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„Effective geometry of fuzzy $\mathbb{C}P_S^2$ and stabilization of fuzzy spaces in matrix models"

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ABSTRACT

This master thesis is composed of two loosely related topics. In the first part, we discuss properties of the effective geometry obtained in the semi-classical limit $N \to \infty$ of squashed $\mathbb{C}P^2_N$. In particular, we obtain an expression for the effective metric in toric coordinates. The second part deals with fluctuations of both, the squashed $\mathbb{C}P^2_N$ and S^2_N background. We show that, on the one hand, for the bosonic sector of the IKKT-model with an additional mass term and quartic interaction, the fluctuation operator of the squashed $\mathbb{C}P^2_N$ exhibits unphysical negative modes. On the other hand however, the vector fluctuation operator for the bosonic sector of the IKKT-model augmented by a radially symmetric potential can indeed be stabilized.

ZUSAMMENFASSUNG

Diese Masterarbeit behandelt zwei lose zusammenhängende Themenfelder. Im ersten Teil widmen wir uns den Eigenschaften der effektiven Geometrie, die sich im semi-klassischen Grenzwert $N \to \infty$ des „squashed $\mathbb{C}P^2_N$“ ergeben. Speziell konstruieren wir torische Koordinaten, in denen sich die effektive Metrik effizient explizit angeben lässt. Im zweiten Teil behandelen wir Fluktuationen über sowohl „sqashed $\mathbb{C}P^2_N$“ als auch über S^2_N. Es zeigt sich, dass – im Falle des bosonischen Sektors des IKKT-Modells mit ergänztem Masseterm und quartischem Wechselwirkungsterm – der Fluktuationsoperator, der sich für den „squashed $\mathbb{C}P^2_N$“ ergibt, unphysikalische negative Eigenwerte besitzt. Bemerkenswerterweise weißt der Fluktuationsoperator für Modelle mit allgemeinem Potential und S^2_N-Hintergrund im Gegensatz dazu ausschließlich nicht-negative Eigenwerte auf.

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1

INTRODUCTION

In the twentieth century the physics community witnessed a couple of major breakthroughs that shaped our understanding of nature on a fundamental level. The framework of general relativity, developed by Albert Einstein, presented a powerful machinery for predicting the motion of celestial objects and further the evolution of space-time itself, in the language of differential geometry. In parallel to these advances in cosmological mechanics, the prevailing conception of the laws of nature on a microscopic level was challenged by quantum mechanics. This ultimately led to the development of quantum field theory and what we now refer to as the standard model.

Despite the great success of both theories, it still remains an open problem how these two frameworks can be consistently combined to an overarching theory containing the interactions of all four fundamental forces. This is partly due to the conceptual differences of quantum field theory and general relativity. Whereas the latter purely relies on geometry, where each event can be localized at an arbitrarily small scale, the former theory implements uncertainties that impose lower bounds on such a localization process. In an oversimplified argument [1] one can argue that localizing an object at a scale Δx requires the wave-numbers to be of the order of magnitude $k \sim 1/\Delta x$ which translates to an energy of the order $E \sim \hbar/\Delta x$. In general relativity, this energy corresponds to a Schwarzschild radius of $r_s = \hbar G/\Delta x$ and hence $\Delta x \geq r_s$. Although this argument has to be taken with a grain of salt, there exist more elaborate arguments that suggest such uncertainty relations for space-time coordinates[2]. In any case, these physically motivated arguments strongly suggest the need for an enhanced theory of fundamental interactions. Unfortunately, the strategies that have shown great success in the electro-weak and strong sector do not translate seamlessly to gravity. In particular, general relativity is non-renormalizable [3]. This turns out to become problematic in the treatment of IR- and UV-divergences in the computation of higher-order Feynman diagrams. This alludes to the conjecture that we are still lacking the missing link between gravity and QFT.

There has been a lot of effort put into finding suitable candidates which may accomplish this task. Some have taken a radically different approach. Among the more popular theories are string theory and quantum loop theory. In this thesis we discuss another model, namely non-commutative geometry as implemented by matrix models. In fact the idea of non-commutative geometry dates back to the very beginnings of quantum mechanics. For an early discussion of a quantized space time, see for instance [4]. Alas, the idea lay dormant for roughly half a century and it was not until the 80s and early 90s that this approach saw a newly increase in interest again with the works of Connes, Woronowicz and others [5, 6, 7].

The most general formalism of non-commutative spaces is expressed in the language of abstract C^*-algebras. In this text, we consider a more tangible subdomain, namely Matrix models. In particular we study certain geometrical aspects of the squashed $\mathbb{C}P^2$, the squashed fuzzy projective plane [8]. On top of that, we study the bosonic sector of the IKKT model [9] complemented by a radial potential. We show that the fuzzy sphere constitutes a stable solution of this model.

This thesis is structured as follows: First we introduce the basic framework of non-commutative geometry in terms of a quantization of symplectic manifolds. We recall the fundamental definitions and give examples of simple matrix models that have been discussed in the literature. As first results, we present some geometrical properties of the semi-classical limit of the squashed fuzzy complex projective plane. In the subsequent sections, we turn to modifications of the IKKT model and study fluctuations around background solutions thereof.

1.1 PRELIMINARIES AND NOTATION

Throughout the thesis, the Einstein sum convention is adopted by default. Exceptions to this rule are explicitly stated.

1.1.1 *Lie algebras and representations*

Since most of this thesis revolves around the theory of Lie-algebras and Lie-algebra representations, we briefly want to skim through the relevant concepts of representation theory. For an in-depth discussion of the mathematical aspects, see [10, 11].

Definition 1. *A Lie algebra $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity.

Another important notion is that of a Lie-algebra representation.

Definition 2. *A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map*

$$\rho : \mathfrak{g} \to \mathrm{End}(V).$$

such that for any two $x, y \in \mathfrak{g}$,

$$\rho([x,y]) = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x).$$

We will denote the elements of (abstract) Lie algebras $x \in \mathfrak{g}$ by lower-case latin characters. When we are given some representation $\rho\colon \mathfrak{g} \to \mathrm{End}\,\mathcal{H}$ on some Hilbert space $\mathcal{H}$ and $x \in \mathfrak{g},\ v \in \mathcal{H}$ we use the shorthand notation

$$Xv := \rho(x)v$$

to declutter expressions, unless there is room for confusion. For the canonical right action of $\mathfrak{g}$ on $\phi \in \mathrm{End}\,\mathcal{H}$ we analogously write

$$\phi X = \phi\rho^\dagger(x),$$

where $\rho^\dagger$ denotes the dual representation.

Since both the fuzzy sphere and the fuzzy $\mathbb{C}P^n$ can be constructed from irreducible representations of Lie-algebras, let us recall the definition of irreducibility.

Definition 3. *A representation $\rho\colon \mathfrak{g} \to \mathrm{End}\,\mathcal{H}$ of a Lie-algebra $\mathfrak{g}$, is called* irreducible, *if the only invariant subspaces*[1] *$V \subseteq \mathcal{H}$ are trivial.*

1 $\forall v \in V, x \in \mathfrak{g}\colon Xv \in V$

1.2 NON-COMMUTATIVE SPACES

Our starting point is the theory of smooth manifolds, since this mathematical framework has turned out to set the stage for most of the modern fundamental theories of nature. Building on the well established knowledge that in classical mechanics, symplectic manifolds arise in a natural way[12], it is reasonable to impose more structure on the geometries under consideration. It is however useful to loosen the definition of a symplectic manifold slightly and to not require the symplectic form to be non-degenerate everywhere. That is where we step into the realm of Poisson manifolds. These objects can be formally defined as follows [13]:

Definition 4. *A Poisson manifold M is a smooth manifold carrying a Poisson structure, i.e. an $\mathbb{R}$-bilinear map*

$$\{\cdot,\cdot\}\colon\ \mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M),\ \ (f,g)\mapsto\{f,g\}$$

satisfying the three properties

(i) Antisymmetry: $\{f,g\}=-\{g,f\}$.

(ii) Jacobi-identity: $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$.

(iii) Leibniz rule: $\{f,gh\}=g\{f,h\}+h\{f,g\}$.

for any choice of smooth functions $f,g,h\in\mathcal{C}^\infty(M)$.

While the first two properties define a Lie-algebra on the manifold, the third property naturally assigns a vector field $X_f\in\Gamma(TM)$ to any function $f\in\mathcal{C}^\infty(M)$, referred to as *Hamiltonian vector field* of f. Clearly $X_f=\{f,\cdot\}$ exhibits all the required properties of a vector field on M. In local coordinates x^μ we can expand any Poisson bracket in the form

$$\{f,g\}=\theta^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\mu},\qquad \theta^{\mu\nu}=\{x^\mu,x^\nu\}. \tag{1}$$

Under the assumption that these local coordinates correspond to a chart on $U\subset M$ where the tensor $\theta^{\mu\nu}$ is non-degenerate, we can furthermore introduce the symplectic form

$$\{f,g\}=\omega(X_g,X_f)=\theta^{\mu\nu}\theta^{\rho\sigma}\frac{\partial g}{\partial x^\mu}\frac{\partial f}{\partial x^\rho}\omega_{\nu\sigma}. \tag{2}$$

By comparing this expression with eq. (1) it becomes clear that $\omega_{\nu\sigma}=\theta^{-1}_{\nu\sigma}$. Note that this also highlights why we needed to presuppose the non-degeneracy of the tensor $\theta^{\mu\nu}$. As we are ultimately interested in studying the semi-classical limits, we introduce an expansion parameter θ and rewrite the Poisson tensor as

$$\theta^{\mu\nu}(x)=\theta\theta_0^{\mu\nu}(x). \tag{3}$$

This expansion parameter becomes essential for a suitable definition of a quantization map.

All the concepts we have discussed so far, arise naturally in the abstract formulation of classical mechanics. To advance beyond the classical domain and obtain a mathematical model featuring “geometrical quantum effects”, we recall that the precision to which two observables can be measured simultaneously is not arbitrary. This observation can be traced back to the non-commutativity of the operators representing these observables. If we now were to interpret the coordinate functions of some symplectic manifold as non-commutative operators, this uncertainty principle carries over to the underlying geometry itself. Unfortunately there is no straightforward prescription to consistently replace the algebra of smooth functions with a non-commutative counterpart and obtain the commutative algebra in some limit, as dictated by the correspondence principle. In order to proceed further we need a precise set of quantisation rules that consistently implement such an identification. It turns out that one needs to tackle this task with a bit of caution. By attempting to solve this problem naïvely, by postulating Dirac's canonical quantization rules, one ultimately is confronted with a contradiction[2]. To overcome this inconvenience, we need to relax our axioms a bit. The result of this endeavor can be summarized as follows [1]:

Definition 5. *Let H be a Hilbert space. Given a Poisson manifold M, a quantization map is a linear mapping*

$$\mathcal{Q}\colon\ \mathcal{C}^\infty(M) \to \mathrm{End}(H),\ \ f \mapsto F \tag{4}$$

such that

(i) $\mathcal{Q}(fg) - \mathcal{Q}(f)\mathcal{Q}(g) \to 0$ *as* $\theta \to 0$

(ii) $\theta^{-1}\left(\mathcal{Q}(i\{f,g\}) - [\mathcal{Q}(f), \mathcal{Q}(g)]\right) \to 0$ *as* $\theta \to 0$

As a side note, it should be mentioned that the precise definition of those limits is non-trivial and should not concern us in this thesis in their full generality. To make things more tangible, we emphasize that the non-commutative algebras we are working with are exclusively matrix algebras, where such limits can be handled and understood.

This definition motivates the following notion: For any $f \in \mathcal{C}^\infty(M)$, let $F = \mathcal{Q}(f)$, i.e. we denote the classical (commutative) algebra elements with lower-case letters and their corresponding non-commutative counterparts with upper-case letters. We my also use the shorthand notation $f \sim F$, or $F \sim f$.

1.3 EFFECTIVE GEOMETRY OF NON-COMMUTATIVE SPACES

The goal of extracting geometric meaning purely from an abstract algebra of functions is by no means a trivial task. The generalized framework was primarily introduced and expanded by Alain Connes, however the vast topic can be made more accessible if one considers only finite-dimensional algebras, i.e. matrix algebras. Furthermore we assume the emerging geometries to be embedded in some $\mathbb{R}^D$. This enables us to view certain elements of the non-commutative algebra as quantized versions of embedding functions. We will follow the terminology established in [1].

Additionally, the commutator of two operators serves as the quantization of the Poisson structure in the dequantized manifold M, effectively defining a symplectic form on M. In particular, for some finite-dimensional vector space we assume the existence of endomorphisms X^a with

$$X^a \sim x^a \in \mathcal{C}^\infty(U; \mathbb{R}^D)$$

2 See theorem of Groenewold and van Hove [14, 15]

where x^a denote the embedding maps for some open subset $U \subset M$. Then by the axioms the following relation holds:

$$i\Theta^{ab} = [X^a, X^b] \sim i\{x^a, x^b\} = i\theta^{\mu\nu}\partial_\mu x^a \partial_\nu x^b. \tag{5}$$

At this point, we introduce the operator $\Box = g^{ab}[X_a, [X_b, \cdot]]$ where depending on whether we choose an Euclidean or Lorentzian signature, $g = \delta$ or $g = \eta$. This operator is commonly refered to as the matrix Laplace operator. It then can be shown [16] that the matrix Laplacian $\Box$ is in the semi-classical limit related to the Laplacian associated with an effective metric G acting on vector fields on the 'classical' counterpart of the manifold to be approximated by the non-commutative embedding functions. To be precise, we have

$$\Box\mathcal{Q}(f) = g^{ab}[X_a, [X_b \mathcal{Q}(f)]] \sim -\eta^{ab}\{x_a, \{x_b, f\}\} = -e^{\sigma}\Delta_G f,$$

with where we defined

$$\begin{aligned} g_{\mu\nu}(x) &:= \partial_\mu x^a \partial_{nu} x_a \quad \text{(induced metric)} \\ G^{\mu\nu}(x) &:= e^{-\sigma}\theta^{\mu\mu'}(x)\theta^{\nu\nu'}g_{\nu\nu'}(x) \quad \text{(effective metric)} \\ e^{-(n-1)\sigma} &:= \frac{1}{\theta^n}|g_{\mu\nu}(x)|^{-\frac{1}{2}}, \quad \theta^n = |\theta^{\mu\nu}|^{\frac{1}{2}}. \end{aligned} \tag{6}$$

While the two metrics g and G in general do not have coincide, there is a criterion for manifolds of dimension four. In this case, $g = G$ if and only if the symplectic form $\omega = \frac{1}{2}\theta^{-1}_{\mu\nu}\,\mathrm{d}\,x^\mu\,\mathrm{d}\,x^\nu$ is (anti-)self-dual [16].

2 EXAMPLES OF FUZZY SPACES

2.1 FUZZY SPHERE

The fuzzy sphere is among the most accessible examples of non-commutative spaces and also one of the most popular and well-studied ones. As a space with a high degree of symmetry and low dimension, it serves as an excellent model for investigating basic properties of fuzzy spaces. The circumstance that under a suitable interpretation, the conventional sphere S^2 can be approximated by a matrix algebra was first introduced and discussed in [17, 18, 19].

In order to define the fuzzy sphere precisely, we consider the classical 2-sphere embedded in $\mathbb{R}^3$ i.e.

$$S^2 = \{(x_i)_{i=1,2,3} \in \mathbb{R}^3 \mid \delta_{ij} x_i x_j = 1\}. \tag{7}$$

We obtain a fuzzy realization of that space by relating the coordinate embedding function x_i to the generators of an irreducible representation of $\mathfrak{su}(2)$. To make this statement more precise, we pick a basis $\{t_i\}_{i=1,2,3} \subseteq \mathfrak{su}(2)$. In the physics literature it is customary to pass instead to the complexified Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ and choose these elements in such a way that they obey the commutation relations

$$[t_i, t_j] = i\epsilon_{ijk} t_k \tag{8}$$

and are normalized with respect to the Killing-form, given by $B(X,Y) := \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$. The theory of finite dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$ tells us, that any irreducible representation may be assigned a unique integer, the Dynkin label. However, more commonly, we may also – up to equivalent representations – uniquely identify the irreducible representations by their respective dimensions. We will denote the N-dimensional irreducible representation simply by (N). As an example, consider the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It can be shown that these matrices span (2), also called the fundamental representation of $\mathfrak{su}(2)$.

For a particular N-dimensional irreducible matrix representation $\rho : \mathfrak{su}(2) \to \mathrm{Mat}(N,\mathbb{C})$, the corresponding Casimir element is then given by

$$C_2 = \sum_{i=1}^{3} \rho(t_i)^2 = \sum_{i=1}^{3} T_i^2 = \frac{1}{4}(N^2 - 1)\,\mathrm{Id}_N\,. \tag{9}$$

To obtain a relation between the "ordinary" manifold S^2 and its fuzzy counterpart, we associate the coordinate functions $x_i \in \mathcal{C}^\infty(S^2)$ with the endomorphisms $T_i = \rho(t_i)$. The quadratic Casimir element then implements the radial constraint $\delta_{ij} x_i x_j = 1$. The fuzzy sphere is constructed by defining the fuzzy embedding functions

$$x_i \sim X_i := \frac{2}{\sqrt{N^2 - 1}} T_i. \tag{10}$$

Note that the normalization factor is chosen such that $X^2 = X_a X_a = \mathrm{Id}_N$, i.e. this reflects the unity of the sphere's radius.

One strength of non-commutative geometries is the ability to retain symmetries of the classical manifold. As with the classical algebra of functions on S^2 that decomposes under the action of $SO(3)$ into spherical harmonics, we can define an action of $SO(3)$ on the matrix algebra that is associated with our fuzzy space. To this end, let $\phi \in \mathrm{Mat}(N, \mathbb{C})$ and consider the N-dimensional representation of $U : SU(2) \to \mathrm{Mat}(N, \mathbb{C})$ acting on $\mathrm{Mat}(N, \mathbb{C})$ in the adjoint, i.e

$$\phi \mapsto U(g)^{-1} \phi U(g). \tag{11}$$

Since the algebra of endomorphisms on $\mathbb{C}^N$ can be interpreted as the tensor product $(\mathbb{C}^N) \otimes (\mathbb{C}^N)^*$, the algebra $\mathrm{Mat}(N, \mathbb{C})$ decomposes into invariant subspaces

$$\mathrm{Mat}(N, \mathbb{C}) \cong \bigoplus_{k=1}^{N} (2k-1), \tag{12}$$

with $SU(2)$ acting irreducibly on $(2k-1)$-dimensional subspaces of $\mathrm{Mat}(N, \mathbb{C})$ in the $(2k-1)$ irreducible representation. Of course, the respective Lie algebra $\mathfrak{su}(2)$ acts on $\mathrm{Mat}(N, \mathbb{C})$ in the adjoint as well, i.e. for the N dimensional irreducible representation ρ, the action on $\mathrm{Mat}(N, \mathbb{C})$ is given by

$$\begin{aligned} [X, \cdot] : \mathrm{Mat}(N, \mathbb{C}) &\to \mathrm{Mat}(N, \mathbb{C}) \\ \phi &\mapsto [X, \phi] = \rho(x)\phi - \phi\rho(x) \end{aligned}$$

for $X = \rho(x),\ x \in \mathfrak{su}(2)$.

As stated in the introduction, the matrix Laplacian is defined as

$$\Box := \sum_{i=1}^{3} [X_i, [X_i, \cdot]]. \tag{13}$$

Note that the matrix Laplacian is invariant under the adjoint action of $SU(2)$.

To emphasize the relation to the conventional spherical harmonics, we pick a set of basis vectors in $\mathrm{End}(\mathcal{H})$, denoted by $\hat{Y}^l_m$. We will call these matrices fuzzy spherical harmonics. For $\hat{Y}^l_m \in (2k-1)$, the index l is set to $l = k-1$ and m is restricted to the range $-l, \ldots, l$. Furthermore, the eigenvalue relations

$$[X_3, \hat{Y}^l_m] = m\hat{Y}^l_m \quad \text{and} \quad \Box\hat{Y}^l_m = l(l+1)\hat{Y}^l_m,$$

are fulfilled. In other words, each of the irreducible subspaces is spanned these fuzzy spherical harmonics,

$$(2k-1) = \left\langle \left\{ \hat{Y}^{k-1}_m | m = -k+1, \ldots, k-1 \right\} \right\rangle.$$

This basis is already by construction orthogonal under the Hilbert-Schmidt norm $\mathrm{End}(\mathcal{H}) \ni \phi \mapsto \mathrm{Tr}(\phi^\dagger \phi)$. We can further assume that these fuzzy spherical harmonics are even normalized under this norm, i.e.

$$\mathrm{Tr}(\hat{Y}^{l_1}_{m_1} \hat{Y}^{l_2}_{m_2}) = \delta_{l_1 l_2} \delta_{m_1 m_2}.$$

and thus we obtain an orthonormal basis for the full space of endomorphisms $\mathrm{End}(\mathcal{H})$.

We have now collected all the necessary ingredients to construct a quantization map and relate the fuzzy spherical harmonics to the conventional spherical harmonics. Recall that any square integrable function on the ordinary 2-sphere, $f \in L^2(S^2)$, can be decomposed into a (countable) sum of spherical harmonics,

$$f(\vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} c_{lm} Y_{lm}(\vartheta, \varphi). \tag{14}$$

Similarly to the fuzzy sphere, the spherical harmonics are eigenfunctions of the Laplace-Beltrami operator. This allows us to define the quantization map by identifying $\hat{Y}^l_m$ with the corresponding spherical harmonic Y^l_m by setting

$$\begin{aligned} \mathcal{Q} : L^2(S^2) &\to \mathrm{Mat}(N, \mathbb{C}) \\ Y_{lm} &\mapsto \begin{cases} \hat{Y}_{lm} & l \leq N \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{15}$$

In other words, we can understand the algebra of functions on the fuzzy space as a truncation of the algebra of functions on S^2, where we regain all the properties of the classical manifold as we let $N \to \infty$.

2.2 FUZZY $\mathbb{C}P^n$

In the discussion on the fuzzy sphere in the previous chapter, we took inspiration from $SO(3)$ acting on the algebra of functions that comes with S^2. A similar approach can be applied to other related spaces that feature a high degree of symmetry. A very broad class of such candidates are homogeneous spaces. In particular, (co-)adjoint orbits provide a rich source of interesting examples that allow a similar construction as we have seen for the fuzzy sphere. In this section we want to restrict ourselves to complex projective spaces.

To start things of, let us repeat the most commonly encountered definition of projective spaces. Consider some vector space V over the field[1] $\mathbb{K}$. We consider the equivalence relation

$$\forall x, y \in V \setminus \{0\} : \; x \propto y \Leftrightarrow \exists \lambda \in \mathbb{K} \setminus \{0\} : x = \lambda y. \tag{16}$$

In other words, two points points in $V \setminus \{0\}$ are equivalent, iff they are contained in the same one-dimensional subspace. The space $P(V) = (V \setminus \{0\})/\propto$ is then called the *projective space of* V. The equivalency classes, corresponding to some representative $x = (x_0, \ldots, x_{n+1}) \in V \setminus \{0\}$, are commonly denoted by $[x_0 : \cdots : x_n] \in P(V)$.

For $V = \mathbb{C}^{n+1}$, we adopt the standard notation $\mathbb{C}P^n := P(\mathbb{C}^{n+1})$. Taking these definitions and notations as our jumping-off point, we want to briefly sketch the idea for how these spaces can be understood as quotients of Lie groups. Based on this construction we then introduce a suitable non-commutative counterpart.

We start by studying orbits under the adjoint[2] representation of $SU(n+1)$ for some fixed group element $t \in \mathfrak{su}(n+1)$,

$$\mathcal{O}(t) = \{\mathrm{Ad}_g \, t \mid g \in SU(n+1)\}. \tag{17}$$

1 Four our purposes, we only consider complex numbers, but this definition makes sense for any field.
2 Equivalently co-adjoint orbits.

To show that for a suitably chosen t, this orbit is diffeomorphic to $\mathbb{C}P^n$ we note that in any $(n+1)$-dimensional matrix representation of $\mathfrak{su}(n+1)$, it is possible to construct an element of the form

$$t_0 = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -n \end{pmatrix} \tag{18}$$

Clearly this matrix is invariant under a subgroup of $SU(n+1)$ that is isomorphic to $SU(n) \times U(1)$. Thus we can identify $\mathcal{O}(\tau_0)$ with the quotient $SU(n+1)/(SU(n) \times U(1))$.

Now consider $SU(n+1)$ acting on $\mathbb{C}^{n+1}$ by matrix multiplication. Since the action is transitive, we clearly obtain the $(2n+1)$-sphere by taking the $SU(n+1)$-orbit of e_{n+1}. On the other hand e_{n+1} is stabilized by

$$G = \left\{ \begin{pmatrix} U & \\ & 1 \end{pmatrix} \mid U \in SU(n) \right\} \subseteq SU(n+1), \tag{19}$$

and thus the group action is not faithful. However, we obtain a one-to-one correspondence only between S^{2n+1} and the coset space $SU(n+1)/G \cong SU(n+1)/SU(n)$. Let us proceed by considering the canonical map taking points on S^{2n+1} to points in the projective space $\mathbb{C}P^n$,

$$\iota\colon\ S^{2n+1} \to \mathbb{C}P^n,\ (z_i)_{i=1,\dots,n+1} \mapsto [z_1 : \cdots : z_{n+1}]. \tag{20}$$

This map clearly is onto and for any two points $p, q \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$, the images under ι coincide if and only if p and q differ by a phase, i.e. there exists some $\zeta \in \mathbb{C}$, $|\zeta| = 1$ such that $p = \zeta q$. Factoring out this residual $U(1)$ freedom, we have constructed a diffeomorphism between the adjoint orbit of τ_0 and the complex projective space,

$$\mathcal{O}(\tau_0) \cong SU(n+1)/(SU(n) \times U(1)) \cong \mathbb{C}P^n. \tag{21}$$

As with the sphere, we want to define the quantization map by relating a basis compatible with the decomposition of the group action to the "harmonics" operating on $\mathbb{C}P^n$. It can be shown[20] that the space of square integrable functions decomposes as

$$L^2(\mathbb{C}P^n) = \bigoplus_{k=0}^{\infty} H_{\Lambda_k} \tag{22}$$

where H_{Λ_k} denotes the irreducible representation with Dynkin label $\Lambda_k = (k, 0, \dots, 0, k)$. To obtain the fuzzy $\mathbb{C}P^n$, we proceed analogously to the case of the fuzzy sphere S_N^2 and consider ρ acting on the Hilbert space $\mathcal{H}$ in the irreducible representation $D(N, 0, \dots, 0)$ of $\mathfrak{su}(n+1)$. The generators $T_i = \rho(t_i),\ t_i \in \mathfrak{su}(n+1)$ can once more be chosen such that they are orthonormal with respect to the Killing form and satisfy the commutation relations

$$[t_i, t_j] = i f_{ijk} t_k,$$

where f_{ijk} are the structure constants of $\mathfrak{su}(n+1)$. We can again pick a orthonormal[3] basis $\{Y^{\Lambda_k}_{\boldsymbol{m},l}\} \subseteq H_{\Lambda_k}$ of weight vectors, where $\boldsymbol{m} = (m_1, \dots, m_{n-1})$ is a weight and l iterates through multiplicities of the respective weight.

3 orthonormal under the Hilbert-Schmidt inner product.

Since $\mathfrak{su}(n+1)$ is isomorphic to $\mathbb{R}^{n(n+2)}$, we can interpret $\mathcal{O}(\tau_0)$ as a sub-manifold embedded in $\mathbb{R}^{n(n+2)}$. In the spirit of the quantization map of $\mathcal{C}^\infty(S^2)$ discussed in the previous section, we identify the coordinate functions x_a with the matrices

$$x_a \sim X_a := \frac{1}{\sqrt{C_2}} T_a,$$

where C_2 is given by the Casimir operator. Once again, $g \in SU(n+1)$ acts on $\phi \in \mathrm{End}(\mathcal{H})$ by

$$\phi \mapsto U(g)^{-1} \phi U(g).$$

It is easy to show that this action on $\mathrm{End}(\mathcal{H}) \cong H \otimes \mathcal{H}^*$ decomposes into the direct sum of irreducible representations

$$\mathrm{End}(\mathcal{H}) \cong \bigoplus_{k=1}^{N} D(\Lambda_k). \tag{23}$$

Note that as representations $D(\Lambda_k)$ and H_{Λ_k} are equivalent. This enables us yet again to choose a basis of weight vectors in $\mathrm{End}(\mathcal{H})$ denoted by $\hat{Y}^{\Lambda_k}_{\boldsymbol{m},l}$. Similarly, those are eigenvectors of the Laplacian

$$\square = \sum_{a=1}^{n^2-1} [X_a, [X_a, \cdot]]. \tag{24}$$

In analogy to the fuzzy sphere, we define the quantization map of $\mathbb{C}P^n$ by

$$\begin{aligned} \mathcal{Q} : L^2(\mathbb{C}P^n) &\to \mathrm{End}(\mathcal{H}) \\ Y^{\Lambda_k}_{\boldsymbol{m},l} &\mapsto \begin{cases} \hat{Y}^{\Lambda_k}_{\boldsymbol{m},l} & k < N \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{25}$$

Since we identify the embedding functions of $\mathbb{C}P^n \subseteq \mathbb{R}^{n(n+2)}$ with the generators of $\mathfrak{su}(n+1)$, it is convenient to derive a polynomial expression in the matrices X_a that precisely produces the orbit $\mathcal{O}(\tau_0)$. To cover different normalizations, we consider the slightly more general case $\mathcal{O}(\eta\tau_0)$ for some non-zero real number η.

Lemma 1. *Let $t \in \mathfrak{su}(n+1)$ and $\eta \in \mathbb{R} \setminus \{0\}$. Then $t \in \mathcal{O}(\eta\tau_0)$ iff $(t-\eta)(t+\eta n) = 0$.*

Proof. Let $\mathcal{T} \subseteq \mathrm{End}(\mathcal{H})$ be the set of solutions of $(t-\eta)(t+\eta n) = 0$.

To show $\mathcal{O}(\tau_0) \subseteq \mathcal{T}$, we note that for any $g^{-1}\tau_0 g \in \mathcal{O}(\tau_0)$, we obtain

$$(g^{-1}\eta\tau_0 g - \eta)(g^{-1}\eta\tau_0 g + \eta n) = \eta g^{-1}(\tau_0 - \eta)(\tau_0 + \eta n) g = 0$$

directly from eq. (18). Conversely, let $t \in \mathcal{T}$. In order for the equation to hold, t must have exactly two eigenvalues[4], namely η and $-\eta n$. Hence, the only possibility of obtaining a traceless matrix, is when $t = g^{-1}\eta\tau_0 g$. ∎

4 We can identify the elements of $\mathfrak{su}(n)$ by their matrix representations. Since those matrices are hermitian, a (real) eigenvalue decomposition is always possible.

2.3 THE COMPLEX PROJECTIVE PLANE

Building upon what we have established in the previous section, we now want to restrict ourselves to $n = 2$. This projective space is also commonly referred to as the complex projective plane. The orbit $\mathcal{O}(\tau)$ is then generated by $SU(3)$. The corresponding Lie-algebra $\mathfrak{su}(3)$ is eight-dimensional. Similarly to the situation for $\mathfrak{su}(3)$, it is customary to consider its complexification $\mathfrak{sl}(3,\mathbb{C})$ and pick a basis $\{t_i\}$, such that

$$[t_i, t_j] = if_{ijk}t_k,$$

where f_{ijk} denote the completely anti-symmetric structure constants, given by

$$\begin{gathered} f_{123} = 1, \\ f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}, \\ f_{458} = f_{678} = \frac{\sqrt{3}}{2}. \end{gathered}$$

In this basis, the root generators are given by

$$t_A^{\pm} := t_1 \pm it_2, \quad t_B^{\pm} := t_4 \pm it_5, \quad t_A^{\pm} := t_6 \pm it_7.$$

To enumerate the irreducible representations, we denote the respective Dynkin labels by $D(p,q)$ for $p,q \in \mathbb{N}$. As an example, consider the fundamental representation $D(1,0)$. We may choose the set of Gell-Mann matrices as matrix representations of $\{t_i\}$ on $\mathbb{C}^3$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}$$

To construct the complex projective plane, choose τ_0 to be the eighth Gell-Mann matrix. Lemma 1 allows us to obtain a set of polynomials in $\mathbb{R}[x_1,\ldots,x_8]$ with $\mathbb{C}P^2 \subseteq \mathbb{R}^8$ being its zero-locus. In order to accomplish this, we let $t = x^a\lambda_a \in \mathfrak{su}(3)$, where $x_a \in \mathbb{R}$ and plug this into $(t - 1/\sqrt{3})(t + 2/\sqrt{3}) = 0$. In the computation we make use of the symmetric structure constants $d_{ijk} = 2\,\mathrm{Tr}(\lambda_i\{\lambda_j,\lambda_k\})$. Explicitly, d_{ijk} takes the values

$$\begin{gathered} d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, \\ d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \\ d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}. \end{gathered}$$

Note that the product of two Gell-Mann matrices can then be expressed as a linear combination of the identity matrix and the Gell-Mann matrices[5] themselves by

$$\lambda_a\lambda_b = \frac{1}{2}\left(\{\lambda_a,\lambda_b\} + [\lambda_a,\lambda_b]\right) = \frac{2}{3}\delta_{ab} + (d_{abc} + if_{abc})\lambda_c. \tag{26}$$

5 The space of three-by-three matrices is nine-dimensional and the eight Gell-Mann matrices together with the identity matrix form a linearly independent set.

With this identity in mind, we can expand the polynomial from lemma 1 and collect the coefficients for each λ_c and the coefficient of the identity matrix,

$$\begin{aligned}\left(t-\frac{1}{\sqrt{3}}\right)\left(t+\frac{2}{\sqrt{3}}\right) =& x_a x_b \lambda_a \lambda_b + \frac{1}{\sqrt{3}} x_a \lambda_a - \frac{2}{3} = \\ &= \left(\frac{2}{3} x_a x_b - \frac{2}{3}\right) + \left(\frac{1}{\sqrt{3}} x_c + (d_{abc} + i f_{abc}) x_a x_b\right) \lambda_c = \\ &= \left(\frac{2}{3} x_a x_a - \frac{2}{3}\right) + \left(\frac{1}{\sqrt{3}} x_c + d_{abc} x_a x_b\right) \lambda_c = 0\end{aligned}$$

where we used the anti-symmetry of the structure constants f_{abc} to arrive at the last line. By equating coefficients, we obtain

$$\begin{gathered} x_a x^a - 1 = 0 \\ \frac{1}{\sqrt{3}} x_c + d_{abc} x^a x^b = 0, \end{gathered} \tag{27}$$

in other words, the complex projective plane can be embedded in $\mathbb{R}^8$ as the zero-locus of the polynomials on the left-hand-side of the equations in (27).

We now aim to find a more convenient identification and rearrange the variables such that we obtain a mapping from $\mathbb{C}^2$ into patches on the complex projective plane. The computation is done by explicitly plugging in the symmetric structure constants. The final result then reads

$$\begin{aligned} x_4 x_6 + x_5 x_7 &= -\frac{1}{\sqrt{3}} x_1 (2x_8 + 1) \\ x_5 x_6 - x_4 x_7 &= -\frac{1}{\sqrt{3}} x_2 (2x_8 + 1) \\ x_4^2 + x_5^2 - x_6^2 - x_7^2 &= -\frac{2}{\sqrt{3}} x_3 (2x_8 + 1) \\ x_2 x_7 - x_1 x_6 &= x_4 \left(x_3 - \frac{1}{\sqrt{3}}(x_8 - 1)\right) \\ x_2 x_6 + x_1 x_7 &= x_5 \left(-x_3 + \frac{1}{\sqrt{3}}(x_8 - 1)\right) \\ x_1 x_4 + x_2 x_5 &= x_6 \left(x_3 + \frac{1}{\sqrt{3}}(x_8 - 1)\right) \\ x_2 x_4 - x_1 x_5 &= x_7 \left(-x_3 - \frac{1}{\sqrt{3}}(x_8 - 1)\right) \\ x_4^2 + x_5^2 + x_6^2 + x_7^2 &= -\frac{2}{3}(x_8 - 1)(2x_8 + 1). \end{aligned}$$

By interpreting the individual embedding functions x_i as real and imaginary parts of the complex parameters

$$z_1 = x_4 + i x_5, \quad z_2 = x_6 + i x_7, \quad z_3 = x_1 + i x_2 \tag{28}$$

the equations above can be cast into the more symmetric and easy-to-work-with form

$$\begin{aligned} z_1 \overline{z_2} &= \frac{1}{\sqrt{3}} z_3 (2x_8 + 1) \\ z_2 z_3 &= -\frac{1}{\sqrt{3}} z_1 \left(x_3 - \frac{1}{\sqrt{3}}(x_8 - 1)\right) \\ \overline{z_1} z_3 &= \frac{1}{\sqrt{3}} \overline{z_2} \left(x_3 + \frac{1}{\sqrt{3}}(x_8 - 1)\right) \end{aligned} \qquad \begin{aligned} |z_1|^2 - |z_2|^2 &= -\frac{2}{\sqrt{3}} x_3 (2x_8 + 1) \\ |z_1|^2 + |z_2|^2 &= -\frac{2}{3}(x_8 - 1)(2x_8 + 1). \end{aligned} \tag{29}$$

Notice that the last of the five equations is independent of x_3 as well as z_3. This allows us to rearrange the equation to find an expression for x_8

$$x_8 = \frac{1}{4}\left(1 \pm 3\sqrt{1 - \frac{4}{3}(|z_1|^2 + |z_2|^2)}\right).$$

Calling $w = x_3 + ix_8$ and solving the above equations for x_3 and z_3 we obtain a parametrization of two patches on the complex projective plane in terms of z_1 and z_2

$$\begin{aligned} w = x_3 + ix_8 &= \frac{1}{\sqrt{3}} \frac{|z_1|^2 - |z_2|^2}{1 \pm \sqrt{1 - \frac{4}{3}(|z_1|^2 + |z_2|^2)}} - \frac{i}{4}\left(1 \pm 3\sqrt{1 - \frac{4}{3}(|z_1|^2 + |z_2|^2)}\right) \\ z_3 = x_1 + ix_2 &= \frac{2}{\sqrt{3}} \frac{\overline{z_2} z_1}{1 \pm \sqrt{1 - \frac{4}{3}(|z_1|^2 + |z_2|^2)}} \end{aligned} . \quad (30)$$

The domain where these identities can be evaluated, is only restricted by the square root and the fraction. Assuming the plus sign is chosen, note that the denominator is always positive as long as the discriminant is greater or equal to zero. Thus we only need to guarantee the existence of the square roots. On the other hand, if we pick the minus sign, the denominator vanishes if and only if $|z_1| = |z_2| = 0$. In conclusion, the identities, corresponding to the case were the positive sign is chosen, are well-defined as long as z_1 and z_2 are contained in the closed ball

$$\overline{B}_{\sqrt{3}/2} = \{(z_1, z_2) \in \mathbb{C}^2\,;\, |z_1|^2 + |z_2|^2 \leq 3/4\}.$$

Conversely, for the negative sign, we need to exclude the origin and obtain the punctured closed ball $\overline{B}_{\sqrt{3}/2} \setminus \{(0,0)\}$. The maps

$$\chi^+_{4567} : \ B_{\sqrt{3}/2} \to \mathbb{R}^8, \ \begin{pmatrix} x_4 + ix_5 \\ x_6 + ix_7 \end{pmatrix} \mapsto (x_a)_{a=1,\ldots,8}$$

$$\chi^-_{4567} : \ B_{\sqrt{3}/2} \setminus \{(0,0)\} \to \mathbb{R}^8, \ \begin{pmatrix} x_4 + ix_5 \\ x_6 + ix_7 \end{pmatrix} \mapsto (x_a)_{a=1,\ldots,8},$$

where x_1, x_2, x_3 and x_8 are the functions of $x_4, \ldots, x_7$ determined implicitly by eq. (30), denote diffeomorphisms between open patches on $\mathbb{C}P^2 \subseteq \mathbb{R}^8$ and open 4-balls of radius $\sqrt{3}/2$ in $\mathbb{R}^4$ around the origin. Note that due to symmetry, we can find two more such parametrizations, namely one with respect to z_1 and z_3 and another one with respect to z_2 and z_3. We may think of these maps as charts covering "hemi-subspaces" in analogy to the hemispheres of S^2.

The polynomial equations eq. (29) also pave the way to as set of toric coordinates for this manifold. We merely need to express z_i in polar form. We first define the two angles

$$\phi_3 = \arg z_1 - \arg z_2, \qquad \phi_8 = \arg z_1 + \arg z_2. \quad (31)$$

Utilizing eq. (29) once more, we can express the modulus of each of the z_i purely in terms of x_3 and x_8. In particular, we compute

$$\begin{aligned} |z_1|^2 &= \frac{1}{2}(|z_1|^2 - |z_2|^2 + |z_1|^2 + |z_2|^2) = -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}(x_8 - 1) + x_3\right)(2x_8 + 1), \\ |z_2|^2 &= \frac{1}{2}(|z_1|^2 + |z_2|^2 - |z_1|^2 + |z_2|^2) = -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}(x_8 - 1) - x_3\right)(2x_8 + 1), \\ |z_3|^2 &= \frac{3}{(2x_8 + 1)^2}|z_1|^2|z_2|^2 = \frac{1}{3}(x_8 - 1)^2 - x_3^2. \end{aligned}$$

Putting everything together, the map

$$\overline{\psi}:\overline{\Delta}\times T^2\to\mathbb{C}^4,\ (x_3,x_8,\phi_3,\phi_8)\mapsto\begin{pmatrix} z_1(x_3,x_8,\phi_3,\phi_8)\\ z_2(x_3,x_8,\phi_3,\phi_8)\\ z_3(x_3,x_8,\phi_3,\phi_8)\\ x_3+ix_8\end{pmatrix} \tag{32}$$

then parametrizes $\mathbb{C}P^2$ embedded in $\mathbb{C}^4\cong\mathbb{R}^8$ where $T^2=S^1\times S^1$ denotes the Clifford torus, $\overline{\Delta}\subseteq\mathbb{R}^2$ is the (closed) equilateral triangle centered at the origin

$$\overline{\Delta}=\left\{\alpha_1\begin{pmatrix}0\\1\end{pmatrix}+\frac{\alpha_2}{2}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix}-\frac{\alpha_3}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}\ \middle|\ \sum_{i=1}^{3}\alpha_i=1,\ \alpha_i\geq 0\right\}$$

and

$$\begin{aligned} z_1(x_3,x_8,\phi_3,\phi_8)&=\sqrt{-\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}(x_8-1)+x_3\right)(2x_8+1)}\exp\left(i\frac{\phi_8+\phi_3}{2}\right),\\ z_2(x_3,x_8,\phi_3,\phi_8)&=\sqrt{-\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}(x_8-1)-x_3\right)(2x_8+1)}\exp\left(i\frac{\phi_8-\phi_3}{2}\right),\\ z_3(x_3,x_8,\phi_3,\phi_8)&=-\sqrt{\frac{1}{3}(x_8-1)^2-x_3^2}\exp(i\phi_3)\,. \end{aligned}$$

We can make a few noteworthy observations. The restriction $\psi=\overline{\psi}|_{\Delta}$ to the open

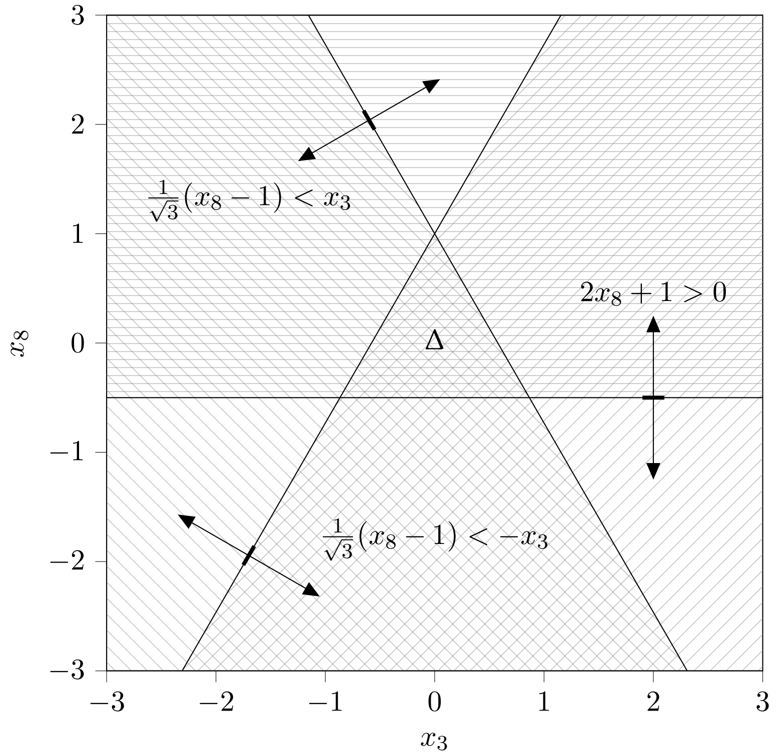

Figure 1: Domain of arguments x_3 and x_8 of the parametrization $\overline{\psi}$. The lines separate areas where the respective inequality flip its validity. Within Δ, each discriminant is positive.

triangle $\Delta\subseteq\overline{\Delta}$ is a diffeomorphism. Furthermore, the shape of Δ is no coincidence. In fact, the convexity of the pre-image of ψ^{-1} is a direct consequence of the theorems discovered by Schur, Horn and Konstant [21, 22, 23]. The idea has even be generalized further by Guillemin, Sternberg and Atiyah [24, 25].

2.3.1 *Poisson structure*

Locally, the Poisson structure can be obtained from (local) embedding functions. This can be most easily seen by recalling the local representation of the Poisson bracket introduced in eq. (1) and evaluating it on the embedding functions $x_a(x_4, x_5, x_6, x_7)$ for $a = 1, 2, 3, 8$,

$$\{x_a, x_b\} = \sum_{c,d \in \{4,5,6,7\}} \theta_{cd} \frac{\partial x_a}{\partial x_c} \frac{\partial x_b}{x_d} = \theta_{ab}.$$

A convenient way to obtain θ_{ab} is via the quantization map, since the coordinate functions are related to generators of $\mathfrak{su}(3)$. From eq. (5), we can immediately see that

$$\{f, g\} = f_{abc} x_c \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}$$

where f_{abc} are the anti-symmetric structure coordinates of $\mathfrak{su}(3)$.

We claim that the toric coordinates obtained in the previous section are almost in Darboux form. Of course we need to prove this claim. Let us first express the vector fields $\{x_3, \cdot\}$ and $\{x_8, \cdot\}$ in local coordinates. We once more turn to the commutators of the quantized embedding functions $\{x_a, x_b\} \sim -i[X_a, X_b]$ and then take the semi-classical limit. We obtain

$$\begin{aligned} -i[X_3, X_4] &= \frac{1}{2} X_5, & -i[X_8, X_5] &= -\frac{\sqrt{3}}{2} X_4, \\ -i[X_3, X_5] &= -\frac{1}{2} X_4, & -i[X_8, X_4] &= \frac{\sqrt{3}}{2} X_5, \\ -i[X_3, X_6] &= -\frac{1}{2} X_7, & -i[X_8, X_6] &= \frac{\sqrt{3}}{2} X_7, \\ -i[X_3, X_7] &= \frac{1}{2} X_6, & -i[X_8, X_7] &= -\frac{\sqrt{3}}{2} X_6. \end{aligned}$$

In other words, the two vector fields

$$\begin{aligned} \{x_3, \cdot\} &= \frac{1}{2}(x_5 \partial_4 - x_4 \partial_5 - (x_7 \partial_6 - x_6 \partial_7)) = \frac{1}{2}(L_{45} - L_{67}), \\ \{x_8, \cdot\} &= \frac{\sqrt{3}}{2}(x_5 \partial_4 - x_4 \partial_5 + x_7 \partial_6 - x_6 \partial_7) = \frac{\sqrt{3}}{2}(L_{45} + L_{67}) \end{aligned}$$

can be interpreted as the linear combination of the two "angular momentum" operators L_{45} and L_{67} generating a torus action on $\mathbb{C}P^2$. Clearly, the two vector fields are linearly independent and the two functions x_3 and x_8 are in involution, i.e. $\{x_3, x_8\} = 0$. For given values $\boldsymbol{\xi} = (\xi_3, \xi_8) \in \mathbb{R}^2$, the level sets

$$M_{\boldsymbol{\xi}} = \{(x_i)_{i=1,\ldots,8} \in \mathbb{C}P^2 \subseteq \mathbb{R}^8 \mid x_3 = \xi_3,\ x_8 = \xi_8\}$$

are diffeomorphic to the torus $T^2 = S^1 \times S^1$. This is an explicit example of Liouville's theorem[12]. If we further introduce the affine transformation

$$x_8 = \sqrt{3} h_8 + 1, \quad x_3 = h_3$$

of the coordinate functions x_8 and x_3, we obtain the parametrization

$$\psi_D(h_3, h_8, \phi_3, \phi_8) = \begin{pmatrix} \sqrt{-(h_8+h_3)(2h_8+\sqrt{3})}\exp\left(i\frac{\phi_8+\phi_3}{2}\right) \\ \sqrt{-(h_8-h_3)(2h_8+\sqrt{3})}\exp\left(i\frac{\phi_8-\phi_3}{2}\right) \\ -\sqrt{h_8^2-h_3^2}\exp\left(i\phi_3\right) \\ h_3+i(\sqrt{3}h_8+1) \end{pmatrix}.$$

The Poisson bracket of the coordinate functions h_3, h_8, ϕ_3 and ϕ_8 are then given by $\{h_3, \phi_3\} = 1 = \{h_8, \phi_8\}$ and vanishes for the other entries, i.e. the symplectic form is in standard form

$$\theta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and the coordinate system h_3, h_8, ϕ_3 and ϕ_8 constitutes a Darboux coordinate system.

2.3.2 *Induced and effective metric*

Recall that the induced metric is given by the pull-back of the metric of the embedding space. Since we work in an euclidean setting, the induced metric for some embedded manifold is given by

$$g_{ij} = \sum_{a,b=1}^{m} \frac{\partial\varphi_a}{\partial\xi_i}\frac{\partial\varphi_b}{\partial\xi_j}\delta_{ab}, \quad 1 \leq i, j \leq n$$

for some immersion $\varphi : U \to \mathbb{R}^m$, were $U \subseteq \mathbb{R}^n$ is open and $n < m$. We want to compute the induced metric of $\mathbb{C}P^2$ as a real four-dimensional manifold. Given the identification eq. (28) together with $z_4 = x_3 + ix_8 = h_3 + i(\sqrt{3}h_8 + 1)$, we note that the euclidean scalar product on $\mathbb{R}^8$ is related to the standard inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^4$ through

$$\sum_{k=1}^{8} x_k x_k' = \sum_{k=1}^{4}\Big(\operatorname{Re}(z_k)\operatorname{Re}(z_k') + \operatorname{Im}(z_k)\operatorname{Im}(z_k')\Big) = \sum_{k=1}^{4}\operatorname{Re}(z_k\overline{z}_k') = \operatorname{Re}\langle z, z'\rangle$$

for any $x, x' \in \mathbb{R}^8$. In toric coordinates, we can therefore compute the components of the metric tensor simply by computing the respective gradients of ψ and then taking the real part of the scalar product:

$$\begin{aligned}
g\left(\frac{\partial}{\partial h_3}, \frac{\partial}{\partial h_3}\right) &= \mathrm{Re}\left\langle \frac{\partial \psi_D}{\partial h_3}, \frac{\partial \psi_D}{\partial h_3}\right\rangle = 1 + \sum_{k=1}^{3}\left(\frac{\partial |z_k|}{\partial h_3}\right)^2 = \\
&= 1 + \frac{1}{4}\left(\frac{1}{|z_1|^2} + \frac{1}{|z_2|^2}\right)(2h_8 + \sqrt{3})^2 + \frac{h_3^2}{|z_3|^2} = \\
&= 1 + \frac{1}{4}\left(|z_1|^2 + |z_2|^2\right)\frac{1}{|z_3|^2} + \frac{h_3^2}{|z_3|^2} = -\frac{\sqrt{3}}{2}\frac{h_8}{|z_3|^2} \\
g\left(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_8}\right) &= \mathrm{Re}\left\langle \frac{\partial \psi_D}{\partial x_3}, \frac{\partial \psi_D}{\partial x_8}\right\rangle = \sum_{k=1}^{3}\frac{\partial |z_k|}{\partial x_3}\frac{\partial |z_k|}{\partial x_8} = \frac{\sqrt{3}}{2}\frac{h_3}{|z_3|^2} \\
g\left(\frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_8}\right) &= \mathrm{Re}\left\langle \frac{\partial \psi_D}{\partial x_8}, \frac{\partial \psi_D}{\partial x_8}\right\rangle = 3 + \sum_{k=1}^{3}\left(\frac{\partial |z_k|}{\partial x_8}\right)^2 = -\frac{\sqrt{3}}{2}\frac{2h_3^2 + \sqrt{3}h_8}{|z_3|^2(2h_8 + \sqrt{3})} \\
g\left(\frac{\partial}{\partial \phi_3}, \frac{\partial}{\partial \phi_3}\right) &= \frac{1}{4}(|z_1|^2 + |z_2|^2) + |z_3|^2 = -h_3^2 - \frac{\sqrt{3}}{2}h_8 \\
g\left(\frac{\partial}{\partial \phi_3}, \frac{\partial}{\partial \phi_8}\right) &= \frac{1}{4}(|z_1|^2 - |z_2|^2) = -\frac{1}{2}h_3(2h_8 + \sqrt{3}) \\
g\left(\frac{\partial}{\partial \phi_8}, \frac{\partial}{\partial \phi_8}\right) &= \frac{1}{4}(|z_1|^2 + |z_2|^2) = -\frac{1}{2}h_8(2h_8 + \sqrt{3})
\end{aligned}$$

All other components vanish since their respective inner product of the gradients are purely imaginary. To summarize, the induced metric tensor in Darboux coordinates takes the form

$$[g] = -\frac{\sqrt{3}}{2}\begin{pmatrix} \frac{h_8}{|z_3|^2} & -\frac{h_3}{|z_3|^2} & 0 & 0 \\ -\frac{h_3}{|z_3|^2} & \frac{2h_3^2+\sqrt{3}h_8}{|z_3|^2(2h_8+\sqrt{3})} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}}(2h_3^2 + \sqrt{3}h_8) & \frac{h_3}{\sqrt{3}}(2h_8 + \sqrt{3}) \\ 0 & 0 & \frac{h_3}{\sqrt{3}}(2h_8 + \sqrt{3}) & \frac{h_8}{\sqrt{3}}(2h_8 + \sqrt{3}) \end{pmatrix}. \tag{33}$$

To compute the effective metric, we additionally need to compute the pre-factor $e^{-\sigma}$, defined in eq. (6). The block-matrix structure of the metric in the chosen coordinates allows us to compute the determinants of the two diagonal blocks separately and multiply them together in order to obtain $|g_{\mu\nu}|$. We can make the important observation that the upper block and the lower block are in fact the inverse of each other, since

$$\det \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}}(2h_3^2 + \sqrt{3}h_8) & \frac{h_3}{\sqrt{3}}(2h_8 + \sqrt{3}) \\ \frac{h_3}{\sqrt{3}}(2h_8 + \sqrt{3}) & \frac{h_8}{\sqrt{3}}(2h_8 + \sqrt{3}) \end{pmatrix}}_{:=h} = \frac{1}{\sqrt{3}}|z_3|^2(2h_8 + \sqrt{3})$$

and thus we can immediately see that

$$\begin{pmatrix} \frac{h_8}{|z_3|^2} & -\frac{h_3}{|z_3|^2} \\ -\frac{h_3}{|z_3|^2} & \frac{2h_3^2+\sqrt{3}h_8}{|z_3|^2(2h_8+\sqrt{3})} \end{pmatrix} = \frac{\sqrt{3}}{|z_3|^2(2h_8 + \sqrt{3})}\,\mathrm{adj}\, h = h^{-1}$$

Therefore, the metric tensor in these particular coordinates simplifies to the convenient block matrix expression

$$[g] = -\frac{\sqrt{3}}{2}\left(\begin{array}{c|c} h^{-1} & \\ \hline & h \end{array}\right)$$

where the off-diagonal blocks vanish in all entries. Hence the computation of the determinant is straightforward and we obtain

$$|g| = \frac{9}{16}|h^{-1}| \cdot |h| = \frac{9}{16}.$$

Since in these coordinates, it is obvious that $|\theta|^2 = 1$, and therefore the pre-factor in the definition of the effective metric reduces to

$$e^{-\sigma} = \frac{1}{\theta^2}|g|^{-\frac{1}{2}} = \frac{4}{3}.$$

Last but not least the effective metric is given By

$$G^{\mu\nu} = e^{-\sigma}\theta^{\mu\mu'}\theta^{\nu\nu'}g_{\mu'\nu'} \quad \Leftrightarrow \quad [G]^{-1} = \frac{4}{3}(\theta^T[g]\theta) = -\frac{2}{\sqrt{3}}\left(\begin{array}{c|c} h & \\ \hline & h^{-1} \end{array}\right) = [g]^{-1}.$$

In other words $G_{\mu\nu} = g_{\mu\nu}$. In fact this identity holds on any four-dimensional Kähler manifold and since the complex projective plane belongs to this class of manifolds this is to be expected.

2.4 SQUASHED SPACES

Squashing (fuzzy) spaces[8] enable us to obtain new solutions for matrix models. In this section we review, based on the sections on the fuzzy sphere and fuzzy $\mathbb{C}P^2$, basic geometric properties of these squashed spaces. To get an intuitive understanding, we briefly want to mention how the squashed fuzzy sphere is constructed. It is implemented as a projection of the sphere along one coordinate axis. Here we choose a projection along x_3. This gives us a two-sided disc. With regards to the fuzzy sphere, squashing amounts to dropping the generator X_3. This manifests itself for instance in the Laplacian operator, where the sum is restricted to

$$\Box = [X_1, [X_1, \cdot]] + [X_2, [X_2, \cdot]].$$

Clearly this alters the spectrum of the operators governing fluctuations and the equations of motion and thus the physical implications of these background solutions.

2.4.1 *Geometry of squashed* $\mathbb{C}P^2$

The idea of projecting the fuzzy space along the Cartan generator can be generalized to the fuzzy complex projective plane by squashing the fuzzy space along the Cartan generators. This carries over to the classical counterpart as a projection along the x_3 and x_8 axis[6].

As a first result, we note that squashed $\mathbb{C}P^2$ is related to Steiners surface $\mathcal{S}$. In fact, $\mathcal{S}$ emerges as the real subset

$$\mathrm{Re}\,\mathbb{C}P^2_S = \{z \in \mathbb{C}P^2_S | \,\mathrm{Im}\, z = 0\} \tag{34}$$

of the squashed complex projective plane.[7] To confirm this claim we first recall the

6 For the embedding under consideration.
7 Up to a possible scaling.

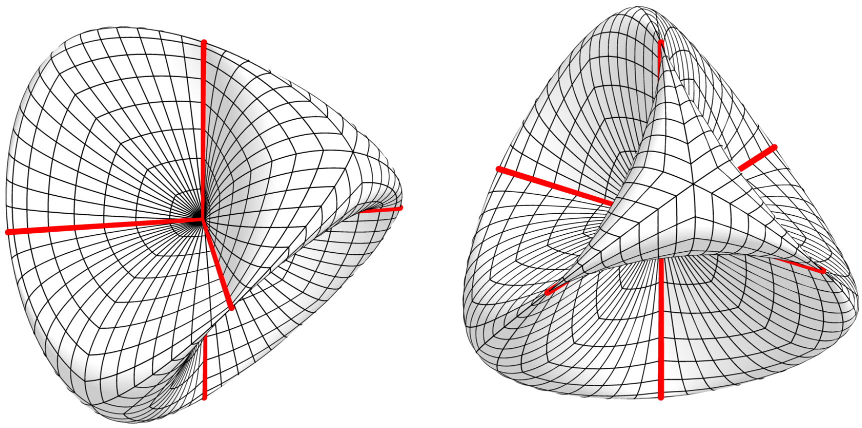

Figure 2: Steiners surface from different viewing angles. The self-intersections are highlighted by red lines. The pinching points are located at the outer ends of the self-intersection, while there is a triple intersection at the origin.

construction of $\mathcal{S}$. Consider again the unit sphere $S^2 \subseteq \mathbb{R}^3$ centered at the origin. $\mathcal{S}$ is given by the image of the map

$$F : S^2 \to \mathbb{R}^3,\ (x_1, x_2, x_3) \mapsto (x_2 x_3, x_1 x_3, x_1 x_2).$$

For any $(x_1, x_2, x_3) \in \mathcal{S}$, we ma also obtain the implicit definition

$$(x_1 x_2)^2 + (x_2 x_3)^2 + (x_1 x_3)^2 - x_1 x_2 x_3 = 0. \tag{35}$$

Applying the transformation F to the parametrization in terms of spherical coordinates, we obtain

$$\begin{aligned} x_1 &= \cos\theta \sin\theta \sin\varphi \\ x_2 &= \cos\theta \sin\theta \cos\varphi \\ x_3 &= \cos^2\theta \cos\varphi \sin\varphi \end{aligned} \tag{36}$$

as possible parametrization of Steiners surface. Next we want to briefly review the observation that $\mathcal{S}$ is an immersion up to six pinching points. To this end, let $\hat{F}$ be the extension of F to $\mathbb{R}^3$. The differential map of $\hat{F}$ is given by

$$\mathrm{d}\hat{F} = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}. \tag{37}$$

Since $\det \mathrm{d}\hat{F} = 2x_1 x_2 x_3$, the differential map is an isomorphism if and only if (x_1, x_2, x_3) does not reside on any of the three coordinate planes. To locate the points where the restriction $\mathrm{d}F = \mathrm{d}\hat{F}|_{S^2}$ is not an isomorphism, we seek out those points where the kernel of $\mathrm{d}\hat{F}$ intersects with the tangent plane to S^2 non-trivially. W.l.o.g, let $x_1 = 0$ and observe that

$$\ker \mathrm{d}\hat{F}|_{x_1=0} = \left\langle \{(0, -x_2, x_3)^T\} \right\rangle.$$

This is a sub-space of the spheres tangential plane exactly at the points where it is normal to the spheres surface normal, i.e.

$$(x_1, x_2, x_3)^T \begin{pmatrix} 0 \\ -x_2 \\ x_3 \end{pmatrix} = x_3^2 - x_2^2 = 0 \Leftrightarrow x_2^2 = x_3^2.$$

Plugging everything into $x_1^2 + x_2^2 + x_3^2 = 1$, we obtain the critical points. Repeating this procedure for $x_2 = 0$ and $x_3 = 0$, we obtain similar conditions and in total, after mapping these twelve points with the aforementioned map F, we find the six singular points of $\mathcal{S}$ also denoted pinch points. They are highlighted in fig. 2.

We now want to present a proof for the claim that the real subset of squashed $\mathbb{C}P^2$ coincides with Steiners surface.

Theorem 1. *Steiners surface $\mathcal{S}$ and the real subset $\operatorname{Re}\mathbb{C}P^2_{S}$ of the squashed complex projective plane coincide up to a uniform scaling factor.*

Proof. In order to align the scaling of the two surfaces, consider the affine transformation

$$\alpha = \frac{1}{\sqrt{3}}(x_8 - 1) + x_3, \quad \beta = \frac{1}{\sqrt{3}}(x_8 - 1) - x_3. \tag{38}$$

In terms of α and β, the action angle coordinates can be written as

$$x_1 = \sqrt{\frac{\alpha\beta}{3}}, \quad x_4 = \sqrt{-\left(\frac{\alpha+\beta}{3} + \frac{1}{\sqrt{3}}\right)\alpha}, \quad x_6 = \sqrt{-\left(\frac{\alpha+\beta}{3} + \frac{1}{\sqrt{3}}\right)\beta}.$$

One can verify by direct computation that x_1, x_4 and x_6 are a solution to eq. (35). Indeed we have

$$x_4^2 x_6^2 + x_1^2 x_6^2 + x_1^2 x_4^2 = \left(\frac{\alpha+\beta}{3} + \frac{1}{\sqrt{3}}\right)\frac{\alpha\beta}{\sqrt{3}} = x_1 x_4 x_6,$$

thus confirming that $\operatorname{Re}\mathbb{C}P^2_S \subseteq \mathcal{S}$.

In order to see the equality of the two sets, we consider the immersion

$$\begin{aligned} \psi\colon \mathbb{R}^4 &\to \mathbb{C}^3 \\ (x_3, x_8, \phi_3, \phi_8) &\to (z_1, z_2, z_3), \end{aligned} \tag{39}$$

constructed from the relations eq. (32). Without loss of generality, let L be the lobe[8] of $\mathcal{S}$ for which $\psi(x_3, x_8, 0, 0) \in L$. The proof for the other three lobes is analogous. Clearly the lobes of Steiners surface are homeomorphic so S^2. Recall that the domain Δ of the two parameters x_3 and x_8 is an equilateral triangle centered in $\mathbb{R}^2$. Restricted to the interior of the triangle, the map $\psi_0(x_3, x_8) = \psi(x_3, x_8, 0, 0)$ is a homeomorphism. On the boundary we find the three different cases:

- The three vertices are all mapped to the origin
- Each edge center is mapped to a unique point in the image of ψ_0.
- Apart from the vertices and the edge centers, ψ_0 is two-to-one.

The quotient space $\Delta/\sim$, where the $\sim$ denotes the equivalence relation induced by ψ_0,

$$x \sim y \Leftrightarrow \psi_0(x) = \psi_0(y),$$

is then homeomorphic to the image of ψ_0. Note that this equivalence relation is most transparently interpreted as folding an equilateral triangle along the dotted lines illustrated in fig. 3 and gluing the edges together. The resulting tetrahedron is homeomorphic to a 2-sphere, and thus so is $\psi_0(\Delta)$.

8 This set includes the self-intersection at the coordinate axes and the origin

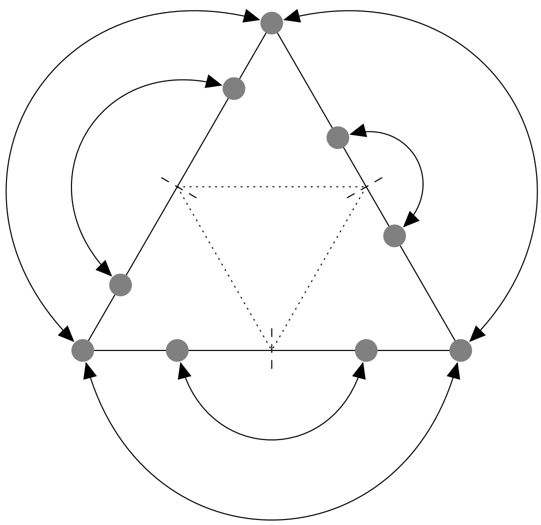

Figure 3: Illustration of the equivalence relation $\sim$

Figure 4: Visualization of the real subset of the graphs generated by B_i.

This however means that $\psi_0(\Delta) \subseteq L$ cannot be a proper subset, as S^2 does not contain proper subsets that are homeomorphic to itself[9] and therefore the two sets must indeed be the same. ■

The squashed $\mathbb{C}P^2$ can be recovered from $\psi_0(\Delta)$ by "dragging" the points contained in the lobe through $\mathbb{C}^3$ by means of the torus action that we obtained from the action angle parametrization:

$$\psi_0(\Delta) \mapsto \exp\left(i\frac{\varphi_8}{2\sqrt{3}}\begin{pmatrix}1 & & \\ & 1 & \\ & & 0\end{pmatrix}\right)\exp\left(i\varphi_3\begin{pmatrix}\sqrt{3} & & \\ & -\sqrt{3} & \\ & & 1\end{pmatrix}\right)\psi_0(\Delta) \tag{40}$$

While we can immediately conclude that the pinching points will remain singularities under this map, resulting in three independent tori[10]. This is analogous to the circle connecting the two disks of the squashed sphere. It remains to be shown that these tori contain all points that exhibit a singularity. While for any point away from the coordinate axes, the action angle coordinates provide four linearly independent vectors spanning tangent spaces, the computation for points located on the tori intersecting the coordinate axes requires a different chart. Conveniently we already constructed a suitable parametrization. Recall eq. (30) and consider the maps

$$\mathcal{B}_i : \mathbb{C}^2 \supset D_{\sqrt{3}/2}(0) \to \mathbb{C}^4, (z_i)_{i=1,2} = \begin{pmatrix} x_4 + ix_5 \\ x_6 + ix_7 \end{pmatrix} \mapsto \begin{cases} (z_1, z_2, z_3^+) & \ldots i = 1 \\ (z_1, z_3^+, z_2) & \ldots i = 2 \\ (z_3^+, z_1, z_2) & \ldots i = 3 \end{cases}. \tag{41}$$

9 This is in fact a corollary of a more general result: Any n-sphere cannot have a n-sphere as a proper subset: assume $A \subset S^n \subset \mathbb{R}^{n+1}$ were homeomorphic to S^n. W.l.o.g assume the north-pole to be in $S^n \setminus A$ and let p the stereographic projection of $S^n \supset A$ to $\mathbb{R}^n$, then $P|_A$ would be a bijective and continuous map from $A \cong S^n$ to $\mathbb{R}^n$, in contradiction to the Borsuk-Ulam theorem.

10 Note that there are always two pinching points belonging to the same pinching torus.

They are clearly differentiable with respect to the real and imaginary components of z_1 and z_2 with a differentiable inverse[11]. The tangent spaces are also of rank four. Most importantly, any coordinate axis is within the domain of any two of those functions.

2.4.2 *Effective metric of squashed* $\mathbb{C}P^2_S$

To compute the effective metric we build upon the results from the previous chapter. The projection

$$(z_i)_{i=1,\ldots,4} \to (z_i)_{i=1,\ldots,3}$$

affects the induced metric in precisely two entries of the metric tensor, namely

$$g_S\left(\frac{\partial}{\partial h_3}, \frac{\partial}{\partial h_3}\right) = \sum_{i=1}^{3}\left(\frac{\partial |z_i|}{\partial h_3}\right)^2 = g\left(\frac{\partial}{\partial h_3}, \frac{\partial}{\partial h_3}\right) - 1,$$
$$g_S\left(\frac{\partial}{\partial h_8}, \frac{\partial}{\partial h_8}\right) = \sum_{i=1}^{3}\left(\frac{\partial |z_i|}{\partial h_8}\right)^2 = g\left(\frac{\partial}{\partial h_8}, \frac{\partial}{\partial h_8}\right) - 3$$

All the other entries are untouched by the squashing procedure. In other words, we can express the induced metric of the squashed complex projective plane as follows

$$\begin{aligned}[g_S] =&[g] - \left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 3 & & \\ \hline & & 0 & 0 \\ & & 0 & 0 \end{array}\right) = \\ =& -\frac{\sqrt{3}}{2}\left(\begin{array}{c|c} h^{-1} + \frac{2}{\sqrt{3}}\begin{pmatrix} 1 & 0 \\ 0 & 3\end{pmatrix} & \\ \hline & h \end{array}\right) =: -\frac{\sqrt{3}}{2}\left(\begin{array}{c|c} \tilde{h} & \\ \hline & h \end{array}\right)\end{aligned}$$

Lemma 2. *Let $A, B \in \mathrm{Mat}(2, \mathbb{C})$ two invertible matrices, such that the sum $A + B$ is invertible as well. Then*

$$(A+B)^{-1} = \frac{|A|}{|A+B|}A^{-1} + \frac{|B|}{|A+B|}B^{-1}.$$

Proof. Clearly the adjoint of a 2×2-matrix is linear and thus

$$|A+B|(A+B)^{-1} = \operatorname{adj}(A+B) = \operatorname{adj} A + \operatorname{adj} B = |A|A^{-1} + |B|B^{-1}. \qquad \blacksquare$$

By analogous arguments as in the previous chapter we obtain the effective metric by evaluating the expression

$$\begin{aligned}G_S =& e^{\sigma}(\theta^T g_S \theta)^{-1} = -\frac{2}{\sqrt{3}}\sqrt{|g_S|}\left(\begin{array}{c|c} h^{-1} & \\ \hline & \tilde{h}^{-1}\end{array}\right) = \\ =& -\frac{\sqrt{3}}{2}\sqrt{|h\tilde{h}|}\left(\begin{array}{c|c} h^{-1} & \\ \hline & \frac{|h^{-1}|}{|\tilde{h}|}h \end{array}\right) - \frac{\sqrt{|g_S|}}{|\tilde{h}|}\left(\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & & \\ \hline & & 3 & 0 \\ & & 0 & 1 \end{array}\right) = \\ =& -\frac{\sqrt{3}}{2}\left(\begin{array}{c|c} \eta^{-1} & \\ \hline & \eta \end{array}\right) - \sqrt{\frac{|h|}{|\tilde{h}|}}\left(\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & & \\ \hline & & 3 & 0 \\ & & 0 & 1 \end{array}\right)\end{aligned}$$

11 The inverse is given by the respective projections to z_1 and z_2.

where we used lemma 2 to expand the inverse of $\tilde{h}$ and introduced the rescaling of the matrix h

$$\eta := \frac{h}{\sqrt{|h\tilde{h}|}}.$$

Finally we can obtain an expression for the determinant $|\tilde{h}|$. To this end, let us derive the following useful formula.

Lemma 3. *Let $A, B \in \mathrm{Mat}(2, \mathbb{C})$ and assume further that A is invertible. Then the formula for the determinant of the sum of A and B*

$$|A + B| = |A| + |B| + |A| \cdot \mathrm{Tr}(A^{-1}B)$$

holds.

Proof. Consider any $B \in \mathrm{Mat}(2, \mathbb{C})$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ the two diagonal entries in its Jordan normal form. Then

$$|I_2 + B| = (1 + \lambda_1)(1 + \lambda_2) = 1 + |B| + \mathrm{Tr}(B).$$

Now, for any invertible matrix A, we find

$$\begin{aligned} |A + B| =& |A(I_2 + A^{-1}B)| = |A| \cdot |I_2 + A^{-1}B| = \\ =& |A| \cdot (1 + |A^{-1}B| + \mathrm{Tr}(A^{-1}B)) = \\ =& |A| + |B| + |A| \cdot \mathrm{Tr}(A^{-1}B). \end{aligned}$$

■

Therefore, we find

$$\begin{aligned} |\tilde{h}| =& \left| h^{-1} + \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \right| = 4 + |h^{-1}| + 2\sqrt{3}\,\mathrm{Tr}\left(\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}^{-1} h^{-1} \right) = \\ =& 4 + |h|^{-1} + 2\sqrt{3} \frac{h_8}{|z_3|^2} + \frac{2}{\sqrt{3}} \frac{2h_3^2 + \sqrt{3}h_8}{|z_3|^2(2h_8 + \sqrt{3})}. \end{aligned}$$

This concludes the computation the effective metric of squashed $\mathbb{C}P^2$. At this point, we want to stress that for the squashed $\mathbb{C}P^2$, the effective and induced metric do not coincide anymore. This reflects the fact that squashing breaks the Kähler nature of the space.

3 SQUASHED $\mathbb{C}P^2_N$ SOLUTIONS OF MATRIX MODELS

3.1 SYM WITH CUBIC POTENTIAL

The fuzzy squashed complex projective plane was introduced in [8] in an effort to include chiral modes into matrix models. These modes can appear at the intersections of fuzzy branes, which marks a significant step forward towards more realistic matrix models. To start with, we briefly summarize the scenario presented in [8].

The initial model is best described in terms of a 10-dimensional SYM, reduced to four dimensions. Additionally a cubic potential is added to the Lagrangian. Given those preliminaries, the action takes the form

$$\begin{aligned} S_{YM} = \int \mathrm{d}^4x \operatorname{tr}_N \Big(&-\frac{1}{4g^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}D^\mu\Phi^a D_\mu\Phi_a + \\ &+ \frac{1}{4}g^2[\Phi^a,\Phi^b][\Phi_a,\Phi_b] + \frac{i}{2}g_{abc}\Phi^a\Phi^b\Phi^c \\ &+ \overline{\Psi}\gamma^\mu(i\partial_\mu + [\mathcal{A}_\mu, .])\Psi + g\overline{\Psi}\Gamma^a[\Phi_a,\Psi]\Big) . \end{aligned} \tag{42}$$

The notation is in accordance to standard quantum field theory. $\mathcal{F}$ denotes the field strength tensor explicitly given by

$$\mathcal{F}_{\mu\nu} = \partial\mathcal{A}_\mu - \partial\mathcal{A}_\nu + g[\mathcal{A}_\mu,\mathcal{A}_\nu]$$

The field Φ corresponds to the scalar fields of the theory with indices $a \in \mathcal{I} = \{1,\ldots,6\}$ and Ψ corresponds to a (matrix-valued) Majorana-Weyl spinor. The gamma matrices γ^μ and Γ^a generate two different Clifford algebras, the former being simply the four Dirac matrices, and the latter $Cl(6,0)$.

While the theory contains a $U(N)$ gauge symmetry, which acts in the adjoint on all involved fields, the $SO(6)$ symmetry acting on Φ by $SO(6) \ni R,\ (R \triangleright \Phi)^a = R^{ab}\Phi_b$ is – apart from a finite subgroup – broken by the cubic term. In addition, due to the total anti-symmetry of the coupling coefficients f, the translational symmetry $\Phi_a \to \Phi_a + c_a,\ c_a \in \mathbb{C}$ is preserved[1].

We are only interested in static solutions, and therefore we omit the kinetic terms. Also, to further emphasize the geometric content of the scalar fields, we denote them X instead of Φ. The matrix model action derived from eq. (42) is then given by

$$S_6[X,\Psi] = \frac{1}{4}\operatorname{Tr}\Big([X_a,X_b][X^a,X^b] + 2ig_{abc}X^aX^bX^c + 4\overline{\Psi}\Gamma_a[X^a,\Psi]\Big), \tag{43}$$

where X_a denote elements of $\mathrm{Mat}(N,\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^N)$. As a consequence of the cyclic invariance of the trace, the action is invariant under transformations $X^a \to UX^aU^{-1}$ for $U \in \mathrm{GL}(N,\mathbb{C})$ and in particular invariant under $U(N)$.

On top of only regarding the static solutions, we will dismiss the fermion term in this thesis and focus for all purposes on the solutions of the model $S_6[X] := S_6[X,0]$. We briefly recapitulate that squashed $\mathbb{C}P^2$ is in fact a solution to the corresponding equations of motion. As usual, we find the e.o.m. in analogy to calculus of variations.

1 To see this, observe that $\operatorname{Tr}(g_{abc}(X^aX^bc^c + X^aX^bc^c + X^aX^bc^c)) = g_{abc}\operatorname{Tr}(X^aX^b)c^c = 0$. The last identity follows from the anti-symmetry of f in a and b, while $\operatorname{Tr}(X^aX^b)$ is symmetric in those indices.

Consider small fluctuations[2] $X_a \to X_a + \phi_a$, for $\phi_a \in \mathrm{Mat}(N, \mathbb{C})$, around some given matrix configuration X^a. According to protocol, we assume stationarity of the action around X^a and deduce

$$\begin{aligned} 0 &= \delta S_6[X+\phi] = S_6[X+\phi] - S_6[X] = \\ &= \frac{1}{4}\,\mathrm{Tr}\left(4[X_a, X_b][X^a, \phi^b] + 6ig_{abc}X^aX^b\phi^c\right) + \mathcal{O}(\phi^2). \end{aligned} \tag{44}$$

Using the cyclic property of the trace and the identity

$$\mathrm{Tr}([A,B][C,D]) = \mathrm{Tr}(-D[C,[A,B]]), \tag{45}$$

eq. (44) easily can be further simplified to

$$0 = \mathrm{Tr}\left(\left(-[X^a,[X_a,X_c]] + \frac{3}{2}ig_{abc}X^aX^b\right)\phi^c\right) + \mathcal{O}(\phi^2). \tag{46}$$

By employing the standard argument that this identity is invariant under any small perturbation ϕ and therefore the first factor in the trace ought to vanish, the equations of motions are found to be

$$\Box X_c = \frac{3}{2}ig_{abc}X^aX^b, \tag{47}$$

where we once again make use of the definition of the matrix Laplacian.

With regard to the fuzzy sphere, consider the N-dimensional representations $\bar{X}_a$ of generators of $x_a \in \mathfrak{su}(2)$ as introduced in chapter two, such that $\bar{X}_a\bar{X}^a = \frac{1}{4}(N^2-1)$. We may allow for an additional radial freedom by setting $X_a = r\bar{X}_a$. Plugging everything into eq. (47) a direct computation leads us to

$$\Box_{S^2}X_c = r^3[\bar{X}_a,[\bar{X}^a,\bar{X}_c]] = \begin{cases} r^3[\bar{X}_2,[\bar{X}_2,\bar{X}_1]] = -ir^2X_1 & \dots c=1 \\ r^3[\bar{X}_1,[\bar{X}_1,\bar{X}_2]] = -ir^2X_2 & \dots c=2 \end{cases}.$$

for the left-hand side.

Correspondingly, we find for the equation's right-hand side the expressions

$$\frac{3}{2}ir^2g_{abc}\bar{X}^a\bar{X}^b = \begin{cases} \frac{3}{2}i(r^2g_{231}\bar{X}_2\bar{X}_3 + r^2g_{321}\bar{X}_3\bar{X}_2) = \frac{3}{2}irg_{231}X_1 & \dots c=1 \\ \frac{3}{2}i(r^2g_{132}\bar{X}_1\bar{X}_3 + r^2g_{312}\bar{X}_3\bar{X}_1) = -\frac{3}{2}irg_{132}X_2 & \dots c=2 \end{cases}.$$

The two sides equal each other, if we set $g_{abc} = -\frac{2}{3}r\epsilon_{abc}$.

In the same fashion, the squashed $\mathbb{C}P^2_N$ constitutes a solution to the matrix model. To prove that the matrices X_a for $a \in \mathcal{I} := \{1,2,4,5,6,7\}$ are a solution of the e.o.m., we can in a first step compute the matrix Laplacian. Consider the root generators

$$\begin{aligned} X_1^{\pm} &= \frac{1}{2}(X_4 \pm iX_5), \\ X_2^{\pm} &= \frac{1}{2}(X_6 \mp iX_7), \\ X_3^{\pm} &= \frac{1}{2}(X_1 \pm iX_2) = \pm[X_1^{\pm}, X_2^{\pm}]. \end{aligned} \tag{48}$$

2 Small in the sense of a small Hilbert-Schmidt norm.

Making use of the bilinearity of the commutator we derive, by direct computation, the three identities

$$
\begin{aligned}
{[X_1^+,[X_1^-,\cdot]]+[X_1^-,[X_1^+,\cdot]]} &= \frac{1}{2}\left([X_4,[X_4,\cdot]]+[X_5,[X_5,\cdot]]\right), \\
{[X_2^+,[X_2^-,\cdot]]+[X_2^-,[X_2^+,\cdot]]} &= \frac{1}{2}\left([X_6,[X_6,\cdot]]+[X_7,[X_7,\cdot]]\right), \\
{[X_3^+,[X_3^-,\cdot]]+[X_3^-,[X_3^+,\cdot]]} &= \frac{1}{2}\left([X_1,[X_1,\cdot]]+[X_2,[X_2,\cdot]]\right).
\end{aligned}
$$

Using these, we can express the matrix Laplacian in terms of these root generators,

$$
\Box_S = 2\sum_{c=1}^{3}([X_c^+,[X_c^-,\cdot]]+[X_c^-,[X_c^+,\cdot]]). \tag{49}
$$

It is now fairly straightforward to compute $\Box_S X_a^\pm$, by making use of the root generator's commutation relations. In any case one finds

$$
\Box_S X_a^\pm = \frac{2}{C_2}X_a^\pm \quad \forall a = 1,\dots,3. \tag{50}
$$

From the linear independence of the generators $X_a,\ a\in\mathcal{I}$, it thus follows that $\Box_S X_a = \frac{2}{C_2}X_a$ for all $a\in\mathcal{I}$.

As for the right-hand side, we first provide the useful identity

$$
\sum_{a,b\in\mathcal{I}} f_{abc}f_{abd} = \delta_{cd}. \tag{51}
$$

Let us briefly verify that this equation is indeed satisfied. Given the structure constants of $\mathfrak{su}(3)$, we can explicitly state the two matrices $(F_3)_{ab} := -if_{3ab}$ and $(F_8)_{ab} := -if_{8ab}$,

$$
F_3 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
F_8 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Since $(F_a)_{bc} = -if_{abc}$ can be considered as matrices acting in the adjoint representation on $\mathbb{C}^8$, we know from the quadratic Casimir operator that

$$
3\delta_{cd} = \sum_{a=1}^{8}\sum_{b=1}^{8}(F_a)_{cb}(F_a)_{bd} = \sum_{a,b=1}^{8} f_{abc}f_{abd}
$$

and splitting up this sum we obtain the equation

$$
3\delta_{cd} = \sum_{a,b\in\mathcal{I}} f_{abc}f_{abd} + 2\sum_{a\in\{3,8\}}\sum_{b=1}^{8} f_{abc}f_{abd} - 2f_{38c}f_{38d}
$$

From the matrices F_3 or F_8, it becomes immediately clear that the third term vanishes for any choice of c and d. The second term can be further condensed into the expression

$$
\sum_{a\in\{3,8\}}\sum_{b=1}^{8} f_{abc}f_{abd} = \sum_{b=1}^{8} f_{3bc}f_{3bd} + \sum_{b=1}^{8} f_{8bc}f_{8bd} = -(F_3^2+F_8^2)_{cd}.
$$

Restricting the indices c, d to the set $\mathcal{J}$, the right-hand side of the equation then simplifies to

$$-(F_3^2 + F_8^2)_{cd} = \delta_{cd}.$$

Plugging everything back into eq. (52), we obtain

$$3\delta_{cd} = \sum_{a,b\in\mathcal{I}} f_{abc} f_{abd} + 2 \sum_{a\in\{3,8\}} \sum_{b=1}^{8} f_{abc} f_{abd} = \sum_{a,b\in\mathcal{I}} f_{abc} f_{abd} + 2\delta_{cd}$$

This proves eq. (51).

To proceed with the proof that squashed $\mathbb{C}P^2$ is a solution to the equation of motion eq. (47), it is straightforward to evaluate $\frac{3}{2} i g_{abc} X^a X^b$. Making use of the anti-symmetry property of g_{abc} and defining $f_{abc} = -\frac{8}{3\sqrt{C_2}} f_{abc}$

$$g_{abc} X^a X^b = \frac{1}{2} g_{abc} [X^a, X^b] = \frac{i}{2\sqrt{C_2}} g_{abc} f_{abd} X_d = -\frac{i}{3} g_{abc} g_{abd} X_d = -\frac{i}{3} X_c. \tag{52}$$

By comparing this with eq. (50), we finally obtain an equation of the same form as eq. (47),

$$\Box_S X_c = \frac{3}{2} i f_{abc} X^a X^b. \tag{53}$$

Allowing for an additional radial freedom with parameter r alters the definition of the antisymmetric coefficients by an additional factor r.

3.2 SYM WITH MASS TERM AND QUARTIC POTENTIAL

Inspired by the previous chapter, we drop the cubic term and consider an action with mass and quartic interaction term

$$\begin{aligned} S[X] &= \frac{1}{4} \operatorname{Tr} \left([X_a, X_b][X^a, X^b] - 2m^2 X_a X^a - 2\lambda (X_a X^a)^2 \right) = \\ &= S_{\text{kin}}[X] + S_{\text{mass}}[X] + S_{\text{quart}}[X] \end{aligned} \tag{54}$$

The equations of motion are derived in the same way as eq. (47) by varying the background $X_a \to X_a + \phi_a$ and collecting terms of first order in ϕ_a. The result is then given by

$$\Box X^a + m^2 X^a + \lambda \{X_c X^c, X^a\} = 0. \tag{55}$$

Note that in the sum $X^2 = X_c X^c$, the indices run only over $\mathcal{I}$. In order for squashed $\mathbb{C}P_N^2$ to be a solution, either of the following two conditions must be fulfilled:

(i) $\lambda = 0$ and $m^2 = -2$.

(ii) $N = 1$ and $m^2 + 2\lambda = -2$.

In contrast to the fuzzy sphere, were we can retain a non-vanishing quartic interaction term for any irreducible representation, the situation for squashed fuzzy $\mathbb{C}P^2$ is a lot more restrictive.

To see this, once more it is convenient to work with the matrix Laplacian in the form introduced in eq. (49). Working out the commutators, all but three terms are non-vanishing, namely

$$\begin{aligned}
[X_1^+,[X_1^-,X_1^+]] &= \frac{1}{2}X_1^+,\\
[X_3^+,[X_3^-,X_1^+]] &= \frac{1}{4}X_1^+,\\
[X_2^-,[X_2^+,X_1^+]] &= \frac{1}{4}X_1^+.
\end{aligned}$$

This computation can be repeated for any $X_k^\pm$, $k=1,\dots,3$ with similar results. Plugging everything back into eq. (49), the action of the matrix Laplacian on the root generators can be summarized by

$$\Box X_k^\pm = 2X_k^\pm, \quad k=1,\dots,3. \tag{56}$$

To work out the anti-commutator term in the equation of motion, note that we are able to rewrite the X^2 term using the quadratic casimir operator,

$$X^2 = C_2 - X_3^2 - X_8^2. \tag{57}$$

If we require that the squashed fuzzy $\mathbb{C}P^2$ is a solution to eq. (55), it is clear that $\{X^2, X_k^\pm\} \propto X_k^\pm$. This is only possible however, if and only if $X^2 \propto \mathrm{Id}_{\mathcal{H}}$. One can choose a basis such that the X_3 and X_8 are diagonal and properly normalized w.r.t. the Killing form. For $N=1$, we can see, by considering the Gell-Mann matrices, that

$$X^2 = \mathrm{Id}_3.$$

This of course implies that

$$\Box X_k^\pm + m^2 X_k^\pm + \lambda\{X_cX^c, X_k^\pm\} = (2+m^2+2\lambda)X_k^\pm = 0 \tag{58}$$

and thus we obtain the condition $m^2+2\lambda=-2$ as advertised.

Conversely, for $N>1$, the main issue is that X^2 fails to be proportional to the identity operator and thus the squashed fuzzy $\mathbb{C}P^2$ cannot be a solution to this model anymore, unless we drop the interaction term, i.e. we set $\lambda=0$. In the next section, we study fluctuations around this background for these two cases.

3.3 BACKGROUND FLUCTUATIONS FOR $N>1$

In this section we study fluctuations of the background and aim to understand the spectrum of the operator that determines the behavior of these fluctuations. This operator can be derived from $S[X]$ by once again plugging in small variations

$$X_a \to X_a + A_a$$

of the background, but in contrast to selecting the terms of linear order in A_a, we consider the second-order perturbations. To simplify the formulas, we introduce the shorthand notation $\delta_{A^2}S[X]$ referring to the terms quadratic in $A_a \in \mathbb{C}^6 \otimes \mathrm{End}\,\mathcal{H}$. Let us first consider the more difficult case $N > 1$ and $\lambda = 0$ and come back to $N = 1$ later. For the second-order perturbations we find

$$\begin{aligned}
\delta_{A^2} S[X] = \operatorname{Tr}(\frac{1}{4}(&[A_a, A_b][X_a, X_b] + [X_a, X_b][A_a, A_b] + \\
&[A_a, X_b][A_a, X_b] + [X_a, A_b][X_a, A_b] + \\
&[A_a, X_b][X_a, A_b] + [X_a, A_b][A_a, X_b]) - \frac{1}{2} m^2 A_a A^a) = \\
= \operatorname{Tr}(\frac{1}{2}(&[A_a, A_b][X_a, X_b] + \\
&[A_a, X_b][A_a, X_b] + \\
&[A_a, X_b][X_a, A_b]) - \frac{1}{2} m^2 A_a A^a) = \\
= \frac{1}{2} \operatorname{Tr}\Big(&- A_a \Box A_a - 2 A_a [[X_a, X_b], A_b] - m^2 A_a A^a + f^2 \Big),
\end{aligned} \tag{59}$$

where we defined $f = i[X_a, A_a]$. Fixing this last term[3] implements a gauge fixing condition. We can conveniently choose $f^2 = 0$ and by further reordering the terms, the second order term can be written as

$$\begin{aligned}
\delta_{A^2} S[X] &= -\frac{1}{2} \operatorname{Tr}(A_a(\delta_{ab}(\Box + m^2) + 2[[X_a, X_b], \cdot]) A_b) = \\
&= -\frac{1}{2} \operatorname{Tr}(A_a (\mathcal{D}^2)_{ab} A_b)
\end{aligned} \tag{60}$$

As a first result in this chapter, we will show that the spectrum of the operator $\mathcal{D}^2$ is non-negative if and only if $m^2 \geq 0$. For the squashed fuzzy $\mathbb{C}P^2$, we have already established that $m^2 = -2$. Consequently, this model is unfortunately not stable for this background. Nevertheless, we can understand this operator even better, by providing a method to find the eigenvalues of this operator by restricting the domain to carefully chosen subspaces in the next section of this chapter.

Let us reiterate that the tensor product $\operatorname{End}(\mathcal{H}) \cong D(N, 0) \otimes D(N, 0)^*$ decomposes as

$$\operatorname{End}(\mathcal{H}) \cong \bigoplus_{p=1}^{N} D(p, p).$$

This means that there is a similarity transformation

$$U : \bigoplus_{p=0}^{N} \mathbb{C}^6 \to \mathbb{C}^6 \otimes \operatorname{End} \mathcal{H}$$

such that $U^{-1} \mathcal{D}^2 U$ is block-diagonal with the blocks similar to the maps

$$\begin{aligned}
&D_p^2 : \mathbb{C}^6 \otimes D(p, p) \to \mathbb{C}^6 \otimes D(p, p) \\
&\qquad (v_a)_{a \in J} \mapsto \left(\sum_{c \in J} \rho_p(t_c)^2 v_a + m^2 v_a + 2 \sum_{b \in J} \sum_{c=1}^{8} i f_{abc} \rho_p(t_c) v_b \right)_{a \in J}
\end{aligned} \tag{61}$$

3 This can always be achieved [26].

where p ranges from 0 tot N. We can rewrite this expression in terms of tensor products, namely

$$\begin{aligned}(D_p^2)_{ab} =&\delta_{ab}\left(\sum_{c\in J}\rho_p(t_c)^2 + m^2\,\mathrm{Id}_{D(p,p)}\right) + 2\sum_{c=1}^{8} if_{abc}\rho_p(t_c) = \\ =&(\mathrm{Id}_6)_{ab}\otimes\sum_{c\in J}\rho_p(t_c)^2 - 2\sum_{c=1}^{8}(F_c)_{ab}\otimes\rho_p(t_c) + m^2(\mathrm{Id}_6)_{ab}\otimes\mathrm{Id}_{D(p,p)}\end{aligned} \tag{62}$$

where $(F_c)_{ab} = -if_{abc}$. To understand the full vector fluctuation operator $\mathcal{D}^2$ better, we may therefore study the spectrum of the sub-operators D_p^2.

3.3.1 *Spectrum of the vector fluctuation operator*

In this section we provide a proof for the positivity of the vector fluctuation operator $\mathcal{D}^2$. We will utilize the well know Weyl's inequality, which we recall here for convenience.

Theorem 2. ***(Weyl's inequality*** *[27]) Let M,N and P be three hermitian matrices, s.t. $M = N + P$ and*

$$\begin{aligned} &\mu_1 \le \mu_2 \ldots \mu_{n-1} \le \mu_n \text{ the eigenvalues of } M \\ &\nu_1 \le \nu_2 \ldots \nu_{n-1} \le \nu_n \text{ the eigenvalues of } N \\ &\rho_1 \le \rho_2 \ldots \rho_{n-1} \le \rho_n \text{ the eigenvalues of } P \end{aligned}$$

then for $i = 1,\ldots,n$ we find $\nu_i + \rho_1 \le \mu_i \le \nu_i + \rho_n$. ■

We can now show the main result of this section. Since the gauge term does not introduce negative modes, we want to only consider the gauge-fixed vector fluctuation operator here. This differs from [8] insofar as here we did not include a cubic flux term in the action.

Theorem 3. *The vector fluctuation operator*

$$\begin{aligned}\mathcal{D}^2 : \mathbb{C}^6\otimes\mathrm{End}(\mathcal{H}) &\to \mathbb{C}^6\otimes\mathrm{End}(\mathcal{H}), \\ (A_a)_{a\in J} &\mapsto \left(\Box A_a + 2\sum_{b\in J}[[X_a, X_b], A_b] + m^2 A_a\right)_{a\in J}\end{aligned}$$

governing fluctuations around a squashed $\mathbb{C}P^2$ background for the Yang-Mills matrix model

$$S[X] = \frac{1}{4}\,\mathrm{Tr}\left([X_a, X_b][X_a, X_b] - 2m^2 X_a X^a\right),$$

where X_a denote the six fuzzy embedding functions of squashed $\mathbb{C}P^2_N$, is

(i) positive definite if and only if $m^2 > 0$.

(ii) negative definite if and only if $m^2 < 0$.

(iii) positive semi-definite otherwise.

Proof. Since $A_a \mapsto m^2 A_a$ only shifts the spectrum by the constant value m^2, we can without loss of generality set $m^2 = 0$.

Let us now consider the decomposition $(D_p^2)_{ab}$ introduced in eq. (61). Recall that by squashing, we restrict a and b to $J = \{1, 2, 4, 5, 6, 7\}$. This means that $\operatorname{span}\{F_a\}_{a\in J}$ is not closed under the Lie-bracket anymore. To get around this issue we introduce an auxiliary operator $\bar{D}_p^2$ where $a, b = 1, \ldots, 8$, i.e.

$$\bar{D}_p^2 : \mathbb{C}^8 \otimes D(p,p) \to \mathbb{C}^8 \otimes D(p,p),\ v \mapsto \left(\mathrm{Id}_8 \otimes \sum_{c\in J} \rho_p(t_c)^2 - 2\sum_{c=1}^{8} F_c \otimes \rho_p(t_c) \right) v$$

and make the squashing explicit by introducing the projection operator

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{Id}_{D(p,p)}.$$

Thus we can recover the vector fluctuation operator in terms of $\bar{D}_p^2$ as follows:

$$D_p^2 \sim P^T \left(\mathrm{Id}_8 \otimes \sum_{i\in J} \rho_p(t_i)^2 - 2\sum_{c=1}^{8} F_c \otimes \rho_p(t_c) \right) P =: P^T \bar{D}_p^2 P.$$

Since the auxiliary operator lives in $D(1,1) \otimes D(p,p)$, we can utilize the full machinery of representation theory.

First we are going to argue that $\bar{D}_p^2$ is non-negative. To this end let

$$M_1 = \mathrm{Id}_8 \otimes \sum_{i\in J} \rho_p(t_i)^2, \quad M_2 = -2\sum_{c=1}^{8} F_c \otimes \rho_p(t_c)$$

For M_1 we can easily state an explicit formula for the eigenvalues. Indeed,

$$M_1 = \mathrm{Id}_8 \otimes (p(p+2)\,\mathrm{Id}_{D(p,p)} - (\rho_p(t_3)^2 + \rho_p(t_8)^2))$$

with the eigenvalues of $\rho_p(t_3)^2 + \rho_p(t_8)^2$ given by the weights of the representation. We can also read of the smallest eigenvalue for M_1. This is of course the one for the highest weight of the representation and we find

$$\lambda_{\min}(M_1) = p(p+2) - \left(\frac{p}{2}\right)^2 - \left(\frac{p\sqrt{3}}{2}\right)^2 = 2p$$

To find a lower bound for the eigenvalues of M_2, we consider the tensor product $W = D(1,1) \otimes D(p,p)$. The tensor product decomposition can, for example, be easily evaluated by means of Young tableaus and the result for $p > 1$ is

$$\begin{aligned} D(1,1) \otimes D(p,p) = D(p+1,p+1) + D(p+2,p-1) + D(p-1,p+2) + \\ + D(p,p) + D(p+1,p-2) + D(p-2,p+1) + D(p-1,p-1), \end{aligned}$$

while for $p = 1$, we find

$$D(1,1) \otimes D(1,1) = D(0,0) \oplus D(1,1) \oplus D(3,0) \oplus D(0,3) \oplus D(2,2).$$

The Casimir operator of the corresponding Lie-algebra representation

$$C_W = \sum_{c=1}^{8} (F_c \otimes \mathrm{Id}_{D(p,p)} + \mathrm{Id}_{D(1,1)} \otimes \rho_p(t_c))^2 \tag{63}$$

then decomposes by Schur's lemma into the blocks proportional to the identity operator. The full spectrum is given by

$$\begin{aligned} \mathrm{spec}(C_W) = \{(1+p)(3+p), \quad (1+p)(2+p), \quad (1+p)(2+p), \\ p(2+p), \quad p(1+p), \quad p(1+p), \quad (-1+p)(1+p)\} \end{aligned} \tag{64}$$

for the case $p > 1$. A similar result is obtained for $p = 1$. To proceed, we expand the square in eq. (63) and compute the Casimir operator for the respective irreducible representations $D(1,1)$ and $D(p,p)$,

$$\begin{aligned} C_W &= \sum_{c=1}^{8} (F_c^2 \otimes \mathrm{Id}_{D(p,p)} + 2F_c \otimes \rho_p(t_c) + \mathrm{Id}_{D(1,1)} \otimes \rho_p(t_c)^2) = \\ &= (3 + p(p+2))\, \mathrm{Id}_{D(1,1)} \otimes \mathrm{Id}_{D(p,p)} + 2\sum_{c=1}^{8} F_c \otimes \rho_p(t_c) \\ \Leftrightarrow \quad -2\sum_{c=1}^{8} F_c \otimes \rho_p(t_c) &= (3 + p(p+2))\, \mathrm{Id}_{D(1,1)} \otimes \mathrm{Id}_{D(p,p)} - C_W, \end{aligned}$$

which is exactly the matrix M_2. Thus the smallest eigenvalue of M_2 can be computed by taking the largest eigenvalue of C_W. From eq. (64) it is clear that this value is given by $\lambda_{\max}(C_W) = (1+p)(3+p)$ and by plugging this into C_W we obtain,

$$\lambda_{\min}(M_2) = 3 + p(p+2) - (1+p)(3+p) = -2p.$$

By employing Weyl's identity, it is now clear that $\bar{D}_p^2$ is at least positive semi-definite, since

$$0 = \lambda_{\min}(M_1) + \lambda_{\min}(M_2) \leq \lambda_{\min}(M_1 + M_2) = \lambda_{\min}(\bar{D}_p^2).$$

Finally, note that for any $v \in \mathbb{C}^6 \otimes D(p,p)$

$$v^T D_p^2 v = v^T P^T \bar{D}_p^2 P v = w^T \bar{D}_p^2 w \geq 0,$$

where we set $w = Pv$ for clarity. Since this applies to all $p > 0$, the fluctuation operator $\mathcal{D}^2$ is at least positive semi-definite. Note that we have already seen that $\mathcal{D}^2$ possesses zero modes and therefore, the operator is not positive for $m^2 = 0$. ■

As already mentioned, we would need to set $m^2 = -2$ in order to obtain the squashed $\mathbb{C}P_N^2$ as a solution to the equation of motion eq. (55). This however pulls the spectrum down and negative modes appear. Alas, these instabilities invalidates the viability as a proper physical model.

Computation of the Spectrum

Having found that the spectrum of the vector fluctuation operator is positive definite, we can refine our results by computing – at least partially – the exact eigenvalues of the vector fluctuation operator. Given a particular $(p+1)^3$ dimensional matrix representation, i.e. with Dynkin labels $D(p,p)$, this certainly can be carried out numerically. However, performing the computation naively has the obvious drawback of cubically growing matrix dimensions. Furthermore, if we want to understand the operator as $p \to \infty$, numerically computing the eigenvalues for the first couple N might not be illuminating. Unfortunately deriving a closed formula for the eigenvalues turns out to be not such a trivial task.

In this section we present a partial solution to this problem by rewriting D_p^2, given in eq. (61), in terms of root generators and then we develop a machinery for constructing eigenvectors with the help of weight diagrams. The main idea resides in the aforementioned decomposition of $\mathrm{End}(\mathcal{H})$ into a direct sum of irreducible representations, namely

$$\mathrm{End}(\mathcal{H}) \cong D(N,0) \otimes D(0,N) \cong \bigoplus_{p=0}^{N} D(p,p)$$

To this end, we will break up the matrices[4] $(F_c)_{ab} = -i f_{abc}$ further into Kronecker products of 2×2 and 3×3 matrices.

Consider the Pauli matrices enriched by the identity matrix, denoted by σ_0,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

enriched by the identity matrix, denoted by σ_0. Similarly, we let $\lambda_k,\ k = 1,\dots,8$ be the eight Gell-Mann matrices and set λ_0 to be the 3×3 identity matrix. In order to demonstrate how the matrices F_a can be decomposed in terms of Pauli- and Gell-Mann matrices, we present the case F_1 explicitly. Observe that

$$F_1 = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 & 0 & \frac{i}{2} & 0 \\ \hline 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 \end{array}\right) = \frac{1}{2}\lambda_6 \otimes \sigma_2.$$

4 Note that here the indices a and b are restricted to $J = \{1,2,4,5,6,7\}$ as a consequence of the squashing procedure.

where $\otimes$ denotes the Kronecker product. Repeating this decomposition for the remaining seven matrices F_c, we find the formulas

$$\begin{aligned}
F_1 &= \frac{1}{2}\lambda_6 \otimes \sigma_2 & F_2 &= \frac{1}{2}\lambda_7 \otimes \sigma_0 \\
F_4 &= -\frac{1}{2}\lambda_5 \otimes \sigma_1 & F_5 &= -\frac{1}{2}\lambda_5 \otimes \sigma_3 \\
F_6 &= -\frac{1}{2}\lambda_1 \otimes \sigma_2 & F_7 &= \frac{1}{2}\lambda_2 \otimes \sigma_0 \\
F_3 &= \frac{1}{2}\left(\frac{2}{3}\lambda_0 + \frac{1}{2}\lambda_3 + \frac{5}{2\sqrt{3}}\lambda_8\right) \otimes \sigma_2 & F_8 &= \frac{1}{2}\left(\frac{2}{\sqrt{3}}\lambda_0 - \frac{\sqrt{3}}{2}\lambda_3 + \frac{1}{2}\lambda_8\right) \otimes \sigma_2
\end{aligned} \tag{65}$$

and by plugging these into eq. (62), the vector fluctuation operator takes the form

$$\begin{aligned}
D_p^2 = \lambda_0 \otimes \sigma_0 \otimes \left(\sum_{c\in J} \rho_p(t_c)^2\right) + \\
&- \left(\frac{2}{3}\lambda_0 + \frac{1}{2}\lambda_3 + \frac{5}{2\sqrt{3}}\lambda_8\right) \otimes \sigma_2 \otimes \rho_p(t_3) - \left(\frac{2}{\sqrt{3}}\lambda_0 - \frac{\sqrt{3}}{2}\lambda_3 + \frac{1}{2}\lambda_8\right) \otimes \sigma_2 \otimes \rho_p(t_8) + \\
&- \lambda_6 \otimes \sigma_2 \otimes \rho_p(t_1) - \lambda_7 \otimes \sigma_0 \otimes \rho_p(t_2) + \lambda_5 \otimes \sigma_1 \otimes \rho_p(t_4) - \lambda_5 \otimes \sigma_3 \otimes \rho_p(t_5) + \\
&+ \lambda_1 \otimes \sigma_2 \otimes \rho_p(t_6) - \lambda_2 \otimes \sigma_0 \otimes \rho_p(t_7).
\end{aligned}$$

The six Gell-Mann matrices $\lambda_k,\ k \in J$ can then be expressed as linear combinations of the root generators by reversing the definitions

$$\lambda_A^{\pm} = \frac{1}{2}(\lambda_1 \pm i\lambda_2), \quad \lambda_B^{\pm} = \frac{1}{2}(\lambda_4 \pm i\lambda_5), \quad \lambda_C^{\pm} = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$$

and, denoting the quadratic Casimir operator by

$$C_2 = \sum_{c=1}^{8} \rho_p(t_c)^2,$$

we obtain, after rearranging terms, the expression we are going to use to study the spectrum:

$$\begin{aligned}
D_p^2 =& \lambda_0 \otimes \sigma_0 \otimes (C_2 - \rho_p(t_3)^2 - \rho_p(t_8)^2) - \lambda_0 \otimes \sigma_2 \otimes \left(\frac{2}{3}\rho_p(t_3) + \frac{2}{\sqrt{3}}\rho_p(t_8)\right) + \\
&- \lambda_3 \otimes \sigma_2 \otimes \left(\frac{1}{2}\rho_p(t_3) - \frac{\sqrt{3}}{2}\rho_p(t_8)\right) - \lambda_8 \otimes \sigma_2 \otimes \left(\frac{5}{2\sqrt{3}}\rho_p(t_3) - \frac{1}{2}\rho_p(t_8)\right) + \\
&+ \lambda_A^- \otimes (\sigma_2 \otimes \rho_p(t_6) - i\sigma_0 \otimes \rho_p(t_7)) + \lambda_A^+ \otimes (\sigma_2 \otimes \rho_p(t_6) + i\sigma_0 \otimes \rho_p(t_7)) + \\
&+ \lambda_B^- \otimes (i\sigma_1 \otimes \rho_p(t_4) - i\sigma_3 \otimes \rho_p(t_5)) - \lambda_B^+ \otimes (i\sigma_1 \otimes \rho_p(t_4) - i\sigma_3 \otimes \rho_p(t_5)) + \\
&- \lambda_C^- \otimes (\sigma_2 \otimes \rho_p(t_1) + i\sigma_0 \otimes \rho_p(t_2)) - \lambda_C^+ \otimes (\sigma_2 \otimes \rho_p(t_1) - i\sigma_0 \otimes \rho_p(t_2)).
\end{aligned} \tag{66}$$

At first glance, this might appear rather cumbersome. However, if we pick any vector of the form

$$V^{\pm} = w \otimes s^{\pm} \otimes v, \quad w \in \mathbf{C}^3,\ v \in D(p,p),$$

where we singled out the two eigenvalues of σ^2,

$$s^+ = \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \text{and} \quad s^- = \begin{pmatrix} i \\ 1 \end{pmatrix},$$

with eigenvalues ± 1 respectively, the terms $D_p^2 V^\pm$ further simplify to

$$\left.\begin{aligned} D_p^2 V^+ =& v \otimes s^+ \otimes \left(\left(C_2 - \rho_p(t_3)^2 - \rho_p(t_8)^2 - \frac{2}{3}\rho_p(t_3) - \frac{2}{\sqrt{3}}\rho_p(t_8) \right) w \right) + \\ &+(\lambda_3 v) \otimes s^+ \otimes \left(\left(-\frac{1}{2}\rho_p(t_3) + \frac{\sqrt{3}}{2}\rho_p(t_8) \right) w \right) + \\ &+(\lambda_8 v) \otimes s^+ \otimes \left(\left(-\frac{5}{2\sqrt{3}}\rho_p(t_3) + \frac{1}{2}\rho_p(t_8) \right) w \right) + \end{aligned}\right\} L_d V^+$$
$$\left.\begin{aligned} &+(\lambda_A^- v) \otimes s^+ \otimes (T_C^- w) + (\lambda_A^+ v) \otimes s^+ \otimes (T_C^+ w) + \\ &+(\lambda_B^- v) \otimes s^- \otimes (T_B^+ w) - (\lambda_B^+ v) \otimes s^- \otimes (T_B^+ w) + \\ &-(\lambda_C^- v) \otimes s^+ \otimes (T_A^+ w) - (\lambda_C^+ v) \otimes s^+ \otimes (T_A^- w) \end{aligned}\right\} L_s V^+$$

and

$$\left.\begin{aligned} D_p^2 V^- =& v \otimes s^- \otimes \left(\left(C_2 - \rho_p(t_3)^2 - \rho_p(t_8)^2 + \frac{2}{3}\rho_p(t_3) + \frac{2}{\sqrt{3}}\rho_p(t_8) \right) w \right) + \\ &-(\lambda_3 v) \otimes s^- \otimes \left(\left(-\frac{1}{2}\rho_p(t_3) + \frac{\sqrt{3}}{2}\rho_p(t_8) \right) w \right) + \\ &-(\lambda_8 v) \otimes s^- \otimes \left(\left(-\frac{5}{2\sqrt{3}}\rho_p(t_3) + \frac{1}{2}\rho_p(t_8) \right) w \right) + \end{aligned}\right\} L_d V^-$$
$$\left.\begin{aligned} &-(\lambda_A^- v) \otimes s^- \otimes (T_C^+ w) - (\lambda_A^+ v) \otimes s^- \otimes (T_C^- w) + \\ &-(\lambda_B^- v) \otimes s^+ \otimes (T_B^- w) + (\lambda_B^+ v) \otimes s^+ \otimes (T_B^- w) + \\ &+(\lambda_C^- v) \otimes s^- \otimes (T_A^- w) + (\lambda_C^+ v) \otimes s^- \otimes (T_A^+ w), \end{aligned}\right\} L_s V^-$$

where we introduced the shorthand notation $T_A^\pm := \rho_p(t_A^\pm)$, $T_B^\pm := \rho_p(t_B^\pm)$ and $T_C^\pm := \rho_p(t_C^\pm)$ and made use of the identities

$$\sigma_1 s^\pm = \mp i s^\mp, \quad \sigma_3 s^\pm = -s^\mp.$$

We can separate the operator D_p^2 into the two parts L_d and L_s. These two parts behave nicely if they act on vectors of the form

$$V_{i,k,l}^\pm := e_i \otimes s^\pm \otimes w_{k,l}, \tag{67}$$

where e_i denotes the standard basis of the vector space $\mathbb{C}^3$ and $w_{k,l}$ corresponds to a weight vector with weight given by $\omega = k\alpha_1 + l\alpha_2$, where α_i are the two simple roots, i.e.

$$\rho_p(t_3) w_{k,l} = \left(k - \frac{1}{2} l \right) w_{k,l}, \quad \rho_p(t_8) w_{k,l} = \frac{\sqrt{3}}{2} l w_{k,l}.$$

Now observe that by construction any such $V_{i,k,l}^\pm$ is an eigenvector of L_d. For this reason we will refer to L_d as the diagonal action of D_p^2. The second part of the fluctuation operator, L_s will be denoted the shifting action, since it transports vectors of the form $V_{i,k,l}^\pm$ into vectors with either k, l or both incremented respectively decremented. To understand the shifting action a bit better, we will identify proper invariant subspaces $W \subseteq D(p,p)$, i.e. $L_s W \subseteq W$. Having found these, we then can try to construct eigenvectors of L_s within these invariant subspaces.

To this end, we fix some terminology. Letting V_ω be the weight space corresponding to the weight ω of the irreducible representation $D(p,p)$, we define the vector spaces

$$W_{k,l} = \begin{cases} V_{k\alpha_1 + l\alpha_2} & \text{if } k\alpha_1 + l\alpha_2 \text{ is a valid weight} \\ \{0\} & \text{otherwise} \end{cases}$$

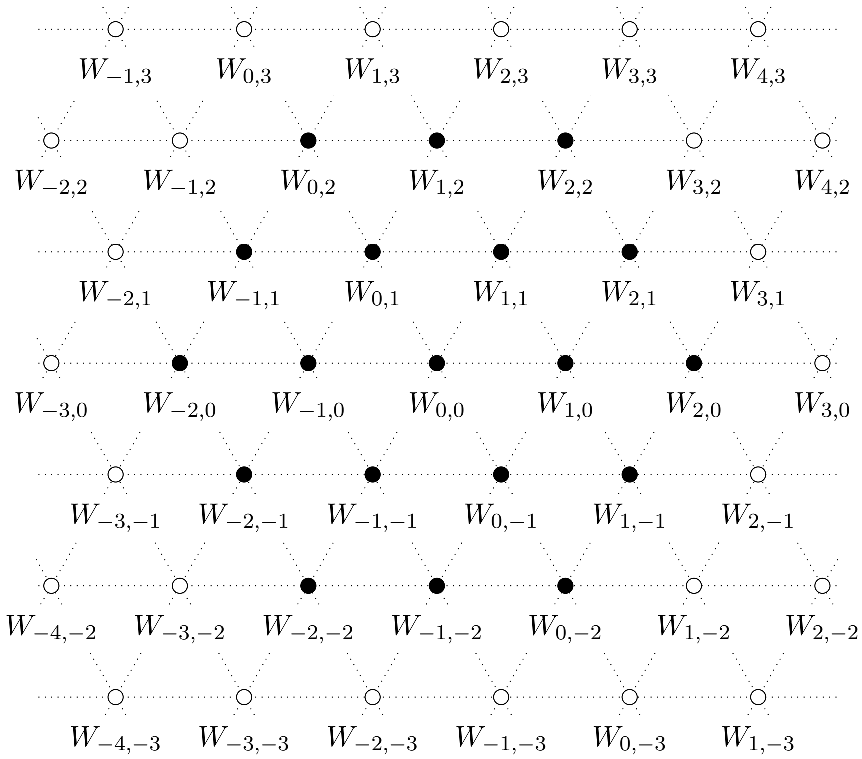

Figure 5: Any white grid point is associated with the trivial vector space $\{0\}$, while the filled dots represent the different weight spaces of the – in this example $p = 2$ – representation $D(p, p)$

where $k, l \in \mathbb{Z}$. Schematically, these spaces can be arranged in a grid. For the case $p = 2$, this is illustrated in fig. 5. We also introduce the notion of an *extended weight space*, given by the subspaces,

$$W^{\pm}_{i,k,l} := \{e_i \otimes s^{\pm} \otimes w \mid w \in W_{k,l}\} \subseteq \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes W_{k,l}$$

where $i = 1, ..., 3$ and $k, l \in \mathbb{Z}$. Equipped with this terminology, we are now ready to better understand the structure of the subspaces that are preserved by the shifting action.

Lemma 4. *Given any pair of integers (k, l), the vector space*

$$\begin{aligned} \mathcal{V}_{k,l} =& W^{-}_{1,k+1,l} \oplus W^{-}_{2,k+1,l+1} \oplus W^{-}_{3,k,l+1} \oplus \\ &\oplus W^{+}_{1,k-1,l} \oplus W^{+}_{2,k-1,l-1} \oplus W^{+}_{3,k,l-1} \subseteq \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes D(p,p) \end{aligned} \tag{68}$$

is preserved by the shifting action L_s.

Proof. We pick any weight (k', l'). Acting with L_s on $e_1 \otimes s^- \otimes w$ for any $w \in W_{k',l'}$, only the two terms

$$-e_2 \otimes s^- \otimes (T^+_C w) - e_3 \otimes s^+ \otimes (T^-_B w) \in W^-_{2,k',l'+1} \oplus W^+_{3,k'-1,l'-1}$$

survive, or in other words,

$$L_s W^-_{1,k',l'} \subseteq W^-_{2,k',l'+1} \oplus W^+_{3,k'-1,l'-1}.$$

Repeating this step recursively for each of the two vector spaces we can trace the action of L_s across the extended weight spaces and take note of any newly encountered extended weight space. This can be visualized diagrammatically for $W^-_{1,k',l'}$ as follows[5]

$$\begin{tikzcd}
 & W^-_{3,k'-1,l'+1} \arrow[r, bend right] \arrow[dl, bend right] & W^-_{2,k',l'+1} \arrow[l, bend right] \arrow[dr, bend left] & \\
W^+_{1,k'-2,l'} \arrow[ur, bend right] \arrow[dr, bend right] & & & W^-_{1,k',l'} \arrow[ul, bend right] \arrow[dl, bend right] \\
 & W^+_{2,k'-2,l'-1} \arrow[ul, bend right] \arrow[r, bend right] & W^+_{3,k'-1,l'-1} \arrow[l, bend right] \arrow[ur, bend right] &
\end{tikzcd}$$

Setting $k = k' - 1$ and $l = l'$, we obtain the claimed assertion. Repeating this computation for the remaining combinations of unit vectors e_i and sign choices, yields the same picture. ■

Given a subspace of the form eq. (68), we can now ask whether we are able to construct an eigenvalue of D_p^2 restricted to any available $\mathcal{V}_{k,l}$.

As for the diagonal action we can show the following lemma:

Lemma 5. *The sub-spaces*

$$\begin{aligned} \mathcal{V}_{k,l} =& W^-{}_{1,k+1,l} \oplus W^-{}_{2,k+1,l+1} \oplus W^-{}_{3,k,l+1} \oplus \\ &\oplus W^+{}_{1,k-1,l} \oplus W^+{}_{2,k-1,l-1} \oplus W^+{}_{3,k,l-1} \subseteq \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes W_{k,l} \end{aligned}$$

are eigensubspaces of L_d. In particular

$$L_d V = \Big((1+p)^2 - kl - (k-l)^2 \Big) V, \quad \forall V \in \mathcal{V}_{k,l}. \tag{69}$$

Proof. Recall that

$$\rho_p(t_3)|_{W_{k,l}} = \left(k - \frac{l}{2} \right) \mathrm{Id}_{W_{k,l}}, \quad \rho_p(t_8)|_{W_{k,l}} = \frac{\sqrt{3}}{2} l \,\mathrm{Id}_{W_{k,l}},$$

we can plug these identities into the definition of L_d, we obtain for any $w \in W_{k,l}$ after collecting terms

$$\begin{aligned} L_d(e_1 \otimes s^\pm \otimes w) &= \Big((p+2)p - k^2 - l^2 + k(l \mp 2) \pm l \Big) e_1 \otimes s^\pm \otimes w \\ L_d(e_2 \otimes s^\pm \otimes w) &= \Big((p+2)p - k^2 - l^2 + k(l \mp 1) \mp l \Big) e_2 \otimes s^\pm \otimes w \\ L_d(e_3 \otimes s^\pm \otimes w) &= \Big((p+2)p - k^2 - l^2 + k(l \pm 1) \mp 2l \Big) e_3 \otimes s^\pm \otimes w. \end{aligned}$$

Evaluating these identities on any of the six terms in the direct sum yields the claimed statement, e.g. for $V \in W^-_{1,k+1,l}$,

$$\begin{aligned} L_d V &= \Big((p+2)p - (k+1)^2 - l^2 + (k+1)(l+2) - l \Big) = \\ &= \Big((1+p)^2 - kl - (k-l)^2 \Big) V. \end{aligned}$$

■

5 Note that this diagram is not commutative.

This previous result tells us that to find the eigenvectors of D_p^2 restricted to $\mathcal{V}_{k,l}$ it suffices to compute the eigenvectors of the shifting operator, restricted to $\mathcal{V}_{k,l}$. The eigenvalues of $\mathcal{D}^2$ are then obtained by adding the eigenvalue of L_d to each of the eigenvalues of the shifting operator.

We will now demonstrate how the eigenvalues on the lower dimensional weight spaces can be computed. To be precise, we restrict ourselves to the situations where the dimension of any of the weight spaces in the construction

$$\begin{aligned}\mathcal{V}_{k,l} =& {W^-}_{1,k+1,l} \oplus {W^-}_{2,k+1,l+1} \oplus {W^-}_{3,k,l+1} \oplus \\ & \oplus {W^+}_{1,k-1,l} \oplus {W^+}_{2,k-1,l-1} \oplus {W^+}_{3,k,l-1}\end{aligned}$$

is at most two. This leaves us with the following cases

(i) Only one of the spaces is non-trivial:

$$\begin{array}{lll} \mathcal{V}_{p+1,0} & \mathcal{V}_{p+1,p+1} & \mathcal{V}_{0,p+1} \\ \mathcal{V}_{-p-1,0} & \mathcal{V}_{-p-1,-p-1} & \mathcal{V}_{0,-p-1} \end{array}$$

(ii) Two of the spaces are are one-dimensional while the other four are trivial:

$$\begin{array}{lll} \mathcal{V}_{p+1,k} & \mathcal{V}_{k,p+1} & \mathcal{V}_{-p-1+k,k} \\ \mathcal{V}_{-p-1,-p-1+k} & \mathcal{V}_{-p-1+k,-p-1} & \mathcal{V}_{k,-p-1+k} \end{array}$$

where $k = 1, \dots, p$.

(iii) Two spaces are one-dimensional and one is two-dimensional while the other three are trivial:

$$\begin{array}{lll} \mathcal{V}_{p,0} & \mathcal{V}_{p,p} & \mathcal{V}_{0,p} \\ \mathcal{V}_{-p,0} & \mathcal{V}_{-p,-p} & \mathcal{V}_{0,-p} \end{array}$$

(iv) Two spaces are two-dimensional, two are one-dimensional and the rest is trivial:

$$\begin{array}{lll} \mathcal{V}_{p,k} & \mathcal{V}_{k,p} & \mathcal{V}_{-p+k,k} \\ \mathcal{V}_{-p,-p+k} & \mathcal{V}_{-p+k,-p} & \mathcal{V}_{k,-p+k} \end{array}$$

where $k = 1, \dots, p-1$.

(v) For $p = 1$, we can furthermore examine the vector space $\mathcal{V}_{0,0}$.

Schematically we can depicted these as in fig. 6. Also note that by symmetry, the eigenspaces for any of the five give cases are all related through the Weyl group.

Let us begin with computing the eigenvalues for case (i). There is only one (one-dimensional) extended weight space contributing to $\mathcal{V}_{k,l}$. To exploit the underlying symmetry, we only need to consider $\mathcal{V}_{p+1,0} \cong W^+_{1,p,0}$ and take each eigenvalue to have a sixfold multiplicity. In fact in this case there is not much to be done. Since L_s maps any $w \in W^+_{1,p,0}$ to the null space, we can immediately see that these spaces correspond to zero modes of L_s. In addition to that the diagonal action is vanishing as well, which can be readily seen by evaluating eq. (69). Thus we have found six zero modes of D_p^2 for all $p \in \mathbb{N}$.

For case (ii), first recall that each pair of root generators – i.e. either of the three pairs $T_A^\pm$, $T_B^\pm$ or $T_C^\pm$ – induce an $\mathfrak{su}(2)$ Lie-sub-algebra. Given such a pair of $\mathfrak{su}(2)$ root

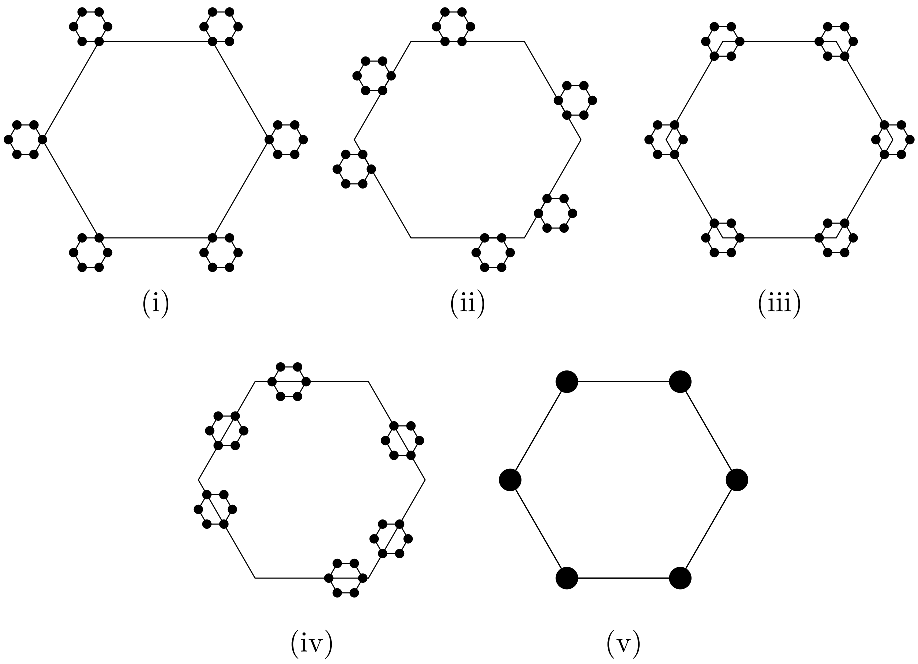

Figure 6: Schematic representation of the five lowest-dimensional configurations of $\mathcal{V}_{k,l}$. The black dots represent the weight spaces in the definition of $\mathcal{V}_{k,l}$.

generators $T^{\pm}$ and picking a vector $v_0 \in D(p,p)$ such that $T^+ v_0 = 0$ we may define the sequence of vectors $v_n := (T^-)^n v_0$ for $n \in \mathbb{N}$. In accordance to prevailing notation there exists a $j \in \mathbb{N}$, s.t. $v_n = 0$ for $n > 2j+1$. Then the norm of v_n under the action of $T^{\pm}$ changes according to the two formulas

$$\|T^{\pm} v_n\| = \sqrt{j(j+1) - (j-n)(j-n\pm 1)}\|v_n\|. \tag{70}$$

Now note that the subspace $\mathcal{V}_{p+1,l}$ collapses to

$$\mathcal{V}_{p+1,l} \cong W^+_{1,p,l} \oplus W^+_{2,p,l-1}.$$

We can pick any non-zero $V_a = e_2 \otimes s^+ \otimes w \in W^+_{2,p,l-1}$ and let[6] $V_b = e_1 \otimes s^+ \otimes T^+_C w$. Acting with the shifting operator on the two vectors V_a and V_b we obtain

$$\begin{aligned} L_s V_a &= (\lambda^+_A e_2) \otimes s^+ \otimes (T^+_C w) = V_b, \\ L_s V_b &= (\lambda^-_A e_1) \otimes s^+ \otimes (T^-_C T^+_C w). \end{aligned}$$

Referring back to eq. (70), the two values j and n are determined by $j = p/2$ and $n = p - l$ and therefore the length of the vector $T^-_C T^+_C w$ with respect to w is given by

$$\frac{\langle w|T^-_C T^+_C|w\rangle}{\|w\|^2} = \frac{\|T^+_C w\|^2}{\|w\|^2} = (p-l)(l+1)$$

and therefore $L_s V_b = (p-l)(l+1)V_a$. Together with eq. (69), we immediately arrive at

$$\begin{aligned} D_p^2(\xi_a V_a + \xi_b V_b) =& L_d(\xi_a V_a + \xi_b V_b) + L_s(\xi_a V_a + \xi_b V_b) = \\ =& (p-l)(l+1)(\xi_a V_a + \xi_b V_b) + \xi_a V_b + \xi_b (p-l)(l+1)V_a = \\ =& (p-l)(l+1)(\xi_a + \xi_b)V_a + ((p-l)(l+1)\xi_b + \xi_a)V_b = \\ =& (p-l)(l+1)\frac{\xi_a + \xi_b}{\xi_a}\left(\xi_a V_a + \frac{(p-l)(l+1)\xi_b + \xi_a}{(p-l)(l+1)(\xi_a+\xi_b)}\frac{\xi_a}{\xi_b}\xi_b V_b\right) \end{aligned}$$

6 Note that V_b does not vanish, since l ranges from 1 to p.

for any $\xi_1, \xi_2 \in \mathbb{R}$. Our ansatz $\xi_a V_a + \xi_b V_b$ is an eigenvector of D_p^2 if and only if

$$\frac{(p-l)(l+1)\xi_b + \xi_a}{(p-l)(l+1)(\xi_a+\xi_b)} \frac{\xi_a}{\xi_b} = 1$$

Since we do not care about the norm of the eigenvector, we can without loss of generality set $\xi_a = 1$ and solve the equation

$$(p-l)(l+1)\xi_b + 1 = (p-l)(l+1)(1+\xi_b)\xi_b.$$

This quadratic equation has the two solutions

$$\xi_b = \pm \frac{1}{\sqrt{(p-l)(l+1)}},$$

with the corresponding eigenvalues of D_p^2

$$\lambda_{p,l}^{(\mathrm{ii})\pm} := (p-l)(l+1) \pm \sqrt{(p-l)(l+1)}. \tag{71}$$

For case (iii), we can, as we did for the previous two cases, exploit the Weyl symmetry and restrict ourselves to working out the eigenvalues situated in $\mathcal{V}_{p,p}$. A convenient basis for this space can be introduced by picking any non-zero $w \in W_{p,p}$ and define

$$\begin{aligned} V_1 &= e_1 \otimes s^+ \otimes T_A^- w & \qquad U_1 &= e_2 \otimes s^+ \otimes T_B^- w \\ V_2 &= e_3 \otimes s^+ \otimes T_C^- w & \qquad U_2 &= e_2 \otimes s^+ \otimes T_A^- T_C^- w. \end{aligned}$$

Clearly, the vectors V_1 and V_2 span the two extended weight spaces $W_{1,p-1,p}^+$ and $W_{3,p,p-1}^+$ since these spaces are one-dimensional. It remains to be shown that $U_1 \in W_{2,p-1,p-1}^+$ and $U_2 \in W_{2,p-1,p-1}^+$ are indeed linearly independent. To that end, recall that $T_B^- w$ and $T_A^- T_C^- w$ are linearly dependent if and only if the two sides of the Cauchy-Schwarz inequality,

$$|\langle w | T_B^+ T_A^- T_C^- | w \rangle| \leq \|T_B^- w\| \|T_A^- T_C^- w\|,$$

are equal to each other. To show linear independence, eq. (70) comes in handy again. It follows that

$$\|T_B^- w\| = \sqrt{2p},$$

since in this case, $j = p$ and $n = 0$. We can now exploit the commutation relations $[T_A^-, T_C^-] = -T_B^-$ and $[T_B^+, T_C^-] = T_A^+$ for simplifying the left hand side of the Cauchy-Schwarz inequality, yielding

$$\begin{aligned} |\langle w | T_B^+ T_A^- T_C^- | w \rangle| &= |\langle w | T_B^+ T_C^- T_A^- | w \rangle - \langle w | T_B^+ T_B^- | w \rangle| = \\ &= |\|T_A^- w\|^2 - \|T_B^- w\|^2| = |p - 2p| = p. \end{aligned}$$

Similarly, the right hand side reduces to

$$\|T_B^+ w\| = \sqrt{2p}, \quad \|T_A^- T_C^- w\| = \sqrt{p(p+1)} \ \Rightarrow\ \|T_B^+ w\| \|T_A^- T_C^- w\| = p\sqrt{2(p+1)},$$

confirming that the inequality is indeed strict since $1 < \sqrt{2(p+1)}$ and thus U_1 and U_2 must be linearly independent.

Having found a basis for the four-dimensional vector space $\mathcal{V}_{p,p}$, we are now ready to seek out suitable linear combinations of V_i and U_i that make up eigenvectors of D_p^2. Recall that we split the problem into the diagonal action and the shifting action,

$$\begin{aligned} D_p^2 \sum_{i=1,2} (\xi_{V_i} V_i + \xi_{U_i} U_i) =& (1+2p) \sum_{i=1,2} (\xi_{V_i} V_i + \xi_{U_i} U_i) + \\ &+ \sum_{i=1,2} (\xi_{V_i} L_s V_i + \xi_{U_i} L_s U_i), \end{aligned}$$

for some real coefficients $\xi_{V_i}, \xi_{U_i} \in \mathbb{R}$. Again we merely need to evaluate the action of L_s restricted to $\mathcal{V}_{p,p}$. Employing the rules derived above, we gather the four expressions

$$\begin{aligned} L_s V_1 &= e_2 \otimes s^+ \otimes T_C^- T_A^- w = e_2 \otimes s^+ \otimes T_B^- w + e_2 \otimes s^+ \otimes T_A^- T_C^- w = U_1 + U_2, \\ L_s V_2 &= -e_2 \otimes s^+ \otimes T_A^- T_C^- w = -U_2, \\ L_s U_1 &= e_1 \otimes s^+ \otimes T_C^+ T_B^- w - e_3 \otimes s^+ \otimes T_A^+ T_B^- w = \\ &= e_1 \otimes s^+ \otimes T_A^- w + e_3 \otimes s^+ \otimes T_C^- w = V_1 + V_2, \\ L_s U_2 &= e_2 \otimes s^+ \otimes T_C^+ T_A^- T_C^- w - e_3 \otimes s^+ \otimes T_A^+ T_A^- T_C^- w = \\ &= e_2 \otimes s^+ \otimes T_A^- T_C^+ T_C^- w - e_3 \otimes s^+ \otimes T_A^+ T_A^- T_C^- w = \\ &= pV_1 - (p+1) V_2. \end{aligned}$$

where we used the commutation relations

$$[T_C^+, T_B^-] = T_A^-, \quad [T_B^-, T_A^+] = T_C^-, \quad [T_C^+, T_A^-] = 0$$

and in the last step eq. (70). In total the vector fluctuation operator restricted to the sub-space $\mathcal{V}_{p,p}$ maps the vector $\sum_{i=1,2} (\xi_{V_i} V_i + \xi_{U_i} U_i)$ to

$$\begin{aligned} D_p^2 \sum_{i=1,2} (\xi_{V_i} V_i + \xi_{U_i} U_i) =& (1+2p) \sum_{i=1,2} (\xi_{V_i} V_i + \xi_{U_i} U_i) + \xi_{V_1} (U_1 + U_2) - \xi_{V_2} U_2 + \\ &+ \xi_{U_1} (V_1 + V_2) + \xi_{U_2} (pV_1 - (p+1) V_2) = \\ =& (\xi_{V_1} (1+2p) + \xi_{U_1} + p \xi_{U_2}) V_1 + \\ &+ (\xi_{V_2} (1+2p) + \xi_{U_1} - (p+1) \xi_{U_2}) V_2 + \\ &+ (\xi_{U_1} (1+2p) + \xi_{V_1}) U_1 + \\ &+ (\xi_{U_2} (1+2p) + \xi_{V_1} - \xi_{V_2}) U_2. \end{aligned} \tag{72}$$

Without loss of generality we may once more assume that $\xi_{V_1} = 1$. Pulling out the coefficient of V_1 we obtain the non-linear system of equations

$$\begin{aligned} \frac{\xi_{V_2} (1+2p) + \xi_{U_1} - (p+1) \xi_{U_2}}{1 + 2p + \xi_{U_1} + p \xi_{U_2}} =& \xi_{V_2} \\ \frac{\xi_{U_1} (1+2p) + 1}{1 + 2p + \xi_{U_1} + p \xi_{U_2}} =& \xi_{U_1} \\ \frac{\xi_{U_2} (1+2p) + 1 - \xi_{V_2}}{1 + 2p + \xi_{U_1} + p \xi_{U_2}} =& \xi_{U_2}, \end{aligned}$$

for which we find the following set of four different solutions:

(iii.1)	$\xi_{V_2} = 1$	$\xi_{U_1} = 1$	$\xi_{U_2} = 0$
(iii.2)	$\xi_{V_2} = 1$	$\xi_{U_1} = -1$	$\xi_{U_2} = 0$
(iii.3)	$\xi_{V_2} = -1$	$\xi_{U_1} = -\frac{1}{\sqrt{2p+1}}$	$\xi_{U_2} = -\frac{2}{\sqrt{2p+1}}$
(iii.4)	$\xi_{V_2} = -1$	$\xi_{U_1} = \frac{1}{\sqrt{2p+1}}$	$\xi_{U_2} = \frac{2}{\sqrt{2p+1}}$

with their corresponding eigenvalues

$$\begin{aligned} \lambda_p^{(\mathrm{iii}.1)} &= 2(p+1) & \lambda_p^{(\mathrm{iii}.2)} &= 2p \\ \lambda_p^{(\mathrm{iii}.3)} &= 2p + \sqrt{2p+1} + 1 & \lambda_p^{(\mathrm{iii}.4)} &= 2p - \sqrt{2p+1} + 1 \end{aligned} \tag{73}$$

For scenario (iv), we can proceed analogously to case (iii). We first pick any $w \in W_{k,l}$ and define

$$\begin{aligned} V_1 &= e_3 \otimes s^- \otimes T_C^+ w & U_1 &= e_1 \otimes s^+ \otimes T_A^- w & U_3 &= e_2 \otimes s^+ \otimes T_B^- w \\ V_2 &= e_3 \otimes s^+ \otimes T_C^- w & U_2 &= e_1 \otimes s^+ \otimes T_B^- T_C^+ w & U_4 &= e_2 \otimes s^+ \otimes T_A^- T_C^- w. \end{aligned}$$

Both pairs, U_1, U_2 and U_3, U_4, are linearly independent. The proof runs as before: it is straightforward to check that for the two pairs, the Cauchy-Schwarz inequality is strict and thus the two vectors are necessarily linearly independent. The eigenequation of the diagonal action reads

$$L_d\left(\sum_{i=1}^{2} \xi_{V_i} V_i + \sum_{i=1}^{4} \xi_{U_i} U_i\right) = (p + (1+l)(1+p-l))\left(\sum_{i=1}^{2} \xi_{V_i} V_i + \sum_{i=1}^{4} \xi_{U_i} U_i\right)$$

and the images of the basis elements under the operator L_s are given by

$$\begin{aligned} L_s V_1 &= U_2, \\ L_s V_2 &= -U_4, \\ L_s U_1 &= -V_1 + U_3 + U_4, \\ L_s U_2 &= (p+l+1)V_1 + (l+1)(p-l)U_3, \\ L_s U_3 &= U_1 + U_2 + V_2, \\ L_s U_4 &= l(p-l+1)U_1 - (2p-l+1)V_2 \end{aligned}$$

Putting everything together and collecting coefficients, the vector fluctuation operator on the sub-spaces $\mathcal{V}_{p,l}$ can be written as

$$\begin{aligned} D_p^2\left(\sum_{i=1}^{2} \xi_{V_i} V_i + \sum_{i=1}^{4} \xi_{U_i} U_i\right) =& (-\xi_{U_1} + (1+l+p)\xi_{U_2} + (p+(1+l)(1-l+p))\xi_{V_1})V_1 \\ &+ (\xi_{U_3} + (-1+l-2p)\xi_{U_4} + (p+(1+l)(1-l+p))\xi_{V_2})V_2 \\ &+ ((p+(1+l)(1-l+p))\xi_{U_1} + \xi_{U_3} + l(1-l+p)\xi_{U_4})U_1 \\ &+ ((p+(1+l)(1-l+p))\xi_{U_2} + \xi_{U_3} + \xi_{V_1})U_2 \\ &+ (\xi_{U_1} + (1+l)(-l+p)\xi_{U_2} + (p+(1+l)(1-l+p))\xi_{U_3})U_3 \\ &+ (\xi_{U_1} + (p+(1+l)(1-l+p))\xi_{U_4} - \xi_{V_2})U_4, \end{aligned}$$

however here it is more convenient to write down the operator L_s restricted to $\mathcal{V}_{p,l}$ as a matrix with respect to the chosen bases,

$$[L_s|_{\mathcal{V}_{p,l}}] = \begin{pmatrix} 0 & 0 & -1 & l+p+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & l-2p-1 \\ 0 & 0 & 0 & 0 & 1 & l(-l+p+1) \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & (l+1)(p-l) & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues can be computed using a computer algebra system and since L_d acts as a scalar on this subspace, the eigenvalues we are looking for can be obtained by adding $p + (1 + l)(1 + p - l)$ to each of the eigenvalues of L_s. In summary, the final result for case (iv) is given by[7]

$$\begin{aligned} \lambda^{(\text{iv.1})\pm}_{p,l} =& p \pm 1 + (1 + l)(1 + p - l) \\ \lambda^{(\text{iv.2})\pm}_{p,l} =& p + (1 + l)(1 + p - l) \pm \sqrt{p + (1 + l)(1 + p - l)}. \end{aligned}$$

Last but not least, case (v) can be worked out analogously. In fact, since the extended weight spaces contributing to the subspace $\mathcal{V}_{0,0}$ are all one-dimensional, it even requires less work than the previous two cases. We pick any non-zero $w \in W_{0,0}$, demanding the additional constraint that $w \notin \ker T^+_k$ for $k = A, B, C$. This is indeed possible. To see this, recall that $D(1, 1)$ is just the adjoint representation. Then Gell-Mann matrix λ_3 fulfills this requirement, whereas λ_8 does not, since $[\lambda^+_A, \lambda_8]$ vanishes.

Under these assumptions, neither of the six vectors

$$\begin{aligned} V_1 &= e_1 \otimes s^- \otimes T^+_A w & V_4 &= e_1 \otimes s^+ \otimes T^-_A w \\ V_2 &= e_2 \otimes s^- \otimes T^+_B w & V_5 &= e_2 \otimes s^+ \otimes T^-_B w \\ V_3 &= e_3 \otimes s^- \otimes T^+_C w & V_6 &= e_3 \otimes s^+ \otimes T^-_C w \end{aligned}$$

vanishes and thus we can use them as a basis of $\mathcal{V}_{0,0}$. The diagonal and shifting actions are then determined by

$$\begin{aligned} L_s V_1 &= -V_2 - \frac{1}{2} V_6 & L_s V_3 &= 2V_2 - 2V_4 & L_s V_5 &= -V_4 + \frac{1}{2} V_6 \\ L_s V_2 &= -V_1 + \frac{1}{2} V_3 & L_s V_4 &= -\frac{1}{2} V_3 - V_5 & L_s V_6 &= -2V_1 + 2V_5 \\ & & L_d V_i &= 4V_i. \end{aligned}$$

The matrix representation of D^2_1 is therefore given by

$$[D^2_1|_{\mathcal{V}_{0,0}}] = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & -2 \\ -1 & 4 & 2 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 4 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 4 & 2 \\ -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 4 \end{pmatrix}.$$

The eigenvalues of this matrix are given by

$$\lambda^{(\text{v.1})} = 6 \quad \lambda^{(\text{v.2})} = 5 \quad \lambda^{(\text{v.3})} = 3 \quad \lambda^{(\text{v.4})} = 2$$

where $\lambda^{(\text{v.2})}$ and $\lambda^{(\text{v.3})}$ have multiplicity two.

Computation using Gelfand-Tsetlin patterns

There is a well known relation between Young-Tableaus and irreducible representations of the special unitary group [10]. Probably not so popular are so called Gelfand-Tsetlin patterns. It has been established that any irreducible matrix representation of $SU(n)$ can be explicitly constructed in the space of these patterns[28, 29]. We will reiterate the core concepts presented in [29] before we demonstrate how these patterns can be used to aid us with finding the spectrum of L_s.

7 Note that the latter pair of eigenvalues has multiplicity two.

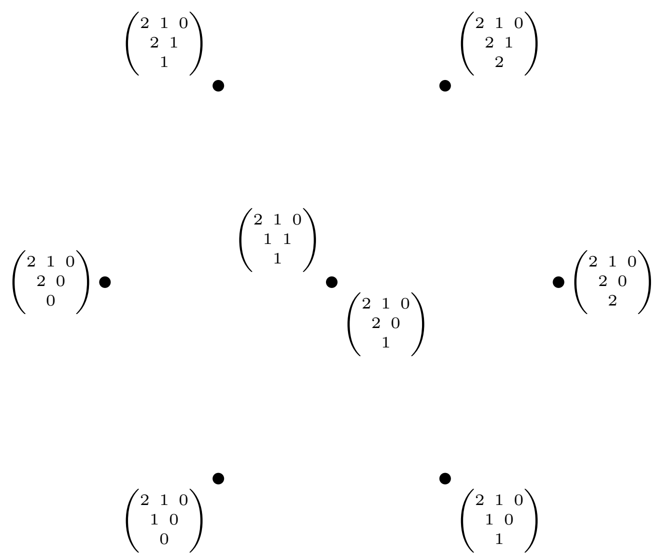

Figure 7: Arrangement of Gelfand-Tsetlin patterns with top row $(2,1,0)$ within the weight lattice. $\mathfrak{su}(3)$ acts in the irreducible representation $D(1,1)$.

Definition 6. *A Gelfand-Tsetlin pattern is a triangular array of integers*

$$\mathbf{a} = (a_{i,j}) = \begin{pmatrix} a_{1,N} & & a_{2,N} & & \cdots & & a_{N-1,N} & & a_{N,N} \\ & a_{1,N-1} & & \ddots & & & & a_{N-1,N-1} & \\ & & \ddots & & \ddots & & \iddots & & \\ & & & a_{1,2} & & a_{2,2} & & & \\ & & & & a_{1,1} & & & & \end{pmatrix}$$

with the additional constraint that $a_{k,l+1} \geq a_{k,l} \geq a_{k+1,l+1}$. We will refer to this constraint as the betweenness condition.

In order to limit the scope of this thesis, we will assume $N = 3$ from here on. However, note that all these arguments can be straightforwardly generalized to any choice of N.

To make the connection of Gelfand-Tsetlin patterns to Dynkin labels, we set the first row to

$$a_{1,3} = p + q + a_{3,3}, \qquad a_{2,3} = q + a_{3,3}.$$

Given a set of integers $(a_{1,3}, a_{2,3}, a_{3,3})$, the number of allowed patterns is clearly independent of $a_{3,3}$ and finite due to the betweenness-condition. In fact it turns out that we can, without loss of generality, assume $a_{3,3} = 0$. Diagrams of this form will be referred to as normalized diagrams. Let $\mathcal{P}_{p,q}$ be the set of all possible normalized Gelfand-Tsetlin patterns with first row set to $(p+q, q, 0)$. We consider the vector space $V(\mathcal{P}_{p,q})$ of all formal linear combinations of elements of $\mathcal{P}_{p,q}$ equipped with the scalar product

$$\langle \mathbf{a}, \mathbf{b} \rangle = \begin{cases} 1 & \text{if } a_{i,j} = b_{i,j} \quad \forall i, j \\ 0 & \text{otherwise} \end{cases}, \qquad \forall \mathbf{a}, \mathbf{b} \in \mathcal{P}_{p,q}$$

and define the following linearly extended operators acting on $V(\mathcal{P}_{p,q})$

$$\mathbf{T}_3 : V(\mathcal{P}_{p,q}) \to V(\mathcal{P}_{p,q}), \ \mathbf{a} \mapsto \left(a_{1,1} - \frac{1}{2}(a_{1,2} + a_{2,2}) \right) \mathbf{a}$$

$$\mathbf{T}_8 : V(\mathcal{P}_{p,q}) \to V(\mathcal{P}_{p,q}), \ \mathbf{a} \mapsto \left(\frac{\sqrt{3}}{2}(a_{1,2} + a_{2,2}) - \frac{1}{\sqrt{3}}(p + 2q) \right) \mathbf{a}$$

Additionally, we may define raising and lowering operators. For convenience, we introduce the following shorthand notation

$$\mathbf{a} \pm \mathbf{1}_{k,l} = \begin{cases} (a_{i,j} \pm \delta_{i,k}\delta_{j,l}) & \text{if } (a_{i,j} \pm \delta_{i,k}\delta_{j,l}) \text{ is a valid pattern} \\ 0 & \text{otherwise} \end{cases}$$

Then, the raising and lowering operators are defined by [29, 30]

$$\langle \mathbf{a} + \mathbf{l}_{k,l}, J_l^+ \mathbf{a} \rangle = \left(- \frac{\prod_{k'=1}^{l+1} (a_{k',l+1} - a_{k,l} + k - k') \prod_{k'=1}^{l-1} (a_{k',l-1} - a_{k,l} + k - k' - 1)}{\prod_{\substack{k'=1 \\ k' \neq k}}^{l} (a_{k',l} - a_{k,l} + k - k')(a_{k',l} - a_{k,l} + k - k' - 1)} \right)^{1/2}$$

and

$$\langle \mathbf{a} - \mathbf{l}_{k,l}, J_l^- \mathbf{a} \rangle = \left(- \frac{\prod_{k'=1}^{l+1} (a_{k',l+1} - a_{k,l} + k - k' + 1) \prod_{k'=1}^{l-1} (a_{k',l-1} - a_{k,l} + k - k')}{\prod_{\substack{k'=1 \\ k' \neq k}}^{l} (a_{k',l} - a_{k,l} + k - k' + 1)(a_{k',l} - a_{k,l} + k - k')} \right)^{1/2}$$

respectively, where $1 \leq k \leq l \leq 2$. It turns out, that $\mathbf{T}_3$ and $\mathbf{T}_8$, together with $J_l^{\pm}$ act on $V(\mathcal{P}_{p,q})$ in the $D(p,q)$ representation of $\mathfrak{su}(3)$.

In order to stay consistent with our notation, we define the six root generators

$$\mathbf{T}_A^{\pm} = J_1^{\pm}, \quad \mathbf{T}_C^{\pm} = J_2^{\pm}, \quad \mathbf{T}_B^{\pm} = \pm[J_1^{\pm}, J_2^{\pm}],$$

which together with $\mathbf{T}_3$ and $\mathbf{T}_8$ generate the $D(p,q)$ irreducible representation of $\mathfrak{su}(3)$ on the vector space $V(\mathcal{P}_{p,q})$.

Having introduced Gelfand-Tsetlin patterns and their relation to representation theory of $\mathfrak{su}(3)$, we can return to our initial goal of For a given weight index (k,l), the system of equations

$$\mathbf{T}_3\mathbf{a} = \left(a_{1,1} - \frac{1}{2}(a_{1,2} + a_{2,2}) \right) \mathbf{a} = \left(k - \frac{l}{2} \right) \mathbf{a}$$
$$\mathbf{T}_8\mathbf{a} = \left(\frac{\sqrt{3}}{2}(a_{1,2} + a_{2,2}) - \frac{1}{\sqrt{3}}(p + 2q) \right) \mathbf{a} = \frac{\sqrt{3}}{2} l \mathbf{a}$$

determine the Gelfand-Tsetlin patterns with weight $(k - l/2, \sqrt{3}l/2)$. Solving these for $a_{1,2}$, $a_{1,1}$ and $a_{2,2}$, we obtain

$$a_{1,2} + a_{2,2} = l + \frac{2}{3}(p + 2q), \quad a_{1,1} = k + \frac{1}{3}(p + 2q).$$

Together with the betweenness-condition, we can construct all Gelfand-Tsetlin patterns $\mathcal{P}_{k,l}$ spanning the weight space $W_{k,l} \subseteq V(\mathcal{P}_{p,q})$. A basis for the L_s-invariant spaces $\mathcal{V}_{k,l}$ can then be labeled by the Gelfand-Tsetlin patterns

$$\mathcal{B} = \mathcal{P}_{k-1,l} \cup \mathcal{P}_{k-1,l-1} \cup \mathcal{P}_{k,l-1} \cup \mathcal{P}_{k+1,l} \cup \mathcal{P}_{k+1,l+1} \cup \mathcal{P}_{k,l+1}$$

For any $\mathbf{a} \in \mathcal{B}$, the shifting action is then computed from the formulas

$$\begin{aligned}
&\mathcal{P}_{k-1,l} \ni \mathbf{a} \mapsto (\mathbf{T}_C^- + \mathbf{T}_B^+)\mathbf{a}, && \mathcal{P}_{k-1,l-1} \ni \mathbf{a} \mapsto (\mathbf{T}_C^+ - \mathbf{T}_A^+)\mathbf{a} \\
&\mathcal{P}_{k,l-1} \ni \mathbf{a} \mapsto (-\mathbf{T}_B^+ - \mathbf{T}_A^-)\mathbf{a}, && \mathcal{P}_{k+1,l} \ni \mathbf{a} \mapsto (-\mathbf{T}_C^+ - \mathbf{T}_B^-)\mathbf{a} \\
&\mathcal{P}_{k+1,l+1} \ni \mathbf{a} \mapsto (-\mathbf{T}_C^- + \mathbf{T}_A^-)\mathbf{a}, && \mathcal{P}_{k,l+1} \ni \mathbf{a} \mapsto (\mathbf{T}_B^- + \mathbf{T}_A^+)\mathbf{a}.
\end{aligned}$$

and after dropping any Gelfand-Tsetlin patterns that are not within $\mathcal{B}$, the resulting linear combinations can be gathered to build the matrix representation $[L_s|_{\mathcal{V}_{k,l}}]$ whose eigenvalues are, by construction, exactly those of L_s restricted to $\mathcal{V}_{k,l}$. Contrary to computing the eigenvalues for the full operator D_p^2 or even $\mathcal{D}^2$, the dimension of these matrices scale linearly with p, i.e. for

$$[L_s|_{\mathcal{V}_{k,l}}] \in \mathrm{Mat}(n_{k,l} \times n_{k,l}, \mathbb{C})$$

where $n_{k,l} = \dim \mathcal{V}_{k,l}$, we observe that $n_{k,l} \leq 6p$ whereas $\dim \mathbb{C}^6 \otimes D(p,p) = 6(p+1)^3$. In other words, instead of computing the eigenvalues of a $6(p+1)^3$-dimensional matrix, we can compute the eigenvalues of multiple smaller matrices, with dimensions bounded by $6p$.

Validation

To make sure all the computations were carried out correctly, we computed the eigenvalues for the lower dimensional representations numerically using Mathematica. The spectral decomposition was on the one hand carried out numerically for the full operator $\mathcal{D}^2$ and on the other hand symbolically by using the procedure based on the Gelfand-Tsetlin patterns. The results are summarized in appendix A.

3.4 BACKGROUND FLUCTUATIONS FOR $N = 1$

Let us conclude this chapter by discussing the remaining $N = 1$ case for the model eq. (54). In this case, the quartic interaction term

$$S_{\text{quart}}[X] = -\frac{1}{2}\,\mathrm{Tr}(\lambda(X^2)^2)$$

is not forbidden by the equation of motion and therefore we can reintroduce the corresponding terms to the vector fluctuation operator. Again we perform a perturbation $X_a \to X_a + A_a$ and compute the second order terms analogously to eq. (59):

$$\begin{aligned}
\delta_{A^2} S_{\text{quart}}[X] = &-\frac{\lambda}{2}\,\mathrm{Tr}(X_aX_aA_bA_b + X_aA_aX_bA_b + A_aX_aX_bA_b + \\
&\qquad X_aA_aA_bX_b + A_aX_aA_bX_b + A_aA_aX_bX_b) = \\
= &-\frac{1}{2}\,\mathrm{Tr}(\lambda A_a\{X^2, A_a\} + \lambda A_b\{X_b, \{X_a, A_a\}\}).
\end{aligned}$$

Thus the full vector fluctuation operator for the quartic action reads

$$\mathcal{D}^2_{ab} = \delta_{ab}(\Box + m^2 + \lambda\{X^2,\cdot\}) + 2[[X_a, X_b],\cdot] - [X_a,[X_b,\cdot]] + \lambda\{X_a,\{X_b,\cdot\}\}.$$

Assuming that squashed $\mathbb{C}P^2$ is a solution to its equation of motion, we have seen in the previous section that this forces $m^2 = -2(\lambda + 1)$. Since $N = 1$ is fixed, we may compute the spectrum using a computer algebra software like Mathematica[31]. The results are summarized in table 1.

Table 1: List of numerically computed eigenvalues and their explicitly computed counterparts

unfixed gauge		gauge fixed: $[X_a, A_a] = 0$	
Eigenvalue	Multiplicity	Eigenvalue	Multiplicity
-2	12	-2	12
0	13	0	21
2	6	$\lambda+1$	2
3	2	$4(\lambda+1)$	1
$\lambda+1$	2	$q_1(\lambda)/4$	6
$4(\lambda+1)$	1	$q_2(\lambda)/4$	6
$q_1(\lambda)/4$	6	$q_3(\lambda)/4$	6
$q_2(\lambda)/4$	6		
$q_3(\lambda)/4$	6		

In table 1, the function $q_i(\lambda)$ denote the three roots of the polynomial

$$f(t) = t^3 - 8\lambda t^2 - (96 + 32\lambda)t - 256.$$

Note that as with the case for $N > 1$, the model unfortunately cannot be stabilized, even though we may adjust the coupling constant.

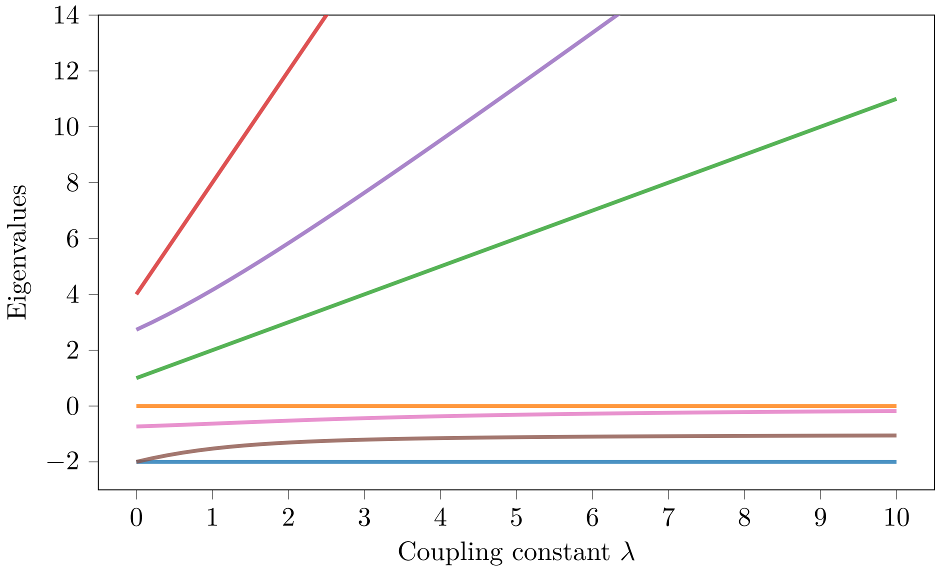

Figure 8: Progression of vector fluction operator spectrum w.r.t. λ for the non-gauge-fixed model.

3.5 SUMMARY

In this chapter we have focused on squashed $\mathbb{C}P^2_N$ solutions of the IKKT-inspired matrix model

$$S[X] = \frac{1}{4}\operatorname{Tr}\left([X_a, X_b][X_a, X_b] - 2m^2 X_a X_a - 2\lambda(X^2)^2\right).$$

We have shown that the quartic interaction term can only be retained if the fuzzy embedding functions X_a are in the $D(1,0)$ representation of $\mathfrak{su}(3)$. Otherwise, to satisfy the equation of motion, the coupling term has to vanish. Furthermore the equations of motion force the mass parameter to $m^2 = -2$.

In order to study the vector fluctuations around a squashed $\mathbb{C}P^2_N$ background, we presented a method for reducing the computational complexity of the spectrum of $\mathcal{D}^2$. To this end, we constructed the spaces

$$\mathcal{V}_{k,l} \subseteq \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes D(p,p).$$

Using these subspaces, we were able to partially find analytic expressions for a certain class of subspaces $\mathcal{V}_{k,l}$. These expressions, together with a wider range of accessible numerical results, suggest that the eigenvalues of $\mathcal{D}^2$ are of a particular form. We conjecture, that any eigenvalue of the shifting operator L_s constructed from the $D(p,p)$ irreducible representation of $\mathfrak{su}(3)$ is either ± 1 or

$$\pm\sqrt{-(k-l)^2 - kl + (p+1)^2},$$

where k, l are integers labeling the corresponding non-trivial $\mathcal{V}_{k,l}$. This was tested numerically up to $N = 25$. Therefore, the eigenvalues of the vector fluctuation operator $\mathcal{D}^2$ are given by

$$\begin{aligned} &-(k-l)^2 - kl + (p+1)^2 \pm 1 \\ \text{and } &-(k-l)^2 - kl + (p+1)^2 \pm \sqrt{-(k-l)^2 - kl + (p+1)^2}. \end{aligned}$$

Unfortunately, this can only produce the eigenvalues and not their multiplicities. To complete the picture and provide a formal proof of these numerically observed patterns more future work is required.

4 GENERAL POTENTIAL WITHOUT MASS TERM

As demonstrated in the previous chapter, choosing the squashed fuzzy projective plane as solution to the modified IKKT model (54) leads to unphysical negative modes of the vector fluctuation operator. Furthermore, this solution is not compatible with a quartic interaction term in the semi-classical limit. The main culprit is the fact that

$$X^2 = \sum_{c \in J} X_c X_c$$

is not proportional to the identity operator for the fuzzy embedding function X_c in the $D(N,0)$ irreducible representation with $N > 1$. For the fuzzy sphere, this limitation does not occur. Moreover, we can even generalize the potential in the action to any radially symmetric real analytic function. In this chapter we will demonstrate that for such a potential, the spectrum of the vector fluctuation operator does not contain negative modes, given that $N > 1$ and the coupling constant η is sufficiently large.

To this end, consider the action

$$S[X] = \frac{1}{4} \operatorname{Tr}\Big([X_a, X_b][X^a, X^b] - \eta V(X_a X^a)\Big) = S_{\text{kin}}[X] + \eta S_V[X]. \tag{74}$$

We pick an irreducible representation (N) of $\mathfrak{su}(2)$ on some N-dimensional Hilbert space $\mathcal{H}$. The matrices $\bar{X}_a \in \operatorname{End}(\mathcal{H})$ denote the group-theoretically normalized generators[1] and we define $X_a = r\bar{X}_a$ for some arbitrary real scaling parameter $r \in \mathbb{R}$. Then, the following relations hold:

$$\begin{aligned} X_a X^a = \mathcal{R}^2 = r^2 R_N^2 \mathrm{Id}_{\mathcal{H}}, \qquad R_N^2 = \frac{1}{4}(N^2 - 1) \\ \bar{X}_a \bar{X}^a = R_N^2 \mathrm{Id}_{\mathcal{H}}. \end{aligned} \tag{75}$$

As discussed in section 2.1, this set of matrices generates the fuzzy sphere of radius r. Once more, we introduce a small variation $X_a \to X_a + \phi_a$, $\phi_a \in \operatorname{End}(\mathcal{H})$ and compute the terms up to second order:

$$\begin{aligned} S[X+\phi] = S[X] &- \underbrace{\frac{1}{4} \operatorname{Tr}\Big(\phi_b \Box_X X^b + \eta V'(\mathcal{R}^2)\{X_a, \phi^a\}\Big)}_{S^{(1)}[X,\phi]} + \\ &- \underbrace{\frac{1}{2} \operatorname{Tr}\Big(\phi^a \Big((\mathcal{D}_0^2)_{ab} + \eta V''(\mathcal{R}^2)\{X_a, \{X_b, \cdot\}\} + 2\eta V'(\mathcal{R}^2)\delta_{ab}\Big) \phi^b\Big)}_{S^{(2)}[X,\phi]} + \mathcal{O}(\phi^3), \end{aligned}$$

where

$$(\mathcal{D}_0^2)_{ab} = \delta_{ab}\Box + 2[[X_a, X_b], \cdot] - [X_a, [X_b, \cdot]].$$

From the first-order terms, we once more obtain the equation of motion

$$\Box_X X^b + 2\eta V'(\mathcal{R}^2) X^b = 0 \tag{76}$$

1 We also assume them to be orthogonal under the Killing-form.

and the second order term defines the vector fluctuation operator of this particular model,

$$\mathcal{D}^2_{ab} = (\mathcal{D}^2_0)_{ab} + \eta V''(\mathcal{R}^2)\{X_a, \{X_b, \cdot\}\} + 2\eta V'(\mathcal{R}^2)\delta_{ab}. \tag{77}$$

By plugging X_a into the equation of motion the first Taylor-coefficient $V'(\mathcal{R}^2)$ is fixed:

$$\begin{aligned} \Box_X X^b + 2\eta V'(\mathcal{R}^2) X^b &= r^3 \Box_{\bar{X}} \bar{X}^b + 2\eta V'(\mathcal{R}^2) X^b = \\ &= 2r^3 \bar{X}^b + 2r\eta V'(\mathcal{R}^2)\bar{X}^b = 0 \Leftrightarrow V'(\mathcal{R}^2) = -\frac{r^2}{\eta}. \end{aligned}$$

This can be plugged back into the vector fluctuation operator

$$\begin{aligned} &\sum_{c=1}^{3}[X_c, [X_c, \cdot]] + 2[[X_a, X_b], \cdot] - [X_a, [X_b, \cdot]] + \mu\{X_a, \{X_b, \cdot\}\} - 2r^2\delta_{ab} = \\ &r^2\left(\sum_{c=1}^{3}[\bar{X}_c, [\bar{X}_c, \cdot]] + 2[[\bar{X}_a, \bar{X}_b], \cdot] - [\bar{X}_a, [\bar{X}_b, \cdot]] + \mu\{\bar{X}_a, \{\bar{X}_b, \cdot\}\} - 2\delta_{ab}\right), \end{aligned}$$

where we set $\mu = \eta V''(\mathcal{R}^2)$. Conveniently, the scaling factor r factors out.

4.0.1 *Stabilization of the model*

A stable model requires the spectrum of the vector fluctuation operator to be non-negative. Numerical computations (fig. 9) suggest that for the fuzzy sphere this can in fact be achieved as long as $N > 1$ and the coupling constant μ is sufficiently large.

To prove this claim, we loosely follow the approach presented in [32] for finding the vector harmonics on the fuzzy 4-sphere. Consider the ansatz

$$\begin{aligned} \mathcal{A}^{(1)}_a &:= \{X_a, \phi\}_+, \\ \mathcal{A}^{(2)}_a &:= [X_a, \phi], \\ \mathcal{A}^{(3)}_a &:= \varepsilon_{abc} X_b [X_c, \phi] \end{aligned} \tag{78}$$

and the operator

$$\mathcal{D}^2_{ab} = \delta_{ab}\Box + 2[[X_a, X_b], \cdot] - (1 - g_0)[X_a, [X_b, \cdot]] + \mu\{X_a, \{X_b, \cdot\}\} + M^2\delta_{ab}. \tag{79}$$

Note that we have introduced the additional parameter g_0 that lets us keep track of the gauge fixing. Setting $g_0 = 1$ fixed the gauge $[X_b, A_b] = 0$. Also, for the fuzzy sphere case, we have seen that $M^2 = -2$. To obtain the spectrum of $\mathcal{D}^2_{ab}$, we straightforwardly compute the action of $\mathcal{D}^2_{ab}$ on the ansatz eq. (78).

We merely state the final results here and refer the reader to the appendix for the detailed computation:

$$\begin{aligned} \mathcal{D}^2_{ab}\mathcal{A}^{(1)}_b[\phi] &= \mathcal{A}^{(1)}_a[(1-\mu)\Box\phi + (4\mu R_N^2 + M^2 + 6)\phi] + \mathcal{A}^{(2)}_a[4\phi] + \mathcal{A}^{(3)}_a[8i\phi] \\ \mathcal{D}^2_{ab}\mathcal{A}^{(2)}_b[\phi] &= \mathcal{A}^{(2)}_a[g_0\Box\phi + (2 + M^2)\phi] \\ \mathcal{D}^2_{ab}\mathcal{A}^{(3)}_b[\phi] &= \mathcal{A}^{(1)}_a\left[i\left(\frac{\mu}{2} - 2\right)\Box\phi\right] + \mathcal{A}^{(2)}_a\left[i\frac{g_0 - 1}{2}\Box\phi\right] + \mathcal{A}^{(3)}_a[(2 + M^2)\phi + \Box\phi] \end{aligned} \tag{80}$$

Now, from eq. (12), recall the decomposition into irreducible representations, i.e. given some $\phi \in (p)$, we have

$$\Box\phi = R^2_{2p}\phi =: \lambda\phi.$$

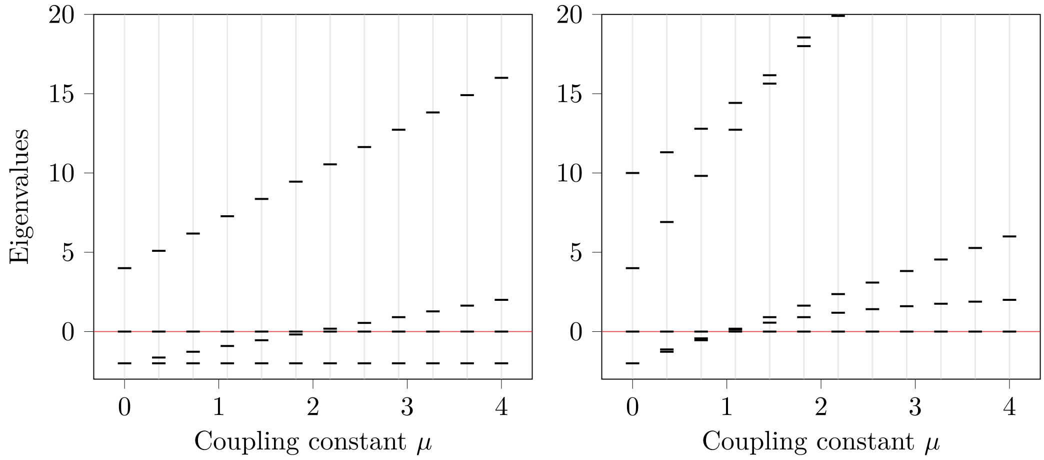

Figure 9: Numerically determined eigenvalues for $N = 1$ (left) and $N = 2$ (right). While in the left figure, there exists a constant negative mode, the vector fluctuation operator for $N = 2$ becomes stable for sufficiently high values of μ.

Plugging this back into eq. (80) the right hand side can be written as a linear combination of the ansatz eq. (78). This allows us to interpret the action of $\mathcal{D}^2_{ab}$ on our ansatz as the 3×3 matrix equation

$$\begin{pmatrix} \mathcal{D}^2_{ab} & & \\ & \mathcal{D}^2_{ab} & \\ & & \mathcal{D}^2_{ab} \end{pmatrix} \begin{pmatrix} \mathcal{A}^{(1)}_b[\phi] \\ \mathcal{A}^{(2)}_b[\phi] \\ \mathcal{A}^{(3)}_b[\phi] \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} (1-\mu)\lambda + (4\mu R_N^2 + M^2 + 6) & 4 & 8i \\ 0 & g_0\lambda + 2 + M^2 & 0 \\ i\lambda\left(\frac{\mu}{2} - 2\right) & i\lambda\frac{g_0 - 1}{2} & \lambda + 2 + M^2 \end{pmatrix}}_{M} \begin{pmatrix} \mathcal{A}^{(1)}_a[\phi] \\ \mathcal{A}^{(2)}_a[\phi] \\ \mathcal{A}^{(3)}_a[\phi] \end{pmatrix}. \tag{81}$$

The eigenvalues of the mixing matrix M are in fact candidates for eigenvalues of D^2_{ab}. To see this note that we may choose a a suitable unitary matrix U such that $D_M = U^{-1}MU$ becomes diagonal. Then eq. (81) is equivalent to

$$\begin{pmatrix} \mathcal{D}^2_{ab} & & \\ & \mathcal{D}^2_{ab} & \\ & & \mathcal{D}^2_{ab} \end{pmatrix} U \begin{pmatrix} \mathcal{A}^{(1)}_b[\phi] \\ \mathcal{A}^{(2)}_b[\phi] \\ \mathcal{A}^{(3)}_b[\phi] \end{pmatrix} = D_M U \begin{pmatrix} \mathcal{A}^{(1)}_a[\phi] \\ \mathcal{A}^{(2)}_a[\phi] \\ \mathcal{A}^{(3)}_a[\phi] \end{pmatrix}. \tag{82}$$

For our particular model, we may set $M^2 = -2$ again. The eigenvalues can be computed straightforwardly for instance using a computer algebra system. The three expressions for the eigenvalues read

$$\begin{aligned} \lambda_{M,1}[\lambda] &= g_0\lambda, \\ \lambda_{M,2}[\lambda] &= \lambda\frac{2-\mu}{2} - \sqrt{\frac{\lambda^2\mu^2}{4} - 2\lambda\left(3\mu + \mu^2 R_N^2 - 8\right) + 4(\mu R_N^2 + 1)^2 + 2\mu R_N^2 + 2}, \\ \lambda_{M,3}[\lambda] &= \lambda\frac{2-\mu}{2} + \sqrt{\frac{\lambda^2\mu^2}{4} - 2\lambda\left(3\mu + \mu^2 R_N^2 - 8\right) + 4(\mu R_N^2 + 1)^2 + 2\mu R_N^2 + 2}. \end{aligned}$$

The model is stabilized once all eigenvalues are non-negative for any possible choice of λ. Clearly, $\lambda_{M,1}$ is by construction positive or vanishes, since $g_0 \in \{0, 1\}$ and $\lambda \geq 0$.

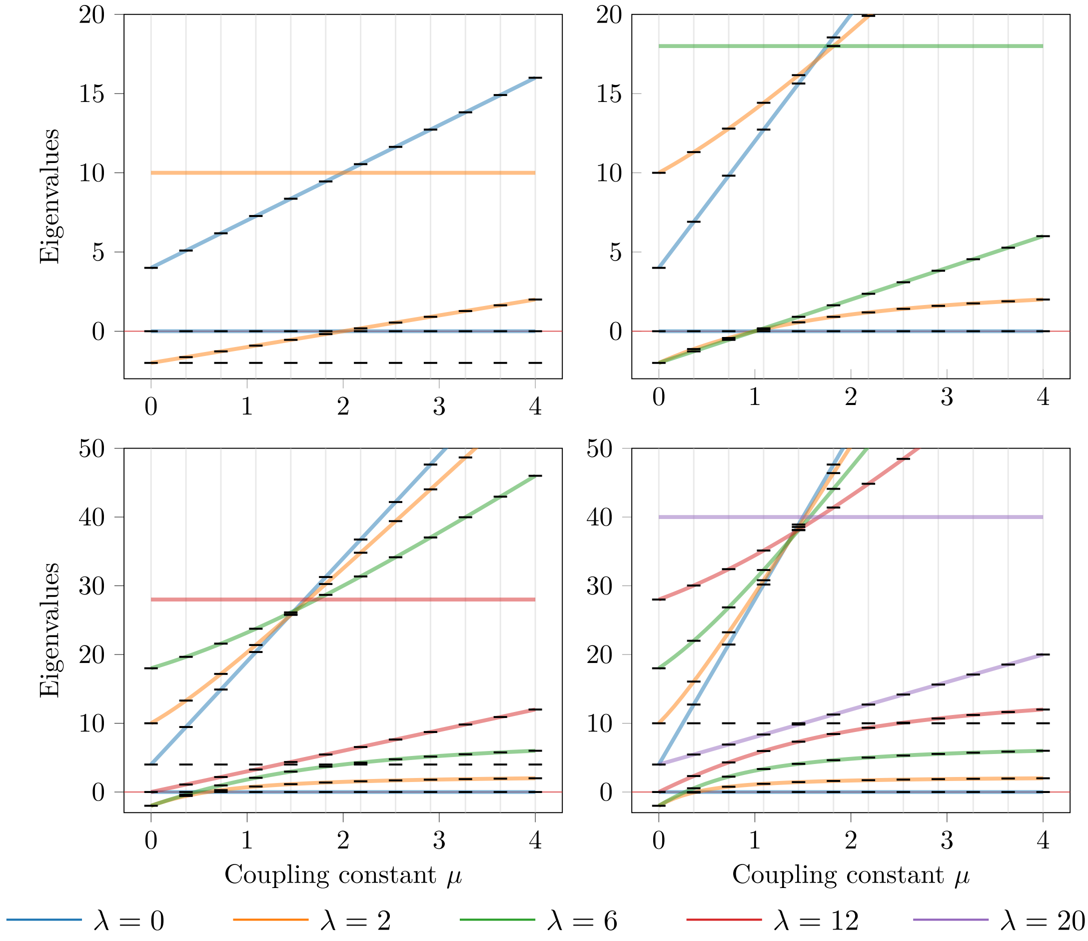

Figure 10: Comparison of numerically and analytically computed eigenvalues for $N = 1$ (top-left), $N = 2$ (top-right), $N = 3$ (bottom-left) and $N = 4$ (bottom-right).

Thus we only need to turn our attention to the other two expressions, $\lambda_{M,2}$ and $\lambda_{M,3}$. We can solve the inequalities $\lambda_{M,i} \geq 0$ for μ to obtain the critical values of μ, where all eigenvalues become positive. This yields the single inequality

$$\mu_{crit} = \frac{12 - \lambda}{4(R_N^2 + 1) - \lambda} = \frac{12 - \lambda}{N^2 + 3 - \lambda}. \tag{83}$$

Note that since $0 \leq \lambda \leq N(N-1)$, μ_{crit} is bounded from above by $12/(N^2 + 3)$ and thus, for a particular choice of the coupling constant $\nu = \mu/V''(\mathcal{R}^2)$, the model remains stable in the semi-classical limit $N \to \infty$.

What remains to be done is to confirm that these eigenvalues of the mixing matrix are in fact true eigenvalues of $\mathcal{D}^2$ and that we do not miss any eigenvalues. To get an idea of the situation, we compare the analytically obtained expressions with numerical computations for the first few representations. The respective plots are given in figure fig. 10. We clearly see that in each plot we find an eigenvalue of the mixing matrix independent of μ that is fact not appearing in the numerical computation. Additionally, there is also an eigenvalue missing, apart from the case $N = 2$, since there this missing eigenvalue coincides with zero.

4.0.2 *Validity of the ansatz*

In general, the ansatz eq. (78) only provides candidates for eigenvalues of the operator $\mathcal{D}^2$. Two scenarios are conceivable:

(i) The expressions $U_{ij}\mathcal{A}_a^{(j)}[\phi]$ vanish, i.e. – assuming each $\mathcal{A}_a^{(j)}[\phi]$ is non-vanishing – the ansatz is not linearly independent. The corresponding entry in D_M can then take any value, since both sides of eq. (82) vanish in that particular entry.

(ii) The ansatz might miss eigenvalues.

In fact both scenarios need to be taken care of for the present case. The ansatz in eq. (78) is valid up to the highest spin components in the decomposition

$$\mathrm{Mat}(N,\mathbb{C}) \cong (N) \otimes (N)^* \cong \bigoplus_{p=1}^{N} (2p-1). \tag{84}$$

First and foremost, we can shown that for any $\phi \in (2N-1)$, the three expressions $\mathcal{A}_a^{(i)}[\phi]$ are linearly dependent and

$$U_{2i}\mathcal{A}_a^{(i)}[\phi] = 0.$$

To see this, we consider $\mathcal{A}_a^{(3)}[\phi]$ and expand it

$$\begin{aligned}\mathcal{A}_a^{(3)}[\phi] =& \varepsilon_{abc}X_b[X_c,\phi] = \varepsilon_{abc}X_bX_c\phi - \varepsilon_{abc}X_b\phi X_c = \\ =& iX_a\phi - \varepsilon_{abc}X_b\phi X_c = \frac{i}{2}\mathcal{A}_a^{(1)}[\phi] + \frac{i}{2}\mathcal{A}_a^{(2)}[\phi] - \varepsilon_{abc}X_b\phi X_c.\end{aligned}$$

Let $\phi_l^{N-1} \in \mathrm{Mat}(N,\mathbb{C})$, such that

$$\begin{aligned}\Box\phi_l^{N-1} &= N(N-1)\phi_l^{N-1}, \\ [X_3,\phi_l^{N-1}] &= l\phi_l^{N-1}, \quad l = -N+1,\dots,N-1\end{aligned}$$

Under these assumptions, we claim that the identity

$$i\varepsilon_{abc}X_b\phi_l^{N-1}X_c = \frac{N-1}{2}\{X_a,\phi_l^{N-1}\} \tag{85}$$

holds. In order to show this, we collect a few observations about the left and right-hand side of this equation. The right-hand side is of course a multiple of $\mathcal{A}_a^{(1)}[\phi_l^{N-1}]$, while the left-hand side motivates the map $\mathcal{B}_a[\phi] := i\varepsilon_{abc}X_b\phi X_c$. We can interpret $\mathcal{A}^{(1)}$, as well as $\mathcal{B}$, as maps taking elements of the tensor product $\mathfrak{su}(2) \otimes \mathrm{Mat}(N,\mathbb{C})$ to $\mathrm{Mat}(N,\mathbb{C})$. More precisely, we introduce the linear maps

$$\begin{aligned}\mathcal{A}^{\pm} : \mathfrak{su}(2) \otimes \mathrm{Mat}(N,\mathbb{C}) &\to \mathrm{Mat}(N,\mathbb{C}) \\ x \otimes \phi \mapsto \mathcal{A}^{\pm}(x \otimes \phi) &= \begin{cases}\{X,\phi\} & \text{for } \mathcal{A}^+ \\ [X,\phi] & \text{for } \mathcal{A}^-\end{cases}\end{aligned} \tag{86}$$

and

$$\mathcal{B} : \mathfrak{su}(2) \otimes \mathrm{Mat}(N,\mathbb{C}) \to \mathrm{Mat}(N,\mathbb{C}),\ x \otimes \phi \mapsto [X_c, X]\phi X_c. \tag{87}$$

Using these definitions, we can recover our ansatz via the formulas

$$\begin{aligned}
\mathcal{A}^+(x_a \otimes \phi) &= \mathcal{A}_a^{(1)}[\phi], \\
\mathcal{A}^-(x_a \otimes \phi) &= \mathcal{A}_a^{(2)}[\phi], \\
-i\mathcal{B}(x_a \otimes \phi) &= -i[X_c, X_a]\phi X_c = \epsilon_{abc} X_b \phi X_c.
\end{aligned}$$

Conveniently, the maps $\mathcal{A}^\pm$ and $\mathcal{B}$ preserve the Lie algebra structure of the tensor product given by

$$(x, y \otimes \phi) \mapsto x \triangleright (y \otimes \phi) = [x, y] \otimes \phi + y \otimes [X, \phi]$$

and the adjoint action on the N-dimensional matrices.

Lemma 6. *The maps $\mathcal{A}^\pm$ and $\mathcal{B}$, as defined above, intertwine the two actions*

$$(x, y \otimes \phi) \mapsto x \triangleright (y \otimes \phi), \qquad (x, \phi) \mapsto [X, \phi],$$

i.e., for $\psi = \mathcal{A}^\pm$ or $\psi = \mathcal{B}$,

$$\psi(x \triangleright y \otimes \phi) = [X, \psi(y \otimes \phi)].$$

Proof. Linearity of the three maps is clear. Also, by construction, the claim is immediate for $\mathcal{A}^-$ from the Jacobi identity. The statement for $\mathcal{A}^+$ can be derived by direct computation:

$$\begin{aligned}
\mathcal{A}^+(x \triangleright (y \otimes \phi)) =& \mathcal{A}^+([x, y] \otimes \phi + y \otimes [X, \phi]) = \\
=& \{[X, Y], \phi\} + \{Y, [X, \phi]\} = [X, \{Y, \phi\}] = [X, \mathcal{A}^+(y \otimes \phi)].
\end{aligned}$$

To carry out the proof for $\mathcal{B}$ consider the identity

$$[X_c, X_a]\phi[X_b, X_c] = [i\epsilon_{bcd} X_c, X_a]\phi X_d = [X_a, [X_b, X_d]]\phi X_d.$$

From linearity of the Lie brackets, this is equivalent to

$$[X_c, X]\phi[Y, X_c] = [X, [Y, X_c]]\phi X_c. \tag{88}$$

Then, on the one hand, we have

$$\begin{aligned}
\mathcal{B}(x \triangleright (y \otimes \phi)) &= \mathcal{B}([x, y] \otimes \phi + y \otimes [X, \phi]) = \\
&= [X_c, [X, Y]]\phi X_c + [X_c, Y][X, \phi]X_c
\end{aligned}$$

and conversely

$$\begin{aligned}
[X, \mathcal{B}(y \otimes \phi)] =& [X, [X_c, Y]\phi X_c] = \\
=& [X, [X_c, Y]]\phi X_c + [X_c, Y][X, \phi]X_c + [X_c, Y]\phi[X, X_c] = \\
=& [X, [X_c, Y]]\phi X_c + [X_c, Y][X, \phi]X_c + [Y, [X, X_c]]\phi X_c = \\
=& [X_c, Y][X, \phi]X_c - [X_c, [Y, X]]\phi X_c = \mathcal{B}(x \triangleright (y \otimes \phi)),
\end{aligned}$$

where we first made use of eq. (88) and then of the Jacobi identity. ∎

This means in particular that we need to show eq. (85) only for the highest-weight vector ϕ^{N-1}_{N-1}. This highest weight vector can readily be constructed from the weight vectors of the N-dimensional representation of $\mathfrak{su}(2)$. Recall that the highest weight of the N-dimensional representation is $\omega = (N-1)/2$. Using the Dirac notation for the respective eigenvectors, we claim that

$$\phi^{N-1}_{N-1} = |\omega\rangle\langle -\omega|.$$

Indeed a straightforward computation shows

$$[X_3, |\omega\rangle\langle -\omega|] = (N-1)|\omega\rangle\langle -\omega|. \tag{89}$$

From the decomposition of $\mathrm{Mat}(N,\mathbb{C})$ into irreducible representations, this already suffices to conclude that $|\omega\rangle\langle -\omega|$ exclusively lives within $(2N-1)$. We can now verify that identity eq. (85) holds for $\phi := \phi^{N-1}_{N-1}$. Note that

$$\begin{aligned} X_b\phi X_b =&(X^+ + X^-)\phi(X^+ + X^-) - (X^+ - X^-)\phi(X^+ - X^-) + X_3\phi X_3 = \\ =&X_3\phi X_3 = -\left(\frac{N-1}{2}\right)^2 \phi. \end{aligned} \tag{90}$$

Instead of working with the basis $\{x_a\}$, it is more convenient to work with raising and lowering operators. Thus we compute the three cases for each of the basis elements $\{x_3, x^{\pm}\}$, where $x^{\pm} = x_1 \pm ix_2$ as usually:

Case x_3:
$$\begin{aligned} \mathcal{A}^+(x_3 \otimes \phi) &= X_3\phi + \phi X_3 = \frac{N-1}{2}\phi - \frac{N-1}{2}\phi = 0. \\ \mathcal{B}(x_3 \otimes \phi) &= [X_c, X_3]\phi X_c = X_c X_3 \phi X_c - X_3 X_c \phi X_c = \\ &= \frac{N-1}{2} X_c\phi X_c + \left(\frac{N-1}{2}\right)^2 X_3\phi = 0, \end{aligned}$$

where we used eq. (90) in the last line.

Case x^+:
$$\begin{aligned} \mathcal{A}^+(x^+ \otimes \phi) &= X^+\phi + \phi X^+ = 0. \\ \mathcal{B}(x^+ \otimes \phi) &= [X_c, X^+]\phi X_c = X_c X^+ \phi X_c - X^+ X_c \phi X_c = \\ &= \left(\frac{N-1}{2}\right)^2 X^+\phi = 0. \end{aligned}$$

The final case is a bit more involved. Let us first present the following intermediate result before we tackle the computation of $\mathcal{B}(x^- \otimes \phi)$:

$$\begin{aligned} X_c X^- \phi X_c =&(X^+ + X^-)\phi(X^+ + X^-) - (X^- + X^-)\phi(X^+ - X^-) + X_3\phi X_3 = \\ =&(X^+ + X^-)X^-\phi X^- + (X^+ - X^-)X^-\phi X^- + X_3 X^- \phi X_3 = \\ =&2X^+X^-\phi X^- - \frac{N-1}{2} X_3 X^- \phi = \\ =&[X^+, X^-]\phi X^- - \frac{N-1}{2}([X_3, X^-] + X^- X_3)\phi = \\ =&X_3\phi X^- - \frac{N-1}{2}(-X^- + \frac{N-1}{2}X^-)\phi = \\ =&\frac{N-1}{2}\{X^-, \phi\} - \left(\frac{N-1}{2}\right)^2 X^-\phi. \end{aligned}$$

Equipped with this identity and together with eq. (90) it follows immediately that for the last remaining case eq. (85) holds as well.

$$\begin{aligned} \textit{Case } x^-: \quad \mathcal{B}(x^- \otimes \phi) &= [X_c, X^-]\phi X_c = X_c X^- \phi X_c - X^- X_c \phi X_c = \\ &= \frac{N-1}{2}\mathcal{A}^+(x^- \otimes \phi) \end{aligned}$$

This shows that eq. (85) holds for the highest weight vector, and since $\mathcal{A}^+$ and $\mathcal{B}$ are intertwiners, this identity can be extended to $(2N-1) \subseteq \mathrm{Mat}(N, \mathbb{C})$.

We now turn our attention back to the mixing matrix and pick eigenvectors of the Laplacian with eigenvalue $\lambda = N(N-1)$, i.e. we take $\phi \in (N-1)$. After plugging everything into eq. (81), we obtain

$$M = \begin{pmatrix} N(N+\mu-1)+4-\mu & 4 & 8i \\ 0 & 0 & 0 \\ \frac{1}{2}iN(N-1)(\mu-4) & -\frac{1}{2}i(N-1)n & N(N-1) \end{pmatrix}.$$

The diagonalization of $M = U^{-1}D_M U$ was carried out with *Mathematica* [31] and is explicitly given by

$$\begin{aligned} D_M &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & N(N+3) & 0 \\ 0 & 0 & (N-1)(N+\mu-4) \end{pmatrix}, \\ U &= -\frac{i(\mu-4)(N-1)}{8(1-2N)-2\mu(N+1)} \begin{pmatrix} 0 & \frac{-N(\mu-8)+\mu-4}{(N-1)(\mu-4)} & 0 \\ N & 1 & 2i \\ -N & -\frac{4N}{(N-1)(\mu-4)} & -\frac{8iN}{(N-1)(\mu-4)} \end{pmatrix}. \end{aligned}$$

We claim that $N(N+3)$ is not a real eigenvalue of $\mathcal{D}^2$. Indeed for the first couple of N, this is confirmed by numerically checking the eigenvalues of the vector fluctuation operator. It can appear as an eigenvalue of the mixing matrix if and only if $U_{2i}\mathcal{A}_a^{(i)}[\phi]$ vanishes. Indeed, using eq. (85), we

$$U_{2i}\mathcal{A}_a^{(i)}[\phi] \propto N\mathcal{A}_a^{(1)}[\phi] + \mathcal{A}_a^{(2)}[\phi] + 2i\mathcal{A}_a^{(3)}[\phi] \tag{91}$$

vanishes. Indeed using the property derived above, we find that

$$\begin{aligned} 0 &= \frac{i}{2}\mathcal{A}_a^{(1)}[\phi] + \frac{i}{2}\mathcal{A}_a^{(2)}[\phi] - \varepsilon_{abc}X_b\phi X_c - \mathcal{A}_a^{(3)}[\phi] = \\ &= \frac{i}{2}\mathcal{A}_a^{(1)}[\phi] + \frac{i}{2}\mathcal{A}_a^{(2)}[\phi] + i\frac{N-1}{2}\mathcal{A}_a^{(1)}[\phi] - \mathcal{A}_a^{(3)}[\phi] = \\ &= \frac{i}{2}N\mathcal{A}_a^{(1)}[\phi] + \frac{i}{2}\mathcal{A}_a^{(2)}[\phi] - \mathcal{A}_a^{(3)}[\phi]. \end{aligned}$$

Multiplying both sides with $2i$ confirms that eq. (91) indeed equals zero and thus, for $\phi \in (N-1)$, our ansatz produces redundances.

To find the eigenvalues that were not produced by the proposed ansatz, we set $\phi := \phi_{N-1}^{N-1}$ and claim that

$$\mathcal{V} := \begin{pmatrix} -i\phi \\ \phi \\ 0 \end{pmatrix}$$

is an eigenvector of $\mathcal{D}^2$ and its eigenvalue is precisely the missing eigenvalue in fig. 10. To demonstrate this, we first note – by again using eq. (85) – that

$$\begin{aligned}\{X_b, \mathcal{V}_b\} =& -i\{X_1, \phi\} + \{X_2, \phi\} = \frac{2}{N-1}(\varepsilon_{1bc}X_b\phi X_c + i\varepsilon_{2bc}X_b\phi X_c) = \\ =&\frac{2}{N-1}(X_2\phi X_3 - X_3\phi X_2 + iX_3\phi X_1 - iX_1\phi X_3) = \\ =&\frac{2i}{N-1}(X_3\underbrace{\phi X^+}_{=0} - \underbrace{X^+\phi}_{=0} X_3) = 0.\end{aligned}$$

Hence it follows immediately, that the interaction term $\{X_a, \{X_b, \mathcal{V}_b\}\}$ in $\mathcal{D}^2$ does not contribute. For the remaining terms of the vector fluctuation operator, we then find for each of the three coordinates:

$$\begin{aligned}\mathcal{D}^2_{1b}\mathcal{V}_b =& -i\Box\phi + i[X_1, [X_1, \phi]] + 2i\phi + 2[[X_1, X_2], \phi] - [X_1, [X_2, \phi]] = \\ =& -iN(N-1)\phi + i[X_1, [X^+, \phi]] + 2i\phi + 2i[X_3, \phi] = N(N-3)\mathcal{V}_1 \\ \mathcal{D}^2_{2b}\mathcal{V}_b =&\Box\phi - 2i[[X_2, X_1], \phi] + i[X_2, [X_1, \phi]] - 2\phi - [X_2, [X_2, \phi]] = \\ =&N(N-1)\phi + i[X_2, [X^+, \phi]] - 2\phi - 2[X_3, \phi] = N(N-3)\mathcal{V}_2 \\ \mathcal{D}^2_{3b}\mathcal{V}_b =& -2i[[X_3, X_1], \phi] + 2[[X_3, X_2], \phi] + i[X_3, [X_1, \phi]] - [X_3, [X_2, \phi]] = \\ =& -2i[X^+, \phi] + i[X_3, [X^+, \phi]] = 0,\end{aligned}$$

where we used

$$\delta_{ab}\Box\mathcal{V}_b = \Box\mathcal{V}_a = N(N-1)\mathcal{V}_a.$$

To address the question of eigenvalue multiplicities and better understand the underlying cause for the the wrong/missing eigenvalues discussed above, we take a look at the decomposition

$$\mathfrak{su}(3) \otimes \mathrm{Mat}(N, \mathbb{C}) \cong (1) \oplus (3)^{\oplus 3} \oplus \cdots \oplus (2N-3)^{\oplus 3} \oplus (2N-1)^{\oplus 2} \oplus (2N+1)$$

Recall that the maps $\mathcal{A}^{\pm}$ and $\mathcal{B}$ map this tensor product space to

$$\mathrm{Mat}(N, \mathbb{C}) \cong (1) \oplus (3) \oplus \cdots \oplus (2N-3) \oplus (2N-1)$$

and thus, irreducibility dictates that $(2N+1)$ necessarily maps to (1). In other words, any $\mathcal{A}^{(k)}_a[\phi]$ fails to produce the explicitly constructed eigenvalue $\mathcal{V} \in (2N+1)$. Also note that the linear dependency highlighted in eq. (91) is a consequence of the twofold multiplicity of $(2N-1)$ appearing in the decomposition of $\mathfrak{su}(3) \otimes \mathrm{Mat}(N, \mathbb{C})$. Furthermore, since the unit matrix spans $(1) \subseteq \mathrm{Mat}(N, \mathbb{C})$, the two intertwiners $\mathcal{A}^{(2)}_a[\phi]$ and $\mathcal{A}^{(3)}_a[\phi]$ vanish for $\phi \in (1)$ and therefore do not contribute to the set of eigenvectors. These results are summarized in table 2.

Table 2: Summary of eigenvalues in spectrum of vector fluctuation operator.

	Representation	Multiplicity	Eigenvalue	Eigenvalue multiplicity
$N=1$	(3)	1	-2	3
$N=2$	(1)	1	$4+3\mu$	1
	(3)	2	$2g_0$	3
			$\mu-2$	3
	(5)	1	-2	5
$N\geq 3$	(1)	1	$4+(N^2-1)\mu$	1
	(3)	3	$\lambda_{M,1}\,[2]$	3
			$\lambda_{M,2}\,[2]$	3
			$\lambda_{M,3}\,[2]$	3
	$\vdots$			
	$(2p-1)$	3	$\lambda_{M,1}\,[p(p-1)]$	$2p-1$
			$\lambda_{M,2}\,[p(p-1)]$	$2p-1$
			$\lambda_{M,3}\,[p(p-1)]$	$2p-1$
	$\vdots$			
	$(2N-3)$	3	$\lambda_{M,1}\,[(N-1)(N-2)]$	$2N-3$
			$\lambda_{M,2}\,[(N-1)(N-2)]$	$2N-3$
			$\lambda_{M,3}\,[(N-1)(N-2)]$	$2N-3$
	$(2N-1)$	2	$N(N-1)g_0$	$2N-1$
			$(N-1)(N+\mu-4)$	$2N-1$
	$(2N+1)$	1	$N(N-3)$	$2N+1$

4.0.3 *Summary*

In this section, we have obtained explicit expressions for the computation of the full spectrum of the vector fluctuation operator $\mathcal{D}^2$ and the respective multiplicities for an IKKT-inspired matrix model with radial potential and a fuzzy sphere background. These results are summarized in table 2. We have shown that all fluctuation modes can be made positive, if the parameter μ is chosen large enough as long as $\mu \geq \mu_{crit}$.

5 CONCLUSION

In this thesis, we have covered two loosely related topics. In the first part, we obtained several parametrizations for the space emerging in the semi-classical limit of the squashed $\mathbb{C}P^2_N$. In particular, the action-angle coordinates prove to be convenient to compute effective geometrical properties. In terms of these coordinates, we computed the effective metric emerging on squashed $\mathbb{C}P^2_N$, which is in turn essential to understanding the dynamics of the fields living on the underlying space.

The second part focused on fluctuations of the background around a given solution of the bosonic section of the IKKT model. The two solutions in consideration where again the squashed $\mathbb{C}P^2_N$ and in addition to that the fuzzy S^2. For the former solution, we augmented the model by a mass term and a quartic interaction term. For the latter solution we were able to consider the more general case of a generic potential introduced to the action. We have presented a machinery to more effectively compute the fluctuation modes for the squashed $\mathbb{C}P^2_N$ background. In principle, this approach should be generalizable to other fuzzy spaces that are build from irreducible representations of Lie algebras. Working out these issues is left for future research. While we have shown that the fluctuations modes of the squashed $\mathbb{C}P^2_N$ background contain negative modes and thus exhibit instabilities, the fuzzy S^2 can be stabilized. This gives hope for a larger class of possibly more physically meaningful models to be discovered in further investigations.

A

APPENDIX

A.1 EIGENVALUES OF VECTOR FLUCTUATION OPERATOR

In this section, we list the eigenvalues of the vector fluctuation operator (61) for $p = 1, \ldots, 3$ and $M^2 = 0$. This serves primarily as a validation for the analytical computation proposed in section three.

Table 3: List of numerically computed eigenvalues and their explicitly computed counterparts

	Numerical Reference		**Explicit calculation**	
	Eigenvalue	Multiplicity		Eigenvalue
$p = 0$	0	6	$\lambda^{(\mathrm{i})}$	0
$p = 1$	6.	1	$\lambda^{(\mathrm{v.1})}$	6
	5.	2	$\lambda^{(\mathrm{v.2})}$	5
	4.73205	6	$\lambda^{(\mathrm{iii.3})}$	$3+\sqrt{3}$
	4.	6	$\lambda^{(\mathrm{iii.1})}$	4
	3.	2	$\lambda^{(\mathrm{v.3})}$	3
	2.	13	$\lambda^{(\mathrm{v.4})}$	2
			$\lambda^{(\mathrm{iii.2})}$	2
			$\lambda^{(\mathrm{ii})+}$	2
	1.26795	6	$\lambda^{(\mathrm{iii.4})}$	$3-\sqrt{3}$
	0.	12	$\lambda^{(\mathrm{i})}$	0
			$\lambda^{(\mathrm{ii})-}$	0
$p = 2$	12.	3		12
	10.8284	18		$8+\sqrt{8}$
	10.	3		10
	9.	12		9
	8.4495	12	$\lambda^{(\mathrm{iv.2}+)}$	$6+\sqrt{6}$
	8.	3		8
	7.2361	6	$\lambda^{(\mathrm{iii.3})}$	$5+\sqrt{5}$
	7.	18	$\lambda^{(\mathrm{iv.1})+}$	7
	6.	9	$\lambda^{(\mathrm{iii.1})}$	6
	5.1716	18		$8-\sqrt{8}$
	5.	6	$\lambda^{(\mathrm{iv.1})-}$	5
	4.	6	$\lambda^{(\mathrm{iii.2})}$	4
	3.5505	12	$\lambda^{(\mathrm{iv.2}-)}$	$6-\sqrt{6}$
	3.4142	12	$\lambda^{(\mathrm{ii})+}$	$2+\sqrt{2}$
	2.7639	6	$\lambda^{(\mathrm{iii.4})}$	$5-\sqrt{5}$
	0.5858	12	$\lambda^{(\mathrm{i})-}$	$2-\sqrt{2}$
	0.	6	$\lambda^{(\mathrm{i})}$	0
$p = 3$	20.	5		20
	18.873	30		$15+\sqrt{15}$
	17.	4		17
	16.6056	24		$13+\sqrt{13}$

16.	18	16
15.4641	18	$12+\sqrt{12}$
15.	4	15
14.	30	14
13.	12	13
12.	41	12
11.127	30	$15-\sqrt{15}$
11.	12	11
10.	12	10
9.6458	6	$7+\sqrt{7}$
9.3944	24	$13-\sqrt{13}$
8.5359	18	$12-\sqrt{12}$
8.	18	8
6.	36	6
4.7321	12	$3+\sqrt{3}$
4.3542	6	$7-\sqrt{7}$
2.	6	2
1.2679	12	$3-\sqrt{3}$
0.	6	0

A.2 DETAILED COMPUTATION OF $\mathcal{D}^2_{ab}\mathcal{A}^{(i)}_b$

In order to tidy up the computations, we first want to state a few useful formulas that will come in handy in this section. We want to emphasize that we consider the set of $\mathfrak{sl}(2,\mathbb{C})$ generators $\{x_a\}_{a=1,\dots 3}$ obeying the commutation relations

$$[x_a, x_b] = i\epsilon_{abc}x_c.$$

To reiterate, note that we denote elements of the (abstract) Lie-algebra $\mathfrak{sl}(2,\mathbb{C})$ by lower-case letters, i.e. $x \in \mathfrak{sl}(2,\mathbb{C})$, while capital letters X are the images of $x \in \mathfrak{sl}(2,\mathbb{C})$ under a particular representation ρ, i.e. $X = \rho(x)$. In this section we take ρ to be the N-dimensional irreducible representation on some N-dimensional Hilbert space $\mathcal{H}$. The endomorphisms X_a are normed s.t. the quadratic Casimir operator is given by

$$\delta_{ab}X_aX_b = R_N^2\,\mathrm{Id}_{\mathcal{H}}, \quad R_N^2 = \frac{1}{4}(N^2-1).$$

For the upcoming computation it turns out to be useful to expand double commutators in terms of linear combinations of $\{x_a\}_{a=1,2,3}$. By virtue of the fundamental property

$$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

of the Levi-Civita symbol we can derive the relation

$$\begin{aligned}[x_a,[x_b,x_c]] =& i\epsilon_{bcd}[x_a,x_d] = i\epsilon_{bcd}i\epsilon_{ade}x_e = \epsilon_{bcd}\epsilon_{aed}x_e = \\ =&(\delta_{ba}\delta_{ce} - \delta_{be}\delta_{ca})x_e = \delta_{ab}x_c - \delta_{ac}x_b.\end{aligned} \tag{92}$$

Lemma 7. *Consider an irreducible representation on some N-dimensional Hilbert space $\mathcal{H}$ such that*

$$X^2 = \delta_{ab}X_aX_b = R_N^2\,\mathrm{Id}_{\mathcal{H}}, \ \textit{where } R_N^2 = \frac{1}{4}(N^2-1)$$

Then, for any $\phi \in \mathrm{End}(\mathcal{H})$, the fuzzy Laplacian takes the form

$$\Box\phi = 2(R_N^2\phi - X_c\phi X_c) = 2X_b[X_b, \phi].$$

Proof. This is a simple matter of expanding the commutator and collecting terms

$$[X_c, [X_c, \phi]] = X_c(X_c\phi - \phi X_c) - (X_c\phi - \phi X_c)X_c = 2(R_N^2\phi - X_c\phi X_c).$$

On the other hand

$$2X_b[X_b, \phi] = 2(R_N^2\phi - 2X_c\phi X_c). \qquad \blacksquare$$

Since, to derive the proposed result eq. (80), we turn to $\mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{Mat}(N,\mathbb{C})$, recall that $\mathfrak{sl}(2,\mathbb{C})$ acts on this tensor product as follows:

$$\begin{aligned} x \triangleright : \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{Mat}(N,\mathbb{C}) &\to \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{Mat}(N,\mathbb{C}) \\ y \otimes \phi &\mapsto x \triangleright (y \otimes \phi) = [x,y] \otimes \phi + y \otimes [X, \phi], \end{aligned}$$

for any choice $x \in \mathfrak{sl}(2,\mathbb{C})$. In this representation, the Casimir operator takes the form

$$\begin{aligned} \Omega : \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{End}(\mathcal{H}) &\to \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{End}(\mathcal{H}) \\ x \otimes \phi &\mapsto \sum_{a=1}^{3} x_a \triangleright x_a \triangleright (x \otimes \phi) \end{aligned}$$

and, by using the identity for the double commutator, Ω can be expanded as follows:

$$\begin{aligned} \Omega(x \otimes \phi) &= [x_a, [x_a, x]] \otimes \phi + x \otimes \Box\phi + 2[x_a, x] \otimes [X_a, \phi] = \\ &= x \otimes \Box\phi + 2(x \otimes \phi + [x_a, x] \otimes [X_a, \phi]). \end{aligned} \tag{93}$$

Here we used the identity $[x_a, [x_a, x]] = 2x$.

Also, recall the definition of the intertwiners

$$\mathcal{A}^{\pm} : \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{Mat}(N,\mathbb{C}) \to \mathrm{Mat}(N,\mathbb{C}) \qquad \mathcal{B} : \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{Mat}(N,\mathbb{C}) \to \mathrm{Mat}(N,\mathbb{C})$$

stated in eq. (86) and eq. (87). The Casimir operator of the representation given on the tensor product then clearly makes the diagram

$$\begin{array}{ccc} \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{End}(\mathcal{H}) & \xrightarrow{\Omega} & \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{End}(\mathcal{H}) \\ \downarrow{\scriptstyle \mathcal{A}^{\pm},\mathcal{B}} & & \downarrow{\scriptstyle \mathcal{A}^{\pm},\mathcal{B}} \\ \mathrm{End}(\mathcal{H}) & \xrightarrow{\quad\Box\quad} & \mathrm{End}(\mathcal{H}) \end{array}$$

commute.

Lemma 8. *For any choice $x \in \mathfrak{sl}(2,\mathbb{C})$, the three intertwiners $\mathcal{A}^{\pm}$ and $\mathcal{B}$ defined above fulfill the following three identifies:*

(i) $\displaystyle\sum_{c=1}^{3} \mathcal{A}^{-}([x_c, x] \otimes [X_c, \phi]) = -\mathcal{A}^{-}(x \otimes \phi).$

(ii) $\displaystyle\sum_{c=1}^{3} \mathcal{A}^{+}([x_c, x] \otimes [X_c, \phi]) = -2\mathcal{B}(x \otimes \phi) - \mathcal{A}^{+}(x \otimes \phi).$

(iii) $$\sum_{c=1}^{3}\mathcal{B}([x_c,x]\otimes[X_c,\phi]) = -\frac{1}{2}\mathcal{A}^+(x\otimes\Box\phi).$$

Proof. To obtain the first formula, note that $\Box$ commutes with the action $[X,\cdot]$ by construction and thus

$$\Box(\mathcal{A}^-(x\otimes\phi)) = \Box[X,\phi] = [X,\Box\phi] = \mathcal{A}^-(x\otimes\Box\phi).$$

On the other hand, since $\mathcal{A}^-$ intertwines $\Box$ and Ω, we have

$$\begin{aligned}\mathcal{A}^-(x\otimes\Box\phi) =&\Box\mathcal{A}^-(x\otimes\phi) = \mathcal{A}^-(\Omega(x\otimes\phi)) = \\ =&\mathcal{A}^-(x\otimes\Box\phi) + 2\mathcal{A}^-(x\otimes\phi) + 2\sum_{c=1}^{3}\mathcal{A}^-([x_c,x]\otimes[X_c,\phi]).\end{aligned}$$

Solving for $\sum_{c=1}^{3}\mathcal{A}^-([x_a,x]\otimes[X_a,\phi])$ reveals the assertion of the lemma.

In order to derive the latter two identities we can without loss of generality – due to linearity – let $x = x_a$. Then, by virtue of eq. (92) and lemma 7, we find

$$\begin{aligned}\mathcal{A}^+([x_c,x_a]\otimes[X_c,\phi]) =&[X_c,X_a]X_c\phi - [X_c,X_a]\phi X_c + \\ &+\underbrace{X_c\phi[X_c,X_a]}_{=-[X_c,X_a]\phi X_c} - \phi X_c[X_c,X_a] = \\ =& -2\mathcal{B}(x_a\otimes\phi) + \{(X_cX_aX_c - X_aX^2),\phi\}_+ = \\ =& -2\mathcal{B}(x_a\otimes\phi) - \mathcal{A}^+(x_a\otimes\phi)\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}([x_c,x_a]\otimes[x_c,\phi]) =&[X_c,[X_b,X_a]][X_b,\phi]X_c = (\delta_{cb}X_a - \delta_{ca}X_b)[X_b,\phi]X_c = \\ =&X_a(X_b\phi X_b - \phi X^2) - (X^2\phi - X_b\phi X_b)X_a = \\ =& -\frac{1}{2}(X_a(\Box\phi) + (\Box\phi)X_a) = -\frac{1}{2}\mathcal{A}^+(x_a\otimes\Box\phi).\end{aligned}$$ ∎

This lemma enables us to determine how the terms of the fluctuation operator act on the ansatz proposed in eq. (78).

LAPLACIAN TERM: As already mentioned in the proof of lemma 7, $\Box$ commutes with $[X,\cdot]$ by construction and thus

$$\Box\mathcal{A}^-(x\otimes\phi) = \Box[X,\phi] = [X,\Box\phi] = \mathcal{A}^-(x\otimes\Box\phi).$$

For the two remaining cases $\mathcal{A}^+$ and $\mathcal{B}$, we take the expansion eq. (93) of Ω and by making use of lemma 8, we obtain for $\Box\circ\mathcal{A}^+$ the result

$$\begin{aligned}\Box\mathcal{A}^+(x\otimes\phi) =&\mathcal{A}^+(\Omega(x\otimes\phi)) = \\ =&\mathcal{A}^+(x\otimes\Box\phi) + 2\mathcal{A}^+(x\otimes\phi) + 2\mathcal{A}^+([x_a,x]\otimes[X_a,\phi]) = \\ =&\mathcal{A}^+(x\otimes\Box\phi) + 2\mathcal{A}^+(x\otimes\phi) - 4\mathcal{B}(x\otimes\phi) - 2\mathcal{A}^+(x\otimes\phi) = \\ =&\mathcal{A}^+(x\otimes\Box\phi) - 4\mathcal{B}(x\otimes\phi).\end{aligned}$$

Similarly, $\Box\circ\mathcal{B}$ expands to

$$\begin{aligned}\Box\mathcal{B}(x\otimes\phi) =&\mathcal{B}(\Omega(x\otimes\phi)) = \\ =&\mathcal{B}(x\otimes\Box\phi) + 2\mathcal{B}(x\otimes\phi) + 2\mathcal{B}([x_a,x]\otimes[X_a,\phi]) = \\ =&\mathcal{B}(x\otimes\Box\phi) + 2\mathcal{B}(x\otimes\phi) - \mathcal{A}^+(x\otimes\Box\phi).\end{aligned}$$

OFF-DIAGONAL OPERATOR $[[X_a, X_b], \cdot]$: For any intertwiner $\psi : \mathfrak{sl}(2,\mathbb{C}) \otimes \mathrm{End}(\mathcal{H}) \to \mathrm{End}(\mathcal{H})$, note that

$$\begin{aligned}
[[X_a, X_b], \psi(x_b \otimes \phi)] =& \psi(\underbrace{[[x_a, x_b], x_b]}_{=2x_a} \otimes \phi + x_b \otimes [[X_a, X_b], \phi]) = \\
=& \psi(2x_a \otimes \phi + (i\epsilon_{abc}x_b) \otimes [X_c, \phi]) = \\
=& 2\psi(x_a \otimes \phi) + \psi([x_c, x_a] \otimes [X_c, \phi]).
\end{aligned}$$

In particular, for the three intertwiners $\mathcal{A}^\pm$ and $\mathcal{B}$, we can directly apply the formulas derived in lemma 8.

GAUGE TERM: The expressions for $\mathcal{A}^\pm$ and $\mathcal{B}$ can be obtained by turning to lemma 7. As for the first two intertwiners, $\mathcal{A}^\pm$, the gauge term simplifies to

$$\begin{aligned}
[X_b, \mathcal{A}^\pm(x_b \otimes \phi)] =& X_b(X_b\phi \pm \phi X_b) - (X_b\phi \pm \phi X_b)X_b = \\
=& X^2\phi \pm X_b\phi X_b - X_b\phi X_b \mp \phi X^2 = \begin{cases} 0 & \text{for } \mathcal{A}^+ \\ \Box\phi & \text{for } \mathcal{A}^- \end{cases}.
\end{aligned}$$

The expression $[X_b, \mathcal{B}(x_b \otimes \phi)]$ vanishes as well, since

$$\begin{aligned}
[X_b, \mathcal{B}(x_b \otimes \phi)] =& \mathcal{B}(x_b \otimes [X_b, \phi]) = X_cX_b[X_b, \phi]X_c - X_bX_c[X_b, \phi]X_c = \\
=& X_c\left(\frac{1}{2}\Box\phi\right)X_c - X_bX^2[X_b, \phi] - \frac{1}{2}\Box[X_b, \phi] = \\
=& X^2\left(\frac{1}{2}\Box\phi\right) - \frac{1}{4}\Box(\Box\phi) - X^2\left(\frac{1}{2}\Box\phi\right) + \frac{1}{2}X_b[X_b, \Box\phi] = 0.
\end{aligned}$$

Therefore, the gauge term only acts non-trivially on $\mathcal{A}^-$, namely

$$[X_a, [X_b, \mathcal{A}^-(x_b \otimes \phi)]] = [X_a, \Box\phi] = \mathcal{A}^-(x_a \otimes \Box\phi).$$

INTERACTION TERM: Due to the structural similarity to the gauge term, we proceed in the same fashion and first compute $\{X_b, \mathcal{A}^\pm(x_b \otimes \phi)\}$. Employing lemma 7 once more leads us to

$$\begin{aligned}
\{X_b, \mathcal{A}^\pm(x_b \otimes \phi)\} =& X_b(X_b\phi \pm \phi X_b) + (X_b\phi \pm \phi X_b)X_b = \\
=& X^2\phi \pm X_b\phi X_b + X_b\phi X_b \pm \phi X^2 = \begin{cases} 4R_N^2\phi - \Box\phi & \text{for } \mathcal{A}^+ \\ 0 & \text{for } \mathcal{A}^- \end{cases}.
\end{aligned}$$

As for the remaining intertwiner $\mathcal{B}$, first note that lemma 7 implies that the following anti-commutator vanishes:

$$\{X_b, [X_b, \phi]\} = X_b[X_b, \phi] + [X_b, \phi]X_b = \frac{1}{2}\Box\phi - \frac{1}{2}\Box\phi = 0.$$

Additionally, we can expand the anti-commutator of a product by means of the formula

$$\{\phi, \{\psi_1, \psi_2\}\} = \{\phi, \psi_1\}\psi_2 - \psi_1[\phi, \psi_2].$$

Last but not least, we can evaluate the final ingredient that we need for working out how the fluctuation operator $\mathcal{D}^2$ acts on the ansatz proposed in eq. (78),

$$\begin{aligned}
\{X_b, [X_c, X_b]\phi X_c\} =& \underbrace{\{X_b, [X_c, X_b]\}}_{=0}\phi X_c - [X_c, X_b][X_b, \phi X_c] = \\
=& -[X_c, X_b][X_b, \phi]X_c - [X_c, X_b]\phi[X_b, X_c] = \\
=& -\underbrace{\mathcal{B}(x_b \otimes [X_b, \phi])}_{=[X_b, \mathcal{B}(x_b \otimes \phi)]=0} - \underbrace{[X_c, X_b]\phi[X_b, X_c]}_{=2X_c\phi X_c} = \Box\phi - 2R_N^2\phi.
\end{aligned}$$

In conclusion, we obtain the following results for the interaction term

$$
\begin{aligned}
\{X_a, \{X_b, \mathcal{A}^+(x_b \otimes \phi)\}\} &= 4R_N^2 \mathcal{A}^+(x_a \otimes \phi) - \mathcal{A}^+(x_a \otimes \Box\phi), \\
\{X_a, \{X_b, \mathcal{A}^-(x_b \otimes \phi)\}\} &= 0 \\
\{X_a, \{X_b, \mathcal{B}(x_b \otimes \phi)\}\} &= -2R_N^2 \mathcal{A}^+(x_a \otimes \phi) + \mathcal{A}^+(x_a \otimes \Box\phi)
\end{aligned}
$$

SUMMARY: Let us briefly collect the individual pieces we computed in this section and write down the image of the vector fluctuation operator of the three intertwiners $\mathcal{A}^\pm$ and $\mathcal{B}$. Collecting all terms, we obtain

$$
\begin{aligned}
\mathcal{D}^2_{ab}\mathcal{A}^-(x_b \otimes \phi) &= \mathcal{A}^-\left(x_a \otimes (g_0\Box\phi + (M^2+2)\phi)\right) \\
\mathcal{D}^2_{ab}\mathcal{A}^+(x_b \otimes \phi) &= \mathcal{A}^+\left(x_a \otimes ((1-\mu)\Box\phi + (M^2+2+4\mu R_N^2)\phi)\right) - 8\mathcal{B}(x_a \otimes \phi) \\
\mathcal{D}^2_{ab}\mathcal{B}(x_b \otimes \phi) &= \mathcal{A}^+\left(x_a \otimes ((\mu-2)\Box\phi - 2\mu R_N^2\phi\right) + \mathcal{B}\left(x_a \otimes (\Box\phi + (M^2+6)\phi)\right)
\end{aligned}
$$

To retrieve the equivalent result expressed in terms of $\mathcal{A}^{(k)}_a[\phi]$ given in eq. (78), we need to perform another straightforward computation by substituting the intertwiner $\mathcal{B}$ by the expression

$$
\mathcal{B}(x_a \otimes \phi) = -\frac{1}{2}(\mathcal{A}^{(1)}_a[\phi] + \mathcal{A}^{(2)}_a) - i\mathcal{A}^{(3)}_a[\phi]. \tag{94}
$$

(i) $\mathcal{D}^2_{ab}\mathcal{A}^{(1)}_b[\phi] = \mathcal{A}^{(1)}_a[(1-\mu)\Box\phi + (M^2+4\mu R_N^2+6)\phi] + \mathcal{A}^{(2)}_a[4\phi] + \mathcal{A}^{(3)}_a[8i\phi]$.

(ii) $\mathcal{D}^2_{ab}\mathcal{A}^{(2)}_b[\phi] = \mathcal{A}^{(2)}_a[g_0\Box\phi + (M^2+2)\phi]$.

(iii) For the last expression, we obtain on the one hand

$$
\begin{aligned}
\mathcal{D}^2_{ab}\mathcal{B}(x_b \otimes \phi) =& -\frac{1}{2}\mathcal{D}^2_{ab}\mathcal{A}^{(1)}_b[\phi] - \frac{1}{2}\mathcal{D}^2_{ab}\mathcal{A}^{(2)}_b[\phi] - i\mathcal{D}^2_{ab}\mathcal{A}^{(3)}_b[\phi] = \\
=& -i\mathcal{D}^2_{ab}\mathcal{A}^{(3)}_b[\phi] \; -\frac{1}{2}\mathcal{A}^{(1)}_a[(1-\mu)\Box\phi + (M^2+4\mu R_N^2+6)\phi] + \\
& -\frac{1}{2}\mathcal{A}^{(2)}_a[g_0\Box\phi + (M^2+6)\phi] - 4i\mathcal{A}^{(3)}_a[\phi]
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
\mathcal{D}^2_{ab}\mathcal{B}(x_b \otimes \phi) =& \mathcal{A}^+\left(x_a \otimes ((\mu-2)\Box\phi - 2\mu R_N^2\phi\right) + \mathcal{B}\left(x_a \otimes (\Box\phi + (M^2+6)\phi)\right) = \\
=& \frac{1}{2}\mathcal{A}^{(1)}_a[(2\mu-5)\Box\phi - (M^2+4\mu R_N^2+6)\phi] + \\
& -\frac{1}{2}\mathcal{A}^{(2)}_a[\Box\phi + (M^2+6)\phi] + \\
& -i\mathcal{A}^{(3)}_a[\Box\phi + (M^2+6)\phi]
\end{aligned}
$$

Equating these two expressions and solving for the desired term $\mathcal{D}^2_{ab}\mathcal{A}^{(3)}_b[\phi]$ yields the final result

$$
\mathcal{D}^2_{ab}\mathcal{A}^{(3)}_b[\phi] = \mathcal{A}^{(1)}_a\left[i\left(\frac{\mu}{2}-2\right)\Box\phi\right] + \mathcal{A}^{(2)}_a\left[i\frac{g_0-1}{2}\Box\phi\right] + \mathcal{A}^{(3)}_a[\Box\phi + (M^2+2)\phi]
$$

in full aggrement with the proposed identities in eq. (80).

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