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Tobias Beran, BSc

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Ἀγεωμέτρητος μηδεὶς εἰσιτω [1]

[1]Said to have been written above the door of Plato's academy

Abstract

Lorentzian pre-length spaces are introduced by analogy to metric spaces, giving a synthetic generalization of spacetimes based on the time distance. Some causality conditions are transferred to this setting. A way of making Lorentzian pre-length spaces intrinsic is shown (in analogy to length spaces). Regularly localizable Lorentzian pre-length spaces are introduced to get length increasing push-up of causal curves.

For intrinsic Lorentzian pre-length spaces, curvature comparison by timelike triangles is introduced by analogy to triangle comparison in intrinsic metric spaces. Hyperbolic angles in intrinsic Lorentzian pre-length spaces are defined by analogy to hyperbolic angles in Lorentzian manifolds and angles in intrinsic metric spaces. The no branching result in spaces of curvature bounded below is reproducible using further weak conditions.

Zusammenfassung

Lorentzsche vor-Längenräume werden per Analogie zu metrischen Räumen eingeführt, sie sind eine synthetische Verallgemeinerung von Raumzeiten, der auf Zeit-Abständen beruht. Hier werden einige Kausalitätsbedingungen auf Lorentzsche vor-Längenräume übertragen. Es wird auch eine Art angegeben, Lorentzsche vor-Längenräume intrinsisch zu machen (bei metrischen Räumen entspricht das den Längenräumen). Es werden regulär lokalisierbare Lorentzsche vor-Längenräume eingeführt, in diesen können kausale Kurven von positiver Länge zu einer zeitartigen Kurve verlängert werden.

Für intrinsische Lorentzsche vor-Längenräume werden Krümmungsvergleiche mit zeitartigen Dreiecken definiert (wie Dreiecksvergleiche in Längenräumen). Man kann auch hyperbolische Winkel definieren. Unter schwachen Bedingungen kann das Resultat, dass in Räumen mit unterer Krümmungsschranke keine Verzweigungen von Geodäten auftreten kann, reproduziert werden.

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1 Defining Lorentzian length spaces

Here, a very short introduction to length spaces and Lorentzian manifolds is given, for the purpose of analogy. For more details on length spaces, see [BBI01], for more details on Lorentzian and semi-Riemannian geometry see [O'N83].

Definition 1.0.1 (Reminder)**.** Let X be a topological space. A function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is called *lower semicontinuous* if for each $a \in \mathbb{R} \cup \{\pm\infty\}$, the set $f^{-1}((a, +\infty])$ is open. For metric spaces, this is equivalent to $\forall p_n \to p$, $\liminf_n f(p_n) \geq f(p)$, i.e. f can jump down in the limit, but not up. $\Diamond$

1.1 Riemann manifolds

Definition 1.1.1. A *Riemannian manifold* is a smooth manifold M together with a *Riemannian inner product* $g \in \Gamma(S^2TM \to M)$: at each point $p \in M$ we have a symmetric positive definite symmetric bilinear form $g|_p : T_pM \times T_pM \to \mathbb{R}$, varying smoothly with the point.

For (M, g) a Riemannian manifold and γ a piecewise C^1 curve, we define the length $L(\gamma) = \int_{\mathrm{dom}(\gamma)} \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}$.

Now we can make each Riemannian manifold into a metric space: for $p, q \in M$, we define $d_L(p, q) = \inf\{L(\gamma) : \gamma$ is a piecewise C^1 curve from p to $q\}$. $\Diamond$

But there is also a notion of length of curves in metric spaces, so we have two notions of length of curves: The differential length L and the variational length $L_{d_L}(\gamma) = \sup\{\sum_i d_L(t_i, t_{i+1}) : t_i < t_{i+1}$ is a partitioning of $\mathrm{dom}(\gamma)\}$.

Proposition 1.1.2. *The two notions of length of curves on Riemannian manifolds agree, i.e.* $\forall \gamma$ *piecewise* C^1 *curve,* $L(\gamma) = L_{d_L}(\gamma)$*.*

Proof. Note that L is lower semicontinuous (i.e. for $\gamma_n \to \gamma$ pointwise, $\liminf(L(\gamma_n)) \geq L(\gamma)$) and see [BBI01, 2.4.3] $\square$

We can also iterate the construction of the distance: $d_{L_{d_L}}(p, q) = \inf\{L_{d_L}(\gamma) : \gamma$ is a rectifiable curve from p to $q\}$

Proposition 1.1.3. *The two notions of distance of points on Riemannian manifolds agree, i.e.* $\forall p, q \in M$*,* $d_{L_{d_L}}(p, q) = d_L(p, q)$*.*

Proof. Both distances are defined via the infimum of the length of curves. By the previous proposition $L_{d_L}(\gamma) = L(\gamma)$ for piecewise C^1 curves. In the infimum in d_L, only such curves are allowed. But in the infimum in $d_{L_{d_L}}$, all curves are allowed. So we get $d_{L_{d_L}}(p, q) \leq d_L(p, q)$.

For the other inequality see [BBI01, 2.4.1] $\square$

1.2 Length spaces

(based on [BBI01]) Inspired by this, we can define a new metric on a metric space:

Definition 1.2.1. Let (X, d) be a metric space (we allow d to attain ∞). Then d induces a length function L_d, which in turn induces the so-called *induced intrinsic metric* $\widehat{d}(p,q) = d_{L_d}(p,q) = \inf\{L_d(\gamma) : \gamma$ a rectifiable curve from p to $q\}$. If there is no such curve between p and q, we set $\widehat{d}(p,q) = \infty$.

We call (X, d) *intrinsic* or a *length space* if $d = \widehat{d}$, and *strictly intrinsic* if additionally the infimum is always attained, i.e. $\forall p, q \in X \exists \gamma \colon p \rightsquigarrow q : L_d(\gamma) = d(p,q)$. Such a curve γ is called *distance realizing* if its length is finite. Restrictions of distance realizing curves are distance realizing as well. $\lozenge$

Theorem 1.2.2. *A Riemannian manifold M viewed as a metric space with d_L is a length space.*

Proof. This is proposition 1.1.3. $\square$

Proposition 1.2.3. *Let (X, d) be a metric space. The induced intrinsic metric $\widehat{d}$ makes X a length space, i.e. $\widehat{\widehat{d}} = \widehat{d}$.*

Proof. see [BBI01, 2.4.1] $\square$

1.3 Spacetimes

(based on [BBI01])

Definition 1.3.1. A *semi-Riemannian manifold* is a smooth manifold M together with a C^2 *semi-Riemannian scalar product* $g \in \Gamma(S^2TM \to M)$: at each point $p \in M$ we have a nondegenerate (but not necessarily positive definite) symmetric bilinear form $g|_p : T_pM \times T_pM \to \mathbb{R}$, varying smoothly with the point. $\lozenge$

The possible structures of the tangent space can be classified easily:

Proposition 1.3.2 (Sylvester's law of inertia)**.** *Let V be a finite dimensional $\mathbb{R}$-vector space and $b : V \times V \to \mathbb{R}$ a symmetric bilinear form on it. Then there exists a basis (v_i) of V such that the basis representation of b is $b(v_i, v_j) = \delta_{i,j} c_i$ ($c_i = 1, 0, -1$), or equivalently, a diagonal matrix with just ones, minus ones and zeros on the diagonal.*

Any two such bilinear forms are related by an isomorphism if and only if they have the same number k of $+1$'s, l of 0's and m of -1's. The signature *of b is the tuple (l, k, m).*

Proof. [Cap15, 9.11] $\square$

So each tangent space T_pM with $g|_p$ is isomorphic to $\mathbb{R}^{n,m}$, which is $\mathbb{R}^{n+m}$ with $b : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$, $b(v, w) = -\sum_{i=1}^{m} v_i w_i + \sum_{i=m+1}^{n+m} v_i w_i$ (as $g|_p$ is nondegenerate, there can be no zeroes).

Proposition 1.3.3. *Let M be a semi-Riemannian manifold. Then the signature of $g|_p$ is constant on connected components.*

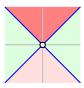
Proof. Consider this in a chart. The signature of $g|_p$ is $(0, p, q)$ exactly when the matrix $g|_p$ has p positive eigenvalues and q negative ones. As the value of the eigenvalues depend continuously on p and 0 is excluded, the signs have to stay the same. □

Definition 1.3.4. An $n + 1$-dimensional semi-Riemannian manifold is called *Lorentzian manifold* if for all $p \in M$ the signature of $g|_p$ is $(0, n, 1)$, i.e. the basis representation has just one -1. Thus, T_pM with $g|_p$ is isomorphic to $\mathbb{R}^{n,1}$: ♢

Example 1.3.5 (Minkowski space)*.* The space $\mathbb{R}^{n,1}$ is called $n + 1$-dimensional Minkowski-space. In this case, one begins numbering the components with 0 (i.e. $v \in \mathbb{R}^{n,1}$ has components $v_0, \cdots, v_n$).

It decomposes into 3 parts: let $v \in \mathbb{R}^{n,1}$. If $b(v, v) > 0$, v is called *spacelike* (green), if $b(v, v) = 0$ and $v \neq 0$, v is called *null* or *lightlike* (blue) and if $b(v, v) < 0$, v is called *timelike* (red). The term *causal* vector is also used ($b(v, v) \leq 0$ and $v \neq 0$). The timelike part splits into two segments, one of which (with $v_0 > 0$) is the *future* part (dark red). This also splits the null vectors and the causal vectors into future and past. The spacelike part does not split into parts if $n \geq 2$.

The future timelike vectors form the *future timecone*, the future null vectors form the *future null cone*, and correspondingly for past directed vectors.



In pictures the "timelike" v_0-component is drawn vertically. ♢

Definition 1.3.6. From their definition in Minkowski space one can also define g-spacelike, g-timelike, g-null and g-causal vectors in the tangent space of a Lorentzian manifold: Let $\varphi : T_pM \to \mathbb{R}^{n,1}$ be an isomorphism (mapping $g|_p$ to b), then $v \in T_pM$ is called *g-spacelike* resp. *g-timelike* resp. *g-null* if $\varphi(v)$ is such. We get the two *timecones* in T_pM as the connected components of the g-timelike vectors at p. These notions do not depend on the choice of φ.

But defining the future is harder here: Flipping the sign of the first coordinate ($v \mapsto (-v_0, v_1, \cdots, v_n)$) is an isomorphism on Minkowski space interchanging future and past, so we have to make a choice:

A *time orientation* of a Lorentzian manifold is a continuous nowhere zero global g-timelike vector field, considered up to homotopy[2] of such. This vector field is viewed as pointing into the future.

A *spacetime* is a connected time-oriented Lorentzian manifold. In a spacetime we get notions of future g-timelike resp. future g-causal vectors, cones and curves: of the two timecones, the global g-timelike vector field contains a vector in the *future* one, and a g-causal vector points into the *future* if it lies in the future timecone, else it is pointing into the *past*.

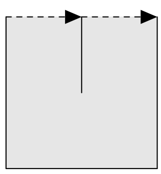
A piecewise C^1 curve γ is *future directed g-causal* resp. *future directed g-timelike* if its velocity γ' is always a future g-causal resp. future g-timelike vector. The *length* or *eigentime* of a future directed g-causal curve γ is $L_g(\gamma) = \int_{\mathrm{dom}(\gamma)} \sqrt{-g|_{\gamma(t)}(\gamma'(t), \gamma'(t))}dt$.

Correspondingly, we define past (directed) g-causal and past g-timelike curves. We will only consider future directed curves here. $\diamondsuit$

Remark 1.3.7. Defining time orientation up to homotopy makes it "more unique": If M is a connected Lorentzian manifold, either there doesn't exist a time orientation (see example 1.3.9) or there exist exactly two, with exchanged notions of future and past. $\diamondsuit$

Example 1.3.8. Considering Minkowski space $\mathbb{R}^{n,1}$ as a manifold makes it a Lorentzian manifold, and the future pointing vector field is usually chosen as constant $(1, 0, \cdots, 0)$. It could also be chosen as constant $(-1, 0, \cdots, 0)$, and all other choices of nowhere zero (g-)timelike vector fields are homotopic to one of these. $\diamondsuit$

Example 1.3.9. Take the subset $(-1, 1] \times (-1, 1) \backslash (0, 1] \times \{0\}$ of Minkowski space and glue $\{1\} \times (-1, 0)$ with $\{1\} \times (0, 1)$ in a distance preserving way.



This is still a Lorentzian manifold, but not time orientable. $\diamondsuit$

We will only consider spacetimes from now on.

Definition 1.3.10. If $M \subseteq \mathbb{R}^n$ is an open subset, we have a basis of coordinate vectors (vector fields) $\partial_i \in T_pM$ (parallel to the ith axis) and the corresponding dual basis $dx_i \in T_p^*M$ ("covectors"). Furthermore, we have the coordinate functions x_i (the coordinate decomposition of the identity). With these, we can

[2]any type gives the same result as long as it contains affine homotopies, as the two parts are convex

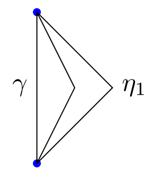
write a semi-Riemannian scalar product g in a basis: $g|_p = \sum_{i,j} g_{i,j}(p) dx_i \otimes dx_j$, usually one writes $dx_i dx_j$ for $\frac{1}{2}(dx_i \otimes dx_j + dx_j \otimes dx_i)$. ♢

The length of curves behaves quite differently from the length of curves in Riemannian geometry: There exist non-trivial curves of length 0 (the null curves), and future directed g-causal geodesics are not locally distance minimizing:

Example 1.3.11 (Twin paradox). The two points $p = (-1, 0), q = (1, 0) \in \mathbb{R}^{1,1}$ in Minkowski space are connected by a unique future directed causal geodesic $\gamma(s) = (s, 0)$. It has $L_g(\gamma) = \int_{-1}^{1} 1 = 2$. The alternative future directed causal curves

$$\eta_x(s) = \begin{cases} (s, x(1+s)) & s \le 0 \\ (s, x(1-s)) & s \ge 0 \end{cases}$$

for $x \in [0, 1]$ have $L(\eta_x) = 2\sqrt{1 - x^2}$ and are thus shorter. The curve η_1 even has a length of 0.



♢

But in contrast to Riemannian geometry, future directed g-causal geodesics are locally distance maximizing curves:

Definition 1.3.12. A *(geodesically) convex set* U in a semi-Riemannian manifold M is an open set, such that any two points $p, q \in U$ can be connected by a unique geodesic contained in U. ♢

Theorem 1.3.13 (Short geodesics maximize time)**.** *Let M be Lorentzian. Then each point has a convex neighbourhood.*

For each convex neighbourhood U and $p, q \in U$ connected by a g-timelike curve within U, the unique geodesic γ contained in U and connecting p to q is g-timelike, and all future directed g-causal curves $\tilde{\gamma}$ contained in U from p to q have a smaller eigentime (i.e. $L_g(\gamma) \ge L_g(\tilde{\gamma})$).

Proof. See [O'N83, 5.7] and [O'N83, 5.34]. □

So to obtain a distance function, we take the supremum instead of the infimum of the length of curves, which is usually finite:

Definition 1.3.14. Let M be a spacetime. We define the *time separation function* as $\tau(p, q) = \sup\{L_g(\gamma) : \gamma : p \rightsquigarrow q \text{ is piecewise } C^1 \text{ future directed } g\text{-causal}\}$ for $p, q \in M$ (which can be infinite). If there is no future directed g-causal curve from p to q, we set $\tau(p, q) = 0$. ♢

Proposition 1.3.15. *For points where the involved curves exist, τ satisfies the reverse triangle inequality: for $p, q, r \in M$, $\tau(p, r) \geq \tau(p, q) + \tau(q, r)$. Furthermore, τ is lower semicontinuous.*

Proof. In the case where future directed g-causal curves exist, this follows by concatenating the curves.

If there exists a future directed g-causal curve from p to q and q to r, there also exists a future directed g-causal curve from p to r.

If there doesn't exist a future directed g-causal curve from p to q or q to r, the inequality can fail.

For the lower semicontinuity of τ see [O'N83, 14.17] or [Gal14, 4.4] $\square$

Definition 1.3.16 (Comparing two Lorentzian metrics)**.** Let M be a manifold and g, h two spacetime structures (i.e. Lorentzian metrics and a time orientation). Then we say that g has *strictly narrower lightcones* than h, $g \prec h$, resp. g has *narrower lightcones* than h, $g \preceq h$, if each future g-causal vector is future h-timelike resp. h-causal. In particular, future directed g-causal curves are future directed h-timelike (resp. h-causal). $\Diamond$

1.4 Causality in spacetimes

(based on [Gal14]) In this section, we are interested when future directed g-timelike or g-causal curves exist:

Definition 1.4.1. Let M be a spacetime.

- $q \in M$ lies in the *timelike future* of $p \in M$, $p \ll q$, if there exists a future directed g-timelike curve from p to q which is not just a point[3].
- $q \in M$ lies in the *causal future* of $p \in M$, $p \leq q$, if $p = q$ or there exists a future directed g-causal curve from p to q.
- The timelike future of $p \in M$ is $I^+(p) = \{q \in M : p \ll q\}$.
- The causal future of $p \in M$ is $J^+(p) = \{q \in M : p \leq q\}$.
- The timelike past of $q \in M$ is $I^-(q) = \{p \in M : p \ll q\}$.
- The causal past of $q \in M$ is $J^-(q) = \{p \in M : p \leq q\}$.
- The *timelike diamond* of $p, q \in M$ is $I(p, q) = I^+(p) \cap I^-(q)$.
- The *(causal) diamond* of $p, q \in M$ is $J(p, q) = J^+(p) \cap J^-(q)$.

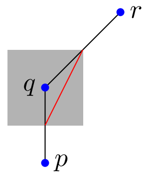
$\Diamond$

[3] To exclude $\mathrm{dom}(\gamma) = [a, a] = \{a\}$

These relations are automatically transitive, and $p \ll q \Rightarrow p \leq q$. Furthermore, they satisfy:

Proposition 1.4.2 (Push-up)**.** *Let M be a spacetime. If $p \ll q \leq r$ or $p \leq q \ll r$, then $p \ll r$.*

Proof-idea. (A proof is in e.g. [Chr11, 2.4.14])
In the first case, we get a future directed g-timelike curve $\gamma_1 : p \rightsquigarrow q$ and a future directed g-causal curve $\gamma_2 : q \rightsquigarrow r$. In a convex neighbourhood U of q, we can deform the parts of γ_1 and γ_2 contained in U to a future directed g-timelike curve. We add it to γ_1 and shorten the curve γ_2. Iterating and using compactness, we get a future directed g-timelike curve $\tilde{\gamma} : p \rightsquigarrow r$.



$\square$

But these relations are not automatically order relations:

Definition 1.4.3 (Causality conditions)**.** Let M be a spacetime. A subset $U \subseteq M$ is *(curve-)causally convex* if no future directed g-causal curve meets U in a disconnected way. [4]

We say that M is:

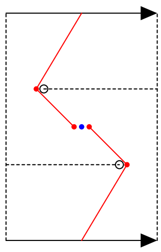
- *chronological* if there are no closed future directed g-timelike curves (or equivalently, $\ll$ is an order relation),
- *causal* if there are no closed future directed g-causal curves (or equivalently, $\leq$ is an order relation),
- *strongly causal* if every point has a neighbourhood base of causally convex open sets. Colloquially, no sequence of future directed g-causal curves is closing in the limit (in a non-trivial way).
- *internally compact* if all the causal diamonds are compact and
- *globally hyperbolic* if it is both strongly causal and internally compact.

$\diamondsuit$

[4] Compare: convex: there exists a connecting geodesic inside, causally convex: all connecting causal curves inside.

Example 1.4.4 (Lorentz cylinder). Glue the top with the bottom of $[0,1] \times (0,1)$ in Minkowski space $\mathbb{R}^{1,1}$. This spacetime is not causal: the future directed g-timelike curve $\gamma(t) = (t, \frac{1}{2})$ is closed. $\Diamond$

Example 1.4.5. We take the following subset of Minkowski space $\mathbb{R}^{1,1}$:



We glue the top with the bottom. Then the red curve will close in the limit (so the blue point has no small causally closed neighbourhoods), but there is no closed future directed g-causal curve as the two white points and the dotted lines are removed. $\Diamond$

Proposition 1.4.6. *Let M be a spacetime, $p \in M$. Then $I^+(p)$ and $I^-(p)$ are open. If M is globally hyperbolic, $J^+(p)$ and $J^-(p)$ are closed.*

Proof. See [Gal14, 2.2,4.3]. $\square$

Sources

Example 1.4.4 comes from [KS18, 2.20].

1.5 Causal spaces

(based on [KS18]) We try to mimic this causality behaviour:

Definition 1.5.1. A set X with two binary relations $\ll, \leq$, where both are transitive, $\leq$ is reflexive, and $\forall p, q \in X \; p \ll q \Rightarrow p \leq q$ (sometimes denoted as $\ll \subseteq \leq$), is called a *causal space.*

Then we can define $I^+(p), I^-(q), J^+(p), J^-(q), I(p,q), J(p,q)$ as above. $\Diamond$

We get two natural topologies:

Definition 1.5.2. Let X be a causal space. The *chronological topology* on X has $\{I^{\pm}(p) : p \in X\}$ as a subbase.

The *Alexandrov topology* on X has $\{I(p,q) : p, q \in X\}$ as a subbase. $\Diamond$

We see that the Alexandrov topology is in general coarser than the chronological topology. But in general the converse is not true:

Example 1.5.3. Let $X = \{1,2\}$ and set $\ll = \{(1,2)\}$ (i.e. $1 \ll 2$). Then the chronological topology is the discrete topology, whereas the Alexandrov topology is the indiscrete topology. $\Diamond$

1.6 Lorentzian pre-length spaces

(mostly based on [KS18]) Here we mimic the behaviour of τ. We include a metric to have sufficiently nice notions of convergence.

Definition 1.6.1. A *Lorentzian pre-length space* is a causal space $(X, \ll, \leq)$ together with a metric d on X and a map $\tau : X \times X \to [0, \infty]$ satisfying:

- τ is lower semicontinuous (w.r.t. d),
- $\tau(p, r) \geq \tau(p, q) + \tau(q, r)$ for $p \leq q \leq r$ (reverse triangle inequality) and
- $\tau(p, q) > 0 \Leftrightarrow p \ll q$.

$\Diamond$

This is a generalization of the notion of spacetimes:

Proposition 1.6.2. *A spacetime M together with its $\ll, \leq$ and τ and the metric induced by a complete Riemannian inner product is a Lorentzian pre-length space.*

Proof. In proposition 1.3.15 we have seen that τ is lower semicontinuous and satisfies the reverse triangle inequality.

For the third property, we note: if $\gamma : p \rightsquigarrow q$ is a future directed g-causal curve with $L_g(\gamma) = \int_{\mathrm{dom}(\gamma)} \sqrt{-g(\gamma', \gamma')} > 0$, we get a subinterval where the integrand is positive, so γ is future directed g-timelike there. Now we apply push-up (1.4.2) twice to get a future directed g-timelike curve $\tilde{\gamma} : p \rightsquigarrow q$. $\square$

Another method to create Lorentzian pre-length spaces is to use absolute time functions:

Example 1.6.3 (From absolute time function). Let X be a metric space and $t : X \to \mathbb{R}$ a continuous function (the absolute time function). We define the causality: $p \leq q \Leftrightarrow t(p) \leq t(q)$ and $p \ll q \Leftrightarrow t(p) < t(q)$. Then

$$\tau(p, q) = \begin{cases} 0 & t(p) \geq t(q) \\ t(q) - t(p) & t(p) \leq t(q) \end{cases}$$

makes X a Lorentzian pre-length space with continuous $\tau < \infty$ and equality in the reverse triangle inequality. $\Diamond$

We consider Lorentzian pre-length spaces made from an absolute time function as not very interesting (as they have a rather trivial causal space), but they can be used for counterexamples.

Proposition 1.6.4 (Push-up)**.** *Let X be a Lorentzian pre-length space. Then $p \leq q \ll r$ or $p \ll q \leq r$ implies $p \ll r$*

Proof. Let $p \leq q \ll r$ (the other case follows from time reversal). Then $\tau(p,r) \geq \tau(p,q) + \tau(q,r) \geq \tau(q,r) > 0$, so $p \ll r$. $\square$

Proposition 1.6.5. *Let X be a Lorentzian pre-length space. Then $I^+(p)$ and $I^-(p)$ are open.*

Proof. The time separation function τ is lower semicontinuous, so $\tau^{-1}((0,\infty]) = \{(p,q) : \tau(p,q) > 0\}$ is open in $X \times X$. Fixing p_0 and taking its slice, we see $\{q : \tau(p_0,q) > 0\} = \{q : p_0 \ll q\} = I^+(p_0)$ is open. Analogously for I^-. $\square$

Sources

The notion of Lorentzian pre-length spaces was introduced in [KS18, 2.8]. Propositions 1.6.2, 1.6.4 and 1.6.5 are from [KS18, 2.10-2.12]. Example 1.6.3 is new.

1.6.1 Causal curves

To speak about intrinsic time separation functions, we define the length of curves:

Definition 1.6.6. Let $(X, \ll, \leq, d, \tau)$ be a Lorentzian pre-length space. A curve γ in X is just a curve in the metric space (X,d).

A non-constant curve γ is *future directed causal* or *future directed timelike* if $\forall t_1 < t_2,\ \gamma(t_1) \ll \gamma(t_2)$ or $\gamma(t_1) \leq \gamma(t_2)$, respectively. The *causal character* of a future directed causal curve is "timelike" if it is timelike and "null" if $\forall t_1 < t_2,\ \gamma(t_1) \not\ll \gamma(t_2)$. Otherwise, we say that the causal character changes.

For past causal and past timelike, we reverse these relations (but we won't use such curves here. Upon parameter reversal, they are future directed causal / timelike).

We can define the length of future directed causal curves as in metric spaces, replacing the supremum with an infimum: Let $\mathfrak{p} = (t_i)$ be a partition of $[a,b]$. We define the *length approximations* $V_\mathfrak{p}(\gamma) = \sum_i \tau(\gamma(t_i), \gamma(t_{i+1}))$, and the *length* $L_\tau(\gamma) = \inf\{V_\mathfrak{p}(\gamma) : \mathfrak{p} \text{ a partition of } \mathrm{dom}(\gamma)\}$. For easier handling of parts of curves we define $L_\tau(\gamma,s,t) = L_\tau(\gamma|_{[s,t]})$ for $s \leq t$ and $L_\tau(\gamma,s,t) = -L_\tau(\gamma,t,s)$ for $s \geq t$ (which agrees for $t = s$ if $\tau(\gamma(t),\gamma(t)) = 0$).

$\Diamond$

Note that the definition of future directed timelike curves does not match the definition of future directed g-timelike curves in Lorentzian manifolds, and therefore also the definition of the causal character is different:

Example 1.6.7. The null spiral $t \mapsto (t, \cos(t), \sin(t))$ in Minkowski space $\mathbb{R}^{2,1}$ is not g-timelike, but is future directed timelike when considered in the corresponding Lorentzian pre-length space. $\Diamond$

Proposition 1.6.8. *When considering a strongly causal spacetime and its induced Lorentzian pre-length space, the definitions of future directed causal curves agree for piecewise C^1 curves γ with γ' never zero, as well as the length of such curves.*

Future directed g-timelike curves in a spacetime are automatically future directed timelike in the induced Lorentzian pre-length space.

Proof. If γ is a (piecewise C^1) future directed g-causal curve, then $\gamma(t_1) \leq \gamma(t_2)$ for $t_1 < t_2$, as $\gamma|_{[t_1,t_2]}$ is a future directed g-causal curve.

Let γ be a piecewise C^1 future directed causal curve in the induced Lorentzian pre-length space, but t_0 where w.l.o.g. the left sided derivative $\gamma'^-(t_0)$ is not a future g-causal vector. So γ is C^1 on $[t_0 - \varepsilon, t_0]$. As γ is future directed causal in the induced Lorentzian pre-length space, we get future directed g-causal curves $\tilde{\gamma}_n : \gamma(t_0 - \frac{1}{n}) \rightsquigarrow \gamma(t_0)$ and we can assume they are geodesics. Seen through the exponential chart at $\gamma(t_0)$, $\tilde{\gamma}_n' = \frac{\gamma(t_0)-\gamma(t_0-\frac{1}{n})}{\frac{1}{n}} \to \gamma'^-(t_0)$. As $\tilde{\gamma}_n'$ is g-causal, also $\gamma'^-(t_0)$ has to be g-causal (the set of g-causal vectors plus 0 is closed in TM, and $\gamma'^-(t_0) = 0$ has been excluded) $\lightning$.

For the length, let γ be a C^1 curve. Then $L_g(\gamma) = \sum_i L_g(\gamma|_{[t_i,t_{i+1}]}) \leq \sum_i \tau(\gamma(t_i), \gamma(t_{i+1}))$. Taking the infimum over the partitions (t_i), we get $L_g(\gamma) \leq L_\tau(\gamma)$.

For the other direction for length, see [KS18, 2.32]

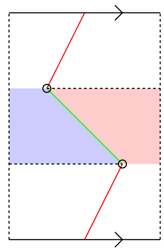
The statement about timelike curves is clear from the definition of $\ll$. $\square$

By mapping partitions we see that length is preserved by bijective strictly monotonically increasing reparametrizations (i.e. for a future directed causal curve $\gamma : [a,b] \to X$ and a bijective strictly monotonically increasing $\psi : [c,d] \to [a,b]$, the curve $\tilde{\gamma} = \gamma \circ \psi$ is future directed causal and has equal length).

Causal curves in non-causal spaces can behave quite badly:

Example 1.6.9 (Lorentz cylinder). Take the Lorentz cylinder (example 1.4.4). It is not causal, and there are future directed g-timelike curves between any two points. Therefore, we have $p \ll q$ for all points, making all curves future directed timelike. $\Diamond$

Example 1.6.10. Take the (non-strongly causal) spacetime from example 1.4.5:



Then the definitions of future directed causal curves agree for piecewise C^1 curves γ with γ' never zero (as removing the green line changes neither $\leq$ nor τ within the red and within the blue areas), and they have the same length: for C^1 future directed causal curves crossing the green line, the point on the green line has to be included in the partition, similarly to 1.6.14.

But τ behaves strangely: It is not continuous "near the diagonal" on small neighbourhoods of a point on the green line: If α is the red line (the "short-cut"), then for p_n on the blue side and q_n on the red side converging to the same point $p = q$ on the green line, $\lim_n \tau(p_n, q_n) = L(\alpha)$, but $\tau(p, p) = 0$. ♢

Proposition 1.6.11 (Generalized reverse triangle inequality)**.** *Let X be a Lorentzian pre-length space, $\gamma : p \rightsquigarrow q$ a future directed causal curve. Then $\tau(p, q) \geq L_\tau(\gamma)$.*

Proof. By definition $L_\tau(\gamma) = \inf\{\sum_i \tau(\gamma(t_i), \gamma(t_{i+1}))\}$. One of these partitions (t_i) is $(t_1 = a, t_2 = b)$, and the corresponding term in the infimum is $\tau(p, q)$. Therefore, the infimum is $\leq \tau(p, q)$. □

Remark 1.6.12 (Pullback)*.* When considering only one curve, it is often enough to "pull back" τ to the domain of the curve: If $\gamma : [a, b] \to X$ is a future directed causal curve in a strongly causal Lorentzian pre-length space, we can make $[a, b]$ a Lorentzian pre-length space by setting $\leq$ to be the normal order relation $\leq$, setting $s \ll t \Leftrightarrow \gamma(s) \ll \gamma(t)$ (which is $\ll=<$ if γ is timelike), and setting $\tau(s, t) = \tau(\gamma(s), \gamma(t))$. There, we have the "canonical" curve $\mathrm{id}(t) = t$ which we identify with γ. Properties "along the curve" like the length of (restrictions of) these curves or being distance realizing agree, whereas properties like maximality of the curve (see 1.7.1) get lost. Using this, we can drop the γs in expressions like $\tau(\gamma(t_i), \gamma(t_{i+1}))$. ♢

Proposition 1.6.13 (Convergence of length)**.** *Let X be a Lorentzian pre-length space, $\gamma : p \rightsquigarrow q$ a future directed causal curve. Then $L_\tau(\gamma)$ is the monotone (directed) limit of $V_{\mathfrak{p}}(\gamma)$ with inclusions of partitions.*

Proof. Partitions with inclusions form a directed partial order: for two partitions $(t_i), (s_j)$ we can form their union, containing both of them. We now pull back τ to $\mathrm{dom}(\gamma)$.

Let the partition (t_i) be included in a partition (s_j), i.e. $s_{j_i} = t_i$. Then for each i, $\tau(t_i, t_{i+1}) \geq \sum_{j_i \leq j < j_{i+1}} \tau(s_j, s_{j+1})$ by the reverse triangle inequality. Summing over i, we get $V_{(s_j)}(\gamma) \leq V_{(t_i)}(\gamma)$, so the limit exists, and the $V_{\mathfrak{p}}(\gamma)$ converge monotonously. □

Contrary to the length in metric spaces, the length of a curve in a Lorentzian pre-length space need not be the limit with respect to partition fineness:

Example 1.6.14. Let $X = [0,1]$, with causality $\leq, <$. Set $M = \{(x,y) : x < \frac{1}{2} < y\}$. We modify the standard Minkowski time separation function $\tau(x,y) = y - x$ if $x < y$ to the new time separation function

$$\tilde{\tau}(x,y) = \begin{cases} \tau(x,y) & (x,y) \notin M \\ \tau(x,y) + 1 & (x,y) \in M \end{cases}.$$

Colloquially, we add 1 to the distance of points on different sides of the point $\frac{1}{2}$.

As $M \subseteq \ll$, the $\ll$-condition is satisfied. As M is open, $\tilde{\tau}$ is still lower semicontinuous. For the reverse triangle inequality, we note that if $(x,z) \in M$ and $x < y < z$ at most one of the (x,y) and (y,z) can be in M, and if $(x,z) \notin M$ none of them is in M, so the number of added 1s in the triangle inequality is bigger on the left side.

We consider the "canonical" future directed timelike curve $\gamma(p) = p$: In the Minkowski time separation function τ, we have $L_\tau(\gamma) = V^\tau_{\mathfrak{p}}(\gamma) = 1$. But for the modified $\tilde{\tau}$, we have to distinguish whether $\frac{1}{2} \in \mathfrak{p}$ or not: If it is, there is no term in M (let i be such that $t_i = \frac{1}{2}$. Then neither (t_i, t_{i+1}) nor (t_{i+1}, t_{i+2}) is in M), and we get the same result as for τ. If it is not, we have exactly one term in M (let i be such that $t_i < \frac{1}{2} < t_{i+1}$. Then $(t_i, t_{i+1}) \in M$.) and we add 1 to the result for τ. So we see the length agrees, but for $\tilde{\tau}$ there are arbitrarily fine partitions with a "bad" length approximation. $\Diamond$

Proposition 1.6.15 (Additivity of length)**.** *Let X be a Lorentzian pre-length space and γ a future directed causal curve. Let $a < b < c$, $[a,c] \subseteq \mathrm{dom}(\gamma)$. Then $(L_\tau(\gamma) =) L_\tau(\gamma, a, c) = L_\tau(\gamma, a, b) + L_\tau(\gamma, b, c)$.*

Proof. For each partition of $[a,c]$, we can add b (by the previous proposition the corresponding approximation will be better). Restricting such partitions $\mathfrak{p} \subset [a,c]$ to $[a,b]$ and $[b,c]$ forms a bijection $\mathfrak{p} \mapsto (\mathfrak{p} \cap [a,b], \mathfrak{p} \cap [b,c])$ between $\{$partitions of $[a,c]$ containing $b\}$ and $\{$partitions of $[a,b]\} \times \{$partitions of $[b,c]\}$, and one easily checks that the corresponding approximations of $L_\tau(\gamma, a, b)$ and $L_\tau(\gamma, b, c)$ add up to the approximation of $L_\tau(\gamma, a, c)$. $\square$

Proposition 1.6.16 (Length continuity)**.** *Let X be a Lorentzian pre-length space and let γ be a future directed causal curve. If τ is finite along γ or $L_\tau(\gamma) < \infty$, the length of the restriction of γ is continuous in the restriction endpoints, i.e. $L_\tau(\gamma, s, t)$ is continuous in both s and t.*

Proof. We can assume that γ is defined on a compact interval $[a,b]$. Now let $s_n \to s$ and $t_n \to t$. We have $s_n \leq t_n$ and $s \leq t$. If $L_\tau(\gamma) = \infty$, we have $\tau(\gamma(a), \gamma(b)) = \infty$ by the generalized reverse triangle inequality $\lightning$, so $L_\tau(\gamma)$ and all its restrictions are finite.

By additivity of length and finiteness we have $L_\tau(\gamma, t_1, t_2) = L_\tau(\gamma, a, t_2) - L_\tau(\gamma, a, t_1)$ and only need $f(t) = L_\tau(\gamma, a, t)$ to be continuous.

Again by additivity of length $f(t)$ is monotonously increasing. So if f were discontinuous at t_0, f jumps up (i.e. $\lim_{t\nearrow t_0} f(t) < f(t_0)$ or $\lim_{t\searrow t_0} f(t) > f(t_0)$). By additivity of length and reversing time we can assume $h = f(t_0) - \lim_{t\nearrow t_0} f(t) > 0$.

We pull back τ along γ (1.6.12). Approximating the length with partitions, we get a "precise" partition $\mathfrak{p}$ of $[a, t_0]$ such that

$$0 \leq V_{\mathfrak{p}}(\gamma|_{[a,t_0]}) - L_\tau(\gamma, a, t_0) < \frac{h}{2}. \tag{1.1}$$

Let the two biggest entries of $\mathfrak{p}$ be $s_0 < t_0$ (i.e. $\mathfrak{p} = \{s_{-n} < s_{-n+1} < \cdots < s_0 < t_0\}$). For each $s_0 < t < t_0$, we define the partition $\mathfrak{p}_t = \mathfrak{p}\backslash\{t_0\} \cup \{t\} = \{s_{-n} < s_{-n+1} < \cdots < s_0 < t\}$ of $[a, t]$.

<u>Claim</u>: This partition also satisfies $0 \leq V_{\mathfrak{p}_t}(\gamma|_{[a,t]}) - L_\tau(\gamma, a, t) < \frac{h}{2}$
By monotone convergence of length (1.1) becomes:

$$\frac{h}{2} > V_{\mathfrak{p}}(\gamma|_{[a,t_0]}) - L_\tau(\gamma, a, t_0) \geq V_{\mathfrak{p}\cup\{t\}}(\gamma|_{[a,t_0]}) - L_\tau(\gamma, a, t_0)$$

Splitting off the $[t, t_0]$ part, this is:

$$V_{\mathfrak{p}_t}(\gamma|_{[a,t]}) + \tau(t, t_0) - L_\tau(\gamma, a, t) - L_\tau(\gamma, t, t_0)$$

As $\tau(t, t_0) \geq L_\tau(\gamma, t, t_0)$, we get the desired

$$\frac{h}{2} > V_{\mathfrak{p}_t}(\gamma|_{[a,t]}) - L_\tau(\gamma, a, t) \geq 0. \tag{1.2}$$

Subtracting (1.2) from (1.1), we get

$$V_{\mathfrak{p}}(\gamma|_{[a,t_0]}) - V_{\mathfrak{p}_t}(\gamma|_{[a,t]}) > L_\tau(\gamma, t, t_0) - \frac{h}{2}$$

But the formulae of the two length approximations almost agree and we are left with

$$\tau(s_0, t_0) - \tau(s_0, t) > L_\tau(\gamma, t, t_0) - \frac{h}{2} \geq \frac{h}{2}$$

as $L_\tau(\gamma, t, t_0) \geq h$. We now let $t \to t_0$, yielding

$$\tau(s_0, t_0) - \frac{h}{2} \geq \liminf_{t\to t_0} \tau(s_0, t)$$

- a contradiction to lower semicontinuity of τ. □

Example 1.6.17 (Counterexample if τ attains ∞). Let $X = [0, 1]$ with causality

$\leq, <$. We define the absolute time function $t(x) = -\frac{1}{x}$ (and $t(0) = -\infty$). Then we get τ as above (we set $\tau(0,0) = 0$, making τ only lower semicontinuous, and have $\tau(0,x) = \infty$ for $x > 0$). This makes X a Lorentzian pre-length space.

Then the canonical curve $\gamma : [0,1] \to X, \gamma(t) = t$ has $L_\tau(\gamma, 0, 0) = 0$, but $L_\tau(\gamma, 0, t) = \infty$ for all $t > 0$. ◊

Example 1.6.18 (Counterexample if τ attains ∞). Let $X = [0,1]$ with causality $\leq, <$. We define $M = \{(x,y) : y - x > 1 - y\}$ and set

$$\tau(x,y) = \begin{cases} 0 & (x,y) \notin M \\ \infty & (x,y) \in M \end{cases}$$

(colloquially, we consider the three points $x \leq y \leq 1$. If y is closer to 1, $\tau(x,y)$ is infinite, if y is closer to x it is zero). It is lower semicontinuous as M is open. It fulfils the reverse triangle inequality: splitting $(x,z) \notin M$ at $x < y < z$, the distance between the involved points will get smaller and the distance to 1 can only increase, so neither (x,y) nor (y,z) can be in M.

If $x < y < 1$, we can split this into pieces $x = t_1 < \cdots < t_n = y$, such that all $\tau(t_i, t_{i+1}) = 0$, whereas when $x < y = 1$, we cannot. Therefore the canonical curve $\gamma : [0,1] \to X, \gamma(t) = t$ has $L_\tau(\gamma, s, t) = 0$ if $t \neq 1$ and $L_\tau(\gamma, s, 1) = \infty$ if $s \neq 1$.

One may also add the Minkowski time separation function (making $L_\tau(\gamma, s, t) = t - s$ if $t \neq 1$). ◊

Definition 1.6.19. Let $\gamma : [a,b) \to X$ be a future directed causal curve ($b \leq \infty$). It is *future inextensible* if we cannot extend it to $[a,b]$ in a continuous future directed causal way (for $b = \infty$, we need to reparametrize it with $-\frac{1}{t}$). Similarly, we define *past inextensible* future directed causal curves (with domain $(a,b]$). ◊

Definition 1.6.20. A future directed causal curve is *rectifiable* if for all $t_1 < t_2$, $L_\tau(\gamma, t_1, t_2) > 0$. If $L_\tau(\gamma) < \infty$, this is equivalent to $t \mapsto L_\tau(\gamma, a, t)$ being strictly monotonically increasing. Especially, γ is future directed timelike.

A future directed causal curve is *parametrized by (τ-)arclength* if for all $t_1 < t_2$, $L_\tau(\gamma, t_1, t_2) = t_2 - t_1$. Especially, it is rectifiable. ◊

Proposition 1.6.21 (Parametrization by τ-arclength)**.** *Let γ be a future directed causal rectifiable curve with $L_\tau(\gamma) < \infty$. We define $\psi(t) = L_\tau(\gamma, a, t)$. Then we can form ψ^{-1}, and $\tilde{\gamma} = \gamma \circ \psi^{-1}$ is parametrized by arclength.*

Proof. As γ is rectifiable, $\psi(t) = L_\tau(\gamma, a, t)$ is strictly monotonically increasing. As $L_\tau(\gamma) < \infty$, ψ is continuous (1.6.16). Therefore it is a bijection with its image, and ψ^{-1} is strictly monotonic as well.

To prove $L_\tau(\tilde{\gamma}, s_1, s_2) = s_2 - s_1$ ($s_1 < s_2$), we find $t_1 = \psi^{-1}(s_1)$ and $t_2 = \psi^{-1}(s_2)$. We note $\psi|_{[t_1,t_2]} : [t_1, t_2] \to [s_1, s_2]$ is a strictly monotonously increasing bijection and $\tilde{\gamma}|_{[s_1,s_2]} = \gamma|_{[t_1,t_2]} \circ \psi|_{[t_1,t_2]}^{-1}$. As length agrees when reparametrizing, $L_\tau(\tilde{\gamma}, s_1, s_2) = L_\tau(\gamma, t_1, t_2) = \psi(t_2) - \psi(t_1) = s_2 - s_1$. $\square$

Sources

The notion of causal and timelike curves and their length was introduced in [KS18, 2.18,2.24], inextensible curves were introduced in [KS18, 3.10], rectifiable curves in [KS18, 2.29]. The example 1.6.7 is [KS18, 2.31], and 1.6.14, 1.6.17 and 1.6.18 are new. The statement 1.6.8 is [KS18, 2.21,2.32], 1.6.11 is [O'N83, 14.16.(2)] and 1.6.15 is [KS18, 2.25]. 1.6.16 is an improved version of [KS18, 3.33], 1.6.21 is an improved version of [KS18, 3.34]. 1.6.13 is new. The tool 1.6.12 is new as well.

1.6.2 Causality conditions

I will distinguish three types of conditions: The compatibility with d, the existence of certain curves and the (standard) causality conditions.

Definition 1.6.22 (Compatibility with d)**.** Let X be a Lorentzian pre-length space. An open set U is *causally closed* if the subset $\leq \subset \overline{U} \times \overline{U}$ is closed, i.e. for $p_n \to p, q_n \to q \in \overline{U}$ with $p_n \leq q_n$ also $p \leq q$. U is called an open *d-compatible* set if the d-length of future directed causal curves within U is bounded.

- X is *locally causally closed* if X is covered by causally closed sets.
- X is *globally causally closed* if X itself is a causally closed set.
- X is *d-compatible* if it is covered by open d-compatible sets.[5]
- X is *non-totally imprisoning* if for every compact set K there is a bound on the d-length of future directed causal curves within K.

$\Diamond$

A locally causally closed Lorentzian pre-length space which is proper as a metric space is enough to use limits in a meaningful way:

Proposition 1.6.23 (Inextensibility and limits)**.** *Let X be a locally causally closed Lorentzian pre-length space and $\gamma : [a, b) \to X$ ($b \leq \infty$) a future directed causal curve. Then γ is future inextensible exactly when $\lim_{t \nearrow b} \gamma(t)$ does not exist.*

[5]Distinguish open d-compatible sets from d-compatible spaces! d-compatible spaces are covered by d-compatible sets.

Proof. If $p = \lim_{t \nearrow b} \gamma(t)$ exists, we extend γ by $\gamma(b) = p$, making it a continuous curve. As X is causally closed near p, this is still a future directed causal curve, so γ was not inextensible.

If γ is not inextensible, there exists an extension to the closed interval $[a, b]$. By continuity of this extension $\lim_{t \nearrow b} \gamma(t) = \gamma(b)$ exists. ☐

Theorem 1.6.24 (Limit curve theorem)**.** *Let X be a proper locally causally closed Lorentzian pre-length space and $\gamma_n : [a, b] \to X$ (or any open or infinite interval) future directed causal curves that are uniformly Lipschitz continuous. Assume there exists $c \in [a, b]$ such that $\gamma_n(c)$ converges. Then there is a subsequence of γ_n converging (locally) uniformly to a (possibly constant) future directed causal curve with the same Lipschitz bound.*

Proof. We assume a closed finite interval (otherwise we find subsequences of subsequences and use a diagonal argument).

As the curves are uniformly Lipschitz continuous (with Lipschitz constant L) and $\gamma_n(c)$ converges, for large enough n all curves are contained in a ball (e.g. $K = \bar{B}_{L(b-a)+1}(\lim_n \gamma_n(c))$). By properness the closed ball K is compact, so we can use the Arzelà-Ascoli theorem to get a subsequence of the γ_n uniformly converging to a curve γ, w.l.o.g. this is the sequence γ_n itself.

γ is, if not constant, a future directed causal curve: As X is locally causally closed, we can cover the trace of γ with open causally closed sets U. For $t < s$ with $\gamma(t), \gamma(s) \in U$, also $\gamma_n(t), \gamma_n(s)$ is in U for large enough n. So we see: $\gamma_n(t) \leq \gamma_n(s)$, and taking the limit we have $\gamma(t) \leq \gamma(s)$. Using transitivity of $\leq$, we see γ is a future directed causal curve. ☐

d-compatibility makes causal curves Lipschitz:

Proposition 1.6.25 (Lipschitz reparametrization)**.** *Let X be a d-compatible Lorentzian pre-length space and γ a future directed causal curve. Then we can reparametrize γ to be Lipschitz.*

Proof. We need to show γ has locally finite d-length (then we can reparametrize it w.r.t. d-arclength). So w.l.o.g., we restrict γ to a compact interval $[a, b]$.

We cover X by open d-compatible sets U. Then the sets $\gamma^{-1}(U) \subseteq [a, b]$ are open. We split them into (open) connected components I_i. (Then $[a, b] = \bigcup_i I_i$). As $[a, b]$ is compact, we only have finitely many is. By d-compatibility $L_d(\gamma|_{I_i})$ is finite. So $L_d(\gamma) \leq \sum_i L_d(\gamma|_{I_i}) < \infty$. ☐

Example 1.6.26. Equip $\mathbb{R} \times \{t \neq 0\} \cup \{(0, 0)\}$ with the standard metric d and τ induced from the Lorentz metric $g = t^2(dx)^2 - 4(dt)^2$. We consider the curve $\gamma(t) = (t \sin(\frac{1}{t}), t)$. It is g-timelike. But the d-length is infinity near $(0, 0)$, so there is no open d-compatible set containing the point $(0, 0)$. ♢

Example 1.6.27. Consider 2-dimensional Minkowski space with a point removed: $M = \mathbb{R}^{1,1} \backslash \{(0,0)\}$. This space is not globally causally closed: the points $p_n = (-1, -1 + \frac{1}{n})$ and $q_n = (1, 1 + \frac{1}{n})$ satisfy $p_n \leq q_n$, but in the limit $p = (-1,-1) \not\leq q = (1,1)$, as the only connecting causal curve $\gamma(t) = (t,t)$ passes through the excluded point $\{(0,0)\}$.

But it is locally causally closed: the sets $\{(t,x) : x > 0\}$, $\{(t,x) : x < 0\}$, $\{(t,x) : t > 0\}$, $\{(t,x) : t < 0\}$ are causally closed (note $(0,0)$ is not allowed as a limit point, as it is not in the metric space) and cover M. ♢

Remark 1.6.28. The limit curve theorem (1.6.24) is usually used in locally causally closed, curve-causally convex (i.e. the definition from the causality conditions (1.4.3) transferred to Lorentzian pre-length spaces) and d-compatible neighbourhoods U, or in locally causally closed, curve-causally convex and relatively compact subsets U of non-totally imprisoning spaces: There, for $p_n \to p, q_n \to q \in U$ and $\gamma_n : p_n \rightsquigarrow q_n$ (which are automatically contained in U by curve-causal convexity), the d-length of γ_n has a bound, so we can reparametrize γ_n w.r.t. d-arclength and extend it constantly so all are defined on the same finite interval. With the limit curve theorem, we now get a future directed causal curve $p \rightsquigarrow q$ in U which is the limit of a subsequence of the reparametrized γ_n. ♢

Definition 1.6.29 (Existence of curves)**.** Let X be a Lorentzian pre-length space. It is called

- *causally path connected* if for each $p \ll q$ we can find a future directed timelike curve connecting them, and for each $p \leq q$ we can find a future directed causal curve (possibly constant for $p = q$) connecting them (i.e. each causal / timelike relation is "realized" by a curve),
- *causally length connected* if for each $p \ll q$ we can find a future directed causal curve of positive length connecting them, and for each $p \leq q$ we can find a future directed causal curve (possibly constant for $p = q$) connecting them (i.e. each causal / timelike relation is "realized" by the length of a curve) and
- *length continuous* if every future directed causal curve $\gamma : [a,b] \to X$ has continuous length, i.e. $L_\tau(\gamma, t_1, t_2)$ is continuous in t_1 and t_2. By proposition 1.6.16 this is automatically the case unless $L_\tau(\gamma) = \infty$.

♢

The difference between causal path connectedness and causal length connectedness consists of timelike curves with zero length like the null spiral (1.6.7) and of curves of changing causal character failing to be "pushed up" to a future directed timelike curve.

Definition 1.6.30. A subset U of a causal space is called *(relation-)causally convex* if $\forall p, q \in U\ J(p,q) \subseteq U$. $\Diamond$

In particular, $I(p,q)$ and $J(p,q)$ and intersections of relation-causally convex sets are relation-causally convex.

Definition 1.6.31 (Causality conditions)**.** A causal space is:

- *chronological*, if $\ll$ is irreflexive (and by transitivity antisymmetric),
- *causal*, if $\leq$ is antisymmetric.
- It has no $\ll$*-isolated points* if for all $p \in X$ there exist $p^+, p^- \in X$ where $p^- \ll p \ll p^+$, i.e. there exist no point in the "beginning" or "end" of time. Equivalently, both $I^+(p), I^-(p)$ are always non-empty.

A Lorentzian pre-length space is:

- *strongly causal* if the Alexandrov topology is the topology induced by d (i.e. timelike diamonds $I(p,q)$ form a subbasis),
- *internally compact* if all causal diamonds are compact and
- *globally hyperbolic* if it is both non-totally imprisoning and internally compact.

$\Diamond$

These definitions agree with the old ones for (strongly causal) spacetimes, and are stronger than the transferred old ones for Lorentzian pre-length spaces:

Proposition 1.6.32 (Curve vs. relation causal convexity)**.** *A subset U of a causally path connected Lorentzian pre-length space X which is curve-causally convex is also relation-causally convex. This also works for spacetimes with (piecewise C^1) future directed g-causal curves.*

A subset U of a Lorentzian pre-length space X which is relation-causally convex is also curve-causally convex.

Proof. Let $p \leq q \leq r$ be points contradicting the relation-causal convexity of U (i.e. $p, r \in U$, $q \notin U$). By causal path connectedness (or the definition of $\leq$ in spacetimes) we find a future directed (g-)causal curve $\gamma : p \rightsquigarrow q \rightsquigarrow r$. But this future directed causal curve meets U in a disconnected way (it begins at $p \in U$, leaves U as $q \notin U$, and ends in $r \in U$), contradicting curve-causal convexity.

Let γ be a future directed causal curve contradicting curve-causal convexity of U, i.e. there are $t_0 < t_1 < t_2$, such that $p = \gamma(t_0)$ and $r = \gamma(t_2)$ are in U, but $q = \gamma(t_1)$ is not. But γ is future directed causal, so $p \leq q \leq r$, and p, q, r contradict relation-causal convexity. $\square$

Theorem 1.6.33. *Let X be a Lorentzian pre-length space.*

(1) If it is strongly causal (with this new definition, 1.6.31), every point has a neighbourhood base of relation-causally convex sets. If furthermore no open set has $\ll$-isolated points, the converse is also true. In particular, if X is strongly causal (with this new definition), every point has a neighbourhood base of curve-causally convex sets (i.e. the old definition of strong causality, 1.4.3, transferred to Lorentzian pre-length spaces).
(2) If it is non-totally imprisoning and causally path connected, it is causal.
(3) If it is strongly causal, locally causally closed and d-compatible, it is non-totally imprisoning.
(4) If it is internally compact and contains no $\ll$-isolated points, $J^+(p)$ is closed for all points $p \in X$.
(5) If it is strongly causal and consists of more than one point, it has no $\ll$-isolated points.

Proof. (1) If X is strongly causal, the sets $\bigcap_i I(p_i, q_i)$ $(i \in \{1, \cdots, n\})$ form a basis of the topology. But one immediately sees these sets are relation-causally convex open sets, so we get the old definition (1.4.3).

On the other hand, if U_i is a basis of the topology consisting of relation-causally convex open sets, we know $\forall p, q \in U_i$ the open $I(p, q) \subseteq U_i$. So if there is no $\ll$-isolated point in U_i, all points in U_i are contained in one of the $I(p, q) \subset U_i$ and we get that the open sets $I(p, q)$ form a basis of the topology.

For a curve-causally convex basis, we note that by proposition 1.6.32 relation-causally convex sets are curve-causally convex, so $V = \bigcap_i I(p_i, q_i)$ $(i \in \{1, \cdots, n\})$ still works.

(2) If X was not causal, we could find $p \leq q \leq p$ $(p \neq q)$. By causal path connectedness we get a future directed causal curve $\alpha : p \rightsquigarrow q$ and a future directed causal curve $\beta : q \rightsquigarrow p$. Concatenating, we get a (non-constant) closed future directed causal curve γ. Its image $\mathrm{im}(\gamma)$ is compact. Repeating γ, we get a curve "going around in circles" of infinite d-length contained in the compact $\mathrm{im}(\gamma)$, contradicting non-total imprisonment.

(3) If X was not non-totally imprisoning, we would have a future directed causal curve γ of infinite d-length contained in a compact set K. It cannot accumulate infinite d-length in a compact domain: cover the compact image with finitely many open d-compatible sets U_i. The connected components of the

pre-images of these sets cover the compact domain, so finitely many are enough. But each of those segments of γ can only contribute a finite length. So w.l.o.g., we assume $\mathrm{dom}(\gamma) = [0, \infty)$.

By strong causality we assume the U_i to be timelike diamonds: $U_i = I(p_i, q_i)$. On each of the finitely many U_i, γ can only gather finite length, so it has to visit a U_i twice, i.e. leave a U_i and enter it again: We have $t_0 < t_1 < t_2$, such that $\gamma(t_0), \gamma(t_2) \in I(p_i, q_i)$, but $\gamma(t_1) \notin I(p_i, q_i)$. But this means $p_i \ll \gamma(t_0) \leq \gamma(t_1) \leq \gamma(t_2) \ll q_i$, so $\gamma(t_1) \in I(p_i, q_i) \lightning$.

(4) Let $q_n \to q$ with $p \leq q_n$. As X contains no $\ll$-isolated points, we find $q \ll \tilde{q}$. The open set $I^-(\tilde{q})$ contains q and thus all but finitely many q_n. As $J(p, \tilde{q})$ is compact and contains the q_n, it also contains q. In particular, $q \in J^+(p)$, so $J^+(p)$ is closed.

(5) Let p be a $\ll$-isolated point, and let q be another point. As X is Hausdorff, there exists an open neighbourhood U of p not containing q. But by strong causality we find an intersection of timelike diamonds $O = \bigcap_i I(x_i, y_i)$ with $p \in O \subseteq U$. This intersection is not the trivial intersection as $q \notin O$, so there exists an i and $x_i \ll p \ll y_i$ and p was not $\ll$-isolated. $\square$

Theorem 1.6.34. *Let M be a spacetime (with an additional Riemannian metric h), and X be the induced Lorentzian pre-length space.*

(1) X is d-compatible.
(2) No open subset of X has $\ll$-isolated points.
(3) If M is strongly causal, X is locally causally closed.
(4) If M is strongly causal, X is non-totally imprisoning.
(5) If M is strongly causal (i.e. every point has a neighbourhood base of curve-causally convex sets, 1.4.3), X is also strongly causal (i.e. timelike diamonds form a subbasis, 1.6.31).
(6) If M is globally hyperbolic, τ is continuous.
(7) Let M be a strongly causal spacetime. Considered as a Lorentzian pre-length space, it is causally path and length connected and length continuous.

Proof. (1) At each point $p \in M$, we take an orthonormal basis $e_i \in T_pM$ and a chart (x^i), such that $g|_p = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2$ and $x^i(p) = 0$. Then the alternative Lorentzian metric $g_\varepsilon = -(1+\varepsilon)(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2$ has strictly narrower lightcones ($g|_p \prec g_\varepsilon|_p$) and so we find a small neighbourhood U of p where we still have $g|_U \prec g_\varepsilon|_U$ (and g_ε was extended constantly). We restrict U to a relatively compact open subset V with $x^0(V) \subseteq (-c, c)$, which will be the open d-compatible set.

Now let γ be a future directed g-causal curve contained in V. Then γ is future directed g_ε-timelike, so the component $x^0(\gamma(t))$ is strictly increasing. By reparametrizing we can assume $x^0(\gamma(t)) = t$. We set $\tilde\gamma(t) = (x^i(\gamma(t)))_{i>0}$ to be the remaining components. As γ' is g_ε-timelike, we know $\|\tilde\gamma'\|^2_{\mathrm{eucl}} < (1+\varepsilon)$. Then the euclidean length of γ can be estimated: $L_{\mathrm{eucl}}(\gamma) = \int_{\mathrm{dom}(\gamma)} \sqrt{1 + \|\tilde\gamma'\|^2_{\mathrm{eucl}}} < 2c\sqrt{2+\varepsilon}$. As all Riemannian metrics can be estimated against each other on relatively compact subsets, we also get a constant estimate of the d-length of future directed causal curves in V, so V is an open d-compatible set.

(2) Let $p \in U \subseteq M$ be a $\ll$-isolated point. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be any timelike geodesic with $\gamma(0) = p$. But then $\gamma(-\delta) \ll p \ll \gamma(\delta)$ are all contained in U for small $\delta > 0$, so p is not $\ll$-isolated.

(5) The sets $I(p,q) = I^+(p) \cap I^-(q)$ are open by proposition 1.6.5. We still need that for each point r and every neighbourhood U, there is such a set contained in U, containing r: But we already know we can find a curve-causally convex neighbourhood $V \subseteq U$ of r, and that every open set has no $\ll$-isolated points. By proposition 1.6.32 (a spacetime is always causally path connected) V is covered by sets of the form $I(p,q)$, i.e. $I(p,q)$ form a basis.

(3) By (1) we can find a cover A of M consisting of open d-compatible neighbourhoods whose closure is compact. By (5) timelike diamonds form a basis and we can refine the cover A to a cover B consisting of timelike diamonds, giving relatively compact relation-causally closed d-compatible neighbourhoods. We now refine B to C, consisting of open sets whose closure is contained in one of the B. We claim C is the covering by causally closed neighbourhoods.

So let $U \in C$, $\bar U \subseteq V \in B$ and $p_n \leq q_n \in \bar U$ converge: $p_n \to p$, $q_n \to q$. We get future directed g-causal curves $\gamma_n : p_n \rightsquigarrow q_n$, which are automatically contained in V as V is relation-causally convex. As V is a d-compatible neighbourhood, we can reparametrize the γ_n to be uniformly Lipschitz and can apply [Chr11, 2.6.1] to get a Lipschitz curve $\gamma : p \rightsquigarrow q$ with γ' future causal almost everywhere. By [KS18, 5.9] we get a piecewise C^1 curve with the same endpoints, and we get the desired $p \leq q$.

(4) now follows from the last theorem (1.6.33.(3)).

For (6), see [O'N83, 14.21]

(7) As $\ll$ and $\leq$ are defined by the existence of future directed g-timelike and (g-)causal curves, X is automatically causally path connected. Causal length connectedness follows, as g-timelike curves in a spacetime have positive length. Length continuity follows directly from the integral formula of the length (the lengths agree by proposition 1.6.8). □

Example 1.6.35. A Lorentzian pre-length space coming from an absolute time function is always causal (but in general not strongly causal). ◊

Example 1.6.36. Take the (non-strongly causal) spacetime from example 1.4.5. Considered as a Lorentzian pre-length space, it is not locally causally closed:

There is no neighbourhood U of any point x on the green line (see graphic in 1.6.10) which is causally closed: Take $p_n \in U$ above x in the blue area converging to p on the green line, $q_n \in U$ below x in the red area converging to q on the green line. Then $p_n \leq q_n$, but $p \not\leq q$. ♢

Example 1.6.37. A globally hyperbolic locally causally closed, but not globally causally closed Lorentzian pre-length space can be explicitly defined as follows:

We begin by taking points $p_n \to p$, $q_n \to q$ with $p_n \leq q_n$ but $p \not\leq q$. To be specific, we say $p_n \not\leq q_m$ for $n \neq m$ and $\tau(p_n, q_n) = 0$ (i.e. they are null related) and no further causal relations. But the set of these points is not strongly causal yet.

To make a Lorentzian pre-length space consisting of more than one point strongly causal, there must not be $\ll$-isolated points (1.6.33.(5)). To satisfy this with a quite small number of additional points, we add further "time shifted" points p^k and p_n^k, $k \in \mathbb{Z}$, with $p = p^0$ and $\tau(p^k, p^l) = \max(l-k, 0)$ etc., and do the same thing for q. Note that p_n^k does not converge to p^k as $n \to \infty$.

We want the points $p_n \to p$ to converge, and so we need timelike diamonds to see this: We introduce further "time shifted" points $p^{\pm\frac{1}{k}}$ with $p_n \in I(p^{-\frac{1}{k}}, p^{\frac{1}{k}})$ iff $n \geq k$. To make notation more intuitive, we will assume $p^1 = p^{\frac{1}{1}}$ and $p^{-1} = p^{-\frac{1}{1}}$. To be specific, we define $\tau(p_n^{-\frac{1}{n}}, p) = \tau(p_n, p^{\frac{1}{n}}) = \frac{1}{n}$ and $\tau(p^x, p^y) = \max(y-x, 0)$ for $x, y \in \{0, \pm k, \pm\frac{1}{k} : k \in \mathbb{N}\backslash\{0\}\}$ (and extend by the smallest value allowed by the reverse triangle inequality). We add the same points for q. In total, we define X as the set:

$$\{p, q, p_n, q_n, p^x, q^x, p_n^m, q_n^m : n \in \mathbb{N}, m \in \mathbb{Z}, x = \pm k \text{ or } x = \pm\frac{1}{k} \text{ for } k \in \mathbb{N}\backslash\{0\}\}$$

For the distance d, we define the distance between any two points $x \neq y$ to be $d(x, y) = 1$ except for the convergence of points we can set $d(p_n, p) = d(p^{\pm\frac{1}{n}}, p) = \frac{1}{n}$ (and the further values required by the triangle inequality, e.g. $d(p^{\frac{1}{n}}, p_m) = \frac{1}{n} + \frac{1}{m} = d(p^{\frac{1}{n}}, p) + d(p, p_m)$), and the same distances for the points converging to q.

τ is continuous: we only have to check a few convergent points: e.g. $\tau(p^{-\frac{1}{n}}, q) = \frac{1}{n} \to 0 = \tau(p, q)$. X is strongly causal by construction, and for internal compactness we note the only convergent sequences are contained in $p_n \to p$, $p^{-\frac{1}{n}} \to p$, $p^{\frac{1}{n}} \to p$ (and correspondingly for q, but w.l.o.g. the converging sequence converges to p). So we need when infinitely many of these points are in $J(x, y)$, also p (resp. q) is in $J(x, y)$. But there are not many cases for x to be even before two of these points: only p^{-k} and $p^{-\frac{1}{n}}$ come into consideration. Similarly for y: only p^k, $p^{\frac{1}{n}}$, q^k and $q^{\frac{1}{n}}$ are possible. But p lies between these. And X is locally causally closed: The points with p in their name form a causally closed set, similarly with q. But X is not globally causally closed: $p_n \leq q_n$, but not

$p \leq q$.[6]

$\lozenge$

Sources

The notions locally causally closed, d-compatible, non-totally imprisoning and causal path connectedness come from [KS18, 3.4,3.13,2.35,3.1], the notions globally causally closed, causal length connectedness and length continuity are new. The causality conditions are taken from [KS18, 2.35], whereas the (non-standard) notion of internal compactness stems from [Gal14, 4.1]. Inextensibility and limits (1.6.23) and the limit curve theorem (1.6.24) come from [KS18, 3.12,3.7]. The Lipschitz reparametrization (1.6.25) and comparison of causal convex sets (1.6.32) are new. The example 1.6.26 is new. The summarizing result 1.6.33 splits up as follows: (1), (4) and (5) are new, (2) and (3) is [KS18, 3.26.(ii,iii)], the other summarizing result 1.6.34 splits up as: (1), (2), (6) and (7) are new, (3) is [KS18, 3.5], (4) is [O'N83, 14.13] and (5) is [KS18, 2.38.(iii)]. The example 1.6.37 is new.

1.7 Lorentzian length spaces

1.7.1 Intrinsic time separation functions

Definition 1.7.1 (Maximal curves and intrinsic spaces)**.** Let X be a Lorentzian pre-length space. A future directed causal curve $\gamma : p \rightsquigarrow q$ is called *distance maximizing* or *maximal* if all other future directed causal curves $\tilde{\gamma} : p \rightsquigarrow q$ have a smaller length, i.e. $L_\tau(\gamma) \geq L_\tau(\tilde{\gamma})$. A future directed causal curve $\gamma : p \rightsquigarrow q$ is called *distance realizing* if $L_\tau(\gamma) = \tau(p, q) < \infty$. By the generalized reverse triangle inequality distance realizing curves are automatically maximal curves. Restrictions of maximal / distance realizing curves are also maximal / distance realizing (by length additivity and the reverse triangle inequality).

Like for intrinsic metric spaces we want to define a new time separation function based on the length of curves: We define $\hat{\tau}(p, q) = \sup\{L_\tau(\gamma) : \gamma \text{ is a future directed causal curve } p \rightsquigarrow q\}$, and $\hat{\tau}(p, q) = 0$ if no such curves exist. We note that maximal curves are maximizers in this supremum.

X is called *intrinsic*, if $\tau = \hat{\tau}$ (especially $p \ll q \Leftrightarrow$ there exists a future directed causal curve $p \rightsquigarrow q$ of positive length) and $p \leq q \Leftrightarrow$ there exists a future directed causal curve $p \rightsquigarrow q$. (Colloquially, everything (except d) comes from the length of curves.) An intrinsic X is automatically causally length connected. If X is intrinsic, finite maximal curves are automatically distance realizing.

[6]This set cannot be realized as a subset of Minkowski space with induced $\tau, \ll, \leq$, as Minkowski space is globally causally closed.

X is called *strictly intrinsic* or *geodesic* if it is intrinsic and the supremum is always attained, i.e. there exists a future directed distance realizing curve between any two points $p \leq q$ with $\tau(p,q) < \infty$ and a future directed curve of infinite length between any two points $p \leq q$ with $\tau(p,q) = \infty$.

X is called *timelike strictly intrinsic* if there exist distance realizing curves between any two points $p \ll q$ with $0 < \tau(p,q) < \infty$ and a future directed curve of infinite length between any two points $p \leq q$ with $\tau(p,q) = \infty$. (This does not imply intrinsic, as null curves need not exist.) $\Diamond$

Remark 1.7.2. The pull-back (1.6.12) of τ along a distance realizing curve γ of finite length in a causal Lorentzian pre-length space comes from an absolute time function (1.6.3): We pick an $a \in \mathrm{dom}(\gamma)$ and define for all $p \in \mathrm{dom}(\gamma)$: $t(p) = L(\gamma, a, p)$. As γ is distance realizing and of finite length, so are its restrictions, and for $p \geq q$ we have $t(p) - t(q) = L(\gamma, a, p) - L(\gamma, a, q) = L(\gamma, q, p) = \tau(q, p)$. For $p < q$, we use that our space is causal, and thus $\tau(q,p) = 0$. $\Diamond$

Under certain conditions, $\hat{\tau}$ induces a new Lorentzian pre-length space:

Theorem 1.7.3 (Intrinsification)**.**

(1) If $X = (X, \ll, \leq, d, \tau)$ is causally length connected and length continuous, $\hat{\tau}$ is a new time separation function, making $\hat{X} = (X, \ll, \leq, d, \hat{\tau})$ an intrinsic Lorentzian pre-length space (i.e. $\hat{\tau} = \hat{\hat{\tau}}$).

(2) If $X = (X, \ll, \leq, d, \tau)$ is just length continuous (and not necessarily causally length connected), we define $p \mathrel{\widehat{\ll}} q \Leftrightarrow \hat{\tau}(p,q) > 0$ and $p \mathrel{\widehat{\leq}} q \Leftrightarrow$ there is a future directed causal curve $p \rightsquigarrow q$ or $p = q$. If $I^+_{\widehat{\ll}}(p)$ and $I^-_{\widehat{\ll}}(p)$ is always open, this makes $\hat{X} = (X, \widehat{\ll}, \widehat{\leq}, d, \hat{\tau})$ an intrinsic Lorentzian pre-length space. (1) is a special case of this.

Proof. To handle both statements at once, we set $\widehat{\ll} = \ll$ and $\widehat{\leq} = \leq$ in (1) (which agrees with the definition in (2) by length connectedness). Note that in (2), if $p \mathrel{\widehat{\leq}} q$ (respectively $p \mathrel{\widehat{\ll}} q$) and $p \neq q$, there exists a future directed causal $\gamma : p \rightsquigarrow q$ (of positive length), i.e. we have causal length connectedness (with $\widehat{\ll}$ and $\widehat{\leq}$) in both statements.

The reverse triangle inequality works via concatenating curves: If $p \mathrel{\widehat{\leq}} q \mathrel{\widehat{\leq}} r$ and none of them are equal, we have future directed causal curves between them (by causal length connectedness). We take future directed causal curves $\alpha : p \rightsquigarrow q$, $\beta : q \rightsquigarrow r$ realizing distance up to ε, concatenate them and see $\hat{\tau}(p,r) \geq \hat{\tau}(p,q) + \hat{\tau}(q,r) + 2\varepsilon$. Taking the limit $\varepsilon \to 0$, we get the reverse triangle inequality.

The condition $\hat{\tau}(p,q) > 0 \Leftrightarrow p \ll q$ is just causal length connectedness.

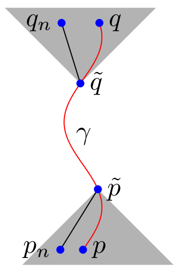
For lower semicontinuity: Let $p_n \to p$, $q_n \to q$ and let $C < \hat{\tau}(p,q)$ be arbitrarily large.[7] If $p \mathrel{\widehat{\not\ll}} q$, $\hat{\tau}(p,q) = 0$ and we have nothing to show. So

[7] To handle $\hat{\tau}(p,q) = \infty$ at the same time. If $\hat{\tau}(p,q) < \infty$, one can use $C = \hat{\tau}(p,q) - \varepsilon$

let $p \mathrel{\widehat{\ll}} q$. We get a future directed causal curve $\gamma : p \rightsquigarrow q$ on the domain $[a, b]$ of length $L_\tau(\gamma) > C$. We pick parameter values $a < t_1 < t_2 < b$, such that $L_\tau(\gamma, t_1, t_2) > C - \varepsilon$ (they exist, as X is length continuous). We define the points $\tilde{p} = \gamma(t_1), \tilde{q} = \gamma(t_2)$. The set $I^-_{\widehat{\ll}}(\tilde{p})$ is open (this was assumed for the second statement) and contains p. As $p_n \to p$, we get an n_0 such that $\forall n \geq n_0,\ p_n \in I^-_{\widehat{\ll}}(\tilde{p})$, i.e. $p_n \mathrel{\widehat{\ll}} \tilde{p}$. We do the same thing (with I^+) at q and pick the larger n_0. By causal length connectedness (with $\widehat{\ll}$) we find future directed causal curves $\alpha_n : p_n \rightsquigarrow \tilde{p}$, $\beta_n : \tilde{q} \rightsquigarrow q_n$. We concatenate $\alpha_n, \gamma|_{[t_1,t_2]}, \beta_n$ and obtain a future directed causal curve $\eta_n : p_n \rightsquigarrow q_n$. It has length:

$$\hat{\tau}(p_n, q_n) \geq L_\tau(\eta_n) \geq L_\tau(\gamma, t_1, t_2) > C - \varepsilon$$

Taking $\varepsilon \to 0$ and $C \to \hat{\tau}(p, q)$, we get $\liminf_n \hat{\tau}(p_n, q_n) \geq \hat{\tau}(p, q)$, i.e. lower semicontinuity.



For (2), we need $\widehat{\ll} \subseteq \widehat{\lesssim} \subseteq \leq$: The first inclusion is clear, the second follows directly from the definition of future directed causal curves.

Switching from $\leq$ to $\widehat{\lesssim}$ does not change the notion of future directed causal curves: $\widehat{\lesssim} \subseteq \leq$ and if $\gamma : [a, b] \to X$ is future directed $\leq$-causal and $t_1 < t_2 \in [a, b]$, $\gamma|_{[t_1,t_2]}$ is a future directed $\leq$-causal curve $\gamma(t_1) \rightsquigarrow \gamma(t_2)$, so $\gamma(t_1) \mathrel{\widehat{\lesssim}} \gamma(t_2)$ and γ is $\widehat{\lesssim}$-future directed causal.

For future directed timelike curves, we have $\widehat{\ll} \subseteq \ll$, but only a rectifiable future directed timelike γ is automatically $\widehat{\ll}$-timelike.

It is intrinsic: we show $L_{\hat{\tau}} = L_\tau$ (as the notion of future directed causal curves agree, $\hat{\tau} = \hat{\hat{\tau}}$ follows):

For $p \mathrel{\widehat{\ll}} q$ and a future directed causal curve $\gamma : p \rightsquigarrow q$, the reverse triangle inequality yields $L_\tau(\gamma) \leq \tau(p, q)$. Taking the supremum over $\gamma : p \rightsquigarrow q$, we see $\hat{\tau} \leq \tau$, and so $L_{\hat{\tau}}(\gamma) = \inf\{\sum_i \hat{\tau}(\gamma(t_i), \gamma(t_{i+1}))\} \leq \inf\{\sum_i \tau(\gamma(t_i), \gamma(t_{i+1}))\} = L_\tau(\gamma)$, and we have $L_{\hat{\tau}} \leq L_\tau$.

For the other direction, let $\gamma : [a, b] \to X$ be a future directed causal curve. Then $L_{\hat{\tau}}(\gamma) = \inf\{\sum_i \hat{\tau}(\gamma(t_i), \gamma(t_{i+1})) : (t_i) \text{ is a partition of } [a, b]\}$, so let $\mathfrak{p} = (t_i)$ be a partition of $[a, b]$ such that $L_{\hat{\tau}}(\gamma) + \varepsilon \geq \sum_i \hat{\tau}(\gamma(t_i), \gamma(t_{i+1}))$. Define $\gamma_i = \gamma|_{[t_i,t_{i+1}]}$. We have $L_\tau(\gamma_i) \leq \hat{\tau}(\gamma(t_i), \gamma(t_{i+1}))$ by definition of $\hat{\tau}$. Summing over i, we get

$$L_{\hat{\tau}}(\gamma) + \varepsilon \geq \sum_i \hat{\tau}(\gamma(t_i), \gamma(t_{i+1})) \geq \sum_i L_\tau(\gamma_i) = L_\tau(\gamma)$$

Taking $\varepsilon \to 0$, we get $L_{\hat{\tau}} \geq L_\tau$. ☐

Proposition 1.7.4. *Let X be a Lorentzian pre-length space, and $\hat{\tau}$ its intrinsified time separation function. Let γ be a curve. Then γ is future directed $\leq$-causal exactly when it is future directed $\widehat{\leq}$-causal. In this case, the lengths agree: $L_\tau(\gamma) = L_{\hat{\tau}}(\gamma)$.*

Proof. This was already proven in the proof of intrinsification (1.7.3). ☐

Proposition 1.7.5. *Let M be a strongly causal spacetime. Then its induced Lorentzian pre-length space is intrinsic.*

Proof. We know $L_g(\alpha) = L_\tau(\alpha)$ for future directed causal piecewise C^1 curves α. Now let $p \leq q$. Then $\hat{\tau}(p,q) \leq \tau(p,q)$ by the generalized reverse triangle inequality.

For the converse inequality, we take $\varepsilon > 0$ and a "long" future directed causal C^1 curve $\gamma : p \rightsquigarrow q$, $L_g(\gamma) \geq \tau(p,q) - \varepsilon$. But we have $\hat{\tau}(p,q) \geq L_\tau(\gamma) = L_g(\gamma) \geq \tau(p,q) - \varepsilon$. Letting $\varepsilon \to 0$, we get the desired $\hat{\tau} = \tau$.

By proposition 1.6.34.(7) we note M is causally length connected, so the $\leq$-condition is automatically satisfied. ☐

Definition 1.7.6. If $(X, \widehat{\ll}, \widehat{\leq}, d, \hat{\tau})$ is a Lorentzian pre-length space, we call it the *intrinsification* of (X, τ).

Its *causality agrees*, if $\widehat{\ll} = \ll$ and $\widehat{\leq} = \leq$. ◊

Example 1.7.7. The intrinsification of a non-causally length connected space can be quite different from the space itself: Let $X = \mathbb{Q}^{n+1} \subset \mathbb{R}^{n,1}$ be the rational points in Minkowski. Then no non-constant curves exist, and $p \mathrel{\widehat{\leq}} q$ only holds for $p = q$. ◊

Theorem 1.7.8. *Let X be a Lorentzian pre-length space having an intrinsification $\hat{X}$. Then:*

(1) $\hat{X}$ is causally length connected.
(2) If X is proper, strongly causal, locally causally closed and d-compatible, $\hat{X}$ is locally causally closed as well.
(3) The properties d-compatibility, non-total imprisonment, length continuity and being chronological / causal are inherited.

(4) If X is causally length connected (i.e. causality agrees), the properties of strong causality, having no $\ll$-isolated points and internal compactness are inherited.

Proof. (1) holds, as $\hat{X}$ is intrinsic ("all properties are given by the length of curves").

(2) By further restricting to timelike diamonds (by strong causality this is possible) we can assume the open d-compatible sets U are relation-causally convex. These will be the causally closed neighbourhoods in $\hat{X}$. So let $p_n \to p$ and $q_n \to q$ be all contained in U with $p_n \mathrel{\widehat{\leq}} q_n$, i.e. we have future directed causal curves $\gamma_n : p_n \rightsquigarrow q_n$. As in 1.6.28, we can reparametrize γ_n and use the limit curve theorem (1.6.24) to get a future directed causal curve $\gamma : p \rightsquigarrow q$, making $p \mathrel{\widehat{\leq}} q$.

(3) By proposition 1.7.4 the notions of future directed causal curves and their lengths agree. Therefore inheritance of d-compatibility, non-total imprisonment and length continuity is clear.

If the new space wasn't chronological resp. causal, we would have $p \neq q$ violating antisymmetry of $\mathrel{\widehat{\ll}}$ resp. $\mathrel{\widehat{\leq}}$. But both relations are contained in the old ones: $\mathrel{\widehat{\ll}} \subseteq \ll$ and $\mathrel{\widehat{\leq}} \subseteq \leq$, thus $p \neq q$ also violate antisymmetry of the old relation $\ll$ resp. $\leq$ and X wasn't chronological resp. causal either.

(4) As X is causally length connected, the relations agree. $\square$

Example 1.7.9 (Counterexample not causally path connected). Take X to be the trace of the null spiral γ in Minkowski space. We take two consecutive points $\gamma(s) = p \ll \gamma(t) = q$ in X. After intrinsifying, we get $\hat{\tau}(p, q) = L_\tau(\gamma, s, t) = 0$ as the length of the null spiral is 0, and $p \mathrel{\widehat{\not\ll}} q$. Therefore causality will change. $\Diamond$

Example 1.7.10 (Counterexample not causally path connected). Take $X = [-1, 0] \times \{0\} \cup \{(t, t) : t \in \mathbb{R}\}$ in Minkowski space, i.e. a union of a timelike curve and a null geodesic. Then $(-1, 0) \mathrel{\widehat{\ll}} (1, 1)$, but there is no future directed timelike curve between them. $\Diamond$

Example 1.7.11. In a Lorentzian pre-length space coming from an absolute time function, all future directed causal curves are distance realizers (by the triangle equality). If any $p \leq q$ has a future directed causal curve $p \rightsquigarrow q$ (i.e. one of the requirements of causally path connected), it is automatically strictly intrinsic. $\Diamond$

In the next two results we study the case of τ being upper semicontinuous as well:

Proposition 1.7.12 (Length is upper semicontinuous if τ is)**.** *Let X be a locally causally closed Lorentzian pre-length space where τ is upper semicontinuous as well (making it continuous). Then if the future directed causal curves γ_n*

converge pointwise to a curve γ, γ is (if not constant) future directed causal and $L_\tau(\gamma) \geq \limsup_n L_\tau(\gamma_n)$.

In particular, L_τ is upper semicontinuous.

Proof. Let $\mathfrak{p} = (t_i)$ be a partition of $[a, b]$. If τ is upper semicontinuous, we have $\limsup_n \tau(\gamma_n(t_i), \gamma_n(t_{i+1})) \leq \tau(\gamma(t_i), \gamma(t_{i+1}))$. Summing over i, we get $\limsup_n V_\mathfrak{p}(\gamma_n) \leq V_\mathfrak{p}(\gamma)$, i.e. $V_\mathfrak{p}$ is upper semicontinuous as well (for a fixed partition $\mathfrak{p}$).

Let $\varepsilon > 0$. We choose $\mathfrak{p}$ fine enough such that $L_\tau(\gamma) + \varepsilon \geq V_\mathfrak{p}(\gamma)$. By upper semicontinuity of $V_\mathfrak{p}$ we have an n_0 such that for all $n > n_0$, $V_\mathfrak{p}(\gamma_n) \leq V_\mathfrak{p}(\gamma) + \varepsilon$. So we have:

$$\begin{aligned} &L_\tau(\gamma) + 2\varepsilon \geq V_\mathfrak{p}(\gamma) + \varepsilon \geq V_\mathfrak{p}(\gamma_n) \geq L_\tau(\gamma_n) \\ \Rightarrow\ &L_\tau(\gamma) + 2\varepsilon \geq \sup_{n \geq n_0} L_\tau(\gamma_n) \geq \inf_{n_0} \sup_{n \geq n_0} L_\tau(\gamma_n) = \limsup_n L_\tau(\gamma_n) \end{aligned}$$

We let $\varepsilon \to 0$ and get upper semicontinuity. □

Proposition 1.7.13 ($\hat{\tau}$ is upper semicontinuous if τ is)**.** *Let X be a locally causally closed, d-compatible and proper Lorentzian pre-length space where τ is upper semicontinuous as well (making it continuous), then $\hat{\tau}$ is upper semicontinuous within open curve-causally convex d-compatible sets U (making $\hat{\tau}|_{U \times U}$ continuous).*

Proof. Let $p_n, q_n \in U$ be converging $p_n \to p, q_n \to q$. We need $\hat{\tau}(p, q) \geq \limsup_n \hat{\tau}(p_n, q_n)$. W.l.o.g., we can assume that either all $\hat{\tau}(p_n, q_n) = \infty$ or none. By the definition of $\hat{\tau}$, for $C_n < \hat{\tau}(p_n, q_n)$ arbitrarily large, [8] we get future directed causal curves $\gamma_n : p_n \rightsquigarrow q_n$ of $L_\tau(\gamma_n) = C_n$. As in 1.6.28, we can reparametrize γ_n and use the limit curve theorem (1.6.24) to get a subsequence uniformly converging to some (possibly constant) future directed causal curve $\gamma : p \rightsquigarrow q$ defined on the same interval, w.l.o.g. we assume $\gamma_n \to \gamma$ uniformly.

By the upper semicontinuity of L_τ:

$$\hat{\tau}(p, q) \geq L_\tau(\gamma) \geq \limsup_n L_\tau(\gamma_n) \geq \limsup_n C_n$$

Now we let $C_n \to \hat{\tau}(p_n, q_n)$ uniformly (as either all or none of the $\hat{\tau}(p_n, q_n) = \infty$, this is possible). The right hand side becomes $\limsup_n \hat{\tau}(p_n, q_n)$ and we get upper semicontinuity of $\hat{\tau}$. □

Definition 1.7.14 (Substructures)**.** A subset $Y \subseteq X$ in a Lorentzian pre-length space $(X, \ll, \leq, d, \tau)$ gets an *induced structure* of a Lorentzian pre-length space by restricting $\ll, \leq, d$ and τ to Y.

[8] To handle $\hat{\tau}(p_n, q_n) = \infty$ at the same time. If $\hat{\tau}(p_n, q_n) < \infty$, one can use $C_n = \hat{\tau}(p_n, q_n) - \varepsilon$

If this restriction satisfies the conditions of proposition 1.7.3.(2), it induces an *induced intrinsic structure* by intrinsification. $\Diamond$

Example 1.7.15 (Causal funnel). Let $\gamma : p \rightsquigarrow q$ be a future directed causal curve in Minkowski space $\mathbb{R}^{n,1}$. Then the subset $J^-(p) \cup \gamma(\mathrm{dom}(\gamma)) \cup J^+(q)$ of $\mathbb{R}^{n,1}$ together with its intrinsified τ (causality agrees) and the restricted metric d of $\mathbb{R}^{n+1}$ is a globally hyperbolic, d-compatible, locally causally closed and strictly intrinsic Lorentzian pre-length space, a *causal funnel*. We note the special cases where γ is timelike (then X is causally path connected, and there is no future directed null curve $p \rightsquigarrow q$) and where γ is null (then X is not causally path connected: there are $p_- \ll q_+$ where all future directed causal curves connecting them have a null segment). $\Diamond$

Sources

The notions of maximal and distance realizing curves, intrinsic and strictly intrinsic spaces are taken from [KS18, 3.22,2.33], the notion of an induced substructure is new. The statement 1.7.5 is from [KS18, 3.24], whereas intrinsification (1.7.3), 1.7.8 and upper semicontinuity of L_τ (1.7.12) and of $\hat{\tau}$ (1.7.13) are new results. Causal funnels (1.7.15) are taken from [KS18, 3.19].

1.7.2 Localizable spaces

We now say what we expect from a nice neighbourhood in Lorentzian geometry:

Definition 1.7.16. A Lorentzian pre-length space X is called *local*, if:

- it is strictly intrinsic,
- τ is upper semicontinuous (making τ continuous),
- there exist no $\ll$-isolated points and
- it is a d-compatible set.

It is called *regularly local* if, additionally, distance realizing curves do not change their causal character. $\Diamond$

Remark 1.7.17. Again note that the causal character does not agree with the one in spacetimes: null tangents at a single point are ignored, and the null spiral (1.6.7) is g-null but timelike in the Lorentzian pre-length space. $\Diamond$

Proposition 1.7.18 (Small neighbourhoods in spacetimes are local)**.** *The Lorentzian pre-length space induced by an open convex normal d-compatible set U in a spacetime is a regularly local space. In particular, small open convex normal sets are regularly local.*

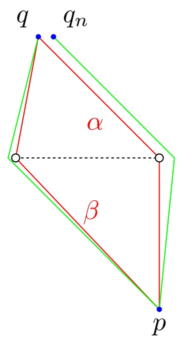
Proof. By proposition 1.3.13 future directed causal geodesics exist exactly when future directed causal curves exist, in which case they are distance realizing, making U strictly intrinsic. Geodesics never change their causality.

If we have sequences $p_n \to p$ and $q_n \to q$ in U with $p_n \leq q_n$, we get future directed causal geodesics $\gamma_n : p_n \rightsquigarrow q_n$. The g-lengths can be determined through the exponential map: $L(\gamma_n) = \left\|\exp_{p_n}^{-1}(q_n)\right\|$. As the map $E : TM \to M \times M$, $E(v_p) = (p, \exp_p(v))$ (see [O'N83, 5.4]) is a diffeomorphism near $\{0 \in T_pM : p \in M\} \subseteq TM$, we get $L(\gamma_n) = \left\|\exp_{p_n}^{-1}(q_n)\right\| \to \left\|\exp_p^{-1}(q)\right\| = L(\gamma)$, so τ is continuous.

In charts we immediately see that there are no $\ll$-isolated points (as the domain is open).

By 1.6.34.(1) small open neighbourhoods are d-compatible sets, thus small open convex normal sets satisfy the conditions of this proposition. $\square$

Example 1.7.19. There exist spacetimes where τ is not continuous (and thus not local): Let $M = \mathbb{R}^{1,1} \backslash (\{0\} \times [\varepsilon, 2])$ in Minkowski space. We set $p = (-2, 2)$ and $q = (1 + \varepsilon, 1 - \varepsilon)$. For the nearly shortest curves, we have to consider two types: The null polygonal line $\alpha : p \rightsquigarrow (0, 2) \rightsquigarrow q$ of length 2 which is excluded as $(0, 2) \notin M$ (and all allowed variations are not causal), and the polygonal line $\beta : p \rightsquigarrow (0, \varepsilon) \rightsquigarrow q$ of small length which is excluded as $(0, \varepsilon) \notin M$, but we have nearby causal curves. Therefore, $\tau(p, q) = L(\beta)$ is small, but we have some sequence of points $q_n \to q$ where $\tau(p, q_n) \to L(\alpha) = 2$.



$\Diamond$

Example 1.7.20. $[0, 1] \times [0, 1]$ in Minkowski space is not local: all the points in $\{0, 1\} \times [0, 1]$ (the future and past boundary) are $\ll$-isolated. $\Diamond$

Definition 1.7.21. A Lorentzian pre-length space is called:

- *localizable*, if it can be covered by open sets U_i with $\widehat{U_i}$ being local and having agreeing causality,
- *regularly localizable*, if it can be covered by open sets U_i with $\widehat{U_i}$ being regularly local and having agreeing causality,

- *strongly localizable*, if there is a basis U_i of the topology with $\widehat{U_i}$ being local and having agreeing causality and
- *strongly regularly localizable*, if there is a basis U_i of the topology with $\widehat{U_i}$ being regularly local and having agreeing causality.

$\Diamond$

Localizable Lorentzian pre-length spaces are automatically d-compatible, have no $\ll$-isolated points, and the strong versions are compatible with taking open subsets.

Proposition 1.7.22 (L_τ is upper semicontinuous in localizable spaces)**.** *In a locally causally closed localizable Lorentzian pre-length space X, L_τ is upper semicontinuous on future directed causal curves with compact domain, i.e. if the future directed causal curves $\gamma_n : [a_n, b_n] \to X$ converge pointwise to a future directed causal curve $\gamma : [a, b] \to X$, then $L_\tau(\gamma) \geq \limsup_n L_\tau(\gamma_n)$.*

Proof. W.l.o.g., all these curves have the same domain $[a, b]$. We cover the compact image of γ by finitely many local spaces U_i $(i = 1, \cdots, n)$ and split up the domain $[a, b] = \bigcup_i [t_i, t_{i+1}]$ such that $\gamma|_{[t_i,t_{i+1}]}$ is contained in U_i. Then for all but finitely many n, $\gamma_n|_{[t_i,t_{i+1}]}$ are contained in U_i. As the length agrees when intrinsifying (1.7.4), we only need to consider the lengths in the local spaces, so we only consider these restrictions and w.l.o.g., we are in a local space. But here, τ is continuous, so length is upper semicontinuous (1.7.12) and $L_\tau(\gamma) \geq \limsup_n L_\tau(\gamma_n)$. $\square$

Proposition 1.7.23 (Length continuity)**.** *A localizable Lorentzian pre-length space X is length continuous, and every future directed causal curve with compact domain has finite length.*

Proof. Let γ be a curve. First, we check the one sided length continuity at t_0 (i.e. $L_\tau(\gamma, a, t)$ is continuous at $t = t_0$). Let U be a local neighbourhood of $\gamma(t_0)$, take $\varepsilon > 0$ such that $\gamma|_{[t_0-\varepsilon,t_0]}$ is still contained in U. By length additivity and as length agrees when intrinsifying (1.7.4) we can work in the local space U ($L_\tau(\gamma, a, t) = L_\tau(\gamma, a, t_0 - \varepsilon) + L_\tau(\gamma, t_0 - \varepsilon, t)$, where the second term can be computed in U). As $\widehat{\tau|_U}$ is finite by locality, we have length continuity (1.6.16).

For the two-sided case (at s_0, t_0), we apply this local argument at both $\gamma(t_0)$ and time-reversed at $\gamma(s_0)$. We have:

$$L_\tau(\gamma, s, t) = L_\tau(\gamma, s, s_0 + \varepsilon) + L_\tau(\gamma, s_0 + \varepsilon, t_0 - \varepsilon) + L_\tau(\gamma, t_0 - \varepsilon, t).$$

If $\forall \varepsilon > 0$ $s_0 + \varepsilon > t_0 - \varepsilon$, we have $s_0 = t_0$ and everything is contained in a local neighbourhood anyway.

For finite length we cover the image of $\gamma : [a,b] \to X$ by finitely many local spaces U_i. We can split up the domain $[a,b] = \bigcup_i [t_i, t_{i+1}]$ such that $\gamma_{[t_i,t_{i+1}]}$ is contained in U_i. As length agrees when intrinsifying, we only need to consider the length in the local spaces, and there we have $L_\tau(\gamma, t_i, t_{i+1}) \leq \widehat{\tau|_{U_i}}(\gamma(t_i), \gamma(t_{i+1})) < \infty$. Summing up, we get $L_\tau(\gamma) < \infty$. □

Theorem 1.7.24 (Push-up of curves). *Let X be a regularly localizable Lorentzian pre-length space. Let $\gamma : p \rightsquigarrow q$ be a future directed causal curve with $L_\tau(\gamma) > 0$ and $t_1 < t_2$ such that $\gamma(t_1) \not\ll \gamma(t_2)$. Then there is a longer future directed timelike curve $\tilde{\gamma} : p \rightsquigarrow q$.*

If X is even strongly regularly localizable, we can even find such a $\tilde{\gamma}$ in every neighbourhood of the image of γ.

Proof. Let $\operatorname{dom}(\gamma) = [a,b]$. W.l.o.g., we assume $t_1 = a$. As $L_\tau(\gamma, t_1, t_2) \leq \tau(\gamma(t_1), \gamma(t_2)) = 0$, it is null there. We set $s = \sup\{t : L_\tau(\gamma, a, t) = 0\}$ (then $0 < s < b$). By length continuity (1.7.23) $L_\tau(\gamma, a, s) = 0$. Let U be a regularly local neighbourhood of $\gamma(s)$, and $s_1 < s < s_2$ such that $\gamma|_{[s_1,s_2]}$ is contained in U. By definition of s we have $L_\tau(\gamma, s_1, s_2) > 0$, but $L_\tau(\gamma, s_1, s) = 0$.

The segment $\gamma|_{[s_1,s_2]}$ is not distance realizing in U: If it were, so would be its restriction $\gamma|_{[s_1,s]}$. But $L_\tau(\gamma, s_1, s_2) > 0$, so $\gamma|_{[s_1,s_2]}$ is timelike, and $L_\tau(\gamma, s_1, s) = 0$, so $\gamma|_{[s_1,s]}$ is null $\lightning$. So we can replace the segment $\gamma|_{[s_1,s_2]}$ by a (future directed timelike) distance realizing curve in U, making it longer.

By compactness we can iterate this procedure to get a longer future directed timelike curve $\tilde{\gamma} : p \rightsquigarrow q$.

If X is strongly regularly localizable and V is a neighbourhood of the image of γ, we can take all U's to be in V. The resulting $\tilde{\gamma}$ will then be contained in V. □

Example 1.7.25 (Causal funnels). A relatively compact open causally convex subset U of a causal funnel (see 1.7.15) X containing all of γ (or none of it) is a local space (making X localizable). If γ contains a null segment, X is not regularly localizable. If γ is timelike, any relatively compact open causally convex subset is regularly local, making X strongly regularly localizable. ◊

Sources

The notions of local and localizable spaces stem from [KS18, 3.16] (but local spaces are not explicitly mentioned there). The example of a spacetime with non-continuous τ (1.7.19) comes from [O'N83, 14.18], locality in spacetimes (1.7.18), upper semicontinuity of L_τ (1.7.22) and push-up of curves (1.7.24) are from [KS18, 3.24.(i),3.17,3.20]. Length continuity in localizable spaces (1.7.23) is new.

1.7.3 Lorentzian length spaces

Definition 1.7.26. A *Lorentzian length space* is a locally causally closed, causally path connected, intrinsic and localizable Lorentzian pre-length space.

If it is regularly localizable, it is a *regular* Lorentzian length space. $\diamondsuit$

Remark 1.7.27. A Lorentzian length space is both causally path connected and causally length connected (as it is intrinsic). $\diamondsuit$

Proposition 1.7.28 (Globally hyperbolic lorentzian length spaces are strictly intrinsic)**.** *Let X be globally hyperbolic Lorentzian length space. Then:*

(1) τ is finite,
(2) τ is continuous and
(3) X is strictly intrinsic.

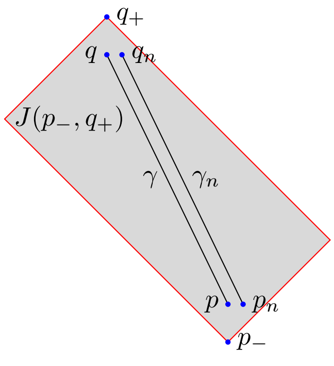
Proof. (2) τ is continuous: We know it is lower semicontinuous. If it were not upper semicontinuous, there would exist $p_n \to p$, $q_n \to q$ with $\limsup_n \tau(p_n, q_n) > \tau(p,q)$. This cannot be the case if $\tau(p,q) = \infty$. So we get $\varepsilon > 0$ such that $\tau(p_n, q_n) \geq \tau(p,q) + \varepsilon$. As X has no $\ll$-isolated points, we find $p_- \ll p$ and $q \ll q_+$. Then p and q and thus all but finitely many p_n and q_n are contained in the open set $I(p_-, q_+) \subseteq J(p_-, q_+)$.

As $\varepsilon > 0$, we have $p_n \ll q_n$, and as X is intrinsic we can take future directed causal curves $\gamma_n : p_n \rightsquigarrow q_n$ of length $L_\tau(\gamma_n) \geq \tau(p_n, q_n) - \frac{\varepsilon}{2}$, which are automatically contained in $I(p_-, q_+)$. As in 1.6.28, we can reparametrize γ_n and apply the limit curve theorem (1.6.24) to get a subsequence of the γ_n converging uniformly to a future directed causal or constant curve $\gamma : p \rightsquigarrow q$ (w.l.o.g., this subsequence is γ_n itself).

By upper semicontinuity of length in localizable spaces (1.7.22)

$$\tau(p,q) \geq L_\tau(\gamma) \geq \limsup_n L_\tau(\gamma_n) \geq \limsup_n \tau(p_n, q_n) - \frac{\varepsilon}{2} \geq \tau(p,q) + \frac{\varepsilon}{2} \quad \lightning$$

so τ is upper semicontinuous.



(1) τ is finite: We indirectly assume there exist $p \leq q$ with $\tau(p,q) = \infty$. As X is intrinsic, we find future directed causal curves $\gamma_n : p \rightsquigarrow q$ with $L_\tau(\gamma_n) \to \infty$. The compact set $J(p,q)$ contains all these curves and is non-totally imprisoning, so as in 1.6.28, we can reparametrize and use the limit curve theorem (1.6.24) to get a subsequence converging to a future directed causal curve $\gamma : [0,K] \to X, p \rightsquigarrow q$.

As X is localizable, we get upper semicontinuity of length (1.7.22) and have $L_\tau(\gamma) \geq \lim_n L_\tau(\gamma_n) = \infty$, so this curve gathers infinite τ-length in a finite interval. But this cannot happen in a localizable space by proposition 1.7.23.

(3) Distance realizing curves exist: Let $p \leq q$. As X is intrinsic, we get future directed causal curves $\gamma_n : p \rightsquigarrow q$ with $L_\tau(\gamma_n) \to \tau(p,q)$. As above, we make them uniformly Lipschitz and apply the limit curve theorem in the compact set $J(p,q)$ to get a future directed causal curve $\gamma : p \rightsquigarrow q$. By upper semicontinuity of length we get $L_\tau(\gamma) \geq \tau(p,q)$. The other inequality is just the generalized reverse triangle inequality. □

This does not work for non-localizable spaces: See example 1.6.17.

Example 1.7.29 (Causal funnel). As we saw in 1.7.15 and 1.7.25, causal funnels with timelike γ are globally hyperbolic Lorentzian length spaces. ◊

Sources

Lorentzian length spaces were introduced in [KS18, 3.22]. The statement 1.7.28 is a summary of [KS18, 3.28,3.30].

2 Curvature comparison

In this section, we will define bounds on the sectional curvatures of semi-Riemannian manifolds, and generalize this to the setting of length spaces and Lorentzian pre-length spaces.

Definition 2.0.1 (Reminder: Sectional curvature). Let (M,g) be a semi-Riemannian manifold, $p \in M$ a point and $P \subseteq T_pM$ a nondegenerate 2-dimensional subspace ("plane")[9]. Let $v, w \in P$ be linearly independent. Let R be the Riemann curvature tensor. Then the value $\frac{g(R(v,w)w,v)}{g(v,v)g(w,w)-g(v,w)^2}$ doesn't depend on the choice of v, w (but on P and p) and is called the *sectional curvature* of M at p on the plane P. ◊

[9] I.e. $(g|_p)|_P : P \times P \to \mathbb{R}$ is a nondegenerate symmetric bilinear map.

2.1 Curvature comparison for length spaces

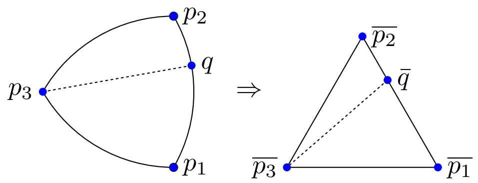
2.1.1 Triangle comparison for $k = 0$

We start with an intuition that on spheres and other positively curved spaces, triangles are "fatter" than normal, and in negatively curved spaces, they are "thinner" than normal. To make this precise, we define the following:

Definition 2.1.1. A *triangle* $\Delta p_1 p_2 p_3$ in a strictly intrinsic length space (X, d) consists of three points $p_i \in X$ and three distance realizing curves $\alpha_{ij} : p_i \rightsquigarrow p_j$.

Here we denote the standard metric on $\mathbb{R}^2$ by $\bar{d}$ to distinguish it from the length space (X, d). A *comparison triangle* to $\Delta p_1 p_2 p_3$ is a triangle with vertices $\overline{p_1}, \overline{p_2}, \overline{p_3}$ in $\mathbb{R}^2$ with agreeing distances, i.e. $d(p_i, p_j) = \bar{d}(\overline{p_i}, \overline{p_j})$. We say the vertex p_i corresponds to $\overline{p_i}$ and the side $\alpha_{ij} : p_i \rightsquigarrow p_j$ corresponds to the straight line $\overline{\alpha_{ij}} : \overline{p_i} \rightsquigarrow \overline{p_j}$.

For a point q on α_{12} we define the corresponding point $\overline{q}$ on $\overline{\alpha_{12}}$ (or correspondingly any other side) by requiring equal distances to the ends of this curve: $d(p_1, q) = \bar{d}(\overline{p_1}, \overline{q})$ (and then automatically, $d(p_2, q) = \bar{d}(\overline{p_2}, \overline{q})$ as α_{12} was distance realizing). Note that it might be necessary to specify which side q should be considered to be on (as two sides can partially overlap).



The length of the solid lines agree, the length of the dashed line will in general not agree and is compared.

We can now locally compare the distance of q to the "opposite" point p_3 with the situation in the comparison situation:

Let X be a strictly intrinsic length space. An open subset U is a *non-negative curvature comparison neighbourhood* resp. *non-positive curvature comparison neighbourhood* if for all triangles $\Delta p_1 p_2 p_3$ in U and q on the $p_1 p_2$-side and all (any) comparison situations $\Delta \overline{p_1 p_2 p_3}$, $\overline{q}$ in $\mathbb{R}^2$, the distance satisfies

$$d(q, p_3) \geq \bar{d}(\overline{q}, \overline{p_3}) \text{ resp. } d(q, p_3) \leq \bar{d}(\overline{q}, \overline{p_3}).$$

Now we call the strictly intrinsic length space X *non-negatively curved* if it is covered by non-negative curvature comparison neighbourhoods. Likewise, X is *non-positively curved* if it is covered by non-positive curvature comparison neighbourhoods. $\Diamond$

Here, we compared the "thickness" of triangles in X to the "thickness" of the comparison triangle in flat space, giving curvature bounded below / above by 0. Using comparison spaces different than flat space, we can do more:

2.1.2 Riemannian comparison spaces

Definition 2.1.2. The Riemannian *constant curvature spaces* or *space forms* are[10]:

- positive curvature k: $K_k^{n,0} = \{v \in \mathbb{R}^{n+1,0} : b(v,v) = \frac{1}{k^2}\} = \frac{1}{k}S^n$, except for $n = 0$ and $n = 1$, where we take a connected component resp. the universal cover (there, the set is two points resp. a circle).
- negative curvature $-k$: $K_{-k}^{n,0}$ is a connected component of $\{v \in \mathbb{R}^{n,1} : b(v,v) = -\frac{1}{k^2}\} = \frac{1}{k}H^n$ (the set is a two-sheeted "hyper"hyperboloid).
- curvature 0 (flat): $K_0^{n,0} = \mathbb{R}^{n,0}$

They are smooth n-dimensional submanifolds of $\mathbb{R}^{n+1,0}$ or $\mathbb{R}^{n,1}$, respectively. The inner product g of $\mathbb{R}^{n+1,0}$ or $\mathbb{R}^{n,1}$ restricts to a Riemannian inner product on these spaces.

In the case $n = 2$, it is called the *k-plane* or the *comparison space* of constant sectional curvature k.

The diameter of $K_k^{n,0}$ is $D_k = \frac{\pi}{\sqrt{k}}$ for $k > 0$ and $D_k = \infty$ for $k \leq 0$. Three values $a_1, a_2, a_3 > 0$ satisfying all three triangle inequalities are said to satisfy the *size bounds for k* if $a_1 + a_2 + a_3 < 2D_k$.[11] This is useful for constructing triangles with prescribed side-lengths in the k-plane. ♢

Example 2.1.3. The k-planes are:

- $k > 0$: $K_k^{2,0} = \frac{1}{\sqrt{k}}S^2$ (the sphere of radius $\frac{1}{\sqrt{k}}$),
- $k = 0$: $K_0^{2,0} = \mathbb{R}^2$ (the Euclidean plane), and
- $k < 0$: $K_k^{2,0} =: \frac{1}{\sqrt{-k}}H$ (the so-called hyperbolic space of radius $\frac{1}{\sqrt{-k}}$).

♢

Proposition 2.1.4 (Constant curvature spaces indeed have constant sectional curvature)**.** *For all $k \in \mathbb{R}$, all $p \in K_k^{n,0}$ and all planes $P \subseteq T_pK_k^{n,0}$, the sectional curvature of the plane P is k. (Note this only makes sense for $n \geq 2$, otherwise, there are no planes.)*

[10]The cases $n = 0$ and $n = 1$ (the exceptions) are not interesting, they just give a point resp. a line

[11]$a_1 + a_2 + a_3 < 2D_k$ is trivial if $k \leq 0$

Proof. For $k = 0$, we have the Riemann tensor $R = 0$ and so $K = 0$. Else, we consider $M = K_k^{n,0}$ as a submanifold of $\bar{M} = \mathbb{R}^{n+1,0}$ resp. $\mathbb{R}^{n,1}$. For two vectors $v, w \in T_pM$ spanning a plane, we have the sectional curvature $K(v,w)$ in M, the sectional curvature $\bar{K}(v,w) = 0$ in $\bar{M}$ and the shape operator $S(v)$ of $M \subset \bar{M}$. The sign ε of $M \subset \bar{M}$ is the sign of k. By [O'N83, 4.20]

$$K(v,w) = \bar{K}(v,w) + \varepsilon \frac{g(S(v),v)g(S(w),w) - g(S(v),w)^2}{g(v,v)g(w,w) - g(v,w)^2}$$

By [O'N83, 4.27]we have $S(v) = -\sqrt{|k|}v$, giving

$$K(v,w) = \varepsilon|k| \frac{g(v,v)g(w,w) - g(v,w)^2}{g(v,v)g(w,w) - g(v,w)^2} = \varepsilon|k| = k.$$

□

Proposition 2.1.5 (Triangles exist)**.** *Given three values $a_1, a_2, a_3 \geq 0$ satisfying all three triangle inequalities and the size bounds for k, there exists a triangle $\Delta p_1p_2p_3$ in a normal neighbourhood in the k-plane such that $d(p_i, p_j) = a_k$ (for i, j, k distinct).*

Any two such triangles $\Delta p_1p_2p_3$, $\Delta q_1q_2q_3$ (for the same a_i) in the k-plane are related by an isometry φ mapping one to the other. φ is unique unless all $a_i = 0$ or there is an equality in one of the triangle inequalities.

Proof. See [AB08, 2.1], points 1. and 2. □

Remark 2.1.6. The values $a_i \geq 0$ are realized as side-lengths of a triangle in the k-plane exactly if they satisfy the triangle inequalities and the weak size bounds for k (i.e. allowing for equality). In the case of equality in the weak size bounds and no equalities in the triangle inequalities, the triangle is however not unique up to an isometry, and in the case of equality in the weak size bounds and an equality in one of the triangle inequalities or all $a_i = 0$, it is unique up to a non-unique isometry. ◊

Proof. By 2.1.5 we only have to consider $k > 0$, i.e. the sphere of radius $r = \frac{1}{\sqrt{k}}$. As the weak size bounds are automatically satisfied on the sphere, we only need to check the size bounds having equality.

We assume $a_1 \geq a_2 \geq a_3$. By the triangle inequality and the equality in the size bounds $2r\pi = a_1 + a_2 + a_3 \geq 2a_1$.

In case of equality, we can set p_2, p_3 to be antipodal points, and can divide a half great circle to get the splitting in $a_2 + a_3 = r\pi$. We still have to choose the shortest curve between the antipodal points which makes it non-unique up to isometry.

In case of inequality, all $a_i < r\pi$ and this triangle can be realized (only) on a great circle: The closed unit speed curve along the great circle has length $2\pi r = a_1 + a_2 + a_3$ and its restriction to subintervals of length less than $r\pi$ is the unique distance minimizing curve between its endpoints. Then we choose three points according to a_1 and a_2 (so two of the lengths are as desired), and by the length of the great circle, the third distance is $2r\pi - a_1 - a_2 = a_3$. This realization is not unique up to isometry, but there is a non-trivial isometry mapping it to itself: the reflection about the great circle. □

Now we can define curvature bounded below / above by any $k \in \mathbb{R}$ by replacing flat space $\mathbb{R}^2$ by the k-plane in the above definition:

2.1.3 Triangle comparison

Definition 2.1.7. Let (X, d) be a metric space, $(K_k^{2,0}, \bar{d})$ be the k-plane and $\Delta p_1 p_2 p_3$ be a triangle in X. A *comparison triangle* to $\Delta p_1 p_2 p_3$ in the k-plane is a triangle with vertices $\overline{p_1}, \overline{p_2}, \overline{p_3}$ in the k-plane with agreeing distances, i.e. $d(p_i, p_j) = \bar{d}(\overline{p_i}, \overline{p_j})$. We say the vertex p_i corresponds to $\overline{p_i}$ and the side $\alpha_{ij} : p_i \rightsquigarrow p_j$ corresponds to the side $\overline{\alpha_{ij}} : \overline{p_i} \rightsquigarrow \overline{p_j}$.

For a point q on α_{12} we define the corresponding point $\overline{q}$ on $\overline{\alpha_{12}}$ by requiring equal distances to the ends of this curve: $d(p_1, q) = \bar{d}(\overline{p_1}, \overline{q})$ (and then automatically, $d(p_2, q) = \bar{d}(\overline{p_2}, \overline{q})$ as α_{12} was distance realizing). We can now compare the distance of q to the "opposite" point p_3 with the situation in the comparison situation:

Let X be a strictly intrinsic length space. An open subset U is called a $\geq k$*-comparison neighbourhood* resp. a $\leq k$*-comparison neighbourhood* if for all triangles $\Delta p_1 p_2 p_3$ in U satisfying the size bounds for k and q on the $p_1 p_2$-side and all (any) comparison situations $\Delta \overline{p_1 p_2 p_3}$ and $\overline{q}$ in the k-plane, the distance satisfies

$$d(q, p_3) \geq \bar{d}(\overline{q}, \overline{p_3}) \text{ resp. } d(q, p_3) \leq \bar{d}(\overline{q}, \overline{p_3}).$$

We say X has *curvature bounded below by k* if it is covered by $\geq k$-comparison neighbourhoods. Likewise, its *curvature is bounded above by k* if it is covered by $\leq k$-comparison neighbourhoods. ♢

Remark 2.1.8 (Automatic size bounds). If X is strictly intrinsic and has curvature bounded above / below, we can restrict ourselves to comparison neighbourhoods U where the size bounds are automatically satisfied, and there exists a basis of the topology of such neighbourhoods. ♢

Proof. Let X be covered by comparison neighbourhoods U. We cover U by balls of radius $\frac{D_k}{6}$ or smaller. In those, the size bounds are automatically satisfied. □

Proposition 2.1.9 (Toponogov, triangle comparing)**.** *Let (M, g) be a Riemannian manifold. Then its sectional curvature is bounded below / above by k if and only if its curvature is bounded below / above by k in the above sense. Moreover, the $\leq k$ or $\geq k$ comparison neighbourhoods can be chosen to be normal neighbourhoods.*

Proof. See [AB08, 5.1,5.2,5.3]. □

In particular, the k-plane has curvature bounded below by any $\tilde{k} \leq k$ and bounded above by any $\tilde{k} \geq k$.

Proposition 2.1.10 (Transitivity of curvature bounds)**.** *A strictly intrinsic length space X having curvature bounded below (resp. above) by k implies having it bounded below (resp. above) by any $\tilde{k} \leq k$ (resp. $\tilde{k} \geq k$).*

Proof. For simplicity, we only do the bounded below case, the other case is analogous. Let $\tilde{k} \leq k$ and the curvature of X bounded below by k, i.e. X is covered by $\geq k$-comparison neighbourhoods. We can use 2.1.8 to get comparison neighbourhoods U where the size bounds for both k and $\tilde{k}$ are automatically satisfied. We claim U is also a $\geq \tilde{k}$-comparison neighbourhood:

We need to check the condition for a triangle $\Delta p_1 p_2 p_3$ and a point q on α_{12} in U: We get a comparison situation $\Delta\overline{p_1 p_2 p_3}$, $\overline{q}$ in a normal neighbourhood in the k-plane, and another comparison situation $\Delta\widetilde{p_1}\widetilde{p_2}\widetilde{p_3}$, $\widetilde{q}$ in the $\tilde{k}$-plane (we denote the metric in the $\tilde{k}$-plane by $\tilde{d}$).

As U is a $\geq k$-comparison neighbourhood, we know $d(q, p_3) \leq \bar{d}(\overline{q}, \overline{p_3})$. As the normal neighbourhood in the k-plane is a $\geq \tilde{k}$-comparison neighbourhood (2.1.9), $\Delta\overline{p_1 p_2 p_3}$, $\overline{q}$ in the k-plane has $\Delta\widetilde{p_1}\widetilde{p_2}\widetilde{p_3}$, $\widetilde{q}$ as a comparison situation in the $\tilde{k}$-plane, so $\bar{d}(\overline{q}, \overline{p_3}) \leq \tilde{d}(\widetilde{q}, \widetilde{p_3})$.

Using both these inequalities, we get $d(q, p_3) \leq \tilde{d}(\widetilde{q}, \widetilde{p_3})$, making U a $\geq \tilde{k}$-comparison neighbourhood and the curvature of X bounded below by $\tilde{k}$. □

2.1.4 Angles

Definition 2.1.11 (Angles)**.** In the Euclidean vector space $\mathbb{R}^n$, we define the *angle* $\alpha = \sphericalangle(v, w) \in [0, \pi]$ between two non-zero vectors $v, w \in \mathbb{R}^n$ by requiring $\langle v, w\rangle = \cos(\alpha)\,\|v\|\,\|w\|$. By the Cauchy Schwarz inequality we have $\frac{|\langle v,w\rangle|}{\|v\|\|w\|} \leq 1$, and thus we can solve for α. It is independent of scaling v and w with positive constants and swapping the vectors.

For two regular piecewise C^1 curves α, β in a Riemannian manifold M with $\alpha(0) = \beta(0) = p$, we define $\sphericalangle(\alpha, \beta)$ to be $\sphericalangle(\alpha'(0), \beta'(0))$ (using an isometry $T_pM \to \mathbb{R}^n$).

For p_1, p_2, p_3 in a strictly intrinsic length space X and a comparison triangle $\Delta\overline{p_1 p_2 p_3}$ in the k-plane we define the *comparison angle* $\tilde{\sphericalangle}_k(p_1, p_2, p_3)$ as the angle of the triangle $\overline{p_1}, \overline{p_2}, \overline{p_3}$ at $\overline{p_2}$. For $k = 0$, we set $\tilde{\sphericalangle} = \tilde{\sphericalangle}_0$.

For two curves $\alpha, \beta : [0, \varepsilon) \to X$ with $\alpha(0) = \beta(0) = p$, we define the *angle* between them as $\sphericalangle(\alpha, \beta) = \lim_{t\to 0, s\to 0} \tilde{\sphericalangle}_0(\alpha(s), p, \beta(t))$, if it exists. $\diamondsuit$

Lemma 2.1.12 (Law of cosines)**.** *The definition of the comparison angle leads to the* law of cosines *for angles: Let p, q, r be three points in the k-plane. Define the distances $a = d(p, q), b = d(q, r), c = d(p, r)$, the angle $\gamma = \tilde{\sphericalangle}_k(p, q, r)$ (which is just the angle of the triangle) and the scaling factor $s = \sqrt{|k|}$. Then we have:*

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(\gamma) && \text{for } k = 0 \\ \cosh(sc) &= \cosh(sa)\cosh(sb) - \sinh(sa)\sinh(sb)\cos(\gamma) && \text{for } k < 0 \\ \cos(sc) &= \cos(sa)\cos(sb) + \sin(sa)\sin(sb)\cos(\gamma) && \text{for } k > 0 \end{aligned}$$

In particular, for $a \neq 0, b \neq 0$ and k fixed, c is a strictly increasing function of γ.

Proof. For $k = 0$, we are in $\mathbb{R}^2$ and the sides of this triangle are just straight lines. [AB08, 2.3] gives us $c^2 = a^2 + b^2 - 2ab\cosh(\omega)$, which is the desired formula.

If $k \neq 0$, the sides of the triangle are the geodesics $\alpha_{pq}, \alpha_{qr}, \alpha_{pr}$, which we assume to be defined on the domain $[0, 1]$. The energy $E(\alpha_{p_i p_j})$ is given by $d(p_i, p_j)^2$ if $p_i \ll p_j$. Now we apply [Kir18, 3.1.3]:

$$\begin{aligned} \cos(\sqrt{kE(\alpha_{pr})}) &= \cos(\sqrt{kE(\alpha_{pq})})\cos(\sqrt{kE(\alpha_{qr})}) \\ &\quad + \frac{\langle -\alpha'_{pq}(1), \alpha'_{qr}(0)\rangle}{\sqrt{E(\alpha_{pq})}\sqrt{E(\alpha_{qr})}} \sin(\sqrt{kE(\alpha_{pq})})\sin(\sqrt{kE(\alpha_{qr})}) \end{aligned}$$

Note this uses imaginary numbers if $k < 0$. In that case, $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$.
For $k > 0$, this equation reads:

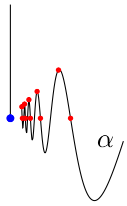
$$\cos(sc) = \cos(sa)\cos(sb) + \cos(\gamma)\sin(sa)\sin(sb)$$

For $k < 0$, this equation reads:

$$\cosh(sc) = \cosh(sa)\cosh(sb) - \cos(\gamma)\sinh(sa)\sinh(sb)$$

$\square$

Example 2.1.13 (Nonexistent angles). In $\mathbb{R}^2$, consider the curves $\alpha(t) = (t, t\sin(\frac{1}{t}))$ and $\beta(t) = (0, t)$ meeting at $p = \alpha(0) = \beta(0) = (0, 0)$. Then for $s_n = \frac{1}{2\pi n} \to 0$, $\Delta\alpha(s_n)p\beta(t)$ has a right angle at 0, but for $s_n = \frac{1}{2\pi n + \frac{\pi}{2}} \to 0$, $\Delta\alpha(s_n)p\beta(t)$ has an angle of 45° ($\frac{\pi}{4}$).



◊

Proposition 2.1.14 (Angles agree)**.** *Let $\alpha, \beta : [0, \varepsilon) \to M$ be two regular (piecewise) C^1 curves in a Riemannian manifold (M, g). Then $\sphericalangle(\alpha, \beta)$ is the same as the standard Riemannian angle*[12]

Proof. W.l.o.g., we can assume α and β to be C^1 (by just considering the piece at 0). As α and β are regular, $v = \alpha'(0)$ and $w = \beta'(0)$ do not vanish and the Riemannian angle makes sense.

By the law of cosines (2.1.12, $k = 0$) we get the comparison angle:

$$\cos(\tilde{\sphericalangle}_0(\alpha(s), p, \beta(t))) = \frac{d(p, \alpha(s))^2 + d(p, \beta(t))^2 - d(\alpha(s), \beta(t))^2}{2d(p, \alpha(s))d(p, \beta(t))}.$$

In a convex neighbourhood U of p all these distances are given by geodesics smoothly varying with the points. The distances appearing in the law of cosines behave as follows when $s, t \to 0$:

- $d(p, \alpha(s))^2 = s^2 g(v, v) + o(s^2)$,
- $d(p, \beta(t))^2 = t^2 g(w, w) + o(t^2)$,
- $d(\alpha(s), \beta(t))^2 = g\,(sv - tw, sv - tw) + o((s + t)^2)$.

The first and second equation are special cases of the third one: replace s resp. t by zero.

To prove the third equation, we use the map $E : TM \to M \times M$ defined by $x_p \mapsto (p, exp_p(x))$. We can shrink U so that E restricts to a diffeomorphism with image $U \times U$. Let γ_x be the geodesic with initial velocity x, then $d(\alpha(s), \beta(t))$ is the length of the distance minimizing geodesic $\gamma_{E^{-1}(\alpha(s), \beta(t))}$ on the domain $[0, 1]$ (as U is convex), which is the length of the initial velocity $E^{-1}(\alpha(s), \beta(t))$, giving $d(\alpha(s), \beta(t))^2 = g(E^{-1}(\alpha(s), \beta(t)), E^{-1}(\alpha(s), \beta(t)))$. In a normal chart, we have $\lim_{t\to 0} \frac{E^{-1}(p, \gamma_y(t)) - E^{-1}(p,p)}{t} = \lim_{t \to 0} \frac{ty}{t} = y$ and $\lim_{t\to 0} \frac{E^{-1}(\gamma_x(t), p) - E^{-1}(p,p)}{t} = \lim_{t\to 0} \frac{-tx}{t} = -x$ as the geodesics between these two points is given by γ_y and γ_{-x}, respectively.

We expand into a Taylor series at $s = t = 0$: The $O(1)$ term is 0, the $O(t)$ term is $\partial g(0, 0) + 2g(\partial_t|_0 E^{-1}(\alpha(0), \beta(t)), 0) = 0$ and similarly for the $O(s)$ term.

[12] I.e. given by $g|_p(\alpha'(0), \beta'(0)) = \sqrt{g|_p(\alpha'(0), \alpha'(0)) g|_p(\beta'(0), \beta'(0))} \cos(\theta)$

The $O(t^2)$ term is

$$\frac{1}{2}(\partial\partial g(0,0) + \cdots + 2g(\partial_t|_0 E^{-1}(\alpha(0), \beta(t)), \partial_t|_0 E^{-1}(\alpha(0), \beta(t)))) = g(w, w),$$

the $O(s^2)$ term is symmetric to this one, the $O(st)$ term is

$$\partial\partial g(0,0) + \cdots + 2g(\partial_t|_0 E^{-1}(\alpha(0), \beta(t)), \partial_s|_0 E^{-1}(\alpha(s), \beta(0))) = 2g(-v, w).$$

In total we get the Taylor expansion of $d(\alpha(s), \beta(t))^2$:

$$0 + 0s + 0t + g(-v, -v)s^2 + g(w, w)t^2 + 2g(-v, w)st + o((s+t)^2) =$$

$$g(sv - tw, sv - tw) + o((s+t)^2)$$

Plugging these approximations into the law of cosines and ignoring the $o((t+s)^2)$ terms (if v and w are non-zero), we are left with

$$\cosh(\tilde{\sphericalangle}_0(\alpha(s), p, \beta(t)))\frac{s^2 g(v,v) + t^2 g(w,w) - g(sv - tw, sv - tw)}{2st\sqrt{g(v,v)}\sqrt{g(w,w)}}.$$

Expanding the difference term and cancelling, we get $\frac{g(v,w)}{\sqrt{g(v,v)}\sqrt{g(w,w)}}$, which is the formula of the Riemannian angle. $\square$

Proposition 2.1.15 (All $\tilde{\sphericalangle}_k$ converge to $\sphericalangle$)**.** *Let X be a length space, α, β two curves with $\alpha(0) = \beta(0) = p$. Then for all k, $\lim_{t,s\to 0} \tilde{\sphericalangle}_k(\alpha(s), p, \beta(t)) = \sphericalangle(\alpha, \beta)$ and one exists if and only if the other exists.*

Proof. We restrict to t, s small enough to have the size bounds automatically satisfied. By the law of cosines (2.1.12) we have (with $l = \sqrt{|k|}$ and the sides $a = d(p, \alpha(s)), b = d(p, \beta(t)), c = d(\alpha(s), \beta(t))$): if $k > 0$,

$$\cos(\tilde{\sphericalangle}_k(\alpha(s), p, \beta(t))) = \frac{\cos(lc) - \cos(la)\cos(lb)}{\sin(la)\sin(lb)}$$

Expanding the cosines and sines, this is:

$$\frac{1 - (lc)^2/2 + o((lc)^2) - (1 - (la)^2/2 + o((la)^2))(1 - (lb)^2/2 + o((lb)^2))}{l^2 ab + o(l^4(ab)^2)} =$$

$$\frac{(la)^2/2 + (lb)^2/2 - (lc)^2/2 - l^4a^2b^2/4 + o((la)^2) + o((lb)^2) + o((lc)^2)}{l^2 ab + o(l^4(ab)^2)} =$$

$$\frac{a^2 + b^2 - c^2 - l^2a^2b^2/2 + o(a^2) + o(b^2) + o(c^2)}{2ab + o((lab)^2)} =$$

$$\frac{a^2 + b^2 - c^2}{2ab} - \frac{l^2a^2b^2/2 + o(a^2) + o(b^2) + o(c^2) + o((lab)^2)}{2ab}$$

As all $a, b, c \to 0$, this converges if and only if

$$\cos(\tilde{\sphericalangle}_0(\alpha(s), p, \beta(t))) = \frac{a^2 + b^2 - c^2}{2ab}$$

converges, and they converge to the same value, and similarly for $k < 0$. $\square$

Proposition 2.1.16 (Angles exist in spaces of bounded curvature)**.** *Let X be a strictly intrinsic length space with curvature bounded above or below. Then for any two distance realizing curves $\alpha, \beta : [0, \varepsilon] \to X$ with $\alpha(0) = \beta(0) = p$, the angle $\sphericalangle(\alpha, \beta)$ exists.*

Proof. We only consider the case with curvature bounded below by k, the other case is analogous. We define the comparison angle function for α and β as $\theta_k(s, t) = \tilde{\sphericalangle}_k(\alpha(s), p, \beta(t))$.
<u>Claim</u>: θ_k is monotonously decreasing in s and t for s and t small.
We only prove θ_k is monotonous in s, the other statement follows analogously. So let $s > s_0 > 0$ and $t > 0$ be small enough to lie in a $\geq k$-comparison neighbourhood where the size bounds for k are automatically satisfied 2.1.8. To calculate $\theta_k(s, t)$ we need a comparison triangle $\bar{\Delta}_1 = \Delta\overline{\alpha(s)\beta(t)}\overline{p}$ to $\Delta_1 = \Delta\alpha(s)p\beta(t)$, and for $\theta_k(s_0, t)$ we need a comparison triangle $\tilde{\Delta}_2$ to $\Delta_2 = \Delta\alpha(s_0)p\beta(t)$ in the k-plane. But as $\alpha(s_0)$ lies on a side of Δ_1, we can do triangle comparison:

The situation $\Delta\alpha(s)\beta(t)p, \alpha(s_0)$ has a comparison situation $\bar{\Delta}_1, \overline{\alpha(s_0)}$. We get $d(\alpha(s_0), \beta(t)) \geq \bar{d}(\overline{\alpha(s_0)}, \overline{\beta(t)})$.

As the sides α and β of Δ_2 are segments of the sides α and β of Δ_1, the sub-triangle $\Delta\overline{\alpha(s_0)\beta(t)}\overline{p}$ of $\bar{\Delta}_1$ and $\tilde{\Delta}_2$ have agreeing side-lengths except the side opposite p. That side satisfies $\bar{d}(\widetilde{\alpha(s_0)}, \widetilde{\beta(t)}) \geq \bar{d}(\overline{\alpha(s_0)}, \overline{\beta(t)})$.

But by the law of cosines (2.1.12) the angle is a strictly increasing function in the opposite side-length. This implies

$$\theta_k(s_0, t) = \tilde{\sphericalangle}_k(\alpha(s_0), p, \beta(t)) \geq \tilde{\sphericalangle}_k(\alpha(s), p, \beta(t)) = \theta_k(s, t),$$

so θ_k is monotonously decreasing in s. Analogously, we get θ_k is monotonously decreasing in t.

As monotone limits exist, $\lim_{s,t\to 0} \theta_k(s, t)$ exists, and as all $\tilde{\sphericalangle}_k$ converge to $\sphericalangle$ (2.1.15), the limit is $\sphericalangle(\alpha, \beta)$. $\square$

Sources

Triangle comparison in length spaces is described in [BBI01, 4], the Riemannian space forms (2.1.2) were taken from [O'N83, 8.22]. The existence of triangles (2.1.5 and Toponogov (2.1.9) is from [AB08, 2.1, 1.1]. Angles (2.1.11) come from [BBI01, 3.6.25], the law of cosines (2.1.12) is mainly [Kir18, 3.1.3]. The agreement of angles (2.1.14) and the convergence of all $\tilde{\measuredangle}_k$ to $\measuredangle$ (2.1.15) seems to be new. The existence of angles (2.1.16) is made analogously to [BBI01, 4.3.5].

2.2 Curvature comparison for Lorentzian length spaces

In Lorentzian length spaces we only have the time distance, so we restrict ourselves to causal triangles:

Definition 2.2.1. A *timelike (geodesic) triangle* $\Delta p_1 p_2 p_3$ in a Lorentzian pre-length space X consists of three points $p_1 \ll p_2 \ll p_3 \in X$ (with $\tau(p_i, p_j) < \infty$ for $i < j$) and three future directed causal distance realizing curves $\alpha_{ij} : p_i \rightsquigarrow p_j$ (for $i < j$).

An *admissible causal (geodesic) triangle* $\Delta p_1 p_2 p_3$ in a Lorentzian pre-length space X consists of three points $p_1 \ll p_2 \leq p_3$ or $p_1 \leq p_2 \ll p_3 \in X$ (with $\tau(p_i, p_j) < \infty$ for $i < j$) and three possibly constant[13] future directed causal distance realizing curves $\alpha_{ij} : p_i \rightsquigarrow p_j$ (for $i < j$). We call the sides between two vertices $p_i \ll p_j$ a *timelike side* (although it need not be a timelike curve).

We call p_1 the *past endpoint* and p_3 the *future endpoint* of the triangle. A causal or timelike triangle is called *non-degenerate* if the reverse triangle inequality $\tau(p, r) \geq \tau(p, q) + \tau(q, r)$ is strict, and it is called *degenerate in the strict sense* if the sides α_{12} and α_{23} are (reparametrized) parts of the longest side α_{13}. $\Diamond$

Now we need the spaces that are to contain the comparison triangle:

2.2.1 Comparison spaces

Example 2.2.2 (Revision). $\mathbb{R}^{n,m}$ is the vector space $\mathbb{R}^{n+m}$ together with the inner product $b(v, w) = -\sum_{i=1}^{m} v_i w_i + \sum_{i=m+1}^{n+m} v_i w_i$. It can also be viewed as a semi-Riemannian manifold. $\Diamond$

Definition 2.2.3. The semi-Riemannian *constant curvature spaces* or *space forms* are:

- positive curvature k: $K_k^{n-1,m} = \{v \in \mathbb{R}^{n,m} : b(v, v) = \frac{1}{k^2}\}$, except for $n-1 = 1$ and $n-1 = 0$, where we take the universal cover resp. a connected component (there the set is a one- resp. two-sheeted hyperboloid)
- negative curvature $-k$: $K_{-k}^{n,m-1} = \{v \in \mathbb{R}^{n,m} : b(v, v) = -\frac{1}{k^2}\}$ except for $m - 1 = 1$ and $m - 1 = 0$, where we take the universal cover resp. a connected component (there the set is a one- resp. two-sheeted hyperboloid)
- curvature 0 (flat): $K_0^{n,m} = \mathbb{R}^{n,m}$

[13] Of course, if $p_1 \leq p_2 \ll p_3$, only $\alpha_{1,2}$ can be constant etc.

They are smooth submanifolds of $\mathbb{R}^{n,m}$. The semi-Riemannian inner product g of $\mathbb{R}^{n,m}$ restricts to a semi-Riemannian inner product on these spaces, with signature given in the upper index (i.e. $(n-1,m)$, $(n,m-1)$ and (n,m), respectively).

In the case $n+m=2$, we call the $K_k^{n,m}$ *k-planes* or the *comparison spaces* of constant sectional curvature k (and signature n,m).

The (finite timelike) diameter of $K_k^{1,1}$ is $D_k = \frac{\pi}{\sqrt{-k}}$ ($D_k = \infty$ for $k \geq 0$)[14].

Three numbers $a_{12}, a_{23}, a_{13} \geq 0$ satisfying the reverse triangle inequality $a_{12} + a_{23} \leq a_{13}$ (making a_{13} the largest) are said to satisfy the *size bounds for k* if $a_{13} < D_k$.[15] This is useful for constructing causal triangles with prescribed side-lengths in the Lorentzian k-plane. ◊

Example 2.2.4. For the Lorentzian case with dimension $n=m=1$, the k-planes are:

- $k>0$: $K_k^{1,1} = \frac{1}{\sqrt{k}} S_1^2$ (scaled versions of the *de Sitter* spacetime),
- $k=0$: $K_0^{1,1} = \mathbb{R}^{1,1}$ (the Minkowski space), and
- $k<0$: $K_k^{1,1} =: \frac{1}{\sqrt{-k}} H_1^2$ (scaled versions of the *anti-de Sitter* spacetime).

◊

Proposition 2.2.5 (Constant curvature spaces indeed have constant sectional curvature)**.** *Let $k \in \mathbb{R}$, $p \in K_k^{n,m}$ be a point and $P \subseteq T_p K_k^{n,m}$ be a plane in the tangent space where $g|_P$ is non-degenerate. Then the sectional curvature of the plane P is k. If $g|_P$ is degenerate, nothing can be said as the denominator in sectional curvature is zero.*
(Note this only makes sense for $n+m \geq 2$, otherwise, there are no planes.)

Proof. For $k=0$, we have the Riemann tensor $R=0$ and so $K=0$. Else, we consider $M = K_k^{n,m}$ as a submanifold of $\bar{M} = \mathbb{R}^{n+1,m}$ resp. $\mathbb{R}^{n,m+1}$. For two vectors $v, w \in T_pM$ spanning a non-degenerate plane, we have the sectional curvature $K(v,w)$ in M, the sectional curvature $\bar{K}(v,w) = 0$ in $\bar{M}$ and the shape operator $S(v)$ of $M \subset \bar{M}$. The sign ε of $M \subset \bar{M}$ is the sign of k. By [O'N83, 4.20]

$$K(v,w) = \bar{K}(v,w) + \varepsilon \frac{g(S(v),v)g(S(w),w) - g(S(v),w)^2}{g(v,v)g(w,w) - g(v,w)^2}$$

By [O'N83, 4.27]we have $S(v) = -\sqrt{|k|}v$, giving

$$K(v,w) = \varepsilon |k| \frac{g(v,v)g(w,w) - g(v,w)^2}{g(v,v)g(w,w) - g(v,w)^2} = \varepsilon |k| = k.$$

□

[14]This is the maximum time distance which is finite.
[15]$a_{13} < D_k$ is trivial if $k \geq 0$.

But sectional curvature comparison is more difficult in the semi-Riemannian case:

Definition 2.2.6 (Sectional curvature comparing)**.** Let M be a semi-Riemannian manifold. Let $p \in M$ be a point and $P \subseteq T_pM$ be a plane. Then P is called *spacelike* if $g|_p$ is positive or negative definite on P, *timelike* if $g|_p$ is nondegenerate and indefinite on P.

M has *sectional curvature bounded below (resp. above) by* k if for all points p and all spacelike planes $P \subseteq T_pM$, the sectional curvature is $\geq k$ (resp. $\leq k$) and for all timelike planes $P \subseteq T_pM$, the sectional curvature is $\leq k$ (resp. $\geq k$). $\quad\lozenge$

Remark 2.2.7. The sectional curvature bounds are not transitive: if M has sectional curvature bounded below / above by k, it does not automatically have it bounded below / above for any $\tilde{k} \neq k$. But if M has timelike sectional curvature bounded below (resp. above) by k, it is automatically bounded below (resp. above) by any $\tilde{k} \geq k$ (resp. $\tilde{k} \leq k$) (but note it is the intuitively wrong direction!). $\lozenge$

Example 2.2.8. The constant curvature space $K_k^{n,m}$ $(n, m \neq 0)$ have curvature bounded above and below by k, but not by any other $\tilde{k} \neq k$. $\lozenge$

Proposition 2.2.9 (Triangles exist)**.** *Given three values $a_{12}, a_{23}, a_{13} \geq 0$ satisfying the reverse triangle inequality $a_{12} + a_{23} \leq a_{13}$ and the size bounds for k, there exists a causal triangle $\Delta p_1p_2p_3$ in a normal neighbourhood in the k-plane such that $\tau(p_i, p_j) = a_{ij}$ (for $i < j$).*

Any two such triangles $\Delta p_1p_2p_3$, $\Delta q_1q_2q_3$ (for the same a_{ij}) in the k-plane are related by an isometry φ mapping one to the other. φ is unique unless $a_{13} = 0$ (making all $a_{ij} = 0$) or the reverse triangle inequality has equality.

Proof. We want to invoke [AB08, 2.1]. Note that there timelike geodesics have negative lengths. So we need side-lengths $(-a_{12}, -a_{23}, -a_{13})$. This result now corresponds exactly to the points 2 and 3 from [AB08, 2.1]. $\square$

Remark 2.2.10. The values $a_{ij} \geq 0$ are realized as side-lengths of a causal triangle in the k-plane exactly if they satisfy the reverse triangle inequality and the weak size bounds for k (i.e. allowing for equality). In the case of equality in the weak size bounds, the causal triangle is however not unique up to an isometry, and in the case all $a_{ij} = 0$, the isometry is not unique. $\lozenge$

Proof. By 2.2.9 we only have to consider $k < 0$, i.e. in the universal cover of the anti-de Sitter space. An important property of the universal cover of anti-de Sitter space is that future directed timelike geodesics are only distance minimizing up to length $\frac{\pi}{\sqrt{-k}}$. At this length all future directed timelike geodesics emanating at the same point p meet again at a point q (seen in the base space of this cover, this is the antipodal point $-p$). Any two timelike related points which are not connected by a distance minimizing timelike geodesic have connecting timelike curves with arbitrarily large length. (See [Chr08, p.3])

So the longest side of a triangle cannot be longer than $\frac{\pi}{\sqrt{-k}}$. 2.2.9 covers side-lengths shorter than this, we only have to consider equality in the size bounds, i.e. $a_{13} = \frac{\pi}{\sqrt{-k}}$. We can pick p_1 and p_3 to be opposite points (lifted to the universal cover such that $\tau(p_1, p_3) = \frac{\pi}{\sqrt{-k}}$). Every future directed timelike geodesic starting at p_1 goes to p_3, and has length $\frac{\pi}{\sqrt{-k}}$. Now we can pick a point on any of these geodesics and we get any degenerate side-lengths. For any triangle with $a_{13} = \frac{\pi}{\sqrt{-k}}$, the geodesic connecting p_1 with p_2 also meets p_3, and thus every triangle with p_1, p_3 antipodal is degenerate. $\square$

2.2.2 Triangle comparison

Definition 2.2.11 (Timelike curvature bounds)**.** Let (X, τ) be a Lorentzian pre-length space, $(K_k^{1,1}, \bar{\tau})$ be the k-plane and $\Delta p_1 p_2 p_3$ be a timelike triangle in X. A *comparison triangle* to $\Delta p_1 p_2 p_3$ in the Lorentzian k-plane is a triangle with vertices $\overline{p_1} \ll \overline{p_2} \ll \overline{p_3}$ in the k-plane with agreeing distances, i.e. $\tau(p_i, p_j) = \bar{\tau}(\overline{p_i}, \overline{p_j})$ for $i < j$. We say the vertex p_i corresponds to $\overline{p_i}$ and the side $\alpha_{ij} : p_i \rightsquigarrow p_j$ $(i < j)$ corresponds to the side $\overline{\alpha_{ij}} : \overline{p_i} \rightsquigarrow \overline{p_j}$.

For a point q on some side α_{ij} of the triangle we define the corresponding point $\overline{q}$ on $\overline{\alpha_{ij}}$ by requiring equal distances to the ends of the curve it is on: we require $\tau(p_i, q) = \bar{\tau}(\overline{p_i}, \overline{q})$ (and then automatically, $\tau(q, p_j) = \bar{\tau}(\overline{q}, \overline{p_j})$ as α_{ij} and $\overline{\alpha_{ij}}$ are distance realizing). Note that it might be necessary to specify which side q should be considered to be on (as two sides can partially overlap). If we take two such points $q_1 \leq q_2$ (usually on different sides), we can compare the distance of q_1 to q_2 with the distance of the corresponding points in the comparison situation (see the proof of 2.2.28 for graphics):

Let X be a Lorentzian pre-length space. An open subset U is called a *timelike* $\geq$ *k-comparison neighbourhood* (resp. *timelike* $\leq$ *k-comparison neighbourhood*) if:

- τ is finite and continuous on U,
- U is timelike strictly intrinsic (i.e. it contains distance realizers between any $p \ll q$) and
- for all timelike triangles $\Delta p_1 p_2 p_3$ in U satisfying the size bounds for k, $q_1 \leq q_2$ two points on the sides α and β and all (any) comparison situations $\Delta\overline{p_1 p_2 p_3}, \overline{q_1}, \overline{q_2}$ in the k-plane, the time distance satisfies

$$\tau(q_1, q_2) \leq \bar{\tau}(\overline{q_1}, \overline{q_2}) \text{ resp. } \tau(q_1, q_2) \geq \bar{\tau}(\overline{q_1}, \overline{q_2})$$

Note that in the bound below, $q_1 \ll q_2$ implies $\overline{q_1} \ll \overline{q_2}$, and in the bound above, $\overline{q_1} \ll \overline{q_2}$ implies $q_1 \ll q_2$.

We say X has *timelike curvature bounded below by k* if it is covered by timelike $\geq k$-comparison neighbourhoods. Likewise, its *timelike curvature is bounded above by k* if it is covered by timelike $\leq k$-comparison neighbourhoods. $\Diamond$

Remark 2.2.12 (Automatic size bounds). If X is strongly causal and has timelike curvature bounded above / below, we can restrict ourselves to comparison neighbourhoods U where the size bounds are automatically satisfied, and there exists a basis of the topology of such neighbourhoods. $\Diamond$

Proof. Let X be covered by timelike comparison neighbourhoods U. By continuity of τ on U we can cover U by open subsets V where the size bounds are automatically satisfied. We cover these V again by timelike diamonds W. These W are again timelike comparison neighbourhoods: τ is still finite and continuous, W is strictly intrinsic, as the distance realizers in U cannot leave the timelike diamond, and the comparison property is trivially satisfied. $\square$

Definition 2.2.13 (Causal curvature bounds)**.** Let X be a Lorentzian pre-length space. An open subset U is called a *causal $\geq k$-comparison neighbourhood* resp. a *causal $\leq k$-comparison neighbourhood* if:

- τ is finite and continuous on U,
- U is strictly intrinsic (i.e. it contains (possibly constant) distance realizers between any $p \leq q$).
- For all admissible causal triangles $\Delta p_1p_2p_3$ in U satisfying the size bounds for k and two points q_1 and q_2 on timelike sides. We require that for all (any) comparison situations $\Delta\overline{p_1p_2p_3}$ and $\overline{q_1}, \overline{q_2}$ in the k-plane, the time distance satisfies

$$\tau(q_1, q_2) \leq \bar{\tau}(\overline{q_1}, \overline{q_2}) \text{ resp. } \tau(q_1, q_2) \geq \bar{\tau}(\overline{q_1}, \overline{q_2}).$$

  Note that $q_1 \ll q_2$ implies $\overline{q_1} \ll \overline{q_2}$ resp. $\overline{q_1} \ll \overline{q_2}$ implies $q_1 \ll q_2$.

As above, we now say X has *causal curvature bounded below by k* if it is covered by causal $\geq k$-comparison neighbourhoods and its *causal curvature is bounded above by k* if it is covered by causal $\leq k$-comparison neighbourhoods. $\Diamond$

The remark on automatic size bounds (2.2.12) is still true for causal curvature bounds.

Proposition 2.2.14 (Toponogov, triangle comparing)**.** *Let (M, g) be a strongly causal spacetime with sectional curvature bounded below / above by k. Then it also has timelike and causal curvature bounded below / above by k in the above sense. Moreover, the timelike and causal $\geq k$ / $\leq k$ comparison neighbourhoods can be chosen to be convex normal neighbourhoods.*

Proof. See [AB08, 5.1,5.2,5.3]. □

In particular, the k-plane has causal curvature bounded below by any $\tilde{k} \geq k$ and bounded above by any $\tilde{k} \leq k$ (note the intuitively wrong direction). But it is unknown if there are spacetimes with timelike but not causal curvature bounded and causal curvature bounded but not the sectional curvature bounded.

Proposition 2.2.15 (Transitivity of curvature bounds)**.** *If a (timelike) strictly intrinsic strongly causal Lorentzian pre-length space X has timelike / causal curvature bounded below (resp. above) by k, it is also bounded below (resp. above) by any larger value $\tilde{k} \geq k$ (resp. smaller value $\tilde{k} \leq k$). Note this is the intuitively wrong direction.*

Proof. We only check that timelike curvature bounded below by k implies it is bounded below by any $\tilde{k} \geq k$ (the other cases are similar). Let $\tilde{k} \geq k$ and the timelike curvature of X be bounded below by k, i.e. X is covered by timelike $\geq k$-comparison neighbourhoods U. As X is strongly causal, we can use 2.2.12, to get comparison neighbourhoods U where both the size bounds for k and the size bounds for $\tilde{k}$ are automatically satisfied. We claim U is also a timelike $\geq \tilde{k}$-comparison neighbourhood:

The first two properties of a timelike $\geq \tilde{k}$-comparison neighbourhood get inherited from U being a timelike $\geq k$-comparison neighbourhood. By 2.2.28 we only need to check the one-sided comparisons: Let $\Delta p_1 p_2 p_3$ be a timelike triangle in U and q be a point on one of the sides. W.l.o.g., we want to compare $q \ll p_i$: We get a comparison situation $\Delta\overline{p_1 p_2 p_3}, \overline{q}$ in a normal neighbourhood in the Lorentzian k-plane, and another comparison situation $\Delta\widetilde{p}_1\widetilde{p}_2\widetilde{p}_3, \widetilde{q}$ in a normal neighbourhood in the $\tilde{k}$-plane (with time separation function $\tilde{\tau}$).

As U is a timelike $\geq k$-comparison neighbourhood, we know $\tau(q, p_i) \leq \bar{\tau}(\overline{q}, \overline{p_i})$. By 2.2.14 we get that the normal neighbourhood in the k-plane is a timelike $\geq \tilde{k}$-comparison neighbourhood (again note the intuitively wrong direction: $\tilde{k} \geq k$). But $\Delta\overline{p_1 p_2 p_3}, \overline{q}$ in the k-plane has $\Delta\widetilde{p}_1\widetilde{p}_2\widetilde{p}_3, \widetilde{q}$ as a comparison situation in the $\tilde{k}$-plane, so $\bar{\tau}(\overline{q}, \overline{p_i}) \leq \tilde{\tau}(\widetilde{q}, \widetilde{p}_i)$.

Using both these inequalities, we get $\tau(q, p_i) \leq \tilde{\tau}(\widetilde{q}, \widetilde{p}_i)$, making U a timelike $\geq \tilde{k}$-comparison neighbourhood, and the timelike curvature of X bounded below by $\tilde{k}$. □

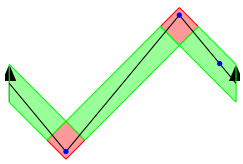
Example 2.2.16. Let $I = (\sqrt{K}t, \infty)$ as a oriented Riemannian manifold (metric h_1), $\mathbb{R}$ the real line as a Riemannian manifold (with h_2) and $f(t) = t^2$, $f : I \to \mathbb{R}^+$. Then we define the spacetime $(-I) \times_f \mathbb{R}$ to be $I \times \mathbb{R}$ as a manifold with Lorentzian metric $g|_{(t,x)} = -h_1|_t + f(t)h_2|_x$ and time orientation given by the orientation of I. This satisfies the conditions of [AB08, 7.1] for sectional curvature bounded below by K (and thus also timelike and causal curvature bounded below by K), but not the condition for sectional curvature bounded above by any $\tilde{K}$. $\Diamond$

Definition 2.2.17. A Lorentzian pre-length space X has *timelike / causal curvature unbounded above / below* if it has an open set U which satisfies the first two properties of a timelike / causal $\geq k$ / $\leq k$-comparison neighbourhood, but the third property (the comparison property) fails for all $k \in \mathbb{R}$. In both cases we say that X has a *curvature singularity*. Obviously, timelike curvature unboundedness implies causal curvature unboundedness. $\Diamond$

Definition 2.2.18. A Lorentzian pre-length space X has *timelike / causal curvature strongly unbounded above / below* if it has a point p and a neighbourhood base of open sets U which satisfy the first two properties of a timelike / causal $\geq k$ / $\leq k$-comparison neighbourhood, but the third property (the comparison property) fails for all $k \in \mathbb{R}$. In both cases we say that X has a *real curvature singularity*. Obviously, strong timelike curvature unboundedness implies strong causal curvature unboundedness. $\Diamond$

Curvature bounds and having curvature strongly unbounded (both in the same direction) exclude each other, but not with the curvature weakly unbounded:

Example 2.2.19 (Non-global bounds). Consider the following subset of Minkowski space and glue the arrows together:



The red and green regions form causal $\geq k$ and $\leq k$-comparison neighbourhoods. But in the whole space, there exist triangles where the triangle bound below-condition is not satisfied: The blue points and black line form a degenerate (but not in the strong sense) triangle where the middle point does not lie on the longest side. Thus for suitably chosen points q_1 and q_2 on the sides of the triangle, $\tau(q_1, q_2) = 0$ as they lie in the different green areas, but $\bar{\tau}(\overline{q_1}, \overline{q_2}) > 0$ in all comparison spaces (as the triangle there is degenerate in the strict sense). Thus $\bar{\tau}(\overline{q_1}, \overline{q_2}) \not\leq \tau(q_1, q_2)$ and the whole space does not satisfy the $\geq k$-comparison property, but the other conditions for being a $\geq k$-comparison neighbourhood. Therefore, this space has curvature bounded below by 0, but at the same time has timelike curvature unbounded below. $\Diamond$

It is unknown whether this is also possible with curvature unbounded above, or if there is a globalization property of the bounds.

Example 2.2.20. Causal funnels with λ timelike have timelike curvature unbounded below: It is l.u.g., so otherwise proposition 2.2.33 would yield it has no timelike branching points. $\Diamond$

Example 2.2.21. The Schwarzschild black hole (including the singularity) has timelike curvature unbounded below. See [KS18, 4.22]. $\Diamond$

2.2.3 Hyperbolic angles

Now for the analogon of angles:

Definition 2.2.22 (Rapidity and speed)**.** In special relativity rapidity is an alternative measure of relative speed: In Minkowski space $(\mathbb{R}^{n,1}, b)$, we define the *rapidity* or *hyperbolic angle* $\omega = \measuredangle(v, w) \geq 0$ between two timelike vectors $v, w \in \mathbb{R}^{n,1}$ by $b(v, w)^2 = \cosh(\omega)^2 b(v, v)b(w, w)$. By the reverse Cauchy Schwarz inequality for future timelike vectors (see [Gal14, 1.1 (1)]) we have $\frac{b(v,w)^2}{b(v,v)b(w,w)} \geq 1$, and thus we can solve for $\omega \geq 0$. It is independent of scaling and reversing v and w.

Using this we can also define rapidity in Lorentzian manifolds M: Let $\alpha, \beta : (-\varepsilon, \varepsilon) \to M$ be two timelike curves with $p = \alpha(0) = \beta(0)$, then the rapidity between them is $\measuredangle(\alpha'(0), \beta'(0))$ (where we identified T_pM with $\mathbb{R}^{n,1}$).

For $p_1 \ll p_2 \ll p_3$ with finite τ-distance in a Lorentzian pre-length space and a comparison triangle $\Delta\overline{p_1 p_2 p_3}$ in the k-plane, we can define the *comparison rapidities* or *hyperbolic comparison angles* $\tilde{\measuredangle}_k(p_2, p_1, p_3)$, $\tilde{\measuredangle}_k(p_1, p_2, p_3)$ and $\tilde{\measuredangle}_k(p_1, p_3, p_2)$ as the rapidities in the corresponding vertex of the comparison triangle. (Like for angles, the second point denoted is the point the hyperbolic angle is at.) For $k = 0$, we set $\tilde{\measuredangle} = \tilde{\measuredangle}_0$.

For a past directed curve $\alpha : [0, \varepsilon) \to X$ and a future directed curve $\beta : [0, \varepsilon) \to X$ with $\alpha(0) = \beta(0) = p$ in a strictly intrinsic Lorentzian pre-length space, we define the *rapidity* or *hyperbolic angle* between them as

$$\measuredangle(\alpha, \beta) = \lim_{t \to 0, s \to 0} \tilde{\measuredangle}_0(\alpha(s), p, \beta(t)),$$

if it exists.[16]

[16]Note requiring α to be past directed and β to be future directed gives $\Delta\alpha(s)p\beta(t)$ a fixed time order. If they were e.g. both future directed in a spacetime, we would find s, t such that $\alpha(s)$ and $\beta(t)$ were not even $\leq$-comparable.

To get the relative speed of two physical particles described by future directed causal curves α and β meeting at $\alpha(0) = \beta(0)$ with rapidity $\omega = \measuredangle(\alpha, \beta)$, the formula $v = \tanh(\omega)$ (in units of the speed of light) is used. $\Diamond$

Lemma 2.2.23 (Law of cosines)**.** *The definition of the comparison rapidity leads to the* hyperbolic law of cosines*: For p, q, r in the Lorentzian k-plane forming a finite timelike triangle (not necessarily in this order), let $a = \max(\tau(p,q), \tau(q,p))$, $b = \max(\tau(q,r), \tau(r,q))$, $c = \max(\tau(p,r), \tau(r,p))$ be the side-lengths, $\omega = \tilde{\measuredangle}_k(p,q,r)$ be the hyperbolic angle at q and the scaling factor $s = \sqrt{|k|}$. If q is a future or past endpoint, we set $\varepsilon = -1$, if it is not, $\varepsilon = 1$*[17]*. Then we have:*

$$\begin{aligned} a^2 + b^2 &= c^2 - 2ab\varepsilon\cosh(\omega) && \textit{for } k = 0 \\ \cos(sc) &= \cos(sa)\cos(sb) + \varepsilon\cosh(\omega)\sin(sa)\sin(sb) && \textit{for } k < 0 \\ \cosh(sc) &= \cosh(sa)\cosh(sb) - \varepsilon\cosh(\omega)\sinh(sa)\sinh(sb) && \textit{for } k > 0 \end{aligned}$$

In particular, fixing a, b and s, $\varepsilon\omega$ is a strictly increasing function in c.

Proof. For $k = 0$, we are in Minkowski space and the sides of this triangle are just the straight lines. [AB08, 2.3] gives us $-c^2 = -a^2 - b^2 - 2ab\varepsilon\cosh(\omega)$, which is the desired formula.

If $k \neq 0$, the sides of the triangle are future or past directed geodesics $\gamma_{pq}, \gamma_{qr}, \gamma_{pr}$, which we assume to be defined on the domain $[0,1]$. The energy $E(\gamma_{p_i p_j})$ is given by $-\tau(p_i, p_j)^2$ if $p_i \ll p_j$. Now we apply [Kir18, 3.1.3]:

$$\begin{aligned} \cos(\sqrt{kE(\gamma_{pr})}) = {} & \cos(\sqrt{kE(\gamma_{pq})})\cos(\sqrt{kE(\gamma_{qr})}) \\ & + \langle \gamma'_{qp}(0), \gamma'_{qr}(0)\rangle \frac{\sin(\sqrt{kE(\gamma_{pq})})}{\sqrt{E(\gamma_{pq})}} \frac{\sin(\sqrt{kE(\gamma_{qr})})}{\sqrt{E(\gamma_{qr})}}. \end{aligned}$$

Note that this uses imaginary numbers if $k > 0$. In that case, $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$.
For $k < 0$, this equation reads:

$$\cos(sc) = \cos(sa)\cos(sb) + \varepsilon\cosh(\omega)\sin(sa)\sin(sb),$$

where ε is the sign of the inner product (which is as described in the statement).
For $k > 0$, this equation reads:

$$\cosh(sc) = \cosh(sa)\cosh(sb) - \varepsilon\cosh(\omega)\sinh(sa)\sinh(sb),$$

where again the minus comes from an i^2 in the denominator. $\square$

Similarly to above, we can construct examples where the rapidity between curves does not exist:

[17] ε is just the sign of the inner product of the initial velocities of the two sides at q

Example 2.2.24 (Nonexistent rapidities). In $\mathbb{R}^{1,1}$, consider the curves $\alpha(t) = (t, \frac{1}{2}\sin(\log(t)))$ and $\beta(t) = (t, \frac{t}{2})$ meeting at $p = \alpha(0) = \beta(0) = (0,0)$. Then $\alpha'(t) = (1, \underbrace{\frac{1}{2}\sin(\log(t)) + \frac{1}{2}\cos(\log(t))}_{\in(-1,1)})$ is always future timelike, so both α and β are future directed timelike.
For $s_n = e^{-2\pi n} \to 0$, $\cosh(\tilde{\measuredangle}_0(\alpha(s_n), p, \beta(t)))^2 = \frac{4}{3}$ (corresponding to $v = \frac{1}{2}$), but for $s_n = e^{-2\pi n - \pi/2} \to 0$, $\cosh(\tilde{\measuredangle}_0(\alpha(s_n), p, \beta(t)))^2 = 1$ (corresponding to $v = 0$). $\Diamond$

Proposition 2.2.25 (Rapidities agree)**.** *Let $\alpha : [0,\varepsilon) \to M$ be a past directed and $\beta : [0,\varepsilon) \to M$ a future directed C^1 regular g-timelike curve in a strongly causal spacetime M. Then the rapidity $\measuredangle(\alpha, \beta)$ in the Lorentzian pre-length space is the same as the Lorentzian rapidity $\measuredangle(\alpha', \beta')$.*

Proof. We try to modify the proof of 2.1.14 to this setting:

As α and β are regular and g-timelike, $v = \alpha'(0)$ and $w = \beta'(0)$ are past resp. future timelike vectors, so the Lorentzian rapidity formula makes sense.

By the law of cosines (2.2.23, $k = 0, \varepsilon = 1$) we get the comparison rapidity:

$$\cosh(\tilde{\measuredangle}_0(\alpha(s), p, \beta(t))) = \frac{\tau(\alpha(s), \beta(t))^2 - \tau(\alpha(s), p)^2 - \tau(p, \beta(t))^2}{2\tau(\alpha(s), p)\tau(p, \beta(t))}.$$

In a convex normal neighbourhood U of p, all these distances are given by geodesics smoothly varying with the points contained in U. The distances appearing behave as follows when $s, t \to 0$:

- $\tau(\alpha(s), p)^2 = -s^2 g(v, v) + o(s^2)$
- $\tau(p, \beta(t))^2 = -t^2 g(w, w) + o(t^2)$
- $\tau(\alpha(s), \beta(t))^2 = -g(-sv + tw, -sv + tw) + o((s + t)^2)$

The first and second equation are special cases of the third one: replace s resp. t by zero.

To prove the third equation, we use the map $E : TM \to M \times M$ defined by $x_p \mapsto (p, \exp_p(x))$. We can shrink U so that E restricts to a diffeomorphism with image $U \times U$. Let γ_x be the geodesic with initial velocity x, then $\tau(\alpha(s), \beta(t))$ is the length of the distance maximizing geodesic $\gamma_{E^{-1}(\alpha(s),\beta(t))}$ on the domain $[0,1]$ (as U is convex), which is the (time-)length of the initial velocity $E^{-1}(\alpha(s), \beta(t))$, giving $\tau(\alpha(s), \beta(t))^2 = -g(E^{-1}(\alpha(s), \beta(t)), E^{-1}(\alpha(s), \beta(t)))$. In a normal chart, we have $\lim_{t\to 0} \frac{E^{-1}(p,\gamma_y(t)) - E^{-1}(p,p)}{t} = \lim_{t\to 0} \frac{ty}{t} = y$ and $\lim_{t\to 0} \frac{E^{-1}(\gamma_x(t),p) - E^{-1}(p,p)}{t} = \lim_{t\to 0} \frac{-tx}{t} = -x$ as the geodesics between these points are given by γ_y and γ_{-x}, respectively.

We expand into a Taylor series at $s = t = 0$: The $O(1)$ term is 0, the $O(t)$ term is $-\partial g(0,0) - 2g(\partial_t|_0 E^{-1}(\alpha(0), \beta(t)), 0) = 0$ and similarly for the $O(s)$ term. The $O(t^2)$ term is

$$\frac{1}{2}(-\partial\partial g(0,0) + \cdots - 2g(\partial_t|_0 E^{-1}(\alpha(0), \beta(t)), \partial_t|_0 E^{-1}(\alpha(0), \beta(t)))) = -g(w, w),$$

the $O(s^2)$ term is symmetric to this one, the $O(st)$ term is

$$-\partial\partial g(0,0) + \cdots - 2g(\partial_t|_0 E^{-1}(\alpha(0), \beta(t)), \partial_s|_0 E^{-1}(\alpha(s), \beta(0))) = -2g(-v, w)$$

In total, we get the Taylor expansion of $\tau(\alpha(s), \beta(t))^2$:

$$0 + 0s + 0t - g(-v, -v)s^2 - g(w, w)t^2 - 2g(-v, w)st + o((s+t)^2) =$$

$$-g(sv - tw, sv - tw) + o((s+t)^2)$$

Plugging these approximations into the law of cosines and ignoring the $o((t+s)^2)$ terms (if v and w are timelike), we are left with

$$\cosh(\tilde{\measuredangle}_0(\alpha(s), p, \beta(t))) = \frac{-g(sv - tw, sv - tw) + g(sv, sv) + g(-tw, -tw)}{2\sqrt{-g(sv, sv)}\sqrt{-g(-tw, -tw)}}.$$

Expanding the difference term and cancelling, we get $\frac{g(v,w)}{\sqrt{-g(v,v)}\sqrt{-g(w,w)}}$, which is the formula of the Lorentzian angle. ☐

Proposition 2.2.26 (All $\tilde{\measuredangle}_k$ converge to $\measuredangle$)**.** *Let X be an intrinsic Lorentzian pre-length space which is strongly causal, α, β a future and a past directed curve with $\alpha(0) = \beta(0) = p$. Then for all k, $\lim_{t,s\to 0} \tilde{\measuredangle}_k(\alpha(s), p, \beta(t)) = \measuredangle(\alpha, \beta)$ (or a corresponding permutation in $\tilde{\measuredangle}_k$) and one exists if and only if the other exists.*

Proof. We restrict to t, s small enough to have the size bounds automatically satisfied (2.2.12). By the law of cosines (2.2.23) we have (with $l = \sqrt{|k|}$, $\varepsilon = \pm 1$ as in the law of cosines and the side-lengths $a = \max(\tau(p, \alpha(s)), \tau(\alpha(s), p)), b = \max(\tau(p, \beta(t)), \tau(\beta(t), p)), c = \max(\tau(\alpha(s), \beta(t)), \tau(\beta(t), \alpha(s)))$): if $k > 0$,

$$\cosh(\tilde{\measuredangle}_k(\alpha(s), p, \beta(t))) = \varepsilon \frac{\cosh(la)\cosh(lb) - \cosh(lc)}{\sinh(la)\sinh(lb)}$$

or the corresponding permutation. Expanding the hyperbolic cosines and sines, this is:

$$\varepsilon \frac{(1 + (la)^2/2 + o(la)^2)(1 + (lb)^2/2 + o(lb)^2) - (1 + (lc)^2/2 + o(lc)^2)}{l^2ab + o(l^2ab)} =$$

$$\varepsilon \frac{1 + (la)^2/2 + (lb)^2/2 + l^4a^2b^2/4 - (lc)^2/2 + o(la)^2 + o(lb)^2 + o(lc)^2}{l^2ab + o(l^2ab)^2} =$$

$$\varepsilon \frac{a^2 + b^2 - c^2 + l^2a^2b^2/2 + o(a)^2 + o(b)^2 + o(c)^2}{2ab + o(lab)^2} =$$

$$\varepsilon \frac{a^2 + b^2 - c^2}{2ab} + \varepsilon \frac{l^2a^2b^2/2 + o(a)^2 + o(b)^2 + o(c)^2 + o(lab)^2}{2ab}$$

As all $a, b, c \to 0$, this converges if and only if

$$\cosh(\tilde{\measuredangle}_0(\alpha(s), p, \beta(t))) = \frac{a^2 + b^2 - c^2}{2ab}$$

(or the corresponding permutation) converges, and they converge to the same value, and similarly for $k < 0$. $\square$

Proposition 2.2.27 (Angles exist in spaces of bounded curvature)**.** *Let X be a strongly causal Lorentzian pre-length space with causal / timelike curvature bounded above or below. Then for any past directed causal / timelike distance realizing curve $\alpha : [0, \varepsilon)$ and future directed timelike curve $\beta : [0, \varepsilon)$ with $\alpha(0) = \beta(0) = p$, the angle $\measuredangle(\alpha, \beta)$ exists. (And of course, an analogous statement holds for α timelike and β causal.)*

Proof. We only consider the case with curvature bounded below by k, the other case is analogous. We define the comparison angle function for α and β as $\theta_k(s, t) = \tilde{\measuredangle}_k(\alpha(s), p, \beta(t))$.
<u>Claim</u>: θ_k is monotonously increasing in s and t for s and t small.
We only prove θ_k is monotonous in s, the other statement follows analogously. So let $s > s_0 > 0$ and $t > 0$ be small enough to lie in a (causal / timelike) $\geq k$-comparison neighbourhood where the size bounds for k are automatically satisfied (2.2.12). To calculate $\theta_k(s, t)$ we need a comparison triangle $\bar{\Delta}_1 = \Delta\overline{\alpha(s)\beta(t)p}$ to $\Delta_1 = \Delta\alpha(s)p\beta(t)$, and for $\theta_k(s_0, t)$ we need a comparison triangle $\tilde{\Delta}_2$ to $\Delta_2 = \Delta\alpha(s_0)p\beta(t)$ in the Lorentzian k-plane. But as $\alpha(s_0)$ lies on a side of Δ_1, we can do triangle comparison:

The situation $\Delta\alpha(s)\beta(t)p, \alpha(s_0)$ has a comparison situation $\bar{\Delta}_1, \overline{\alpha(s_0)}$. We get $\tau(\alpha(s_0), \beta(t)) \leq \bar{\tau}(\overline{\alpha(s_0)}, \overline{\beta(t)})$.

As the sides α and β of Δ_2 are segments of the sides α and β of Δ_1, the sub-triangle $\Delta\overline{\alpha(s_0)\beta(t)p}$ of $\bar{\Delta}_1$ and $\tilde{\Delta}_2$ have agreeing side-lengths except the side opposite p. That side satisfies $\bar{\tau}(\widetilde{\alpha(s_0)}, \widetilde{\beta(t)}) \leq \bar{\tau}(\overline{\alpha(s_0)}, \overline{\beta(t)})$.

But by the law of cosines (2.2.23, $\varepsilon = 1$) the hyperbolic angle is a strictly increasing function in the opposite side-length. This implies

$$\theta_k(s_0, t) = \tilde{\measuredangle}_k(\alpha(s_0), p, \beta(t)) \leq \tilde{\measuredangle}_k(\alpha(s), p, \beta(t)) = \theta_k(s, t),$$

so θ_k is monotonously increasing in s. Analogously, we get θ_k is monotonously increasing in t.

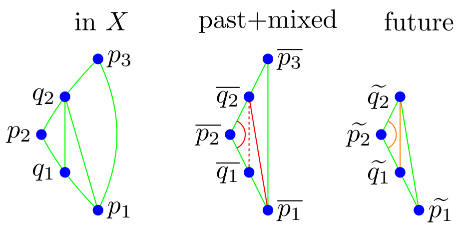
As monotone limits exist, $\lim_{s,t\to 0} \theta_k(s,t)$ exists, and as all $\tilde{\measuredangle}_k$ converge to $\measuredangle$ (2.2.26), the limit is $\measuredangle(\alpha, \beta)$. $\square$

Remark 2.2.28 (One-sided comparison). The definition of timelike and causal curvature bounds with triangle comparisons involving two points q_1 and q_2 on two sides is equivalent to using a *one-sided comparison* of one point q on a side to a vertex p_i. We distinguish *future comparison* if $p_i = p_3$, *past comparison* if $p_i = p_1$ and *across comparison* if $p_i = p_2$. $\Diamond$

Proof. By picking one of the q_i to be a vertex one immediately sees one-sided comparisons are two-sided comparisons.

For the other direction we consider the two-sided comparison (q_1, q_2) and decompose it into two one-sided comparisons: We only do the case of curvature bounded below, the other case follows nearly analogously. We note that the size bounds of a causal triangle get inherited by any causal sub-triangle (i.e. having the vertices on the sides of the triangle), so we need not care about them.

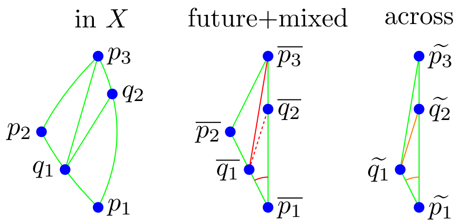
We need to distinguish on which sides the q lie: If both lie on the same side, we have nothing to check (the side is distance realizing and comparisons always have equality). If q_1 is on α_{12} and q_2 is on α_{23} (i.e. the two shorter sides), we need to check the condition on the comparison situation $\Delta\overline{p_1p_2p_3}, \overline{q_1}, \overline{q_2}$. We automatically have $q_1 \ll q_2$ and $\overline{q_1} \ll \overline{q_2}$, so we only have to check the condition on $\tau(q_1, q_2)$. By past comparison for $\Delta p_1p_2p_3, q_2$ we get $\tau(p_1, q_2) \leq \bar{\tau}(\overline{p_1}, \overline{q_2})$, by future comparison for $\Delta p_1p_2q_2, q_1$ (this is a timelike / causal triangle as all points on the side α_{23} are in the timelike future of p_1, except possibly p_2) we get a future comparison triangle $\Delta\widetilde{p_1}\widetilde{p_2}\widetilde{q_2}, \widetilde{q_1}$ which satisfies $\tau(q_1, q_2) \leq \bar{\tau}(\widetilde{q_1}, \widetilde{q_2})$). We need to compare $\tau(q_1, q_2)$ with $\bar{\tau}(\overline{q_1}, \overline{q_2})$: Note that the situation $\Delta\overline{p_1p_2q_2}, \overline{q_1}$ (inside the comparison situation $\Delta\overline{p_1p_2p_3}$) looks just like the future comparison situation $\Delta\widetilde{p_1}\widetilde{p_2}\widetilde{q_2}, \widetilde{q_1}$, except that one side is of different length: $\bar{\tau}(\widetilde{p_1}, \widetilde{q_2}) = \tau(p_1, q_2) \leq \bar{\tau}(\overline{p_1}, \overline{q_2})$. But as the length of the "longest side" $\bar{\tau}(\widetilde{p_1}, \widetilde{q_2})$ increases to $\bar{\tau}(\overline{p_1}, \overline{q_2})$, the angle $\tilde{\measuredangle}_k(\widetilde{q_2}\widetilde{p_2}\widetilde{p_1}) = \tilde{\measuredangle}_k(\widetilde{q_2}\widetilde{p_2}\widetilde{q_1})$ increases to $\tilde{\measuredangle}_k(\overline{q_2p_2p_1}) = \tilde{\measuredangle}_k(\overline{q_2p_2q_1})$ by the law of cosines (2.2.23, $\varepsilon = 1$) and the distance $\bar{\tau}(\widetilde{q_1}, \widetilde{q_2})$ increases to $\bar{\tau}(\overline{q_1}, \overline{q_2})$, yielding the desired $\tau(q_1, q_2) \leq \bar{\tau}(\widetilde{q_1}, \widetilde{q_2}) \leq \bar{\tau}(\overline{q_1}, \overline{q_2})$.



(Sides between equally named points have equal length if they are drawn in the same colour)

Otherwise, we can w.l.o.g. assume q_1 is on α_{12} and q_2 is on α_{13}. As q_1 and q_2 do not automatically have a time ordering, we need to check the condition on both $\tau(q_1, q_2)$ and $\tau(q_2, q_1)$. First we do $\tau(q_1, q_2)$: We want to show the condition on the comparison situation $\Delta\overline{p_1p_2p_3}, \overline{q_1}, \overline{q_2}$ by decomposing it into two one-sided comparisons: By future comparison for $\Delta p_1p_2p_3, q_1$ we get $\tau(q_1, p_3) \leq \bar{\tau}(\overline{q_1}, \overline{p_3})$. By across comparison for $\Delta p_1q_1p_3, q_2$ (this is a timelike / causal triangle as all points on the side α_{12} are in the timelike past of p_3) we get an across comparison situation $\Delta\widetilde{p_1}\widetilde{q_1}\widetilde{p_3}, \widetilde{q_2}$ which satisfies $\tau(q_1, q_2) \leq \bar{\tau}(\widetilde{q_1}, \widetilde{q_2})$.

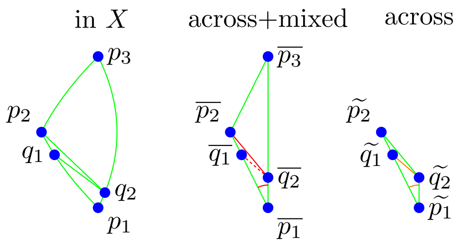
We need to compare $\tau(q_1, q_2)$ with $\bar{\tau}(\overline{q_1}, \overline{q_2})$. We note that the situation $\Delta\overline{p_1q_1p_3}, \overline{q_2}$ looks just like the across comparison situation $\Delta\widetilde{p_1}\widetilde{q_1}\widetilde{p_3}, \widetilde{q_2}$, except that one side is of different length: $\bar{\tau}(\widetilde{q_1}, \widetilde{p_3}) = \tau(q_1, p_3) \leq \bar{\tau}(\overline{q_1}, \overline{p_3})$. But as this "future side" $\bar{\tau}(\widetilde{q_1}, \widetilde{p_3})$ increases, the angle $\tilde{\measuredangle}_k(\widetilde{p_3}\widetilde{p_1}\widetilde{q_1}) = \tilde{\measuredangle}_k(\widetilde{q_2}\widetilde{p_1}\widetilde{q_1})$ increases to $\tilde{\measuredangle}_k(\overline{p_3p_1q_1}) = \tilde{\measuredangle}_k(\overline{q_2p_1q_1})$ by the law of cosines (2.2.23, $\varepsilon = -1$) and the distance $\bar{\tau}(\widetilde{q_1}, \widetilde{q_2})$ increases to $\bar{\tau}(\overline{q_1}, \overline{q_2})$, yielding the desired $\tau(q_1, q_2) \leq \bar{\tau}(\widetilde{q_1}, \widetilde{q_2}) \leq \bar{\tau}(\overline{q_1}, \overline{q_2})$.



For the condition on $\tau(q_2, q_1)$, we need to check the condition on the comparison situation $\Delta\overline{p_1p_2p_3}, \overline{q_2}, \overline{q_1}$ by decomposing it into two one-sided comparisons: By across comparison for $\Delta p_1p_2p_3, q_2$ we get $\tau(q_2, p_2) \leq \bar{\tau}(\overline{q_2}, \overline{p_2})$.

If $q_2 \not\ll p_2$, also $q_2 \not\ll q_1$ (as $q_1 \leq p_2$ are on the same side) and $\tau(q_2, q_1) = 0 \leq \bar{\tau}(\overline{q_2}, \overline{q_1})$ (this is only needed for the bound below. For the bound above, we have $\tau(q_2, p_2) \geq \bar{\tau}(\overline{q_2}, \overline{p_2})$ and so also $\overline{q_2} \not\ll \overline{p_2}$. Therefore both $\tau(q_2, q_1)$ and $\bar{\tau}(\overline{q_2}, \overline{q_1})$ are zero). If $q_2 \ll p_2$, we can use across comparison for $\Delta p_1q_2p_2, q_1$ (this is a timelike / causal triangle if $q_2 \ll p_2$), we get an across comparison situation $\Delta\widetilde{p_1}\widetilde{q_2}\widetilde{p_2}, \widetilde{q_1}$ which satisfies $\tau(q_2, q_1) \leq \bar{\tau}(\widetilde{q_2}, \widetilde{q_1})$.

We need to compare $\tau(q_2, q_1)$ with $\bar{\tau}(\overline{q_2}, \overline{q_1})$. We note that the situation $\Delta\overline{p_1q_2p_2}, \overline{q_1}$ looks just like the across comparison situation $\Delta\widetilde{p_1}\widetilde{q_2}\widetilde{p_2}, \widetilde{q_1}$, except that one side is of different length: $\bar{\tau}(\widetilde{q_2}, \widetilde{p_2}) = \tau(q_2, p_2) \leq \bar{\tau}(\overline{q_2}, \overline{p_2})$. But as this "future side" $\bar{\tau}(\widetilde{q_2}, \widetilde{p_2})$ increases to $\bar{\tau}(\overline{q_2}, \overline{p_2})$, the angle $\tilde{\measuredangle}_k(\widetilde{q_2}\widetilde{p_1}\widetilde{p_2}) = \tilde{\measuredangle}_k(\widetilde{q_2}\widetilde{p_1}\widetilde{q_1})$ decreases to $\tilde{\measuredangle}_k(\overline{q_2p_1p_2}) = \tilde{\measuredangle}_k(\overline{q_2p_1q_1})$ by the law of cosines (2.2.23, $\varepsilon = -1$) and the distance $\bar{\tau}(\widetilde{q_2}, \widetilde{q_1})$ increases to $\bar{\tau}(\overline{q_2}, \overline{q_1})$, yielding the desired $\tau(q_2, q_1) \leq \bar{\tau}(\widetilde{q_2}, \widetilde{q_1}) \leq \bar{\tau}(\overline{q_2}, \overline{q_1})$.



□

2.2.4 Branching

Definition 2.2.29. A maximal future directed causal curve $\gamma : [a, b] \to X$ in a Lorentzian pre-length space has a *future branching point* $p = \gamma(t)$ ($p \neq \gamma(a)$) if there exists a maximal future directed causal curve $\tilde{\gamma} : [a, t + \varepsilon]$ (for some small $\varepsilon > 0$) , such that:

- they agree up to t (i.e. $\gamma|_{[a,t]} = \tilde{\gamma}|_{[a,t]}$),
- they are non-trivial from t to $t+\varepsilon$ (i.e. $\gamma(t) \neq \gamma(t+\varepsilon)$ and $\tilde{\gamma}(t) \neq \tilde{\gamma}(t+\varepsilon)$), but
- they do not meet shortly after t (i.e. $\gamma((t, t + \varepsilon]) \cap \tilde{\gamma}((t, t + \varepsilon]) = \emptyset$)

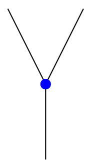
Correspondingly, one defines *past branching points* where they agree after t and do not meet shortly before t. A branching point is either of these.

A *timelike branching point* $x \in X$ is a future / past branching point of a future directed timelike maximal curve γ where also $\tilde{\gamma}$ is future directed timelike.

A Lorentzian pre-length space X is *timelike locally uniquely geodesic* (*l.u.g.*) if it is covered by open sets U where for any $p \ll q \in U$ there exists a unique (up to reparametrization) maximal future directed causal curve $p \rightsquigarrow q$ contained in U. In particular, X is locally timelike path connected. ◊

Example 2.2.30. A causal funnel with timelike γ (1.7.15) is l.u.g. (even globally uniquely geodesic), but has timelike branching points: the end point of γ is future timelike branching, and the starting point of γ is past timelike branching. ◊

Example 2.2.31. The subset $[-1, 0] \times \{0\} \cup \{(t, \frac{1}{2}t) : t \in [0, 1]\} \cup \{(t, -\frac{1}{2}t) : t \in [0, 1]\}$ in Minkowski space has $(0, 0)$ as a future timelike branching point, but is l.u.g. (even globally uniquely geodesic).



$\diamondsuit$

Example 2.2.32. On $X = \mathbb{R}^2 = \{(t,x) \in \mathbb{R}^2\}$, we take $(t,x) \mapsto t$ as a absolute time function to make it a Lorentzian pre-length space. Then every curve with strictly increasing t-component is a distance realizing future directed timelike curve. In particular, every future directed timelike curve is distance realizing and branching everywhere, and X is not timelike locally uniquely geodesic. $\diamondsuit$

Theorem 2.2.33 (Curvature bounded below has no branching)**.** *Let X be a strongly causal Lorentzian pre-length space with timelike curvature bounded below. If either:*

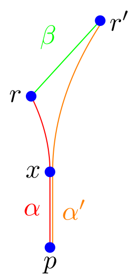
(i) X is covered by relatively compact, causally closed and open U where for $p \ll q \in U$ there is a maximal future directed timelike curve γ contained in U, and every future directed causal curve in U which contains a null segment is strictly shorter, or

(ii) X is l.u.g.,

then X has no timelike branching points.

Proof. Let x be a (w.l.o.g. future) timelike branching point. Then we get future directed timelike curves $\alpha, \alpha' : [a,b] \to X$ with $x = \alpha(t)$ as in the definition of branching. As (i), (ii) and being a timelike comparison neighbourhood is preserved when taking a timelike diamond inside, we can find a neighbourhood U of x which is a timelike comparison neighbourhood in the form of a timelike diamond satisfying (i) or (ii). We now construct a non-degenerate timelike triangle in U with segments of α and α' as sides, including x on both:

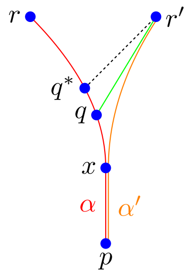
If U satisfies (ii), choose points $p = \alpha(s) \ll x \ll r = \alpha(u) \in U$ and $x \ll r' = \alpha'(u')$ in U with $r \ll r'$, with parameters $s < t < u, u'$ and r not on α'. We now restrict α and α' to this interval: W.l.o.g., $\alpha : p \rightsquigarrow x \rightsquigarrow r$ and $\alpha' : p \rightsquigarrow x \rightsquigarrow r'$. We also have a unique (by l.u.g.) maximal future directed causal $\beta : r \rightsquigarrow r'$ of positive length (U is timelike strictly intrinsic). This gives the desired timelike triangle $\Delta prr'$ which is non-degenerate by the l.u.g. property (α concatenated with β is also $p \rightsquigarrow r'$ and thus shorter than α').



If U satisfies (i), choose points $p = \alpha(s) \ll x$ and $r = \alpha(u)$, $r' = \alpha'(u')$ with $\tau(x,r) = \tau(x,r')$ in U and parameters $s < t < u, u'$ (this is possible as τ is continuous on U). By the reverse triangle inequality $\tau(r,r') = 0$ and $\tau(x,r') > 0$. Going backward from r on α, we set $u^* = \sup\{u : \tau(\alpha(u), r') > 0\} \in [t, u]$. We get $u_n \to u^*$ where there exist future directed timelike $\gamma_n : \alpha(u_n) \rightsquigarrow r'$.

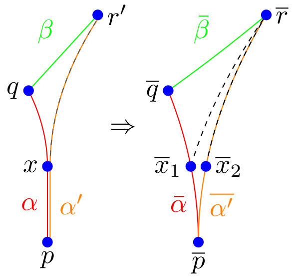
As X is strongly causal, d-compatible and causally path connected, it is non-totally imprisoning by 1.6.33.(3). As U is causally closed, relatively compact, we use the limit curve theorem (1.6.28) to get $q^* = \alpha(u^*)$ is null before r' (the dotted line below). Now $q^* \ll r$, otherwise $q^* = r$ and the future directed causal curve going from $p \rightsquigarrow q^*$ with α and then from $q^* \rightsquigarrow r'$ has the same length but contains a null segment, $\lightning$ (i).

Moving a bit backwards from u^*, we find a point $x \ll q \ll q^*$ on α where $\tau(q,r') < \tau(q^*,r)$ (as τ is continuous). Then $p \ll q \ll r'$ forms the desired timelike triangle which is non-degenerate: $\tau(p,q) + \tau(q,r') < \tau(p,q) + \tau(q^*,r) = \tau(p,r) = \tau(p,r')$.



In both cases, we get a non-degenerate timelike triangle $\Delta pqr'$ and find the corresponding comparison triangle $\Delta\overline{p}\overline{q}\overline{r'}$ in the comparison space. We now get two comparison points for the point x, as it is on two sides of $\Delta pqr'$: $\overline{x}_1$ on the side $\overline{p} \rightsquigarrow \overline{q}$ and $\overline{x}_2$ on the side $\overline{p} \rightsquigarrow \overline{r'}$. We have $\bar{\tau}(\overline{x}_1, \overline{r'}) < \bar{\tau}(\overline{x}_2, \overline{r'})$, as otherwise the broken geodesic $\overline{p} \rightsquigarrow \overline{x}_1 \rightsquigarrow \overline{r'}$ would be at least as long as the geodesic $\overline{p} \rightsquigarrow \overline{r'}$ (via $\overline{x}_2$) in the uniquely geodesic comparison space.

As the timelike curvature is bounded from below, the future comparison is $\bar{\tau}(\overline{x}_1, \overline{r'}) \geq \tau(x,r')$, yielding $\tau(x,r') = \bar{\tau}(\overline{x}_2, \overline{r'}) > \bar{\tau}(\overline{x}_1, \overline{r'}) \geq \tau(x,r')$ $\lightning$.



The dotted line is compared. Both dotted lines on the right correspond to the distance $\tau(x, r)$, i.e. the dotted line on left.

$\square$

Sources

Triangle comparison in Lorentzian pre-length spaces is described in [KS18, 4.7] for timelike bounds and [KS18, 4.14] for causal bounds, branching is taken from [KS18, 4.10]. The Lorentzian constant curvature spaces were taken from [O'N83, 8.22,8.24]. The existence of triangles (2.2.9) and Toponogov (2.2.14) are from [AB08, 2.1,1.1]. The example 2.2.16 is done analogously to [AB08, 7.1], example 2.2.19 is new. The law of cosines is mainly [Kir18, 3.1.3]. The agreement of rapidities and the convergence of all $\tilde{\measuredangle}_k$ to $\measuredangle$ (2.2.26) are probably new results. The existence of angles (2.2.27) and one-sided comparison (2.2.28) are new. Curvature bounded below has no branching (2.2.33) is taken from [KS18, 4.12].

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