 := n(2m_1+1)$. We need to show that $S_{n_0}^i \subseteq O_j \setminus A_1$, or by using the above equalities $S_{n(2m_0)}^{\psi(2m_0)} \subseteq O_{\mu(2m_0)} \setminus A_1$. By construction $S_{n_0}^{\psi(2m_0)} \subseteq O_{\mu(2m_0)}$ and $b \notin S_{n_0}^{\psi(2m_0)}$. So it suffices to show that $S_{n_0}^{\psi(2m_0)}$ is contained in A_0. The construction yields on one hand that $S_{n_0}^{\psi(2m_0)} \subseteq W_{2m_0}$. On the other hand if $s \in S_{n_0}^{\psi(2m_0)}$, then s cannot lie in V_k, so $s \notin W_{2m_0+1}$. This clearly implies that $S_{n_0}^{\psi(2m_0)}$ is contained in A_0. Of course one can show analogoulsy $S_{n_1}^i \subseteq O_j \setminus A_0$. □

The following lemma is a special case of [6, Lemma 3.4]. We identify 0 and 1 with the corresponding constant sequences in $2^\omega = \{0,1\}^\omega$.

Lemma 2.1.5. *Let $A_0, A_1 \subseteq \mathbb{C}$ be closed subsets covering $\mathbb{C}$, which intersect in a single point $p \in \mathbb{C}$. Let $Y := A_0 \times \{0\} \cup A_1 \times \{1\} \subseteq \mathbb{C} \times \mathbb{C}$ and denote by $\pi : Y \to X$ the projection. Suppose $S = \langle S_i : i \in I \rangle$ is a sequence of closed subsets of $\mathbb{C}$ and (A_0, A_1) preserves S. If $x \in \mathbb{C}$ is a strong limit point of S, then every point in $\pi^{-1}(x)$ is a strong limit point of $\langle \pi^{-1}[S_i] : i \in I \rangle$.*

Proof. Let $(x, j) \in Y$ be any point in $\pi^{-1}(x)$ such that x is a strong limit point of the sequence S. To show that (x, j) is a strong limit point of $\langle \pi^{-1}[S_i] : i \in I \rangle$, let $U \times \{j\}$ an open neighborhood of (x, j), where U is an open neighborhood of x. We have to deal with two cases here.

If $x \neq p$, then $x \notin A_{1-j}$, and there is an open neighborhood V such that $x \in V \subseteq U$ and $A_{1-j} \cap V = \emptyset$. Because x is a strong limit point of S, we may find $i \in I$ such that $S_i \subseteq V \setminus \{x\}$. In case $x = p$, we have that $x \in A_0 \cap A_1$ is a strong limit point of S and (A_0, A_1) preserves S, we can find some $i \in I$ such that $S_i \subseteq U \setminus A_{1-j}$.

If (c, d) lies in $\pi^{-1}[S_i]$, then $c \in S_i$, which in both cases implies that $c \in U \setminus \{x\}$ and c cannot lie in A_{1-j}, hence d can only be j, i.e. $(c, d) \in \big(U \times \{j\}\big) \setminus \{(x, j)\}$. We showed in both cases that $\pi^{-1}[S_i]$ is contained in $\big(U \times \{j\}\big) \setminus \{(x, j)\}$. □

2.2 The Diamond Principle

The Diamond Principle is a set-theoretic principle first introduced by Jensen in 1972. It was shown to be independent from ZFC by two results. The first one is that the Diamond Principle holds in Gödel's Constructible Universe L, so it is consistent with ZFC. The second result is the fact that the Diamond Principle implies the Continuum Hypothesis, which is known to be independent from ZFC by the method of forcing. This means that in the model for ZFC where the Continuum Hypothesis fails, the Diamond Principle does not hold either. In this section, we formulate the Diamond Principle and mention some facts, which will be useful for this paper.

On ω_1, the set of all countable ordinals, there is a canonical topology induced by the order $\in$ on ω_1. It is called the *order topology* and is generated by the sets of the form $\{x \in \omega_1 : \alpha \in x\}$, $\{x \in \omega_1 : x \in \beta\}$ and $]\alpha, \beta[:= \{x \in \omega_1 : \alpha \in x \text{ and } x \in \beta\}$ where $\alpha, \beta \in \omega_1$. In this topology, a subset $A \subseteq \omega_1$ is closed if and only if for all limit ordinals $\alpha \in \omega_1$ such that $A \cap]\zeta, \alpha[\neq \emptyset$ for any $\zeta < \alpha$, we have that $\alpha \in A$. Recall also that a subset $A \subseteq \omega_1$ is called *unbounded*, if for each $\alpha \in \omega_1$ we can find some $\beta \in A$ such that $\alpha \leq \beta$.

Definition 2.2.1. A set $A \subseteq \omega_1$ is called

- ▷ a *club set*, if it is closed (in the order topology of ω_1) and unbounded.
- ▷ *stationary*, if for each club $C \subseteq \omega_1$ we have that A is not disjoint from C.

The *Diamond Principle* is the following statement:

$\diamondsuit$ There is a sequence $\langle A_\alpha : \alpha \in \omega_1 \rangle$ such that $A_\alpha \subseteq \alpha$ for all $\alpha \in \omega_1$ and for each $A \subseteq \omega_1$ the set $\{\alpha \in \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary.

Such a sequence is often called a $\diamondsuit$*-sequence*.

The following lemma easily follows from the corresponding definitions.

Lemma 2.2.2.

(a) The set of all limit ordinals below ω_1 is a club set.

(b) Every stationary set is unbounded.

(c) The intersection of two club sets is a club set.

(d) The intersection of a club set with a stationary set is stationary.

Note that the statements (a), (c) and (d) imply that whenever a club or a stationary set is given, by passing to a smaller set if necessary, we can assume that this set consists only of limit ordinals.

We formulate in Lemma 2.2.4 an alternative way to state the Diamond Principle in a more practical way for the construction of our topological space. This formulation can be found in [6, Section 4], in addition a similar approach is presented in [9, §2]. The proof we give is based on an exercise in [20, Exercise III.7.9.], which is partially solved in [14]. This yields the following lemma, which we prove in the same fashion.

Lemma 2.2.3. *Let $f : \omega_1 \to \omega \times (\{-1\} \cup \omega_1) \times \omega_1$ be a bijection. Then the set $C = \{\alpha \in \omega_1 : f[\alpha] = \omega \times (\{-1\} \cup \alpha) \times \alpha\}$ is a club set.*

Proof. By a simple calculation, it is quite easy to see that C is closed.

We show that C is unbounded in ω_1. For this let $\alpha \in \omega_1$. The aim is to find a $\beta \geq \alpha$ which is in C. In this proof, we shall denote by $p_1 : \omega \times (\{-1\} \cup \omega_1) \times \omega_1 \to (\{-1\} \cup \omega_1)$ and $p_2 : \omega \times (\{-1\} \cup \omega_1) \times \omega_1 \to \omega_1$ the obvious projections.

We define inductively sequences $\langle \alpha_n : n \in \omega \rangle$ and $\langle \beta_n : n \in \omega \rangle$ as follows. Put $\alpha_0 := \alpha$ and suppose that each α_k for $k \leq n$ and all β_k for $k < n$ are defined. If $\alpha_n \in C$ we let $\beta_n := \alpha_n$ and if not, we let

$$\beta_n := \max\{\alpha_n, \sup p_1[f[\alpha_n]], \sup p_2[f[\alpha_n]]\} + 1.$$

In addition, if $\beta_n \in C$ we define the next member of the first sequence to be $\alpha_{n+1} := \beta_n$ and if not, we let $\alpha_{n+1} := \max\{\beta_n, \sup f^{-1}[\omega \times (\{-1\} \cup \beta_n) \times \beta_n]\} + 1$. By these inductive definitions we can observe:

(a) $\alpha_n \leq \beta_n < \omega_1$.

(b) $\beta_n \leq \alpha_{n+1} < \omega_1$

(c) If $\alpha_n \notin C$, then $f[\alpha_n] \subseteq \omega \times (\{-1\} \cup \beta_n) \times \beta_n$.

(d) If $\beta_n \notin C$, then $\omega \times (\{-1\} \cup \beta_n) \times \beta_n \subseteq f[\alpha_{n+1}]$.

Items (a) and (b) just follow by the definition and because the image of a countable set under any function is countable. To see (c), observe that if $\alpha_n \notin C$, then $y \in f[\alpha_n]$ implies $p_i[y] \leq \sup p_i[f[\alpha_n]] < \beta_n$ for $i \in \{0, 1\}$. To show (d), we assume $\beta_n \notin C$ and show that $f^{-1}[\omega \times (\{-1\} \cup \beta_n) \times \beta_n] \subseteq \alpha_{n+1}$ by seeing that if y is in the first set, then $y \leq \sup f^{-1}[\omega \times (\{-1\} \cup \beta_n) \times \beta_n] < \alpha_{n+1}$.

We define $\beta := \sup\{\alpha_n : n \in \omega\} \geq \alpha_0$, which by (a) and (b) coincides with $\sup\{\beta_n : n \in \omega\}$. We show that β lies inside C. There are two cases: if some α_n lies in C, then all $\alpha_m = \alpha_n$ for $m \geq n$ and hence $\beta = \alpha_n \in C$. If no α_n lies in C, then in particular no β_n can lie in C either. From our observations (c) and (d) we get $f[\alpha_n] \subseteq \omega \times (\{-1\} \cup \beta_n) \times \beta_n \subseteq f[\alpha_{n+1}]$ for each natural number n.

By taking unions on every side of this inclusions over all $n \in \omega$, and noticing that on one hand the union commutes with the image of the function and on the other hand that the union of ordinals is just their supremum, we conclude that $f[\beta] \subseteq \omega \times (\{-1\} \cup \beta) \times \beta \subseteq f[\beta]$. This implies the equality between these sets, and hence it follows that $\beta \in C$, which finishes the proof of the fact that C is unbounded. □

For $\beta \leq \alpha \leq \omega_1$ we denote by $\pi^\alpha_\beta : \mathbb{C} \times \mathbb{C}^\alpha \to \mathbb{C} \times \mathbb{C}^\beta$ the canonical projection. Moreover, we let $\pi_\beta := \pi^{\omega_1}_\beta$ for $\beta \leq \omega_1$, i.e. in case $\alpha = \omega_1$ we will not write the above index. We will label the very first coordinate of elements of $\mathbb{C} \times \mathbb{C}^\alpha$ by -1.

Lemma 2.2.4. *Assume $\diamondsuit$. Then there are sequences $\langle z^\alpha_\gamma : \gamma \in \alpha\rangle$ in $\mathbb{C} \times \mathbb{C}^\alpha$, where $\alpha \in \omega_1$, such that for every sequence $\langle x_\gamma : \gamma \in \omega_1\rangle$ in $\mathbb{C} \times \mathbb{C}^{\omega_1}$ the set $\{\alpha \in \omega_1 : \forall \gamma \in \alpha\ (\pi_\alpha(x_\gamma) = z^\alpha_\gamma)\}$ is stationary.*

Proof. The idea is to code sequences $\langle s_\gamma : \gamma \in \alpha\rangle$ in $\mathbb{C} \times \mathbb{C}^\alpha$ for $\alpha \leq \omega_1$ by three-dimensional matrices of size $\omega \times (\{-1\} \cup \alpha) \times \alpha$ with entries zero or one. Here, the size does not refer to a cardinality but to an order type in each of the three dimensions of the matrix. If we look at the entry indexed by (n, β, γ) of this matrix, the third coordinate γ will indicate the member s_γ of the sequence, the second coordinate β will indicate the β-th coordinate of s_γ, denoted by $s_\gamma(\beta)$. Notice that $s_\gamma(\beta)$ is itself a 0-1-sequence, whose n-th entry is the entry of the matrix at (n, β, γ). This entry will be denoted by $s_\gamma(\beta)_n$.

We fix a bijection $f : \omega_1 \to \omega \times (\{-1\} \cup \omega_1) \times \omega_1$ such that for each ordinal $\alpha \in \omega_1$ one has $f[\alpha] \subseteq \omega \times (\{-1\} \cup \alpha) \times \alpha$. Furthermore we let $B_\alpha := f[A_\alpha]$, where A_α is given by $\diamondsuit$. Each B_α is a subset of $\omega \times (\{-1\} \cup \alpha) \times \alpha$, so it can be seen as a three-dimensional matrix, which codes a sequence $\langle z^\alpha_\gamma : \gamma \in \alpha\rangle$ in $\mathbb{C} \times \mathbb{C}^\alpha$ as described above. Formally, this sequence can be defined by putting $z^\alpha_\gamma(\beta)_n := 1$ whenever $(n, \beta, \gamma) \in B_\alpha$ and otherwise $z^\alpha_\gamma(\beta)_n := 0$.

Let $\langle x_\gamma : \gamma \in \omega_1\rangle$ be a sequence in $\mathbb{C} \times \mathbb{C}^{\omega_1}$ and let $B \subseteq \omega \times (\{-1\} \cup \omega_1) \times \omega_1$ be its corresponding matrix defined by stipulating $(n, \beta, \gamma) \in B$ if and only if $x_\gamma(\beta)_n = 1$. Set $A := f^{-1}[B]$ and apply $\diamondsuit$ in its classical form to see that $\{\alpha \in \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary. Let $C := \{\alpha \in \omega_1 : f[\alpha] = \omega \times (\{-1\} \cup \alpha) \times \alpha\}$, which by Lemma 2.2.3

is a club set, and define also $S := \{\alpha \in \omega_1 : B \cap (\omega \times (\{-1\} \cup \alpha) \times \alpha) = B_\alpha\}$. By Lemma 2.2.2 we get that in particular the set

$$\begin{aligned}\{\alpha \in \omega_1 : A \cap \alpha = A_\alpha\} \cap C &= \{\alpha \in \omega_1 : f(A) \cap f(\alpha) = f(A_\alpha)\} \cap C \\ &= S \cap C\end{aligned}$$

is stationary, hence S is also stationary. For $\alpha \in S$, we have the following equivalences

$$\begin{aligned}&x_\gamma(\beta)_n = 1 \text{ and } n \in \omega, \beta \in \{-1\} \cup \alpha, \gamma \in \alpha \\ \iff& (n, \beta, \gamma) \in B \cap (\omega \times (\{-1\} \cup \alpha) \times \alpha) \\ \iff& (n, \beta, \gamma) \in B_\alpha \\ \iff& z_\gamma^\alpha(\beta)_n = 1.\end{aligned}$$

This implies that for each γ smaller than such an α, the first α coordinates of x_γ coincide with the ones from z_γ^α. In other words, we have the inclusion (even the equality holds) $S \subseteq \{\alpha \in \omega_1 : \forall \gamma \in \alpha \ \pi_\alpha(x_\gamma) = z_\gamma^\alpha\}$, and therefore the latter is stationary. This ends the proof. □

The following lemma is known as Fodor's Lemma, for a proof see [20, Lemma III.6.14].

Lemma 2.2.5 (Fodor)**.** *Let $S \subseteq \omega_1$ be stationary and $\xi : S \to \omega_1$ be a function such that $\xi(\alpha) < \alpha$ for each $\alpha \in S$. Then there is a stationary set $S' \subseteq S$ and $\gamma \in \omega_1$ such that $\xi(\alpha) = \gamma$ for all $\alpha \in S'$.*

Lemma 2.2.6. *Let $\beta : \omega_1 \to \omega_1$ be a function. Then there is a club $C \subseteq \omega_1$ such that for each $\alpha \in C$ and for each $\xi < \alpha$ we have that $\beta(\xi) < \alpha$.*

Proof. Let us define a function $c : \omega_1 \to \omega_1$ as follows. For each $\mu \in \omega_1$, we put $\mu^{(0)} := \mu$ and if $\mu^{(n)}$ is already defined we set $\mu^(n+1) := \mu^{(n)} + \sup\{\beta(\xi) : \xi < \mu^{(n)}\} + 1$. Finally, let $c(\mu) := \mu^{(\omega)} := \sup\{\mu^{(n)} : n \in \omega\}$. This defines the function c as wanted, because ω_1 is regular (countable unions of countable ordinals are countable).

Define C to be the closure (with respect to the order topology of ω_1) of the range of the function c, i.e. $C := \overline{\{c(\mu) : \mu \in \omega_1\}}$. We claim that C is the desired club set. Let $\alpha \in C$ and $\xi < \alpha$. Because α lies in the closure of the range of c, we can find some $\mu \in \omega_1$ such that $\xi < c(\mu) \leq \alpha$ (because the interval $]\xi, \alpha]$ is an open neighborhood of α in ω_1). By the definition of the function c, there is some natural number n such that $\xi < \mu^{(n)} < c(\mu) \leq \alpha$. Then it follows that

$$\begin{aligned}\mu^{(n+1)} &= \mu^{(n)} + \sup\{\beta(\xi') : \xi' < \mu^{(n)}\} + 1 \\ &> \mu^{(n)} + \beta(\xi) \\ &\geq \beta(\xi)\end{aligned}$$

and so we have $\beta(\xi) < \mu^{(n+1)} \leq c(\mu) \leq \alpha$, as desired.

The set C is a club, because it is clearly closed by definition, and it is also unbounded, as for $\mu \in \omega_1$ we have $c(\mu) \geq \mu$ and $c(\mu) \in C$. □

2.3 Inverse limits

Inverse limits are very general objects, which are also considered in category theory. Here, the definition of inverse limits will remain somewhat special, as it is adapted to our needs. Our goal was namely to construct a superrigid Efimov space, therefore we want to get rid of homeomorphisms between compact disjoint subspaces of the desired space. Inverse limits are very practical for this, because – as we will show in Proposition 2.3.5 – such homeomorphisms reflect down to club many places, where then these homeomorphisms can be eliminated in the induction step of the construction of our space.

Definition 2.3.1. Let $\gamma \leq \omega_1$ and $\langle X_\alpha : \alpha \in \gamma\rangle$ be a sequence such that $X_\alpha \subseteq \mathbb{C}\times\mathbb{C}^\alpha$ is such that $\pi^\alpha_\beta[X_\alpha] = X_\beta$ for each $\beta \leq \alpha < \gamma$. Such a sequence will be called a γ-*inverse system*. If γ is a limit ordinal, the γ-*inverse limit* of $\langle X_\alpha : \alpha \in \gamma\rangle$ is the space $X = \bigcap_{\alpha\in\gamma}(\pi^\gamma_\alpha)^{-1}[X_\alpha]$, which is a subspace of $\mathbb{C} \times \mathbb{C}^\gamma$.

An *inverse system* is an ω_1-inverse system and an *inverse limit* is an ω_1-inverse limit, for which we sometimes write X_{ω_1}.

Note that in the classical literature, as e.g. in [10, Section 2.5], the inverse limit of topological spaces is defined as a certain subspace of their product. Formally, this is not what is done here. If we followed this classical definition, we would not say that X as above is an inverse limit of $\langle X_\alpha : \alpha \in \omega_1\rangle$, but the set $\{\langle\pi_\alpha(x) : \alpha \in \omega_1\rangle : x \in X\}$. We just drop away unnecessary (repetitive) information, so that the inverse limit, as a formal object, gets simpler. However, we can still use the following facts, which can be derived from [10, Theorem 3.2.13 and Corollary 3.2.15] also for the formal definition used here.

Lemma 2.3.2. *Let $\gamma \leq \omega_1$ and X_γ the γ-inverse limit of $\langle X_\alpha : \alpha \in \gamma \rangle$, where each X_α is non-empty and compact. Then X_γ is non-empty, compact and furthermore for every $\alpha \leq \omega_1$ the projection π^γ_α is surjective from X_γ onto X_α, i.e. $\pi^\gamma_\alpha[X_\gamma] = X_\alpha$.*

One could also prove the above lemma directly by an inductive argument using the compactness at each limit stage. Notice that without compactness, inverse limits may die out, i.e. it can happen that the inverse limit of non-empty spaces is empty.

Remark 2.3.3. The product topology on $\mathbb{C} \times \mathbb{C}^\alpha$ is generated by the sets of the form $\prod_{\delta \in \{-1\} \cup \alpha} O_\delta$, where $O_\delta = \mathbb{C}$ for almost all $\delta \in \{-1\} \cup \omega_1$ and for only finitely many δ's O_δ is any open set in $\mathbb{C}$. Recall that $\mathbb{C}$ has a basis consisting of clopen sets, therefore each O_δ contains a clopen set U_δ such that if $O_\delta = \mathbb{C}$, then $U_\delta = \mathbb{C}$.

Therefore every open set in $\mathbb{C} \times \mathbb{C}^\alpha$ contains a clopen set of the form $\prod_{\delta \in \{-1\} \cup \alpha} U_\delta$, where U_δ is a clopen set in $\mathbb{C}$ and $U_\delta \neq \mathbb{C}$ for only finitely many δ. It is easy to see that the sets of this form generate the topology of $\mathbb{C} \times \mathbb{C}^\alpha$, that is why we will call those sets *basic clopen sets* in this context. These sets give also rise to the basic clopen sets of any subspace of this space in the usual way. We may also refer to these as basic clopen sets of a given subspace.

In the proof of the following lemma, we will use a fact about compact spaces. A point is called a *full accumulation point* of some subset A, if for every open neighborhood U of this point we have $|A| = |A \cap U|$. In every compact space, every infinite subset A has a full accumulation point (see [10, Problem 3.12.1]). This fact also has an "indexed" version, namely if X is compact and $\langle a_i : i \in U \rangle \in X^I$, where I is infinite, then there exists some $x \in X$ such that $|\{i \in I : a_i \in U\}| = |I|$ for any open U containing x.

Lemma 2.3.4. *Let X be an inverse limit of $\langle X_\alpha : \alpha \in \omega_1 \rangle$. If Y and Z are compact disjoint subsets of X, there is a $\beta \in \omega_1$ such that for all $\alpha \geq \beta$ we have that $Y_\alpha := \pi_\alpha[Y]$ is disjoint from $Z_\alpha := \pi_\alpha[Z]$.*

Proof. This statement is equivalent to the following: there is a $\alpha \in \omega_1$ such that Y_α is disjoint from Z_α. We show that if this fails, then Y cannot be disjoint from Z.

If for all $\alpha \in \omega_1$ we have $\pi_\alpha[Y] \cap \pi_\alpha[Z] \neq \emptyset$, we may pick for each α an x_α in this intersection. This means that there are $y_\alpha \in Y, z_\alpha \in Z$ such that $\pi_\alpha(y_\alpha) = \pi_\alpha(z_\alpha) = x_\alpha$. Using the "indexed" version of the existence of a full accumulation point find $(y, z) \in Y \times Z$ such that for each basic neighborhood $U \times V$ of (y, z) we have $|\{\alpha \in \omega_1 : (y_\alpha, z_\alpha) \in U \times V\}| = \omega_1$.

We show that $y = z$, which then implies that Y and Z are not disjoint. If not, find some $\beta \in \omega_1$ such that $\pi_\beta(y) \neq \pi_\beta(z)$: Moreover find a basic clopen subset $U \subseteq X_\beta$ such that $\pi_\beta(y) \in U$, but $\pi_\beta(z) \notin U$. Then $(\pi_\beta)^{-1}[U] \times (X \setminus (\pi_\beta)^{-1}[U])$ is a basic open neighborhood of (y, z), so $|\{\alpha \in \omega_1 : (y_\alpha, z_\alpha) \in (\pi_\beta)^{-1}[U] \times (X \setminus (\pi_\beta)^{-1}[U])\}| = \omega_1$. Hence, we can pick $\alpha \geq \beta$ such that $(y_\alpha, z_\alpha) \in (\pi_\beta)^{-1}[U] \times (X \setminus (\pi_\beta)^{-1}[U])$. On one hand this yields $\pi_\beta(y_\alpha) \in U$ and $\pi_\beta(z_\alpha) \notin U$, but on the other hand we had $\pi_\alpha(y_\alpha) = \pi_\alpha(z_\alpha)$, which in particular yields $\pi_\beta(y_\alpha) = \pi_\beta(z_\alpha)$, a contradiction. Hence $y = z \in Y \cap Z$, and thus Y cannot be disjoint from Z. □

The following fact is mentioned in [6, Section 4] without a proof, we give a proof for it.

Proposition 2.3.5. *Let X be a compact inverse limit of $\langle X_\alpha : \alpha \in \omega_1 \rangle$, Y and Z compact subspaces of X and $h : Y \to Z$ a homeomorphism. Then there is a club $Q_h \subseteq \omega_1$ such that for each $\alpha \in Q_h$ there is a homeomorphism $h_\alpha : \pi_\alpha[Y] \to \pi_\alpha[Z]$ such that for each $x \in Y$ we have that $\pi_\alpha(h(x)) = h_\alpha(\pi_\alpha(x))$.*

Proof. We need to show that there is a club P_h such that for any $\alpha \in P_h$, the following defines a function: For any $y \in \pi_\alpha[Y]$ pick an $x \in Y$ with $\pi_\alpha(x) = y$ and define $h_\alpha(y) = \pi_\alpha(h(x))$. Once this is proved, we let $Q_h := P_h \cap P_{h^{-1}}$. Then it is not hard to see that h_α is a homeomorphism with inverse $(h^{-1})_\alpha$ for all $\alpha \in Q_h$. So we need to prove that for club many $\alpha \in P_h$, this is a well-defined function.

By contradiction, suppose this is not the case and let $S := \{\alpha \in \omega_1 : h_\alpha$ is ill-defined$\}$. Then there is no club which is disjoint from S, in other words: S must be stationary. Without loss of generality we can assume that S consists of limit ordinals. By the definition of S, we know that for each $\alpha \in S$, there are $y_\alpha \in \pi_\alpha[Y]$ and x^0_α and x^1_α in Y such that $\pi_\alpha(x^0_\alpha) = \pi_\alpha(x^1_\alpha) = y_\alpha$ but $\pi_\alpha(h(x^0_\alpha)) \neq \pi_\alpha(h(x^1_\alpha))$.

Find a basic clopen $U_\alpha \subseteq \mathbb{C} \times \mathbb{C}^\alpha$ such that $\pi_\alpha(h(x^0_\alpha)) \in U_\alpha$ and $\pi_\alpha(h(x^1_\alpha)) \notin U_\alpha$. Because every $\alpha \in S$ is a limit ordinal, we can find some $\xi(\alpha) < \alpha$ and a clopen $U'_\alpha \subseteq \mathbb{C} \times \mathbb{C}^{\xi(\alpha)}$ such that $U_\alpha = (\pi^\alpha_{\xi(\alpha)})^{-1}[U'_\alpha]$. We can see ξ as a function on the stationary set S. By Fodor's Lemma (2.2.5), there is a stationary set $S' \subseteq S$ and a $\gamma \in \omega_1$ such that $\xi(\alpha) = \gamma$ for all $\alpha \in S'$. As U'_α is a basic clopen subset of $\mathbb{C} \times \mathbb{C}^\gamma$ for $\alpha \in S'$, there are only countably many possibilities for choosing U'_α. But S' is uncountable, therefore by an uncountable version of the Pigeonhole Principle there is an uncountable (hence unbounded) $S'' \subseteq S'$ and a clopen $U \subseteq X_\gamma$ such that $U = U'_\alpha$ for each $\alpha \in S''$.

Now we set $V := h^{-1}[Z \cap \pi_\gamma^{-1}[U]]$ and $W := h^{-1}[Z \cap \pi_\gamma^{-1}[X_\gamma \setminus U]] = Y \setminus V$. These sets are compact, as they are closed in a compact set and so Lemma 2.3.4 implies that there is some ν such that for each $\alpha \geq \nu$, we have that $\pi_\alpha(V)$ and $\pi_\alpha(W)$ are disjoint. Now pick $\alpha \in S''$ such that $\alpha \geq \nu$. Then $y_\alpha = \pi_\alpha(x^0_\alpha) = \pi_\alpha(x^1_\alpha) \in \pi_\alpha(V) \cap \pi_\alpha(W)$. This is a contradiction, therefore the assumption that there is no club on which the above construction defines a function is wrong. □

2.4 Sufficient conditions for topological properties of inverse limits

Remember that our goal is to construct under $\diamondsuit$ a superrigid Efimov space, i.e. an Efimov space containing no two infinite, compact, disjoint sets. This space will be built as an inverse limit of a sequence $\langle X_\alpha : \alpha \in \omega_1 \rangle$. Before we do this, we need to study sufficient conditions on the sequence $\langle X_\alpha : \alpha \in \omega_1 \rangle$ which imply that its inverse limit has the properties we are interested in. These sufficient conditions will afterwards be implemented in the construction, so that our inverse limit becomes a superrigid Efimov space.

Definition 2.4.1. An infinite compact space is called

- ▷ *hereditarily separable (HS)* if every subspace is separable.
- ▷ *Fedorčuk* if it is HS and if it contains no non-trivial convergent sequence.
- ▷ *rigid* if its only autohomeomorphism is the identity map.

Lemma 2.4.2. *Every zero-dimensional, compact superrigid space with no isolated points is rigid. Every Fedorčuk space is Efimov.*

Proof. Suppose there is an autohomeomorphism h of X which is not the identity function. Then there is some $x \in X$ such that $h(x) \neq x$. Separate x from $h(x)$ by clopen neighborhoods U and V such that $V \subseteq h[U]$. Now consider the homeomorphic clopen neighborhoods $h^{-1}[V]$ and V of x and $h(x)$ respectively. As x and $h(x)$ are not isolated, these neighborhoods must be infinite and because X is compact, they are also compact. This contradicts the superrigidity of the space.

The second statement just follows from the fact that $\beta\omega \setminus \omega$ is not separable, hence it can not be contained in an HS space. To see this fact, we let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be an uncountable maximal almost disjoint family, i.e. $A \cap B$ is finite for each $A, B \in \mathcal{A}$ and the family is maximal with respect to inclusion. The existence of such a family is not hard to prove, see e.g. [15, Proposition 8.6]. Consider the open subset $U_A := \{p \in \beta\omega \setminus \omega : A \in p\}$ of the remainder $\beta\omega \setminus \omega$, where $A \in \mathcal{A}$. It is easy to see that for $A \neq B$ in $\mathcal{A}$, the set U_A is disjoint from U_B, because if there was a free ultrafilter $p \in U_A \cap U_B$ it would contain the finite set $A \cap B$, which is impossible for free ultrafilters. If D was a dense subset of $\beta\omega \setminus \omega$, then D contains at least one point from each U_A ($A \in \mathcal{A}$). Since $\{U_A : A \in \mathcal{A}\}$ is an uncountable family of pairwise disjoint open subsets of $\beta\omega \setminus \omega$, D must be uncountable too. Consequently $\beta\omega \setminus \omega$ can not be separable. □

From now on, we assume $\diamondsuit$, so in light of Lemma 2.2.4 we fix sequences $\langle z_\gamma^\alpha : \gamma \in \alpha\rangle$ in $\mathbb{C} \times \mathbb{C}^\alpha$, where $\alpha \in \omega_1$, such that for every sequence $\langle x_\gamma : \gamma \in \omega_1\rangle$ in $\mathbb{C} \times \mathbb{C}^{\omega_1}$ the set $S = \{\alpha \in \omega_1 : \forall \gamma \in \alpha \ \pi_\alpha(x_\gamma) = z_\gamma^\alpha\}$ is stationary. We introduce sufficient conditions for topological properties in the same fashion as in [6].

Definition 2.4.3. Let X be an inverse limit of $\langle X_\alpha : \alpha \in \omega_1\rangle$, where each X_α is non-empty and compact. We abbreviate some statements as follows:

(C3) For all $\beta \leq \alpha \leq \omega_1$, if the z_γ^β's are all distinct, where $\gamma < \beta$ and lie in X_β, then for every $x \in X_\alpha$ for which $\pi_\beta^\alpha(x)$ is a limit point of $\langle z_\gamma^\beta : \gamma \in \beta\rangle$, we have that x is a strong limit point of $\langle X_\alpha \cap (\pi_\beta^\alpha)^{-1}(z_\gamma^\beta) : \gamma \in \beta\rangle$.

(C4) If $\{z_n^\alpha : n \in \omega\}$ is a non-trivial convergent sequence in X_α with limit point $x \in X_\alpha$, then $|X_{\alpha+1} \cap (\pi_\alpha^{\alpha+1})^{-1}(x)| > 1$.

(C7) For all $\alpha \in \omega_1$, if $\{z_n^\alpha : n \in \omega\}$ is a dense subset of X_α, $\{z_{2n}^\alpha : n \in \omega\}$ and $\{z_{\omega+2n}^\alpha : n \in \omega\}$ have disjoint closures, say Y_α and Z_α respectively, and $f : Y_\alpha \to Z_\alpha$ is a homeomorphism such that $f(z_{2n}^\alpha) = z_{\omega+2n}^\alpha$, then there are sequences $\langle a_n^\alpha : n \in \omega\rangle$ in Y_α converging to a^α and $\langle b_n^\alpha : n \in \omega\rangle$ in Z_α converging to b^α such that, if by $I_7 \subseteq \omega_1$ we denote the set of those α for which these premises are fulfilled, we have

  (R1) For each $\alpha \in I_7$ and each compact $R \subseteq X_{\alpha+1}$ such that $\pi_\alpha^{\alpha+1}[R] = Z_\alpha$ there is an $x \in R \cap (\pi_\alpha^{\alpha+1})^{-1}(b^\alpha)$ which is not a strong limit point of the sequence $\langle R \cap (\pi_\alpha^{\alpha+1})^{-1}(b_{2n}^\alpha) : n \in \omega\rangle$.

  (R2) For all $\beta < \alpha \leq \omega_1$ such that $\beta \in I_7$ and for all $x \in X_\alpha \cap (\pi_\beta^\alpha)^{-1}(a^\beta)$, we have that x is a strong limit point of $\langle X_\alpha \cap (\pi_\beta^\alpha)^{-1}(a_{2n}^\beta) : n \in \omega\rangle$.

  (R3) $f(a_n^\alpha) = b_n^\alpha$ for all $n \in \omega$.

Let us shortly rephrase, what these conditions mean intuitively. The condition (C3) means that under given conditions all limit points of the sequence given by $\diamondsuit$ are preserved by lifting it up. The condition (C4) says that whenever the initial segment of length ω of the sequence given by $\diamondsuit$ converges, its limit point is split in the next space. The statement (C7) says that when the premises are satisfied, then we can find a convergent sequence on each "side" of the homeomorphism, such that the limit point of one sequence is preserved by lifting it up, while for the other sequence it is not.

These statements will be shown to be sufficient conditions for some topological properties as superrigidity, HS or containing no non-trivial convergent sequences.

Definition 2.4.4. Let $Y = \{y_\alpha : \alpha \in \omega_1\} \subseteq X$. Then Y is called left-separated if for each $\alpha \in \omega_1$ the set $\{y_\gamma : \gamma \geq \alpha\}$ is open in Y.

Lemma 2.4.5. *Every space X, which is not HS contains a left-separated subspace $Y = \{y_\alpha : \alpha \in \omega_1\}$.*

Proof. Let $Z \subseteq X$ be non-separable. By induction pick $y_\alpha \in Z$ such that $y_\alpha \notin \overline{\{y_\beta : \beta < \alpha\}}$. It is easy to see that $\{y_\alpha : \alpha \in \omega_1\}$ is left separated. □

The next result is mentioned without a proof in [6, Section 4] but with a reference to a proof of a similar fact in [9, Lemma 2.2].

Proposition 2.4.6. *Every inverse limit of HS-spaces which satisfies condition (C3) is HS.*

Proof. Suppose that X is not HS, so by Lemma 2.4.5 X contains a left-separated $Y = \{y_\alpha : \alpha \in \omega_1\}$.

Claim. There is a club C such that for all $\alpha \in C$ we have that $\{\pi_\alpha(y_\gamma) : \gamma \in \alpha\}$ is dense in $\pi_\alpha(Y)$.

Proof of Claim. For any α use that X_α is HS to pick $\beta(\alpha)$ such that $\{\pi_\alpha(y_\gamma) : \gamma \in \beta(\alpha)\}$ is dense in $\{\pi_\alpha(y_\xi) : \xi \in \omega_1\}$. Let C be the club given by Lemma 2.2.6 applied to the function β, and without loss of generality C consists only of limit ordinals. It suffices to show that every basic clopen subset U hitting $\pi_\alpha[Y]$ also hits $\{\pi_\alpha(y_\gamma) : \gamma \in \alpha\}$.

There is some $\xi \in \alpha$ such that $U = (\pi^\alpha_\xi)^{-1}[\pi^\alpha_\xi[U]]$. By definition of $\beta(\xi)$ we have that $\{\pi_\xi(y_\gamma) : \gamma < \beta(\xi)\}$ and hence $\{\pi_\xi(y_\gamma) : \gamma < \alpha\}$ (as $\beta(\xi) < \alpha$) is dense in $\pi_\xi[Y]$. As $\pi_\xi[U]$ is an open set, it follows that there is some $\gamma < \alpha$ such that $\pi_\xi(y_\gamma) \in \pi^\alpha_\xi[U]$. The claim follows now by observing that

$$\pi_\alpha(y_\gamma) \in (\pi^\alpha_\xi)^{-1}[\pi_\xi(y_\gamma)] \subseteq (\pi^\alpha_\xi)^{-1}[\pi^\alpha_\xi[U]] = U. \qquad \square_{\text{Claim}}$$

We know that $S := \{\alpha \in \omega_1 : \forall\gamma \in \alpha\ \pi_\alpha(y_\gamma) = z^\alpha_\gamma\}$ is stationary, so it hits the club C. Pick $\alpha \in S \cap C$. Then $\pi_\alpha(y_\alpha)$ lies inside the closure of $\{z^\alpha_\gamma : \gamma < \alpha\}$, i.e. $\pi_\alpha(y_\alpha)$ is a limit point of $\{z^\alpha_\gamma : \gamma < \alpha\}$. By (C3), we have that y_α is a strong limit point of $\langle X \cap (\pi_\alpha)^{-1}(\pi_\alpha(y_\gamma)) : \gamma \in \alpha\rangle$, which implies that y_α is a limit point of $\{y_\gamma : \gamma < \alpha\}$. This is a contradiction to the fact that $\{y_\gamma : \gamma < \alpha\}$ is closed in Y. Hence X must be HS. □

A rather sketchy proof of the next fact is given by de la Vega in [6, Section 4]. In our proof, we will give some more details. The results following afterwards are also inspired by the ideas of de la Vega.

Proposition 2.4.7. *Let X be the inverse limit of $\langle X_\alpha : \alpha \in \omega_1\rangle$, where each X_α is non-empty, compact and has no isolated points. If X satisfies conditions (C3) and (C4), then X contains no non-trivial convergent sequence.*

Proof. Suppose $\langle x_n : n \in \omega\rangle$ was a non-trivial convergent sequence in X. Then because X has a basis of clopen sets, it is possible to find inductively mutually different points x_α (where $\omega \leq \alpha < \omega_1$) such that $\overline{\{x_n : n \in \omega\}}$ is disjoint from $\overline{\{x_\alpha : \alpha \geq \omega\}}$.

Indeed, the complement of the closure of the first set contains some non-empty basic clopen set A, so we can pick the x_α's for $\alpha \geq \omega$ in A. Pick them such that they are mutually different. Note that the set A cannot be countable, because otherwise for any $\alpha \in \omega_1$ we would have that $Q := \pi_\alpha[A]$ was a countable clopen subspace of X_α and the latter is homeomorphic to the metrizable space $\mathbb{C}$, by Brouwer's Theorem (2.1.2). Because Q is open, it can have no isolated points and thus Q being countable, metrizable and having no isolated points, it must be a copy of the rationals (see eg. [5]). However, Q is closed, thus compact, and the rationals are not.

It is clear that then $\langle x_n : n \in \omega\rangle$ and $\langle x_\alpha : \alpha \geq \omega\rangle$ have disjoint closures. By Lemma 2.3.4 there is some $\beta \in \omega_1$ such that for all $\alpha \geq \beta$ we also have that $\langle \pi_\alpha(x_n) : n \in \omega\rangle$ and $\langle \pi_\alpha(x_\gamma) : \gamma \geq \omega\rangle$ have disjoint closures.

There is a club C such that for each $\alpha \in C$ all $\pi_\alpha(x_\gamma)$, where $\gamma \in \alpha$, are mutually different. Indeed, for each $\xi \in \omega_1$ we may pick some $\beta(\xi)$ such that all $\pi_{\beta(\xi)}(x_\gamma)$, where $\gamma \in \xi$, are mutually different. By Lemma 2.2.6 we find a club C such that for each $\alpha \in C$ and for each $\xi < \alpha$ we have that $\beta(\xi) < \alpha$, i.e. all $\pi_\alpha(x_\gamma)$, where $\gamma \in \xi$, are mutually different. By assuming that C consists only of limit ordinals, this implies that for each $\alpha \in C$ all $\pi_\alpha(x_\gamma)$, where $\gamma \in \alpha$, are mutually different.

Let $S := \{\alpha \in \omega_1 : \forall\gamma \in \alpha\ (\pi_\alpha(x_\gamma) = z^\alpha_\gamma)\}$ and pick $\alpha \in S \cap C$ such that $\alpha \geq \beta$. Then we have that $z^\alpha_\gamma = \pi_\alpha(x_\gamma)$ for all $\gamma \in \alpha$ and $\langle \pi_\alpha(x_n) : n \in \omega\rangle = \langle z^\alpha_n :$

$n \in \omega\rangle$ is a non-trivial convergent sequence with limit point $\pi_\alpha(x) =: y$. Fix any $\widetilde{x} \in (\pi_\alpha)^{-1}(y) \cap X$, i.e. $\pi_\alpha(\widetilde{x}) = \pi_\alpha(x) =: y$. Notice that y is also a limit point of $\langle z_\gamma^\alpha : \gamma \in \alpha\rangle$ and all z_γ^α are mutually different, so (C3) implies that $\widetilde{x}$ is a strong limit point of $\langle X \cap (\pi_\alpha)^{-1}(z_\gamma^\alpha) : \gamma \in \alpha\rangle$.

We show that $\widetilde{x}$ is a strong limit point of $\langle X \cap (\pi_\alpha)^{-1}(z_n^\alpha) : n \in \omega\rangle$. For this let U be a neighborhood of $\widetilde{x}$. Notice that $\widetilde{x} \in (\pi_\alpha)^{-1}[\pi_\alpha[\overline{\{x_n : n \in \omega\}}]]$, and because we chose $\alpha \geq \beta$ the latter is disjoint from $(\pi_\alpha)^{-1}[\pi_\alpha[\overline{\{x_\gamma : \gamma \geq \omega\}}]]$. Hence $U \setminus (\pi_\alpha)^{-1}[\pi_\alpha[\overline{\{x_\gamma : \gamma \geq \omega\}}]]$ is an open neighborhood of x and because $\widetilde{x}$ is a strong limit point of $\langle X \cap (\pi_\alpha)^{-1}(z_\gamma^\alpha) : \gamma \in \alpha\rangle$ one can find some $\gamma \in \alpha$ such that

$$
\begin{aligned}
\widetilde{x} \notin X \cap (\pi_\alpha)^{-1}(z_\gamma^\alpha) &\subseteq U \setminus \overline{\{x_\xi : \xi \geq \omega\}} \\
&\subseteq U \setminus \{x_\xi : \xi \geq \omega\}.
\end{aligned}
$$

It is clear that $\gamma < \omega$, because $x_\gamma \in X \cap (\pi_\alpha)^{-1}(z_\gamma^\alpha)$ implies that $x_\gamma \notin \{x_\xi : \xi \geq \omega\}$. In conclusion, $\widetilde{x}$ is a strong limit point of $\langle X \cap (\pi_\alpha)^{-1}(z_n^\alpha) : n \in \omega\rangle$, and it follows that $\widetilde{x}$ is also a limit point of $\langle x_n : n \in \omega\rangle$.

Remember that we chose $\widetilde{x}$ arbitrarily in $X \cap (\pi_\alpha)^{-1}(y)$, hence any point in this set is a limit point of our convergent sequence $\langle x_n : n \in \omega\rangle$. This means that there can be at most one point in this set. However, by (C4) it follows that $|X_{\alpha+1} \cap (\pi_\alpha^{\alpha+1})^{-1}(x)| > 1$. By choosing two different points in there and using the fact that $X_{\alpha+1} = \pi_{\alpha+1}(X)$ (see Lemma 2.3.2) to lift them to different points in $X \cap (\pi_\alpha)^{-1}(y)$, we see that $|X \cap (\pi_\alpha)^{-1}(y)| > 1$, which is a contradiction. □

Lemma 2.4.8. *Let X be the inverse limit of $\langle X_\alpha : \alpha \in \omega_1\rangle$ satisfying the property (C7). Then we have:*

- ($\star$) *For all $\alpha \in \omega_1$, if $\{z_n^\alpha : n \in \omega\}$ is dense in X_α, $\{z_{2n}^\alpha : n \in \omega\}$ and $\{z_{\omega+2n}^\alpha : n \in \omega\}$ are subsets of X_α and have disjoint closures, say Y_α and Z_α respectively, and $f : Y_\alpha \to Z_\alpha$ is a homeomorphism such that $f(z_{2n}^\alpha) = z_{\omega+2n}^\alpha$, then for all compact, disjoint $Y, Z \subseteq X$ such that $\pi_\alpha[Y] = Y_\alpha$ and $\pi_\alpha[Z] = Z_\alpha$ there is no homeomorphism $h : Y \to Z$ such that $\alpha \in Q_h$ and $h_\alpha = f$.*

Proof. Indeed, if there was such a homeomorphism $h : Y \to Z$, the members of the sequence $\langle Y \cap (\pi_\alpha)^{-1}(a_{2n}^\alpha) : n \in \omega\rangle$ would be mapped by h onto the members of $\langle Z \cap (\pi_\alpha)^{-1}(b_{2n}^\alpha) : n \in \omega\rangle$ and the image of $Y \cap (\pi_\alpha)^{-1}(a^\alpha)$ under h would be $Z \cap (\pi_\alpha)^{-1}(b^\alpha)$. Now the set $\pi_{\alpha+1}[Z]$ is compact and satisfies $\pi_\alpha^{\alpha+1}[\pi_{\alpha+1}[Z]] = Z_\alpha$, so by condition (R1) we can find some $x \in \pi_{\alpha+1}(Z) \cap (\pi_\alpha^{\alpha+1})^{-1}(b^\alpha)$ which is not a strong limit point of $\langle \pi_{\alpha+1}[Z] \cap (\pi_\alpha^{\alpha+1})^{-1}(b_{2n}^\alpha) : n \in \omega\rangle$. Let $z \in Z$ such that $\pi_{\alpha+1}(z) = x$.

Then $z \in Z \cap (\pi_\alpha)^{-1}(b^\alpha)$ is not a strong limit point of $\langle Z \cap (\pi_\alpha)^{-1}(b^\alpha_{2n}) : n \in \omega\rangle$. Hence $h^{-1}(z) \in X \cap (\pi_\alpha)^{-1}(a^\alpha)$ is not a strong limit point of $\langle Y \cap (\pi_\alpha)^{-1}(a^\alpha_{2n}) : n \in \omega\rangle$ and in particular not of $\langle X \cap (\pi_\alpha)^{-1}(a^\alpha_{2n}) : n \in \omega\rangle$, which clearly contradicts (R2). □

Proposition 2.4.9. *Let X be the inverse limit of $\langle X_\alpha : \alpha \in \omega_1\rangle$ and assume that all X_α are non-empty and compact. If X is HS and satisfies condition (C7), then X is also superrigid.*

Proof. The space X is compact and infinite by Lemma 2.3.2. We show that X has no infinite compact disjoint homeomorphic sets. By contradiction suppose that we have two infinite compact disjoint $Y, Z \subseteq X$ and a homeomorphism $h : Y \to Z$, then by Proposition 2.3.5 there is a club Q_h, such that for each $\alpha \in Q_h$ there is a homeomorphism $h_\alpha : \pi_\alpha[Y] \to \pi_\alpha[Z]$ satisfying $\pi_\alpha(h(x)) = h_\alpha(\pi_\alpha(x))$ for all $x \in Y$.

Now let $\{x_{2n} : n \in \omega\}$ be dense in Y (we know that X is HS) and let $\{x_{2n+1} : n \in \omega\}$ be dense in X (hence $\{x_n : n \in \omega\}$ is also dense in X). Let $x_{\omega+2n} := h(x_{2n})$, and let all other x_γ be arbitrary points in X. Then the set $S = \{\alpha \geq \omega + \omega : \forall \gamma \in \alpha : (\pi_\alpha(x_\gamma) = z^\alpha_\gamma)\}$ is stationary, and its intersection with Q_h is unbounded in ω_1. It is not too hard to see that $Y_\alpha := \overline{\{z^\alpha_{2n} : n \in \omega\}} = \pi_\alpha[Y]$ and $Z_\alpha := \overline{\{z^\alpha_{\omega+2n} : n \in \omega\}} = \pi_\alpha[Z]$ for all $\alpha \in S$. (For this just notice that $\pi_\alpha : X \to X_\alpha$ is a closed and continuous map, which implies that $\pi_\alpha[\overline{\{x_{2n} : n \in \omega\}}] = \overline{\{\pi_\alpha(x_{2n}) : n \in \omega\}}$.)

Because Y and Z are disjoint, by Lemma 2.3.4 there is a $\beta \in \omega_1$ such that for all $\alpha \geq \beta$ we have that Y_α is disjoint from Z_α. Furthermore for each $\alpha \geq \beta$ in $Q_h \cap S$ we get that $h_\alpha(z^\alpha_{2n}) = h_\alpha(\pi_\alpha(x_{2n})) = \pi_\alpha(h(x_{2n})) = \pi_\alpha(x_{\omega+2n}) = z^\alpha_{\omega+2n}$. Therefore $(\star)$ from the above lemma does not hold. This contradicts (C7), so there are no two infinite compact disjoint homeomorphic subsets of X. □

To sum up the results from this section we formulate the following corollary:

Corollary 2.4.10. *Suppose X is the inverse limit of $\langle X_\alpha : \alpha \in \omega_1\rangle$, where all X_α are infinite and compact. If X has satisfies (C3), (C4) and (C7), then X is a superrigid Fedorčuk space.*

Proof. By Lemma 2.3.2 X is compact and infinite. To see that X is HS, apply Proposition 2.4.6. Note that each subspace of X_α has a countable base, so each X_α must be HS. Furthermore, X contains no non-trivial convergent sequence by Proposition 2.4.7. So X is Fedorčuk. Finally, by Proposition 2.4.9 we can also conclude that X is superrigid. □

2.5 A superrigid Fedorčuk space

The goal of this section is to prove the following result:

Theorem 2.5.1. *($\diamondsuit$) There is a superrigid Fedorčuk space.*

Proof. The construction of this space follows the idea of [6, Section 4], but with slight changes and simplification. As mentioned, the desired space X will be constructed as an inverse limit of an inverse system $\langle X_\alpha : \alpha \in \omega_1 \rangle$. This system will be built by induction over α, such that

(C1) X_α is closed in $\mathbb{C} \times \mathbb{C}^\alpha$, infinite and has no isolated points,

(C2) $\pi^\alpha_\beta[X_\alpha] = X_\beta$ for every $\beta \leq \alpha$,

and conditions (C3), (C4) and (C7) from Definition 2.4.3 hold. By Corollary 2.4.10, the space X will be a superrigid Fedorčuk space.

Note that the statements (C4) and (C7) are basically statements about the next step of our construction, that is why we will satisfy them whenever we build the space $X_{\alpha+1}$ from X_α. By setting $X_0 := \mathbb{C} \times \{\emptyset\}$, these conditions hold at step 0 of the construction.

If α is a limit ordinal and for all $\gamma < \alpha$ the space X_γ is already constructed and satisfies the above conditions, then we just let $X_\alpha := \bigcap_{\gamma\in\alpha}(\pi^\alpha_\gamma)^{-1}[X_\gamma]$, which is the α-inverse limit of $\langle X_\gamma : \gamma \in \alpha\rangle$. Then X_α is clearly closed in $\mathbb{C} \times \mathbb{C}^\alpha$ and by Lemma 2.3.2 we have $\pi^\alpha_\beta[X_\alpha] = X_\beta$ for every $\beta \leq \alpha$, which implies that X_α is infinite.

We also need to check that X_α has no isolated points. Suppose by contradiction it had an isolated point x, then we can find a basic open set $U \subseteq \mathbb{C} \times \mathbb{C}^\alpha$ which contains only one point from X_α, namely x. Note that there is a $\gamma \in \alpha$ such that $U = (\pi^\alpha_\gamma)^{-1}[\pi^\alpha_\gamma[U]]$. We check that $\pi^\alpha_\gamma(x)$ is an isolated point of X_γ, which is a contradiction. Let $y \in X_\gamma \cap \pi^\alpha_\gamma[U]$, then we can find some $v \in U$ such that $\pi^\alpha_\gamma(v) = y$. Because $\pi^\alpha_\gamma(v) \in X_\gamma = \pi^\alpha_\gamma[X_\alpha]$, there is a $u \in X_\alpha$ such that $\pi^\alpha_\gamma(u) = \pi^\alpha_\gamma(v)$. This implies that u is not only in X_α, but also in U, and therefore $u = x$. Hence $y = \pi^\alpha_\gamma(x)$ is an isolated point in X_γ, contradicting the induction hypothesis.

To preserve the properties (C3) and (R2) in the limit case, it is important to notice that limit points are preserved in the following sense: Suppose that $x \in X_\alpha$ and for some $\beta < \alpha$, S is a sequence of points in X_β such that for all $\gamma \in \alpha \setminus \beta$, $\pi^\alpha_\gamma(x)$ is a strong limit point of $\langle X_\gamma \cap (\pi^\gamma_\beta)^{-1}(s) : s \in S\rangle$, then x is a strong limit point of $\langle X_\alpha \cap (\pi^\alpha_\beta)^{-1}(s) : s \in S\rangle$. Indeed if $\beta < \alpha$, let U be a basic open neighborhood of x.

Then there is some $\gamma \in \alpha \setminus \beta$ such that $U = (\pi^\alpha_\gamma)^{-1}[\pi^\alpha_\gamma[U]]$. As $\pi^\alpha_\beta(x) = \pi^\gamma_\beta(\pi^\alpha_\gamma(x))$, by induction hypothesis $\pi^\alpha_\gamma(x)$ is a strong limit point of $\langle X_\gamma \cap (\pi^\gamma_\beta)^{-1}(s) : s \in S\rangle$. Therefore there is some $s \in S$ such that $\pi^\alpha_\gamma(x) \notin X_\gamma \cap (\pi^\gamma_\beta)^{-1}(s) \subseteq \pi^\alpha_\gamma[U]$, which then implies that

$$\begin{aligned} x \notin X_\alpha \cap (\pi^\alpha_\beta)^{-1}(s) &\subseteq (\pi^\alpha_\gamma)^{-1}[X_\gamma] \cap (\pi^\alpha_\gamma)^{-1}[(\pi^\gamma_\beta)^{-1}(s)] \\ &= (\pi^\alpha_\gamma)^{-1}[X_\gamma \cap (\pi^\gamma_\beta)^{-1}(s))] \subseteq (\pi^\alpha_\gamma)^{-1}[\pi^\alpha_\gamma[U]] = U. \end{aligned}$$

The interesting case is the successor case. Suppose that X_α with the above properties was built and let us build $X_{\alpha+1}$. First note that the hypotheses of (C7) and (C4) cannot hold at the same time, because a convergent sequence in X_α (in this case $\langle z^\alpha_n : n \in \omega\rangle$) cannot be dense. So we will care about three cases.

The first case is the one in which the hypothesis of (C7) is fulfilled, i.e. $\alpha \in I_7$. The construction of the required sequences is also similar to de la Vega's construction in [6, Section 4]. We gather inside a set $\mathcal{S}''$ all sequences $\langle X_\alpha \cap (\pi^\alpha_\beta)^{-1}(z^\beta_\gamma) : \gamma \in \beta\rangle$ where $\beta \leq \alpha$ is such that the z^β_γ are mutually distinct and lie inside X_β, and also all the sequences $\langle X_\alpha \cap (\pi^\alpha_\beta)^{-1}(a^\beta_{2n}) : n \in \omega\rangle$, where $\beta \in \alpha \cap I_7$ satisfies the conditions of (C7).

Fix any non-isolated $a^\alpha \in Y_\alpha$ and let $b^\alpha := f(a^\alpha)$. Note that any compact infinite set contains a non-isolated point. Furthermore fix a sequence $\langle O_n : n \in \omega\rangle$ of non-empty closed pairwise disjoint subsets of Y_α converging to a^α, such that $a^\alpha \notin O_n$. This can be done as follows: Fix a countable clopen neighborhood basis $\{B_0, B_1, ...\}$ of a^α. This point is not isolated, so there is some $y_n \in ((B_n \setminus (O_0 \cup ... \cup O_{n-1})) \setminus \{a^\alpha\}) \cap Y_\alpha$. Then there is a clopen neighborhood $C_n \subseteq B_n \setminus (O_0 \cup ... \cup O_{n-1})$ of y_n, which does not contain a^α. Then $O_n := C_n \cap Y_\alpha$ is as desired.

Now we define $\mathcal{S}' := \mathcal{S}'' \cup \{\langle f[O_n] : n \in \omega\rangle\}$. Note that by Brouwer's Theorem (2.1.2), X_α is homeomorphic to $\mathbb{C}$ and therefore we can apply the Splitting Lemma (2.1.4) to the constant sequence b^α to find closed $A'_0, A'_1 \subseteq X_\alpha$ covering X_α such that (A'_0, A'_1) preserves each sequence from $\mathcal{S}'$.

Find infinitely many natural numbers n_i, $i \in \omega$, such that for all i we have $f[O_{n_i}] \subseteq A'_0$. This can be done as follows. If $n_0, ..., n_k$ were already found, let U be the complement of $\bigcup_{i=0}^k f[O_{n_i}]$, hence U is open. Because (A'_0, A'_1) preserves $\langle f[O_n] : n \in \omega\rangle$, we can find some n_{k+1} such that $f[O_{n_{k+1}}] \subseteq U \setminus A'_1 \subseteq A'_0$. It is also clear that n_{k+1} must be different from all $n_0, ..., n_k$.

We may now choose $b^\alpha_i \in f[O_{n_i}]$ and $a^\alpha_i = f^{-1}(b^\alpha_i)$. Then the sequence $\langle b^\alpha_i : i \in \omega\rangle$ avoids $\mathcal{S}'$, because if $S = \langle S_i : i \in I\rangle \in \mathcal{S}'$, then (A'_0, A'_1) preserves S. It follows that whenever b^α is a strong limit point of S and U an open neighborhood of b^α, there is some $i \in I$ such that $S_i \subseteq U \setminus A'_0 \subseteq U \setminus (\{b_i^\alpha : i \in \omega\} \cup \{b^\alpha\})$. Furthermore the sequence $\langle b_i^\alpha : i \in \omega\rangle$ also avoids $\{\langle a_{2n}^\alpha : n \in \omega\rangle\}$, as $b^\alpha \neq a^\alpha$ implies that b^α cannot be a strong limit point of $\{\langle a_{2n}^\alpha : n \in \omega\rangle\}$.

We can hence apply the Splitting Lemma (2.1.4) to the sequence of distinct points $\langle b_i^\alpha : i \in \omega\rangle$ and to $\mathcal{S} := \mathcal{S}' \cup \{\langle a_{2n}^\alpha : n \in \omega\rangle\}$, by which we get closed $A_0, A_1 \subseteq X_\alpha$ such that

(i) $A_0 \cup A_1 = X_\alpha$,

(ii) $A_0 \cap A_1 = \{b^\alpha\}$,

(iii) $\{b_{2i}^\alpha : i \in \omega\} \subseteq A_0$ and $\{b_{2i+1}^\alpha : i \in \omega\} \subseteq A_1$

(iv) (A_0, A_1) preserves each sequence in $\mathcal{S}$.

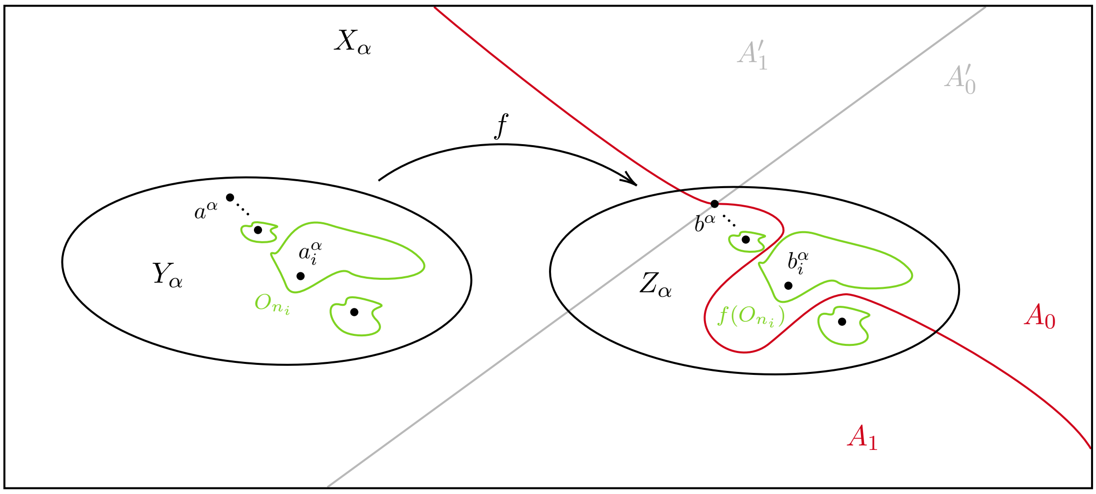

Figure 2.1: Splitting the space X_α and the sequence $\langle b_i : i \in \omega\rangle$ into A_0 and A_1

We then let $X_{\alpha+1} := A_0 \times \{0\} \cup A_1 \times \{1\}$ where by 0 resp. 1 we mean the constant sequence of zeroes resp. ones. We check that (R1), (R2) and (R3) hold:

(R1) Let $R \subseteq X_{\alpha+1}$ be compact such that $\pi_\alpha^{\alpha+1}[R] = Z_\alpha$. The sequence $\langle (b_{2n+1}^\alpha, 1) : n \in \omega\rangle$, whose members all lie in R, converges to the point $x := (b^\alpha, 1)$ and because R is closed this point must lie in R. It is not a strong limit point of the sequence $\langle R \cap (\pi_\alpha^{\alpha+1})^{-1}(b_{2n}^\alpha) : n \in \omega\rangle = \langle \{(b_{2n}^\alpha, 0)\} : n \in \omega\rangle$, because this sequence can only have one limit point, which is clearly $(b^\alpha, 0)$. That is why $(b^\alpha, 1)$ can not be a limit point of this sequence.

(R2) Suppose $\beta < \alpha + 1$ and $x \in X_{\alpha+1} \cap (\pi^{\alpha+1}_{\beta})^{-1}(a^{\beta})$. By Lemma 2.1.5, it suffices to show that $\pi^{\alpha+1}_{\alpha}(x)$ is a strong limit point of $\langle X_{\alpha} \cap (\pi^{\alpha}_{\beta})^{-1}(a^{\beta}_{n}) : n \in \omega\rangle$. If $\beta = \alpha$, this is trivial. If $\beta < \alpha$, this just follows from the induction hypothesis and the observation that $\pi^{\alpha+1}_{\alpha}(x) \in X_{\alpha} \cap (\pi^{\alpha}_{\beta})^{-1}(a^{\beta})$.

(R3) holds trivially.

The conditions (C1) and (C2) are trivially satisfied, and the condition (C3) is preserved by the same argument as for (R2).

Let us consider the second case, in which the premise of (C4) holds, i.e. $\langle z^{\alpha}_{n} : n \in \omega\rangle$ converges to some point x. Apply the Splitting Lemma (2.1.4) to the constant sequence x and the collection of sequences $\mathcal{S}''$ from above. This gives us closed A_0, A_1 such that $A_0 \cap A_1 = \{x\}$ and as before, we can again define $X_{\alpha+1} := A_0 \times \{0\} \cup A_1 \times \{1\}$, which clearly implies the statement (C4). For the same reasons as in the previous case, the conditions (C1), (C2) and (C3) are also preserved here.

In the third case, where the premises of (C7) and (C4) both fail to hold, the same is done with any point $x \in X_{\alpha}$. □

Chapter 3

Limitations on the construction of Efimov spaces

After we have presented in the previous chapter our modification of de la Vega's streamlined construction of an Efimov space allowing to control its autohomeomorphisms, it is natural to try to weaken the set-theoretic assumption $\diamondsuit$ used there. This way one gets a hereditarily separable Efimov space of weight ω_1.

In this chapter we present a number of set-theoretic assumptions which imply that no Efimov space can be hereditarily separable or have weight ω_1, respectively. These assumptions are complemented nicely by the classical Čech-Pospišil theorem which demonstrates that often one cannot get Efimov spaces of size $\mathfrak{c}$. To complete the picture we also present Fedorčuk's construction of an Efimov space of size $\mathfrak{c}$ under assumptions which hold, e.g., after adding at least ω_2 many Cohen reals to a model of GCH.

3.1 Limitations on the weight of the space

Definition 3.1.1. For a topological space X, the *weight* of X is the least size of a base for the topology of X. It is denoted by $w(X)$, i.e.

$$w(X) := \min\{|B| : B \text{ is a base for the topology of } X\}.$$

It is clearly well-defined, as the collection of all open sets is trivially a base for the topology of X.

We will show that there are sufficient conditions on the weight of a compact

Hausdorff space, which imply that the space contains a non-trivial convergent sequence.

Let us state two facts related to this definition. The first fact is obvious, while the second one is more tedious. Its proof can be found in [10, Theorem 3.2.2].

Lemma 3.1.2.

(a) The weight of an infinite Hausdorff space is infinite.

(b) Let X be a compact Hausdorff space such that $w(X)$ is infinite. Then X is the continuous image of a closed subspace of $2^{w(X)}$.

The splitting number is one of so-called cardinal characteristics. These are usually critical cardinalities of subsets or families of subsets of the real line with certain properties. These properties could be combinatorial, topological, measure-theoretic or have some other nature.

Definition 3.1.3. Let x and y be infinite subsets of ω. We say that x *splits* y, if both $y \cap x$ and $y \setminus x$ are infinite. A family $\mathcal{S}$ of infinite subsets of ω is called *splitting*, if for each infinite $y \subseteq \omega$ there is some $x \in \mathcal{S}$ such that x splits y. The *splitting number* $\mathfrak{s}$ is defined as the least size of a splitting family, i.e.

$$\mathfrak{s} := \min\{|\mathcal{S}| : \mathcal{S} \text{ is splitting.}\}.$$

This is well defined because the collection of all infinite subsets of ω is splitting.

For infinite spaces, the following notion implies the existence of a non-trivial convergent sequence.

Definition 3.1.4. A topological space is called *sequentially compact* if every sequence of its elements has a convergent subsequence.

We first observe the following fact, whose proof is not difficult and thus omitted.

Lemma 3.1.5. *The continuous image of a sequentially compact space is also sequentially compact.*

The next result is due to Booth [3, Theorem 2], whose ideas we will follow here.

Theorem 3.1.6 (Booth)**.** *If X is an infinite compact Hausdorff space such that $w(X) < \mathfrak{s}$, then X is sequentially compact. In particular X contains a non-trivial convergent sequence, and can therefore not be Efimov.*

Proof. Let X be an infinite compact Hausdorff space such that $w(X) < \mathfrak{s}$. We first show that the space $2^{w(X)}$ is sequentially compact. For this let $\langle x_n : n \in \omega \rangle$ be a sequence in $2^{w(X)}$. Define $a_\alpha := \{n \in \omega : x_n(\alpha) = 1\}$ for $\alpha < w(X)$. Because $w(X) < \mathfrak{s}$, the family $\{a_\alpha : \alpha < w(X) \wedge |a_\alpha| = \omega\}$ is not splitting, hence there is an infinite $b \subseteq \omega$ such that for each $\alpha < w(X)$ with $|a_\alpha| = \omega$ we have that either $b \cap a_\alpha$ or $b \setminus a_\alpha$ is finite. We define $x \in 2^{w(X)}$ by the stipulation $x(\alpha) = 1$ if and only if $x_n(\alpha) = 1$ for almost all $n \in b$.

We show that the subsequence $\langle x_n : n \in b \rangle$ converges to the point x. By the definition of the topology on $2^{w(X)}$ it suffices to show that for each $\alpha < w(X)$ we have $x_n(\alpha) = x(\alpha)$ for almost all $n \in b$. If a_α is finite, it is clear that $x_n(\alpha) = 0 = x(\alpha)$ for almost all $n \in \omega$, so also for almost all $n \in b$. If a_α is infinite, there are two cases. If $x(\alpha) = 1$, then by definition $x_n(\alpha) = 1 = x(\alpha)$ for almost all $n \in b$. If $x(\alpha) = 0$, we have $x_n(\alpha) = 0$ for infinitely many $n \in b$, which can be equivalently expressed by $|b \setminus a_\alpha| = \omega$. In this case we must have that $b \cap a_\alpha$ is finite, which means that $x_n(\alpha) = 0 = x(\alpha)$ for almost all $n \in b$. Consequently, the sequence $\langle x_n : n \in \omega \rangle$ has a convergent subsequence and as the sequence was arbitrary, the space $2^{w(X)}$ is sequentially compact.

From this, we can deduce that X is sequentially compact. By Lemma 3.1.2, X is the continuous image of a closed subset of $2^{w(X)}$. This closed subset is clearly sequentially compact. By Lemma 3.1.5 we conclude that X is sequentially compact. □

Let us go on to the second limitation related to the weight of a space. For this, a definition and a related lemma will be needed.

Definition 3.1.7. A topological space is called *scattered* if every non-empty closed subspace A contains an isolated point in A.

In his PhD thesis [23, Lemma 2.3.4 and Proposition 2.3.5], Sobota presents the following two results, which are due to Geschke, see [13, Theorem 2.1]. Here we present the proofs in a more detailed way, based on these two sources.

Lemma 3.1.8. *If X is a scattered infinite compact Hausdorff space, then X contains a non-trivial convergent sequence.*

Proof. We construct a countable $A \subseteq X$ by defining inductively A_n for natural numbers n, which will be guaranteed to be finite and open. Let $A_0 = \emptyset$ and whenever A_n is a given finite open set, put $A_{n+1} := A_n \cup \{p\}$, where p is an isolated point of $X \setminus A_n$ (such a point exists, because X is infinite and scattered). The set $A := \bigcup_{n\in\omega} A_n$ is clearly a countable open set.

The boundary ∂A of A is a non-empty closed set, because if it was empty, then A must be closed. The compactness of X would imply that A is compact. However A consists of isolated points, therefore it must be finite, which is a contradiction.

Hence the closed set ∂A contains an isolated point x in ∂A. Separate x from the closed set $\partial A \setminus \{x\}$ by open subsets V and W respectively. Note that x cannot lie in A, because A is open, so $X \setminus A$ is closed and thus $\partial A = \partial(X \setminus A) \subseteq X \setminus A$

Let $\langle x_n : n \in \omega\rangle$ be an enumeration of the countable set $V \cap A$. We will show that $\langle x_n : n \in \omega\rangle$ converges to the point x. For this, we first observe that $\overline{V \cap A} = (V \cap A) \cup \{x\}$. We show that the latter is the smallest closed set containing $V \cap A$, more precisely that $V \cap A$ is not closed, while $(V \cap A) \cup \{x\}$ is.

The set $V \cap A$ cannot be closed, because $x \notin V \cap A$ while $x \in \partial(V \cap A)$: If U is an open neighborhood of x, we have that $U \cap V \cap A$ and $(U \cap V) \setminus A$ are non-empty, in particular $U \setminus A$ and hence $U \setminus (V \cap A)$ are non-empty, which means that x lies in $\partial(V \cap A)$, as required. We check that $(V \cap A) \cup \{x\}$ is a closed set. It is clear that $(V \cap A) \cup \{x\}$ is contained in $\overline{V \cap A}$, because $x \in \partial(V \cap A)$. Now suppose that y lies in $\overline{V \cap A}$ but not inside $V \cap A$. This implies that for each open neighborhood U of y we have that $U \cap V \cap A$ is non-empty, and clearly also $U \setminus (V \cap A)$. We can apply this to the open neighborhood $U \cap V$ of y to obtain that for each such U, $U \cap V \cap A$ and $(U \cap V) \setminus (V \cap A) = (U \cap V) \setminus A$ are non-empty. This implies in particular the non-emptiness of $U \cap A$ and $U \setminus A$ for every open neighborhood U of y, in other words $y \in \partial A$. Observe also that $V \cap A \subseteq X \setminus W$ and as the latter is closed, we also have $\overline{V \cap A} \subseteq X \setminus W$. This means that y does not lie in W, but since it is in ∂A, it follows that $y = x$. We have thus shown that $\overline{V \cap A} \subseteq (V \cap A) \cup \{x\}$ and hence $\overline{V \cap A} = (V \cap A) \cup \{x\}$.

It follows that $K := (V \cap A) \cup \{x\}$ is a one-point-compactification of $V \cap A$. We show that $\langle x_n : n \in \omega\rangle$ converges to the point x. If U is an open neighborhood of x in X, then $U \cap K$ is open in K and so $K \setminus U$ must be compact. However, every point in there is isolated and by compactness we obtain that $K \setminus U$ is finite. This means that only finitely many x_n's lie outside of U, which yields the claim. □

As we will see, the statement which was just proven is in fact true also for a non-scattered space if its weight satisfies some condition. To prove this, we will need some terminology which we are going to introduce now.

Martin's Axiom is a statement which is (as $\diamondsuit$ and the Continuum Hypothesis) independent from ZFC, i.e. it can be neither proven nor disproven by these axioms. The definition of this statement requires several other definitions, which we will recall. Let $(P, \leq)$ be a partially ordered set (shortly, a poset) and $A \subseteq P$ be non-empty. We say that A is *dense* if for each $p \in P$ there is some $a \in A$ such that $a \leq p$. We say that A is a *filter* on P if for each $a, b \in A$ there is some $c \in A$ such that $c \leq a$ and $c \leq b$ and if for each $a \in A$ and $b \in P$ such that $a \leq b$ we have $b \in A$. Two members p, q of P are said to be *incompatible* if there is no $r \in P$ such that $r \leq p$ and $r \leq q$. A subset $A \subseteq P$ is called an *antichain*, if for each $a, b \in A$ we have that a and b are incompatible. The poset P is said to have the *countable chain condition* (*ccc* for short), if every antichain is countable.

Definition 3.1.9. First, we formulate Martin's Axiom for κ-many dense sets (MA_κ) as follows:

> For each ccc poset $(P, \leq)$, and for each family $\mathcal{D}$ of dense sets in $(P, \leq)$ with $|\mathcal{D}| \leq \kappa$, there is some filter $F \subseteq P$ which meets every member of $\mathcal{D}$.

Martin's Axiom (MA) is the statement $\forall \kappa < \mathfrak{c} : \mathrm{MA}_\kappa$. Furthermore, we also define Martin' Axiom for κ-many dense sets and countable posets ($\mathrm{MA}_\kappa(\aleph_0)$) as follows:

> For each <u>countable</u>, ccc poset $(P, \leq)$, and for each family $\mathcal{D}$ of dense sets in $(P, \leq)$ with $|\mathcal{D}| \leq \kappa$, there is some filter $F \subseteq P$ which meets every member of $\mathcal{D}$.

Of course, ccc may be dropped in $\mathrm{MA}_\kappa(\aleph_0)$, because every countable poset is trivially ccc.

Another cardinal characteristic (as the splitting number $\mathfrak{s}$) is the covering number of the meager ideal. Recall that a subset of the real line is called *nowhere dense*, if the interior of its closure is empty. A countable union of nowhere dense sets is called *meager*. The collection $\mathcal{M}$ of all meager subsets of the real line forms obviously an ideal, meaning that the collection is closed under finite unions, subsets and that the empty set is meager. (Of course, the word ideal has not the same meaning as in Section 1.3).

Definition 3.1.10. The *covering number of the meager ideal* $\mathcal{M}$ is the least number of meager sets needed to cover the real line, or more precisely

$$\mathsf{cov}(\mathcal{M}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M} \wedge \bigcup \mathcal{A} = \mathbb{R}\}.$$

The following proposition will be very practical. Among other things, it gives a characterisation of a cardinal being less than $\mathsf{cov}(\mathcal{M})$ in terms of a version of Martin's Axiom.

Proposition 3.1.11.

(a) MA_ω *is always true.*

(b) $\mathrm{MA}_{\mathfrak{c}}$ *is always false.*

(c) Let κ *be a cardinal. Then* $\kappa < \mathsf{cov}(\mathcal{M})$ *if and only if* $MA_\kappa(\aleph_0)$ *holds.*

Proof. The first two statements can be found in [20, Lemmas III.3.13 and III.3.14] and for the proof of (c) we refer the reader to [2, Theorem 7.13]. □

With all these terminologies and characterisations in our pockets, we are now able to prove a theorem by Geschke. As already mentioned, the idea of the proof can be found in [23, Proposition 2.3.5] and [13, Theorem 2.1].

Recall that a family of sets $\mathcal{A}$ has the finite intersection property if for each finite $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcap \mathcal{B} \neq \emptyset$. A space is compact if and only if every family of closed sets with the finite intersection property has non-empty intersection.

Definition 3.1.12. Let X be a topological space and $\mathcal{P}$ be a collection of open subset of X. We say that $\mathcal{P}$ forms a π-*base* for X, if for each non-empty open set $U \subseteq X$ there is some $B \in \mathcal{P}$ such that $B \subseteq U$.

Theorem 3.1.13 (Geschke)**.** *If* X *is an infinite compact Hausdorff space such that* $w(X) < \mathsf{cov}(\mathcal{M})$*, then* X *contains a non-trivial convergent sequence.*

Proof. The case in which X is scattered was proven in Lemma 3.1.8 (without using any condition on the weight of X).

If X is not scattered, then without loss of generality we can assume that X has no isolated points, otherwise we just work inside a non-empty closed subspace of X which has no isolated points (such a space exists, as X is not scattered).

We first construct a closed $L_0 \subseteq X$ and a surjection $f_0 : L_0 \to \mathbb{C}$ as follows. Let $\langle O_s : s \in 2^{<\omega} \rangle$ be a sequence of open subsets of X such that $\overline{O_{s^\frown 0}}$ is disjoint from $\overline{O_{s^\frown 1}}$ and $O_{s^\frown i} \subseteq O_s$ for each $s \in 2^{<\omega}$ and $i \in \{0, 1\}$. Such a sequence can be built inductively over the length of s: $O_\emptyset := X$ and whenever O_s is given, it cannot be a singleton and thus we can pick two different points inside O_s, separate them by disjoint open neighborhoods V_0 and V_1. Because our compact Hausdorff space is also T_3, we may find $O_{s^\frown i} \subseteq V_i$ such that $\overline{O_{s^\frown i}} \subseteq V_i$. It is clear that the requirements are fulfilled by this construction.

We let $L_0 := \bigcap_{n\in\omega}\bigcup_{s\in 2^n}\overline{O_s}$ and f_0 is defined by putting $f_0^{-1}(y) = \bigcap_{n\in\omega}\overline{O_{y\restriction n}}$ for $y \in \mathbb{C}$. This function is well-defined, because for different $y, y' \in \mathbb{C}$ we have clearly that $\bigcap_{n\in\omega} O_{y\restriction n}$ and $\bigcap_{n\in\omega} O_{y'\restriction n}$ are disjoint, so there is no $x \in L_0$ which belongs to $f_0^{-1}(y)$ and $f_0^{-1}(y')$ for different $y, y' \in \mathbb{C}$. The domain of f_0 is clearly $\bigcup_{y\in\mathbb{C}}\bigcap_{n\in\omega}\overline{O_{y\restriction n}} = L_0$. The function f_0 is surjective, because the family $\{\overline{O_{y\restriction n}} : n \in \omega\}$ has the finite intersection property and compactness yields that $f_0^{-1}(y) = \bigcap_{n\in\omega}\overline{O_{y\restriction n}} \neq \emptyset$ for each $y \in \mathbb{C}$. Note that $f_0^{-1}(y)$ is closed for each $y \in \mathbb{C}$.

Using Zorn's Lemma, we show that there is some closed $L \subseteq L_0$ such that $f := f_0 \restriction L$ is irreducible in the sense that for each closed proper subset F of L we have that the restriction $f \restriction F$ is not onto $\mathbb{C}$. For this, let

$$\mathcal{W} := \{L \subseteq L_0 : L \text{ is closed and } f_0 \restriction L \text{ is onto } \mathbb{C}\}$$

be a poset, ordered by inclusion. We show that every decreasing chain has a lower bound in $\mathcal{W}$, so that by Zorn's Lemma $\mathcal{W}$ has a minimal element L, which is obviously as desired. Let $\langle L^\gamma : \gamma \in \alpha\rangle$ be a decreasing chain in $\mathcal{W}$ and define $B := \bigcap_{\gamma\in\alpha} L^\gamma$. It suffices to show that $B \in \mathcal{W}$. Because $L^\gamma \in \mathcal{W}$, we have for each $y \in \mathbb{C}$ that $L^\gamma \cap f_0^{-1}(y) \neq \emptyset$. Using compactness of X, it follows that $B \cap f_0^{-1}(y) = \bigcap_{\gamma\in\alpha}(L^\gamma \cap f_0^{-1}(y)) \neq \emptyset$ for each $y \in \mathbb{C}$, because the family $\{L^\gamma \cap f_0^{-1}(y) : \gamma \in \alpha\}$ consists of closed sets and has the finite intersection property. Hence the given decreasing chain has a lower bound $B \in \mathcal{W}$, as required.

We claim that L contains no isolated points. If L contained an isolated point x, then $L' := L \setminus \{x\}$ is a closed proper subspace of L, so $f \restriction L'$ is not onto $\mathbb{C}$, which means that $y := f(x)$ is not hit by any member of L'. However, the family $\{L \cap \overline{O_{y\restriction n}} \setminus \{x\} : n \in \omega\}$ consists of closed sets and has the finite intersection property, so by compactness we get $f^{-1}(y) \setminus \{x\} = (L \cap \bigcap_{n\in\omega}\overline{O_{y\restriction n}}) \setminus \{x\} \neq \emptyset$, a contradiction.

Moreover, we also get that L has a countable π-base consisting of clopen sets. Let $\mathcal{B}$ be a countable base for the Cantor space $\mathbb{C}$, consisting of clopen subsets of $\mathbb{C}$. We show that the countable collection $\mathcal{P} := \{f^{-1}[B] : B \in \mathcal{B}\}$ forms a π-base for L. For this, let U be a non-empty open subset of L. We want to show that there is some $B \in \mathcal{B}$ such that $f^{-1}[B] \subseteq U$. For this we define $O := \{y \in \mathbb{C} : \exists n \in \omega\ L \cap \overline{O_{y\restriction n}} \subseteq U\}$. This is an open set because $O = \bigcup_{n\in\omega}\bigcup_{s\in Z_n}\{y \in \mathbb{C} : s \subseteq y\}$, where $Z_n := \{s \in 2^n : L \cap \overline{O_s} \subseteq U\}$. Furthermore, we observe that O is non-empty, because $L \setminus U$ is a closed proper subset of L, which implies that $f \restriction (L \setminus U)$ is not onto $\mathbb{C}$. This means that $L \cap \bigcap_{n\in\omega}\overline{O_{y\restriction n}} = f^{-1}(y) \subseteq U$ for some $y \in \mathbb{C}$. Using compactness, we obtain that there is some $y \in \mathbb{C}$ and some $n \in \omega$ such that $L \cap \overline{O_{y\restriction n}} \subseteq U$, i.e.

O is non-empty. Therefore we can find some $B \in \mathcal{B}$ with $B \subseteq O$. This implies that for each $y \in B$ there is some natural number n such that $\overline{O_{y \restriction n}} \subseteq U$, which yields $f^{-1}[B] = \bigcup_{y \in B} f^{-1}(y) = \bigcup_{y \in B} \left(L \cap \bigcap_{n \in \omega} \overline{O_{y \restriction n}}\right) \subseteq U$, as required.

Now that we observed all these things, we can go to the meat of the proof. Fix a base $\mathcal{B}'$ of size $w(L) \leq w(X) < \mathsf{cov}(\mathcal{M})$ and consider $\mathcal{P}$ with the partial order $\subseteq$. For each $U \in \mathcal{B}'$, we let $D_U := \{P \in \mathcal{P} : P \subseteq U \text{ or } P \subseteq L \setminus U\}$. This set is clearly dense in $\mathcal{P}$.

By Proposition 3.1.11(c) we have that $MA_{w(L)}(\aleph_0)$ holds and so we can find a filter $\mathcal{F} \subseteq \mathcal{P}$, which hits every D_U. Moreover, we can find some $x \in L$ with $\bigcap \mathcal{F} = \{x\}$, because by compactness $\bigcap \mathcal{F} \neq \emptyset$ and if two distinct points x_0, x_1 lie in all $F \in \mathcal{F}$, we could separate them by disjoint basic open sets U_0 and U_1 in $\mathcal{B}'$. We know that there are $P_i \in \mathcal{F}$ such that $P_i \in D_{U_i}$. However, we would then have $P_i \subseteq U_i$, because $x_i \in P_i$, but this contradicts the fact that $x_i \in P_{1-i}$. By enumerating $\mathcal{F}$ and making it decreasing, one can obtain by compactness a countable neighborhood base for x (For more details, we refer to Lemma 3.2.4 where an analogous argument is done). Because x is non-isolated this implies the existence of a non-trivial convergent sequence $\langle x_n : n \in \omega \rangle$ with limit x. This ends the proof. □

It is well-known that under MA, we have that $\mathsf{cov}(\mathcal{M}) = \mathfrak{s} = \mathfrak{c}$ (cf. [15, Theorem 13.6 together with Theorems 8.11, 8.13, 8.16 and 21.5]), therefore we get the following corollary under MA.

Corollary 3.1.14. *Under* MA*, each infinite compact Hausdorff space X with $w(X) < \mathfrak{c}$ contains a non-trivial convergent sequence.*

3.2 The Szentmiklóssy Theorem

In this section we present the proof of a theorem of Szentmiklóssy asserting that under Martin's Axiom for ω_1 many dense subsets, each HS compact has plenty of convergent sequences. More precisely, MA_{ω_1} implies that HS compacts are hereditarily Lindelöf and hence first-countable. As a conclusion, we will get that the construction under $\diamondsuit$ from the previous chapter cannot be made under the assumption MA_{ω_1}, since the constructed space was an infinite HS compact Hausdorff space with no convergent sequences, which by the above is impossible whenever we assume MA_{ω_1}.

This result of Szentmiklóssy was published in [24, Theorem 3], on which this section will be mainly based. First we introduce a topological notion.

Definition 3.2.1. A space X is called *hereditarily Lindelöf (HL)* if every subspace Y of X is Lindelöf, i.e. every open cover of Y has a countable subcover.

Definition 3.2.2. Let $Y = \{y_\alpha : \alpha \in \omega_1\} \subseteq X$. Then Y is called right-separated if for each $\alpha \in \omega_1$ the set $\{y_\gamma : \gamma < \alpha\}$ is open in Y.

Lemma 3.2.3. *Every space X, which is not HL contains a right-separated subspace $Y = \{y_\alpha : \alpha \in \omega_1\}$.*

Proof. Fix a non-Lindelöf $Z \subseteq X$ and an open cover $\{U_\gamma : \gamma \in \kappa\}$ of Z with no countable subcovers. By replacing for $\gamma < \omega_1$ the set U_γ by $\bigcup_{\delta \leq \gamma} U_\gamma$ and deleting recurring members of the cover if needed, we may assume that the first ω_1-many U_γ's are strictly increasing, i.e. for $\delta < \gamma < \omega_1$ we have $U_\delta \subset U_\gamma$. For each $\alpha \in \omega_1$, we may therefore pick $y_\alpha \in U_{\alpha+1} \setminus U_\alpha$, which implies that $Y = \{y_\alpha : \alpha \in \omega_1\}$ is right-separated. □

The following lemma will be very interesting when it comes to finding convergent sequences.

Lemma 3.2.4. *If X is a compact Hausdorff space which in addition is HL, then every point in X has a countable neighborhood base.*

Proof. Let $x \in X$ and let $\mathcal{K}$ be the collection of all closed neighborhoods of x. Recall that any compact Hausdorff space is T_3 and thus the closed neighborhoods of x form a neighborhood basis for x. The collection $\{X \setminus K : K \in \mathcal{K}\}$ is an open cover of $X \setminus \{x\}$, because for every $y \neq x$ we can find some basic $K \in \mathcal{K}$ which does not contain y, i.e. $X \setminus K$ contains y. By assumption, the subspace $X \setminus \{x\}$ is Lindelöf and therefore there is a countable subset $\{U_n : n \in \omega\}$ of $\{X \setminus K : K \in \mathcal{K}\}$ which covers $X \setminus \{x\}$.

By replacing U_n by $U_0 \cup ... \cup U_n$ if needed, we can assume that the U_n's are increasing. Notice that since $\bigcup_{n\in\omega} U_n = X \setminus \{x\}$, we have also $\bigcap_{n\in\omega} X \setminus U_n = \{x\}$. We show that the decreasing collection $\{X \setminus U_n : n \in \omega\}$ is a countable neighborhood basis at x. If $K \in \mathcal{K}$ we have $\bigcap_{n\in\omega} X \setminus U_n = \{x\} \subseteq K$. This implies that $\bigcap_{n\in\omega} X \setminus (K \cup U_n) = \emptyset$ and because each $X \setminus (K \cup U_n)$ is closed in X it follows that $\bigcap_{n\in a} X \setminus (K \cup U_n) = \emptyset$ for some finite $a \in \omega$. This implies $\bigcap_{n\in a} X \setminus U_n \subseteq K$ and since the $X \setminus U_n$'s are decreasing we get $X \setminus U_{\max a} \subseteq K$. Moreover we clearly have $x \notin U_{\max a}$, and it was thus shown that $\{X \setminus U_n : n \in \omega\}$ forms a countable neighborhood basis at x. □

The following definition and the corresponding technical lemma below are due to Szentmiklóssy [24, Section 3] and are a crucial ingredient for the proof of Theorem 3.2.13.

Definition 3.2.5. A family $\mathfrak{B}$ of subsets of ω_1 has property P if it is closed under finite unions and there exists a sequence $\langle (a_\xi, B_\xi) : \xi \in \omega_1 \rangle$ and a natural number n such that:

(P1) a_ξ is a subset of ω_1 of size n for all $\xi \in \omega_1$,

(P2) B_ξ is countable and lies in $\mathfrak{B}$ for all $\xi \in \omega_1$,

(P3) $\max a_\xi < \min a_\eta$ for all $\xi < \eta < \omega_1$,

(P4) $a_\xi \cap B_\eta \neq \emptyset$ for all $\xi < \eta < \omega_1$.

Lemma 3.2.6 (Szentmiklóssy)**.** *If $\mathfrak{B}$ satisfies property P, then there is an uncountable $A \subseteq \omega_1$ and for all $\xi \in \omega_1$ there is some countable $B_\xi \in \mathfrak{B}$ such that for each uncountable $C \subseteq A$ there is an ordinal $\alpha \in \omega_1$ such that $\{B_\xi \cap C : \xi \in \omega_1, \xi \geq \alpha\}$ has the finite intersection property.*

Proof. Let n be the minimal natural number such that there is a sequence $\langle (a_\xi, B_\xi) : \xi \in \omega_1 \rangle$ with $|a_\xi| = n$ for each $\xi \in \omega_1$, satisfying the properties (P1)-(P4). This minimal number is clearly well-defined, because $\mathfrak{B}$ satisfies the property P. By (P2) each B_ξ is countable and lies in $\mathfrak{B}$. Furthermore we define $A := \{\min a_\xi : \xi \in \omega_1\}$, which by condition (P3) is clearly uncountable.

By contradiction, suppose that there is an uncountable $C' \subseteq A$ such that for each ordinal $\alpha \in \omega_1$ the family $\{B_\xi \cap C' : \xi \in \omega_1, \xi \geq \alpha\}$ does not have the finite intersection property, i.e. there is some finite collection b'_α of countable ordinals $\xi \geq \alpha$ such that $\bigcap_{\xi \in b'_\alpha} B_\xi \cap C' = \emptyset$. The goal will be to construct a sequence $\langle (a'_\xi, B'_\xi) : \xi \in \omega_1 \rangle$ with $|a'_\xi| = n - 1$ for each $\xi \in \omega_1$ and satisfying properties (P1)-(P4). This would contradict the minimality of n with this property.

Inductively we first construct an uncountable subset $C \subseteq C'$ and $\{b_\alpha : \alpha \in \omega_1\} \subseteq \{b'_\alpha : \alpha \in \omega_1\}$ such that for each $\alpha < \beta < \omega_1$ we have

$$\max a^*_\alpha < \min b_\beta \leq \max b_\beta \leq c_\beta,$$

where $C = \{c_\alpha : \alpha \in \omega_1\}$ is the strictly increasing enumeration of C and a^*_α is defined as follows. Because $C \subseteq A$, for each $\alpha \in \omega_1$ there is a unique $\eta(\alpha) \in \omega_1$ such that $c_\alpha = \min a_{\eta(\alpha)}$ and we define $a^*_\alpha := a_{\eta(\alpha)}$. It is then clear that $\max a^*_\alpha < \min a^*_\beta$ for each $\alpha < \beta < \omega_1$ as well as $\bigcap_{\xi \in b_\alpha} B_\xi \cap C = \emptyset$.

The proof is by induction on β. If $\beta = 0$, we just need to define $b_0 := b'_0$ and $c_0 := \min\{c \in C : c \geq \max b_0\}$ to satisfy the required statement. If $\beta > 0$ and b_α, c_α were already defined for $\alpha < \beta$ such that the above inequality holds, we can define b_β and c_β as follows. Fix an ordinal $\mu \in \omega_1$ such that $\bigcup_{\alpha<\beta} a^*_\alpha \subseteq \mu$, which is possible because the left hand side is a countable union of finite subsets of ω_1, hence countable. We define $b_\beta := b'_\mu$ and $c_\beta := \min\{c \in C : c \geq \max b_\beta\}$. For each $\alpha < \beta$ we have that

$$\max a^*_\alpha < \mu \leq \min b'_\mu = \min b_\beta \leq c_\beta,$$

as required.

We now put $a'_\alpha := a^*_\alpha \setminus \{c_\alpha\}$ and $B'_\alpha := \bigcup_{\xi \in b_\alpha} B_\xi$. As $c_\alpha \in a^*_\alpha$ and each a^*_α has size n, it is obvious that a'_α is of size $n-1$ for each $\alpha \in \omega_1$. We show that (P1)-(P4) are fulfilled, towards a contradiction. The item (P1) was already shown, for (P2) we just observe that B'_α is countable, because it is a finite union of countable B_ξ's. Moreover $B'_\alpha \in \mathfrak{B}$, because the B_ξ's lie all in $\mathfrak{B}$, which is closed under finite unions. To see (P3), just observe $\max a'_\alpha = \max a^*_\alpha < \min a^*_\beta = c_\beta < \min a'_\beta$ for $\alpha < \beta < \omega_1$. Note that as we will show (P4), we will also have $a'_\alpha \neq \emptyset$, which implies that all these maxima and minima are well-defined.

Finally, to check (P4) let $\alpha < \beta$ and let us show that $a'_\alpha \cap B'_\beta \neq \emptyset$. Because $c_\alpha \in C$ and $\bigcap_{\xi \in b_\beta} B_\xi \cap C = \emptyset$, there is at least one $\xi \in b_\beta$ such that $c_\alpha \notin B_\xi$. We ensured that $\min b_\beta > \max a^*_\alpha$, and as $\xi \in b_\beta$ we also have $\xi > \max a^*_\alpha = \max a_{\eta(\alpha)}$. Moreover we clearly have $\max a_{\eta(\alpha)} \geq \eta(\alpha)$, so $\xi > \eta(\alpha)$. Using that $\langle (a_\xi, B_\xi) : \xi \in \omega_1 \rangle$ satisfies (P4) we get that $a_{\eta(\alpha)} \cap B_\xi \neq \emptyset$. However

$$a_{\eta(\alpha)} \cap B_\xi = a^*_\alpha \cap B_\xi = a'_\alpha \cap B_\xi,$$

because $c_\alpha \notin B_\xi$, and moreover we also have $B_\xi \subseteq B'_\beta$, because $\xi \in b_\beta$, and hence $a'_\alpha \cap B_\xi \subseteq a'_\alpha \cap B'_\beta$. We conclude that $a'_\alpha \cap B_\xi$ and therefore $a'_\alpha \cap B'_\beta$ is not empty, as required. □

Before we prove the theorem of Szentmiklóssy, we need to introduce (a special case of) the notion of the precaliber of a poset.

Definition 3.2.7. Let P be a poset and $A \subseteq P$. Then A is called *centered* if for each natural number n and for each $p_1, ..., p_n \in A$ there is some $p \in P$ such that $p \leq p_i$ for each $1 \leq i \leq n$.

We say that P has *precaliber* ω_1 if each subset of P of size ω_1 contains a centered subset of size ω_1.

Lemma 3.2.8. *Assume* MA_{ω_1}*. Then every ccc poset has precaliber* ω_1*.*

Proof. We follow [17, Lemma 23.13]. Let P be a ccc poset and $W = \{w_\alpha : \alpha \in \omega_1\} \subseteq P$ uncountable. We first show that there exists a $q \in W$ such that for each $p \leq q$ there are uncountably many w_α's which are compatible with p, i.e.

$$\exists \alpha \in \omega_1 \forall v_\alpha \leq w_\alpha \exists A \subseteq \omega_1 : \Big(|A| = \omega_1 \wedge \forall \beta \in A : (v_\alpha \text{ is compatible with } w_\beta) \Big).$$

If not, then by negating the sentence we get

$$\forall \alpha \in \omega_1 \exists v_\alpha \leq w_\alpha \forall A \subseteq \omega_1 : \Big(|A| = \omega_1 \rightarrow \exists \beta \in A : (v_\alpha \text{ is incompatible with } w_\beta) \Big),$$

or equivalently

$$\forall \alpha \in \omega_1 \exists v_\alpha \leq w_\alpha \exists \beta_\alpha \forall \mu \geq \beta_\alpha (v_\alpha \text{ is incompatible with } w_\mu),$$

This allows us to inductively choose for each $\delta \in \omega_1$ an α_δ such that the v_{α_δ}'s are pairwise incompatible, which is clearly a contradiction to P being ccc.

More precisely, fix $\alpha_0 := 0$ and suppose that the α_γ's are given for all γ which are smaller than some δ such that the v_{α_γ}'s are pairwise incompatible. For each α_γ, by the above observation we may find some $\beta_\gamma \geq \alpha_\gamma$ such that for each $\mu \geq \beta_\gamma$ we have v_{α_γ} is incompatible with w_μ. Let $\alpha_\delta := \sup\{\beta_\gamma : \gamma < \delta\}$. Since $\alpha_\delta \geq \beta_\gamma$ for each $\gamma < \delta$ we have v_{α_γ} is incompatible with w_{α_δ}, for each $\gamma < \delta$. Since $v_{\alpha_\delta} \leq w_{\alpha_\delta}$, this implies also that v_{α_γ} is incompatible with v_{α_δ} for all $\gamma < \delta$. This ends our inductive construction of an uncountable set of pairwise incompatible elements and thus leads to the contradiction.

Consider the poset $Q := \{p \in P : p \leq q\}$ with the induced order and set $D_\alpha := \{p \in Q : p \leq w_\gamma \text{ for some } \gamma \geq \alpha\}$ for each $\alpha \in \omega_1$. This set is dense in Q, because for each $p \in Q$ we can find uncountably many w_α's which are compatible with p, which means that there is some $\gamma \geq \alpha$ such that p and w_γ have a common lower bound r. So $r \leq p$ and $r \in D_\alpha$, thus showing the density of D_α. Note that Q is clearly ccc, and so using MA_{ω_1} we can find some filter $F \subseteq Q$ which intersects each D_α. For each α, we may fix $x_\alpha \in F \cap D_\alpha$, which implies that $x_\alpha \leq w_\gamma$ for some $\gamma \geq \alpha$. It follows that this w_γ must lie in $F \cap W$. Because for each $\alpha \in \omega_1$ there is some $\gamma \geq \alpha$ such that $w_\gamma \in F \cap W$, it follows that $F \cap W$ is uncountable. Moreover, $F \cap W$ is clearly centered, because it is a subset of a filter. □

An additional topological notion will be needed.

Definition 3.2.9. A topological space is said to be countably tight, if for each $A \subseteq X$ we have $\overline{A} = \bigcup\{\overline{B} : B \text{ is a countable subset of } A\}$.

Lemma 3.2.10.

(a) If X is countably tight and $A \subseteq X$ is such that for each countable $B \subseteq A$ we have $\overline{B} \subseteq A$, then A is closed.

(b) If X is HS, then X is countably tight.

Proof.

(a) follows directly from the definition of countably tightness.

(b) If $A \subseteq X$, we may pick a countable set D, which is dense in A, and get that $\overline{A} = \overline{D} \subseteq \bigcup\{\overline{B} : B \text{ is a countable subset of } A\}$. Furthermore, the inclusion $\overline{A} \supseteq \bigcup\{\overline{B} : B \text{ is a countable subset of } A\}$ always holds and thus we showed that X is countably tight. □

The proof of Szentmiklóssy's theorem (3.2.13) will also use the well-known Δ-system Lemma.

Definition 3.2.11. A family $\mathcal{A}$ of finite sets is called a Δ-*system*, if there is some r such that $a \cap b = r$ for each $a \neq b \in \mathcal{A}$. The set r is called the *root* of the Δ-system $\mathcal{A}$.

Lemma 3.2.12 (Δ-system Lemma)**.** *Let $\mathcal{A}$ be a family of finite sets such that $|\mathcal{A}| = \omega_1$. Then there is a family $\mathcal{B} \subseteq \mathcal{A}$ of size ω_1, which forms a Δ-system, and a natural number m such that $|b| = m$ for each $b \in \mathcal{B}$.*

Proof. See Kunen's book [20, Lemma III.2.6]. □

Finally, we have all ingredients to prove Szentmiklóssy's theorem. We provide more details as compared to [24, Theorem 2].

Theorem 3.2.13 (Szentmiklóssy)**.** *Assume* MA_{ω_1}. *Then every HS compact Hausdorff space is HL.*

Proof. Because X is HS, it is also countably tight by Lemma 3.2.10(b). If X was non-HL, then we could find by Lemma 3.2.3 a right-separated $Y \subseteq X$. We will identify the underlying set of Y with ω_1 so that α is open for all α. Note that here the topology of Y (which is identified with the set ω_1) is different from the order topology of ω_1, because the latter is not separable.

Let $F_\xi := \overline{\omega_1 \setminus \xi}$ for all $\xi < \omega_1$ and define $F := \bigcap_{\xi \in \omega_1} F_\xi$ as well as

$$\mathfrak{B} := \{K \cap \omega_1 : K \text{ is compact and disjoint from } F\}.$$

Claim. The collection $\mathfrak{B}$ has property P.

Proof of Claim. It is clear that $\mathfrak{B}$ is closed under finite unions. For each $\xi \in \omega_1$ we have $\xi \notin F_{\xi+1}$ and therefore we can find a closed (hence compact) neighborhood K_ξ of the point ξ which is disjoint from $F_{\xi+1}$. Notice that, because also $\omega_1 \cap K_{\xi+1}$ is disjoint from $F_{\xi+1} \supseteq \omega_1 \setminus (\xi + 1)$, which means that $K_\xi \cap \omega_1 \subseteq \xi + 1$, we have that $K_\xi \cap \omega_1$ is countable for each $\xi \in \omega_1$. Moreover, each $K_\xi \cap \omega_1$ lies in $\mathfrak{B}$, because K_ξ is a compact set disjoint from $F_{\xi+1} \supseteq F$.

We define a poset

$$\mathbb{P} := \{a \subseteq \omega_1 : a \text{ is finite and } \forall \xi, \eta \in a : \xi < \eta \to \xi \notin K_\eta\},$$

where for $a, b \in \mathbb{P}$ we define $a \leq b$ if and only if $a \supseteq b$.

This poset is not ccc. If it was ccc, it would have precaliber ω_1 by Lemma 3.2.8 (here MA_{ω_1} is used). Note that $A := \{\{\xi\} : \xi \in \omega_1\} \subseteq \mathbb{P}$ so there is some uncountable centered $D' \subseteq A$. Let $D := \{\xi \in \omega_1 : \{\xi\} \in D'\}$. The goal will be to show that this is a discrete subspace of the HS space X. This will then lead to a contradiction, because D has a countable dense set Q, i.e. Q meets every open set. However if D is discrete, each singleton is open and therefore the countable Q must contain every point of D, which is impossible as D is uncountable.

So let us show that every singleton in D is open. Because for each $\xi, \eta \in D$ we have some $a \in \mathbb{P}$ such that $a \leq \{\xi\}$ and $a \leq \{\eta\}$ we have that $a \supseteq \{\xi, \eta\}$. It is easy to see that $a \in \mathbb{P}$ implies that $\{\xi, \eta\} \in \mathbb{P}$. It follows that whenever $\xi, \eta \in D$ are such that $\eta < \xi$, then $\eta \notin K_\xi$. On the other hand, if $\xi < \eta$, we have that $\eta \in F_{\xi+1}$ and as the latter is disjoint from K_ξ we have that $\eta \notin K_\xi$. We can now conclude that for each $\xi \in D$ the singleton $\{\xi\}$ is open: Because K_ξ is a compact neighborhood of ξ, there is some open $U \subseteq X$ such that $\xi \in U \subseteq K_\xi$. The set $U \cap D$ is an open neighborhood of ξ in D. However, we just showed that for $\eta \neq \xi$ in D (i.e. whenever $\eta < \xi$ or $\xi < \eta$) we have $\eta \notin K_\xi$, hence $\eta \notin U \cap D$ and therefore $U \cap D = \{\xi\}$. Thus we have shown that every singleton $\{\xi\}$ in D is open, i.e. D is discrete, which leads to the contradiction described above. We conclude that the poset $\mathbb{P}$ cannot be ccc.

So it follows that there is a sequence $\langle a_\alpha : \alpha \in \omega_1 \rangle$ of pairwise incompatible elements of $\mathbb{P}$. Using the Δ-system Lemma, we show that the a_α's can be chosen such that they are all of size n for some natural number n and such that $\max a_\alpha < \min a_\beta$ for all $\alpha < \beta$, which implies (P1) and (P3). If we put $\mathcal{A} := \{a_\alpha : \alpha \in \omega_1\}$, by the Δ-system Lemma, there is some $\mathcal{B} \subseteq \mathcal{A}$ which forms a Δ-system of size ω_1 and some natural number m such that $|b| = m$ for each $b \in \mathcal{B}$. Let r be the root of the Δ-system $\mathcal{B}$ and put $\widetilde{\mathcal{B}} := \{b \setminus r : b \in \mathcal{B}\}$, which we enumerate by $\widetilde{\mathcal{B}} = \{b_\alpha : \alpha \in \omega_1\}$. Note that by eliminating the root r, we ensured that the b_α's are pairwise disjoint, and also the members of $\widetilde{\mathcal{B}}$ still have all the same size, say n.

Inductively, we may find an uncountable subset $\widetilde{\mathcal{C}} = \{c_\gamma : \gamma \in \omega_1\} = \{b_{\alpha_\gamma} : \gamma \in \omega_1\}$ of $\widetilde{\mathcal{B}}$ such that $\max c_\delta < \min c_\gamma$ for all $\delta < \gamma < \omega_1$. Indeed, whenever c_δ is defined for each $\delta < \gamma$ such that $\max c_{\delta_1} < \min c_{\delta_2}$ holds for all $\delta_1 < \delta_2 < \gamma$, put $s := \sup\{\max c_\delta : \delta < \gamma\} + 1 < \omega_1$. Because the b_α's are pairwise disjoint, there can only be countably many b_α's containing an element smaller than s. Therefore there must be some b_α, which has no elements below s. Pick c_γ to be exactly this b_α, which implies $\max c_\delta < \min c_\gamma$ for all $\delta < \gamma$.

It remains to show that the members of $\widetilde{\mathcal{C}}$ are still pairwise incompatible. For this, pick two different elements $b_1 \setminus r$ and $b_2 \setminus r$ of $\widetilde{\mathcal{C}}$, where $b_1, b_2 \in \mathcal{B}$. As b_1 and b_2 are incompatible we have $b_1 \cup b_2 \notin \mathbb{P}$, which means that there are $\xi < \eta$ both in $b_1 \cup b_2$ such that $\xi \in K_\eta$. However b_1 and b_2 both lie in $\mathbb{P}$ and so neither ξ not η can lie in the root r. It follows that $(b_1 \cup b_2) \setminus r = (b_1 \setminus r) \cup (b_2 \setminus r) \notin \mathbb{P}$ and hence $b_1 \setminus r$ and $b_2 \setminus r$ are incompatible. If necessary, replace $\mathcal{A}$ by $\widetilde{\mathcal{C}}$ to obtain a sequence of pairwise incompatible elements of the same size such that $\max a_\alpha < \min a_\beta$ for all $\alpha < \beta$ so that (P1) and (P3) become true.

For each $\alpha \in \omega_1$ we define $B_\alpha := \omega_1 \cap \bigcup_{\eta \in a_\alpha} K_\eta$. To see that the item (P2) holds, we check that each B_α is countable by recalling that each $K_\xi \cap \omega_1$ is countable and realising that each B_α is the union of finitely many such countable sets. Furthermore each B_α lies in $\mathfrak{B}$ because each $K_\xi \cap \omega_1$ does and $\mathfrak{B}$ is closed under finite unions.

It remains to show that (P4) is true. If $\alpha < \beta < \omega_1$, then because a_α and a_β are incompatible, it must be the case that $a_\alpha \cup a_\beta \notin \mathbb{P}$. Therefore there must be some $\xi, \eta \in a_\alpha \cup a_\beta$ such that $\xi < \eta$ and $\xi \in K_\eta$. However, because a_α, a_β both lie in $\mathbb{P}$, it can neither happen that ξ and η lie both in a_α, nor that both lie in a_β. Since $\alpha < \beta$, we have $\max a_\alpha < \min a_\beta$ and as $\xi < \eta$ we have that $\xi \in a_\alpha$ and $\eta \in a_\beta$. Because $\xi \in K_\eta$, we have in particular

$$\xi \in a_\alpha \cap \omega_1 \cap \bigcup_{\eta \in a_\beta} K_\eta = a_\alpha \cap B_\beta \neq \emptyset,$$

and thus item (P4) is proven. $\square_{\text{Claim}}$

Then using Lemma 3.2.6, we can get an uncountable $A \subseteq \omega_1$ and for each $\xi \in \omega_1$ some countable $B_\xi \in \mathfrak{B}$ with the property that for each uncountable $C \subseteq A$ there is an ordinal $\alpha \in \omega_1$ such that $\{B_\xi \cap C : \xi \in \omega_1, \xi \geq \alpha\}$ has the finite intersection property.

Because each B_ξ lies in $\mathfrak{B}$, it can be written as $L_\xi \cap \omega_1$ for some compact L_ξ, which is disjoint from F. Define $H_\xi := \bigcap_{\xi \leq \eta < \omega_1} L_\eta$ and $H := \bigcup_{\xi \in \omega_1} H_\xi$. It is clear that the H_ξ's are closed and increasing, i.e. $H_\xi \subseteq H_\zeta$ for each $\xi \leq \zeta$. Moreover H_ξ is contained in L_ξ and hence, each H_ξ is also disjoint from F, thus $H \cap F = \emptyset$. Because X is countably tight, it follows that H is also closed by using Lemma 3.2.10(a): If B is a countable subset of H, then because the H_ξ's are increasing, there must be some $\xi \in \omega_1$ such that $B \subseteq H_\xi$. As H_ξ is closed we also have $\overline{B} \subseteq H_\xi \subseteq H$, and thus Lemma 3.2.10(a) implies that H is closed.

Note that $\bigcap_{\xi \in \omega_1} H \cap F_\xi = H \cap F = \emptyset$ and because each $H \cap F_\xi$ is closed, by compactness and because $\langle H \cap F_\xi : \xi \in \omega_1 \rangle$ is decreasing, we can find some $\xi_0 \in \omega_1$ such that $H \cap F_{\xi_0} = \emptyset$.

Define

$$C := F_{\xi_0} \cap A = \overline{\omega_1 \setminus \xi_0} \cap A = \overline{\omega_1 \setminus \xi_0}^A = \overline{\omega_1 \setminus \xi_0}^{\omega_1} \cap A = (\omega_1 \setminus \xi_0) \cap A = A \setminus \xi_0,$$

which is an uncountable subset of A. Here, we use that ω_1 is right-separated. There is some ordinal $\alpha \in \omega_1$ such that $\{B_\xi \cap C : \alpha \leq \xi < \omega_1\}$ has the finite intersection property, and since $B_\xi \cap C \subseteq L_\xi \cap C \subseteq L_\xi \cap F_{\xi_0}$, also $\{L_\xi \cap F_{\xi_0} : \alpha \leq \xi < \omega_1\}$ has the finite intersection property. Moreover each $L_\xi \cap F_{\xi_0}$ is closed and thus using compactness we know that $\bigcap_{\alpha \leq \xi < \omega_1} L_\xi \cap F_{\xi_0} \neq \emptyset$. However

$$\bigcap_{\alpha \leq \xi < \omega_1} L_\xi \cap F_{\xi_0} = F_{\xi_0} \cap H_\alpha \subseteq F_{\xi_0} \cap H$$

implies that $F_{\xi_0} \cap H$ is non-empty, a contradiction.

Therefore X cannot have a right-separated subspace and thus it must be HL. $\square$

As promised, we get from this theorem as a corollary a sufficient condition for compact spaces containing non-trivial convergent sequences.

Corollary 3.2.14. *Assume* MA_{ω_1}. *Let X be a HS compact Hausdorff space and $x \in X$ non-isolated. Then there is a non-trivial convergent sequence with limit point x.*

Proof. By Szentmiklóssy's Theorem (3.2.13) we get that X is HL and because X is a compact Hausdorff space too, Lemma 3.2.4 implies that x has a countable base. Since x is also non-isolated there is a non-trivial convergent sequence with limit x. □

As already mentioned, we thus showed under MA_{ω_1} that infinite compact HS Hausdorff spaces contain plenty of non-trivial convergent sequences (for each non-isolated point at least one). In particular, the construction under $\diamondsuit$ from Chapter 2 is impossible under MA_{ω_1}, since it yielded an infinite compact HS Hausdorff space with no non-trivial convergent sequences. It is important to notice that there is no contradiction here, as $\diamondsuit$ implies CH and thus under $\diamondsuit$ the statement MA_{ω_1} would be equivalent to $\mathrm{MA}_{\mathfrak{c}}$, which is always wrong, see Proposition 3.1.11(b). In other words $\diamondsuit$ implies the negation of MA_{ω_1}.

3.3 The Čech-Pospišil Theorem

In 1938, Čech and Pospišil showed a theorem relating the least size of a neighborhood base at each point of a compact Hausdorff space to the size of the space. From this result, we can get a sufficient condition for a compact Hausdorff space to contain a convergent sequence. This condition is related to the size of the space, namely it suffices that its size is less than 2^{ω_1}. Therefore, there can be no Efimov spaces of size less than 2^{ω_1}, and so the size of the space from Chapter 2 can be determined.

Definition 3.3.1. For a topological space X and a point $x \in X$, the character of x is the least size of a neighborhood base at x. It is denoted by $\chi(x, X)$, i.e.

$$\chi(x, X) := \min\{|B| : B \text{ is a neighborhood base at } x\}.$$

It is clearly well-defined, as the collection of all open neighborhoods of x is trivially a neighborhood base of this point.

Theorem 3.3.2 (Čech-Pospišil). *Let X be a compact Hausdorff space and suppose that for each $x \in X$, we have $\chi(x, X) \geq \kappa \geq \aleph_0$. Then the size of X is at least 2^κ.*

Proof. We follow the hint in Exercise 3.12.11 in [10]. For each $s \in 2^{<\kappa}$ we define inductively over the length of s an open set $V(s)$ such that:

(i) $\overline{V(s ^\frown i)} \subseteq V(s)$ for each $s \in 2^{<\kappa}$, $i \in \{0, 1\}$,

(ii) $\overline{V(s ^\frown 0)} \cap \overline{V(s ^\frown 1)} = \emptyset$ for each $s \in 2^{<\kappa}$,

(iii) $\bigcap_{\beta\leq\alpha} V(s\restriction\beta)\neq\emptyset$ for each $s\in 2^{<\kappa}$ with length α,

(iv) $V(s)=X$ whenever the length of $s\in 2^{<\kappa}$ is a limit ordinal.

If the length of s is 0 or a limit ordinal, we put $V(s):=X$. In this case we only need to check that (iii) is preserved. If $\bigcap_{\beta\leq\alpha} V(s\restriction\beta)$ was empty, then so was $\bigcap_{\beta<\alpha} V(s\restriction\beta)$. Using the induction hypothesis (i), we also see that $\bigcap_{\beta<\alpha} \overline{V(s\restriction(\beta+1))}$ is empty and hence also $\bigcap_{\beta<\alpha}\overline{V(s\restriction\beta)}$. By compactness $\bigcap_{\beta\leq\gamma}\overline{V(s\restriction\beta)}=\emptyset$ for some $\gamma<\alpha$ and therefore also $\bigcap_{\beta\leq\gamma} V(s\restriction\beta)=\emptyset$, which is a contradiction to the induction hypothesis.

Suppose $V(s)$ was defined such that the above properties hold and let us define $V(s^\frown i)$ for $i\in\{0,1\}$. We know that $\bigcap_{\beta\leq\alpha} V(s\restriction\beta)\neq\emptyset$, so we can fix some point p_0 inside this intersection. The set $\bigcap_{\beta\leq\alpha} V(s\restriction\beta)$ cannot be a singleton, because otherwise we would have $\chi(p_0,X)\leq|\alpha|<\kappa$, a contradiction. So we can pick some $p_1\neq p_0$ in $\bigcap_{\beta\leq\alpha} V(s\restriction\beta)$. Note that $\bigcap_{\beta\leq\alpha} V(s\restriction\beta)\subseteq V(s)$ and thus p_0 and p_1 lie in $V(s)$. Because X is a compact Hausdorff space, we may separate these two points by open neighborhoods $V(s^\frown 0), V(s^\frown 1)\subseteq V(s)$ such that $\overline{V(s^\frown 0)},\overline{V(s^\frown 1)}\subseteq V(s)$ and $\overline{V(s^\frown 0)}\cap\overline{V(s^\frown 1)}=\emptyset$. Then (i) and (ii) are clearly satisfied, and (iii) can be checked by computing

$$\bigcap_{\beta\leq\alpha+1} V((s^\frown i)\restriction\beta)=\bigcap_{\beta\leq\alpha} V(s\restriction\beta)\cap V(s^\frown i),$$

which contains p_i and hence is non-empty.

By compactness we get that $\bigcap_{\beta<\kappa}\overline{V(f\restriction\beta)}\neq\emptyset$ for each $f\in 2^\kappa$. For each such f let x_f be some point in this intersection. By (ii), we get that if $f,g\in 2^\kappa$ such that $f(\alpha)\neq g(\alpha)$ for some successor ordinal $\alpha\in\kappa$, then $\bigcap_{\beta<\kappa}\overline{V(f\restriction\beta)}$ is disjoint from $\bigcap_{\beta<\kappa}\overline{V(g\restriction\beta)}\neq\emptyset$ and thus $x_f\neq x_g$.

Observe that the set S of all successor ordinals smaller than κ has size κ, because $\alpha\mapsto\alpha+1$ is a bijection from κ to S. For each $g\in 2^S$ we let $f_g\in 2^\kappa$ such that $f_g\restriction S=g$. We showed that then the map $g\mapsto x_{f_g}$ is an injection from 2^S to X. It follows that $|X|\geq|2^S|=2^{|S|}=2^\kappa$, as required. □

As mentioned, we get two corollaries relating the existence of non-trivial convergent sequences to the size of the space.

Corollary 3.3.3. *Every compact Haudorff space X with $\omega\leq|X|<2^{\omega_1}$ contains a non-trivial convergent sequence. In particular, if $\mathfrak{c}<2^{\omega_1}$, then every compact Haudorff space X with $\omega\leq|X|\leq\mathfrak{c}$ contains a non-trivial convergent sequence.*

Proof. The space X is either scattered or has a closed subspace without isolated points. The case of scattered spaces has been treated in Lemma 3.1.8. Now suppose that X is not scattered and let $A \subseteq X$ be a closed subspace without isolated points.

We claim that there is some point $x \in A$, such that $\chi(x, A) \leq \omega$. If for each point x in A we had $\chi(x, A) > \omega$, i.e. $\chi(x, A) \geq \omega_1$, then by the Čech-Pospišil-Theorem (3.3.2), A and thus X would have size at least 2^{ω_1}, which contradicts our hypothesis. Note that the Čech-Pospišil-Theorem can be applied because A is compact.

The point $x \in A$ is not isolated and satisfies $\chi(x, A) \leq \omega$, so there must be a non-trivial convergent sequence of points in A with limit x. In particular, X contains also such a sequence.

The second statement follows obviously from the first one. □

We showed therefore that there can be no Efimov space of size less than 2^{ω_1} and in particular under the assumption $\mathfrak{c} < 2^{\omega_1}$, there are no Efimov spaces of size less or equal than $\mathfrak{c}$. However, Fedorčuk showed that under the assumption $\mathfrak{c} = 2^{\omega_1} \wedge \mathfrak{s} = \omega_1$ there is an Efimov space of size $\mathfrak{c}$. Before we show his construction, let us add one more remark on the size of the Efimov space from Chapter 2.

Corollary 3.3.4. *(CH) There are no Efimov spaces of size less or equal than* $\mathfrak{c}$*. Moreover the size of the space* X *from Theorem 2.5.1 is* $2^{\omega_1} > \omega_1 = \mathfrak{c}$*.*

Proof. Note that CH implies $2^{\omega_1} > \omega_1 = \mathfrak{c}$, and by Corollary 3.3.3 the first statement holds.

It remains to show the second assertion. The space X is a compact Hausdorff space with no non-trivial convergent sequences and so $|X| \geq 2^{\omega_1}$. Moreover X is a subset of $\mathbb{C} \times \mathbb{C}^{\omega_1}$, which has size $\mathfrak{c} \cdot \mathfrak{c}^{\omega_1} = \mathfrak{c} \cdot (2^{\omega})^{\omega_1} = \mathfrak{c} \cdot 2^{\omega \cdot \omega_1} = 2^{\omega_1}$ (here $\cdot$ obviously denotes cardinal multiplication), so $|X| \leq 2^{\omega_1}$. All in all, we conclude $|X| = 2^{\omega_1} > \omega_1 = \mathfrak{c}$. □

3.4 An Efimov space of size $\mathfrak{c}$

In this last section, we will show that under the assumption $\mathfrak{s} = \omega_1 \wedge \mathfrak{c} = 2^{\omega_1}$ it is possible to construct an Efimonv space of size $\mathfrak{c}$. This assumption is consistent with ZFC, for a proof we refer to [12, Theorem 1]. The construction presented here is due to Fedorčuk and can be found in [12], on which this whole section relies. We improve his notation slightly, as for example at his time the splitting number was not yet used. We also present some more details in the proofs.

The space Fedorčuk constructs is in fact an inverse limit of some inverse system. We already encountered this concept in Chapter 2. However, a more general notion of inverse limits will be needed here. For this section, we use the following terminology.

Definition 3.4.1. Let $\alpha \leq \omega_1$. An *inverse system* is a sequence $\Sigma = \langle X_\gamma : \gamma \in \alpha \rangle$ of topological spaces, together with so-called *bonding mappings* $\pi^\delta_\gamma : X_\delta \to X_\gamma$, $\gamma \leq \delta < \alpha$, which are continuous and satisfy $\pi^\gamma_\beta \circ \pi^\delta_\gamma = \pi^\delta_\beta$ for $\beta \leq \gamma \leq \delta < \alpha$. If not mentioned otherwise, the bonding mappings will always be denoted as in this definition.

If Σ is an inverse system of length $\alpha \leq \omega_1$, let $\pi_\gamma : \prod_{\delta<\alpha} X_\delta \to X_\gamma$ be the obvious projection. If α is a limit ordinal, we say that the space

$$\lim \Sigma := \{z \in \prod_{\delta<\alpha} X_\delta : \pi_\gamma(z) = \pi^\delta_\gamma \circ \pi_\delta(z) \text{ for } \gamma \leq \delta < \alpha\}$$

is the *inverse limit* of Σ. Note that in this case, $\langle X_\gamma : \gamma \leq \alpha \rangle$ is also an inverse system, where the bonding mappings π^δ_γ, $\gamma \leq \delta < \alpha$, come from the system Σ and $\pi^\alpha_\gamma = \pi_\gamma$ is the projection mentioned above.

As in the previous chapter an inverse system will be built inductively. In the successor step of the construction, the following definition introduced by Fedorčuk will play an important role.

Definition 3.4.2. Let X be a topological space. If $t = (F, f)$ is a pair, where $F \subseteq X$ is closed and $f : X \setminus F \to 2$ is continuous, we let

$$K(X, t) := (F \times 2) \cup f \subseteq X \times 2,$$

where we identify the function f with its graph. Furthermore we will write $\tau_t : K(X, t) \to X$ for the projection to the first coordinate.

If $\mathfrak{A}$ is a set of pairs as just described, we denote by $p_t : \prod_{b\in\mathfrak{A}} K(X, b) \to K(X, t)$ the obvious projection and we let

$$X^{\mathfrak{A}} := \{z \in \prod_{t\in\mathfrak{A}} K(X, t) : \tau_t \circ p_t(z) = \tau_s \circ p_s(z) \text{ for all } t, s \in \mathfrak{A}\}.$$

By this definition, it makes sense to define $\pi^{\mathfrak{A}} : X^{\mathfrak{A}} \to X$ by putting $\pi^{\mathfrak{A}} := \tau_t \circ p_t$ where t is any element of $\mathfrak{A}$, whose choice is clearly not influencing the definition of $\pi^{\mathfrak{A}}$.

Definition 3.4.3. Let $\mathfrak{A}$ be a collection of pairs as described above. The collection $\mathfrak{A}$ is called *point-countable*, if for each point $x \in X$, the set $\mathfrak{B}$ of all pairs $(F, f) \in \mathfrak{A}$ such that $x \in F$ is countable. (Notice that in the classical literature, one would rather call the collection $\{F : (F, f) \in \mathfrak{A}$ for some function $f\}$ point-countable.)

Lemma 3.4.4. *Let $\mathfrak{A}$ be a point-countable collection of pairs as described above.*

(a) If X is compact, then so is $X^{\mathfrak{A}}$.

(b) If X is first-countable, then so is $X^{\mathfrak{A}}$.

(c) If X is zero-dimensional, then so is $X^{\mathfrak{A}}$.

Proof. To see the first assertion, just notice that $X^{\mathfrak{A}}$ is a closed subspace of a compact space, because $X^{\mathfrak{A}} = \bigcap_{s,t\in\mathfrak{A}}\{z \in \prod_{t\in\mathfrak{A}} K(X, t) : \tau_t \circ p_t(z) = \tau_s \circ p_s(z)\}$ and the complement of $\{z \in \prod_{t\in\mathfrak{A}} K(X, t) : \tau_t \circ p_t(z) = \tau_s \circ p_s(z)\}$ is easily seen to be open in the product topology, for all $s, t \in \mathfrak{A}$. It follows that $X^{\mathfrak{A}}$ is compact.

Let us show the second statement. Fix $z \in X^{\mathfrak{A}}$ and let $x := \pi^{\mathfrak{A}}(x)$. Since X is first-countable, we may fix a countable neighborhood base $\{\widetilde{B_n} : n \in \omega\}$ of $x \in X$. Because $\mathfrak{A}$ is point-countable, the set $\mathfrak{B}$ of all pairs $(F, f) \in \mathfrak{A}$ such that $x \in F$ is countable. For each finite $\mathfrak{C} \subseteq \mathfrak{B}$, $q \in 2^{\mathfrak{C}}$ and $n \in \omega$ we define $B_n^{\mathfrak{C},q} := X^{\mathfrak{A}} \cap \prod_{t\in\mathfrak{A}}(\widetilde{B_n} \times R_t)$, where

$$R_{(F,f)} := \begin{cases} \{f(z)\}, & \text{for } z \notin F, \\ \{q(F, f)\}, & \text{for } z \in F \text{ and } (F, f) \in \mathfrak{C}, \\ \{0, 1\}, & \text{for } z \in F \text{ and } (F, f) \notin \mathfrak{C} \end{cases}$$

for each $t = (F, f) \in \mathfrak{A}$. The collection $\{B_n^{\mathfrak{C},q} : \mathfrak{C} \subseteq \mathfrak{B} \text{ finite}, q \in 2^{\mathfrak{C}}, n \in \omega\}$ is clearly countable. It is not difficult to check that this collection forms a neighborhood base of z in $X^{\mathfrak{A}}$.

The last statement just follows from the fact that subspaces and products of zero-dimensional spaces are zero-dimensional. □

The following notions and the related results were discovered by Fedorčuk (see again [12]) in order to construct the inverse system in such a way that its limit does not contain any convergent sequence.

Definition 3.4.5. Let Y and X be topological spaces, $f : Y \to X$ a function and $\langle x_n : n \in \omega\rangle$ a non-trivial convergent sequence in X. We say that f kills $\langle x_n : n \in \omega\rangle$ if there is no convergent sequence $\langle y_n : n \in \omega\rangle$ in Y such that $f(y_n) = x_n$ for each natural number n.

Lemma 3.4.6. *Let X, Y and Z be topological spaces, $f : Y \to X$ continuous, $g : Z \to Y$ any function and $\langle x_n : n \in \omega \rangle$ a non-trivial convergent sequence in X. If f kills $\langle x_n : n \in \omega \rangle$, then $f \circ g$ kills $\langle x_n : n \in \omega \rangle$ too.*

Proof. Suppose that $f \circ g$ does not kill $\langle x_n : n \in \omega \rangle$, then there is a convergent sequence $\langle z_n : n \in \omega \rangle$ in Z such that $f(g(z_n)) = x_n$. However, then $\langle y_n : n \in \omega \rangle := \langle g(z_n) : n \in \omega \rangle$ is a convergent sequence in Y such that $f(y_n) = x_n$. □

Definition 3.4.7. Let $\Sigma = \langle X_\alpha : \alpha \in \omega_1 \rangle$ be an inverse system. We say that Σ kills all convergent sequences if for each $\alpha \in \omega_1$ and every non-trivial convergent sequence $\langle x_n : n \in \omega \rangle$ there is some $\beta \geq \alpha$ such that π^β_α kills $\langle x_n : n \in \omega \rangle$.

Lemma 3.4.8. *Let Σ be an inverse system which kills all convergent sequences. Then the inverse limit X of Σ contains no non-trivial convergent sequences.*

Proof. If there was a non-trivial convergent sequence $\langle x_n : n \in \omega \rangle$ with limit point $x = x_\omega$ in X, we can find a countable ordinal α such that all $\pi_\alpha(x_n)$ are mutually different for $n \leq \omega$. More precisely, if for each $n \leq \omega$ we write x_n as $\langle x^\gamma_n : \gamma \in \omega_1 \rangle$, where $x^\gamma_n = \pi_\gamma(x_n)$, we can find α such that all $\langle x^\gamma_n : \gamma \in \alpha \rangle$ differ for $n \leq \omega$. This means that for indices $n, m \leq \omega$ there is some $\gamma < \alpha$ such that $x^\gamma_n \neq x^\gamma_m$. Since each x_n lies in $X = \lim \Sigma$, we get that in fact

$$\pi^\alpha_\gamma(x^\alpha_n) = \pi^\alpha_\gamma(\pi_\alpha(x_n)) = \pi_\gamma(x_n) \neq \pi_\gamma(x_m) = \pi^\alpha_\gamma(\pi_\alpha(x_m)) = \pi^\alpha_\gamma(x^\alpha_m),$$

which implies that $\pi_\alpha(x_n) = x^\alpha_n \neq x^\alpha_m = \pi_\alpha(x_m)$.

This means that $\langle \pi_\alpha(x_n) : n \in \omega \rangle$ is a non-trivial convergent sequence, so π_α cannot kill the sequence $\langle \pi_\alpha(x_n) : n \in \omega \rangle$. On the other hand we can find some $\beta \geq \alpha$ such that π^β_α kills $\langle \pi_\alpha(x_n) : n \in \omega \rangle$. By Lemma 3.4.6, it follows that $\pi^\beta_\alpha \circ \pi_\beta = \pi_\alpha$ kills the sequence $\langle \pi_\alpha(x_n) : n \in \omega \rangle$, a contradiction. □

To shorten the notation, we introduce the following definition.

Definition 3.4.9. A topological space is called *suitable* if it is first-countable, zero-dimensional, compact and Hausdorff.

From now on, we use the assumption that $\mathfrak{s} = \omega_1$. Therefore we may fix a splitting family $\mathcal{S} = \{s_\alpha : \alpha \in \omega_1\}$ of size ω_1. Note that the whole construction, which is done from now on, depends on the choice of this splitting family of size ω_1. In fact, the order in which this family is enumerated also influences the construction. The ideas presented are due to Fedorčuk, cf. [12].

Definition 3.4.10. If X is any suitable space and $x \in X$, we may fix a decreasing countable neighborhood base $\{O_m(x) : m \in \omega\}$ consisting of clopen subsets of X such that $O_0(x) = X$. If $\alpha \in \omega_1$, one can define $f^{\alpha}_{(x,X)} : X \setminus \{x\} \to \{0,1\}$ by stipulating

$$f^{\alpha}_{(x,X)}(y) = 0 :\Longleftrightarrow y \in \bigcup_{m \in s_\alpha} O_m(x) \setminus O_{m+1}(x).$$

Note that $\{O_m(x) \setminus O_{m+1}(x) : m \in \omega\}$ forms a partition of $X \setminus \{x\}$, for each $x \in X$.

Definition 3.4.11. Let X be a suitable space and $\langle x_n : n \in \omega \rangle$ be a non-trivial convergent sequence with limit $x \in X$. Moreover, we let N be the infinite set of all natural numbers m such that $O_m(x) \setminus O_{m+1}(x)$ contains at least one x_n. The ordinal

$$\alpha = \min\{\beta \in \omega_1 : N \cap s_\beta \text{ and } N \setminus s_\beta \text{ are both infinite.}\}$$

is called the index of the sequence $\langle x_n : n \in \omega \rangle$.

Under $\mathfrak{s} = \omega_1$, it is clear that this notion is well-defined and that the index of each non-trivial convergent sequence is a countable ordinal. Note that this definition depends of course not only on the choice of the splitting family $\mathcal{S}$ of size ω_1, but also on its enumeration $\{s_\alpha : \alpha \in \omega_1\}$.

Lemma 3.4.12. *($\mathfrak{s} = \omega_1$) Let X be a compact Hausdorff space and Y a suitable space. Moreover, let $\langle y_n : n \in \omega \rangle$ be a non-trivial convergent sequence of index $\gamma < \omega_1$ in Y with limit point y and $g : X \to Y$ be continuous. If $t = (g^{-1}(y), f^{\gamma}_{(y,Y)} \circ g)$, then $g \circ \tau_t : K(X,t) \to Y$ kills $\langle y_n : n \in \omega \rangle$.*

Proof. Suppose that there was a convergent sequence $\langle z_n : n \in \omega \rangle$ with limit point z in $K(X,t)$ such that $g \circ \tau_t(z_n) = y_n$ for each natural number n. Let again N be the infinite set of all natural numbers m such that $O_m(y) \setminus O_{m+1}(y)$ contains a member of the sequence $\langle y_n : n \in \omega \rangle$. Note that because the index of $\langle y_n : n \in \omega \rangle$ is γ, we have that $s_\gamma \cap N$ and $N \setminus s_\gamma$ are both infinite.

Let $a_0 := \{n \in \omega : y_n \in \bigcup_{m \in s_\gamma} O_m(y) \setminus O_{m+1}(y)\}$. Note that for each $m \in N \cap s_\gamma$ there is some member of the sequence which lies in $O_m(y) \setminus O_{m+1}(y)$, and as $N \cap s_\gamma$ is infinite, and the $O_m(y) \setminus O_{m+1}(y)$'s are pairwise disjoint, there are infinitely many members of the sequence lying inside $\bigcup_{m \in s_\gamma} O_m(y) \setminus O_{m+1}(y)$. So $\langle y_n : n \in a_0 \rangle$ forms an infinite subsequence.

If we let $a_1 := \omega \setminus a_0$, we can again note that for each $m \in N \setminus s_\gamma$ there is some member of the sequence which lies in $O_m(y) \setminus O_{m+1}(y)$, and the same argument yields that there are infinitely many members of the sequence lying inside $\bigcup_{m\in\omega\setminus s_\gamma} O_m(y)\setminus O_{m+1}(y)$. This implies that infinitely many members of the sequence do not lie in $\bigcup_{m\in s_\gamma} O_m(x)\setminus O_{m+1}(x)$, where $m\in s_\gamma$. In other words, $\langle y_n : n\in a_1\rangle$ forms an infinite subsequence too.

It follows that $\langle z_n : n\in a_0\rangle$ and $\langle z_n : n\in a_1\rangle$ are infinite subsequences of $\langle z_n : n\in\omega\rangle$, which therefore converge both towards z. We show that for each $n\in a_i$ (where $i\in\{0,1\}$ we have that $z_n=(\tau_t(z_n),i)$. First note that $\tau_t(z_n)\notin g^{-1}(y)$, because otherwise $y_n=g(\tau_t(z_n))=y$. Thus $\tau_t(z_n)\in \operatorname{dom} f^\gamma_{(y,Y)}\circ g$ and by the definition of $K(X,t)$, it suffices to check that $f^\gamma_{(y,Y)}\circ g\circ\tau_t(z_n)=i$, i.e. $f^\gamma_{(y,Y)}(y_n)=i$. This just follows by the definition of $f^\gamma_{(y,Y)}$: If $i=0$ (so $n\in a_0$) we have $y_n\in\bigcup_{m\in s_\gamma} O_m(y)\setminus O_{m+1}(y)$ and thus $f^\gamma_{(y,Y)}(y_n)=0$ and if $i=1$ (so $n\in a_1$) we have $y_n\notin\bigcup_{m\in s_\gamma} O_m(y)\setminus O_{m+1}(y)$ and thus $f^\gamma_{(y,Y)}(y_n)=1$, as required.

From this we get a contradiction, namely that $\langle z_n : n\in a_0\rangle$ converges to $(\tau_t(z),0)$, whereas $\langle z_n : n\in a_1\rangle$ converges to $(\tau_t(z),1)$, which is impossible as these subsequences must have the same limit. $\square$

With all these auxiliary results in our pockets, we may start the inductive construction of an inverse system $\Sigma=\langle X_\alpha : \alpha<\omega_1\rangle$, whose limit is going to be the desired Efimov space. Put $X_0:=\mathbb{C}$ and suppose that for $\alpha<\omega_1$ we already defined an inverse system $\Sigma_\alpha:=\langle X_\beta:\beta<\alpha\rangle$. We need to define define X_α.

If α is a limit ordinal, we just let X_α be the inverse limit of Σ_α and for each $\beta<\alpha$ the bonding mapping π^α_β is given by the projection $X_\alpha\to X_\beta$. If α is a successor ordinal, say $\alpha=\delta+1$, we define

$$\mathfrak{A}_\alpha=\mathfrak{A}_{\delta+1}:=\{\left((\pi^\delta_\beta)^{-1}(x), f^\gamma_{(x,X_\beta)}\circ\pi^\delta_\beta\right):\beta,\gamma\in\alpha, x\in X_\beta\}$$

and we put $X_\alpha=X_{\delta+1}:=X_\delta^{\mathfrak{A}_{\delta+1}}$. We also need to define the bonding mappings $\pi^\alpha_\delta:=\pi^{\mathfrak{A}_\alpha}$ and $\pi^\alpha_\beta:=\pi^\delta_\beta\circ\pi^\alpha_\delta$ for $\beta\leq\delta$. The sequence $\Sigma_{\alpha+1}:=\langle X_\beta:\beta\leq\alpha\rangle$ is also an inverse system.

From this we get an inverse system $\Sigma=\langle X_\alpha:\alpha<\omega_1\rangle$, whose inverse limit will be denoted by $\mathbb{D}$. Let us note that by Lemma 3.4.4 we have that each X_α is suitable. The next fact will provide an upper bound for the size of $\mathbb{D}$.

Theorem 3.4.13 (Arkhangel'skii)**.** *If X is a first-countable compact Hausdorff space, then $|X|\leq\mathfrak{c}$.*

Proof. See Engelking's book [10, Corollary 3.1.30]. $\square$

We are now able to show that under the assumption $\mathfrak{s} = \omega_1 \wedge \mathfrak{c} = 2^{\omega_1}$ the above construction defines a space with the desired properties.

Theorem 3.4.14 (Fedorčuk)**.** *Assume $\mathfrak{s} = \omega_1$ and $\mathfrak{c} = 2^{\omega_1}$. The space $\mathbb{D}$ is a compact Hausdorff space of size $\mathfrak{c}$ with no non-trivial convergent sequences and no isolated points. In particular, $\mathbb{D}$ is an Efimov space.*

Proof. Throughout the proof, $\cdot$ will denote cardinal multiplication. The size of $\mathbb{D}$ can be determined as follows: Because $X_0 = \mathbb{C}$ has size $\mathfrak{c}$ and by Lemma 2.3.2 the map π_0 is surjective from $\mathbb{D}$ onto $\mathbb{C}$ we must have that $|\mathbb{D}| \geq \mathfrak{c}$. (Note that this lemma was stated for the more specific terminology of Chapter 2, but it holds also for the notion of inverse limits used here.)

On the other hand, each X_α is first-countable and compact, thus by Arkhangel'skii's Theorem (3.4.13) $|X_\alpha| \leq \mathfrak{c}$ for each α. The space $\mathbb{D}$ is a subspace of the product of all X_α, where $\alpha \in \omega_1$. Therefore we have

$$|\mathbb{D}| \leq \mathfrak{c}^{\omega_1} = (2^{\omega})^{\omega_1} = 2^{\omega \cdot \omega_1} = 2^{\omega_1} = \mathfrak{c}.$$

We conclude that the space $\mathbb{D}$ has size $\mathfrak{c}$.

To show that $\mathbb{D}$ has no isolated points, we first note that for each $\alpha \in \omega_1$ and $x \in X_\alpha$ we have $|(\pi_\alpha^{\alpha+1})^{-1}(x)| \geq 2$. Put

$$t := \left(\{x\}, f^\alpha_{(x,X_\alpha)}\right) = \left((\pi^\alpha_\alpha)^{-1}(x), f^\alpha_{(x,X_\alpha)} \circ \pi^\alpha_\alpha\right) \in \mathfrak{A}_{\alpha+1}.$$

We clearly have that $x \notin \operatorname{dom} f^\alpha_{(x,X_\alpha)}$ and therefore both $(x,0)$ and $(x,1)$ lie in $K(X,t)$. For all other $s = (F_s, f_s) \in \mathfrak{A}_{\alpha+1}$, $s \neq t$, it is not difficult to find some $k \in K(X,s)$ such that $p_s(k) = x$, just because either x lies in the domain of f_s then $k = (x, f_s(x))$ or x does not lie in the domain of f_s, then $k = (x,0)$. We obtain therefore two members y_0, y_1 of $X_\alpha^{\mathfrak{A}_{\alpha+1}} = X_{\alpha+1}$ such that for $i \in \{0,1\}$ we have

$$\pi_\alpha^{\alpha+1}(y_i) = \pi^{\mathfrak{A}_{\alpha+1}}(y_i) = \tau_t \circ p_t(y_i) = \tau_t(x,i) = x.$$

The above observation yields that $\mathbb{D}$ has no isolated points. If there was an isolated point $x \in \mathbb{D}$, we could find a basic open U such that $U = \{x\}$. Because U is basic there is some α such that for each $\beta \geq \alpha$ we have $\pi_\beta[U] = X_\beta$. The point $\pi_\alpha(x) \in X_\alpha$ is split into at least two different points y_0, y_1 lying both in $X_{\alpha+1} = \pi_{\alpha+1}[U]$. It follows that U contains at least two points, a contradiction.

It remains to show that the space $\mathbb{D}$ contains no non-trivial convergent sequences. We first observe that the inverse system Σ kills all convergent sequences. If for some

$\beta \in \omega_1$ there is a non-trivial convergent sequence $\langle y_n : \alpha \in \omega \rangle$ in X_β of index $\gamma < \omega_1$ with limit point y, then take $\alpha := \max\{\beta, \gamma\}$. All X_β's are suitable ($\beta < \omega_1$), and thus Lemma 3.4.12 can be applied to $X = X_\alpha, Y = X_\beta$; $g := \pi^\alpha_\beta : X_\alpha \to X_\beta$ and $t := ((\pi^\alpha_\beta)^{-1}(y), f^\gamma_{(y,Y)} \circ \pi^\alpha_\beta)$, so that the map $\pi^\alpha_\beta \circ \tau_t : K(X_\alpha, t) \to X_\beta$ kills the sequence $\langle y_n : n \in \omega \rangle$. Hence, by Lemma 3.4.6, also $\pi^\alpha_\beta \circ \tau_t \circ p_t$ kills this sequence. However, because $\beta, \gamma < \alpha + 1$ we have $t \in \mathfrak{A}_{\alpha+1}$ and hence

$$\pi^\alpha_\beta \circ \tau_t \circ p_t = \pi^\alpha_\beta \circ \pi^{\mathfrak{A}_{\alpha+1}} = \pi^\alpha_\beta \circ \pi^{\alpha+1}_\alpha = \pi^{\alpha+1}_\beta,$$

which means that $\pi^{\alpha+1}_\beta$ kills the sequence $\langle y_n : n \in \omega \rangle$. Thus for each $\beta \in \omega_1$ and each convergent sequence in X_β we have found $\alpha + 1 > \beta$ such that $\pi^{\alpha+1}_\beta$ kills this sequence, i.e. Σ kills all convergent sequences. By Lemma 3.4.8, this implies that its inverse limit $\mathbb{D}$ contains no non-trivial convergent sequences.

The space $\mathbb{D}$ is Efimov because it also cannot contain a copy of $\beta\omega$: $|\beta\omega| = 2^{\mathfrak{c}} > \mathfrak{c} = |\mathbb{D}|$ as there are $2^{\mathfrak{c}}$-many ultrafilters on ω (see [17, Theorem 56]). $\square$

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