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Abstract

This master's thesis is dedicated to the proof of a local version of Toponogov's triangle comparison theorem for semi-Riemannian manifolds. We show that curvature bounds on semi-Riemannian manifolds imply local triangle comparison and that if, conversely, triangle comparison holds on normal neighborhoods of each point, curvature bounds follow. As a byproduct, the constant curvature model spaces in semi-Riemannian geometry are studied in detail. Moreover, a general form of the law of cosines is derived that applies to surfaces of arbitrary curvature and index and is needed for showing realizability of triangles in the model spaces. The hinge and straightening lemmas of Alexandrov are generalized to the semi-Riemannian setting. A modified distance function and a self adjoint-modified shape operator are introduced, which satisfies a differential equation of Riccati type and a theory for comparison of families of self-adjoint linear maps is developed, which allows us to show the main result. The starting point and inspiration for this thesis is a recent paper by Stephanie B. Alexander and Richard L. Bishop titled "Lorentz and Semi-Riemannian Spaces with Alexandrov Curvature Bounds" [1].

Abstract

Diese Masterarbeit beschäftigt sich mit einer verallgemeinerten lokalen Version von Toponogov's Satz für Dreiecksvergleiche auf semi-Riemannsche Mannigfaltigkeiten. Es wird gezeigt, dass Krümmungsschranken die Dreiecksvergleichseigenschaft implizieren, und umgekehrt, wenn die Dreiecksvergleichseigenschaft für alle Punkte in einer Umgebung für Riemannsche Normalkoordinaten erfüllt ist, folgen die entsprechenden Krümmungsschranken. Diese Arbeit beinhaltet eine detaillierte Untersuchung der semi-Riemannschen Modellräume konstanter Krümmung. Außerdem wird eine allgemeine Version des Kosinussatzes bewiesen, die für Flächen mit beliebiger konstanter Krümmung und Index gilt. Dieser Satz ist essentiell für den Beweis der Realisierbarkeit von Dreiecken in den Modellräumen. Klassische Sätze aus der Alexandrov-Geometrie werden auf den semi-Riemannschen Fall verallgemeinert. Weiters wird eine modifizierte Distanzfunktion eingeführt und der dadurch induzierte modifizierte Formoperator, eine selbstadjungierte lineare Abbildung, welche eine Riccati-Differentialgleichung erfüllt. Die notwendige Theorie zum Vergleich von Familien selbstadjungierter linearer Abbildungen wird entwickelt, was uns erlaubt, das Hauptresultat zu beweisen. Ausgangspunkt und Inspiration für diese Arbeit ist ein Artikel von Stephanie B. Alexander und Richard L. Bishop mit dem Titel "Lorentz and Semi-Riemannian Spaces with Alexandrov Curvature Bounds" [1].

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Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

In the field of triangle comparison, one studies same sized triangles in two different Riemannian manifolds and relates the information so obtained to properties of the Riemannian manifolds. This approach is particularly fruitful if one of the two spaces is some kind of model space, where much is known. In flat Euclidean space, triangles are given by three points connected by straight lines, which are the unique geodesics connecting the vertices. So to generalize triangles to Riemannian manifolds, one needs uniqueness for the geodesics connecting the vertices. This is guaranteed in normal (convex) neighborhoods. The property that we will be interested in is sectional curvature. In terms of curvature the simplest spaces, our model spaces, are spaces of constant curvature. One obtains a model space for any real number K, and qualitatively it is sufficient to study the cases $K < 0$, $K = 0$ and $K > 0$. The model spaces for these three cases are given by the hyperbolic space ($K < 0$), the Euclidean space ($K = 0$), and the sphere ($K > 0$), respectively. So if we consider a triangle Δ in the plane and compare it to a triangle on the sphere, which has the same side lengths as Δ, a so called comparison triangle, then one makes the observation

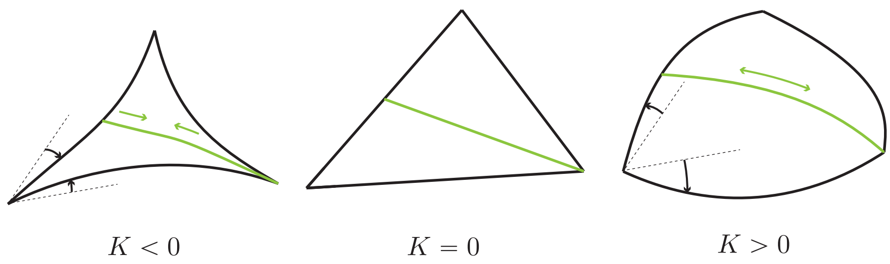

Figure 1.1: Triangles with the same side lengths realized spaces of different curvature K.

that the comparison triangle is "fatter" than the original triangle in the plane, and that the angles at the vertices increase. On the other hand, in hyperbolic space, the comparison triangle is "slimmer" and angles decrease. The statement of Toponogov's (triangle comparison) theorem, in a non rigorous form, is as follows:

Theorem 1.1.1 (Toponogov)**.** *Let M be a Riemannian manifold with sectional curvature $\mathcal{K}$ bounded below by some $\delta \in \mathbb{R}$, $\delta \leqslant \mathcal{K}$, then every triangle Δ in M has a comparison triangle Δ_δ in a space of constant curvature δ and the angles at the vertices of Δ are bigger than the corresponding angles in Δ_δ. If M has sectional curvature bounded above by some $\kappa \in \mathbb{R}$, then every sufficiently small triangle has a comparison triangle Δ_κ in a space of constant curvature κ and the angles at the vertices of Δ are smaller than the corresponding angles in Δ_κ.*

A proof of Toponogov's theorem can be found in [2, 4.1,4.2]. Equivalently one sees that distances between points on the sides of Δ are bigger than distances between the corresponding points on Δ_δ, respectively in case of upper curvature bounds we get that distances between points on the sides of Δ are less then the distances between the corresponding points on Δ_κ. This master's thesis is inspired by a recent paper by Stephanie B. Alexander and Richard L. Bishop titled "Lorentz and Semi-Riemannian Spaces with Alexandrov Curvature Bounds" [1], which generalizes Toponogov's theorem in two substantial ways. First, triangles in semi-Riemannian manifolds are considered, where the length is given by a signed distance, which accounts for the causal character of a geodesic segment connecting two points. Secondly, also a local converse of Toponogov's theorem is show, which is as follows: suppose in a normal coordinate neighborhood U all triangles Δ together with comparison triangles $\tilde{\Delta}$ exhibit the same behavior as one might expect from triangles in a space with curvature bounded above by some K, then the sectional curvature of M is bounded above by K on U. So if this holds for all points with corresponding normal coordinate neighborhoods, then the curvature of M is bounded above by K and the same hold for bounds from below. For this one needs an extended family of model spaces of constant curvature, in particular model spaces with non zero index, which will be introduced in Chapter 1 and studied in depth in Chapter 3. In particular, we will prove a general form of the law of cosines for our model surfaces and show realizability for comparison triangles. Our main tool, which helps us to compare triangles in M with triangles in model spaces will be a so called modified distance function and the modified shape operator induced by it, which is the main topic of Chapter 4. These modified shape operators are self-adjoint linear maps on the tangent space and along geodesics one gets families of these maps. The comparison theory of families of self-adjoint linear maps which we need is developed in Chapter 2. This chapter can be read independently of the rest of the text. Then the main result is shown in Chapter 5. This thesis is written such that no previous knowledge in comparison geometry is needed. It is assumed that the reader knows the basics of Riemannian geometry as is presented in [3] or [4], and has some familiarity with semi-Riemannian geometry, see [5], which will be our main reference.

1.2 Riemannian Curvature Tensor and Curvature Bounds

A ***semi-Riemannian manifold*** is a smooth manifold M together with a scalar product denoted by $\langle\ ,\ \rangle$. A scalar product or ***semi-Riemannian metric*** is defined as a non-degenerate symmetric bilinear form $T_pM \times T_pM \to \mathbb{R}$ which is smoothly defined on M and is a $\binom{0}{2}$ tensor. If $\langle\ ,\ \rangle$ is positive definite we call $\langle\ ,\ \rangle$ an ***inner product*** or ***Riemannian metric***. Then M together with an inner product $\langle\ ,\ \rangle$ is called a ***Riemannian manifold***. A subspace of the tangent space is called ***non-degenerate*** if the restriction of the metric to this subspace is again a non-degenerate bilinear form. For Riemannian manifolds this is always the case, while in the indefinite case there are always sections on which the metric is degenerate. On any semi-Riemannian manifold there is a unique connection compatible with the metric, called the Levi-Civita connection. Let ∇ be the Levi-Civita connection for M, then the ***Riemannian-curvature tensor*** $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ is defined by

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z, \tag{1.1}$$

where $[X, Y]$ denotes the Lie-bracket and $[\nabla_X, \nabla_Y]Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z$. Further we set $R(X, Y, Z, W) := \langle R_{XY}Z, W\rangle$ and we call R_{XY} the ***curvature operator***. As usual, one has to be aware that there are different sign conventions for the Riemannian-curvature tensor. So objects that are derived from the Riemannian curvature tensor may have different signs in the literature. Here we use the same convention as in O'Neil's book [5], which will be our main reference for facts concerning semi-Riemannian geometry.

Definition 1.2.1. *Let M be a semi-Riemannian manifold and Π a non-degenerate plane in T_pM with basis given by $v, w \in T_pM$. The* **sectional curvature** *$\mathcal{K}$ of Π is defined as*

$$\mathcal{K}(v, w) := \frac{R(v, w, v, w)}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2} \tag{1.2}$$

One can easily show that this definition is independent of the choice of basis for Π. For details see [5, p. 77]. We call a non-degenerate plane Π in T_pM ***spacelike***, if the restriction of the metric to Π is positive or negative definite. If the restriction is indefinite we call it ***timelike***. In Riemannian geometry curvature bounds from below or above are defined by one sided bounds on the sectional curvature by some number $K \in \mathbb{R}$. For semi-Riemannian manifolds with indefinite metric, it turns out that one sided bounds force the curvature to be constant, which is shown by the following theorem. Sectional curvature is not defined for degenerate planes, but we can still compute $\langle R_{vw}v, w\rangle$. By the proof of [5, Lemma 3.39 p. 77] we know that $\langle R_{vw}v, w\rangle$ is independent of the choice of basis up to a positive factor. So we get a well defined function $\mathcal{N}$ which maps degenerate planes of T_pM to $\{-1, 0, 1\}$, corresponding to the sign of $\langle R_{vw}v, w\rangle$ and to 0 if this expression vanishes.

Theorem 1.2.2 (Kulkarni et al.). *[5, Proposition 8.28] Let M be a semi-Riemannian manifold with indefinite metric, then for T_pM the following are equivalent*

- *The sectional curvature $\mathcal{K}$ is constant*

- $\mathcal{N} = 0$
- *$a \leqslant \mathcal{K}$ or $\mathcal{K} \leqslant b$ for $a, b \in \mathbb{R}$*
- *$a \leqslant \mathcal{K} \leqslant b$ on timelike planes*
- *$a \leqslant \mathcal{K} \leqslant b$ on spacelike planes*

Proof. See [5]. □

Thus in the indefinite case, manifolds which satisfy one sided bounds on sectional curvature turn out to be basically just the model spaces of constant curvature, which will be discussed soon. The following definition as used in [1, p. 252] turned out to be better suited for the indefinite case.

Definition 1.2.3 (Semi-Riemannian Curvature Bounds)**.** *Let K be a real number. The* **sectional curvature** *$\mathcal{K}$ of a semi-Riemannian manifold M is said to be* **bounded above by** *K if $\mathcal{K} \leqslant K$ for all spacelike planes and $\mathcal{K} \geqslant K$ for timelike planes. For short, we write $\mathcal{R} \leqslant K$. The* **sectional curvature** *$\mathcal{K}$ is said to be* **bounded below by** *K if $\mathcal{K} \geqslant K$ for all spacelike planes and $\mathcal{K} \leqslant K$ for timelike planes. In short we write $\mathcal{R} \geqslant K$.*

Remark

1. For Riemannian manifolds this is just the usual definition for one sided curvature bounds.
2. Note that in general $\mathcal{R} \leqslant K \leqslant K'$ respectively $\mathcal{R} \geqslant K \geqslant K'$ does not imply $\mathcal{R} \leqslant K'$ or $\mathcal{R} \geqslant K'$. This implication is only true for Riemannian manifolds.

Lemma 1.2.4. *Having an upper bound on the sectional curvature $\mathcal{R} \leqslant K$ as in Definition 1.2.3 is equivalent to*

$$R(v, w, v, w) \leqslant K(\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2) \tag{1.3}$$

for all p in M, and all $v, w \in T_pM$ spanning a non-degenerate plane. In the case of a lower bound $\mathcal{R} \geqslant K$ the above inequality reverses.

Proof. First suppose M has curvature bounded above by K then

$$\mathcal{K}(v, w) = \frac{R(v, w, v, w)}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2} \leqslant K \text{ or } \geqslant K$$

where $\leqslant$ holds if v and w span a spacelike plane Π, and $\geqslant$ if the span is timelike. In the spacelike case, both v and w are spacelike or both are timelike. Since the sectional curvature $\mathcal{K}$ does not depend on the choice of basis for Π, we can suppose that v and w are orthogonal. Then the denominator on the left hand side is positive[1], so by bringing it to the other side we do not change the inequality. In the indefinite case, w.l.o.g. suppose that v is timelike and orthogonal to w, which is spacelike. So $\langle v, v\rangle\langle w, w\rangle < 0$, which shows that the denominator is < 0 thus multiplication changes the direction of the inequality. □

[1] If v and w are spacelike and span a spacelike plane this also follows by the Cauchy-Schwarz inequality. This does not hold if the metric restricted to the plane spanned by v and w is not positive definite. For example in $\mathbb{R}^2_1$ the spacelike vectors $v = (1, \frac{1}{2})$ and $w = (1, 0)$ are a basis of $\mathbb{R}^2_1$ and $\langle v, w\rangle^2 = 1 > \frac{3}{4} = \langle v, v\rangle\langle w, w\rangle$.

1.3 Gradient and Hessian

In this section we will take a look at the gradient and Hessian of smooth functions on a semi-Riemannian manifold M and prove some properties which will be used frequently later on. First, recall that a ***geodesic*** is a smooth curve $c : I \to M$, where I is some interval, which satisfies the geodesic equation

$$\nabla_{c'} c' = 0. \tag{1.4}$$

If I is of the form $I = [a, b]$ with $a, b \in \mathbb{R}$, $c(a) := q$ and $c(b) := p$, then we call c ***geodesic segment*** connecting q with p. If $b = \infty$, then c is called a ***geodesic ray*** starting at q, if further $b = \infty$, c is called a ***geodesic line***. By the ***causal character*** of a geodesic we mean the causal character of $c'(t)$, which is the same for all times $t \in I$ since

$$\frac{d}{dt}\langle c', c'\rangle = 2\langle \nabla_{c'} c', c'\rangle \overset{(1.4)}{=} 0. \tag{1.5}$$

So $\langle c', c'\rangle$ is constant and $\langle c', c'\rangle = 0$ if and only if c is a null geodesic. Thus the ***sign*** of a geodesic σ is well defined by setting

$$\operatorname{sgn}(\sigma) := \begin{cases} +1 & \text{if } \langle \sigma', \sigma'\rangle > 0 \\ \ \ 0 & \text{if } \langle \sigma', \sigma'\rangle = 0 \\ -1 & \text{if } \langle \sigma', \sigma'\rangle < 0. \end{cases} \tag{1.6}$$

Definition 1.3.1. *Let $f \in \mathcal{C}^\infty(M)$, then the* **gradient** $\operatorname{grad} f \in \mathfrak{X}(M)$ *is defined as the metrically equivalent vector field to the differential $df \in \Omega^1(M)$. So the gradient is characterized by*

$$\langle \operatorname{grad} f, X\rangle = df(X) = Xf$$

for all $X \in \mathfrak{X}(M)$. For $p \in M$, denote by $\operatorname{grad}_p f$ the value of the gradient vector field at p.

Lemma 1.3.2. *Given $f \in \mathcal{C}^\infty(M)$ and a smooth function $g : \mathbb{R} \to \mathbb{R}$, then*

$$\operatorname{grad}(g \circ f) = g' \operatorname{grad} f.$$

Proof. Note that $g \circ f$ is a smooth function on M. So given $X \in \mathfrak{X}(M)$, by definition of the gradient together with the chain rule for the differential we get

$$\langle \operatorname{grad}(g \circ f), X\rangle = d(g \circ f)(X) = dg(df(X)) = g'\langle \operatorname{grad} f, X\rangle.$$

Since X was arbitrary we are done. □

Definition 1.3.3. *The* **Hessian** H^f *of a function $f \in \mathcal{C}^\infty(M)$ is the second covariant derivative of f, $H^f := \nabla\nabla f$. As shown in [5, Lemma 3.49, p. 86], the Hessian is a symmetric $\binom{0}{2}$ tensor field, and for all vector fields $X, Y \in \mathfrak{X}(M)$ it holds that*

$$H^f(X, Y) = XYf - (\nabla_X Y)f = \langle \nabla_X \operatorname{grad} f, Y\rangle. \tag{1.7}$$

Lemma 1.3.4. *Let c be a geodesic and $f \in \mathcal{C}^\infty(M)$, then along c*

$$H^f(c', c') = (f \circ c)''. \tag{1.8}$$

Proof. This follows from

$$(f \circ c)'' = \nabla_{c'}(f \circ c)' = \nabla_{c'} df(c')$$
$$= \nabla_{c'}\langle \operatorname{grad} f, c'\rangle = \langle \nabla_{c'} \operatorname{grad} f, c'\rangle + \langle \operatorname{grad} c', \underbrace{\nabla_{c'} c'}_{=0}\rangle \overset{(1.7)}{=} H^f(c', c').$$

□

One may ask if some vector field V is of the form $V = \operatorname{grad} f$, for some smooth function f. The next lemma shows that we can find such a f locally, if V satisfies some additional condition.

Lemma 1.3.5. *Let $V \in \mathfrak{X}(M)$ satisfy*

$$\langle \nabla_X V, Y\rangle = \langle X, \nabla_Y V\rangle$$

for all $X, Y \in \mathfrak{X}(M)$. Then locally $V = \operatorname{grad} f$ for some $f \in C^\infty(M)$.

Proof. First, for $X, Y \in \mathfrak{X}(M)$ we know that

$$X\langle V, Y\rangle = \langle \nabla_X V, Y\rangle + \langle V, \nabla_X Y\rangle$$
$$Y\langle X, V\rangle = \langle \nabla_Y X, V\rangle + \langle X, \nabla_Y V\rangle.$$

Subtracting the second equation from the first one, then since by assumption $\langle \nabla_X V, Y\rangle - \langle X, \nabla_Y V\rangle = 0$, we obtain

$$X\langle V, Y\rangle - Y\langle X, V\rangle = \langle V, \underbrace{\nabla_X Y - \nabla_Y X}_{=[X,Y]}\rangle.$$

Let φ be a chart defined on some neighborhood U in M, where we may assume U to be smoothly contractible to some point $p_0 \in M$. Let $\partial_1, \dots, \partial_n$ be the coordinate vector fields for this chart. Since $[\partial_i, \partial_j] = 0$ for all $i, j \in \{1, \dots, n\}$, inserting the coordinate vector fields into the above equation shows

$$\partial_i\langle V, \partial_j\rangle - \partial_j\langle \partial_i, V\rangle = 0.$$

So for $V = \sum_i V^i \partial_i$, by expanding the above we get

$$\partial_i\underbrace{(\sum_k V^k g_{kj})}_{=:\omega_j} - \partial_j\underbrace{(\sum_k V^k g_{ki})}_{=:\omega_i} = 0.$$

where $\omega_1, \dots, \omega_n$ are the coefficients of the metrically equivalent one-form to V. Then, since $\partial_i \omega_j = \partial_j \omega_i$ we get by [6, Corollary 7.18 p. 306] that $\omega_i = \partial_i f$ for some $f \in C^\infty(U)$ and thus

$$V^l = \sum_{i=1}^n g^{li}\omega_i = \sum_{i=1}^n g^{li}\partial_i f = (\operatorname{grad} f)^l,$$

so we are done. □

1.4 Semi-Riemannian Hypersurfaces

In this section we will collect some results about semi-Riemannian Hypersurfaces, which are semi-Riemannian submanifolds of codimension one. The next pages are based on [5, p. 97-108]. In the following, V and M will be manifolds with $V \subset M$ and inclusion map $j : V \to M$. Further, M is assumed to have dimension n and to be equipped with a semi-Riemannian metric denoted by g_M. We call V a ***submanifold*** of M if it is a topological subspace and the inclusion map j is smooth with injective differential $T_p j$ for all points p of V. If the pullback metric $j^* g_M =: g_V$ induces a semi-Riemannian metric on V, we call V a ***semi-Riemannian submanifold*** of M. If the dimension of V is $n-1$ it is called a ***semi-Riemannian hypersurface***. We can write $T_pM = T_pV \oplus T_pV^\perp$. The ***sign*** of a semi-Riemannian hypersurface V in M is defined by

- 1 if $\langle v, v\rangle > 0$ for all v in $T_pV^\perp$
- -1 if $\langle v, v\rangle < 0$ for all v in $T_pV^\perp$.[2].

The next proposition shows that semi-Riemannian hypersurfaces can be constructed as preimages of smooth maps with non-vanishing gradient.

Proposition 1.4.1. *[5, Proposition 4.17 , p. 106] Let $f \in \mathcal{C}^\infty(M)$ be a smooth function on M, and c a value of f. Then $V := f^{-1}(c)$ is a semi-Riemannian hypersurface of M, if and only if $\langle \operatorname{grad} f, \operatorname{grad} f\rangle$ is > 0 or < 0 on V. Then the sign of V is equal to the sign of* grad f. *A unit normal vector field to V is given by $\frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}$ and $f^{-1}(c)$ is called a* **level set** *of f.*

Proof. See [5, p. 106] □

Let ∇ be the Levi-Civita connection on M, then for X and Y in $\mathfrak{X}(V)$ the Levi-Civita connection $\tilde{\nabla}$ on V is given by orthogonal projection onto TV, so

$$\tilde{\nabla}_X Y := \tan(\nabla_\mathrm{X} \mathrm{Y})$$

and we call $\tilde{\nabla}$ the ***induced connection*** on V. The ***second fundamental form*** $II : \mathfrak{X}(V) \times \mathfrak{X}(V) \to \mathfrak{X}(V)^\perp$ is defined by by $\mathrm{II(X, Y)} := \mathrm{nor}((\nabla_\mathrm{X}\mathrm{Y}))$. So the induced connection $\nabla : \mathfrak{X}(V) \times \mathfrak{X}(V) \to \bar{\mathfrak{X}}(V)$ can be written as

$$\nabla_X Y = \tilde{\nabla}_X Y + \mathrm{II(X, Y)}\,.$$

For more details see [5, p. 97-102].

Definition 1.4.2. *The* **shape operator** *or* **Weingarten-map** *W of a semi-Riemannian hypersurface V in M with unit normal vector field U is defined by*

$$\langle W(X), Y\rangle := -\langle \mathrm{II(X, Y)}, U\rangle \text{ for all } X, Y \in \mathfrak{X}(V).$$

A unit normal U exists at least locally and the shape operator is then uniquely determined up to the sign. Here we use the opposite sign convention as in [5].

[2]If the sign is 1 the co-index is 0, so the index of V is the same as the index of M. For sign -1, the co-index of V is 1, so the index of V is -1 the index of M.

Lemma 1.4.3. *Let M be a semi-Riemannian manifold and V a semi-Riemannian hypersurface of M. If U is a unit normal vector field to V (locally), then the shape operator of V is given by*

$$W(X) = \nabla_X U, \tag{1.9}$$

where ∇ is the Levi-Civita connection of M. At each point p in V where W is defined it gives a self adjoint linear map on T_pV.

Proof. Since $\langle U, U\rangle = \pm 1$ we compute $0 = \nabla_X \langle U, U\rangle = 2\langle \nabla_X U, U\rangle$ so $\nabla_X U$ is in the tangent space of V for all X in $\mathfrak{X}(V)$. Now for $Y \in \mathfrak{X}(V)$ first note that since $\langle Y, U\rangle = 0$ we get

$$0 = \nabla_X \langle Y, U\rangle = \langle \nabla_X Y, U\rangle + \langle Y, \nabla_X U\rangle.$$

Thus $\langle \nabla_X Y, U\rangle = -\langle Y, \nabla_X U\rangle$. So

$$\langle W(X), Y\rangle = -\langle \mathrm{II(X, Y)}, U\rangle = -\langle \nabla_X Y, U\rangle = \langle \nabla_X U, Y\rangle,$$

which shows (1.9). Since II is symmetric it follows that W is self adjoint. □

1.5 Exponential Map, Normal Neighborhoods and Gauss-Lemma

Let q be a point in a semi-Riemannian manifold M and $v \in T_qM$, then there exists an unique geodesic $\gamma_v : I \to M$ with $\gamma_v(0) = q$ and $\gamma_v'(0) = v$, and such that the interval I is maximal. Set $\mathcal{D}_q := \{v \in T_qM \mid \gamma_v \text{ is at least defined on } [0, 1]\}$, then $\mathcal{D}_q$ is an open subset of T_qM. The ***exponential map*** at q, $\exp_q : \mathcal{D}_q \to M$, is defined by $\exp_q(v) := \gamma_v(1)$.

Proposition 1.5.1. *The set $\mathcal{D}_q$ is an open star-shaped subset of T_qM and $\exp_q$ is smooth on $\mathcal{D}_p$. Further,*

$$\gamma_{tv}(1) = \gamma_v(t) \tag{1.10}$$

for all t such that both sides are defined. There exists a star-shaped neighborhood $\tilde{U}$ of 0 in T_qM such that the exponential map $\exp_q : \tilde{U} \to U$ is a diffeomorphism.

Proof. See [5, Proposition 3.30. p. 71]. □

Let U and $\tilde{U}$ be star-shaped sets as in Proposition 1.5.1 such that the exponential map is a diffeomorphism. Then U is called a ***normal coordinate neighborhood*** of p in U. If U is a normal coordinate neighborhood of all of its points we call U ***normal neighborhood***[3]. The existence of normal neighborhoods is guaranteed by [5, Proposition 5.7, p. 130]. Given an coordinate basis $\partial_1, \ldots, \partial_n$ for T_qM and U a normal coordinate neighborhood of $q \in M$, then by Proposition 1.5.1 every point p in U has a unique representation as

$$\exp_q^{-1}(p) = \sum_i x^i(p)\partial_i$$

and $(x^1, \ldots, x^n)$ are called ***Riemannian normal coordinates***. The next lemma collects some important properties of the exponential map.

[3]In literature different names are used for normal neighborhoods, like totally normal, convex, geodesically convex or uniformly normal [3]. To be consistent with [1] we will use the name normal neighborhood.

Lemma 1.5.2. *Let U be normal coordinate neighborhood of $q \in M$, then for $p \in U$ the following holds*

(i) There exists a unique geodesic $\gamma_{qp} : [0,1] \to M$ connecting q with p and $\gamma'_{qp}(0) = \exp_q^{-1}(p)$.

(ii) Let $v = \sum_i v^i \partial_i \in T_qM$ for an basis $\partial_1, \ldots, \partial_n$ then in normal coordinates the geodesic γ_v has the form

$$t \mapsto (tv^1, \ldots, tv^n)$$

for all t such that $\gamma_v(t)$ is in U.

Proof. To show (i) set $v := \exp_q^{-1}(p)$ and let γ_v be the unique geodesic such that $\gamma_v(0) = q$ and $\gamma'_v(0) = v$. Then $\gamma_v(1) = \exp_q(\exp_q^{-1}(p)) = p$. Now (ii) holds since $\exp_q^{-1}(\gamma_v(t)) = \exp_q^{-1}(\exp_q(tv)) = tv = \sum_{i=1}^n tv^i \partial_i$.

□

Definition 1.5.3. *The image of a line through* 0 *in $\mathcal{D}_q$ under the exponential map* $\exp_q$ *is called a* **radial geodesic** *or* **geodesic line** *through q.*

Along radial directions the exponential map behaves especially nice, as the following important and well known result shows.

Theorem 1.5.4 (Gauss-Lemma)**.** *Let M a semi-Riemannian manifold, $p \in M$, and $0 \neq x \in \mathcal{D}_p \subset T_pM$. Then given $v_x, w_x \in T_x(T_pM)$ with v_x radial, where radial means $v_x = \lambda x$ for $0 \neq \lambda \in \mathbb{R}$, it holds that*

$$\langle v_x, w_x \rangle = \langle (\,d\exp_p)_x(v_x), (\,d\exp_p)_x(w_x) \rangle.$$

Thus the exponential map is an isometry along radial geodesics.

Proof. See [5, Lemma 5.1, p. 127] or [4, Theorem 2.1.21, p. 43]. □

1.6 Pregeodesics and Affine Parametrizations

Recall that a smooth curve $c : I \to M$ is called a ***pregeodesic*** if it has a ***reparametrization*** $\theta : J \to I$ such that $c \circ \theta$ is a geodesic. Here, θ is assumed to be a strictly-monotonic smooth map between intervals J and I. We will show a lemma which will help us to detect if some curve is a pregeodesic. To show this, we need the following result.

Lemma 1.6.1. *[5, exercise 3, p. 93] Let $c : I \to M$ be a smooth curve and $\theta : J \to I$ be a reparametrization of c. Let $Z \in \mathfrak{X}(c)$ be a vector field along c, then*

$$(Z \circ \theta)' = \frac{d\theta}{dt} Z' \circ \theta \tag{1.11}$$

and

$$(Z \circ \theta)'' = \frac{d^2\theta}{dt^2} Z' \circ \theta + (\frac{d\theta}{dt})^2 Z'' \circ \theta. \tag{1.12}$$

Proof. W.l.o.g. assume that c is contained in a chart domain U, then $Z(t) = \sum_k Z^k(t)\frac{\partial}{\partial x^k}|_{c(t)}$. So $Z(\theta(t)) = \sum_k Z^k(\theta(t))\frac{\partial}{\partial x^k}|_{c\circ\theta(t)}$ and $Z \circ \theta \in \mathfrak{X}(c \circ \theta)$. The local formula for the induced covariant derivative [5, p. 66] then yields

$$\begin{aligned}\frac{\nabla}{dt}(Z\circ\theta)^k &= \frac{d(Z^k\circ\theta)}{dt} + \sum_{i,j}\Gamma^k_{ij}Z^i(\theta(t))\frac{d(c^j\circ\theta)}{dt} \\ &= \frac{d\theta}{dt}\left(\frac{dZ^k}{dt}(\theta(t)) + \sum_{i,j}\Gamma^k_{ij}Z^i(\theta(t))\frac{dc^j}{dt}(\theta(t))\right) \\ &= \frac{d\theta}{dt}\frac{\nabla}{dt}Z^i|_{\theta(t)}\end{aligned}$$

which shows (1.11). Applying the product rule for the induced covariant derivative further differentiation yields

$$(Z\circ\theta)'' \overset{(1.11)}{=} (\frac{d\theta}{dt}Z'\circ\theta)' = \frac{d^2\theta}{dt^2}Z'\circ\theta + \frac{d\theta}{dt}(Z'\circ\theta)' \overset{(1.11)}{=} \frac{d^2\theta}{dt^2}Z'\circ\theta + (\frac{d\theta}{dt})^2 Z''\circ\theta$$

and we are done. □

Lemma 1.6.2. *[5, exercise 19, p. 95] Given a smooth curve $c : I \to M$ which is regular* [4]*, if*

$$\nabla_{c'(t)}c'(t) = \lambda(t)c'(t) \tag{1.13}$$

for some smooth function $\lambda : I \to \mathbb{R}$ then c is a pregeodesic and the reparametrized curve $\tilde{c} = c \circ \theta$ is a geodesic if and only if

$$\theta'' + \lambda(\theta)(\theta')^2 = 0.^5 \tag{1.14}$$

Further it holds that $\langle c', c'\rangle = 0$ everywhere or $\langle c', c'\rangle \neq 0$ for all times. If c is a nonnull pregeodesic with $\langle c'(t), c'(t)\rangle \neq 0$ for all $t \in I$, then every constant speed reparametrization of c is a geodesic.

Proof. Let c be a smooth curve such that (1.13) holds. Let $\tilde{c} = c \circ \theta$ be an arbitrary reparametrization of c, then

$$\tilde{c}'' \overset{(1.11)}{=} \theta''(c'\circ\theta) + (\theta')^2(c''\circ\theta) \overset{(1.13)}{=} (\theta'' + \lambda(\theta)(\theta')^2)(c'\circ\theta), \tag{*}$$

so if $\theta'' + \lambda(\theta)(\theta')^2 = 0$ it follows that $\tilde{c}$ is a geodesic since then $\nabla_{\tilde{c}'}\tilde{c}' = \tilde{c}'' = 0$. Since c is regular, if c can be reparametrized as a geodesic then (*) shows that $\theta'' + \lambda(\theta)(\theta')^2$ has to vanish. We can always find such a θ since (1.14) is a second order nonlinear ODE which can be solved at least locally. Further, $\langle c', c'\rangle$ satisfies following ODE

$$\frac{d}{dt}\langle c', c'\rangle = 2\langle c'', c'\rangle = 2\lambda\langle c', c'\rangle,$$

which shows that $\langle c', c'\rangle = ae^{2\int\lambda\,dt}$ for some $a \in \mathbb{R}$. Thus, since the exponential is always greater than zero the claim follows. It remains to show that for a nonnull pregeodesic c, every constant speed reparametrization is a geodesic. So

[4] $c'(t) \neq 0$ for all $t \in I$.

[5] If θ satisfies this equation clearly also $a\theta + b$ is a solution for all $a, b \in \mathbb{R}$.

let θ be such that $\tilde{c} := c \circ \theta$ has constant speed $\langle \tilde{c}', \tilde{c}' \rangle := k \neq 0$. Then we get $\tilde{c}'(t) = \theta'(t)c'(\theta(t))$ and

$$0 = \frac{d}{dt}\langle \tilde{c}', \tilde{c}' \rangle = 2\langle \nabla_{\tilde{c}'}\tilde{c}', \tilde{c}' \rangle.$$

This together with (*) yields

$$0 = \langle \nabla_{\tilde{c}'}\tilde{c}', \tilde{c}' \rangle = \theta'(\theta'' + \lambda(\theta)(\theta')^2)\langle c' \circ \theta, c' \circ \theta \rangle.$$

But since by assumption c is nonnull $\langle c', c' \rangle \neq 0$. Also $\theta' \neq 0$, since θ is assumed to be strictly monotonic. Thus $\theta'' + \lambda(\theta)(\theta')^2 = 0$, which shows that $\tilde{c} = c \circ \theta$ is a geodesic.

□

Remarks

(i) In particular if c is already a geodesic, that is $\lambda = 0$, then any reparametrization $c \circ \theta$ such that $\theta'' = 0$ is again a geodesic, which is solved by $\theta(t) = \alpha t + \beta$ with $\alpha, \beta \in \mathbb{R}$. If $\alpha = 0$ we just get a curve which assigns the same point to any parameter value t. The choice of a affine reparametrization is called ***affine parameter*** for c. If $\alpha > 0$ then the reparametrized curve is traversed in the same direction and in case $\alpha < 0$ the direction is switched.

(ii) We already know that the causal character of geodesics is always the same, above lemma shows that this also holds for pregeodesics.

Definition 1.6.3. *Let c be a pregeodesic in a semi-Riemannian manifold M then a reparametrization $\tilde{c}$ of c such that $\tilde{c}$ is a geodesic is called an* **affine reparametrization** *of c.*

If $c : [a, b] \to M$ a geodesic then we get a parametrization by $[0, 1]$ by setting $\tilde{c}(t) := c((b-a)t+a)$. We will mostly deal with geodesics parametrized by $[0, 1]$. Let $c_1 : [0, 1] \to M$ and $c_1 : [0, 1] \to \tilde{M}$ be smooth curves into semi-Riemannian manifolds M and $\tilde{M}$. Two points on $p_1 \in c_1([0, 1])$ and $p_2 \in c_2([0, 1])$ are called ***corresponding points*** if there exists $t_0 \in [0, 1]$ such that $c_1(t_0) = p_1$ and $c_2(t_0) = p_2$, and we say that p_1 and p_2 have the same ***affine parameter***. If c_1 and c_2 are not parametrized by $[0, 1]$ we can still speak of corresponding points if we take affine reparametrizations of both curves which don't change the direction, such that they are defined on $[0, 1]$. This also explains the name affine parameter.

1.7 Length and Distance in Semi-Riemannian Manifolds

By a ***semi-Euclidean*** vector space we mean a vector space equipped with a non-degenerate symmetric bilinear form. Every semi-Euclidean space is isometric to some $\mathbb{R}^n_\nu$. Since the scalar product can also take negative values, we can't just define the length of a tangential vector as the square root of the scalar product with itself. Since we are interested in the causal character we will use the following definition.

Definition 1.7.1 (Signed Length)**.** *Let V be a semi-Euclidean vector space. Then the* **signed length** *of a vector $v \in V$ is defined as*

$$|v|_\pm := \mathrm{sgn(v)} \sqrt{|\langle v, v\rangle|} \tag{1.15}$$

where the **sign**, $\mathrm{sgn}(.) : V \to \{-1, 1\}$ *is given by*

$$\mathrm{sgn(v)} := \begin{cases} +1 & \text{if } \langle v, v\rangle \geqslant 0 \\ -1 & \text{if } \langle v, v\rangle < 0. \end{cases} \tag{1.16}$$

By **unsigned length** *or* **norm** *we mean $\|v\| := \sqrt{|\langle v, v\rangle|}$, which corresponds to the usual 2-norm in the case of inner product spaces.*

In the positive-definite case we have $|v|_\pm = \|v\|$. In particular since the real line has index $\nu = 0$ signed length there just gives the usual modulus.

Definition 1.7.2. *Let M be a semi-Riemannian manifold, $q, p \in M$ such that p is contained in a normal coordinate neighborhood of q. Then let $\gamma_{qp} : [0, 1] \to M$ be the unique geodesic connecting q with p, that is $\gamma_{qp}(0) = q$ and $\gamma_{qp}(1) = p$. Then by*

$$|qp|_\pm := |\gamma_{qp}|_\pm := |\gamma'_{qp}(0)|_\pm$$

we denote the **signed distance** *between q and p. The* **energy** *of γ_{qp} is defined as*

$$E(\gamma_{qp}) := E_q(p) := \langle \gamma'_{qp}(0), \gamma'_{qp}(0)\rangle.$$

Then the energy $E_q(p)$ is related to the signed length $|qp|_\pm$ by

$$E_q(p) = \mathrm{sgn}(\gamma_{\mathrm{qp}})|\gamma_{qp}|_\pm{}^2 = \mathrm{sgn}(\gamma_{\mathrm{qp}})|qp|_\pm{}^2, \tag{1.17}$$

so they have the same sign. Also if we vary the point p, if the signed distance increases then also the energy increases. By Lemma 1.5.2(i) we see that

$$E_q(p) = \langle \exp_q^{-1}(p), \exp_q^{-1}(p)\rangle, \tag{1.18}$$

which is well defined for all p in a normal coordinate neighborhood of q. By (1.5) we know that

$$E(\gamma_{qp}) = \langle \gamma'_{qp}(t), \gamma'_{qp}(t)\rangle \tag{1.19}$$

for any $t \in [0, 1]$. Let σ be a radial geodesic through q defined on a normal coordinate neighborhood of q. Then there exists a $v \in T_qM$ such that $\sigma(t) = \exp_q(tv)$ and so

$$E_q(\sigma(t)) = \langle \exp_q^{-1}(\sigma(t)), \exp_q^{-1}(\sigma(t))\rangle = t^2\langle v, v\rangle. \tag{1.20}$$

Lemma 1.7.3. *Let U be a normal coordinate neighborhood of a point q, then*

$$E_q(p) = E_p(q). \tag{1.21}$$

for all $p \in U$.

Proof. Set $\gamma_{pq}(t) := \gamma_{qp}(1 - t)$, then $\gamma_{pq} : [0, 1] \to M$ is a smooth curve connecting p with q, further γ_{pq} is geodesic since it is obtained by an affine reparametrization of a geodesic, given by $\theta(t) := 1 - t$. Finally, $\gamma'_{pq} = (\gamma_{qp} \circ \theta)' = -(\gamma'_{qp} \circ \theta)$ and so

$$E_q(p) = \langle \gamma'_{qp}(0), \gamma'_{qp}(0)\rangle = \langle \gamma'_{pq}(1), \gamma'_{pq}(1)\rangle \overset{(1.5)}{=} \langle \gamma'_{pq}(0), \gamma'_{pq}(0)\rangle = E_p(q).$$

□

This shows that if we deal with quantities which only depend on the energy of a curve, it does not matter in which direction the curve is parametrized.

Lemma 1.7.4. *Let $\sigma : I \to M$ be a geodesic parametrized in such a way that $0 \in I \subset \mathbb{R}$ and $\sigma(0) = q$, then for all $\lambda \in I$ it holds that*

$$E_q(\sigma(\lambda)) = \lambda^2 \langle \sigma'(0), \sigma'(0) \rangle. \tag{1.22}$$

Proof. Let $\hat{\sigma}(t) := \sigma(\lambda t)$, then $\hat{\sigma} : [0,1] \to M$ is again a geodesic since it is given by an affine reparametrization of a geodesic and further $\hat{\sigma}(0) = q$, $\hat{\sigma}(1) = \sigma(\lambda)$. Then $\hat{\sigma}'(t) = (\sigma(\lambda t))' = \lambda \sigma'(\lambda t)$ and so

$$E_q(\sigma(\lambda)) = \langle \hat{\sigma}'(0), \hat{\sigma}'(0) \rangle = \lambda^2 \langle \sigma'(0), \sigma'(0) \rangle.$$

□

1.8 Semi-Riemannian Manifolds of Constant Curvature

In this section we will define our model spaces and study their properties. These model spaces are simply-connected semi-Riemannian manifolds with constant sectional curvature. For triangle comparison, the two dimensional model spaces (surfaces) will be of particular importance. Up to isometry these spaces are uniquely determined by their curvature $K \in \mathbb{R}$, dimension n and index $0 \leqslant \nu \leqslant n$. The constant curvature model spaces can be realized as hypersurfaces of $\mathbb{R}^n_\nu$, the so called hyperquadrics.

Constant Curvature Spaces and Space Forms

Definition 1.8.1. *A semi-Riemannian manifold M has* **constant curvature** *if the sectional curvature $\mathcal{K}$ as defined in Definition 1.2.1 is constant.*

In more detail there is a number $K \in \mathbb{R}$ such that, for every $p \in M$ and all $v, w \in T_pM$ for which $\mathcal{K}$ is defined, $\mathcal{K}(v, w) = K$. If M has constant curvature K, then the action of the curvature operator R_{vw} is given by

$$R_{vw}x = K(\langle v, x \rangle w - \langle w, x \rangle v) \tag{1.23}$$

for all $p \in M$ and all $v, w \in T_pM$ spanning a nondegenerate 2-plane. For a proof see [5, Corollary 3.43 p. 80].

Definition 1.8.2 (Space Form)**.** *A* **space form** *is a geodesically complete and connected semi-Riemannian manifold of constant sectional curvature. If we don't want to be specific about the index and dimension we will denote space forms of curvature K by $\mathcal{Q}_K$.*

Proposition 1.8.3. *[5, Proposition 8.23 , p. 227] Simply connected space forms are isometric if and only if their dimension, index and curvature agree.*

So to study properties of space forms, it suffices to work with one realization of constant curvature spaces.

Hyperquadrics

Hyperquadrics can be thought of as generalizations of spheres to $\mathbb{R}^n_\nu$, with the usual spheres S^n in Euclidean space $\mathbb{R}^n$ as a special cases. The unit ***sphere*** in $\mathbb{R}^n$ is given by the equation $\langle x, x\rangle = 1$, and is a Riemannian submanifold of dimension $n-1$, index $\nu = 0$ and constant sectional curvature 1. By scaling the unit sphere S^n by some factor $r > 0$ (the radius) we get spaces of constant curvature $\frac{1}{r^2}$ defined by the equation $\langle x, x\rangle = r^2$. A space of constant negative curvature $K < 0$ is obtained by the equation $\langle x, x\rangle_1 = -r^2$ in $\mathbb{R}^n_1$, which yields the ***hyperbolic space*** of dimension $n-1$, index $\nu = 0$ (the index of the surrounding space minus 1) and constant negative curvature $K = -\frac{1}{r^2}$. In fact we can construct "spheres" of any dimension, index and curvature. First we will collect several facts, which we will need for defining our model spaces. More details can be found in [5, p. 108-114]. Recall that in a vector space V, the tangent space is just a copy of V at each point. Thus the following definition makes sense.

Definition 1.8.4. *Let $e_1, \dots e_n$ be a basis for $\mathbb{R}^n_\nu$, then $x \in \mathbb{R}^n_\nu$ can be written as $x = \sum_i x^i e_i$. The* **position vector field** $\tilde{P} \in \mathfrak{X}(\mathbb{R}^n_\nu)$ *is defined by* $x \mapsto \sum_i x^i \partial_i \in T_x\mathbb{R}^n_\nu \cong \mathbb{R}^n_\nu$. *Further we define the* **quadratic form** $\tilde{\boldsymbol{q}} : \mathbb{R}^n_\nu \to \mathbb{R}$ *by* $\tilde{\boldsymbol{q}} : x \mapsto \langle \tilde{P}(x), \tilde{P}(x)\rangle$.

So the position vector field just takes the position vector of a point $p \in V$ and "moves" it to the tangent space T_pV of p, which is again V.

Lemma 1.8.5. *Let V be a semi-Euclidean vector space. For $\tilde{\boldsymbol{q}}$ and $\tilde{P}$ as in Definition 1.8.4 we have*

$$\operatorname{grad} \tilde{\boldsymbol{q}} = 2\tilde{P}. \tag{1.24}$$

Proof. Given $v \in V$, then by definition of the gradient we get

$$\langle \operatorname{grad} \tilde{\boldsymbol{q}}, v\rangle = d\tilde{\boldsymbol{q}}(v) = v\tilde{\boldsymbol{q}} = v\langle \tilde{P}, \tilde{P}\rangle = 2\langle \nabla_v \tilde{P}, \tilde{P}\rangle = 2\langle v, \tilde{P}\rangle.$$

To see $\nabla_v \tilde{P} = v$, note that the Christoffel symbols Γ^k_{ij} vanish in flat space. Then by the coordinate formula for the induced covariant derivative [5, Proposition 3.13(1), p. 62]

$$\nabla_v \tilde{P} = \sum_i v x^i \partial_i = \sum_i v^i \partial_i = v.$$

Since v was arbitrary the claim follows. □

Now we are able to define hyperquadrics as preimages of the quadratic form.

Definition 1.8.6 (Hyperquadrics)**.** *Fix $r > 0$ and $\varepsilon = \pm 1$, and let $\tilde{\boldsymbol{q}}$ be the quadratic form on $\mathbb{R}^n_\nu$. Then the set $\tilde{\boldsymbol{q}}^{-1}(\varepsilon r^2)$ is called a* **hyperquadric**. *Further set $\tilde{\boldsymbol{q}}^{-1}(0) := \tilde{\Lambda}$, where $\tilde{\Lambda}$ is the null cone*[6] *of $\mathbb{R}^n_\nu$.*

Lemma 1.8.7. *For $n \geqslant 2$, $0 \leqslant \nu \leqslant n$, $r > 0$ and $\varepsilon = \pm 1$ the subsets $\tilde{\boldsymbol{q}}^{-1}(\varepsilon r^2)$ of $\mathbb{R}^{n+1}_\nu$ are semi-Riemannian hypersurfaces of dimension n, unit normal given by $\frac{1}{r}\tilde{P}$ and sign ε. Further, $\tilde{\boldsymbol{q}}^{-1}(0) = \tilde{\Lambda}$ is not a semi-Riemannian hypersurface since the induced metric is degenerate.*

[6]The nullcone $\tilde{\Lambda}$ is defined as the set of all $v \in V$ such that $v \neq 0$ and $\langle v, v\rangle = 0$. Vectors in $\tilde{\Lambda}$ are called null vectors. For index $\nu = 0$ we get $\tilde{\Lambda} = \varnothing$ since by definition 0 is not a null vector.

Proof. See [5, p. 108,109] □

Definition 1.8.8. *Let $n \geqslant 2$ and $0 \leqslant \nu \leqslant n$ and $r > 0$. Then the* **pseudosphere** *of radius r and dimension n is defined as the semi-Riemannian hypersurface of $\mathbb{R}^{n+1}_\nu$ given by*

$$S^n_\nu(r) := \left\{p \in \mathbb{R}^{n+1}_\nu \mid \langle p, p\rangle = r^2\right\} = \tilde{\boldsymbol{q}}^{-1}(r^2).$$

The **pseudohyperbolic space** *of radius r, index ν and dimension n is the semi-Riemannian hypersurface $\mathbb{R}^{n+1}_{\nu+1}$ is given by*

$$H^n_\nu(r) := \left\{p \in \mathbb{R}^{n+1}_{\nu+1} \mid \langle p, p\rangle = -r^2\right\} = \tilde{\boldsymbol{q}}^{-1}(-r^2).$$

Note that $H^n_\nu(r)$ is a subset of $\mathbb{R}^{n+1}_{\nu+1}$, while $S^n_\nu(r)$ lies in $\mathbb{R}^{n+1}_\nu$. For $\nu = 0$ we recover the usual sphere, respectively the hyperbolic space of dimension n. By Proposition 1.4.1 it is easy to see that $S^n_\nu(r)$ and $H^n_\nu(r)$ are semi-Riemannian manifolds. Pseudospheres have constant positive sectional curvature given by $K = \frac{1}{r^2}$, whereas the sectional curvature of pseudohyperbolic spaces is negative and given by $K = -\frac{1}{r^2} < 0$, see [5, Proposition 4.29, p. 113]. There exists an anti isometry between $S^n_\nu(r)$ and $H^n_{n-\nu}(r)$. A map σ between two semi-Riemannian manifolds is called ***anti-isometry*** if $\langle\, d\sigma(v), d\sigma(w)\rangle = -\langle v, w\rangle$ for all v and w. So if a result holds for $S^n_\nu(r)$, we get the analogous result for $H^n_{n-\nu}(r)$ simply by exchanging the words timelike and spacelike. We also know how the geodesics look, and we have criteria for when two points can be connected by a geodesic.

Lemma 1.8.9. *[5, Proposition 4.28, p. 112] Let γ be a nonconstant geodesic in $S^n_\nu(r)$, then γ coincides with one connected component of the intersection of $S^n_\nu(r)$ with a plane through 0. On the other hand, every connected component of such an intersection is a geodesic if it is parametrized with constant speed. In particular:*

1. *If γ is timelike, it is given by a parametrization of one branch of a hyperbola in $\mathbb{R}^{n+1}_\nu$.*

2. *If γ is null, it is given by a straight line in $\mathbb{R}^{n+1}_\nu$.*

3. *If γ is spacelike, it is given by a periodic parametrization of an ellipse in $\mathbb{R}^{n+1}_\nu$.*

The analogous result holds for pseudohyperbolic spaces $H^n_\nu(r)$ with the words timelike and spacelike exchanged.

Proof. [5, p. 112] □

Now that we know what the geodesics look like, it is easy to explicitly write down the exponential map for hyperquadrics.

Lemma 1.8.10. *Let $S^n_\nu(r)$ be a pseudosphere, q a point on $S^n_\nu(r)$ and $v \in T_qS^n_\nu(r)$ a tangent vector such that $\langle v, v\rangle \in \{1, 0, -1\}$. Then the exponential map $\exp_q : T_qS^n_\nu(r) \to S^n_\nu(r)$ is given by*

$$exp_q(tv) = \gamma_v(t)$$

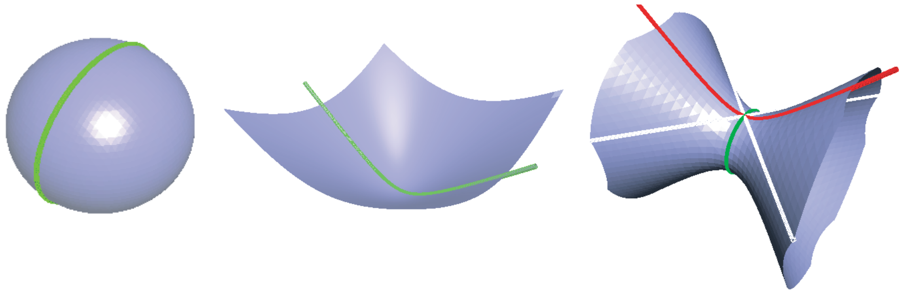

Figure 1.2: Geodesics on the sphere, hyperbolic plane and the de Sitter space, where spacelike geodesics are green, null geodesics are white and timelike geodesics are red.

where γ_v is defined by

$$\gamma_v(t) := \begin{cases} \cos\left(\frac{t}{r}\right)q + r\sin\left(\frac{t}{r}\right)v & \text{if } \langle v,v\rangle = \ \ 1 \\ \cosh\left(\frac{t}{r}\right)q + r\sinh\left(\frac{t}{r}\right)v & \text{if } \langle v,v\rangle = -1 \\ q + tv & \text{if } \langle v,v\rangle = \ \ 0. \end{cases} \tag{1.25}$$

If v is spacelike or timelike, the radial geodesics γ_v are of unit speed, so the parametrization is by arc length. We get the result for $H^n_\nu(r)$ by exchanging the the cases for timelike and spacelike v in (1.25)*.*

Proof. By Lemma 1.8.9 and the corresponding proof in [5, p. 112], we know that the geodesics are of the form as in (1.25), only the factor r is missing for the cos and cosh terms, since we have to multiply q by $\frac{1}{r}$ to scale it to unit length. By definition $\gamma_v(0) = q$. The derivative of γ_v is given by

$$\frac{d}{dt}\gamma_v(t) = \begin{cases} -\frac{1}{r}\sin\left(\frac{t}{r}\right)q + \cos\left(\frac{t}{r}\right)v \\ \ \ \frac{1}{r}\sinh\left(\frac{t}{r}\right)q + \cosh\left(\frac{t}{r}\right)v \\ \ \ v, \end{cases}$$

and so $\gamma_v'(0) = v$. If $\langle v,v\rangle = 1$ we get

$$\langle\gamma_v'(t),\gamma_v'(t)\rangle = \frac{1}{r^2}\sin^2\left(\frac{t}{r}\right)\underbrace{\langle q,q\rangle}_{=r^2} + \cos^2\left(\frac{t}{r}\right)\underbrace{\langle v,v\rangle}_{=1} = 1,$$

and for $\langle v,v\rangle = -1$

$$\langle\gamma_v'(t),\gamma_v'(t)\rangle = \frac{1}{r^2}\sinh^2\left(\frac{t}{r}\right)\underbrace{\langle q,q\rangle}_{=r^2} + \cosh^2\left(\frac{t}{r}\right)\underbrace{\langle v,v\rangle}_{=-1} = -1.$$

So timelike or spacelike geodesics γ_v are indeed parametrized by unit speed. □

Corollary 1.8.11. *Let $\mathcal{Q}_K$ be a hyperquadric and $q, p \in \mathcal{Q}_K$ distinct points. Suppose q and p are connected by a unique nonnull geodesic $\gamma_{qp} : [0,1] \to \mathcal{Q}_K$, and set $v := \frac{\gamma'_{qp}(0)}{\sqrt{E(\gamma_{qp})}}$. Then the point p can be written as*

$$p = \cos\left(\sqrt{E(\gamma_{qp})K}\right) q + \frac{1}{\sqrt{K}} \sin\left(\sqrt{E(\gamma_{qp})K}\right) v, \tag{1.26}$$

where we use the convention $\sqrt{a} := i\sqrt{|a|}$ for $a < 0$.

Proof. Note that v satisfies

$$\langle v, v\rangle = \frac{\overbrace{\langle \gamma'_{qp}(0), \gamma'_{qp}(0)\rangle}^{=E(\gamma_{qp})}}{\sqrt{E(\gamma_{qp})}\sqrt{E(\gamma_{qp})}} = 1,$$

which is clear for $E(\gamma_{qp}) > 0$. For $E(\gamma_{qp}) < 0$ this follows by

$$\langle v, v\rangle = \frac{E(\gamma_{qp})}{\sqrt{E(\gamma_{qp})}\sqrt{E(\gamma_{qp})}} = \frac{1}{i^2}\frac{E(\gamma_{qp})}{|E(\gamma_{qp})|} = \frac{|E(\gamma_{qp})|}{|E(\gamma_{qp})|} = 1.$$

In case $E(\gamma_{qp}) < 0$, v is of the form $v = i\tilde{v}$ with $\tilde{v} = \frac{\gamma'_{qp}(0)}{\sqrt{|E(\gamma_{qp})|}}$. Then $\tilde{v}$ is of unit length and has the same causal character as γ_{qp}, since it is a rescaling of $\gamma'_{qp}(0)$. By definition $\gamma'_{qp}(0) = \sqrt{|E(\gamma_{qp})|}\tilde{v}$, so we see that

$$p = \exp_q(\gamma'_{qp}(0)) = \gamma_{\gamma'_{qp}(0)}(1) \overset{(1.10)}{=} \gamma_{\tilde{v}}(\sqrt{|E(\gamma_{qp})|}).$$

Now we insert $t = \sqrt{|E(\gamma_{qp})|}$ and $\tilde{v}$ into (1.25) and show that (1.26) follows. We have to check four cases depending on the sign of K and $E(\gamma_{qp})$. The curvature K and radius r are related by $K = \frac{1}{r^2}$ for pseudospheres and $K = -\frac{1}{r^2}$ for pseudohyperbolic spaces. Then for example for $K < 0$ and $E(\gamma_{qp}) < 0$, we have $r = \frac{1}{\sqrt{|K|}}$, then since in this case

$$\frac{\sqrt{|E(\gamma_{qp})|}}{r} = \sqrt{|E(\gamma_{qp})||K|} = \sqrt{E(\gamma_{qp})}\sqrt{K},$$

we conclude that

$$\begin{aligned} p &= \gamma_{\tilde{v}}\left(\sqrt{|E(\gamma_{qp})|}\right) \\ &= \cos\left(\frac{\sqrt{|E(\gamma_{qp})|}}{r}\right) q + r\sin\left(\frac{\sqrt{|E(\gamma_{qp})|)}}{r}\right)\tilde{v} \\ &= \cos\left(\sqrt{E(\gamma_{qp})K)}\right) q + \frac{1}{\sqrt{|K|}}\sin\left(\sqrt{E(\gamma_{qp})K}\right) iv \\ &= \cos\left(\sqrt{E(\gamma_{qp})K)}\right) q + \frac{1}{\sqrt{K}}\sin\left(\sqrt{E(\gamma_{qp})K}\right) v \end{aligned}$$

The other cases are show similarly by utilizing the identities $\cos ix = \cosh x$ and $\sin ix = i\sinh x$.

$\square$

Lemma 1.8.12. *[5, Proposition 5.38 p. 149,150] Let p and q be points in $S^n_\nu(r)$ with $p \neq -q$ (which are not antipodal), then:*

(i) If $\langle p, q\rangle > r^2$, then p and q lie on a unique geodesic, which is timelike and one-to-one.

(ii) If $\langle p, q\rangle = r^2$, then p and q lie on a unique geodesic, which is also a null geodesic in the ambient space $\mathbb{R}^{n+1}_\nu$

(iii) If $-r^2 < \langle p, q\rangle < r^2$, the p and q lie on a unique geodesic, which is spacelike and periodic.

(iv) If $\langle p, q\rangle \leqslant r^2$, then there is no geodesic connecting p with q.

The corresponding results for the pseudohyperbolic spaces $H^n_\nu(r)$ are obtained by exchanging the words spacelike and timelike.

Proof. See [5, p. 150]. □

Lemma 1.8.13. *Let $\mathcal{Q}_K$ be a hyperquadric with radius $r > 0$. For $R \in [-\infty, \pi r)$ and $q \in \mathcal{Q}_K$, denote by $S_R(q)$ the distance sphere of radius R in $\mathcal{Q}_K$ around q. Then $S_R(q)$ is obtained by translating the tangent plane $T_q\mathcal{Q}_K$ along the vector q and intersecting it with $\mathcal{Q}_K$, where we think of $T_q\mathcal{Q}_K$ as a linear subspace through 0 in the ambient space of $\mathcal{Q}_K$, so*

$$S_R(p) = (T_q\mathcal{Q}_K + \mu q) \cap \mathcal{Q}_K.$$

For pseudospheres $S^n_\nu(r)$ the factor μ is given by

$$\mu = \begin{cases} \cos\left(\frac{R}{r}\right) & \textit{for } R \geqslant 0 \\ \cosh\left(\frac{R}{r}\right) & \textit{for } R < 0 \end{cases}$$

and for pseudohyperbolic spaces cos *and* cosh *are interchanged.*

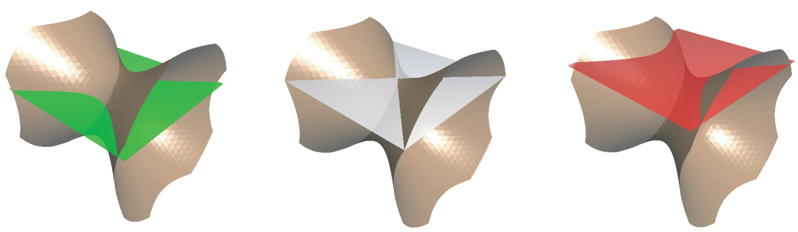

Figure 1.3: Distance spheres around a point in the de Sitter space S^2_1 obtained by intersection with a translate of the tangent plane. From left to right: $R > 0$, $R = 0$ and $R < 0$.

Proof. For $v \in T_q\mathcal{Q}_K$ define γ_v as in Lemma 1.8.10. By (1.22) we see that the the signed length of $\gamma_v : [0,R] \to \mathcal{Q}_K$ is given by $|q\gamma_v(R)|_\pm = \mathrm{sgn}(\mathrm{v})|R|\sqrt{|\langle v,v\rangle|} = \mathrm{sgn}(\mathrm{v})|R|$, as long as $\gamma_v(R)$ is contained in a normal coordinate neighborhood of q. We have to find all vectors $v \in T_q\mathcal{Q}_K$ such that their signed length is R. First consider $R \neq 0$, then $\mathrm{sgn}(\mathrm{v})\sqrt{|\langle v,v\rangle|} = R$ has to be satisfied. Since by (1.10) $\gamma_v(t) = \gamma_{tv}(1)$ and $\mathrm{sgn}(\mathrm{v})\sqrt{|\langle tv,tv\rangle|} = |t|\,\mathrm{sgn}(\mathrm{v})\sqrt{|\langle v,v\rangle|}$, we can by scaling choose v to be of unit length, thus by Lemma 1.8.10 γ_v is parametrized by arc length. For $R > 0$ and $\mathcal{Q}_K = S^n_\nu(r)$, the corresponding sphere of radius R is thus given by the image under the exponential map of $\{Rv \in T_q\mathcal{Q}_K \mid \langle v,v\rangle = 1\}$ which by Lemma 1.8.10 is of the form

$$\left\{\cos\left(\frac{R}{r}\right)q + r\sin\left(\frac{R}{r}\right)v \mid v \in T_q\mathcal{Q}_K \text{ s.t. } \langle v,v\rangle = 1\right\}. \tag{1.27}$$

To see when this geodesic is unique we compute

$$\langle q,\gamma_v(R)\rangle = \cos\left(\frac{R}{r}\right)\underbrace{\langle q,q\rangle}_{=r^2} + r\sin\left(\frac{R}{r}\right)\underbrace{\langle q,v\rangle}_{=0} = r^2\cos\left(\frac{R}{r}\right).$$

By Lemma 1.8.12(iii), if $-r^2 < \langle q,\gamma_v(R)\rangle < r^2$, then q and $\gamma_v(R)$ lie on a unique geodesic, which is spacelike, periodic and is even one-to-one in case $R < \pi r$. To see that (1.27) is given by the intersection of the tangent plane with the hyperquadric, just note that the first summand is fixed while the second one consists of rescaled tangent vectors. For $R < 0$ the image of $\{|R|v \in T_pM \mid \langle v,v\rangle = -1\}$ under the exponential map is given by

$$\{\cosh\left(\frac{|R|}{r}\right)q + r\sinh\left(\frac{|R|}{r}\right)v \mid \langle v,v\rangle = -1\}$$

and since cosh is an even function, $\mu = \cosh\frac{R}{r}$. Also

$$\langle q,\gamma_v(|R|)\rangle = \cosh\left(\frac{|R|}{r}\right)\underbrace{\langle q,q\rangle}_{=r^2} + r\sinh\left(\frac{|R|}{r}\right)\underbrace{\langle q,v\rangle}_{=0} = r^2\cosh\left(\frac{|R|}{r}\right)$$

and so $\langle q,\gamma_v(|R|)\rangle > r^2$ for all $R < 0$ and by Lemma 1.8.12(i), q and $\gamma_v(|R|)$ lie on a unique geodesic which is timelike and one-to-one. Also for pseudohyperbolic spaces $H^n_\nu(r)$ the result is easily seen. □

Corollary 1.8.14. *For $K > 0$ let $S_K := S^n_0(\frac{1}{\sqrt{K}})$ be the sphere of radius $r = \frac{1}{\sqrt{K}}$ Then for $q \in S_K$ the closed distance ball $B_R(q)$ of radius $R > 0$ around q is a normal neighborhood (convex) if $R < \frac{\pi}{2\sqrt{K}}$. For $R \geqslant \frac{\pi}{2\sqrt{K}}$ the closed ball $B_R(q)$ is not convex.*

Proof. Two points $p, r \in S_K$ can be joined by a unique geodesic if $|pr|_\pm = d(p,r) < \frac{\pi}{\sqrt{K}}$ by (1.25). This geodesic is part of a great arc on the sphere and is obtained by intersection of the positive cone $\{\lambda p + \mu r \mid \lambda + \mu \geqslant 1\}$ with S_K. Consider $p, r \in S_K$ contained in the closed ball $B_R(q)$, where $q \in S_K$ and $R < \frac{\pi}{2\sqrt{K}}$. Then for $m \in S_K$, the number $\langle q,m\rangle$ is the length of the projection of m onto q, so by Lemma 1.8.13 we know that a point m belongs to $B_R(q)$ if and only if $\langle q, m\rangle \geqslant \cos\left(\sqrt{K}R\right)$, which in particular holds for p and r. Then for $\lambda + \mu \geqslant 1$ we get

$$\langle \lambda p + \mu r, q\rangle = \lambda\langle p, q\rangle + \mu\langle r, q\rangle \geqslant (\lambda + \mu)\cos\left(\sqrt{K}R\right) \geqslant \cos\left(\sqrt{K}R\right)$$

where the last inequality holds since $\cos\left(\sqrt{K}R\right) > 0$ for $R < \frac{\pi}{2\sqrt{K}}$. So the unique geodesic connecting p with r is contained in $B_R(q)$. If $R = \frac{\pi}{2\sqrt{K}}$, then by Lemma 1.8.13 we know that the sphere of radius R is obtained by intersection of the tangent space of q with S_K, where we think of the tangent space as a linear subspace of the ambient space of S_K through 0. But then by Lemma 1.8.9 we obtain a great circle on the sphere on which antipodal points cannot be connected by unique geodesics. □

Surfaces of Constant Curvature

We will call two-dimensional space forms ***surfaces of constant curvature***. Such spaces can be realized as hyperquadrics in the case $K \neq 0$, respectively their universal coverings in case the corresponding hyperquadric is not simply connected, or the choice of one connected component if the hyperquadric is not connected. By Proposition 1.8.3 we know that simply connected space forms are determined up to isometry by their dimension, index and curvature. We distinguish the cases $K < 0$, $K = 0$ and $K > 0$ and the three possible values for the index $\nu = 0, 1, 2$.

curvature	$\nu = 0$	$\nu = 1$	$\nu = 2$
$K > 0$	$S_0^2\left(\frac{1}{\sqrt{K}}\right)$	$\tilde{S}_1^2\left(\frac{1}{\sqrt{K}}\right)$	$H_2^2\left(\frac{1}{\sqrt{K}}\right)$
$K = 0$	$\mathbb{R}^2$	$\mathbb{R}_1^2$	$\mathbb{R}_2^2$
$K < 0$	$c(H_0^2\left(\frac{1}{\sqrt{-K}}\right))$	$\tilde{H}_1^2\left(\frac{1}{\sqrt{-K}}\right)$	$c(S_2^2\left(\frac{1}{\sqrt{-K}}\right))$
name of spaces in column	S_K	M_K	$-S_K$

Since there exists an anti-isometry between $S_\nu^n(r)$ and $H_{n-\nu}^n(r)$, the spaces in the rows for the columns S_K and $-S_K$ are essentially the same, that is up to the sign of the metric. The spaces $\tilde{S}_1^2\left(\frac{1}{\sqrt{K}}\right)$ and $\tilde{H}_1^2\left(\frac{1}{\sqrt{-K}}\right)$ are even isomorphic, as one can easily check. By $c(H_0^2\left(\frac{1}{\sqrt{-K}}\right))$ and $c(S_2^2\left(\frac{1}{\sqrt{-K}}\right))$ we denote the choice of the connected component that contains $\left(\frac{1}{\sqrt{-K}}, 0, 0\right)$ respectively $\left(-\frac{1}{\sqrt{-K}}, 0, 0\right)$. $c(H_0^2(1))$ is known as the ***upper embedding of the hyperbolic plane*** in $\mathbb{R}_1^3$. Further, by $\tilde{S}_1^2$ and $\tilde{H}_1^2$ we denote the universal covering manifolds of these hyperquadrics, which exist by [5, Theorem A.12 p. 444], since both spaces are diffeomorphic to $\mathbb{R} \times S^1$ and hence are connected. The space $\tilde{S}_1^2$ is often called two dimensional ***de Sitter space***, while $\tilde{H}_1^2$ is known as the two dimensional ***anti de Sitter space***. Note that the metric on both de Sitter spaces is indefinite. Further, $S_0^2\left(\frac{1}{\sqrt{K}}\right)$ is a sphere of radius $\frac{1}{\sqrt{K}}$ with the induced metric of $\mathbb{R}^3$. Finally, $\mathbb{R}^2$ is the usual ***Euclidean plane*** and $\mathbb{R}^2_1$ is called ***semi-Euclidean plane*** or 2-dimensional ***Minkowski space***.

1.9 Local Hyperquadrics

For every semi-Riemannian manifold, the tangent space is a semi-Euclidean vector space. Thus we can define the position vector field and the quadratic form as in Definition 1.8.4, which then allows us to define hyperquadrics on the tangent space. Locally we can transport these objects on the tangent space to the manifold by the exponential map.

Definition 1.9.1. *Given a semi-Riemannian manifold M of dimension n, index ν and $q \in M$, then denote by $\tilde{P}$ and $\tilde{\boldsymbol{q}}$ the position vector field and the quadratic form on $T_qM \cong \mathbb{R}^n_\nu$. Let U be a normal coordinate neighborhood of q, $\tilde{U}$ such that $\exp_q : \tilde{U} \to U$ is diffeomorphism. Then we set $P := d\exp_q \circ \tilde{P} \circ \exp_q^{-1}$, the* **local position vector field** *at q and $\boldsymbol{q} := \tilde{\boldsymbol{q}} \circ \exp_q^{-1}$ is the* **local quadratic form** *at q. For $c \neq 0$ we call $\boldsymbol{q}^{-1}(c)$* **local hyperquadric** *in U and $\Lambda := \boldsymbol{q}^{-1}(0)$ is called the* **local null cone** *at q.*

Lemma 1.9.2. *Let $\boldsymbol{q}$ be the local quadratic form, then*

$$\operatorname{grad} \boldsymbol{q} = 2P \tag{1.28}$$

and since $\boldsymbol{q}(p) = \langle \exp_q^{-1}(p), \exp_q^{-1}(p)\rangle$, the above equation is equivalent to

$$\operatorname{grad} E_q = 2P.$$

Proof. This is a consequence of the Gauss-lemma together with $\operatorname{grad} \tilde{\boldsymbol{q}} = 2P$, which is shown in Lemma 1.8.5. So let $\tilde{v} \in \tilde{U}$ and $v \in U$ such that $d\exp_q(\tilde{v}) = v$, then

$$\begin{aligned}\langle \operatorname{grad} \boldsymbol{q}, v\rangle &= d\boldsymbol{q}(v) = d\boldsymbol{q}(d\exp_q(\tilde{v})) = d(\boldsymbol{q} \circ \exp_q)(\tilde{v}) \\ &= \tilde{v}(\boldsymbol{q} \circ \exp_q) = \tilde{v}\tilde{\boldsymbol{q}} = \langle \operatorname{grad} \tilde{\boldsymbol{q}}, \tilde{v}\rangle \overset{(1.24)}{=} 2\langle \tilde{P}, \tilde{v}\rangle = 2\langle P, v\rangle,\end{aligned}$$

where we used Theorem 1.5.4 in the last step. ☐

Lemma 1.9.3. *For $r \neq 0$ the sets $\boldsymbol{q}^{-1}(r)$. are semi-Riemannian hypersurfaces of U and are obtained as*

$$\exp_q(Q_{\pm\frac{1}{r^2}} \cap \tilde{U}),$$

the images of the corresponding hyperquadrics in the tangent space under the exponential map. Further the local position vector field P at q is orthogonal to every local hyperquadric at q and P is both tangential and orthogonal to the local null cone Λ at q. Finally it holds that

$$\langle P, P\rangle = \langle \tilde{P}, \tilde{P}\rangle. \tag{1.29}$$

Proof. On U, by definition

$$\boldsymbol{q}^{-1}(r) = (\tilde{\boldsymbol{q}} \circ \exp_q^{-1})^{-1}(r) = \exp_q(\tilde{\boldsymbol{q}}^{-1}(r)) = \exp_q(Q_{\pm\frac{1}{r^2}}).$$

Orthogonality then holds since $\tilde{P}$ is orthogonal to hyperquadrics, thus the claim follows by the Gauss-lemma since the exponential map is a radial isometry.

Equation (1.29) also follows by the Gauss-lemma. Given $v \in \tilde{U}$ non null and $p = \exp_q(v)$, we use (1.28) and (1.29) to conclude

$$\begin{aligned}\langle \operatorname{grad}_p \boldsymbol{q}, \operatorname{grad}_p \boldsymbol{q}\rangle &= 4\langle (\,d\exp_q)_v(\tilde{P}(v)), (\,d\exp_q)_v(\tilde{P}(v))\rangle \\ &= 4\langle \tilde{P}(v), \tilde{P}(v)\rangle = 4\langle v, v\rangle \neq 0\end{aligned}$$

since $\langle v, v\rangle \neq 0$, so by Proposition 1.4.1 we get that $\boldsymbol{q}^{-1}(r)$ is a semi-Riemannian hypersurface for $r \neq 0$. □

Lemma 1.9.4. *Let q be a point in a semi-Riemannian manifold M and let P be the local position vector field at q. Then for every radial geodesic σ starting at q we have*

$$(P \circ \sigma)'(t) = \sigma'(t) \tag{1.30}$$

and

$$P(\sigma(t)) = t\sigma'(t), \tag{1.31}$$

where $'$ denotes the induced covariant derivative along σ.

Proof. First note that since σ is a radial geodesic, it is of the form

$$t \mapsto \exp_q tv$$

for some v in T_qM with $v = \sigma'(0)$, since $(\exp_q^{-1} \circ \sigma)(t) = tv$. The local position vector field P is defined by $P = \,d\exp_q \circ \tilde{P} \circ \exp_q^{-1}$. Now Lemma 1.5.2(ii) shows that in normal coordinates, σ is of the form $t \mapsto tv$. So we see that

$$\sigma^k(t) = tv^k$$

and $(P \circ \sigma)^i(t) = \sigma^i(t)$. Now by differentiating the above twice with respect to t we get

$$0 = \frac{d^2}{dt^2}(tv^k) = \frac{d^2\sigma^i}{dt^2} = -\sum_{ij} \Gamma^k_{ij}(\sigma(t))\frac{d\sigma^i}{dt}\frac{d\sigma^j}{dt},$$

where the last equality is due to the geodesic equation for σ. So by the above and the coordinate formula for the induced covariant derivative [5, p. 66] we get

$$(P\circ\sigma)'(t) = \sum_k \left\{ \frac{d\sigma^k}{dt} + \sum_{ij} \Gamma^k_{ij}(\sigma(t))\frac{d\sigma^i}{dt}\frac{d\sigma^j}{dt} \right\} \partial_k = \sum_k \frac{d\sigma^k}{dt}\partial_k = \sigma'(t), \tag{1.32}$$

which shows (1.30). To show (1.31), set $\tilde{\sigma}(t) := (\exp_q^{-1} \circ \sigma)(t) = (t \mapsto tv)$, then $\tilde{P}(\tilde{\sigma}(t)) = (tv)_{tv} = t\tilde{\sigma}'(t)$ and applying $\,d\exp_q(tv)$ to both sides of the last equation yields

$$d\exp_q \circ \tilde{P} \circ \exp_q^{-1}(\sigma(t)) = t\,d\exp_q \circ \,d\exp_q^{-1}(\sigma'(t)) = t\sigma'(t).$$

Since by definition, the right hand side is P, we get $P(\sigma(t)) = t\sigma'(t)$, which shows (1.31). □

Remark This does not hold along curves which are not geodesics. Take for example the curve $c : [0,1] \to \mathbb{R}^2$ into flat space given by $c(t) = (t, t^2)$ then $c'(t) = (1, 2t)$ and $(P \circ c)(t) = (t, t^2)$. Thus $(P \circ c)'(t) = c'(t)$ but since $tc'(t) = (t, 2t^2)$ we see that $P(c(t)) \neq tc'(t)$. In fact equation (1.30) always holds in flat space since there the Christoffel symbols vanish and thus (1.32) shows the result without any previous considerations. So if one wants to construct a counterexample to (1.30) it has to be a non geodesic curve in non flat space. On the other hand the proof of (1.31) uses the form of geodesics in flat space (respectively their image under the exponential map) and therefore only shows the equation for geodesics.

1.10 Distance Functions

Definition 1.10.1. *Let M be a semi-Riemannian manifold and let $U \subset M$ be open. A smooth function $f : U \to \mathbb{R}$ is called* **distance function** *if* $\|\operatorname{grad} f\| = 1$.

Examples

1. Let $x_0 \in \mathbb{R}^n$ with the usual inner product, then $r_{x_0}(x) := d(x_0, x) = \sqrt{\langle x - x_0, x - x_0 \rangle}$ is smooth on $U := \mathbb{R}^n \backslash \{x_0\}$ and a simple calculation shows that $\|\operatorname{grad} r_{x_0}\| = 1$ on U, thus r_{x_0} is a distance function on U.

2. Let S^n be the unit sphere in $\mathbb{R}^n$, then $r_{S^n}(x) := \inf\{r_{x_0}(x) | x_0 \in S^n\}$ is smooth on $U := \mathbb{R}^n \backslash (S^n \cup 0)$ and $\|\operatorname{grad} r_{S^n}\| = 1$, which shows that r_{S^n} is a distance function on U.

3. More generally, let M be any submanifold of $\mathbb{R}^n$, and as before set $r_M(x) := \inf\{r_{x_0}(x) \mid x \in M\}$. Then one can show that there exists a neighborhood U of M such that r_M is a distance function. See [7, Example 22 p. 42].

Lemma 1.10.2. *Let $f : U \to \mathbb{R}$ be a distance function, then*

$$\nabla_{\operatorname{grad} f} \operatorname{grad} f = 0. \tag{1.33}$$

Every integral curve of $\operatorname{grad} f$ *is a unit speed geodesic. For every t in the image of f, the level sets $V_t := f^{-1}(t)$ are semi-Riemannian hypersurfaces, with unit normal given by* $\operatorname{grad} f$ *and shape operator*

$$W_t(X) = \nabla_X \operatorname{grad} f, \tag{1.34}$$

where $X \in \mathfrak{X}(V_t)$.

Proof. For $X \in \mathfrak{X}(M)$ arbitrary,

$$\begin{aligned}\langle \nabla_{\operatorname{grad} f} \operatorname{grad} f, X \rangle &= H^f(\operatorname{grad} f, X) = H^f(X, \operatorname{grad} f) \\ &= \langle \nabla_X \operatorname{grad} f, \operatorname{grad} f \rangle = \frac{1}{2} X \underbrace{\langle \operatorname{grad} f, \operatorname{grad} f \rangle}_{=\pm 1} = 0,\end{aligned}$$

where the first equality is the definition of the Hessian (1.7). The second equality follows by symmetry of the Hessian. To see that that any integral curve c of $\operatorname{grad} f$ is a geodesic, note that integral curves are defined by the equation $c'(t) = \operatorname{grad}_{c(t)} f$ and so $\nabla_{c'(t)} c'(t) = 0$ follows by (1.33). Since $\langle c', c' \rangle = \langle \operatorname{grad}_{c(t)} f, \operatorname{grad}_{c(t)} f \rangle = \pm 1$, we see that c has unit speed. By Proposition 1.4.1 all level sets $V_t := f^{-1}(t)$ of a distance function are semi-Riemannian hypersurfaces, where the unit normal given by restriction of $\operatorname{grad} f$, finally by Lemma 1.4.3 the shape operator for V_t is of the form (1.34). $\square$

Definition 1.10.3. *Let $q \in M$ be a point with normal coordinate neighborhood U. Then the* **radius function** $r_q : U \to \mathbb{R}$ *is defined as*

$$r_q(p) := |qp|_{\pm}.$$

Remark For Riemannian manifolds $r_q(p) = d(q, p)$, where $d(\ , \ )$ is the Riemannian distance function.

Lemma 1.10.4. *Let U be a normal coordinate neighborhood of q, P the local position vector field at q and Λ the local null cone. Then the function r_q is smooth on $\hat{U}$ and the gradient of r_q is given by*

$$\operatorname{grad}_p r_q = \frac{\mathrm{sgn(E_q(p))}}{\sqrt{|E_q(p)|}} P(p), \tag{1.35}$$

where $\operatorname{grad}_p r_q$ is smoothly defined on the open set $\hat{U} = U \backslash (\Lambda \cup \{0\})$. Also $\|\operatorname{grad} r_q\| = 1$, on $\hat{U}$, so r_q is a distance function on $\hat{U}$. The level sets of $(r_q)^{-1}(t)$ are orthogonal to P and the integral curves of $\operatorname{grad} r_q$ are unit speed radial geodesics through q.

Proof. By Lemma 1.9.2 we know that $\operatorname{grad} E_q = 2P$ and Lemma 1.3.2 yields

$$\operatorname{grad}_p r_q = \frac{\mathrm{sgn(E_q(p))}}{\sqrt{|E_q(p)|}} P(p).$$

For P the local position vector field at q we see that

$$\|P(p)\|^2 = |\langle P(p), P(p) \rangle| \overset{(1.29)}{=} \left|\langle \tilde{P}(p), \tilde{P}(p) \rangle\right| = |E_q(p)|,$$

and so for p such that $E_q(p) \neq 0$

$$\|\operatorname{grad} r_q(p)\| = \frac{1}{\sqrt{|E_q(p)|}} \|P(p)\| = 1.$$

Clearly $\operatorname{grad} r_q$ is smooth away from $\Lambda \cup \{0\}$. For $c \neq 0$ we know that $r_q^{-1}(c) \in U \backslash (\Lambda \cup \{0\})$, so by Lemma 1.10.2 we see that $r_q^{-1}(c)$ is a semi-Riemannian hypersurface with unit normal vector field given by $\operatorname{grad} r_q$, which is a multiple of the position vector field, thus P is orthogonal to $r_q^{-1}(c)$. Also by Lemma 1.10.2 we know that integral curves of $\operatorname{grad} r_q$ are of unit speed.

$\square$

Chapter 2

Comparison of Self Adjoint Linear Maps

2.1 Self Adjoint Linear Maps

In this section we will study families $A(t)$ of self-adjoint linear maps in a semi-Euclidean space V, where A depends smoothly on a parameter t in $\mathbb{R}$ and by semi-Euclidean we mean spaces isometric to $\mathbb{R}^n_\nu$. So in particular the following results can be applied to operators defined on the tangent spaces of semi-Riemannian manifolds.

Definition 2.1.1. *Let A be a linear map on a semi-Euclidean space V. The* **adjoint** *A^* of A is defined by $\langle Av, w\rangle = \langle v, A^*w\rangle$ for all v and w in V. If $A = A^*$ we call A* **self-adjoint***. A* **positive semi definite** *linear map is characterized by the property $\langle Av, v\rangle \geqslant 0$ for all $v \neq 0$ and* **positive definite** *if $>$ holds for $v \neq 0$. For B another self adjoint linear operator we say $A \geqslant B$ or $A > B$ if $A - B$ is positive semi-definite or positive definite, respectively.*

Remarks

(i) Adjoints always exist and are unique for isomorphisms and it holds that

$$(AB)^* = B^*A^* \text{ and } (A^{-1})^* = (A^*)^{-1}. \tag{2.1}$$

(ii) Note that in case $\nu > 0$ the identity map I is not positive definite.

(iii) In case $\nu = 0$ all the eigenvalues of a positive (semi-)definite linear map are real and nonnegative. This does not hold in the indefinite case.

(iv) Let V a semi-Euclidean space and W a nondegenerate subspace of V. Then by [5, Lemma 2.23 p. 49] nondegeneracy of W is equivalent to $V = W \bigoplus W^\perp$. Let A and B self adjoint linear maps on V. If $A|_W = B|_W$ and $A \leqslant B$ on $W^\perp$, then $A \leqslant B$ on V. To see this write $v \in V$ as

$v = v_1 + v_2$ with $v_1 \in W$ and $v_2 \in W^\perp$, then

$$\begin{aligned}\langle (B-A)v, v\rangle &= \langle \underbrace{(B-A)v_1}_{=(A-A)v_1=0}, v_1\rangle + \underbrace{\langle (B-A)v_1, v_2\rangle}_{=0} \\ &\quad + \underbrace{\langle (B-A)v_2, v_1\rangle}_{=\langle v_2,(B-A)v_1\rangle=0} + \underbrace{\langle (B-A)v_2, v_2\rangle}_{\geqslant 0} \geqslant 0.\end{aligned}$$

From now on self-adjointness will be assumed if we compare linear operators by inequalities. The next basic result is well known for inner product spaces and also holds in the semi-Euclidean setting.

Lemma 2.1.2. *[8, Lemma 3.1 p. 18] If $A \geqslant 0$ and $\langle Ax, x\rangle = 0$ for some fixed x then $Ax = 0$.*

Proof. Since A is positive semi-definite we have for any $h \in V$ that

$$0 \leqslant \langle A(x+h), x+h\rangle = 2\langle Ax, h\rangle + \langle Ah, h\rangle$$

where we used that A is self adjoint and $\langle Ax, x\rangle = 0$. Suppose $Ax \neq 0$. Since $\langle , \rangle$ is non-degenerate we find some h_0 such that $\langle Ax, h_0\rangle \neq 0$ and we can choose h_0 such that $\langle Ax, h_0\rangle < 0$. Scaling h_0 by $\lambda \in \mathbb{R}$ and inserting into the above inequality we get

$$0 \leqslant 2\lambda\langle Ax, h_0\rangle + \lambda^2\langle Ah_0, h_0\rangle.$$

Now $\langle Ah_0, h_0\rangle \geqslant 0$ since A is positive definite. This gives a contradiction since the right hand side gets negative for λ small enough. So it holds that $Ax = 0$. □

2.2 Comparison Theorems for Self-Adjoint Linear Maps

By $M_n(\mathbb{R})$ we denote the set of $n \times n$ matrices with real entries.

Lemma 2.2.1. *[9, Lemma 1.2.14 p. 32] Let $F : [0, b] \to M_n(\mathbb{R})$ be smooth with $F(0) = 0$ and $F'(0)$ invertible. Then there exists a smooth map $\bar{F}$ such that $F(t) = t\bar{F}(t)$ on $[0, b]$ such that $\bar{F}(0) = F'(0)$ is invertible.*

Remark F is a real valued function for $n = 1$ and invertibility just means that F is nonzero.

Proof. Using that F is smooth and that $F(0) = 0$ we write

$$F(t) = F(t) - F(0) = \int_0^1 \frac{d}{ds} F(st)\, ds = t \int_0^1 F'(st) ds =: t\bar{F}(t)$$

where $\bar{F}$ is smooth since F is smooth. Moreover, $\bar{F}(0) = \int_0^1 F'(0) ds = F'(0)$ which is invertible by assumption.

□

Lemma 2.2.2. *[1, Lemma 4.2 p. 265] Let $R(t)$ and $F(t)$ be linear maps on a semi-Euclidean space V with $R, F \in C^2[0,b]$ where $b > 0$ and $R(t)$ self-adjoint for all $t \in [0,b]$. Further suppose that F satisfies $F(0) = 0$, $F'(0)$ is invertible and $F(t)$ is invertible for $t \in (0,b]$ and the following equation holds for all $t \in [0,b]$:*

$$F''(t) + R(t)F(t) = 0. \tag{2.2}$$

Given a smooth function $g : [0,b] \to \mathbb{R}$ which is constant 1 or with $g(0) = 0$, $g'(0) = 1$ and $g > 0$ on $(0,b]$, define S by

$$g(t)F'(t) = S(t)F(t) \text{ for } t \in (0,b] \tag{2.3}$$

and

$$S(0) = I. \tag{2.4}$$

Then S is a smooth self-adjoint map defined on $[0,b]$, and satisfies

$$gS' + S^2 - g'S + g^2R = 0. \tag{2.5}$$

Proof. First we show that (2.5) holds on $(0,b]$. Since F is invertible on this interval, (2.3) can be written as

$$S = gF'F^{-1} \tag{2.6}$$

and since also $g > 0$ we have, using our equations and the product rule

$$\begin{aligned} S'F + \frac{S^2F}{g} &\overset{(2.6)}{=} S'F + SF' = (SF)' \overset{(2.3)}{=} (gF')' \\ &= g'F' + gF'' \overset{(2.2)}{=} g'F' - gRF \overset{(2.3)}{=} \frac{g'}{g}SF - gRF. \end{aligned}$$

Now (2.5) is obtained by multiplying the left and right hand side of the above equation by gF^{-1} from the right.

To show that S is smooth note that by Lemma 2.2.1 we can write $g(t) = t\bar{g}(t)$ and $F(t) = t\bar{F}(t)$ for smooth maps $\bar{g}$ and $\bar{F}$ with $\bar{F}(0) = F'(0)$ invertible and $\bar{g}(0) = g'(0) = 1$. Inserting this into (2.3) we get $t\bar{g}(t)F'(t) = tS(t)\bar{F}$ on $(0,b]$. Dividing by t and then setting $t = 0$ one gets $g'(0)F'(0) = S(0)F'(0)$, which shows by (2.4) that S is smooth everywhere.

We now show that S is self-adjoint. This is clear for $t = 0$ since by definition $S(0) = I$. For $t \in (0,b]$, $F(t)$ is invertible, so we have $S(t) = g(t)F'(t)F^{-1}(t)$. Showing that S is self-adjoint on $(0,t]$ is equivalent to showing that

$$\frac{d}{dt}((F')^*F - F^*F') = 0 \text{ for } t \in (0,b] \text{ and } S(0) - S^*(0) = 0$$

since $S = S^*$ on $(0,b]$ is equivalent to

$$g(F'F^{-1})^* = gF'F^{-1}.$$

Since $g > 0$ on $(0,b]$ it can be canceled, then using (2.1) yields

$$(F')^*F - F^*F' = 0.$$

Now $S(0) - S^*(0) = I - I = 0$ by definition, so if we show that the derivative of the above equation on $(0,b]$ is zero we show that $S(t) - S^*(t) = 0$ on $[0,b]$ since we have shown before that S is smooth. This can be done using the product rule and (2.2):

$$\begin{aligned}\frac{d}{dt}((F')^*F - F^*F') &= (F^*)'F' - F^*F'' + (F'')^*F - (F')^*F' \\ &\overset{(2.2)}{=} -F^*(-RF) + (-RF)^*F = 0.\end{aligned}$$

One easily checks that the proof also works in case of $g = 1$.

□

Theorem 2.2.3. *[1, Theorem 4.3 p. 266] For $i = 1, 2$ let g, R_i, F_i be as in Lemma 2.2.2. Then corresponding to R_i and F_i, let S_i be the smooth self-adjoint map defined by* (2.3) *and* (2.4) *in Lemma 2.2.2. Further assume that*

$$g''(0) = 0. \tag{2.7}$$

If the self-adjoint linear maps R_i satisfy $R_1(t) \leqslant R_2(t)$ for all $t \in [0, b]$, then for S_i the inequality reverses. That is $S_1(t) \geqslant S_2(t)$ for $t \in [0, b]$. If $S_1(b) = S_2(b)$ it follows that $S_1(t) = S_2(t)$ and $R_1(t) = R_2(t)$ for $t \in [0, b]$. Further for $i = 1$ or $i = 2$ set $S = S_i$ and $R = R_i$, then we have

$$S'(0) = 0 \tag{2.8}$$

and

$$S''(0) = \frac{1}{3}(g'''(0)I - 2R(0)). \tag{2.9}$$

Before proving above theorem we show the following result which is similar to [8, Theorem 3.2].

Lemma 2.2.4. *Let S_1, S_2 be as in Theorem 2.2.3 corresponding to R_1 respectively R_2. Suppose that S_1 and S_2 are defined on $[t_0, b]$ for $0 < t_0 < b$ and set $A_1 = S_1(t_0)$ and $A_2 = S_2(t_0)$. If $R_1(t) \leqslant R_2(t)$ on $[t_0, b]$ and $A_1 \geqslant A_2$ then $S_1 \geqslant S_2$ on $[t_0, b]$.*

Proof. At first consider the case $A_1 > A_2$ and $R_1(t) < R_2(t)$ for $t \in [t_0, b]$. We show that then $S_1(t) > S_2(t)$ for all $t \in [t_0, b]$. Suppose the claim were false. Then there exists $\bar{t} \in (t_0, b]$ such that $S_1(t) > S_2(t)$ for $t < \bar{t}$ and $S_1(\bar{t}) \geqslant S_2(\bar{t})$ but $S_1(\bar{t}) - S_2(\bar{t})$ is not positive definite. So there exists $\bar{x} \neq 0$ such that

$$\langle (S_1(\bar{t}) - S_2(\bar{t}))\bar{x}, \bar{x}\rangle = 0. \tag{2.10}$$

So by Lemma 2.1.2 it follows that

$$S_1(\bar{t})\bar{x} = S_2(\bar{t})\bar{x}. \tag{2.11}$$

Setting $f(t) := \langle (S_1(t) - S_2(t))\bar{x}, \bar{x}\rangle$ and g as in Theorem 2.2.3 we compute

$$\begin{aligned} g(\bar{t})f'(\bar{t}) &= \langle (S_1(\bar{t}) - S_2(\bar{t}))\bar{x}, \bar{x}\rangle \\ &\overset{(2.5)}{=} g(\bar{t})\,\underbrace{\langle S_2'(\bar{t})\bar{x}, S_2'(\bar{t})\bar{x}\rangle - \langle S_1(\bar{t})\bar{x}, S_1(\bar{t})\bar{x}\rangle}_{=0 \text{ by } (2.11)} \\ \underbrace{+\langle g'(\bar{t})(S_1(\bar{t}) - S_2(\bar{t}))\bar{x}, \bar{x}\rangle}_{=0 \text{ by } (2.10)} &+ g(\bar{t})^2\langle (R_2(\bar{t}) - R_1(\bar{t}))\bar{x}, \bar{x}\rangle \\ &= \underbrace{g(\bar{t})^2\langle (R_2(\bar{t}) - R_1(\bar{t}))\bar{x}, \bar{x}\rangle > 0}_{\text{since } R_1 < R_2}. \end{aligned}$$

Since $f(t) > 0$ on $(t_0, \bar{t})$ and $f(\bar{t}) = 0$ it follows that $f'(\bar{t}) \leqslant 0$ but since $g(\bar{t}) > 0$ by definition we conclude $g(\bar{t})f'(\bar{t}) \leqslant 0$, which gives a contradiction. Now for the general case let $\delta > 0$ and S_δ be a solution to (2.3) on $[t_0, b]$ with respect to $R_\delta = R_2 + \delta B$, where B is a fixed positive definite linear map and set $A_\delta = S_\delta(t_0) = A_2 - \delta B$. Then S_δ exists on $[t_0, b]$ for δ small enough by [10, 10.5.6 p.285] and S_δ depends smoothly on δ. Then we have $R_1 < R_\delta$ and $A_1 > A_\delta$ since for any $x \neq 0$

$$\langle (R_\delta(t) - R_1(t))x, x\rangle = \underbrace{\langle (R_2(t) - R_1(t))x, x\rangle}_{\geqslant 0} + \delta\underbrace{\langle Bx, x\rangle}_{>0} > 0$$

for t in $[t_0, b]$ and

$$\langle (A_\delta - A_1)x, x\rangle = \underbrace{\langle (A_1 - A_2)x, x\rangle}_{\geqslant 0} + \delta\underbrace{\langle Bx, x\rangle}_{>0} > 0$$

so by our first result $S_1(t) \geqslant S_\delta(t)$ on $[t_0, b]$ so $\delta \to 0$ gives $S_1(t) \geqslant S_2(t)$ on $[t_0, b]$.

□

Proof of Theorem 2.2.3. To see (2.8) differentiating (2.5) results in

$$g'S' + gS'' + S'S + SS' - g''S - g'S' + (g^2R)' = 0.$$

Then (2.8) follows by setting $t = 0$ and applying (2.7) and the initial data for S and g from Lemma 2.2.2. Now cancel the $g'S'$ term in the last equation and differentiate again to obtain

$$g'S'' + gS''' + 2S'S' + S''S + SS'' - g'''S - g''S' + (g^2R)'' = 0$$

where

$$(g^2R)'' = 2g''gR + 2(g')^2R + 4g'gR' + g^2R''$$

so again by setting $t = 0$ and using the properties of g and the initial condition $S(0) = I$ we obtain (2.9).

Now to apply Lemma 2.2.4 we need to show that for perturbed R_2 we find a neighborhood of 0 such that the solutions disagree. For $\delta > 0$ and some positive definite linear map B define $R_\delta = R + \delta B$ and let S_δ be the solution to (2.5) on $[0, b]$ with $S_\delta(0) = I$. This solution S_δ on $[0, b]$ exists by [10, 10.5.6 p.285] for δ small enough. Now inserting into (2.9) gives

$$S_1''(0) = \frac{1}{3}(g'''(0)I - 2R_1(0)) \text{ and } S_\delta''(0) = \frac{1}{3}(g'''(0)I - 2R_\delta(0))$$

so we have

$$S_1''(0) - S_\delta''(0) = \frac{2}{3}(R_2(0) - R_1(0) + \delta B).$$

But then since $R_1(t) \leqslant R_2(t)$ and B is positive definite it follows that $S_1''(0) > S_\delta''(0)$. Note that since S is self adjoint also all derivatives are. Further we have that $S_1(0) = I = S_\delta(0)$ by definition and by (2.8) $S_1'(0) = 0 = S_\delta'(0)$. So their Taylor series at 0 agree for the constant and linear term. So we can find some $a > 0$ such that $S_1 > S_2$ on $(0, a)$. To obtain $S_1(t) \geqslant S_2(t)$ for $t \in [0, b]$ pick some t_0 in $(0, a)$ and set $A_1 = S_1(t_0)$, $A_2 = S_2(t_0)$ so by Lemma 2.2.4 we have

$S_1(t) \geqslant S_2(t)$ on $[t_0, b]$ which together with $S_1(t) > S_2(t)$ on $[0, t_0]$ gives the result.

Now consider the case $S_1(b) = S_2(b)$. Then for $\underline{t} := \inf\{t \in [0, b] \mid S_1(t) = S_2(t)\}$ it remains to show that $\underline{t} = 0$. So suppose that $\underline{t} \neq 0$ then there is $0 \leqslant t_0 < \underline{t}$ such that $S_1(t_0) \neq S_2(t_0)$ and $|S_1(t_0) - S_2(t_0)|$ is small enough such that the solution S of (2.5) with respect to R_2 and with $S(t_0) = S_2(t_0)$ exists on $[t_0, b]$ [10, 10.5.6 p.285]. But then by Lemma 2.2.4 we have

$$S_2(t) \leqslant S(t) \leqslant S_1(t) \text{ for all } t \in [t_0, b].$$

Then since $S_1(b) = S_2(b)$ it follows by above inequality that $S_1(b) = S(b)$. Hence S_1 and S satisfy the same ODE on $[t_0, b]$ with the same initial value at b, so they agree on $[t_0, b]$. In particular, $S_1(t_0) = S(t_0)$ which gives a contradiction since $S(t_0) = S_2(t_0) \neq S_1(t_0)$. So we have $\underline{t} = 0$ and by (2.5) also $R_1 = R_2$ on $[0, b]$.

□

Chapter 3

Triangle Comparison in Semi-Riemannian Manifolds and Triangle Lemmas

3.1 Triangles in Semi-Riemannian Manifolds

In Euclidean-space, geodesic triangles are given by three distinct points connected by straight lines, which are the unique minimizing geodesics connecting these points. On the other hand on the sphere it is not immediately clear how to define geodesic triangles since antipodal points can be connected by infinitely many geodesics, all of the same length. So we need to require uniqueness of geodesics connecting the vertices, to get a good definition for geodesic triangles.

Definition 3.1.1. *Let M be a semi-Riemannian manifold and $p, q, r \in M$ such that $\gamma_{qp}, \gamma_{qr} : [0,1] \to M$ are the unique geodesic segments connecting the point q with p, respectively q with r. We call p, q, r together with γ_{qp} and γ_{qr} a* **hinge**. *The* **non-normalized angle** *of this hinge at q is defined as*

$$\angle pqr := \langle \gamma'_{qp}(0), \gamma'_{qr}(0) \rangle.$$

If there exists a unique geodesic $\gamma_{pr} : [0,1] \to M$ connecting p with r, then the **vertices** *p, r, q together with the* **sides** *given by $\gamma_{qp}, \gamma_{qr}, \gamma_{pr}$ are called a* **triangle**[1]*, in short Δpqr. We also suppose for triangles that the triple of signed lengths $(|\gamma_{qr}|_\pm, |\gamma_{qp}|_\pm, |\gamma_{pr}|_\pm) = (|qr|_\pm, |qp|_\pm, |pr|_\pm) \in \mathbb{R}^3$ is not 0. The numbers $|qr|_\pm, |qp|_\pm, |pr|_\pm$ are called the* **side lengths** *of Δpqr. From the perspective of the vertex q we call γ_{pr} the* **opposite side***, and γ_{qp},γ_{qr}* **adjacent sides***, and analogously for the other vertices.*

Remarks

1. In a normal neighborhood, a hinge is uniquely determined by three point with one of them "marked", so that we know where the angle sits. A triangle is uniquely determined by three points such that at least one side length is non zero.

[1] This is also called geodesic triangle in the literature.

2. In semi-Riemannian manifolds of non-zero index, the sides of Δpqr can have any causal character, thus also sides with negative signed lengths and sides of length 0 are possible.

3. Unless explicitly noted otherwise, angles are always assumed to be non-normalized.

Lemma 3.1.2. *Let Δpqr be a triangle contained in a normal neighborhood U, then*

$$(E_q \circ \gamma_{pr})'(0) = -2\angle qpr, \tag{3.1}$$

where γ_{pr} is the unique geodesic parametrized by $[0,1]$ connecting p with r and $\angle qpr$ denotes the non-normalized angle.

Proof. Set $x(t,s) := \exp_q(t \exp_q^{-1}(\gamma_{pr}(s)))$, which is defined for s and t in $[0,1]$. This holds since U is a normal neighborhood of Δpqr and thus any two points on the sides of Δpqr can be connected by a unique geodesic. We have

$$x_t(0,s) = \exp_q^{-1}(\gamma_{pr}(s)) = \gamma'_{q\gamma_{pr}(s)}(0)$$

by Lemma 1.5.2[(i)], and $E_q(\gamma_{pr}(s)) = \langle \gamma'_{q\gamma_{pr}(s)}(0), \gamma'_{q\gamma_{pr}(s)}(0)\rangle$ by definition. Combining these results and differentiating with respect to s we obtain

$$(E_q \circ \gamma_{pr})'(0) = \partial_s|_0 \langle x_t(0,s), x_t(0,s)\rangle = 2\langle x_{ts}(0,0), x_t(0,0)\rangle. \tag{3.2}$$

So it suffices to compute $x_t(0,0)$ and $x_{ts}(0,0)$ to get the result. To do so, choose Riemannian normal coordinates around q. Then the corresponding chart is given by $\varphi := \exp_q^{-1}$ and we set $x^\varphi := \varphi \circ x$. Then $x^\varphi(t,s) = t\exp_q^{-1}(\gamma_{pr}(s))$ and so we see that $x_t^\varphi(t,s) = \exp_q^{-1}(\gamma_{pr}(s))$ and $x_t^\varphi(0,0) = \exp_q^{-1}(p)$. But in our chart φ, radial geodesics are straight lines by Lemma 1.5.2[(ii)], and we know that $\exp_q^{-1}(p)$ is the vector that defines a geodesic from q to p which is parametrized by $[0,1]$. So the negative of this vector defines via the exponential map a geodesic connecting p with q. So

$$x_t^\varphi = -\left(\gamma_{pq}^\varphi\right)'(0). \tag{3.3}$$

By the coordinate formula for the covariant derivative [5, p. 66], and since at q the Christoffel symbols vanish in Riemannian normal coordinates, we get

$$(x_{ts}^\varphi)^k(0,0) = \partial_s|_0(\exp_q^{-1}(\gamma_{pr}(s))^k) + \sum_{i,j} \underbrace{\Gamma_{ij}^k(\overbrace{x(0,0)}^{=q})}_{=0} \exp_q^{-1}(\gamma_{pr}(s))^i \exp_q^{-1}(\gamma_{pr}(s))^j$$

and so

$$x_{ts}^\varphi(0,0) = \partial_s|_0 \underbrace{\exp_q^{-1}(\gamma_{pr}(s))}_{=\gamma_{pr}^\varphi(s)} = (\gamma_{pr}^\varphi)'(0). \tag{3.4}$$

Then the result follows by inserting

$$(E_q \circ \gamma_{pr})'(0) \overset{(3.2)}{=} 2\langle x_{ts}(0,0), x_t(0,0)\rangle \overset{(3.3)+(3.4)}{=} -2\langle \left(\gamma_{pq}^\varphi\right)'(0), \left(\gamma_{pr}^\varphi\right)'(0)\rangle = -2\angle qpr.$$

□

Remark: To see the geometric meaning of this result, first note that the function $E_q \circ \gamma_{pr} : [0,1] \to \mathbb{R}$ measures the signed distance from q to points on the opposite side of the triangle, going from p to q. This can be seen since by (1.17) the energy is a strictly increasing function of the signed length. Then the right hand side of (3.1) tells us if this function is increasing or decreasing locally at p, depending on the non-normalized angle at p.

Theorem 3.1.3. *For $K \neq 0$, let $\mathcal{Q}_K$ be a hyperquadric, U a normal neighborhood in $\mathcal{Q}_K$ and Δpqr a triangle in U with nonnull side lengths. Then the law of cosines is*

$$\begin{aligned}\cos\sqrt{KE(\gamma_{pr})} &= \cos\sqrt{KE(\gamma_{qp})}\cos\sqrt{KE(\gamma_{qr})}\\ &\quad+\frac{\angle pqr}{\sqrt{E(\gamma_{qp})}\sqrt{E(\gamma_{qr})}}\sin\sqrt{KE(\gamma_{qp})}\sin\sqrt{KE(\gamma_{qr})}.\end{aligned} \tag{3.5}$$

Here $\gamma_{pr}, \gamma_{qp}, \gamma_{qr} : [0,1] \to \mathcal{Q}_K$ are the nonnull geodesics connecting the vertices of Δpqr. Further, $\angle pqr$ denotes the nonnormalized angle at the vertex q.

Proof. Here we combine the proofs of the spherical law of cosines and the hyperbolic law of cosines found in [11, 2.2,2.7, p.17, p. 20] and also give a generalization which applies to semi-Riemannian surfaces with curvature $K \neq 0$. The unified law of cosines (3.5) is from [1, Remark 2.3, p. 260], where it is presented without a proof. We set $v := \frac{\gamma'_{qp}(0)}{\sqrt{E(\gamma_{qp})}}$ and $u := \frac{\gamma'_{qr}(0)}{\sqrt{E(\gamma_{qr})}}$. Then by equation (1.26) in Corollary 1.8.11 we see that

$$p = \cos\left(\sqrt{KE(\gamma_{qp})}\right)q + \frac{1}{\sqrt{K}}\sin\left(\sqrt{KE(\gamma_{qp})}\right)v$$

and

$$r = \cos\left(\sqrt{KE(\gamma_{qr})}\right)q + \frac{1}{\sqrt{K}}\sin\left(\sqrt{KE(\gamma_{qr})}\right)u.$$

Since $\langle q,u\rangle = 0$ and $\langle q,v\rangle = 0$[2] the above equations yield

$$\begin{aligned}\langle p,r\rangle &= \cos\left(\sqrt{KE(\gamma_{qp})}\right)\cos\left(\sqrt{KE(\gamma_{qr})}\right)\langle q,q\rangle\\ &\quad+\frac{1}{K}\sin\left(\sqrt{KE(\gamma_{qp})}\right)\sin\left(\sqrt{KE(\gamma_{qr})}\right)\langle v,u\rangle.\end{aligned} \tag{3.6}$$

Using the definition of the nonnormalized angle we see that

$$\angle pqr = \langle \gamma'_{qp}(0), \gamma'_{qr}(0)\rangle = \sqrt{E(\gamma_{qp})}\sqrt{E(\gamma_{qr})}\langle v,u\rangle. \tag{*}$$

Now we set $w := \frac{\gamma'_{pr}(0)}{\sqrt{E(\gamma_{pr})}}$, then by (1.26) it follows that the point r can be written as

$$r = \cos\left(\sqrt{KE(\gamma_{pr})}\right)p + \frac{1}{\sqrt{K}}\sin\left(\sqrt{KE(\gamma_{pr})}\right)w.$$

[2] By $\langle\ ,\ \rangle$ we mean the scalar product of the ambient space of the hyperquadric, which restricted to the hyperquadric is the metric of $\mathcal{Q}_K$.

First we suppose that $\mathcal{Q}_K$ is a pseudosphere of curvature $K > 0$, then all points in $\mathcal{Q}_K$ have inner product $\frac{1}{K}$ with themselves and in case of pseudohyperbolic spaces the inner product is also $\frac{1}{K}$, but here $K < 0$. Then since $\langle p, w \rangle = 0$ and $\langle p, p \rangle = \frac{1}{K}$, by the above equation we get

$$\langle p, r \rangle = \frac{1}{K} \cos\left(\sqrt{K E(\gamma_{pr})}\right). \tag{**}$$

So we can insert (**) as the left hand side of (3.6) and use (*) to substitute for $\langle v, u \rangle$. Also $\langle q, q \rangle = \frac{1}{K}$ in (3.6). Then the law of cosines follows by multiplication with K.

□

As an immediate consequence of the law of cosines we are able to prove a version of the hinge lemma for hyperquadrics with curvature $K \neq 0$.

Corollary 3.1.4. *Let Δpqr be a triangle contained in a normal coordinate U with nonnull side lengths. Then the non normalized angle $\angle pqr$ is a decreasing function of the signed length of $|pr|_\pm$.*

Proof. This is a consequence of the law of cosines (3.5) if we vary $E(\gamma_{pr})$, while keeping $E(\gamma_{qp})$ and $E(\gamma_{qr})$ fixed. First, for $K > 0$ it is easy to see that the function $t \mapsto \cos(\sqrt{K t})$ is a strictly monotonically decreasing function on $[-\infty, \frac{\pi^2}{\sqrt{K}})$. Then the factor belonging to $\angle pqr$ is always positive if $E(\gamma_{qp}), E(\gamma_{qr}) < \frac{\pi^2}{\sqrt{K}}$, which is obvious if both are > 0, and if for example $E(\gamma_{qp}) < 0$, then

$$\frac{\sin\sqrt{K E(\gamma_{qp})}}{\sqrt{E(\gamma_{qp})}} = \frac{i \sinh\sqrt{K |E(\gamma_{qp})|}}{i\sqrt{|E(\gamma_{qp})|}} = \frac{\sinh\sqrt{K |E(\gamma_{qp})|}}{\sqrt{|E(\gamma_{qp})|}} > 0.$$

So if $E(\gamma_{pr})$ increases, $\angle pqr$ decreases. For $K < 0$, the function $t \mapsto \cosh\sqrt{|K| t}$ is a strictly monotonically increasing function on $[-\infty, \frac{\pi^2}{\sqrt{K}})$. Also,

$$\frac{\sin\sqrt{K E(\gamma_{qp})}}{\sqrt{E(\gamma_{qp})}} \frac{\sin\sqrt{K E(\gamma_{qr})}}{\sqrt{E(\gamma_{qr})}} = -\frac{\sinh\sqrt{|K| E(\gamma_{qp})}}{\sqrt{E(\gamma_{qp})}} \frac{\sinh\sqrt{|K| E(\gamma_{qr})}}{\sqrt{E(\gamma_{qr})}} < 0,$$

which holds independent of the causal character of γ_{qp} and γ_{qr}. By this we see that $\angle pqr$ has to decrease for increasing $E(\gamma_{pr})$.

□

3.2 Realizability Lemma

Given a triangle in a semi-Riemannian manifold, then under certain assumptions we can find a triangle in a model spaces of curvature K which has the same side lengths. This so-called comparison triangle in a model space of constant curvature is then unique up to isometry.

Definition 3.2.1. *Given three numbers $a, b, c \geqslant 0$ such that $a > 0$, we say that a, b, c satisfy the* **strict triangle inequality** *if $b + c > a$. Denote the subset of $\mathbb{R}^3$ consisting of all such triples by T^+ and set $T^- := -T^+$. Positive triples which satisfy $b + c = a$ are called* **degenerate** *triples. The corresponding subset is denoted by D^+ and we set $D^- := -D^+$. Further, denote the complement of $T^+ \cup T^- \cup D^+ \cup D^- \cup 0$ by $C_{T,D}$.*

Remarks Let V be a vector space, then a subset C of V is called ***cone*** if for all $x, y \in C$ also $x + y \in C$ and for all $\lambda > 0$ also $\lambda x \in C$. It is easy to see that the sets T^+, T^-, D^+, D^- are cones.

Lemma 3.2.2. *Given a positive triple in $(a, b, c) \in \mathbb{R}^3\backslash 0$, then up to isometry there is a unique realizing triangle Δpqr in the Euclidean plane $\mathbb{R}^2 = S_0$ if $(a, b, c) \in T^+ \cup D^+$. If the triple (a, b, c) is negative, then up to isometry there is a unique realizing triangle in $-S_0$ if $(a, b, c) \in T^- \cup D^-$.*

Remark It follows that, given three positive side lengths, they constitute a triangle in $\mathbb{R}^2$ if the triangle inequality is satisfied. Then the side lengths can be arbitrarily big.

Proof. Since in Euclidean space distances between points are always positive, a, b, c are $\geqslant 0$. W.l.o.g. suppose that $a \geqslant b$ and $a \geqslant c$, then since the triple is non null, $a > 0$. A sphere of radius b around 0 is given by the equation

$$\langle x, x\rangle = b^2. \tag{1}$$

A sphere at $\hat{a} := (a, 0)$ with radius c is defined by

$$\langle x - \hat{a}, x - \tilde{a}\rangle = c^2. \tag{2}$$

For $x := (x_1, x_2) \in \mathbb{R}^2$, the last equation is equivalent to $\langle x, x\rangle - 2x_1 a + a^2 = c^2$. Thus using (1) yields

$$x_1 = \frac{b^2 + a^2 - c^2}{2a}.$$

Also, equation (1) is equivalent to

$$x_2^2 = b^2 - x_1^2, \tag{*}$$

where the right hand side is $\geqslant 0$ exactly if $b + c \geqslant a$. To see this, note that by a simple calculation,

$$b^2 - x_1^2 = \left(\frac{(a+b)^2 - c^2}{2a}\right)\left(\frac{c^2 - (a-b)^2}{2a}\right).$$

The first factor is always > 0 since $a \geqslant c, b$, and the second factor is $\geqslant 0$ if $c + b \geqslant a$. Also $b^2 - x_1^2 = 0$ exactly if $a = b + c$ and the unique solution to (*) is given by $x_2 = 0$. In this case we thus get a degenerate triangle where all three vertices lie on one line. Otherwise there are two solutions $x_2 = \pm\sqrt{b^2 - x_1^2}$. But the triangles obtained by the different solutions are isometric by reflection at the x_1-axis. Clearly triangles in $\mathbb{R}^2$ which have the same side lengths are isometric. □

Lemma 3.2.3. *Given a positive triple in $(a, b, c) \in \mathbb{R}^3\backslash 0$, then up to isometry there is a unique realizing triangle Δpqr in the Minkowski plane $\mathbb{R}^2_1 = M_0$ if $(a, b, c) \in \mathbb{R}^3\backslash(T^+ \cup T^- \cup 0)$.*

Proof. Again w.l.o.g. assume that $a \geqslant b, c$ and first suppose $a > 0$. Then we construct circles of radius b and c around the points 0 respectively $\hat{a} := (0, a)$. These circles are given by

$$\langle x, x\rangle_1 = \mathrm{sgn(b)}\, b^2 \tag{*}$$

and

$$\langle x - \hat{a}, x - \hat{a} \rangle_1 = \mathrm{sgn}(\mathrm{c})\, c^2, \tag{**}$$

where $x := (x_1, x_2) \in \mathbb{R}^2$ and $\langle x, x \rangle_1 = -x_1^2 + x_2^1$. Then

$$x_1^2 = x_2^2 - \mathrm{sgn}(\mathrm{b})\, b^2 \tag{3.7}$$

and

$$x_2 = \frac{\mathrm{sgn}(\mathrm{b})\, b^2 + a^2 - \mathrm{sgn}(\mathrm{c})\, c^2}{2a}. \tag{3.8}$$

So we have to check when $x_2^2 - \mathrm{sgn}(\mathrm{b})\, b^2$ is $\geqslant 0$. First consider the case $\mathrm{sgn}(\mathrm{b}) = -1$, then $x_2^2 + b^2 \geqslant 0$ independent of $\mathrm{sgn}(\mathrm{c})$. In the case that $\mathrm{sgn}(\mathrm{b}) = 1$ we get

$$b^2 - x_1^2 = -\left(\frac{(a+b)^2 - \mathrm{sgn}(\mathrm{c})\, c^2}{2a}\right)\left(\frac{\mathrm{sgn}(\mathrm{c})\, c^2 - (a-b)^2}{2a}\right).$$

For $\mathrm{sgn}(\mathrm{c}) = -1$ this expression is always positive. If $\mathrm{sgn}(\mathrm{c}) = -1$ the first factor is always positive, so the second factor is required to be $\leqslant 0$. This condition holds exactly for $b + c \leqslant a$. If $a < 0$ the calculations are similar, one just has to set $\hat{a} := (|a|, 0)$. Finally if $a = 0$ we have to solve

$$\begin{aligned} \langle x, x \rangle_1 &= \mathrm{sgn}(\mathrm{b})\, b^2 \\ \langle x, x \rangle_1 &= \mathrm{sgn}(\mathrm{c})\, c^2. \end{aligned}$$

These two hyperbolas only intersect if they agree, so $\mathrm{sgn}(\mathrm{b}) = \mathrm{sgn}(\mathrm{c})$ and $b = c$. In this case there are infinitely many solutions. These degenerate triangles are all isometric by Lorentz transformations in $\mathcal{L}(2)$, even if they don't lie on the same connected component of the hyperbola. For more details see [5, Examples 9.4 p. 236]. That two triangles which have the same side lengths are isometric will be shown in the proof of the next lemma. ☐

Lemma 3.2.4 (Realizability)**.** *[1, Lemma 2.1 p. 257] A point in $\mathbb{R}^3 \backslash 0$ can be realized by a triangle in the model spaces of constant curvature K as follows*

(i) A point in T^+ can be realized by a triangle in S_K if the sum of the side lengths is $< \frac{2\pi}{\sqrt{K}}$. A point in T^- can be realized in $-S_K$ if the sum of its side lengths is $> -\frac{2\pi}{\sqrt{K}}$. In particular the largest side length has to be $< \frac{\pi}{\sqrt{K}}$, respectively the smallest side length $> -\frac{\pi}{\sqrt{K}}$.

(ii) A point in D^+ can be realized by a triangle in S_K and M_K if the largest side length is $< \frac{\pi}{\sqrt{K}}$. A point in D^- can be realized by a triangle in $-S_K$ and M_K if the smallest coordinate is $> -\frac{\pi}{\sqrt{K}}$.

(iii) Every point in $C_{T,D}$ is realized by a unique triangle in $M_0 = \mathbb{R}^2_1$. For $K > 0$ a realizing triangle exists if the largest coordinate is $< \frac{\pi}{\sqrt{K}}$. For $K < 0$ a realizing triangle exists if the smallest coordinate is $> -\frac{\pi}{\sqrt{-K}}$.

These realizations are unique up to isometry of the model spaces and for $K \leqslant 0$ we set $\frac{\pi}{\sqrt{K}} := \infty$, so no size bounds apply.

Proof. For $K = 0$ we have seen in Lemma 3.2.2 that for any non null triple there is a unique realizing triangle in S_0 up to isometry. The same holds for non null negative triples which are realized in $-S_0$. The result for M_0 is shown in *Lemma* 3.2.3. Let Δpqr be a triangle in $\mathcal{Q}_K$, and set $a := |pr|_\pm$, $b := |pq|_\pm$, $c := |qr|_\pm$ with sides parametrized by geodesics γ_{pr}, γ_{pq} and γ_{qr}. We now show which side lengths (a, b, c) can be realized in a given surface of constant curvature K and which conditions have to hold. For $K \neq 0$, we first consider the Riemannian case, so all the side lengths are $\geqslant 0$. Then w.l.o.g. we may suppose that $a \geqslant b \geqslant c \geqslant 0$ and in particular $a > 0$, since by definition at least one side length has to be non zero. But then also $b > 0$, since otherwise the triple of side lengths would be 0. The side lengths are in D^+ if our triangle is given by three points on a geodesic segment of length $a < \frac{\pi}{\sqrt{K}}$, which implies that $a = b + c$. The other kind of degenerate triangle on the sphere is given by a great circle. In this case, one has to assume $a < \frac{\pi}{\sqrt{K}}$ to exclude antipodal points, which then implies $c > 0$. We then have $a = b + c$ and $a + b + c = \frac{2\pi}{\sqrt{K}}$. Now assume that all sides have non zero length, then by the law of cosines (3.5) we get

$$\cos\left(\sqrt{K}a\right) = \cos\left(\sqrt{K}b\right)\cos\left(\sqrt{K}c\right) + \frac{\angle pqr}{bc}\sin\left(\sqrt{K}b\right)\sin\left(\sqrt{K}c\right). \tag{3.9}$$

Since $\gamma'_{qp}(0)$ and $\gamma'_{qr}(0)$ are contained in a spacelike plane and $bc \neq 0$, we see that

$$\cos(\alpha) = \frac{\angle pqr}{bc}$$

for a unique $\alpha \in [0, \pi]$. Now we keep b and c constant. Then in case $K > 0$, if we increase α from 0 to π, the right hand side of (3.9) monotonically decreases from $\cos(\sqrt{K}(b - c))$ to $\cos(\sqrt{K}(b + c))$ [3]. So the law of cosines shows that $\cos(\sqrt{K}a) \geqslant \cos(\sqrt{K}(b + c))$ and so $a \leqslant b + c$. We have to assume $a < \frac{\pi}{\sqrt{K}}$, since otherwise there is no unique geodesic connecting p with r. To see the bound on the sum of the side lengths, let m be the center of the circumscribing circle of Δpqr. The point m is obtained as follows: First, let I be the intersection of the sets $\{x \in S_K \mid d(p, x) = d(q, x)\}$, $\{x \in S_K \mid d(r, x) = d(q, x)\}$ and $\{x \in S_K \mid d(p, x) = d(r, x)\}$, then define $m := \min_{x\in I} d(p, x)$. The definition of m implies that $d(p, m) = d(q, m) = d(r, m) =: R$. In case $R = \frac{\pi}{2\sqrt{K}}$ it follows that p, q, r lie on one great circle, thus $(a, b, c) \in D^+$. First we show that $R \leqslant \frac{\pi}{2\sqrt{K}}$, since suppose $R > \frac{\pi}{2\sqrt{K}}$, then the points p, q, r are contained in a closed ball of radius $< \frac{\pi}{2\sqrt{K}}$ around the antipodal point $-m$ to m, which is convex by Corollary 1.8.14. But $-m \in I$, which contradicts the minimality of m. Now consider three radial geodesics starting at m going through the points p, q, r and let p', q', r' be the intersections of these geodesics with the distance sphere $S_{\frac{\pi}{2\sqrt{K}}}(m)$ around m, which is a great circle on the sphere. Then the law of cosines shows that $a = d(p, r) < d(p', r')$, $b = d(p, q) < d(p', q')$ and $c = d(q, r) < d(q', r')$. This shows $a + b + c < \frac{2\pi}{\sqrt{K}}$ since $d(p', q') + d(p', r') + d(q', r') = \frac{2\pi}{\sqrt{K}}$. For $K < 0$ equation (3.9) transforms to

$$\begin{aligned}\cosh\left(\sqrt{K}a\right) &= \cosh\left(\sqrt{|K|}b\right)\cosh\left(\sqrt{|K|}c\right) \\ &\quad - \cos(\alpha)\sinh\left(\sqrt{|K|}b\right)\sinh\left(\sqrt{|K|}c\right),\end{aligned} \tag{3.10}$$

[3] This follows by the sum rule $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

so if we increase α from 0 to π, we see that the right hand side of the above equation strictly increases from $\cosh(\sqrt{K}(b-c))$ to $\cosh(\sqrt{K}(b+c))$ [4]. But then by (3.10), $\cosh(\sqrt{K}a) \leqslant \cosh(\sqrt{K}(b+c))$ and so $a \leqslant b+c$. In hyperbolic spaces triangles of arbitrary side length can be realized. This is seen by the form of the exponential map (1.25), which allows geodesic segments of arbitrary length, which then are the unique geodesics between their endpoints. So we can construct hinges of arbitrary size and by varying the angle of the hinge, we get a triangle of the required third side length by the law of cosines (3.5). Triples in D^+ are realized in the hyperbolic space by three points on a common geodesic. The analogous statements for $-S_K$ then follow since it is anti-isometric to S_K. Now for the indefinite constant curvature surface M_K, first suppose that all side lengths of Δpqr are non null. If tangent vectors v and w are both timelike, the ***hyperbolic angle*** α between them is defined by

$$\langle v, w\rangle = \begin{cases} -\|v\|\|w\|\cosh(\alpha) & \text{if } v, w \text{ are in the same timecone} \\ \|v\|\|w\|\cosh(\alpha) & \text{if } v, w \text{ are in opposite timecones} \end{cases}$$

for a unique $\alpha \geqslant 0$. If both vectors are spacelike and nonnull, the hyperbolic angle is given by

$$\langle v, w\rangle = \begin{cases} \|v\|\|w\|\cosh(\alpha) & \text{if } \langle v, w\rangle > 0 \\ -\|v\|\|w\|\cosh(\alpha) & \text{if } \langle v, w\rangle < 0 \end{cases}$$

for a unique $\alpha \geqslant 0$. First we consider the case that all side lengths are positive, then $\gamma'_{pq}(0)$ and $\gamma'_{qr}(0)$ are in opposite timecones since γ_{pq} is parametrized towards q, while γ_{qr} is parametrized starting at q. So in this case we have

$$\angle pqr = -bc\cosh(\alpha)$$

for $\alpha \geqslant 0$. Then inserting this into (3.9) yields

$$\cos\left(\sqrt{K}a\right) = \cos\left(\sqrt{K}b\right)\cos\left(\sqrt{K}c\right) - \cosh(\alpha)\sin\left(\sqrt{K}b\right)\sin\left(\sqrt{K}c\right).$$

Now again fix the side lengths b and c. First, for $K > 0$, the largest side length has to satisfy $a < \frac{\pi}{\sqrt{K}}$. For $\alpha = 0$ the right hand side of the above equation is $\cos(\sqrt{K}(b+c))$ and decreases for growing α, since $\sin(\sqrt{K}b)\sin(\sqrt{K}c) \geqslant 0$ for $b, c \in [0, \frac{\pi}{\sqrt{K}})$. This shows that $\cos(\sqrt{K}a) \leqslant \cos(\sqrt{K}(b+c))$ and so $b+c \leqslant a$, thus $(a,b,c) \in C_{T,D}$. Also it holds that $(a,b,c) \in D^+$ if the triangle lies on the equator of the hyperquadric, which is a spacelike periodic geodesic and has circumference $\frac{2\pi}{\sqrt{K}}$. Thus we conclude that triangles on the equator of M_K have the same properties as triangles on a great arc of S_K. For $K < 0$ we get that $\cosh(\sqrt{|K|}(b+c)) \leqslant \cosh(\sqrt{|K|}a)$, so the inverse triangle inequality $a \geqslant b+c$ has to hold, and there are no size bounds to a. Recall that for $K < 0$ the spacelike geodesics are branches of hyperbolas, see Lemma 1.8.9. The degenerate triangles are thus given by three points on a common hyperbola, so they are the same as in the hyperbolic plane. In the case that two sides have positive length and one side has negative length, then w.l.o.g. we can assume that $a \geqslant b > 0 > c$. Then inserting into the law of cosines yields

$$\cosh\left(\sqrt{K}\sqrt{|E(\gamma_{qr})|}\right) = \cos\left(\sqrt{K}a\right)\cos\left(\sqrt{K}b\right) + \cosh(\gamma)\sin\left(\sqrt{K}a\right)\sin\left(\sqrt{K}b\right).$$

[4] Since $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.

Then the right hand side for fixed a, b and $K > 0$ is $\cos(\sqrt{K}(a-b))$ for $\gamma = 0$, and increases for growing γ. This shows that $\cosh(\sqrt{K}\sqrt{|E(\gamma_{qr})|}) \geqslant \cos(\sqrt{K}(a-b))$, which is always satisfied. So in this case one only has to assume that $a < \frac{\pi}{\sqrt{K}}$. For $K < 0$ we get

$$\begin{aligned}\cos\left(\sqrt{|K|}\sqrt{|E(\gamma_{qr})|}\right) &= \cosh\left(\sqrt{|K|}a\right)\cosh\left(\sqrt{|K|}b\right)\\ &\quad - \cosh(\gamma)\sinh\left(\sqrt{|K|}a\right)\sinh\left(\sqrt{|K|}b\right).\end{aligned}$$

Then for $\gamma = 0$ the right hand side is $\cosh(\sqrt{K}(a-b))$ and is decreasing for growing γ. This shows $\cos(\sqrt{|K|}\sqrt{|E(\gamma_{qr})|}) \leqslant \cosh(\sqrt{K}(a-b))$, which again is always satisfied and a can be arbitrary big since $K < 0$. One only has to assume that $c > \frac{-\pi}{\sqrt{-K}}$ since in pseudohyperbolic space timelike geodesics are periodic, see Lemma 1.8.9. Now in case that two side lengths are negative we may w.l.o.g. assume that $a > 0 > b > c$. then for $K > 0$ we get

$$\begin{aligned}\cos\left(\sqrt{K}a\right) &= \cosh\left(\sqrt{K}\sqrt{|E(\gamma_{pq})|}\right)\cosh\left(\sqrt{K}\sqrt{|E(\gamma_{qr})|}\right)\\ &\quad - \cosh(\alpha)\sinh\left(\sqrt{K}\sqrt{|E(\gamma_{pq})|}\right)\sinh\left(\sqrt{K}\sqrt{|E(\gamma_{qr})|}\right).\end{aligned}$$

Then since $E(\gamma_{pq}) = -|pq|_\pm = -b$ and $E(\gamma_{qr}) = -|qr|_\pm = -c$, we see that for $K > 0$ the right hand side for $\alpha = 0$ is $\cosh(-\sqrt{K}(b-c))$ and decreases for growing α. We get $\cos(\sqrt{K}a) \leqslant \cosh(-\sqrt{K}(b-c))$, which is always satisfied, so only $a < \frac{\pi}{\sqrt{K}}$ has to hold. The case $K < 0$ is shown analogously and similar calculations also show the case, where all the side lengths are negative. Now the only case remaining is the null case, so given $a > 0$ suppose $c = 0$. Then we draw the side a on the equator of the hyperquadric. The null geodesics starting at q are given by a pair of straight lines in the hyperquadric, which are also straight lines in the ambient space. Then the distance sphere of radius b around p is given by a translate of the tangent plane at p. These lines and the plane intersect in three-dimensional space if they are not parallel, so $a < \frac{\pi}{\sqrt{K}}$ has to hold in the case $K < 0$. Note that these two points of intersection then are in the hyperquadric since the null lines are. Since by assumption $b \leqslant a$, the inverse triangle inequality holds. That the two realizing triangles are isometric will be shown next. Also for $a < 0$ there is nothing to show since then by assumtion the other sides cannot be null and the case $a = 0$ is as in Lemma 3.2.3.

Now it only remains to show that that triangles with the same side lengths are isometric. Here we will also prove the case for $K = 0$. To do so, let Δpqr and $\Delta\tilde{p}\tilde{q}\tilde{r}$ be triangles in $\mathcal{Q}_K$ such that all three side lengths agree. Set $v_1 := \gamma'_{pq}(0)$, and $v_2 := \gamma'_{pr}(0)$, $w_1 := \gamma'_{\tilde{p}\tilde{q}}(0)$, and $w_2 := \gamma'_{\tilde{p}\tilde{r}}(0)$. Now using the Gram-Schmidt process, first set $e_1 := v_1$, $f_1 := w_2$ and then

$$\begin{aligned} e_2 &= v_2 - \frac{\langle v_1, v_2\rangle}{\langle v_1, v_1\rangle} v_1\\ f_2 &= w_2 - \frac{\langle w_1, w_2\rangle}{\langle w_1, w_1\rangle} w_1.\end{aligned} \tag{3.11}$$

By definition $\langle v_1, v_2\rangle = \langle \gamma'_{pq}(0), \gamma'_{pr}(0)\rangle = \angle qpr$ and $\langle w_1, w_2\rangle = \langle \gamma'_{\tilde{p}\tilde{q}}(0), \gamma'_{\tilde{p}\tilde{r}}(0)\rangle =$

$\angle\tilde{q}\tilde{p}\tilde{r}$. Then the law of cosines[5] shows that $\angle qpr = \angle\tilde{q}\tilde{p}\tilde{r}$, respectively $\langle v_1, v_2\rangle = \langle w_1, w_2\rangle$, and so the factors on the right hand side of (3.11) in front of v_1 respectively w_2 agree. We set $\omega := \frac{\langle v_1, v_2\rangle}{\langle v_1, v_1\rangle}$. Since the side lengths of the two triangles agree, also $\langle v_1, v_1\rangle = \langle w_1, w_1\rangle$ and $\langle v_2, v_2\rangle = \langle w_2, w_2\rangle$, so one concludes that also $\langle e_1, e_1\rangle = \langle f_1, f_1\rangle$. Using this we see that

$$\langle e_2, e_2\rangle = \omega^2 \underbrace{\langle v_1, v_1\rangle}_{=\langle w_1, w_1\rangle} - 2\omega \underbrace{\langle v_1, v_2\rangle^2}_{=\langle w_1, w_2\rangle^2} + \underbrace{\langle v_2, v_2\rangle}_{=\langle w_2, w_2\rangle} = \langle f_2, f_2\rangle.$$

Then by Gram-Schmid we get orthogonal bases $e_1, \ldots, e_n$ for $T_p\mathcal{Q}_K$ and $f_1, \ldots, f_n$ for $T_{\tilde{p}}\mathcal{Q}_K$, where $n = \dim \mathcal{Q}_K$, and such that $\langle e_i, e_i\rangle = \langle f_i, f_i\rangle$ for $i = 1, \ldots, n$. By [5, Proposition 4.30 p. 113] there exists an unique isometry $\phi : T_p\mathcal{Q}_K \to T_{\tilde{p}}\mathcal{Q}_K$ of $\mathcal{Q}_K$ such that $d\phi(e_i) = f_i$ for $i = 1, \ldots, n$. Hence

$$\begin{aligned} d\phi(v_1) &= w_1 \\ d\phi(v_2) &= d\phi(e_1) + \omega\, d\phi(e_2) = f_1 + \omega f_2 = w_2, \end{aligned}$$

so ϕ gives an isometry between the two hinges and hence also between Δpqr and $\Delta\tilde{p}\tilde{q}\tilde{r}$. □

Remark One interesting way to visualize the degenerate triangles is as follows: Consider a sphere, the de Sitter space and the hyperbolic plane, all of the same radius, embedded in the same three dimensional space. The sphere and the de Sitter space intersect in a great arc on the sphere. Then consider a triangle on the sphere where one side, which we keep fixed, lies on this great arc. Then we decrease the other two side lengths until the triangle gets degenerate, so it lies on this great arc. Now if we want to decrease the two side lengths even further we cannot continue on the sphere since the triangle inequality will be violated. Thus we have to continue in the de Sitter space where the reverse triangle inequality holds and also negative side lengths are possible. The de Sitter space and the hyperbolic plane intersect in one hyperbola, where degenerate triangles are realized in both spaces.

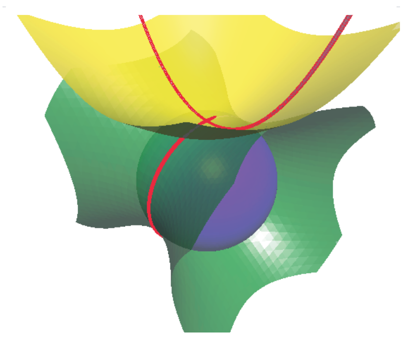

Figure 3.1: The Hyperbolic plane (yellow), the sphere (blue) and de Sitter space (green), where intersections are marked by red lines.

[5] For $K = 0$ law of cosines will be a byproduct of the proof of Lemma 3.3.1 and is given by (3.12).

As a Corollary to Lemma 3.2.4 we get the law of cosines for the Riemannian surfaces of constant curvature.

Corollary 3.2.5. *Given a triangle on a sphere of radius r with side lengths a, b, c, let γ denote the angle at the vertex opposite to the side with length c. Then the curvature is given by $K = \frac{1}{r^2}$ and the law of cosines takes of the form*

$$\cos \sqrt{K} c = \cos \sqrt{K} a \cos \sqrt{K} b + \cos \gamma \sin \sqrt{K} a \sin \sqrt{K} b.$$

For the hyperbolic plane of radius r, the curvature is given by $K = -\frac{1}{r^2}$. Then the law of cosines is

$$\cosh \sqrt{-K} c = \cosh \sqrt{-K} a \cosh \sqrt{-K} b - \cos \gamma \sinh \sqrt{-K} a \sinh \sqrt{-K} b.$$

The Euclidean plane $\mathbb{R}^2$ is of constant curvature $K = 0$ and the law of cosines is of the form

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

Proof. This follows from the proof of Lemma 3.2.4. For $K = 0$ the result follows from equation (3.12) in the proof Lemma 3.3.1, which we will prove in the following section and which is independent of this corollary. $\square$

Definition 3.2.6. *Let M be a semi-Riemannian manifold and U a normal neighborhood in M. Then U is called* **normal for** *K if every triangle Δpqr in U, with side lengths $(|qp|_\pm, |qr|_\pm, |pr|_\pm) \in \mathbb{R}^3 \backslash 0$, can be realized in a corresponding model space $\mathcal{Q}_K$ of curvature K. Suppose Δpqr satisfies size bounds for K. Let $\Delta \tilde{p}\tilde{q}\tilde{r}$ in $\mathcal{Q}_K$ be a triangle such that*

$$(|\tilde{q}\tilde{p}|_\pm, |\tilde{q}\tilde{r}|_\pm, |\tilde{p}\tilde{r}|_\pm) = (|qp|_\pm, |qr|_\pm, |pr|_\pm),$$

then $\Delta \tilde{p}\tilde{q}\tilde{r}$ is called **comparison triangle** *or* **Alexandrov triangle** *to Δpqr.*

Remark: So U is normal for K if all possible triples of side lengths for triangles in U satisfy the corresponding bounds in Lemma 3.2.4.

3.3 Hinge Lemma and Straightening Lemma For Shoulder Angles

Lemma 3.3.1 (Hinge lemma)**.** *[1, Lemma 2.2 p. 259] Let $(|pq|_\pm, |qr|_\pm, |pr|_\pm) \in \mathbb{R}^3 \backslash 0$ be a non-negative triple, which can be realized by a triangle Δpqr in a model space $\mathcal{Q}_K$ of constant curvature K. If we vary the size of the third coordinate $|pr|_\pm$ while keeping $|pq|_\pm$ and $|qr|_\pm$ fixed, then*

(i) The angle $\angle pqr$ is a decreasing function of $|pr|_\pm$.

(ii) The angles $\angle qpr$ and $\angle qrp$ are increasing functions of $|pr|_\pm$.

Remarks

1. *Let σ_1, σ_2 be geodesics parametrized by $[0,1]$. Recall that points on both geodesics are called corresponding if they have the same affine parameter.*

2. *By varying one side length, the model space which realizes the triangle can change.*

3. *This result may seem counterintuitive at first, since in statement (i), one might expect $\angle pqr$ to be an increasing function of $|pr|_\pm$, and similar for (ii). This is because angles are by definition nonnormalized. For example in flat Euclidean space, if we consider the usual definition of angle,*

$$\cos(\alpha) = \frac{\angle pqr}{\|pq\|\|qr\|},$$

then if α is increasing on $[0,\pi]$, $\angle pqr$ is decreasing since $\cos$ *is decreasing on $[0,\pi]$.*

4. *The angle $\angle pqr$ is called the* **included angle**, *while $\angle qpr$ and $\angle qrp$ are called* **shoulder angles**.

Proof. First we will show the claim for $K = 0$. Then the model surfaces are $\mathbb{R}^2$, either equipped with the usual Euclidean metric, the Euclidean metric with sign switched, or with indefinite scalar product, which yields the 2-dimensional Minkowski space. In all three spaces geodesics are given by straight lines and thus the sides of Δpqr can be represented by vectors. Let $P \in \mathbb{R}^2$ be the side from q to p, and $R \in \mathbb{R}^2$ the side connecting q with r. Then the side connecting p with r is described by $R - P$. The unique geodesics connecting the vertices $\gamma_{qp}, \gamma_{qr}, \gamma_{pr} : [0,1] \to \mathbb{R}^2$ are given by $\gamma_{qp}(t) := q + tP$, $\gamma_{qr}(t) := q + tR$ and $\gamma_{pr}(t) = p + t(R-P)$. The nonnormalized angle $\angle pqr$ then can be computed by $\angle pqr = \langle \gamma'_{qp}(0), \gamma'_{qr}(0)\rangle = \langle P, R\rangle$. The other two angles are given by $\angle qpr = \langle -P, R-P\rangle = \langle P, P-R\rangle$ and $\angle qrp = \langle -R, P-R\rangle = \langle R, R-P\rangle$. Further, by definition $E_q(p) = \langle P, P\rangle$, $E_q(r) = \langle R, R\rangle$ and $E_p(r) = \langle R-P, R-P\rangle$. By (1.17) we know that the energy and the signed length are related by $E_q(p) = \mathrm{sgn}(\gamma_{\mathrm{qp}})|qp|_\pm{}^2$ and so

$$\begin{aligned} |pr|_\pm{}^2 &= \mathrm{sgn(R\text{-}P)}\langle R-P, R-P\rangle \\ &= \mathrm{sgn(R\text{-}P)}\left(\langle P,P\rangle + \langle R,R\rangle - 2\langle R,P\rangle\right) \\ &= \mathrm{sgn(R\text{-}P)}\left(\mathrm{sgn(P)}|qp|_\pm{}^2 + \mathrm{sgn(R)}|qr|_\pm{}^2 - 2\angle pqr\right). \end{aligned} \tag{3.12}$$

Note that the above equation is the law of cosines for flat space. Since $|qp|_\pm$ and $|qr|_\pm$ are assumed to be fixed, we set $k := \mathrm{sgn(P)}|qp|_\pm{}^2 + \mathrm{sgn(R)}|qr|_\pm{}^2$. First suppose that $\mathrm{sgn(R\text{-}P)} = 1$, then $|pr|_\pm \geqslant 0$ and so if $|pr|_\pm$ increases also $|pr|_\pm{}^2$ increases and (3.12) becomes

$$|pr|_\pm{}^2 = k - 2\angle pqr.$$

So since k is a constant, it follows that for increasing $|pr|_\pm$, $\angle pqr$ has to decrease. On the other hand, if $\mathrm{sgn(R\text{-}P)} = -1$, then $|pr|_\pm < 0$, and so for $|pr|_\pm$ increasing, $|pr|_\pm{}^2$ is decreasing. But then

$$|pr|_\pm{}^2 = 2\angle pqr - k.$$

shows that $\measuredangle pqr$ also has to decrease, so we have shown (i). To see (ii) for $K = 0$, note that

$$\measuredangle qrp = \langle -R, P - R\rangle = -\langle P, R\rangle + \langle R, R\rangle = -\measuredangle pqr + \mathrm{sgn(R)}|qr|_{\pm}{}^2.$$

So since by (i), if $|pr|_{\pm}$ increases then $\measuredangle pqr$ decreases, thus $\measuredangle qrp$ increases since $|qr|_{\pm}$ is assumed to be constant. And similarly one sees that $\measuredangle qpr$ is an increasing function of $|pr|_{\pm}$. Next we consider the case $K > 0$, w.l.o.g. assuming that $K = 1$. So let Δpqr be a comparison triangle in a model space of constant curvature K, with geodesics γ_{qr}, γ_{qp} and γ_{pr} connecting the corresponding vertices. Then we set $R := \gamma'_{qr}(0)$, $P := \gamma'_{qp}(0)$ and $L := \gamma'_{pr}(0)$. The three points p, q, r also span a plane Π in the ambient space of the hyperquadric, and if the sides of the triangle are nonnull the metric is nondegenerate on Π. So Π is isometric to one of the model surfaces of constant curvature 0. Denote by R' the vector in this plane connecting q with r, and let P' be defined analogously. Further, let L' be the vector connecting p with r. By Lemma 1.8.13 we know that the distance spheres around q are given by parallel translation of the tangent plane $T_q\mathcal{Q}_K$, which is then intersected with $\mathcal{Q}_K$. Thus R' can be written as a linear combination of $R \in T_q\mathcal{Q}_K$ and q which is orthogonal to $T_q\mathcal{Q}_K$ by Lemma 1.8.7. So $R' = \alpha R - \mu_1 q$, $P' = \beta P - \mu_2 q$ and $L' = \lambda L - \mu_3 p$ for $\mu_1, \mu_2, \mu_3 \geqslant 0$ and $\alpha, \beta, \lambda > 0$. To show that $\langle L', L'\rangle$ is an increasing function of $|pr|_{\pm}$, first suppose that $0 \leqslant |pr|_{\pm} \leqslant \pi$, then by Lemma 1.8.13 we see that $\mu_1 = 1 - \cos|pr|_{\pm}$ and so

$$\begin{aligned}\langle L', L'\rangle &= \lambda^2\langle L, L\rangle + (1 - \cos|pr|_{\pm})^2 \overbrace{\langle p, p\rangle}^{=1} \\ &= \lambda^2|pr|_{\pm}{}^2 + (1 - \cos|pr|_{\pm})^2.\end{aligned}$$

Then the claim follows since $(1-\cos(t))^2$ is an increasing function on $[0, \pi]$, and so $\langle L', L'\rangle$ is an increasing function of $|pr|_{\pm}$ independent of λ. For $|pr|_{\pm} < 0$, μ_1 is given by $\mu_1 = \cosh|pr|_{\pm} - 1$ and $\lambda = \sinh\|pr\|$, which is seen by Lemma 1.8.10. Then

$$\begin{aligned}\langle L', L'\rangle &= (\sinh(\|pr\|))^2\langle L, L\rangle + (\cosh|pr|_{\pm} - 1)^2 \overbrace{\langle p, p\rangle}^{=1} \\ &= -(\sinh(\|pr\|))^2|pr|_{\pm}{}^2 + (\cosh|pr|_{\pm} - 1)^2\end{aligned} \tag{3.13}$$

and the claim follows since the right hand side is an increasing function of $|pr|_{\pm}$ for $|pr|_{\pm} < 0$. But this then also shows that $\langle R', R'\rangle$ and $\langle P', P'\rangle$ are increasing functions of $|qr|_{\pm}$ respectively $|qp|_{\pm}$. In Π the angle between P' and R' is given by

$$\langle P', R'\rangle = \alpha\beta\langle P, R\rangle + \mu_1\mu_2.$$

Then since $\measuredangle pqr = \langle P, R\rangle$,

$$\measuredangle pqr = \frac{1}{\alpha\beta}(\langle P', R'\rangle - \mu_1\mu_2).$$

If $|pr|_{\pm}$ increases, also $\langle L', L'\rangle$ increases and so $\langle P', R'\rangle$ decreases by (i) for the case $K = 0$. But then by the above equation also $\measuredangle pqr$ has to decrease. Now for the shoulder angles,

$$\langle P', L'\rangle = \alpha\lambda\langle P, L\rangle + \mu_2\mu_3\langle q, p\rangle, \tag{3.14}$$

so since $\langle -P, L\rangle = \angle qpr$ we have

$$\angle qpr = \frac{1}{\alpha\lambda}(\langle -P', L'\rangle - \mu_2\mu_3\langle q, p\rangle). \tag{3.15}$$

Again, if $|pr|_\pm$ increases also $\langle L', L'\rangle$ increases and so $\langle -P', L'\rangle$ increases by (ii) for $K = 0$. Then the above equation shows that $\angle qpr$ increases. The case $K < 0$ is shown in a similar fashion. Just note that the pseudohyperbolic spaces H_0^2 and H_1^2 are semi-Riemannian hypersurfaces of sign -1, so every normal vector is timelike. In particular any point q in either space satisfies $\langle q, q\rangle < 0$. The realizing space for Δpqr changes when the line $t \mapsto (|pq|_\pm, |qr|_\pm, t)$ passes through D^+. In D^+ angles agree in both realizing model spaces as we have seen in the proof of Lemma 3.2.4. Then since (i) and (ii) hold on every closed segment in one model space, the claim is true along the whole line.

□

Lemma 3.3.2 (Straightening Lemma for Shoulder Angles - Alexandrov Lemma)**.** *[1, Lemma 2.4 p. 261] Let Δpqr be a triangle in a semi-Riemannian manifold M such that the triple $(|pq|_\pm, |qr|_\pm, |rp|_\pm) \neq 0$ satisfies size bounds for some $K \in \mathbb{R}$. Let $\gamma_{pr} : [0,1] \to M$ be the unique geodesic connecting p with r, and for $\lambda \in [0,1]$ set $m := \gamma_{pr}(\lambda)$. Let $\Delta q_1p_1m_1$ and $\Delta q_2m_2r_2$ be triangles in a corresponding model space of curvature K such that $|q_1m_1|_\pm = |q_2m_2|_\pm = |qm|_\pm$, $|q_1p_1|_\pm = |qp|_\pm$, $|q_2r_2|_\pm = |qr|_\pm$, $|p_1m_1|_\pm = |pm|_\pm$ and $|m_2r_2|_\pm = |mr|_\pm$. If $\Delta q_1p_1m_1$ and $\Delta q_2m_2r_2$ satisfy size bounds for K and*

$$(1-\lambda)\angle p_1m_1q_1 + \lambda\angle r_2m_2q_2 \geqslant 0, \tag{3.16}$$

then

$$\angle qpm \geqslant \angle q_1p_1m_1 \quad \text{and} \quad \angle qrm \geqslant \angle q_2r_2m_2. \tag{3.17}$$

The above statement with inequalities in (3.16) *and* (3.17) *reversed is also true.*

Proof. First we show that

$$(1-\lambda)\angle pmq + \lambda\angle rmq = 0. \tag{3.18}$$

To see this, note that in the definition of the nonnormalized angle geodesics are assumed to be parametrized by $[0,1]$. Now $\gamma_{mr}(t) := \gamma_{pr}(\lambda + (1-\lambda)t)$ is a geodesic connecting m with r if parametrized by $[0,1]$. And $\gamma_{mp} : [0,1] \to M$, $\gamma_{mp}(t) := \gamma_{pr}(\lambda - \lambda t)$ connects m with p. Then $\gamma'_{mr}(0) = (1-\lambda)\gamma'_{pr}(\lambda)$, $\gamma'_{mp}(0) = -\lambda\gamma'_{pr}(\lambda)$, and so

$$\begin{aligned}(1-\lambda)\angle pmq + \lambda\angle rmq &= (1-\lambda)\langle\gamma'_{mp}(0), \gamma'_{mq}(0)\rangle + \lambda\langle\gamma'_{mr}(0), \gamma'_{mq}(0)\rangle \\ &= (1-\lambda)(-\lambda)\langle\gamma'_{pr}(\lambda), \gamma'_{mq}(0)\rangle + \lambda(1-\lambda)\langle\gamma'_{pr}(\lambda), \gamma'_{mq}(0)\rangle \\ &= 0\end{aligned}$$

shows (3.18). Subtracting (3.18) from (3.16) yields

$$(1-\lambda)(\angle p_1m_1q_1 - \angle pmq) + \lambda(\angle r_2m_2q_2 - \angle rmq) \geqslant 0.$$

From this we get $\angle p_1m_1q_1 \geqslant \angle pmq$ or $\angle r_2m_2q_2 \geqslant \angle rmq$. To see this, suppose $\angle p_1m_1q_1 < \angle pmq$ and $\angle r_2m_2q_2 < \angle rmq$, then since $\lambda, 1-\lambda > 0$ we see

$$(1-\lambda)(\angle p_1m_1q_1 - \angle pmq) + \lambda(\angle r_2m_2q_2 - \angle rmq) < 0,$$

which gives a contradiction. By Lemma 3.3.1 we know that the shoulder non-normalized angle in a triangle is an increasing function of the signed lengths of the sides enclosing this angle. So suppose that $\angle p_1 m_1 q_1 \geqslant \angle pmq$, then we get $|q_1 m_1|_\pm \geqslant |qm|_\pm$. Since by assumption $|q_1 m_1|_\pm = |q_2 m_2|_\pm$, also $|q_2 m_2|_\pm \geqslant |qm|_\pm$ is true. By statement (i) in Lemma 3.3.1, in a triangle the nonnormalized angles are a decreasing function of the signed length of the side opposite to it. Thus we get (3.17).

□

3.4 Equivalent Definitions for Triangle Comparison

Lemma 3.4.1. *Suppose Δpqr satisfies size bounds for K. Then let $\Delta\tilde{p}\tilde{q}\tilde{r}$ be a comparison triangle in the corresponding model space $\mathcal{Q}_K$. Given a point m on the side γ_{pq} with affine parameter λ, and the corresponding point $\tilde{m}$ on $\gamma_{\tilde{p}\tilde{q}}$, then if $\angle rpq \leqslant \angle\tilde{r}\tilde{p}\tilde{q}$ also $\angle rpm \leqslant \angle\tilde{r}\tilde{p}\tilde{m}$. The result with all inequalities reversed also holds.*

Proof. The unique geodesic $\gamma_{pm} : [0,1] \to U$ connecting p with m is given by $\gamma_{pm}(t) := \gamma_{pq}(\lambda t)$, and $\gamma'_{pm}(0) = \lambda\gamma'_{pq}(0)$ And for $\gamma_{\tilde{p}\tilde{m}}(t) := \gamma_{\tilde{p}\tilde{q}}(\lambda t)$ we get $\gamma'_{\tilde{p}\tilde{m}}(0) = \lambda\gamma'_{\tilde{p}\tilde{m}}(0)$, so

$$\begin{aligned}\angle rpm &= \langle\gamma'_{pr}(0), \gamma'_{pm}(0)\rangle \\ &= \lambda\langle\gamma'_{pr}(0), \gamma'_{pq}(0)\rangle = \lambda\angle rpq \leqslant \lambda\angle\tilde{r}\tilde{p}\tilde{q} = \ldots = \angle\tilde{r}\tilde{p}\tilde{q}\end{aligned}$$

which shows the claim. □

Proposition 3.4.2. *[1, Proposition 5.1 p. 270] Let M be a semi-Riemannian manifold and U be a normal neighborhood in M which is normal for K. If we choose $\mathcal{Q}_K$ with the same dimension and index as M, then the following conditions on all triangles Δpqr in U with comparison triangle in $\Delta\tilde{p}\tilde{q}\tilde{r}$ in $\mathcal{Q}_K$ are equivalent:*

(i) The signed distance between points on the sides of Δpqr is $\geqslant(\leqslant)$ the signed distance of the corresponding points in $\Delta\tilde{p}\tilde{q}\tilde{r}$.

(ii) The signed distance from a vertex of Δpqr to a point on the opposite side is $\geqslant(\leqslant)$ the signed distance of the corresponding points in $\Delta\tilde{p}\tilde{q}\tilde{r}$.

(iii) The nonnormalized angles in Δpqr are $\leqslant(\geqslant)$ the corresponding nonnormalized angles in $\Delta\tilde{p}\tilde{q}\tilde{r}$.

Remark It is essential that the conditions hold for all triangles in U.

Proof. (i)⇒(ii): since the inequality holds for all points, it also holds if we fix one point to be a vertex of Δpqr. To see the reverse implication (ii)⇒(i), let Δpqr be a triangle in U with comparison triangle $\Delta\tilde{p}\tilde{q}\tilde{r}$. Then let m be a point on the side γ_{pr} with affine parameter value λ_m, and n a point on the side γ_{pq} with parameter λ_n. Further, let $\tilde{m}$ and $\tilde{n}$ be points on $\gamma_{\tilde{p}\tilde{r}}$ and $\gamma_{\tilde{p}\tilde{q}}$ with affine parameter values λ_m and λ_n. On $\gamma_{\check{p}\check{q}}$ let $\check{n}$ be the point with affine parameter value λ_n. Then consider the triangles $\Delta pqm, \Delta pnm$ with corresponding comparison triangles

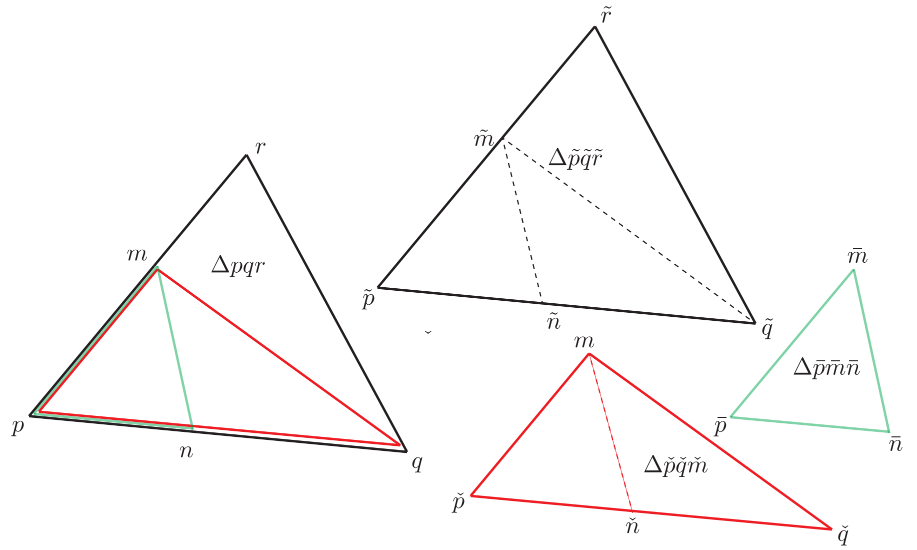

Figure 3.2: A triangle Δpqr in M and three model triangles in corresponding constant curvature model spaces.

$\Delta\check p\check q\check m$ and $\Delta\bar p\bar m\bar m$. All comparison triangles may lie in different model spaces of curvature K. Now $|\bar m\bar n|_\pm = |mn|_\pm \geqslant |\check m\check n|_\pm$. The equality holds since $\Delta\bar p\bar m\bar n$ is a comparison triangle for Δpmn, so the side lengths agree, and the inequality follows by (ii) since $\Delta\check p\check m\check q$ is a comparison triangle for Δpmq thus the signed distances to corresponding points opposite to a vertex decrease. Then Lemma 3.3.1(Hinge) applied to $\Delta\check p\check m\check n$ and $\Delta\bar p\bar m\bar n$[6] implies

$$\angle\bar m\bar p\bar n \leqslant \angle\check m\check p\check n. \tag{3.19}$$

Then apply (ii) to Δpqr and $\Delta\tilde p\tilde q\tilde r$, which yields $|\check m\check q|_\pm = |mq|_\pm \geqslant |\tilde m\tilde q|_\pm$. So applying hinge to $\Delta\check p\check m\check q$ and $\Delta\tilde p\tilde m\tilde q$ yields

$$\angle\check m\check p\check q \leqslant \angle\tilde m\tilde p\tilde q$$

so by Lemma 3.4.1 also $\angle\check m\check p\check n \leqslant \angle\tilde m\tilde p\tilde n$. Combining this with (3.19) we get

$$\angle\bar m\bar p\bar n \leqslant \angle\check m\check p\check n \leqslant \angle\tilde m\tilde p\tilde n.$$

Hence, applying hinge once more to the above, we get $|mn|_\pm = |\bar m\bar n|_\pm \geqslant |\tilde m\tilde n|_\pm$, and since m and n were arbitrary, the implication follows.
<u>(ii) $\Rightarrow$ (iii)</u>: By Lemma 3.1.2 we know that

$$(E_q \circ \gamma_{pr})'(0) = -2\angle qpr, \tag{*}$$

where γ_{pr} is the side connecting p with r parametrized by $[0,1]$. Let $\gamma_{\tilde p\tilde r}$ be the corresponding side in $\Delta\tilde p\tilde q\tilde r$, then by (ii) and since $\mathrm{sgn}(\gamma_{\mathrm{qp}})|qp|_\pm{}^2 = E_q(p)$ we get

$$(E_q \circ \gamma_{pr})(t) \geqslant (E_{\tilde q} \circ \gamma_{\tilde p\tilde r})(t) \tag{**}$$

[6] This can be done since by definition $|\check p\check m|_\pm = |\bar p\bar m|_\pm$ and $|\check p\check n|_\pm = |\bar p\bar n|_\pm$.

for all $t \in [0,1]$. By definition of our comparison triangle we get

$$(E_q \circ \gamma_{pr})(0) = E_q(p) = E_{\tilde{q}}(\tilde{p}) = (E_{\tilde{q}} \circ \gamma_{\tilde{p}\tilde{r}})(0).$$

Consequently,

$$\begin{aligned}
-2\angle qpr &\overset{(*)}{=} (E_q \circ \gamma_{pr})'(0) = \lim_{t\to 0} \frac{1}{t}(E_q \circ \gamma_{pr}(t) - E_q \circ \gamma_{pr}(0)) \\
&\overset{(**)}{\geqslant} \lim_{t\to 0} \frac{1}{t}(E_{\tilde{q}} \circ \gamma_{\tilde{p}\tilde{r}}(t) - E_{\tilde{q}} \circ \gamma_{\tilde{p}\tilde{r}}(0)) = (E_{\tilde{q}} \circ \gamma_{\tilde{p}\tilde{r}})'(0) \overset{(*)}{=} -2\angle \tilde{q}\tilde{p}\tilde{r},
\end{aligned}$$

which shows $\angle qpr \leqslant \angle \tilde{q}\tilde{p}\tilde{r}$. For the remaining implication (iii)⇒(i), subdivide Δpqr into two triangles Δpmq and Δmqr with m as in (ii)⇒(i), with comparison triangles $\Delta\check{p}\check{m}\check{q}$ and $\Delta p'q'n'$. Further, let $\Delta\tilde{p}\tilde{q}\tilde{r}$ be a comparison triangle for Δpqr. Then by (iii) $\angle pmq \leqslant \angle\check{p}\check{m}\check{q}$ and $\angle qmr \leqslant \angle q'm'r'$. As in the proof of Lemma 3.3.2 we see that $(1-\lambda_m)\angle pmq + \lambda_m \angle qmr = 0$. So the above inequalities imply that $(1-\lambda_m)\angle\check{p}\check{m}\check{q} + \lambda_m\angle q'm'r' \geqslant 0$. Therefore we can apply Lemma 3.3.2(Straightening Lemma for Shoulder Angles) to get $\angle\tilde{q}\tilde{p}\tilde{m} \geqslant \angle\check{q}\check{p}\check{m}$. But then by hinge applied to $\Delta\tilde{p}\tilde{q}\tilde{m}$ and $\Delta\check{p}\check{q}\check{m}$ we get $|qm|_\pm = |\check{q}\check{m}|_\pm \geqslant |\tilde{q}\tilde{m}|_\pm$. □

Let U be a neighborhood which is normal for K, then if one of the equivalent conditions on all triangles in U is satisfied, we say that ***triangle comparison holds*** on U. Our goal will be to show that the triangle comparison property is equivalent to curvature bounds $\mathcal{R} \geqslant K$ respectively $\mathcal{R} \leqslant K$.

Chapter 4

Modified Distance Function and Modified Shape Operator

4.1 Modified Distance Function

In this section we define a so called modified distance function for semi-Riemannian manifolds. This modified distance function is not a distance function in the sense of Definition 1.10.1, but still shares some similar features and will be an important tool in proving our main results. First we will study properties of the modified distance function on general semi-Riemannian manifolds and then its properties on model spaces of constant curvature.

Definition 4.1.1. *Let M be a semi-Riemannian manifold and $q \in M$ with normal coordinate neighborhood U. Then the* **modified distance function** $h_{K,q} : U \to \mathbb{R}$ *at q as in [1] is defined by*

$$h_{K,q}(p) := \begin{cases} (1 - \cos\sqrt{KE_q(p)})/K = \sum_{n=1}^{\infty} \frac{(-K)^{n-1}(E_q(p))^n}{(2n)!} & K \neq 0 \\ E_q(p)/2 & K = 0 \end{cases} \tag{4.1}$$

and further for $K \neq 0$ we define $\ell_{K,q} : U \to \mathbb{R}$ as

$$\ell_{K,q}(p) = \frac{1}{K}\cos\sqrt{KE_q(p)}. \tag{4.2}$$

Remarks

(i) In particular if $K \neq 0$ then by definition

$$h_{K,q} = -\ell_{K,q} + \frac{1}{K}. \tag{4.3}$$

(ii) Note that the term $KE_q(p)$ can be < 0 in case K is negative and q and p are connected by a spacelike geodesic or if K is > 0 and $E_q(p)$ is negative, that is q and p are connected by a timelike geodesic. Still $\cos\sqrt{KE_q(p)}$ is always real valued since cos transforms to cosh by the identity

$$\cos ix = \cosh x$$

which can be easily seen by looking at the corresponding power series.

Lemma 4.1.2. *The gradient $G := \operatorname{grad} h_{K,q}$ of the modified distance function $h_{K,q}$ is given by*

$$G = \operatorname{grad} h_{K,q} = \begin{cases} P, & \text{for } K = 0 \\ \frac{\sin(\sqrt{KE_q})}{\sqrt{KE_q}} P, & \text{for } K \neq 0 \text{ and } E_q \neq 0 \\ P, & \text{for } K \neq 0 \text{ and } E_q = 0 \end{cases} \tag{4.4}$$

which holds on a normal coordinate neighborhood U of q where $h_{K,q}$ is defined and G is smooth on U.

Remark Note that for some geodesic σ starting at q we see that

$$\phi(t) := \frac{\sin(\sqrt{KE_q(\sigma(t))})}{\sqrt{KE_q(\sigma(t))}}$$

is a real-valued function even in case $KE_q(p) < 0$ since $\sin(it) = i\sinh(t)$ for all $t \in \mathbb{R}$ and thus i cancels. Further, ϕ is positive in case $KE_q(\sigma(t)) > 0$ as long as $|E_q(\sigma(t))| < \pi^2|K|^{-1}$ and if either $K < 0$ or $E_q(\sigma(t)) < 0$ then $\phi \geqslant 0$ along σ for all t since $\sinh(t) \geqslant 0$ for $t \geqslant 0$ and $\phi(0) = 1$ as seen in the following proof.

Proof. Let P be the image of the position vector field $\tilde{P}$ in $T_qM \cong \mathbb{R}^n_\nu$ under the differential of the exponential map at q on some normal coordinate neighborhood U of q (see Definition 1.9.1). Then we know by Lemma 1.9.2 that $\operatorname{grad} E_q = 2P$ thus equation (4.1.2) for $K = 0$ follows immediately. Now in case $K \neq 0$, we know by Lemma 1.3.2 that $\operatorname{grad}(g \circ f) = g' \operatorname{grad} f$. Now set $f := E_q$ and $g(t) := (1 - \cos\sqrt{Kt})/K$ with $g'(t) = \frac{1}{2}\frac{\sin\sqrt{Kt}}{\sqrt{Kt}}$ then we obtain

$$\begin{aligned} \operatorname{grad} h_{K,q} &= \operatorname{grad}(1 - \cos\sqrt{KE_q})/K = \operatorname{grad}(g \circ E_q) \\ &= g' \operatorname{grad} E_q = g' 2P = \frac{\sin(\sqrt{KE_q})}{\sqrt{KE_q}} P. \end{aligned}$$

Since P is a smooth vector field and $E_q(p)$ depends smoothly on p, to show that $\operatorname{grad} h_{K,q}$ is smooth it suffices to show that $\frac{\sin\sqrt{KE_q}}{\sqrt{KE_q}}$ is smooth in case $E_q(p) = 0$. Thus we expand sin as a series and get

$$\frac{\sin\sqrt{t}}{\sqrt{t}} = t^{-\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \frac{(t^{\frac{1}{2}})^{2n+1}}{(2n+1)!} = 1 - \frac{t}{3!} + \frac{t^2}{5!} - \frac{t^3}{7!} + \dots$$

which is smooth at 0 and shows that along null geodesics $\phi(t) = 1$.

□

Remark The gradient of the modified distance function $h_{K,q}$ is defined on a whole normal coordinate neighborhood U of q and is smooth there. This is not the case for the radius function r_q, which cannot be smoothly defined on the whole of U, with the exception of the Riemannian case, see Lemma 1.10.4.

Lemma 4.1.3. *Let $q \in M$ and U a normal coordinate neighborhood of q such that*

$$|E_q(p)| < \frac{\pi^2}{\sqrt{K}} \tag{4.5}$$

in case $E_q(p)K > 0$. Let σ be a unit speed geodesic starting at q and set

$$\phi(t) := \begin{cases} \frac{\sin\sqrt{KE_q(\sigma(t))}}{\sqrt{KE_q(\sigma(t))}} & \text{for } K \neq 0 \\ 1 & \text{for } K = 0. \end{cases} \tag{4.6}$$

Then $\phi(t) \geqslant 0$ on U. For $G := \operatorname{grad} h_{K,q}$ as above set

$$g(t) := \|G(\sigma(t))\| = \sqrt{|\langle G(\sigma(t)), G(\sigma(t))\rangle|} \tag{4.7}$$

then along σ it holds that

(i) $G(\sigma(t)) = t\phi(t)\sigma'(t)$

(ii) $g(t) = t\phi(t)$ *and* $g'(t) = \cos\sqrt{KE_q(\sigma(t))} = 1 - Kh_{K,q}(\sigma(t))$

(iii) $g(0) = 0$, $g'(0) = 1$ *and* $g''(0) = 0$.

In particular the modified distance function $h_{K,q}$ is not a distance function in the sense of Definition 1.10.1.

Remark Note that (4.5) is always satisfied in normal coordinate neighborhoods in our model spaces $\mathcal{Q}_K$ which can be seen by the form of the exponential map as in Lemma 1.8.10. Thus for our local comparison results this condition will always be satisfied.

Proof. Let σ be a nonnull radial geodesic starting at q. Then by Lemma 4.1.2 we know that in a normal coordinate neighborhood of q we have $G(\sigma(t)) = \operatorname{grad}_{\sigma(t)} h_{K,q} = \phi(t)P(\sigma(t))$ where ϕ is the smooth function given by (4.6) and P is the local position vector field at q. Since σ is a radial geodesic we can apply Lemma 1.9.4 and by (1.31) we know that $P(\sigma(t)) = t\sigma'(t)$, which yields

$$G(t) = t\phi(t)\sigma'(t). \tag{*}$$

Thus we have shown (i). Since by the above remark $\phi(t) \geqslant 0$ we get

$$g(t) \overset{(*)}{=} \sqrt{|\langle t\phi(t)\sigma'(t), t\phi(t)\sigma'(t)\rangle|} = t\phi(t)\underbrace{\sqrt{|\langle\sigma'(t), \sigma'(t)\rangle|}}_{=1} = t\phi(t) \tag{**}$$

which is > 0 for $t > 0$ and $= 0$ for $t = 0$. Further,

$$g'(t) = \phi(t) + t\phi'(t).$$

In the case $K = 0$ and along null geodesics for arbitrary K we have $\phi = 1$ and thus $g'(0) = 1$. If $K \neq 0$

$$\phi(t) = \frac{\sin \sqrt{KE_q}}{\sqrt{KE_q}},$$

so choose $v \in T_qM$ such that $\sigma(t) = \exp_q(tv)$ and note that by (1.22)

$$E_q(\sigma(t)) = \langle \exp_q^{-1} \sigma(t), \exp_q^{-1} \sigma(t) \rangle = t^2 \langle v, v \rangle.$$

Set $B := \sqrt{K\langle v, v\rangle} \in \mathbb{C}\backslash\{0\}$. Then

$$\phi'(t) = (\frac{\sin(Bt)}{Bt})' = \frac{Bt\cos(Bt) - \sin(Bt)}{Bt^2},$$

so we get

$$g'(t) = \phi(t) + t\phi'(t) = \frac{\sin(Bt)}{Bt} + t\frac{Bt\cos(Bt) - \sin(Bt)}{Bt^2} = \cos Bt,$$

which shows that $g'(0) = 1$ even if B is of the form $B = i|B|$ since then $\cos(Bt) = \cosh(|B|t)$ and $\cosh 0 = 1$ as well. Now $g''(0)$ is 0 since $g''(t)$ is just a scalar multiple of $\sin Bt$ respectively $\sinh |B|t$ and both terms vanish for $t = 0$. Now since $\left\|\operatorname{grad}_{\sigma(t)} h_{K,q}\right\| = \|G(\sigma(t))\| \overset{(**)}{=} t\phi(t)$ one sees that the norm can't be constant 1, which shows that $h_{K,q}$ is not a distance function. $\square$

Remark If $\tilde{\sigma} := \sigma(at + b)$ is a affine reparametrization of σ, $a, b \in \mathbb{R}$ and $a \neq 0$, then

(i) $G(t) = t\phi(t)\tilde{\sigma}'(t) = t\phi(t)|a|\sigma'(at + b)$

(ii) $g(t) = t|a|\phi(t)$ and $g'(t) = |a| \cos \sqrt{KE_q(\sigma(t))}$

(iii) $g(0) = 0$ and $g'(0) = |a|$.

thus above lemma also holds for parametrizations which are not of unit speed.

Lemma 4.1.4. *Let $q \in M$ and U a normal neighborhood of q which is normal for K. Then along any radial geodesic σ starting at q, $h_{K,q} \circ \sigma$ is a monotonically increasing function of the signed length $|q\sigma|_\pm$ as long as $|q\sigma|_\pm \leqslant \frac{\pi}{\sqrt{K}}$. Along null geodesics, $h_{K,q}$ and $|q\sigma|_\pm$ are both constant with value 0.*

Proof. Let $\sigma : [0, 1] \to U$ be a nonnull radial geodesic starting at q. Then for $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, set $p_1 := \sigma(t_1)$ and $p_2 := \sigma(t_2)$. We need to show that if $|qp_1|_\pm < |qp_2|_\pm$ it follows that $h_{K,q}(p_1) < h_{K,q}(p_2)$. Note that p_1 and p_2 lie on the same geodesic which has constant causal character, so $E_q(p_1)$ and $E_q(p_2)$ have the same sign. Also in case p_1, p_2 are situated on a timelike geodesic, then from $E_q(p_1) < E_q(p_2) < 0$ it follows that $0 < |E_q(p_2)| < |E_q(p_1)|$, thus while the energy is increasing the modulus is decreasing. In case $K = 0$ we have $h_{K,q} = \frac{1}{2}E_q$ and $|qp|_\pm = \mathrm{sgn}(\mathrm{E_q(p)}) \sqrt{|E_q(p)|}$. Thus if $|qp_1|_\pm < |qp_2|_\pm$ we get

$$\mathrm{sgn}(\mathrm{E_q(p_1)}) \sqrt{|E_q(p_1)|} < \mathrm{sgn}(\mathrm{E_q(p_2)}) \sqrt{|E_q(p_2)|},$$

so for $\mathrm{sgn}(\mathrm{E_q(p_1)}) = 1$ the claim follows. Now if $\mathrm{sgn}(\mathrm{E_q(p_1)}) = -1$ we get $\sqrt{|h_{K,q}(p_1)|} > \sqrt{|h_{K,q}(p_2)|}$ thus the claim follows by the above considerations. For $K \neq 0$ we have

$$h_{K,q} = \frac{1}{K}(1 - \cos\sqrt{KE_q})$$

with cos transforming to cosh for negative values under the square root. We have to distinguish four cases depending on all possible signs for $E_q(p_1)$ respectively $E_q(p_2)$ and K.

(i) $K > 0$ and $E_q > 0$: cos is monotonically decreasing on $[0, \pi]$ then together with the result for $K = 0$ we see that $\sqrt{E_q(p_1)} < \sqrt{E_q(p_2)} < \frac{\pi}{\sqrt{K}}$.

(ii) $K < 0$ and $E_q > 0$: We get

$$h_{K,q} = \frac{1}{K}(1 - \cosh\sqrt{|K|E_q})$$

and since cosh is monotonically increasing on $[0, \infty)$, $1 - \cosh$ is decreasing but $h_{K,q}$ is increasing since the factor $\frac{1}{K}$ is negative.

(iii) $K > 0$ and $E_q < 0$: We have

$$h_{K,q} = \frac{1}{K}(1 - \cosh\sqrt{K|E_q|})$$

so for E_q increasing $|E_q|$ is decreasing and thus $1 - \cosh\sqrt{K|E_q|}$ is increasing

(iv) $K < 0$ and $E_q < 0$: here the signs under the square root cancel, thus

$$h_{K,q} = \frac{1}{K}(1 - \cos\sqrt{|K||E_q|})$$

and analogous reasoning again shows that $h_{K,q}$ is increasing with respect to signed length.

□

4.2 Modified Distance Function In The Model Spaces

As shown in Section 1.8 spaces $\mathcal{Q}_K$ of constant curvature $K \neq 0$ of fixed dimension n and index ν can be realized as the universal covers of hyperquadrics which are semi-Riemannian submanifolds of $\mathbb{R}^{n+1}_\nu$. Every point q in $\mathbb{R}^{n+1}_\nu$ can be identified with a linear functional

$$\tilde{\ell}_{K,q} := \langle q, p \rangle. \tag{4.8}$$

We will show that $\tilde{\ell}_{K,q}$ agrees with (4.2) if restricted to our model spaces of constant curvature and is K-affine, which is defined as follows:

Definition 4.2.1. *[1, Definition 3.1 p. 262] A smooth real-valued function f on a semi-Riemannian manifold is called K-**affine** if the restriction to any geodesic γ satisfies*

$$(f \circ \gamma)'' + K\langle \gamma', \gamma' \rangle (f \circ \gamma) = 0, \tag{4.9}$$

*further f is called K-**concave** if*

$$(f \circ \gamma)'' + K\langle \gamma', \gamma' \rangle (f \circ \gamma) \leqslant 0 \tag{4.10}$$

*and K-**convex** if the reversed inequality holds.*

Proposition 4.2.2. *[1, Proposition 3.2 3.1 p. 262] For $K \neq 0$, let*

$$\ell_{K,q} := \tilde{\ell}_{K,q}\big|_{\mathcal{Q}_K}$$

be the restriction of $\tilde{\ell}_{K,q} = \langle q, p \rangle$, as in (4.8) to $\mathcal{Q}_K$. Then $\ell_{K,q}$ is K-affine along every geodesic γ in $\mathcal{Q}_K$:

$$(\ell_{K,q} \circ \gamma)''(t) = -K\langle \gamma', \gamma' \rangle \ell_{K,q}(\gamma(t)) \tag{4.11}$$

and for two points $q, p \in \mathcal{Q}_K$ joined by a geodesic in $\mathcal{Q}_K$ we have

$$\ell_{K,q}(p) = \frac{1}{K} \cos \sqrt{K E_q(p)}, \tag{4.12}$$

where imaginary arguments of cos *are possible.*

Proof. The tangent space at a point in $\mathbb{R}^{n+1}_\nu$ is just a copy of $\mathbb{R}^{n+1}_\nu$ at this point and the tangent space to a point in $\mathcal{Q}_K \subseteq \mathbb{R}^{n+1}_\nu$ is an n-dimensional linear subspace of $\mathbb{R}^{n+1}_\nu$. In $\mathbb{R}^{n+1}_\nu$ the gradient of the function $x \mapsto \langle q, x \rangle$ is given by q [1], where we think of q as a vector field on $\mathbb{R}^{n+1}_\nu$ which assigns the vector q at every point of $\mathbb{R}^{n+1}_\nu$. For p in $\mathcal{Q}_K$ we first show that the projection $\pi_p : T_p\mathbb{R}^{n+1}_\nu \to T_p\mathcal{Q}_K$ is given by

$$\pi_p(v) = v - K\langle p, v \rangle p. \tag{4.13}$$

Note that p is normal to $T_p\mathcal{Q}_K$ since $\mathcal{Q}_K$ is orthogonal to the position vector field P and $P(p) = p$. Now

$$\langle p, \pi_p(v) \rangle = \langle p, v - K\langle v, p \rangle p \rangle = \langle p, v \rangle - K\langle p, v \rangle \langle p, p \rangle \overset{\langle p,p \rangle = \frac{1}{K}}{=} 0$$

which shows that $\pi_p(v)$ indeed maps to $T_p\mathcal{Q}_K$ and if $w \in T_p\mathcal{Q}_K$ then

$$\pi_p(w) = w - K \underbrace{\langle p, w \rangle}_{=0} p = w$$

thus $\pi_p|_{T_p\mathcal{Q}_K} = \mathrm{Id}_{\mathcal{Q}_K}$. One easily checks that $\pi_p(p) = 0$ and $\pi_p \circ \pi_p = \pi_p$. Now we will show that

$$\operatorname{grad}_p \ell_{K,q} = \pi_p q. \tag{4.14}$$

[1] Since for $f(x) = \langle q, x \rangle$ we have $\frac{\partial f}{\partial x^i} = q^i$ and thus by [5, Definition 3.47 p. 85] since $g^{ij} = \delta_{ij}$ in $\mathbb{R}^{n+1}_\nu$ we get $\operatorname{grad} f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \partial_j = \sum_i q^i \partial_i = q$.

Let $v \in T_p\mathcal{Q}_K$ then $(d\ell_{K,q})_p$ is the restriction of $(d\tilde{\ell}_{K,q})_p$ to $T_p\mathcal{Q}_K$ and $(d\tilde{\ell}_{K,q})_p(v) = \langle q, v\rangle$ since it is just the usual derivative of a real valued function on $\mathbb{R}^{n+1}_\nu$. Thus we compute

$$\langle \operatorname{grad}_p \ell_{K,q}, v\rangle = (d\ell_{K,q})_p(v) = (d\tilde{\ell}_{K,q})_p(v) = \langle q, v\rangle = \langle \pi_p(q), v\rangle,$$

where the last equality holds since v is in $T_p\mathcal{Q}_K$. Since this holds for any $v \in T_p\mathcal{Q}_K$ we get (4.14). Let D be the Levi-Civita connection on $\mathbb{R}^{n+1}_\nu$ and ∇ the Levi-Civita connection on $\mathcal{Q}_K$. Then by [5, Lemma 4.3 p. 99] they are related by projection onto the tangent space of $\mathcal{Q}_K$. So $\nabla_v X = \pi_p(D_v X)$ for v in $T_p\mathcal{Q}_K$ and $X \in \mathfrak{X}(\mathcal{Q}_K)$. Now given a geodesic $\gamma : I \to \mathcal{Q}_K$, for $t \in I$ set $p = \gamma(t)$ and $v = \gamma'(t)$. Then to show that $\ell_{K,q}$ is K-affine we compute

$$\begin{aligned}(\ell_{K,q} \circ \gamma)''(t) &\overset{(1.8)}{=} \langle \nabla_v \operatorname{grad} \ell_{K,q}, v\rangle \\ &\overset{(4.14)}{=} \langle \pi_p(D_v \pi_p(q)), v\rangle \overset{(4.13)}{=} \langle \pi_p(D_v(q - K\langle p, q\rangle p)), v\rangle \\ &= \langle \pi_p(D_v q - K(D_v(\langle p, q\rangle)p - K\langle q, p\rangle D_v p), v\rangle \\ &= \langle -K\langle q, v\rangle \pi_p(p) - K\langle q, p\rangle v, v\rangle \overset{\pi_p(p)=0}{=} -K\langle v, v\rangle \ell_{K,q}(\gamma(t))\end{aligned} \tag{4.15}$$

which shows (4.11). Here, the second to last equation holds since $D_v q = D_{\gamma'(t)} q = 0$ since q is a constant vector field on $\mathbb{R}^{n+1}_\nu$. Also

$$(d\ell_{K,q})_q(v) = \langle \operatorname{grad} \ell_{K,q}(q), v\rangle \overset{(4.14)}{=} \langle \underbrace{\pi_q(q)}_{=0}, v\rangle = 0,$$

which shows that $d\ell_{K,q}$ vanishes at q. Thus for any geodesic γ in $\mathcal{Q}_K$ with $\gamma(0) = q$ and $v := \gamma'(0)$ we get $(\ell_{K,q} \circ \gamma)'(0) = (d\ell_{K,q})_q(\gamma'(0)) = 0$. Further, $(\ell_{K,q} \circ \gamma)(0) = \ell_{K,q}(q) = \langle q, q\rangle = \frac{1}{K}$ and

$$(\ell_{K,q} \circ \gamma)''(0) \overset{(4.11)}{=} -K\langle v, v\rangle \ell_{K,q}(q) \overset{\langle q,q\rangle = \frac{1}{K}}{=} -\langle v, v\rangle.$$

With these initial conditions the solution to the linear differential equation (4.11) which is of the form $x'' + K\langle v, v\rangle x = 0$ is given by

$$(\ell_{K,q} \circ \gamma)(t) = \frac{\cos \sqrt{K\langle v, v\rangle} t}{K}$$

which shows (4.12). □

Proposition 4.2.3. *For q in $\mathcal{Q}_K$ let $h_{K,q}$ be the modified distance function defined by* (4.1). *Then $h_{K,q}$ satisfies along every geodesic γ in $\mathcal{Q}_K$ which is parametrized by $[0, 1]$ with $\gamma(0) = q$ and $\gamma(1) = p$ that*

$$(h_{K,q} \circ \gamma)'' + K\langle \gamma', \gamma'\rangle (h_{K,q} \circ \gamma) = \langle \gamma', \gamma'\rangle \tag{4.16}$$

and in particular

$$(h_{K,q} \circ \gamma)'' + KE_q(p)(h_{K,q} \circ \gamma) = E_q(p) \tag{4.17}$$

where $E_q(p)$ is defined by (1.18).

Proof. For $K \neq 0$ recall that $h_{K,q} = -\ell_{K,q} + \frac{1}{K}$. Thus by Proposition 4.2.2 we get

$$
\begin{aligned}
(h_{K,q} \circ \gamma)'' &\overset{(4.3)}{=} (-\ell_{K,q} \circ \gamma + \frac{1}{K})'' = -(\ell_{K,q} \circ \gamma)'' \\
&\overset{(4.11)}{=} K\langle \gamma', \gamma' \rangle (\ell_{K,q} \circ \gamma) \overset{(4.3)}{=} \langle \gamma', \gamma' \rangle - K\langle \gamma', \gamma' \rangle (h_{K,q} \circ \gamma).
\end{aligned}
$$

Now in the remaining case $K = 0$ by definition $h_{K,q} = \frac{1}{2}E_q$ and (4.16) reduces to $(h_{K,q} \circ \gamma)'' = \langle \gamma', \gamma' \rangle$. Note that for γ parametrized by $[0,1]$ by (1.20) we have that

$$E_q(\gamma(t)) = t^2 \langle \gamma'(0), \gamma'(0) \rangle = t^2 E(\gamma)$$

and $E(\gamma)$ does not depend on t. Thus

$$(h_{K,q} \circ \gamma''(t) = \left(\frac{E_q(\gamma(t))}{2} \right)'' = \left(\frac{t^2 E(\gamma)}{2} \right)'' = E(\gamma).$$

Since γ is a geodesic, $\langle \gamma'(t), \gamma'(t) \rangle$ is constant so our claim follows by

$$E(\gamma) = \langle \gamma'(0), \gamma'(0) \rangle = \langle \gamma'(t), \gamma'(t) \rangle.$$

□

4.3 Modified Shape Operator

Definition 4.3.1. *Let M be a semi-Riemannian manifold, q in M, and $h_{K,q}$ the modified distance function as given by Definition 4.1.1. Let U be a normal coordinate neighborhood of q, then the* **modified shape operator** *$S_{K,q}$ is the self-adjoint operator $S_{K,q} : TU \to TU$ defined as*

$$S_{K,q}(v) := \nabla_v \operatorname{grad} h_{K,q}. \tag{4.18}$$

To see that $S_{K,q}$ is self adjoint note that the Hessian of any smooth function is a symmetric tensor (see Definition 1.3.3) and so for $p \in U$ and $v, w \in T_pM$ we get

$$
\begin{aligned}
\langle S_{K,q}(v), w \rangle &= \langle \nabla_v \operatorname{grad} h_{K,q}, w \rangle = H^{h_{K,q}}(v, w) \\
&= H^{h_{K,q}}(w, v) = \langle \nabla_w \operatorname{grad} h_{K,q}, v \rangle = \langle v, S_{K,q}(w) \rangle.
\end{aligned}
$$

Next we show that in our model spaces of constant curvature K the modified shape operator is given as a multiple of the identity Id on each tangent space.

Lemma 4.3.2. *On any hyperquadric $\tilde{Q}_K$ the modified shape operator $S_{K,q}$ is given by*

$$S_{K,q} = \begin{cases} \mathrm{Id} & \text{if } K = 0 \\ K\ell_{K,q}\mathrm{Id} & \text{if } K \neq 0 \end{cases} \tag{4.19}$$

Proof. The case $K \neq 0$ can be seen by using (4.11). Let γ be a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$, then

$$\langle -K\ell_{K,q}(p)v, v \rangle = (\ell_{K,q} \circ \gamma)''(0) \overset{(1.8)}{=} \langle \nabla_v \operatorname{grad} \ell_{K,q}, v \rangle$$

and so since $h_{K,q} = -\ell_{K,q} + \frac{1}{K}$ we see that $\operatorname{grad} h_{K,q} = -\operatorname{grad} \ell_{K,q}$, which results in

$$\nabla_v \operatorname{grad} h_{K,q} = -\nabla_v \operatorname{grad} \ell_{K,q} = K\ell_{K,q}\mathrm{Id}.$$

In the case $K = 0$ we have $h_{0,q} = \frac{1}{2}E_q$, so

$$S_{0,q}(v) = \nabla_v \operatorname{grad} h_{0,q} \overset{(1.24)}{=} \nabla_v P = v$$

and thus $S_{0,q} = \mathrm{Id}$. □

Proposition 4.3.3. *Let q be a point in a semi-Riemannian manifold, $K \in \mathbb{R}$ and σ a radial geodesic with $\sigma(0) = q$. Then along σ the modified shape operator $S_{K,q}$ satisfies*

$$\left(\nabla_{\sigma'(t)} \operatorname{grad} h_{K,q}\right)_{\sigma(t)} = (1 - Kh_{K,q}(\sigma(t)))\sigma'(t) = K\ell_{K,q}(\sigma(t))\sigma'(t), \qquad (4.20)$$

where the last equation is only defined in case $K \neq 0$.

Remarks

(i) To simplify notation, as in [1] set $G := \operatorname{grad} h_{K,q}$, $N := \sigma'(t)$, $h := h_{K,q}$, $\ell := \ell_{K,q}$ and $S := S_{K,q}$ if there is no ambiguity regarding K and q. So in particular (4.20) then reads as

$$\nabla_N G = (1 - Kh)N = K\ell N.$$

(ii) By Lemma 1.3.4, along a geodesic σ the Hessian of some function f is given by $(f \circ \sigma)'' = H^f(\sigma', \sigma')$, so by the definition of the Hessian together with (4.20) we compute

$$\begin{aligned}(h_{K,q} \circ \sigma)'' = H^{h_{K,q}}(\sigma', \sigma') &= \langle \nabla_{\sigma'} \operatorname{grad} h_{K,q}(\sigma'), \sigma' \rangle \\ &= \langle \sigma', \sigma' \rangle - K\langle \sigma', \sigma' \rangle (h_{K,q} \circ \sigma),\end{aligned}$$

which is equivalent to

$$(h_{K,q} \circ \sigma)'' + K\langle \sigma', \sigma' \rangle (h_{K,q} \circ \sigma) = \langle \sigma', \sigma' \rangle. \qquad (4.21)$$

Since for geodesics the causal-character doesn't change we have for σ radial starting at q:

-) $\langle \sigma', \sigma' \rangle > 0$: $h_{K,q}$ is K-convex along spacelike geodesics

-) $\langle \sigma', \sigma' \rangle = 0$: $h_{K,q}$ is K-affine along null geodesics

-) $\langle \sigma', \sigma' \rangle < 0$: $h_{K,q}$ is K-concave along timelike geodesics

Proof. First consider the case $K = 0$. Then $G = \operatorname{grad} h \overset{(4.4)}{=} P$. Let $'$ denote the induced covariant derivative along curves, then by [5, 3.18(3), p. 65] we know that $\nabla_{\sigma'(t)} P = (P \circ \sigma)'(t)$, so

$$\nabla_N G = \nabla_{\sigma'(t)} P = (P \circ \sigma)'(t) \overset{(1.30)}{=} \sigma'(t) = N \overset{K=0}{=} (1 - Kh)N$$

which proves the case $K = 0$. For $K \neq 0$ again by [5, p. 65] $\nabla_{\sigma'(t)} G = (G \circ \sigma)'(t)$ and we know by Lemma 4.1.2 that along radial geodesics $G = \phi(t)P$ with

$\phi(t) = \frac{\sin(\sqrt{KE_q})}{\sqrt{KE_q}}$ for $E_q \neq 0$ and $\phi = 1$ along null geodesics. Thus along any radial geodesic

$$(G \circ \sigma)(t) = \phi(t)P(\sigma(t)).$$

By the product rule for the induced covariant derivative [5, 3.18(2) p. 65] we get

$$(G \circ \sigma)'(t) = (\phi(t)P(\sigma(t)))' = \phi'(t)P(\sigma(t)) + \phi(t)(P \circ \sigma)'(t).$$

Then inserting (1.31) and (1.30) yields

$$(G \circ \sigma)'(t) = \phi'(t)t\sigma'(t) + \phi(t)\sigma'(t).$$

If σ is null we are done since then $\phi = 1$ and $\phi' = 0$. Otherwise $(G \circ \sigma)'(t) = \alpha(t)\sigma'(t)$ for some real valued function α and it remains to show that $\alpha(t) = 1 - Kh(\sigma(t))$. Since we know by the above that $(G \circ \sigma)'(t)$ is a multiple of $\sigma'(t) = N$ it suffices to show that

$$\langle \nabla_N G, N \rangle = \langle (1 - Kh)N, N \rangle. \tag{4.22}$$

Now we have

$$\langle \nabla_N G, N \rangle = \langle \nabla_{\sigma'(t)} \operatorname{grad} h_{K,q}, \sigma'(t) \rangle \stackrel{(1.8)}{=} (h_{K,q} \circ \sigma)''(t).$$

By setting $\lambda := \langle \sigma'(0), \sigma'(0) \rangle$ and since $E_q(\sigma(t)) = t^2 \langle \sigma'(0), \sigma'(0) \rangle = \lambda t^2$ by (1.20) we see that

$$(h_{K,q} \circ \sigma)(t) = \frac{1}{K}(1 - \cos\sqrt{KE_q(\sigma(t))}) = \frac{1}{K}(1 - \cos(\sqrt{K\lambda}t)).$$

Differentiating this expression yields

$$(h_{K,q} \circ \sigma)'(t) = \sqrt{\frac{\lambda}{K}} \sin(\sqrt{K\lambda}t)$$

and

$$(h_{K,q} \circ \sigma)''(t) = \lambda \cos(\sqrt{K\lambda}t). \tag{4.23}$$

Since geodesics are always parametrized with constant speed $\langle \sigma'(t), \sigma'(t) \rangle = \langle \sigma'(0), \sigma'(0) \rangle = \lambda$ the right hand side of (4.22) can be rewritten as

$$(1 - Kh_{K,q})\langle N, N \rangle = \cos(\sqrt{K\lambda}t)\langle \sigma'(t), \sigma'(t) \rangle \stackrel{(4.23)}{=} (h_{K,q} \circ \sigma)''(t)$$

and this shows (4.22) and hence the claim for $K \neq 0$. □

As a corollary we see will see that the integral curves of G are pregeodesics.

Corollary 4.3.4. *The integral curves of* $G = \operatorname{grad} h_{K,q}$ *are pregeodesics.*

Proof. To show that an integral curve c of G is a pregeodesic by Lemma 1.6.2 it suffices to show that $c'(t)$ and $c''(t)$ are colinear for all t. Recall that integral curves of G are characterized by the equation

$$c'(t) = G(c(t)) = \operatorname{grad} h_{K,q}(c(t)). \tag{4.24}$$

Set $g := \sqrt{|\langle G, G\rangle|}$ then $c' = gN$ where N is a unit vector field along c. Also N is orthogonal to the level sets of $h_{K,q}$ and therefore also to the level sets of r_q and thus since $\langle N, N\rangle = \pm 1$ depending on the causal character of c we see by *Lemma* 1.10.4 that $N = \operatorname{grad} r_q$ and the integral curves of N are unit speed radial geodesics. Thus we compute

$$\begin{aligned} c''(t) &= \nabla_{c'(t)} c'(t) \overset{(4.24)}{=} \nabla_{c'(t)} \operatorname{grad} h_{K,q}(c(t)) \overset{(4.18)}{=} S(c'(t)) \\ &= gS(N) \overset{(4.20)}{=} g(1 - Kh)N = (1 - Kh)c'(t) \end{aligned}$$

and by Lemma 1.6.2 c is a pregeodesic. □

Remark: Thus for integral curves c of $\operatorname{grad} h_{K,q}$ the function λ as in Lemma 1.6.2 is given by $\lambda(t) = 1 - Kh_{K,q}(\sigma(t)) = K\ell_{K,q}(\sigma(t))$. So if $K = 0$ we can find a reparametrization θ such that $c \circ \theta$ is a geodesic by solving

$$\theta'' + (\theta')^2 = 0$$

which for $t \geqslant 0$ is solved by $\ln(t)$. In case $K \neq 0$ we have to distinguish if our integral curve is timelike or spacelike. In the spacelike case we have to solve

$$\theta''(t) + \cos(\sqrt{K}t)(\theta')^2 = 0$$

whereas in the timelike case

$$\theta''(t) + \cosh(\sqrt{K}t)(\theta')^2 = 0.$$

4.4 Ricci Operator and Jacobi Fields

Definition 4.4.1. *The* **Ricci operator***, also called* **tidal force operator***, $R_X : T_pM \to T_pM$ is defined as $R_X(v) := R(X, v)X$ with $X \in \mathfrak{X}(M)$, $v \in T_pM$ and R the Riemannian curvature operator.*

Note that $R_X(v)$ is not $\mathbb{R}$-linear in the X component and thus the Ricci operator is not a $(1, 2)$-tensor.

Lemma 4.4.2. *The Ricci operator is self-adjoint and $R_v(v) = 0$ for $v \in T_pM$. The orthogonal complement $v^\perp$ is invariant under R_v.*

Proof. That R_X is self adjoint follows by pair-symmetry of the Riemannian curvature tensor, see [5, Proposition 3.36 p. 75]:

$$\langle R_X v, w\rangle = \langle R(X, v)X, w\rangle = \langle R(X, w)X, v\rangle = \langle v, R_X(w)\rangle.$$

Next, $R_v(v) = R_{vv}v = 0$ by skew-symmetry of the Riemannian curvature operator. Let $w \in v^\perp$ and $z \in T_pM$, which we can write as $z = \tilde{z} + \lambda v$ with $\tilde{z} \in v^\perp$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} \langle R_v(w), z\rangle &= \langle R_v(w), \tilde{z} + \lambda v\rangle = \langle w, R_v(\tilde{z}) + \lambda R_v(v)\rangle \\ &= \langle w, R_v(\tilde{z})\rangle = \langle R_v(w), \tilde{z}\rangle \end{aligned}$$

Since this holds for any z we have shown that $R_v(w)$ lies in $v^\perp$. □

Remarks

(i) If M has constant curvature K, then by (1.23) the Ricci operator is given by

$$R_v(w) = \begin{cases} 0 & \text{if } v = w \text{ or } v \text{ null} \\ K\langle v, v\rangle w & \text{for } v \perp w \\ K(\langle v, v\rangle w - \langle v, w\rangle v) & \text{otherwise.} \end{cases} \tag{4.25}$$

(ii) If M is a Riemannian manifold the eigenvalues $R_X(Y) = \lambda Y$ of the Ricci operator R_X give bounds for the sectional curvature $\mathcal{K}$. Let λ_- be the smallest eigenvalue of R_X and λ_+ the largest eigenvalue. Both of them are real since R_X is self adjoint. Note that this only holds in the Riemannian case and does not apply if the scalar product is indefinite. So choose X such that $\langle X, X\rangle = 1$ and some non null eigenvector Y corresponding to some eigenvalue λ with $\langle Y, X\rangle = 0$. Then we see that

$$\mathcal{K}(Y, X) = \frac{\langle R_X(Y), Y\rangle}{\langle Y, Y\rangle} = \lambda$$

and thus conclude

$$\lambda_- \leqslant \mathcal{K}(Y, X) \leqslant \lambda_+.$$

In particular, for our constant curvature spaces of index $\nu = 0$, all eigenvalues are equal to K, so $\lambda_-(R_N) = \lambda_+(R_N) = K$ and so $\mathcal{K}(X, N) = K$, as expected.

Let σ be a geodesic, then a vector field J along σ is called a ***Jacobi field*** if it satisfies the ***Jacobi equation***

$$J'' + R(\sigma', J)\sigma' = 0. \tag{4.26}$$

Using the Ricci operator, the Jacobi equation can be written as

$$J'' + R_{\sigma'}(J) = 0.$$

In particular, if our space has constant sectional curvature K and $J \perp \sigma'$ span a nondegenerate 2-plane, then by (4.25) the Jacobi equation is given by

$$J'' + \langle\sigma', \sigma'\rangle KJ = 0.$$

The Jacobi equation is a linear second order differential equation. Let $\sigma(0) = p$ and given $v, w \in T_pM$, then by ODE-theory there exist a unique Jacobi field such that $J(0) = v$ and $J'(0) = w$. Jacobi fields along some geodesic σ are exactly the variational vector fields of geodesic variations σ_s of σ. Let $\sigma(0) = p$, then variations which fix p correspond to Jacobi fields J along σ with $J(0) = 0$.

Lemma 4.4.3. *Let σ be a geodesic with $\sigma(0) = q$ and $\sigma'(0) = w$. Given $w \in T_qM$ then*

$$J(t) = (\,d\exp)_{tw}(tv). \tag{4.27}$$

is the unique Jacobi field along σ with $J(0) = 0$ and $J'(0) = v$.

Proof. See [12, Lemma 5.3, p. 18]. □

Lemma 4.4.4. *Let σ be a geodesic in a semi Riemannian manifold M with $\sigma(0) = q$ and J a Jacobi field along σ with $J(0) = 0$ and $J'(0) = v$. Then J is given by*

$$\begin{aligned} J(t) &= \mathcal{P}_0^t(vt - \frac{1}{6}R_{\sigma'(0)v}(\sigma'(0))t^3 + O(t^4)) \\ &= \mathcal{P}_0^t(vt - \frac{1}{6}R_{\sigma'(0)}(v)t^3 + O(t^4)) \end{aligned} \tag{4.28}$$

where $\mathcal{P}_0^t$ denotes the parallel transport along σ from $\sigma(0) = q$ to $\sigma(t)$, in particular $\mathcal{P}_0^0 = \mathrm{Id}_{T_qM}$.

Proof. Let $E_1, \dots E_n$ be an orthonormal basis for T_qM, then by parallel transport we get an orthonormal frame $E_i(t)$ along σ given by $E_i(t) := \mathcal{P}_0^t(E_i)$, with $E_i(0) = E_i$ for $i = 1, \dots n$. Thus for $J(t) = \sum_{i=1}^n J^i(t)E_i(t)$ we see that

$$(\mathcal{P}_0^t)^{-1}J(t) = \sum_{i=1}^n J^i(t)E_i(0).$$

Now we compute the Taylor expansion of J at $t = 0$. By assumption $J(0) = 0$, $J'(0) = v$ and by the Jacobi equation

$$J''(0) = -R_{\sigma'(0)J(0)}\sigma'(0) \overset{J(0)=0}{=} 0.$$

Then since $\nabla_{\sigma'}\sigma' = 0$ we see that

$$\begin{aligned} J''' &= \frac{\nabla}{dt}J'' = -\nabla_{\sigma'}R_{\sigma'J}(\sigma') \\ &= -(\nabla_{\sigma'}R)_{\sigma'J}(\sigma') - R_{\nabla_{\sigma'}\sigma'J}(\sigma') - R_{\sigma'\nabla_{\sigma'}J}(\sigma') - R_{\sigma'J}(\nabla_{\sigma'}\sigma') \\ &= -(\nabla_{\sigma'}R)_{\sigma'J}(\sigma') - R_{\sigma'\nabla_{\sigma'}J}(\sigma'). \end{aligned}$$

Now since $\nabla_{\sigma'}R$ is tensorial and $J(0) = 0$ we get $(\nabla_{\sigma'}R)_{\sigma'(0)J(0)}\sigma'(0) = 0$ and thus

$$J'''(0) = -R_{\sigma'(0)J'(0)}\sigma'(0) = -R_{\sigma'(0)v}\sigma'(0).$$

So the result follows by Taylor's theorem.

□

4.5 Riccati Equation

This section follows [12, p.6-11]. Let M be a semi-Riemannian manifold. The geodesic equation $\nabla_{\sigma'}\sigma' = 0$ is a nonlinear differential equation. One can view the Jacobi equation as a linearization of the geodesic equation as follows: let σ_s be a geodesic variation of σ, so $\sigma_0 = \sigma$ and σ_s is a geodesic for every s in some open interval containing 0. Let $N := \frac{\partial \sigma_s}{\partial t}$ and $F := \frac{\partial \sigma_s}{\partial s}$ be the tangent respectively variational vector fields of a geodesic variation of σ. Then F is a Jacobi field along σ by [5, Lemma 8.3, p. 216], so

$$F'' + R_N(F) = 0.$$

As a next step, we will split this second order linear equation into a system of first order differential equations. To do so, we will look at all variations of σ, which are orthogonal to some semi-Riemannian hypersurface V of M. Let

$$\sigma : V \times I \to M$$

be a smooth map such that $\sigma(p,t) := \sigma_p(t)$ is a geodesic for fixed p starting at p and $\sigma_p'(0) \perp T_pV$. By [5, Proposition 7.26 p. 200] there exists a normal coordinate neighborhood U of V. So every point in U lies on exactly one geodesic starting at V. On U we can look at the vector field $d\sigma(\frac{\partial}{\partial t}) = N|_V$. Then N satisfies $\nabla_N N = 0$. Let F be a Jacobi field along σ such that $F(0) \in T_{\sigma(0)}V$, which can be obtained as the variational vector field of a geodesic variation in V, that is $s \mapsto \sigma_s(t_0)$ is a smooth curve in V and $\sigma(t_0) \in V$. Let $(x^2, \dots, x^n)$ be chart for V, where we can suppose that $\frac{\partial}{\partial x^2}|_0 = F(0)$. Then we define a chart on U by $(t, x^2, \dots, x^n) \mapsto \exp^\perp(tN(\phi^{-1}(x^2, \dots, x^n))$. So since coordinate vector fields commute, $[F, N] = 0$ follows. By this we see $\nabla_N F = \nabla_F N$ and since the right hand side is linear in F, we get a linear map $S := \nabla N$ given by $SX = \nabla_X N$. So our first equation is given by

$$\nabla_N F = SF. \tag{4.29}$$

Hence given S we can compute F by solving a first order system. Next we will derive a differential equation for S. By the product rule for tensor fields we obtain

$$(\nabla_N S)X = \nabla_N(SX) - S\nabla_N X,$$

so inserting the definition of S we compute

$$\begin{aligned}
(\nabla_N S)X &= \nabla_N\nabla_X N - S(\nabla_X N + [N, X]) \\
&= \nabla_X\nabla_N N + R(X, N)N - \nabla_{[X,N]}N - S(SX) - S([N, X]) \\
&= \nabla_X\nabla_N N - R_N(X) + \nabla_{[N,X]}N - S^2X - \nabla_{[N,X]}N \\
&= \nabla_X\nabla_N N - R_N(X) - S^2X.
\end{aligned}$$

Since $\nabla_N N = 0$, we obtain a ***Riccati equation*** for S

$$\nabla_N S + R_N + S^2 = 0. \tag{4.30}$$

We know form Lemma 1.4.3 that $S = \nabla N$ is self adjoint, so

$$\langle \nabla_X N, Y\rangle = \langle X, \nabla_Y N\rangle$$

for all vector fields $X, Y \in \mathfrak{X}(U)$. Then by lemma 1.3.5 N is locally a gradient of some smooth map $f \in C^\infty(U)$ and further

$$X\langle N, N\rangle = 2\langle \nabla_X N, N\rangle = 2\langle X, \underbrace{\nabla_N N}_{=0}\rangle = 0,$$

so $\langle N, N\rangle$ is constant. We may assume that $1 = \|N\| = \|\operatorname{grad} f\|$ and thus f is a distance function on U. In particular the level sets $V_t = \{x \in M \mid f(x) = t\}$ are semi-Riemannian hypersurfaces by Proposition 1.4.1, where the unit normal vector field is given by N. If $V = V_{t_0}$ then $\sigma : V \times I \to M$ is given by $\sigma(p,t) = \exp_p((t - t_0)N(p))$. Then we set $\phi_t(p) := \sigma(p,t)$ and see that $V_t = \phi_t(V)$, so V_t is a family of ***equidistant*** hypersurfaces. If we restrict ∇N to the tangent bundle TV_t of our level sets, then since N is a unit normal we see by lemma 1.4.3 that

$$W_t = \nabla N|_{V_t}, \tag{4.31}$$

where W_t is the Weingarten-map or shape operator of the semi-Riemannian hypersurface V_t.

Remark In Lemma 2.2.2, note that for $g = 1$ equation (2.5) is the Riccati equation (4.30) and (2.2) is the Jacobi equation if we choose R to be the Ricci operator along some geodesic.

4.6 Riccati Equation for the Modified Shape Operator

Now we are ready to show one of the main results presented in the paper by Alexander and Bishop [1], which shows that the modified shape operator $S_{K,q}$ satisfies a certain matrix Riccati equation. Riccati equations are first order non-linear differential equations of the form

$$y'(t) = g_0(t) + g_1(t)y(t) + g_2(t)y(t)^2, \tag{4.32}$$

hence are second order ODEs with a quadratic term.

Proposition 4.6.1. *[1, Proposition 4.1] Let M be a semi-Riemannian manifold and in a normal coordinate neighborhood U of a point q in M let σ be a radial unit speed geodesic at q. Then the modified shape operator $S = S_{K,q} = \nabla_v \operatorname{grad} h_{K,q}$ and $\hat{S}$ as obtained by Lemma 2.2.2 are the same if there we choose $R = R_{\sigma'}$ and $g = \langle G, G\rangle$. Further, S satisfies the following first order partial differential equation of Riccati type*

$$\nabla_G S + S^2 - (1 - Kh)S + R_G + K\, dh \otimes G = 0, \tag{4.33}$$

where we use the notation as in the remark after Proposition 4.3.3 and R_G is the Ricci operator, see Definition 4.4.1.

Remark

- Since $h = h_{K,q}$ is a smooth real valued function we see that dh is a one-form and $G = \operatorname{grad} h_{K,q}$ is a vector field. Thus $dh \otimes G$ is a $\binom{1}{1}$-tensor. Application of $dh \otimes G$ to some vector field V is given by $dh \otimes G(V) = dh(V)G = V(h)G$ and so the result is again a vector field.
- Comparing (4.33) with (4.32) we observe

$$\underbrace{\nabla_G S}_{y'} = \underbrace{-R_G - K\, dh \otimes G}_{g_0} + \underbrace{(1 - Kh)S}_{g_1 y} + \underbrace{(-1)S^2}_{g_2 y^2}.$$

Proof. We show that (4.33) holds along nonnull radial unit speed geodesics radiating from a point q in M. The general case for null geodesics then follows by continuity. So let σ be a nonnull radial unit speed geodesic starting at q and set $N := \sigma'(t)$, which is a parallel unit vector field along σ since σ is a geodesic. First we show that (4.33) holds in directions tangent to σ. By Proposition 4.3.3 we know that along σ the modified shape operator S is of the form

$$SN = \nabla_N G = (1 - Kh)N = K\ell N. \tag{4.34}$$

By Lemma 4.1.3(i), on a normal coordinate neighborhood of q every radial unit speed geodesic starting at q satisfies

$$G(t) = t\phi(t)\sigma'(t) \tag{4.35}$$

for some smooth function ϕ and thus

$$\nabla_G N = \nabla_{t\phi\sigma'}\sigma' = t\phi \overbrace{\nabla_{\sigma'}\sigma'}^{=0} = 0.$$

By $\mathcal{C}^\infty(M)$-linearity and skew-symmetry of the Riemannian curvature tensor we get

$$R_G(N) = R(t\phi N, N)(t\phi N) = t^2\phi^2 \overbrace{R(N,N)}^{=0} N = 0.$$

Next we show that

$$(\nabla_G S)(N) = -K(Gh)N$$

which can be seen since S is a $\binom{1}{1}$-tensor [2] and applying the product rule for tensor derivations [5, Proposition 2.13, p. 44] yields together with $\nabla_G N = 0$ as shown above that

$$\begin{aligned}(\nabla_G S)(N) &= \nabla_G(S(N)) - S(\overbrace{\nabla_G N}^{=0}) \overset{(4.34)}{=} \nabla_G((1-Kh)N) \\ &= (\nabla_G(1-Kh))N + (1-Kh)\overbrace{\nabla_G N}^{=0} = -K(Gh)N.\end{aligned}$$

By combing the above results we get

$$\begin{aligned}&(\nabla_G S + S^2 - (1-Kh)S + R_G + K\, dh \otimes G)N \\ &\quad = -K(Gh)N + (1-Kh)^2 N - (1-Kh)^2 N + 0 + K(Nh)G \\ &\quad = -K(\phi Nh)N + K(Nh)\phi N = 0,\end{aligned}$$

where we used $G = \phi N$. Now we will show that $\hat{S}$ and S agree and use this to show that the Riccati equation (4.33) holds on $V := V_{\sigma(t)} := \sigma'(t)^\perp = G(\sigma(t))^\perp$, the normal space to σ at time t. Since σ is nonnull, $V_{\sigma(t)}$ is a non-degenerate subspace of $T_{\sigma(t)}M$ and hence of dimension $(n-1)$. First we choose a parallel frame along σ by parallel transport of some orthonormal basis of T_qM along σ and identify T_qM with $\mathbb{R}^n_k$. In this way we can identify a linear operator on $T_{\sigma(t)}M$ with a smooth family of linear operators on $\mathbb{R}^n_k$ and linear operators on $V_{\sigma'(t)}$ are identified with linear operators on an $(n-1)$-dimensional subspace of $\mathbb{R}^n_k$. We now use Lemma 2.2.2 and then (4.31) to show that $\hat{S} = S_{K,q}$. Since R_N is self adjoint and smooth by Lemma 4.4.2 we can set $R := R_N$ in Lemma 2.2.2 and so (2.2) is

$$F'' + R_N(F) = 0,$$

which is the Jacobi-equation (4.26) for a Jacobi-field F orthogonal to σ since by Lemma 4.4.2, R_N leaves $N^\perp = V$ invariant . Since we assume $F'(0) \neq 0$ it follows that F is a nontrivial Jacobi field along σ, so in particular $F(t)$ is invertible for all t since σ is contained in a normal coordinate neighborhood of q, so no conjugate points can occur along σ. Now if we set $g := 1$ in Lemma 2.2.2 then since $g' = 0$ we see that (2.5) is the Riccati equation (4.30), that is

$$S' + R_N + S^2 = 0.$$

[2]Since the modified shape operator $S : \mathfrak{X}(U) \to \mathfrak{X}(U)$ is a $\mathcal{C}^\infty(M)$-linear map.

Let S_W be the solution to this equation, then by Lemma 2.2.2 we have

$$S_W(t) = F'(t)F(t)^{-1},$$

which is a self adjoint and linear map with $S_W(0) = I$. Also the Weingarten- or shape operator $W_t(v) := W_{\sigma(t)}(v) = (\nabla_v \operatorname{grad} f)(\sigma(t))$ of $V_{\sigma(t)}$, where f is a distance function as in Section 4.5, solves the above Riccati equation with $W_0 = I$. Thus W and S_W agree and

$$S_W(t) = W_t = F'(t)F(t)^{-1} \tag{4.36}$$

are the Weingarten maps of equidistant hypersurfaces from q. On the other hand, set $g := \sqrt{|\langle G, G\rangle|}$, then by Lemma 4.1.3 (ii)+(iii) it follows that g satisfies the perquisites of Lemma 2.2.2 and further

$$g'(t) = \cos\sqrt{KE_q(\sigma(t))} = (1 - Kh_{K,q}(\sigma(t))).^3 \tag{4.37}$$

Comparing (4.35) and Lemma 4.1.3(ii) shows $G = gN$. Also for any $v \in V_{\sigma(t)}$ we have $v(g) = 0$ since g is constant on level sets $V_{\sigma(t)}$. To see this just note that G only depends on $E_q(\sigma(t))$ and $P(\sigma(t))$, which are constant on the level sets $V_{\sigma(t)}$. For $R = R_N$ as above, F in (2.2) remains unchanged and (2.3) yields for $t > 0$

$$\hat{S}(t) := g(t)\overbrace{F'(t)F(t)^{-1}}^{=W_t \text{ by } (4.36)} = g(t)W_t$$

and

$$\hat{S}(v) = gW(v) = g\nabla_v N \overset{v(g)=0}{=} \nabla_v(gN) = \nabla_v G = S_{K,q}(v).$$

Now for $t = 0$ we have $h_{K,q}(\sigma(0)) = 0$ and thus (4.34) reduces to $S_{K,q}(\sigma'(0)) = \sigma'(0)$. Using the exponential map we can realize any unit vector $v \in T_qM$ as a geodesic $c(t) := \exp(tv)$ with $c'(0) = v$. By this it follows that $S_{K,q}(0) = I$ has to be the identity, so $S_{K,q}$ also satisfies (2.4), which implies that the modified shape operator $S_{K,q} = \nabla \operatorname{grad}_{h_{K,q}}$ and $\hat{S}$ are the same if we choose $R = R_N$ and $g = \langle G, G\rangle$. So now it only remains to show that the Riccati equation (4.33) holds for $S = S_{K,q}$ restricted to $V = \sigma'^{\perp}$. We will do so by utilizing (2.5), which tells us that S satisfies

$$gS' + S^2 - g'S + g^2R = 0. \tag{4.38}$$

Thus if we show that on V the left hand side of (4.33) is equal to the left hand side of the above equation we are done. Thus for $v \in V$ we compute

$$(\nabla_G S)(v) = (\nabla_{gN} S)(v) = g\,\overbrace{(\nabla_{\sigma'} S)}^{=S'}(v) = gS'(v)$$

and

$$R_G(v) = R(G, v)G \overset{G=gN}{=} g^2R(N, v)N = g^2R(v).$$

[3]Another way to prove this is by seeing that $g = \sqrt{\pm\langle G, G\rangle}$ where the sign depends on the causal character of σ and thus $g' = \frac{1}{2g}2\pm\langle\nabla_N G, G\rangle \overset{(4.34)}{=} \pm(1-Kh)g^{-1}g\underbrace{\langle N, N\rangle}_{=\pm 1} = (1-Kh)$.

We have $v(h) = 0$ since $h(\sigma(t)) = h_{K,q}(\sigma(t))$ is constant on the level set and since $v \in V_{\sigma(t)}$, which is the tangent space of the level set in $\sigma(t)$. It follows that

$$K\,dh \otimes G(v) = Kv(h)G = 0.$$

At last

$$(1 - Kh)S(v) \overset{(4.37)}{=} g'(t)S(v)$$

and so all the terms of the Riccati equation (4.33) are dealt with and we are done by (4.38).

$\square$

Chapter 5

Local Triangle Comparison Theorems

5.1 Prerequisites

Here we will study the differential inequality $\psi'' + \kappa\psi \leqslant 0$. Under certain assumptions the solutions are positive functions.

Lemma 5.1.1. *For $\kappa \in \mathbb{R}$, let $\psi : [0, L] \to \mathbb{R}$ be a smooth function such that*

$$\psi'' + \kappa\psi \leqslant 0, \tag{5.1}$$

$\psi(0) = 0$ and $\psi(L) = 0$. For $\kappa > 0$, assume that $L < \frac{\pi}{\sqrt{\kappa}}$. Then $\psi(t) \geqslant 0$ for all $t \in [0, L]$.

Proof. First we prove the result for $\kappa > 0$. The proof for this case is due to [12, p. 23,24]. So suppose to the contrary that there is a point $\bar{t} \in (0, L)$ such that $\psi(\bar{t}) < 0$, and pick $\bar{t}_{\min}$ such that $\psi(\bar{t}_{\min}) = \min_{t\in[0,L]} \psi(t)$. Since by assumption $L < \frac{\pi}{\sqrt{\kappa}}$ there exist $k > 0$ and $\varepsilon > 0$ with $k > \kappa$, such that

$$L < \frac{\pi}{\sqrt{k}} - \varepsilon < \frac{\pi}{\sqrt{k}} < \frac{\pi}{\sqrt{\kappa}} \tag{5.2}$$

is satisfied. For $c_0 > 0$ set $s_0(t) := -c_0 \sin(\sqrt{k}(t + \varepsilon))$, then s_0 satisfies the differential equation $s_0'' + ks_0 = 0$ together with $s_0(-\varepsilon) = 0$ and $s_0(\frac{\pi}{\sqrt{k}} - \varepsilon) = 0$. Further, $s_0(t) < 0$ for all $t \in (-\varepsilon, \frac{\pi}{\sqrt{k}} - \varepsilon)$, thus by (5.2) also on $[0, L]$. Note that this holds for any $c_0 > 0$. By choosing c_0 big enough, we can have $s_0(t) < \psi(\bar{t}_{\min}) \leqslant \psi(t)$ for all $t \in [0, L]$. Now we scale s_0 by some $c \in \mathbb{R}$ with $0 < c \leqslant 1$ in such a way that the graphs of cs_0 and ψ touch for the first time, where c is decreased starting at 1. The graphs intersect since we can choose c in such a way that cs_0 is arbitrarily small at any point and ψ is assumed to have negative values. So choose $c > 0$ such that $s(t) := cs_0(t)$ satisfies $s(t) \leqslant \psi(t)$ for $t \in [0, L]$ and $s(t_0) = \psi(t_0)$ for some $t_0 \in [0, L]$. Now by the above s again satisfies

$$s'' = -ks. \tag{5.3}$$

The difference $\psi - s$ then has a minimum at t_0, and since $s(t_0) = \psi(t_0) < 0$

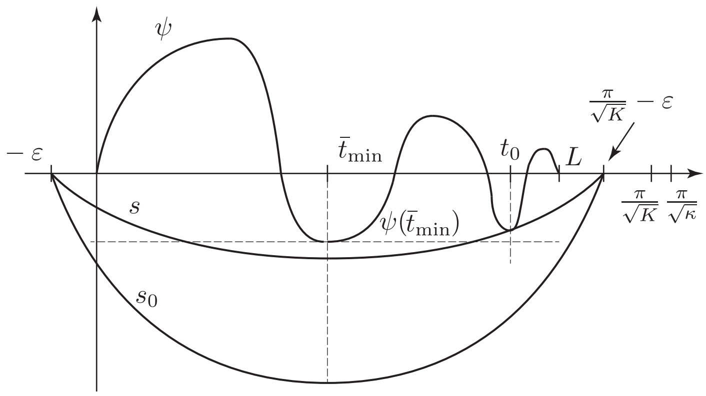

Figure 5.1: Setup in the proof for the case $\kappa > 0$.

$$(\psi - s)''(t_0) \overset{(5.1)+(5.3)}{\leqslant} \underbrace{(k-\kappa)}_{>0}\psi(t_0) < 0.$$

But this contradicts $\psi - s$ having a minimum at t_0. For $\kappa \leqslant 0$, again suppose there is a point $\bar{t} \in (0, L)$ such that $\psi(\bar{t}) < 0$. Since the values of ψ at the endpoints are 0 and by continuity, there exists a whole interval such that $\psi(t) \leqslant 0$ for $t \in [t_{\min}, t_{\max}]$, where $0 \leqslant t_{\min} < \bar{t} < t_{\max} \leqslant L$, and such that $\psi(t_{\min}) = \psi(t_{\max}) = 0$. So (5.1) shows for $t \in [t_{\min}, t_{\max}]$ that

$$\psi''(t) \leqslant -\underbrace{\kappa\psi(t)}_{\geqslant 0} \leqslant 0.$$

This implies that ψ restricted to $[t_{\min}, t_{\max}]$ is a concave function. Therefore, for all $\alpha \in [0, 1]$ by definition of concavity

$$\psi((1-\alpha)t_{min} + \alpha t_{max}) \geqslant (1-\alpha)\psi(t_{min}) + \alpha\psi(t_{max}) = 0$$

holds, which is a contradiction to $\psi(\bar{t}) < 0$.

□

5.2 Curvature Bounds Imply Local Triangle Comparison

Definition 5.2.1. *Let M and $\tilde{M}$ be semi-Riemannian manifolds, σ and $\tilde{\sigma}$ geodesics in M respectively $\tilde{M}$. Let σ and $\tilde{\sigma}$ be defined on the same interval I, then σ and $\tilde{\sigma}$ are called* **corresponding geodesics** *(corresponding geodesic segments/lines/rays) if*

$$\langle \sigma'(t), \sigma'(t) \rangle = \langle \tilde{\sigma}'(t), \tilde{\sigma}'(t) \rangle$$

for all $t \in I$. If two points on σ and $\tilde{\sigma}$ have the same affine parameter value they are called **corresponding points***.*

This definition allows us to to compare linear operators along geodesics in certain situations. Namely suppose that M and $\tilde{M}$ have the same dimension n and index ν, and let σ and $\tilde{\sigma}$ be corresponding geodesics defined on some interval I. Now for some fixed $t_0 \in I$ let $q := \sigma(t_0)$ and $\tilde{q} := \tilde{\sigma}(t_0)$ be corresponding points. Then we can identify the tangent spaces T_qM and $T_{\tilde{q}}\tilde{M}$ with each other by an isometry $\mathcal{I} : T_{\tilde{q}}\tilde{M} \to T_qM$ which identifies the tangent vectors $\tilde{c}'(t_0)$ and $c'(t_0)$. Further let $\mathcal{J} : T_qM \to \mathbb{R}^n_\nu$ be an isometry. By choosing parallel frames along both curves we can identify linear operators along σ and $\tilde{\sigma}$ with families of linear operators on T_qM and $T_{\tilde{q}}\tilde{M}$, which depend smoothly on $t \in I$. This is done in the same way as in the proof of Lemma 4.4.4. So let $A(t)$ and $\tilde{A}(t)$ be families of linear maps on T_qM and $T_{\tilde{q}}\tilde{M}$. Then A and $\tilde{A}$ can be compared by looking at $\mathcal{J} \circ A \circ \mathcal{J}^{-1}$ and $\mathcal{J} \circ \mathcal{I} \circ \tilde{A} \circ \mathcal{I}^{-1} \circ \mathcal{J}^{-1}$, which are booth families of self-adjoint linear maps on $\mathbb{R}^n_\nu$. We will use this construction in the following proofs without mentioning $\mathcal{J}$ and $\mathcal{I}$, to keep notation simple.

Lemma 5.2.2. *[1, Corollary 4.5 p. 268,269] Let M and $\tilde{M}$ be semi-Riemannian manifolds which have the same index and dimension. Given corresponding non-null geodesic rays $\sigma : [0,b] \to M$ and $\tilde{\sigma} : [0,b] \to \tilde{M}$ radiating from points $q := \sigma(0) \in M$ and $\tilde{q} := \tilde{\sigma}(0) \in \tilde{M}$, assume that no conjugate points exist along σ and $\tilde{\sigma}$ (e.g. that they are contained in a normal coordinate neighborhood of q respectively $\tilde{q}$). If the Ricci operators satisfy*

$$R_{\sigma'} \geqslant \tilde{R}_{\tilde{\sigma}'}$$

for corresponding points, then the modified shape operators satisfy

$$S_{K,q} \leqslant \tilde{S}_{K,\tilde{q}}$$

at corresponding points, where comparison of self-adjoint linear operators along geodesics is done as explained above. If we reverse all inequalities, the result is also true.

Proof. Let σ and $\tilde{\sigma}$ be corresponding geodesic rays, then by Lemma 1.7.4 we see that

$$E_q(\sigma(t)) = t^2\langle\sigma'(0), \sigma'(0)\rangle = t^2\langle\tilde{\sigma}'(0), \tilde{\sigma}'(0)\rangle = E_{\tilde{q}}(\tilde{\sigma}(t)). \tag{5.4}$$

But then, by definition of the modified distance function also $h_{K,q}(\sigma(t)) = h_{K,\tilde{q}}(\tilde{\sigma}(t))$. Set $h(t) := h_{K,q}(\sigma(t)))$ and $\tilde{h}(t) := h_{K,\tilde{q}}(\tilde{\sigma}(t))$. As shown in Proposition 4.3.3, along radial directions the modified shape operator $S := S_{K,q}$ is of the form

$$S(\sigma') = \nabla_{\sigma'} \operatorname{grad} h_{K,q} = (1 - Kh)\sigma'$$

for any semi-Riemannian manifold. The analogous result also holds for $\tilde{S} := \tilde{S}_{K,\tilde{q}}$ along $\tilde{\sigma}$. By again using the assumption that σ and $\tilde{\sigma}$ are corresponding geodesics, we get by the above

$$\langle S(\sigma'), \sigma'\rangle = (1 - Kh)\langle\sigma', \sigma'\rangle = (1 - K\tilde{h})\langle\tilde{\sigma}', \tilde{\sigma}'\rangle = \langle\tilde{S}(\tilde{\sigma}'), \tilde{\sigma}'\rangle$$

which shows that

$$S|_{\sigma'} = \tilde{S}|_{\tilde{\sigma}'}. \tag{*}$$

By Proposition 4.6.1 the action of S on $V := \sigma'^{\perp}$, respectively that of $\tilde{S}$ on $\tilde{V} := \tilde{\sigma}'^{\perp}$, is described by Lemma 2.2.2. Denote by P the local position vector field at q, and by $\tilde{P}$ the local position vector field at $\tilde{q}$. Then for $G := \operatorname{grad} h_{K,q}$ and $\tilde{G} := \operatorname{grad} h_{K,\tilde{q}}$, we will show that $\langle G, G\rangle = \langle \tilde{G}, \tilde{G}\rangle$. Since

$$\langle P(\sigma(t)), P(\sigma(t))\rangle \overset{(1.31)}{=} t^2\langle \sigma'(t), \sigma'(t)\rangle = t^2\langle \tilde{\sigma}'(t), \tilde{\sigma}'(t)\rangle \overset{(1.31)}{=} \langle \tilde{P}(\tilde{\sigma}(t)), \tilde{P}(\tilde{\sigma}(t))\rangle,$$

by *Lemma* 4.1.2 the result for $K = 0$ follows immediately. For $K \neq 0$ note that $\phi(t)$, as in *Lemma* 4.1.2 such that $(G \circ \sigma)(t) = \phi(t)(P \circ \sigma)(t)$, only depends on $E_q(\sigma(t))$, and $E_q(\sigma(t)) = E_{\tilde{q}}(\tilde{\sigma}(t))$ by (5.4). In Theorem 2.2.3, set $R_1 := R_{\tilde{\sigma}'}$, $R_2 := R_{\sigma'}$ and

$$g := \sqrt{|\langle G, G\rangle|} = \sqrt{|\langle \tilde{G}, \tilde{G}\rangle|}.$$

Then by Lemma 4.1.3(iii), g satisfies the prerequisites of Theorem 2.2.3, which yields

$$S|_V \leqslant \tilde{S}|_{\tilde{V}}. \tag{**}$$

Since σ and $\tilde{\sigma}$ are nonull we get $T_{\sigma(t)}M = \sigma'(t) \oplus V$ and $T_{\tilde{\sigma}(t)}\tilde{M} = \tilde{\sigma}'(t) \oplus \tilde{V}$. So we can apply Remark(iv) after *Definition* 2.1.1 to (*) and (**) to obtain $S \leqslant \tilde{S}$.

□

Lemma 5.2.3. *[1, Corollary 4.6 p. 269] Let M be a semi-Riemannian manifold such that $\mathcal{R} \geqslant K$ and $\mathcal{Q}_K$ be a constant curvature space of the same dimension and index as M. For $q, p \in M$ connected by a geodesic segment σ which has no conjugate points, if a corresponding geodesic segment in $\mathcal{Q}_K$ has no conjugate points then at p, the modified shape operator $S_{K,q}$ satisfies*

$$\left(S_{K,q}\right)_p \leqslant (1 - K h_{K,q}(p))(\mathrm{Id})_{T_pM}. \tag{5.5}$$

Here $(\mathrm{Id})_{T_pM}$ denotes the identity on T_pM. For $\mathcal{R} \leqslant K$ the same statement holds with the inequality in (5.5) *reversed.*

Proof. By Lemma 1.2.4 we know that $\mathcal{R} \geqslant K$ is equivalent to

$$\langle R_{vw}v, w\rangle \geqslant K(\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2) \tag{*}$$

for all $v, w \in T_pM$ which span a non-degenerate plane. Since T_pM and $T_{\tilde{p}}\mathcal{Q}_K$ are identified by an isometry, we will denote the images of v and w under this isometry with v and w as well. In the constant curvature space $\mathcal{Q}_K$ with curvature tensor $\tilde{R}$,

$$K = \frac{\langle \tilde{R}_{vw}v, w\rangle}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2}.$$

So by (*) we obtain

$$\langle R_{vw}v, w\rangle \geqslant \langle \tilde{R}_{vw}v, w\rangle.$$

But then

$$\langle (R_{vw} - \tilde{R}_{vw})v, w\rangle \geqslant 0,$$

which shows that at corresponding points of σ and σ', the self adjoint Ricci operators satisfy $R_{\sigma'} \geqslant \tilde{R}_{\tilde{\sigma}'}$. Denote the modified shape operator on $\mathcal{Q}_K$ by $\tilde{S}_{K,\tilde{q}}$ and the identity on $T_{\tilde{p}}\mathcal{Q}_K$ by $(\tilde{\mathrm{Id}})_{T_{\tilde{p}}\mathcal{Q}}$. Then by Lemma 5.2.2, since by assumption there are no conjugate points along both geodesics,

$$(S_{K,q})_p \leqslant (\tilde{S}_{K,\tilde{q}})_{\tilde{p}} = \begin{cases} (\tilde{\mathrm{Id}})_{T_{\tilde{p}}\mathcal{Q}_K} & \text{if } K = 0 \\ K\ell_{K,\tilde{q}}(\tilde{p})(\tilde{\mathrm{Id}})_{T_{\tilde{p}}\mathcal{Q}_K} & \text{if } K \neq 0. \end{cases}$$

This shows (5.5) for $K = 0$. If $K \neq 0$ the result follows by (4.19).

□

Theorem 5.2.4. *[1, Proposition 5.2 p. 271] Let U be a neighborhood normal for K*[1]*. If $\mathcal{R} \geqslant K$ on U, then triangle comparison holds on U: For all triangles Δpqr in U with comparison triangle $\Delta\tilde{p}\tilde{q}\tilde{r}$, the signed distance between points in Δpqr are $\geqslant$ the signed distance between corresponding points in $\Delta\tilde{p}\tilde{q}\tilde{r}$. The result with switched inequalities is also true.*

Remark: We can use any of the equivalent conditions for triangle comparison in Proposition 3.4.2.

Proof. Let Δpqr be a triangle in U with comparison triangle $\Delta\tilde{p}\tilde{q}\tilde{r}$ in a model space $\mathcal{Q}_K$ of constant curvature K with the same dimension and index as M. We will show that $\mathcal{R} \geqslant K$ ($\mathcal{R} \leqslant K$) implies that the signed distance from q to any point m on the geodesic connecting p with r is $\geqslant$ ($\leqslant$) than the distance between the corresponding points $\tilde{q}$ and $\tilde{m}$ in the model triangle $\Delta\tilde{p}\tilde{q}\tilde{r}$. Let $\sigma : [0,1] \to U$ be the unique geodesic connecting p with r and $\tilde{\sigma} : [0,1] \to \mathcal{Q}_K$ the corresponding geodesic in our model space connecting $\tilde{p}$ with $\tilde{r}$. The function $h_{K,q} \circ \sigma : [0,1] \to \mathbb{R}$ gives the modified distance from q to $\sigma(t)$. By Lemma 4.1.4 the modified distance function is a monotonically increasing function of the signed distance, so it suffices to show that $\mathcal{R} \geqslant K$ ($\mathcal{R} \leqslant K$) implies $h_{K,q} \circ \sigma \geqslant \tilde{h}_{K,\tilde{q}} \circ \tilde{\sigma}$ ($h_{K,q} \circ \sigma \leqslant \tilde{h}_{K,\tilde{q}} \circ \tilde{\sigma}$). First suppose that $\mathcal{R} \geqslant K$. Set $m := \sigma(s)$ for some $s \in [0,1]$, then by Lemma 5.2.3 we know that the modified shape operator $S_{K,q}$ satisfies

$$(S_{K,q})_m \leqslant (1 - Kh_{K,q}(\sigma(s)))(\mathrm{Id})_{T_mM}, \tag{*}$$

since q and m can be joined by unique geodesic without conjugate points. This holds since U is assumed to be a normal neighborhood for K, so in particular U is a normal neighborhood by definition. Using the above we compute

$$\begin{aligned}(h_{K,q} \circ \sigma)''(s) &\overset{(1.8)}{=} H^{h_{K,q}}(\sigma'(s), \sigma'(s)) \overset{(1.7)}{=} \langle \nabla_{\sigma'(s)} \operatorname{grad} h_{K,q}, \sigma'(s)\rangle \\ &\overset{(4.18)}{=} \langle (S_{K,q})_{\sigma(s)}(\sigma'(s)), \sigma'(s)\rangle \overset{(*)}{\leqslant} (1 - Kh_{K,q}(\sigma(s)))\langle \sigma'(s), \sigma'(s)\rangle.\end{aligned}$$

Since σ is parametrized by $[0,1]$ we know by (1.19) that $E(\sigma) = \langle \sigma'(t), \sigma'(t)\rangle$ for all $t \in [0,1]$. So set $h(t) := (h_{K,q} \circ \sigma)(t)$, then by the above h satisfies the differential inequality

$$h'' + KE(\sigma)h \leqslant E(\sigma). \tag{5.6}$$

Denote by $\tilde{h}_{K,\tilde{q}}$ the modified distance function in the model space $\mathcal{Q}_K$, and set $\tilde{h}(t) := (\tilde{h}_{K,\tilde{q}} \circ \tilde{\sigma})(t)$, then by (4.17) even equality holds

$$\tilde{h}'' + KE(\tilde{\sigma})\tilde{h} = E(\tilde{\sigma}). \tag{5.7}$$

[1]Remember that this means that U is a normal neighborhood and every triangle in U satisfies size bounds for K, which gives a unique realization in some model space of curvature K as shown in Lemma 3.2.4.

Since σ and $\tilde{\sigma}$ are corresponding geodesics,

$$E(\sigma) = \langle \sigma'(0), \sigma'(0) \rangle = \langle \tilde{\sigma}'(0), \tilde{\sigma}'(0) \rangle = E(\tilde{\sigma}).$$

Set

$$\psi := h - \tilde{h} \tag{5.8}$$

and then subtract (5.7) from (5.6) to get

$$\psi'' + KE(\sigma)\psi \leqslant 0. \tag{5.9}$$

At the endpoints, ψ has the values

$$\psi(0) = \tilde{h}_{K,\tilde{q}}(\tilde{\sigma}(0)) - h_{K,q}(\sigma(0)) = \tilde{h}_{K,\tilde{q}}(\tilde{p}) - h_{K,q}(p) = 0$$

since $E_q(p) = E_{\tilde{q}}(\tilde{p})$, which follows from $|pq|_\pm = |\tilde{q}\tilde{p}|_\pm$. Also,

$$\psi(1) = \tilde{h}_{K,\tilde{q}}(\tilde{\sigma}(1)) - h_{K,q}(\sigma(1)) = \tilde{h}_{K,\tilde{q}}(\tilde{r}) - h_{K,q}(r) = 0.$$

So for $\kappa := KE(\sigma)$, to apply Lemma 5.1.1 to (5.9) it only remains to show that in case $\kappa > 0$ the bounds for L hold. In our case $L = 1$ since our geodesics are assumed to be parametrized by $[0, 1]$ and the size bounds are then of the form

$$1 < \frac{\pi}{\sqrt{KE(\sigma)}}.$$

If $K > 0$ and $E(\sigma) > 0$, then $\sqrt{E(\sigma)} = |pr|_\pm = d(p, r)$ is the Riemannian distance between p and r, so

$$|pr|_\pm < \frac{\pi}{\sqrt{K}} \tag{1}$$

has to be satisfied. Now in case $K < 0$ and $E(\sigma) < 0$ the signed distance is given by $|pr|_\pm = -\sqrt{-E(\sigma)}$, and so the requirement then is

$$|pr|_\pm > \frac{-\pi}{\sqrt{-K}}. \tag{2}$$

Since U is normal for K, the signed side lengths of triangles in U have to satisfy the bounds in Lemma 3.2.4. So in particular (1) and (2) are satisfied. Then Lemma 5.1.1 shows that $\psi(t) \geqslant 0$ on $t \in [0, 1]$, which by (5.8) is equivalent to

$$(h_{K,q} \circ \sigma)(t) \geqslant (\tilde{h}_{K,\tilde{q}} \circ \tilde{\sigma})(t)$$

for all $t \in [0, 1]$ and we are done since Lemma 4.1.4 implies

$$|q\sigma(t)|_\pm \geqslant |\tilde{q}\tilde{\sigma}(t)|_\pm$$

Finally, the case $\mathcal{R} \leqslant K$ is shown the same way. To apply Lemma 5.1.1 in this case one has to set $\psi := \tilde{h} - h$ to get $\leqslant$ in the differential inequality for ψ.

□

5.3 From Triangle Comparison to Curvature Bounds

Theorem 5.3.1. *[1, Proposition 5.3.] Let U be a neighborhood which is normal for K. If for every triangle Δpqr in U, the signed distance between any two points is $\geqslant$ ($\leqslant$) the signed distance of the corresponding points in a realizing triangle $\Delta\tilde{p}\tilde{q}\tilde{r}$ in a model space of curvature K, then $\mathcal{R} \geqslant K$ ($\mathcal{R} \leqslant K$).*

Remark Instead of comparison of distances between any two points on corresponding triangles, any of the equivalent conditions of Proposition 3.4.2 can be used.

Proof. Let σ be a nonnull geodesic segment $\sigma : [0,1] \to U$. Then choose some $v \in T_{\sigma(0)}M$ which is orthogonal to $\sigma'(0)$. The subspace V spanned by $\sigma'(0)$ and v is two-dimensional since both vectors are nonnull. Let $\mathcal{Q}_K$ be the two-dimensional model space of curvature K with the same index as V. Choose a geodesic $\tilde{\sigma} : [0,1] \to \mathcal{Q}_K$ such that $\langle \sigma'(t), \sigma'(t)\rangle = \langle \tilde{\sigma}'(t), \tilde{\sigma}'(t)\rangle$ for all $t \in [0,1]$, and $\tilde{v} \in T_{\tilde{\sigma}(0)}\mathcal{Q}_K$ with $\langle \tilde{\sigma}'(0), \tilde{v}\rangle = 0$ and $\langle v, v\rangle = \langle \tilde{v}, \tilde{v}\rangle$. By Lemma 4.4.3 there is a unique Jacobi field along σ such that $J(0) = 0$ and $J'(0) = v$. Also along $\tilde{\sigma}$ there is a unique Jacobi field $\tilde{J}$ such that $\tilde{J}(0) = 0$ and $\tilde{J}'(0) = \tilde{v}$. Let τ be the geodesic variation of σ given by

$$\tau(t,s) := \sigma_s(t) := \exp_{\sigma(0)}(t(\sigma'(0) + sv))$$

and let $\tilde{\tau}$ be the corresponding geodesic variation of $\tilde{\sigma}$

$$\tilde{\tau}(t,s) := \tilde{\sigma}_s(t) := \exp_{\tilde{\sigma}(0)}(t(\tilde{\sigma}'(0) + s\tilde{v})).$$

The Jacobi fields J and $\tilde{J}$ are obtained as variational vector fields of these geodesic variations, $J(t) = \frac{\partial}{\partial s}|_{s=0}\sigma_s(t)$ and $\tilde{J}(t) = \frac{\partial}{\partial s}|_{s=0}\tilde{\sigma}_s(t)$. Further $\frac{\partial \tau}{\partial t}(0,s) = \sigma'(0) + sv$ and $\frac{\partial \tilde{\tau}}{\partial t}(0,s) = \tilde{\sigma}'(0) + s\tilde{v}$. So since $v \perp \sigma'(0)$ and $\tilde{v} \perp \tilde{\sigma}'(0)$,

$$\begin{aligned}\Big\langle \frac{\partial \tau}{\partial t}(0,s), \frac{\partial \tau}{\partial t}(0,0)\Big\rangle &= \langle \sigma'(0) + sv, \sigma'(0)\rangle = \langle \sigma'(0), \sigma'(0)\rangle \\ &= \langle \tilde{\sigma}'(0), \tilde{\sigma}'(0)\rangle = \langle \tilde{\sigma}'(0) + s\tilde{v}, \tilde{\sigma}'(0)\rangle \\ &= \Big\langle \frac{\partial \tilde{\tau}}{\partial t}(0,s), \frac{\partial \tilde{\tau}}{\partial t}(0,0)\Big\rangle.\end{aligned}$$

This shows that

$$\langle \sigma'(0), \sigma_s'(0)\rangle = \langle \tilde{\sigma}'(0), \tilde{\sigma}_s'(0)\rangle. \tag{5.10}$$

So the nonnormalized angle of the hinge given by σ and σ_s is the same as the nonnormalized angle of the corresponding hinge in $\mathcal{Q}_K$. Set $\alpha := \langle \sigma'(0), \sigma_s'(0)\rangle$. Now we show that

$$J(t) = \frac{1}{s}\exp^{-1}_{\sigma_0(t)}(\sigma_s(t)) + O(s). \tag{5.11}$$

This can be seen by looking at the Taylor expansion of $\exp^{-1}_{\sigma_0(t)}(\sigma_s(t))$ with respect to s, which is given by

$$\exp^{-1}_{\sigma_0(t)}(\sigma_s(t)) = \exp^{-1}_{\sigma_0(t)}(\sigma_0(t)) + s\,\frac{\partial}{\partial s}\Big|_{s=0}(\exp^{-1}_{\sigma_0(t)}(\sigma_s(t))) + O(s^2).$$

Now since $\exp^{-1}_{\sigma_0(t)}(\sigma_0(t)) = 0$ and

$$\frac{\partial}{\partial s}\Big|_{s=0}(\exp^{-1}_{\sigma_0(t)}(\sigma_s(t))) = \underbrace{(\, d\exp^{-1}_{\sigma_0(t)})_{\sigma_0(t)}}_{\mathrm{Id}}\Big(\frac{\partial}{\partial s}\Big|_{s=0}\sigma_s(t)) = J(t)$$

which shows (5.11). So we get

$$\begin{aligned}
|J(t)|_{\pm} &= \mathrm{sgn(J(t))}\sqrt{|\langle J(t), J(t)\rangle|} \\
&= \mathrm{sgn(J(t))}\lim_{s\to 0}\frac{1}{s}\sqrt{\left|\langle \exp^{-1}_{\sigma_0(t)}(\sigma_s(t)), \exp^{-1}_{\sigma_0(t)}(\sigma_s(t))\rangle\right|} \qquad (5.12)\\
&= \lim_{s\to 0}\frac{1}{s}|\sigma_0(t)\sigma_s(t)|_{\pm}
\end{aligned}$$

since $\mathrm{sgn(J(t))} = \mathrm{sgn}(\exp^{-1}_{\sigma_0(\mathrm{t})}(\sigma_\mathrm{s}(\mathrm{t})))$. Suppose that triangle comparison as in Proposition 3.4.2(3) holds, that is nonnormalized angles in triangles in U are $\geqslant$ the corresponding angles in a comparison triangle. Then this implies

$$|\sigma_0(t)\sigma_s(t)|_{\pm} \leqslant |\tilde{\sigma}_0(t)\tilde{\sigma}_s(t)|_{\pm}. \tag{5.13}$$

Since, suppose to the contrary that $|\sigma_0(t)\sigma_s(t)|_{\pm} > |\tilde{\sigma}_0(t)\tilde{\sigma}_s(t)|_{\pm} =: \tilde{d}_{new}$. Consider the triangle Δ_{new} in $\mathcal{Q}_K$ with side lengths $(|\sigma_0|_{\pm}, |\sigma_s|_{\pm}, \tilde{d}_{new})$ and let α_{new} be the nonnormalized angle between the sides with lengths $|\sigma_0|_{\pm}$ and $|\sigma_s|_{\pm}$. Then by the Hinge Lemma Lemma 3.3.1 the nonnormalized angle α_{new} is a decreasing function of the opposite side length. Since $|\sigma_0(t)\sigma_s(t)|_{\pm} > \tilde{d}_{new}$ this implies $\alpha < \alpha_{new}$, which contradicts Proposition 3.4.2(3). By combining (5.12) and (5.13) we get

$$\begin{aligned}
\langle J(t), J(t)\rangle &= \lim_{s\to 0}\frac{1}{s^2}\langle \exp^{-1}_{\sigma_0(t)}(\sigma_s(t)), \exp^{-1}_{\sigma_0(t)}(\sigma_s(t))\rangle \\
&= |\sigma_0(t)\sigma_s(t)|_{\pm} \leqslant |\tilde{\sigma}_0(t)\tilde{\sigma}_s(t)|_{\pm} \qquad (5.14)\\
&= \lim_{s\to 0}\frac{1}{s^2}\langle \exp^{-1}_{\tilde{\sigma}_0(t)}(\tilde{\sigma}_s(t)), \exp^{-1}_{\tilde{\sigma}_0(t)}(\tilde{\sigma}_s(t))\rangle = \langle \tilde{J}(t), \tilde{J}(t)\rangle.
\end{aligned}$$

By Lemma 4.4.4 we can write J and $\tilde{J}$ as

$$\begin{aligned}
J(t) &= \mathcal{P}_0^t(vt - \frac{1}{6}R_{\sigma'(0)v}(\sigma'(0))t^3 + O(t^4)) \\
\tilde{J}(t) &= \tilde{\mathcal{P}}_0^t(\tilde{v}t - \frac{1}{6}\tilde{R}_{\tilde{\sigma}'(0)\tilde{v}}(\tilde{\sigma}'(0))t^3 + O(t^4)).
\end{aligned}$$

where $\mathcal{P}_0^t$ and $\tilde{\mathcal{P}}_0^t$ denote the parallel transport along σ and $\tilde{\sigma}$, respectively. In the constant curvature model space

$$\tilde{R}_{\tilde{\sigma}'(0)\tilde{v}}(\tilde{\sigma}'(0)) \overset{(1.23)}{=} K(\underbrace{\langle\tilde{\sigma}'(0), \tilde{\sigma}'(0)\rangle}_{=\langle\sigma'(0),\sigma'(0)\rangle}\tilde{v} - \underbrace{\langle\tilde{v}, \tilde{\sigma}'(0)\rangle}_{=0}\tilde{\sigma}'(0)) = K\langle\sigma'(0), \sigma'(0)\rangle\tilde{v}.$$

Parallel transport is an isometry, so we get

$$\begin{aligned}
\langle J(t), J(t)\rangle &= \langle v, v\rangle t^2 - \frac{1}{3}\langle R_{\sigma'(0)v}\sigma'(0), v\rangle t^4 + O(t^5) \\
\langle \tilde{J}(t), \tilde{J}(t)\rangle &= \langle \tilde{v}, \tilde{v}\rangle t^2 - \frac{1}{3}K\langle\sigma'(0), \sigma'(0)\rangle\langle\tilde{v}, \tilde{v}\rangle t^4 + O(t^5).
\end{aligned}$$

Since $\langle v, v\rangle = \langle\tilde{v}, \tilde{v}\rangle$ by definition, the coefficients for the t^2-terms agree, we obtain from (5.14) via $t \to 0$ that

$$\langle R_{\sigma'(0)v}\sigma'(0), v\rangle \geqslant K\langle\sigma'(0), \sigma'(0)\rangle\langle v, v\rangle.$$

Any non-degenerate plane Π can be obtained by a suitable choice of σ and v[2]. So we get $\mathcal{R} \geqslant K$ by Lemma 1.2.4. □

[2]Let e_1, e_2 be an orthonormal basis for $\Pi \subset T_qM$ for $q \in U$. Set $\sigma(t) := \exp_q(te_1)$ and $v := e_2$, then $\sigma'(0) = e_1$.

5.4 Main Theorem

Now can collect our results in the main theorem:

Theorem 5.4.1. *[1, Theorem 1.1, p. 253] Let M be a semi-Riemannian manifold and U a neighborhood which is normal for K. Then $\mathcal{R} \geqslant K$ ($\mathcal{R} \leqslant K$) if the signed distance between any two points on a triangle in U is $\geqslant$ ($\leqslant$) the signed distance of corresponding points on a realizing triangle in $\mathcal{Q}_K$[3]. If, on the other hand, triangle comparison holds on a normal coordinate neighborhood U, then the corresponding size bounds $\mathcal{R} \geqslant K$ or $\mathcal{R} \leqslant K$ on U follow. So, in particular, if this holds for all points then curvature bounds for M follow.*

Proof. By *Theorem* 5.2.4 and *Theorem* 5.3.1. □

We end this thesis with a short recapitulation of the steps which have led us to the proof of Theorem 5.4.1. The main tool was the modified distance function $h_{K,q}$, which is K-convex along spacelike geodesics, K-concave along timelike geodesics and K-affine for null geodesics, as shown in Proposition 4.3.3. The modified distance function then induces a modified shape operator $S_{K,q}$, which is smoothly defined on all radial geodesics even if they are null. On hyperquadrics $\mathcal{Q}_K$ the shape operator is just a scalar multiple of the identity on each tangent space. In Proposition 4.6.1 we show that the modified shape operator can also be obtained from the Ricci operator using Lemma 2.2.2 and satisfies a Riccati equation. Further, in the proof of Proposition 4.6.1 have seen that the modified shape operator is a rescaling of the Weingarten map of an equidistant hypersurface of q. Along radial geodesics, inequalities for the Ricci operators then imply inequalities for the modified shape operators in Lemma 5.2.2, where we use Theorem 2.2.3 for the comparison of self adjoint linear maps. If the sectional curvature of M is bounded above (below) by K, then by Lemma 5.2.2 we get a bound on the modified shape operator, given by a rescaling of the identity on the tangent space by $(1 - Kh_{K,q})$. By this we see in the proof of Theorem 5.2.4 that $h = h_{K,q}$ satisfies $h'' + KE(\sigma) \leqslant E(\sigma)$. Furthermore in the model spaces of constant curvature K even equality holds for the corresponding modified distance function $\tilde{h}$. Using Lemma 5.1.1 we then conclude that triangle comparison holds. The equivalent definitions for triangle comparison on neighborhoods which are normal for K are shown in Proposition 3.4.2, using Lemma 3.3.1(hinge) and Lemma 3.3.2(straightening). The reverse direction, where we suppose that triangle comparison holds, is shown by comparing Jacobi fields along radial geodesics and utilizing the Taylor expansion for Jacobi fields.

[3]So triangle comparison holds on U, which means all of the equivalent conditions in Proposition 3.4.2 hold.

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