DISSERTATION / DOCTORAL THESIS

Titel der Dissertation /Title of the Doctoral Thesis

,,Geometric Invariants for the Resolution of Curve Singularities and for the Problem of the Moduli Space of n Points on the Projective Line‘‘

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Doktorin der Naturwissenschaften (Dr. rer. nat.)

Wien, 2020 / Vienna, 2020

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 796 605 405
Dissertationsgebiet lt. Studienblatt / field of study as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Univ.-Prof. Mag. Dr. Herwig Hauser

Acknowledgements

First, I would like to express my special appreciation and thanks to my advisor Herwig Hauser who not only awoke my interest in the fascinating problem of resolution of singularities but whose ideas gave rise to the core concept of my PhD research. Thank you for sharing your ideas with me!

Besides my advisor, I am deeply indebted to Alin Bostan not only for his comments and valuable discussions with him which led to a new insight on the problems of my PhD thesis, but also for his moral support, especially during the finishing phase of the PhD program. Thank you for your time and patience!

I would also like to express my sincere gratitude to Matteo Gallet, Hiraku Kawanoue, Stefan Perlega and Josef Schicho for giving me the opportunity to work with them and for sharing so much of their vast knowledge. This thesis has also profited from many (personal and email) discussions. In particular, my sincere thanks goes to Maria Alberich, Mariemi Alonso, Angélica Benito, Michel Brion, Francisco-Jesús Castro-Jiménez, Hanspeter Kraft, Michael Singer, Bernard Teissier and Orlando Villamayor.

Part of this work was carried out during my visit of Eduardo Casas-Alvero in Barcelona, Ana Bravo in Madrid and Alin Bostan and Mohab Safey El Din in Paris who I want to thank for their invitations and hospitality during my research stays and also for the provided financial support.

I gratefully acknowledge the financial support I received mainly from the Austrian Academy of Sciences ÖAW (in form of the doc-fellowship FA506081) and partially also from the Austrian Science Fund FWF (through the projects P25652, P31061 and P31338) which have enabled the realization of my research project at the University of Vienna.

Furthermore, let me thank Ana Bravo, Luis Narváez Macarro and Josef Schicho for their willingness to serve as possible referees for the present PhD thesis.

I am also grateful to my colleagues and friends Giancarlo Castellano, Christopher Chiu, Chiara Novarini, Stefan Perlega and Sergey Yurkevich for both, all the beautiful mathematics we shared and all the fun we had.

Finally, my sincere thanks goes to my family. I appreciate all that you have done to encourage me and wrap me in all the positive energy. Děkuji, rodičové, že jste to se mnou vydrželi!

Especially thankful I am to my husband. He was a permanent source of encouragement and support during hard times. Děkuji, Pepi, za vše!

Contents

Introduction

The underlying topic of my PhD thesis is the problem of resolution of singular curves. My thesis aims to present a geometrically flavoured approach to resolution of unibranched singular algebraic curves. The goal is to construct geometrically inspired modifications of varieties which may be able to advance in the resolution of surface singularities and to provide a geometric approach complementing the known more algebraic algorithms.

The key players to achieve this are so-called *algebraic curvatures* representing "higher order tangent spaces". They describe each curve in its smooth points completely and thus provide already information on how the curve runs into a singular point. Finally, using the limits of the "higher order tangent spaces" when running into a singularity, I define by means of algebraic curvatures the center of the blowup which already resolves the singularity.

The second topic of this thesis is another application of algebraic curvatures outside algebraic geometry. The algebraic curvatures represent a complete system of so-called *geometric invariants*, rational expressions in parametrizations and their derivatives of analytic varieties that are equivariant under the natural action of the group of reaparametrizations. I show that the "minimal" geometric invariants are in one-to-one correspondence with rational functions

$$\mathbb{C}(\mathbb{x}) := \mathbb{C}(x_{i,j} : i = 1, \dots, m; j = 1, \dots, n)$$

that are invariant under the action of $\mathrm{GL}_m(\mathbb{C})$. As such they already determine the invariant field $\mathbb{C}(\mathbb{x})^{\mathrm{GL}_m}$ and yield thus a new insight into the First Fundamental Theorem for $\mathrm{GL}_m(\mathbb{C})$. A proof of the First Fundamental Theorem for $\mathrm{SL}_m(\mathbb{C})$ is also provided in my thesis as well as proofs for $\mathrm{GL}_m(K)$ and $\mathrm{SL}_m(K)$ for an arbitrary infinite field K.

Let me give a brief description of the results of this thesis starting with results gained towards resolution of singularities of algebraic curves.

The history of resolution of singularities of algebraic curves goes back more than 150 years to the work of M. Noether [Noe71, Noe75] who used it in order to find a formula for the genus of plane algebraic curves. More on the analytic side, at that time, the concept of Puiseux parametrizations was known as well — first discovered by I. Newton [Ne36, pp. 191-209] and later rediscovered by V. A. Puiseux [Pu50] while studying the solution space of $f(x, y(x)) = 0$ — yielding an analytic form of resolution. Nowadays, several methods for resolution of singular algebraic curves are available (see J. Kollár's book [Ko07]): Successive blowups of the singular points eventually resolve all singularities since it can be shown that certain well chosen local invariants improve under each blowup. By induction on these invariants — usually lexicographically ordered string of integers — one is done after finitely many steps. A one step resolution is obtained by normalization but here, the geometric intuition is hidden behind commutative algebra machinery, see [MZ39]. As the resolution of each algebraic curve is unique up to isomorphisms and the normalization is a finite map, it can also be used together with the result of A. Nobile [No75] saying the Nash modification is an isomorphism on whole curve X if and only if X is smooth, to prove that performing successively the Nash modification yields resolution after finitely many steps.

Let us now fix the setting: Let $X \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ be an algebraic space curve with a singularity at the origin $0 \in X$ defined by polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n, y]$. Assume that X is unibranched at the origin. Let us for each $x \in X$ denote by $[x]$ its corresponding projective point in $\mathbb{P}^n_{\mathbb{C}}$. A comparison of the standard blowup of X at 0 with its Nash modification shows immediately that the Nash modification is a more refined approach to improve singularities. With the standard blowup of X, one associates to each smooth point x on X the slope of the secant going through x and the origin:

$$X \backslash \{0\} \to \mathbb{P}^n_{\mathbb{C}},\ x \mapsto [x]$$

and finally takes the Zariski closure of the graph of this map in $\mathbb{A}^{n+1}_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$. The Nash modification looks into the local geometry of X at a point more closely. One associates to each smooth point $x \in X$ the tangent line $s(x) = T_x X$ of X at x as an element of the projective space $\mathbb{P}^n_{\mathbb{C}}$ and takes then the Zariski closure of the graph of the map

$$X \backslash \{0\} \to \mathbb{P}^n_{\mathbb{C}},\ x \mapsto s(x)$$

in $\mathbb{A}^{n+1}_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$. This corresponds, for plane curves, to the blowup of the curve in the Jacobian ideal of the defining equation of the curve. Thus, the Nash modification represents already a more geometric treatment of curve singularities. However, at the same time, in general many repetitions are necessary to achieve the resolution.

As any composition of blowups can be seen as a single blowup in a (in general) non-radical and very complicated ideal, one knows, by Hironaka's result (see [Hi64a, Hi64b]), that the resolution of singularities can be obtained by a single blowup. One "just" has to define the correct center. But as far as I know, aside from trivial examples, the choice of such a center is completely unknown and mysterious.

In this thesis, I present a more refined procedure to resolve singularities based on the consideration of algebraic curvatures — a variation of the classical curvature known from differential geometry — which captures more accurately than the tangent lines how the curve runs into a singular point. The main trick I use to establish a resolution of X with one blowing up is to use local parametrizations of X at the origin providing very precise information about the complexity of the singularity itself: Look at X at 0 from the perspective of parametrizations. Let

$$\gamma\colon t \mapsto (x_1(t), \dots, x_n(t), y(t))$$

be an analytic parametrization of X at 0. Construct a rational expression $z(t) = \frac{z_1(t)}{z_2(t)}$ in $x_1(t), \dots, x_n(t), y(t)$ and their derivatives such that:

(i) $z(t)$ is a power series of order one,

(ii) $z(t)$ admits a rational expression as a formula in the polynomials defining X (and their partial derivatives), i.e., there exists

$$\widetilde{z} = \frac{\widetilde{z}_1}{\widetilde{z}_2} \in \mathbb{C}[\partial^i f_j : i \in \mathbb{N}^{n+1}, j = 1, \dots, r] \subseteq \mathbb{C}[x_1, \dots, x_n, y],$$

such that the equality

$$z(t) = \widetilde{z}(\gamma(t))$$

is fulfilled. Here, for $i = (i_1, \dots, i_{n+1}) \in \mathbb{N}^{n+1}$, by ∂^i we denote $\partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \partial_y^{i_{n+1}}$.

It is then not hard to see that the graph of the "height function"

$$\phi : X\backslash\{0\} \to \mathbb{P}^1_{\mathbb{C}},\ x \mapsto (\widetilde{z}_1(x) : \widetilde{z}_2(x))$$

defines a quasi-affine space curve. Let us denote its Zariski closure by $\widetilde{X}$.

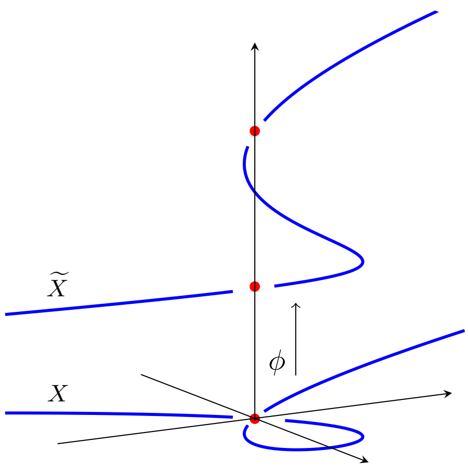

Figure 1: Resolution of singularities of the node given by the equation $y^2 - y^3 = x^3$.

The curve $\widetilde{X}$ together with the morphism $\pi : \widetilde{X} \to X$ induced by the projection onto the first $n+1$ components $\mathbb{A}^{n+1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{A}^{n+1}_{\mathbb{C}}$ is the blowup of X in the ideal $(\widetilde{z}_1, \widetilde{z}_2)$. It follows by the properties of blowups that

$$\pi : \widetilde{X}\backslash E \to X\backslash\{0\}$$

is an isomorphism outside the preimage of the singular point $E = \pi^{-1}(0)$ (by our assumption on X to be unibranched at the origin, the morphism π is injective on X), let us denote it by $\widetilde{x}$. Moreover, by the property (ii), in one of the affine charts, $\widetilde{X}$ is parametrized at $\widetilde{x}$ by the vector

$$(x_1(t), \ldots, x_n(t), y(t), z(t)),$$

which is, according to (i) an order-one parametrization. This shows, that $\widetilde{X}$ is locally at $\widetilde{x}$ biholomorphic to the germ $(\mathbb{C}, 0)$ and thus smooth. Thus, $\widetilde{X}$ together with the projection morphism defines already resolution of singularities of X.

The goal now is, therefore, to construct a rational expression $z(t)$ in $x_1(t), \ldots, x_n(t), y(t)$ and their derivatives satisfying (i) and (ii) and thus defining the resolving height function ϕ. Now, it is time to explain what the algebraic curvatures are: Inspired by the differential geometric notion of the slope of the tangent vector, curvature and torsion of space curves in $\mathbb{A}^3_{\mathbb{C}}$, we define

$$\kappa_{0,j}(t) := \frac{x_j'(t)}{y'(t)}, \quad \text{for } j = 1, \ldots, n,$$

to be the *slopes (of the tangent vector)* or 0-*th algebraic curvatures* of X. Further, we define the *first* and *higher algebraic curvatures* of X via

$$\kappa_{1,j}(t) := \frac{x_j''(t)y'(t) - x_j'(t)y''(t)}{y'(t)} \text{ and } \kappa_{i,j}(t) := \frac{\partial_t \kappa_{i-1,j}(t)}{y'(t)} \text{ for } i \geq 2, j = 1, \ldots, n,$$

respectively. The observation now is that each of the algebraic curvatures is equivariant under reparametrizations and thus defines a quantity of the curve which does not depend on a choice of parametrization. As such, intuitively, they should admit also an implicit description as a rational function in the implicit equations $f_1, \ldots, f_r$ of X and their partial derivatives. In fact, this intuition is confirmed by a rigorous proof. I have even proven a more general statement (see Theorems 1.1.5, 1.2.2 and 1.1.12, 1.2.6 for curves and 1.3.4, 1.4.5 and 1.3.6, 1.4.7 for their generalization to higher dimensional varieties):

Let us call each rational function in $x_1(t), \ldots, x_n(t), y(t)$ and their derivatives that is equivariant under reparametrizations, a *geometric invariant* of X.

Theorem. *Consider an algebraic space curve $X \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ with parametrization*

$$\gamma(t) = (x_1(t), \ldots, x_n(t), y(t)) \in \mathbb{C}[\![t]\!]^{n+1}.$$

Let $p(t)$ be a rational function in $x_1(t), \ldots, x_n(t), y(t)$. Then the following statements are equivalent:

(i) $p(t)$ is a geometric invariant.

(ii) $p(t)$ admits a representation as a rational function in $x_1(t), \ldots, x_n(t), y(t)$ and $\kappa_{i,j}(t)$ for $i \in \mathbb{N}$ and $j = 1, \ldots, n$.

(iii) $p(t)$ admits an implicit expression as a rational function in the defining equations of X and their partial derivatives.

Now, based on this theorem, the algebraic curvatures seem already as natural candidates for the searched rational expression $z(t)$ establishing resolution. Actually, each of the algebraic curvatures looks more closely into the local geometry of the curve at the point. In fact, these curvatures determine the curve locally and as such they provide a new way to look at singularities, see Corollaries 1.1.13 and 1.2.7. One could, hence, think of testing successively the height functions defined by the algebraic curvatures. In general, the algebraic curvatures themselves, however, do not yield resolution of X immediately. This is due to the fact that parametrizations usually do not have only one characteristic pair but admit more characteristic exponents. This can be seen on the following example:
Let us consider the plane curve

$$X = \{-x^3 + (3y^2 - 6y + 1)x^2 + (-3y^4 - 2y^3)x + y^6 = 0\}$$

with parametrization $\gamma(t) = (t^6, t^2 + t^3)$ at the origin. Then, each $\kappa_{i,1}(t)$ has an even order as $\mathrm{ord}(\kappa_{i,1}(t)) = 6 - 2(i + 1)$ for all $i \in \mathbb{N}$.

In order to be able to construct a geometric invariant of order one, one needs to extract from the parametrization $\gamma(t)$ of X all *characteristic exponents* — supposing that $\gamma(t)$ is of the form $(x_1(t), \ldots, x_n(t), t^m)$ (which can always be reached with a linear coordinate change), these are the "minimal" elements $a_1, \ldots, a_k$ of the union

$$\cup_{j=1}^{n} \mathrm{supp}(x_j(t))$$

of supports of all power series $x_j(t), j = 1, \ldots, n$, satisfying the condition

$$\gcd(m, a_1, \ldots, a_k) = 1.$$

An algorithm providing a systematic treatment of all characteristic exponents by means of algebraic curvatures and finally constructing a geometric invariant of order one as a rational function in the algebraic curvatures is established in this thesis. As such the following theorem (see Theorems 2.1.11 and 2.2.6) is proven:

Theorem. *Let $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ be an algebraic space curve with a singularity $0 \in X$. Assume that X is unibranched at the origin and that $\gamma(t)$ is a parametrization of the branch at 0. Then, there exists $z(t)$, a rational function in algebraic curvatures of X, let $\widetilde{z} = \frac{\widetilde{z}_1}{\widetilde{z}_2}$ be its implicit expression in terms of the defining polynomials of X, such that the blowup of X in the ideal $(\widetilde{z}_1, \widetilde{z}_2)$ yields resolution of singularities of X.*

Once resolution of one point on a curve is established, one proceeds inductively with curves with multiple singular points in order to resolve all their singularities as proven in Theorem 2.3.3.

The concept of geometric invariants of varieties of arbitrary dimensions as well as their complete characterization are presented in Section 1.4 of this thesis. Moreover, in Sections 3.1 and 3.2 of this thesis, I describe their application to the problem of the moduli space of n points in the $(m-1)$-dimensional projective space $\mathbb{P}_{\mathbb{C}}^{m-1}$ and to the First Fundamental Theorems for

$\mathrm{GL}_m(\mathbb{C})$ and $\mathrm{SL}_m(\mathbb{C})$.

Let me present briefly the main constructions towards this problem on the simple case of the moduli space of n points on the projective line $\mathbb{P}^1_{\mathbb{C}}$. The techniques used in the general case are then more technical but they follow the same argument.

Let us consider the set of n points on the projective line over $\mathbb{C}$,

$$\mathbb{P}^{1\times n} := \{\big((x_1 : y_1), \ldots, (x_n : y_n)\big) \mid (x_i : y_i) \in \mathbb{P}^1_{\mathbb{C}}\}.$$

The projective linear group $\mathrm{PGL}_2(\mathbb{C})$ acts naturally from the left on the set $\mathbb{P}^{1\times n}$ by the usual matrix-vector multiplication on each of the projective points $(x_i : y_i)$. The goal is to determine the structure of the geometric invariant theory (GIT) quotient $\mathbb{P}^{1\times n}/\!/\,\mathrm{PGL}_2(\mathbb{C})$.

In order to study the scheme structure of $\mathbb{P}^{1\times n}/\!/\,\mathrm{PGL}_2(\mathbb{C})$, we have to consider the invariant field $\mathbb{C}(x_i, y_i \,:\, 1 \leq i \leq n)^{\mathrm{GL}_2}$, where the general linear groups acts on the field $\mathbb{C}(x_i, y_i \,:\, 1 \leq i \leq n)$ from the right by the matrix-vector multiplication on the pairs of variables (x_i, y_i).

It is not difficult to find an element of the invariant field. Let us set

$$f_{i,j} := x_i y_j - x_j y_i.$$

It is obvious that each polynomial $f_{i,j}$ satisfies the equality $\sigma \cdot f_{i,j} = \det(\sigma) f_{i,j}$. Hence, it is invariant under the action of $\mathrm{SL}_2(\mathbb{C})$ and semi-invariant under the action of $\mathrm{GL}_2(\mathbb{C})$. Thus, the rational functions $\frac{f_{i,j}}{f_{k,l}}$ are invariant under the action of $\mathrm{GL}_2(\mathbb{C})$. Actually, we can translate each rational function $\frac{f_{i,j}}{f_{k,l}}$ into a "minimal" geometric invariant of surfaces and the condition of being invariant under the action of $\mathrm{GL}_2(\mathbb{C})$ into being equivariant under linear transformations as a rational function on a parametric surface. In this way, I created a bridge between the problem of the moduli space and the problem of the classification of "minimal" geometric invariants of surfaces. Finally, using the fact that the "minimal" geometric invariants of a parametric surface $(x_1(t,s), \ldots, x_n(t,s)) \in \mathbb{C}[\![t,s]\!]^n$ are generated over $\mathbb{C}$ by the algebraic curvatures of minimal order given by

$$\frac{\partial_t x_i \cdot \partial_s x_j - \partial_s x_i \cdot \partial_t x_j}{\partial_t x_k \cdot \partial_s x_l - \partial_s x_k \cdot \partial_t x_l}, \text{ with } 1 \leq i, j, k, l \leq n,$$

I proved that the function field of the set $\mathbb{P}^{1\times n}/\!/\,\mathrm{PGL}_2(\mathbb{C})$ contains only rational functions in $f_{i,j}$ whose numerator and denominator are homogeneous of the same degree. Moreover, from this I concluded the First Fundamental Theorems for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{C})$ stating that the polynomials $f_{i,j}$ and the rational functions $\frac{f_{i,j}}{f_{k,l}}$ generate the whole invariant ring and field, respectively:

First Fundamental Theorem for $\mathrm{SL}_2(\mathbb{C})$. *The invariant polynomial ring under the action of $\mathrm{SL}_2(\mathbb{C})$ is generated over $\mathbb{C}$ by the polynomials $f_{i,j}$ for $1 \leq i < j \leq n$, i.e., we have the equality*

$$\mathbb{C}[x_i, y_i : 1 \leq i \leq n]^{\mathrm{SL}_2} = \mathbb{C}[f_{i,j} : 1 \leq i < j \leq n].$$

First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$**.** *The field of invariant rational functions under the action of* $\mathrm{GL}_2(\mathbb{C})$ *is generated over* $\mathbb{C}$ *by* $\frac{f_{i,j}}{f_{k,l}}$*, i.e., we have*

$$\mathbb{C}(x_i, y_i : 1 \leq i \leq n)^{\mathrm{GL}_2} = \mathbb{C}\left(\frac{f_{i,j}}{f_{k,l}} : 1 \leq i, j, k, l \leq n\right).$$

The statements of the First Fundamental Theorems remain valid also over an arbitrary infinite field K. However, they are wrong for any finite field K. Counterexamples are provided.

The general proofs of the First Fundamental Theorems for $\mathrm{SL}_m(K)$ and $\mathrm{GL}_m(K)$ for arbitrary $m \geq 2$ and an infinite field K as well as counter examples for their statements over finite fields are provided in Section 3.2.

Chapter 1

Geometric Invariants

In this chapter I introduce the concept of the so-called *geometric invariants* of algebraic varieties. They are a very powerful tool in the problem of resolution of algebraic curves which is a classical and very famous problem in algebraic geometry, and also the objective of the second chapter of this thesis. The geometric invariants admit applications outside algebraic geometry as well. It turns out that they describe completely the structure of the moduli space of n points on the projective line and, even more, they give a geometric explanation for it. The problem of the moduli space, itself a subfield of invariant and group theory, and its connection to the geometric invariants is the topic of the third chapter.

First we investigate geometric invariants of (plane) curves, as the techniques used here are very instructive and also apply in the higher dimensional case. Since the geometric invariants of space curves are just a generalization of the concept of geometric invariants of plane curves, we will start with the study of the plane curve case in detail. The results about geometric invariants of plane curves will then be extended to the space curves case and later, also to surfaces and higher dimensional varieties. Throughout this chapter we will frequently use the existence of D-transcendental power series and D-algebraically independent families of power series whose basic properties are listed in Section 4.1.

1.1 Geometric Invariants of Plane Curves

The aim of this section is to introduce the geometric invariants of plane curves and to study their basic properties. In this section we will extend the concept of under reparametrizations equivariant rational functions in the components of a parametrization $\gamma(t) = (x(t), y(t)) \in \mathbb{C}[\![t]\!]^2$, of a plane algebraic curve and their derivatives to a more general setting. We replace the derivatives $\partial_t^i x(t)$ and $\partial_t^i y(t)$ for $i \in \mathbb{N}$ by new variables $x^{(i)}$ and $y^{(i)}$, respectively, and translate the property of being equivariant under reparametrizations into a new invariance property on these variables. We study the structure of the field of these invariants and provide with Theorem 1.1.5 a minimal countable system of generators over $\mathbb{C}$. Further, we show in Theorem 1.1.12 that these invariants can be described in terms of the implicit equation of the plane curve and that they already uniquely determine its smooth analytic branches, see Corollary 1.1.13.

Let us consider two sets of countably many variables $x^{(i)}$ and $y^{(i)}, i \in \mathbb{N}$ (we think of $x^{(i)}$ as a symbolic derivative of $x^{(i-1)}$). We consider the field

$$F := \mathbb{C}(x^{(i)}, y^{(i)} : i \in \mathbb{N})$$

generated by all $x^{(i)}, y^{(i)}$, equipped with the $\mathbb{C}$-derivation

$$\begin{aligned} \partial : F &\to F \\ x^{(i)} &\mapsto x^{(i+1)} \\ y^{(i)} &\mapsto y^{(i+1)}. \end{aligned}$$

It thus becomes a differential field (F, ∂). Let $\varphi^{(i)}, i \in \mathbb{N}$, be another set of variables (they play a different role than $x^{(i)}, y^{(i)}$). Let

$$L := F(\varphi^{(i)} : i \in \mathbb{N}) = \mathbb{C}(x^{(i)}, y^{(i)}, \varphi^{(i)} : i \in \mathbb{N}),$$

and extend ∂ to L by $\partial(\varphi^{(i)}) = \varphi^{(i+1)}$. On L we simulate the chain rule by another $\mathbb{C}$-derivation:

$$\begin{aligned} \chi : L &\to L \\ x^{(i)} &\mapsto x^{(i+1)}\varphi^{(1)} \\ y^{(i)} &\mapsto y^{(i+1)}\varphi^{(1)} \\ \varphi^{(i)} &\mapsto \varphi^{(i+1)}. \end{aligned}$$

We think of φ as a symbol for reparametrization of a parametrized curve $(x(t), y(t))$. More precisely, given a parametrized curve $(x(t), y(t)) \in \mathbb{C}[\![t]\!]^2$, we associate

$$\begin{aligned} x^{(0)} &\leftrightarrow x(t), \\ y^{(0)} &\leftrightarrow y(t), \end{aligned}$$

and

$$\begin{aligned} x^{(i)} &\leftrightarrow \partial_t^i x(t), \\ y^{(i)} &\leftrightarrow \partial_t^i y(t), \end{aligned}$$

for $i \geq 1$. Let $\varphi \in \mathrm{Aut}(\mathbb{C}[\![t]\!])$ be an algebra automorphism. We call each such φ a *reparametrization*. Note that φ is given by a power series $\varphi(t) \in \mathbb{C}[\![t]\!]$ with $\mathrm{ord}(\varphi(t)) = 1$. The automorphism group $\mathrm{Aut}(\mathbb{C}[\![t]\!])$ acts then from the right on $\mathbb{C}[\![t]\!]^2$ via

$$\begin{aligned} \mathrm{Aut}(\mathbb{C}[\![t]\!]) \times \mathbb{C}[\![t]\!]^2 &\to \mathbb{C}[\![t]\!]^2 \\ \big(\varphi, (x(t), y(t))\big) &\mapsto \big(x(\varphi(t)), y(\varphi(t))\big). \end{aligned}$$

From now on, by a reparametrization φ we always mean the power series reprezentation $\varphi(t)$ of the automorphism $\varphi \in \mathrm{Aut}(\mathbb{C}[\![t]\!])$. We associate now

$$\varphi^{(0)} \leftrightarrow \varphi(t)$$

and

$$\varphi^{(i)} \leftrightarrow \partial_t^i \varphi(t)$$

for $i \geq 1$. With this, the derivation χ reflects the chain rule

$$\partial_t\big((\partial_t^i x)(\varphi(t))\big) = (\partial_t^{i+1} x)(\varphi(t)) \cdot \varphi'(t).$$

Next we define a $\mathbb{C}$-morphism, i.e., a field homomorphism whose fixed field equals $\mathbb{C}$, on L by

$$\begin{aligned} \Lambda : L &\to L \\ x^{(i)} &\mapsto \chi^i(x^{(0)}) \\ y^{(i)} &\mapsto \chi^i(y^{(0)}) \\ \varphi^{(i)} &\mapsto \chi^i(\varphi^{(0)}). \end{aligned}$$

where χ^i denotes the composition $\underbrace{\chi \circ \cdots \circ \chi}_{i-\text{times}}$. In terms of power series, this means

$$\begin{aligned} \partial_t^i x(t) &\mapsto \partial_t^i (x \circ \varphi)(t), \\ \partial_t^i y(t) &\mapsto \partial_t^i (y \circ \varphi)(t), \\ \partial_t^i \varphi(t) &\mapsto \partial_t^i \varphi(t), \end{aligned}$$

So for $x(t)$ and $y(t)$, their higher derivatives are replaced by the derivatives of the compositions $(x \circ \varphi)(t)$ and $(y \circ \varphi)(t)$, respectively, for which the iterated chain rule applies. From now on we will denote the vectors $(x^{(0)}, x^{(1)}, \dots)$ and $(y^{(0)}, y^{(1)}, \dots)$ by $\underline{x}$ and $\underline{y}$, respectively, and for $p(x^{(i)}, y^{(i)} : i \in \mathbb{N})$ we will use the notation $p(\underline{x}, \underline{y})$.

Definition 1.1.1. A rational expression $p \in F$ is called a *geometric invariant of plane curves* if it is fixed under Λ, namely

$$\Lambda(p) = p.$$

In terms of power series (parametrizations) this means

$$p\big(\partial_t^i x(t), \partial_t^i y(t) \ : \ i \in \mathbb{N}\big) \circ \varphi(t) = p\big(\partial_t^i (x \circ \varphi)(t), \partial_t^i (y \circ \varphi)(t) \ : \ i \in \mathbb{N}\big). \tag{1.1}$$

Being a geometric invariant is hence a property which reflects the equivariance of a rational expression in power series $x(t), y(t)$ and their derivatives under reparametrizations as the next proposition shows. We denote the vectors $(x(t), \partial_t x(t), \partial_t^2 x(t), \dots)$ and $(y(t), \partial_t y(t), \partial_t^2 y(t), \dots)$ by $\underline{x(t)}$ and $\underline{y(t)}$, respectively.

Proposition 1.1.2. *Let* $p(\underline{x}, \underline{y}) = \frac{g(\underline{x}, \underline{y})}{h(\underline{x}, \underline{y})}$ *be an element of* $F = \mathbb{C}(x^{(i)}, y^{(i)} \ : \ i \in \mathbb{N})$*. Then the following statements are equivalent:*

(i) p *is a geometric invariant of plane curves.*

(ii) The equality

$$p\big(\underline{x(t)}, \underline{y(t)}\big) \circ \varphi(t) = p\big(\underline{(x \circ \varphi)(t)}, \underline{(y \circ \varphi)(t)}\big)$$

holds for all power series $x(t), y(t) \in \mathbb{C}[\![t]\!]$ with $h\big(\underline{x(t)}, \underline{y(t)}\big) \neq 0$ and all reparametrizations $\varphi(t)$, i.e., $p\big(\underline{x(t)}, \underline{y(t)}\big)$ is equivariant under reparametrizations.

Proof. $(ii) \Rightarrow (i)$: Let $x(t), y(t), \varphi(t) \in \mathbb{C}[\![t]\!]$ be a family of D-algebraically independent power series (see Appendix 4.1 for the definition and basic properties of D-algebraically independent power series) satisfying the condition $\partial_t\varphi(0) \neq 0$, i.e., $\varphi(t)$ is a reparametrization. Then the higher derivatives $\partial_t^i x, \partial_t^i y$ and $\partial_t \varphi$ do not satisfy any polynomial equation and clearly this remains true also after reparametrization. Thus, they can be considered as variables $x^{(i)}, y^{(i)}, \varphi^{(i)}$. Then according to the chain rule and after rewriting the derivatives $\partial_t^i x \circ \varphi, \partial_t^i y \circ \varphi, \partial_t^i \varphi$ as $x^{(i)}, y^{(i)}, \varphi^{(i)}$, respectively, we have

$$\begin{aligned} p(\underline{x}, \underline{y}) &= p\big(\underline{x(t)}, \underline{y(t)}\big) \circ \varphi = p\big(\underline{(x \circ \varphi)(t)}, \underline{(y \circ \varphi)(t)}\big) \\ &= \Lambda(p)\big(\underline{x(t)} \circ \varphi, \underline{y(t)} \circ \varphi, \underline{\varphi(t)}\big) = \Lambda(p)(\underline{x}, \underline{y}, \underline{\varphi}). \end{aligned}$$

From this we conclude $p = \Lambda(p)$, which shows that p is a geometric invariant of plane curves.
$(i) \Rightarrow (ii)$: This follows from the fact that reparametrizations act exactly in the same way as Λ. $\square$

Lemma 1.1.3. *For an element $p \in F$ we have the following two equalities*

(i) $\chi(p) = \partial(p)\varphi^{(1)}$,

(ii) $\Lambda(\partial(p)) = \chi(\Lambda(p))$.

Proof. (i): As χ is a derivation and acts on the generators of F by $\chi(x^{(i)}) = \partial(x^{(i)})\varphi^{(1)}$ and $\chi(y^{(i)}) = \partial(y^{(i)})\varphi^{(1)}$, the claimed equality follows.
(ii): Take an element $p \in F$. Since the maps Λ, ∂ and χ are additive, we may assume that p is of the form

$$p = \prod_{i \in I} x^{(i)} \prod_{j \in J} y^{(j)}$$

with some index sets I and J. A short computation shows then

$$\begin{aligned} \Lambda(\partial(p)) =& \Lambda\left(\sum_{i \in I}\big(x^{(i+1)} \prod_{k \in I \setminus \{i\}} x^{(k)} \prod_{j \in J} y^{(j)}\big) + \sum_{j \in J}\big(y^{(j+1)} \prod_{i \in I} x^{(k)} \prod_{k \in J \setminus \{j\}} y^{(k)}\big)\right) \\ =& \sum_{i \in I}\big(\chi^{(i+1)}(x^{(0)}) \prod_{k \in I \setminus \{i\}} \chi^{(k)}(x^{(0)}) \prod_{j \in J} \chi^{(j)}(y^{(0)})\big) + \\ &+ \sum_{j \in J}\big(\chi^{(j+1)}(y^{(0)}) \prod_{i \in I} \chi^{(k)}(x^{(0)}) \prod_{k \in J \setminus \{j\}} \chi^{(k)}(y^{(0)})\big) \\ =& \chi\big(\prod_{i \in I} \chi^{(i)}(x^{(0)}) \prod_{j \in J} \chi^{(j)}(y^{(0)})\big) = \chi\big(\Lambda\big(\prod_{i \in I} x^{(i)} \prod_{j \in J} y^{(j)}\big)\big) = \chi(\Lambda(p)). \end{aligned}$$ $\square$

Using the last proposition and lemma, we can construct a whole family of geometric invariants:

Example 1.1.4. The variables $x^{(0)}, y^{(0)}$ are of course geometric invariants. But there are more interesting examples.

(1) The slope of the tangent vector
$$s(t) = \frac{x'(t)}{y'(t)}$$
of the parametrized curve $\gamma(t)$ is obviously equivariant under reparametrizations. Hence,
$$\kappa_0 := \frac{x^{(1)}}{y^{(1)}}$$
is a geometric invariant, called the *slope* (*of the tangent vector*).

(2) The formula for the classical curvature
$$\kappa(t) = \frac{x''(t)y'(t) - x'(t)y''(t)}{\sqrt{\left(x'(t)^2 + y'(t)^2\right)^3}}$$
does not yield a geometric invariant in our sense (although it is equivariant under reparametrizations) since we do not allow square roots in our definition. However, a little modification of the denominator leads to the rational expression
$$\frac{x''(t)y'(t) - x'(t)y''(t)}{\left(x'(t) + y'(t)\right)^3}$$
which is also equivariant under reparametrizations. Note that any linear combination $ax'(t) + by'(t)$ with $a \neq 0$ or $b \neq 0$ in the denominator also yields a geometric invariant. (For further computations it is convenient to choose $a = 0, b = 1$.) We set
$$\kappa_1 := \frac{x^{(2)}y^{(1)} - x^{(1)}y^{(2)}}{(y^{(1)})^3}.$$

(3) The rational expressions
$$\kappa_{i+1} := \frac{\partial(\kappa_i)}{y^{(1)}} \text{ for any } i \geq 1,$$
are again geometric invariants. This follows directly from Lemma 1.1.3.

The slope of the tangent vector $s(t)$ and the curvature $\kappa(t)$ of parametric plane and also space curves over $\mathbb{R}$ are well-known and standard notions in differential geometry. The classical literature (for instance, R. Goldman in [Go05], or M. P. do Carmo in [dCa76, Chapter 1, §5]) often refers to them as "invariants (under reparametrizations)". So the classical curvature κ is the differential geometric analog to the more algebraically defined geometric invariant κ_1. Actually, the origin of the definition of geometric invariants as rational functions, that are equivariant under reparametrizations, comes exactly from the fact that being "invariant" is a crucial property of the classical curvature. The classical curvature is clasically defined as the reciprocal of the radius of the osculating circle of the curve at a given point (see e.g. [Sp99, Chapter 1], [Be02, §1.3.2], [Ei09, Chapter 1,§7], or [CH99, §26]). Nowadays, there are several equivalent definitions for the curvature of plane curves in $\mathbb{A}^2_{\mathbb{R}}$:

- intrinsic definition as reciprocal of the radius of the osculating circle,
- in terms of a general parametrization $\gamma(t) = (x(t), y(t))$ (for instance in [Pr01, Proposition 2.1], [Ei09, Chapter 1, §4-§7], or [Ha99, Example 3.1.2]):
$$\kappa(t) = \frac{x''(t)y'(t) - x'(t)y''(t)}{\sqrt{(x'(t)^2 + y'(t)^2)^3}},$$
- rate of change of the tangent direction, in terms of an arc-length parametrization $\gamma(t) = (x(t), y(t))$ (see e.g. [Pr01, Definition 2.1], [dCa76, Chapter 1,§5], or [Sp99, Chapter 1]):
$$\kappa(t) = |\gamma''(t)| = \sqrt{x''(t)^2 + y''(t)^2},$$
- in terms of the implicit equation $f \in \mathbb{R}[x, y]$ (for example in [Go05, Proposition 3.1], or [Ha99, Example 3.1.1]):
$$\kappa(f) = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{\sqrt{(f_x^2 + f_y^2)^3}}$$
- via the Frenet-Serret formula for a curve parametrized by arc-length $\gamma(t)$ (see e.g. [Go05] or [Sp99, Chapter 1]):
$$\gamma''(t) = \kappa(t) \cdot n(t),$$
where $n(t)$ is the unit normal vector in the direction of $\gamma''(t)$.

We will show later that, similarly to the classical curvature, each $\kappa_i, i \geq 0$ admits an expression in terms of the implicit equation $f \in \mathbb{C}[x, y]$ of a given plane curve.

Since the geometric invariant κ_1 constructed above was derived from the formula for the classical curvature, we call κ_1 the *first algebraic curvature* (*of plane curves*) and $\kappa_i, i > 1$ the *higher algebraic curvatures* (*of plane curves*). Notice that since Λ is a field homomorphism, the geometric invariants of plane curves form a field. We set

$$I_F := \text{field of geometric invariants (of plane curves).}$$

The algebraic curvatures do not only represent a family of geometric invariants, but even more, they generate the whole field I_F.

Theorem 1.1.5. *The field of geometric invariants of plane curves is generated over $\mathbb{C}$ by the variables $x^{(0)}, y^{(0)}$, the slope of the tangent vector and the first and higher algebraic curvatures, i.e., we have*
$$I_F = \mathbb{C}(x^{(0)}, y^{(0)}, \kappa_i : i \in \mathbb{N}).$$

We prove the theorem in two different manners. The first proof presented here uses differential field extensions. This proof was done in collaboration with J. Schicho and M. Gallet. The second proof follows a different strategy. It uses Proposition 1.1.2 and the correspondence between geometric invariants and rational expressions in parametrizations and their higher derivatives which are equivariant under reparametrizations. The second proof will show us even more. Given a geometric invariant p, the second proof tells us how to find the representation of p as a rational function in the generators $x^{(0)}, y^{(0)}$ and $\kappa_i, i \in \mathbb{N}$.

Let us now start with the first proof. Set

$$J := \mathbb{C}(x^{(0)}, y^{(0)}, \kappa_i \,:\, i \in \mathbb{N}) \subseteq F \quad \text{ and } \quad Y := \{y^{(i)} \,:\, i \in \mathbb{N}, i \neq 0\} \subseteq F.$$

Lemma 1.1.6. *The field F is generated by Y over the subfield J, i.e., $F = J(Y)$.*

Proof. From $\partial(\kappa_i) = \kappa_{i+1} y^{(1)} \in J(Y)$ and $\partial(Y) \subseteq Y$, it follows that $J(Y)$ is closed under ∂, i.e., $\partial(J(Y)) \subseteq J(Y)$. Further, since $x^{(0)}, y^{(0)} \in J(Y)$, all higher derivatives $x^{(i)}, y^{(i)}, i \geq 1$ are elements of $J(Y)$ as well. Hence, the generators of F lie in $J(Y)$ and so $F \subseteq J(Y)$. Since $J(Y) \subseteq F$ by construction, the statement is proven. □

For $i \geq 1$ set $V_i := F[\varphi^{(j)} \,:\, 1 \leq j \leq i] \subseteq L$. Notice that for each i we have the inclusion $\chi(V_i) \subseteq V_{i+1}$.

Lemma 1.1.7. *For each $i \geq 1$ we have $\Lambda(y^{(i)}) - \varphi^{(i)} y^{(1)} \in V_{i-1}$.*

Proof. We proceed by induction. The induction base follows immediately from $\Lambda(y^{(1)}) = \varphi^{(1)} y^{(1)}$. For $i \geq 2$ we use Lemma 1.1.3 and obtain

$$\Lambda(y^{(i+1)}) - \varphi^{(i+1)} y^{(1)} = \Lambda\big(\partial(y^{(i)})\big) - \varphi^{(i+1)} y^{(1)} = \chi\big(\Lambda(y^{(i)})\big) - \varphi^{(i+1)} y^{(1)}.$$

According to the induction hypothesis, we have $\Lambda(y^{(i)}) = \varphi^{(i)} y^{(1)} + v_{i-1}$ for some element $v_{i-1} \in V_{i-1}$. Thus, we can express

$$\chi\big(\Lambda(y^{(i)})\big) - \varphi^{(i+1)} y^{(1)} = \chi(\varphi^{(i)} y^{(1)} + v_{i-1}) - \varphi^{(i+1)} y^{(1)} = \varphi^{(i)} \varphi^{(1)} y^{(2)} + \chi(v_{i-1}).$$

But $\varphi^{(i)} \varphi^{(1)} y^{(2)} + \chi(v_{i-1}) \in V_i$ and the claim follows. □

Proposition 1.1.8. *The set Y is algebraically independent over I_F.*

Proof. Let us assume indirectly that Y is algebraically dependent over I_F. Let $m \in \mathbb{N}$ be the minimal positive integer such that there exists a polynomial in m variables

$$g(w) \in I_F[w] = I_F[w_1, \ldots, w_m], g(w) \neq 0 \;\text{ with }\; g(Y) := g(y^{(1)}, \ldots, y^{(m)}) = 0.$$

Let us denote by w' the vector $(w_1, \ldots, w_{m-1})$ and write

$$g(w) = \sum_j \alpha_j(w') w_m^j$$

as a polynomial in w_m with coefficients in $I_F[w'] = I_F[w_1, \dots, w_{m-1}]$. Set

$$\beta_j := \Lambda\big(\alpha_j(Y)\big) = \Lambda\big(\alpha_j(y^{(1)}, \dots, y^{(m-1)})\big).$$

Then we get the equality

$$0 = \Lambda\big(g(Y)\big) = \sum_j \beta_j \cdot \big(\Lambda(y^{(m)})\big)^j.$$

Notice that $\beta_j \in V_{m-1}$ and with Lemma 1.1.7 it holds also $\Lambda(y^{(m)}) - \varphi^{(m)} y^{(1)} \in V_{m-1}$. All this applied to the above equality gives us

$$V_{m-1} \ni \sum_j \beta_j \cdot \big((\Lambda(y^{(m)}))^j - (\varphi^{(m)})^j (y^{(1)})^j\big) = 0 - \sum_j \beta_j \cdot (\varphi^{(m)})^j (y^{(1)})^j,$$

whence follows $\beta_j = 0$ for all $j \geq 1$ because no power $\varphi^{(m)}$ belongs to V_{m-1}. Since Λ is a field homomorphism, it is injective, so $\alpha_j(Y) = 0$ for all $j \geq 1$. But as m was chosen minimal, we have $\alpha_j(w') = 0$ for any $j \geq 1$. Further, as $g(Y) = 0$, g cannot have a constant term and therefore $g(w) = 0$, a contradiction. $\square$

First Proof of Theorem 1.1.5. The inclusion $J \subseteq I_F$ is clear. It thus remains to show the inclusion $I_F \subseteq J$. Let $p \in I_F$ be a geometric invariant of plane curves. Since I_F is generated as a field over J by Y, there exist $f, g \in J[w] = J[w_1, \dots, w_n], g(w) \neq 0$, polynomials in n variables for some $n \in \mathbb{N}$, with $p = \frac{f(Y)}{g(Y)}$. From $J \subseteq I_F$ we conclude that $0 = f(w) - pg(w) \in I_F[w]$. Now, as $g(w) \neq 0$, the comparison of coefficients in the equality $0 = f(w) - pg(w)$ yields $p \in \mathrm{Quot}(J) = J$ which finishes the proof. $\square$

Now we move to the second and more geometric proof of Theorem 1.1.5. Consider the $\mathbb{C}$-morphism

$$\begin{aligned} i_\kappa \colon F &\to F \\ x^{(0)} &\mapsto x^{(0)}, x^{(i)} \mapsto \kappa_{i-1} \ \text{ for all } i \geq 1, \\ y^{(0)} &\mapsto y^{(0)}, y^{(1)} \mapsto 1, y^{(i)} \mapsto 0 \ \text{ for all } i \geq 2. \end{aligned}$$

The goal is to prove that each geometric invariant stays invariant under i_κ.

Proposition 1.1.9. *For each geometric invariant of plane curves $p \in I_F$ we have the following equality*

$$p = i_\kappa(p).$$

Once the Proposition 1.1.9 is proven, the statement of Theorem 1.1.5 follows immediately. Let us mention that the key argument in the following proof is strongly inspired by the idea used by J.-P. Demailly in the proof of Theorem 6.8 in [De97].

Second Proof of Theorem 1.1.5 and Proposition 1.1.9. Let $p \in I_F$ be a geometric invariant of plane curves. Then for all power series $x(t), y(t) \in \mathbb{C}[\![t]\!]$, equality (1.1) is satisfied by p. Let us choose $x(t)$ and $y(t)$ to be D-algebraically independent and such that $\operatorname{ord}(y(t)) = 1$. Denote by $\varphi(t)$ the inverse of $y(t)$, i.e., the power series satisfying $(y \circ \varphi)(t) = t$. Applying the chain rule to $y \circ \varphi$ yields

$$\varphi'(t) = \frac{1}{y'(\varphi(t))}$$

and so for the first derivative of $x \circ \varphi$ we have the equality

$$(x \circ \varphi)'(t) = \frac{x'(\varphi(t))}{y'(\varphi(t))} = \kappa_0(\underline{x(t)}, \underline{y(t)}) \circ \varphi(t) = \kappa_0\big(\underline{(x \circ \varphi)(t)}, \underline{(y \circ \varphi)(t)}\big).$$

For the first and the higher derivatives of $y \circ \varphi$ we have obviously

$$\begin{aligned} &\partial_t(y \circ \varphi)(t) = 1, \\ &\partial_t^i(y \circ \varphi)(t) = 0 \ \text{ for all } i \geq 2, \end{aligned}$$

and for $x \circ \varphi$, with the iterated chain rule, by induction we get

$$\partial_t^i(x \circ \varphi) = \kappa_{i-1}\big(\underline{(x \circ \varphi)(t)}, \underline{(y \circ \varphi)(t)}\big) \ \text{ for all } i \geq 2.$$

For $x(t), y(t)$ and $\varphi(t)$ as above, equality (1.1) composed with φ^{-1} on both sides becomes then

$$\begin{aligned} p(\underline{x(t)}, \underline{y(t)}) &= p(\underline{x(t)}, \underline{y(t)}) \circ (\varphi \circ \varphi^{-1})(t) = p\big(\underline{(x \circ \varphi)(t)}, \underline{(y \circ \varphi)(t)}\big) \circ \varphi^{-1}(t) \\ &= i_\kappa(p)\big(\underline{(x \circ \varphi)(t)}, \underline{(y \circ \varphi)(t)}\big) \circ \varphi^{-1}(t) = i_\kappa(p)(\underline{x(t)}, \underline{y(t)}) \circ (\varphi \circ \varphi^{-1})(t) \\ &= i_\kappa(p)(\underline{x(t)}, \underline{y(t)}) \end{aligned}$$

and thus,

$$\big(p - i_\kappa(p)\big)(\underline{x(t)}, \underline{y(t)}) = 0.$$

But since $x(t), y(t)$ were chosen to be D-algebraically independent, it follows

$$p - i_\kappa(p) = 0,$$

which finishes the proof. □

So for a given geometric invariant of plane curves $p \in I_F$, Theorem 1.1.5 ensures that it can be written as a rational function in the slope of the tangent vector and algebraic curvatures and Proposition 1.1.9 explains how to construct such a rational function. Namely, one can replace each $x^{(i)}$, with $i \geq 1$, by κ_{i-1}, the element $y^{(1)}$ by 1, and all $y^{(i)}$ for $i \geq 2$ by 0 to obtain the required representation of p as a rational function in the generators $x^{(0)}, y^{(0)}, \kappa_i$ of I_F.

Remark 1.1.10. It is clear that the field of geometric invariants of plane curves can be generated over $\mathbb{C}$ also by elements $x^{(0)}, y^{(0)}, \widetilde{\kappa}_i, i \in \mathbb{N}$, where

$$\begin{aligned} \widetilde{\kappa}_0 &:= \kappa_0^{-1} \\ \widetilde{\kappa}_i &:= \frac{\partial(\widetilde{\kappa}_{i-1})}{x^{(1)}}. \end{aligned}$$

But it is less clear how to represent a given geometric invariant as a rational function in $x^{(0)}, y^{(0)}$ and $\widetilde{\kappa}_i, i \in \mathbb{N}$. However, if we in the second proof of Theorem 1.1.5 require $\operatorname{ord}(x(t)) = 1$ instead of $\operatorname{ord}(y(t)) = 1$, denote by $\varphi(t)$ the inverse of $x(t)$ and adapt the argument of the proof to this new setting, i.e., we substitute $\frac{1}{x'(\varphi(t))}$ for $\varphi'(t)$, we will see that each geometric invariant of plane curves is invariant under the following $\mathbb{C}$-morphism:

$$\begin{aligned} i_{\widetilde{\kappa}} : F &\to F \\ x^{(0)} &\mapsto x^{(0)}, x^{(1)} \mapsto 1, x^{(i)} \mapsto 0 \ \text{ for all } i \geq 2, \\ y^{(0)} &\mapsto y^{(0)}, y^{(i)} \mapsto \widetilde{\kappa}_{i-1} \ \text{ for all } i \geq 1. \end{aligned}$$

Thus, we obtain the following extension of Proposition 1.1.9:

Lemma 1.1.11. *For each geometric invariant of plane curves $p \in I_F$ we have the following equalities*

$$p = i_\kappa(p) = i_{\widetilde{\kappa}}(p).$$

Implicit description of geometric invariants (of plane curves)

As next, we will discuss the interaction between the parametric and implicit representation of plane curves and its impact on (the implicit formulas for) geometric invariants. The equation (1.1) shows that for a parametrized curve $\gamma(t) = (x(t), y(t))$, each geometric invariant yields a geometric numeral which does not depend on a chosen parametrization. Hence, it should be possible to describe each such geometric numeral given by a geometric invariant also without using local parametrizations of a plane algebraic curve, namely, just by its defining implicit equation. We will now prove that such a description in terms of the defining implicit equation is always possible, and, moreover, we will also present implicit formulas for the slope of the tangent vector and algebraic curvatures, the generators of the field of geometric invaraints. Once their implicit expressions are known, one is able to find an implicit expression for an arbitrary geometric invariant.

Consider a plane algebraic curve $X = V(f), f \in \mathbb{C}[x, y]$, defined by a square-free polynomial f with $f(0,0) = 0$. We assume that 0 is a smooth point of the curve X. Thus X is necessarily analytically irreducible at the origin. Let us w.l.o.g. assume $f_x(0) \neq 0$. Let $\gamma(t)$ be a parametrization of X at the origin, i.e., $\gamma(t) = (x(t), y(t)) \in \mathbb{C}[\![t]\!]^2$ a pair of power series for which the ring map

$$\begin{aligned} \gamma^* \colon \mathbb{C}[\![x, y]\!]/(f) &\to \mathbb{C}[\![t]\!] \\ x &\mapsto x(t) \\ y &\mapsto y(t) \end{aligned}$$

is injective (one possibility would be to take a Puiseux parametrization – this kind of parametrizations is discussed in more detail in Appendix 4.2). Differentiating now both sides of the equality $f(x(t), y(t)) = 0$ with respect to t gives us

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0. \tag{1.2}$$

Notice, that from $f_x(0) \neq 0$ it follows that every parametrization $\gamma(t)$ satisfies $f_x(\gamma(t)) \neq 0$. Therefore, from equality (1.2), we immediately see

(1) $\kappa_0(\underline{x(t)}, \underline{y(t)}) = -\dfrac{f_y}{f_x}(x(t), y(t)),$

(2) $\kappa_1(\underline{x(t)}, \underline{y(t)}) = -\dfrac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_x^3}(x(t), y(t)),$

(3) $\kappa_i(\underline{x(t)}, \underline{y(t)}) = \dfrac{\partial_y\kappa_{i-1}(f) \cdot f_x - \partial_x\kappa_{i-1}(f) \cdot f_y}{f_x}(x(t), y(t)),$ for $i \geq 2$.

Hence, we set

$$\begin{aligned} \kappa_0\big(f\big)(x, y) &:= -\frac{f_y}{f_x}, \\ \kappa_1\big(f\big)(x, y) &:= -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_x^3}, \\ \kappa_i\big(f\big)(x, y) &:= \frac{\partial_y\kappa_{i-1}(f) \cdot f_x - \partial_x\kappa_{i-1}(f) \cdot f_y}{f_x}, \quad \text{for } i \geq 2, \end{aligned}$$

to be the implicit expressions of the algebraic curvatures. Together with the fact that the field of geometric invariants of plane curves is generated over $\mathbb{C}$ by $x^{(0)}, y^{(0)}$, the slope of the tangent vector and the (first and the higher) algebraic curvatures, we obtain the following theorem:

Theorem 1.1.12. *Given $p = \frac{p_1}{p_2} \in I_F$ a geometric invariant of plane curves, then there exist polynomials $p\big(f\big)_1(x, y)$, $p\big(f\big)_2(x, y)$ in f and its partial derivatives, i.e.,*

$$p\big(f\big)_1, p\big(f\big)_2 \in \mathbb{C}[\partial_x^i\partial_y^j f \,:\, i, j \in \mathbb{N}] \subseteq \mathbb{C}[x, y]$$

such that

$$p(\underline{x(t)}, \underline{y(t)}) = \frac{p\big(f\big)_1(x(t), y(t))}{p\big(f\big)_2(x(t), y(t))}$$

for all parametrizations $(x(t), y(t))$ of X with $p_2(\underline{x(t)}, \underline{y(t)}) \neq 0$. In other words, each geometric invariant (of a given plane curve) admits an implicit description (in terms of its defining equation and its derivatives).

Finally, we prove that we are able to reconstruct the analytic branch of X at the origin from the values of the implicit expressions of the slope and of algebraic curvatures of plane curves $\kappa_i\big(f\big)(x, y)$ at $(0, 0)$.

Corollary 1.1.13. *Let us assume that $\kappa_i\big(f\big)(0) < \infty$ for all $i \in \mathbb{N}$. Then the equation*

$$x - \sum_{i \geq 0} \frac{\kappa_i\big(f\big)(0)}{(i+1)!} \cdot y^{i+1} = 0$$

defines the analytic branch of X at the origin.

Proof. Notice first that by the assumption $\kappa_i\big(f\big)(0) < \infty$, the case $f_x(0) \neq 0$ is excluded. Hence, the Implicit Function Theorem applies to f and guarantees the existence of a parametrization of the form $(x(t), t) \in \mathbb{C}[\![t]\!]^2$ of X at 0, where $x(t)$ is even a convergent power series (since it is algebraic). The analytic branch of X at 0 is thus defined by the equation $g = x - x(y)$. The power series $x(t)$ can be expressed by the Taylor expansions as

$$x(t) = \sum_{i \geq 1} \frac{\partial_t^i x(0)}{i!} \cdot t^i = \sum_{i \geq 0} \frac{\kappa_i(\underline{x(t)}, \underline{t})|_{t=0}}{(i+1)!} \cdot t^{i+1}.$$

After rewriting each $\kappa_i(\underline{x(t)}, \underline{t})$ as $\kappa_i\big(f\big)(\underline{x(t)}, \underline{t})$ and substituting $t = 0$, we obtain the claimed equality. $\square$

Remark 1.1.14. Notice that for each plane algebraic curve $X = \{f = 0\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ that is smooth at the origin, either $f_x(0) \neq 0$ or $f_y(0) \neq 0$ holds. Therefore, in the case of $f_x(0) = 0$, we can just use coordinate change $x \mapsto y, y \mapsto x$, in order to reach the assumptions of Corollary 1.1.13. In such case, the theorem gives us for the analytic branch of X at the origin an analytic equation of the form $y - y(x) = 0$.

1.2 Geometric Invariants of Space Curves

The goal of this section is to generalize the concept of geometric invariants to space curves of arbitrary embedding dimensions and to introduce for them the concept of algebraic curvatures. Further, with Theorem 1.2.2, we show that the algebraic curvatures together with the slopes (of the tangent vector) generate the field of geometric invariants completely. We also provide their implicit formulas in terms of the defining polynomial equations of algebraic curves and their partial derivatives from which we conclude that an arbitrary geometric invariant admits and implicit expression as well, see Theorem 1.2.6. Finally, we explain with Corollary 1.2.7 how to reconstruct analytic branches of space curves from their algebraic curvatures.

As already mentioned, the concept of geometric invariants and the ideas and techniques used in the case of plane curves can be easily extended to space curves in $\mathbb{A}^{n+1}_{\mathbb{C}}$ as well. There are only few technicalities we have to deal with and which have to be carried out explicitly. Hence, as the proofs of the most results about geometric invariants of space curves follow the same punch line as in the plane curve case, we will not repeat them completely. Instead of that we will often refer to the corresponding statements and proofs from the previous section and will rather concentrate on fixing new difficulties which appear when considering higher embedding dimensions $n + 1$ with $n \geq 2$.

For each $n \in \mathbb{N}, n \geq 1$ let us consider the set of variables $x_j^{(i)}, y^{(i)}$ for $i, j \in \mathbb{N}, 1 \leq j \leq n$ and the field

$$F_n := \mathbb{C}(x_j^{(i)}, y^{(i)} \ : \ i, j \in \mathbb{N}, 1 \leq j \leq n).$$

The integer $n + 1$ stands for the embedding dimension of the space curves. We extend the derivation ∂ to F_n by $\partial(x_j^{(i)}) = x_j^{(i+1)}$ and thus, obtain the differential field (F_n, ∂). Let us define

$$L_n := F_n(\varphi^{(i)}, i \in \mathbb{N}) = \mathbb{C}(x_j^{(i)}, y^{(i)}, \varphi^{(i)} \,:\, i, j \in \mathbb{N}, 1 \leq j \leq n).$$

Further, we extend the symbolic chain rule χ and the field homomorphism Λ to L_n by

$$\chi(x_j^{(i)}) = x_j^{(i+1)}\varphi^{(1)} \text{ and } \Lambda(x_j^{(i)}) = \chi^{i+1}(x_j^{(0)}),$$

respectively and define geometric invariants of space curves as those rational expressions that are invariant under Λ.

Definition 1.2.1. We call a rational expression $p(\underline{x_j}, \underline{y} \,:\, 1 \leq j \leq n)$ in $x_j^{(0)}, y^{(0)}$ and their higher symbolic derivatives a *geometric invariant of algebraic space curves (of embedding dimension* $n+1$*)* if it stays fixed under Λ, i.e., if the following equality is fulfilled

$$p = \Lambda(p).$$

By I_{F_n} we denote the corresponding invariant field, the field of all geometric invariants of space curves of embedding dimension $n+1$.

Let us mention at this point, that the notion of tangent vector and curvature exists also for real parametrized space curves in $\mathbb{A}^3_{\mathbb{R}}$. The curvature, in literature often called the first curvature, is intrinsically defined as the reciprocal of the radius of the osculating circle (or osculating sphere) defined with help of the osculating plane (see e.g. [CH99, §27], [Kue06, §2C], or [Sp99, Chapter 1]). Another definition of the curvature uses a local parametrization $\gamma(t) \in \mathbb{R}[\![t]\!]^3$ of the curve:

$$\kappa(t) = \frac{|\gamma''(t) \times \gamma'(t)|}{|\gamma'(t)|^3},$$

and can be found for example by A. Pressley in [Pr01, Proposition 2.1] or R. Goldman in [Go05, §2.1], where also its corresponding expression in terms of the implicit equations can be found. Another source mentioning the implicit expression for the curvature is for instance the work by T. J. Willmore [Wi59, §5]. In the case that $\gamma(t)$ is an arc-length parametrization, the curvature is given by $\kappa(t) = |\gamma''(t)|$ as discussed for example by L. P. Eisenhart in [Ei09, §6-§11] where the author describes also a connection to the other definitions. This definition of curvature can be extended even to space curves in $\mathbb{A}^n_{\mathbb{R}}$ parametrized by arc-length. In [Kue06, §2A] W. Kühnel introduces the so-called *osculating conic* for space curves in $\mathbb{A}^n_{\mathbb{R}}$ that are parametrized by arc-length. Further, Kühnel defines in §2D of his book the concept of *Frenet curvatures* and *Frenet torsion* of space curves in $\mathbb{A}^n_{\mathbb{R}}$. There is also the notion of *torsion*, often called the second curvature, for curves in $\mathbb{A}^3_{\mathbb{R}}$. It should measure the “deviation of the curve from the osculating plane” and in terms of a parametrization $\gamma(t)$ it is given by

$$\tau = \frac{\det\begin{pmatrix}\gamma'(t) & \gamma''(t) & \gamma'''(t)\end{pmatrix}}{|\gamma'(t) \times \gamma''(t)|^2}.$$

For more details on torsion see for instance the paper by R. Goldman [Go05], where the author provides also an implicit description. For other, equivalent definitions of the torsion one can have look into the works by M. P. do Carmo [dCa76, Chapter 1, §5] or L. P. Eisenhart [Ei09, §10].

It can be shown that the classical curvature and also the torsion are equivariant under reparametrizations. Therefore, they would be natural candidates also for geometric invariants. However, here we have the same problem as in the plane curve case, namely the normalization factor containing the square root which does not allow us to consider the classical curvature and the torsion as geometric invariants of space curves in our sense.

We follow here a different strategy for the construction of geometric invariants of space curves. We will use geometric invariants of plane curves, which we have already studied and whose basic properties are already known to us. Notice that each geometric invariant of plane curves, except for polynomials in $y^{(0)}$ with coefficients in $\mathbb{C}$, gives rise to n different geometric invariants of space curves of embedding dimension $n+1$: Each geometric invariant p of plane curves is a rational function in variables $x^{(i)}, y^{(i)}, i \in \mathbb{N}$. Let us emphasize the set of variables by writing $p(\underline{x}, \underline{y})$ instead of just p. Hence, replacing each variable $x^{(i)}$ in $p(\underline{x}, \underline{y})$ by $x_j^{(i)}$, for some j, does not disturb the invariance and yields therefore a geometric invariant of algebraic space curves. Using this substitution we define

$$\kappa_{i,j} := \kappa_i(\underline{x_j}, \underline{y}),$$

and call these expressions again the *slopes (of the tangent vector)* in the case $i = 0$ and (the *first* if $i = 1$ and the *higher* for $i \geq 2$) *algebraic curvatures* (*of space curves*) otherwise. In this way we obtain the following system of geometric invariants of space curves:

$$x_j^{(0)}, y^{(0)}, \kappa_{i,j}, \qquad \text{where } i, j \in \mathbb{N} \text{ and } 1 \leq j \leq n.$$

It turns even out that they represent a complete system of generators of the field of geometric invariants of space curves of embedding dimension $n+1$ and that each geometric invariant of space curves can be written as a rational function in the algebraic curvatures when applying the following $\mathbb{C}$-morphism to it

$$\begin{aligned} &i_\kappa : F_n \to F_n \\ &x_j^{(0)} \mapsto x_j^{(0)}, x_j^{(i)} \mapsto \kappa_{i-1,j} \ \text{ for all } \ i, j \geq 1 \\ &y^{(0)} \mapsto y^{(0)}, y^{(1)} \mapsto 1, y^{(i)} \mapsto 0 \ \text{ for all } \ i \geq 2. \end{aligned}$$

More precisely:

Theorem 1.2.2. *The field of geometric invariants of space curves of embedding dimension $n+1$ is generated over $\mathbb{C}$ by the variables $x_j^{(0)}, y^{(0)}$, the slopes and the first and higher algebraic curvatures $\kappa_{i,j}, i, j \in \mathbb{N}, 1 \leq j \leq n$, i.e.,*

$$I_{F_n} = \mathbb{C}(x_j^{(0)}, y^{(0)}, \kappa_{i,j} \ : \ i, j, \in \mathbb{N}, 1 \leq j \leq n). \tag{1.3}$$

Moreover, for each geometric invariant $p(\underline{x_1}, \ldots, \underline{x_n}, \underline{y})$ the following equality is fulfilled

$$p = i_\kappa(p). \tag{1.4}$$

There are again two proofs of this theorem. If we just replace the field F by F_n and $J = \mathbb{C}(x^{(0)}, y^{(0)}, \kappa_i : i \in \mathbb{N})$ by the field $J_n := \mathbb{C}(x_j^{(0)}, y^{(0)}, \kappa_{i,j} : i,j \in \mathbb{N}, 1 \leq j \leq n)$ in Lemma 1.1.6, Proposition 1.1.8 and the first proof of Theorem 1.1.5, we obtain already one proof of the equality (1.3). However, this proof does not explain the second part of Theorem 1.2.2, namely the equality (1.4). For this we need again the trick from the second proof presented in the previous section.

Proof. The proof follows the same line as the second proof of Theorem 1.1.5. We consider again $x_1(t), \ldots, x_n(t), y(t) \in \mathbb{C}[\![t]\!]$ a family of D-algebraically independent power series with the property that $\operatorname{ord}(y(t)) = 1$. Using equation (1.1), which obviously holds also for $(n+1)$-tuples of power series, the equality

$$p(\underline{x_1(t)}, \ldots, \underline{x_n(t)}, \underline{y(t)}) = i_\kappa(p)(\underline{x_1(t)}, \ldots, \underline{x_n(t)}, \underline{y(t)})$$

can be shown with the same trick as in the proof of Theorem 1.1.5. Finally, we use the D-algebraically independence of the power series $x_1(t), \ldots, x_n(t), y(t)$ and conclude the required equality in the field of geometric invariants of space curves. $\square$

Remark 1.2.3. Notice that, given a system of generators of the field of geometric invariants of space curves, we can always produce another set of generators by switching the roles of the variables $y^{(i)}$ and $x_j^{(i)}$ in the generators. Let us set $\widetilde{\kappa}_{i,j,k} := \kappa_i(\underline{x_j}, \underline{x_k})$ and $\widetilde{\kappa}_{i,n+1,k} := \kappa_i(\underline{y}, \underline{x_k})$ for each $k = 1, \ldots, n$ and obtain n different sets of generators, namely for each $k = 1, \ldots, n$, the set

$$x_j^{(0)}, y^{(0)}, \widetilde{\kappa}_{i,j,k}, i \in \mathbb{N}, j \in \{1, \ldots, n+1\} \backslash \{k\}.$$

We define for each $k = 1, \ldots, n$, the following $\mathbb{C}$-morphism:

$$\begin{aligned}
i_{\widetilde{\kappa}_k} &: F_n \to F_n \\
&x_j^{(0)} \mapsto x_j^{(0)}, x_j^{(i)} \mapsto \widetilde{\kappa}_{i-1,j,k} \ \text{ for all } \ i \geq 1, j \in \{1, \ldots, n\} \backslash \{k\} \\
&x_k^{(0)} \mapsto x_k^{(0)}, x_k^{(1)} \mapsto 1, x_k^{(i)} \mapsto 0 \ \text{ for all } \ i \geq 2 \\
&y^{(0)} \mapsto y^{(0)}, y^{(i)} \mapsto \widetilde{\kappa}_{i-1,n+1,k} \ \text{ for all } \ i \geq 1,
\end{aligned}$$

where $1 \leq k \leq n$. With the last remark, we are able to extend Lemma 1.1.11:

Lemma 1.2.4. *For each geometric invariant of space curves $p \in I_{F_n}$ we have the following equalities*

$$p = i_\kappa(p) = i_{\widetilde{\kappa}_k}(p),$$

for all $k = 1, \ldots, n$.

Implicit description of geometric invariants (of space curves)

Similarly to the plane curve case, geometric invariants of space curves do admit implicit expressions in terms of the defining implicit equations of space curves and their higher derivatives as well. To find these implicit expressions we differentiate again the composition of the implicit equations with a parametrization and use the chain rule.

More precisely, consider an algebraic space curve $X = V(I) \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ defined by a radical ideal I. Further assume 0 being a smooth point on X. Let $\gamma(t)$ be a parametrization of X at 0. Here by a parametrization we again mean an $(n+1)$-tuple of univariate power series $\gamma(t) = (x_1(t), \dots, x_n(t), y(t)) \in \mathbb{C}[\![t]\!]^{n+1}$ for which the corresponding ring map γ^* between the local rings $\mathbb{C}[\![x_1, \dots, x_n]\!]/I$ and $\mathbb{C}[\![t]\!]$ is injective. Let us write $I = (f_1, \dots, f_r)$, with $f_j \in \mathbb{C}[x_1, \dots, x_n, y]$ the generators of I for some $r \geq n$. If we differentiate the equalities $f_j(\gamma(t)) = 0$ for all $1 \leq j \leq r$ with respect to t, we get the following system of equations:

$$J_{f_1,\dots,f_r}(\gamma(t)) \cdot \gamma'(t) = 0, \tag{1.5}$$

where

$$J_{f_1,\dots,f_r}(\gamma(t)) = \begin{pmatrix} \partial_{x_1} f_1(\gamma(t)) & \cdots & \partial_{x_n} f_1(\gamma(t)) & \partial_y f_1(\gamma(t)) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} f_r(\gamma(t)) & \cdots & \partial_{x_n} f_r(\gamma(t)) & \partial_y f_r(\gamma(t)) \end{pmatrix}$$

denotes the evaluation at $\gamma(t)$ of the Jacobian matrix of the polynomials $f_1, \dots, f_r$. The strategy to find implicit expressions for the slopes $\kappa_{0,j}$ is to eliminate all components of the vector $\gamma'(t)$ except for $x_j'(t)$ and $y'(t)$ in equation (1.5), or, in other words, to express each $x_j'(t)$ as a linear function in the parameter $y'(t)$. For this purpose we need only n rows of the Jacobian matrix $J_{f_1,\dots,f_r}$ that are linearly independent, let us say the first n rows.

Remark 1.2.5. Recall that for any point $a \in X$ we have $\operatorname{rk}(J_{f_1,\dots,f_r}(a)) \leq \operatorname{codim}(X) = n$ and the equality holds if and only if a is a smooth point of X. Hence, as 0 is a smooth point of X, if it happens that the first n rows of the Jacobian matrix $J_{f_1,\dots,f_r}$ are linearly dependent at 0, we can always reorder the generators of the ideal I, let us say $f_{\sigma(1)}, \dots, f_{\sigma(r)}$ for some permutation $\sigma \in S_r$, so that $J_{f_{\sigma(1)},\dots,f_{\sigma(n)}}$ is invertible at 0. Thus, we can always w.l.o.g. assume that $\det(J_{f_1,\dots,f_n}(0)) \neq 0$.

Let us rewrite now the first n rows of the equation (1.5) into

$$\mathcal{J}(\gamma(t)) \cdot \left(\frac{x_j'(t)}{y'(t)}\right)_{j=1}^{n} = -\big(\partial_y f_j(\gamma(t))\big)_{j=1}^{n},$$

with

$$\mathcal{J} := \begin{pmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_n} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_n & \cdots & \partial_{x_n} f_n \end{pmatrix}.$$

Applying the Cramer's rule to the above system of linear equations yields for each $j = 1, \dots, n$ the equality

$$\kappa_{0,j}(\underline{x_j(t)}, \underline{y(t)}) = \frac{x_j'(t)}{y'(t)} = \frac{\det \mathcal{J}_j(\gamma(t))}{\det \mathcal{J}(\gamma(t))},$$

where $\mathcal{J}_j$ is the matrix formed by replacing the j-th column of $\mathcal{J}$ by the column vector $-(\partial_y f_j)_{j=1}^n$, i.e.,

$$\mathcal{J}_j \coloneqq \begin{pmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_{j-1}} f_1 & -\partial_y f_1 & \partial_{x_{j+1}} f_1 & \cdots & \partial_{x_n} f_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \partial_{x_1} f_n & \cdots & \partial_{x_{j-1}} f_n & -\partial_y f_n & \partial_{x_{j+1}} f_n & \cdots & \partial_{x_n} f_n \end{pmatrix}.$$

We set

$$\kappa_{0,j}\big(f_1, \ldots, f_r\big)(x_1, \ldots, x_n, y) \coloneqq \frac{\det \mathcal{J}_j}{\det \mathcal{J}}$$

to be an implicit expression for the slope $\kappa_{0,j}$.

Let us assume that we have already computed $\kappa_{i,j}\big(f_1, \ldots, f_r\big)$, an implicit expression for $\kappa_{i,j}$. Then, for the higher algebraic curvatures $\kappa_{i,j}, i \geq 1$ we have by definition

$$\begin{aligned} \kappa_{i+1,j}(\underline{x_j(t)}, \underline{y(t)}) =& \frac{1}{y'(t)} \cdot \partial_t \kappa_{i,j}(\underline{x_j(t)}, \underline{y(t)}) \\ =& \frac{1}{y'(t)} \left(\sum_{k=1}^{n} \partial_{x_k} \kappa_{i,j}\big(f_1, \ldots, f_r\big)(\gamma(t)) x'_k(t) + \partial_y \kappa_{i,j}\big(f_1, \ldots, f_r\big)(\gamma(t)) y'(t) \right) \\ =& \sum_{k=1}^{n} \partial_{x_k} \kappa_{i,j}\big(f_1, \ldots, f_r\big)(\gamma(t)) \cdot \kappa_{0,k}\big(f_1, \ldots, f_r\big)(\gamma(t)) \\ &+ \partial_y \kappa_{i,j}\big(f_1, \ldots, f_r\big)(\gamma(t)). \end{aligned}$$

We set

$$\kappa_{i+1,j}\big(f_1, \ldots, f_r\big) \coloneqq \sum_{k=1}^{n} \partial_{x_k} \kappa_{i,j}\big(f_1, \ldots, f_r\big) \cdot \kappa_{0,k}\big(f_1, \ldots, f_r\big) + \partial_y \kappa_{i,j}\big(f_1, \ldots, f_r\big)$$

to be an implicit expression for the algebraic curvature $\kappa_{i+1,j}$.

Now, since the algebraic curvatures generate the whole field of geometric invariants of space curves, we conclude the existence of an implicit expression for all geometric invariants of space curves.

Theorem 1.2.6. *For each geometric invariant $p = \frac{p_1}{p_2} \in I_{F_n}$ of space curves of embedding dimension $n+1$ there exist polynomials $p\big(f_1, \ldots, f_r\big)_1, p\big(f_1, \ldots, f_r\big)_2$ in $f_1, \ldots f_r$ and their partial derivatives, i.e.,*

$$p\big(f_1, \ldots, f_r\big)_i, \in \mathbb{C}[\partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \partial_y^{i_{n+1}} f_k \,:\, i_0, \ldots, i_n \in \mathbb{N}, 1 \leq k \leq r] \subseteq \mathbb{C}[x_1, \ldots, x_n, y],$$

with $i = 1, 2$, such that the equality

$$p\big(\underline{x_1(t)}, \ldots, \underline{x_n(t)}, \underline{y(t)}\big) = \frac{p\big(f_1, \ldots, f_r\big)_1(\gamma(t))}{p\big(f_1, \ldots, f_r\big)_2(\gamma(t))}$$

is satisfied for all parametrizations $\gamma(t)$ of X for which $p_2(\underline{\gamma(t)}) \neq 0$. In other words, each geometric invariant (of a given space curve) admits an implicit description (in terms of its defining equations).

As in the case of plane curves, the values of the (higher) algebraic curvatures of a given space curve at a smooth point describe the curve completely. More precisely:

Corollary 1.2.7. *Suppose that* $\kappa_{i,j}\big(f_1,\ldots,f_n\big)(0) < \infty$ *for all* $i \in \mathbb{N}, 1 \leq j \leq n$. *Then then analytic branch of* X *at* 0 *is defined by the ideal*

$$\mathcal{I}_X = \left(x_j - \sum_{i\geq 0} \frac{\kappa_{i,j}\big(f_1,\ldots,f_n\big)(0)}{(i+1)!} y^{i+1} \,:\, 1 \leq j \leq n \right).$$

Proof. Since 0 is a smooth point on X, the curve X is locally at 0 biholomorphic to an open subset of $\mathbb{C}$ containing 0. Thus, X can be parametrized at 0 by a parametrization $\gamma(t) = (x_1(t),\ldots,x_n(t),y(t))$ with at least one component of order equal to one. Further, the first n rows of equality (1.5) can be written as

$$x_j'(t)\cdot\det\mathcal{J}(\gamma(t)) = y'(t)\cdot\det\mathcal{J}_j(\gamma(t)). \tag{1.6}$$

From the assumption $\kappa_{i,j}\big(f_1,\ldots,f_n\big)(0) < \infty$ it follows that $\det\mathcal{J}(0) \neq 0$. Evaluating both sides of equality (1.6) at $t = 0$ and using $\det\mathcal{J}(0) \neq 0$, we see that $\operatorname{ord}(y(t)) = 1$. Therefore, X admits a parametrization $\gamma(t)$ of the form $(x_1(t),\ldots,x_n(t),t) \in \mathbb{C}\{t\}^{n+1}$ at 0. Once we have determined the components $x_j(t), j = 1,\ldots,n$, of the parametrization, we conclude immediately that the analytic branch of X at 0 is contained in the analytic variety defined by the ideal $(x_j - x_j(y) : 1 \leq j \leq n)$. But this ideal has height equal to n and is a prime ideal. Hence, it defines already the analytic branch of X at 0. To determine the components $x_j(t), 1 \leq j \leq$ of the parametrization, we argue again with the Taylor expansion:

$$x_j(t) = \sum_{i\geq 0} \frac{\partial_t^i x_j(0)}{i!}\cdot t^i = \sum_{i\geq 0} \frac{\kappa_{i,j}(\underline{x_j(t),t})|_{t=0}}{(i+1)!}\cdot t^{i+1}$$

for all $1 \leq j \leq n$. Rewriting each $\kappa_{i,j}(\underline{x_j(t),t})$ as $\kappa_{i,j}\big(f_1,\ldots,f_r\big)(\gamma(t))$ and substituting $t = 0$ finishes now the proof. $\square$

1.3 Geometric Invariants of Surfaces

Whereas the step from the geometric invariants of plane curves to the geometric invariants of space curves was rather an "easy" generalization — we just had to rewrite the definitions and statements to the multivariate case — the step from the geometric invariants of curves to the geometric invariants of surfaces will be more tricky as the Krull dimension increases by one. In this section we show that the construction of the normal vector of a parametric surface is the key tool in the construction process of a complete system of generators of the field of geometric invariants of surfaces. We provide then again a ring map under which each geometric invariant is stable and which determines a representation of a given geometric invariant as a rational function in the elements of the generating system, see Theorem 1.3.4. Finally, we present implicit expression for the generators of the field of geometric invariants and conclude Corollary 1.3.6

saying that each geometric invariant admits such an implicit expression.

Let $x^{(i,j)}, y^{(i,j)}, z^{(i,j)}, i, j \in \mathbb{N}$ be three sets of countably many variables and set

$$F := \big(\mathbb{C}(x^{(i,j)}, y^{(i,j)}, z^{(i,j)} : i, j \in \mathbb{N}); \partial_1, \partial_2\big),$$

to be the differential field generated by all $x^{(i,j)}, y^{(i,j)}, z^{(i,j)}$ equipped with the two commutative $\mathbb{C}$-derivations

$$\partial_1 : F \to F, x^{(i,j)} \mapsto x^{(i+1,j)}, y^{(i,j)} \mapsto y^{(i+1,j)}, z^{(i,j)} \mapsto z^{(i+1,j)},$$

and

$$\partial_2 : F \to F, x^{(i,j)} \mapsto x^{(i,j+1)}, y^{(i,j)} \mapsto y^{(i,j+1)}, z^{(i,j)} \mapsto z^{(i,j+1)}.$$

Let us consider another set of variables $\varphi_k^{(i,j)}, i, j \in \mathbb{N}, k = 1, 2$ (similarly to the curve case, they play a different role than the variables $x^{(i,j)}, y^{(i,j)}, z^{(i,j)}$). Set

$$L := F(\varphi_k^{(i,j)} : i, j \in \mathbb{N}, k = 1, 2) = \mathbb{C}(x^{(i,j)}, y^{(i,j)}, z^{(i,j)}, \varphi_k^{(i,j)} : i, j \in \mathbb{N}, k = 1, 2)$$

and extend ∂_1 and ∂_2 to L by $\partial_1(\varphi_k^{(i,j)}) = \varphi_k^{(i+1,j)}$ and $\partial_2(\varphi_k^{(i,j)}) = \varphi_k^{(i,j+1)}$. On L we simulate the chain rule with respect to the two $\mathbb{C}$-derivations by

$$\begin{aligned}
&\chi_1 : L \to L,\\
&x^{(i,j)} \mapsto x^{(i+1,j)}\varphi_1^{(1,0)} + x^{(i,j+1)}\varphi_2^{(1,0)}\\
&y^{(i,j)} \mapsto y^{(i+1,j)}\varphi_1^{(1,0)} + y^{(i,j+1)}\varphi_2^{(1,0)},\\
&z^{(i,j)} \mapsto z^{(i+1,j)}\varphi_1^{(1,0)} + z^{(i,j+1)}\varphi_2^{(1,0)},\\
&\varphi_k^{(i,j)} \mapsto \varphi_k^{(i+1,j)},
\end{aligned}$$

and

$$\begin{aligned}
&\chi_2 : L \to L\\
&x^{(i,j)} \mapsto x^{(i+1,j)}\varphi_1^{(0,1)} + x^{(i,j+1)}\varphi_2^{(0,1)},\\
&y^{(i,j)} \mapsto y^{(i+1,j)}\varphi_1^{(0,1)} + y^{(i,j+1)}\varphi_2^{(0,1)},\\
&z^{(i,j)} \mapsto z^{(i+1,j)}\varphi_1^{(0,1)} + z^{(i,j+1)}\varphi_2^{(0,1)},\\
&\varphi_k^{(i,j)} \mapsto \varphi_k^{(i,j+1)}.
\end{aligned}$$

We think of φ_1 and φ_2 as symbols for the components of a reparametrization of a parametrized surface $\big(x(t,s), y(t,s), z(t,s)\big)$. To be more precise let us consider a parametrized surface

$$\gamma(t,s) = \big(x(t,s), y(t,s), z(t,s)\big) \in \mathbb{C}[\![t,s]\!]^3.$$

We then associate

$$\begin{aligned}x^{(0,0)} &\leftrightarrow x(t,s)\\ y^{(0,0)} &\leftrightarrow y(t,s)\\ z^{(0,0)} &\leftrightarrow z(t,s)\end{aligned}$$

and

$$\begin{aligned}x^{(i,j)} &\leftrightarrow \partial_t^i\partial_s^j x(t,s)\\ y^{(i,j)} &\leftrightarrow \partial_t^i\partial_s^j y(t,s)\\ z^{(i,j)} &\leftrightarrow \partial_t^i\partial_s^j z(t,s)\end{aligned}$$

for $i+j \geq 1$. Let $\varphi \in \mathrm{Aut}(\mathbb{C}[\![t,s]\!])$ be a (local) algebra automorphism, also called *reparametrization*. Notice that φ is given by a pair of power series $\varphi(t,s) = (\varphi_1(t,s), \varphi_2(t,s)) \in \mathbb{C}[\![t,s]\!]^2$ with linearly independent vectors of linear terms. We then associate

$$\varphi^{(0,0)} \leftrightarrow \varphi(t,s),$$

and

$$\varphi^{(i,j)} \leftrightarrow \partial_t^i\partial_s^j \varphi(t,s),$$

for $i+j \geq 1$. The derivation χ_1 reflects the chain rule with respect to t:

$$\partial_t\big((\partial_t^i\partial_s^j\gamma)\circ\varphi\big) = \partial_t\varphi_1\cdot(\partial_t^{i+1}\partial_s^j\gamma)\circ\varphi + \partial_t\varphi_2\cdot(\partial_t^i\partial_s^{j+1}\gamma)\circ\varphi,$$

and analogously, the derivation χ_2 stimulates the chain rule with respect to s. Next, we define a $\mathbb{C}$-morphism on L by

$$\begin{aligned}&\Lambda : L \to L,\\ &x^{(i,j)} \mapsto \chi_1^i\chi_2^j(x^{(0,0)}),\\ &y^{(i,j)} \mapsto \chi_1^i\chi_2^j(y^{(0,0)}),\\ &z^{(i,j)} \mapsto \chi_1^i\chi_2^j(z^{(0,0)}),\\ &\varphi_k^{(i,j)} \mapsto \varphi_k^{(i,j)},\end{aligned}$$

which in terms of power series means

$$\begin{aligned}\partial_t^i\partial_s^j\gamma(t,s) &\mapsto \partial_t^i\partial_s^j\big((\gamma\circ\varphi)(t,s)\big),\\ \partial_t^i\partial_s^j\varphi(t,s) &\mapsto \partial_t^i\partial_s^j\varphi(t,s).\end{aligned}$$

Let us use again the notation $p(\underline{x},\underline{y},\underline{z})$ for $p(x^{(i,j)}, y^{(i,j)}, z^{(i,j)} : i,j \in \mathbb{N})$ and let us denote the evaluation $p(\partial_t^i\partial_s^j x(t,s), \partial_t^i\partial_s^j y(t,s), \partial_t^i\partial_s^j z(t,s) : i,j \in \mathbb{N})$ at the triple $(x(t,s), y(t,s), z(t,s)) \in \mathbb{C}[\![t,s]\!]^3$ shortly by $p(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)})$.

Definition 1.3.1. We call an element p of F a *geometric invariant of surfaces* if it is fixed under Λ :

$$\Lambda(p) = p,$$

and set

$$I_F := \text{the field of geometric invariants of surfaces.}$$

For geometric invariants of surfaces, we have following extension of Proposition 1.1.2:

Proposition 1.3.2. *Let* $p(\underline{x}, \underline{y}, \underline{z}) = \frac{f(\underline{x},\underline{y},\underline{z})}{g(\underline{x},\underline{y},\underline{z})} \in F = \mathbb{C}(x^{(i,j)}, y^{(i,j)}, z^{(i,j)} : i, j \in \mathbb{N})$*. Then the following statements are equivalent:*

(i) p *is a geometric invariant of surfaces, i.e.,* $p \in I_F$.

(ii) The equality

$$p(\underline{(x \circ \varphi)(t,s)}, \underline{(y \circ \varphi)(t,s)}, \underline{(z \circ \varphi)(t,s)}) = p(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)}) \circ \varphi \tag{1.7}$$

holds for all power series $x, y, z \in \mathbb{C}[\![t,s]\!]$ *with* $g(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)}) \neq 0$ *and all reparametrizations* $\varphi \in \mathrm{Aut}(K[\![t,s]\!])$*, i.e.,* $p(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)})$ *is equivariant under reparametrizations.*

More on differential geometry side, there are two examples of invariants, the so-called *Gaussian* and *mean* curvature. They were first investigated by L. Euler in 1760. The idea was to reduce the problem of describing points on surfaces to the problem of studying plane curves by intersecting the surface with various normal planes at a given point. Euler observed that, if all the intersections do not have the same curvature, then there exists an intersection curve, which has minimal curvature k_1 at the intersection point, and another intersection curve which has maximal curvature k_2. These are called the *principal curvatures* and as they are defined intrinsically as curvatures of certain plane curves, they are invariant under rotations and translations. The *Gaussian curvature* κ_G is defined as their product k_1k_2 and the *mean curvature* κ_M is defined as their average $(k_1 + k_2)/2$ (for more details see for instance the work by M. Spivak [Sp99, Chapater 2] or M. P. do Carmo [dCa76, Chapater 3, §2]). Both curvatures can be expressed also in terms of a parametrization or implicit equation. If $\gamma(t,s) \in \mathbb{R}[\![t,s]\!]^3$ is a parametric surface, then the Gaussian and mean curvature are respectively given by

$$\kappa_G = \frac{\langle(\gamma_t \times \gamma_s), (n_t \times n_s)\rangle}{|\gamma_t \times \gamma_s|^2} \quad \text{and} \quad \kappa_M = \frac{\langle(\gamma_t \times \gamma_s), ((n_s \times \gamma_t) - (n_t \times \gamma_s))\rangle}{2|\gamma_t \times \gamma_s|^2},$$

where $n(t,s) = (\gamma_s \times \gamma_t)/|\gamma_s \times \gamma_t|$ is the unit normal vector. These formulas can be found for example by R. Goldman [Go05] together with their corresponding implicit equations. For the implicit definitions we should mention also the paper by E. Hartmann [Ha99, Chapter 4].

We will present now another system of "invariants" of surfaces which is strongly inspired by the concept of the classical Gaussian and mean curvature. Let us consider the following two

$\mathbb{C}$-derivations on F:

$$\begin{aligned}\Delta_1 : F &\to F\\ q &\mapsto \partial_1(q) \cdot \frac{y^{(0,1)}}{x^{(1,0)}y^{(0,1)} - x^{(0,1)}y^{(1,0)}} - \partial_2(q) \cdot \frac{y^{(1,0)}}{x^{(1,0)}y^{(0,1)} - x^{(0,1)}y^{(1,0)}}\end{aligned}$$

and

$$\begin{aligned}\Delta_2 : F &\to F\\ q &\mapsto -\partial_1(q) \cdot \frac{x^{(0,1)}}{x^{(1,0)}y^{(0,1)} - x^{(0,1)}y^{(1,0)}} + \partial_2(q) \cdot \frac{x^{(1,0)}}{x^{(1,0)}y^{(0,1)} - x^{(0,1)}y^{(1,0)}}.\end{aligned}$$

Using the fact that ∂_1 and ∂_2 commute, it can be shown by a computation that the drivations Δ_1 and Δ_2 commute as well, i.e.,

$$\Delta_1\Delta_2 = \Delta_2\Delta_1.$$

It should be clear from the construction of the derivations Δ_1 and Δ_2 that for a given geometric invariant p, the elements $\Delta_1(p)$ and $\Delta_2(p)$ are again geometric invariants. Thus, the derivations Δ_1 and Δ_2 generate a whole system of geometric invariants, more precisely we have the following geometric invariants:

Example 1.3.3.

0) The elements $x^{(0,0)}, y^{(0,0)}, z^{(0,0)}$ are obviously geometric invariants.

(1) As the normal vector

$$n_\gamma = \partial_t\gamma \times \partial_s\gamma$$

defined by the cross product of the two tangent vectors $\partial_t\gamma$ and $\partial_s\gamma$ is semi-equivariant under reparametrizations. Namely, for any $\varphi(t,s) = (\varphi_1(t,s), \varphi_2(t,s)) \in \mathbb{C}[\![t,s]\!]^2$ representing a reparametrization, we have

$$n_{\gamma\circ\varphi} = \det(\mathrm{J}_\varphi) \cdot n_\gamma \circ \varphi.$$

Hence, all the components of the normal vector are semi-equivariant with the same character $\det(\mathrm{J}_\varphi)$. Here

$$\mathrm{J}_\varphi = \begin{pmatrix} \partial_t\varphi_1 & \partial_s\varphi_1 \\ \partial_t\varphi_2 & \partial_s\varphi_2 \end{pmatrix}$$

denotes the Jacobian matrix of φ. Therefore, the rational expressions

$$\begin{aligned}\kappa_{1,0} &:= \frac{y^{(0,1)}z^{(1,0)} - y^{(1,0)}z^{(0,1)}}{x^{(1,0)}y^{(0,1)} - x^{(0,1)}y^{(1,0)}} = \Delta_1(z^{(0,0)}),\\ \kappa_{0,1} &:= \frac{x^{(1,0)}z^{(0,1)} - x^{(0,1)}z^{(1,0)}}{x^{(1,0)}y^{(0,1)} - x^{(0,1)}y^{(1,0)}} = \Delta_2(z^{(0,0)}),\end{aligned}$$

are geometric invariants.

(2) For $i, j \in \mathbb{N}, i+j \geq 2$ the rational expressions

$$\kappa_{i+1,j} := \Delta_1(\kappa_{i,j}), \kappa_{i,j+1} := \Delta_2(\kappa_{i,j})$$

are geometric invariants again.

We call, analogously to the curve case, the elements $\kappa_{i,j}$ the *algebraic curvatures of surfaces*. The advantage of the curvatures is that they already generate the whole field of geometric invariants. Moreover, for an arbitrary geometric invariant, we can determin its representation as a rational function in the algebraic curvatures easily by applying the following $\mathbb{C}$-morphism:

$$\begin{aligned}
&i_\kappa : F \to F\\
&x^{(0,0)} \mapsto x^{(0,0)}, x^{(1,0)} \mapsto 1, x^{(0,1)} \mapsto 0, x^{(i,j)} \mapsto 0 \text{ for all } i+j>1,\\
&y^{(0,0)} \mapsto y^{(0,0)}, y^{(1,0)} \mapsto 0, y^{(0,1)} \mapsto 1, y^{(i,j)} \mapsto 0 \text{ for all } i+j>1,\\
&z^{(0,0)} \mapsto z^{(0,0)}, z^{(i,j)} \mapsto \kappa_{i,j} \text{ for all } i+j \geq 1.
\end{aligned}$$

Theorem 1.3.4. *The field of geometric invariants of surfaces is generated over $\mathbb{C}$ by the variables $x^{(0,0)}, y^{(0,0)}, z^{(0,0)}$ and the algebraic curvatures, i.e.,*

$$I_F = \mathbb{C}(x^{(0,0)}, y^{(0,0)}, z^{(0,0)}, \kappa_{i,j} : i, j \in \mathbb{N}, i+j \geq 1).$$

Moreover, for each geometric invariant $p \in I_F$ we have

$$p = i_\kappa(p).$$

Recall that in the case of geometric invariants of (plane and space) curves, we proved the first part of this theorem in two different ways. The first proof was more combinatorial and used transcendence basis of a certain differential field extensions. The second proof used the idea by Demailly and provided also a geometrical explanation of the nature of the geometric invariants. We will prove now this theorem using the techniques of the second proof. Unfortunately, the combinatorial arguments contained in the first proof are much more technical and involved in the surface case than in the curve case, so we have not managed to generalize this proof to the higher dimensional case yet and it still remains on a list with open question (see the Section 5.1).

Proof. Let p be a geometric invariant of surfaces. Then, according to Lemma 1.3.2, p satisfies equality (1.7) for arbitrary power series $x(t,s), y(t,s), z(t,s) \in \mathbb{C}[\![t,s]\!]$ (for which the denominator of p does not vanish). Let us choose now the power series in such a way that the family $x(t,s), y(t,s), z(t,s), \varphi_1(t,s), \varphi_2(t,s)$ is D-algebraically independent and such that $(x_s y_t - x_t y_s)(0,0) \neq 0$. Let $\varphi(t,s) = (\varphi_1(t,s), \varphi_2(t,s))$ be the unique reparametrization satisfying

$$(x \circ \varphi, y \circ \varphi) = (t,s). \tag{1.8}$$

Applying the chain rule to equality (1.8) with respect to the variables t and s yields the following system of linear equations:

$$\begin{pmatrix} x_t \circ \varphi & x_s \circ \varphi & 0 & 0\\ y_t \circ \varphi & y_s \circ \varphi & 0 & 0\\ 0 & 0 & x_t \circ \varphi & x_s \circ \varphi\\ 0 & 0 & y_t \circ \varphi & y_s \circ \varphi \end{pmatrix} \begin{pmatrix} \partial_t \varphi_1\\ \partial_t \varphi_2\\ \partial_s \varphi_1\\ \partial_s \varphi_2 \end{pmatrix} = \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix}.$$

Solving this system for $\partial_t\varphi_1, \partial_t\varphi_2, \partial_s\varphi_1$ and $\partial_s\varphi_2$ gives us

$$\begin{aligned}
\partial_t\varphi_1 &= \frac{y_s \circ \varphi}{(x_t y_s - x_s y_t) \circ \varphi}, \\
\partial_s\varphi_1 &= \frac{-x_s \circ \varphi}{(x_t y_s - x_s y_t) \circ \varphi}, \\
\partial_t\varphi_2 &= \frac{-y_t \circ \varphi}{(x_t y_s - x_s y_t) \circ \varphi}, \\
\partial_s\varphi_2 &= \frac{x_t \circ \varphi}{(x_t y_s - x_s y_t) \circ \varphi}.
\end{aligned}$$

After substituting the above equalities for $\partial_t\varphi_1, \partial_t\varphi_2, \partial_s\varphi_1$ and $\partial_s\varphi_2$ into the partial derivatives of compositions $x \circ \varphi, y \circ \varphi$ and $z \circ \varphi$, we obtain

$$\begin{gathered}
\partial_t(x \circ \varphi) = 1, \partial_s(x \circ \varphi) = 0, \\
\partial_t(y \circ \varphi) = 0, \partial_s(y \circ \varphi) = 1,
\end{gathered}$$

and

$$\begin{aligned}
\partial_t(z \circ \varphi) &= \left(\frac{y_s z_t - y_t z_s}{x_t y_s - x_s y_t}\right) \circ \varphi = \kappa_{1,0}\big(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)}\big) \circ \varphi, \\
\partial_s(z \circ \varphi) &= \left(\frac{x_t z_s - x_s z_t}{x_t y_s - x_s y_t}\right) \circ \varphi = \kappa_{0,1}\big(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)}\big) \circ \varphi.
\end{aligned}$$

By induction we then have

$$\partial_t^i \partial_s^j (x \circ \varphi) = 0, \partial_t^i \partial_s^j (y \circ \varphi) = 0,$$

and

$$\partial_t^i \partial_s^j (z \circ \varphi) = \kappa_{i,j}\big(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)}\big) \circ \varphi,$$

for all $i + j > 1$. With these equalities, equation (1.7) becomes

$$p\big(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)}\big) = p\big(\underline{(x \circ \varphi)}, \underline{(y \circ \varphi)}, \underline{(z \circ \varphi)}\big) \circ \varphi^{-1} =$$

$$= p\left((x \circ \varphi), 1, 0, \ldots, (y \circ \varphi), 0, 1, 0, \ldots, (z \circ \varphi), \kappa_{1,0}\big(\underline{(x \circ \varphi)}, \underline{(y \circ \varphi)}, \underline{(z \circ \varphi)}\big), \ldots\right) \circ \varphi^{-1} =$$

$$= p\left(x(t,s), 1, 0, \ldots, y(t,s), 0, 1, 0, \ldots, z(t,s), \kappa_{i,j}\big(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)}\big) : i + j \geq 1\right) =$$

$$= i_\kappa(p)\big(\underline{x(t,s)}, \underline{y(t,s)}, \underline{z(t,s)}\big).$$

Now, as the power series $x(t,s), y(t,s), z(t,s)$ and $\varphi_i(t,s)$, for $i = 1, 2$ were chosen to be D-algebraically independent, we can consider them as variables and conclude the equality

$$p(\underline{x}, \underline{y}, \underline{z}) = p(x^{(0,0)}, 1, 0 \ldots, y^{(0,0)}, 0, 1, 0, \ldots, z^{(0,0)}, \kappa_{i,j} : i + j \geq 1) = i_\kappa(p)(\underline{x}, \underline{y}, \underline{z}),$$

which finishes the proof. □

Let us notice that the invariance of each $\kappa_{i,j}$ is preserved under any permutation of variables. More precisely, there are 6 possibilities how to permute the triple (x, y, z) and hence also $(\underline{x}, \underline{y}, \underline{z})$. We denote by $(\sigma(x), \sigma(y), \sigma(z))$ the one of these possibilities which corresponds to the permutation $\sigma \in S_3$. The vector $(\underline{x}, \underline{y}, \underline{z})$ maps then under σ to $(\underline{\sigma(x)}, \underline{\sigma(y)}, \underline{\sigma(z)})$. It is obvious that each $\kappa_{i,j}(\underline{\sigma(x)}, \underline{\sigma(y)}, \underline{\sigma(z)})$ defines again a geometric invariant of surfaces. Moreover, considering all possible permutations of the variables gives us 5 other sets of generators for the field of geometric invariants, namely for each $\sigma \in S_3$ the following one:

$$x^{(0,0)}, y^{(0,0)}, z^{(0,0)}, \kappa_{i,j}(\underline{\sigma(x)}, \underline{\sigma(y)}, \underline{\sigma(z)}).$$

We can now use the same method as in the proof of Theorem 1.3.4, just adapted to the permuted set of variables, to construct for each $\sigma \in S_3$ the $\mathbb{C}$-morphism

$$\begin{aligned}
&i_{\kappa,\sigma} : F \to F\\
&\sigma(x)^{(0,0)} \mapsto \sigma(x)^{(0,0)}, \sigma(x)^{(1,0)} \mapsto 1, \sigma(x)^{(0,1)} \mapsto 0, \sigma(x)^{(i,j)} \mapsto 0 \text{ for all } i+j>1,\\
&\sigma(y)^{(0,0)} \mapsto \sigma(y)^{(0,0)}, \sigma(y)^{(1,0)} \mapsto 0, \sigma(y)^{(0,1)} \mapsto 1, \sigma(y)^{(i,j)} \mapsto 0 \text{ for all } i+j>1,\\
&\sigma(z)^{(0,0)} \mapsto \sigma(z)^{(0,0)}, \sigma(z)^{(i,j)} \mapsto \kappa_{i,j}(\underline{\sigma(x)}, \underline{\sigma(y)}, \underline{\sigma(z)}) \text{ for all } i+j \geq 1
\end{aligned}$$

and to conclude the following lemma:

Lemma 1.3.5. *Each geometric invariant of surfaces $p \in I_F$ satisfies for any $\sigma \in S_3$ the following equality:*

$$p = i_\kappa(p) = i_{\kappa,\sigma}(p).$$

Implicit description of geometric invariants (of surfaces)

As in the curve case, geometric invariants of surfaces admit implicit expressions. Let $X = V(f)$ be an algebraic surface defined by a square-free polynomial $f \in \mathbb{C}[x, y, z]$. Assume that $f(0, 0, 0) = 0$. Let us further assume that X is analytically irreducible at the origin and that it can be parametrized at the origin by $\gamma(t, s) = (x(t, s), y(t, s), z(t, s))$. By a parametrization $\gamma(t, s)$, we mean here a triple of power series for which the induced map $\gamma* : \mathbb{C}[x, y, z]/(f) \mapsto \mathbb{C}[\![t, s]\!]$ is injective. For further purposes, for any $q \in F$ we will denote $q(\underline{x(t, s)}, \underline{y(t, s)}, \underline{z(t, s)})$ shortly by $q(\underline{\gamma})$. Now, differentiating the equality $f(\gamma) = 0$ w.r.t. t and s yields according to the chain rule:

$$\begin{aligned}
f_x(\gamma)x_t + f_y(\gamma)y_t + f_z(\gamma)z_t = 0,\\
f_x(\gamma)x_s + f_y(\gamma)y_s + f_z(\gamma)z_s = 0,
\end{aligned}$$

from which we conclude that the vector $(f_x(\gamma), f_y(\gamma), f_z(\gamma))$ and the cross product of the two tangent vectors $\partial_t\gamma(t, s) \times \partial_s\gamma(t, s)$ are parallel. Finally, we obtain

(1)

$$\begin{aligned}
\kappa_{0,1}(\underline{\gamma}) &= \frac{x_t z_s - x_s z_t}{x_t y_s - x_s y_t}(\gamma) = -\frac{f_y}{f_z}(\gamma),\\
\kappa_{1,0}(\underline{\gamma}) &= \frac{y_s z_t - y_t z_s}{x_t y_s - x_s y_t}(\gamma) = -\frac{f_x}{f_z}(\gamma),
\end{aligned}$$

for the first algebraic curvatures and hence we set

$$\kappa_{0,1}(f) \coloneqq -\frac{f_y}{f_z} \quad \text{and} \quad \kappa_{1,0}(f) \coloneqq -\frac{f_x}{f_z}$$

to be the implicit expression of the first algebraic curvatures of X. Further we have

(2)

$$\kappa_{i+1,j}(\underline{\gamma}) =$$

$$= \frac{(\partial_x\kappa_{i,j}(f)x_t + \partial_y\kappa_{i,j}(f))y_t + \partial_z\kappa_{i,j}(f)z_t)y_s - (\partial_x\kappa_{i,j}(f)x_s + \partial_y\kappa_{i,j}(f)y_s + \partial_z\kappa_{i,j}(f)z_s)y_t}{x_ty_s - x_sy_t}(\gamma) =$$

$$= \left(\partial_x\kappa_{i,j}(f) + \partial_z\kappa_{i,j}(f)\cdot\kappa_{1,0}(f)\right)(\gamma),$$

and

$$\kappa_{i,j+1}(\underline{\gamma}) =$$

$$= \frac{(\partial_x\kappa_{i,j}(f)x_s + \partial_y\kappa_{i,j}(f)y_s + \partial_z\kappa_{i,j}(f)z_s)x_t - (\partial_x\kappa_{i,j}(f)x_t + \partial_y\kappa_{i,j}(f)y_t + \partial_z\kappa_{i,j}(f)z_t)x_s}{x_ty_s - x_sy_t}(\gamma)$$

$$= \left(\partial_y\kappa_{i,j}(f) + \partial_z\kappa_{i,j}(f)\cdot\kappa_{0,1}(f)\right)(\gamma)$$

for the higher algebraic curvatures. Therefore, we set

$$\kappa_{i+1,j}(f) \coloneqq \partial_x\kappa_{i,j}(f) + \partial_z\kappa_{i,j}(f)\cdot\kappa_{1,0}(f),$$

$$\kappa_{i,j+1}(f) \coloneqq \partial_y\kappa_{i,j}(f) + \partial_z\kappa_{i,j}(f)\cdot\kappa_{0,1}(f),$$

to be the implicit expressions of the higher algebraic curvatures of X. As the field of geometric invariants of surfaces is generated by the algebraic curvatures, we conclude that every geometric invariant admits an implicit description in terms of the defining polynomial f and its partial derivatives:

Theorem 1.3.6. *For every geometric invariant $p = \frac{p_1}{p_2} \in I_F$ of surfaces there exist polynomials $p(f)_1, p(f)_2$ in f and its partial derivatives, i.e.,*

$$p(f)_1, p(f)_2 \in \mathbb{C}[\partial_x^i\partial_y^j\partial_z^k f \ : \ i,j,k \in \mathbb{N}] \subseteq \mathbb{C}[x,y,z],$$

such that

$$p(\underline{\gamma}) = \frac{p(f)_1(\gamma)}{p(f)_2(\gamma)}$$

is fulfilled for all parametrizations $\gamma(t,s)$ of X for which $p_2(\underline{\gamma}) \neq 0$. In other words, each geometric invariant (of a given surface) admits an implicit expression as a rational function in the defining equation of the surface and its partial derivatives.

Notice, that with the same argument as in Remark 1.2.5, for a a regular point on X, we can w.l.o.g. assume $f_z(a) \neq 0$. Thus, the algebraic curvatures are well defined at regular points of surfaces.

1.4 Geometric Invariants of Higher Dimensional Varieties

In this section we show that the the concept of geometric invariants can be extended to arbitrary dimensions. We crate again with Proposition 1.4.2 a bridge between geometric invariants and rational expressions in parametrizations and their partial derivatives that are equivariant under reparametrizations. We then introduce the concept of algebraic curvatures of higher dimensional varieties and show with Theorem 1.4.5 that they generate the whole field of geometric invariants. Finally, the existence of their implicit expression is provided by Theorem 1.4.7.

Let us consider the set $x_k^{(i_1,\dots,i_m)}, k \in [n], i_j \in \mathbb{N}$ of countably many variables. Here for a positive integer n we denote by $[n]$ the set $\{1,\dots,n\}$. Set

$$F := \mathbb{C}(x_k^{(i_1,\dots,i_m)} \ : \ k \in [n], i_j \in \mathbb{N}, \text{ for } j \in [m]),$$

to be the differential field generated by all $x_k^{(i_1,\dots,i_m)}$ and equipped with the following m commutative $\mathbb{C}$-derivations

$$\begin{aligned} &\partial_j : F \to F \\ &x_k^{(i_1,\dots,i_m)} \mapsto x_k^{(i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_m)}, \ \text{ for } k \in [n], \end{aligned}$$

for $j \in [m]$. Let us consider another set of countably many variables $\varphi_l^{(i_1,\dots,i_m)}$ with $l \in [m]$ and $i_j \in \mathbb{N}$ and set

$$L := F(\varphi_l^{(i_1,\dots,i_m)} \ : \ l \in [m], i_j \in \mathbb{N}) = \mathbb{C}(x_k^{(i_1,\dots,i_m)}, \varphi_l^{(i_1,\dots,i_m)} \ : \ k \in [n], l \in [m], i_j \in \mathbb{N}).$$

We extend the derivations ∂_j to L by

$$\partial_j : \varphi_l^{(i_1,\dots,i_m)} \mapsto \varphi_l^{(i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_m)},$$

for $l \in [m]$ and simulate on L the chain rule with respect to each derivation ∂_j via the following $\mathbb{C}$-derivations:

$$\begin{aligned} &\chi_j : L \to L \\ &x_k^{(i_1,\dots,i_m)} \mapsto \sum_{l=1}^{m} \partial_l x_k^{(i_1,\dots,i_m)} \partial_j \varphi_l^{(0,\dots,0)} \\ &\varphi_l^{(i_1,\dots,i_m)} \mapsto \partial_j \varphi_l^{(i_1,\dots,i_m)}, \end{aligned}$$

where $j, l \in [m], k \in [n]$.

We think of each $x_k^{(i_1,\dots,i_m)}$ again as a symbol for the higher partial derivatives of the k-th coordinate of a parametrized variety $\gamma(\underline{t}) = (x_1(\underline{t}),\dots,x_n(\underline{t})) \in \mathbb{C}[\![\underline{t}]\!]^n$, where $\underline{t} = (t_1,\dots,t_m)$, i.e., for each $k \in [n]$ we associate

$$\begin{aligned} &x_k^{(0,\dots,0)} \leftrightarrow x_k(\underline{t}), \\ &x_k^{(i_1,\dots,i_m)} \leftrightarrow \partial_{t_1}^{i_1} \cdots \partial_{t_m}^{i_m} x_k(\underline{t}). \end{aligned}$$

From now on we will use the notation $p(\underline{x(\underline{t})})$ or $p(\underline{\gamma})$ for $p(\partial_{t_1}^{i_1}\cdots\partial_{t_m}^{i_m}x_k(\underline{t}) : k\in[n], i_j\in\mathbb{N})$, and $p(x_k^{(i_1,\ldots,i_m)} : k\in[n], i_j\in\mathbb{N})$ will be shortly denoted by $p(\underline{x})$. Let $\varphi\in\mathrm{Aut}(\mathbb{C}[\![\underline{t}]\!])$ be a (local) algebra automorphism. Notice that φ is given by an m - tuple of power series

$$\varphi(\underline{t}) = (\varphi_1(\underline{t}),\ldots,\varphi_m(\underline{t}))\in\mathbb{C}[\![\underline{t}]\!]^m$$

with linearly independent vectors of linear terms. We then associate

$$\begin{aligned}
&\varphi_l^{(0,\ldots,0)} \leftrightarrow \varphi_l(\underline{t}),\\
&\varphi_l^{(i_1,\ldots,i_m)} \leftrightarrow \partial_{t_1}^{i_1}\cdots\partial_{t_m}^{i_m}\varphi_l(\underline{t})
\end{aligned}$$

for each $l\in[m]$. Notice that in terms of parametrizations, the derivation χ_l reflects the chain rule with respect to the variable t_l. Namely, for each $p\in\mathbb{C}(\underline{x(\underline{t})})$ we have

$$\partial_{t_l}(p\circ\varphi)(\underline{t}) = \sum_{j=1}^{m}(\partial_{t_j}p\circ\varphi)(\underline{t})\cdot\partial_{t_l}\varphi_j(\underline{t}).$$

Next we define a $\mathbb{C}$-morphism on L

$$\begin{aligned}
&\Lambda : L\to L,\\
&x_k^{(i_1,\ldots,i_m)} \mapsto \chi_1^{i_1}\cdots\chi_m^{i_m}(x_k^{(0,\ldots,0)}),\\
&\varphi_l^{(i_1,\ldots,i_m)} \mapsto \chi_1^{i_1}\cdots\chi_m^{i_m}(\varphi_l^{(0,\ldots,0)}) = \varphi_l^{(i_1,\ldots,i_m)}.
\end{aligned}$$

In terms of power series, we can think of Λ as of the following assignment:

$$\begin{aligned}
&\partial_{t_1}^{i_1}\cdots\partial_{t_m}^{i_m}x_k(\underline{t}) \mapsto \partial_{t_1}^{i_1}\cdots\partial_{t_m}^{i_m}(x_k\circ\varphi)(\underline{t}),\\
&\partial_{t_1}^{i_1}\cdots\partial_{t_m}^{i_m}\varphi_l(\underline{t}) \mapsto \partial_{t_1}^{i_1}\cdots\partial_{t_m}^{i_m}\varphi_l(\underline{t}).
\end{aligned}$$

Definition 1.4.1. We call an element p of F a *geometric invariant* if it is fixed under Λ, i.e., if the equality

$$\Lambda(p) = p$$

is fulfilled.

In terms of power series (parametrizations) this means the following equality

$$p(\underline{\gamma})\circ\varphi = p(\underline{\gamma\circ\varphi}). \tag{1.9}$$

We define

$$I_F := \text{the field of geometric invariants.}$$

Notice that Proposition 1.1.2 describing a connection between geometric invariants of plane curves and the parametric expressions, that are equivariant under reparametrizations, generalizes not only to space curves and surfaces (Proposition 1.3.2) but also to the higher dimensional case (with exactly the same proof):

Proposition 1.4.2. *Let* $p(\underline{x}) = \frac{f(\underline{x})}{g(\underline{x})} \in F = \mathbb{C}(\underline{x})$. *Then the following statements are equivalent:*

(i) $p \in I_F$, *i.e.,* p *is a geometric invariant.*

(ii) The equality

$$p(\underline{(x \circ \varphi)(\underline{t})}) = p(\underline{x(\underline{t})}) \circ \varphi \tag{1.10}$$

holds for all n*-tuples of power series* $x(\underline{t}) = (x_1(\underline{t}), \ldots, x_n(\underline{t})) \in \mathbb{C}[\![\underline{t}]\!]^n$ *with* $g(\underline{x(\underline{t})}) \neq 0$ *and all reparametrizations* $\varphi \in \mathrm{Aut}(\mathbb{C}[\![\underline{t}]\!])$, *i.e.,* $p(\underline{x(\underline{t})})$ *is equivariant under reparametrizations.*

Let us now denote by ∂x the following $m \times m$ matrix:

$$\partial x := \begin{pmatrix} \partial_1 x_{n-m+1}^{(0,\ldots,0)} & \cdots & \partial_m x_{n-m+1}^{(0,\ldots,0)} \\ \vdots & \ddots & \vdots \\ \partial_1 x_n^{(0,\ldots,0)} & \cdots & \partial_m x_n^{(0,\ldots,0)}, \end{pmatrix}$$

and set $\partial x_{(i,j)}$ to be the cofactor of the (i,j) entry of ∂x. Further, let us consider the following $\mathbb{C}$-derivations on F:

$$\begin{aligned} \Delta_j &: F \to F \\ q &\mapsto \frac{1}{\det(\partial x)} \sum_{l=1}^{m} (-1)^{l+j} \partial_l q \cdot \det(\partial x_{(j,l)}) \end{aligned}$$

for $j \in [m]$. As the derivations ∂_j commute with each other, the derivations Δ_j are commutative as well, i.e., we have

$$\Delta_i \Delta_j = \Delta_j \Delta_i$$

for all $i, j \in [m]$.

Remark 1.4.3. Notice that $\det(\partial x) \cdot \Delta_j(q) = \sum_{l=1}^{m} (-1)^{l+j} \partial_l q \cdot \det(\partial x_{(j,l)})$ is nothing else then the Laplace expansion along the j - th row of the matrix ∂x with the j-th row replaced by the vector $(\partial_1 q, \ldots, \partial_m q)$, i.e.,

$$\det(\partial x) \cdot \Delta_j(q) = \det \begin{pmatrix} \partial_1 x_{n-m+1}^{(0,\ldots,0)} & \cdots & \partial_m x_{n-m+1}^{(0,\ldots,0)} \\ \vdots & & \vdots \\ \partial_1 x_{n-m+j-1}^{(0,\ldots,0)} & \cdots & \partial_m x_{n-m+j-1}^{(0,\ldots,0)} \\ \partial_1 q & \cdots & \partial_m q \\ \partial_1 x_{n-m+j+1}^{(0,\ldots,0)} & \cdots & \partial_m x_{n-m+j+1}^{(0,\ldots,0)} \\ \vdots & & \vdots \\ \partial_1 x_n^{(0,\ldots,0)} & \cdots & \partial_m x_n^{(0,\ldots,0)} \end{pmatrix}.$$

Now, using the derivations Δ_j, with $j \in [m]$ iteratively, we define a whole system of geometric invariants:

Example 1.4.4. The variables $x_k^{(0,\dots,0)}, k \in [n]$ are obviously geometric invariants. But more interesting are the following examples:

(1) The determinant $\det(\partial x)$ is semi-invariant with character $\det(\partial\varphi)$ under Λ since

$$\Lambda(\det(\partial x)) = \det(\Lambda(\partial x)) = \det(\partial x \cdot \partial\varphi) = \det(\partial x) \cdot \det(\partial\varphi),$$

where

$$\partial\varphi = \begin{pmatrix} \partial_1\varphi_1^{(0,\dots,0)} & \cdots & \partial_m\varphi_1^{(0,\dots,0)} \\ \vdots & \ddots & \vdots \\ \partial_1\varphi_m^{(0,\dots,0)} & \cdots & \partial_m\varphi_m^{(0,\dots,0)} \end{pmatrix}.$$

(2) Given a geometric invariant $p \in I_F$, according to Remark 1.4.3 and the observation in the previous example, the sum

$$\sum_{l=1}^{m} (-1)^{j+l} \partial_l p \cdot \det(\partial x_{(j,l)})$$

is semi-invariant under Λ with character $\det(\partial\varphi)$.

(3) From Examples (1) and (2) we conclude the invariance under Λ of each expression

$$\kappa_{k,(i_1,\dots,i_m)}(\underline{x}) := \Delta_1^{i_1} \cdots \Delta_m^{i_m}(x_k^{(0,\dots,0)}).$$

Let us compare these geometric invariants with the geometric invariants of plane curves, i.e,. the case $n = 2, m = 1$ (let us for simplicity denote the variables $x_1^{(i)}$ and $x_2^{(i)}$ by $x^{(i)}$ and $y^{(i)}$). In the plane curve case, with the derivation Δ_1 we obtain the following geometric invariants:

$$\begin{aligned} \Delta_1(x^{(0)}) &= \frac{x^{(1)}}{y^{(1)}} = \kappa_0, \\ \Delta_1^2(x^{(0)}) &= \Delta_1\left(\frac{x^{(1)}}{y^{(1)}}\right) = \frac{x^{(2)}y^{(1)} - x^{(1)}y^{(2)}}{(y^{(1)})^3} = \kappa_1, \\ &\vdots \\ \Delta_1^{i+1}(x^{(0)}) &= \Delta_1(\kappa_{i-1}) = \frac{\partial(\kappa_{i-1})}{y^{(1)}} = \kappa_i, \end{aligned}$$

which are exactly the slope (of the tangent vector) and the (first and the higher) algebraic curvatures of plane curves. Therefore, we call each $\kappa_{k,(i_1,\dots,i_m)}$ with $i_1 + \cdots + i_m = 1$ a *slope* and all expressions $\kappa_{k,(i_1,\dots,i_m)}$ with $i_1 + \cdots + i_m > 1$ the *algebraic curvatures*. Consider now the following $\mathbb{C}$-morphism

$$\begin{aligned} &i_\kappa : F \to F \\ &x_k^{(0,\dots,0)} \mapsto x_k^{(0,\dots,0)} \quad \text{for all } k \in [n], \\ &x_k^{(i_1,\dots,i_m)} \mapsto \kappa_{k,(i_1,\dots,i_m)} \quad \text{for all } k \in [n] \text{ with } i_1 + \cdots + i_m > 1. \end{aligned}$$

Notice that by definition, $\kappa_{k,(i_1,\dots,i_m)} = 0$ for all $k = n - m + 1, \dots, n$. Analogously to the plane curve case, these algebraic curvatures generate the whole field of geometric invariants:

Theorem 1.4.5. *The field of geometric invariants is generated over $\mathbb{C}$ by the variables $x_k^{(0,\ldots,0)}$, $k \in [n]$, the slopes and the algebraic curvatures $\kappa_{l,(i_1,\ldots,i_m)}$, $l \in [n-m]$, i.e.,*

$$I_F = \mathbb{C}(x_k^{(0,\ldots,0)}, \kappa_{l,(i_1,\ldots,i_m)}(\underline{x}) : k \in [n], l \in [n-m], i_j \in \mathbb{N}).$$

Moreover, for each geometric invariant $p \in I_F$ we can find its representation as a rational function in the generators via applying the $\mathbb{C}$-morphism i_κ :

$$p = i_\kappa(p).$$

Proof. Let $p \in I_F$ be a geometric invariant. Then p satisfies the equality (1.10) for arbitrary power series $x_k(\underline{t}) \in \mathbb{C}[\![\underline{t}]\!], k \in [n]$. Let us pick a vector of n D-algebraically independent power series $x(\underline{t}) := (x_1(\underline{t}), \ldots, x_n(\underline{t}))$ for which the matrix

$$\partial x(\underline{t}) := \begin{pmatrix} \partial_{t_1} x_{n-m+1}(\underline{t}) & \cdots & \partial_{t_m} x_{n-m+1}(\underline{t}) \\ \vdots & \ddots & \vdots \\ \partial_{t_1} x_n(\underline{t}) & \cdots & \partial_{t_m} x_n(\underline{t}) \end{pmatrix}$$

is invertible at 0. Let $\varphi(\underline{t}) = (\varphi_1(\underline{t}), \ldots, \varphi_m(\underline{t})) \in \mathbb{C}[\![\underline{t}]\!]^m$ be the unique vector of power series (defining a reparametrization) satisfying the equality

$$(x_{n-m+1}, \ldots, x_n) \circ \varphi = (\underline{t}). \tag{1.11}$$

Differentiating the equality (1.11) with respect to the variables $t_1, \ldots, t_m$ yields according to the chain rule:

$$\begin{pmatrix} (\partial x \circ \varphi)(\underline{t}) & & \mathbf{0} \\ & (\partial x \circ \varphi)(\underline{t}) & \\ \mathbf{0} & & \ddots \\ & & & (\partial x \circ \varphi)(\underline{t}) \end{pmatrix} \cdot \begin{pmatrix} \partial_{t_1}\varphi(\underline{t})^T \\ \partial_{t_2}\varphi(\underline{t})^T \\ \vdots \\ \partial_{t_m}\varphi(\underline{t})^T \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix},$$

where $(\partial x \circ \varphi)(\underline{t})$ denotes just the matrix $\partial x(\underline{t})$ evaluated at the point $\varphi(\underline{t})$ and each $e_j, j \in [m]$ denotes the j-th standard basis column vector. To solve this system of equations for $\partial_{t_j}\varphi_k(\underline{t})$ with $j, k \in [m]$, we apply Cramer's rule to each subsystem

$$(\partial x \circ \varphi)(\underline{t}) \cdot \partial_{t_j}\varphi(\underline{t})^T = e_j.$$

This gives us

$$\partial_{t_j}\varphi_k(\underline{t}) = (-1)^{j+k} \frac{\det\big((\partial x_{(j,k)} \circ \varphi)(\underline{t})\big)}{\det\big((\partial x \circ \varphi)(\underline{t})\big)}.$$

Hence we obtain

$$\partial_{t_j}(x \circ \varphi)(\underline{t}) = e_j,$$

and

$$\partial_{t_j}(x_l \circ \varphi)(\underline{t}) = \sum_{i=1}^{m} (\partial_{t_i} x_l \circ \varphi)(\underline{t}) \cdot \partial_{t_j}\varphi_i(\underline{t}) = \kappa_{l,e_j}(\underline{x(\underline{t})}) \circ \varphi$$

for all $j \in [m], l \in [n]$. By induction we see that

$$\partial_{t_1}^{i_1} \cdots \partial_{t_m}^{i_m} (x_l \circ \varphi)(\underline{t}) = \kappa_{l,(i_1,\dots,i_m)}(\underline{x(\underline{t})}) \circ \varphi, \text{ for } i_1 + \cdots + i_m > 1$$

and, thus, we have the following equality for p:

$$p((\underline{x \circ \varphi)(\underline{t}})) = i_\kappa(p)(\underline{x(\underline{t})}) \circ \varphi.$$

With this, equality (1.10) transforms into

$$p(\underline{x(\underline{t})}) = p(\underline{x(\underline{t})}) \circ (\varphi \circ \varphi^{-1}) = p((\underline{x \circ \varphi)(\underline{t}})) \circ \varphi^{-1} = i_\kappa(p)(\underline{x(\underline{t})}) \circ (\varphi \circ \varphi^{-1}) = i_\kappa(p)(\underline{x(\underline{t})}).$$

Since the power series $x_1(\underline{t}), \dots, x_n(\underline{t})$ are D-algebraically independent, we can consider them as variables in the above equality. This gives the following equality in the field F:

$$p(\underline{x}) = i_\kappa(p)(\underline{x}).$$

□

As in the case of geometric invariants of (plane and space) curves and surfaces, we will again use the fact that the invariance of each $\kappa_{l,(i_1,\dots,i_m)}$ is preserved under any permutation of variables to produce further systems of generators of the field of geometric invariants. More precisely, let $\sigma \in S_n$ be a permutation of n elements. Then after applying σ to the variables $x_k^{(i_1,\dots,i_m)}, k \in [n]$ in each curvature $\kappa_{l,(i_1,\dots,i_m)}$, we obtain the following system of generators of the field of geometric invariants:

$$x_k^{(0,\dots,0)}, \kappa_{l,(i_1,\dots,i_m)}(x_{\sigma(k)}^{(j_1,\dots,j_m)} : k \in [n], j_i \in \mathbb{N}),\ k \in [n], l \in [n-m].$$

Using now the same techniques as in the proof of Theorem 1.4.5, but adapted to the new set of variables that we obtain after applying a permutation $\sigma \in S_n$, yields the $\mathbb{C}$-morphism

$$\begin{aligned}
&i_{\kappa,\sigma} : F \to F\\
&x_{\sigma(k)}^{(0,\dots,0)} \mapsto x_{\sigma(k)}^{(0,\dots,0)} \text{ for all } k \in [n],\\
&x_{\sigma(k)}^{(i_1,\dots,i_m)} \mapsto \kappa_{\sigma(k),(i_1,\dots,i_m)}(x_{\sigma(k)}^{(j_1,\dots,j_m)} : k \in [n], j_l \in \mathbb{N}) \text{ for all } k \in [n] \text{ and } i_1 + \cdots + i_m > 1
\end{aligned}$$

and deduce the following lemma:

Lemma 1.4.6. *Each geometric invariant $p \in I_F$ satisfies the following equality:*

$$p = i_\kappa(p) = i_{\kappa,\sigma}(p),$$

for any $\sigma \in S_n$.

Implicit description of geometric invariants (of higher dimensional varieties)

Let us now discuss the description of geometric invariants in terms of the defining implicit equations of a certain variety. Analogously to the previous cases, also here we can find an implicit formula for each geometric invariant of a given algebraic variety. Consider an algebraic variety $X = V(I) \subseteq \mathbb{A}^n, I \subseteq \mathbb{C}[x_1, \dots, x_n]$, of Krull dimension m and embedding dimension n with $m \le n-1$ defined by a radical ideal $I = \sqrt{I}$. Let us fix a set $f_1, \dots, f_r, r \ge n-m$, of generators for I. Assume that $0 \in X$ is a non-singular point on X and that X is analytically irreducible at the origin. Let us consider a regular parametrization $\gamma(\underline{t}) = (x_1(\underline{t}), \dots, x_n(\underline{t})) \in \mathbb{C}[\![\underline{t}]\!]^n$, with $\underline{t} = (t_1, \dots, t_m)$, of X at the origin. Here, under a parametrization we understand again a vector of n power series $\gamma(\underline{t})$ for which the corresponding ring map $\gamma^*\colon \mathbb{C}[\![x_1, \dots, x_n]\!]/I \to \mathbb{C}[\![\underline{t}]\!]$ is injective. As $\gamma(\underline{t})$ is a regular parametrization of X at 0, we may w.l.o.g. assume that $\det\big(\partial x(\underline{t})\big)|_{\underline{t}=0} \neq 0$.

We differentiate now the equalities $f_k(\gamma) = 0, k \in [r]$ w.r.t. all variables $t_j, j \in [m]$ and obtain (according to the chain rule) the following system of equations

$$\sum_{i=1}^n \partial_{x_i} f_k(\gamma) \cdot \begin{pmatrix} \partial_{t_1} x_i(\underline{t}) \\ \vdots \\ \partial_{t_m} x_i(\underline{t}) \end{pmatrix} = 0. \tag{1.12}$$

As next, for each $j \in [m]$ we consider the vector

$$\big((-1)^{j+1} \det(\partial x(\underline{t})_{(j,1)}), \dots, (-1)^{j+m} \det(\partial x(\underline{t})_{(j,m)})\big)^T$$

and consider the scalar product of this vector with the left hand side of equality (1.12):

$$0 = \left\langle \sum_{i=1}^n \partial_{x_i} f_k(\gamma) \cdot \begin{pmatrix} \partial_{t_1} x_i(\underline{t}) \\ \vdots \\ \partial_{t_m} x_i(\underline{t}) \end{pmatrix}, \begin{pmatrix} (-1)^{j+1} \det(\partial x(\underline{t})_{(j,1)}) \\ \vdots \\ (-1)^{j+m} \det(\partial x(\underline{t})_{(j,m)}) \end{pmatrix} \right\rangle =$$

$$= \sum_{i=1}^n \partial_{x_i} f_k(\gamma) \left\langle \begin{pmatrix} \partial_{t_1} x_i(\underline{t}) \\ \vdots \\ \partial_{t_m} x_i(\underline{t}) \end{pmatrix}, \begin{pmatrix} (-1)^{j+1} \det(\partial x_{(j,1)}(\underline{t})) \\ \vdots \\ (-1)^{j+m} \det(\partial x_{(j,m)}(\underline{t})) \end{pmatrix} \right\rangle =$$

$$= \det\big(\partial x(\underline{t})\big) \sum_{i=1}^n \partial_{x_i} f_k(\gamma) \big(\Delta_j(x_i^{(0,\dots,0)})\big)(\underline{\gamma}) = \det\big(\partial x(\underline{t})\big) \sum_{i=1}^n \partial_{x_i} f_k(\gamma) \kappa_{i,e_j}(\underline{\gamma}),$$

where e_j denotes the j-standard basis vector and where $k = 1, \dots, r$. Using now the relations $\Delta_j(x_i^{(0,\dots,0)}) = 0$ for all $i \in \{n-m+1, \dots, n\} \backslash \{n-m+j\}$, and the assumption $\det\big(\partial x(\underline{t})\big) \neq 0$, we can for each $j \in [m]$ rewrite the above system of equations as

$$\begin{pmatrix} \partial_{x_1} f_1(\gamma) & \cdots & \partial_{x_{n-m}} f_1(\gamma) \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_r(\gamma) & \cdots & \partial_{x_{n-m}} f_r(\gamma) \end{pmatrix} \cdot \begin{pmatrix} \kappa_{1,e_j}(\underline{\gamma}) \\ \vdots \\ \kappa_{n-m,e_j}(\underline{\gamma}) \end{pmatrix} = - \begin{pmatrix} \partial_{n-m+j} f_1(\gamma) \\ \vdots \\ \partial_{n-m+j} f_r(\gamma) \end{pmatrix}.$$

The goal now is to solve this system of linear equations for $\kappa_{l,e_j}(\underline{\gamma}), l = 1, \ldots, n - m$. To do so, it is enough to consider only $n-m$ equations defined by this system, let us say the equations that are defined by the first $n-m$ rows. Hence, we define the following matrix

$$\mathcal{J} := \begin{pmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_{n-m}} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_{n-m} & \cdots & \partial_{x_{n-m}} f_{n-m} \end{pmatrix}.$$

Let us denote by $\mathcal{J}_{l,e_j}$ the matrix formed from $\mathcal{J}$ by replacing the l-th column by the column vector $(\partial_{n-m+j} f_i(\gamma))_{i=1}^{n-m}$. Now Cramer's rule applies and yields

$$\kappa_{l,e_j}(\underline{t}) = \frac{\det \mathcal{J}_{l,e_j}(\gamma)}{\det \mathcal{J}(\gamma)}$$

for all $j \in [m], l \in [n-m]$. We set

$$\kappa_{l,e_j}(f_1, \ldots, f_r) := \frac{\det \mathcal{J}_{l,e_j}}{\det \mathcal{J}}$$

to be the implicit expressions for the slopes. For the algebraic curvatures $\kappa_{l,(i_1,\ldots,i_m)}$ with $i_1 + \cdots + i_m > 1$, we then inductively obtain

$$\begin{aligned} \kappa_{l,(i_1,\ldots,i_m)}(\underline{\gamma}) &= \frac{1}{\partial x(\underline{t})} \sum_{u=1}^{m} (-1)^{u+l} \partial_{t_u}(\kappa_{l,(i_1,\ldots,i_m)-e_l}(\underline{\gamma})) \det(\partial x(\underline{t})_{(l,u)}) \\ &= \frac{1}{\partial x(\underline{t})} \sum_{u=1}^{m} \sum_{v=1}^{n} \partial_{x_v}(\kappa_{l,(i_1,\ldots,i_m)-e_l}(f_1, \ldots, f_r))(\gamma)(-1)^{l+u} \partial_{t_u} x_v(\underline{t}) \det(\partial x(\underline{t})_{(l,u)}) \\ &= \sum_{v=1}^{n} \partial_{x_v}(\kappa_{l,(i_1,\ldots,i_m)-e_l}(f_1, \ldots, f_r))(\gamma) \cdot \left(\Delta_l(x_v^{(0,\ldots,0)})\right)(\underline{\gamma}) \\ &= \sum_{v=1}^{n} \partial_{x_v}(\kappa_{l,(i_1,\ldots,i_m)-e_l}(f_1, \ldots, f_r))(\gamma) \cdot \kappa_{v,e_l}(f_1, \ldots, f_r)(\gamma). \end{aligned}$$

Hence, for each $l \in [n-m]$ and $i_1, \ldots, i_m \in \mathbb{N}$ with $i_1 + \cdots + i_m > 1$, we define

$$\kappa_{l,(i_1,\ldots,i_m)}(f_1, \ldots, f_r) := \sum_{v=1}^{n} \partial_{x_v}(\kappa_{l,(i_1,\ldots,i_m)-e_l}(f_1, \ldots, f_r)) \cdot \kappa_{v,(e_l)}(f_1, \ldots, f_r)$$

to be the implicit expressions of the algebraic curvature $\kappa_{l,(i_1,\ldots,i_m)}$. Using Theorem 1.4.5, we conclude that each geometric invariant of a given variety can be implicitly described via its defining equations and their partial derivatives:

Theorem 1.4.7. *For each geometric invariant $p = \frac{p_1}{p_2} \in I_F$ there exist polynomials $p(f_1, \ldots, f_r)_1$ and $p(f_1, \ldots, f_r)_2$ in the polynomials $f_1, \ldots, f_r$ defining X and their partial derivatives, i.e.,*

$$p(f_1, \ldots, f_r)_i \in \mathbb{C}[\partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} f_k \,:\, i_j \in \mathbb{N}, 1 \leq k \leq r] \subseteq \mathbb{C}[x_1, \ldots, x_n],$$

for $i = 1, 2$, such that the equality

$$p(\underline{\gamma}) = \frac{p(f_1, \ldots, f_r)_1(\gamma)}{p(f_1, \ldots, f_r)_2(\gamma)}.$$

holds for all parametrizations $\gamma(\underline{t})$ of X for which $p_2(\underline{\gamma}) \neq 0$. In other words, every geometric invariant admits an implicit description in terms of the defining equations.

Recall that the rank of the Jacobian matrix $J_{f_1,\ldots,f_r}$ evaluated at a point a equals $n - m$ if and only if a is a non-singular point of X, as discussed in Remark 1.2.5. In this case, we may w.l.o.g assume that the first minor $\mathcal{J}$ is invertible at a. Hence, each $\kappa_{l,(i_1,\ldots,i_m)}(f), l \in [n], i_j \in \mathbb{N}$ is well defined at any non-singular point of X.

Let me stress, that the notion of geometric invariants of (plane and space) curves, surfaces and also higher dimensional varieties was introduced over the field of complex numbers and that we investigated the properties of them only over $\mathbb{C}$ through the whole chapter. The reason for that is the following: Whereas the definition of a geometric invariant can be blindly done over an arbitrary field in the same way as above, the validity of Theorems 1.1.5, 1.2.2, 1.3.4 and 1.4.5, and hence also the basic properties of geometric invariants, is strongly dependent on the existence of D-transcendental power series and D-algebraically independent families of power series. Since we don't have a proof for the existence of D-transcendental power series and D-algebraically independent families of power series over fields different from $\mathbb{C}$ yet (see the list with unsolved problems in Section 5.1), we also don't know whether a generalization of the theory of geometric invariants to other fields is possible. Hence we put this problem on a list with open questions.

Chapter 2

Resolution of Singular Curves via Geometric Invariants

This chapter of my thesis is dedicated to the problem of resolution of singular curves in $\mathbb{A}^n_{\mathbb{C}}$ for $n \geq 2$. Our ultimate goal is to use their geometric properties described by the (higher) algebraic curvatures in order to prove that each singularity of an analytically irreducible singular curve can be resolved just by one blowing up in a suitable center defined in terms of algebraic curvatures.

The problem of resolution of singularities is the following: Given a singular algebraic variety X one tries to find a non-singular variety $\widetilde{X}$ together with a proper birational morphism $\pi : \widetilde{X} \to X$. Such a variety $\widetilde{X}$ is called a *resolution of singularities of* X.

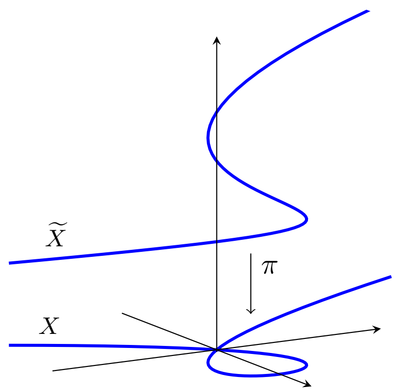

Figure 2.1: Resolution of singularities of the node given by the equation $y^2 - y^3 = x^3$.

Even in simple examples, the construction of a suitable $\widetilde{X}$ from X is highly non-trivial and has aroused interest of mathematicians since the middle of the 19th century. The big step forward happened in 1964 when H. Hironaka proved the existence of resolution of singularities for all dimensions, supposing that the ground field is of characteristic 0. This spectacular work — the proof is 200 pages long and was considered of the time as nearly inaccessible — has been used since then in numerous settings (for instance for the computation of the ζ-function [Fk07a], or the minimal model program [KM98]). For more applications see [Fk07b, Fk12, FP05]. Hironaka was awarded the Fields Medal for this theorem [Hi64a, Hi64b].

Since then, many contributions towards strengthening Hironaka's result and simplifications of his original proof have been done, among them by A. Benito and O. Villamayor [BV12], by E. Bierstone and P. Milman [BM91], by A. Bravo and O. Villamayor [BV01], or by H. Hauser together with S. Encinas [EH02], by S. Encinas and O. Villamayor [EV03], H. Hauser [Ha03], O. Villamayor [Vi89], or by J. Włodarczyk [Wl05].

Nevertheless, the arguments are still very complicated, involve several mutually interwoven inductions, and do not allow a direct formula for the construction of $\widetilde{X}$ and π. And, above all, they are of purely algebraic nature not allowing any geometric interpretation or insight. In most algorithms, the construction of the morphism π is done purely algebraically by means of some numerical invariants attached to the variety and its defining equations. As such they conceal the geometric content and flavour of the resolution process.

Let us mention here that Hironaka's proof is strictly restricted to fields of characteristic 0, and that in positive characteristic the existence of resolution was so far proven only for curves (see J. Kollár's book [Ko07]), surfaces (e.g. by S. S. Abhyankar [Ab1], S. D. Cutkosky [Cu04], or by S. Perlega [Pe17]) and recently also threefolds (by V. Cossart and O. Piltant in [CP08], [CP09] or by H. Kawanoue and K. Matsuki in [KM16]), but is widely open for higher dimensional varieteis.

Starting from this unsatisfactory situation, there have been several attempts to define a geometrically motivated resolution process. The most prominent among these approaches is the Nash modification (see the works by G. González-Sprinberg [Gs87], H. Hironaka [Hi3], V. Rebassoo [Re77] and M. Spivakovsky [Sp90]). Whereas a classical blowup is defined algebraically, Nash modification has a strong differential geometric feature: Divide X into two parts $X^0 = \mathrm{Reg}(X)$ and $\mathrm{Sing}(X)$ where the first collects all smooth points of X, i.e., the points where X is locally a manifold. At each smooth point x of X, there is a well defined tangent space T_xX of X and the tangent bundle TX^0 over X^0 forms a quasi-affine variety inside the tangent bundle TX. Define now X' as the Zariski-closure of TX^0 in TX. This X' maps naturally to X and the map $\nu : X' \to X$ is an isomorphism over $\mathrm{Reg}(X)$. Over a singular point $x \in \mathrm{Sing}(X)$, the fiber of ν consists of all limiting positions of tangent spaces T_yX at smooth points y as these tend to x. This can already been illustrated in the simple example of the node:

Example 2.0.1. Let X be defined by the equation $x^2 + x^3 = y^2$, where $(0, 0)$ is the only singular point (the intersection point of the two analytic branches) and where $x = \pm y$ are the two limiting positions of the tangent lines. The curve X' now lies in $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and, by construction, separates the two branches. It is thus already a resolution of X. The main issue here is the observation that sending a smooth point $z \in X$ to the point $(z, [z, T_zX]) \in \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ tears apart the curve X nearby $(0, 0)$. In fact, points on the lower branch will have a tangent with slope close to $+1$ (the blue dashed line), whereas on the upper branch the slope is close to -1 (the red dashed line).

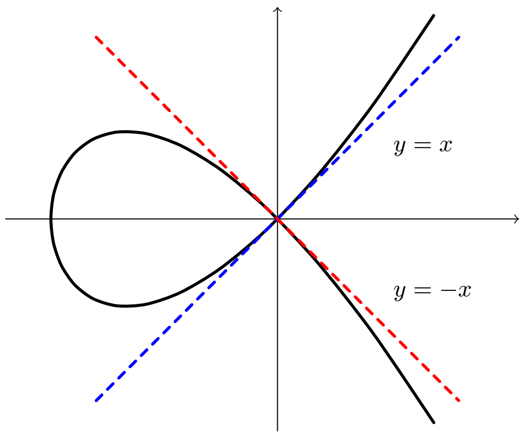

Figure 2.2: The tangent space of the node at the origin (singular point) — defined as the union of the tangent spaces of its smooth branches.

It can be shown that the Nash modification of a hypersurface X defined by one polynomial equation $f(x_1, \ldots, x_n) = 0$ is a blowup with center defined by the Jacobian ideal $J(f) = (\partial_{x_1} f, \ldots, \partial_{x_n} f)$. As such, it is a more geometric object. However, its disadvantage lies in the algebraic complexity of computations. It is not too hard to show that an iteration of Nash modifications eventually resolves any singular curve, but again, in general many repetitions are necessary. For surfaces, the situation is already much more difficult, and it is not known whether a resolution is guaranteed by Nash modifications. A famous result of M. Spivakovsky asserts that an iteration of Nash modifications followed by normalization always suffices to resolve. The proof is very involved and requires the classification of so-called sandwiched singularities [Sp90]. So, again the actual knowledge is rather limited. It is, however, known by results of A. Nobile that the Nash modification is an isomorphism (on whole X) if and only if X is smooth [No75].

A more refined method was presented by T. Yasuda in [Ya07, Ya09] where he introduced the concept of higher Nash blowups and proved a stronger statement, namely that each curve can be resolved by its N-th higher Nash blowup, for N large enough. One decade later, D. Duarte translated the Yasuda's abstract definition of higher Nash blowups into a more computational and algorithmic language which allows to interpret each higher Nash blowup as a blowup in the center given by a suitable higher Jacobian matrix, see [Du17].

In this context we propose a new and alternative approach based on more refined methods and the use of “higher order tangent spaces" defined by means of geometric invariants.

2.1 Analytically Irreducible Plane Curves with one Singularity

Let us fix a plane algebraic curve $X \subseteq \mathbb{A}^2_{\mathbb{C}}$ with only one singular point $0 \in X$. Let $f \in \mathbb{C}[x,y]$ be its defining polynomial. Assume f to be irreducible. Let us further assume that X is unibranched at the origin. In this section we prove the existence of a resolution of singularities of X by establishing an algorithm which constructs a geometric invariant

$$\widetilde{\kappa} = \frac{\widetilde{\kappa}_1}{\widetilde{\kappa}_2}$$

with the property that the blowup of X in the ideal $(\widetilde{\kappa}(f)_1, \widetilde{\kappa}(f)_2)$ gives a smooth curve $\widetilde{X}_{\widetilde{\kappa}}$. Or equivalently (see [Ha14, §4]), with the property that the Zariski closure of the graph of the height function induced by $\widetilde{\kappa}$:

$$\begin{aligned} \phi_{\widetilde{\kappa}} \colon X \backslash Z &\to \mathbb{P}^1_{\mathbb{C}} \\ x &\mapsto \big(\widetilde{\kappa}(f)_1(x) : \widetilde{\kappa}(f)_2(x)\big) \end{aligned} \tag{2.1}$$

is smooth. Here $\widetilde{\kappa}(f)_1$ and $\widetilde{\kappa}(f)_2$ denote the minimal numerator and denominator (by minimality we mean with no common divisor) of an implicit expression of $\widetilde{\kappa}$ in terms of f, respectively, and $Z = V(f, \widetilde{\kappa}(f)_1, \widetilde{\kappa}(f)_2)$ the vanishing set of the ideal $(f, \widetilde{\kappa}(f)_1, \widetilde{\kappa}(f)_2)$. From now on, when talking about the numerator and denominator of an implicit expression of a geometric invariant, we will always mean two polynomials with no common divisor.

Definition 2.1.1. We call a geometric invariant $\widetilde{\kappa}$ a *crucial curvature* of X (at $(0,0)$) if the blowup of X in the ideal $(\widetilde{\kappa}(f)_1, \widetilde{\kappa}(f)_2)$ defined by the numerator and denominator of the implicit expression of $\widetilde{\kappa}$ yields already resolution of X.

Remark that a crucial curvature is not unique as we can see already on the simplest examples:

Example 2.1.2. It is not hard to show that the singularity of the cusp defined by the polynomial $f = x^2 - y^3$ can be resolved by both, the standard blowup, i.e., the monomial blowup in the ideal (x,y), and by the Nash modification which is defined by the blowup in the ideal $(f_x, f_y) = (x, y^2)$. These two correspond to the crucial curvatures

$$\widetilde{\kappa} = \frac{x^{(0)}}{y^{(0)}} \quad \text{and} \quad \hat{\kappa} = \frac{x^{(1)}}{y^{(1)}},$$

respectively. However, they are not equal.

At this point, it is very instructive to look at X from the perspective of parametrization. So let us consider a parametrization $\gamma(t) = (x(t), y(t)) \in \mathbb{C}\{t\}^2$ of X at 0 (one can always construct a convergent parametrization according to the Newton-Puiseux algorithm, see Section 4.2). The evaluation $\widetilde{\kappa}(\underline{\gamma(t)})$ of a crucial curvature at $\underline{\gamma(t)}$ gives us the pair $\big(\widetilde{\kappa}_1(\underline{\gamma(t)}), \widetilde{\kappa}_2(\underline{\gamma(t)})\big) \in \mathbb{C}\{t\}^2$ of power series in t. Intuitively, the vector of power series

$$\gamma_{\widetilde{\kappa}}(t) = \gamma(t) \times \big(\widetilde{\kappa}_1(\underline{\gamma(t)}) : \widetilde{\kappa}_2(\underline{\gamma(t)})\big)$$

defines a parametrization of $\widetilde{X}_{\widetilde{\kappa}}$, and we will prove this also rigorously with Lemma 2.1.3. Moreover, with the same lemma we will discuss basic properties of $\gamma_{\widetilde{\kappa}}(t)$ and their relation to the properties of $\widetilde{\kappa}$ which will serve as the most important indicator for recognizing crucial curvatures.

Throughout this section, we use for geometric invariants the same notation as in Section 1.1. Let I_F again denote the field of all geometric invariants of plane curves. Further, for an arbitrary geometric invariant $p \in I_F$, by $\underline{p}$ we denote the vector $(\partial^i p)_{i\geq 0}$.

Let us prove now a more general statement about maps of type (2.1) induced by a geometric invariant $p \in I_F$.

Lemma 2.1.3. *Let $Y \subseteq \mathbb{A}^2_{\mathbb{C}}$ be a plane algebraic curve and defined by a polynomial $f \in \mathbb{C}[x,y]$. Assume that Y is analytically irreducible at each point. Further, let $p = \frac{p_1}{p_2} \in I_F$ be a geometric invariant satisfying $p_i(\underline{\gamma(t)}) \neq 0$, for $i = 1, 2$, and for any parametrization $\gamma(t)$ of Y and let $p(f) = \frac{p(f)_1}{p(f)_2}$ be its implicit expression in terms of f and its partial derivatives. Consider the by p induced map*

$$\begin{aligned} \phi_p \colon Y\backslash Z &\to \mathbb{P}^1_{\mathbb{C}} \\ y &\mapsto (p(f)_1(y) : p(f)_2(y)), \end{aligned}$$

where $Z = V(f, p(f)_1, p(f)_2)$ is the vanishing set of the ideal generated by f and the numerator and denominator of $p(f)$. The Zariski closure $\widetilde{Y}_p \subseteq \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ of the graph of ϕ_p then satisfies:

(i) The projection map $\pi : \widetilde{Y}_p \to Y$ induced by the first projection $\pi : \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ is a proper birational morphism which is an isomorphism $\pi : \widetilde{Y}_p\backslash E \to Y\backslash Z$ outside $E = \pi^{-1}(Z)$.

(ii) The projection map $\pi : \widetilde{Y}_p \to Y$ is injective.

(iii) $\widetilde{Y}_p$ is analytically irreducible at each point.

(iv) Let $\gamma(t)$ be a parametrization of Y at $y \in Y$. The vector

$$\gamma(t) \times \big(p_1(\underline{\gamma(t)}) : p_2(\underline{\gamma(t)})\big)$$

parametrizes $\widetilde{Y}_p$ at $\tilde{y} = \pi^{-1}(y)$.

(v) We have the inclusion $\operatorname{Sing}(\widetilde{Y}_p) \subseteq \pi^{-1}(\operatorname{Sing}(Y))$.

Proof. (ii): Let us consider an arbitrary point $y \in Y$. After a coordinate change we may assume that $y = 0$. Let $\gamma(t) \in \mathbb{C}\{t\}^2$ be a parametrization of Y at 0 (notice that according to the Newton-Puiseux algorithm we can always suppose a parametrization to be convergent, see Section 4.2). Notice that there exists a small neighbourhood $U \subseteq \mathbb{C}^2$ of 0 such that each point of

$Y \cap U$ is uniquely determined by $\gamma(\underline{s})$ for some $s \in \mathbb{C}$ as Y is analytically irreducible. Hence to show that there is only one point on $\widetilde{Y_p}$ lying over 0 it is enough to show that the preimage under π of the point $0 = \gamma(0)$ lying on the arc given by $\gamma(t)$ is uniquely determined by the height function ϕ_p. Let us consider the two rational mappings into the affine charts of $\mathbb{P}^1_{\mathbb{C}}$ induced by ϕ_p:

$$\begin{aligned}\phi_{p,1}\colon Y\backslash Z &\to \mathbb{A}^1_{\mathbb{C}}\\ y &\mapsto \frac{p(f)_1(y)}{p(f)_2(y)}\end{aligned}$$

and

$$\begin{aligned}\phi_{p,2}\colon Y\backslash Z &\to \mathbb{A}^3_{\mathbb{C}},\\ y &\mapsto \frac{p(f)_2(y)}{p(f)_1(y)}\end{aligned}$$

and the Zariski closures $\widetilde{Y}_{p,1}$ and $\widetilde{Y}_{p,2}$ in $\mathbb{A}^3_{\mathbb{C}}$ of their respective graphs. A parametric point $\gamma(t)$ on Y maps under $\phi_{p,1}$ and $\phi_{p,2}$ to

$$\frac{p_1(\underline{\gamma(t)})}{p_2(\underline{\gamma(t)})} \quad \text{and} \quad \frac{p_2(\underline{\gamma(t)})}{p_1(\underline{\gamma(t)})},$$

respectively. Consider now the vectors of Laurent series

$$\gamma_1(t) = \left(\gamma(t), \frac{p_1(\underline{\gamma(t)})}{p_2(\underline{\gamma(t)})}\right) \text{ and } \gamma_2(t) = \left(\gamma(t), \frac{p_2(\underline{\gamma(t)})}{p_1(\underline{\gamma(t)})}\right),$$

respectively. If it happens that the power series $p_1(\underline{\gamma(t)}), p_2(\underline{\gamma(t)})$ have the same order, then both their quotients define a power series with a non-vanishing constant term and the evaluation at 0 of both of them is well defined. In this case $\gamma_1(0)$ and $\gamma_2(0)$ define both the same point $\tilde{y}$ on $\widetilde{Y}$, the only point lying over $(0,0)$. In the case that they have distinct orders, one of their quotients is a power series, let us say $p(\underline{\gamma(t)}) = \frac{p_1(\underline{\gamma(t)})}{p_2(\underline{\gamma(t)})} \in \mathbb{C}\{t\}$, and so its evaluation at 0 is well defined, and the other quotient is a Laurent series. There is again only one point $\tilde{y}$ on $\widetilde{Y_p}$ lying over $(0,0)$, namely the one which has affine coordinates $\big(\gamma(0), p(\underline{\gamma}(t))|_{t=0}\big) = (0,0,0)$ in the first chart and which is not visible in the second chart.
(i): Let r be the minimum of the radii of convergence of the power series $p_i(\underline{\gamma(t)})$, for $i = 1,2$, and let us by B_r denote the open ball of radius r centred at 0. Notice that as $p_i(\underline{\gamma(t)}), i = 1,2$ are both convergent power series and different from zero, hence holomorphic functions, they can vanish simultaneously only at finitely many point of B_r. Set

$$B_r^\circ \coloneqq \{s \in B_r \;:\; \big(p_1(\underline{\gamma(s)}), p_2(\underline{\gamma(s)})\big) \neq (0,0)\}.$$

Then, B_r° is an open and dense subset of B_r and hence $\gamma(B_r^\circ)$ is an open and dense subset of Y, since Y is analytically irreducible at each point and hence irreducible as an algebraic curve. As the map ϕ_p is defined on $\gamma(B_r^\circ)$, it follows that $\pi : \widetilde{Y_p} \to Y$ is a birational morphism with its inverse map given by

$$Y\backslash Z \to \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}},\ y \mapsto y \times \phi_p(y)$$

on $\gamma(B_r^\circ)$. Hence, Z is a finite subset of Y. So, $\widetilde{Y_p}$ is together with the projection morphism $\pi : \widetilde{Y_p} \to Y$ the blowup of Y in the ideal $(f, p(f)_1, p(f)_2)$. It follows now from [Ha77, Chapter II, Proposition 7.16] that π is a proper birational morphism. That its restriction $\pi\colon \widetilde{Y_p}\backslash E \to Y\backslash Z$ is an isomorphism outside $E = \pi^{-1}(Z)$, follows now immediately.
(iii): It is clear, that $\widetilde{Y_p}$ is analytically irreducible outside $E = \pi^{-1}(Z)$ as $\pi\colon \widetilde{Y_p}\backslash E \to Y\backslash Z$ is an isomorphism. As π is a birational morphism, and even an isomorphism outside the finite set E, the analytic branches of $\widetilde{Y_p}$ at the point $\pi^{-1}(0) = \tilde{y} \in \widetilde{Y_p}$ are uniquely determined by the images of analytic branches of Y at 0 under the map

$$Y\backslash Z \to \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}},\ y \mapsto y \times \phi_p(y).$$

But as Y has only one branch at 0, so has $\widetilde{Y_p}$ at $\tilde{y}$ which proves the analytical irreducibility of $\widetilde{Y_p}$.
(iv): As the set $\gamma_1(B_r)$ defines an open and dense subset of $\widetilde{Y}_{p,1}$, the triple $(\gamma(t), p(t))$ parametrizes the affine chart expression $\widetilde{Y}_{p,1}$ of $\widetilde{Y_p}$ and the claim follows.
(v): Notice first that as the restriction

$$\pi : \widetilde{Y}\backslash E \to Y\backslash Z$$

is an isomorphism outside $E = \pi^{-1}(Z)$, then in the case $0 \notin Z$ we have:

$$\pi^{-1}(0) = \tilde{y} \in \mathrm{Sing}(\widetilde{Y}) \text{ if and only if } 0 \in \mathrm{Sing}(Y).$$

Hence, it remains to discuss whether in the case that $0 \in Z$, the point $\tilde{y}$ can be singular although $0 \in Y$ is not. Let us assume that 0 is a smooth point of Y. As $\widetilde{Y_p}$ is analytically irreducible at $\tilde{y}$, in order to prove that $\tilde{y}$ is a smooth point of $\widetilde{Y_p}$, it is enough to show that $\widetilde{Y_p}$ admits a regular parametrization at $\widetilde{y}$. Then, $\widetilde{Y_p}$ is locally at $\widetilde{y}$ a manifold and the claim follows. From the smoothness of Y at 0 it follows that its branch at 0 is biholomorphic to an open subset $V \subseteq \mathbb{C}$ containing 0 and so Y can be parametrized at 0 by a pair of power series $\gamma_0(t) = (z_1(t), z_2(t))$ with one component of order one, let us say $z_1(t)$. According to (iii), the vector $\gamma_0(t) \times \big(p_1(\underline{\gamma_0(t)}), p_1(\underline{\gamma_0(t)})\big)$ parametrizes $\widetilde{Y_p}$ at $\tilde{y}$, and moreover, it has one component of order one which yields a regular parametrization of $\widetilde{Y_p}$ at $\widetilde{y}$ and thus finishes the proof. □

Remark 2.1.4. Let us mention that in the proof of item (ii) of Lemma 2.1.3, the fact that Z is a finite subset of $\mathbb{C}$ could be concluded also directly from Bezout's theorem as

$$\gcd(p(f)_1, p(f)_2) = 1.$$

This argument, in contrary to the argument we use in the proof of Lemma 2.1.3, doesn't work for polynomials in more than two variables. For instance, polynomials $f = x$ and $g = y$ have no common divisor in the power series ring $\mathbb{C}[x, y, z]$, their vanishing set $V(f, g)$, however, is an infinite set as it contains the whole z-axis.

With Lemma 2.1.3, we obtain the following sufficient and necessary condition on a crucial curvature $\widetilde{\kappa}$:

Corollary 2.1.5. *A geometric invariant $p \in I_F$ is a crucial curvature of X at 0 if and only if there exists a parametrization $\gamma(t)$ of X at 0 with*

$$\operatorname{ord}\big(p(\underline{\gamma(t)})\big) = 1.$$

Proof. It follows from (i) of Lemma 2.1.3 that the projection $\pi : \widetilde{X}_p \to X$ is a proper birational morpism. Further, as X is analytically irreducible at the origin, we conclude with (ii) of the same lemma that there is only one point $\widetilde{x} \in \widetilde{X}_p$ lying over 0 and according to the item (v), $\widetilde{X}_p$ is smooth outside this point. Now, since $\operatorname{ord}\big(p(\underline{\gamma(t)})\big) = 1$, so is $\tilde{x}$ visible only in one affine chart and the affine expression of $\widetilde{X}_p$ in this chart is parametrized by the triple $\big(\gamma(t), p(\underline{\gamma(t)})\big) \in \mathbb{C}\{t\}^3$, which is a regular parametrization and thus the smoothness follows.
For the other implication let us assume that $\widetilde{X}_p$ is smooth. Let $\tilde{x}$ be the point on $\widetilde{X}_p$ lying over 0. Investigate now one affine chart expression $\widetilde{X}_{\text{aff}}$ of $\widetilde{X}_p$ in which $\tilde{x}$ is visible. Let us w.l.o.g. assume that $\tilde{x} = 0$ there. Then, as $\widetilde{X}_{\text{aff}}$ is smooth at 0, it is locally at 0 biholomorphic to an open subset $U \subseteq \mathbb{C}$ containing 0 and as such it can be parametrized at 0 by a triple of convergent power series $(x(t), y(t), z(t)) \in \mathbb{C}\{t\}^3$ with one component of order one. However, as the pair $\gamma(t) = (x(t), y(t))$ defines a parametrization of X at the singular point 0, we have $\min\{\operatorname{ord}(x(t)), \operatorname{ord}(y(t))\} > 1$. Thus, $\operatorname{ord}(z(t)) = 1$ follows. Now the equality $z(t) = p(\underline{\gamma(t)})$ concludes the proof. ☐

Thus, to resolve X is equivalent to construct a geometric invariant $\widetilde{\kappa} \in I_F$ which satisfies

$$\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = 1$$

for a suitably chosen parametrization $\gamma(t)$ of X. The construction of a crucial curvature $\widetilde{\kappa}$ is the objective of the remaining part of this section.

Let $\gamma(t)$ be a Puiseux parametrization of X at 0. Notice, that since f is analytically irreducible and defines a singular curve, we have $f \neq x$ as well as $f \neq y$ and so according to Theorem 4.2.1 and Corollary 4.2.5, the Puiseux parametrization $\gamma(t)$ is of the form

$$\gamma(t) = (x(t), y(t)) = (t^n, y(t)) \in \mathbb{C}\{t\}^2,$$

with $m = \operatorname{ord}(y(t)) \geq 2$.
Let us first discuss the behavior of the orders of algebraic curvatures evaluated at $\underline{\gamma(t)}$. This will have significant impact on our algorithm as the algebraic curvatures represent a generating system of geometric invariants:

Remark 2.1.6. By induction, one can show the following equalities:

$$\operatorname{ord}\big(\kappa_0(\underline{\gamma(t)})\big) = \operatorname{ord}\left(\frac{x'(t)}{y'(t)}\right) = n - m,$$

$$\operatorname{ord}\big(\kappa_i(\underline{\gamma(t)})\big) = \operatorname{ord}\left(\frac{\partial_t \kappa_{i-1}(\underline{\gamma(t)})}{y'(t)}\right) = n - (i+1)m, \text{ for all } i \geq 1.$$

So for each higher algebraic curvature, the order of its evaluation at $\underline{\gamma(t)}$ drops by m compared to the order of the previous one. We can iteratively even construct a geometric invariant $\hat{\kappa}$ with

$$\operatorname{ord}\big(\hat{\kappa}(\underline{\gamma(t)})\big) = \gcd(n, m).$$

Euclidean Algorithm for Geometric Invariants Construction of a geometric invariant of order at $\underline{\gamma(t)}$ equal to the greatest common divisor of orders at $\underline{\gamma(t)}$ of two other geometric invariants

---

Input two geometric invariants p and q & pair of power series $\gamma(t) \in \mathbb{C}[\![t]\!]^2$
Output geometric invariant $\hat{\kappa}$ satisfying $\operatorname{ord}\big(\hat{\kappa}(\underline{\gamma(t)})\big) = \gcd(n, m)$, where $n = \operatorname{ord}\big(p(\underline{\gamma(t)})\big)$ and $m = \operatorname{ord}\big(q(\underline{\gamma(t)})\big)$, or FAIL if either n and m both equal zero or $p(\underline{\gamma(t)}) \cdot q(\underline{\gamma(t)}) =$ 0 or if at least one of the integers n and m is negative
procedure $\textsc{gcd}(p, q; \gamma(t))$

1st step:
Set

$$\mathsf{n} := \operatorname{ord}\big(p(\underline{\gamma(t)})\big) \text{ and } \mathsf{m} := \operatorname{ord}\big(q(\underline{\gamma(t)})\big).$$

If $0 \leq \mathsf{n}, \mathsf{m} < \infty$ not both equal to 0, then $p(\underline{\gamma(t)}), q(\underline{\gamma(t)}) \in \mathbb{C}\{t\}$ are both power series different from zero and at least one of them has no constant coefficient. In this case we continue with the 2nd step of the algorithm.
Otherwise, **return** FAIL.

2nd step:
We proceed according to the Euclidean algorithm. Let us write the Euclidean algorithm for n and m as the sequence of following equations:

$$\begin{aligned}
\mathsf{m} &= q_0\mathsf{n} + r_0 \\
\mathsf{n} &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
r_1 &= q_3 r_2 + r_3 \\
&\vdots \\
r_{N-2} &= q_N r_{N-1} + r_N \\
r_{N-1} &= q_{N+1} r_N + r_{N+1},
\end{aligned}$$

where $0 = r_{N+1} < r_N = \gcd(\mathsf{n}, \mathsf{m}) < r_{N-1} < \cdots < r_0 < \mathsf{n}$.

Define now geometric invariants $z_i \in I_F$ for $1 \leq i \leq N$ according to the Euclidean algorithm and investigate with help of Remark 2.1.6 the order of their evaluation at $\underline{\gamma(t)}$:

$$\begin{aligned}
z_0 &:= \kappa_{q_o-1}(\underline{q}, \underline{p}) & \operatorname{ord}\big(z_0(\underline{\gamma(t)})\big) &= r_0, \\
z_1 &:= \kappa_{q_1-1}(\underline{p}, \underline{z_0}) & \operatorname{ord}\big(z_1(\underline{\gamma(t)})\big) &= r_1, \\
z_2 &:= \kappa_{q_2-1}(\underline{z_0}, \underline{z_1}) & \operatorname{ord}\big(z_2(\underline{\gamma(t)})\big) &= r_2, \\
&\vdots & &\vdots \\
z_N &:= \kappa_{q_N-1}(\underline{z_{N-2}}, \underline{z_{N-1}}) & \operatorname{ord}\big(z_N(\underline{\gamma(t)})\big) &= r_N,
\end{aligned}$$

where in the case $q_0 = 0$ we set

$$\kappa_{-1}(\underline{q}, \underline{p}) := q.$$

We **return**

$$\hat{\kappa} := z_N.$$

Example 2.1.7. Let us consider the geometric invariants $p = y^{(0)}$ and $q = \kappa_0$ and a Puiseux parametrization $\gamma(t) = (t^8, t^3)$. Let us for the orders

$$\begin{aligned}
5 &= \operatorname{ord}\big(q(\underline{\gamma(t)})\big) = \operatorname{ord}\left(\frac{x'(t)}{y'(t)}\right) = \operatorname{ord}\left(\frac{8t^7}{3t^2}\right) = \operatorname{ord}\left(\frac{8}{3}t^5\right), \\
3 &= \operatorname{ord}\big(p(\underline{\gamma(t)})\big) = \operatorname{ord}(y(t)) = \operatorname{ord}(t^3),
\end{aligned}$$

write the Euclidean algorithm and construct:

$$\begin{aligned}
5 &= 1 \cdot 3 + 2 & z_0 &= \kappa_0(\underline{\kappa_0}, \underline{y^{(0)}}) \\
3 &= 1 \cdot 2 + 1 & z_1 &= \kappa_0(\underline{y^{(0)}}, \underline{z_0}) \\
2 &= 2 \cdot 1 + 0
\end{aligned}$$

Then

$$z_0 = \frac{\partial \kappa_0}{y^{(1)}} = \kappa_1,$$

and

$$\hat{\kappa} = z_1 = \kappa_0(\underline{y^{(0)}}, \underline{\kappa_1}) = \frac{y^{(1)}}{\partial \kappa_1} = \kappa_2^{-1}.$$

With Remark 2.1.6 we see immediately that $\operatorname{ord}\big((\hat{\kappa})(\underline{\gamma(t)})\big) = 1$.

Notice, that Remark 2.1.6 guarantees an improvement of the singularity after each blowup in the ideal defined by a (higher) algebraic curvature (as the order of the power series parametrization drops). Our wish is to see these improvements also implicitly just by means of transformations of the defining equation of X, without the need of a parametrization. This we do not know how to do at the moment. Hence, we refer to this problem also on the list with open questions in Section 5.2.

Next, we discuss about how the polydromy order of Puiseux series behaves under triangular coordinate changes:

Proposition 2.1.8. *Let $y(t^{\frac{1}{n}}) \in \mathbb{C}[\![t^{\frac{1}{n}}]\!]$ be a Puiseux series of polydromy order n.*

(i) The polydromy order of $(y \circ \varphi)(t^{\frac{1}{n}})$ is equal to n for any reparametrization $\varphi \in \mathrm{Aut}(\mathbb{C}[\![t]\!])$. Here the action of $\mathrm{Aut}(\mathbb{C}[\![t]\!])$ on $\mathbb{C}[\![t^{\frac{1}{n}}]\!]$ is defined by $\varphi(t^{\frac{1}{n}}) := \varphi(t)|_{t=t^{\frac{1}{n}}}$.

(ii) Let $x(t) \in \mathbb{C}[\![t]\!]$ be a power series of order n. Let $\varphi \in \mathrm{Aut}(\mathbb{C}[\![t]\!])$ be the unique reparametrization with $(x \circ \varphi)(t) = t^n$. Let us write

$$(y \circ \varphi)(t) = a_1 t^n + \cdots + a_k t^{kn} + \hat{y}(t),$$

for some $k \in \mathbb{N}$ and $\hat{y} \in \mathbb{C}[\![x]\!]$ a power series with $\mathrm{ord}(\hat{y}(t)) > kn$ and $n \nmid \mathrm{ord}(\hat{y}(t))$, i.e., $\mathrm{ord}(\hat{y}(t)) = m_1$ is the first characteristic exponent of $(y \circ \varphi)(t^{\frac{1}{n}})$. Let us set $n_1 := \gcd(n, m_1)$. Then after the triangular coordinate change

$$\mathsf{y}(t) := y(t) - \sum_{i=1}^{k} a_i x(t)^i, \tag{2.2}$$

the Puiseux series $\mathsf{y}(t^{\frac{1}{n_1}})$ has polydromy order n_1.

Proof. (i): Let n be the polydromy order of $y(t^{\frac{1}{n}})$. Let $\varphi \in \mathrm{Aut}(\mathbb{C}[\![t]\!])$ be an arbitrary reparametrization. Let us assume by contradiction that the polydromy order of $(y \circ \varphi)(t^{\frac{1}{n}})$ equals $m < n$. Then m is a divisor of n and we can write $n = \alpha \cdot m$ for some $\alpha \in \mathbb{N}$. Thus, the degree of each term of $(y \circ \varphi)(t^{\frac{1}{n}})$ is a multiple of α and so

$$(y \circ \varphi)(t^{\frac{1}{n}}) = \tilde{y}(t^{\frac{\alpha}{n}}) = \tilde{y}(t^{\frac{1}{m}})$$

for some power series $\tilde{y} \in \mathbb{C}[\![x]\!]$. But, applying the inverse reparametrization φ^{-1} to $\tilde{y}$ would give us

$$y(t^{\frac{1}{n}}) = (\tilde{y} \circ \varphi^{-1})(t^{\frac{1}{m}}) \in \mathbb{C}[\![t^{\frac{1}{m}}]\!] \text{ with } m < n,$$

which is a contradiction to the minimality of n.
(ii): Let $x(t)$ be a power series with $\mathrm{ord}(x(t)) = n$ and let φ be the unique reparametrization satisfying $(x \circ \varphi)(t) = t^n$. It follows from (i) of the proposition that $(y \circ \varphi)(t^{\frac{1}{n}})$ has again polydromy order n. Let $m_1, \ldots, m_l$ be its characteristic exponents. Recall that they satisfy by definition $\gcd(n, m_1, \ldots, m_l) = 1$. It is obvious, that after the triangular coordinate change (2.2), $\{m_1, \ldots, m_l\}$ is still a subset of the support of $(\mathsf{y} \circ \varphi)(t)$. Further, we have $\gcd(n_1, m_1, m_2, \ldots, m_l) = \gcd(n, m_1, \ldots, m_l) = 1$, where $n_1 = \gcd(n, m_1)$. Therefore, n_1 is the polydromy order of $(\mathsf{y} \circ \varphi)(t^{\frac{1}{n_1}})$. Now, we use again (i) of this proposition to conclude that n_1 is also the polydromy order of $\mathsf{y}(t^{\frac{1}{n_1}})$. $\square$

Remark, that whereas the polydromy order is stable under reparametrizations, this is in general no longer true for the characteristic exponents. We demonstrate this phenomenon on the following example:

Example 2.1.9. Let us consider the Puiseux series

$$s(x^{\frac{1}{6}}) = x^{\frac{2}{6}} - 2x^{\frac{3}{6}}.$$

Its polydromy order equals 6 and the characteristic exponents are 2 and 3. Consider the reparametrization $\varphi(x) = x - x^2$. The composition $(s \circ \varphi)(x)$ defines the Puiseux series

$$(s \circ \varphi)(x^{\frac{1}{6}}) = x^{\frac{2}{6}} - 5x^{\frac{4}{6}} + 6x^{\frac{5}{6}} - 2x^{\frac{6}{6}},$$

whose polydromy order is again 6. However, the characteristic exponents of $(s \circ \varphi)(x^{1/6})$ are now 2 and 5.

We present now an algorithm for a construction of a crucial curvature. Our algorithm extracts step by step from the support of $y(t)$ a subset which has the same property as the characteristic exponents of $y(t^{\frac{1}{n}})$, namely that the greatest common divisor of n and this subset equals one. We proceed exactly according to the definition of characteristic exponents (4.6).

Resolution algorithm for plane curves with one singularity

Resolution Algorithm for Plane Curves - One Singularity Construction of a Crucial Curvature

Input parametrization $\gamma(t) = (t^n, y(t)) \in \mathbb{C}[\![t]\!]^2$ of a plane algebraic curve at 0 of polydromy order equal to $\frac{n}{\alpha}$ for some $\alpha \in \mathbb{N}$
Output geometric invariant $\widetilde{\kappa}$ satisfying $\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = \alpha$
procedure $\textsc{PlaneCurvature}(\gamma(t))$

1st step: Variable declaration:

Set

$$\mathsf{x}_0 := x^{(0)},\ \mathsf{y}_0 := y^{(0)} \ \text{ and } \ \mathsf{m}_0 := \operatorname{ord}(y(t)),\ \mathsf{n}_0 := n.$$

Further, check whether $\mathsf{y}_0(\underline{\gamma(t)}) \in \mathbb{C}[\![t^{\mathsf{n}_0}]\!]$:

If $\mathsf{y}_0(\underline{\gamma(t)}) \in \mathbb{C}[\![t^{\mathsf{n}_0}]\!]$, then **return** $\widetilde{\kappa} := \mathsf{x}_0$.

Otherwise continue with the next step of the algorithm.

2nd step: Construction of a geometric invariant y_i of order equal to the first exponent m_i of $\mathsf{y}_{i-1}(\underline{\gamma(t)})$ that is not divisible by n_{i-1}:

Consider the pair of power series

$$\big(\mathsf{x}_{i-1}(\underline{\gamma(t)}), \mathsf{y}_{i-1}(\underline{\gamma(t)})\big) \in \mathbb{C}[\![t]\!]^2$$

with

$$\operatorname{ord}\big(\mathsf{x}_{i-1}(\underline{\gamma(t)})\big) = \mathsf{n}_{i-1} \ \text{ and } \ \operatorname{ord}\big(\mathsf{y}_{i-1}(\underline{\gamma(t)})\big) = \mathsf{m}_{i-1}.$$

Let $\varphi \in \operatorname{Aut}(\mathbb{C}[\![t]\!])$ be the unique reparametrization satisfying

$$\mathsf{x}_{i-1}\big(\underline{(\gamma \circ \varphi)(t)}\big) = t^{\mathsf{n}_{i-1}}.$$

Let us write

$$\mathsf{y}_{i-1}\big(\underline{(\gamma \circ \varphi)(t)}\big) = a_1 t^{\mathsf{n}_{i-1}} + \cdots + a_k t^{k\mathsf{n}_{i-1}} + \hat{y}(t) \tag{2.3}$$

for some $k \in \mathbb{N}$ and $\hat{y}(t) \in \mathbb{C}[\![t]\!]$ such that $\operatorname{ord}(\hat{y}(t)) > k\mathsf{n}_{i-1}$ and $\mathsf{n}_{i-1} \nmid \operatorname{ord}(\hat{y}(t))$. Apply now a triangular coordinate change to $\mathsf{x}_{i-1}\big(\underline{(\gamma \circ \varphi)(t)}\big)$ and $\mathsf{y}_{i-1}\big(\underline{(\gamma \circ \varphi)(t)}\big)$ in order to eliminate all terms of $\mathsf{y}_{i-1}\big(\underline{(\gamma \circ \varphi)(t)}\big)$ which are of degree strictly smaller than $\operatorname{ord}(\hat{y}(t))$ and define the geometric invariant

$$\mathsf{y}_i \coloneqq \mathsf{y}_{i-1} - \sum_{i=1}^{k} a_i (\mathsf{x}_{i-1})^i.$$

Set

$$\mathsf{m}_i \coloneqq \operatorname{ord}\big(\mathsf{y}_i(\underline{\gamma(t)})\big).$$

3rd step: Construction of a geometric invariant of order equal to $\gcd(\mathsf{n}_{i-1}, \mathsf{m}_i)$**:**

We set

$$\mathsf{x}_i \coloneqq \mathrm{GCD}(\mathsf{y}_i, \mathsf{x}_{i-1}; \gamma(t)).$$

Then, the geometric invariant x_i satisfies

$$\operatorname{ord}\big(\mathsf{x}_i(\underline{\gamma(t)})\big) = \gcd(\mathsf{n}_{i-1}, \mathsf{m}_i)$$

and we set

$$\mathsf{n}_i \coloneqq \operatorname{ord}\big(\mathsf{x}_i(\underline{\gamma(t)})\big).$$

4th step: Test whether the 2nd and 3rd step of the algorithm can be applied to x_i**:**

Check whether $\mathsf{y}_i(\underline{\gamma(t)}) \in \mathbb{C}[\![t^{\mathsf{n}_i}]\!]$:

If $\mathsf{y}_i(\underline{\gamma(t)}) \in \mathbb{C}[\![t^{\mathsf{n}_i}]\!]$, then **return** $\widetilde{\kappa} \coloneqq \mathsf{x}_i$.

Otherwise repeat the 2nd, 3rd and 4th step of the algorithm.

Termination and Correctness of the algorithm PLANECURVATURE**:**

Let us first discuss the case when $\gamma(t) = (t^n, y(t))$ is a Puiseux parametrization, i.e., n is the polydromy order of $y(t^{\frac{1}{n}})$, and show that the algorithm constructs in that case a crucial curvature of X at the origin. The general case then follows by a suitable variable substitution.

1. **$\gamma(t)$ is a Puiseux parametrization:**

   Observe first that the algorithm terminates in the 1^{st} step if and only if $n = 1$, which would mean that the curve parametrized by $\gamma(t)$ is smooth at the origin and the variable x_0 is a crucial curvature of the curve at 0. Let us therefore consider a Puiseux parametrization $\gamma(t) = (t^n, y(t))$ with $n > 1$.

   It is sufficient to verify the following two claims:

   1. Each Puiseux series $\mathsf{y}_i(\underline{\gamma}(t^{\frac{1}{\mathsf{n}_i}}))$ has polydromy order equal to n_i.
      $\dashrightarrow$ Then the algorithm terminates in the 4^{th} step if and only if $\mathsf{n}_i = \operatorname{ord}\big(\mathsf{x}_i(\underline{\gamma(t)})\big) = 1$.
   2. $\mathsf{n}_i > \mathsf{n}_{i+1} \geq 1$ for each $i \geq 0$.
      $\dashrightarrow$ It follows then that the algorithm terminates as the set of positive integers is a well-ordered set.

   We proceed by induction on i. Notice first that m_1 is by construction the first characteristic exponent of $\mathsf{y}_0(\underline{\gamma(t^{\frac{1}{\mathsf{n}_0}})}) = y(t^{\frac{1}{n}})$. At the same time, n_0 is the polydromy order of $\mathsf{y}_0(\underline{\gamma(t^{\frac{1}{\mathsf{n}_0}})}) = y(t^{\frac{1}{n}})$. Hence, we have

   $$\mathsf{n}_0 > \mathsf{n}_1 = \gcd(\mathsf{n}_0, \mathsf{m}_1) \geq 1,$$

   and the Puiseux series $\mathsf{y}_1(\underline{\gamma(t^{\frac{1}{\mathsf{n}_1}})})$ has polydromy order n_1 according to Proposition 2.1.8. Let us now suppose

   $$\mathsf{n}_{i-1} > \mathsf{n}_i = \gcd(\mathsf{n}_{i-1}, \mathsf{m}_i) > 1,$$

   where m_i is the order of $\mathsf{y}_i(\underline{\gamma(t)})$. Suppose further the Puiseux series $\mathsf{y}_i(\underline{\gamma(t^{\frac{1}{\mathsf{n}_i}})})$ having polydromy order equal to $\mathsf{n}_i = \gcd(\mathsf{n}_{i-1}, \mathsf{m}_i)$. Now, Proposition 2.1.8 applies and shows that for any reparametrization φ, the polydromy order of $\mathsf{y}_i\big(\underline{(\gamma \circ \varphi)(t^{\frac{1}{\mathsf{n}_i}})}\big)$ equals n_i as well. Hence as $\mathsf{n}_i > 1$, the Puiseux series $\mathsf{y}_i\big(\underline{(\gamma \circ \varphi)(t^{\frac{1}{\mathsf{n}_i}})}\big)$ is not contained in the power series ring and thus has at least one characteristic exponent which shows that the decomposition (2.3) of the 2^{nd} step of the algorithm is well defined. Actually, m_{i+1} is by construction its first characteristic exponent. Now, we again use Proposition 2.1.8 to see that $\mathsf{y}_{i+1}(\underline{\gamma(t^{\frac{1}{\mathsf{n}_{i+1}}})})$ has polydromy order equal to $\mathsf{n}_{i+1} = \gcd(\mathsf{n}_i, \mathsf{m}_{i+1})$. But m_{i+1} is the first characteristic exponent of the Puiseux series $\mathsf{y}_i\big(\underline{(\gamma \circ \varphi)(t^{\frac{1}{\mathsf{n}_i}})}\big)$ which has polydromy order equal to n_i. Hence, $\mathsf{n}_{i+1} = \gcd(\mathsf{n}_i, \mathsf{m}_{i+1}) < \mathsf{n}_i$. Thus, we conclude

   $$\mathsf{n}_{i-1} > \mathsf{n}_i > \mathsf{n}_{i+1} \geq 1,$$

   which guarantees that the algorithm terminates in finitely many steps.

2. **$\gamma(t)$ is not a Puiseux parametrization:**

   In this case there exists some positive integer $\alpha \in \mathbb{N}, \alpha > 1$, with $\alpha | n$ so that

   $$y(t) = \sum_{i \geq 0} a_i t^{i \cdot \alpha}.$$

Let α be maximal with this property. Then, the polydromy order of $y(t^{\frac{1}{n}})$ equals $\frac{n}{\alpha}$. Thus,

$$\gamma(t^{\frac{1}{\alpha}}) = (t^{\frac{n}{\alpha}}, y(t^{\frac{1}{\alpha}})) \in \mathbb{C}\{t\}^2$$

is a Puiseux parametrization and

$$\widetilde{\kappa}_1 = \text{PLANECURVATURE}(\gamma(t^{\frac{1}{\alpha}}))$$

satisfies

$$\operatorname{ord}\big(\widetilde{\kappa}_1(\underline{\gamma(t^{\frac{1}{\alpha}})})\big) = 1.$$

However, if we construct

$$\widetilde{\kappa}_2 := \text{PLANECURVATURE}(\gamma(t)),$$

we see that after finitely many iterations of the 2nd up to the 4th step of the algorithm, the algorithm necessarily constructs a geometric invariant x_i of order α at $\underline{\gamma(t)}$ in its 3rd step. But at the same time $\mathsf{y}_i \in \mathbb{C}[\![t^\alpha]\!]$. More precisely,

$$\widetilde{\kappa}_2(\underline{\gamma(t)}) = \widetilde{\kappa}_1(\underline{\gamma(t^{\frac{1}{\alpha}})})|_{t=t^\alpha},$$

and so

$$\operatorname{ord}\big(\widetilde{\kappa}_2(\underline{\gamma(t)})\big) = \alpha.$$

Example 2.1.10 (Construction of a crucial curvature with the algorithm PLANECURVATURE)**.** Consider the curve

$$X = \{-x^3 + (3y^2 - 6y + 1)x^2 + (-3y^4 - 2y^3)x + y^6 = 0\}.$$

The pair

$$\gamma(t) = (t^6, t^2 + t^3)$$

defines a Puiseux parametrization of X at 0. We construct now $\widetilde{\kappa} = \text{PLANECURVATURE}(\gamma(t))$ according to the algorithm.

1st step: $\mathsf{x}_0 = x^{(0)}, \mathsf{y}_0 = y^{(0)}, \mathsf{m}_0 = \operatorname{ord}(t^2 + t^3) = 2, \mathsf{n}_0 = \operatorname{ord}(t^6) = 6.$
And since $\mathsf{y}_0 \notin \mathbb{C}[\![t^{\mathsf{n}_0}]\!] = \mathbb{C}[\![t^6]\!]$ we continue with the 2nd step of the algorithm.

2nd step: No triangular coordinate change is needed as $\mathsf{n}_0 = 6 > 2 = \mathsf{m}_0$. Hence,

$$\mathsf{y}_1 = \mathsf{y}_0 \text{ and } \mathsf{m}_1 = 2.$$

3rd step: We write the Euclidean algorithm for 6 and 2 and compute the corresponding geometric invariants according to the algorithm GCD:

$$6 = 2 \cdot 2 + 2 \qquad \mathsf{x}_1 = \kappa_1(\mathsf{x}_0, \mathsf{y}_1) = \kappa_1, \qquad \mathsf{n}_1 = \operatorname{ord}\big(\mathsf{x}_1(\underline{\gamma(t)})\big) = 2$$
$$2 = 1 \cdot 2 + 0$$

4th step: As $\mathsf{n}_1 \neq 1$ and $\mathsf{y}_1(\underline{\gamma(t)}) = t^2 + t^3 \notin \mathbb{C}[\![t^2]\!]$ (notice that $\mathsf{n}_1 = 2$), we go again through the 2nd - 4th step of the algorithm.

5th step: $\mathsf{x}_1(\underline{\gamma(t)})$ satisfies $\mathsf{x}_1\big(\underline{(\gamma \circ \varphi)(t)}\big) = t^2$ for the reparametrization

$$\varphi(t) = \frac{\sqrt{6}}{6}t + \frac{9}{32}t^2 + \frac{229\sqrt{6}}{2048}t^3 + \dots .$$

At the same time we have

$$\mathsf{y}_1\big(\underline{(\gamma \circ \varphi)(t)}\big) = \frac{1}{6}t^2 + \frac{35\sqrt{6}}{288}t^3 + \dots$$

Hence, in order to eliminate the factor $\frac{1}{6}t^2$, we set

$$\mathsf{y}_2 = \mathsf{y}_1 - \frac{1}{6}\mathsf{x}_1 \quad \text{and} \quad \mathsf{m}_2 = \operatorname{ord}\big(\mathsf{y}_2(\underline{\gamma(t)})\big) = 3.$$

6th step: We write the Euclidean algorithm for the integers 3 and 2 and compute the corresponding geometric invariant according to the algorithm GCD:

$$\begin{array}{lll} 3 = 1 \cdot 2 + 1 & \qquad \mathsf{x}_2 = \kappa_0(\mathsf{y}_2, \mathsf{x}_1) & \qquad \mathsf{n}_2 = \operatorname{ord}\big(\mathsf{x}_2(\underline{\gamma(t)})\big) = 1 \\ 2 = 1 \cdot 2 + 0 & & \end{array}$$

The algorithm stops here, as we have already constructed a crucial curvature

$$\widetilde{\kappa} = \mathsf{x}_2 = \frac{\partial \mathsf{y}_2}{\partial \mathsf{x}_1} = \frac{\partial(y^{(0)} - \frac{1}{6}\kappa_1)}{\partial \kappa_1} = \frac{y^{(1)} - \frac{1}{6}\kappa_2 \cdot y^{(1)}}{\kappa_2 \cdot y^{(1)}} = \frac{1}{\kappa_2} - \frac{1}{6}$$

of order one at $\underline{\gamma(t)}$.

Thus, for a plane algebraic curve $X \subseteq \mathbb{A}^2_{\mathbb{C}}$ with only one singular point $0 \in X$ and only one analytic branch at the origin, we have just proven the following theorem:

Theorem 2.1.11. *For any Puiseux parametrization $\gamma(t)$ of X at the origin, the algorithm* PLANECURVATURE *constructs a crucial curvature of X, i.e., a geometric invariant $\widetilde{\kappa}$ that satisfies*

$$\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = 1.$$

Moreover, if $f \in \mathbb{C}[x, y]$ is a defining polynomial of X, then the blowup of X in the ideal $(\widetilde{\kappa}(f)_1, \widetilde{\kappa}(f)_2)$ defines a resolution of singularities of X.

In this section we established an algorithm only for resolution of plane algebraic curves with only one singular point. The resolution of plane curves with several singularities is discussed in Section 2.3 of this note.

2.2 Analytically irreducible Space Curves with one Singularity

We present in this section an algorithm for resolution of analytically irreducible space curves with a single singular point. Our algorithm is, as in the plane curve case, based on the existence of characteristic exponents of Puiseux parametrizations. Given an algebraic space curve $X \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ defined by a radical ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n, y]$ with only one singularity and only one analytic branch at the origin, our algorithm constructs a geometric invariant of space curves $\widetilde{\kappa}$ satisfying the property

$$\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = 1,$$

for at least one parametrization $\gamma(t) \in \mathbb{C}[\![t]\!]^{n+1}$ of X at the origin.

Definition 2.2.1. We call a geometric invariant $\widetilde{\kappa}$ a *crucial curvature* of X (at the origin) if the blowup of X in the ideal $(\widetilde{\kappa}\big(I\big)_1, \widetilde{\kappa}\big(I\big)_2)$ defined by the (minimal) numerator and denominator of the implicit expression of $\widetilde{\kappa}$ resolves X.

Given a Puiseux parametrization $\gamma(t)$ of X at 0, the strategy of our algorithm is to project the space curve X to coordinate planes and using the algorithm PLANECURVATURE there to construct for each projection a geometric invariant of minimal possible order (when evaluating at $\underline{\gamma(t)}$). The orders of the evaluations of these geometric invariants at $\underline{\gamma(t)}$ satisfy by construction the same property as the set of the characteristic exponents of $\gamma(t)$, namely that the greatest common divisor of these orders and the polydromy order of the parametrization $\gamma(t)$ equals one. Hence, it should be possible to construct with them a geometric invariant whose evaluation at $\underline{\gamma(t)}$ has order equal to 1.

To be more precise: As in the case of plane curves, we will construct a resolution $\widetilde{X}$ of X via a blowup in a suitable ideal $(\widetilde{\kappa}\big(I\big)_1, \widetilde{\kappa}\big(I\big)_2)$ defined in X by the numerator and denominator of an implicit expression of a crucial curvature $\widetilde{\kappa}$. The resolution $\widetilde{X}$ again equals the Zariski closure $\widetilde{X}_{\widetilde{\kappa}}$ of the graph of the map induced by the crucial curvature $\widetilde{\kappa}$:

$$\begin{aligned} \phi_{\widetilde{\kappa}} \colon X \backslash Z &\to \mathbb{P}^1_{\mathbb{C}} \\ x &\mapsto (\widetilde{\kappa}\big(I\big)_1(x) : \widetilde{\kappa}\big(I\big)_2(x)), \end{aligned}$$

with $Z = V\big(I + (\widetilde{\kappa}\big(I\big)_1, \widetilde{\kappa}\big(I\big)_2)\big)$. This property together with generalization of Lemma 2.1.3 to space algebraic curves turns out to be crucial for the construction of resolution of singularities.

Let us at this point recall the notation I_{F_n} for the field of geometric invariants of space curves in $\mathbb{A}^{n+1}_{\mathbb{C}}$ and also other standard notation from Section 1.2 which will be used throughout this section. Further, for an a geometric invariant $p \in I_{F_n}$, we again denote the vector $(\partial^i p)_{i \geq 0}$ by $\underline{p}$.

Lemma 2.2.2. *Let $Y \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ be an algebraic space curve defined by the ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n, y]$. Assume that Y is analytically irreducible at each point. Further, let $p = \frac{p_1}{p_2} \in I_{F_n}$ be a geometric invariant of space curves satisfying $p_2(\underline{\gamma(t)}) \neq 0$, for any parametrization $\gamma(t)$ of Y. Let*

$p(I) = \frac{p(I)_1}{p(I)_2}$ *be an implicit expression of* p *in terms of* I*. Consider the map*

$$\begin{aligned} \phi_p \colon Y \backslash Z &\to \mathbb{P}^1_{\mathbb{C}} \\ y &\mapsto \big(p(I)_1(y) : p(I)_2(y)\big) \end{aligned}$$

induced by p*, with* $Z = V\big(I + (p(I)_1, p(I)_2)\big)$*. Further, let* $\widetilde{Y_p} \subseteq \mathbb{A}^{n+1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ *be the Zariski closure of its graph. Then the following holds:*

(i) The projection map $\pi : \widetilde{Y_p} \to Y$ *induced by the projection* $\mathbb{A}^{n+1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{A}^{n+1}_{\mathbb{C}}$ *onto the first* $n+1$ *components is a proper birational morphism which is an isomorphism* $\pi : \widetilde{Y} \backslash E \to Y \backslash Z$ *outside* $E = \pi^{-1}(Z)$*.*

(ii) For each $y \in Y$*, the fibre* $\pi^{-1}(y)$ *is only one point and* $\widetilde{Y_p}$ *is analytically irreducible at this point.*

(iii) Let $\gamma(t)$ *be a parametrization of* Y *at* y*. Then the vector*

$$\gamma(t) \times \big(p_1(\underline{\gamma(t)}) : p_2(\underline{\gamma(t)})\big)$$

parametrizes $\widetilde{Y_p}$ *at* $\tilde{y} = \pi^{-1}(y)$*.*

(iv) We have the inclusion $\mathrm{Sing}(\widetilde{Y_p}) \subseteq \pi^{-1}(\mathrm{Sing}(Y))$*.*

Proof. The proof goes along the same line as the one of Lemma 2.1.3. □

With this last lemma we obtain:

Corollary 2.2.3. *The condition*

$$\text{“}\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = 1 \text{ for at least one parametrization } \gamma(t) \text{ of } X \text{ at } 0\text{”}$$

is a necessary and sufficient condition for $\widetilde{\kappa}$ *being a crucial curvature of* X *at* 0*.*

The last corollary gives us a characterization for crucial curvatures in terms of parametrizations. However, as we at the end want to construct a resolution of X as the blowup in the ideal corresponding to a crucial curvature, we would like to have also an equivalent characterization for the ideals defined by crucial curvatures. Such an implicit characterization is, however, not known to us at the moment and we hence put this problem on the list with open questions in Section 5.2.

Let us now assume that X is not contained in the hyperplane $\{y = 0\}$ and let

$$\gamma(t) = (x_1(t), \ldots, x_n(t), t^l) \in \mathbb{C}\{t\}^{n+1}$$

be its Puiseux parametrization (the existence of such a parametrization is guaranteed by Theorem 4.2.8). As already mention, the strategy of our algorithm is to project the curve X with the n projections

$$\begin{aligned} \pi_i \colon \mathbb{A}^{n+1}_{\mathbb{C}} &\to \mathbb{A}^2_{\mathbb{C}} \\ (x_1, \ldots, x_n, y) &\mapsto (y, x_i) \end{aligned}$$

to plane curves X_i parametrized by $\gamma_i(t) = (t^l, x_i(t))$ at the origin and to apply the algorithm PLANECURVATURE to these projections in order to construct for each of the parametrizations $\gamma_i(t)$ a geometric invariant of minimal possible order. More precisely:

Remark 2.2.4. The polydromy order l of $\gamma(t) = (x_1(t), \ldots, x_n(t), t^l)$ is the product of the polydromy orders l_i of the parametrizations $\gamma_i(t) = (t^l, x_i(t))$ (see Section 4.2), i.e., $l = \prod_i l_i$. By construction we have

$$\operatorname{ord}\Big(\big(\textsc{PlaneCurvature}(\gamma_i(t))\big(\underline{\gamma_i(t)}\big)\Big) = \frac{l}{l_i}.$$

Resolution algorithm for space curves with one singularity

Resolution Algorithm for Space Curves - one Singularity Construction of a Crucial Curvature

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Input Puiseux parametrization $\gamma(t) = (x_1(t), \ldots, x_n(t), t^l) \in \mathbb{C}[\![t]\!]^{n+1}$ of an algebraic space curve at 0 of polydromy order l & the embedding dimension $N = n + 1$
Output geometric invariant $\widetilde{\kappa}$ satisfying $\operatorname{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = 1$
procedure $\textsc{SpaceCurvature}(\gamma(t); N)$

1st step: Variable declaration:

Set

$$\mathsf{x}_1 := x_1^{(0)}, \ \ldots \ , \mathsf{x}_{N-1} := x_{N-1}^{(0)}, \ \mathsf{y} := y^{(0)}.$$

Further let l_i be the polydromy order of $x_i(t^{\frac{1}{l}})$, for $i = 1, \ldots, N-1$.

2nd step: Construction of curvatures of minimal order of each projection $(\mathsf{y}, \mathsf{x}_i)$:

For each $i = 1, \ldots, N-1$, set

$$z_i := \textsc{PlaneCurvature}\big(\mathsf{y}(\underline{\gamma(t)}), \mathsf{x}_i(\underline{\gamma(t)})\big).$$

Each z_i is per construction (due to the 1st step "variable declaration" of PLANECURVATURE) a geometric invariant of plane curves, i.e., an invariant rational function in variables $x^{(j)}$ and $y^{(j)}$ for $j \in \mathbb{N}$, and according to Remark 2.2.4 satisfies

$$\operatorname{ord}\Big(z_i\big(\underline{(y(t), x_i(t))}\big)\Big) = \frac{l}{l_i}.$$

For each $i = 1, \ldots, N-1$, we apply the substitution

$$\begin{aligned} \lambda_i \colon \mathbb{C}(x^{(j)}, y^{(j)} \ : \ j \in \mathbb{N}) &\to \mathbb{C}(x_i^{(j)}, y^{(j)} \ : \ i, j \in \mathbb{N}, 1 \leq i \leq n) \\ x^{(j)} &\mapsto \partial^j \mathsf{y} \\ y^{(j)} &\mapsto \partial^j \mathsf{x}_i \end{aligned}$$

in order to obtain geometric invariants of space curves

$$\mathsf{z}_i := \lambda_i(z_i)$$

of order at $\underline{\gamma(t)}$ equal to

$$\mathsf{n}_i := \mathrm{ord}(\mathsf{z}_i(\underline{\gamma(t)})) = \mathrm{ord}\big(z_i\big(\underline{(y(t), x_i(t))}\big)\big) = \frac{l}{l_i}.$$

3rd step: Construction of a crucial curvature:

Define iteratively and observe

$$\begin{array}{ll} \widetilde{\mathsf{z}}_2 := \mathrm{GCD}(\mathsf{z}_1, \mathsf{z}_2; \gamma(t)) & \mathrm{ord}\big(\widetilde{\mathsf{z}}_2(\underline{\gamma(t)})\big) = \gcd(\mathsf{n}_1, \mathsf{n}_2), \\ \widetilde{\mathsf{z}}_3 := \mathrm{GCD}(\widetilde{\mathsf{z}}_2, \mathsf{z}_3; \gamma(t)) & \mathrm{ord}\big(\widetilde{\mathsf{z}}_3(\underline{\gamma(t)})\big) = \gcd(\mathsf{n}_1, \mathsf{n}_2, \mathsf{n}_3), \\ \vdots & \vdots \\ \widetilde{\mathsf{z}}_{N-1} := \mathrm{GCD}(\widetilde{\mathsf{z}}_{N-2}, \mathsf{z}_{N-1}; \gamma(t)) & \mathrm{ord}\big(\widetilde{\mathsf{z}}_{N-1}(\underline{\gamma(t)})\big) = \gcd(\mathsf{n}_1, \dots, \mathsf{n}_{N-1}). \end{array}$$

Finally, **return** $\widetilde{\kappa} := \widetilde{\mathsf{z}}_{N-1}$.

Correctness of the algorithm SPACECURVATURE**:**
In fact, by construction, the order of $\widetilde{\kappa} = \textsc{SpaceCurvature}(\gamma(t); N)$ at $\underline{\gamma(t)}$ satisfies

$$\mathrm{ord}(\widetilde{\kappa}(\underline{\gamma(t)})) = \gcd\left(\frac{l}{l_1}, \dots, \frac{l}{l_n}\right) = 1.$$

Remark 2.2.5. Remark that for a Puiseux parametrization $\gamma(t) = (t^n, y(t)) \in \mathbb{C}\{t\}^2$ of a plane algebraic curve in $\mathbb{A}^2_{\mathbb{C}}$ we have

$$\textsc{PlaneCurvature}((t^n, y(t))) = \textsc{SpaceCurvature}((y(t), t^n); 2).$$

Thus, for an algebraic space curve $X \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ with only one singularity $0 \in X$ and only one analytic branch at the origin, we have just proven the following theorem:

Theorem 2.2.6. *For any Puiseux parametrization $\gamma(t) = (x_1(t), \dots, x_n(t), t^l)$ of X at the origin, the algorithm* SPACECURVATURE *constructs a crucial curvature of X, i.e., a geometric invariant $\widetilde{\kappa}$ that satisfies*

$$\mathrm{ord}\big(\widetilde{\kappa}(\underline{\gamma(t)})\big) = 1.$$

Moreover, for $I \subseteq \mathbb{C}[x_1, \dots, x_n, y]$ a defining ideal of X, the blowup of X in the ideal

$$(\widetilde{\kappa}(I)_1, \widetilde{\kappa}(I)_2)$$

defines a resolution of singularities of X.

However, the algorithm SPACECURVATURE constructs a resolution only for analytically irreducible space curves with only one singular point. If X has more than one singular point, iterations of the algorithm SPACECURVATURE are needed as we will see in the next section.

2.3 Analytically irreducible Plane and Space Curves with multiple Singularities

Let us fix a plane or space curve $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$, i.e., $n \geq 1$, with m singular points

$$\operatorname{Sing}(X) = \{a_1, \ldots, a_m\}.$$

Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n, y]$ be a defining ideal of X. Let us assume that X is analytically irreducible at each point. The goal of this section is to present an algorithm for construction of a resolution of X based on the algorithms PLANECURVATURE and SPACECURVATURE presented in Section 2.1 and 2.2, respectively.

Let us fix for each $i = 1, \ldots, m$, a Puiseux parametrization $\gamma_i(t) \in \mathbb{C}\{t\}^{n+1}$ of X at the singular point a_i. Further consider for each i the following coordinate change:

$$\begin{aligned} \lambda_{a_i} : \mathbb{A}_{\mathbb{C}}^{n+1} &\to \mathbb{A}_{\mathbb{C}}^{n+1} \\ (x_1, \ldots, x_n, y) &\mapsto (x_1, \ldots, x_n, y) - a_i, \end{aligned}$$

under which a_i moves to the origin. Let us further denote by X_{a_i} the image of the curve X under λ_{a_i}. By definition, $\gamma_i(t) - a_i$ is a Puiseux parametrization of X_{a_i} at 0. Let us w.l.o.g. assume that it is of the form

$$\gamma_i(t) - a_i = (x_{i,1}(t), \ldots, x_{i,n}(t), t^{l_i})$$

with l_i the polydromy order and so that $x_{i,j}(t) \neq 0$ is fulfilled for all i, j (otherwise we could embed X in $\mathbb{A}_{\mathbb{C}}^{n}$).

For each singular point a_i on X, our algorithm constructs with SPACECURVATURE a crucial curvature $\widetilde{\kappa}_i = \frac{\widetilde{\kappa}_{i,1}}{\widetilde{\kappa}_{i,2}}$ of X at a_i. The claim is that, for $\gamma(t)$ a parametrization of X, the curve in $\mathbb{A}_{\mathbb{C}}^{n+1} \times (\mathbb{P}_{\mathbb{C}}^1)^m$ parametrized by the vector

$$\widetilde{\gamma}(t) = (\gamma) \times \big(\widetilde{\kappa}_{i,1}(\underline{\gamma(t)}) : \widetilde{\kappa}_{i,2}(\underline{\gamma(t)})\big) \times \cdots \times \big(\widetilde{\kappa}_{m,1}(\underline{\gamma(t)}) : \widetilde{\kappa}_{m,2}(\underline{\gamma(t)})\big)$$

defines a resolution of singularities of X.

Resolution algorithm for curves with multiple singularities

Resolution Algorithm - Multiple Singularities Construction of a crucial curvature at each singular point

Input number of singularities m & for each $i = 1, \ldots, m$ a Puiseux parametrization $\gamma_i(t) = a_i + (x_{i,1}(t), \ldots, x_{i,n}(t), t^{l_i}) \in \mathbb{C}\{t\}^{n+1}$ of X at a_i with l_i its polydromy order & the embedding dimension $N = n + 1$

Output vector $\widetilde{\kappa} = (\widetilde{\kappa}_1, \ldots, \widetilde{\kappa}_m)$ of geometric invariants satisfying $\operatorname{ord}\big(\widetilde{\kappa}_i(\underline{\gamma_i(t)})\big) = 1$

procedure CURVATURES$(\gamma_1(t), \ldots, \gamma_m(t)\,; m, N)$

For each $i = 1, \dots, m$ compute

$$\widetilde{\kappa}_i := \text{SPACECURVATURE}(\gamma_i(t) - \gamma_i(0)\,; N).$$

Finally **return** the list

$$\widetilde{\kappa} := (\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_k).$$

Correctness of the algorithm CURVATURES**:**

We have

$$\widetilde{\kappa}(\underline{\gamma_i(t)}) = \widetilde{\kappa}(\underline{\gamma_i(t) - \gamma_i(0)}),$$

for each $i = 1, \dots, n$, and so the i-th component of $\widetilde{\kappa}$ is of order one when evaluating at $\underline{\gamma_i(t)}$.

Let $\widetilde{\kappa} = (\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_m) = \text{CURVATURES}(\gamma_1(t), \dots, \gamma_k(t); m, n+1)$ be the list of crucial curvatures produced by the algorithm CURVATURES. Consider the map

$$\begin{aligned} \phi_{\widetilde{\kappa}} \colon X \backslash Z &\to (\mathbb{P}^1_{\mathbb{C}})^m \\ x &\mapsto \big(\widetilde{\kappa}_1\big(I\big)_1(x) : \widetilde{\kappa}_1\big(I\big)_2(x)\big) \times \cdots \times \big(\widetilde{\kappa}_m\big(I\big)_1(x) : \widetilde{\kappa}_m\big(I\big)_2(x)\big), \end{aligned}$$

where $Z = V\big(I + (\widetilde{\kappa}_i\big(I\big)_j \,:\, 1 \le i \le m, j = 1, 2)\big)$ and where $\widetilde{\kappa}_i\big(I\big)_j$ denote the (minimal) numerator for $j = 1$ and denominator for $j = 2$ of an implicit expression of $\widetilde{\kappa}_i$, respectively. Let $\widetilde{X}_{\widetilde{\kappa}}$ denote the Zariski closure of the graph of $\phi_{\widetilde{\kappa}}$.

Proposition 2.3.1. *The projection morphism*

$$\pi : \widetilde{X}_{\widetilde{\kappa}} \to X$$

induced by the projection $\pi \colon \mathbb{A}^{n+1}_{\mathbb{C}} \times (\mathbb{P}^1_{\mathbb{C}})^m \to \mathbb{A}^{n+1}_{\mathbb{C}}$ *onto the first* $n+1$ *components is a birational and projective morphism which is an isomorphism*

$$\pi \colon \widetilde{X}_{\widetilde{\kappa}} \backslash E \to X \backslash Z$$

outside $E = \pi^{-1}(Z)$*, where* $Z = V\big(I + (\widetilde{\kappa}_i\big(I\big)_j \,:\, 1 \le i \le m, j = 1, 2)\big)$.

Proof. For each $1 \le l \le m$, we denote by $\widetilde{X}^l_{\widetilde{\kappa}}$ the Zariski closure of the graph of the map

$$\begin{aligned} \phi^l_{\widetilde{\kappa}} \colon X \backslash Z_l &\to (\mathbb{P}^1_{\mathbb{C}})^l \\ z &\mapsto (\widetilde{\kappa}_1\big(I\big)_1(z) : \widetilde{\kappa}_1\big(I\big)_2(z)) \times \cdots \times (\widetilde{\kappa}_l\big(I\big)_1(z) : \widetilde{\kappa}_l\big(I\big)_2(z)), \end{aligned}$$

with $Z_l = V\big(I + (\widetilde{\kappa}_i\big(I\big)_j \,:\, 1 \le i \le l, j = 1, 2)\big)$. The same argument as in the proof of (i) of Lemma 2.1.3 shows that each projection $\pi_l : \widetilde{X}^l_{\widetilde{\kappa}} \to X$ induced by the projection morphism $\mathbb{A}^{n+1}_{\mathbb{C}} \times (\mathbb{P}^1_{\mathbb{C}})^l \to \mathbb{A}^{n+1}_{\mathbb{C}}$ defines a birational morphism and, moreover, an isomorphism $\pi_l : \widetilde{X}^l_{\widetilde{\kappa}} \backslash E_l \to X \backslash Z_l$ outside $E_l = \pi_l^{-1}(Z_l)$ with the inverse map given by $x \mapsto x \times \phi^l_{\widetilde{\kappa}}$. We proceed now by induction on l to show that π_l is projective for each $l = 1, \dots, m$. For $l = 1$,

$X^1_{\widetilde{\kappa}}$ is the blowup of X in the ideal $(\widetilde{\kappa}_1(I)_1, \widetilde{\kappa}_1(I)_2)$ and the claim follows directly from the definition. For $l+1$, we observe that $\widetilde{X}^{l+1}_{\widetilde{\kappa}}$ is the Zariski closure of the image of the map

$$\widetilde{X}^l_{\widetilde{\kappa}} \backslash V\big(I + (\widetilde{\kappa}_{l+1}(I)_1, \widetilde{\kappa}_{l+1}(I)_2)\big) \to \mathbb{A}^{n+1}_{\mathbb{C}} \times (\mathbb{P}^1_{\mathbb{C}})^l \times \mathbb{P}^1_{\mathbb{C}}$$

induced by the geometric invariant $\widetilde{\kappa}_{l+1}$. The claim follows now immediately from the induction hypothesis on $\widetilde{X}^l_{\widetilde{\kappa}}$. □

Moreover, using [Ha77, Chapter II, Theorem 7.17.] the following corollary follows:

Corollary 2.3.2. *There exists an ideal $J \subseteq \mathbb{C}[x_1, \ldots, x_n, y]$ such that the curve $\widetilde{X}_{\widetilde{\kappa}}$ together with the projection $\pi : \widetilde{X}_{\widetilde{\kappa}} \to X$ is a blowup of X in the ideal J.*

Resolution of singularities of X via the height function $\phi_{\widetilde{\kappa}}$ follows now immediately:

Theorem 2.3.3. *The Zariski closure $\widetilde{X}_{\widetilde{\kappa}}$ of the graph of $\phi_{\widetilde{\kappa}}$ defines together with the projection morphism $\pi : \widetilde{X}_{\widetilde{\kappa}} \to X$ a resolution of singularities of X.*

Proof. According to (iii) of Lemma 2.2.2, for each singular point a_i, the vector

$$\widetilde{\gamma}_i(t) = (\gamma_i) \times \big(\widetilde{\kappa}_{i,1}(\underline{\gamma_i(t)}) : \widetilde{\kappa}_{i,2}(\underline{\gamma_i(t)})\big) \times \cdots \times \big(\widetilde{\kappa}_{m,1}(\underline{\gamma_i(t)}) : \widetilde{\kappa}_{m,2}(\underline{\gamma_i(t)})\big)$$

defines a parametrization of $\widetilde{X}_{\widetilde{\kappa}}$ at $\pi^{-1}(a_i)$. Moreover, it follows by construction that $\widetilde{\gamma}_i(t)$ is a regular parametrization of $\widetilde{X}_{\widetilde{\kappa}}$ at the point $\pi^{-1}(a_i)$. Therefore, $\widetilde{X}_{\widetilde{\kappa}}$ is smooth at each point $\pi^{-1}(a_i)$. Further, by (v) of Lemma 2.2.2, $\widetilde{X}_{\widetilde{\kappa}}$ is smooth outside the set $\{\pi^{-1}(a_i) \,:\, 1 \leq i \leq k\}$. Finally, Proposition 2.3.1 applies and shows that π is a birational morphism and projective, hence proper by [Ha77, Chapter II, Theorem 4.9]. □

Theorem 2.3.3 together with Corollary 2.3.2 present already the main results of my PhD thesis. With this theorem we proved that any analytically irreducible curve $X \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ can be resolved either in one blowup in an ideal defined by a crucial curvature or that its resolution can be defined by as many suitably chosen geometric invariants as the number of singular points, namely crucial curvatures of X at the singular points. Moreover, Corollary 2.3.2 allows a representation of the smooth curve $\widetilde{X}_{\widetilde{\kappa}}$ (constructed as the Zariski closure of the image of $\phi_{\widetilde{\kappa}}$) as a blowup in a suitable ideal J. However, although the construction of $X_{\widetilde{\kappa}}$ is not hard to do, it is not clear at the moment to us how to construct the center J defining the blowup leading to $X_{\widetilde{\kappa}}$. We therefore put this problem on a list with open questions (see Section 5.2).

2.4 Analytically Reducible Singular Curves

An obstacle with impact on practical computations is the fact that our algorithms do not give a resolution of curves with several analytic branches at their singular points. Whereas, it is very easy to use the minimal curvatures, namely the slopes (of the tangent vectors), to separate two transversal branches, it is in general not clear how to use the (higher) algebraic curvatures to separate two analytic branches that meet tangentially.

Example 2.4.1. Consider the curve $X = \{x^2 - y^4 = 0\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ with a singularity at the origin and observe that X is a union of two horizontal parabolas:

$$X = X_1 \cup X_2 \quad \text{with} \quad X_1 = \{x - y^2 = 0\} \text{ and } X_2 = \{x + y^2 = 0\},$$

with their respective parametrizations

$$\gamma_1(t) = (t^2, t) \quad \text{and} \quad \gamma_2(t) = (-t^2, t)$$

at the origin.

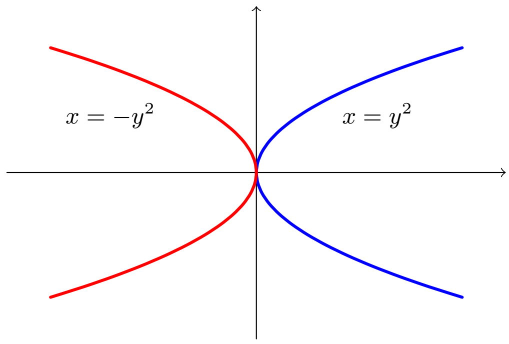

Figure 2.3: Two symmetric horizontal parabolas $x = y^2$ (blue) and $x = -y^2$ (red) meeting at the origin.

Then each algebraic curvature $\kappa_i(\gamma_j(t))$, with $j = 1, 2$, is a Laurent series of order $-2i - 1$ and, moreover, we have the equality

$$\kappa_i(\gamma_1(t)) = (-1)^{i+1}\kappa_i(\gamma_2(t)). \tag{2.4}$$

The two branches of the blowup of X in the ideal corresponding to one κ_i can be parametrized by

$$\widetilde{\gamma_1}_i(t) = \gamma_1(t)\times\big(\kappa_{i,1}(\underline{\gamma_1(t)}) : \kappa_{i,2}(\underline{\gamma_1(t)})\big) \quad \text{and} \quad \widetilde{\gamma_2}_i(t) = \gamma_2(t)\times\big(\kappa_{i,1}(\underline{\gamma_2(t)}) : \kappa_{i,2}(\underline{\gamma_2(t)})\big),$$

respectively. In the affine chart, in which the point lying over 0 is visible, the branches are parametrized by the triples

$$(\gamma_1(t), \kappa_i(\underline{\gamma_1(t)})^{-1}) \quad \text{and} \quad (\gamma_2(t), \kappa_i(\underline{\gamma_2(t)})^{-1}).$$

The equality (2.4), however, shows that they meet at 0 again. The (higher) algebraic curvatures are therefore not able to distinguish between both curves at the origin and hence are also not able to tear them apart.

We therefore put the problem about resolution of (analytically) reducible curves on a list with open questions (see Section 5.2).

Another point on our list with unsolved problems is the question about validity of our results gained in this chapter over other fields, fields of characteristic zero different from $\mathbb{C}$ and fields of positive characteristic. There are many problems connected with this question. The first problem is that the concept of geometric invariants was introduced only over the field of complex numbers (already mentioned in Section 5.1). Another problem is that the concept of Puiseux parametrizations, the key tool of this of this chapter, does not exist over an arbitrary field, especially not over fields of positive characteristic or over fields that are not algebraically closed. Hence, there will be an additional clue needed to generalize some of the statements of this chapter also to (some) other characteristic.

One possibility to approach the problem of positive characteristic would be to interpret the statements and proofs of this chapter implicitly, i.e., in terms of the defining ideal of a curve instead of using Puiseux parametrizations. The search for an implicit proof of resolution via geometric invariants is also one of the tasks of our list in Section 5.3.

Finally, the next big step is to carry the results and constructions for singular curves over to singular surfaces. We are aware that this is a major challenge which will require a very detailed analysis of the difficulties which make the surface case much more delicate. Let us mention that a decisive advance in the resolution of surface singularities would present a long-awaited and certainly highly acclaimed break through (which would also open the door for attacking the higher-dimensional case).

Chapter 3

The Moduli Space of n Points on the Projective Line and the First Fundamental Theorems for $\mathrm{GL}_m(K)$ and $\mathrm{SL}_m(K)$ for an Infinite Field K

The origin of the moduli problem is dated to the early 1960s and attributed to D. Mumford who developed a method to study equivalence classes defined by group actions on algebraic objects and to find their scheme structure and to parametrize them (see the original work by D. Mumford [FM82] or by P. Deligne and D. Mumford [DM69]). This method is known under the name *Geometric Invariant Theory* (or shortly GIT). For the foundation of GIT, Mumford drew inspiration from ideas of D. Hilbert [Hi93] and turned some of the classical techniques and results obtained by Hilbert into revolutionary tools which can be applied to answer modern algebraic geometric questions.

The moduli space of n points on the projective line is one of the classical and most important examples of a GIT quotient and can be found in many books as one of the first and most instructive examples, e.g. by I. Dolgachev [Do03, Chapter 11,§2], or in his book with D. Ortland [DO88, Chapter 1], or by D. Mumford and K. Suominen [MS72, Chapter 2]. Among many other examples of moduli spaces (see e.g. [Fu69b], [Ha77], [Kr84], [HM98], [Mu03], [Va03], [Br10], . . .), the problem of the moduli space of n points on the projective line became very popular and it is still very popular nowadays. For the study and understanding of its scheme structure, often fusions of various mathematical fields are demanded, which makes the problem very attractive. For example to determine the generators of the coordinate ring of the moduli space when considered as an affine variety, one can use properties of integrally closed rings in combination with very advanced algebraic and differential geometric techniques like J.-I. Igusa did in his paper [Ig54], or play with suitably chosen differential operators combined with invariant theory arguments like by I. Dolgachev in [Do03, Chapter 2], or another possibility is to use combinatorial arguments together with elimination theory modulo Gröbner basis [St08, §3.1, §3.2], etc.

Let us discuss first the case $K = \mathbb{C}$. Whereas the generators, let us say $g_1, \ldots, g_k \in \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$, of the coordinate ring of the moduli space were known since 1894 (A. B. Kempe [Ke94] proved that the coordinate ring is generated by the lowest degree invariants, the so-called *Kempe generators*), the question about algebraic relations

$$F_j(g_1, \ldots, g_k) = 0,$$

for $j = 1, \ldots, l$, among them, which would allow us to write the coordinate ring of the moduli space as a quotient of a polynomial ring

$$\mathbb{C}[z_1, \ldots, z_k]/\{F_1, \ldots, F_l\},$$

remained open for long time. Very recently, in years 2006-2009, B. Howard, J. Millson, A. Snowden and R. Vakil collected in their papers [HMSV1, HMSV2, HMSV3, HMSV4, HMSV5] several observations from their investigations of the relations between the Kempe generators. We have to mention also their nice graph theoretical representation of the relations for the cases $n \leq 6$ presented in these papers, which enlarges again the spectrum of mathematical methods and fields contributing to this moduli problem. The true breakthrough was made by B. Howard, J. Millson, A. Snowden and R. Vakil in [HMSV4, Theorem 1.1] where the authors proved that the relations between the Kempe generators are generated by quadratic binomial relations if $n \neq 6$ and by the Segre cubic relations in the case $n = 6$.

3.1 First Fundamental Theorem for $\mathrm{SL}_2(K)$ and $\mathrm{GL}_2(K)$

In this section we will study the *moduli space of n points on the complex projective line* given by the equivalence classes of n-tuples of points $(\mathsf{x} : \mathsf{y}) \in \mathbb{P}^1_{\mathbb{C}}$ modulo the action of the projective special linear group, i.e., the orbit space $(\mathbb{P}^1_{\mathbb{C}})^n / \mathrm{PSL}_2(\mathbb{C})$. This set has structure of an affine algebraic variety whose coordinate ring equals $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{\mathrm{SL}_2}$, the invariant ring under the action of $\mathrm{SL}_2(\mathbb{C})$, and whose function field is a subfield of the invariant field under the action of $\mathrm{GL}_2(\mathbb{C})$, i.e, $\mathrm{F}((\mathbb{P}^1_{\mathbb{C}})^n / \mathrm{PSL}_2(\mathbb{C})) \subseteq \mathbb{C}(x_1, y_1, \ldots, x_n, y_n)^{\mathrm{GL}_2}$ (as we will show later). The generators of this ring and field are known and given by the First Fundamental Theorem (we will refer to it shortly as FFT) for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{C})$, respectively.

I will present in this section of my thesis two different new strategies how to prove the FFT for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{C})$ and thus, also describe in two different ways how to find the (Kempe) generators of the coordinate ring and of the function field of $(\mathbb{P}^1_{\mathbb{C}})^n / \mathrm{PSL}_2(\mathbb{C})$. Whereas the first method is based on results gained in Section 1.4 of the first chapter of my thesis explaining the properties of geometric invariants, especially on Proposition 1.4.2, and as such provides a geometrical explanation for the FFT's, the second method is rather combinatorial and completely independent of the results of the first and second chapter of my thesis. Let me mention from the very beginning that the second proof was obtained in collaboration with S. Yurkevich (University of Vienna, Austria).

The advantage of the second proof of the FFT's for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{C})$ is that it is valid over all infinite fields as well. Thus, we can conclude the FFT's also for $\mathrm{SL}_2(K)$ and $\mathrm{GL}_2(K)$, where K is an arbitrary infinite field. Moreover, our techniques for proving the FFT's for $\mathrm{SL}_2(K)$ and $\mathrm{GL}_2(K)$, for K an infinite field, can be generalized easily to polynomial rings and fields of rational functions in $m \cdot n$ variables and the respective group actions of $\mathrm{SL}_m(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{C})$, so that we prove even the FFT's for $\mathrm{SL}_m(K)$ and $\mathrm{GL}_m(K), m \geq 2$, and obtain as corollary also the generators of the coordinate ring and the function field of the moduli space of n points in the $(m-1)$-dimensional projective space. Further, we provide a family of counter-examples for the FFT's over finite fields.

To be more precise, let $(\mathbb{P}^1_{\mathbb{C}})^n = \{\big((\mathsf{x}_1 : \mathsf{y}_1), \ldots, (\mathsf{x}_n : \mathsf{y}_n)\big) \,:\, (\mathsf{x}_i : \mathsf{y}_i) \in \mathbb{P}^1_{\mathbb{C}}\}$ be the set of n points on the projective line. The projective general linear group $\mathrm{PSL}_2(\mathbb{C})$ acts on $(\mathbb{P}^1_{\mathbb{C}})^n$ by acting from the left on each point separately:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\mathsf{x}_i : \mathsf{y}_i) = (a\mathsf{x}_i + b\mathsf{y}_i : c\mathsf{x}_i + d\mathsf{y}_i). \tag{3.1}$$

This group action defines an equivalence relation on $(\mathbb{P}^1_{\mathbb{C}})^n$ and so, the natural question appears: How does the appropriate moduli space $\mathcal{M} = (\mathbb{P}^1_{\mathbb{C}})^n / \mathrm{PSL}_2(\mathbb{C})$ look like? In order to determine the structure of the moduli space itself, we will investigate the polynomial and rational functions on it.

Consider the polynomial ring in pairs of variables $\mathbb{C}[\mathbb{x}, \mathbb{y}] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ over $\mathbb{C}$ and its quotient field $\mathbb{C}(\mathbb{x}, \mathbb{y})$. The general linear group $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{C}[\mathbb{x}, \mathbb{y}]$, and thus also on $\mathbb{C}(\mathbb{x}, \mathbb{y})$, from the right by the usual matrix-vector multiplication on the pairs of variables (x_i, y_i) as described by (3.1). We denote the corresponding invariant field by $\mathbb{C}(\mathbb{x}, \mathbb{y})^{\mathrm{GL}_2}$. Clearly, no polynomial, except for the constant ones, can be invariant under the action of $\mathrm{GL}_2(\mathbb{C})$, i.e., $\mathbb{C}[\mathbb{x}, \mathbb{y}]^{\mathrm{GL}_2} = \mathbb{C}$. However, it is easy to construct polynomials that are invariant under the action of $\mathrm{SL}_2(\mathbb{C})$. Let us denote the corresponding invariant ring by $\mathbb{C}[\mathbb{x}, \mathbb{y}]^{\mathrm{SL}_2}$. We set

$$f_{i,j} := x_i y_j - y_i x_j \in \mathbb{C}[\mathbb{x}, \mathbb{y}], \quad 1 \leq i, j \leq n,$$

and denote by $\mathbb{C}[f_{ij}]$ the polynomial ring generated over $\mathbb{C}$ by all $f_{i,j}, 1 \leq i, j \leq n$ and by $\mathbb{C}(f_{ij})$ its quotient field. Notice that these polynomials satisfy the straightforward equalities

$$f_{i,i} = 0, \quad f_{j,i} = -f_{i,j}, \tag{3.2}$$

and also the *Plücker relation*

$$f_{i,j} f_{k,l} = f_{i,k} f_{j,l} - f_{i,l} f_{j,k}, \tag{3.3}$$

for all $1 \leq i, j, k, l \leq n$. Moreover, it is obvious that each $f_{i,j}$ is *semi-invariant* under the action of $\mathrm{GL}_2(\mathbb{C})$, i.e.,

$$G \cdot f_{i,j} = \det(G) f_{i,j},$$

for any $G \in \mathrm{GL}_2(\mathbb{C})$, and hence invariant under the action of $\mathrm{SL}_2(\mathbb{C})$, justifying the inclusions $\mathbb{C}[\mathbb{x},\mathbb{y}]^{\mathrm{SL}_2} \supseteq \mathbb{C}[f_{ij}]$ and $\mathbb{C}(\mathbb{x},\mathbb{y})^{\mathrm{GL}_2} \supseteq \mathbb{C}\left(\frac{f_{i,j}}{f_{k,l}} : 1 \leq i,j,k,l \leq n\right)$.

The goal is to understand the structure of the ring $\mathbb{C}[f_{ij}]$ and the field $\mathbb{C}\left(\frac{f_{i,j}}{f_{k,l}}\right)$ and to prove the First Fundamental Theorems for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{C})$:

First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$**.** *An element $q \in \mathbb{C}(\mathbb{x},\mathbb{y})$ is invariant under the action of $\mathrm{GL}_2(\mathbb{C})$ if and only if q can be written as a rational function in $f_{i,j}/f_{k,l}, 1 \leq i,j,k,l \leq n$, i.e.,*

$$\mathbb{C}(\mathbb{x},\mathbb{y})^{\mathrm{GL}_2} = \mathbb{C}\left(\frac{f_{i,j}}{f_{k,l}} : 1 \leq i,j,k,l \leq n\right).$$

Moreover, an invariant q admits the following representation in the generators $f_{i,j}/f_{k,l}$'s:

$$q(x_i, y_i) = q\left(\frac{f_{1,i}}{f_{1,2}}, \frac{f_{2,i}}{f_{1,2}}\right).$$

First Fundamental Theorem for $\mathrm{SL}_2(\mathbb{C})$**.** *An element $p \in \mathbb{C}[\mathbb{x},\mathbb{y}]$ is invariant under the action of $\mathrm{SL}_2(\mathbb{C})$ if and only if p can be written as a polynomial in $f_{i,j}, 1 \leq i,j \leq n$, i.e.,*

$$\mathbb{C}[\mathbb{x},\mathbb{y}]^{\mathrm{SL}_2} = \mathbb{C}[f_{ij}].$$

The history of the First Fundamental Theorem is long and complex. Depending on the source, it is first attributed to A. Clebsch [Cl70], H. Weyl [We39], W. V. D. Hodge [Ho43] or J.-I. Igusa [Ig54]. We will explain the contribution of these authors as well as provide insight into more recent approaches.

Indeed, D. R. Richman [Ri89, page 44] recognized the oldest reference [Cl70, page 51] from 1870, in which A. Clebsch considered $\mathrm{GL}_2(K)$ semi-invariant polynomials by working with the so-called *Aronhold operator*. Note that this proof works for all fields K of characteristic 0. Some 30 years later, H. Grace and A. Young found an easier proof of Clebsch's theorem using the *Cayley Ω-operator* and compared it with the original one [GY03, §28, §35]. The first reference for the $\mathrm{GL}_n(K)$ semi-invariant polynomials for any $n \in \mathbb{N}$ is [Tu29, §5] in which W. Turnbull generalized the preceding ideas of the Ω-operator. Then H. Weyl made a first breakthrough in this area by employing *Capelli identities* in [We39, Theorem 2.6.A].

Soon after in 1953, J.-I. Igusa [Ig54, Theorem 4] proved the FFT for $\mathrm{SL}_n(K)$, where K is any *universal domain*, by placing it in a completely different, geometric, setting. Embedding the invariant ring into the coordinate ring of a Grassmann variety and using tools from abstract algebraic geometry, he was the first one who showed the theorem for any algebraically closed field K.

The next major change in perspective was done by P. Doubilet, G.-C. Rota and J. Stein in [DRS74, page 200-202] where the authors first introduced the combinatorial *straightening lemma* (see Lemma 3.1.2) and *double tableaux*, and then proved the FFT for $\mathrm{GL}_n(K)$ and

$\mathrm{SL}_n(K)$ in a different but equivalent setting for arbitrary infinite fields. It shall be noted that the straightening lemma (while not named like this) was already proven by W. V. D. Hodge [Ho43, page 27], who attributed ideas to A. Young [Yo28, Theorem 1]. Two years later, C. De Concini and C. Procesi noted that the paper [DRS74] had a gap and fixed it [dCP76, Theorem 1.2]. A decade later, M. Barnabei and A. Brini [BB86] published an article with a more elementary proof, again for infinite fields, where they managed to avoid double tableaux.

An even more recent and new proof for all infinite fields was found by D. R. Richman [Ri89, §3], in which the author described a reduction to the case $n = 2$. Then Richman showed that polynomial invariants under the action of $\mathrm{SL}_2(K)$ are equal to the ones under the action of the special upper triangular matrices. It turned out that the latter can be described easier.

Moreover, we mention the paper [SW89] in which B. Sturmfels and N. White presented the straightening algorithm using reduction modulo Gröbner bases. In his book [St08, §3.2], Sturmfels explained how this algorithm can be used to show the FFT for $\mathrm{SL}_n(\mathbb{C})$. Moreover, it turns out that the direct straightening algorithm approach is rather slow for practical computations and so in the same book the author gave another more efficient algorithm for the representation of $\mathrm{SL}_n(\mathbb{C})$ invariants in terms of the generators of the invariant ring.

Finally, a more recent proof is due to H. Kraft and C. Procesi [KP96, §8], in which they deduce the FFT for $\mathrm{SL}_n(K)$ and $\mathrm{GL}_n(K)$ for infinite fields from a generalization of Weyl's Theorems. We also refer to [Do03] for extended bibliographic notes and a well-written proof using Cayley's Ω-operator.

Proofs of the First Fundamental Theorems

Let us start with two proofs of the First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$. The strategy of our first proof of the FFT for $\mathrm{GL}_2(\mathbb{C})$ is to use Proposition 1.4.2 in order to interpret $\mathrm{GL}_2(\mathbb{C})$-invariant rational functions as "minimal" geometric invariants. Then Theorem 1.4.5 applies and gives us a representation of q as a rational function in the generators of the invariant field.

From now on we will refer to each rational function $q = q(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{C}(\mathbb{x}, \mathbb{y})$ as $q(x_i, y_i)$, in order to shorten the notation.

First Proof of the First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$. Let $q = \frac{q_1}{q_2} \in \mathbb{C}(\mathbb{x}, \mathbb{y})$ be invariant under the action of $\mathrm{GL}_2(\mathbb{C})$. Then q satisfies the following equality for all $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$:

$$q_1(ax_i + by_i, cx_i + dy_i) \cdot q_2(x_i, y_i) - q_1(x_i, y_i) \cdot q_2(ax_i + by_i, cx_i + dy_i) = 0. \tag{3.4}$$

By Weyl's, principle we conclude the equality even for arbitrary $a, b, c, d \in \mathbb{C}$ which allows us to consider them even as variables in the above equality. Hence the equality (3.4) still holds if for any family of bivariate power series $u_i \in \mathbb{C}[\![t, s]\!]$ and any reparametrization $\varphi \in \mathrm{Aut}(\mathbb{C}[\![t, s]\!])$ we perform the substitution

$$x_i \mapsto \partial_t u_i, \; y_i \mapsto \partial_s u_i,$$

$$a \mapsto \partial_t \varphi_1,\ b \mapsto \partial_t \varphi_2,\ c \mapsto \partial_s \varphi_1,\ d \mapsto \partial_s \varphi_2.$$

Thus, we see that $q(\partial_t u_i, \partial_s u_i)$ satisfies the equality (1.10) of Proposition 1.4.2. But as the power series u_i were arbitrary (they only have to satisfy the condition $q_2(\partial_t u_i, \partial_s u_i) \neq 0$) it follows from Proposition 1.4.2, that $q(x_i^{(1,0)}, x_i^{(0,1)})$ is a geometric invariant. Now Theorem 1.4.5 applies and proves the following equality

$$q(x_i^{(1,0)}, x_i^{(0,1)}) = q(\kappa_{i,(1,0)}, \kappa_{i,(0,1)}) = q\left(\frac{x_i^{(1,0)} x_n^{(0,1)} - x_i^{(0,1)} x_n^{(1,0)}}{x_{n-1}^{(1,0)} x_n^{(0,1)} - x_{n-1}^{(0,1)} x_n^{(1,0)}}, \frac{x_i^{(0,1)} x_{n-1}^{(1,0)} - x_i^{(1,0)} x_{n-1}^{(0,1)}}{x_{n-1}^{(1,0)} x_n^{(0,1)} - x_{n-1}^{(0,1)} x_n^{(1,0)}}\right).$$

According to Corollary 1.4.6, we can apply the substitution

$$x_1^{(i,j)} \mapsto x_{n-1}^{(i,j)},\ x_2^{(i,j)} \mapsto x_n^{(i,j)},\ x_{n-1}^{(i,j)} \mapsto x_1^{(i,j)},\ x_n^{(i,j)} \mapsto x_2^{(i,j)},$$

with $i + j = 1$, to the right-hand side of the above equality in order to find the representation of the geometric invariant $q(x_i^{(1,0)}, x_i^{(0,1)})$ in another system of generators. This substitution yields, after renaming the variables, the equality

$$q(x_i, y_i) = q\left(-\frac{f_{2,i}}{f_{1,2}}, \frac{f_{1,i}}{f_{1,2}}\right). \tag{3.5}$$

Finally, we let the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

act on both sides of the equation (3.5) and use the invariance of q to conclude that

$$q(x_i, y_i) = q\left(\frac{f_{1,i}}{f_{1,2}}, \frac{f_{2,i}}{f_{1,2}}\right).$$

□

Notice that the invariant field $\mathbb{C}(\mathbb{x}, \mathbb{y})^{\mathrm{GL}_2}$ is equal to the field of minimal geometric invariants. Here, with "minimal" we mean "of minimal order" with respect to the two derivations ∂_1 and ∂_2 that generate the differential field $F = \mathbb{C}(x_k^{(i,j)} : i, j \in \mathbb{N})$ (see Section 1.4). More precisely, the minimal geometric invariants are the rational functions in $\partial_i x_j^{(0,0)}$, with $j = 1, \ldots, n, i = 1, 2$.

The second proof of the First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$ we present in this chapter, is self-contained and very elementary:

Second Proof of the First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$. Let $q \in \mathbb{C}(\mathbb{x}, \mathbb{y})$ be invariant under the action of $\mathrm{GL}_2(\mathbb{C})$. Then for any matrix

$$G = \frac{1}{ad - bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying $ad - bc \neq 0$ we have the equality

$$q(x_i, y_i) = G \cdot q(x_i, y_i) = q\left(\frac{ax_i + by_i}{ad - bc}, \frac{cx_i + dy_i}{ad - bc}\right). \tag{3.6}$$

Now, the equality (3.6) holds on an Zariski open subset of $\mathbb{C}^{2n+4}$, namely on all $(2n+4)$-tuples $(x_1, y_1, \ldots, x_n, y_n, a, b, c, d)$ for which $ad - bc \neq 0$ and the denominator of q does not vanish. Moreover, since $|\mathbb{C}| = \infty$, it follows by Weyl's principle that we can consider a, b, c, d to be variables. Let us now substitute $a = -y_1, b = x_1, c = -y_2, d = x_2$ into (3.6) and obtain:

$$q(x_i, y_i) = q\left(\frac{f_{1,i}}{f_{1,2}}, \frac{f_{2,i}}{f_{1,2}}\right).$$

This proves that $K(\mathbb{x}, \mathbb{y})^{\mathrm{GL}_2} \subseteq \mathbb{C}\left(\frac{f_{ij}}{f_{k,l}}\right)$. The other inclusion is clear from our considerations before. $\square$

Remark 3.1.1. Whereas the first proof of the First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$ uses the properties of geometric invariants which were proven only over $\mathbb{C}$ (and it is not clear at the moment whether they can be generalized also to other fields (see a list with open problems in Section 5.3)), our second proof is based on Weyl's principle of the irrelevance of algebraic inequalities which applies to (in-)equalities over any infinite field K. Therefore, if we replace $\mathbb{C}$ by an infinite field K in our second proof, the proof will remain vaild and we obtain even the statement for $\mathrm{GL}_2(K)$ for an arbitrary infinite field K.

Let us now move to the First Fundamental Theorem for $\mathrm{SL}_2(\mathbb{C})$. We start with the explanation of the main ideas of the two proofs of FFT for $\mathrm{SL}_2(\mathbb{C})$ presented here.

We will reprove and use Hodge's straightening lemma [Ho43] and draw inspiration from C. De Concini and C. Procesi [dCP76]. Given an invariant polynomial $p \in \mathbb{C}[\mathbb{x}, \mathbb{y}]^{\mathrm{SL}_2}$, one may assume that p is homogeneous in the variables $x_1, \ldots, x_n$, let us say of degree m. We then:

- for the first proof use the First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$,
- for the second proof let a suitably chosen $\mathrm{GL}_2(\mathbb{C})$-matrix act on p

in order to show that $f_{1,2}^m \cdot p \in \mathbb{C}[f_{ij}]$. However, from this one cannot immediately conclude that $p \in \mathbb{C}[f_{ij}]$. The problem here is the fact that relations between the elements of this ring exist. Hence, p does not admit a unique representation as a polynomial in the generators $f_{i,j}$. Therefore, we investigate first the ring $\mathbb{C}[f_{ij}]$, study its structure, when considered as a $\mathbb{C}$-algebra, and construct a $\mathbb{C}$-basis. Only then we can eliminate possible relations and deduce that $p \in \mathbb{C}[f_{ij}]$.

Let us now start with the investigation of the ring $\mathbb{C}[f_{ij}]$. Any product of the form $f_{i_1,j_1} f_{i_2,j_2} \cdots f_{i_m,j_m}$ can be associated with the following diagram

$$\begin{bmatrix} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & j_m \end{bmatrix},$$

where $i_k < j_k$ for all k (using relations (3.2)). If we can permute the columns of the diagram in such a way that $i_1 \leq i_2 \leq \cdots \leq i_m$ and $j_1 \leq j_2 \leq \cdots \leq j_m$, the diagram becomes a standard Young tableau and the corresponding product $f_{i_1,j_1} f_{i_2,j_2} \cdots f_{i_m,j_m}$ is called a *standard product*. Notice, that each product $f_{i_1,j_1} f_{i_2,j_2} \cdots f_{i_m,j_m}$ can be transformed into a sum of standard product just by applying iteratively the Plücker relation.

Lemma 3.1.2 (Straightening lemma)**.** *The monic standard products form a $\mathbb{C}$-basis of $\mathbb{C}[f_{ij}]$.*

This lemma was first proven by Hodge [Ho43], who attributes the idea to consider standard tableaux to Young [Yo28]. However, Hodge's proof is lengthy and so we will provide a simpler argument.

Proof. It follows from the Plücker relations that the monic standard products form a generating system. Consider now the monomial ordering on $\mathbb{N}^{2n}$ given by $x_1 \prec y_1 \prec x_2 \prec \cdots \prec y_n$. In this way, different standard products have different leading monomials which proves their linear independence. $\square$

Hence, any $p \in \mathbb{C}[f_{ij}]$ can be uniquely written as $p = \sum_{\alpha \in I} c_\alpha F_\alpha$, for some index set $I \subseteq (\{1, \ldots, n\} \times \{1, \ldots, n\})^N$ where $N \in \mathbb{N}$, $c_\alpha \in \mathbb{C}$ and the F_α's are standard products.

Lemma 3.1.3. *Let $p = \sum_{\alpha \in I} c_\alpha F_\alpha$ be a $\mathbb{C}$-linear combination of standard products F_α. If for some i, the polynomial p vanishes after the substitution $(x_i, y_i) = (0, 0)$, i.e.,*

$$p|_{(x_i, y_i) = (0,0)} = 0,$$

then each summand F_α is divisible by f_{i,r_α} for some $r_\alpha \in \{1, \ldots, n\}$.

Proof. Let us assume by contradiction that there are standard products $F_{\alpha_1}, \ldots, F_{\alpha_k}, \alpha_j \in I$ which are not divisible by any $f_{i,r}, r \in \{1, \ldots, n\}$. Notice that these standard products satisfy $F_{\alpha_j} = F_{\alpha_j}|_{(x_i, y_i) = (0,0)}$ for each j. Then evaluating p at $(x_i, y_i) = (0, 0)$ gives

$$0 = \sum_{i=1}^{k} c_{\alpha_i} F_{\alpha_i},$$

which contradicts the linear independence of standard products. $\square$

Lemma 3.1.4. *Let $q \in \mathbb{C}[\mathrm{x}, \mathrm{y}]$ be a polynomial satisfying*

$$f_{1,2} \cdot q \in \mathbb{C}[f_{ij}].$$

Then q already belongs to the ring $\mathbb{C}[f_{ij}]$.

Proof. Write $p = f_{1,2} \cdot q$ uniquely as a linear combination of the standard products

$$p = \sum_{\alpha \in I} c_\alpha F_\alpha.$$

We prove that each F_α is divisible by $f_{1,2}$.

Notice that $p|_{(x_1,y_1)=(0,0)} = 0$ and also $p|_{(x_2,y_2)=(0,0)} = 0$. Thus, Lemma 3.1.3 applies and guarantees that each F_α is divisible by some $f_{1,r}$ and some $f_{2,s}$. We claim that one can pick $r = 2$. Assume by contradiction that $F_{\alpha_1}, \dots, F_{\alpha_k}$ are not divisible by $f_{1,2}$ and write

$$p = f_{1,2} \cdot P + \sum_{l=1}^{k} c_{\alpha_l} F_{\alpha_l} \tag{3.7}$$

for some $P \in \mathbb{C}[f_{i,j}]$, a sum of standard products of smaller degree than the degree of p, and some coefficients $c_{\alpha_l} \in \mathbb{C}$. We enlarge now the polynomial ring $\mathbb{C}[\mathbb{x}, \mathbb{y}]$ to $\mathbb{C}[\mathbb{x}, \mathbb{y}, \lambda]$ by adding a new variable λ which will be used for weighting the standard products. The goal now is to carry out a suitable substitution such that, whereas the left-hand side of the equation (3.7) and also the summand $f_{1,2} \cdot P$ of the right-hand side vanish, the sum $\sum_{l=1}^{k} c_{\alpha_l} F_{\alpha_l}$ transforms into another sum of standard weighted products. This would yield a contradiction.

Let λ be a variable. We perform the substitution $^* : (x_1, y_1) \mapsto (\lambda x_2, \lambda y_2)$ under which $f_{1,k} \mapsto \lambda f_{2,k}$ for all $k \geq 2$ and $f_{i,j} \mapsto f_{i,j}$ if $i, j \neq 1$. Obviously, all standard products (and arbitrary polynomials), that are divisible by $f_{1,2}$, become zero after this substitution. Further, each standard product F_{α_l} with the corresponding diagram

$$\begin{bmatrix} \mathbf{1} & \cdots & \mathbf{1} & 2 & \cdots & 2 & i_{b_l+1} & \cdots \\ j_1 & \cdots & j_{a_l} & j_{a_l+1} & \cdots & j_{b_l} & i_{b_l+1} & \cdots \end{bmatrix}$$

transforms under * into the standard product $F_{\alpha_l}|_{(x_1,y_1)=(x_2,y_2)}$, with the corresponding diagram

$$\begin{bmatrix} \mathbf{2} & \cdots & \mathbf{2} & 2 & \cdots & 2 & i_{b_l+1} & \cdots \\ j_1 & \cdots & j_{a_l} & j_{a_l+1} & \cdots & j_{b_l} & i_{b_l+1} & \cdots \end{bmatrix},$$

multiplied by λ^{a_l}, where a_l is the number of columns with 1 on the top in the diagram of F_{α_l}. It follows that the substitution * acts injectively on the standard products that are not divisible by $f_{1,2}$. After applying *, the equality (3.7) becomes

$$0 = p^* = \sum_{l=1}^{k} c_{\alpha_l} F^*_{\alpha_l}.$$

Now, since the substitution * is injective on F_{α_l}'s, we conclude $\sum_{l=1}^{k} c_{\alpha_l} F_{\alpha_l} = 0$. □

Before we move to the proof of the First Fundamental Theorem for $\mathrm{SL}_2(\mathbb{C})$, let us discuss the behaviour of an $\mathrm{SL}_2(\mathbb{C})$-invariant polynomial under the action of $\mathrm{GL}_2(\mathbb{C})$.

Lemma 3.1.5. *Let $p \in \mathbb{C}[\mathbb{x}, \mathbb{y}]^{\mathrm{SL}_2}$ be an invariant polynomial.*

(i) Then each of its homogeneous parts is itself invariant under the action of $\mathrm{SL}_2(\mathbb{C})$.

(ii) If p is homogeneous, then it is homogeneous of an even degree $2m$, for some $m \in \mathbb{N}$. Moreover, p is homogeneous of degree m as a polynomial in $x_1, \dots, x_n$ and the same holds also for p considered as a polynomial in variables $y_1, \dots, y_n$.

Proof. (i) First observe that, if $p \in \mathbb{C}[\mathbb{x}, \mathbb{y}]^{\mathrm{SL}_2}$ is an invariant polynomial, then by the linearity of the action of $\mathrm{SL}_2(\mathbb{C})$ it follows that each of its homogeneous summands must be invariant as well.

(ii) Let $p \in \mathbb{C}[\mathbb{x}, \mathbb{y}]^{\mathrm{SL}_2}$ be homogeneous of some degree k. Let us now consider the matrix

$$S = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

where $t \in \mathbb{C}^*$ is arbitrary. Then, as S is an $\mathrm{SL}_2(\mathbb{C})$-matrix, p stays invariant under its action:

$$p(x_i, y_i) = S \cdot p(x_i, y_i) = p(tx_i, t^{-1}y_i). \tag{3.8}$$

Because $|\mathbb{C}| = \infty$ and this equality holds for all $t \neq 0$, the principle of Weyl applies and we can consider t as a variable. Now, comparison of terms of equal degree on the left- and the right-hand sides of (3.8) shows that each term of p must contain as many variables from the set $\{x_1, \dots, x_n\}$ as from the set $\{y_1, \dots, y_n\}$, if counted with multiplicity. Hence, p is not only homogeneous of an even degree $k = 2m$, for some m, but it is also homogeneous in the variables $x_1, \dots, x_n$ of degree m and also in the variables $y_1, \dots, y_n$ of the same degree.

□

Lemma 3.1.6. *Let $p \in \mathbb{C}[\mathbb{x}, \mathbb{y}]^{\mathrm{SL}_2}$ be a homogeneous invariant polynomial of degree $2m$. Then p is semi-invariant under the action of $\mathrm{GL}_2(\mathbb{C})$ with character $\det(G)^m$, i.e., for any invertible matrix $G \in \mathrm{GL}_2(\mathbb{C})$, p satisfies the equality*

$$G \cdot p = \det(G)^m p.$$

Proof. First notice that according to Lemma 3.1.5, each homogeneous invariant polynomial $p \in \mathbb{C}[\mathbb{x}, \mathbb{y}]^{\mathrm{SL}_2}$ of degree $2m$ is homogeneous in the variables $x_1, \dots, x_n$ of degree m and also homogeneous in the variables $y_1, \dots, y_n$ of degree m.
Let us consider an invertible matrix $G \in \mathrm{GL}_2(\mathbb{C})$ and examine its action on p. We write the matrix in the following way:

$$G = G \cdot \begin{pmatrix} \det(G)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \det(G) & 0 \\ 0 & 1 \end{pmatrix}.$$

Obviously, the product

$$G \cdot \begin{pmatrix} \det(G)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

is an element of $\mathrm{SL}_2(\mathbb{C})$. Hence, as p is invariant under the action of $\mathrm{SL}_2(\mathbb{C})$, the action of G on p reduces to the action of

$$\tilde{G} \coloneqq \begin{pmatrix} \det(G) & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, we obtain the equality

$$\begin{aligned} p(ax_i+by_i, cx_i+dy_i) &= G\cdot p(x_i,y_i) = \tilde{G}\cdot p(x_i,y_i) = p(\det(G)x_i, y_i) = \\ &= \det(G)^m p(x_i,y_i) = (ad-bc)^m p(x_i,y_i) = \qquad (3.9)\\ &= \det(G)^m p(x_i,y_i), \qquad (3.10)\end{aligned}$$

using that p has degree equal to m in the variables $x_1,\dots,x_n$. $\square$

We are now in a position to prove the First Fundamental Theorem for $\mathrm{SL}_2(\mathbb{C})$.

First Proof of the First Fundamental Theorem for $\mathrm{SL}_2(\mathbb{C})$. Let $p \in \mathbb{C}[\mathbb{x},\mathbb{y}]^{\mathrm{SL}_2}$ be an invariant polynomial. Let us w.l.o.g. assume that p is homogeneous of degree $2m$ (use Lemma 3.1.5). Then according to Lemma 3.1.6, for any invertible matrix $G \in \mathrm{GL}_2(\mathbb{C})$, the equation $G\cdot p = \det(G)^m p$ is fulfilled. As $f_{1,2}^m$ is also semi-invariant under the action of $\mathrm{GL}_2(\mathbb{C})$, and even with the same character as p, it follows that

$$q := \frac{p}{f_{1,2}^m}$$

is invariant under the action of $\mathrm{GL}_2(\mathbb{C})$. Now, First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$ applies and gives us the equality

$$\frac{p(x_i,y_i)}{f_{1,2}^m} = q(x_i,y_i) = q\left(\frac{f_{1,i}}{f_{1,2}}, \frac{f_{2,i}}{f_{1,2}}\right) = \frac{p(f_{1,i},f_{2,i})}{f_{1,2}^{2m}},$$

using that p is homogeneous of degree $2m$. Multiplying both sides of this equality with $f_{1,2}^{2m}$ yields

$$f_{1,2}^m \cdot p = p(f_{1,i}, f_{2,i}).$$

Now, Lemma 3.1.4 applies and finishes the proof. $\square$

Notice, that in this proof we used the First Fundamental Theorem for $\mathrm{GL}_2(\mathbb{C})$. However, we will present also a second proof of the First Fundamental Theorem for $\mathrm{SL}_2(\mathbb{C})$ which is elementary and independent of the FFT for $\mathrm{GL}_2(\mathbb{C})$.

Second Proof of the First Fundamental Theorem for $\mathrm{SL}_2(\mathbb{C})$. If $p \in \mathbb{C}[\mathbb{x},\mathbb{y}]^{\mathrm{SL}_2}$ is an invariant polynomial, then according to Lemma 3.1.5 we can w.l.o.g. consider it as a homogeneous polynomial of degree $2m$ for some $m \in \mathbb{N}$. Further, according to Lemma 3.1.6, p satisfies the equality

$$G\cdot p = \det(G)^m p = (ad-bc)^m p,$$

where $G \in \mathrm{GL}_2(\mathbb{C})$ is an arbitrary invertible matrix and $a,b,c,d \in \mathbb{C}$ its entries. This equality holds on the Zariski open set

$$\{(x_1,y_1,\dots,x_n,y_n,a,b,c,d) \in \mathbb{C}^{2n+4} : ad-bc \neq 0\}.$$

As $|\mathbb{C}| = \infty$, Weyl's principle applies and we can consider a,b,c,d as variables. After substituting $a=-y_1, b=x_1, c=-y_2, d=x_2$, the equality (3.9) becomes

$$f_{1,2}^m \cdot p(x_i,y_i) = p\left(f_{1,i}, f_{2,i}\right) \in \mathbb{C}[\mathbb{x},\mathbb{y}]. \qquad (3.11)$$

Now, the claim follows by Lemma 3.1.4. $\square$

Remark 3.1.7. Firstly, remark that proofs of Lemma 3.1.2, Lemma 3.1.3, Lemma 3.1.4, Lemma 3.1.5, Lemma 3.1.6 remain all valid when replacing $\mathbb{C}$ by an arbitrary infinite field K.
Secondly, due to the fact that our first proof of the FFT for $\mathrm{SL}_2(\mathbb{C})$ is based on these lemmata and on the FFT for $\mathrm{GL}_2(\mathbb{C})$, whose validity over an arbitrary infinite field was justified as well (see Remark 3.1.1), our first proof of FFT for $\mathrm{SL}_2(\mathbb{C})$ can be applied to any infinite field K.
Thirdly, since our second proof of FFT for $\mathrm{SL}_2(\mathbb{C})$ uses additionally only Weyl's princpile of the irrelevance of algebraic inequalities which applies also to (in-)equalities over any infinite field K, also this proof is valid over K with $|K| = \infty$.
Thus we conclude, that we have even two proofs for the First Fundamental Theorem for $\mathrm{SL}_2(K)$, where K is an infinite field.

Let K be an infinite field. Notice that the equality (3.11) also shows that each invariant polynomial p belongs to the intersection $K[\mathbb{x}, \mathbb{y}] \cap K(f_{1,i}, f_{2,i})$, where $K(f_{1,i}, f_{2,i}) = K(f_{1,i}, f_{2,i} : i = 3, \ldots, n)$. Thus, one could intuitively think that p does not only lie in the polynomial ring $K[f_{i,j}]$ but that it belongs already to the subring $K[f_{1,i}, f_{2,i}] = K[f_{1,i}, f_{2,i} : i = 3, \ldots, n]$. However, the inclusion $K[\mathbb{x}, \mathbb{y}] \cap K(f_{1,i}, f_{2,i}) \subseteq K[f_{1,i}, f_{2,i}]$ is wrong. For example, because of the Plücker relation, the polynomial $f_{3,4}$ can be written as

$$f_{3,4} = \frac{f_{1,3}f_{2,4} - f_{1,4}f_{2,3}}{f_{1,2}}.$$

Therefore, $f_{3,4}$ is obviously an element of $K[\mathbb{x}, \mathbb{y}] \cap K(f_{1,i}, f_{2,i})$ but there is no reason for it to be contained in $K[f_{1,i}, f_{2,i}]$. In fact, we can easily show the following equality first observed by Igusa in [Ig54, Theorem 3]:

Corollary 3.1.8 (Igusa)**.** *It holds that* $K(f_{ij}) \cap K[\mathbb{x}, \mathbb{y}] = K[f_{ij}]$.

Proof. The First Fundamental Theorem ensures that $K(f_{ij}) \cap \mathbb{C}[\mathbb{x}, \mathbb{y}] \subseteq K[\mathbb{x}, \mathbb{y}]^{\mathrm{SL}_2} = K[f_{ij}]$. The other inclusion is obvious. □

We have already proven how the coordinate ring of $(\mathbb{P}^1_{\mathbb{C}})^n / \mathrm{SL}_2(\mathbb{C})$ is generated as a $\mathbb{C}$-algebra, namely by the polynomials $f_{i,j}$. We can now also conclude with the First Fundamental Theorems for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{C})$ that the invariant field $\mathbb{C}(\mathbb{x}, \mathbb{y})^{\mathrm{GL}_2}$ contains the function field of $(\mathbb{P}^1_{\mathbb{C}})^n / \mathrm{SL}_2(\mathbb{C})$.

Corollary 3.1.9. *The function field of* $(\mathbb{P}^1_{\mathbb{C}})^n / \mathrm{SL}_2(\mathbb{C})$ *is a subfield of the invariant field* $\mathbb{C}(\mathbb{x}, \mathbb{y})^{\mathrm{GL}_2}$.

Proof. Notice first that each element q in the function field of $(\mathbb{P}^1_{\mathbb{C}})^n / \mathrm{SL}_2(\mathbb{C})$ is a rational function with numerator and denominator both invariant under the action of $\mathrm{SL}_2(\mathbb{C})$. Due to the fact that q defines a function on $(\mathbb{P}^1_{\mathbb{C}})^n$, its numerator and denominator of must be both homogeneous of the same degree (even multihomogeneous), otherwise q would not be well defined . Let us denote their degree by m. Further, Lemma 3.1.6 tells us that both, the numerator and the denominator of q, are semi-invariant under the action of $\mathrm{GL}_2(\mathbb{C})$ and with character $\det(G)^m$. Thus, q itself is invariant under $\mathrm{GL}_2(\mathbb{C})$. □

Counterexamples to the First Fundamental Theorems over finite fields

Note that the statements of the First Fundamental Theorems for $\mathrm{GL}_2(K)$ and $\mathrm{SL}_2(K)$ are wrong for any finite field K. In fact, for any prime power q, the polynomials

$$p_i := x_i^q y_i - x_i y_i^q, \quad 1 \leq i \leq n,$$

are semi-invariant under the action of $\mathrm{GL}_2(\mathbb{F}_q)$, i.e., we have

$$G \cdot p_i = \det(G) p_i. \tag{3.12}$$

Therefore, it follows that $p_i \in \mathbb{F}_q[\mathtt{x}, \mathtt{y}]^{\mathrm{SL}_2}$ and $p_i/p_j \in \mathbb{F}_q(\mathtt{x}, \mathtt{y})^{\mathrm{GL}_2}$, for $i \neq j$. However, it is obvious that $p_i \notin \mathbb{F}_q[f_{ij}]$, showing that $\mathbb{F}_q(f_{ij}) \subsetneq \mathbb{F}_q(\mathtt{x}, \mathtt{y})^{\mathrm{GL}_2}$ and $\mathbb{F}_q[f_{ij}] \subsetneq \mathbb{F}_q[\mathtt{x}, \mathtt{y}]^{\mathrm{SL}_2}$.

3.2 First Fundamental Theorem for $\mathrm{SL}_m(K)$ and $\mathrm{GL}_m(K)$

The goal of this section is to extend the techniques and methods developed in the previous section to the ring

$$\mathbb{C}[\mathtt{x}] := \mathbb{C}[x_{i,j} : i = 1, \ldots, m; j = 1, \ldots, n]$$

and its quotient field $\mathbb{C}(\mathtt{x}) := \mathrm{Quot}(\mathbb{C}[\mathtt{x}])$ in order to investigate the correspoding invariant ring and field under the actions of $\mathrm{SL}_m(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{C})$, respectively.

To keep the notation simple, we will denote the set $\{1, \ldots, n\}$ shortly by $[n]$ and we will write $q(x_{i,j})$ for each element $q(x_{i,j} : i \in [m], j \in [n]) \in \mathbb{C}(\mathtt{x})$. We let $\mathrm{GL}_m(\mathbb{C})$ act on the ring $\mathbb{C}[\mathtt{x}]$ from the right by the usual matrix-vector multiplication on each of the column vectors $(x_{1,j}, \ldots, x_{m,j})^T$, $j \in [n]$ and extend this to an action on the field $\mathbb{C}(\mathtt{x})$. By $\mathbb{C}(\mathtt{x})^{\mathrm{GL}_m}$ we denote the corresponding invariant field. Obviously, the constant polynomials are the only polynomial invariants under the action of $\mathrm{GL}_m(\mathbb{C})$, i.e., we have $\mathbb{C}[\mathtt{x}]^{\mathrm{GL}_m} = \mathbb{C}$. Hence, we consider the action of $\mathrm{SL}_m(\mathbb{C})$ on the polynomial ring $\mathbb{C}[\mathtt{x}]$, which gives us already more flexibility for the construction of invariant elements, and study the structure of the invariant ring $\mathbb{C}[\mathtt{x}]^{\mathrm{SL}_m}$.

Let us mention, that the invariant ring $\mathbb{C}[\mathtt{x}]^{\mathrm{SL}_m}$ is again equal to the coordinate ring of the GIT quotient $(\mathbb{P}_{\mathbb{C}}^{m-1})^n / \mathrm{SL}_m(\mathbb{C})$, namely the moduli space of n points on the $(m-1)$-dimensional projective space. And we will also see later that the function field of $(\mathbb{P}_{\mathbb{C}}^{m-1})^n / \mathrm{SL}_m(\mathbb{C})$ is a subfield of $\mathbb{C}(\mathtt{x})^{\mathrm{GL}_m}$.

To each vector $(i_1, \ldots, i_m)$, with $1 \leq i_j \leq n$, we associate the matrix

$$X_{i_1,\ldots,i_m} = \begin{pmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_m} \\ x_{2,i_1} & x_{2,i_2} & \cdots & x_{2,i_m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,i_1} & x_{m,i_2} & \cdots & x_{m,i_m} \end{pmatrix}$$

and set

$$f_{i_1,\ldots,i_m} := \det(X_{i_1,\ldots,i_m}) \in \mathbb{C}[\mathtt{x}].$$

By $\mathbb{C}[f_{i_1,\dots,i_m}]$ we denote the polynomial ring generated over $\mathbb{C}$ by all $f_{i_1,\dots,i_m}$, $1 \leq i_j \leq n$, and by $\mathbb{C}(f_{i_1,\dots,i_m})$ its quotient field. Obviously, the following relations between the polynomials $f_{i_1,\dots,i_m}$'s are satisfied:

$$f_{i_1,\dots,i_m} = 0 \ \text{ if } i_j = i_k \text{ for some } j \neq k, \tag{3.13}$$

$$f_{i_1,\dots,i_m} = \operatorname{sgn}(\sigma) f_{i_{\sigma(1)},\dots,i_{\sigma(m)}} \ \text{ for any permutation } \sigma \in S_m, \tag{3.14}$$

where $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. We also have the *Plücker relation*

$$\sum_{l=1}^{m} (-1)^l f_{i_1,\dots,i_{m-1},j_l} \cdot f_{j_1,\dots,j_{l-1},j_{l+1},\dots,j_{m+1}} = 0. \tag{3.15}$$

For a short proof of Plücker relations see for instance the paper by W. V. D. Hodge [Ho43, p. 24]. Further, for any invertible matrix $G \in \mathrm{GL}_m(\mathbb{C})$, the following equality is satisfied:

$$G \cdot f_{i_1,\dots,i_m} = \det(G) f_{i_1,\dots,i_m}.$$

Thus, each polynomial $f_{i_1,\dots,i_m}$ is *semi-invariant* under the action of $\mathrm{GL}_m(\mathbb{C})$ and invariant under the action of $\mathrm{SL}_m(\mathbb{C})$ justifying the inclusions

$$\mathbb{C}\left(\frac{f_{i_1,\dots,i_m}}{f_{j_1,\dots,j_m}} \ : \ 1 \leq i_k, j_l \leq n\right) \subseteq \mathbb{C}(\mathbb{x})^{\mathrm{GL}_m} \text{ and } \mathbb{C}[f_{i_1,\dots,i_m} \ : \ 1 \leq i_k \leq n] \subseteq \mathbb{C}[\mathbb{x}]^{\mathrm{SL}_m}.$$

Let us denote the field $\mathbb{C}\left(\frac{f_{i_1,\dots,i_m}}{f_{j_1,\dots,j_m}} \ : \ 1 \leq i_k, j_l \leq n\right)$ and the ring $\mathbb{C}[f_{i_1,\dots,i_m} \ : \ 1 \leq i_k \leq n]$ shortly by $\mathbb{C}\left(\frac{f_{i_1,\dots,i_m}}{f_{j_1,\dots,j_m}}\right)$ and $\mathbb{C}[f_{i_1,\dots,i_m}]$, respectively. To understand their structure and to prove the First Fundamental Theorems for $\mathrm{SL}_m(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{C})$ is the aim of this section.

First Fundamental Theorem for $\mathrm{GL}_m(\mathbb{C})$. *An element $q \in \mathbb{C}(\mathbb{x})$ is invariant under the action of $\mathrm{GL}_m(\mathbb{C})$ if and only if q can be written as a rational function in $\frac{f_{i_1,\dots,i_m}}{f_{j_1,\dots,j_m}}$, $1 \leq i_k, j_l \leq n$, i.e.,*

$$\mathbb{C}(\mathbb{x})^{\mathrm{GL}_m} = \mathbb{C}\left(\frac{f_{i_1,\dots,i_m}}{f_{j_1,\dots,j_m}}\right).$$

Moreover, an invariant q admits the following representation in the generators $\frac{f_{i_1,\dots,i_m}}{f_{j_1,\dots,j_m}}$'s:

$$q(x_{i,j}) = q\left(\frac{f_{1,\dots,i-1,j,i+1,\dots,m}}{f_{1,\dots,m}}\right).$$

First Fundamental Theorem for $\mathrm{SL}_m(\mathbb{C})$. *An element $p \in \mathbb{C}[\mathbb{x}]$ is invariant under the action of $\mathrm{SL}_m(\mathbb{C})$ if and only if p can be written as a polynomial in $f_{i_1,\dots,i_m}$, $1 \leq i_j \leq n$, i.e.,*

$$\mathbb{C}[\mathbb{x}]^{\mathrm{SL}_m} = \mathbb{C}[f_{i_1,\dots,i_m}].$$

Proofs of the First Fundamental Theorems

Let us again start with the proofs of the First Fundamental Theorem for $\mathrm{GL}_m(\mathbb{C})$. In our first proof we use exactly the same strategy as in the first proof of the FFT for $\mathrm{GL}_2(\mathbb{C})$, just extended to the larger set of variables. Namely, we interpret invariants under the action of $\mathrm{GL}_m(\mathbb{C})$ again as minimal geometric invariants.

First Proof of the First Fundamental Theorem for $\mathrm{GL}_m(\mathbb{C})$. Let $q = \frac{q_1}{q_2} \in \mathbb{C}(\mathbb{x})^{\mathrm{GL}_m}$ be an invariant rational function. Further, let us consider n power series $u_i \in \mathbb{C}[\![t_1, \ldots, t_m]\!]$, with $i = 1, \ldots, n$ in m variables. Using the same arguments as in the first proof of the FFT for $\mathrm{GL}_2(\mathbb{C})$, we show that $q(\partial_{t_i} u_j)$ satisfies equation (1.10) from Proposition 1.4.2 which means that $q(x_j^{e_i}), i \in [m], j \in [n]$ is a geometric invariant. Here, e_i denotes the i-th standard basis vector. Now, we conclude with Theorem 1.4.5 that q satisfies the equality

$$q(x_j^{e_i}) = q(\kappa_{j,e_i}),$$

which after renaming the variables can be read as

$$q(x_{i,j}) = q\left(\frac{f_{n-m+1,\ldots,n-m+i-1,j,n-m+i+1,\ldots,n}}{f_{n-m+1,\ldots,n}}\right).$$

Now, Corollary 1.4.6 applies and justifies the equality

$$q(x_{i,j}) = q\left(\frac{f_{1,\ldots,i-1,j,i+1,\ldots,m}}{f_{1,\ldots,m}}\right).$$

□

As in the situation of the FFT for $\mathrm{GL}_2(\mathbb{C})$, also here the first proof of the FFT for $\mathrm{GL}_m(\mathbb{C})$ cannot be extended to a proof over fields different from $\mathbb{C}$. The reason is the same as in the case of $\mathrm{GL}_2(\mathbb{C})$, namely the fact that we defined geometric invariants and proved their basic properties only over $\mathbb{C}$. We put the question about possible generalizations of this proof also to infinity fields on a list with open questions, see Section 5.3. We present now a second proof of the FFT for $\mathrm{GL}_m(\mathbb{C})$ which is valid over any infinite field K.

Second Proof of the First Fundamental Theorem for $\mathrm{GL}_m(\mathbb{C})$. Let $q \in \mathbb{C}(\mathbb{x})$ be invariant under the action of $\mathrm{GL}_m(\mathbb{C})$. Then for any invertible matrix $G \in \mathrm{GL}_m(\mathbb{C})$ we have the equality

$$q(x_{i,j}) = G \cdot q(x_{i,j}) = q\left(\left(G \cdot (x_{1,j}, \ldots, x_{m,j})^T\right)_i\right), \tag{3.16}$$

where $(G \cdot (x_{1,j}, \ldots, x_{m,j})^T)_i$ means the i-th entry of the vector that we obtain after performing the matrix-vector multiplication $G \cdot (x_{1,j}, \ldots, x_{m,j})^T$. Since G is an arbitrary invertible matrix, as in the second proof of FFT for $\mathrm{GL}_2(\mathbb{C})$ we conclude with Weyl's principle that the entries $g_{i,j}$ of the matrix G can be considered as variables in the equality (3.16). Now, we perform for G the substitution

$$G \mapsto \frac{1}{f_{1,\ldots,m}} \cdot \mathrm{adj}(X_{1,\ldots,m}), \tag{3.17}$$

where $\text{adj}(X_{1,\dots,m})$ denotes the adjoint of the matrix $X_{1,\dots,m}$. Let us denote the cofactor of the (j,i) entry of the matrix $X_{1,\dots,m}$ by $(X_{1,\dots,m})_{(j,i)}$. After the substitution (3.17), each

$$(G \cdot (x_{1,j}, \dots, x_{m,j})^T)_i = \sum_{l=0}^{m} g_{i,l} x_{l,j}$$

becomes

$$\frac{1}{f_{1,\dots,m}} \sum_{l=0}^{m} (-1)^{i+l} \det\big((X_{1,\dots,m})_{(l,i)}\big) x_{l,j} = \frac{1}{f_{1,\dots,m}} \det(X_{1,\dots,i-1,j,i+1,\dots,m}) = \frac{f_{1,\dots,i-1,j,i+1,\dots,m}}{f_{1,\dots,m}}.$$

□

In order to prove the First Fundamental Theorem for $\mathrm{SL}_m(\mathbb{C})$, we have again to study the properties of the ring $\mathbb{C}[f_{i_1,\dots,i_m}]$. We start with the introduction of standard products in this larger set of variables.

To any product $f_{i_{1,1},\dots,i_{m,1}} \cdots f_{i_{1,k},\dots,i_{m,k}}$ we associate the following diagram

$$\begin{bmatrix} i_{1,1} & i_{1,2} & \cdots & i_{1,k} \\ i_{2,1} & i_{2,2} & \cdots & i_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ i_{m,1} & i_{m,2} & \cdots & i_{m,k} \end{bmatrix},$$

with $i_{k,j} < i_{l,j}$ for all j and $k < l$ (using relations (3.13) and (3.14)). We call the product $f_{i_{1,1},\dots,i_{m,1}} \cdots f_{i_{1,k},\dots,i_{m,k}}$ a *standard product* if we manage to permute the columns of the diagram in such a way that $i_{j,1} \le i_{j,2} \le \cdots \le i_{j,m}$ holds for all j, i.e., if with the permutation of columns we manage to transform the diagram to a standard Young tableau. Notice, that by applying iteratively the Plücker relation, each product $f_{i_{1,1},\dots,i_{m,1}} \cdots f_{i_{1,k},\dots,i_{m,k}}$ can be transformed into a sum of standard product (for more details see [Ho43, p. 25]).

We prove now the Straightening lemma (compare with Lemma 3.1.2), which implies that any element $p \in \mathbb{C}[f_{i_1,\dots,i_m}]$ can be uniquely written as a linear combination of standard products. The Straightening lemma will be again crucial for the proof of the FFT for $\mathrm{SL}_m(\mathbb{C})$, in this larger set of variables.

Lemma 3.2.1 (Straightening lemma)**.** *The monic standard products form a $\mathbb{C}$-basis of the ring $\mathbb{C}[f_{i_1,\dots,i_m}]$.*

Proof. The proof goes along the same line as the proof of Lemma 3.1.2. Firstly, that the monic standard products form a generating system of the ring $\mathbb{C}[f_{i_1,\dots,i_m}]$, is clear from the Plücker relations. Secondly, that they are linearly independent over $\mathbb{C}$ can be shown again by using the monomial ordering $\prec$ on $\mathbb{N}^{mn}$ given by

$$x_{1,1} \prec x_{2,1} \prec \cdots \prec x_{m,1} \prec x_{1,2} \prec x_{2,2} \prec \cdots \prec x_{m,2} \prec \cdots \prec x_{m,m}.$$

According to this ordering, different standard products have different leading monomials. □

Using now the same argument as in the proof of Lemma 3.1.3 in combination with the Straightening lemma 3.2.1, this gives us the following:

Lemma 3.2.2. *Let $p = \sum_{\alpha \in I} c_\alpha F_\alpha$ be a $\mathbb{C}$-linear combination of standard products F_α. If for some i, the polynomial p vanishes after the substitution $(x_{1,j}, \ldots, x_{m,j}) = (0, \ldots, 0)$, i.e.,*

$$p|_{(x_{i,j})_{i=1}^m = 0} = 0,$$

then each summand F_α is divisible by $f_{j, r_{2,\alpha}, \ldots, r_{m,\alpha}}$ for some $r_{2,\alpha}, \ldots, r_{m,\alpha} \in \{1, \ldots, n\}$.

Lemma 3.2.3. *Let $q \in \mathbb{C}[\mathbb{x}]$ be a polynomial satisfying*

$$f_{1,\ldots,m} \cdot q \in \mathbb{C}[f_{i_1,\ldots,i_m}].$$

Then q already belongs to the ring $\mathbb{C}[f_{i_1,\ldots,i_m}]$.

Proof. If we write $p = f_{1,\ldots,m} \cdot q$ uniquely as a linear combination of the standard products

$$p = \sum_{\alpha \in I} c_\alpha F_\alpha,$$

then using Lemma 3.2.2, we show by the same argument as in the proof Lemma 3.1.4 that each F_α is divisible by $f_{j, r_{2,j}, \ldots, r_{m,j}}$ for all $j = 1, \ldots, m$ and some $r_{i,j} \in [n]$. The goal is to show that for each $k = 2, \ldots, m$, we can always take $r_{k,j} = k$, which would mean that each F_α is divisible by $f_{1,\ldots,m}$. We assume by contradiction that $F_{\alpha_1}, \ldots, F_{\alpha_s}$ are not divisible by $f_{1,\ldots,m}$ and write

$$p = f_{1,\ldots,m} \cdot P + \sum_{l=1}^{s} c_{\alpha_l} F_{\alpha_l} \tag{3.18}$$

for some $P \in \mathbb{C}[f_{i_1,\ldots,i_m}]$, a sum of standard products of smaller degree than the degree of p, and some coefficients $c_{\alpha_l} \in \mathbb{C}$. Let us enlarge the polynomial ring $\mathbb{C}[\mathbb{x}]$ to $\mathbb{C}[\mathbb{x}, \lambda_2, \ldots, \lambda_m]$ by adding new variables $\lambda_2, \ldots, \lambda_m$ and consider the following substitution

$$^* : (x_{1,1}, \ldots, x_{m,1}) \mapsto \sum_{j=2}^{m} \lambda_j (x_{1,j}, \ldots, x_{m,j}).$$

Obviously, each polynomial $f_{i_1,\ldots,i_m}$ transforms under * as follows:

$$\begin{aligned} f_{1,\ldots,m} &\mapsto 0 \\ f_{1,i_2,\ldots,i_m} &\mapsto \sum_{j=2}^{m} \lambda_j f_{j,i_2,\ldots,i_m} \\ f_{i_1,\ldots,i_m} &\mapsto f_{i_1,\ldots,i_m} \quad \text{for all } 1 < i_1, \ldots, i_m. \end{aligned}$$

Moreover, we show using the trick by C. de Concini and C. Procesi [dCP76, p. 333-334] that * acts injectively on standard products that are not divisible by $f_{1,\ldots,m}$: Each such a standard product

$$\mathrm{F}_\alpha = f_{i_{1,1},\ldots,i_{m,1}} \cdots f_{i_{1,k},\ldots,i_{m,k}},$$

whose corresponding diagram we denote by D_{F_α}, transforms under * into the sum of

$$\lambda_2^{h_2}\cdots\lambda_m^{h_m}(f_{i_{1,1},\ldots,i_{m,1}}|_{(x_{j,1})_{j=1}^m=(x_{j,a_1})_{j=1}^m}\cdots f_{i_{1,k},\ldots,i_{m,k}}|_{(x_{j,1})_{j=1}^m=(x_{j,a_k})_{j=1}^m}),$$

where $a_1,\ldots,a_k\in\{2,\ldots,m\}$. Let us look at the summand $\lambda_2^{h_2}\cdots\lambda_m^{h_m}\overline{F}_\alpha$ of the maximal degree $(h_2,\ldots,h_m)$ in $\lambda_2,\ldots,\lambda_m$ with respect to the ordering $\lambda_m\prec\cdots\prec\lambda_2$ on $\mathbb{N}^{m-1}$, study its corresponding diagram $D_{\overline{F}_\alpha}$ and explain its connection to the diagram D_{F_α} of the standard product F_α. Obviously,

$$\begin{aligned}
h_2 = &\text{ number of columns in } D_{F_\alpha} \text{ which start with } 1\ \nu,\ \text{with } \nu>2,\\
h_3 = &\text{ number of columns in } D_{F_\alpha} \text{ which start with } 1\ 2\ \nu,\ \text{with } \nu>3,\\
\vdots&\\
h_m = &\text{ number of columns in } D_{F_\alpha} \text{ which start with } 1\ 2\ \ldots\ m-1\ \nu,\ \text{with } \nu>m.
\end{aligned}$$

Therefore, after replacing each 1 by j in the h_j rows starting with $1\ 2\ \ldots\ j-1\ \nu,\nu>j$, of the diagram D_{F_α} of F_α

$$\begin{bmatrix}
\overbrace{\begin{matrix}1 & \cdots & 1\end{matrix}}^{h_m} & \overbrace{\begin{matrix}1 & \cdots & 1\end{matrix}}^{h_{m-1}} & \cdots & \overbrace{\begin{matrix}1 & \cdots & 1\end{matrix}}^{h_3} & \overbrace{\begin{matrix}1 & \cdots & 1\end{matrix}}^{h_2} & \cdots\\
\begin{matrix}2 & \cdots & 2\end{matrix} & \begin{matrix}2 & \cdots & 2\end{matrix} & \cdots & \begin{matrix}2 & \cdots & 2\end{matrix} & \begin{matrix}* & \cdots & *\end{matrix} & \cdots\\
\begin{matrix}3 & \cdots & 3\end{matrix} & \begin{matrix}3 & \cdots & 3\end{matrix} & \cdots & \begin{matrix}* & \cdots & *\end{matrix} & \begin{matrix}* & \cdots & *\end{matrix} & \cdots\\
\begin{matrix}\vdots & & \vdots\end{matrix} & \begin{matrix}\vdots & & \vdots\end{matrix} & & \begin{matrix}\vdots & & \vdots\end{matrix} & \begin{matrix}\vdots & & \vdots\end{matrix} & \\
\begin{matrix}m-2 & \cdots & m-2\end{matrix} & \begin{matrix}m-2 & \cdots & m-2\end{matrix} & \cdots & \begin{matrix}* & \cdots & *\end{matrix} & \begin{matrix}* & \cdots & *\end{matrix} & \cdots\\
\begin{matrix}m-1 & \cdots & m-1\end{matrix} & \begin{matrix}* & \cdots & *\end{matrix} & \cdots & \begin{matrix}* & \cdots & *\end{matrix} & \begin{matrix}* & \cdots & *\end{matrix} & \cdots\\
\begin{matrix}* & \cdots & *\end{matrix} & \begin{matrix}* & \cdots & *\end{matrix} & \cdots & \begin{matrix}* & \cdots & *\end{matrix} & \begin{matrix}* & \cdots & *\end{matrix} & \cdots\\
\begin{matrix}\vdots & & \vdots\end{matrix} & \begin{matrix}\vdots & & \vdots\end{matrix} & & \begin{matrix}\vdots & & \vdots\end{matrix} & \begin{matrix}\vdots & & \vdots\end{matrix} &
\end{bmatrix},$$

the first row of D_{F_α} transforms into

$$(\overbrace{m\ \cdots\ m}^{h_m}\ \overbrace{m-1\ \cdots\ m-1}^{h_{m-1}}\ \cdots\ \overbrace{3\ \cdots\ 3}^{h_3}\ \overbrace{2\ \cdots\ 2}^{h_2}\ \cdots).$$

This arrangement no longer looks like an arrangement of a diagram corresponding to a standard product. However, it follows from the shape of the diagram D_{F_α} that we can turn this according to the relations (3.14) into a standard Young tableau. Namely, by moving to the top the block of the second up to the $(j-1)$ - st row of the h_j columns starting now with $j\ 2\ 3\ \ldots\ j-1\ \nu,\nu>j$ and by moving the first row $j\ \ldots\ j$ in these h_j columns to the $(j-1)$ - st row. After performing this transformation for each $j=2,\ldots,m$, we obtain the diagram $D_{\overline{F}_\alpha}$

$$
\begin{array}{c}
\overbrace{\qquad\qquad\qquad}^{h_m}\qquad\quad\overbrace{\qquad\qquad\qquad}^{h_{m-1}}\qquad\qquad\qquad\overbrace{\qquad\quad}^{h_3}\ \overbrace{\qquad\quad}^{h_2}\qquad\quad \\
\begin{bmatrix}
2 & \cdots & 2 & 2 & \cdots & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 & \cdots \\
3 & \cdots & 3 & 3 & \cdots & 3 & \cdots & 3 & \cdots & 3 & * & \cdots & * & \cdots \\
4 & \cdots & 4 & 4 & \cdots & 4 & \cdots & * & \cdots & * & * & \cdots & * & \cdots \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \\
m-1 & \cdots & m-1 & m-1 & \cdots & m-1 & \cdots & * & \cdots & * & * & \cdots & * & \cdots \\
m & \cdots & m & * & \cdots & * & \cdots & * & \cdots & * & * & \cdots & * & \cdots \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots &
\end{bmatrix},
\end{array}
$$

which is the diagram of $\overline{F}_\alpha$. Therefore, $\overline{F}_\alpha$ is a standard product. Notice that, we can easily go back from $D_{\overline{F}_\alpha}$ to D_{F_α}. Thus, it follows that two different standard products $F_\alpha \neq F_\beta$ that are both not divisible by $f_{1,\ldots,m}$, have after the substitution * different standard products $\overline{F}_\alpha \neq \overline{F}_\beta$ of maximal degree in $\lambda_2, \ldots, \lambda_m$ and the injectivity of * follows.

Applying now * to the equality (3.18) yields a linear relation between the standard products of maximal degree in $\lambda_2, \ldots, \lambda_m$, which must be trivial as the standard products are linearly independent. The claim follows now by the injectivity of *. □

We need two other technical lemmata in order to prove the First Fundamental Theorem for $\mathrm{SL}_m(\mathbb{C})$.

Lemma 3.2.4. *Let $p \in \mathbb{C}[\mathbb{x}]^{\mathrm{SL}_m}$ be an invariant polynomial.*

(i) Then each of its homogeneous parts is itself invariant under the action of $\mathrm{SL}_m(\mathbb{C})$.

(ii) If p is homogeneous, then it is homogeneous of degree ml, for some $l \in \mathbb{N}$. Moreover, p is homogeneous of degree l as a polynomial in $x_{j,1}, \ldots, x_{j,n}$ for each $j = 1, \ldots, m$.

Proof. Once we have defined the matrix

$$
S = \begin{pmatrix}
t_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & t_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & t_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & 0 & \cdots & t_{n-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & (t_1 \cdots t_{n-1})^{-1}
\end{pmatrix}
$$

the proof goes along the same line as the proof of Lemma 3.1.5. □

Using Lemma 3.2.4 and the argument in the proof of Lemma 3.1.6, we obtain the following generalization of Lemma 3.1.6:

Lemma 3.2.5. *Let $p \in \mathbb{C}[\mathbb{x}]^{\mathrm{SL}_m}$ be a homogeneous invariant polynomial of degree ml. Then p is semi-invariant under the action of $\mathrm{GL}_m(\mathbb{C})$ with character $\det(G)^l$, i.e., for any $G \in \mathrm{GL}_m(\mathbb{C})$, p satisfies the equality*

$$G \cdot p = \det(G)^l p.$$

Finally, we move to the proof of the First Fundamental Theorem for $\mathrm{SL}_m(\mathbb{C})$:

First Proof of the First Fundamental Theorem for $\mathrm{SL}_m(\mathbb{C})$. Let us w.l.o.g consider a homogeneous $\mathrm{SL}_m(\mathbb{C})$-invariant polynomial p of degree ml. Use Lemma 3.2.4 and Lemma 3.2.5 and construct from p the following $\mathrm{GL}_m(\mathbb{C})$-invariant rational function:

$$q := \frac{p}{f_{1,\ldots,m}^l}.$$

Now the First Fundamental Theorem for $\mathrm{GL}_m(\mathbb{C})$ applies and shows the equality

$$f_{1,\ldots,m}^l \cdot p \in \mathbb{C}[f_{i_1,\ldots,i_m}].$$

The claim follows now from Lemma 3.2.3. □

Second Proof of the First Fundamental Theorem for $\mathrm{SL}_m(\mathbb{C})$. Setting $G := \mathrm{adj}(X_{1,\ldots,m})$, where $\mathrm{adj}(X_{1,\ldots,m})$ denotes the adjoint of $X_{1,\ldots,m}$ (notice that $\det(\mathrm{adj}(X_{1,\ldots,m})) = f_{1,\ldots,m}^{m-1}$), and using the same strategy as in the second proof of FFT for $\mathrm{SL}_2(\mathbb{C})$ proves the FFT for $\mathrm{SL}_m(\mathbb{C})$. □

Let us point out that both our proofs work over any infinite field K as well (for the reason as in the case of $\mathrm{SL}_2(\mathbb{C})$). Further, we conclude Igusa's Theorem [Ig54, Theorem 3] over infinite fields:

Corollary 3.2.6 (Igusa)**.** *It holds that* $K(f_{i_1,\ldots,i_m}) \cap K[\mathbb{x}] = K[f_{i_1,\ldots,i_m}]$.

Further, using the fact that each element p of the function field of $(\mathbb{P}_{\mathbb{C}}^{m-1})^n/\mathrm{SL}_m(\mathbb{C})$ has (as a function on the projective space $\mathbb{P}_{\mathbb{C}}^{m-1}$) homogeneous denominator and numerator of the same degree and using the fact the the numerator and denominator are both $\mathrm{SL}_m(\mathbb{C})$-invariant, we conclude with Lemma 3.2.5 that p is even invariant under the action of $\mathrm{GL}_m(\mathbb{C})$:

Corollary 3.2.7. *The function field of* $(\mathbb{P}_{\mathbb{C}}^{m-1})^n/\mathrm{SL}_m(\mathbb{C})$ *is a subfield of the invariant field* $\mathbb{C}(\mathbb{x})^{\mathrm{GL}_m}$.

Counterexamples to the First Fundamental Theorems over finite fields

Note that the statements of the First Fundamental Theorems for $\mathrm{GL}_m(K)$ and $\mathrm{SL}_m(K)$ are wrong for any finite field K. For any prime power q, let us associate to any $(m-1)$ - tuple $(i_1,\ldots,i_{m-1})$, $i_j \in [n]$, of distinct numbers the matrix

$$X_{i_1,\ldots,i_{m-1}}^q := \begin{pmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_{m-1}} & x_{1,i_1}^q \\ x_{2,i_1} & x_{2,i_2} & \cdots & x_{2,i_{m-1}} & x_{2,i_1}^q \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m,i_1} & x_{m,i_2} & \cdots & x_{m,i_{m-1}} & x_{m,i_1}^q \end{pmatrix}.$$

According to the fact that each element $z \in K$ is a fixed point of the iterate of the Frobenius endomorphism $z \mapsto z^q$, we have $(G \cdot x_{i,j})^q = G \cdot x_{i,j}^q$ for each $G \in \mathrm{GL}_m(\mathbb{C})$. Thus, the polynomial

$$p_{i_1,\ldots,i_{m-1}} := \det(X_{i_1,\ldots,i_{m-1}}^q)$$

is then semi-invariant under the action of $\mathrm{GL}_m(\mathbb{F}_q)$, i.e., we have

$$G \cdot p_{i_1,\ldots,i_{m-1}} = \det(G) p_{i_1,\ldots,i_{m-1}}. \tag{3.19}$$

Thus, it follows that $p_{i_1,\ldots,i_{m-1}} \in \mathbb{F}_q[\mathbb{x}]^{\mathrm{SL}_m}$ and $p_{i_1,\ldots,i_{m-1}}/p_{j_1,\ldots,j_{m-1}} \in \mathbb{F}_q(\mathbb{x})^{\mathrm{GL}_m}$, for any $(i_1,\ldots,i_{m-1}) \neq (j_1,\ldots,j_{m-1})$. However, $p_{i_1,\ldots,i_{m-1}}$ is obviously not an element of the ring $\mathbb{F}_q[f_{i_1,\ldots,i_m}]$ from which we conclude $\mathbb{F}_q\left(\frac{f_{i_1,\ldots,i_m}}{f_{j_1,\ldots,j_m}}\right) \subsetneq \mathbb{F}_q(\mathbb{x})^{\mathrm{GL}_m}$ and $\mathbb{F}_q[f_{i_1,\ldots,i_m}] \subsetneq \mathbb{F}_q[\mathbb{x}]^{\mathrm{SL}_m}$.

This shows that in the case $K = \mathbb{F}_q$ a finite field, the ring $K[f_{i_1,\ldots,i_m}]$ and the field $K\left(\frac{f_{i_1,\ldots,i_m}}{f_{j_1,\ldots,j_m}}\right)$ are strictly contained in the $\mathrm{SL}_m(K)$-invariant ring and in the $\mathrm{GL}_m(K)$-invariant field, respectively. Thus, the question now is: What are the generators of the invariant ring $K[\mathbb{x}]^{\mathrm{SL}_m}$ and the invariant field $K(\mathbb{x})^{\mathrm{GL}_m}$ in the case that K is a finite field? We put this question on a list with unsolved problems (see Section 5.3).

Chapter 4

Appendix

This appendix serves as a preparation for the main topics of the thesis. Its aim is to collect some of the basic results about the most important objects frequently used in this thesis. The proofs and techniques listed here are partially very well known and standard. Some of them are, however, new and containing innovative ideas.

4.1 D-transcendental Power Series

In this section, we collect some facts about differentially algebraic power series which play an important role in the theory of geometric invariants. The theorems and their proofs, that we list here, are either classical results and techniques by J. F. Ritt and E. Gourin [GR27], which can be found also in Rubel's survey [Ru89], or their generalizations to the multivariate case which use the concept of the generalized Wronskian determinant introduced by Ostrowski [Os19]. To our knowledge, some of these extensions are new, and we have obtained them in collaboration with A. Bostan (Inria, Saclay, France).

Recall that a power series $f \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ in n variables is called *algebraic* if it is a solution of $p(x_1, \dots, x_n, f(x_1, \dots, x_n)) = 0$ for some nonzero polynomial $p \in \mathbb{C}[x_1, \dots, x_n, y]$ in $n + 1$ variables, and that f is called *transcendental* otherwise. But it may happen that, even though f is transcendental, it satisfies an algebraic differential equation, strictly speaking a partial differential equation, i.e.,

$$q(x_1, \dots, x_n, \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} f : 0 \le i_1, \dots, i_n \le k - 1) = 0 \tag{4.1}$$

is fulfilled by f for some nonzero polynomial $q \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_{k^n}]$ in $k^n + n$ variables, with $k \in \mathbb{N}$ some positive integer. In this case we say that f is *differentially algebraic* or *D-algebraic* and otherwise we call f *transcendentally transcendental* or *hypertranscendental* or *D-transcendental*. Our definition is due to E. Kolchin [Ko73, Chapter I, §6] and this concept of D-algebraicity appears for example also in works by L. A. Rubel [Ru92], and T. Dreyfus and C. Hardouin [DH19]. However, this definition is not considered to be the "classical" one. In 1920 A. Ostrowski [Os20, §6] defined differentially algebraic power series in several variables, we will refer to them *classically differentially algebraic* or *CD-algebraic*, as those which are D-algebraic w.r.t. each variable when the other variables are held fixed, i.e., a power series $f \in \mathbb{C}[\![x_1,\dots,x_n]\!]$ is called CD-algebraic if for each $i = 1,\dots,n$ the family of derivatives $\partial_{x_i}^j f$, with $j \in \mathbb{N}$ satisfies an algebraic equation with coefficients in $\mathbb{C}[x_1,\dots,x_n]$. This definition is more natural as it extends the classical notion of multivariate D-finite power series (see [Li89]). We call *classically differentially transcendental* or *CD-transcendental* each power series that is not CD-algebraic. This "classical" notion of differential algebraic power series is used for instance by L. A. Rubel and M. F.Singer [RS85], T. Dreyfus, C. Hardouin, J. Roques and M. F. Singer [DHRS18], T. Dreyfus and K. Raschel [DR19] and J. van der Hoeven [Ho19]. For CD-algebraic power series we have the following characterization which was shown by A. Ostrowski [Os20, §5]:

Theorem 4.1.1. *A power series $f \in \mathbb{C}[\![x_1,\dots,x_n]\!]$ is CD-algebraic if and only if for all n-tuples $(z_1(t),\dots,z_n(t)) \in \mathbb{C}[\![t]\!]$ of D-algebraic power series, $f(z_1(t),\dots,z_n(t)) \in \mathbb{C}[\![t]\!]$ is D-algebraic as a univariate power series.*

As the following example shows, the above definitions are equivalent only in the univariate case. Let $f(x) \in \mathbb{C}[\![x]\!]$ be a D-transcendental power series in one variable. Further, set $F(x,y) = f(x) + y$. Then as F satisfies the algebraic differential equation $\partial_x\partial_y F = 0$, it is D-algebraic. However, $F(x,y)$ is not D-algebraic w.r.t. the variable x, and thus, it is CD-transcendental.

One can easily find examples of D-algebraic and also CD-algebraic power series. Notice, that to construct a CD-algebraic power series is also rather easy. Actually, once we have a univariate D-transcendental power series $f(t)$, we can easily construct a CD-transcendental power series in n variables by setting $F = f(x_1)+x_2+\cdots+x_n$ as already discussed above. However, to construct a multivariate D-transcendental power series is more tricky and requires a deeper understanding of the concept. Our goal is to prove the existence of D-transcendental power series (and families of power series).

Before we come to the proof, let us recall the notion of the (*generalized*) *Wronskian determinant*. The *Wronskian matrix* of the family $g_1,\dots,g_m \in \mathbb{C}[\![x]\!]$ of m power series in one variable is defined as

$$\begin{pmatrix} g_1 & \cdots & g_m \\ \partial_x g_1 & \cdots & \partial_x g_m \\ \vdots & \ddots & \vdots \\ \partial_x^{m-1} g_1 & \cdots & \partial_x^{m-1} g_m \end{pmatrix}.$$

The determinant of this matrix is called the *Wronskian determinant* of this family. Obviously, the Wronskian determinant of a linearly dependent family of power series equals zero. And it can be shown (see e.g. [Bo01, pp. 90-92]) that the converse is true as well. So we have the following result:

Lemma 4.1.2. *A family of finitely many power series in one variable is linearly dependent over $\mathbb{C}$ if and only if its Wronskian determinant equals zero.*

Let us now consider the behavior of the Wronskian determinant in the multivariate case. Let $f_1, \dots, f_m \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ be a family of power series in n variables. If the family is linearly dependent, then its Wronskian with respect to any variable $x_i, 1 \leq i \leq n$ obviously equals zero. One could naively think now that, if the Wronsikan determinant with respect to each variable $x_i, 1 \leq i \leq n$, equals zero, the family $f_1, \dots, f_m$ must be necessarily linearly dependent. But this is not the case:

Example 4.1.3. Let $f_1 = x_1, f_2 = x_2, f_3 = 1 \in \mathbb{C}[\![x_1, x_2]\!]$. Denote by W_1 and W_2 the Wronskian of the family f_1, f_2, f_3 with respect to the variables x_1 and x_2, respectively. Then we have

$$W_1 = \begin{pmatrix} x_1 & x_2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} x_1 & x_2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which have both determinant equal to zero. However, the family f_1, f_2, f_3 is linearly independent over $\mathbb{C}$.

In the multivariate case, the notion of the so-called *generalized Wronskian determinant* is needed. Let us consider a family $f_1, \dots, f_m \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ of power series in n variables and $m-1$ differential operators $\Delta_0, \dots, \Delta_{m-1}$, where $\Delta_s = (\partial_{x_1}^{j_1} \dots \partial_{x_n}^{j_n})$ for some $j_1, \dots, j_n \in \mathbb{N}$ with $j_1 + \dots + j_n \leq s$. The *generalized Wronskian determinant* associated to $\Delta_0, \dots, \Delta_{m-1}$ of the family $f_1, \dots, f_m$ is defined to be the determinant of the following *generalized Wronskian matrix*:

$$\begin{pmatrix} \Delta_0(f_1) & \dots & \Delta_0(f_m) \\ \Delta_1(f_1) & \dots & \Delta_1(f_m) \\ \vdots & \ddots & \vdots \\ \Delta_{m-1}(f_1) & \dots & \Delta_{m-1}(f_m) \end{pmatrix}.$$

Similarly to the univariate case, it holds also here that all generalized Wronskian determinants of a family of power series equal zero as soon as the family is linearly dependent. Moreover, using the concept of the generalized Wronskian determinant, the statement of Lemma 4.1.2 can be generalized to the multivariate case:

Theorem 4.1.4. *A family of finitely many power series $f_1, \dots, f_m \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ is linearly independent over $\mathbb{C}$ if and only if at least one of the generalized Wronskian determinants of $f_1, \dots, f_m$ is not equal to zero.*

This generalization is for instance proven by M. Hindry and J. H. Silverman [HS00, Lemma D.6.1.] and appears also by E. Kolchin in [Ko73, Chapter II, Theorem 1]. Another proof was done by A. Bostan and Ph. Dumas in [BD10, Theorem 3], where also more historical information is provided.

Now we use Theorem 4.1.4 in order to prove the following statement:

Theorem 4.1.5. *Let $f \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ be a D-algebraic power series. Then f satisfies an algebraic differential equation with integer coefficients.*

Let us mention that the proof of Theorem 4.1.5 presented here is a multivariate extension of the proof by E. Gourin and J. F. Ritt [GR27, §2] of the univariate case with generalized Wronskian determinants instead of classical univariate Wronskian determinants.

In order to shorten the notation, let us use the following notation for $i \in \mathbb{N}^n$:

$$\underline{x} = (x_1, \dots, x_n),\ \underline{x}^i = x_1^{i_1} \cdots x_n^{i_n},\ i! = i_1! \cdots i_n!,\ \partial^i = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n},\ |i| = \sum_{j=1}^{n} i_j.$$

Lemma 4.1.6. *Let $k, N \in \mathbb{N}$ be two positive integers. Consider arbitrary $2N$ vectors*

$$a^r = (a_1^r, \dots, a_n^r) \in \mathbb{C}^n \ \text{ and } \ b^r = (b_i^r \, : \, i \in \mathbb{N}^n, 0 \leq i_1, \dots, i_n \leq k-1) \in \mathbb{C}^{k^n},$$

for $1 \leq r \leq N$. Then there always exists a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ satisfying for each $1 \leq r \leq N$ and for each $i \in \mathbb{N}^n$ with $0 \leq i_1, \dots, i_n \leq k-1$ the equality

$$\partial^i f(a^r) = b_i^r. \tag{4.2}$$

In other words, for any number of points in $\mathbb{C}^n$, we can always construct a polynomial f with any given values of itself and its first (according to a lexicographical order) $k^n - 1$ partial derivatives $\partial^i f$ at these points.

Proof. We prove the claim by induction on N. The induction base follows immediately by setting

$$f := \sum_{i \in \mathbb{N}^n} \frac{b_i^1}{i!} (\underline{x} - a^1)^i.$$

Let now $g \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial satisfying

$$\partial^i g(a^r) = b_i^r,$$

for all $i \in \mathbb{N}^n$ with $0 \leq i_1, \dots, i_n \leq k-1$ and for all $1 \leq r \leq N-1$. We then define

$$f := g + \underbrace{\prod_{r=1}^{N-1} (\underline{x} - a^r)^{(k,\dots,k)} \cdot \sum_{\substack{i \in \mathbb{N}^n \\ 0 \leq i_1, \dots, i_n \leq k-1}} \frac{\tilde{b}_i}{i!} (\underline{x} - a^N)^i}_{=: \tilde{g}},$$

where the $\tilde{b}_i$'s can be iteratively computed as the solution of the following system of linear equations:

$$b_i^N = \partial^i g(a^N) + \partial^i \tilde{g}(a^N),$$

where $i \in \mathbb{N}^n$ with $0 \leq i_1, \dots, i_n \leq k-1$. Notice that each $\tilde{b}_i$ is uniquely given as a linear combination of $\tilde{b}_j$'s with $j \prec i$ w.r.t. a lexicographical order $\prec$ on $\mathbb{N}^n$. The polynomial f then satisfies obviously the condition (4.2). $\square$

Proof of Theorem 4.1.5. Any algebraic differential equation of the form (4.1) satisfied by a power series f can be written as

$$\sum_{j\in J, l\in L} c_{(j,l)} \underline{x}^j \underline{f}^l = 0, \tag{4.3}$$

where $J \subseteq \mathbb{N}^n$ and $L \subseteq \mathbb{N}^{kn}$, for some $k \in \mathbb{N}$, and $\underline{f}^l = \prod_i (\partial^i f)^{l_i}$. By construction, all the expressions $\underline{x}^j \underline{f}^l$ are distinct from each other. This means that the family of power series $\underline{x}^j \underline{f}^l$ is linearly dependent over $\mathbb{C}$ and thus all its generalized Wronskian determinants must equal zero. As the generalized Wronskian determinants are polynomials in $\Delta_s \underline{x}^j \underline{f}^l$ with integer coefficients, they are also polynomials in $\underline{x}$ and f and their higher partial derivatives with integer coefficients. Therefore, it is enough to show that at least one generalized Wronskian determinant is a non-zero polynomial to obtain an algebraic differential equation with integer coefficients as claimed. Let us assume by contradiction that all generalized Wronskian determinants are identically zero in $\underline{x}$ and $\partial^i f$ for any power series f. According to Theorem 4.1.4, this would mean (again for any power series f) that $\underline{x}^j \underline{f}^l$'s are linearly dependent over $\mathbb{C}$ and so they satisfy an equation like (4.3), let us say

$$\sum_{j\in J, l\in L} a_{(j,l)} \underline{x}^j \underline{f}^l(\underline{x}) = 0. \tag{4.4}$$

Further, according to Lemma 4.1.6, for $N \in \mathbb{N}$ sufficiently large, we can always construct a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ with the property that the vectors

$$\begin{pmatrix} \underline{1}^j \underline{f}^l(\underline{1}) \\ \vdots \\ \underline{N}^j \underline{f}^l(\underline{N}) \end{pmatrix}, \quad \text{for } j \text{ and } l \text{ as in } (4.4) \in J, l \in L,$$

are linearly independent over $\mathbb{C}$. Here for any $s \in \mathbb{C}$, by $\underline{s}$ we denote the vector $(s, \dots, s)$. But this contradicts the linear relation (4.4). $\square$

Using the same argument as in the proof above, Theorem 4.1.5 can be shown also for CD-algebraic power series:

Theorem 4.1.7. *Let $f \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ be a CD-algebraic power series. Then for each $i = 1, \dots, n$, the family $\partial^j_{x_i} f, j = 0, 1, \dots$ satisfies an algebraic equation with coefficients in $\mathbb{Z}[x_1, \dots, x_n]$.*

Actually, Theorem 4.1.7 is equivalent to the following one ([Ho19, Chapter 5, Proposition 25]):

Theorem 4.1.8. *A power series $f \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ in n variables is CD-algebraic if and only if there exists a field F of finite transcendence degree over $\mathbb{Q}$ that contains f and all its higher partial derivatives.*

Theorem 4.1.7 can be obtained from Theorem 4.1.8 in the following way: Let us consider a CD-algebraic power series $f \in \mathbb{C}[\![x_1, \ldots, x_n]\!]$ and let $F \supseteq \mathbb{Q}(\partial^i f : i \in \mathbb{N}^n)$ be a field extension of finite transcendence degree over $\mathbb{Q}$ which contains f and all its higher partial derivatives. Let us now consider the following chain of field extensions:

$$\mathbb{Q} \subseteq \mathbb{Q}(x_1, \ldots, x_n) \subseteq \mathbb{Q}(x_1, \ldots, x_n, \partial^i f : i \in \mathbb{N}^n) \subseteq F(x_1, \ldots, x_n).$$

As $F(x_1, \cdots, x_n)$ has finite transcendence degree over F and F is of finite transcendence degree over $\mathbb{Q}$, it follows that $\mathbb{Q}(x_1, \ldots, x_n, \partial^i f : i \in \mathbb{N}^n)$ has finite transcendence degree over $\mathbb{Q}(x_1, \ldots, x_n)$. Thus, an arbitrary family of higher partial derivatives of f satisfies an algebraic equation over $\mathbb{Q}(x_1, \ldots, x_n)$ and after cleaning denominators even over $\mathbb{Z}[x_1, \ldots, x_n]$.

We will show now that Theorem 4.1.8 follows directly from Theorem 4.1.7.

Proof of Theorem 4.1.8. The reverse implication is straightforward. So let $f \in \mathbb{C}[\![x_1, \ldots, x_n]\!]$ be a CD-algebraic power series. By Theorem 4.1.7 the derivatives of f satisfy the following system of algebraic equations:

$$\sum_{l \in \mathbb{N}^k} c_{j,l} \prod_{m=0}^{k-1} (\partial_{x_j}^m f)^{l_m} = 0, \quad \text{for } j = 1, \ldots, n,$$

with coefficients $c_{j,l} \in \mathbb{Z}[x_1, \ldots, x_n]$ and for some $k \in \mathbb{N}$. Differentiating iteratively these equations with respect to all possible variables yields then algebraic equations with coefficients in $\mathbb{Z}[x_1, \ldots, x_n]$ for all partial derivatives $\partial^i f$ with $|i| \geq N$ for some $N \in \mathbb{N}$ large enough. Hence, the transcendence degree of $\mathbb{Q}(x_1, \ldots, x_n, \partial^i f : i \in \mathbb{N}^n)$ over $\mathbb{Q}(x_1, \ldots, x_n)$ is finite. Obviously, the same holds also for the transcendence degree of $\mathbb{Q}(x_1, \ldots, x_n)$ over $\mathbb{Q}$. Thus, the field $F = \mathbb{Q}(x_1, \ldots, x_n, \partial^i f : i \in \mathbb{N}^n)$ cannot have infinite transcendence degree over $\mathbb{Q}$. $\square$

Let us remark that, whereas Theorem 4.1.7 holds also for D-algebraic power series (this is just Theorem 4.1.5), Theorem 4.1.8 is no longer true when replacing the term "CD-algebraic" by "D-algebraic". Namely, let us take a univariate D-transcendental power series $f(x)$ and consider the bivariate power series $F = f(x) + y \in \mathbb{C}[\![x, y]\!]$. Then F is D-algebraic, as it satisfies the algebraic differential equation $\partial_x \partial_y F = 0$, but its higher derivatives w.r.t. x equal $\partial_x^j f(x)$ and thus, do not satisfy any algebraic equation as $f(x)$ is D-transcendental. Therefore, any field containing all the higher derivatives of F w.r.t. x has an infinite transcendence degree over $\mathbb{Q}$.

The next step is to prove the existence of D-transcendental power series. Notice that, the first proof of the existence of (univariate) D-transcendental power series is due to O. Hölder [Ho87], who showed that the Gamma function $\Gamma(x)$ is D-transcendental. We follow here a different strategy from Hölder's and use our previous results for the construction of a multivariate D-transcendental power series.

Theorem 4.1.9. *The set of D-transcendental power series in n variables is non-empty for any positive integer n.*

Proof. Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be any sequence of complex numbers that has an infinite transcendence degree over $\mathbb{Q}$ and let $\Phi : \mathbb{N}^n \to \mathbb{N}\backslash\{0, \ldots, n-1\}$ be a bijection between $\mathbb{N}^n$ and $\mathbb{N}\backslash\{0, \ldots, n-1\}$. Set now f to be

$$f := \sum_{i \in \mathbb{N}^n} \frac{\alpha_{\Phi(i)}}{i!} (\underline{x} - \underline{\alpha})^i,$$

where $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{n-1})$. We now conclude the proof by showing that f is D-transcendental. Let us assume by contradiction that f is differentially algebraic. Then by Theorem 4.1.5, it satisfies an algebraic differential equation with integer coefficients. However, notice that the partial derivatives of f satisfy $\partial^i f(\underline{\alpha}) = \alpha_{\Phi(i)}$ and so substituting $x_i \mapsto \alpha_{i-1}$ into the algebraic equation would yield an algebraic equation with integer coefficients for the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$. But this contradicts the fact that the sequence has infinite transcendence degree over $\mathbb{Q}$. $\square$

Notice that in the univariate case, several examples of D-transcendental power series are known. One of them is the already mentioned Gamma function $\Gamma(x)$. Another D-transcendental power series, namely

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n^n)!},$$

was presented by A. Hurwitz [Hu89]. In 1896, E. Moore [Mo96] showed that for any positive integer $k \geq 2$, the series

$$f(x) = \sum_{n=0}^{\infty} x^{n^k}$$

is D-transcendental. Later then, K. Mahler provided in his paper [Ma30] more examples of univariate D-transcendental power series. Among them e.g.

$$f(x) = \sum_{n=0}^{\infty} x^{2^n}, \quad f(x) = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^n}}, \quad f(x) = \frac{1}{1-x} \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 + x^{2^n}}.$$

M. Heins [He55] constructed the so-called Blaschke product which then later turned out to be D-transcendental. Another example of a D-transcendental power series would be the generating function of Bell numbers whose D-transcendence was proved by M. Klazar [Kl03]. Since there are many examples of univariate D-transcendental power series, it is natural to ask whether starting with a univariate D-transcendental power series it is possible to construct a D-transcendental power series in n variables out of it. The answer to this question is not clear to us yet, hence we put this question on a list with open questions (see Section 5.1). We should also mention the method for constructions of univariate D-transcendental power series in one variable presented in the paper by J. F. Ritt and E. Gourin [RG27] where the authors use Theorem 4.1.5 (originally, in the one variable case established by them) and the countability of algebraic numbers. A different way to construct univariate D-transcendental power series is to make the gaps between its coefficients sufficiently large. This was proven for example by A. Ostrowski [Os20, Theorem 12], L. Lipschitz and L. A. Rubel [LR86], or by K. Mahler [Ma76, §13, Theorem 16]. Many more examples and methods on constructions of univariate D-transcendental power series as well as an historical overview can be found in Rubel's survey [Ru89]. For further examples and methods see also the papers by R. D. Carmichael [Ca13] and J. F. Ritt [Ri26].

Concerning the multivariate case, we should mention the paper [Gr98] by H. Grönwall where the author proves that the multivariate series

$$f(\underline{x}) = \sum_{i=0}^{\infty} x_1^i x_2^{i^i} \cdots x_n^{i^{i^{\cdot^{\cdot^{\cdot^i}}}}}$$

is D-transcendental. However, it seems that not much is known about D-algebraic power series on several variables. Thus, it would be natural, and it is also one of the problems on a list with open questions in Section 5.1, to study their basic properties and, once this is done, to construct examples of multivariate D-transcendental power series using these properties.

The concept of D-algebraicity can be extended also to families of power series. We call a family $f_1, \ldots, f_l \subseteq \mathbb{C}[\![x_1, \ldots, x_n]\!]$ of power series in n variables *differentially algebraically dependent* or *D-algebraically dependent* if there exists a polynomial $q \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_{k^n l}]$ in $k^n l + n$ variables, for some $k \in \mathbb{N}$, such that

$$q(x_1, \ldots, x_n, \partial^i f_j : 0 \leq i_1, \ldots, i_n \leq k-1, 1 \leq j \leq l) = 0.$$

Otherwise, we call the family *differentially algebraically independent* or *D-algebraically independent*. Let us remark that the concept of D-algebraic dependent families of power series was used for example by E. Kolchin in [Ko73, Chapter II, §7] or by C. Hardouin and M. F. Singer in [HS08, §3.1].

Notice first that, similarly to the case of D-algebraic power series (and it can be proven using the same argument as in the proof of Theorem 4.1.5), a family of D-algebraically dependent power series satisfies an algebraic differential equation with integer coefficients.

Theorem 4.1.10. *Let $f_1, \ldots, f_l \in \mathbb{C}[\![x_1, \ldots, x_n]\!]$ be a D-algebraically dependent family of power series. Then the power series $f_1, \ldots, f_l$ satisfy an algebraic differential equation with integer coefficients.*

The question now is whether it is always possible to construct arbitrarily large D-algebraically independent families of power series. The answer is "yes" as stated in the following theorem:

Theorem 4.1.11. *For any given positive integers $l, n \in \mathbb{N}$, there exists a D-algebraically independent family of l power series in n variables.*

Proof. Assume by contradiction that any family $f_1, \ldots, f_l$ of l power series in n variables is D-algebraically dependent, i.e., the power series satisfy an algebraic differential equation

$$q(x_1, \ldots, x_n, \partial^i f_j : 0 \leq i_1, \ldots, i_n \leq k-1, 1 \leq j \leq l) = 0, \tag{4.5}$$

for some $k \in \mathbb{N}$. Then by Theorem 4.1.10, we may w.l.o.g. assume that q has integer coefficients. Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be any sequence of complex numbers that has infinite transcendence degree over $\mathbb{Q}$. Further, let $\Phi : \{1,\dots,l\}\times\mathbb{N}^n \to \mathbb{N}\backslash\{0,\dots,n-1\}$ be a bijection between $\{1,\dots,l\}\times\mathbb{N}^n$ and $\mathbb{N}\backslash\{0,\dots,n-1\}$. For each $j = 1,\dots,l$ we define

$$f_j := \sum_{i\in\mathbb{N}^n} \frac{\alpha_{\Phi(j,i)}}{i!}(\underline{x}-\underline{\alpha})^i,$$

where $\underline{\alpha} = (\alpha_0,\dots,\alpha_{n-1})$. Hence, each partial derivative of f_j satisfies $\partial^i f_j(\underline{\alpha}) = \alpha_{\Phi(j,i)}$. But substituting $x_i \mapsto \alpha_{i-1}$ into the equation (4.5) yields an algebraic equation over $\mathbb{Q}$ for α_j's which is a contradiction to the infinite transcendence degree over $\mathbb{Q}$ of the sequence $\alpha_0,\alpha_1,\alpha_2,\dots$. □

Notice, that we have considered only algebraic differential equations with complex coefficients and their solutions in the complex power series ring $\mathbb{C}[\![x_1,\dots,x_n]\!]$. A natural question would be what happens if we replace $\mathbb{C}$ by any other field K (e.g. an infinite field, finite field of large characteristic,...). Which of the above mentioned results will still remain true? And what about the existence of D-transcendental power series or D-algebraically independent families of power series in that case? This question is one among the unsolved problems in Section 5.1.

4.2 Puiseux Parametrizations

This section serves as a preparation for the theory of resolution of singular curves via geometric invariants. We collect here basic facts about Puiseux parametrizations of algebraic curves (plane curves in $\mathbb{A}^2_{\mathbb{C}}$ and space curves in $\mathbb{A}^{n+1}_{\mathbb{C}}$) which play a crucial role in our resolution algorithms presented in Sections 2.1, 2.2 and 2.3. All the results, and also their proofs, about Puiseux parametrizations of plane algebraic curves and their analytic branches listed here are standard and can be found in E. Casas-Alvero's book [Ca00, Chapter 1]. The results concerning Puiseux parametrizations of space curves are then extensions and generalizations of the classical results to the case of higher embedding dimensions.

Let us start by recalling some facts about the classical Newton-Puiseux algorithm for the search of y-roots of bivariate power series. Let $f \in \mathbb{C}[\![x,y]\!]$ be a bivariate power series satisfying $f(0,0) = 0$. Recall that a univariate power series $y(x) \in \mathbb{C}[\![x]\!]$ is called a *y-root* of f if it satisfies the equality $f(x,y(x)) = 0$. The existence of a y-root in the case that $f_y(0) \neq 0$ is guaranteed by the Implicit Function Theorem. Here f_y denotes the partial derivative $\partial_y f$. However, if $f_y(0) = 0$, there is no reason for f to have a power series y-root. In fact, for an irreducible power series $f \in \mathbb{C}[\![x,y]\!]$ with $f_y(0) = 0$, it is impossible to construct a y-root which would be again a power series, as we will see in Corollary 4.2.5. For instance the polynomial $f(x,y) = x^3 - y^2$ defining a cusp obviously does not have any power series y-root. In order to be able to find for an arbitrary $f \in \mathbb{C}[\![x,y]\!]$ some $y(x)$ satisfying $f(x,y(x)) = 0$, we need to enlarge the ring in which we search for solutions. In our example, the polynomial $f(x,y) = x^3 - y^2$ has the fractionary power series $y_1(x) = x^{\frac{3}{2}}$ and $y_2(x) = -x^{\frac{3}{2}}$ as y-roots. The fractionary power series, i.e., power series with fractional exponents with bounded denominator $y(x^{1/n}) \in \mathbb{C}[\![x^{1/n}]\!]$ for some $n \in \mathbb{N}$, are called *Puiseux series*. The Puiseux series ring is defined as the union of all the rings $\mathbb{C}[\![x^{\frac{1}{n}}]\!]$, i.e.,

$$\text{Puiseux series ring } := \bigcup_{n=1}^{\infty} \mathbb{C}[\![x^{\frac{1}{n}}]\!].$$

Puiseux series were discovered in 1671 by I. Newton [Ne36, pp. 191-209] who observed that they necessarily appear when searching for y-roots of bivariate polynomials. Later, in 1850, V. A. Puiseux [Pu50] rediscovered Puiseux series again while studying the solution space of $f(x, y(x)) = 0$. Using the Newton-Puiseux algorithm for construction of y-roots of formal power series (see e.g. [Ca00, §1.4], [Te07, Chapter 3], [Wa78, §3]), one proves the following theorem:

Theorem 4.2.1. *If $f \in \mathbb{C}[\![x, y]\!]$ with $f(0, 0) = 0$ is irreducible and $f \neq x^m$ for all $m \in \mathbb{N}$, then there is a Puiseux series $s(x)$ which is a y-root of f, i.e., $f(x, s(x)) = 0$.*

If we instead of formal power series in x and y consider power series in y with coefficients rational functions in x, the Newton-Puiseux algorithm gives rise even to Puiseux Laurent series as y-roots:

Newton-Puiseux theorem. *The algebraic closure of the field $\mathbb{C}((x)) = \operatorname{Quot}(\mathbb{C}[\![x]\!])$ is the union of the fields $\mathbb{C}\left(\left(x^{\frac{1}{n}}\right)\right)$ for $n \geq 1$, i.e.,*

$$\overline{\mathbb{C}((x))} = \bigcup_{n=1}^{\infty} \mathbb{C}\left(\left(x^{\frac{1}{n}}\right)\right).$$

Remark 4.2.2. It can be shown that if $f \in \mathbb{C}\{x, y\}$ is a convergent power series, so are all its y-roots (for more details see [Ca00, §1.7]) and so we obtain even convergent versions of Theorem 4.2.1 and of the Newton-Puiseux theorem.

Let $X \subseteq \mathbb{A}^2_{\mathbb{C}}$ be a plane algebraic curve. We call a pair of formal power series $\gamma(t) = (x(t), y(t)) \in \mathbb{C}[\![t]\!]^2$ a *parametrization* of X if the ring map

$$\begin{aligned} \gamma\colon \mathbb{C}[x, y]/\mathcal{I}(X) &\to \mathbb{C}[\![t]\!] \\ x &\mapsto x(t) \\ y &\mapsto y(t) \end{aligned}$$

is injective. Here $\mathcal{I}(X)$ denotes the vanishing ideal of X. Notice that reducible curves admit no parametrizations according to this definition. Fruther, if we additionally have $\gamma(0) = z$ for some point $z \in X$, we say that $\gamma(t)$ parametrizes X at the point z.

Corollary 4.2.3. *Let X and Y be two irreducible plane algebraic curves. The curves have a common parametrization $(x(t), y(t)) \in \mathbb{C}[[t]]^2$ if and only if $X = Y$.*

Proof. This follows directly from the fact that the vanishing ideals of X and Y determine both the kernel of the map

$$\begin{aligned} \gamma\colon \mathbb{C}[x, y] &\to \mathbb{C}[\![t]\!] \\ x &\mapsto x(t) \\ y &\mapsto y(t) \end{aligned}$$

and thus, they are equal. ∎

We conclude now, that two different irreducible polynomials do not have the same y-roots:

Corollary 4.2.4. *Let $F_1, F_2 \in \mathbb{C}[x,y]$ be two irreducible polynomials. Then the sets of their y-roots have a non-empty intersection if and only if $F_1 = F_2$.*

For a Puiseux series $s(x)$, we call the minimal $n \in \mathbb{N}$, for which $s(x) \in \mathbb{C}[\![x^{\frac{1}{n}}]\!]$ holds, the *polydromy order* of $s(x)$. Let us wirte

$$s(x) = \sum_{i \geq 0} a_i x^{\frac{i}{n}}.$$

For a primitive n-th root of unity ξ and each $j = 1, \ldots, n$, the substitution $x^{\frac{1}{n}} \mapsto \xi^j x^{\frac{1}{n}}$ induces an automorhpism σ_{ξ^j} of $\mathbb{C}[\![x^{\frac{1}{n}}]\!]$. We call each Puiseux series

$$\sigma_{\xi^j}(s) = \sum_{i \geq 0} a_i (\xi^j)^i x^{\frac{i}{n}}.$$

a *conjugate* of $s(x)$.

Notice that f having a y-root $s(x)$ means that f is divisible by $(y - s(x))$ in the Puiseux series ring. Moreover, if $s(x)$ is a y-root of f, so are also all its conjugates $\sigma_{\xi^j}(s), j = 1, \ldots, n-1$, where n is the polydromy order of $s(x)$ and ξ is a primitive n-th root of unity. Furthermore, the product

$$\prod_{j=1}^{n} (y - \sigma_{\xi^j}(s))$$

is an element of $\mathbb{C}[\![x,y]\!]$, as it is invariant under conjugation (see [Ca00, Lemma 1.2.1]), and thus we obtain a unique decomposition of f:

Corollary 4.2.5. *Each irreducible power series $f \in \mathbb{C}[\![x,y]\!]$ with $f(0,0) = 0$ can be uniquely written as*

$$f = u \cdot \prod_{j=1}^{n} (y - \sigma_{\xi^j}(s)),$$

where $u \in \mathbb{C}[\![x,y]\!]^*$ is an invertible power series, $s(x)$ is a y-root of f and n is the polydromy order of $s(x)$.

It follows now that two different irreducible power series $f_1, f_2 \in \mathbb{C}[\![x,y]\!]$, $f_1 \neq f_2$ have a common y-root if and only if $f_1 = u \cdot f_2$ for some $u \in \mathbb{C}[\![x,y]\!]^*$. Therefore, if we consider an irreducible convergent power series $f \in \mathbb{C}\{x,y\}$, each y-root $s(x)$ of f of polydromy order n gives rise to precisely n different parametrizations, the *Puiseux parametrizations*, of the analytic curve defined by f:

$$\gamma_j(t) = (t^n, \sigma_{\xi^j}(s)|_{x=t^n}) \ \text{ for } j = 1, \ldots, n,$$

where ξ is a primitive n-th root of unity. Here, an analytic curve defined by f is the germ $(V(f), 0)$ of the vanishing set of f in the neighborhood of 0. Further, under a parametrization of the analytic curve $(V(f), 0)$, we understand a pair $\gamma(t) = (x(t), y(t)) \in \mathbb{C}[\![t]\!]^2$ of power series for which the ring homomorphism

$$\begin{aligned} \hat{\gamma}\colon \mathbb{C}[\![x,y]\!]/f &\to \mathbb{C}[\![t]\!] \\ x &\mapsto x(t) \\ y &\mapsto y(t) \end{aligned}$$

is injective. If $\gamma(t) = (t^n, y(t))$ is a Puiseux parametrization at 0, i.e., n is minimal or in other words n is the polydromy order of $y(t^{\frac{1}{n}})$, we call n also the *polydromy order of* $\gamma(t)$. We call also a parametrization of the form $(x(t), t^n) \in \mathbb{C}[\![t]\!]^2$ a Puiseux parametrization if n is the polydromy order of $x(t^{\frac{1}{n}})$. Under a Puiseux parametrization of an analytic curve at a point $z \neq 0$ we understand the pair of power series $\gamma(t) = (z_1 + t^n, z_2 + y(t))$ for which $\gamma(t) - (z_1, z_2) = (x(t), y(t))$ is a Puiseux parametrization at 0 (i.e., either $x(t) = t^n$ or $y(t) = t^n$ with n minimal).

Remark 4.2.6. Notice that given a pair of convergent power series $\gamma(t) = (x(t), y(t)) \in \mathbb{C}\{t\}^2$, with the same argument as in the proof of Corollary 4.2.3, we always find a reparametrization $\varphi \in \mathbb{C}\{t\}$ which transforms $\gamma(t)$ into $(t^n, \widetilde{y}(t))$, where $n = \operatorname{ord}(x(t))$ and $\widetilde{y}(t) \in \mathbb{C}\{t\}$. So, $\widetilde{y}(t^{\frac{1}{n}})$ is a y-root of

$$g(x,y) := \prod_{j=1}^{n} (y - \widetilde{y}(\xi^j t^{\frac{1}{n}})),$$

where ξ is a primitive n-th root of unitiy. As g is invariant under all substitutions $x^{\frac{1}{n}} \mapsto \xi^j x^{\frac{1}{n}}$ for $j = 1, \ldots, n$, it already belongs to the convergent power series ring $g \in \mathbb{C}\{x, y\}$ (see [Ca00, Lemma 1.2.1]). Moreover, g is irreducible according to Corollary 4.2.5. Hence $\gamma(t)$ parametrizes only the analytic curve given by the vanishing set of g.

The Newton-Puiseux algorithm constructs all y-roots of a power series $f \in \mathbb{C}[\![x, y]\!]$. Hence, to a given irreducible plane algebraic curve $X \subseteq \mathbb{A}^2_{\mathbb{C}}$ defined by the polynomial $F \in \mathbb{C}[x, y]$ with factorization $F = \prod_i f_i$ into its irreducible convergent factors $f_i \in \mathbb{C}\{x, y\}$, the Newton-Puiseux algorithm constructs all y-roots of the power series f_i's appearing in the factorization of F. Thus, it gives rise also to all parametrizations of all analytic branches of X at the origin. Here each f_i defines an analytic curve which is called an *analytic branch* of X at 0. Moreover, in this way we obtain also all parametrizations of X itself at 0.

Let us now consider a Puiseux series $s(x) = \sum_{i \geq 0} a_i x^{\frac{i}{n}} \in \mathbb{C}[\![x^{\frac{1}{n}}]\!]$ of polydromy order n. Then, by minimality of n, the integer n and the set of integers i with $a_i \neq 0$ have no common factor and we can define the characteristic exponents of $s(x)$. Let us set

$$\begin{aligned} m_1 &:= \min\{i : a_i \neq 0 \text{ and } n \nmid i\}, \\ n_j &:= \gcd(n, m_1, \ldots, m_j), \\ m_{j+1} &:= \min\{i : a_i \neq 0 \text{ and } n_j \nmid i\}, \end{aligned} \tag{4.6}$$

and let k be the minimal integer satisfying $n_k = 1$. We define the *characteristic exponents* of $s(x)$ to be the set $\{m_1, \ldots, m_k\}$, the "minimal" subset of the support of $s(x^n)$, which has no common divisor with n. We call the set $m_1, \dots, m_k$ also the characteristic exponents of the Puiseux parametrization $\gamma(t) = (t^n, s(t^n))$ (or of the parametrization $\gamma(t) = (s(t^n), t^n)$).

Example 4.2.7. Let ξ be a primitive 6-th root of unity. Then the polynomial

$$\begin{aligned} f(x,y) =& \prod_{i=1}^{6}(y - \xi^{3i}x^{1/2} - 2\xi^{4i}x^{2/3} - 3\xi^{5i}x^{5/6} - 5\xi^{7i}x^{7/6}) =, \\ =& - 15625x^7 + 12000x^6 + 9000x^5y + 1275x^4y^2 - 300x^3y^3 - 90x^2y^4 + y^6 - 5014x^5 - \\ & - 3312x^4y - 846x^3y^2 - 88x^2y^3 - 3xy^4 + 16x^4 + 24x^3y + 3x^2y^2 - x^3 \end{aligned}$$

is analytically irreducible and admits only one parametrization at the origin (up to conjugation)

$$\gamma(t) = (x(t), y(t)) = (t^6, t^3 - 2t^4 - 3t^5 - 5t^7).$$

Notice that $\gcd(6, 3, 4) = 1$, hence, the characteristic exponents are $3, 4$.

The classical Newton-Puiseux algorithm constructs parametrizations only for plane algebraic curves and their analytic branches. However, since Puiseux' study of fractional power series, several generalizations of the algorithm for solving more general systems of polynomial equations were established. In 1980, J. Maurer gave in his paper [Ma80] a constructive proof for the existence of parametrizations of space curves, i.e., he solved the problem of finding y_i-roots of a system of polynomials

$$F_j(x, y_1, \dots, y_n) = 0 \tag{4.7}$$

defining an algebraic space curve. Maurer replaced the Newton polygon used in the classical Newton-Puiseux algorithm by the Newton polyhedron and used its edges in order to compute the so-called tropisms of F_j's corresponding to the approximations of parametrizations of analytic branches of a space curve. Another approach for solving (4.7) is for instance by means of tropical geometry methods as A. N. Jensen, H. Markwig and T. Markwig described in their paper [JMM08]. Another extension of the Newton-Puiseux algorithm, namely to polynomial equations of the form

$$F(x_1, \dots, x_n, y) = 0,$$

was done for example by J. McDonald [Md95] and by M. J. Soto and J. L. Vicente [SV11]. In the case that F is quasi-ordinary w.r.t. the variable y, the existence of Puiseux y-roots of such F is guaranteed also by the Abhyankar-Jung theorem (see e.g. [Lu83], [Zu93],[PR12]). Further, for the special case of F defining a quasi-ordinary surface we mention the works by J. Lipman [Li65] and P. D. González Pérez [Gp00]. For statements and algorithms solving the even more general system of polynomial equations

$$F_j(x_1, \dots, x_l, y_1, \dots, y_k) = 0$$

we refer e.g. to [Md02], [Ar04], [AIdM10].

The method we use to construct parametrizations of space curves follows the following strategy: we project a space curve to all possible coordinate planes where the classical Newton-Puiseux algorithm applies and constructs the Puiseux parametrizations of the projections. Then we glue all the projections together and reconstruct from all their parametrizations a parametrization of the space curve. To investigate this construction procedure in more detail is the objective of the remaining part of this section.

Let us consider a space curve $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$. As in the plane curve case, we call here an $(n+1)$-tuple of power series $\gamma(t) = (x(t), y_1(t), \dots, y_n(t)) \in \mathbb{C}[\![t]\!]^{n+1}$ a *parametrization* of X if the ring homomorphism

$$\begin{aligned} \gamma^* \colon \mathbb{C}[x_1, y_1, \dots, y_n]/\mathcal{I}(X) &\to \mathbb{C}[\![t]\!] \\ x &\mapsto x(t) \\ y_j &\mapsto y_j(t) \ \text{ for } 1 \leq j \leq n, \end{aligned}$$

is injective. The notation $\mathcal{I}(X)$ stands here again for the vanishing ideal of X. We call a parametrization γ a *parametrization of X at a point $z \in X$*, if $\gamma(0) = z$ is satisfied.

Let us now assume that X is irreducible and that $0 \in X$. Whereas its vanishing ideal $\mathcal{I}(X)$ is a prime ideal in the polynomial ring $\mathbb{C}[x, y_1, \dots, y_n]$, this is in general no longer true when considering $\mathcal{I}(X)$ as an ideal in the convergent power series ring $\mathbb{C}\{x, y_1, \dots, y_n\}$. This phenomenon shows that an irreducible algebraic space curve can locally at a point be viewed as a union of analytic curves. More precisely, an irredundant primary decomposition

$$\mathcal{I}(X) = Q_1 \cap \dots \cap Q_k$$

of $\mathcal{I}(X)$ into primary ideals $Q_i \subseteq \mathbb{C}\{x, y_1, \dots, y_n\}$ defines a decomposition of X into a union of irreducible analytic curves defined by Q_i's or, equivalently, their associated prime ideals $P_i = \sqrt{Q_i}$ for $i = 1, \dots, k$. Here again, an analytic curve defined by an ideal P_i is a germ $(V(P_i), 0)$ at the origin of the vanishing set of P_i, called also an *(analytic) branch* of X at 0. We call an $(n+1)$-tuple $\gamma(t) = (x(t), y_1(t), \dots, y_n(t)) \in \mathbb{C}[\![t]\!]^{n+1}$ a parametrization of the branch $(V(P_i), 0)$, if the corresponding ring map

$$\begin{aligned} \hat{\gamma} \colon \mathbb{C}\{x, y\}/P_i &\to \mathbb{C}[\![t]\!] \\ x &\mapsto x(t) \\ y_j &\mapsto y_j(t) \ \text{ for } 1 \leq j \leq n, \end{aligned}$$

is injective.

Theorem 4.2.8. *Let $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ be an irreducible algebraic space curve with $0 \in X$. Assume that its vanishing ideal $\mathcal{I}(X)$ does not contain the polynomial x, i.e., $x \notin \mathcal{I}(X)$. Then X can be parametrized at the origin by*

$$\gamma(t) = (t^l, y_1(t), \dots, y_n(t)),$$

for some $l \in \mathbb{N}$ and convergent power series $y_j(t) \in \mathbb{C}\{t\}$ for $j = 1, \dots, n$.

Lemma 4.2.9. *Let X and Y be two irreducible algebraic space curves in $\mathbb{A}^{n+1}_{\mathbb{C}}$. The curves have a common branch, if and only if $X = Y$.*

Proof. Assume w.l.o.g. that $0 \in X \cap Y$. Let us assume that X and Y have a common branch at 0 and let $P \subseteq \mathbb{C}\{x, y_1, \dots, y_n\}$ be its defining prime ideal. Then the Zariski closure of $V(P)$ has dimension 1 and defines a subset of X. Hence, as X is irreducible, it is equal to X. Analogously for Y. Thus we obtain $X = \overline{V(P)} = Y$. □

Proof of Theorem 4.2.8. Notice first that the Zariski closure of the image of X under the projection morphisms

$$\begin{aligned} \pi_j \colon \mathbb{A}^{n+1}_{\mathbb{C}} &\to \mathbb{A}^2_{\mathbb{C}} \\ (a, b_1, \dots, b_n) &\mapsto (a, b_j), \end{aligned}$$

is irreducible and contained in a plane algebraic curve for all $j = 1, \dots, n$ (see e.g. [Mu88, Proposition 1, pp. 68]). According to the assumption that $x \notin \mathcal{I}(X)$ and that X is irreducible, it is a consequence of Lemma 4.2.9 that X cannot have an analytic branch entirely contained in the hyperplane $\{x = 0\}$. Let $\widetilde{X} \subseteq X$ be a a dense subset of X defining a branch of X at 0. The image of $\widetilde{X}$ under each projection π_j is contained in $\pi_j(X)$, and we also have $|\pi_j(\widetilde{X})| = \infty$ for all j. Hence, for each $j = 1, \dots, n$, the Zariski closures of the image $\pi_j(\widetilde{X})$ must be an irreducible plain algebraic curve and equal to the Zariski closure of $\pi_j(X)$. Denote by $\widetilde{X_j}$ the branch of $\overline{\pi_j(X)}$ defined by $\pi_j(\widetilde{X})$. The power series defining $\widetilde{X_j}$ obviously satisfies the assumptions of Theorem 4.2.1 and so the existence of a Puiseux parametrization $\gamma_j(t) = (t^{l_j}, y_j(t)) \in \mathbb{C}\{t\}^2$, with some $l_j \in \mathbb{N}$, of $\widetilde{X_j}$ is guaranteed. We claim now that the $(n+1)$-tuple

$$\gamma(t) = \left(t^l, y_1(t^{\frac{l}{l_1}}), \dots, y_1(t^{\frac{l}{l_n}})\right),$$

with $l = l_1 \cdots l_n$, is a parametrization of X. To prove the claim we need to show firstly that each polynomial $g \in \mathcal{I}(X)$ satisfies $g\big(t^l, y_1(t^{\frac{l}{l_1}}), \dots, y_1(t^{\frac{l}{l_n}})\big) = 0$ and secondly that the corresponding ring homomorpism γ^* is injective. Let us define

$$S \coloneqq \{s \in \mathbb{C} \,:\, \gamma_j(s) \in \pi_j(X) \text{ for all } j = 1, \dots n\}.$$

Then obviously $\gamma(s) \in X$ for all $s \in S$ from which we conclude

$$g\big(\gamma(t)\big) = g\big(t^l, y_1(t^{\frac{l}{l_1}}), \dots, y_1(t^{\frac{l}{l_n}})\big) = 0$$

for all $g \in \mathcal{I}(X)$. Using now the fact that the set $\{\gamma(s) \,:\, s \in S\}$ defines infinitely many points on X und thus a dense subset of X (actually an analytic branch of X), the injectivity of γ^* follows from Lemma 4.2.9. □

We call a parametrization $\gamma(t)$ of X at 0, which is (up to permutation of the components of the parametrization) of the form $\gamma(t) = (t^l, y_1(t), \dots, y_n(t))$, a *Puiseux parametrization* of X at 0 if l is minimal, i.e., l is the product of the polydromy orders of the Puiseux series $y_j(t^{\frac{1}{l}})$. We call this l the *polydromy order of* γ. Further, let $m_{j,1}, \ldots, m_{j,k_j}$ be the characteristic exponents of $y_j(t^{\frac{1}{l}})$ for $j = 1, \ldots, n$. Then l and the set

$$\bigcup_{j=1}^{n} \{m_{j,1}, \ldots, m_{j,k_j}\}$$

have no common factor by the minimality of l. We call the elements of the union $\bigcup_{j=1}^{n} \{m_{j,1}, \ldots, m_{j,k_j}\}$ of sets of characteristic exponents of $y_j(t^{\frac{1}{l}})$ *the characteristic exponents* of $\gamma(t)$. As in the plane curve case, an $(n+1)$-tuple of power series $\gamma(t) \in \mathbb{C}[\![t]\!]^{n+1}$ is a Puiseux parametrization of X at a point $z \neq 0$, if additionally to the condition $\gamma(0) = z$, the vector of power series $\gamma(t) - z$ has one component equal to t^l with l minimal.

Analogously to the plane curve case, also here we have that each parametrization of an algebraic space curve is convergent. Before proving this, let us recall Artin's Approximation Theorem.

Artin Approximation Theorem. *Let $f_1, \ldots, f_k \in \mathbb{C}\{x, y_1, \ldots, y_n\}$ be convergent power series with $f_i(0) = 0$ for all i. Consider a formal solution $y(x) = (y_1(x), \ldots, y_n(x)) \in \mathbb{C}[\![x]\!]^n$ of the system*

$$f_i(x, \underline{y}) = f_i(x, y_1, \ldots, y_n) = 0, \quad \textit{for } i = 1, \ldots, k. \tag{4.8}$$

Then for each integer $\alpha \geq 1$, there exists a convergent solution

$$y_\alpha(x) = (y_{1,\alpha}(x), \ldots, y_{n,\alpha}(x)) \in \mathbb{C}\{t\}^n$$

of the system (4.8) which coincides with $y(x)$ up to degree α, i.e.,

$$y(x) \equiv y_\alpha(x) \mod (x^\alpha).$$

Nowadays, many different techniques for proving the Artin approximation theorem are known. Some of them can be found for example by M. J. Greenberg in [Gr66, Theorem 1], by M. Artin in [Ar68], in J. M. Ruiz's book [Ru93, Proposition 3.1], or by H. Hauser [Ha17] and G. Rond in [Ro18, Theorem 1.2].

Proposition 4.2.10. *Let $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ be an algebraic space curve. Consider a parametrization $\gamma(t) = (x(t), y_1(t), \ldots, y_n(t)) \in \mathbb{C}[\![t]\!]^{n+1}$ of X at the origin. Then $\gamma(t)$ is already convergent, $\gamma(t) \in \mathbb{C}\{t\}^{n+1}$.*

Proof. The strategy of the proof is first to show using the Artin approximation theorem that the pairs of the components of each parametrization of a space curve $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ define already parametrizations of plane algebraic curves. Then we use the theory of Puiseux parametrizations of plane curves to conclude the convergence.
Let $\gamma(t) = (x(t), y_1(t), \ldots, y_n(t)) \in \mathbb{C}[\![t]\!]^{n+1}$ be a parametrization of X at 0. Notice first that at least one component of $\gamma(t)$ is different from zero (otherwise, we could embed the curve in

$\mathbb{A}^n_{\mathbb{C}}$). Let us w.l.o.g. assume $x(t) \neq 0$. Use the same trick as in the proof of Corollary 4.2.3 to transform $\gamma(t)$ into $\widetilde{\gamma}(t)$ which still parametrizes X at the origin and which is of the form

$$\widetilde{\gamma}(t) = (\widetilde{x}(t), \widetilde{y}_1(t), \ldots, \widetilde{y}_n(t)) = (t^l, \widetilde{y}_1(t), \ldots, \widetilde{y}_n(t)),$$

with $l = \mathrm{ord}(x(t))$ and formal power series $\widetilde{y}_j(t) \in \mathbb{C}[\![t]\!]$, for all $j = 1, \ldots, n$. According to the Artin approximation theorem, for each integer $\alpha \geq 1$ there exists an n-tuple of convergent power series $y_\alpha(t) = (y_{1,\alpha}(t), \ldots, y_{n,\alpha}(t)) \in \mathbb{C}\{t\}^n$ with the property that $(\widetilde{x}(t), y_\alpha(t))$ parametrizes X at the origin and satisfying

$$\widetilde{\gamma}(t) \equiv (\widetilde{x}(t), y_\alpha(t)) \mod (t^\alpha).$$

Consider the projection morphisms

$$\begin{aligned} \pi_j \colon \mathbb{A}^{n+1}_{\mathbb{C}} &\to \mathbb{A}^2_{\mathbb{C}} \\ (a, b_1, \ldots, b_n) &\mapsto (a, b_j), \end{aligned}$$

for $j = 1, \ldots, n$. Let us pick an $\alpha \geq 1$ and let r_α be the minimum of the radii of convergence of all $y_{j,\alpha}$'s. The set $(\widetilde{x}(B_{r_\alpha}), y_\alpha(B_{r_\alpha})) := \{(\widetilde{x}(z), y_\alpha(z)) : |z| < r_\alpha\}$ defines a branch of X at 0. Here B_{r_α} denotes an open ball of radius r_α centred at 0. Hence, for each $j = 1, \ldots, n$, the image $\pi_j(\widetilde{x}(B_{r_\alpha}), y_\alpha(B_{r_\alpha})) = (\widetilde{x}(B_{r_\alpha}), y_{j,\alpha}(B_{r_\alpha}))$ defines a branch of the plane algebraic curve $X_j = \overline{\pi_j(X)}$, the Zariski closure of the image $\pi_j(X)$, which is irreducible (compare this with the argument in the proof of Theorem 4.2.8). Let $F_j \in \mathbb{C}[x, y]$ be the polynomial defining X_j. Then we have the following equality

$$F_j(\widetilde{x}(t), y_{j,\alpha}(t)) = F_j(t^l, y_{j,\alpha}(t)) = 0, \tag{4.9}$$

which shows that $y_{j,\alpha}(t^{\frac{1}{l}})$ is a y-root of $F_j(t, y)$.
Let us assume now by contradiction that $\widetilde{\gamma}(t) \neq (\widetilde{x}(t), y_\alpha(t))$ for all $\alpha \geq 1$. This gives us infinitely many convergent parametrizations $(\widetilde{x}(t), y_\alpha(t))$ of X and hence also infinitely many y-roots of only finitely many polynomials $F_j(t, y)$ with $j = 1, \ldots, n$, which is a contradiction to the fact that $\mathbb{C}\{x, y\}$ is a unique factorization domain and to Corollary 4.2.5. □

The classical Newton-Puiseux algorithm constructs y-roots not only for power series with coefficients from $\mathbb{C}$. Even more, it is valid for any bivariate power series $f \in K[\![x, y]\!]$, where K is an algebraically closed field of characteristic 0. Hence all statements and constructions presented in this section remain true when we replace the field of complex numbers $\mathbb{C}$ by any other algebraically closed field K of characteristic zero.

Concerning the positive characteristic case, the Newton-Puiseux theorem holds under certain assumptions also there (see e.g. [AM73, pp. 67]):

Theorem 4.2.11. *Let K be an algebraically closed field of characteristic $p > 0$. Let $f(x, y) \in K((x))[y]$ be a monic polynomial of degree n in y which is irreducible in $K((x))[y]$. Let us assume that p does not divide n. Then there exists a positive integer $m \in \mathbb{N}$ and an element $y(t) \in K((t))$ such that*

$$f(t^m, y(t)) = 0.$$

However, if the assumptions of Theorem 4.2.11 are not satisfied, there is in general no hope for Puiseux Laurent series y-roots in positive characteristic. C. Chevalley noted in his book [Ch51, pp. 64] that for any field K of characteristic $p > 0$, the Artin-Schreier polynomial $f(x,y) = y^p - y - \frac{1}{x}$ has no y-root in $\bigcup_{i\geq 1} K((x^{\frac{1}{n}}))$. Another proof for this can be found also in [Ab2] where S. Abhyankar gave a factorization of f into generalized Puiseux series with unbounded denominator. Later then, in [Ke01] and [Ke17], K. S. Kedlaya showed that in the search for y-roots in positive characteristic, the generalized Puiseux series with unbounded denominator observed by Abhyankar cannot be avoided and, moreover, he constructed explicitly an algebraic closure of $K((x))$ for any field K of positive characteristic in terms of them. More results about Puiseux parametrizations and y-roots in positive characteristic can be found for example in [St82], [Ra68], [Va97], [Va04].

Chapter 5

Open Questions and Problems

Here I collect all problems related to the topics of my PhD thesis which I have encountered during the PhD program and which have not been solved yet. All of these problems and questions are mentioned in context in the thesis. This list is intended to be used as a starting point for the followup postdoctoral research.

5.1 Geometric Invariants

(i) Find basic properties of multivariate D-algebraic power series and use them in order to construct a family of multivariate D-transcendental power series.

(ii) Given a univariate D-trascendental power seires $f(t) \in \mathbb{C}[\![t]\!]$, is it always possible to use $f(t)$ and its properties to construct a D-transcendental power series $\zeta(x_1, \dots, x_n) \in \mathbb{C}[\![x_1, \dots, x_n]\!]$ in n variables?

(iii) Clarify, which of the techniques used in Chapter 1 for construction of D-transcendental power series and D-algebraically independent families of power series can be applied also to other fields different from $\mathbb{C}$. Subsequently, study the basic properties of fields to that the techniques apply.

(iv) Prove or disprove the existence of D-transcendental power series and D-algebraically independent families of power series over fields different from $\mathbb{C}$.

(v) Find basic properties of the fields over which D-transcendental power series and D-algebraically independent families of power series do not exist.

(vi) Generalize the first proof of Theorems 1.1.5 and 1.2.2 also to higher dimensional varieties. In other words: Prove Theorems 1.3.4 and 1.4.5 in a combinatorial flavour just by use of the theory of differential field extensions.

(vii) Make clear, whether the concept of geometric invariants can be generalized also to other fields different from $\mathbb{C}$. In the cases that allow a generalization, generalize it.

5.2 Resolution of Singular Curves via Geometric Invariants

(i) Characterize crucial curvatures of a curve at a singular point by the properties of the defining ideal of the curve.

(ii) Find an implicit interpretation for Remark 2.1.6 showing that the maximal order of a parametrization of an algebraic curve at a singular point drops after each blowup in the ideal corresponding to an algebraic curvature and which thus guarantees an improvement of the singularity after the blowup. In other words: Interpret implicitly, i.e., in terms of transformations of the defining equations of a curve, the improvement of curve singularities under each blowup in the ideal given by an algebraic curvature.

(iii) Find an implicit proof for the resolution of singularities via the geometric invariants constructed with algorithms PLANECURVATURE, SPACECURVATURE and CURVATURES. In other words, prove that the crucial curvatures constructed by these algorithms yield a resolution of singularities of algebraic curves without using parametrizations.

(iv) Clarify which of the results gained in Chapter 2 are valid also over other fields of characteristic zero different from $\mathbb{C}$. In parallel to this question, study the basic properties of those fields in which our results are still valid.

(v) Generalize the results gained in Chapter 2 to fields of positive characteristic fields.

(vi) Find basic properties of the blowups corresponding to (higher) algebraic curvatures of (analytically) reducible algebraic curves.

(vii) For any curve X with more than one singularity, Theorem 2.3.3 guarantees the existence of a resolution $\tilde{X}$ of singularities of X constructed by means of geometric invariants (namely with the algorithm CURVATURES). Construct a polynomial ideal J (which according to Corollary 2.3.2 do always exist) such that $\widetilde{X}$ is the blowup of X in J.

(viii) Carry the constructions and methods established in the curve case over to surfaces.

5.3 The Moduli Space of n Points on the Projective Line and the First Fundamental Theorem for $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$

(i) Can the First Fundamental Theorems for $\mathrm{SL}_n(K)$ and $\mathrm{GL}_n(K)$, where K is an infinite field, be proven using geometric invariants? (Study for this the properties of minimal geometric invariants, i.e., geometric invariants that can be written as rational functions in $x_k^{(i_1,\dots,i_n)}$ with $i_1 + \cdots + i_n = 1$.)

(ii) Determine the generators of the invariant ring $K[\mathbb{x}]^{\mathrm{SL}_n}$ and the invariant field $K(\mathbb{x})^{\mathrm{GL}_n}$ and state the First Fundamental Theorems for $\mathrm{SL}_n(K)$ and $\mathrm{GL}_n(K)$ over a finite field K.

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Abstract

Abstract English

The main objective of this thesis is to introduce new geometric algebraic quantities of algebraic varieties describing their geometry (especially the local geometry of the varieties at their singular points) and having applications in resolution of singularities. The document is organized in three main parts.

The first part of my PhD thesis is crucial and very important for the remaining two parts. I introduce there a new concept of *algebraic curvatures* — a system of generators of so-called *geometric invariants* — of algebraic varieties which are then later the main ingredient of the resolution algorithm for singular curves presented in the second part of this thesis, and also a very important tool concerning the problem of the moduli space of n points on the projective line which is the contents of the third part. Not only the introduction of geometric invariants, but also their complete classification are done in this part of the thesis by using techniques from invariant theory, theory of differentially transcendental power series, and the theory of differential fields and their extensions. Moreover, for plane and space curves, a proof that the algebraic curvatures already determine algebraic curves completely, is provided as well.

In order to understand singularities better, we define by means of algebraic curvatures and with help of the local Puiseux parametrizations of a curve at its singular points parametric expressions that measure how far the singularities are from being smooth. A proof that these parametric expressions can be defined in terms of the implicit polynomial equations of the curve is also part of the first chapter of this thesis.

The content of the second part of this thesis is the application of algebraic curvatures to the problem of resolution of singular curves. In fact, for a given algebraic curve, among all algebraic curvatures there are some whose corresponding blowup of the curve improves the singularities. In the second chapter of this thesis, an algorithm constructing a geometric invariant (by means of algebraic curvatures) whose blowup already resolves a given singularity of a curve is presented.

In the third, and last part, of this thesis, the algebraic curvatures of higher dimensional varieties are used for the proof of the First Fundamental Theorems for $\mathrm{GL}_m(K)$ and $\mathrm{SL}_m(K)$, for $m \geq 2$ and K an arbitrary infinite fields. Actually, I show that the $\mathrm{GL}_m(K)$-invariant

elements are in one-to-one correspondence with the minimal algebraic curvatures. Moreover, counterexamples to the First Fundamental Theorems for $\mathrm{GL}_m(K)$ and $\mathrm{SL}_m(K)$ over finite fields are provided in the third chapter of this thesis as well.

Zusammenfassung deutsch

Das Hauptziel der vorliegenden Dissertation ist es, neue geometrischen algebraischen Größen von algebraischen Varietäten vorzustellen, die ihre Geometrie (vor allem die lokale Geometrie in singulären Punkten) beschreiben und eine Anwendung in der Auflösung von Singularitäten haben. Dieses Dokument ist in drei Kapitel gegliedert.

Das erste Kapitel meiner Dissertation stellt einen Baustein für die restlichen zwei Kapitel dar. Ich führe hier das Konzept von *algebraischen Krümmungen* algebraischer Varietäten ein — sie bilden ein Erzeugendensystem von sogenannten *geometrischen Invarianten*. Die algebraischen Krümmungen spielen eine wichtige Rolle in der Auflösung von singulären Kurven (das Thema des zweiten Kapitels dieser Dissertation) und gleichzeitig können auch als ein sehr nützliches Werkzeug in dem Problem des Modulraums von n Punkten auf der projektiven Gerade verwendet werden. In dem ersten Kapitel dieser Dissertation werden die algebraischen Krümmungen nicht nur definiert, sondern es werden auch ihre Eigenschaften diskutiert. Mit Hilfe von Techniken aus der Invariantentheorie, Theorie der differential-transzendenten Potenzreihen, und der Theorie der Differentialkörper und ihrer Erweiterungen werden alle geometrischen Invarianten klassifiziert. Es wird weiters gezeigt, dass jede algebraische Kurve durch ihre algebraischen Krümmungen bereits eindeutig festgelegt wird.

Um Singularitäten von Kurven besser zu verstehen, definieren wir mithilfe der algebraischen Krümmungen und lokalen Puiseux Parametrisierungen von algebraischen Kurven in ihren Singularitäten parametrisierte Größen von Kurven, die die Komplexität jeder Singularität auf der Kurve messen. Ausserdem, beschreiben wir jede dieser parametrisierten Größen via die impliziten Polynome der Kurve.

Das zweite Kapitel dieser Dissertation behandelt über die Anwendung der algebraischen Krümmungen in der Auflösungsproblematik singulärer Kurven. Es ist nicht schwer zu zeigen, dass es für jede Kurve einige algebraische Krümmungen gibt, sodass die Blowups der Kurve in den durch diese Krümmungen definierten Ideale die Singularitäten der Kurve verbessern. In diesem Kapitel wird ein Algorithmus präsentiert, der für jede Singularität einer gegebenen algebraischen Kurve eine geometrische Invariante aus den algebraischen Krümungen konstruiert, sodass das durch diese geometrische Invariante definierte Blowup bereits die Singularität auflöst.

Im dritten Kapitel der Dissertation wird eine andere Anwendung der algebraischen Krümmungen vorgestellt und zwar, sie werden für den Beweis der Ersten Fundamentalsätze für $\mathrm{GL}_m(K)$ und $\mathrm{SL}_m(K)$ für einen unendlichen Körper K verwendet. Abschliessend präsentiere ich noch Gegenbeispiele über endliche Körper zu den Aussagen der Ersten Fundamentalsätze für $\mathrm{GL}_m(K)$ und $\mathrm{SL}_m(K)$.

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Publications

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