MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

A Relativistic Bell Test within Quantum Reference Frames

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2019 / Vienna, 2019

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 876
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Physik UG2002
Betreut von / Supervisor:	Univ.-Prof. Mag. Dr. Časlav Brukner

CONTENTS

Zusammenfassung

Bell Tests werden allgegenwärtig für eine Vielzahl von Anwendungen eingesetzt und sind ein Grundpfeiler sowohl in der Quanteninformation als auch in den Grundlagen der Quantenmechanik. Die Verletzung der CHSH-Bell Ungleichung durch einen Bell Test zeigt auf grundlegende Art und Weise, dass die Natur mit keiner *lokal realistischen* Theorie vollständig beschrieben werden kann. Damit ist gezeigt, dass Quantenverschränkung tiefgründiger ist als klassische Korrelationen und dies ermöglicht konkrete Anwendungen, wie z.B. die Quantenteleportation, die klassisch unerklärlich sind.

Für einen Bell Test können zwei verschränkte und massive Spin-1/2 Teilchen verwendet werden, deren Spin in raumartig getrennten Gebieten jeweils durch einen Stern-Gerlach-Apparat gemessen wird. Wenn sich die Teilchen jedoch in einer Superposition relativistischer Geschwindigkeiten bewegen, ist die Realisierung eines Bell Tests unbekannt, da in diesem Fall die Spinzustände vom Impuls abhängen und es keine operationelle Definition für die beiden Spinobservablen gibt.

In der vorliegenden Arbeit wird eine Lösung für dieses Problem vorgeschlagen, welche *Quantenbezugssysteme* und die damit ermöglichte Definition des *Ruhesystems eines Quantenobjekts* benützt. Das Ruhesystem eines Quantenobjekts kann nicht mittels üblicher Transformationen zwischen Bezugssystemen definiert werden, da sich Quantenobjekte im Allgemeinen in einer Superposition von Geschwindigkeiten bewegen. Hier wird diese Limitierung mithilfe von Transformationen zwischen Quantenbezugssystemen überwunden. Somit kann im Ruhesystem eines relativistischen Teilchens dessen Spin, wie in der nicht-relativistischen Quantenmechanik, operationell definiert werden. Die entsprechende Beschreibung in einem anderen (Quanten-)Bezugssystem ergibt sich mittels einer relativistischen Transformation zwischen Quantenbezugssystemen. Auf diese Weise ergibt sich eine operationelle Definition der beiden Spinobservablen für den Bell Test im Laborbezugssystem, wo sich die beiden massiven Spin-1/2 Teilchen in einer Superposition relativistischer Geschwindigkeiten bewegen. Des Weiteren wird gezeigt, dass die Verletzung der CHSH-Bell Ungleichung, im Gegensatz zu den Spinobservablen, unabhängig vom Bezugssystem ist. Die (maximale) Verletzung der CHSH-Bell Ungleichung zeigt einen möglichen Weg, Quanteninformationsprotokolle operationell auf das relativistische Milieu zu erweitern, wobei (verschränkte) massive Teilchen, die sich in einer

Superposition relativistischer Geschwindigkeiten bewegen, als Träger der Quanteninformation dienen.

Abstract

Bell tests are ubiquitously used for a variety of applications, and have been a cornerstone both in quantum information and quantum foundations. The violation of the CHSH-Bell inequality, by means of a Bell test, shows fundamentally that nature cannot be entirely described with any *local realistic* theory. This reveals that quantum entanglement is more profound than classical correlations and allows for concrete applications with no classical counterpart, such as quantum teleportation.

For a typical Bell test we can use two massive spin-1/2 particles which are entangled in their spin degrees of freedom and jointly measure their spin states in space-like separated regions with the help of two Stern-Gerlach apparatuses. However, when the particles move in a superposition of relativistic velocities we do not know how to set up a Bell test since, in this case, the spins are momentum-dependent and there is no operational definition for the joint spin observables.

A solution to this problem is proposed in the present thesis, which exploits *quantum reference frames* and the accompanying possibility of defining the *rest frame of a quantum system*. Specifically, quantum reference frame transformations can reveal the rest frame of a general quantum system (moving in a superposition of velocities from the point of view of another physical system) which cannot be achieved by means of standard reference frame transformations. Thus, the rest frame of a relativistic particle can be used to operationally define its spin degree of freedom as in non-relativistic quantum mechanics and the corresponding description in another (quantum) reference frame is found by means of a relativistic quantum reference frame transformation. In this way, we find an operational definition of the joint spin observables for a Bell test in the laboratory frame, where the two massive spin-1/2 particles move in a superposition of relativistic velocities. Furthermore, it is shown that the violation of the CHSH-Bell inequality is frame-independent in contrast to the spin observables which transform in a covariant way. The (maximal) violation of the CHSH-Bell inequality points out a possible way to operationally extend quantum information protocols to the relativistic regime, where (entangled) massive particles moving in a superposition of relativistic velocities are utilized as a resources of quantum information.

Acknowledgments

I would like to express my deepest appreciation to Dr. Flaminia Giacomini (Institute of Quantum Optics Quantum Information, Vienna) for her continuous and immediate support. Thank you for proposing me the interesting topic of my thesis and for the numerous hours you dedicated to help me improving my scientific practice and understanding.

I would like to extend my sincere thanks to Univ.-Prof. Časlav Brukner (Faculty of Physics, University of Vienna) for his broad support. Thank you for giving me the freedom to explore my own thoughts and interests.

Finally, I would like to thank my family and my friends for their continuous encouragement and understanding being.

1. INTRODUCTION

The usual interpretation of coordinate systems[1] rests on the abstraction of idealized physical systems. In the case of spatial coordinates, Albert Einstein made clear that: "The physical interpretation of the spatial co-ordinates presupposed a fixed body of reference, which, moreover, had to be in a more or less definite state of motion (inertial system). In a given inertial system the co-ordinates meant the results of certain measurements with rigid (stationary) rods"[2][1]. In this view, the physical meaning of coordinates is grounded on the relation between two physical systems. Crucially, physical systems are ultimately quantum; therefore, they can be in a superposition or entangled from the point of view of another physical system. However, is it possible to use quantum systems to define a reference frame?

An approach to answer this question was proposed in [2], where reference frames are "attached" to quantum systems and exhibit genuine quantum features, i.e. they can be in a superposition, or entangled, from the point of view of a different physical system. These so-called *quantum reference frames (QRFs)* generalize the notion of usual reference frames as abstract and idealized entities. Moreover, the rest of the world in the perspective of a QRF represents the relational information between the QRF and all physical systems external to it. This is in line with the relational description of physics [3, 4], since a quantum state with its properties, such as superposition and entanglement, is defined only relative to a QRF - i.e. relative to a physical system. As a result of the presented QRF framework [2], it is concluded that superposition and entanglement are frame-dependent features.[3] The central idea about quantum reference frames with the aim of linking the perspectives of different QRFs is pictorially illustrated in figure 1.1.

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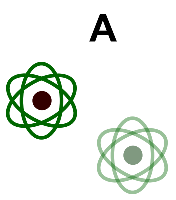
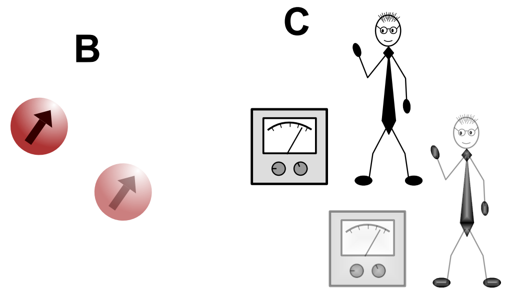
1 Coordinate systems and reference frames are used as synonyms in the present thesis.

2 Albert Einstein's original words in his autobiographical notes [1]: "Die physikalische Deutung der räumlichen Koordinaten setzten einen starren Bezugskörper voraus, der noch dazu von mehr oder minder bestimmten Bewegungszustande (Intertialsystem) sein musste. Bei gegebenem Inertialsystem bedeuteten die Koordinaten Ergebnisse von bestimmten Messungen mit starren (ruhenden) Stäben."

3 It is worth mentioning that independently of the QRF approach [2], it is shown by utilizing a *Gedanken experiment* in a general relativistic setup that superposition has to be understood relationally [5]. This emphasizes the need of non-classical relations between coordinates. In addition, arguments for the relational nature of superposition are given in [6].



(a) The rest of the world from the point of view of a laboratory (or another physical system). Here, an atom A in spatial superposition and a spin particle B are shown.



(b) Since the relation between physical systems entirely defines their properties relative to each other, the spin particle B and the laboratory C must be in a spatial superposition state from the point of view of the atom.

Figure 1.1.: We take the perspective of a physical system by means of a quantum reference frame (QRF) which exhibits genuine quantum features such as superposition and entanglement. With the help of generalized coordinate transformations, so-called QRF transformations, it is possible to change between the perspectives of quantum reference frames. This leads to a frame-dependent notion of superposition and entanglement.

By choosing relative variables, it is possible to find generalized transformations mapping the description of an initially chosen QRF to the perspective of another QRF. These QRF transformations are quantum canonical transformations [7, 8] that take everything external to the initial QRF as input and output the perspective of the new QRF describing everything that is external to the new QRF. This means that no QRF describes its own dynamical degrees of freedom which can be related to the so-called self-reference problem [9]. Furthermore, the considered approach to quantum reference frames [2] is operational since it assigns a central role to primitive laboratory operations (preparations, transformations and measurements). Consequently, the utilized QRF formalism can be fully specified by notions with immediate physical meaning.

It was shown in [2] that a QRF transformation corresponding to a "superposition of boosts" can be defined which outputs the rest frame of a quantum system that is moving in a superposition of velocities from the point of view of an initial QRF. It is crucial to notice that, if a quantum system moves in a superposition of velocities, no standard transformation can take us its rest frame. This generalized view of the rest frame of a quantum system is a key element of the present thesis and allows for concrete applications, where the natural description of internal degrees of freedom in the rest frame is utilized to find the corresponding description in the laboratory reference frame.

Specifically, this thesis focuses on particles with spin as internal degree of freedom moving in a superposition of relativistic velocities from the point of view of the laboratory frame. In contrast to non-relativistic quantum mechanics, the operational definition of spin through a Stern-Gerlach apparatus fails in the relativistic regime, because the spin state of a massive particle, which is moving in a superposition of relativistic velocities, is not covariant under Lorentz transformations [10]. Significantly, this means that in relativistic quantum mechanics, the spin degree of freedom is momentum-dependent and an initially considered pure reduced spin state (obtained by tracing out the momentum) gets mixed if a Lorentz boost is applied (before the momentum is traced out). As a result, the spin degree of freedom cannot be treated as a resource of quantum information in the relativistic regime, because no spin state can be prepared and measured with probability one with respect to different inertial frames. In fact, there are several proposals for a covariant spin operator in the relativistic regime which are mostly inspired through abstract group theoretical considerations and lacking an operational identification [11]. However, in the rest frame of a particle its spin state can be operationally well defined through a Stern-Gerlach apparatus as in non-relativistic quantum mechanics. With the help of a relativistic QRF transformation[4], corresponding to a "superposition of Lorentz boosts", a relativistic version of the Stern-Gerlach experiment in the laboratory frame has been found in [12]. This gives an operational definition of spin in the relativistic regime and allows to treat one massive spin-1/2 particle moving in a superposition of relativistic velocities as a quantum bit - a *qubit*.

The original results of the present thesis show, with the help of the QRF formalism [2, 12], how to devise a Bell test with two massive spin-1/2 particles moving in a superposition of relativistic velocities relative to the laboratory frame. Significantly, Bell tests arose from the philosophical discussion of physical reality [14], since this led John Bell to derive an inequality [15] which is satisfied by any description of nature that obeys *local realism*. The worldview of local realism arose out of our everyday experiences and can be composed into *realism*, i.e. the existence of the properties of a physical system independently of measurement, and *locality*, i.e. the bounded propagation speed of physical influences by the speed of light. Significantly, the violation of the Clauser-Horne-Shimony-Holt version of Bell's inequality [16], referred to as *CHSH-Bell inequality*, made it possible to experimentally [17, 18, 19] rule out any local realistic description of nature which forces us rethink our worldview. So far, however, there is no experimental Bell test utilizing massive spin particles at relativistic velocities. In fact, there are lots of theoretical approaches [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34], where different spin operators and quantum states are utilized. Apart from [34], all the other authors have considered quantum states with sharp momenta; however, the extension to states in a superposition of momenta poses a non-trivial problem which is addressed in the present thesis. In contrast to [20, 23, 24, 25, 26, 29, 30, 32, 34] and in line with [21, 22, 27, 28, 31, 33], we find that the violation of the CHSH-Bell inequality is frame-independent. This follows from the unitarity of the considered relativistic QRF transformation which guarantees

[4] Note that the extension of the QRF approach to the special-relativistic regime [12] is accompanied by abandoning time as an absolute parameter, i.e. abandoning absolute Newtonian time [13], which is underlying in the Galilean treatment of QRFs.

that different observers/QRFs agree on the probabilities of observing the same event. As a consequence, the joint spin observables (Bell observable) differ between QRFs. In particular, we find an operational definition of the Bell observable in a relativistic setting from the point of view of the laboratory frame by exploiting the rest frame of quantum systems where spin measurements can be defined operationally. In this context, the violation of the CHSH-Bell inequality reveals a possible way to operationally use spin entanglement between relativistic massive spin-1/2 particles moving in a superposition of momenta such that quantum communication protocols, e.g. quantum teleportation [35], can be extended operationally to this regime.

The present thesis is organized in the following way. In chapter 2, an introduction to the QRF formalism [2] is given, where examples of QRF transformations corresponding to the "superposition of spatial translations" and the "superposition of boosts" are presented and the generalized notion of the rest frame of a quantum system is defined and exemplified. The problems associated with the definition of spin within a special-relativistic framework of quantum mechanics are explained in chapter 3. Moreover, [12] is reviewed, where a relativistic QRF transformation corresponding to the "superposition of Lorentz boosts" is introduced, and an operational definition of spin in the relativistic regime is given. The original results of the present thesis are shown in chapter 4, where a Bell test with different quantum states of massive spin-1/2 particles moving in a superposition of relativistic velocities is discussed within the framework of quantum reference frames.

2. Quantum Reference Frames

The present chapter reviews a formalism to describe physics from the point of view of a quantum reference frame (QRF) introduced in [2]. In section 2.1, it is shown how to extend the standard reference frame transformation to a QRF transformation, and examples of different transformations, corresponding to the "superposition of spatial translations" and the "superposition of boosts" are presented. Finally, in section 2.2, a novel way of defining the rest frame of a quantum system is discussed.

It is worth mentioning that in [2] the focus is put on the covariance of physical laws and hence on dynamics, and accelerated QRFs are considered as well. However, these topics are skipped here in order to maintain the guiding thread of this work.

2.1. Quantum Reference Frame Transformations

In the following, two quantum systems, A and B, are described from the point of view of a third system C. In general, A and B can be composed quantum systems; for instance, A and B can refer to an atom which is composed of electrons, protons and neutrons. The goal is to define a QRF transformation which allows to "jump" to the perspective of another quantum system. To fix the ideas, the new QRF is chosen to be A and the corresponding change of perspective is pictorially represented in figure 1.1. More precisely, the QRF transformations are found by choosing relative variables defined on a particular basis.

The general procedure in this chapter is to start in the perspective of C describing everything that is external to it, i.e. A and B, and then apply a QRF transformation mapping from C's to A's description. Importantly, the resulting perspective of A only involves B and C as physical degrees of freedom. Moreover, the action of all QRF transformations on the mutually observed system (B) is controlled by the quantum system we are jumping to (A).

In mathematical terms, the QRF transformation $\hat{S}$ is defined as a map between Hilbert spaces and the change from C's to A's perspective is denoted by $\hat{S} : \mathcal{H}_{\mathrm{A}}^{|\mathrm{C}} \otimes \mathcal{H}_{\mathrm{B}}^{|\mathrm{C}} \mapsto \mathcal{H}_{\mathrm{B}}^{|\mathrm{A}} \otimes \mathcal{H}_{\mathrm{C}}^{|\mathrm{A}}$ where $\mathcal{H}_{\mathrm{K}}^{|\mathrm{J}}$ is the Hilbert space of K as seen by J. Crucially, QRF transformations are required to be quantum canonical transformations [7, 8]; this means that the transformation is invertible and mapping the phase space observables $(\hat{x},\hat{p}) \mapsto (\hat{q},\hat{\pi})$ such that the commutation relation is preserved, i.e. $[\hat{x},\hat{p}] = [\hat{q},\hat{\pi}] = \mathrm{i}\hbar$.

For simplicity, we only consider unitary QRF transformations, i.e. $\hat{S}^{-1} = \hat{S}^{\dagger}$ implying $\hat{S}\hat{S}^{\dagger} = \hat{S}^{\dagger}\hat{S} = \mathbb{1}$. Consequently, the reverse QRF transformation, from A's to C's perspective, is simply given by $\hat{S}^{\dagger}$ and the canonicity is immediate since

$$[\hat{q},\hat{\pi}] \equiv [\hat{S}\hat{x}\hat{S}^{\dagger}, \hat{S}\hat{p}\hat{S}^{\dagger}] = \hat{S}\hat{x}\hat{S}^{\dagger}\hat{S}\hat{p}\hat{S}^{\dagger} - \hat{S}\hat{p}\hat{S}^{\dagger}\hat{S}\hat{x}\hat{S}^{\dagger} = \hat{S}[\hat{x},\hat{p}]\hat{S}^{\dagger} = \mathrm{i}\hbar \tag{2.1}$$

where $[\hat{x},\hat{p}] = \mathrm{i}\hbar$ is presupposed. Moreover, the unitarity ensures that the scalar product stays invariant under QRF transformations, i.e.

$$\langle\phi|\psi\rangle^{|\mathrm{C}} = \langle\hat{S}\phi|\hat{S}\psi\rangle^{|\mathrm{A}} \tag{2.2}$$

where the labels $|\mathrm{C}$ and $|\mathrm{A}$ refer to the Hilbert space on which the scalar product is applied; in particular, $|\mathrm{C}$ and $|\mathrm{A}$ refer to $\mathcal{H}_{\mathrm{A}}^{|\mathrm{C}} \otimes \mathcal{H}_{\mathrm{B}}^{|\mathrm{C}}$ and $\mathcal{H}_{\mathrm{B}}^{|\mathrm{A}} \otimes \mathcal{H}_{\mathrm{C}}^{|\mathrm{A}}$, respectively. Notice that the functional form of the two scalar products does not have to be the same since the measure of the Hilbert space is allowed to change.

Until now the very basic framework has been discussed. In the following sections some specific QRF transformations are derived and illustrated to get an intuition of the subject.

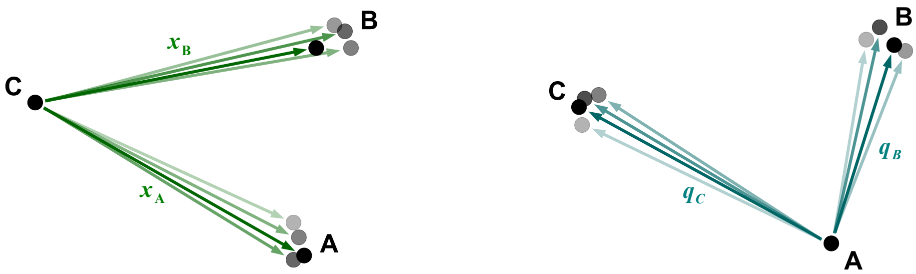
2.1.1. The QRF Transformation of Relative Positions

When reference frames are treated as abstract entities, and not as physical degrees of freedom, the distance between the reference frames is standardly considered to be fixed. Thus, the description of a quantum system, say B, as seen from another (abstract) reference frame at position x_0 relative to the initial frame is obtained by applying the standard translation operator

$$\hat{T}_{x_0} = \mathrm{e}^{\mathrm{i}x_0\hat{p}_{\mathrm{B}}/\hbar} \tag{2.3}$$

which translates the wave function of B by a fixed amount x_0, see appendix A.1 for details.

In contrast, quantum reference frames are treated as physical degrees of freedom and the aim is to switch between relative coordinates from C's to A's perspective as illustrated in figure 2.1. Crucially, the distance between QRFs cannot be fixed due to the possibility of spatial superposition states. Consequently, the change to the quantum reference frame "attached" to the position of A can be obtained by promoting x_0 to the position operator $\hat{x}_{\mathrm{A}}$, which is *coherently translating* the wave function of B by the distance between C and A, and a subsequent *parity-swap operator* $\hat{\mathcal{P}}_{\mathrm{AC}} : \mathcal{H}_{\mathrm{A}}^{|\mathrm{C}} \mapsto \mathcal{H}_{\mathrm{C}}^{|\mathrm{A}}$ transforming the state of A from C's perspective to the state of C from A's perspective. The parity-swap operator is defined by $\hat{\mathcal{P}}_{\mathrm{AC}}\hat{x}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = -\hat{q}_{\mathrm{C}}$ from which follows that $\hat{\mathcal{P}}_{\mathrm{AC}}\hat{p}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = -\hat{\pi}_{\mathrm{C}}$ by canonicity, see appendix A.2 for details.



(a) Perspective of C (b) Perspective of A

Figure 2.1.: Description of the rest of the world from two different perspectives, C and A. The relation between the describing system and the rest of the world is fundamentally quantum and it is preserved through QRF transformations. The multiple arrows indicate the possibility of spatial superposition states with respect to the corresponding perspective.

The QRF transformation for relative positions $\hat{S}_x : \mathcal{H}_{\mathrm{A}}^{|\mathrm{C}} \otimes \mathcal{H}_{\mathrm{B}}^{|\mathrm{C}} \mapsto \mathcal{H}_{\mathrm{B}}^{|\mathrm{A}} \otimes \mathcal{H}_{\mathrm{C}}^{|\mathrm{A}}$ is given by

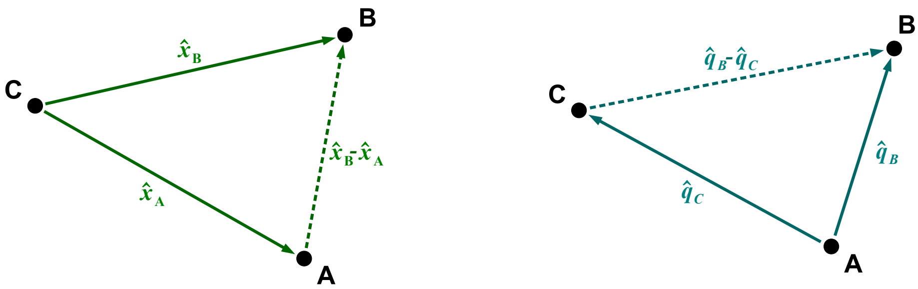
$$\hat{S}_x \equiv \hat{\mathcal{P}}_{\mathrm{AC}}\, \mathrm{e}^{\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar} \tag{2.4}$$

which transforms the considered phase space observables according to

$$\hat{x}_{\mathrm{A}} \mapsto -\hat{q}_{\mathrm{C}}, \qquad \hat{x}_{\mathrm{B}} \mapsto \hat{q}_{\mathrm{B}} - \hat{q}_{\mathrm{C}}, \tag{2.5}$$

$$\hat{p}_{\mathrm{A}} \mapsto -(\hat{\pi}_{\mathrm{B}} + \hat{\pi}_{\mathrm{C}}), \qquad \hat{p}_{\mathrm{B}} \mapsto \hat{\pi}_{\mathrm{B}}, \tag{2.6}$$

where the first line corresponds to the situation illustrated in figure 2.2 and the transfor-



(a) Perspective of C (b) Perspective of A

Figure 2.2.: Spatial observables as seen from different perspectives. In comparison to figure 2.1, the ability of superpositions is indicated by replacing the ordinary distances with their corresponding operators.

mation of momenta follows from the requirement of canonicity. Notice that it is easy to show that a transformation mapping to both relative positions and relative momenta is not canonical [36, 37].

The maps given in the two equations above can be easily verified since

$$\hat{x}_{\mathrm{A}} \mapsto \hat{S}_x \hat{x}_{\mathrm{A}} \hat{S}_x^{\dagger} \equiv \hat{\mathcal{P}}_{\mathrm{AC}}\, \mathrm{e}^{\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar} \hat{x}_{\mathrm{A}} \mathrm{e}^{-\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar}\, \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{x}_{\mathrm{A}} \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \equiv -\hat{q}_{\mathrm{C}} \tag{2.7}$$

and with the help of the well known *Baker-Campbell-Hausdorff* formula

$$\mathrm{e}^{\hat{A}} \hat{B} \mathrm{e}^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \frac{1}{3!}\left[\hat{A}, \left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right]\right] + \frac{1}{4!}\cdots \tag{2.8}$$

we obtain that

$$\begin{aligned} \hat{p}_{\mathrm{A}} \mapsto \hat{S}_x \hat{p}_{\mathrm{A}} \hat{S}_x^{\dagger} &\equiv \hat{\mathcal{P}}_{\mathrm{AC}}\, \mathrm{e}^{\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar} \hat{p}_{\mathrm{A}} \mathrm{e}^{-\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar}\, \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \overset{(2.8)}{=} \hat{\mathcal{P}}_{\mathrm{AC}} \left(\hat{p}_{\mathrm{A}} + [\tfrac{\mathrm{i}}{\hbar}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}, \hat{p}_{\mathrm{A}}] + \ldots\right) \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \\ &= \hat{\mathcal{P}}_{\mathrm{AC}} \left(\hat{p}_{\mathrm{A}} + \tfrac{\mathrm{i}}{\hbar}[\hat{x}_{\mathrm{A}}, \hat{p}_{\mathrm{A}}]\hat{p}_{\mathrm{B}} + 0\right) \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \end{aligned} \tag{2.9}$$

where all higher order commutators are zero because $[\hat{x}_{\mathrm{A}}, \hat{p}_{\mathrm{A}}] = \mathrm{i}\hbar$ is a scalar; thus,

$$\hat{p}_{\mathrm{A}} \mapsto \hat{\mathcal{P}}_{\mathrm{AC}} \left(\hat{p}_{\mathrm{A}} - \hat{p}_{\mathrm{B}}\right) \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{A}} \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} - \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{B}} \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = -\hat{\pi}_{\mathrm{C}} - \hat{\pi}_{\mathrm{B}} = -\left(\hat{\pi}_{\mathrm{B}} + \hat{\pi}_{\mathrm{C}}\right). \tag{2.10}$$

Analogously, the phase space observables of B as seen by C are obtained via

$$\begin{aligned} \hat{x}_{\mathrm{B}} \mapsto \hat{S}_x \hat{x}_{\mathrm{B}} \hat{S}_x^{\dagger} &\equiv \hat{\mathcal{P}}_{\mathrm{AC}}\, \mathrm{e}^{\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar} \hat{x}_{\mathrm{B}} \mathrm{e}^{-\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar}\, \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \overset{(2.8)}{=} \hat{\mathcal{P}}_{\mathrm{AC}} \left(\hat{x}_{\mathrm{B}} + [\tfrac{\mathrm{i}}{\hbar}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}, \hat{x}_{\mathrm{B}}] + \ldots\right) \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \\ &= \hat{\mathcal{P}}_{\mathrm{AC}} \left(\hat{x}_{\mathrm{B}} + \tfrac{\mathrm{i}}{\hbar}\hat{x}_{\mathrm{A}}[\hat{p}_{\mathrm{B}}, \hat{x}_{\mathrm{B}}] + 0\right) \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}} \left(\hat{x}_{\mathrm{B}} + \hat{x}_{\mathrm{A}}\right) \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \\ &= \hat{q}_{\mathrm{B}} - \hat{q}_{\mathrm{C}} \end{aligned} \tag{2.11}$$

and

$$\hat{p}_{\mathrm{B}} \mapsto \hat{S}_x \hat{p}_{\mathrm{B}} \hat{S}_x^{\dagger} \equiv \hat{\mathcal{P}}_{\mathrm{AC}}\, \mathrm{e}^{\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar} \hat{p}_{\mathrm{B}} \mathrm{e}^{-\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar}\, \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{B}} \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = \hat{\pi}_{\mathrm{B}}. \tag{2.12}$$

The QRF transformation $\hat{S}_x$ can also be derived from a gravity inspired symmetry principle which imposes that all physical observables are relational (as in the treatment above). This symmetry principle leads to redundancies when all systems A, B and C are considered jointly from a so-called perspective neutral structure, which contains all reference frames perspectives at once [38]. It is then possible to show that the choice of a reference frame amounts to eliminating the redundancies. The QRF transformation obtained with this method is shown to be equivalent to the one in eq. (2.4).

So far, we have seen that we can find a non-classical transformation mapping relative position observables appropriately; in the following, we show how the QRF transformation acts on quantum states.

Action on Quantum States

If C assigns an arbitrary yet separable state to A and B, i.e.

$$|\psi\rangle_{\mathrm{AB}}^{|C} = |\phi\rangle_{\mathrm{A}} |\xi\rangle_{\mathrm{B}} = \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{A}} \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{B}}\ \phi(x_{\mathrm{A}})\xi(x_{\mathrm{B}})\, |x_{\mathrm{A}}\rangle_{\mathrm{A}} |x_{\mathrm{B}}\rangle_{\mathrm{B}}\,, \tag{2.13}$$

then the transformation to A's perspective via $\hat{S}_x$ yields

$$\begin{aligned} |\psi\rangle_{\mathrm{BC}}^{|A} &= \hat{S}_x\, |\psi\rangle_{\mathrm{AB}}^{|C} \equiv \hat{\mathcal{P}}_{\mathrm{AC}}\, \mathrm{e}^{\mathrm{i}\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar}\, |\psi\rangle_{\mathrm{AB}}^{|C} \\ &= \hat{\mathcal{P}}_{\mathrm{AC}} \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{A}} \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{B}}\ \phi(x_{\mathrm{A}})\xi(x_{\mathrm{B}})\, |x_{\mathrm{A}}\rangle_{\mathrm{A}}\, \mathrm{e}^{\mathrm{i}x_{\mathrm{A}}\hat{p}_{\mathrm{B}}/\hbar}\, |x_{\mathrm{B}}\rangle_{\mathrm{B}} \\ &= \hat{\mathcal{P}}_{\mathrm{AC}} \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{A}} \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{B}}\ \phi(x_{\mathrm{A}})\xi(x_{\mathrm{B}})\, |x_{\mathrm{A}}\rangle_{\mathrm{A}}\, |x_{\mathrm{B}} - x_{\mathrm{A}}\rangle_{\mathrm{B}} \\ &= \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{A}} \int_{-\infty}^{\infty} \mathrm{d}x_{\mathrm{B}}\ \phi(x_{\mathrm{A}})\xi(x_{\mathrm{B}})\, |{-x_{\mathrm{A}}}\rangle_{\mathrm{C}}\, |x_{\mathrm{B}} - x_{\mathrm{A}}\rangle_{\mathrm{B}} \\ &= \int_{-\infty}^{\infty} \mathrm{d}q_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d}q_{\mathrm{C}}\ \xi(q_{\mathrm{B}} - q_{\mathrm{C}})\phi(-q_{\mathrm{C}})\, |q_{\mathrm{B}}\rangle_{\mathrm{B}}\, |q_{\mathrm{C}}\rangle_{\mathrm{C}} \end{aligned} \tag{2.14}$$

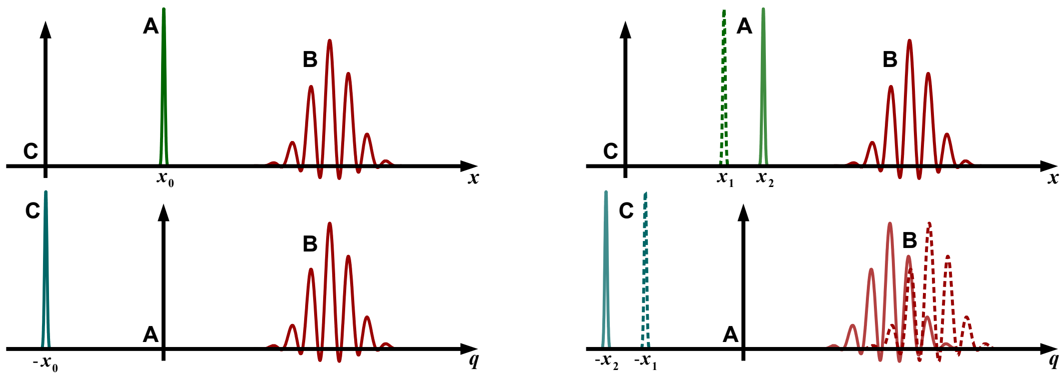
which is, in general, not a separable state. This means that entanglement depends on the frame of reference - it is frame-dependent. (Details of the action of the translation and parity-swap operator utilized here can be found in appendix A.1 and A.2, respectively.)

Illustrative Examples

Let us consider now a few examples to get an intuition about what it means to *coherently translate* a quantum system. Similarly to what has been shown above, we start with a separable state of A and B as seen from the perspective of C, and then transform to the perspective of A, from which we describe systems B and C.

First of all, the QRF transformation $\hat{S}_x$ reduces to a classical reference transformation when system A has a quantum state that is sharply localized in position basis with respect to C, as illustrated in figure 2.3(a). In this case, C just assigns a position eigenstate to A, i.e. $|\phi\rangle_{\mathrm{A}} = |x_0\rangle_{\mathrm{A}}$ and hence $\phi(x_{\mathrm{A}}) = \delta(x_{\mathrm{A}} - x_0)$. Consequently, B gets only translated by a well defined distance x_0. This transformation corresponds to the standard translation operator, eq. (2.3), with an additional switch of the roles of A and C via the parity-swap operator.

In the simplest, non-classical, case system A is in a superposition state of two sharp locations with respect to the reference system C, i.e. $|\phi\rangle_{\mathrm{A}} = (|x_1\rangle_{\mathrm{A}} + |x_2\rangle_{\mathrm{A}})/\sqrt{2}$ and hence $\phi(x_{\mathrm{A}}) = [\delta(x_{\mathrm{A}} - x_1) + \delta(x_{\mathrm{A}} - x_2)]/\sqrt{2}$. Allowing for arbitrary states of B (so that the joint



(a) $\phi(x) = \delta(x - x_0)$

(b) $\phi(x) = (\delta(x - x_1) + \delta(x - x_2))/\sqrt{2}$

Figure 2.3.: Action of the relative coordinate QRF transformation on states; C's perspective is drawn in the first and A's view in the second line. A classical translation between reference frames is shown in (a) where the states are shifted by a fixed distance x_0. Superposition of translations are shown in (b) where the states are coherently translated by x_1 and x_2 yielding to an entangled state in A's perspective which is graphically indicated by the correlation between the patterns of the lines (solid and dashed).

state of A and B is separable) we get in accordance with eq. (2.14)

$$\begin{aligned}|\psi\rangle_{\mathrm{BC}}^{|A} &= \int_{-\infty}^{\infty} \mathrm{d}q_{\mathrm{B}} \int_{-\infty}^{\infty} \mathrm{d}q_{\mathrm{C}}\, \xi(q_{\mathrm{B}} - q_{\mathrm{C}}) \frac{\delta(q_{\mathrm{C}} + x_1) + \delta(q_{\mathrm{C}} + x_2)}{\sqrt{2}} \,|q_{\mathrm{B}}\rangle_{\mathrm{B}} \,|q_{\mathrm{C}}\rangle_{\mathrm{C}} \\ &= \frac{1}{\sqrt{2}} \left(|{-x_1}\rangle_{\mathrm{C}} \int_{-\infty}^{\infty} \mathrm{d}q_{\mathrm{B}}\, \xi(q_{\mathrm{B}} + x_1)\, |q_{\mathrm{B}}\rangle_{\mathrm{B}} + |{-x_2}\rangle_{\mathrm{C}} \int_{-\infty}^{\infty} \mathrm{d}q_{\mathrm{B}}\, \xi(q_{\mathrm{B}} + x_2)\, |q_{\mathrm{B}}\rangle_{\mathrm{B}} \right)\end{aligned} \tag{2.15}$$

which is clearly entangled and graphically illustrated in figure 2.3(b). The reverse transformation, from A's to C's perspective, is simply given by $\hat{S}_x^\dagger$ due to the required unitarity of QRF transformations. Therefore, we could have also started with an entangled state and ended up with a separable state according to $|\psi\rangle_{\mathrm{AB}}^{|\mathrm{C}} = \hat{S}_x^\dagger \,|\psi\rangle_{\mathrm{BC}}^{|\mathrm{A}}$. This very simple example illustrates the relational, i.e. frame-dependent, nature of superposition and entanglement.

In order to see how the transformation acts on general product states, consider now the case where C assigns a Gaussian state to A in position basis, i.e. $\phi(x) = \pi^{-1/4} b^{-1/2} \mathrm{e}^{-\frac{(x-a)^2}{4b^2}}$ with a referring to the expectation value and b^2 to the variance of $|\phi(x)|^2$. The resulting state in A's perspective is entangled in a similar way as in the previous example; however, in this case each standard translation is weighted by the Gaussian. This is illustrated in figure 2.4, where three weighted position eigenstates are highlighted with different patterns in order to graphically illustrate entanglement. Notice that the envelopes shown for C and B in A's perspective are obtained by tracing out B and C, respectively. Conse-

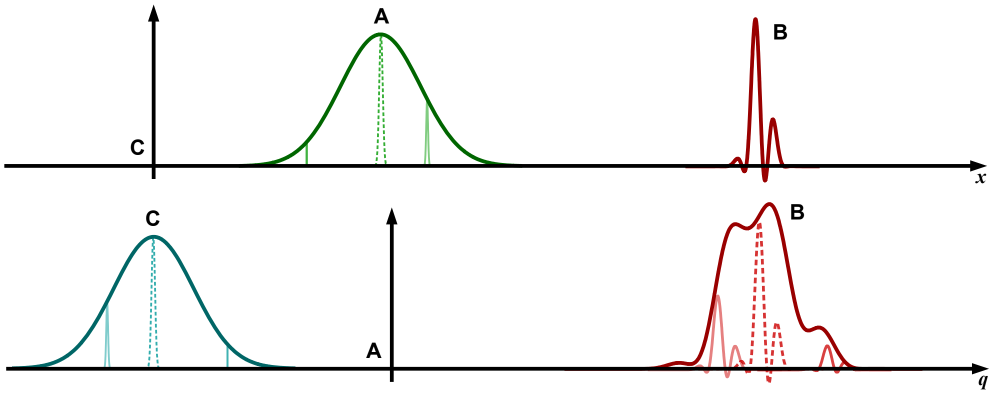


Figure 2.4.: Quantum state of A and B as seen from C (above) and the transformed quantum state of B and C as seen from A (below). In C's reference frame, the quantum state of A has a Gaussian profile in position basis, and it is in a product state with the quantum state of B. Since C sees A as a coherent sum of position eigenstates and not as a single position eigenstate, the QRF transformation coherently translates all states external to A by the single components of the Gaussian. In order to see the entanglement as perceived by A, three of those components are highlighted with different patterns for distinguishability. The drawn envelopes for C and B in A's perspective are obtained by tracing out B and C, respectively.

quently, C's envelope is given by

$$\int_{-\infty}^{\infty} \mathrm{d}q_{\mathrm{B}} |\xi(q_{\mathrm{B}} - q_{\mathrm{C}})|^2 |\phi(-q_{\mathrm{C}})|^2 = |\phi(-q_{\mathrm{C}})|^2 \tag{2.16}$$

where the normalization of $\xi(q_{\mathrm{B}})$ has been exploited and the envelope of B is obtained via

$$\int_{-\infty}^{\infty} \mathrm{d}q_{\mathrm{C}} |\xi(q_{\mathrm{B}} - q_{\mathrm{C}})|^2 |\phi(-q_{\mathrm{C}})|^2 \tag{2.17}$$

which is the convolution product of the absolute squares of the two considered wave functions ξ and ϕ. For illustration purpose $|\phi(-q_{\mathrm{C}})| = \phi(-q_{\mathrm{C}})$ and $\int \mathrm{d}q_{\mathrm{C}}\, \xi(q_{\mathrm{B}} - q_{\mathrm{C}})\phi(-q_{\mathrm{C}})$ have been drawn for the envelopes in figure 2.4 in A's perspective. The most significant aspect here is that transforming to the perspective of a quantum system that is initially described by a Gaussian wave function does not simply result in more broadened states in the new perspective (indicated by the envelopes) but to a highly entangled state (indicated by only three of the infinitely many position eigenstates building up the Gaussian wave function).

2.1.2. The QRF Transformation of Relative Velocities

The aim of this section is to find a transformation to the QRF of a particle moving in a superposition of velocities from the point of view of the initial reference frame.

In the previous section the position basis has been used to express relative positions and by canonicity the corresponding momenta have been calculated. Analogously, the QRF transformation of relative momenta is obtained by utilizing the momentum basis, see appendix A.3 for details. Note that this is the general scheme of finding a QRF transformation: first, choose a basis to express the relative quantities of interest and secondly, use the required canonicity to complete the transformation of the conjugate variables.

For the QRF transformation of relative velocities from A's to C's perspective we choose the momentum basis to express the physical conditions, that the velocity of A as seen by C is opposite to the velocity of C as seen by A, i.e. $\frac{\hat{p}_\mathrm{A}}{m_\mathrm{A}} \mapsto -\frac{\hat{\pi}_\mathrm{C}}{m_\mathrm{C}}$ [1], and that the velocity of the jointly described system B as seen by A is coherently boosted by the velocity of C, i.e. $\hat{p}_\mathrm{B} \mapsto \hat{\pi}_\mathrm{B} - m_\mathrm{B}\frac{\hat{\pi}_\mathrm{C}}{m_\mathrm{C}}$.

In order to swap the velocities of A and C, the parity-swap operator $\hat{\mathcal{P}}_\mathrm{AC}$ mapping $\hat{p}_\mathrm{A} \mapsto -\hat{\pi}_\mathrm{C}$ has to be generalized. Therefore, we introduce the *generalized parity-swap* operator

$$\hat{\mathcal{P}}^{(v)}_\mathrm{AC} \equiv \hat{\mathcal{P}}_\mathrm{AC} \exp\left\{\frac{\mathrm{i}}{\hbar}\log\sqrt{\frac{m_\mathrm{C}}{m_\mathrm{A}}}\left(\hat{x}_\mathrm{A}\hat{p}_\mathrm{A} + \hat{p}_\mathrm{A}\hat{x}_\mathrm{A}\right)\right\} \tag{2.18}$$

which rescales the corresponding momenta by the ratio of the masses of A and C such that $\frac{\hat{p}_\mathrm{A}}{m_\mathrm{A}} \mapsto -\frac{\hat{\pi}_\mathrm{C}}{m_\mathrm{C}}$.

The action of the added scaling operator is shown with the Baker-Campbell-Hausdorff formula eq. (2.8) according to

$$\exp\left\{\eta\left(\hat{x}_\mathrm{A}\hat{p}_\mathrm{A} + \hat{p}_\mathrm{A}\hat{x}_\mathrm{A}\right)\right\}\hat{p}_\mathrm{A}\exp\left\{-\eta\left(\hat{x}_\mathrm{A}\hat{p}_\mathrm{A} + \hat{p}_\mathrm{A}\hat{x}_\mathrm{A}\right)\right\} = \left(\hat{p}_\mathrm{A} + \left[\eta\left(\hat{x}_\mathrm{A}\hat{p}_\mathrm{A} + \hat{p}_\mathrm{A}\hat{x}_\mathrm{A}\right), \hat{p}_\mathrm{A}\right] + \ldots\right) \tag{2.19}$$

where $\eta \equiv \frac{\mathrm{i}}{\hbar}\log\sqrt{\frac{m_\mathrm{C}}{m_\mathrm{A}}} = \frac{\mathrm{i}}{2\hbar}\log\frac{m_\mathrm{C}}{m_\mathrm{A}}$. By recognizing $\hat{x}_\mathrm{A}\hat{p}_\mathrm{A} + \hat{p}_\mathrm{A}\hat{x}_\mathrm{A} = 2\hat{p}_\mathrm{A}\hat{x}_\mathrm{A} + [\hat{x}_\mathrm{A}, \hat{p}_\mathrm{A}] = 2\hat{p}_\mathrm{A}\hat{x}_\mathrm{A} + \mathrm{i}\hbar$, it follows that

$$\left[\eta\left(\hat{x}_\mathrm{A}\hat{p}_\mathrm{A} + \hat{p}_\mathrm{A}\hat{x}_\mathrm{A}\right), \hat{p}_\mathrm{A}\right] = 2\eta[\hat{p}_\mathrm{A}\hat{x}_\mathrm{A}, \hat{p}_\mathrm{A}] = 2\eta\hat{p}_\mathrm{A}[\hat{x}_\mathrm{A}, \hat{p}_\mathrm{A}] = 2\mathrm{i}\hbar\eta\hat{p}_\mathrm{A} \tag{2.20}$$

and hence

$$\begin{aligned}
\hat{p}_\mathrm{A} + \left[\eta\left(\hat{x}_\mathrm{A}\hat{p}_\mathrm{A} + \hat{p}_\mathrm{A}\hat{x}_\mathrm{A}\right), \hat{p}_\mathrm{A}\right] + \ldots &= \hat{p}_\mathrm{A} + 2\mathrm{i}\hbar\eta\hat{p}_\mathrm{A} + \frac{1}{2!}\left[\eta\left(\hat{x}_\mathrm{A}\hat{p}_\mathrm{A} + \hat{p}_\mathrm{A}\hat{x}_\mathrm{A}\right), 2\mathrm{i}\hbar\eta\hat{p}_\mathrm{A}\right] + \ldots \\
&= \hat{p}_\mathrm{A}\left(1 + 2\mathrm{i}\hbar\eta + (2\mathrm{i}\hbar\eta)^2/2! + \ldots\right) \\
&= \hat{p}_\mathrm{A}\mathrm{e}^{2\mathrm{i}\hbar\eta} = \hat{p}_\mathrm{A}\exp\left\{-2\log\sqrt{\frac{m_\mathrm{C}}{m_\mathrm{A}}}\right\} \\
&= \frac{m_\mathrm{A}}{m_\mathrm{C}}\hat{p}_\mathrm{A}.
\end{aligned} \tag{2.21}$$

[1] Note that velocities are given by the ratio between momentum and mass due to the quadratic form of the corresponding Hamiltonian in momenta, see [2].

The subsequent action of the parity-swap operator finally gives the required condition $\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}} \frac{\hat{p}_{\mathrm{A}}}{m_{\mathrm{A}}} \hat{\mathcal{P}}^{(v)\dagger}_{\mathrm{AC}} = -\frac{\hat{\pi}_{\mathrm{C}}}{m_{\mathrm{C}}}$. Notice that the scaling operator $\exp\left\{\eta\left(\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\right)\right\}$ has already been considered as a quantum canonical transformation in [8].

Finally, the QRF transformation between relative velocities $\hat{S}_v : \mathcal{H}^{|\mathrm{C}}_{\mathrm{A}} \otimes \mathcal{H}^{|\mathrm{C}}_{\mathrm{B}} \mapsto \mathcal{H}^{|\mathrm{A}}_{\mathrm{B}} \otimes \mathcal{H}^{|\mathrm{A}}_{\mathrm{C}}$ is given by

$$\hat{S}_v \equiv \hat{\mathcal{P}}^{(v)}_{\mathrm{AC}} \exp\left\{-\frac{\mathrm{i}}{\hbar}\frac{m_{\mathrm{B}}}{m_{\mathrm{A}}}\,\hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{B}}\right\} \tag{2.22}$$

where

$$\hat{x}_{\mathrm{A}} \mapsto -\left(m_{\mathrm{B}}\hat{q}_{\mathrm{B}} + m_{\mathrm{C}}\hat{q}_{\mathrm{C}}\right)/m_{\mathrm{A}}, \qquad \hat{x}_{\mathrm{B}} \mapsto \hat{q}_{\mathrm{B}}, \tag{2.23}$$

$$\hat{p}_{\mathrm{A}} \mapsto -\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\hat{\pi}_{\mathrm{C}}, \qquad \hat{p}_{\mathrm{B}} \mapsto \hat{\pi}_{\mathrm{B}} - \frac{m_{\mathrm{B}}}{m_{\mathrm{C}}}\hat{\pi}_{\mathrm{C}}. \tag{2.24}$$

The map $\hat{p}_{\mathrm{A}} \mapsto -\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\hat{\pi}_{\mathrm{C}}$ has already been proven here, since $\hat{p}_{\mathrm{A}}$ commutes with the operator $\exp\left\{-\frac{\mathrm{i}}{\hbar}\frac{m_{\mathrm{B}}}{m_{\mathrm{A}}}\,\hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{B}}\right\}$ which is responsible for shifting the momentum of B appropriately. The proofs of the other maps are given in appendix A.4.

2.2. Notion of the Rest Frame of a Quantum System

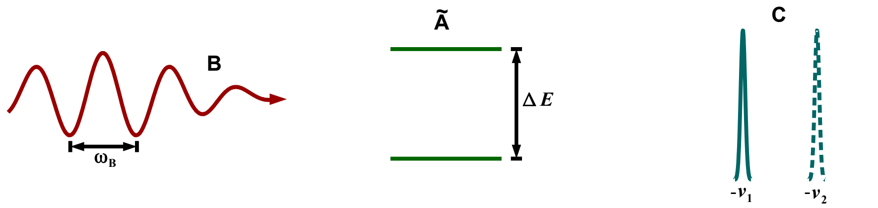
In the previous section, generalized transformations have been introduced which allow to "jump" to a reference frame associated to a quantum system - a quantum reference frame (QRF). Crucially, in section 2.1.2, it has been shown how to transform to a QRF of a quantum system, say A, moving in a superposition of velocities form the point of view of another frame, say the laboratory C; here, the QRF of A is referred to as the *rest frame of a quantum system*, since it is the frame where, in comparison to the laboratory frame, the quantum system is at rest. Consequently, the QRF formalism allows us to define operationally what we mean by the rest frame of a quantum system.

It is worth to emphasize that such transformations to the QRF of a system moving in a superposition of velocities cannot be achieved by using standard reference frame transformations. Only approximately, when the dynamics of the considered quantum system is semi-classical (e.g., assuming that the velocity is sharp), an ordinary coordinate transformation can take us to its rest frame. However, when a quantum system (e.g. a particle) moves in a superposition of velocities, QRF transformations are essential to transform to its rest frame.

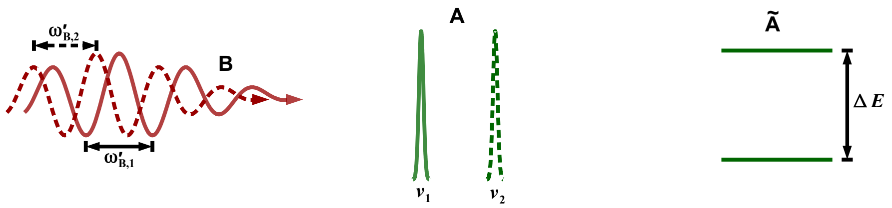
In general, the rest frame is of special interest when composite systems with external as well as internal degrees of freedom are considered. In the rest frame, the physical description of the internal degrees of freedom is the simplest, and the rest frame Hamiltonian alone is responsible for the dynamics of the internal degrees of freedom. We now provide an example of how QRF techniques can be applied to a practical situation in which a particle with internal degrees of freedom moves in a superposition of velocities.

Consider a situation in the laboratory (C) where an interaction between a photon (B) and an atom with external (A) and internal ($\tilde{\mathrm{A}}$) degrees of freedom is investigated. As a consequence of the interaction, the photon may be absorbed by the atom at the end of the process. Importantly, we do not restrict to the case where the atom has a sharp momentum/velocity state. Moreover, $\tilde{\mathrm{A}}$ refers to the internal energy degrees of freedom of the atom described by a two-level system with the energy gap ΔE between the ground and the excited state. The goal is to find the state of the photon and the atom, which has to be prepared in the laboratory, such that the probability of photon absorption is maximized.

This situation can be described easily in the rest frame of the atom, illustrated in figure 2.5(a), where the transition probability is maximal if the photon has the spectral frequency $\omega_{\mathrm{B}} = \Delta E/\hbar$. Since the atom and the laboratory move in a superposition of velocities



(a) **Rest frame of the atom A.** The photon B is in resonance with the internal energy levels of the atom (separated by ΔE) if it has the spectral frequency $\omega_{\mathrm{B}} = \Delta E/\hbar$.



(b) **Laboratory frame C.** Equivalently to the description in **(a)**, the photon is in resonance with the atom. In the laboratory frame, the atom's external degrees of freedom and the photon are entangled, because both are coherently boosted by the velocity v_1 and v_2 and, due to the Doppler effect, the frequency of the photon is transformed to $\omega'_{\mathrm{B},1}$ and $\omega'_{\mathrm{B},2}$ with $\omega'_{\mathrm{B},i} = \omega_{\mathrm{B}}(1 + \frac{v_i}{c})$. Entanglement is graphically illustrated by the correlation between solid and dashed lines of A and B.

Figure 2.5.: The absorption of a photon by an atom in the perspective of **(a)** the rest frame of the atom and **(b)** the laboratory frame. The photon absorption is naturally described in rest frame of the atom A; a QRF transformation leads to an equivalent description from the point of view of the laboratory frame C where the frequency of the photon is entangled with the external degrees of freedom of the atom due to the Doppler effect.

relative to one another, the state of the laboratory C is in a superposition of velocities from the point of view of the atom A. For simplicity, the state of C is considered to move in a superposition of two sharp velocities $-v_1$ and $-v_2$ with respect to A.

A QRF transformation (controlled by the velocity of the laboratory) leads to an equivalent description of this situation in the laboratory frame where the state of the photon

B and the atom's external degrees of freedom A are entangled as illustrated in 2.5(b). The entanglement as seen in the laboratory arises from the superposed relative motion between the photon source and the atom, and the corresponding coherent Doppler-shift of the photon frequency. In general, the Doppler effect occurs when the emitter and the receiver of a photon are moving relative to each other with the non-relativistic velocity v shifting the received frequency according to $\omega'_{\mathrm{B}} = \omega_{\mathrm{B}}(1 + \frac{v}{c})$ where ω_{B} denotes the emitted frequency and $v > 0$ (or $v < 0$) if the motion is towards (or away from) each other. With the mentioned QRF transformation the description of this situation as seen from the laboratory is obtained, where the photon source is at rest and the receiver (atom) of the photon is moving towards the source in a superposition of non-relativistic velocities. Corresponding to the velocity of the atom v_1 and v_2, the frequency received by the atom is coherently Doppler-shifted according to $\omega'_{\mathrm{B},1} = \omega_{\mathrm{B}}(1 + \frac{v_1}{c})$ and $\omega'_{\mathrm{B},2} = \omega_{\mathrm{B}}(1 + \frac{v_2}{c})$, respectively. Consequently, the state of the external degrees of freedom of the atom A and the photon B are entangled from the point of view of the laboratory.

Crucially, if the photon is absorbed by the atom in its rest frame, then it is absorbed in the laboratory frame as well. The frame-independence of the photon absorption is guaranteed by the unitarity of QRF transformations and the transformation of the observables; here, the observable for the photon frequency in A's perspective $\hat{\omega}_{\mathrm{B}}$ is transformed to the corresponding observable in C's perspective $\hat{\omega}_{\mathrm{B}}(1 + \frac{\hat{p}_{\mathrm{A}}}{cm_{\mathrm{A}}})$. Note that the explicit form of the mentioned QRF transformation and comprehensive calculations are given in [2].

The results of the present thesis (chapter 4), as well as section 3.4 and 3.5, rely on this possibility of assigning operational meaning to the internal degrees of freedom of a quantum system moving in a superposition of velocities by transforming to its rest frame. Specifically, relativistic particles with spin as internal degree of freedom are of particular interest here and the reason for that is given in the next chapter before introducing the corresponding relativistic QRF transformation.

3. Spin in Special Relativity

Quantum systems with spin are used as a resource of (quantum) information; specifically, a spin-1/2 system is the prime example of a quantum bit - a *qubit*. Consequently, it is of major interest to use the quantum state encoded in a qubit for communication purposes such as, for instance, quantum teleportation [35]. In this context, it is crucial to enter the domain of relativistic velocities since this increases the communication speed substantially. However, in special relativity, an operational definition of the spin is missing, i.e. there is no relativistic Stern-Gerlach experiment, and this prevents us from using the spin degree of freedom as a resource of quantum information.

This chapter is organized as follows. In section 3.1, the problem of identifying a covariant spin operator is explained and subsequently, in section 3.2, the requirements for a proper relativistic spin operator are given. In section 3.3, the one-particle states of the Dirac equation and Lorentz boosts are introduced. In section 3.4, a proposal for an operational definition of relativistic spin, introduced in [12], is reviewed. This proposal employs a relativistic QRF transformation which is capable to transform between the rest frame of a spin particle and the laboratory reference frame, which move in a superposition of relativistic velocities relative to each other. Finally, with this non-classical transformation a relativistic spin operator is defined operationally in section 3.5.

3.1. The Problem of Defining Spin

In order to utilize spin as information carrier in the relativistic regime an operational definition of spin is needed which gives the operations for preparing and measuring the spin state of a relativistic particle. However, such an operational definition of spin is lacking.

A key issue is the ambiguity of splitting the conserved total angular momentum operator $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ into an external $\hat{\mathbf{L}}$ and an internal $\hat{\mathbf{S}}$ part referred to as orbital angular momentum and spin, respectively. This is related to the definition of a relativistic position operator $\hat{\mathbf{x}}$ because $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$ with the momentum operator $\hat{\mathbf{p}} = -\mathrm{i}\hbar\boldsymbol{\nabla}$; thus, different definitions of $\hat{\mathbf{S}}$ induce different relativistic position operators $\hat{\mathbf{x}}$, and vice versa. [11, 39]

Moreover, in the seminal work [10], the authors show that the spin state of a relativistic spin-1/2 particle (obtained by tracing out the momentum degree of freedom of the total momentum-spin state) moving in a superposition of relativistic velocities is not covariant under Lorentz transformations; whereas, the spin state transforms in a covariant way under standard Lorentz boosts for states with sharp momentum. This has been shown by comparing the entropy of the reduced spin density matrix of a general one-particle state before and after a Lorentz boost. Since entropy can be used to quantify entanglement, it has been concluded that an initially pure reduced spin state gets mixed after applying a Lorentz boost which means that a Lorentz boost entangles the spin with the momentum degree of freedom; however, the reduced spin density matrix stays pure for states with sharp momentum.

Additionally, by questioning the preparation and measurement of relativistic spin states via a Stern-Gerlach apparatus it is shown, by Lorentz boosting the inhomogeneous magnetic field of the apparatus, that the measurable expectation value depends on the momentum of the particle [40].

A possible solution to this problem would be to choose the description of the spin (internal) degree of freedom from the point of view of the rest frame, where the spin can be unambiguously defined, and then transform this description to the laboratory frame. However, a standard reference frame transformation can only capture situations in which the particle moves with a sharp velocity from the point of view of the laboratory. Thus, for general quantum states (in a superposition of velocities), the transformation to the rest frame has to be generalized. As discussed in section 2.2, the QRF formalism can be exploited to transform to the rest frame of a general quantum system. Before introducing the appropriate QRF transformation for relativistic spin states, in the next section, the requirements for a good spin operator are discussed.

3.2. Characterization of a Proper Spin Operator

Essential features of a proper relativistic spin operator are discussed and compared to existing proposals in [11]. The motivation of the authors is the lack of a commonly accepted spin operator in a relativistic setting and, in addition, the focus is set on the ability of defining such an operator by experimental methods.

Following [11], the requirements of a relativistic spin operator for massive spin-1/2 particles are given in the following. The first requirement is that the spin commutes with the free Dirac Hamiltonian (or with the Dirac Hamiltonian for central potentials) ensuring that spin is a conserved quantity as in non-relativistic quantum mechanics. In mathematical terms, a relativistic spin operator $\hat{\mathbf{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ has to satisfy

$$[\hat{H}_0, \hat{\mathbf{S}}] = 0 \tag{3.1}$$

where $\hat{H}_0$ denotes the free Dirac Hamiltonian. Moreover, it has to obey the $\mathfrak{su}(2)$ algebra and give the same eigenvalues as in the non-relativistic limit, i.e.

$$[\hat{S}_i, \hat{S}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{S}_k \quad \text{with eigenvalues} = \pm\hbar/2 \tag{3.2}$$

where ε_{ijk} denotes the Levi-Civita symbol. It is worth mentioning that eq. (3.2) is considered to be *the* fundamental property of angular momentum operators of spin-1/2 particles [41]. Finally, a proper relativistic spin operator has to have the correct non-relativistic limit.

In [11], the properties of seven relativistic spin operators (Pauli, Foldy-Wouthuysen, Czachor, Frenkel, Chakrabarti, Pryce and Fradkin-Good)[1], which are often motivated by abstract group theoretical considerations, are investigated. All of them converge to the same (correct) expectation values in the non-relativistic limit. However, only two, the Foldy-Wouthuysen [42] and the Pryce [43], of the seven investigated spin operators qualify as proper relativistic spin operator since only these two satisfy eq. (3.1) and (3.2) at once.

3.3. One-particle States

Here and in the following, the relativistic momentum-spin states are taken to be the positive-energy solutions of the Dirac equation in the Foldy-Wouthuysen representation [42]. In addition, the full quantum field-theory state is projected onto the one-particle sector. A comprehensive treatment of the one-particle sector can be found, for instance, in [44]. Useful relations and the used notation are introduced in the following.

A relativistic spin state for a Dirac particle with mass $m > 0$ is defined by

$$|\psi\rangle = \sum_{\sigma} \int \mathrm{d}\mu(\mathbf{p})\psi_\sigma(\mathbf{p})\, |\mathbf{p}; \Sigma_{\mathbf{p}}(\sigma)\rangle \tag{3.3}$$

where σ denotes the spin in the rest frame of the Dirac particle, $\mathbf{p}$ the momentum, $\mathrm{d}\mu(\mathbf{p})$ the Lorentz-invariant integration measure and $\psi_\sigma(\mathbf{p}) = \langle \mathbf{p}; \Sigma_{\mathbf{p}}(\sigma)|\psi\rangle$ the wave function. For clarity, the momentum dependent spin label $\Sigma_{\mathbf{p}}(\sigma)$ explicitly denotes the spin-momentum entanglement as shown in [10] and discussed in section 3.1. The basis elements are defined via standard Lorentz boosts according to

$$|\mathbf{p}; \Sigma_{\mathbf{p}}(\sigma)\rangle \equiv \hat{U}(L_{\mathbf{p}})\, |\mathbf{0}; \sigma\rangle \tag{3.4}$$

where $\hat{U}(L_{\mathbf{p}})$ is a unitary representation of a pure boost $L_{\mathbf{p}}$ taking the four-momentum of the Dirac particle from $k^\mu = (mc, \mathbf{0})$ to $p^\mu = (L_{\mathbf{p}})^\mu{}_\nu k^\nu = (p^0, \mathbf{p})$ with $p^0 = p^0(|\mathbf{p}|) =$

[1] References of these spin operators are given exhaustively in [11].

$\sqrt{m^2c^2 + \mathbf{p}^2}$ [2] and the speed of light c. It is important to notice that in the rest frame, $|\mathbf{0}; \Sigma_{\mathbf{0}}(\sigma)\rangle \equiv |\mathbf{0}; \sigma\rangle$, there is no entanglement between momentum and spin, i.e. the one-particle state is separable: $|\mathbf{0}; \sigma\rangle = |\mathbf{0}\rangle\, |\sigma\rangle$. The Lorentz-invariant integration measure for positive energy solutions, i.e. $E = p^0/c > 0$, is given by [44]

$$\mathrm{d}\mu(\mathbf{p}) = \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2p^0} = \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2\sqrt{m^2c^2 + \mathbf{p}^2}}. \tag{3.5}$$

Moreover, the orthogonality relation

$$\langle \mathbf{p}'; \Sigma_{\mathbf{p}'}(\sigma') | \mathbf{p}; \Sigma_{\mathbf{p}}(\sigma)\rangle = (2\pi)^3 (2p^0) \delta_{\sigma\sigma'} \delta^3(\mathbf{p} - \mathbf{p}') \tag{3.6}$$

leads to the scalar product

$$\langle \varphi | \psi \rangle = \sum_{\sigma} \int \mathrm{d}\mu(\mathbf{p}) \varphi^*_{\sigma}(\mathbf{p}) \psi_{\sigma}(\mathbf{p}) \tag{3.7}$$

where $*$ denotes the complex conjugate.

Notice that in the literature, e.g. [45, 46], the spin-momentum entanglement is implicit; however, such a notation might be misleading since

$$c_1\, |\mathbf{p}_1; \sigma\rangle + c_2\, |\mathbf{p}_2; \sigma\rangle \neq (c_1\, |\mathbf{p}_1\rangle + c_2\, |\mathbf{p}_2\rangle)\, |\sigma\rangle \tag{3.8}$$

because the spin state depends on the momentum and σ refers to the spin in the rest frame only. Therefore, we denote basis elements by $|\mathbf{p}; \Sigma_{\mathbf{p}}(\sigma)\rangle \equiv |\mathbf{p}; \Sigma(\sigma)\rangle$[3].

The explicit matrix for pure Lorentz boosts, transforming to a reference frame which is moving with velocity $\mathbf{v}$ relative to the initial frame, is given by

$$L_{\mathbf{v}} = \begin{pmatrix} \gamma & \gamma \frac{\mathbf{v}^\top}{c} \\ \gamma \frac{\mathbf{v}}{c} & \mathbb{1} + \frac{\gamma^2}{\gamma+1} \frac{\mathbf{v}\mathbf{v}^\top}{c^2} \end{pmatrix} {}^{4} \tag{3.9}$$

where $\gamma \equiv (1 - \frac{\mathbf{v}^2}{c^2})^{-1/2}$ [47]. This matrix can be written completely in terms of the momentum $\mathbf{p}$, or more accurately the ratio $\mathbf{p}/m$, by substituting $p^\mu \equiv (p^0, \mathbf{p}) = m\gamma(c, \mathbf{v})$ in eq. (3.9)

$$L_{\mathbf{p}} \equiv L_{\frac{\mathbf{p}}{m}} = \begin{pmatrix} \frac{p^0}{mc} & \frac{\mathbf{p}^\top}{mc} \\ \frac{\mathbf{p}}{mc} & \mathbb{1} + \frac{1}{\gamma+1} \frac{\mathbf{p}\mathbf{p}^\top}{(mc)^2} \end{pmatrix} \tag{3.10}$$

with $\gamma = \gamma_{\frac{\mathbf{p}}{m}} = \sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}}$ [5] and $p^0 = mc\gamma = \sqrt{m^2c^2 + \mathbf{p}^2} = p^0(|\mathbf{p}|)$.

---

[2] Notice that the metric utilized here is $\eta = \mathrm{diag}(1, -1, -1, -1)$ leading to $p^0 = p_0$.

[3] In later calculations, the index $\mathbf{p}$ of Σ is neglected for simplicity; however, Σ still indicates the momentum dependence, where its index unambiguously reflects the same momentum as in the shared ket.

[4] Notice that here the transpose is explicitly denoted for compactness and correctness; however, to avoid cumbersome reading the transpose is not denoted explicitly elsewhere.

[5] $\frac{1}{\gamma^2} = 1 - \frac{\mathbf{v}^2}{c^2} = 1 - \frac{\mathbf{p}^2}{\gamma^2 m^2 c^2} \Leftrightarrow \gamma^2 = 1 + \frac{\mathbf{p}^2}{m^2c^2}$

3.4. Superposition of Lorentz Boosts

As discussed in section 2.2, the QRF formalism can be utilized to define the rest frame of quantum systems via corresponding QRF transformations. Following [12], an extension to special relativity of the formalism introduced in [2] is given now by introducing a transformation corresponding to a "superposition of Lorentz boosts".

In particular, we start in the rest frame of a spin-1/2 particle $\mathbf{A}$, where spin can be defined operationally[6], and then transform to the laboratory C. Consequently, the laboratory C describes the Dirac particle $\mathbf{A} \equiv \mathrm{A}\tilde{\mathrm{A}}$ with its external (momentum) and internal (spin) degree of freedom which are labeled by A and $\tilde{\mathrm{A}}$, respectively. However, in the rest frame of the Dirac particle, i.e. in A's perspective, only the spin $\tilde{\mathrm{A}}$ and the laboratory C are described and a general quantum state is given by

$$|\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathrm{C}} = |\boldsymbol{\sigma}\rangle_{\tilde{\mathrm{A}}} \, |\phi\rangle_{\mathrm{C}} \,, \tag{3.11}$$

where it is assumed that the states of $\tilde{\mathrm{A}}$ and C are separable. In the following, the spin state is written in the eigenbasis $\{|+z\rangle , |-z\rangle\}$ of the Pauli operator $\hat{\sigma}^z = |+z\rangle \langle +z| - |-z\rangle \langle -z| = \mathrm{diag}(1, -1)$[7]

$$|\boldsymbol{\sigma}\rangle_{\tilde{\mathrm{A}}} = \sum_{\lambda=\pm z} c_\lambda \, |\lambda\rangle_{\tilde{\mathrm{A}}} \quad \text{with} \quad \sum_{\lambda=\pm z} |c_\lambda|^2 = 1. \tag{3.12}$$

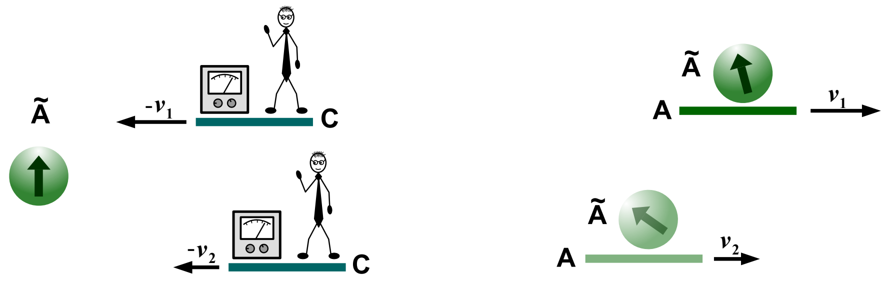
The state $|\phi\rangle_{\mathrm{C}}$ is expressed in the momentum basis and kept general; in particular, C is allowed to move in a superposition of relativistic velocities. For simplicity, the relative motion between A and C is restricted to be relativistic only in one dimension. Thus,

$$|\phi\rangle_{\mathrm{C}} = \int \mathrm{d}\mu_{\mathrm{C}}(\pi) \phi(\pi) \, |\pi\rangle_{\mathrm{C}} \tag{3.13}$$

with the (1+1)-momentum $\pi^\mu = (\pi^0(\pi), \pi)$ where $\pi^0(\pi) = \sqrt{m_{\mathrm{C}}^2 c^2 + \pi^2}$ and $\mathrm{d}\mu_{\mathrm{C}}(\pi)$ is the (1+1)-dimensional version of eq. (3.5) with the mass m_{C} indicated by the index C of the integration measure $\mathrm{d}\mu$. Notice that, in general, the situation discussed here has to be considered in three spatial dimensions due to the spin of the Dirac particle. The extension to three dimension is done by enlarging the Hilbert space of C according to $|\phi\rangle_{\mathrm{C}} \to |\phi\rangle_{\mathrm{C}} \, |\phi^0_y\rangle_{\mathrm{C}} \, |\phi^0_z\rangle_{\mathrm{C}}$ with the non-relativistic states $|\phi^0_i\rangle$ where $i \in \{y, z\}$. However, these extra non-relativistic states can be ignored for what concerns the relativistic treatment of the Dirac particle. The state $|\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathrm{C}}$ as seen by the rest frame of the Dirac particle is shown in figure 3.1(a) where, for illustration purposes, the laboratory moves in a superposition of two sharp relativistic velocities only.

[6] The spin state of a quantum system can, in principle, be tomographically verified by a series of standard Stern-Gerlach measurements in the rest frame of a spin particle (or more precisely for slow velocities).

[7] Equivalently, one could chose (normalized) eigenstates of $\hat{\sigma}^x$ or $\hat{\sigma}^y$.



(a) Rest frame A of the Dirac particle $\mathbf{A}$. (b) Laboratory frame C.

Figure 3.1.: (a) In the rest frame of the Dirac particle, the total state of the spin $\tilde{\mathrm{A}}$ and of the laboratory C is separable. The laboratory has a state which is in a superposition of two sharp relativistic velocities $-v_1$ and $-v_2$. In this QRF, spin can be defined operationally with a Stern-Gerlach experiment. **(b)** In C's perspective, the spin $(\tilde{\mathrm{A}})$ is entangled with the momentum (A) degree of freedom of the Dirac particle (pictorially, this is represented by the correlation between the transparency of the drawn symbols).

The QRF transformation between the momentum degree of freedom and the laboratory $\hat{S}_{\mathrm{L}} : \mathcal{H}^{|\mathrm{A}}_{\tilde{\mathrm{A}}} \otimes \mathcal{H}^{|\mathrm{A}}_{\mathrm{C}} \mapsto \mathcal{H}^{|\mathrm{C}}_{\mathrm{A}} \otimes \mathcal{H}^{|\mathrm{C}}_{\tilde{\mathrm{A}}}$ maps $|\psi\rangle^{|\mathrm{C}}_{\mathrm{A}\tilde{\mathrm{A}}} = \hat{S}_{\mathrm{L}} |\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathrm{C}}$ where

$$\hat{S}_{\mathrm{L}} \equiv \hat{\mathcal{P}}^{(v)}_{\mathrm{CA}} \hat{U}_{\tilde{\mathrm{A}}}(\hat{\pi}_{\mathrm{C}}) \tag{3.14}$$

with the generalized parity-swap operator $\hat{\mathcal{P}}^{(v)}_{\mathrm{CA}} : \mathcal{H}^{|\mathrm{A}}_{\tilde{\mathrm{A}}} \otimes \mathcal{H}^{|\mathrm{A}}_{\mathrm{C}} \mapsto \mathcal{H}^{|\mathrm{C}}_{\mathrm{A}} \otimes \mathcal{H}^{|\mathrm{C}}_{\tilde{\mathrm{A}}}$ given by $\hat{\mathcal{P}}^{(v)}_{\mathrm{CA}} = \hat{\mathcal{P}}_{\mathrm{CA}} \exp\left\{\frac{\mathrm{i}}{\hbar} \log \sqrt{\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}} \left(\hat{q}_{\mathrm{C}}\hat{\pi}_{\mathrm{C}} + \hat{\pi}_{\mathrm{C}}\hat{q}_{\mathrm{C}}\right)\right\}$[8] where $\hat{\mathcal{P}}^{(v)}_{\mathrm{CA}}\ \hat{\pi}_{\mathrm{C}} \hat{\mathcal{P}}^{(v)\dagger}_{\mathrm{CA}} = -\frac{m_{\mathrm{C}}}{m_{\mathrm{A}}} \hat{p}_{\mathrm{A}}$ (see section 2.1.2) and $\hat{U}_{\tilde{\mathrm{A}}}(\hat{\pi}_{\mathrm{C}})$ is a unitary operator depending on the momentum of C and acting on the spin $\tilde{\mathrm{A}}$. The QRF transformation $\hat{S}_{\mathrm{L}}$ can be defined via its action on a basis element of a complete basis via $\hat{S}_{\mathrm{L}} |\lambda\rangle_{\tilde{\mathrm{A}}} |\pi\rangle_{\mathrm{C}} = |-\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi; \Sigma(\lambda)\rangle_{\mathrm{A}\tilde{\mathrm{A}}}$ where $|p; \Sigma(\lambda)\rangle_{\mathbf{A}} \equiv |p; \Sigma_p(\lambda)\rangle_{\mathbf{A}} \equiv \hat{U}_{\mathbf{A}}(L_{p/m_{\mathrm{A}}}) |0; \lambda\rangle_{\mathbf{A}}$ with $\mathbf{A} \equiv \mathrm{A}\tilde{\mathrm{A}}$. In comparison to the previous section, $L_{p/m_{\mathrm{A}}}$ is the (1+1)-dimensional version of eq. (3.10) and $\hat{U}_{\mathbf{A}}(L_{p/m_{\mathrm{A}}})$ denotes the unitary representation of a pure Lorentz boost taking the (1+1)-momentum from $k^{\mu}_{\mathrm{A}} = (m_{\mathrm{A}}c, 0)$ to $p^{\mu}_{\mathrm{A}} = (L_{p/m_{\mathrm{A}}})^{\mu}{}_{\nu} k^{\nu}_{\mathrm{A}} = (p^0_{\mathrm{A}}, p)$ where $p^0_{\mathrm{A}} = p^0_{\mathrm{A}}(|p|) = \sqrt{m^2_{\mathrm{A}}c^2 + p^2}$.

The action of the quantum reference transformation $\hat{S}_{\mathrm{L}}$ on the basis elements can be derived when all appearing systems are considered jointly from an external perspective. In line with the situation above, we start with a Dirac particle $\mathbf{A}$ at rest and with the laboratory C moving with momentum π; thus, a general basis element is given by $|0; \lambda\rangle_{\mathbf{A}} |\pi\rangle_{\mathrm{C}}$. Because $\mathbf{A}$ has zero momentum there is no entanglement between its momentum and spin; hence, the state of $\mathbf{A} \equiv \mathrm{A}\tilde{\mathrm{A}}$ is separable, i.e. $|0; \lambda\rangle_{\mathbf{A}} = |0\rangle_{\mathrm{A}} |\lambda\rangle_{\tilde{\mathrm{A}}}$. It is important to see that, in comparison to the treatment above, only the redundant state $|0\rangle_{\mathrm{A}}$ has been added. To transform from the rest frame A to the laboratory frame C, i.e. to the state in

[8] In order to keep the label C referring to the laboratory, this is the reverse transformation of the generalized parity-swap operator introduced in section 2.1.2, i.e. $\hat{\mathcal{P}}^{(v)}_{\mathrm{CA}} = \hat{\mathcal{P}}^{(v)\dagger}_{\mathrm{AC}}$ and $\hat{\mathcal{P}}_{\mathrm{CA}} = \hat{\mathcal{P}}^{\dagger}_{\mathrm{AC}}$.

which the momentum of the laboratory is zero $|0\rangle_{\mathrm{C}}$, we first have to Lorentz boost $\mathbf{A}$ by a velocity controlled by the momentum of the laboratory C, via

$$\hat{U}_{\mathbf{A}}(L_{-\hat{\pi}_{\mathrm{C}}/m_{\mathrm{C}}})\,|0;\lambda\rangle_{\mathbf{A}}\,|\pi\rangle_{\mathrm{C}} = \hat{U}_{\mathbf{A}}(L_{-\pi/m_{\mathrm{C}}})\,|0;\lambda\rangle_{\mathbf{A}}\,|\pi\rangle_{\mathrm{C}} = \left|-\tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi, \Sigma(\lambda)\right\rangle_{\mathbf{A}}|\pi\rangle_{\mathrm{C}} \tag{3.15}$$

where $\mathbf{A}$ has been boosted to $p_{\mathrm{A}} = (p^0_{\mathrm{A}}(\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi), -\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi)$ and the spin now depends on the momentum of the corresponding state as discussed in section 3.3. To complete the change of perspective, the laboratory state has to be boosted to its rest state. This can be achieved by another Lorentz boost depending on the momentum of $\mathbf{A}$, i.e. a coherent Lorentz boost, given by $\hat{U}^{\dagger}_{\mathrm{C}}(L_{-\hat{p}_{\mathrm{A}}/m_{\mathrm{A}}}) = \hat{U}_{\mathrm{C}}(L_{\hat{p}_{\mathrm{A}}/m_{\mathrm{A}}})$; consequently,

$$\begin{aligned}\hat{U}_{\mathrm{C}}(L_{\hat{p}_{\mathrm{A}}/m_{\mathrm{A}}})\hat{U}_{\mathbf{A}}(L_{-\hat{\pi}_{\mathrm{C}}/m_{\mathrm{C}}})\,|0;\lambda\rangle_{\mathbf{A}}\,|\pi\rangle_{\mathrm{C}} &= \left|-\tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi, \Sigma(\lambda)\right\rangle_{\mathbf{A}}\hat{U}_{\mathrm{C}}(L_{-\pi/m_{\mathrm{C}}})\,|\pi\rangle_{\mathrm{C}} \\ &= \left|-\tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi, \Sigma(\lambda)\right\rangle_{\mathbf{A}}|0\rangle_{\mathrm{C}}\,.\end{aligned} \tag{3.16}$$

Since the two states of zero momentum $|0\rangle_{\mathrm{A}}$ and $|0\rangle_{\mathrm{C}}$ in the QRF of C and A, respectively, do not play any role, we can discard them. Hence, the action of the QRF transformation $\hat{S}_{\mathrm{L}}$ on the basis elements

$$\hat{S}_{\mathrm{L}}\,|\lambda\rangle_{\tilde{\mathrm{A}}}\,|\pi\rangle_{\mathrm{C}} = \left|-\tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi, \Sigma(\lambda)\right\rangle_{\mathbf{A}} \tag{3.17}$$

is obtained.

Consequently, this QRF transformation acts on the total state in the following way

$$\begin{aligned}|\psi\rangle^{|\mathrm{C}}_{\mathrm{A}\tilde{\mathrm{A}}} &= \hat{S}_{\mathrm{L}}\,|\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathrm{C}} \\ &= \sum_{\lambda} c_{\lambda}\int \mathrm{d}\mu_{\mathrm{C}}(\pi)\phi(\pi)\hat{S}_{\mathrm{L}}\,|\lambda\rangle_{\tilde{\mathrm{A}}}\,|\pi\rangle_{\mathrm{C}} \\ &= \sum_{\lambda} c_{\lambda}\int \mathrm{d}\mu_{\mathrm{A}}(p)\;\phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}p\right)|p;\Sigma(\lambda)\rangle\end{aligned} \tag{3.18}$$

where $\mathrm{d}\mu_{\mathrm{X}}(k) = \frac{\mathrm{d}k}{4\pi\sqrt{m_{\mathrm{X}}^2c^2+k^2}}$ and in the last step the substitution $p = -\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi$ has been used. In figure 3.1, the QRF transformation from $|\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathrm{C}}$ to $|\psi\rangle^{|\mathrm{C}}_{\mathrm{A}\tilde{\mathrm{A}}}$ is illustrated where, for simplicity, only two sharp relativistic velocities in superposition are considered.

3.5. An Operational Definition of Spin

With the help of the previously introduced relativistic QRF transformation $\hat{S}_{\mathrm{L}}$, it is now possible to define a relativistic spin operator revealing the spin as seen in the rest frame of the Dirac particle according to

$$\hat{\Xi}^i_{\hat{p}_{\mathrm{A}}} \equiv \hat{S}_{\mathrm{L}}\left(\hat{\sigma}^i_{\tilde{\mathrm{A}}}\otimes\mathbb{1}_{\mathrm{C}}\right)\hat{S}^{\dagger}_{\mathrm{L}} \tag{3.19}$$

which is simply called *relativistic spin operator* in the following and can be written in terms of the manifestly covariant Pauli-Lubański operator $\hat{\Sigma}^{\mu}_{\hat{p}_\mathrm{A}} = (\hat{\Sigma}^0_{\hat{p}_\mathrm{A}}, \hat{\boldsymbol{\Sigma}}_{\hat{p}_\mathrm{A}})$, see appendix A.5.

By definition, this relativistic spin operator obeys the $\mathfrak{su}(2)$ algebra $\left[\hat{\Xi}^i_{\hat{p}_\mathrm{A}}, \hat{\Xi}^j_{\hat{p}_\mathrm{A}}\right] = \mathrm{i}\epsilon_{ijk}\hat{\Xi}^k_{\hat{p}_\mathrm{A}}$. Moreover, $\hat{\Xi}^i_{\hat{p}_\mathrm{A}}$ has the same eigenvalues as the corresponding Pauli operator $\hat{\sigma}^i_{\tilde{\mathrm{A}}}$. This can be shown by considering the state $|\psi^{\pm}\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathrm{C}} = |\pm i\rangle_{\tilde{\mathrm{A}}} |\phi\rangle_{\mathrm{C}}$ in the rest frame of $\mathbf{A}$, where $|\pm i\rangle$ denotes the two eigenstates of the corresponding Pauli operator $\hat{\sigma}^i$, i.e. $\hat{\sigma}^i |\pm i\rangle = \pm |\pm i\rangle$ with $i \in \{x, y, z\}$. Consequently,

$$\begin{aligned}
\hat{\Xi}^i_{\hat{p}_\mathrm{A}} |\psi^{\pm}\rangle^{|\mathrm{C}}_{\mathrm{A}\tilde{\mathrm{A}}} &= \hat{S}_\mathrm{L} \left(\hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \mathbb{1}_\mathrm{C}\right) \hat{S}^{\dagger}_\mathrm{L} \hat{S}_\mathrm{L} |\psi^{\pm}\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathrm{C}} \\
&= \hat{S}_\mathrm{L} \left(\hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \mathbb{1}_\mathrm{C}\right) |\pm i\rangle_{\tilde{\mathrm{A}}} |\phi\rangle_\mathrm{C} \\
&= \pm |\psi^{\pm}\rangle^{|\mathrm{C}}_{\mathrm{A}\tilde{\mathrm{A}}} \quad \text{with} \quad |\psi^{\pm}\rangle^{|\mathrm{C}}_{\mathrm{A}\tilde{\mathrm{A}}} = \int \mathrm{d}\mu_\mathrm{A}(p)\, \phi\left(-\tfrac{m_\mathrm{C}}{m_\mathrm{A}} p\right) |p; \Sigma(\pm i)\rangle \,.
\end{aligned} \tag{3.20}$$

With these operators, it is possible to partition the Hilbert space $\mathcal{H}^{|\mathrm{C}}_\mathrm{A} \otimes \mathcal{H}^{|\mathrm{C}}_{\tilde{\mathrm{A}}}$ into two equivalence classes by choosing one of the Pauli operators with the corresponding eigenvectors (here: $\hat{\sigma}^z$ with $|\pm z\rangle$) according to

$$\begin{aligned}
\mathcal{H}_{+z} &= \left\{ |\psi\rangle_{\mathrm{A}\tilde{\mathrm{A}}} \in \mathcal{H}^{|\mathrm{C}}_\mathrm{A} \otimes \mathcal{H}^{|\mathrm{C}}_{\tilde{\mathrm{A}}} : |\psi\rangle_{\mathrm{A}\tilde{\mathrm{A}}} \sim \hat{S}_\mathrm{L} |+z\rangle_{\tilde{\mathrm{A}}} |\phi\rangle_\mathrm{C} \,, \forall\, |\phi\rangle_\mathrm{C} \in \mathcal{H}^{|\mathrm{A}}_\mathrm{C} \right\}, \\
\mathcal{H}_{-z} &= \left\{ |\psi\rangle_{\mathrm{A}\tilde{\mathrm{A}}} \in \mathcal{H}^{|\mathrm{C}}_\mathrm{A} \otimes \mathcal{H}^{|\mathrm{C}}_{\tilde{\mathrm{A}}} : |\psi\rangle_{\mathrm{A}\tilde{\mathrm{A}}} \sim \hat{S}_\mathrm{L} |-z\rangle_{\tilde{\mathrm{A}}} |\xi\rangle_\mathrm{C} \,, \forall\, |\xi\rangle_\mathrm{C} \in \mathcal{H}^{|\mathrm{A}}_\mathrm{C} \right\}.
\end{aligned} \tag{3.21}$$

This means that, if two states are eigenvectors of $\hat{\Xi}^z_{\hat{p}_\mathrm{A}}$ with the same eigenvalue, then they are considered to be equivalent and the Hilbert space $\mathcal{H}^{|\mathrm{C}}_\mathrm{A} \otimes \mathcal{H}^{|\mathrm{C}}_{\tilde{\mathrm{A}}}$ can be divided in two highly degenerate subspaces where the states in $\mathcal{H}_{+z}$ correspond to "spin up" and the states in $\mathcal{H}_{-z}$ to the "spin down" eigenvalue. [12]

In addition, the relativistic spin operator $\hat{\boldsymbol{\Xi}}_{\hat{p}_\mathrm{A}}$ commutes with the free Dirac Hamiltonian [12]; finally, it has the correct non-relativistic limit since $\hat{S}_\mathrm{L} \to \hat{\mathcal{P}}^{(v)}_{\mathrm{CA}}$, i.e. $\hat{U}_{\tilde{\mathrm{A}}}(\hat{\pi}_\mathrm{C}) \to \mathbb{1}$, if the velocity of C tends to zero. Consequently, this relativistic spin operator satisfies all the criteria for a proper relativistic spin operator discussed in section 3.2.[9]

Moreover, it can be shown that the relativistic spin operator $\hat{\boldsymbol{\Xi}}_{\hat{p}_\mathrm{A}}$ is equivalent to the Foldy-Wouthuysen [42] and the Pryce [43] spin operator. [12]

[9] Notice that the actual non-relativistic spin operator, $\hbar\hat{\boldsymbol{\sigma}}/2$, differs from the Pauli operator $\hat{\boldsymbol{\sigma}}$ only by a factor of $\hbar/2$ and, similarly, the relativistic spin operator $\hat{\boldsymbol{\Xi}}_{\hat{p}_\mathrm{A}}$ introduced here is defined without the factor of $\hbar/2$.

4. Relativistic Bell Test within Quantum Reference Frames

This chapter contains the original results of the present thesis. It is shown how relativistic QRFs can be used in order to set up a relativistic Bell test, i.e. a Bell test with relativistic massive particles. In section 4.1, a short introduction to Bell's theorem is given. In section 4.2, the relativistic QRF transformation $\hat{S}_{\mathrm{L}}$ (introduced in section 3.4) is extended to the two particle case, where another spin-1/2 Dirac particle $\mathbf{B}$ is considered on top of the Dirac particle $\mathbf{A}$ in chapter 3. With such extended QRF transformation $\hat{S}_2$, it is possible to find the observables for a joint spin measurement on the two spin particles in different QRFs. The relativistic Bell test is analyzed in different situations. In section 4.3, the quantum state of the two Dirac particles is taken to be perfectly correlated in velocity, so that the transformation to the rest frame of one particle automatically gives the rest frame of the other particle (shared rest frame). This enables us to define operationally the spin observables for a Bell test and, by applying the QRF transformation $\hat{S}_2$, the corresponding observables in the laboratory frame are obtained. This treatment is generalized by considering a general state for $\mathbf{B}$ in the rest frame of $\mathbf{A}$ in section 4.4 and, on top of that, allowing for non-collinear relative motion between the considered systems in section 4.5. Finally, in section 4.6, it is concluded that the relativistic QRF transformation $\hat{S}_2$ preserves spacetime intervals and thus space-like separation which is relevant for the Bell experiment.

4.1. Bell's Theorem

In 1964, John S. Bell showed that no *local realistic* theory of nature (such as classical mechanics) is able to reproduce all predictions of quantum theory. This worldview changing result follows from the violation of a device-independent[1] inequality which is satisfied by any local realistic theory of nature. [15]

[1] In this context, *device-independent* means that measurement apparatuses are described by "black boxes" with inputs (measurement settings) and outputs (measurement outcomes), where no internal structure of the measurement apparatuses, properties of the measured systems, Hamiltonians, etc. are assumed.

In this context, an event is considered to be *local* if it is not affected by any action taking place in a space-like separated region. On the other hand, physical systems are considered to be *real* if they posses properties prior to and independent of any observation, i.e. the outcome of a measurement is predetermined by the preexisting properties of the system.

In the following, the Clauser-Horne-Shimony-Holt version of Bell's inequality [16], referred to as *CHSH-Bell inequality*, is considered. In particular, the CHSH-Bell inequality sets a bound on the correlations of joint spin[2] measurements on two space-like separated particles, say $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$, with the outcomes ± 1 for each spin measurement according to

$$|S| = |E(\mathbf{x}_1, \mathbf{y}_1) + E(\mathbf{x}_1, \mathbf{y}_2) + E(\mathbf{x}_2, \mathbf{y}_1) - E(\mathbf{x}_2, \mathbf{y}_2)| \leq 2 \tag{4.1}$$

where $E(\mathbf{x}_i, \mathbf{y}_j)$ denotes the expectation value of the joint measurement on $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$, and the measurement settings $\mathbf{x}_1$ and $\mathbf{x}_2$ ($\mathbf{y}_1$ and $\mathbf{y}_2$) refer to the spin measurement on $\tilde{\mathrm{A}}$ ($\tilde{\mathrm{B}}$). In (non-relativistic) quantum mechanics, a joint spin measurement is described by the observable $\mathbf{x}\cdot\hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{A}}}\otimes\mathbf{y}\cdot\hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{B}}}$, where $\hat{\boldsymbol{\sigma}} \equiv (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z) \equiv (\hat{\sigma}^1, \hat{\sigma}^2, \hat{\sigma}^3)$ denotes the Pauli operator and $\mathbf{x} = (x^1, x^2, x^3)$ as well as $\mathbf{y} = (y^1, y^2, y^3)$ with $|\mathbf{x}| = |\mathbf{y}| = 1$ are Bloch vectors referring to the measurement settings. For the singlet state $|\Psi^-\rangle_{\tilde{\mathrm{A}}\tilde{\mathrm{B}}} = (|{+z}\rangle_{\tilde{\mathrm{A}}}\,|{-z}\rangle_{\tilde{\mathrm{B}}} - |{-z}\rangle_{\tilde{\mathrm{A}}}\,|{+z}\rangle_{\tilde{\mathrm{B}}})/\sqrt{2}$ the expectation value for a joint spin measurement is given by

$$E(\mathbf{x}, \mathbf{y}) = \langle\Psi^-|\mathbf{x}\cdot\hat{\boldsymbol{\sigma}}\otimes\mathbf{y}\cdot\hat{\boldsymbol{\sigma}}|\Psi^-\rangle = \sum_{i,j} x^i y^j \,\langle\Psi^-|\hat{\sigma}^i\otimes\hat{\sigma}^j|\Psi^-\rangle = -\mathbf{x}\cdot\mathbf{y} \tag{4.2}$$

and a proper choice of measurement settings, e.g.

$$\mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}, \quad \mathbf{y}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{4.3}$$

leads to $|S| = 2\sqrt{2}$ which violates the CHSH-Bell inequality. Significantly, in 2015, the first experimental *loophole-free*[3] violation of the CHSH-Bell inequality was conducted [17, 18, 19]. Consequently, a local realistic description of nature can be ruled out.

For convenience, we only consider the correlation tensor of the wave function $|\Psi\rangle$ defined via $T^{ij}_{|\Psi\rangle} = \langle\Psi|\hat{\sigma}^i\otimes\hat{\sigma}^j|\Psi\rangle$ in the following. In particular, the correlation tensor for the singlet state $T^{ij}_{|\Psi^-\rangle} \equiv \langle\Psi^-|\hat{\sigma}^i\otimes\hat{\sigma}^j|\Psi^-\rangle = -\delta_{ij}$, where δ_{ij} denotes the Kronecker delta and $i, j \in \{1, 2, 3\}$, has been shown to be sufficient evidence to conclude that the CHSH-Bell inequality can be violated.

[2] Here, we restrict to spin measurements but, in principle, any kind of measurement (with outcomes ± 1) could be used as well.

[3] Except *local realism* and *freedom of choice* (i.e. the measurement settings are not predetermined), there are no further (experimental) assumptions.

In relativistic quantum mechanics, the momentum degrees of freedom of the two particles must be also taken into account, because their spin degrees of freedom depend on their momentum degrees of freedom; in particular, the reduced spin state gets mixed through a Lorentz boost of the full (spin and momentum) state [10]. This means that different inertial observers would not agree on the violation of the CHSH-Bell inequality, eq. (4.1), if all of them apply the same operator $\mathbf{x} \cdot \hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{A}}} \otimes \mathbf{y} \cdot \hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{B}}}$ to their corresponding Lorentz boosted states (after tracing out the momentum degrees of freedom). However, if the joint spin observable is Lorentz transformed in addition to the full state, we find an observer-independent violation of the CHSH-Bell inequality. This means that different observers agree on the probabilities of observing the same event, but disagree on the measured observables.

In the literature, different relativistic spin operators defined on the total (momentum and spin) Hilbert space of each Dirac particle have been utilized for relativistic Bell tests instead of the ordinary Pauli operator. In fact, there are lots of approaches with different conclusions regarding the violation of the CHSH-Bell inequality in a relativistic setting where a variety of spin operators and entangled states have been considered, see [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. In particular, some authors conclude that the violation of the CHSH-Bell inequality is observer-dependent [20, 23, 24, 25, 26, 29, 30, 32, 34] and others claim that it is observer-independent [21, 22, 27, 28, 31, 33]. Notice that all the authors, except [34], were not considering quantum states in a superposition of momenta. In comparison to the present work, the authors of [34] consider a standard Stern-Gerlach apparatus in the laboratory frame with a postselection of the particles' momenta which leads to a momentum dependent violation of the CHSH-Bell inequality, where maximal violation is obtained in the case of sharp momenta.

Apart from the philosophical significance of abandoning local realism on grounds of the violation of Bell-like inequalities, the QRF approach is utilized in the following to find an operational definition of Bell observables in a relativistic setting from the point of view of different QRFs. In this context, the violation of the CHSH-Bell inequality proves the ability to operationally use spin entanglement between relativistic massive spin-1/2 particles such that quantum communication protocols, e.g. quantum teleportation [35], can be extended operationally to the regime of superposed relativistic velocities.

4.2. Inclusion of a Second Dirac Particle

In order to set up a Bell experiment within QRFs, we now extend the formalism for relativistic QRFs given in section 3.4 and 3.5 to include a second Dirac particle $\mathbf{B} \equiv \mathrm{B}\tilde{\mathrm{B}}$, where B refers to external (momentum) and $\tilde{\mathrm{B}}$ to internal (spin) degrees of freedom. In doing so, a (1+1)-dimensional situation is considered, where from the viewpoint of particle A the motion of particle $\mathbf{B}$ and the laboratory C is along the same spatial direction. Consequently, the QRF transformation $\hat{S}_{\mathrm{L}}$ from QRF A to QRF C (laboratory) has to be extended with a boost on system $\mathbf{B}$ controlled by the velocity of the laboratory C. Accordingly, the new QRF transformation $\hat{S}_2 : \mathcal{H}_{\tilde{\mathrm{A}}}^{|\mathrm{A}} \otimes \mathcal{H}_{\mathbf{B}}^{|\mathrm{A}} \otimes \mathcal{H}_{\mathrm{C}}^{|\mathrm{A}} \mapsto \mathcal{H}_{\mathbf{A}}^{|\mathrm{C}} \otimes \mathcal{H}_{\mathbf{B}}^{|\mathrm{C}}$ with $\mathbf{A} \equiv \mathrm{A}\tilde{\mathrm{A}}$ and $\mathbf{B} \equiv \mathrm{B}\tilde{\mathrm{B}}$ is given by

$$\hat{S}_2 \equiv \hat{S}_{\mathrm{L}}\, \hat{U}_{\mathbf{B}}(L_{-\hat{\pi}_{\mathrm{C}}/m_{\mathrm{C}}}) \tag{4.4}$$

where $\hat{U}_{\mathbf{B}}(L_{-\hat{\pi}_{\mathrm{C}}/m_{\mathrm{C}}})$ is a unitary representation of a Lorentz boost controlled by the velocity of C and acting on the Dirac particle $\mathbf{B}$; as introduced in section 3.4, $\hat{S}_{\mathrm{L}} \equiv \hat{\mathcal{P}}_{\mathrm{CA}}^{(v)} \hat{U}_{\tilde{\mathrm{A}}}(\hat{\pi}_{\mathrm{C}})$ is given by the generalized parity-swap operator $\hat{\mathcal{P}}_{\mathrm{CA}}^{(v)}$ mapping $\hat{\mathcal{P}}_{\mathrm{CA}}^{(v)}\, \hat{\pi}_{\mathrm{C}} \hat{\mathcal{P}}_{\mathrm{CA}}^{(v)\dagger} = -\frac{m_{\mathrm{C}}}{m_{\mathrm{A}}} \hat{p}_{\mathrm{A}}$[4] and a unitary operator $\hat{U}_{\tilde{\mathrm{A}}}(\hat{\pi}_{\mathrm{C}})$ controlled by the momentum of C and acting on the spin degree of freedom $\tilde{\mathrm{A}}$.

We now take a basis of the total Hilbert space in the rest frame of $\mathbf{A}$ according to $|a\rangle_{\tilde{\mathrm{A}}} |\pi_B; \Sigma(b)\rangle_{\mathbf{B}} |\pi_{\mathrm{C}}\rangle_{\mathrm{C}}$ where $|a\rangle_{\tilde{\mathrm{A}}}$ is the spin state of $\mathbf{A}$ in its rest frame and $|\pi_{\mathrm{C}}\rangle_{\mathrm{C}}$ is a momentum eigenstate of the laboratory as seen by $\mathbf{A}$ as in chapter 3; here, the state $|\pi_B; \Sigma(b)\rangle_{\mathbf{B}} \equiv \hat{U}_{\mathbf{B}}(L_{\pi_{\mathrm{B}}/m_{\mathrm{B}}}) |0; b\rangle_{\mathbf{B}}$ is added where b refers to the spin of the Dirac particle $\mathbf{B}$ in its rest frame. By utilizing the properties of the Lorentz group (in particular that two successive collinear Lorentz boost are still a Lorentz boost), the action of the QRF transformation $\hat{S}_2$ on the considered basis is given by

$$\begin{aligned} \hat{S}_2 |a\rangle_{\tilde{\mathrm{A}}} |\pi_B; \Sigma(b)\rangle_{\mathbf{B}} |\pi_{\mathrm{C}}\rangle_{\mathrm{C}} &= \hat{S}_{\mathrm{L}} |a\rangle_{\tilde{\mathrm{A}}} |\pi_{\mathrm{C}}\rangle\, \hat{U}_{\mathbf{B}}(L_{-\pi_{\mathrm{C}}/m_{\mathrm{C}}}) |\pi_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \\ &= \hat{S}_{\mathrm{L}} |a\rangle_{\tilde{\mathrm{A}}} |\pi_{\mathrm{C}}\rangle\, \hat{U}_{\mathbf{B}}(L_{-\pi_{\mathrm{C}}/m_{\mathrm{C}}}) \hat{U}_{\mathbf{B}}(L_{\pi_{\mathrm{B}}/m_{\mathrm{B}}}) |0; b\rangle_{\mathbf{B}} \\ &= |-\tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}} \pi_{\mathrm{C}}; \Sigma(a)\rangle_{\mathbf{A}} |L\pi_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \end{aligned} \tag{4.5}$$

where in C's perspective the spin of $\mathbf{A}$ is entangled with its momentum degrees of freedom and $\mathbf{B}$ moves with the booted momentum $L\pi_{\mathrm{B}}$ which refers to the spatial part of the (1+1)-momentum $\left(L_{-\pi_{\mathrm{C}}/m_{\mathrm{C}}}\right)^{\mu}{}_{\nu}\, \pi_{\mathrm{B}}^{\nu} \equiv p_{\mathrm{B}}^{\mu}$ where $\pi_{\mathrm{B}}^{\nu} \equiv (\pi_{\mathrm{B}}^0, \pi_{\mathrm{B}})$ with $\pi_{\mathrm{B}}^0 = \pi_{\mathrm{B}}^0(|\pi_{\mathrm{B}}|) = \sqrt{m_{\mathrm{B}}^2 c^2 + \pi_{\mathrm{B}}^2}$; consequently, the momentum degrees of freedom A and B are correlated as well. It is essential to notice that the spin state $\tilde{\mathrm{B}}$ is different after this transformation since Σ **always** refers to the momentum of the corresponding momentum-spin ket. Moreover, *Wigner rotations* (appendix A.6) do not appear for this configuration, because the relativistic momenta π_{B} and π_{C} are aligned such that the two successive boosts are described by a single boost, i.e. $\hat{U}(L_{-\pi_{\mathrm{C}}}) \hat{U}(L_{\pi_{\mathrm{B}}}) = \hat{U}(L_{L\pi_{\mathrm{B}}})$.[5]

4.3. Shared Rest Frame

In this section, we consider a specific situation where the two Dirac particles $\mathbf{A}$ and $\mathbf{B}$ are entangled in their spin degree of freedom and share the same rest frame. This state can

[4] The generalized parity-swap operator is explicitly given by $\hat{\mathcal{P}}_{\mathrm{CA}}^{(v)} = \hat{\mathcal{P}}_{\mathrm{CA}} \exp\left\{ \frac{\mathrm{i}}{\hbar} \log \sqrt{\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}} \left(\hat{q}_{\mathrm{C}} \hat{\pi}_{\mathrm{C}} + \hat{\pi}_{\mathrm{C}} \hat{q}_{\mathrm{C}} \right) \right\}$ where $\hat{\mathcal{P}}_{\mathrm{AC}} \hat{p}_{\mathrm{A}} \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = -\hat{\pi}_{\mathrm{C}}$. See section 2.1.2 for details.

[5] Here, this is trivially guaranteed since relativistic velocities are only considered in one spatial dimension. Notice that the other two spatial dimensions are neglected for simplicity; however, non-relativistically, they can be easily added as discussed in section 3.4.

be written in A's perspective according to

$$|\psi^-\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}\mathrm{C}} = \frac{1}{\sqrt{2}}\left(|{+z}\rangle_{\tilde{\mathrm{A}}}|0;-z\rangle_{\mathbf{B}} - |{-z}\rangle_{\tilde{\mathrm{A}}}|0;+z\rangle_{\mathbf{B}}\right)|\phi\rangle_{\mathrm{C}}$$
$$= \sum_{\lambda=\pm z} c_\lambda |\lambda\rangle_{\tilde{\mathrm{A}}}|0;-\lambda\rangle_{\mathbf{B}}|\phi\rangle_{\mathrm{C}} \quad \text{with} \quad c_{\pm z} = \pm 1/\sqrt{2}, \tag{4.6}$$

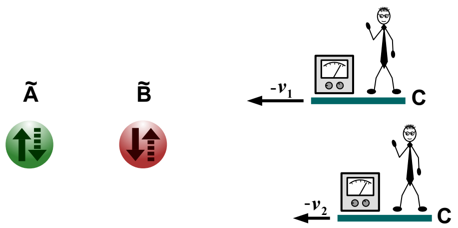
where $|\pm z\rangle$ denote the (normalized) eigenstates of the $\hat{\sigma}^z$ Pauli operator, $|0;\lambda\rangle_{\mathbf{B}} = |0\rangle_{\mathrm{B}}|\lambda\rangle_{\tilde{\mathrm{B}}}$ represents the state of $\mathbf{B}$ at rest as seen by A and

$$|\phi\rangle_{\mathrm{C}} \equiv |\phi\rangle^{|\mathrm{A}}_{\mathrm{C}} = \int \mathrm{d}\mu_{\mathrm{C}}(\pi_{\mathrm{C}})\,\phi(\pi_{\mathrm{C}})\,|\pi_{\mathrm{C}}\rangle_{\mathrm{C}} \quad \text{with} \quad \mathrm{d}\mu_{\mathrm{C}}(\pi_{\mathrm{C}}) = \frac{\mathrm{d}\pi_{\mathrm{C}}}{4\pi\sqrt{m_{\mathrm{C}}^2c^2+\pi_{\mathrm{C}}^2}} \tag{4.7}$$

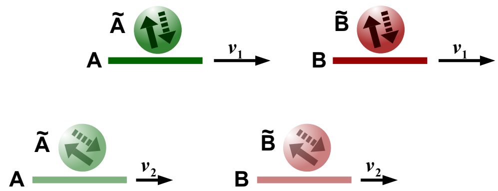
is a general laboratory state in A's perspective expanded in the momentum basis. Since the momentum degree of freedom B factorizes, we can write

$$|\psi^-\rangle^{|A}_{\tilde{\mathrm{A}}\mathbf{B}\mathrm{C}} = \frac{1}{\sqrt{2}}\left(|{+z}\rangle_{\tilde{\mathrm{A}}}|{-z}\rangle_{\tilde{\mathrm{B}}} - |{-z}\rangle_{\tilde{\mathrm{A}}}|{+z}\rangle_{\tilde{\mathrm{B}}}\right)|0\rangle_{\mathrm{B}}|\phi\rangle_{\mathrm{C}} = |\Psi^-\rangle_{\tilde{\mathrm{A}}\tilde{\mathrm{B}}}|0\rangle_{\mathrm{B}}|\phi\rangle_{\mathrm{C}} \tag{4.8}$$

with the singlet state $|\Psi^-\rangle_{\tilde{\mathrm{A}}\tilde{\mathrm{B}}}$ which is pictorially illustrated in figure 4.1(a) where, for simplicity, a superposition state of two sharp momenta/velocities is drawn for $|\phi\rangle_{\mathrm{C}}$.



(a) Shared rest frame of A and B.



(b) Laboratory frame C.

Figure 4.1.: In the shared rest frame **(a)** only the spin degrees of freedom $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$ are entangled (indicated by the correlation between the arrow patterns) and the laboratory state, moving in a superposition of two sharp relativistic velocities $-v_1$ and $-v_2$ relative to the shared rest frame, factorizes. The QRF transformation $\hat{S}_2$ takes us to the corresponding perspective of the laboratory **(b)**, where the total quantum state of the Dirac particles $\mathbf{A}$ and $\mathbf{B}$ is entangled (indicated by the correlation between the patterns of the drawn symbols).

The corresponding state in the laboratory frame is obtained according to

$$|\psi^-\rangle^{|\mathrm{C}}_{\mathbf{AB}} = \hat{S}_2\,|\psi^-\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}\mathrm{C}}$$
$$= \sum_\lambda c_\lambda \int \mathrm{d}\mu_{\mathrm{C}}(\pi)\phi(\pi)\hat{S}_{\mathrm{L}}\,|\lambda\rangle_{\tilde{\mathrm{A}}}\,|\pi\rangle_{\mathrm{C}}\,\hat{U}_{\mathbf{B}}(L_{-\pi/m_{\mathrm{C}}})\,|0;-\lambda\rangle_{\mathbf{B}}$$
$$= \sum_\lambda c_\lambda \int \mathrm{d}\mu_{\mathrm{A}}(p)\,\phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}p\right)|p;\Sigma(\lambda)\rangle_{\mathbf{A}}\,|\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}p;\Sigma(-\lambda)\rangle_{\mathbf{B}} \tag{4.9}$$

where the spin degree of freedom of each Dirac particle is correlated with its momentum degree of freedom; moreover, the coherent Lorentz boost has caused entanglement between the total quantum state of particle $\mathbf{A}$ and $\mathbf{B}$. This is illustrated in figure 4.1(b) where, for simplicity of illustration, the two QRFs A and C move in a superposition of two sharp velocities relative to each other.

4.3.1. Bell Observable in the Shared Rest Frame

Concerning the state $|\psi^-\rangle^{|\mathrm{A}} = |\Psi^-\rangle_{\tilde{\mathrm{A}}\tilde{\mathrm{B}}} |0\rangle_{\mathrm{B}} |\phi\rangle_{\mathrm{C}}$, we can adopt the non-relativistic treatment regarding the CHSH-Bell inequality in $\mathbf{A}$'s rest frame by considering the Bell observable

$$\hat{O}^{ij}_{|\mathrm{A}} \equiv \hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \mathbb{1}_{\mathrm{B}} \otimes \hat{\sigma}^j_{\tilde{\mathrm{B}}} \otimes \mathbb{1}_{\mathrm{C}} \tag{4.10}$$

with the Pauli operators $\hat{\sigma}^i$ because this leads to the same correlation tensor as in the non-relativistic case. This is immediate since the singlet state factorizes from the total state and thus

$$T^{ij} = \left[\langle\psi^-| \, \hat{O}^{ij}_{|\mathrm{A}} \, |\psi^-\rangle\right]^{|\mathrm{A}} = \; {}_{\tilde{\mathrm{A}}\tilde{\mathrm{B}}}\langle\Psi^-| \, \hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \hat{\sigma}^j_{\tilde{\mathrm{B}}} \, |\Psi^-\rangle_{\tilde{\mathrm{A}}\tilde{\mathrm{B}}} = -\delta_{ij} \tag{4.11}$$

where $[\langle\psi^-| \dots |\psi^-\rangle]^{|\mathrm{A}}$ refers to the utilized state $|\psi^-\rangle^{|\mathrm{A}} \equiv |\psi^-\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{BC}}$. With the measurement settings given in eq. (4.3) the CHSH-Bell inequality is violated in the rest frame of the Dirac particle $\mathbf{A}$.

Crucially, the QRF transformation $\hat{S}_2$ leaves the correlation tensor invariant. In particular,

$$T^{ij} = \left[\langle\psi^-| \, \hat{O}^{ij}_{|\mathrm{A}} \, |\psi^-\rangle\right]^{|\mathrm{A}} = \left[\langle\psi^-| \, \hat{O}^{ij}_{|\mathrm{C}} \, |\psi^-\rangle\right]^{|\mathrm{C}}. \tag{4.12}$$

with the observable $\hat{O}^{ij}_{|\mathrm{C}} = \hat{S}_2 \hat{O}^{ij}_{|\mathrm{A}} \hat{S}_2^\dagger$ and state $|\psi^-\rangle^{|\mathrm{C}} = \hat{S}_2 \, |\psi^-\rangle^{|\mathrm{A}}$ as seen by the laboratory C. This ensures that the CHSH-Bell inequality is violated in the laboratory frame C when it is violated in the rest frame of $\mathbf{A}$, and vice versa. Notice that the invariance of the correlation tensor is guaranteed for all possible states and observables since it is based on the unitarity of $\hat{S}_2$ and the accompanied transformation of the observables only. In the next section, we show that the transformed observables take a local form in the Hilbert spaces of the two Dirac particles $\mathbf{A}$ and $\mathbf{B}$, so that a relativistic version of the Bell experiment can be set up in the laboratory frame.

4.3.2. Bell Observable in the Laboratory Frame

In this section, the form of the observable in the laboratory frame $\hat{O}^{ij}_{|\mathrm{C}} = \hat{S}_2 \hat{O}^{ij}_{|\mathrm{A}} \hat{S}_2^\dagger$ is calculated via its action on the corresponding state

$$\begin{aligned} \hat{O}^{ij}_{|\mathrm{C}} \, |\psi^-\rangle^{|\mathrm{C}}_{\mathbf{AB}} &= \hat{S}_2 \hat{O}^{ij}_{|\mathrm{A}} \hat{S}_2^\dagger \hat{S}_2 \, |\psi^-\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{BC}} = \hat{S}_2 \, \hat{O}^{ij}_{|\mathrm{A}} \, |\psi^-\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{BC}} \\ &= \hat{S}_2 \sum_\lambda c_\lambda \int \mathrm{d}\mu_{\mathrm{C}}(\pi) \phi(\pi) \hat{O}^{ij}_{|\mathrm{A}} \, |\lambda\rangle_{\tilde{\mathrm{A}}} \, |0; -\lambda\rangle_{\mathbf{B}} \, |\pi\rangle_{\mathrm{C}} \end{aligned} \tag{4.13}$$

where

$$\begin{aligned}\hat{O}^{ij}_{|\mathrm{A}} \left|\lambda\right\rangle_{\tilde{\mathrm{A}}} \left|0;-\lambda\right\rangle_{\mathbf{B}} \left|\pi\right\rangle_{\mathrm{C}} &= \left(\hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \mathbb{1}_{\mathrm{B}} \otimes \hat{\sigma}^j_{\tilde{\mathrm{B}}} \otimes \mathbb{1}_{\mathrm{C}}\right) \left|\lambda\right\rangle_{\tilde{\mathrm{A}}} \otimes \left|0\right\rangle_{\mathrm{B}} \otimes \left|-\lambda\right\rangle_{\tilde{\mathrm{B}}} \otimes \left|\pi\right\rangle_{\mathrm{C}} \\ &= \hat{\sigma}^i_{\tilde{\mathrm{A}}} \left|\lambda\right\rangle_{\tilde{\mathrm{A}}} \otimes \left|0\right\rangle_{\tilde{\mathrm{B}}} \otimes \hat{\sigma}^j_{\tilde{\mathrm{B}}} \left|-\lambda\right\rangle_{\tilde{\mathrm{B}}} \otimes \left|\pi\right\rangle_{\mathrm{C}} \\ &= \sum_a [\sigma^i]_{a,\lambda} \left|a\right\rangle_{\tilde{\mathrm{A}}} \otimes \left|0\right\rangle_{\tilde{\mathrm{B}}} \otimes \sum_a [\sigma^j]_{b,-\lambda} \left|b\right\rangle_{\tilde{\mathrm{B}}} \otimes \left|\pi\right\rangle_{\mathrm{C}} \\ &= \sum_{a,b} [\sigma^i]_{a,\lambda}[\sigma^j]_{b,-\lambda} \left|a\right\rangle_{\tilde{\mathrm{A}}} \left|0;b\right\rangle_{\mathbf{B}} \left|\pi\right\rangle_{\mathrm{C}}\end{aligned} \tag{4.14}$$

with $a, b = \pm z$ and the matrix elements $[\sigma^k]_{m,n}$ of the Pauli operator $\hat{\sigma}^k$ written in the $\hat{\sigma}^z$-basis as explicitly given in appendix A.5. Consequently,

$$\begin{aligned}\hat{O}^{ij}_{|\mathrm{C}} \left|\psi^-\right\rangle^{|\mathrm{C}}_{\mathbf{AB}} &= \hat{S}_2 \sum_{\lambda,a,b} c_\lambda \, [\sigma^i]_{a,\lambda}[\sigma^j]_{b,-\lambda} \int \mathrm{d}\mu_{\mathrm{C}}(\pi)\phi(\pi) \left|a\right\rangle_{\tilde{\mathrm{A}}} \left|0;b\right\rangle_{\mathbf{B}} \left|\pi\right\rangle_{\mathrm{C}} \\ &= \sum_{\lambda,a,b} c_\lambda \, [\sigma^i]_{a,\lambda}[\sigma^j]_{b,-\lambda} \int \mathrm{d}\mu_{\mathrm{A}}(p)\phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}p\right) \left|p;\Sigma(a)\right\rangle_{\mathbf{A}} \left|\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}p;\Sigma(b)\right\rangle_{\mathbf{B}}\end{aligned} \tag{4.15}$$

with $p = -\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\pi$ and the Lorentz-invariant integration measures $\mathrm{d}\mu_{\mathrm{X}}(k) = \frac{\mathrm{d}k}{4\pi\sqrt{m_{\mathrm{X}}^2 c^2 + k^2}}$.

On the other hand, with the laboratory state $\left|\psi^-\right\rangle^{|\mathrm{C}}_{\mathbf{AB}}$, eq. (4.9), it follows that

$$\hat{O}^{ij}_{|\mathrm{C}} \left|\psi^-\right\rangle^{|\mathrm{C}}_{\mathbf{AB}} = \sum_\lambda c_\lambda \int \mathrm{d}\mu_{\mathrm{A}}(p) \, \phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}p\right) \; \hat{O}^{ij}_{|\mathrm{C}} \left|p;\Sigma(\lambda)\right\rangle_{\mathbf{A}} \left|\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}p;\Sigma(-\lambda)\right\rangle_{\mathbf{B}} \tag{4.16}$$

and, by comparing the previous two equations, we obtain the action of the Bell observable $\hat{O}^{ij}_{|\mathrm{C}}$ on the eigenstates

$$\begin{aligned}\hat{O}^{ij}_{|\mathrm{C}} \left|p;\Sigma(\lambda)\right\rangle_{\mathbf{A}} \left|\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}p;\Sigma(-\lambda)\right\rangle_{\mathbf{B}} &= \sum_{a,b} [\sigma^i]_{a,\lambda}[\sigma^j]_{b,-\lambda} \left|p;\Sigma(a)\right\rangle_{\mathbf{A}} \left|\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}p;\Sigma(b)\right\rangle_{\mathbf{B}} \\ &= \sum_a [\sigma^i]_{a,\lambda} \left|p;\Sigma(a)\right\rangle_{\mathbf{A}} \otimes \sum_b [\sigma^j]_{b,-\lambda} \left|\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}p;\Sigma(b)\right\rangle_{\mathbf{B}} .\end{aligned} \tag{4.17}$$

Notice that the observable $\hat{O}^{ij}_{|\mathrm{C}}$ acts on the total Hilbert space locally in $\mathbf{A}$ and $\mathbf{B}$, i.e. the total observable in the laboratory frame factorizes into $\hat{O}^{ij}_{|\mathrm{C}} = \hat{O}^i_{\mathbf{A}} \otimes \hat{O}^j_{\mathbf{B}}$. Thus, we can consider each particle separately. With the help of the standard Lorentz boost $\hat{U}_{\mathbf{A}}(L_p) \equiv$

$\hat{U}_{\mathbf{A}}(L_{p/m_{\mathrm{A}}})$, we find

$$\begin{aligned}
\sum_a [\sigma^i]_{a,\lambda} \left|p;\Sigma(a)\right\rangle_{\mathbf{A}} &= \sum_a [\sigma^i]_{a,\lambda} \hat{U}_{\mathbf{A}}(L_p)\hat{U}^\dagger_{\mathbf{A}}(L_p) \left|p;\Sigma_p(a)\right\rangle_{\mathbf{A}} \\
&= \hat{U}_{\mathbf{A}}(L_p) \sum_a [\sigma^i]_{a,\lambda} \left|0;a\right\rangle_{\mathbf{A}} \\
&= \hat{U}_{\mathbf{A}}(L_p) \left(\mathbb{1}_{\mathrm{A}} \otimes \hat{\sigma}^i_{\tilde{\mathrm{A}}}\right) \left|0;\lambda\right\rangle_{\mathbf{A}} \\
&= \hat{U}_{\mathbf{A}}(L_p) \left(\mathbb{1}_{\mathrm{A}} \otimes \hat{\sigma}^i_{\tilde{\mathrm{A}}}\right) \hat{U}^\dagger_{\mathbf{A}}(L_p) \left|p;\Sigma(\lambda)\right\rangle_{\mathbf{A}} \\
&= \hat{\Xi}^i_p \left|p;\Sigma(\lambda)\right\rangle_{\mathbf{A}} = \hat{\Xi}^i_{\hat{p}_{\mathrm{A}}} \left|p;\Sigma(\lambda)\right\rangle_{\mathbf{A}} ,
\end{aligned} \tag{4.18}$$

where we have used the unitarity of the standard Lorentz boost $\hat{U}_{\mathbf{A}}(L_p) \equiv \hat{U}_{\mathbf{A}}(L_{p/m_{\mathrm{A}}})$. In appendix A.5, it is shown that the relativistic spin observable $\hat{\Xi}^i_p$ can be specified in terms of the Pauli-Lubański spin operator $\hat{\Sigma}^\nu_p$ according to $\hat{\Xi}^i_p = \hat{U}(L_p)\left(\mathbb{1} \otimes \hat{\sigma}^i\right)\hat{U}^\dagger(L_p) = (L_p)^i_{\ \nu}\hat{\Sigma}^\nu_p$. Analogously, with $p_{\mathrm{B}} \equiv \frac{m_{\mathrm{B}}}{m_{\mathrm{A}}} p$ and $\hat{U}_{\mathbf{B}}(L_{p_{\mathrm{B}}}) \equiv \hat{U}_{\mathbf{B}}(L_{p_{\mathrm{B}}/m_{\mathrm{B}}})$, we obtain

$$\begin{aligned}
\sum_b [\sigma^j]_{b,-\lambda} \left|p_{\mathrm{B}};\Sigma(b)\right\rangle_{\mathbf{B}} &= \hat{U}_{\mathbf{B}}(L_{p_{\mathrm{B}}}) \left(\mathbb{1}_{\mathrm{B}} \otimes \hat{\sigma}^j_{\tilde{\mathrm{B}}}\right) \hat{U}^\dagger_{\mathbf{B}}(L_{p_{\mathrm{B}}}) \left|p_{\mathrm{B}};\Sigma(-\lambda)\right\rangle_{\mathbf{B}} \\
&= \hat{\Xi}^j_{p_{\mathrm{B}}} \left|p_{\mathrm{B}};\Sigma(-\lambda)\right\rangle_{\mathbf{B}} = \hat{\Xi}^j_{\hat{p}_{\mathrm{B}}} \left|p_{\mathrm{B}};\Sigma(-\lambda)\right\rangle_{\mathbf{B}} .
\end{aligned} \tag{4.19}$$

Notice that whenever there is a momentum index the corresponding mass is implicitly taken into account, i.e. $\hat{\Xi}^j_{\hat{p}_{\mathrm{A}}} \equiv \hat{\Xi}^i_{\hat{p}_{\mathrm{A}}/m_{\mathrm{A}}} = \left(L_{\hat{p}_{\mathrm{A}}/m_{\mathrm{A}}}\right)^i_{\ \mu} \hat{\Sigma}^\mu_{\hat{p}_{\mathrm{A}}/m_{\mathrm{A}}}$ and $\hat{\Xi}^j_{\hat{p}_{\mathrm{B}}} \equiv \hat{\Xi}^j_{\hat{p}_{\mathrm{B}}/m_{\mathrm{B}}} = \left(L_{\hat{p}_{\mathrm{B}}/m_{\mathrm{B}}}\right)^j_{\ \mu} \hat{\Sigma}^\mu_{\hat{p}_{\mathrm{B}}/m_{\mathrm{B}}}$.

Consequently,

$$\begin{aligned}
\hat{O}^{ij}_{|\mathrm{C}} \left|\psi\right\rangle^{|\mathrm{C}}_{\mathbf{AB}} &= \sum_\lambda c_\lambda \int \mathrm{d}\mu_{\mathrm{A}}(p)\, \phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}} p\right) \left[\hat{\Xi}^i_{\hat{p}_{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{p}_{\mathrm{B}}}\right] \left|p;\Sigma_p(\lambda)\right\rangle_{\mathbf{A}} \left|\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}} p;\Sigma_{\frac{m_{\mathrm{B}}}{m_{\mathrm{A}}} p}(-\lambda)\right\rangle_{\mathbf{B}} \\
&= \hat{\Xi}^i_{\hat{p}_{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{p}_{\mathrm{B}}} \left|\psi\right\rangle^{|\mathrm{C}}_{\mathbf{AB}} .
\end{aligned} \tag{4.20}$$

Therefore, in the case of a shared rest frame, the laboratory Bell observable is given by

$$\hat{O}^{ij}_{|\mathrm{C}} = \hat{S}_2 \hat{O}^{ij}_{|\mathrm{A}} \hat{S}^\dagger_2 = \hat{S}_2 \left(\hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \mathbb{1}_{\mathrm{B}} \otimes \hat{\sigma}^j_{\tilde{\mathrm{B}}} \otimes \mathbb{1}_{\mathrm{C}}\right) \hat{S}^\dagger_2 = \hat{\Xi}^i_{\hat{p}_{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{p}_{\mathrm{B}}}, \tag{4.21}$$

where the spin and momentum degrees of freedom are not separable in the laboratory frame. However, it is crucial to notice that $\hat{O}^{ij}_{|\mathrm{C}}$ is separable in the Hilbert spaces of the two Dirac particles $\mathbf{A}$ and $\mathbf{B}$; thus, the spin states of $\mathbf{A}$ and $\mathbf{B}$ can be measured separately via $\hat{\Xi}^i_{\hat{p}_{\mathrm{A}}}$ and $\hat{\Xi}^j_{\hat{p}_{\mathrm{B}}}$, respectively. This feature ensures that the two spin measurements can be performed in space-like separated regions such that the locality assumption of Bell's theorem is satisfied.

4.4. General Scenario

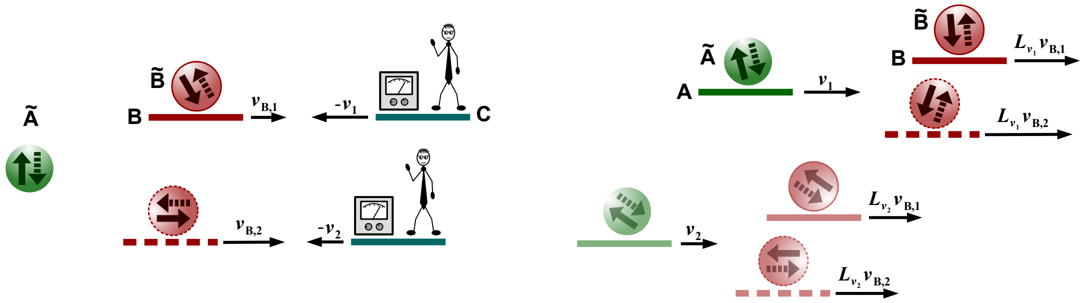
In comparison to the previous section, it is no longer assumed that the state of the two Dirac particles is perfectly correlated in momentum basis. Thus, the total state in the rest frame of $\mathbf{A}$ is described by

$$|\psi\rangle^{|\mathrm{A}} \equiv |\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}C} = |\eta\rangle_{\tilde{\mathrm{A}}\mathbf{B}} |\phi\rangle_{\mathrm{C}} \tag{4.22}$$

where $|\phi\rangle_{\mathrm{C}}$, given in eq. (4.7), is the state of the laboratory degree of freedom from the point of view of particle $\mathbf{A}$, and it factorizes from

$$|\eta\rangle_{\tilde{\mathrm{A}}\mathbf{B}} \equiv |\eta\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}} = \sum_{a,b} c_{ab} \int \mathrm{d}\mu_{\mathrm{B}}(\pi_{\mathrm{B}})\, \eta(\pi_{\mathrm{B}})\, |a\rangle_{\tilde{\mathrm{A}}}\, |\pi_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \tag{4.23}$$

where the Lorentz-invariant integration measure is given by $\mathrm{d}\mu_{\mathrm{B}}(\pi_{\mathrm{B}}) = \frac{\mathrm{d}\pi_{\mathrm{B}}}{4\pi\sqrt{m_{\mathrm{B}}^2 c^2 + \pi_{\mathrm{B}}^2}}$. Hence, in A's perspective the Dirac particle $\mathbf{B}$ moves in a superposition of momenta and can be entangled with the spin state of $\tilde{\mathrm{A}}$ which is graphically illustrated in figure 4.2(a).



(a) Rest frame A of the Dirac particle $\mathbf{A}$.

(b) Laboratory frame C.

Figure 4.2.: (a) In A's perspective, the spin degree of freedom $\tilde{\mathrm{B}}$ of the Dirac particle $\mathbf{B} \equiv \mathrm{B}\tilde{\mathrm{B}}$ depends on its momentum degree of freedom B (indicated by the orientation of dashed and solid lines) since $\mathbf{B}$ is moving in a superposition of two sharp relativistic velocities $v_{\mathrm{B},1}$ and $v_{\mathrm{B},2}$ with respect to the QRF A. Moreover, the state of the laboratory C is moving in a superposition of two sharp relativistic velocities $-v_1$ and $-v_2$ relative to $\mathbf{A}$. In the initial QRF A, we consider entanglement between the spin state of $\tilde{\mathrm{A}}$ and the Dirac particle $\mathbf{B}$ (illustrated by the correlation between the dashed and between the solid lines). The QRF transformation $\hat{S}_2$ coherently boosts the two Dirac particles by the velocity of C and outputs the perspective of the laboratory. **(b)** In the laboratory frame C, the two Dirac particles $\mathbf{A}$ and $\mathbf{B}$ are entangled and both spin degrees of freedom $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$ depend on the corresponding momentum degrees of freedom A and B.

Accordingly, the state in the perspective of the laboratory C is given by

$$\begin{aligned}
|\psi\rangle^{|\mathrm{C}} \equiv |\psi\rangle^{|\mathrm{C}}_{\mathbf{AB}} &= \hat{S}_2 \, |\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}\mathrm{C}} \\
&= \sum_{a,b} c_{ab} \int \mathrm{d}\mu_{\mathrm{B}}(\pi_{\mathrm{B}}) \mathrm{d}\mu_{\mathrm{C}}(\pi_{\mathrm{C}}) \, \eta(\pi_{\mathrm{B}}) \phi(\pi_{\mathrm{C}}) \, \hat{S}_2 \, |a\rangle_{\tilde{\mathrm{A}}} \, |\pi_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \, |\pi_{\mathrm{C}}\rangle_{\mathrm{C}} \\
&\overset{(4.5)}{=\!=} \sum_{a,b} c_{ab} \int \mathrm{d}\mu_{\mathrm{A}}(p_{\mathrm{A}}) \mathrm{d}\mu_{\mathrm{B}}(p_{\mathrm{B}}) \, \eta(L^{-1} p_{\mathrm{B}}) \phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}} p_{\mathrm{A}}\right) |p_{\mathrm{A}}; \Sigma(a)\rangle_{\mathbf{A}} \, |p_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}}
\end{aligned} \tag{4.24}$$

with $p_{\mathrm{A}} = -\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}} \pi_{\mathrm{C}}$, p_{B} referring to $p^{\mu}_{\mathrm{B}} \equiv (p^0_{\mathrm{B}}, p_{\mathrm{B}}) \equiv (L_{-\pi_{\mathrm{C}}/m_{\mathrm{C}}})^{\mu}{}_{\nu} \pi^{\nu}_{\mathrm{B}}$, $L^{-1} p_{\mathrm{B}}$ referring to the spatial part of $\pi^{\mu}_{\mathrm{B}} \equiv (L^{-1}_{p_{\mathrm{A}}/m_{\mathrm{A}}})^{\mu}{}_{\nu} p^{\nu}_{\mathrm{B}}$ and the Lorentz-invariant integration measures $\mathrm{d}\mu_{\mathrm{X}}(k) = \frac{\mathrm{d}k}{4\pi \sqrt{m_{\mathrm{X}}^2 c^2 + k^2}}$. Thus, in C's perspective the two Dirac particles **A** and **B** are entangled and their spin degrees of freedom, $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$, are momentum dependent. This is graphically illustrated in figure 4.2(b).

4.4.1. Bell Observable in the Rest Frame

In the rest frame of the Dirac particle **A** a proper spin observable for its spin state $\tilde{\mathrm{A}}$ is indeed the same as in the non-relativistic case, i.e. $\hat{\sigma}^i_{\tilde{\mathrm{A}}}$. However, $\mathbb{1}_{\mathrm{B}} \otimes \hat{\sigma}^j_{\tilde{\mathrm{B}}}$ is no longer valid as spin observable for the other Dirac particle **B** due to its (non-sharp) relativistic momentum states which leads to spin-momentum entanglement. Therefore, in analogy to section 3.5, the observable $\hat{\Xi}^j_{\hat{\pi}_{\mathrm{B}}}$ is used as relativistic spin observable for **B**. Consequently, the joint spin measurement in **A**'s rest frame is described by

$$\hat{G}^{ij}_{|\mathrm{A}} \equiv \hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{\pi}_{\mathrm{B}}} \otimes \mathbb{1}_{\mathrm{C}}. \tag{4.25}$$

Since the momentum degree of freedom B does no longer factorize from the total state, we have to check that the CHSH-Bell inequality, eq. (4.1), is violated with this choice of observables. This is done by calculating the corresponding (QRF-invariant) correlation tensor in the following section.

4.4.2. Correlation Tensor

Here, the correlation tensor T^{ij} is calculated in the rest frame of the Dirac particle **A**, where the laboratory C separates from the total state such that

$$T^{ij} = \left[\langle\psi| \, \hat{G}^{ij}_{|\mathrm{A}} \, |\psi\rangle\right]^{|\mathrm{A}} = \langle\eta| \, \hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{\pi}_{\mathrm{B}}} \, |\eta\rangle \quad \text{with} \quad |\eta\rangle = |\eta\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}}. \tag{4.26}$$

With the general state $|\eta\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}}$, eq. (4.23), it follows that

$$\hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{\pi}_{\mathrm{B}}} \, |\eta\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}} = \sum_{a,b} c_{ab} \int \mathrm{d}\mu_{\mathrm{B}}(\pi_{\mathrm{B}}) \, \eta(\pi_{\mathrm{B}}) \, \hat{\sigma}^i_{\tilde{\mathrm{A}}} \, |a\rangle_{\tilde{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{\pi}_{\mathrm{B}}} \, |\pi_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \tag{4.27}$$

where $\hat{\sigma}^i_{\tilde{\mathrm{A}}}\,|a\rangle_{\tilde{\mathrm{A}}} = \sum_{a'}[\sigma^i]_{a'a}\,|a'\rangle_{\tilde{\mathrm{A}}}$ and following eq. (4.19) we can write $\hat{\Xi}^j_{\hat{\pi}_\mathrm{B}}\,|\pi_\mathrm{B};\Sigma(b)\rangle_{\mathbf{B}} = \hat{\Xi}^j_{\pi_\mathrm{B}}\,|\pi_\mathrm{B};\Sigma(b)\rangle_{\mathbf{B}} = \hat{U}_{\mathbf{B}}(L_{\pi_\mathrm{B}})\left(\mathbb{1}_\mathrm{B}\otimes\hat{\sigma}^j_{\tilde{\mathrm{B}}}\right)\hat{U}^\dagger_{\mathbf{B}}(L_{\pi_\mathrm{B}})\,|\pi_\mathrm{B};\Sigma(b)\rangle_{\mathbf{B}} = \sum_{b'}[\sigma^j]_{b'b}\,|\pi_\mathrm{B};\Sigma(b')\rangle_{\mathbf{B}}$ with $a,a',b,b' = \pm z$ and the matrix elements $[\sigma^k]_{mn}$ of the Pauli operator $\hat{\sigma}^k$ written in the $\hat{\sigma}^z$-basis as explicitly given in eq. (A.28). Consequently,

$$\hat{\sigma}^i_{\tilde{\mathrm{A}}}\otimes\hat{\Xi}^j_{\hat{\pi}_\mathrm{B}}\,|\eta\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}} = \sum_{a,a',b,b'} c_{ab}[\sigma^i]_{a'a}[\sigma^j]_{b'b}\int\!\mathrm{d}\mu_\mathrm{B}(\pi_\mathrm{B})\;\eta(\pi_\mathrm{B})\,|a'\rangle_{\tilde{\mathrm{A}}}\,|\pi_\mathrm{B};\Sigma(b')\rangle_{\mathbf{B}} \tag{4.28}$$

and, by using the orthogonality relations $\langle\bar{a}|a'\rangle = \delta_{\bar{a}a'}$ as well as $\langle\bar{\pi}_\mathrm{B};\Sigma(\bar{b})|\pi_\mathrm{B};\Sigma(b')\rangle = (2\pi)2\bar{\pi}^0_\mathrm{B}\delta_{\bar{b}b'}\delta(\pi_\mathrm{B}-\bar{\pi}_\mathrm{B})$, the correlation tensor is given by

$$T^{ij} = \sum_{a,a',b,b'} c^*_{a'b'}c_{ab}[\sigma^i]_{a'a}[\sigma^j]_{b'b}\int\!\mathrm{d}\mu_\mathrm{B}(\pi_\mathrm{B})\;|\eta(\pi_\mathrm{B})|^2. \tag{4.29}$$

This can be further simplified by noting that the quantum state is normalized, i.e.

$$\begin{aligned}[\langle\eta|\eta\rangle]^{|\mathrm{A}} &= \sum_{a,b}|c_{ab}|^2\;\int\!\mathrm{d}\mu_\mathrm{B}(\pi_\mathrm{B})\;|\eta(\pi_\mathrm{B})|^2 = 1\\ &\Leftrightarrow \sum_{a,b}|c_{ab}|^2 = 1 \quad\text{and}\quad \int\!\mathrm{d}\mu_\mathrm{B}(\pi_\mathrm{B})\;|\eta(\pi_\mathrm{B})|^2 = 1.\end{aligned} \tag{4.30}$$

This leads to

$$T^{ij} = \sum_{a,a',b,b'} c^*_{a'b'}c_{ab}[\sigma^i]_{a'a}[\sigma^j]_{b'b}. \tag{4.31}$$

From now on, entanglement between the spin $\tilde{\mathrm{A}}$ and the Dirac particle $\mathbf{B}$ as seen from the rest frame of $\mathbf{A}$ is considered by the state

$$\begin{aligned}|\eta^-\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}} &= \sum_{\lambda=\pm z} c_\lambda\int\!\mathrm{d}\mu_\mathrm{B}(\pi_\mathrm{B})\;\eta(\pi_\mathrm{B})\,|\lambda\rangle_{\tilde{\mathrm{A}}}\,|\pi_\mathrm{B};\Sigma(-\lambda)\rangle_{\mathbf{B}} \quad\text{with}\quad c_{\pm z} = \pm 1/\sqrt{2}\\ &= \int\!\mathrm{d}\mu_\mathrm{B}(\pi_\mathrm{B})\;\eta(\pi_\mathrm{B})\left[|{+z}\rangle_{\tilde{\mathrm{A}}}\,|\pi_\mathrm{B};\Sigma(-z)\rangle_{\mathbf{B}} - |{-z}\rangle_{\tilde{\mathrm{A}}}\,|\pi_\mathrm{B};\Sigma(+z)\rangle_{\mathbf{B}}\right]/\sqrt{2}\end{aligned} \tag{4.32}$$

which reflects a relativistic generalization of the singlet state in the case where $\mathbf{B}$ is a relativistic particle. The state $|\eta^-\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}}$ is obtained by inserting $c_{ab} = c_a\delta_{a,-b}$ and $c_{\pm z} = \pm 1/\sqrt{2}$ into $|\eta\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}}$ given in eq. (4.23). Notice that the previous case, where the particles $\mathbf{A}$ and $\mathbf{B}$ are perfectly correlated in momenta, is obtained by taking a sharply localized state in momentum basis around $\pi_B = 0$ for particle $\mathbf{B}$.[6] Finally, the correlation tensor is given by

$$T^{ij} = \sum_{\lambda,\lambda'} c^*_{\lambda'}c_\lambda\;[\sigma^i]_{\lambda'\lambda}[\sigma^j]_{-\lambda',-\lambda} = -\delta_{ij} \tag{4.33}$$

[6] In particular, $|\eta^-\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}} \to |\Psi^-\rangle_{\tilde{\mathrm{A}}\tilde{\mathrm{B}}}\otimes|0\rangle_\mathrm{B}$ for $\pi_\mathrm{B}\to 0$, where $|\Psi^-\rangle_{\tilde{\mathrm{A}}\tilde{\mathrm{B}}}$ denotes the singlet state.

which is calculated straightforwardly by inserting coefficients $c_{\pm z} = \pm 1/\sqrt{2}$ and the matrix elements $[\sigma^k]_{mn}$ given in eq. (A.28). Thus, the CHSH-Bell inequality can be violated in $\mathbf{A}$'s rest frame with the initial state $|\psi'^-\rangle^{|A}_{\tilde{\mathrm{A}}\mathbf{B}\mathrm{C}} \equiv |\eta^-\rangle^{|A}_{\tilde{\mathrm{A}}\mathbf{B}} |\phi\rangle_{\mathrm{C}}$ and the corresponding observable $\hat{G}^{ij}_{|\mathrm{A}} \equiv \hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{\pi}_{\mathrm{B}}} \otimes \mathbb{1}_{\mathrm{C}}$.

As a consequence, the CHSH-Bell inequality can also be violated in the laboratory frame when the corresponding observable $\hat{G}^{ij}_{|\mathrm{C}} = \hat{S}_2 \hat{G}^{ij}_{|\mathrm{A}} \hat{S}^\dagger_2$ is applied.

4.4.3. Bell Observable in the Laboratory Frame

We now derive the general form of the Bell observable in the laboratory frame $\hat{G}^{ij}_{|\mathrm{C}} = \hat{S}_2 \hat{G}^{ij}_{|\mathrm{A}} \hat{S}^\dagger_2$. For this purpose, the considered class of states in the rest frame of $\mathbf{A}$ is kept general - only the laboratory state $|\phi\rangle_{\mathrm{C}}$, given in eq. (4.22), factorizes.

First, notice that the action of $\hat{G}^{ij}_{|\mathrm{C}}$ on that class of states is immediate from the unitarity of $\hat{S}_2$ since

$$\begin{aligned} \hat{G}^{ij}_{|\mathrm{C}} |\psi\rangle^{|\mathrm{C}}_{\mathbf{AB}} &= \hat{S}_2 \hat{G}^{ij}_{|\mathrm{A}} \hat{S}^\dagger_2 \hat{S}_2 |\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}C} = \hat{S}_2 \hat{G}^{ij}_{|\mathrm{A}} |\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}C} = \hat{S}_2 \left(\left[\hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{\pi}_{\mathrm{B}}} |\eta\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}} \right] \otimes |\phi\rangle_{\mathrm{C}} \right) \\ &= \sum_{a,a',b,b'} c_{ab}\, [\sigma^i]_{a'a} [\sigma^j]_{b'b} \int \mathrm{d}\mu_{\mathrm{A}}(p_{\mathrm{A}}) \mathrm{d}\mu_{\mathrm{B}}(p_{\mathrm{B}}) \\ &\qquad \eta(L^{-1} p_{\mathrm{B}}) \phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}} p_{\mathrm{A}} \right) |p_{\mathrm{A}}; \Sigma(a')\rangle_{\mathbf{A}} |p_{\mathrm{B}}; \Sigma(b')\rangle_{\mathbf{B}} \end{aligned} \tag{4.34}$$

where eq. (4.28) followed by eq. (4.24) have been utilized. Consequently,

$$\begin{aligned} \hat{G}^{ij}_{|\mathrm{C}} |\psi\rangle^{|\mathrm{C}}_{\mathbf{AB}} &= \sum_{a,b} c_{ab} \int \mathrm{d}\mu_{\mathrm{A}}(p_{\mathrm{A}}) \mathrm{d}\mu_{\mathrm{B}}(p_{\mathrm{B}})\, \eta(L^{-1} p_{\mathrm{B}}) \phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}} p_{\mathrm{A}} \right) \\ &\qquad \sum_{a'} [\sigma^i]_{a'a} |p_{\mathrm{A}}; \Sigma(a')\rangle_{\mathbf{A}} \otimes \sum_{b'} [\sigma^j]_{b'b} |p_{\mathrm{B}}; \Sigma(b')\rangle_{\mathbf{B}} . \end{aligned} \tag{4.35}$$

where, in analogy to eq. (4.18) and (4.19), we can write

$$\sum_{a'} [\sigma^i]_{a'a} |p_{\mathrm{A}}; \Sigma(a')\rangle_{\mathbf{A}} = \hat{\Xi}^i_{\hat{p}_{\mathrm{A}}} |p_{\mathrm{A}}; \Sigma(a)\rangle_{\mathbf{A}} \tag{4.36}$$

and

$$\sum_{b'} [\sigma^j]_{b'b} |p_{\mathrm{B}}; \Sigma(b')\rangle_{\mathbf{B}} = \hat{\Xi}^j_{\hat{p}_{\mathrm{B}}} |p_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} . \tag{4.37}$$

The operator index of $\hat{\Xi}^i_{\hat{p}_{\mathrm{A}}} \equiv \hat{\Xi}^i_{\hat{p}_{\mathrm{A}}/m_{\mathrm{A}}}$ and $\hat{\Xi}^j_{\hat{p}_{\mathrm{B}}} \equiv \hat{\Xi}^j_{\hat{p}_{\mathrm{B}}/m_{\mathrm{B}}}$ makes it possible to move the two relativistic spin operators outside the integral and we obtain

$$\hat{G}^{ij}_{|\mathrm{C}} |\psi\rangle^{|\mathrm{C}}_{\mathbf{AB}} = \hat{\Xi}^i_{\hat{p}_{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{p}_{\mathrm{B}}} |\psi\rangle^{|\mathrm{C}}_{\mathbf{AB}} . \tag{4.38}$$

Finally, the joint spin measurement as seen by the laboratory C is described with the observable

$$\hat{G}^{ij}_{|\mathrm{C}} = \hat{\Xi}^{i}_{\hat{p}_\mathrm{A}} \otimes \hat{\Xi}^{j}_{\hat{p}_\mathrm{B}}. \tag{4.39}$$

Crucially, the two spin measurements on the two Dirac particles $\mathbf{A}$ and $\mathbf{B}$ can be performed independently in space-like separated regions due to the product form of $\hat{G}^{ij}_{|\mathrm{C}}$; thus, a proper Bell test can be performed in the laboratory C. Notice that the shared rest frame scenario (section 4.3) is a special case of the situation considered here and thus yielding to the same Bell observable in the laboratory frame.

4.5. Extension to Non-collinear Relative Motion

Additionally to section 4.4, we now allow for scenarios where the motion of the Dirac particle $\mathbf{B}$ and the laboratory C is not collinear with respect to the rest frame of particle $\mathbf{A}$. However, we demand the angle ξ between the momentum of the Dirac particle $\boldsymbol{\pi}_\mathrm{B}$ and the laboratory $\boldsymbol{\pi}_\mathrm{C}$ to be fixed, i.e. the two systems do not move in superposed directions. Without loss of generality, we replace the previously considered 1-dimensional momenta by 3-dimensional vectors according to

$$\pi_\mathrm{B} \rightarrow \boldsymbol{\pi}_\mathrm{B} = (\pi_\mathrm{B}, 0, 0) = \pi_\mathrm{B}\, \mathbf{e}_x \tag{4.40}$$

and

$$\pi_\mathrm{C} \rightarrow \boldsymbol{\pi}_\mathrm{C} = \pi_\mathrm{C}\, \mathbf{u}, \quad \text{where} \quad \mathbf{u} = (u_x, u_y, 0),\ |\mathbf{u}| = 1. \tag{4.41}$$

Since $\xi = \sphericalangle(\boldsymbol{\pi}_\mathrm{B}, \boldsymbol{\pi}_\mathrm{C})$ is fixed, we still have a one-dimensional integration for each system, i.e. we are integrating along $\mathbf{e}_x \equiv (1,0,0)$ and $\mathbf{u}$ over the projections $\pi_\mathrm{B} \equiv \boldsymbol{\pi}_\mathrm{B} \cdot \mathbf{e}_x$ and $\pi_\mathrm{C} \equiv \boldsymbol{\pi}_\mathrm{C} \cdot \mathbf{u}$, respectively.

Apart from the replacements $\pi_\mathrm{B} \rightarrow \boldsymbol{\pi}_\mathrm{B}$ and $\pi_\mathrm{B} \rightarrow \boldsymbol{\pi}_\mathrm{C}$, the same state $|\psi\rangle^{|\mathrm{A}} \equiv |\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{B}C} = |\eta\rangle_{\tilde{\mathrm{A}}\mathbf{B}}\, |\phi\rangle_\mathrm{C}$ as in section 4.4 is considered in the following. Thus, we can adopt all statements given in the previous section which refer to the rest frame of the Dirac particle $\mathbf{A}$, since the laboratory state $|\psi\rangle_\mathrm{C}$[7] factorizes. Crucially, this allows to conclude immediately that the CHSH-Bell inequality is violated in the rest frame of particle $\mathbf{A}$ if the observable

$$\mathbf{x} \cdot \hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{A}}} \otimes \mathbf{y} \cdot \hat{\boldsymbol{\Xi}}_{\hat{\boldsymbol{\pi}}_\mathrm{B}} \otimes \mathbb{1}_\mathrm{C} = \sum_{i,j} x_i y_j\, \hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{\boldsymbol{\pi}}_\mathrm{B}} \otimes \mathbb{1}_\mathrm{C} \tag{4.42}$$

with proper measurement settings $\mathbf{x}$ and $\mathbf{y}$, as e.g. given in eq. (4.3), is utilized.

However, the description in the laboratory frame is different now due to an additional Wigner rotation (appendix A.6) of the spin degree of freedom $\tilde{\mathrm{B}}$. The Wigner rotation appears, in contrast to section 4.4, because the laboratory C and the Dirac particle $\mathbf{B}$

[7] Effectively, $|\phi\rangle_\mathrm{C}$ is the only state that has been changed in comparison to the previous treatment. The difference is that now the state $|\phi\rangle_\mathrm{C}$ propagates along $\mathbf{u}$, whereas it was propagating along $\mathbf{e}_x$ before.

do not move in a line with respect to the rest frame of $\mathbf{A}$. Accordingly, the state in the perspective of the laboratory C is given by

$$\begin{aligned}
|\psi\rangle^{|\mathrm{C}} \equiv |\psi\rangle^{|\mathrm{C}}_{\mathbf{AB}} &= \hat{S}_2 \, |\psi\rangle^{|\mathrm{A}}_{\tilde{\mathrm{A}}\mathbf{BC}} \\
&= \hat{S}_2 \sum_{a,b} c_{ab} \int \mathrm{d}\mu_{\mathrm{B}}(\pi_{\mathrm{B}}) \mathrm{d}\mu_{\mathrm{C}}(\pi_{\mathrm{C}}) \, \eta(\boldsymbol{\pi}_{\mathrm{B}}) \phi(\boldsymbol{\pi}_{\mathrm{C}}) \; |a\rangle_{\tilde{\mathrm{A}}} \, |\boldsymbol{\pi}_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \, |\boldsymbol{\pi}_{\mathrm{C}}\rangle_{\mathrm{C}} \\
&= \sum_{a,b} c_{ab} \int \mathrm{d}\mu_{\mathrm{B}}(\pi_{\mathrm{B}}) \mathrm{d}\mu_{\mathrm{C}}(\pi_{\mathrm{C}}) \, \eta(\boldsymbol{\pi}_{\mathrm{B}}) \phi(\boldsymbol{\pi}_{\mathrm{C}}) \; \hat{S}_2 \, |a\rangle_{\tilde{\mathrm{A}}} \, |\boldsymbol{\pi}_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \, |\boldsymbol{\pi}_{\mathrm{C}}\rangle_{\mathrm{C}}
\end{aligned} \tag{4.43}$$

where, by applying the (1+3)-dimensional extension of the previous QRF transformation $\hat{S}_2 \equiv \hat{S}_{\mathrm{L}} \, \hat{U}_{\mathbf{B}}(L_{-\hat{\boldsymbol{\pi}}_{\mathrm{C}}})$, we obtain

$$\begin{aligned}
\hat{S}_2 \, |a\rangle_{\tilde{\mathrm{A}}} \, |\boldsymbol{\pi}_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \, |\boldsymbol{\pi}_{\mathrm{C}}\rangle_{\mathrm{C}} &= \hat{S}_{\mathrm{L}} \, |a\rangle_{\tilde{\mathrm{A}}} \, |\boldsymbol{\pi}_{\mathrm{C}}\rangle_{\mathrm{C}} \, \hat{U}_{\mathbf{B}}(L_{-\boldsymbol{\pi}_{\mathrm{C}}}) \, |\boldsymbol{\pi}_{\mathrm{B}}; \Sigma(b)\rangle_{\mathbf{B}} \\
&= \left|-\tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}} \boldsymbol{\pi}_{\mathrm{C}}; \Sigma(a)\right\rangle_{\mathbf{A}} \hat{U}_{\mathbf{B}}(L_{-\boldsymbol{\pi}_{\mathrm{C}}}) \hat{U}_{\mathbf{B}}(L_{\boldsymbol{\pi}_{\mathrm{B}}}) \, |0; b\rangle_{\mathbf{B}} \, .
\end{aligned} \tag{4.44}$$

The two successive non-collinear ($\boldsymbol{\pi}_{\mathrm{B}} \nparallel \boldsymbol{\pi}_{\mathrm{C}}$) boosts on particle $\mathbf{B}$ result in a boost and a Wigner rotation of the spin degree of freedom $\tilde{\mathrm{B}}$ according to

$$\begin{aligned}
\hat{U}_{\mathbf{B}}(L_{-\boldsymbol{\pi}_{\mathrm{C}}}) \hat{U}_{\mathbf{B}}(L_{\boldsymbol{\pi}_{\mathrm{B}}}) \, |0; b\rangle_{\mathbf{B}} &= \hat{U}_{\mathbf{B}}(L_{L\boldsymbol{\pi}_{\mathrm{B}}}) \left[\mathbb{1}_{\mathrm{B}} \otimes \hat{R}_{\tilde{\mathrm{B}}}(\boldsymbol{\Omega}) \right] |0; b\rangle_{\mathbf{B}} \\
&= \hat{U}_{\mathbf{B}}(L_{L\boldsymbol{\pi}_{\mathrm{B}}}) \, |0; R_{\boldsymbol{\Omega}}(b)\rangle_{\mathbf{B}} \\
&= |L\boldsymbol{\pi}_{\mathrm{B}}; \Sigma(R_{\boldsymbol{\Omega}}(b))\rangle_{\mathbf{B}}
\end{aligned} \tag{4.45}$$

where $L\boldsymbol{\pi}_{\mathrm{B}}$ refers to the spatial part of the four-momentum $\left(L_{-\pi_{\mathrm{C}}/m_{\mathrm{C}}}\right)^{\mu}_{\ \nu} \pi^{\nu}_{\mathrm{B}} \equiv p^{\mu}_{\mathrm{B}} \equiv (p^0_{\mathrm{B}}, \mathbf{p}_{\mathrm{B}})$, i.e. $L\boldsymbol{\pi}_{\mathrm{B}} \equiv \mathbf{p}_{\mathrm{B}}$, and the rotation is specified through $\boldsymbol{\Omega} \equiv \Omega \mathbf{n}, |\mathbf{n}| = 1$. More specifically, the spin $\tilde{\mathrm{B}}$ is rotated around the axis

$$\mathbf{n} = \frac{\boldsymbol{\pi}_{\mathrm{B}} \times \boldsymbol{\pi}_{\mathrm{C}}}{|\boldsymbol{\pi}_{\mathrm{B}} \times \boldsymbol{\pi}_{\mathrm{C}}|} = \mathbf{e}_z \equiv (0, 0, 1) \tag{4.46}$$

by the (Bloch) angle Ω given by

$$\cos\Omega = \frac{1 + \gamma_{\boldsymbol{\pi}_{\mathrm{B}}} + \gamma_{\boldsymbol{\pi}_{\mathrm{C}}} + \gamma_{L\boldsymbol{\pi}_{\mathrm{B}}}}{(1 + \gamma_{\boldsymbol{\pi}_{\mathrm{B}}})(1 + \gamma_{\boldsymbol{\pi}_{\mathrm{C}}})(1 + \gamma_{L\boldsymbol{\pi}_{\mathrm{B}}})} - 1 \tag{4.47}$$

where $\gamma_{\boldsymbol{\pi}_{\mathrm{B}}} = \sqrt{1 + \frac{\boldsymbol{\pi}^2_{\mathrm{B}}}{m^2_{\mathrm{B}} c^2}}$, $\gamma_{\boldsymbol{\pi}_{\mathrm{C}}} = \sqrt{1 + \frac{\boldsymbol{\pi}^2_{\mathrm{C}}}{m^2_{\mathrm{C}} c^2}}$ and

$$\gamma_{L\boldsymbol{\pi}_{\mathrm{B}}} = \gamma_{\boldsymbol{\pi}_{\mathrm{B}}} \gamma_{\boldsymbol{\pi}_{\mathrm{C}}} (1 - \boldsymbol{\beta}_{\boldsymbol{\pi}_{\mathrm{B}}} \cdot \boldsymbol{\beta}_{\boldsymbol{\pi}_{\mathrm{C}}}) = \gamma_{\boldsymbol{\pi}_{\mathrm{B}}} \gamma_{\boldsymbol{\pi}_{\mathrm{C}}} \left(1 - \frac{\boldsymbol{\pi}_{\mathrm{B}}}{\sqrt{m^2_{\mathrm{B}} c^2 + \pi^2_{\mathrm{B}}}} \cdot \frac{\boldsymbol{\pi}_{\mathrm{C}}}{\sqrt{m^2_{\mathrm{C}} c^2 + \pi^2_{\mathrm{C}}}} \right). \tag{4.48}$$

Consequently, the rotation operator can be explicitly written as

$$\hat{R}(\boldsymbol{\Omega}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\Omega} \cdot \hat{\boldsymbol{\sigma}}/2} = \mathrm{e}^{-\mathrm{i}\Omega \hat{\sigma}^z/2} = \mathbb{1} \cos(\Omega/2) - \mathrm{i}\hat{\sigma}^z \sin(\Omega/2). \tag{4.49}$$

Thus, the axis of rotation is fixed and the rotation angle Ω depends on the relative orientation as well as on the magnitude of the two momenta $\boldsymbol{\pi}_{\mathrm{B}}$ and $\boldsymbol{\pi}_{\mathrm{C}}$, i.e. $\Omega = \Omega(\boldsymbol{\pi}_{\mathrm{B}}, \boldsymbol{\pi}_{\mathrm{C}})$. This means that we actually have a superposition of Wigner rotations acting on the spin $\tilde{\mathrm{B}}$ because of the superpositions of momenta. Finally, we obtain the total state in the laboratory frame

$$|\psi\rangle^{|\mathrm{C}} = \sum_{a,b} c_{ab} \int \mathrm{d}\mu_{\mathrm{A}}(p_{\mathrm{A}}) \mathrm{d}\mu_{\mathrm{B}}(p_{\mathrm{B}})\, \eta(L^{-1}\mathbf{p}_{\mathrm{B}}) \phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\mathbf{p}_{\mathrm{A}}\right) |\mathbf{p}_{\mathrm{A}}; \Sigma(a)\rangle_{\mathbf{A}} \,|\mathbf{p}_{\mathrm{B}}; \Sigma(R_{\boldsymbol{\Omega}}(b))\rangle_{\mathbf{B}} \tag{4.50}$$

where $\mathbf{p}_{\mathrm{A}} \equiv -\frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\boldsymbol{\pi}_{\mathrm{C}}$ and $\mathbf{p}_{\mathrm{B}}$ denotes the spatial part of the four-momentum $p^{\mu}_{\mathrm{B}} \equiv (p^0_{\mathrm{B}}, \mathbf{p}_{\mathrm{B}}) \equiv (L_{-\boldsymbol{\pi}_{\mathrm{C}}})^{\mu}{}_{\nu}\, \pi^{\nu}_{\mathrm{B}}$. With this we find $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mathbf{p}_{\mathrm{A}}, \mathbf{p}_{\mathrm{B}})$ in the laboratory frame C, where the rotation is also around the z-axis (as the two momenta $\mathbf{p}_{\mathrm{A}}$ and $\mathbf{p}_{\mathrm{B}}$ lie in the x-y-plane) and the rotation angle is given by

$$\cos\Omega = \frac{1 + \gamma_{\mathbf{p}_{\mathrm{A}}} + \gamma_{\mathbf{p}_{\mathrm{B}}} + \gamma_{L\mathbf{p}_{\mathrm{B}}}}{(1 + \gamma_{\mathbf{p}_{\mathrm{A}}})(1 + \gamma_{\mathbf{p}_{\mathrm{B}}})(1 + \gamma_{L\mathbf{p}_{\mathrm{B}}})} - 1 \tag{4.51}$$

where $L\mathbf{p}_{\mathrm{B}}$ refers to the spatial part of $\left(L_{-\mathbf{p}_{\mathrm{A}}}\right)^{\mu}{}_{\nu}\, p^{\nu}_{\mathrm{B}}$, $\gamma_{\mathbf{p}_{\mathrm{A}}} = \sqrt{1 + \frac{\mathbf{p}^2_{\mathrm{A}}}{m^2_{\mathrm{A}} c^2}}$, $\gamma_{\mathbf{p}_{\mathrm{B}}} = \sqrt{1 + \frac{\mathbf{p}^2_{\mathrm{B}}}{m^2_{\mathrm{B}} c^2}}$ and $\gamma_{L\mathbf{p}_{\mathrm{B}}} = \gamma_{\mathbf{p}_{\mathrm{A}}}\gamma_{\mathbf{p}_{\mathrm{B}}}(1 - \boldsymbol{\beta}_{\mathbf{p}_{\mathrm{A}}} \cdot \boldsymbol{\beta}_{\mathbf{p}_{\mathrm{B}}})$.

Ultimately, the most significant question is the form of the Bell observable in the laboratory frame, $\hat{G}^{\mathbf{xy}}_{|\mathrm{C}} \equiv \hat{S}_2 \left(\mathbf{x} \cdot \hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{A}}} \otimes \mathbf{y} \cdot \hat{\boldsymbol{\Xi}}_{\hat{\boldsymbol{\pi}}_{\mathrm{B}}} \otimes \mathbb{1}_{\mathrm{C}}\right) \hat{S}_2^{\dagger}$, which is calculated in the following through its action on the laboratory state according to

$$\begin{aligned}
\hat{G}^{\mathbf{xy}}_{|\mathrm{C}} |\psi\rangle^{|\mathrm{C}} &= \hat{S}_2 \left(\mathbf{x} \cdot \hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{A}}} \otimes \mathbf{y} \cdot \hat{\boldsymbol{\Xi}}_{\hat{\pi}_{\mathrm{B}}} \otimes \mathbb{1}_{\mathrm{C}}\right) \hat{S}_2^{\dagger}\hat{S}_2 |\psi\rangle^{|\mathrm{A}} \\
&\overset{(4.28)}{=\!=} \hat{S}_2 \sum_{i,j} x_i y_j \sum_{a,a',b,b'} c_{ab} [\sigma^i]_{a'a} [\sigma^j]_{b'b} \int \mathrm{d}\mu_{\mathrm{B}}(\pi_{\mathrm{B}})\, \eta(\boldsymbol{\pi}_{\mathrm{B}})\, |a'\rangle_{\tilde{\mathrm{A}}}\, |\boldsymbol{\pi}_{\mathrm{B}}; \Sigma(b')\rangle_{\mathbf{B}} \otimes |\phi\rangle_{\mathrm{C}} \\
&= \hat{S}_2 \sum_{i,j,a,b,a',b'} x_i y_j c_{ab} [\sigma^i]_{a'a} [\sigma^j]_{b'b} \int \mathrm{d}\mu_{\mathrm{B}}(\pi_{\mathrm{B}}) \mathrm{d}\mu_{\mathrm{C}}(\pi_{\mathrm{C}})\, \eta(\boldsymbol{\pi}_{\mathrm{B}}) \phi\left(\boldsymbol{\pi}_{\mathrm{C}}\right) \\
&\qquad\qquad |a'\rangle_{\tilde{\mathrm{A}}}\, |\boldsymbol{\pi}_{\mathrm{B}}; \Sigma(b')\rangle_{\mathbf{B}}\, |\boldsymbol{\pi}_{\mathrm{C}}\rangle_{\mathrm{C}} \\
&= \sum_{i,j,a,b,a',b'} x_i y_j c_{ab} [\sigma^i]_{a'a} [\sigma^j]_{b'b} \int \mathrm{d}\mu_{\mathrm{A}}(p_{\mathrm{A}}) \mathrm{d}\mu_{\mathrm{B}}(p_{\mathrm{B}})\, \eta(L^{-1}\mathbf{p}_{\mathrm{B}}) \phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\mathbf{p}_{\mathrm{A}}\right) \\
&\qquad\qquad |\mathbf{p}_{\mathrm{A}}; \Sigma(a')\rangle_{\mathbf{A}}\, |\mathbf{p}_{\mathrm{B}}; \Sigma(R_{\boldsymbol{\Omega}}(b'))\rangle_{\mathbf{B}} \\
&= \sum_{i,j,a,b} x_i y_j c_{ab} \int \mathrm{d}\mu_{\mathrm{A}}(p_{\mathrm{A}}) \mathrm{d}\mu_{\mathrm{B}}(p_{\mathrm{B}})\, \eta(L^{-1}\mathbf{p}_{\mathrm{B}}) \phi\left(-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\mathbf{p}_{\mathrm{A}}\right) \\
&\qquad\qquad \sum_{a'} [\sigma^i]_{a'a}\, |\mathbf{p}_{\mathrm{A}}; \Sigma(a')\rangle_{\mathbf{A}} \otimes \sum_{b'} [\sigma^j]_{b'b}\, |\mathbf{p}_{\mathrm{B}}; \Sigma(R_{\boldsymbol{\Omega}}(b'))\rangle_{\mathbf{B}}
\end{aligned} \tag{4.52}$$

where for the Dirac particle $\mathbf{A}$ we know already that $\sum_{a'}[\sigma^i]_{a'a}\,|\mathbf{p}_\mathrm{A};\Sigma(a')\rangle_\mathbf{A} = \hat{\Xi}^i_{\hat{\mathbf{p}}_\mathrm{A}}\,|\mathbf{p}_\mathrm{A};\Sigma(a)\rangle_\mathbf{A}$ and hence

$$\begin{aligned}\hat{G}^{\mathbf{xy}}_{|\mathrm{C}}\,|\psi\rangle^{|\mathrm{C}} = \sum_{a,b} c_{ab}\int & \mathrm{d}\mu_\mathrm{A}(p_\mathrm{A})\mathrm{d}\mu_\mathrm{B}(p_\mathrm{B})\,\eta(L^{-1}\mathbf{p}_\mathrm{B})\phi\left(-\tfrac{m_\mathrm{C}}{m_\mathrm{A}}\mathbf{p}_\mathrm{A}\right) \\ & \mathbf{x}\cdot\hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}_\mathrm{A}}\,|\mathbf{p}_\mathrm{A};\Sigma(a)\rangle_\mathbf{A}\otimes\sum_{j,b'} y_j[\sigma^j]_{b'b}\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b'))\rangle_\mathbf{B}\,.\end{aligned} \tag{4.53}$$

For the Dirac particle $\mathbf{B}$ we find

$$\begin{aligned}\sum_{j,b'} y_j[\sigma^j]_{b'b}\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b'))\rangle_\mathbf{B} &= \sum_{j,b'} y_j[\sigma^j]_{b'b}\hat{U}(L_{\mathbf{p}_\mathrm{B}})\,|0;R_\mathbf{\Omega}(b')\rangle_\mathbf{B} \\ &= \sum_{j,b'} y_j[\sigma^j]_{b'b}\hat{U}(L_{\mathbf{p}_\mathrm{B}})\left[\mathbb{1}_\mathrm{B}\otimes\hat{R}_{\tilde{\mathrm{B}}}(\mathbf{\Omega})\right]|0;b'\rangle_\mathbf{B} \\ &= \hat{U}(L_{\mathbf{p}_\mathrm{B}})\left[\mathbb{1}_\mathrm{B}\otimes\hat{R}_{\tilde{\mathrm{B}}}(\mathbf{\Omega})\right](\mathbb{1}_\mathrm{B}\otimes\mathbf{y}\cdot\hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{B}}})\,|0;b\rangle_\mathbf{B}\end{aligned} \tag{4.54}$$

and by inserting $\mathbb{1}_\mathbf{B} = \mathbb{1}_{\mathrm{B}\tilde{\mathrm{B}}} = [\mathbb{1}_\mathrm{B}\otimes\hat{R}^\dagger_{\tilde{\mathrm{B}}}(\mathbf{\Omega})]\hat{U}^\dagger(L_{\mathbf{p}_\mathrm{B}})\hat{U}(L_{\mathbf{p}_\mathrm{B}})[\mathbb{1}_\mathrm{B}\otimes\hat{R}_{\tilde{\mathrm{B}}}(\mathbf{\Omega})]$ we get

$$\begin{aligned}&\sum_{j,b'} y_j[\sigma^j]_{b'b}\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b'))\rangle_\mathbf{B} \\ &= \hat{U}(L_{\mathbf{p}_\mathrm{B}})\left[\mathbb{1}_\mathrm{B}\otimes\hat{R}_{\tilde{\mathrm{B}}}(\mathbf{\Omega})\right](\mathbb{1}_\mathrm{B}\otimes\mathbf{y}\cdot\hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{B}}})\left[\mathbb{1}_\mathrm{B}\otimes\hat{R}^\dagger_{\tilde{\mathrm{B}}}(\mathbf{\Omega})\right]\hat{U}^\dagger(L_{\mathbf{p}_\mathrm{B}})\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b))\rangle_\mathbf{B} \\ &= \hat{U}(L_{\mathbf{p}_\mathrm{B}})\left[\mathbb{1}_\mathrm{B}\otimes\hat{R}_{\tilde{\mathrm{B}}}(\mathbf{\Omega})(\mathbf{y}\cdot\hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{B}}})\hat{R}^\dagger_{\tilde{\mathrm{B}}}(\mathbf{\Omega})\right]\hat{U}^\dagger(L_{\mathbf{p}_\mathrm{B}})\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b))\rangle_\mathbf{B}\,.\end{aligned} \tag{4.55}$$

Its is straight forward to show that $\hat{R}_{\tilde{\mathrm{B}}}(\mathbf{\Omega})(\mathbf{y}\cdot\hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{B}}})\hat{R}^\dagger_{\tilde{\mathrm{B}}}(\mathbf{\Omega}) = \mathbf{y}^R\cdot\hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{B}}}$ where

$$\mathbf{y}^R = \mathbf{y}^R(\mathbf{\Omega}) = \mathbf{y}\cos\Omega + \mathbf{n}(\mathbf{n}\cdot\mathbf{y})(1-\cos\Omega) + (\mathbf{n}\times\mathbf{y})\sin\Omega \tag{4.56}$$

and $\mathbf{\Omega} = \Omega\mathbf{n}$ [48]. This means that the measurement setting for the Dirac particle $\mathbf{B}$ in the laboratory frame is rotated with respect to the setting in the rest frame of $\mathbf{A}$. With this, it follows that

$$\begin{aligned}\sum_{j,b'} y_j[\sigma^j]_{b'b}\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b'))\rangle_\mathbf{B} &= \hat{U}(L_{\mathbf{p}_\mathrm{B}})(\mathbb{1}_\mathrm{B}\otimes\mathbf{y}^R\cdot\hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{B}}})\hat{U}^\dagger(L_{\mathbf{p}_\mathrm{B}})\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b))\rangle_\mathbf{B} \\ &= \sum_j y^R_j\,\hat{U}(L_{\mathbf{p}_\mathrm{B}})(\mathbb{1}_\mathrm{B}\otimes\hat{\sigma}^j_{\tilde{\mathrm{B}}})\hat{U}^\dagger(L_{\mathbf{p}_\mathrm{B}})\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b))\rangle_\mathbf{B} \\ &= \sum_j y^R_j\,\hat{\Xi}^j_{\hat{\mathbf{p}}_\mathrm{B}}\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b))\rangle_\mathbf{B} \\ &= \mathbf{y}^R\cdot\hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}_\mathrm{B}}\,|\mathbf{p}_\mathrm{B};\Sigma(R_\mathbf{\Omega}(b))\rangle_\mathbf{B}\end{aligned} \tag{4.57}$$

and with $\mathbf{y}^R = \mathbf{y}^R(\boldsymbol{\Omega}) = \mathbf{y}^R(\mathbf{p}_\mathrm{A}, \mathbf{p}_\mathrm{B})$ we obtain

$$\hat{G}^{\mathbf{xy}}_{|\mathrm{C}} \left|\psi\right\rangle^{|\mathrm{C}} = \sum_{a,b} c_{ab} \int \mathrm{d}\mu_\mathrm{A}(p_\mathrm{A}) \mathrm{d}\mu_\mathrm{B}(p_\mathrm{B})\, \eta(L^{-1}\mathbf{p}_\mathrm{B}) \phi\left(-\tfrac{m_\mathrm{C}}{m_\mathrm{A}} \mathbf{p}_\mathrm{A}\right) \\ \mathbf{x} \cdot \hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}_\mathrm{A}} \left|\mathbf{p}_\mathrm{A}; \Sigma(a)\right\rangle_\mathbf{A} \otimes \mathbf{y}^R(\mathbf{p}_\mathrm{A}, \hat{\mathbf{p}}_\mathrm{B}) \cdot \hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}_\mathrm{B}} \left|\mathbf{p}_\mathrm{B}; \Sigma(R_\mathbf{\Omega}(b))\right\rangle_\mathbf{B} \tag{4.58}$$

where $\mathbf{p}_\mathrm{B}$ has been promoted to an operator which is possible since $\hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}_\mathrm{B}}$ does not change the momentum of the Dirac particle $\mathbf{B}$. However, if we promote $\mathbf{p}_\mathrm{A} \to \hat{\mathbf{p}}_\mathrm{A}$ in the argument of $\mathbf{y}^R$, then we cannot split the Bell observable in the laboratory frame into two observables $\hat{G}^{\mathbf{x}}_{\mathrm{A}\tilde{\mathrm{A}}} \in \mathcal{H}_\mathbf{A} \equiv \mathcal{H}_{\mathrm{A}\tilde{\mathrm{A}}}$ and $\hat{G}^{\mathbf{y}^R}_{\mathrm{B}\tilde{\mathrm{B}}} \in \mathcal{H}_\mathbf{B} \equiv \mathcal{H}_{\mathrm{B}\tilde{\mathrm{B}}}$ acting on particle $\mathbf{A}$ and $\mathbf{B}$ separately, i.e.

$$\begin{aligned} \hat{G}^{\mathbf{xy}}_{|\mathrm{C}} &\equiv \hat{S}_2 \left(\mathbf{x} \cdot \hat{\boldsymbol{\sigma}}_{\tilde{\mathrm{A}}} \otimes \mathbf{y} \cdot \hat{\boldsymbol{\Xi}}_{\hat{\boldsymbol{\pi}}_\mathrm{B}} \otimes \mathbb{1}_\mathrm{C}\right) \hat{S}_2^\dagger \\ &= \sum_j \left(\mathbf{x} \cdot \hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}_\mathrm{A}} \otimes \mathbb{1}_{\mathrm{A}\tilde{\mathrm{A}}}\right) \left(y_j^R(\hat{\mathbf{p}}_\mathrm{A}, \hat{\mathbf{p}}_\mathrm{B}) \otimes \mathbb{1}_{\tilde{\mathrm{A}}} \otimes \mathbb{1}_{\tilde{\mathrm{B}}}\right) \left(\mathbb{1}_{\mathrm{A}\tilde{\mathrm{A}}} \otimes \hat{\Xi}^j_{\hat{\mathbf{p}}_\mathrm{B}}\right) \\ &\neq \hat{G}^{\mathbf{x}}_{\mathrm{A}\tilde{\mathrm{A}}} \otimes \hat{G}^{\mathbf{y}^R}_{\mathrm{B}\tilde{\mathrm{B}}}. \end{aligned} \tag{4.59}$$

As a consequence, it is not immediately clear that we can perform a Bell test in the laboratory such that the locality assumption of Bell's theorem is fulfilled. Notice that the measurement setting $\mathbf{y}^R = \mathbf{y}^R(\hat{\mathbf{p}}_\mathrm{A}, \hat{\mathbf{p}}_\mathrm{B})$ is not fixed as it is the case in the rest frame, but it depends on the momenta $\mathbf{p}_\mathrm{A}$ and $\mathbf{p}_\mathrm{B}$ which results from a "coherent rotation" of the fixed measurement direction $\mathbf{y}$ (as seen by the rest frame of $\mathbf{A}$) by the angle $\Omega(\mathbf{p}_\mathrm{A}, \mathbf{p}_\mathrm{B})$ around the z-axis ($\mathbf{n} = \mathbf{e}_z$).

However, $\mathbf{y}^R(\hat{\mathbf{p}}_\mathrm{A}, \hat{\mathbf{p}}_\mathrm{B})$ does not depend on the choice of the spin measurement direction $\mathbf{x}$ (referring to particle $\mathbf{A}$). Thus, it might be possible to set up a proper Bell test, for example, if the momentum and the spin degrees of freedom are measured successively. A possible scheme could be that before the spin measurements are performed the momenta $\mathbf{p}_\mathrm{A}$ and $\mathbf{p}_\mathrm{B}$ are measured by Alice and Bob who are located next to the Dirac particles $\mathbf{A}$ and $\mathbf{B}$, respectively. Then, Alice sends her outcome $\mathbf{p}_\mathrm{A}$ to Bob. With the knowledge of $\mathbf{p}_\mathrm{A}$ and $\mathbf{p}_\mathrm{B}$, Bob can adjust his spin measurement settings, following eq. (4.56), according to

$$\mathbf{y}_1^R = \begin{pmatrix} -\sin\left[\Omega(\mathbf{p}_\mathrm{A}, \mathbf{p}_\mathrm{B})\right] \\ \cos\left[\Omega(\mathbf{p}_\mathrm{A}, \mathbf{p}_\mathrm{B})\right] \\ 0 \end{pmatrix} \quad \text{as well as} \quad \mathbf{y}_2^R = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv \mathbf{y}_1, \tag{4.60}$$

where the measurement settings $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$ and $\mathbf{y}_2$ as given in eq. (4.3) are considered here, and $\Omega(\mathbf{p}_\mathrm{A}, \mathbf{p}_\mathrm{B})$ can be calculated via eq. (4.51). Since the choice of the direction $\mathbf{y}_i^R$ is made after Alice signaled to Bob, the choices between the two measurement settings $(\mathbf{x}_1, \mathbf{x}_2)$ for Alice and $(\mathbf{y}_1^R, \mathbf{y}_2^R \equiv \mathbf{y}_2)$ for Bob are still space-like separated such that the locality assumption of Bell's theorem is satisfied. Notice that, according to this scheme, the momentum measurements will collapse the momentum superposition states such that we obtain sharp momenta and thus a sharp Wigner rotation instead of a superposition of Wigner rotations as the general result, see eq. (4.58) and (4.59), predicts.

4.6. Preservation of Spacetime Intervals

For a proper Bell test the two spin measurements must be performed in space-like separated regions so that there cannot be any signaling between the two spin systems. In the initially considered rest frame A we can set up the Bell experiment and impose that the two spin measurements on $\tilde{\mathrm{A}}$ and $\mathbf{B} \equiv \mathrm{B}\tilde{\mathrm{B}}$ are space-like separated. However, can the corresponding experiment in the laboratory frame also considered to be space-like separated? Or does the superposition of Lorentz boosts in some way mix up the space-like separation?

The QRF transformation $\hat{S}_{\mathrm{L}}$ is derived from an external perspective (section 3.4); consequently, the QRF transformation $\hat{S}_2$ which includes the second Dirac particle $\mathbf{B}$ can be related to an external perspective transforming from A's to C's rest frame via

$$\hat{S}_{\mathrm{ext}} = \hat{U}_{\mathrm{C}}(L_{\hat{\mathbf{p}}_{\mathrm{A}}/m_{\mathrm{A}}})\hat{U}_{\mathbf{A}}(L_{-\hat{\boldsymbol{\pi}}_{\mathrm{C}}/m_{\mathrm{C}}})\hat{U}_{\mathbf{B}}(L_{-\hat{\boldsymbol{\pi}}_{\mathrm{C}}/m_{\mathrm{C}}}). \tag{4.61}$$

This means that the two Dirac particles $\mathbf{A} \equiv \mathrm{A}\tilde{\mathrm{A}}$ and $\mathbf{B} \equiv \mathrm{B}\tilde{\mathrm{B}}$ are coherently boosted by the velocity of the laboratory C; in other words, both boosts are controlled by C's velocity. Thus, the spacetime coordinates assigned for two events (e.g. the two spin measurements in A's perspective) are coherently boosted by the same velocity. Consequently, the QRF transformation $\hat{S}_2 \equiv \hat{S}_{\mathrm{L}}\, \hat{U}_{\mathbf{B}}(L_{-\hat{\boldsymbol{\pi}}_{\mathrm{C}}/m_{\mathrm{C}}})$, derived from $\hat{S}_{\mathrm{ext}}$ and utilized for the Bell tests above, preserves spacetime intervals in a coherent manner. Significantly, this means that two space-like separated events in the initial QRF are also space-like separated in the final QRF.

5. Conclusion

In the present thesis, it has been shown how quantum reference frames can be used to devise a Bell test for two spin-1/2 particles moving in a superposition of relativistic velocities. In particular, the quantum reference frame formalism leads to a generalized notion of the rest frame of a quantum system which can be utilized to operationally define the Bell observable (joint spin observables) in the rest frame of one of the two spin particles. With the help of a relativistic quantum reference frame transformation, the corresponding Bell observable in the laboratory frame, where the spin particles move in a superposition of relativistic velocities, has been calculated for different state configurations. Significantly, it follows that the violation of the CHSH-Bell inequality is frame-independent in contrast to the Bell observable which is frame-dependent.

It has been shown that, as long as the relativistic motion of the two spin particles is collinear, the Bell observable in the laboratory frame factorizes into two momentum-dependent spin observables acting on each particle separately. This means that the joint spin measurements of the Bell test are represented by a product of observables in space-like separated regions such that the locality assumption of Bell's theorem is satisfied. However, if their relativistic motion is not collinear, the appearance of a superposition of Wigner rotations (depending on the superposed momenta of both particles) destroys the above mentioned factorization of the Bell observable as seen in the laboratory frame. For this case, a possible way how to perform a proper Bell test has been outlined, where the choices of the measurement settings are still made in space-like separated regions.

The second scenario, where the motion of the two particles is not collinear, has thrown up many questions in need of further investigation. For example, the agreement with the locality assumption needs to be analyzed more rigorously and the spin measurement in superposed (Wigner rotated) directions should be studied more thoroughly. Apart from this theoretical future line of research, the experimental implementation of the relativistic Stern-Gerlach apparatus would lead to significant insights. Since the rest frame of a quantum system is not directly accessible to us, the relativistic Stern-Gerlach apparatus could be used to experimentally test the generalized notion of the rest frame of a quantum system and the quantum reference frame approach itself.

A. Appendix

A.1. Translation Operator

In quantum mechanics textbooks, the usual translation operator is defined by its action on a position eigenstate via

$$\hat{T}_a |x\rangle \equiv |x-a\rangle \tag{A.1}$$

and can be represented by

$$\hat{T}_a = \mathrm{e}^{\mathrm{i}a\hat{p}/\hbar}. \tag{A.2}$$

This can be shown by utilizing the completeness relation $\mathbb{1} = \int \mathrm{d}p\,|p\rangle\langle p|$ and $\langle x|p\rangle = \mathrm{e}^{\mathrm{i}xp/\hbar}/\sqrt{2\pi\hbar}$ via

$$\begin{aligned}
\hat{T}_a |x\rangle &= \mathrm{e}^{\mathrm{i}a\hat{p}/\hbar}|x\rangle = \int_{-\infty}^{\infty}\mathrm{d}p\,|p\rangle \underbrace{\langle p|\,\mathrm{e}^{\mathrm{i}a\hat{p}/\hbar}}_{=\langle p|\mathrm{e}^{\mathrm{i}ap/\hbar}}|x\rangle = \int_{-\infty}^{\infty}\mathrm{d}p\,\langle p|x\rangle\,\mathrm{e}^{\mathrm{i}ap/\hbar}|p\rangle \\
&= \int_{-\infty}^{\infty}\mathrm{d}p\,\frac{\mathrm{e}^{-i(x-a)p/\hbar}}{\sqrt{2\pi\hbar}}|p\rangle = \int_{-\infty}^{\infty}\mathrm{d}p\,\langle p|x-a\rangle\,|p\rangle = \int_{-\infty}^{\infty}\mathrm{d}p\,|p\rangle\langle p|x-a\rangle \\
&= |x-a\rangle\,.
\end{aligned} \tag{A.3}$$

Moreover, we can shift momentum eigenstates according to

$$\mathrm{e}^{-\mathrm{i}b\hat{x}/\hbar}|p\rangle = |p-b\rangle \tag{A.4}$$

which can be shown analogously to eq. (A.3) by utilizing the completeness of the position basis $\mathbb{1} = \int \mathrm{d}x\,|x\rangle\langle x|$.

Note that the results above can be easily extended to all three spatial dimensions with $\langle\mathbf{x}|\mathbf{p}\rangle = (2\pi\hbar)^{-3/2}\,\mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{p}/\hbar}$ and $\mathbb{1} = \int \mathrm{d}^3\mathbf{x}\,|\mathbf{x}\rangle\langle\mathbf{x}| = \int \mathrm{d}^3\mathbf{p}\,|\mathbf{p}\rangle\langle\mathbf{p}|$ where $\mathbf{x} \equiv (x_1, x_2, x_3)$ and $\mathbf{p} \equiv (p_1, p_2, p_3)$.

A.2. Parity Operator

The unitary parity-swap operator $\hat{\mathcal{P}}_{\mathrm{AC}} : \mathcal{H}_{\mathrm{A}}^{|\mathrm{C}} \mapsto \mathcal{H}_{\mathrm{C}}^{|\mathrm{A}}$ is defined by

$$\hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{x}_{\mathrm{A}} \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \equiv -\hat{q}_{\mathrm{C}}. \tag{A.5}$$

From this definition and the required canonicity, i.e. $\mathrm{i}\hbar = [\hat{x}_{\mathrm{A}}, \hat{p}_{\mathrm{A}}] = [\hat{q}_{\mathrm{C}}, \hat{\pi}_{\mathrm{C}}]$, it follows that

$$\begin{aligned} \mathrm{i}\hbar &= \hat{\mathcal{P}}_{\mathrm{AC}}[\hat{x}_{\mathrm{A}}, \hat{p}_{\mathrm{A}}]\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} - \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \\ &= \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{x}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger}\, \hat{p}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} - \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger}\, \hat{x}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \\ &\overset{(A.5)}{=} -\hat{q}_{\mathrm{C}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger}\, \hat{p}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} + \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{q}_{\mathrm{C}} \overset{!}{=} \hat{q}_{\mathrm{C}}\hat{\pi}_{\mathrm{C}} - \hat{\pi}_{\mathrm{C}}\hat{q}_{\mathrm{C}} \equiv [\hat{q}_{\mathrm{C}}, \hat{\pi}_{\mathrm{C}}] \end{aligned} \tag{A.6}$$

and hence

$$\hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = -\hat{\pi}_{\mathrm{C}}. \tag{A.7}$$

The action of the parity-swap operator on position and momentum eigenstates is given by

$$\hat{\mathcal{P}}_{\mathrm{AC}}\, |x\rangle_{\mathrm{A}} = |{-x}\rangle_{\mathrm{C}} \quad \text{and} \quad \hat{\mathcal{P}}_{\mathrm{AC}}\, |p\rangle_{\mathrm{A}} = |{-p}\rangle_{\mathrm{C}} \tag{A.8}$$

which can be verified with the definition (A.5), and analogously with equation (A.7), via

$$\begin{aligned} \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{x}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} &= \hat{\mathcal{P}}_{\mathrm{AC}} \int_{-\infty}^{\infty} \mathrm{d}x\, |x\rangle_{\mathrm{A}}\langle x|\, \hat{x}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = \int_{-\infty}^{\infty} \mathrm{d}x\, x\, \hat{\mathcal{P}}_{\mathrm{AC}}\ |x\rangle_{\mathrm{A}}\langle x|\, \hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \\ &= \int_{-\infty}^{\infty} \mathrm{d}x\, x\ |{-x}\rangle_{\mathrm{C}}\langle -x| = -\int_{-\infty}^{\infty} \mathrm{d}q\, q\ |q\rangle_{\mathrm{C}}\langle q| = \int_{-\infty}^{\infty} \mathrm{d}x\, x\ |{-x}\rangle_{\mathrm{C}}\langle -x| \\ &= -\int_{-\infty}^{\infty} \mathrm{d}q\, \hat{q}_{\mathrm{C}}\ |q\rangle_{\mathrm{C}}\langle q| = -\hat{q}_{\mathrm{C}} \end{aligned} \tag{A.9}$$

where the completeness relations $\mathbb{1} = \int \mathrm{d}x\, |x\rangle_{\mathrm{A}}\langle x| = \int \mathrm{d}q\, |q\rangle_{\mathrm{C}}\langle q|$ have been utilized.

In order to identify immediately to which perspective the appearing operators belong, the convention

$$\hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{x}_{\mathrm{B}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \equiv \hat{q}_{\mathrm{B}} \quad \text{and} \quad \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{B}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} \equiv \hat{\pi}_{\mathrm{B}}. \tag{A.10}$$

is used and whenever the parity-swap operator takes its action on states the variables of describing the state of B are relabeled, specifically $(x_{\mathrm{B}}, p_{\mathrm{B}}) \to (q_{\mathrm{B}}, \pi_{\mathrm{B}})$.

A.3. The QRF Transformation of Relative Momenta

In section 2.1.1, the position basis has been used to express relative locations and by canonicity the corresponding momenta have been calculated. Analogously, the momentum basis can be used to express the momenta as the relational variables according to

$$\hat{p}_{\mathrm{B}} \mapsto \hat{\pi}_{\mathrm{B}} - \hat{\pi}_{\mathrm{C}} \quad \text{and} \quad \hat{p}_{\mathrm{A}} \mapsto -\hat{\pi}_{\mathrm{C}} \tag{A.11}$$

and from canonicity it follows that $\hat{x}_{\mathrm{B}} \mapsto \hat{q}_{\mathrm{B}}$ and $\hat{x}_{\mathrm{A}} \mapsto -(\hat{q}_{\mathrm{C}} + \hat{q}_{\mathrm{B}})$. It has already been pointed out that momenta are generating translations and due to complementarity it is immediate that shifts in momenta (boosts) are generated by position operators, i.e.

$$\mathrm{e}^{\mathrm{i}a\hat{p}_{\mathrm{B}}/\hbar} |x\rangle_{\mathrm{B}} = |x-a\rangle_{\mathrm{B}} \quad \text{and} \quad \mathrm{e}^{-\mathrm{i}b\hat{x}_{\mathrm{B}}/\hbar} |p\rangle_{\mathrm{B}} = |p-b\rangle_{\mathrm{B}} \,. \tag{A.12}$$

Accordingly, by exchanging the position and momentum operator of A and B in the QRF transformation of relative coordinates $\hat{S}_x$, eq. (2.4), we obtain the QRF transformation between relative momenta

$$\hat{S}_p \equiv \hat{\mathcal{P}}_{\mathrm{AC}}\, \mathrm{e}^{-\mathrm{i}\hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{B}}/\hbar} \tag{A.13}$$

where the parity-swap operator maps $\hat{\mathcal{P}}_{\mathrm{AC}}\hat{p}_{\mathrm{A}}\hat{\mathcal{P}}_{\mathrm{AC}}^{\dagger} = -\hat{\pi}_{\mathrm{C}}$ as derived in appendix A.2.

A.4. The QRF Transformation of Relative Velocities

The QRF transformation of relative velocities $\hat{S}_v : \mathcal{H}_{\mathrm{A}}^{|\mathrm{C}} \otimes \mathcal{H}_{\mathrm{B}}^{|\mathrm{C}} \mapsto \mathcal{H}_{\mathrm{B}}^{|\mathrm{A}} \otimes \mathcal{H}_{\mathrm{C}}^{|\mathrm{A}}$ defined by

$$\hat{S}_v \equiv \hat{\mathcal{P}}_{\mathrm{AC}}^{(v)} \exp\left\{-\tfrac{\mathrm{i}}{\hbar}\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}\, \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{B}}\right\} \tag{A.14}$$

is mapping

$$\hat{x}_{\mathrm{A}} \mapsto -\left(m_{\mathrm{B}}\hat{q}_{\mathrm{B}} + m_{\mathrm{C}}\hat{q}_{\mathrm{C}}\right)/m_{\mathrm{A}}, \qquad \hat{x}_{\mathrm{B}} \mapsto \hat{q}_{\mathrm{B}}, \tag{A.15}$$

$$\hat{p}_{\mathrm{A}} \mapsto -\tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\hat{\pi}_{\mathrm{C}}, \qquad \hat{p}_{\mathrm{B}} \mapsto \hat{\pi}_{\mathrm{B}} - \tfrac{m_{\mathrm{B}}}{m_{\mathrm{C}}}\hat{\pi}_{\mathrm{C}} \tag{A.16}$$

which is shown in the following.

The first operation, $\exp\left\{-\tfrac{\mathrm{i}}{\hbar}\tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}\, \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{B}}\right\}$, is coherently boosting B with the velocity of A. Its action can be calculated with Baker-Campbell-Hausdorff formula (2.8) analogously to the coherent translation case in section 2.1.1; they only differ by flipping the labels A and B plus an additional factor of relative masses. Hence,

$$\begin{aligned}
\hat{x}_{\mathrm{A}} &\mapsto \hat{S}_v\hat{x}_{\mathrm{A}}\hat{S}_v^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}^{(v)}\left(\hat{x}_{\mathrm{A}} - \tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}\hat{x}_{\mathrm{B}}\right)\left(\hat{\mathcal{P}}_{\mathrm{AC}}^{(v)}\right)^{\dagger}, \\
\hat{p}_{\mathrm{A}} &\mapsto \hat{S}_v\hat{p}_{\mathrm{A}}\hat{S}_v^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}^{(v)}\,\hat{p}_{\mathrm{A}}\left(\hat{\mathcal{P}}_{\mathrm{AC}}^{(v)}\right)^{\dagger}, \\
\hat{x}_{\mathrm{B}} &\mapsto \hat{S}_v\hat{x}_{\mathrm{B}}\hat{S}_v^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}^{(v)}\,\hat{x}_{\mathrm{B}}\left(\hat{\mathcal{P}}_{\mathrm{AC}}^{(v)}\right)^{\dagger}, \\
\hat{p}_{\mathrm{B}} &\mapsto \hat{S}_v\hat{p}_{\mathrm{B}}\hat{S}_v^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}^{(v)}\left(\hat{p}_{\mathrm{B}} + \tfrac{m_{\mathrm{B}}}{m_{\mathrm{A}}}\hat{p}_{\mathrm{A}}\right)\left(\hat{\mathcal{P}}_{\mathrm{AC}}^{(v)}\right)^{\dagger}.
\end{aligned} \tag{A.17}$$

In order to keep the calculations compact, consider now only the action of $\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}$ on the phase space observables of C, i.e. on $\hat{x}_{\mathrm{A}}, \hat{p}_{\mathrm{A}}, \hat{x}_{\mathrm{B}}, \hat{p}_{\mathrm{B}}$, and define $\eta \equiv \frac{\mathrm{i}}{\hbar} \log \sqrt{\frac{m_{\mathrm{C}}}{m_{\mathrm{A}}}} = \frac{\mathrm{i}}{2\hbar} \log \frac{m_{\mathrm{C}}}{m_{\mathrm{A}}}$. Thus,

$$\begin{aligned}\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\, \hat{x}_{\mathrm{A}} \left(\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\right)^{\dagger} &= \hat{\mathcal{P}}_{\mathrm{AC}} \exp\left\{\eta\left(\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\right)\right\} \hat{x}_{\mathrm{A}} \exp\left\{-\eta\left(\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\right)\right\} \hat{\mathcal{P}}^{\dagger}_{\mathrm{AC}} \\ &\overset{(2.8)}{=} \hat{\mathcal{P}}_{\mathrm{AC}} \left(\hat{x}_{\mathrm{A}} + \left[\eta\left(\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\right), \hat{x}_{\mathrm{A}}\right] + \ldots\right) \hat{\mathcal{P}}^{\dagger}_{\mathrm{AC}},\end{aligned} \tag{A.18}$$

where by utilizing Heisenberg's indeterminacy relation in accordance with $\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}} = 2\hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}} + [\hat{x}_{\mathrm{A}}, \hat{p}_{\mathrm{A}}] = 2\hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}} + \mathrm{i}\hbar$, it follows that

$$\left[\eta\left(\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\right), \hat{x}_{\mathrm{A}}\right] = 2\eta[\hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}, \hat{x}_{\mathrm{A}}] = 2\eta[\hat{p}_{\mathrm{A}}, \hat{x}_{\mathrm{A}}]\hat{x}_{\mathrm{A}} = -2\mathrm{i}\hbar\eta\hat{x}_{\mathrm{A}} \tag{A.19}$$

and hence

$$\begin{aligned}\hat{x}_{\mathrm{A}} + \left[\eta\left(\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\right), \hat{x}_{\mathrm{A}}\right] + \ldots &= \hat{x}_{\mathrm{A}} - 2\mathrm{i}\hbar\eta\hat{x}_{\mathrm{A}} + \tfrac{1}{2!}\left[\eta\left(\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\right), -2\mathrm{i}\hbar\eta\hat{x}_{\mathrm{A}}\right] + \ldots \\ &= \hat{x}_{\mathrm{A}}\left(1 + (-2\mathrm{i}\hbar\eta) + (-2\mathrm{i}\hbar\eta)^2/2! + \ldots\right) \\ &= \hat{x}_{\mathrm{A}}\mathrm{e}^{-2\mathrm{i}\hbar\eta} = \hat{x}_{\mathrm{A}} \exp\left\{\log \tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\right\} \\ &= \tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\hat{x}_{\mathrm{A}}.\end{aligned} \tag{A.20}$$

Consequently,

$$\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\, \hat{x}_{\mathrm{A}} \left(\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\right)^{\dagger} = \tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\, \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{x}_{\mathrm{A}}\hat{\mathcal{P}}^{\dagger}_{\mathrm{AC}} = -\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\hat{q}_{\mathrm{C}} \tag{A.21}$$

and as already shown by construction of the generalized parity-swap operator in section 2.1.2, it is

$$\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\, \hat{p}_{\mathrm{A}} \left(\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\right)^{\dagger} = \tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\, \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{A}}\hat{\mathcal{P}}^{\dagger}_{\mathrm{AC}} = -\tfrac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\hat{\pi}_{\mathrm{C}}. \tag{A.22}$$

Since $\hat{x}_{\mathrm{B}}$ and $\hat{p}_{\mathrm{B}}$ commute with the scaling operator $\exp\left\{\eta\left(\hat{x}_{\mathrm{A}}\hat{p}_{\mathrm{A}} + \hat{p}_{\mathrm{A}}\hat{x}_{\mathrm{A}}\right)\right\}$ it is immediate that

$$\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\, \hat{x}_{\mathrm{B}} \left(\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\right)^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{x}_{\mathrm{B}}\hat{\mathcal{P}}^{\dagger}_{\mathrm{AC}} = \hat{q}_{\mathrm{B}} \tag{A.23}$$

and

$$\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\, \hat{p}_{\mathrm{B}} \left(\hat{\mathcal{P}}^{(v)}_{\mathrm{AC}}\right)^{\dagger} = \hat{\mathcal{P}}_{\mathrm{AC}}\, \hat{p}_{\mathrm{B}}\hat{\mathcal{P}}^{\dagger}_{\mathrm{AC}} = \hat{\pi}_{\mathrm{B}}. \tag{A.24}$$

A.5. Explicit Form of the Relativistic Spin Operator

In special relativity, the spin operator $\hat{\mathbf{S}}$ in the rest frame of a quantum system can be described with the four-spin $\hat{S}^{\mu}_{\mathrm{R}} = (0, \hat{\mathbf{S}})$; thus, the spin operator as seen from an inertial frame $\hat{S}^{\mu}$ (following classical trajectories) is obtained with a Lorentz boost via

$\hat{S}^{\mu} = (L_{\mathbf{p}})^{\mu}{}_{\nu} \hat{S}^{\nu}_{\mathrm{R}}$ where $L_{\mathbf{p}}$ is defined by

$$L_{\mathbf{p}} \equiv L_{\frac{\mathbf{p}}{m}} = \begin{pmatrix} \frac{p^0}{mc} & \frac{\mathbf{p}^{\top}}{mc} \\ \frac{\mathbf{p}}{mc} & \mathbb{1} + \frac{1}{\gamma+1}\frac{\mathbf{p}\mathbf{p}^{\top}}{(mc)^2} \end{pmatrix} \tag{A.25}$$

with $\gamma = \gamma_{\mathbf{p}} = \gamma_{\mathbf{p}/m} = \sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}}$ and $p^0 = mc\gamma = \sqrt{m^2c^2 + \mathbf{p}^2} = p^0(|\mathbf{p}|)$. [39]

When the Pauli operator $\hat{\boldsymbol{\sigma}} \equiv (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$ is utilized as spin operator[1] in the rest frame, i.e. $\hat{\sigma}^{\mu} = (0, \hat{\boldsymbol{\sigma}})$, then the spin as seen from another inertial frame is given (up to a factor m) by the manifestly covariant Pauli-Lubański spin operator $\hat{\Sigma}^{\mu}_{\mathbf{p}} \equiv (\hat{\Sigma}^0_{\mathbf{p}}, \hat{\boldsymbol{\Sigma}}_{\mathbf{p}}) = (L_{-\mathbf{p}})^{\mu}{}_{\nu}\hat{\sigma}^{\nu}$. Here, the sign of $\mathbf{p}$ has been adjusted for consistency with the main text where the parameter of the Lorentz boost of the Dirac particle **A** has a negative sign, see (3.15). Simply applying the Lorentz boost $L_{-\mathbf{p}}$ leads to the explicit form

$$\hat{\Sigma}^0_{\mathbf{p}} = \gamma_{\mathbf{p}} \left(\boldsymbol{\beta}_{\mathbf{p}} \cdot \hat{\boldsymbol{\sigma}}\right) \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_{\mathbf{p}} = \hat{\boldsymbol{\sigma}} + \frac{\gamma_{\mathbf{p}}^2}{\gamma_{\mathbf{p}} + 1} \left(\boldsymbol{\beta}_{\mathbf{p}} \cdot \hat{\boldsymbol{\sigma}}\right) \boldsymbol{\beta}_{\mathbf{p}} \tag{A.26}$$

where $\boldsymbol{\beta}_{\mathbf{p}} \equiv \boldsymbol{\beta}_{\mathbf{p}/m} = \frac{\mathbf{p}}{\sqrt{m^2c^2+\mathbf{p}^2}}$ and hence $\hat{\Sigma}^{\mu}_{\mathbf{p}} \equiv \hat{\Sigma}^{\mu}_{\mathbf{p}/m}$. It is important to notice that the four-spin does not introduce extra degrees of freedom due to the covariant constraint $\eta_{\mu\nu}p^{\mu}\hat{\Sigma}^{\nu}_{\mathbf{p}} = \eta_{\mu\nu}k^{\mu}\hat{\sigma}^{\nu} = 0$ where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Crucially, this result stays the same when the appearing momenta are promoted to operators, $\mathbf{p} \to \hat{\mathbf{p}}$, which is utilized in this work and noted accordingly with a corresponding index, e.g. $\hat{\Sigma}^{\mu}_{\hat{\mathbf{p}}} \equiv \hat{\Sigma}^{\mu}_{\hat{\mathbf{p}}/m}$.

The explicit form of the relativistic spin operator $\hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}}$ can be derived via its action on a one-particle eigenstate, $|\mathbf{p}; \Sigma(\lambda)\rangle$ with $\lambda = \pm z$, according to

$$\begin{aligned} \hat{\Xi}^i_{\hat{\mathbf{p}}_{\mathrm{A}}} |\mathbf{p}; \Sigma(\lambda)\rangle_{\mathrm{A}\tilde{\mathrm{A}}} &\equiv \hat{S}_{\mathrm{L}}(\hat{\sigma}^i_{\tilde{\mathrm{A}}} \otimes \mathbb{1}_{\mathrm{C}})\hat{S}^{\dagger}_{\mathrm{L}}\hat{S}_{\mathrm{L}} |\lambda\rangle_{\tilde{\mathrm{A}}} \left|-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\mathbf{p}\right\rangle_{\mathrm{C}} \\ &= \hat{S}_{\mathrm{L}} \sum_{a=\pm z} [\sigma^i]_{a,\lambda} |a\rangle_{\tilde{\mathrm{A}}} \left|-\tfrac{m_{\mathrm{C}}}{m_{\mathrm{A}}}\mathbf{p}\right\rangle_{\mathrm{C}} \\ &= \sum_{a=\pm z} [\sigma^i]_{a,\lambda} |\mathbf{p}; \Sigma(a)\rangle_{\mathrm{A}\tilde{\mathrm{A}}} \end{aligned} \tag{A.27}$$

where $[\sigma^i]_{a,\lambda}$ is a matrix element of the corresponding Pauli matrix written in the $\hat{\sigma}^z$-basis, i.e.

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.28}$$

[1] Up to the factor $\hbar/2$.

By applying a standard Lorentz boost, specifically $\hat{U}^\dagger(L_{\mathbf{p}})\,|\mathbf{p};\Sigma(a)\rangle = |\mathbf{0};a\rangle = |\mathbf{0}\rangle\,|a\rangle$, it is

$$\begin{aligned}\hat{\Xi}^i_{\hat{\mathbf{p}}_{\mathrm{A}}}\,|\mathbf{p};\Sigma(\lambda)\rangle_{\mathrm{A}\tilde{\mathrm{A}}} &= \sum_a [\sigma^i]_{a,\lambda}\hat{U}_{\mathbf{A}}(L_{\mathbf{p}})\hat{U}^\dagger_{\mathbf{A}}(L_{\mathbf{p}})\,|\mathbf{p};\Sigma(a)\rangle_{\mathbf{A}} \\ &= \hat{U}_{\mathbf{A}}(L_{\mathbf{p}})\sum_a [\sigma^i]_{a,\lambda}\,|\mathbf{0};a\rangle_{\mathbf{A}} \\ &= \hat{U}_{\mathbf{A}}(L_{\mathbf{p}})\left(\mathbb{1}_{\mathrm{A}}\otimes\hat{\sigma}^i_{\tilde{\mathrm{A}}}\right)|\mathbf{0};\lambda\rangle_{\mathbf{A}}\,,\end{aligned} \tag{A.29}$$

where $\mathbf{A}\equiv\mathrm{A}\tilde{\mathrm{A}}$.[2] Notice that the Pauli-Lubański operator coincides with the non-relativistic Pauli operator when it is applied on a zero-momentum state. It follows that

$$\begin{aligned}\hat{\Xi}^i_{\hat{\mathbf{p}}_{\mathrm{A}}}\,|\mathbf{p};\Sigma(\lambda)\rangle_{\mathrm{A}\tilde{\mathrm{A}}} &= \hat{U}(L_{\mathbf{p}})\hat{\Sigma}^i_{\hat{\mathbf{p}}_{\mathrm{A}}}\,|\mathbf{0};\lambda\rangle_{\mathbf{A}} = \hat{U}(L_{\mathbf{p}})\hat{\Sigma}^i_{\hat{\mathbf{p}}_{\mathrm{A}}}\hat{U}^\dagger(L_{\mathbf{p}})\hat{U}(L_{\mathbf{p}})\,|\mathbf{0};\lambda\rangle_{\mathbf{A}} \\ &= (L_{\mathbf{p}})^i{}_\nu\hat{\Sigma}^\nu_{\hat{\mathbf{p}}_{\mathrm{A}}}\,|\mathbf{p};\Sigma(\lambda)\rangle_{\mathbf{A}} = (L_{\hat{\mathbf{p}}_{\mathrm{A}}})^i{}_\nu\hat{\Sigma}^\nu_{\hat{\mathbf{p}}_{\mathrm{A}}}\,|\mathbf{p};\Sigma(\lambda)\rangle_{\mathbf{A}}\end{aligned} \tag{A.30}$$

where in the last step the parameter of the Lorentz boost has been promoted to an operator; this is possible because the Pauli-Lubański operator leaves the momentum of the one-particle eigenstate invariant.

Consequently, by applying $\hat{\Xi}^i_{\hat{\mathbf{p}}} = (L_{\hat{\mathbf{p}}})^i{}_\nu\hat{\Sigma}^\nu_{\hat{\mathbf{p}}}$ and exploiting the covariant constraint $\eta_{\mu\nu}\hat{p}^\mu\hat{\Sigma}^\nu_{\hat{\mathbf{p}}} = 0$, we find

$$\hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}} = \hat{\boldsymbol{\Sigma}}_{\hat{\mathbf{p}}} - \frac{\gamma_{\hat{\mathbf{p}}}}{\gamma_{\hat{\mathbf{p}}}+1}\left(\hat{\boldsymbol{\Sigma}}_{\hat{\mathbf{p}}}\cdot\boldsymbol{\beta}_{\hat{\mathbf{p}}}\right)\boldsymbol{\beta}_{\hat{\mathbf{p}}}. \tag{A.31}$$

It is important to notice that $\hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}} \equiv \hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}/m}$ acts on both the external (momentum) and internal (spin) degrees of freedom.

A.6. Wigner Rotations

As introduced in section 3.3, a relativistic spin state for a Dirac particle with mass $m>0$ is given by

$$|\psi\rangle = \sum_\sigma \int \mathrm{d}\mu(\mathbf{p})\psi_\sigma(\mathbf{p})\,|\mathbf{p};\Sigma_{\mathbf{p}}(\sigma)\rangle \tag{A.32}$$

where σ denotes the spin in the rest frame of the Dirac particle, $\mathbf{p}$ the momentum, $\mathrm{d}\mu(\mathbf{p})$ the Lorentz-invariant integration measure and $\psi_\sigma(\mathbf{p}) = \langle\mathbf{p};\Sigma_{\mathbf{p}}(\sigma)|\psi\rangle$ the wave function. The basis elements are defined via standard Lorentz boosts according to

$$|\mathbf{p};\Sigma_{\mathbf{p}}(\sigma)\rangle \equiv \hat{U}(L_{\mathbf{p}})\,|\mathbf{0};\sigma\rangle \tag{A.33}$$

where $\hat{U}(L_{\mathbf{p}})$ is a unitary representation of a pure boost $L_{\mathbf{p}}$ taking the four-momentum of the Dirac particle from $k^\mu = (mc,\mathbf{0})$ to $p^\mu = (L_{\mathbf{p}})^\mu{}_\nu k^\nu = (p^0,\mathbf{p})$ with $p^0 = p^0(|\mathbf{p}|) = \sqrt{m^2c^2+\mathbf{p}^2}$. It is important to notice that in the particle's rest frame the one-particle state is separable, i.e. $|\mathbf{0};\sigma\rangle = |\mathbf{0}\rangle\,|\sigma\rangle$.

[2] By inserting an identity, this allows to identify $\hat{\boldsymbol{\Xi}}_{\hat{\mathbf{p}}} = \hat{U}(L_{\hat{\mathbf{p}}})(\mathbb{1}\otimes\hat{\boldsymbol{\sigma}})\hat{U}^\dagger(L_{\hat{\mathbf{p}}})$.

If the state $|\mathbf{p};\Sigma_{\mathbf{p}}(\sigma)\rangle$ gets boosted by the momentum $\mathbf{q}$ we obtain two successive Lorentz boosts with respect to the rest frame of the particle according to $\hat{U}(L_{\mathbf{q}})\hat{U}(L_{\mathbf{p}})\,|\mathbf{0};\sigma\rangle$, where the corresponding four-momentum is given by $(L_{\mathbf{q}})^{\mu}_{\ \nu}\,p^{\nu} \equiv w^{\mu} = (w^0,\mathbf{w})$. Surprisingly, two successive non-collinear boosts, i.e. $\mathbf{q} \nparallel \mathbf{p}$, lead to one boost by the momentum $\mathbf{w}$ and an additional spatial rotation which is called *Wigner rotation*. This follows from a straight forward matrix multiplication of the two Lorentz boosts [47] as well as from group properties of the Poincaré group (see e.g. [44]) according to

$$\hat{U}(L_{\mathbf{q}})\hat{U}(L_{\mathbf{p}}) = \hat{U}(L_{\mathbf{w}})\hat{U}^{-1}(L_{\mathbf{w}})\hat{U}(L_{\mathbf{q}})\hat{U}(L_{\mathbf{p}}) = \hat{U}(L_{\mathbf{w}})\hat{U}(L_{\mathbf{w}}^{-1}L_{\mathbf{q}}L_{\mathbf{p}}), \tag{A.34}$$

where it is immediate that $W \equiv L_{\mathbf{w}}^{-1}L_{\mathbf{q}}L_{\mathbf{p}}$ can at most be a spatial rotation because it takes $k^{\mu} \to p^{\mu} \to w^{\mu} = (L_{\mathbf{q}})^{\mu}_{\ \nu}\,p^{\nu}$ and back to k^{μ}. If the two momenta $\mathbf{p}$ and $\mathbf{q}$ are aligned then $W = \mathbb{1}$, i.e. two successive collinear boosts result in a single boost. The Wigner rotation rotation $W = W(\mathbf{q},\mathbf{p})$ can be specified through the rotation axis

$$\mathbf{n} = \frac{\mathbf{q}\times\mathbf{p}}{|\mathbf{q}\times\mathbf{p}|}, \quad |\mathbf{n}| = 1 \tag{A.35}$$

and the rotation angle Ω given in accordance with

$$\cos\Omega = \frac{1+\gamma_{\mathbf{p}}+\gamma_{\mathbf{q}}+\gamma_{\mathbf{w}}}{(1+\gamma_{\mathbf{p}})(1+\gamma_{\mathbf{q}})(1+\gamma_{\mathbf{w}})} - 1 \tag{A.36}$$

where $\gamma_{\mathbf{p}} = \gamma_{\mathbf{p}/m_p} = \sqrt{1+\frac{\mathbf{p}^2}{m_p^2c^2}}$, $\gamma_{\mathbf{q}} = \gamma_{\mathbf{q}/m_q} = \sqrt{1+\frac{\mathbf{q}^2}{m_q^2c^2}}$ and $\gamma_{\mathbf{w}} = \gamma_{\mathbf{p}}\gamma_{\mathbf{q}}(1+\boldsymbol{\beta}_{\mathbf{p}}\cdot\boldsymbol{\beta}_{\mathbf{q}})$ with $\boldsymbol{\beta}_{\mathbf{p}} \equiv \boldsymbol{\beta}_{\mathbf{p}/m_p} = \frac{\mathbf{p}}{\sqrt{m_p^2c^2+\mathbf{p}^2}}$ and $\boldsymbol{\beta}_{\mathbf{q}} \equiv \boldsymbol{\beta}_{\mathbf{q}/m_q} = \frac{\mathbf{q}}{\sqrt{m_q^2c^2+\mathbf{q}^2}}$, see [47]. Hence, $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mathbf{p},\mathbf{q}) = \Omega\mathbf{n}$ completely determines the Wigner rotation.

In our case the Wigner rotation is applied to a zero-momentum state such that the rotation only affects the spin degree of freedom. Thus,

$$\begin{aligned}\hat{U}(L_{\mathbf{q}})\,|\mathbf{p};\Sigma_{\mathbf{p}}(\sigma)\rangle &\equiv \hat{U}(L_{\mathbf{q}})\hat{U}(L_{\mathbf{p}})\,|\mathbf{0};\sigma\rangle = \hat{U}(L_{\mathbf{w}})\hat{U}(W)\,|\mathbf{0};\sigma\rangle \\ &= \hat{U}(L_{\mathbf{w}})\left[\mathbb{1}\otimes\hat{R}(\boldsymbol{\Omega})\right]|\mathbf{0};\sigma\rangle = \hat{U}(L_{\mathbf{w}})\,|\mathbf{0};R_{\boldsymbol{\Omega}}(\sigma)\rangle \\ &= |\mathbf{w};\Sigma_{\mathbf{w}}(R_{\boldsymbol{\Omega}}(\sigma))\rangle\end{aligned} \tag{A.37}$$

where $R_{\boldsymbol{\Omega}}(\sigma)$ refers to the Wigner rotated spin state and the rotation operator is given by

$$\hat{R}(\boldsymbol{\Omega}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\Omega}\cdot\hat{\boldsymbol{\sigma}}/2} = \mathbb{1}\cos(\Omega/2) - \mathrm{i}(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\sin(\Omega/2) \tag{A.38}$$

with the Pauli operator $\hat{\boldsymbol{\sigma}} \equiv (\hat{\sigma}^x,\hat{\sigma}^y,\hat{\sigma}^z)$.

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