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Klein-Gordon Equation in Cosmological Models

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Abstract

In this work we consider cosmological models which are modeled by PDEs (Partial Differential Equations) based on the dynamics of scalar fields. Using the FLRW ansatz for the metric g of the spacetime, the Einstein equations reduce to the Friedmann equations. In the first section we give a introduction into the cosmic inflation theory. Here a scalar field is postulated which is responsible for a period of rapid expansion in the early universe to which avoid the so called horizon problem. This scalar field is called inflaton and is unobserved until now. In the end of the first section we will discuss the future stability of scalar field configurations and present a very powerful result in [1]. In the second part we are investigating the cosmological model of a Bose-Einstein condensate (BEC). The authors in [2] showed that BEC can expand very rapidly and therefore can explain the isotropic cosmic microwave background (CMB) in the horizon problem too without postulating the existence of the inflaton. BEC attend a additional hydrodynamical description which can be used to obtain the nonrelativistic and semiclassical limit in a analytic rigorous way, as we will do in the end of the second part, following [3].

Zusammenfassung

In der vorliegenden Arbeit betrachten wir kosmologische Modelle die auf skalaren Feldern basieren. Wir verwenden einen FLRW Ansatz für die Metrik g der Raumzeit. Dadurch reduzieren sich die Einstein'schen Feldgleichungen auf die Friedmann Gleichungen. Im ersten Abschnitt beschäftigen wir und mit der Theorie der Inflation, die ein skalares Teilchen, genannt das Inflaton, postuliert das im frühen Universum eine Periode der starken Ausdehnung induziert hat, welche das sogenannte Horizontproblem vermeidet. Weiteres widmen wir uns der Zukunftsstabilität von Lösungen für Kosmologien mit skalaren Feldern und präsentieren ein sehr weitreichendes Theorem aus [1]. Im zweiten Abschnitt der Arbeit nähern wir uns den kosmologischen Modellen von einer anderen Seite indem wir sogenannte Bose-Einstein-Kondensate (BEC) betrachten. Die Autoren in [2] zeigten unter anderem nämlich das sich BEC äußerst rapide räumlich ausdehnen können. Das würde die Isotropie im komischen Strahlenhintergrund (CMB) ebenfalls erklären ohne ein Inflaton zu postulieren. Diese führt schlussendlich nämlich zum Horizont Problem. Bemerkenswerter Weise ermöglichen BEC auch eine hydrodynamische Beschreibung die es erleichtert einen formalen Beweis für den nicht relativistischen und semiklassichen Limes anzugeben. Dabei folgen wir den Ausführungen in [3].

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Contents

Chapter 1

Introduction

In the year 1915 Albert Einstein revolutionized physics by recognizing that gravitation, in contrast to the other fundamental forces, has a different explanation with a deep link to geometry. His principle of relativity and the equivalence of acceleration and a gravitational field was a great jump beyond of Newton's theory. He described the universe as a four dimensional Lorentzian manifold, the *spacetime*. The important thing about spacetime is that it does not have to be a flat metric on it. In his work about the anomalous perihelion advance of Mercury [4], Einstein formulated the famous *Einstein equations* whose solutions determine the geometry on spacetime. They are sixteen coupled nonlinear inhomogeneous PDEs of second order. The source terms of the Einstein equation models the matter distribution in the considered spacetime and the solution describes the corresponding metric on this manifold. Even today only very few solutions for very simple matter distributions to this field equations are known.

At the same time in the beginning of the 20th century strange observations in experiments were made. A famous experiment was done by Otto Stern and Walther Gerlach. The heated up silver atoms in an oven and send them through a inhomogeneous magnetic field towards a screen. What they observed was that only two dots were formed on the screen and not as classical expected continuous distribution. This leads to the fundamental awareness that nature is discretized on small scales. What Stern and Gerlach exactly measured is that *spin* is quantized. Spin is sometimes treated as something mysterious but it simply describes the angular momentum of a subatomic object, this angular moment is quantized in units of $\hbar$.
Many famous physicists at this time started to put things into order in this odd quantum world. One of them was Erwin Schrödinger. He was searching for the equation of motion of an electron which moves in the electric field of the nuclei and he found one, the *Schrödinger equation.* But this equation does not take into account relativistic effects and the equation needs to be modified to be compatible with the theory of relativity.
Historically Schrödinger found a relativistic equation of motion before he de-

rived the his famous Schrödinger equation but dropped it because there were negative energy solutions, too. The equation he discovered was the Klein Gordon equation, a fully relativistic equation, which describes the dynamic of *spin*-0 particles, also called *scalar fields*. Here "relativistic" means that the equation is invariant under Lorentz transformation with a finite speed of light c. A example for a scalar field particle is the Higgs Boson which is interacting with every known particle and "gives them mass".

At this point we should remark that spin divides the known matter in the universe in two parts. On the one hand side we have particles which carry an half integer spin, these are called fermions and on the other hand we have particles which carry an integer spin an they are called bosons. This can be traced back to the fact that the N-body wavefunctions of bosons are symmetric with respect to an exchange of the position arguments. For fermions a pairwise exchange of the position arguments lead an additional minus sign. Following the discourse in [5] the relation to the spin numbers can be verified. These two particle sorts have fundamental different properties. The most important for our context is that bosons are not effected from the Pauli exclusion principle. Which means that several bosons can occupy the same quantum state which is not true for fermions. Here only one fermion can occupy one specific quantum state, i.e. fermions in principle are always distinguishable. On the other hand if many bosons occupy the same quantum state they become indistinguishable.

It is a priori surprising that the Schrödinger equation can be obtained as the non relativistic limit of the Klein Gordon equation since we already noted that one equation describes fermions and the other one describes bosons. This "paradox" can be explained in a setting where the nonrelativistic limit, i.e. $c \to \infty$ is combined the *semiclassical* limit $\hbar \to 0$. In the semiclassical limit the angular momentum is no longer discretized and therefore bosons and fermions become the same.

If the discretization of nature is only present at small scales one should think that they can be neglected at astronomical scales. Einsteins theory of general relativity does not take into account quantum effects. A theory which combines general relativity with the quantum theory is not known yet and it is among the biggest aim for physics nowadays to find such a *Grand Unified Theory (GUT)*.

In cosmology one assumes that the corresponding metric g, which is the solution to the Einstein equations, has the form

$$g = -dt^2 + g_{ij}dx^i dx^j = -dt^2 + a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right). \tag{1.1}$$

Here Ω stands for the solid angle, t for time and r for the radius. The parameter k determines if the curvature is positive, null or negative. In our case we set $k = 0$ since one can assume in a good approximation that the universe is flat

on astronomical scales. Solutions of this kind are known as Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes. The factor a is called the *scale factor* and responsible for the character of propagation of the spacetime. We also observe from the Cosmic Microwave Background CMB that the temperature in the universe is very uniformly distributed. This justifies the assumption that the universe can be considered as isotropic.
Since a couple of years it has become very popular to model the accelerating character of our universe by the dynamics of a scalar fields and so therefore the source term in the Einstein field equations is modeled by a stress energy tensor for a scalar field, see [6],

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right). \tag{1.2}$$

If we in addition assume that the universe is homogeneous in space the Einstein equations reduce to a coupled ODE system which determine the scale factor a and the scalar field ϕ. This ODE system is called the *Friedmann equations*

$$\frac{\dot{a}}{a} - \frac{k}{a^2} = \frac{8\pi G}{3} \rho, \tag{1.3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P), \tag{1.4}$$

Here $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) = T_{00}$ stands for the energy density $P = \frac{1}{2}\dot{\phi}^2 - V(\phi) = T_{ij}$ for $i, j \in \{1, 2, 3\}$ for the momentum density.

We already mentioned that the CMB is very isotropic. The isotropy of the CMB implies that the most spots of the CMB should have overlapping past light cones. But that is not possible in general. More precisely this applies to every two points p and q which are separated by more than one degree, see for example [6]. So each two points which are causally disconnected could not know anything about the temperature of the other point. Therefore no thermodynamic process could happen which achieves the equal temperature of p and q. But the question remains, if there was not enough time for the points p and q to communicate why the CMB is so isotropic? This is called the *Horizon Problem*. An answer to solve this problem is cosmic inflation. Here one postulates a certain period in the early universe of rapid expansion. Such that we "gain" a period in time where the past light cones of the spots in the CMB could overlap and therefore are causal connected.
To model inflation one postulates a scalar field ϕ whose equations of motion are coupled to the Friedmann equations and causes this inflation period in the early universe. A very important aspect which has become a milestone paper in general relativity and related subjects is the proof of Hans Ringström [1] on the stability of scalar field inflation models.

As we already noted before, many bosons can occupy the same state and therefore can be described by one single wave function. A so called Bose Einstein

Condensate (BEC) consists of bosons occupying the same quantum state. The idea is now to consider a BEC and drop the assumption of homogeneity of space in ϕ. The equation of motion for a scalar field is the Klein-Gordon equation

$$\frac{\hbar^2}{2m}\Box_f \phi = \left(\frac{1}{2}mc^2 + V'(\vec{x},t)\right)\phi + U'\phi. \tag{1.5}$$

Here$\Box_f$ stand for d'Alembert operator in a flat space, $V(\vec{x},t)$ stands for some external potential, where $U(\hat{\rho},\lambda_i)$ is a self-interaction term of the form

$$U(\hat{\rho},\lambda_i) = \frac{\lambda_2}{2}\hat{\rho}^2 + \frac{\lambda_3}{6}\hat{\rho}^3 + \ldots\,, \tag{1.6}$$

with $\hat{\rho} = \phi^\dagger\phi$ the density operator of the boson field and the λ_i are the coupling constants of the different interaction types. The prime in (1.5) denotes a derivative with respect to $\hat{\rho}$. The field ϕ is a complex valued scalar field defined on a certain spatial domain $\Omega \subset \mathbb{R}^n$.

To mimic the accelerating character of our universe its enough to require $\rho + 3P < 0$. Physicists would say that the scalar field breaks the so called *strong energy condition*. In fact there are many classical matter configurations violating the strong energy condition, so the explanation of the universe with a BEC does not need to postulate some kind of exotic matter (Dark Matter) like other models do. This is the reason why this approach became so popular.

Similarly to the Schrödinger equation a BEC admits an additional description in terms of fluid dynamic equations, see [7] and [3], [8]. To this we introduce the so-called Madelung transformation,

$$\phi = A\exp\left(\frac{iS}{\hbar}\right). \tag{1.7}$$

Here A stands for the magnitude and $iS/\hbar$ is the phase function. Note that the physical dimension of both S and $\hbar$ is an action. Therefore the exponent is indeed dimensionless. Now we are plugging this ansatz into the KGE (1.5) with $U = 0$ (for simplicity) and split it into a real and imaginary part. This gives us

$$\partial_t S + \frac{1}{2m}|\nabla S|^2 - \frac{1}{2m}\left(\frac{\partial_t S}{c}\right) + V'(A^2) = \frac{\hbar^2}{2m}\frac{\Box_f A}{A}, \tag{1.8}$$

for the real part, and the imaginary part becomes

$$\partial_t A + \frac{A}{2m}\Box_f S + \frac{1}{m}\nabla A\cdot\nabla S - \frac{1}{mc^2}\partial_t A\partial_t S = 0. \tag{1.9}$$

From the hydrodynamical description the non relativistic and semi-classical limits can be obtained , see [3] and [8]. Such relations can indeed be useful for setting up a suitable numerical method. On the other hand this limits are very interesting from a physical point of view too and can help to interpret our results correctly.

In this thesis we additionally treat the before mentioned relation of the nonlinear Klein-Gordon equation and the nonlinear Schrödinger equation. Especially we will consider the nonrelativistic semiclassical limit and give a rigorous proof of convergence to a system of compressible Euler equations in the hydrodynamical description (At least for small times, until the underlying classical fluid dynamics develop "shocks").

Chapter 2

Scalar cosmology

2.1 Cosmic Inflation

This and the following two sections originates from [9]. For further details of the physical background of general relativity we refer to [10]. The energy momentum tensor $T_{\mu\nu}$ determines the geometry of spacetime via the Einstein equations

$$\underbrace{R\mu\nu - \frac{1}{2}g_{\mu\nu}R}_{=:G_{\mu\nu}} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.1}$$

Here $R_{\mu\nu}$ is the Ricci tensor and R the corresponding curvature scalar, which is simply the trace of the Ricci tensor and $T_{\mu\nu}$ can be understood as a source term of curvature. Λ is called the cosmological constant and for our considerations we set $\Lambda = 0$.

2.1.1 Cosmological Principle and FLRW metric

In this section we follow the discourse in [10]. We consider the Friedman-Lemaître-Robertson-Walker (FLRW) metric, using polar coordinates it reads

$$g = -dt^2 + g_{ij}dx^i dx^j = -dt^2 + a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right). \tag{2.2}$$

Here $k \in \{0, \pm 1\}$ such that it satisfies the requirements of the *cosmological principle*. Which satisfy the following assumptions

> On spacetime there exists a time-function t, such that on each of its level sets
>
> $$\mathscr{S}_\tau = \{t = \tau\}, \tag{2.3}$$
>
> the universe looks the same every point in each direction.
>
> We consider a spacetime $(\mathscr{M}, g)$ such that

1. $\mathscr{M} = \mathbb{R} \times N$, for some three-dimensional manifold N.
2. For each $t \in \mathbb{R}$, $(N, g|_{\{t\}\times N})$ is maximally symmetric Riemannian manifold.

To proceed we first need to define the *Hubble constant*

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \tag{2.4}$$

by assuming the differentiability of a with respect to t. Our aim is to derive the *Friedmann equations*, which are the *Einstein field equations* of the FRLW metric in terms of a. But before we can continue with calculating the Einstein tensor $G_{\mu\nu}$ we need to discuss which forms the energy momentum tensor $T_{\mu\nu}$ can takehave and why.

2.1.2 The Energy Momentum Tensor

We follow the presentation in [6]. For models of the universe physicist try to keep things simple, therefore in general relativity it is a usual approach to assume isotropy and homogeneity of the universe. Due to highly developed telescopes it was able to create a temperature map of the whole universe. This famous map is called the *cosmic microwave background* (CMB), see figure 2.1.

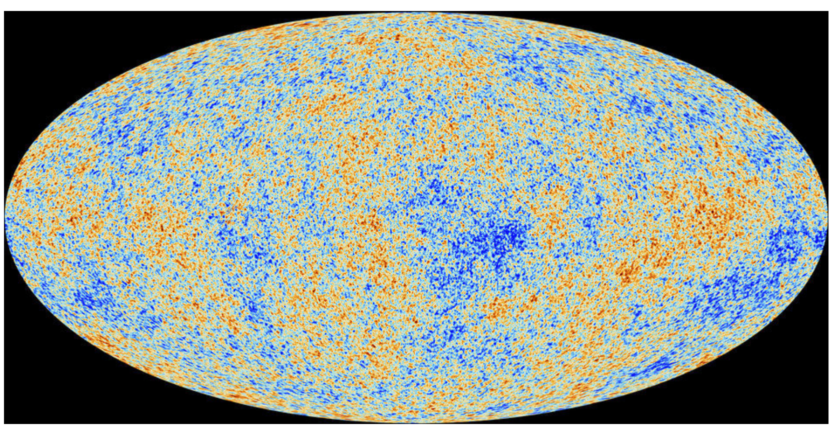

Figure 2.1: The cosmic background radiation. The temperature scale reach from approximately $3K$ (red) to $0K$ (blue).
Source by:`http://www.esa.int/Our_Activities/Space_Science/Highlights/Planck_s_Universe`

The cosmic background radiation suggest that our assumption of isotropy is justified, since the temperature is almost equally distributed. Note that the remarkable phenomenon of this temperature map is that, temperature is also equally distributed in causal disconnected areas of the universe. From this point of view the mean values of the 3-vectors has to vanish i.e. $T_{i0} = T_{0j} = 0$ by isotropy. Moreover, isotropy around a point $\boldsymbol{x} = 0$ requires the mean value of any 3-tensor, such as T_{ij} at that point to be proportional to δ_{ij} and hence to g_{ij}, which is equal to $-a^2(t)\delta_{ij}$ at $\boldsymbol{x} = 0$, such that we have

$$T_{ij}(\boldsymbol{x} = 0) \propto \delta_{ij} \propto g_{ij}(\boldsymbol{x} = 0). \tag{2.5}$$

The homogeneity condition requires the proportionality coefficient to be only a function of time. Since all frames of reference are equal, we have that the energy momentum tensor has to form

$$T_{00} = \rho(t)\ , \quad T_{0i} = Tj0 = 0\ , \quad T_{ij} = -P(t)g_{ij}(t, \boldsymbol{x}). \tag{2.6}$$

The function ρ is called the energy density and P stands for the pressure of the system. The conservation laws of physics translate in general relativity to the covariant equation

$$\nabla_\mu T^\mu_\nu = 0. \tag{2.7}$$

The derivative ∇_μ denotes the covariant derivative. By performing straight forward calculations one obtains

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{2.8}$$

Different sources of the energy momentum tensor

It turns out to be useful to classify different sources of the energy momentum tensor. There are three different types we will briefly discuss now. This different types belong to different ratios of ρ and P.

Matter

We use the term *matter* for all situation where $|P| \ll \rho$. For example this is the case for a non-relativistic gas. Setting $P = 0$ in (2.8) leads to

$$\rho \propto a^{-3}. \tag{2.9}$$

Further one distinguishes between two types

1. **Baryons.** The kind of particles [1] we can observe in experiments.
2. **Dark matter.** A until now unknown species of particles which only interact via gravitation.

[1] The term Baryons is technically incorrect, but since Baryons have much more mass then Leptons cosmologist neglect the other particle species.

Radiation

The term *radiation* covers every case where the absolute value of the pressure is about a third of the energy density, i.e. $|P| \approx \frac{1}{3}\rho$. In this case the conservation law (2.8) reduces to

$$\rho \propto a^{-4}. \tag{2.10}$$

The radiations split in three essential terms. Without going into details we distinguish between *photons*, *neutrinos* and *gravitational waves*.

Dark energy

The last source is called *dark energy*. Due to our obersations of the universe we know that it is dominated by a negative pressure since it still expands. So we classify this case by $P = -\rho$ which is unlike everything we have encountered in the lab so far. By considering (2.8) we find that the energy density ρ has to be constant, i.e.

$$\rho \propto a^{0}. \tag{2.11}$$

One possible explanation comes from quantum field theory (QFT), which predicts a vacuum energy. Unfortunately the predicted size of this effects is approximately 10^{-120} times smaller then the observed effect. This suggests that our model of nature is maybe incomplete and we missed a fundamental mechanism in our descriptions.

2.1.3 Friedmann equations

Again we follow the discourse in [6]. We now calculate the Einstein tensor for the FRLW metric, it is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{2.12}$$

Here $R_{\mu\nu}$ is the Ricci tensor and R the corresponding curvature scalar, which is simply the trace of the Ricci tensor. One calculates that

$$R_{00} = -3\frac{\ddot{a}}{a}, \tag{2.13}$$

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right) + 2\frac{k}{a^2}\right] g_{ij}, \tag{2.14}$$

and for the Ricci scalar one obtains

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right]. \tag{2.15}$$

Therefor the Einstein tensor reads

$$G^0_0 = 3\left[\left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right], \tag{2.16}$$

$$G^i_j = \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right]\delta^i_j. \tag{2.17}$$

From combining the equations (2.16) and (2.17) we obtain the *Friedmann equations*

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{k}{a^2}, \tag{2.18}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right). \tag{2.19}$$

2.1.4 Scalar Field Energy Momentum Tensor with homogeneity and isotropy

In this chapter we will trade Newtons's constant for the reduced Planck mass

$$M_pl = \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18} GeV, \tag{2.20}$$

so the first Friedmann equation takes the form

$$H^2 = \rho_\phi/(3M^2_{pl}). \tag{2.21}$$

As a simple model to describe inflation one considers a scalar field $\phi = \phi(x,t)$. The energy momentum tensor $T_{\mu\nu}$ of this model has the form

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right). \tag{2.22}$$

Here V stand for a potential whose gradient is proportional to the force acting on ϕ. Note that when V is not constant the force acting on ϕ is nonzero. Further we have to assume that the scalar field ϕ depends only on the time t such that $\phi = \phi(t)$ to be consistent with the symmetries in the FRLW metric. If this energy momentum tensor dominates the universe it sources the evolution of the FRLW background.

Analyzing the components of the energy momentum tensor we obtain

$$T^0_0 = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{2.23}$$

Here ρ_ϕ represents the total energy of the system, where the first part stands for the kinetic energy of the field ϕ. The space components of the energy momentum tensor above give the *pressure* of the system, which is the difference of the kinetic energy and the potential term,

$$T^i_j = -P_\phi\delta^i_j, \tag{2.24}$$

with

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.25}$$

Substituting our relation for the total energy which we found above into the first Friedmann equation $H^2 = \rho_\phi/(3M_{pl}^2)$, we get

$$H^2 = \frac{1}{3M_{pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]. \tag{2.26}$$

By taking the time derivative, we obtain

$$2H\dot{H} = \frac{1}{3M_{pl}^2}\left[\dot{\phi}\ddot{\phi} + V'\dot{\phi}\right]. \tag{2.27}$$

Here the derivative V' is understood as derivative with respect to ϕ. Using the second Friedmann equation $\dot{H} + H^2 = -\frac{1}{6M_{pl}^2}(\rho_\phi + 3P_\phi)$ together with the continuity equation $\rho_\phi = 3H(\rho_\phi + 3P_\phi)$ and our relations for ρ_ϕ and P_ϕ we find that the $\dot{H}$ can be interpreted as the kinetic energy density of field with mass M_{pl},

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}}{M_{pl}^2}. \tag{2.28}$$

Combining (2.26), (2.27) and (2.28) we get a *Klein Gordon equation* for the scalar field ϕ,

$$\ddot{\phi} + 3\dot{H}\phi + V' = 0. \tag{2.29}$$

Here $\dot{H}\phi$ can be understood as a friction term, where V' enters the equation like a force. Note that due to the assumption of homogeneity and isotropy in the energy momentum tensor the equation above even reduce to an ODE.

For a possible numerical implementation of the Friedmann equation with a ϕ^4-potential one can use the below scaling, see section 3.3

2.1.5 The Horizon problem

We follow the discourse in [6]. In this section it turns out to be very useful to set $c = 1$. The size of a causal connected area is determined by the distance photons can travel in a certain amount of time. Due to the isotropy of the universe we can define a coordinate system such that photons travel purely along the radial direction, i.e. $\theta = \phi = const.$. Here θ denotes the azimuth angle and ϕ the polar angle. This coordinate system is often called *conformal coordinate system.* Thus the line element in this coordinate system is given by

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2\right]. \tag{2.30}$$

Since photons travel along null-geodesics, i.e. $\mathrm{d}s^2 = 0$, the traveled length easily becomes

$$\Delta\chi(\tau) = \pm\Delta\tau. \tag{2.31}$$

In this coordinate system light rays correspond to straight lines at 45°. Note that τ does not represent the physical time. The reason why we are using the proper time τ is that the cone for the physical time would be curved in a curved spacetime. In the following we are going to define several concepts which we need to proceed. First,

Particle Horizon is the maximal distance light can travel between two times τ_1 and $\tau_2 > \tau_1$ which is simply $\Delta\tau = \tau_2 - \tau_1$. In terms of the physical times the particle horizon becomes

$$\chi_{ph} = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}s}{a(s)}. \tag{2.32}$$

In other word the particle horizon is the maximal distance from were we can receive signals. Everything which is within the past light cone in the age of the universe could have reached us until now. The Particle Horizon can may be visualized as the intersection of the past light cone of an observer with the spacelike hypersurface $\tau = 0$. Or in a more general setup one defines the particle horizon for any spacelike hypersurface $\tau = \tau_i$ which conicides with the expression (2.32). Next we consider the

Event Horizon. Note that there are events in the universe which we can not observe now or at any future time. The event horizon is the maximal distance how far an object can be away from us at time t such that we could it observe at a future time t_f. This distance is given by

$$\chi_{eh}\tau = \tau_f - \tau = \int_{t}^{t_f} \frac{\mathrm{d}s}{a(s)}. \tag{2.33}$$

This concepts is similar to event horizons of black holes. The

Hubble Sphere is the distance where objects beyond this distance recede faster than the speed of light from the observer due to the expansion of the universe.

We measure that the Hubble parameter H is shrinking since the beginning of the measurements. This implies that the Hubble sphere is growing. In literature the Hubble sphere is often treated to be equal to the particle horizon, which is a good approximation. But we will see that this approximation does not hold for a universe with a period of inflation. To see that the Particle Horizon and the Hubble Sphere yield the same values for a growing Hubble sphere rewrite

$$\chi_{ph}(\tau) = \int_{t_i}^{t} \frac{\mathrm{d}t}{a} = \int_{a_i}^{a} \frac{\mathrm{d}a}{a\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1} \,\mathrm{d}\ln a. \tag{2.34}$$

Here $a_i :=$ is defined to be zero at the Big Bang. The expression $(aH)^{-1}$ has the dimension of a length and is called the Hubble radius. For a universe dominated by a fluid with constant equation of state $\omega := P\rho$ we have

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3\omega)}. \tag{2.35}$$

Where H_0 is the Hubble constant measured at the present time. All familiar sources of matter satisfy the condition $1+3\omega > 0$ so it is a justified assumption that the Hubble radius is increasing nowadays. With the above expression we obtain that

$$\chi_{ph}(\tau) = \frac{2H_0^{-1}}{1+3\omega}\left[a^{\frac{1}{2}(1+3\omega)} - a_i^{\frac{1}{2}(1+3\omega)}\right] = \tau - \tau_i. \tag{2.36}$$

Letting $\tau_i \to 0$ is equal to $a_i^{\frac{1}{2}(1+3\omega)} \to 0$, now we obtain for the particle horizon

$$\chi_{ph}(\tau) = \frac{2H_0^{-1}}{1+3\omega}a(t)^{\frac{1}{2}(1+3\omega)} = \frac{2}{1+3\omega}(aH)^{-1}. \tag{2.37}$$

We see that in standard cosmology one has $\chi_{ph} \approx (aH)^{-1}$. This has lead to the confusing practice of calling both the Particle Horizon and the Hubble Sphere the "Horizon".

Above we noted that Cosmic Microwave Background (CMB) is isotropic. About 380 000 years after the Big Bang photons decoupled from the primordial plasma and hydrogen was formed. We observe this event nowadays in form of the CMB. The isotropy of the CMB implies that the most spots of the CMB should have overlapping past light cones. But that is not possible in general. More precicsely this applies to every two points p and q which are separated by more than one degree in sky. So each two points which are causal disconnected could not know anything about the temperature of the other point. Therefore no thermodynamic process could happen which achieves the equal temperature of p and q. But the question remains, if there was not enough time for the points p and q to communicate why the CMB is so isotropic? This is called the *Horizon Problem*. The Horizon Problem is illustrated by the following figure 2.2.

2.1.5.1 The Shrinking Hubble Sphere

Remark that the main difficulty of our considerations of the Horizon Problem was that in standard cosmology we observe a growing Hubble Sphere. The Hubble Sphere in the early universe was "to small" such that a thermodynamic process could have been established. Therefore a simple solution to the Horizon Problem suggests itself. Let us conjecture a phase of a *decreasing Hubble Sphere* in the early universe, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0. \tag{2.38}$$

If this lasts long enough the Horizon Problem can be avoided. Physically the shrinking Hubble Sphere corresponds to the condition $1+3\omega < 0$.
For a shrinking Hubble Sphere the integral in (2.34) is dominated by the lower bound. We now push the Big Bang singularity to the negative conformal time

$$\tau_i = \frac{2H_0^{-1}}{1+3\omega}a_i^{\frac{1}{2}(1+3\omega)} \to -\infty \tag{2.39}$$

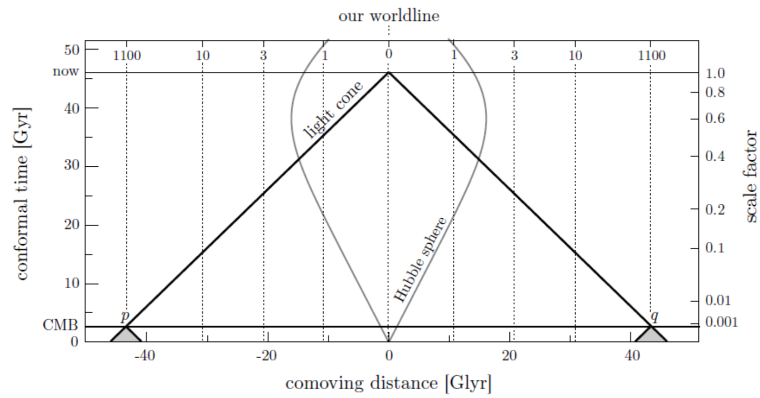

Figure 2.2: The Horizon Problem, the conformal time is measured in [Gyr]=[Gigayears]=tim, the scale factor stands for the value of a, the comoving distance is measured in [Glyr]=[Gigalightyears] = [Gyr] $\times$ velocity
Source: This figure is taken over from [6].

This means that there was "much more conformal time between the Big Bang and the decoupling of the photons then we tought". This epoch is called *inflation*. Now the past light cones of widely separated points p and q had enough time to intersect before the decoupling take place. This can be illustrated as shown in figure 2.3.

We see the key idea of inflation is that $\chi_{ph} \gg (aH)^{-1}$ and therefore it is no longer justified to view the Particle Horizon χ_{ph} as the same as the Hubble Sphere.

The condition for inflation to occur was

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0. \tag{2.40}$$

When we write this as

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\epsilon), \quad \text{where} \quad \epsilon := -\frac{\dot{H}}{H^2}, \tag{2.41}$$

then the shrinking Hubble Sphere corresponds to the condition

$$\epsilon = -\frac{\dot{H}}{H^2} < 1. \tag{2.42}$$

We already noticed that the Horizon Problem could be solved when the inflationary period lasts long enough. One defines the quantity $\mathrm{d}N := \mathrm{d}\ln a = H\,\mathrm{d}t$. Note that

$$a(t) \sim e^{N(t)} \tag{2.43}$$

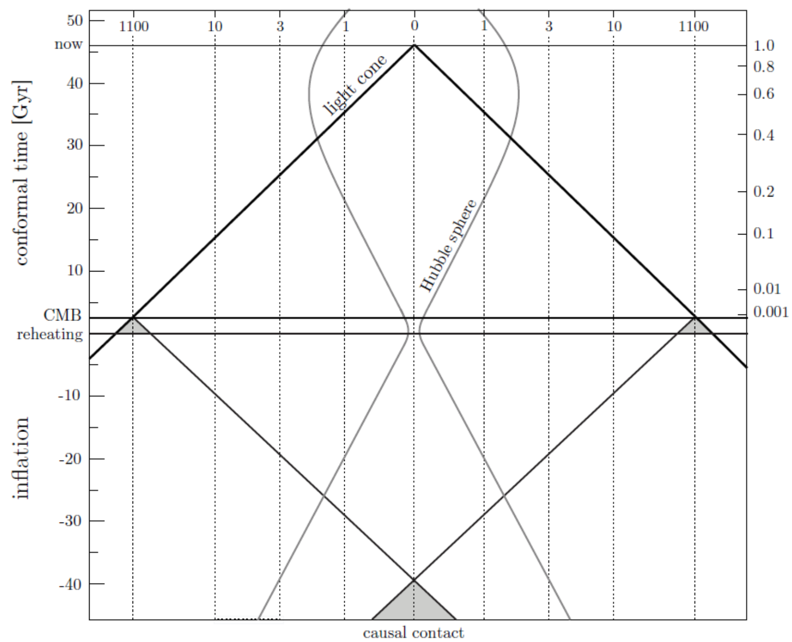

Figure 2.3: The Horizon Problem, the conformal time is measured in [Gyr]=[Gigayears]=time, scale factor stands for the value of a, the comoving distance is measured in [Glyr]=[Gigalightyears] = [Gyr] $\times$ velocity

Source: This figure is taken over from [6].

such that N measures the number of e-folds of inflationary expansion. Equation (2.42) now reads

$$\epsilon = -\frac{\mathrm{d}\ln H}{\mathrm{d}N} < 1. \tag{2.44}$$

This means that the change of the Hubble parameter H per e-fold is small. We also introduce a parameter η which measures the change of ϵ per Hubble time,

$$\eta = \frac{\mathrm{d}\ln\epsilon}{\mathrm{d}N} = \frac{\dot{\epsilon}}{H\epsilon}. \tag{2.45}$$

Note that for $|\eta| < 1$ inflation persists. When we plugging in (2.42) into (2.28) we obtain

$$\epsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2}. \tag{2.46}$$

This implies that inflation ($\epsilon < 1$) can only occur if the kinetic energy $\frac{1}{2}\dot{\phi}^2$ only makes small contributions to the total energy $\rho_\phi = 3M_{pl}^2 H^2$. This situation is called *slow roll inflation*.

The *slow roll approximation* $\epsilon \ll 1$ in turn implies that $\frac{1}{2}\dot{\phi}^2 \ll V$. With this approximation the Friedmann equation (2.26) becomes

$$H^2 \approx \frac{V}{3M_{pl}^2}. \tag{2.47}$$

Next we define the dimensionless acceleration per Hubble time

$$\delta = -\frac{\ddot{\phi}}{H\dot{\phi}}, \tag{2.48}$$

and take the time derivative of ϵ. This yields

$$\dot{\epsilon} = \frac{\dot{\phi}\ddot{\phi}}{M_{pl}^2 H^2} - \frac{\dot{\phi}^2 H}{M_{pl}^2 H^3}. \tag{2.49}$$

A quick comparison with (2.44) gives us

$$\eta = \frac{\dot{\epsilon}}{H\epsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\epsilon - \delta). \tag{2.50}$$

The condition $|\delta| \ll 1$ reduces the Klein-Gordon equation (2.29) to

$$3H\dot{\phi} \approx V'. \tag{2.51}$$

When we plug (2.50) and (2.51) into (2.46), we obtain a new expression for ϵ

$$\epsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2} \approx \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2. \tag{2.52}$$

Taking the time derivative of the reduced Klein-Gordon equation

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}, \tag{2.53}$$

leads to

$$\delta + \epsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} \approx M_{pl}^2 \frac{V''}{V}. \tag{2.54}$$

Note that successful slow-roll inflation occurs only when the parameters $\epsilon_V, |\eta_V| \ll 1$. Here

$$\epsilon_V = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 \quad , \quad |\eta_V| = M_{pl}^2 \frac{|V''|}{V}. \tag{2.55}$$

Further we are interested in the total number of e-folds of accelerated expansion which are given by

$$N_{tot} = \int_{a_I}^{a_E} \mathrm{d}\ln a = \int_{t_I}^{t_E} H(t)\,\mathrm{d}t, \tag{2.56}$$

where t_i and t_E are defined as the times when $\epsilon(t_I) = \epsilon(t_E) = 1$. In the slow roll regime we find

$$H\,\mathrm{d}t = \frac{H}{\dot{\phi}}\,\mathrm{d}\phi = \frac{1}{\sqrt{2\epsilon}}\frac{|\,\mathrm{d}\phi|}{M_{pl}}. \tag{2.57}$$

Therefore the total number of e-folds is given by

$$N_{tot} = \int_{\phi_I}^{\phi_E} \frac{1}{\sqrt{2\epsilon}}\frac{|\,\mathrm{d}\phi|}{M_{pl}}, \tag{2.58}$$

where ϕ_E and ϕ_I are defined as the boundaries where $\epsilon_V < 1$. The largest scales observed in the CMB are produced about $N_{CMB} \approx 60$ e-folds before the end of inflation and therefore a successful solution to the Horizon Problem requires $N_{tot} > N_{CMB}$.

2.2 Scalar field cosmology without the assumption of homogeneity

When we drop the assumption of homogeneity, i.e. the scalar field ϕ is not only a function of time, we end up with another Klein Gordon type equation for the motion of the scalar field ϕ.
To see this it is more convenient to derive the equation from the Hamilton principle (c.l. A.2) and us some formulas for simplification instead of manipulating the system of Einstein equations. Note that the Einstein field equations are exactly the Euler-Lagrange equation in the metric g. So the two approaches are equivalent.

Writing down the action integral of the scalar field

$$S = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L} = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right]. \tag{2.59}$$

Requiring that the action S is invariant under the variation δ_ϕ, i.e. $\delta_\phi S = 0$, gives

$$0 = \delta_\phi S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\delta_\phi\partial_\beta\phi - \frac{\partial V(\phi)}{\partial\phi}\delta_\phi\phi\right]. \tag{2.60}$$

The expression above equals

$$0 = \int \mathrm{d}^4x\left[\partial_\mu(\sqrt{-g}g^{\alpha\beta}\partial_\alpha\phi) - \sqrt{-g}\frac{\partial V(\phi)}{\partial\phi}\right]\delta_\phi\phi. \tag{2.61}$$

So the equation of motion reads

$$\partial_\mu(\sqrt{-g}g^{\alpha\beta}\partial_\alpha\phi) = \sqrt{-g}\frac{\partial V(\phi)}{\partial\phi}. \tag{2.62}$$

Hence that the covariant *d'Alembert* operator is given by

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\alpha\beta}\partial_\alpha\phi\right). \tag{2.63}$$

Thus the equation of motion reduces to

$$\Box\phi = \frac{\partial V(\phi)}{\partial\phi}. \tag{2.64}$$

We are setting $k = 0$ in the FRLW metric, since the flat universe is a good approximation when we look at the experimental data of cosmic background radiation. The determinant of g in this flat FRLW metric becomes $\det g = -a^6 \det(\gamma_{ij})$, where $\det(\gamma_{ij}) = r^4 sin^2\theta$. With this we can express the *d'Alembert* operator as

$$\Box\phi = -\ddot{\phi} - 3H\dot{\phi} + \frac{1}{a^2}\nabla^2\phi. \tag{2.65}$$

Here we defined the Laplacian, $\nabla^2\phi := \partial_i\left(\sqrt{\gamma}\partial^i\phi\right)/\sqrt{\gamma}$. So that we obtain the following system of equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{2.66}$$

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\left(\dot{\phi}^2 - V(\phi)\right), \tag{2.67}$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \frac{\partial V(\phi)}{\partial\phi} = 0. \tag{2.68}$$

Called the *Einstein-Klein-Gordon* system.

2.3 Stability results for the Einstein-Klein-Gordon system

In this section we are going to present a extremely powerful result proven by Hans Ringström in [1] concerning the stability of the Einstein-Klein-Gordon system (2.66), (2.67) and (2.68). It is remarkable that there are no restrictions due to symmetry assumptions of the metric and that this results hold for a very general class of potentials V.

But before we have a closer look on this theorem we need to clarify some preliminaries. First we need the concept of geodesics.

Geodesics are defined as the shortest connection of to points p_1 and p_2 on a manifold M. Therefore let $g(x)$ be the metric at $x \in M$. The geodesic between this two points is then precisely a curve γ whose length $\|\gamma\|_g$ satisfies

$$\|\gamma\|_g = inf_{\tilde{\gamma}}\,\|\tilde{\gamma}\|_g \tag{2.69}$$

For all parametrized curves $\tilde{\gamma} : [0,1] \to M$ which satisfy $\tilde{\gamma}(0) = p_1$ and $\tilde{\gamma}(1) = p_2$. The Lagrange formalism reformulates the minimization problem into solving a PDE instead. The equation is called the geodesic equation and is just the Euler Lagrange Formalism on the manifold M. Let $\{x^\mu\}_{n=1,\dots,n}$ be the coordinates on the manifold M and t the parametrization parameter. The geodesic equation reads

$$\frac{d^2 x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0. \tag{2.70}$$

In a coordinate free representation the geodesic equation simply becomes

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0. \tag{2.71}$$

Where γ is the geodesic one searches for. To understand the meaning of the theorem below we need the following definition.

Definition 2.3.1. *Let M be a Riemannian manifold, the metric g on M is said to be geodesically complete if and only if for every point $p \in M$ and every tangent vector $v \in T_p M$ at p the solution to*

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0, \tag{2.72}$$

with the initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$ is defined for all $t \in \mathbb{R}$. Here t again stands for the curve parameter.

The important point about geodesics is that all physical observers travel along geodesics through the spacetime. So geodesically completeness of course is a necessary condition for a spacetime's metric g to be a physical relevant solution. Otherwise metric singularities can occur in finite time. Therefore when we are interested in solutions to the Einstein-Klein-Gordon system then we should only take into account those which are geodesically complete in an "adequate neighbourhood" around the "true solution". This is exactly what the theorem below guarantees for a suitable class of potentials V.

First we should specify the expression of stability. What is meant here is that a solution of the Einstein-Klein-Gordon system which is future causally geodesically complete and has the property that if we make small perturbations of the initial data to this solution, then the resulting spacetime solution is future causally geodesically complete too. This theorem applies to the Einstein-Klein-Gordon system (2.66), (2.67) and (2.68) for a large class of potentials V. These are characterized by the conditions $V(0) > 0$, $V'(0) = 0$ and $V''(0) > 0$ where V is smooth.

Without going too much into the details of the very detailed work [1] we give a simplified version of the theorem of Hans Ringström.

Theorem 2.3.1. *Let V be a smooth function such that $V(0) > 0$, $V'(0) = 0$ and $V''(0) > 0$. Let $n \geq 3$ be the dimension of the Riemannian manifold Σ with metric h such that $(\Sigma, h, \phi_0, \phi_1)$ are the corresponding initial conditions for the Einstein-Klein-Gordon system (2.66), (2.67) and (2.68). Where ϕ_0 and ϕ_1 stand for $\phi(0)$ and $\phi'(0)$ respectively. Then the solution to the Einstein-Klein-Gordon system is stable in the above sense.*

Proof. The proof can be found in [1]. □

An important remark is that this theorem does not only apply to the Einstein-Klein-Gordon system (2.66), (2.67) and (2.68). It even applies to solutions of the Einstein equations (2.1) with a positive cosmological constant Λ. A positive cosmological constant has the same effect on the spacetime as our scalar field which causes inflation. A positive cosmological constant Λ yields solutions which are accelerating. Therefore we define the quantities H and χ as solutions of

$$nH^2 = \frac{2}{n-1}V(0), \tag{2.73}$$

and

$$\chi = \frac{V''(0)}{H^2}. \tag{2.74}$$

These quantities appear for example in two solutions to the Einstein equations (2.1) with a positive cosmological constant. The first one is given by

$$g_1 = -\,\mathrm{d}t^2 + cosh^2(Ht)g_h, \tag{2.75}$$

and the second one by

$$g_2 = -\,\mathrm{d}t^2 + e^{2Ht}g_h. \tag{2.76}$$

Next we consider the Einstein equations of the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{2.77}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{2.78}$$

For the energy-momentum tensor we assume it has the form

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right), \tag{2.79}$$

where $\phi \in C^\infty(M)$ and $V \in C^\infty(\mathbb{R})$ which satisfies the condition of the above theorem.

By assumption we have that $H, \chi > 0$. With this choice of H and χ, (2.75) and (2.76) are solutions to (2.77) with $\phi = 0$ and a positive cosmological constant given by $\Lambda = n(n-1)\frac{H^2}{2}$.

Chapter 3

The universe as Bose-Einstein-Condensate

3.1 Cosmological model of a Bose-Einstein condensate

When we observe the universe around us we see planets circling around stars, this stellar systems form galaxies and a huge amount of galaxies form galaxy clusters and these are separated by billions of light years. This shows us that the matter in the universe is not uniformly distributed. This phenomenon is addressed to tiny energy density fluctuations in the early universe.
A very challenging task is to formulate a model of our cosmos which is compatible with the structure formation of the universe at all length scales. One of the most delicate problems is, that such a model has to explain the phenomenon of dark matter. Dark matter is a, until now, unobserved object which interacts via gravitation with the "ordinary" matter. One of the clearest hints for dark matter comes from observation of rotating galaxies. According to Kepler's third law the rotation speed of the stars around the center of the galaxy should decrease, but one obverses the exact opposite. This suggests that the distribution of mass around the galaxy center is different from the light emitting matter which can be seen by a telescope. Especially there has to be more matter in the outer regions. This is the reason why we call it dark matter.

The aim is to find a model which explains dark matter on all scales. The first approach in that direction was a "cold" dark matter model but without going into details it does not reflect the observations on very large scales. A different type of dark matter model is the "warm" dark matter model, but again this model can also only partly explain the observations. The main issue occurs during the change of different scales. Nowadays scientists prefer models which have an implemented scale dependence. Over the last few years the Bose

Einstein Condensate (BEC) model has become very popular. The phenomenon of boson condensation was invented by Satyendra Bose [1] approximatly ninety years ago, the experimental proof was delivered by Eric A. Cornell and Carl E. Wieman [2] in 1995. A BEC is a large number of bosons occupying the same state such that they are macroscopic observable when the system is below some critical temperature. In this matter phase the BEC can be described by just one single wave function. Due to quantum fluctuation effects the wave function of the BEC is perturbed and so there are a few higher energy states which are occupied too. A remarkable fact is that during the condensation phase the BEC admits an additional hydrodynamic description in terms of the particle density and the velocity potential resembling the Euler equations of fluid dynamics. The first works of BEC in cosmology include a non-relativistic, time dependent Gross-Pitaevskii equation characterized by an external potential, which should model the gravitational potential, where the link to such a gravitational potential is in general determined by a coupling to the Poisson equation. Such a non-relativistic BEC model of the universe solves some problems which occurred in the traditional dark matter model but fails in other things, see [11] and [12]. We will consider a full relativistic model of a BEC coupled to gravity, including a so called non-minimal coupling. This kind of model pretty much fits the observations.

In this section we will derive the Klein-Gordon equation from a physics point of view. A key question of this section is how this equation behaves under the limits $c \to \infty$ and $\hbar \to 0$. The methods we are going to use are closely related to methods known from other equations in the relativistic quantum mechanic regime. For further details on this topic we refer the reader to [8], [13], [14] and [15].

3.1.1 Relativistic BEC in flat space

Let us begin the discussion about the BEC with an introduction to the formalism we met in [12]. The BEC consists of a boson field which condensates under a critical temperature, call this field $\hat{\phi} = \hat{\phi}(\vec{x}, t)$. The Lagrangian of the system reads

$$\mathcal{L} = \frac{1}{c^2} \frac{\partial \hat{\phi}^\dagger}{\partial t} \frac{\partial \hat{\phi}}{\partial t} - \nabla \hat{\phi}^\dagger \nabla \hat{\phi} - \left(\frac{m^2 c^2}{\hbar^2} + V(\vec{x}, t) \right) \hat{\rho} - U(\hat{\rho}, \lambda_i). \tag{3.1}$$

Here $V(\vec{x}, t)$ stands for some external potential, where $U(\hat{\rho}, \lambda_i)$ is a self-interaction term of the form

$$U(\hat{\rho}, \lambda_i) = \frac{\lambda_2}{2} \hat{\rho}^2 + \frac{\lambda_3}{6} \hat{\rho}^3 + \dots , \tag{3.2}$$

with $\hat{\rho} = \hat{\phi}^\dagger \hat{\phi}$ the density operator of the boson field and the λ_i are the coupling constants of the different interaction types. The first term of U is the usual two

[1] Satyendra Nath Bose was an Indian physicist at the University of Calcutta

[2] Eric Allin Cornell and Carl Edwin Wieman are both US-American physicists. 2001 they earned the Nobel prize in physics for the experimental evidence of a BEC.

body-interaction, the following terms describe possible many body-interactions. What can easily be seen is that the above Lagrangian $\mathcal{L}$ is invariant under the global action of $U(1)$ transformations, since the ordinary field and complex conjugated one occurs together with the same power in each term. By Noether's theorem this leads to a conserved current, which is given by

$$j^\mu = \frac{i}{2}\left(\hat{\phi}^\dagger \partial^\mu \hat{\phi} - \hat{\phi}\partial^\mu \hat{\phi}^\dagger\right). \tag{3.3}$$

The corresponding conserved quantity is the ensemble charge $N - \bar{N}$, where N ($\bar{N}$) stands for the boson (anti-boson) number. When one sets the interaction term U to zero, the ensemble charge can be written as

$$N - \bar{N} = \sum_k n_k - \bar{n}_k, \tag{3.4}$$

where

$$n_k(\mu,\beta) = \frac{1}{e^{\beta(|E_k|-\mu)} - 1}, \qquad \bar{n}_k(\mu,\beta) = \frac{1}{e^{\beta(|E_k|+\mu)} - 1}. \tag{3.5}$$

Here μ stands for the chemical potential, the energy level E_k is defined by $E_k^2 = \hbar^2 c^2 k^2 + m^2 c^4$ with $k \in \mathbb{N}$, β can be expressed as $\beta = k_B T$ where T stands for the temperature and k_B is the Boltzmann constant[3]. We can use this fact to find a relation of the critcal temperature of the BEC and the charge density $n := \frac{N-\bar{N}}{V}$, where V is the volume of the considered system. For the charge density one obtains

$$n = C \int k^2 \frac{sinh(\beta_c mc^2)}{cosh(\beta_c |E_k|) - cosh(\beta_c mc^2)}\, \mathrm{d}k, \tag{3.6}$$

where $\beta_c = T_c k_B$ and T_c is the critical temperature. We have used the fact that the chemical potential is $\mu = mc^2$ at the critical temperature. The normalization constant in front of the integral becomes

$$C = \frac{1}{4\pi^{3/2}\Gamma(3/2)}. \tag{3.7}$$

A straight forward calculation leads to a relation of the critical temperature in the relativistic case

$$k_B T_c = \left[\frac{\hbar^3 c \Gamma(3/2)(2\pi)^3}{4m\pi^{3/2}\Gamma(3)\zeta(2)}\right]^{\frac{1}{2}} n^{\frac{1}{2}}. \tag{3.8}$$

Here ζ stands for the Riemann zeta function, Γ for the gamma function and m is the mass of the boson. When we are instead using the classical energy momentum relation instead of the relativistic one above one ends up with the relation for the critical temperature in the non-relativistic case

$$k_B T_c^{NR} = \frac{2\pi\hbar^2}{m}\left[\frac{n}{\zeta(3/2)}\right]^{\frac{2}{3}}. \tag{3.9}$$

[3] $k_B = 1,38064852(79) \cdot 10^{-23} \frac{J}{K}$ in SI units

The above condition sets an upper limit for the boson mass m. By plugging in our experimental data we obtain $m < 2eV$. This suggests that the main part of our universe should be dominated by very light particles which are non-relativistic. This matter component is also called cold dark matter. In both cases when the temperature $T \gg T_c$ the scalar field behaves as any other self-interacting quantum field. But when $T \ll T_C$ the scalar field condenses and almost all particles end up occupying the ground state. In this phase state the condensate can be described as

$$\hat{\phi} = \phi(1 + \hat{\phi}), \tag{3.10}$$

where ϕ stands for the wave function of the condensate which describes the collective dynamics of the ground state. The mixed term represents the above mentioned fluctuations which came from quantum effects, i.e. these represent the excited states. When one neglects the excitation, the equation of motion becomes a non-linear Klein-Gordon equation (KGE)

$$\frac{\hbar^2}{2m}\Box_f \phi = \left(\frac{1}{2}mc^2 + V'(\vec{x}, t)\right)\phi + U'\phi, \tag{3.11}$$

where the prime denotes a derivative with respect to $\rho := \bar{\phi}\phi$. The field ϕ is a complex valued scalar field defined on a certain spatial domain $\Omega \subset \mathbb{R}^n$. We assume that the potential V is a twice differentiable nonlinear real-valued function over $\mathbb{R}^+$. If $V = const.$ then the equation above reduces to the equation of motion of a free particle. The subscript $_f$ denotes that we are considering a flat metric, i.e. $n_{\mu\nu} = diag[-1, 1, 1, 1]$ such that the d'Alembert operator does not need to be adapted

$$\Box_f \phi = n^{\mu\nu}\partial_\mu \partial_{nu}\phi = -\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi. \tag{3.12}$$

For the sake of completeness one can find a heuristically derivation of the Klein Gordon equation in the appendix A.1, which differs from the derivation above. The model of a cosmic BEC in flat spacetime became very popular in the last several years. This is mainly due to its simplicity as compared to the full relativistic case in non flat spacetimes. In the further analysis of the non-linear Klein Gordon equation (3.11) we set $U = 0$. A further physical plausible assumption is that the potential V only depends on the magnitude of ρ. i.e. $V = V(|\rho|) = V(|\phi|^2)$. We make the following ansatz for the wave function ϕ,

$$\phi(\vec{x}, t) = \Phi(\vec{x}, t)\exp\left(\frac{imc^2 t}{\hbar}\right), \tag{3.13}$$

where the factor $\exp\left(\frac{imc^2 t}{\hbar}\right)$ is the phase factor of the wave function and Φ stands for the magnitude of the wave function. With this ansatz we obtain the modulated version of the Klein Gordon equation

$$i\hbar\partial_t \Phi + \frac{\hbar^2}{2m}\Box_f \Phi - V'(|\Phi|^2)\Phi = 0. \tag{3.14}$$

Non relativistic limit: To obtain the nonrelativistic limit let $c \to \infty$ the the d'Alembert operator $\Box_f$ reduces to the Laplacian operator $\nabla^2 = \Delta$ and the modulated version of the Klein Gordon equation becomes the nonlinear Schrödinger equation

$$i\hbar\partial_t\Phi + \frac{\hbar^2}{2m}\Delta\Phi - V'(|\Phi|^2)\Phi = 0. \tag{3.15}$$

For more detailed discussion on the nonrelativistic limit we refer the reader to [8]. Here the authors proofed that solutions of the Klein-Gordon-Maxwell system on $\mathbb{R}^{1+3}$ converge in the energy space $C([0,T];H_1)$ to solutions of a Schrödinger-Poisson. The Klein-Gordon-Maxwell system can be obtained by choosing V as the electromagnetic potential. Some other related works are [13], [14] and [15]. system

3.1.2 Conservation laws for the Klein Gordon equation

Here we follow the ideas of [3]. For the analysis of the KGE it is important to formulate the following conservation laws of some conserved quantities. The action of the modulated non linear KGE is given by

$$S(\Phi) = \int\int -\frac{\hbar^2}{2m^2c^2}|\partial_t\Phi|^2 - \frac{i\hbar}{2m}\left(\bar{\Phi}\partial_t\Phi - \Phi\partial_t\bar{\Phi}\right) + \frac{\hbar^2}{2m^2}|\Delta\Phi|^2 + \frac{1}{m}V(|\Phi|^2)\,\mathrm{d}\vec{x}\,\mathrm{d}t \tag{3.16}$$

One can easily see that the action above is invariant under the following transformations for Φ. Let ϵ be a real parameter, where we do not distinguish between a scalar and a vector in our notation.

$$\begin{aligned} \Phi(\vec{x},t) &\to e^{i\epsilon}\Phi(\vec{x},t) \;\text{ with generator }\; \delta\Phi = i\Phi, \\ \Phi(\vec{x},t) &\to e^{i\epsilon}\Phi(\vec{x}+\epsilon,t) \;\text{ with generator }\; \delta\Phi = \nabla\Phi, \\ \Phi(\vec{x},t) &\to e^{i\epsilon}\Phi(\vec{x},t+\epsilon) \;\text{ with generator }\; \delta\Phi = \partial_t\Phi. \end{aligned} \tag{3.17}$$

Due to Noether's theorem there are charge quantities corresponding to each symmetry. We summarize this fact in the following theorem.

Theorem 3.1.1. *The modulated nonlinear KGE* (3.14) *admits the following conserved charges:*

$$\int\left[|\Phi|^2 + \frac{i\hbar}{2mc^2}(\bar{\Phi}\partial_t\Phi - \Phi\partial_t\bar{\Phi})\right]\mathrm{d}\vec{x} = C_1 \tag{3.18}$$

$$\int\left[\frac{i\hbar}{2mc^2}(\Phi\nabla\bar{\Phi} - \bar{\Phi}\nabla\Phi) - \frac{\hbar^2}{2m^2c^2}(\partial_t\Phi\nabla\bar{\Phi} + \partial_t\bar{\Phi}\nabla\Phi)\right]\mathrm{d}\vec{x} = C_2 \tag{3.19}$$

$$\int\left[\frac{\hbar^2}{2m^2}(\frac{1}{c^2}|\Phi|^2 + |\nabla\Phi|^2) + \frac{1}{m}V(|\Phi|^2)\right]\mathrm{d}\vec{x} = C_3 \tag{3.20}$$

Proof. *(i)*: Start with the modulated nonlinear KGE (3.14) and multiply it by $\bar{\Phi}$ and then subtract the conjugated version of (3.14) multiplied with Φ. After a few equivalence transformations we obtain the following expression

$$\frac{\partial}{\partial t}\left[|\Phi|^2 + \frac{i\hbar}{2mc^2}(\bar{\Phi}\partial_t\Phi - \Phi\partial_t\bar{\Phi})\right] + \nabla\cdot\left[\frac{i\hbar}{2m}(\Phi\nabla\bar{\Phi} - \bar{\Phi}\nabla\Phi)\right] = 0. \tag{3.21}$$

An integration over the whole space and using the divergence theorem of Gauss yields the expression in (3.18), since we make the standard assumption that the field Φ vanishes at infinity.

(ii): The next step is very similar to the first one. Multiply the (3.14) with $\nabla\bar{\Phi}$ and the complex conjugated version by $\nabla\Phi$ and then subtract both equations. This leads to

$$\begin{aligned}\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m}(\Phi\nabla\bar{\Phi} - \bar{\Phi}\nabla\Phi) - \frac{\hbar^2}{2m^2c^2}(\partial_t\Phi\nabla\bar{\Phi} + \partial_t\bar{\Phi}\nabla\Phi)\right]&\\ + \frac{\hbar^2}{4m^2}\nabla\cdot\left[2(\nabla\Phi\otimes\nabla\bar{\Phi} + \nabla\bar{\Phi}\otimes\nabla\Phi) - \Delta(|\Phi|^2)\right]&\\ + \frac{\hbar^2}{4mc^2}\nabla\partial_t\left[\Phi\partial_t\bar{\Phi} + \bar{\Phi}\partial_t\Phi\right] + \frac{1}{m}\nabla\cdot\left[|\Phi|^2V'(|\Phi|^2) - V(|\Phi|^2)\right] = 0.&\end{aligned} \tag{3.22}$$

Again a integration over the whole space and the usage of the Gauss theorem finally gives us (3.19).

(iii): Finally we multiply the modulated KGE (3.14) with $\partial_t\bar{\Phi}$ and the complex conjugated version by $\partial_t\Phi$. We obtain

$$\frac{\partial}{\partial t}\left[\frac{\hbar^2}{2m}\left(\frac{1}{c^2}|\partial_t\Phi|^2 + |\nabla\Phi|^2\right) + \frac{V(|\Phi|^2)}{m}\right] = \nabla\cdot\left[\frac{\hbar^2}{2m^2}(\nabla\Phi\partial_t\bar{\Phi} + \nabla\bar{\Phi}\partial_t\Phi)\right]. \tag{3.23}$$

The Gauss theorem and a integration over the whole space imply (3.20). □

Next we want to consider the non-relativistic limit of the conserved quantities.

3.1.2.1 The non-relativistic limit

One obtains the conservation laws in the non-relativistic limit when we let $c \to \infty$ in (3.21), (3.25) and (3.26). This leads to following expressions

$$\frac{\partial}{\partial t}\left[|\Phi|^2\right] + \nabla\cdot\left[\frac{i\hbar}{2m}(\Phi\nabla\bar{\Phi} - \bar{\Phi}\nabla\Phi)\right] = 0, \tag{3.24}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m}(\Phi\nabla\bar{\Phi} - \bar{\Phi}\nabla\Phi)\right] + \frac{\hbar^2}{4m^2}\nabla\cdot\left[2(\nabla\Phi\otimes\nabla\bar{\Phi} + \nabla\bar{\Phi}\otimes\nabla\Phi) - \Delta(|\Phi|^2)\right]&\\ + \frac{1}{m}\nabla\cdot\left[|\Phi|^2V'(|\Phi|^2) - V(|\Phi|^2)\right] = 0,&\end{aligned} \tag{3.25}$$

and

$$\frac{\partial}{\partial t}\left[\frac{\hbar^2}{2m}\big(|\nabla\Phi|^2\big)+\frac{V(|\Phi|^2)}{m}\right]=\nabla\cdot\left[\frac{\hbar^2}{2m^2}(\nabla\Phi\partial_t\bar{\Phi}+\nabla\bar{\Phi}\partial_t\Phi)\right]. \tag{3.26}$$

Before we turn over to the hydrodynamical formulation of the KGE, we want to compare the local conservation laws of the non-linear KGE

$$\frac{\hbar^2}{2m}\Box_f\phi-\frac{1}{2}mc^2\phi-V'(|\phi|^2)\phi=0. \tag{3.27}$$

with those of the modulated version of the nonlinear KGE. The main difference lies in the fact that the charge $|\phi|^2$ and the current $\frac{i\hbar}{2m}(\phi\nabla\bar{\phi}-\bar{\phi}\nabla\phi)$ vanish in (3.21) and (3.25) respectively for the nonlinear KGE. In this section we want to analyze the transition of the modulated KGE in the hydrodynamical description from the relativistic regime to the nonrelativistic and semirelativistic domain. For our further purposes we are defining the following quantities. First we state the conserved charge density and the conserved current of the nonlinear Schrödinger equation

$$\rho_S=\phi\bar{\phi}=|\phi|^2,\quad J_S=\frac{i\hbar}{2m}(\phi\nabla\bar{\phi}-\bar{\phi}\nabla\phi), \tag{3.28}$$

where ρ_S and J_S are the corresponding charge and the current of the non-linear Schrödinger equation. ρ_s can be interpreted as the probability density to detect a particle of the field ϕ at the position x at time t. The current density J_S describes the flow of the conserved charge density ρ_S. For the non-linear KGE we again define two very similar quantities

$$\rho_K=\frac{i\hbar}{2mc^2}(\bar{\Phi}\partial_t\Phi-\Phi\partial_t\bar{\Phi}),\quad J_K=\frac{\hbar^2}{2m^2c^2}(\partial_t\Phi\nabla\bar{\Phi}-\partial_t\bar{\Phi}\nabla\Phi). \tag{3.29}$$

The main physical difference between the conserved charges ρ_S and ρ_K is that ρ_K is no longer positive definite. The corresponding conserved quantities for the modulated version of the non-linear KGE become

$$\rho_M=\rho_S+\rho_K=|\phi|^2+\frac{i\hbar}{2mc^2}(\bar{\phi}\partial_t\phi-\phi\partial_t\bar{\phi}), \tag{3.30}$$

$$J_M=J_S+J_K=\frac{i\hbar}{2m}(\phi\nabla\bar{\phi}-\bar{\phi}\nabla\phi)-\frac{\hbar^2}{2m^2c^2}(\partial_t\phi\nabla\bar{\phi}+\partial_t\bar{\phi}\nabla\phi). \tag{3.31}$$

The remarkable thing about the conserved charge ρ_M is that it is positive definite in the limits $c\to\infty$ or $\hbar\to 0$. Since this would imply $\rho_M\to\rho_S$. But only in the non-relativistic limit $c\to\infty$ we also obtain $J_M\to J_S$. This suggests that the non-relativistic limit would be easier to handle and indeed this is the case since the modulated equation formally converges to the non-linear Schrödinger equation in the non-relativistic limit $c\to\infty$.

3.2 Hydrodynamical Description

In this section we follow the discourse in [3]. To introduce the semiclassical limit of the modulated KGE we follow the same strategy which is known from the non-linear Schrödinger equation. To achieve this we introduce the so called *WKB transformation* or also called the *Madelung transformation*,

$$\phi = A \exp\left(\frac{iS}{\hbar}\right). \tag{3.32}$$

Here A stands for the magnitude and $iS/\hbar$ is the phase function. Note that the physical dimension of both S and $\hbar$ is an action. Therefore the exponent is indeed dimensionless. Now we are plugging this ansatz into the KGE (3.11) and split it into a real and imaginary part. This gives us

$$\partial_t S + \frac{1}{2m}|\nabla S|^2 - \frac{1}{2m}\left(\frac{\partial_t S}{c}\right) + V'(A^2) = \frac{\hbar^2}{2m}\frac{\Box_f A}{A}, \tag{3.33}$$

for the real part, and the imaginary part becomes

$$\partial_t A + \frac{A}{2m}\Box_f S + \frac{1}{m}\nabla A \cdot \nabla S - \frac{1}{mc^2}\partial_t A \partial_t S = 0. \tag{3.34}$$

Here we assume the differentiability of A and S such that this two equations are indeed equivalent to the KGE. To continue we need to define the following functions

$$\rho = A^2 = |\phi|^2, \tag{3.35}$$

$$u = \frac{1}{m}\nabla S = \frac{i\hbar}{2m}\frac{1}{|\phi|^2}(\phi\nabla\bar{\phi} - \bar{\phi}\nabla\phi), \tag{3.36}$$

$$\rho_K = \frac{1}{mc^2}A^2\partial_t S = \frac{i\hbar}{2mc^2}(\phi\nabla\partial_t\bar{\phi} - \bar{\phi}\partial_t\phi). \tag{3.37}$$

When we plug this into (3.33) and (3.34) then after some intermediate computation steps one can obtain the following equivalent system

$$\partial_t(\rho - \rho_K) + \nabla\cdot(\rho u) = 0, \quad \partial_t u = c^2\nabla\left(\frac{\rho_K}{\rho}\right), \tag{3.38}$$

$$\begin{aligned}\left(1 - \frac{\rho_K}{\rho}\right)\partial_t u + (u\cdot\nabla)u + \frac{1}{m}V''(\rho)\nabla\rho = \\ \frac{\hbar^2}{2m^2}\left[\nabla\cdot\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{1}{c^2}\nabla\left(\frac{\partial_t^2\sqrt{\rho}}{\sqrt{\rho}}\right)\right].\end{aligned} \tag{3.39}$$

By defining

$$(\rho) := \frac{1}{m}(\rho V'(\rho) - V(\rho)), \tag{3.40}$$

and multiplying (3.39) by ρ and using the continuity equation (3.38) we obtain a conservation law of the quantity P which is interpreted as pressure.

$$\begin{aligned} &\partial_t(\rho u - \rho_K u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) \\ &= \frac{\hbar^2}{4m^2} \nabla \cdot \left(\rho \nabla^2 \log \rho\right) - \frac{\hbar^2}{4m^2c^2} \partial_t \left(\rho\right) \nabla \partial_t \log \rho\right). \end{aligned} \tag{3.41}$$

Note that the equations (3.38) and (3.41) describe a system governing ρ, ρ_K and ρu which is of the form of a perturbation of a relativistic compressible Euler equation with pressure $P(\rho)$. To end this section we finally introduce the functions E_S, the Schrödinger part energy and E_K, the relativistic part energy for the hydrodynamical ansatz.

$$E_S = \frac{1}{2}\rho|u|^2 + \frac{\hbar^2}{8m^2}\frac{|\nabla\rho|^2}{\rho} + \frac{1}{m}V(\rho) = \frac{\hbar^2}{2m^2}|\nabla\phi|^2 + \frac{1}{m}V(|\phi|^2), \tag{3.42}$$

$$E_K = \frac{c^2}{\rho}\frac{\rho_K^2}{\rho} + \frac{\hbar^2}{8m^2c^2}\frac{|\partial_t\rho|^2}{\rho} = \frac{\hbar^2}{2m^2c^2}|\partial_t\phi|^2. \tag{3.43}$$

With this definition we obtain from (3.38) and (3.41) the energy equation

$$\partial_t(E_S + E_K) + \nabla \cdot \big((E_S + P(\rho))u\big) = \frac{\hbar^2}{4m^2}\nabla \cdot \left[u\Delta\rho - \nabla \cdot (\rho u)\frac{\nabla\rho}{\rho}\right]. \tag{3.44}$$

3.2.1 Singular Limits

We introduced the hydrodynamical description for the KGE since it is very easy to discuss the various singular limits from the fluid dynamical formulation (3.38), (3.41) and (3.44).

Non-relativistic limit: First we let $c \to \infty$. From the definition of ρ_K we obtain $\rho_K \to 0$. This finally yields the system

$$\partial\rho + \nabla \cdot (\rho u) = 0, \tag{3.45}$$

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) = \frac{\hbar^2}{4m^2}\nabla \cdot \left[\rho\nabla^2 \log\rho\right], \tag{3.46}$$

$$\partial_t(E_S) + \nabla \cdot \big((E_S + P(\rho))u\big) = \frac{\hbar^2}{4m^2}\nabla \cdot \left[u\Delta\rho - \nabla \cdot (\rho u)\frac{\nabla\rho}{\rho}\right]. \tag{3.47}$$

where the limit hydrodynamic variables ρ, u and E are given in terms of the limit wave function ϕ by

$$\rho = \phi\bar{\phi} = |\phi|^2, \tag{3.48}$$

$$u = \frac{1}{m}\nabla S = \frac{i\hbar}{2m}\frac{1}{|\phi|^2}(\phi\nabla\bar{\phi} - \bar{\phi}\nabla\phi), \tag{3.49}$$

$$E = \frac{1}{2}\rho|u|^2 + \frac{\hbar^2}{4m^2}\frac{|\nabla\rho|^2}{\rho} + \frac{1}{m}V(\rho). \tag{3.50}$$

The expressions above exactly coincident with the fluid dynamical formulation of the non-linear Schrödinger equation.

Semiclassical limit: Next we consider the semiclassical limit where $\hbar \to 0$. A natural conjecture is that the terms with $\mathcal{O}(\hbar^2)$ are negligible as $\hbar \to 0$ and the limiting functions ρ, u and ρ_K statisfy the following equations

$$\partial_t(\rho - \rho_K) + \nabla \cdot (\rho u) = 0, \quad \partial_t u = c^2 \nabla \left(\frac{\rho_K}{\rho} \right), \tag{3.51}$$

$$\partial_t(\rho u - \rho_K u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) = 0, \tag{3.52}$$

$$\partial_t \left(E_S + \frac{c^2}{2} \frac{\rho_K^2}{\rho} \right) + \nabla \cdot ((E_S + P(\rho))u) = 0. \tag{3.53}$$

As mentioned before for more details on the nonrelativistic limit we refer to [8].
Non-relativistic semiclassical limit: The last singular limit we want to consider is the case where $c \to \infty$ and simultaneously $\hbar \to 0$. The limit equations of (3.38), (3.41) and (3.44) will be the following set of compressible Euler equations

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \tag{3.54}$$

$$\partial_t(\rho u) + \cdot(\rho u \otimes u) + \nabla P(\rho) = 0, \tag{3.55}$$

$$\partial_t E_S + \nabla \cdot ((E_S + P(\rho))u) = . \tag{3.56}$$

In our next section we will investigate this limit in more detail and we will follow the authors in [3] and give a rigorous proof.

3.2.2 Non-relativistic semiclassical limit

Now we will have a closer look on the non-reltivistic semiclassical limit, i.e. $c \to \infty$ and $\hbar \to 0$ simultaneously. We predict a relation between c and $\hbar$ of the form $\hbar = \epsilon$ and $c^{-1} = \epsilon^\alpha$. In addition we assume that the potential V behaves such that $V'(|\phi^\epsilon|^2) = |\phi^\epsilon|^{2(\gamma-1)}$ with $\gamma > 2$. Here the superscript ϵ denotes that we are considering the non-relativistic semiclassical limit with the predicted relations between c and $\hbar$ as above. The modulated KGE becomes

$$i\epsilon \partial_t \psi^\epsilon \partial_t^2 \psi^\epsilon + \frac{\epsilon^2}{2} \Delta \psi^\epsilon - |\psi^\epsilon|^{2(\gamma-1)} \psi^\epsilon = 0, \tag{3.57}$$

and the initial conditions are given by

$$\psi^\epsilon(x, 0) = \psi_0^\epsilon(x), \quad \partial_t \psi^\epsilon(x, 0) = \psi_1^\epsilon(x) \tag{3.58}$$

The equations (3.38), (3.41) and (3.44) become in the corresponding hydrodynamical formulation

$$\partial_t(\rho^\epsilon - \rho_K^\epsilon) + \nabla \cdot (\rho^\epsilon u^\epsilon) = 0, \quad \epsilon^{2\alpha} \partial_t u^\epsilon = \nabla \left(\frac{\rho_K^\epsilon}{\rho^\epsilon} \right), \tag{3.59}$$

$$
\begin{aligned}
&\partial_t(\rho^\epsilon u^\epsilon - \rho^\epsilon_K u^\epsilon) + \nabla\cdot(\rho^\epsilon u^\epsilon \otimes u^\epsilon) + \frac{1}{\gamma}\nabla(\rho^\epsilon)^\gamma \\
&= \frac{\epsilon^2}{4}\nabla\cdot\left(\rho^\epsilon \nabla^2 \log\rho^\epsilon\right) - \frac{\epsilon^{2+2\alpha}}{4}\partial_t\left(\rho^\epsilon \nabla\log\rho^\epsilon\right).
\end{aligned}
\tag{3.60}
$$

The conservation of energy implies that $\int (E_S + E_K)\,\mathrm{d}x$ is constant in time. With the definition of V' the Schrödinger part energy E_S and the relativistic part energy E_K read

$$
E^\epsilon_S = \frac{1}{2}\rho^\epsilon|u^\epsilon|^2 + \frac{\epsilon^2}{8}\frac{|\nabla\rho^\epsilon|^2}{\rho^\epsilon} + \frac{1}{\gamma}\left(\rho^\epsilon\right)^\gamma, \tag{3.61}
$$

$$
E^\epsilon_K = \frac{1}{2\epsilon^{2\alpha}}\frac{(\rho^\epsilon_K)^2}{\rho^\epsilon} + \frac{\epsilon^{2+2\alpha}}{8}\frac{|\partial_t\rho^\epsilon|^2}{\rho^\epsilon}. \tag{3.62}
$$

We further assume that the total energy is bounded $\int (E_S + E_K)\,\mathrm{d}x \le C$. In the rest of this section we will follow the ideas in [3] and will present a proof that the corresponding limit equations are the following compressible Euler equations

$$
\partial_t\rho + \nabla\cdot(\rho u) = 0, \tag{3.63}
$$

$$
\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \frac{1}{\gamma}\nabla\rho^\gamma = 0. \tag{3.64}
$$

with the initial conditions

$$
\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x). \tag{3.65}
$$

We summarize this fact in the following theorem

Theorem 3.2.1. *Let $\gamma \ge 2$ and ψ^ϵ be a solution of the modulated KGE* (3.57) *with initial conditions $(\psi^\epsilon_0, \psi^\epsilon_1) \in H^{s+1}(\mathbb{R}^n) \oplus H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 2$, with the total energy $\int (E_S + E_K)\,\mathrm{d}x \le C$ bounded from above and the energy functional H^ϵ fulfilling $H^\epsilon(0) \to 0$ as $\epsilon \to 0$. Then there exists a $T_* > 0$ such that*

$$
\|(\rho^\epsilon - \rho)(\cdot,t)\|_{L^\gamma(\mathbb{R}^n)} \to 0, \quad \|\rho^\epsilon_K(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^n)} \to 0, \tag{3.66}
$$

$$
\|(\rho^\epsilon u^\epsilon - \rho u)\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^n)} \to 0, \tag{3.67}
$$

for $T \in [0, T_*)$ as $\epsilon \to 0$, where (ρ, u) is the unique local smooth solution of (3.63) *and* (3.64) *with the initial conditions* (3.65).

The strategy to prove this statement is based on an estimate of the energy functional H^ϵ which is defined as follows

$$
\begin{aligned}
H^\epsilon(t) = \frac{1}{2}\int \rho^\epsilon|u^\epsilon - u|^2 dx + \frac{\epsilon^2}{2}\int \left|\nabla\sqrt{\rho^\epsilon}\right|\mathrm{d}x \\
+\frac{1}{2\epsilon^{2\alpha}}\int \left|\frac{\rho^\epsilon_K}{\sqrt{\rho^\epsilon}}\right|^2 \mathrm{d}x + \frac{\epsilon^{2+2\alpha}}{2}\int \left|\partial_t\sqrt{\rho^\epsilon}\right|^2\mathrm{d}x + \int \Theta(\rho^\epsilon,\rho)\,\mathrm{d}x,
\end{aligned}
\tag{3.68}
$$

where Θ is defined as

$$
\Theta(\rho^\epsilon,\rho) = \frac{1}{\gamma}\left((\rho^\epsilon)^\gamma - \rho^\gamma\right) - \rho^\gamma(\rho^\epsilon - \rho). \tag{3.69}
$$

Proof. The following proof is taken over from [3]. Using the conservation of energy and the Hölder inequality we obtain

$$\|\rho_K^\epsilon\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^n)} \leq \|\sqrt{\rho^\epsilon}\|_{L^{2\gamma}(\mathbb{R}^n)} \left\|\frac{1}{\sqrt{\rho^\epsilon}}\rho_K^\epsilon\right\|_{L^2(\mathbb{R}^n)} \tag{3.70}$$

this in turn implies that

$$\|\rho_K^\epsilon(\cdot,t)\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3.71}$$

Next we want to investigate the behaviour of the energy functional H^ϵ in time. Differentiating H^ϵ with respect to time yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H^\epsilon(t) = &-\int_{\mathbb{T}^n} u\cdot(\rho^\epsilon u^\epsilon)\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n}\rho^\epsilon|u|^2\,\mathrm{d}x \\ &+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n}\left(\frac{\gamma-1}{\gamma}\rho-\rho^\epsilon\right)\rho^{\gamma-1}\,\mathrm{d}x =: I_1+I_2+I_3.\end{aligned} \tag{3.72}$$

For the first term I_1 integration by parts and the conservation law of momentum (3.60) give us

$$\begin{aligned}I_1 = &-\int_{\mathbb{T}^n}\partial_t u\cdot(\rho^\epsilon u^\epsilon)\,\mathrm{d}x - \int(\rho^\epsilon u^\epsilon\otimes u^\epsilon)\nabla u\,\mathrm{d}x - \int_{\mathbb{T}^n}\frac{\gamma-1}{\gamma}(\rho^\epsilon)^\gamma\nabla\cdot u\,\mathrm{d}x \\ &-\frac{\epsilon^2}{4}\int_{\mathbb{T}^n}(\nabla\sqrt{\rho^\epsilon}\otimes\nabla\sqrt{\rho^\epsilon}):\nabla u + \nabla\rho^\epsilon\cdot(\nabla\nabla\cdot u)\,\mathrm{d}dx \\ &\frac{1}{4}\epsilon^{2+2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n}(\partial_t\rho^\epsilon)\nabla\cdot u\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n}u\cdot\left(\rho_K^\epsilon u^\epsilon+\epsilon^{2+2\alpha}\partial_t\sqrt{\rho^\epsilon}\nabla\sqrt{\rho^\epsilon}\right)\mathrm{d}x \\ &+\frac{1}{4}\epsilon^{2+2\alpha}\int_{\mathbb{T}^n}(\partial_t\rho^\epsilon)\nabla\cdot\partial_t u\,\mathrm{d}x + \int_{\mathbb{T}^n}\partial_t u\left(\rho_K^\epsilon u^\epsilon+\epsilon^{2+2\alpha}\partial_t\sqrt{\rho^\epsilon}\nabla\sqrt{\rho^\epsilon}\right)\mathrm{d}x.\end{aligned} \tag{3.73}$$

Next, the conservation of charge (3.59) and again integration by parts yield

$$\begin{aligned}I_2 = &\int\rho^\epsilon u\cdot\partial_t u\,\mathrm{d}x + \frac{1}{2}\int\nabla|u|^2\cdot(\rho^\epsilon u^\epsilon)\,\mathrm{d}x \\ &+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_K^\epsilon|u|^2\,\mathrm{d}x - \int\rho_K^\epsilon u\cdot\partial_t u\,\mathrm{d}x.\end{aligned} \tag{3.74}$$

For the third term we have

$$\begin{aligned}I_3 = &\int(\gamma-1)\rho^{\gamma-2}(\rho-\rho^\epsilon)\partial_t\rho\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int\rho^{\gamma-1}\rho_K^\epsilon\,\mathrm{d}x \\ &+\int\left(\partial_t\rho^{\gamma-1}\rho_K^\epsilon - \nabla\rho^{\gamma-1}(\rho^\epsilon u^\epsilon)\right)\mathrm{d}x.\end{aligned} \tag{3.75}$$

Our goal is to introduce a correction function G^ϵ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H^\epsilon(t)-G^\epsilon(t)\right) \leq \|\nabla u\|_{L^\infty(\mathbb{R}^n)} H^\epsilon(t) + o(1) \tag{3.76}$$

The function G^ϵ is defined as

$$
\begin{aligned}
G^\epsilon(t) = -\frac{1}{2}\int |u|^2\rho_K^\epsilon \,\mathrm{d}x + \int \rho^{\gamma-1}\rho_K^\epsilon \,\mathrm{d}x + \frac{1}{4}\epsilon^{2+2\alpha}\int \partial_t\rho^\epsilon \nabla\cdot u \,\mathrm{d}x \\
+ \int u\cdot\left(\rho_K^\epsilon u^\epsilon + \epsilon^{2+2\alpha}\partial_t\sqrt{\rho^\epsilon}\nabla\sqrt{\rho^\epsilon}\right)\mathrm{d}x.
\end{aligned}
\tag{3.77}
$$

With G^ϵ the expression $\frac{\mathrm{d}}{\mathrm{d}t}\left(H^\epsilon(t) - G^\epsilon(t)\right)$ becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(H^\epsilon(t) - G^\epsilon(t)\right) = -\frac{\epsilon}{4}\int \left(\nabla\sqrt{\rho^\epsilon}\otimes\nabla\sqrt{\rho^\epsilon}\right) : \nabla u \,\mathrm{d}x \\
+ \int -(\rho^\epsilon u^\epsilon)\cdot\nabla\rho^{\gamma-1} - \frac{\gamma-1}{\gamma}(\rho^\epsilon)^\gamma \nabla\cdot u \,\mathrm{d}x \\
-\int(\rho^\epsilon u^\epsilon - \rho^\epsilon u)\cdot\partial_t u \,\mathrm{d}x + \frac{1}{2}\int(\rho^\epsilon u^\epsilon)\cdot\nabla|u|^2 \,\mathrm{d}x + R_1 + R_2 + R_3.
\end{aligned}
\tag{3.78}
$$

Here

$$
R_1 = -\int(\rho^\epsilon u^\epsilon\otimes u^\epsilon) : \nabla u \,\mathrm{d}x, \tag{3.79}
$$

$$
R_2 = \int(\gamma-1)\rho^{\gamma-2}(\rho 1\epsilon - \rho)\nabla\cdot(\rho u)\,\mathrm{d}x, \tag{3.80}
$$

$$
\begin{aligned}
R_3 = \int \partial_t\rho^{\gamma-1}\rho_K^\epsilon + \partial_t u\cdot\left(\rho_K^\epsilon u^\epsilon + \epsilon^{2+2\alpha}\partial_t\sqrt{\rho^\epsilon}\nabla\sqrt{\rho^\epsilon}\right) - \frac{1}{2}\rho_K^\epsilon\partial_t|u|^2 \\
-\frac{\epsilon^2}{4}\nabla\rho^\epsilon\cdot(\nabla\nabla\cdot u) + \frac{1}{4}\epsilon^{2+2\alpha}\nabla\cdot\partial_t u\partial_t\rho^\epsilon \,\mathrm{d}x.
\end{aligned}
\tag{3.81}
$$

To continue we have to rewrite the term R_1 as

$$
\begin{aligned}
R_1 = -\int\left(\rho^\epsilon(u^\epsilon - u)\otimes(u^\epsilon - u)\right) : \nabla u \,\mathrm{d}x - \int(\rho^\epsilon u\otimes u^\epsilon) : \nabla u \,\mathrm{d}x \\
+ \int(\rho^\epsilon u\otimes u) : \nabla u \,\mathrm{d}x - \int(\rho^\epsilon u^\epsilon\otimes u) : \nabla u \,\mathrm{d}x.
\end{aligned}
\tag{3.82}
$$

We directly calculate

$$
-\int(\rho^\epsilon u\otimes u^\epsilon) : \nabla u \,\mathrm{d}x = \int\frac{1}{2}|u|^2\nabla\cdot(\rho^\epsilon u^\epsilon)\,\mathrm{d}x, \tag{3.83}
$$

and

$$
\begin{aligned}
\int(\rho^\epsilon u\otimes u) : \nabla u \,\mathrm{d}x - \int(\rho^\epsilon u^\epsilon\otimes u) : \nabla u \,\mathrm{d}x \\
= \int[(u\cdot\nabla)u]\cdot(\rho^\epsilon u - \rho^\epsilon u^\epsilon)\,\mathrm{d}x.
\end{aligned}
\tag{3.84}
$$

So the expression R_1 becomes

$$
\begin{aligned}
R_1 = -\int\left(\rho^\epsilon(u^\epsilon - u)\otimes(u^\epsilon - u)\right) : \nabla u \,\mathrm{d}x \\
+ \int\frac{1}{2}|u|^2\nabla\cdot(\rho^\epsilon u^\epsilon)\,\mathrm{d}x + \int[(u\cdot\nabla)u]\cdot(\rho^\epsilon u - \rho^\epsilon u^\epsilon)\,\mathrm{d}x.
\end{aligned}
\tag{3.85}
$$

For the second term R_2 we consider the identity

$$(\gamma-1)\rho^{\gamma-1}\nabla\cdot(\rho u)=\frac{\gamma-1}{\gamma}(\nabla\rho^{\gamma})\cdot u+(\gamma-1)\rho^{\gamma}\nabla\cdot u, \tag{3.86}$$

This in turn implies that

$$-\int(\gamma-1)\rho^{\gamma-1}\nabla\cdot(\rho u)=\frac{\gamma-1}{\gamma}(\nabla\rho^{\gamma})\cdot u+(\gamma-1)\rho^{\gamma}\nabla\cdot u, \tag{3.87}$$

therefore

$$\begin{aligned}&\int(\gamma-1)\rho^{\gamma-2}\rho^{\epsilon}\nabla\cdot(\rho u)\,\mathrm{d}x\\ &=\int(\gamma-1)\rho^{\gamma-1}\rho^{\epsilon}\nabla\cdot u\,\mathrm{d}x+\int\rho^{\epsilon}u\cdot\nabla\rho^{\gamma-1}\,\mathrm{d}x.\end{aligned} \tag{3.88}$$

For R_2 we finally end up with

$$R_2=(\gamma-1)\int\left[\rho^{\gamma-1}\rho^{\epsilon}-\frac{\gamma-1}{\gamma}\rho^{\gamma}\right]\nabla\cdot u\,\mathrm{d}x+\int\rho^{\epsilon}u\cdot\nabla\rho^{\gamma-1}\,\mathrm{d}x. \tag{3.89}$$

For the third term R_3 one can easily find an estimate where $R_3\to 0$ as $\epsilon\to 0$, so we omit the details here. Putting our results together yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(H^{\epsilon}(t)-G^{\epsilon}(t)\right)=&-\int\left(\rho^{\epsilon}(u^{\epsilon}-u)\otimes(u^{\epsilon}-u)\right):\nabla u\,\mathrm{d}x\\ &\frac{\epsilon^2}{4}\int\left(\nabla\sqrt{\rho^{\epsilon}}\otimes\nabla\sqrt{\rho^{\epsilon}}\right):\nabla u\,\mathrm{d}x\\ -(\gamma-1)\int&\left[\frac{1}{\gamma}((\rho^{\epsilon})^{\gamma}-\rho^{\gamma})-\rho^{\gamma-1}(\rho^{\epsilon}-\rho)\right]\nabla\cdot u\,\mathrm{d}x+o(1).\end{aligned} \tag{3.90}$$

Combining the above estimates yield the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H^{\epsilon}(t)-G^{\epsilon}(t)\right)=\|\nabla u\|_{L^{\infty}(\mathbb{R}^n)}H^{\epsilon}(t)+o(1) \tag{3.91}$$

for $t\in[0,T_*)$. From a integration with respect to t we obtain

$$H^{\epsilon}(t)\le H^{\epsilon}(0)+G^{\epsilon}(0)-G^{\epsilon}(t)+C_1\int_0^t H^{\epsilon}(\tau)\,\mathrm{d}\tau+o(1). \tag{3.92}$$

In addition one can show that $G^{\epsilon}(0)-G^{\epsilon}(t)=o(1)$ and this implies

$$H^{\epsilon}(t)\le H^{\epsilon}(0)+C_1\int_0^t H^{\epsilon}(\tau)\,\mathrm{d}\tau+o(1). \tag{3.93}$$

Employing the initial condition $H^{\epsilon}(0)$ and the Grönwall identity one can show that $H^{\epsilon}(t)\to 0$ as $\epsilon\to 0$ and one deduces that

$$\int\rho|\rho^{\epsilon}-u|^2\,\mathrm{d}x\to 0,\quad \int\Theta(\rho^{\epsilon},\rho)\,\mathrm{d}x\to 0 \tag{3.94}$$

as $\epsilon \to 0$. It can easily been shown that

$$\frac{1}{\gamma}|\rho^\epsilon - \rho|^\gamma \leq \Theta(\rho^\epsilon, \rho) \quad \text{if} \quad \gamma \geq 2. \tag{3.95}$$

Hence $\|\rho^\epsilon - \rho\|_{L^\gamma(\mathbb{R}^n)} \to 0$ as $\epsilon \to 0$, this proves the identity (3.66). On the other hand the triangle inequality and the Hölder inequality imply

$$\begin{aligned}
&\|\rho^\epsilon u^\epsilon - \rho u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^n)} \\
&\qquad \leq \|\rho^\epsilon u^\epsilon - \rho^\epsilon u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^n)} + \|(\rho^\epsilon - \rho)u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^n)} \\
&\qquad \leq \left\|\sqrt{\rho^\epsilon}\right\|_{L^{2\gamma}(\mathbb{R}^n)} \left\|\frac{1}{\sqrt{\rho^\epsilon}}(\rho^\epsilon u^\epsilon - \rho^\epsilon u)\right\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^n)} \\
&\qquad + \|\rho^\epsilon - \rho\|_{L^\gamma(\mathbb{R}^n} \|u\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^n)}.
\end{aligned} \tag{3.96}$$

This proves the assertion in (3.67). □

3.2.3 The WKB method and the Wigner transformation

In the last sections we did not present the whole theory behind the semiclassical and nonrelativistic limit of the KGE since that would have led us to discussions which lie far beyond the scope of this work. A widely used method for semiclassical limits is the *WKB method*, which is valid for short times, until the ansatz breaks down (or multi-valued solutions are used, see [16]). Global in time semiclassical limits can be obtained by the *Wigner transform method* – see eg. [15] for the combined semiclassical and nonrelativistic limit from Dirac-Maxwell to Vlasv-Poisson. For the sake of completeness we will give an overview. We follow the discourse in [16] and [17].

WKB method: What we have considered is the so called WKB ansatz, which is generally written as

$$\phi^\epsilon = A^\epsilon \exp\left(\frac{iS}{\epsilon}\right). \tag{3.97}$$

here A^ϵ and S are real valued, $\epsilon \sim \hbar$ and the superscript ϵ of A indicates that $A^\epsilon = A + a_1\epsilon + A_2\epsilon^2 + \ldots$. For the semiclassical limit we send $\epsilon \to 0$. However, the WKB method is only justified for smooth initial data A_I and S_I in

$$\phi_I^\epsilon = A_I \exp\left(\frac{iS_I}{\epsilon}\right). \tag{3.98}$$

The problem with this ansatz is that the obtained hydrodynamical equations do not need to have global solutions (global solutions can only be obtained for special initial data), i.e. the system can break down in some finite time $t_f \in \mathbb{R}$. These singularities are called *focal points* or more general *caustics* and the energy of the wave function ϕ^ϵ will become unbounded there.

Wigner method: Next we turn our attention to the *Wigner method*. We begin with the following definition.

Definition 3.2.1. *For given $f, g \in \mathcal{S}'(\mathbb{R}^d)$ and given $\epsilon \in (0, \epsilon_0]$ we define the Wigner transformation on the scale ϵ by*

$$w^\epsilon(f,g)(x,t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f\left(x - \epsilon\frac{z}{2}\right) \bar{g}\left(x + \epsilon\frac{z}{2}\right) e^{iz\xi}\,\mathrm{d}z. \tag{3.99}$$

The Wigner transformation is a continuous, bilinear map $w^\epsilon : \mathcal{S}'(\mathbb{R}^d_x) \times \mathcal{S}'(\mathbb{R}^d_x) \to \mathcal{S}'(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$. For example the nonlinear Schrödinger equation

$$\epsilon\partial_t\psi^\epsilon - i\frac{\epsilon^2}{2}\Delta\psi^\epsilon + iV((x)\psi^\epsilon = 0, \tag{3.100}$$

becomes

$$\partial_t w^\epsilon + \xi\cdot\nabla_x w^\epsilon - \theta^\epsilon[V]\partial_t w^\epsilon = 0, \quad x, \xi \in \mathbb{R}^d,\ t \in \mathbb{R}, \tag{3.101}$$

where $w^\epsilon := w^\epsilon(\psi^\epsilon, \psi^\epsilon)$ and $\theta^\epsilon[V]\partial_t w^\epsilon$ denotes a pseudo differential operator of the form

$$\theta^\epsilon[V]\partial_t w^\epsilon(x,\xi,t) := \frac{i}{(2\pi)^d\epsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[V\left(x+\epsilon\frac{z}{2}\right) - V\left(x-\epsilon\frac{z}{2}\right)\right]\cdot w^\epsilon(x,\xi,t)e^{iz(\xi-\xi')}\,\mathrm{d}z\,\mathrm{d}\xi'. \tag{3.102}$$

In the case of a free particle, i.e. $V = 0$, one easily sees that the "Wignerized" nonlinear Schrödinger equation becomes a transport type equation

$$\partial_t w^\epsilon + \xi\cdot\nabla_x w^\epsilon = 0, \quad x, \xi \in \mathbb{R}^d,\ t \in \mathbb{R}. \tag{3.103}$$

In the last decades the use of the Wigner transformation became very popular for the analysis of the semiclassical limit of the nonlinear Schrödinger equation. The Wigner transform provides a phase space description which "unfolds" the caustics and therefore leads to global solutions. Additionally one can weaken the regularity assumptions for the initial data A_I and S_I. The disadvantage of the Wigner transform lies in the fact that it describes the problem in the phase space which means that it doubles the dimension of our problem. The authors in [16] presented a proof which shows that Wigner transformations can be used to obtain a semiclassical phase space description.

Appendices

Appendix A

Physical background

A.1 A Heuristic derivation of the Klein Gordon equation

The relativistic energy-momentum relation is given by

$$E^2 - \boldsymbol{p}^2 c^2 = m^2 c^4. \tag{A.1}$$

Where E is the energy of the particle, $\boldsymbol{p}$ its three dimensional momentum, m its mass and c the speed of light. Using the Einstein sum convention and the Minkowski metric with signature $(+, -, -, -)$ we can rewrite this formula as

$$p^\mu p_\mu - m^2 c^2 = g_{\mu\nu} p^\mu p^\nu - m^2 c^2 = 0 \tag{A.2}$$

Where $p = (\frac{E}{c}, \boldsymbol{p})$, $x = (ct, \boldsymbol{x})$ are so called four-vectors and $g_{\mu\nu}$ is the Minkowski metric tensor (more precisely the component of the metric tensor). Due to the correspondence principle from classical quantum physics we make the following replacement

$$p_\mu \to i\hbar\partial_\mu \tag{A.3}$$

where ∂_μ just means the gradient $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ in space-time coordinates and $\hbar$ the reduced Planck constant. The justification for this replacement is that it turns out of Noether's theorem that translations in time cause energy shifts and translations in space change the momentum. Note that translations in time and space are generated by their derivatives. (To be mathematically precise we need to introduce the concepts of the so called position-space and momentum-space and they are related by Fourier transformation. On these spaces we define the position operator and the momentum operator respectively. In fact the Fourier representation of the momentum operator is exactly $i\hbar\partial_\mu$.) In fact we replace the four-vector p_μ by the differential operator $i\hbar\partial_\mu$ and let it act on a scalar function ψ. This is also known as quantisation process and it seems to occur very randomly. Well there is no strict proof for this.

When we apply this to (A.2) we obtain

$$\left(-\hbar^2 \partial^\mu \partial_\mu - m^2 c^2\right)(\psi) = 0 \tag{A.4}$$

which can be expressed as

$$-\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t} + \nabla^2 \psi = \left(\frac{mc}{\hbar}\right)^2 \psi. \tag{A.5}$$

(A.5) is called the Klein Gordon equation and further on we will use the shortcut "KGE" and write

$$\left(\Box - \frac{m^2 c^2}{\hbar^2}\right)(\psi) = 0 \tag{A.6}$$

instead of (A.5). Where $\Box$ is the so called d'Alembert operator and it is defined by

$$\Box \psi = -\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t} + \nabla^2 \psi \tag{A.7}$$

The field ψ corresponds to a particle of spin 0. What precisely is meant by a spin 0 particle is that its corresponding scalar field is invariant under Lorentz transformations, i.e. they transform under the trivial $(0, 0)$ representation of the Lorentz group

A.2 Introduction to the Lagrange formalism

First we need to introduce the so called Hamilton principle. It states that the dynamic of a certain physical system is determined by a variation problem for a functional based on a function called the Lagrange function. The Lagrange function or shorter the Lagrangian contains all physical information concerning the system and the forces acting on it. When we are presenting the theory of the Lagrange formalism we follow the discourse in [5].

The Lagrangian is in general a function which depends on the field ϕ it's four-gradient $\partial_\mu \phi$ and the space-time coordinate x. Note that there is no general analytical way to derive the Lagrangian of a certain physical system. Often energy considerations to approximate the dynamic of the system are used. We denote the Lagrangian by $\mathcal{L}(x, \phi, \partial_\mu \phi)$. Further there is one basic physically reasonable assumption concerning the field ϕ. We assume that the field and its derivatives decay fast enough thus the terms occurring by performing partial integration over an unbounded domain $\Omega \subset \mathbb{R}^4$ have to vanish.

Next we define the action of the physical system to be

$$S[\phi] = \int_\Omega \mathcal{L}(x, \phi, \partial_\mu \phi)\, d^4x. \tag{A.8}$$

the Hamilton principle exactly says

$$\delta S[\phi] = 0. \tag{A.9}$$

Where δ denotes the functional derivative. This is equivalent to say that the action $S[\phi]$ is stationary under a variation in the field

$$\phi(x) \to \phi(x) + \delta\phi(x) = \phi' \quad , \tag{A.10}$$

which vanishes on the boundary $\partial\Omega$ such that

$$\delta\phi(x) = 0, \quad x \in \partial\Omega. \tag{A.11}$$

The physical field configuration in the space-time volume Ω is such that the action S is invariant under variation of the field for fixed boundary conditions. The calculation of the variation of the action yields the Euler-Lagrange equations of motion for the field. Thus we obtain

$$\begin{aligned} \delta S[\phi] &= \int_\Omega \delta\mathcal{L} d^4x \\ &= \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\partial_\mu\phi \right). \end{aligned} \tag{A.12}$$

Hence

$$\begin{aligned} \delta\partial_\mu\phi &= \partial_\mu\phi' - \partial\phi \\ &= \partial_\mu\left(\phi' - \phi\right) = \partial_\mu\delta\phi. \end{aligned} \tag{A.13}$$

Finally a integration by parts yields

$$\begin{aligned} \delta S[\phi] &= \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} \right)\delta\phi + \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\left(\partial_\mu\delta\phi\right) + \partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi \right) \\ &= \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} \right)\delta\phi + \int_\Omega \partial\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi \right). \end{aligned} \tag{A.14}$$

Hence the boundary terms occurring due to integration by parts vanish. The term including the total derivative is zero too, to see this we can use Gauss's divergence theorem to obtain

$$\int_\Omega \partial\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi \right) d^4x = \int_{\partial\Omega} \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi \;\; d^3x = 0, \tag{A.15}$$

since $\delta\phi(x) = 0$ for $x \in \partial\Omega$. So

$$\delta S[\phi] = 0 = \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} \right)\delta\phi \tag{A.16}$$

by Hamilton's principle. Since $\delta\phi$ is arbitrary the integrand has to be identically zero and we obtain

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} = 0. \tag{A.17}$$

These are the Euler-Lagrange equations of motions for the field ϕ.

3.3 Scaling of the Firedmann equation with assumption of homogeneity

A very successful inflation model is determined by the following potential

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2, \tag{3.18}$$

where v is the *vacuum field*. One can derive that

$$v = \pm\frac{m}{\sqrt{\lambda}}, \tag{3.19}$$

with m the mass of the scalar field and λ the coupling of the field. One should take note that

$$[\phi] = L^{-1}, \tag{3.20}$$

$$[m] = L^{-1}. \tag{3.21}$$

Here L stands for a unit length, where λ is a dimensionless parameter. Using the potential V defined as above we have the following set of equations

$$\frac{\dot{a}}{a} - \frac{k}{a^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{3.22}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V(\phi)\right), \tag{3.23}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0. \tag{3.24}$$

Together with the following set of initial data

$$a(0) = 0, \tag{3.25}$$

$$\dot{a}(0) = H_0, \tag{3.26}$$

$$\phi(0) = \phi_0, \tag{3.27}$$

$$\dot{\phi}(0) = \psi_0. \tag{3.28}$$

For a numerical simulation we have four parameters, i.e. $m, \lambda, \phi_0, \psi_0$. But to solve the equations numerically we have to make scale transformations such that the equations become dimensionless. Therefore we introduce the following new variables

$$\tau = H_0 t, \tag{3.29}$$

$$\Phi = \sqrt{\frac{8\pi G}{3}}\phi, \tag{3.30}$$

$$\tilde{m} = \frac{m}{H_0}, \tag{3.31}$$

$$\tilde{\lambda} = \frac{3\lambda}{8\pi G H_0^2}. \tag{3.32}$$

With this new variables we obtain the transformed equations

$$\frac{a''}{a} = -\Phi^2 + V(\Phi), \tag{3.33}$$

$$\Phi'' + 3\frac{a'}{a}\Phi' + \frac{dV}{\Phi} = 0. \tag{3.34}$$

Here the prime stands for the derivative with respect to the dimensionless variable τ.

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