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# „Non-Relativistic Supergravity via Lie Algebra Expansions" 

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#### Abstract

Recent applications for non-relativistic geometric formulations of gravity as in $5-11$ have lead to increased interest in the understanding and development of extended theories in the non-relativistic domain. Pioneering work in 16 managed to derive non-relativistic algebras and corresponding actions by applying the theory of Lie algebra expansion $\sqrt{17}$ to relativistic algebras.

This opens the door to applications in the non-relativistic supergravity sector. While previous work in 13 was limited to an extension/contraction procedure, this work expands upon the strictly bosonic cases considered in [16 by applying the Lie algebra expansion formalism to relativistic superalgebras. We reproduce the results of 15 in a novel way and expand the algebra in 18 to include the supersymmetric sector. Further, we derive the algebra and first order action underlying a theory of 4D non-relativistic supergravity.

Independently and concurrently these findings were also produced in 19 20].


#### Abstract

Aktuelle Anwendungen für nicht-relativistische, geometrische Formulierungen der Gravitationstheorie wie in [5-11], haben zu einem verstärkten Interesse an erweiterten Theorien in diesem Gebiet geführt. Die Arbeit in 16] konnte durch Anwendung der 'Lie Algebra Expansion' Methode - entwickelt in [17] - eine wichtige Grundlage schaffen um neue, nicht-relativistische Algebren auf systematische Art und Weise aus relativistischen Algebren abzuleiten.

Ausgehend von den strikt bosonischen Algebren in (16) kann diese Methode auch auf den supersymmetrischen Sektor erweitert werden. Bislang war die Herleitung von nicht-relativistischen Superalgebren wie in 13 auf Erweiterungen und Kontraktionen von Algebren limitiert, welche in dieser Arbeit durch 'Lie Algebra Expansions' ergänzt werden. Wir reproduzieren die Resultate in 15 auf eine systematische Art und erweitern die Algebra aus 18 um den supersymmetrischen Sektor. Weiters leiten wir eine Algebra und Wirkung in der '1. Ordnung' Formulierung für 4D nicht-relativistische Supergravitation her.

Unabhängig und zeitgleich zu dieser Arbeit wurden diese Resultate in $19 \mid 20$ entwickelt.


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## Chapter 1

## Introduction

It is a well-known result in general relativity, that the spacetime structure necessary to describe the kinematics of the theory can be obtained via a gauging process of an underlying Poincaré algebra [1]. For this procedure we introduce gauge fields and curvatures corresponding to each generator and set the curvature of spacetime translations $R_{\mu \nu}\left(P^{A}\right)$ to zero. This 'conventional constraint' is used to introduce diffeomorphisms into the theory and turns the remaining curvature into the curvature of spacetime in general relativity. In this way we introduce general coordinate transformations as the natural local extension of translations. Simultaneously, the Lorentz transformations are manifest in freefalling frames to comply with Einstein's equivalence principle, which states that we can locally recover the symmetries of special relativity. Further, this procedure can be extended to different relativistic algebras containing the Poincaré algebra as a subalgebra to end up with modified Einstein gravity. Most prominently supergravity can be derived by gauging a Poincaré superalgebra as in [2].

Unrelated to these results, it was discovered by Cartan in 3,4 that Newton's theory of gravity can be reformulated geometrically as the non-relativistic analog to Einstein gravity, called 'Newton-Cartan gravity'. Lately, this has been used in the description of a wide range of physical phenomena as in condensed matter physics, where Newton-Cartan gravity acts as an effective background spacetime geometry [5] , as well as in holography as certain boundary geometries [8 11].

These practical applications increase the interest in a more comprehensive understanding of non-relativistic theories of gravity, as well as an extension to supersymmetric models. The first step in this direction was made in 12 by showing that Newton-Cartan gravity can be derived via a gauging procedure from an underlying non-relativistic algebra - the so-called 'Bargmann algebra'. Interestingly, this approach does not use the Galilean algebra which is conventionally viewed as the symmetry algebra of non-relativistic spacetime, but rather a centrally extended version thereof, which includes an additional generator that corresponds to the Noether charge of mass conservation.

The same paper additionally shows how the Bargmann algebra can be derived from an extended Poincaré algebra via an algebra contraction, thus giving a systematic path from a relativistic algebra to non-relativistic gravity. This approach was then subsequently used in 13 to derive an on-shell version of 3D Newton-Cartan supergravity by starting from a Poincaré superalgebra. An off-shell version was derived in 14 by working out a non-relativistic limit of
the already gauged relativistic algebra. Starting from a previous understanding of 3D Newton-Cartan supergravity, a novel extension of its known underlying superalgebra was constructed in an ad-hoc way via trial and error in [15. This novel algebra is called the ' 3 D extended Bargmann superalgebra' and it contains additional generators compared to the algebra of 3D Newton-Cartan supergravity. Due to these extra generators, the algebra allows for a non-degenerate, invariant bilinear form and as a result a Chern-Simons type action (which is absent in 13,14 ) can be constructed. While this approach cannot straightforwardly be extended to further non-relativistic algebras, it gave a goal to try and derive as a non-relativistic limit from some extended version of the Poincaré algebra. A successful method would then not only further the understanding of the algebra derived in [15], but possibly extend our understanding of the class of non-relativistic algebras in general.

Since the gauging procedure can generally be applied to any non-relativistic algebra, resulting in the kinematics of a non-relativistic theory of gravity, understanding this class of algebras is particularly fruitful in order to extend our knowledge of non-relativistic gravity.

This was precisely the aim of this work as initially posed. Starting from a centrally extended $3 \mathrm{D} \mathcal{N}=2$ Poincaré superalgebra, we redefined the generators with appropriate expansion parameters as was done in 13 for $3 \mathrm{D} \mathcal{N}=2$ Bargmann superalgebra, but going one step further and including second order terms in the power series expansion for spatial rotations and one of the supercharges. This did in fact lead to the algebra derived in 15], but the construction is cumbersome and does not guarantee consistency for the algebra after taking an infinitesimal limit for the expansion parameter. Thus, it necessitates checking the final algebra by hand.

The process was streamlined significantly in 16 who realized that the theory of Lie algebra expansions as developed in 17 could be applied to the Poincaré algebra or extensions thereof to derive whole families of non-relativistic algebras. The formalism then guarantees closure and consistency of the algebra. Furthermore - with some caveats which will be discussed in chapter 5 - the expansion can be directly applied to the relativistic action as well to find the corresponding non-relativistic action in first order formulation.

As a result we adopted the Lie algebra expansion formalism and applied the methods developed in 16 to the $3 \mathrm{D} \mathcal{N}=2$ Poincaré superalgebra, re-deriving the 3D extended Bargmann superalgebra first discovered in 15]. Additionally, we went one step further and derived the supersymmetric extension of a 4D nonrelativistic algebra introduced in 18] - called '4D extended string Bargmann algebra' - as well as a novel 4D extended Bargmann superalgebra with accompanying action. These findings overlap with the work done independently and concurrently in 1920 .

In order to properly understand the main calculations in chapter 5 we need to establish the necessary theoretical groundwork. Chapter 2 quickly introduces the notion of gauge theory in general terms and defines gauge fields and curvatures. Additionally, we introduce differential forms and the Maurer-Cartan formalism, which streamline the notation in the later chapters.

Chapter 3 applies these concepts to gravity in particular, by going through the process of gauging the Poincaré algebra. We will also explain how diffeomorphisms can be introduced by replacing local translations by general coordinate transformations and setting their curvature to zero. The calculation is repeated


Table 1.1: Starting from the Poincaré algebra the Lie algebra expansion formalism leads to a non-relativistic algebra, which in turn implies a non-relativistic theory of gravity via a gauging procedure.
for $\mathcal{N}=1$ supergravity and Newton-Cartan gravity. The more involved calculations for $3 \mathrm{D} \mathcal{N}=2$ Newton-Cartan supergravity can be found in 13 and follow a similar pattern. In principl ${ }^{1}$ this framework can be applied to the algebras derived in chapter5 to be used as a starting point for the development of the corresponding theories of gravity.

Chapter 4 gives a comprehensive derivation of the Lie algebra expansion method developed in 17 as relevant to non-relativistic expansions. Utilizing this formalism in chapter 5 as by the work in [16], we take supersymmetric extensions of the Poincaré algebra and derive an - in principle - infinite collection of corresponding non-relativistic superalgebras.

The different elements that go into this approach can be neatly summarized via the diagram in table 1.1. We show the derivation of the non-relativistic algebras in chapter 5, which in turn give a starting point for the development of theories of gravity.

[^0]
## Chapter 2

## Gauge Theory

In this chapter, we introduce the main concepts of gauge theory in general terms, which will be utilized in chapter 3 to derive the kinematics of a theory of gravity from an underlying symmetry algebra. We follow the steps laid out in 21 and start from global symmetries which - upon making local - lead to the gauge fields and their respective covariant derivatives and curvatures. This serves primarily as a foundation for the later chapters and fixes the used conventions and notation.

Additionally, we include a concise description of differential forms and their application to gauge theory. In particular, they give a natural basis for the Maurer-Cartan formalism which we employ in chapter4 in the Lie algebra expansion method.

### 2.1 Global symmetries

We are interested in the realization of symmetries in the context of a field theory, which is to say that they act as transformations of the field content $\Phi(x)$ defined over a spacetime with coordinates $x$. A general symmetry of a physical system is then given by a transformation of fields

$$
\begin{equation*}
\Phi(x) \rightarrow \Phi^{\prime}\left(x^{\prime}\right), \tag{2.1}
\end{equation*}
$$

that leaves the equations of motion invariant.
As a general definition this encompasses the behaviour we expect from a symmetry but it is not particularly useful in practice. Instead it is more useful to restrict our attention to symmetries which leave the action of the system $S[\Phi]$ invariant, so

$$
\begin{equation*}
S[\Phi]=S\left[\Phi^{\prime}\right] . \tag{2.2}
\end{equation*}
$$

Since the equations of motion can be derived from the action they will also respect the imposed symmetries.

Further, we restrict this discussion to continuous globa $1^{11}$ symmetries which can be described by Lie groups, since only these will be relevant for the gauging

[^1]procedure. Conveniently, this allows us to work with the corresponding infinitesimal transformations at the level of the Lie algebra $\mathcal{G}$, as these are connected to group elements via the exponential map.

The main symmetries under consideration in this work will be spacetimeand supersymmetry transformations, which lead to gravity and supergravity theories. However, the formalism in this chapter is kept general and thus also valid for internal symmetries with Lie algebra structure.

We implement the global symmetry transformations via infinitesimal transformations ${ }^{2}$

$$
\begin{equation*}
\delta(\epsilon) \Phi(x)=\epsilon^{A} T_{A} \Phi(x), \tag{2.3}
\end{equation*}
$$

with $\epsilon^{A}$ the parameter of the transformation which determines the symmetry transformation and $T_{A}$ the field space operator acting on the field content of the theory. The $T_{A}$ are representations of the generators of the Lie algebra and as a consequence obey

$$
\begin{equation*}
\left[T_{A}, T_{B}\right]=f_{A B}{ }^{C} T_{C}, \tag{2.4}
\end{equation*}
$$

where $f_{A B}{ }^{C}$ are the structure constants of the algebra. These constants fully fix the Lie algebra under consideration. The Lie bracket defined in eq. 2.4 has to obey the Jacobi identity

$$
\begin{equation*}
\left[T_{A},\left[T_{B}, T_{C}\right]\right]+\left[T_{C},\left[T_{A}, T_{B}\right]\right]+\left[T_{B},\left[T_{C}, T_{A}\right]\right]=0 \tag{2.5}
\end{equation*}
$$

which complicates the construction of new algebras. Equivalently the identity in eq 2.5 can be written as a condition for the structure constants via eq. 2.4 so

$$
\begin{equation*}
f_{[A B}{ }^{D} f_{C] D}{ }^{E}=0 . \tag{2.6}
\end{equation*}
$$

The action of the field space operators $T_{A}$ depends on the field they are acting on and need not be linear (although it will be for all cases considered in this work).

As an example we might consider an internal symmetry acting linearly on a set of scalar fields $\phi^{i}$ in the matrix representation as

$$
\begin{equation*}
T_{A} \phi^{i}=-\left(t_{A}\right)^{i}{ }_{j} \phi^{j}, \tag{2.7}
\end{equation*}
$$

and on fields in the adjoint representation $\phi^{A}$ via

$$
\begin{equation*}
T_{A} \phi^{B}=-f_{A C}{ }^{B} \phi^{C} . \tag{2.8}
\end{equation*}
$$

Since we want to include supersymmetry as part of our symmetry group we have to extend our notion of a Lie algebra to that of a $\mathbb{Z}_{2}$-graded algebra.

Per definition, a supersymmetry transformation takes a bosonic field to its fermionic partner and vice versa, thus requiring the generators $Q_{\alpha}$ to be fermionic. On the field content, they are realized as operators that satisfy anticommutation (instead of commutation) relations among themselves. Due to the spin statistics theorem, fermionic particles take the structure of spinors, which is briefly explored in appendix A. The infinitesimal transformation

$$
\begin{equation*}
\delta(\epsilon)=\bar{\epsilon}^{\alpha} Q_{\alpha}, \tag{2.9}
\end{equation*}
$$

[^2]however should remain bosonic, such that we take the corresponding parameters $\bar{\epsilon}^{\alpha}$ to be spinor-valued as well.

We define a $\mathbb{Z}_{2}$-graded algebra $\mathcal{G}$ as a vector space $V$ with direct sum decomposition $V=V_{0} \oplus V_{1}$ according to the grading function

$$
|X|=\left\{\begin{array}{lll}
0 & \text { for } & X \in V_{0}  \tag{2.10}\\
1 & \text { for } & X \in V_{1}
\end{array}\right.
$$

equipped with a graded bracket $[\cdot, \cdot\}: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ with symmetry properties

$$
\begin{equation*}
[X, Y\}=(-1)^{|X||Y|+1}[Y, X\}, \quad \text { for } \quad X, Y \in \mathcal{G} \tag{2.11}
\end{equation*}
$$

Since there exists a definite grading for all $X \in \mathcal{G}$, we have a strict dichotomy and can call objects with grading $|X|=0$ bosonic and $|X|=1$ fermionic.

Additionally, this modification requires us to change the Jacobi identity in eq. 2.5 to its graded analog given by

$$
\begin{equation*}
(-1)^{|X||Z|}[X,[Y, Z\}\}+(-1)^{|Z||Y|}[Z,[X, Y\}\}+(-1)^{|Y||X|}[Y,[Z, X\}\}=0 \tag{2.12}
\end{equation*}
$$

For abstract discussions the graded bracket introduced above is quite convenient. However, in the context of this work we are more interested in the practical application and thus we decompose the bracket according to its symmetry properties into commutators and anti-commutators.

We adopt the notation

$$
[X, Y\}=\left\{\begin{array}{lll}
{[X, Y]} & \text { for } & |X||Y|=0  \tag{2.13}\\
\{X, Y\} & \text { for } & |X||Y|=1
\end{array}\right.
$$

At this point we have everything we need to describe a theory with global supersymmetry. The next step will be to gather all the parts needed to make the symmetries local and thus introduce gauge fields.

### 2.2 Gauge Fields

The global symmetry transformations allow for great insight into the properties of a theory (especially via the Noether formalism), but remain quite restrictive in their scope.

As a next step to gravity and supergravity theories we now take our global symmetries to be local, meaning we allow for different symmetry transformations in all points of spacetime. In other words, we allow for a dependence on spacetime coordinates for the parameters $s^{3} \epsilon^{A}(x)$ in eq. 2.3 . It is immediately apparent that this leads to a more general theory, since we can choose the same local transformation for all points and reproduce the global transformations.

At this point the introduction of further complexity does not seem particularly physically motivated and more like a mathematical curiosity. However, the usefulness will become apparent later on, when we produce the kinematics of a theory of gravity from spacetime symmetries.

Letting the parameters of a global theory depend on the coordinates is however not enough to end up with an invariant theory. In order for the theory

[^3]to contain propagating fields, we need at least first order derivatives in the Lagrangian. Thus, for a field $\Phi$ transforming according to eq. 2.3, the Lagrangian has to contain terms of the form $\partial_{\mu} \Phi$, which transform under a gauge transformation as
\[

$$
\begin{equation*}
\delta(\epsilon) \partial_{\mu} \Phi=\partial_{\mu}\left(\epsilon^{A} T_{A} \Phi\right)=\epsilon^{A} \partial_{\mu}\left(T_{A} \Phi\right)+\left(\partial_{\mu} \epsilon^{A}\right) T_{A} \Phi \tag{2.14}
\end{equation*}
$$

\]

We observe that a transformation with a local parameter will invariably lead to additional terms containing derivatives of that parameter. In order to preserve invariance under symmetry, these terms have to be compensated for. This can be achieved by introducing a 'gauge field' with appropriate transformation properties to cancel the extraneous terms.

For every local symmetry we need to include a gauge field $B_{\mu}^{A}(x)$ in the field content of our theory, which transforms under the gauge group as

$$
\begin{equation*}
\delta(\epsilon) B_{\mu}^{A}=\partial_{\mu} \epsilon^{A}+\epsilon^{C} B_{\mu}^{B} f_{B C}^{A} \tag{2.15}
\end{equation*}
$$

In principle (as will often be the case) there can be multiple local symmetries which form a Lie group with structure constants $f_{A B}^{C}$. The sum over the abstract indices $A, B, C, \ldots$ extends over all symmetry labels, which means that in general a gauge field transforms with multiple parameters and gauge fields of the theory.

### 2.3 Covariant Derivatives and Curvatures

The argument leading to eq. 2.14 shows that our usual notion of a derivative is no longer sufficient upon gauging of the symmetry group. Since the offending terms containing derivatives on the parameters only enter via expressions containing derivatives of a field, it is convenient to introduce a modified derivative which transforms covariantly by itself.

We define the 'covariant derivative' as

$$
\begin{equation*}
\mathcal{D}_{\mu}=\partial_{\mu}-\delta\left(B_{\mu}\right) \tag{2.16}
\end{equation*}
$$

where $\delta\left(B_{\mu}\right)$ is defined to be the gauge transformation acting on some field with the parameter of the transformation substituted by the corresponding gauge field $B_{\mu}$.

Checking that the operator in eq. 2.16 in fact transforms covariantly, we take the transformation of

$$
\begin{equation*}
\mathcal{D}_{\mu} \Phi=\left(\partial_{\mu}-\delta\left(B_{\mu}\right)\right) \Phi=\partial_{\mu} \Phi-B_{\mu}^{A} T_{A} \Phi \tag{2.17}
\end{equation*}
$$

The first term transforms according to eq. 2.14 , while the second term gives

$$
\begin{align*}
\delta(\epsilon)\left(B_{\mu}^{A} T_{A} \Phi\right) & =\delta(\epsilon) B_{\mu}^{A} T_{A} \Phi+B_{\mu}^{A} \delta(\epsilon) T_{A} \Phi \\
& =\left(\partial_{\mu} \epsilon^{A}\right) T_{A} \Phi+\epsilon^{C} B_{\mu}^{B} f_{B C}{ }^{A} T_{A} \Phi+\epsilon^{B} B_{\mu}^{A} T_{B} T_{A} \Phi  \tag{2.18}\\
& =\left(\partial_{\mu} \epsilon^{A}\right) T_{A} \Phi+\epsilon^{C} B_{\mu}^{B}\left[T_{B}, T_{C} \Phi+\epsilon^{C} B_{\mu}^{B} T_{C} T_{B} \Phi\right. \\
& =\left(\partial_{\mu} \epsilon^{A}\right) T_{A} \Phi+\epsilon^{A} B_{\mu}^{B} T_{B} T_{A} \Phi
\end{align*}
$$

where we used eq. 2.15 for the second line, eq. 2.4 for the third and strategically renamed some of the dummy indices. Thus, we have for the full transformation

$$
\begin{equation*}
\delta(\epsilon)\left(\mathcal{D}_{\mu} \Phi\right)=\epsilon^{A}\left(\partial_{\mu}-B_{\mu}^{B} T_{B}\right)\left(T_{A} \Phi\right)=\epsilon^{A} \mathcal{D}_{\mu}\left(T_{A} \Phi\right) \tag{2.19}
\end{equation*}
$$

as expected. Replacing all derivatives with covariant derivatives can be used as a method of covariantizing expressions.

Ultimately, the covariant derivative has only a secondary role in this work, as a stepping stone for the definition of the curvature. We will not deal with explicit matter fields on which the gauge transformations act, but rather the underlying structure of the gauge algebra.

It is a well known fact that the curvature for a given gauge field can be defined as the commutator of covariant derivatives (see [21]) via

$$
\begin{align*}
{\left[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}\right] } & =-\delta\left(R_{\mu \nu}\right), \\
\Rightarrow R_{\mu \nu}^{A} & =2 \partial_{[\mu} B_{\nu]}^{A}+B_{\nu}^{C} B_{\mu}^{B} f_{B C}{ }^{A} . \tag{2.20}
\end{align*}
$$

Since this object is then defined via covariant quantities, it is itself covariant. The curvature is useful for the construction of an action of the system, as is the case for the Einstein-Hilbert action of general relativity, electromagnetism or Yang-Mills theory in QFT.

### 2.4 Differential Forms

The aim of this section is to introduce the Maurer-Cartan formalism, which will be used in chapter 4 for the expansion of Lie algebras. This is best done in the language of differential forms, which gives an alternative formalism for many of the recurring objects defined above.

Gauge theory is mainly concerned with parameters $\epsilon^{A}$, gauge fields $B_{\mu}^{A}$ and curvatures $R_{\mu \nu}{ }^{A}$ which are spacetime scalars, covariant vectors and totally antisymmetric covariant 2 -tensors, respectively. Using coordinate differentials $\mathrm{d} x^{\mu}$ we can write them as 1 - and 2 -forms respectively, by defining ${ }^{4}$

$$
\begin{align*}
B^{A} & =B_{\mu}^{A} \mathrm{~d} x^{\mu} \\
R^{A} & =\frac{1}{2} R_{\mu \nu}^{A} \mathrm{~d} x^{\mu} \wedge \mathrm{d} x^{\nu} \tag{2.21}
\end{align*}
$$

For the general case we define a $p$-form as

$$
\begin{equation*}
\omega^{(p)}=\frac{1}{p!} \omega_{\mu_{1} \mu_{2} \cdots \mu_{p}} \mathrm{~d} x^{\mu_{1}} \wedge \mathrm{~d} x^{\mu_{2}} \wedge \cdots \wedge \mathrm{~d} x^{\mu_{p}} \tag{2.22}
\end{equation*}
$$

fixing the convention for the sum. The wedge product $\wedge$ is anti-symmetric by definition, which is exactly what we need for the curvatures.

An important operation is given by the exterior derivative $d$ taking any $p$-form into a $p+1$-form via

$$
\begin{equation*}
\mathrm{d} \omega^{(p)}=\frac{1}{p!} \partial_{\mu} \omega_{\mu_{1} \mu_{2} \cdots \mu_{p}} \mathrm{~d} x^{\mu} \wedge \mathrm{d} x^{\mu_{1}} \wedge \mathrm{~d} x^{\mu_{2}} \wedge \cdots \wedge \mathrm{~d} x^{\mu_{p}} \tag{2.23}
\end{equation*}
$$

[^4]Thus, we can write down the equations for the transformation of gauge fields eq. 2.15 and curvatures eq. 2.20 in a condensed form as

$$
\begin{align*}
\delta B^{A} & =\mathrm{d} \epsilon^{A}+\epsilon^{C} B^{B} f_{B C}^{A} \\
R^{A} & =\mathrm{d} B^{A}-\frac{1}{2} B^{C} \wedge B^{B} f_{B C}{ }^{A} . \tag{2.24}
\end{align*}
$$

A useful property of the exterior derivative is that we have

$$
\begin{equation*}
\mathrm{dd} \equiv 0 \tag{2.25}
\end{equation*}
$$

which can easily be checked using the definition in eq. 2.23 and anti-symmetry in the indices. Calculating the variation of some curvature $\delta R^{A}$, this eliminates the terms containing derivatives of the parameters immediately.

Going to the expanded gauge theories in the later chapters, this formalism proves particularly useful since it puts emphasis on the actual fields and parameters instead of superfluous indices.

Having introduced differential forms, we can now discuss the Maurer-Cartan ('MC') formalism. In general terms, the MC formalism gives a dual description for a given Lie algebra in terms of so-called 'Maurer-Cartan one-forms' $\omega^{A}(g)$ defined via

$$
\begin{equation*}
\omega^{A} T_{A} \equiv g^{-1} \mathrm{~d} g \tag{2.26}
\end{equation*}
$$

with $T_{A}$ generators of the Lie algebra and $g \in G$ an element of the Lie group.
In order to find the structure equation for the MC formalism, which serves as an analog to the commutator of the Lie algebra eq. 2.4, we take the exterior derivative of eq. 2.26 Making the auxiliary statement

$$
\begin{align*}
\mathrm{d}\left(g^{-1} g\right) & =0=\mathrm{d} g^{-1} g+g^{-1} \mathrm{~d} g \\
& \Longleftrightarrow  \tag{2.27}\\
\mathrm{~d} g^{-1} & =-g^{-1} \mathrm{~d} g g^{-1}
\end{align*}
$$

we have

$$
\begin{align*}
\mathrm{d}\left(g^{-1} \mathrm{~d} g\right) & =\mathrm{d} g^{-1} \wedge \mathrm{~d} g  \tag{2.28}\\
& =-g^{-1} \mathrm{~d} g \wedge g^{-1} \mathrm{~d} g
\end{align*}
$$

which via the definition in eq. 2.26 yields

$$
\begin{equation*}
\mathrm{d} \omega^{C} T_{C}=-T_{A} T_{B} \omega^{A} \wedge \omega^{B} \tag{2.29}
\end{equation*}
$$

The structure of the algebra $\mathcal{G}$ is given by the commutator in eq. 2.4. which leads to

$$
\begin{equation*}
\mathrm{d} \omega^{C} T_{C}=-\frac{1}{2} f_{A B}^{C} T_{C} \omega^{A} \wedge \omega^{B} \tag{2.30}
\end{equation*}
$$

and then finally the MC equations in their usual form

$$
\begin{equation*}
\mathrm{d} \omega^{C}(g)=-\frac{1}{2} f_{A B}^{C} \omega^{A}(g) \wedge \omega^{B}(g) \tag{2.31}
\end{equation*}
$$

The Jacobi identity in eq. 2.5 has an analog as well, by requiring that the property in eq. 2.25 be implemented consistently 5 so

$$
\begin{equation*}
\operatorname{dd} \omega^{A} \equiv 0 \tag{2.32}
\end{equation*}
$$

[^5]In this sense the MC equations give a dual formulation for a given Lie algebra. Comparing eq. 2.31 with the curvature in eq. 2.24 , we can see a correspondence when we identify the MC forms with the gauge fields. Schematically we have:

$$
\begin{array}{ccc}
\omega^{A}(g) & \longleftrightarrow & 0=\mathrm{d} \omega^{C}(g)+\frac{1}{2} f_{A B}^{C} \omega^{A}(g) \wedge \omega^{B}(g) \\
\downarrow & & \downarrow  \tag{2.33}\\
B^{A} & \longleftrightarrow & R^{A}=\mathrm{d} B^{A}-\frac{1}{2} B^{C} \wedge B^{B} f_{B C}{ }^{A}
\end{array}
$$

Comparing the equations we can see that the curvature $R^{A}$ can be interpreted as quantifying a deviation from the MC equation. We will make use of this correspondence in chapter 4.

## Chapter 3

## Gravity from Symmetry

This chapter is dedicated to the derivation of kinematics of theories of gravity from some underlying symmetry algebra. While this work is mostly concerned with the properties at the level of the algebra - namely the construction of nonrelativistic algebras from relativistic ones - it is important to understand how this in turn relates to corresponding theories of gravity.

The case of relativistic symmetries is much easier to treat and gives more comprehensible results, which is why we focus on it first. We start by gauging the Poincaré algebra and discuss the subtleties which appear due to the special role of local translations. Then, we are ready to go through the derivation of kinematics for Einstein gravity and the extension to supergravity.

Lastly, we apply the procedure to the Bargmann algebra which leads to a theory that can be viewed as the non-relativistic limit of ordinary Einstein gravity. This gives the general layout for all the algebras derived in chapter 5

### 3.1 Gauging of the Poincaré algebra

Before we can take the step to gravity we have to gauge the underlying symmetry group, described by the $D$-dimensional Poincaré algebra. The general outline for this procedure is already given in section 2.2, however the application to local translations is not entirely straightforward. Thus, it is worth going through the steps in some detail.

The Poincaré algebra consists of the spacetime translations with generators $P_{A}$ and Lorentz transformations with generators $M_{A B}$ with commutation relations $(A=0,1, \ldots, D-1)$

$$
\begin{align*}
{\left[P_{A}, P_{B}\right] } & =0, \\
{\left[P_{A}, M_{B C}\right] } & =2 \eta_{A[B} P_{C]},  \tag{3.1}\\
{\left[M_{A B}, M_{C D}\right] } & =4 \eta_{[A[C} M_{D] B]} .
\end{align*}
$$

Importantly, the diffeomorphism symmetry that we expect for general relativity is absent in the Poincaré algebra. In order to establish the connection, the corresponding transformations should thus appear in the set of symmetry transformations after gauging.

The generators of the Poincaré group act on the field space via differential operators for the translations and the 'orbital part' of the Lorentz transformations. In case we consider only scalar fields ${ }^{1}$, the infinitesimal transformation with corresponding parameters $a^{\mu}$ and $\lambda^{\mu \nu}$ reads

$$
\begin{equation*}
\delta(a, \lambda) \phi(x)=\left(a^{\mu}+\lambda^{\mu \nu} x_{\nu}\right) \partial_{\mu} \phi(x) \tag{3.2}
\end{equation*}
$$

Going to local translations with parameter $a^{\mu}(x)$ as an arbitrary function of the spacetime coordinates $x$, we can absorb the orbital part to produce general coordinate transformations ('gct')

$$
\begin{equation*}
\delta_{\mathrm{gct}}(\xi) \phi(x)=\xi^{\mu}(x) \partial_{\mu} \phi(x), \tag{3.3}
\end{equation*}
$$

by defining $\xi^{\mu}(x)=a^{\mu}(x)+\lambda^{\mu \nu}(x) x_{\nu}$.
Thus, the gauged Poincaré transformations have a natural description in terms of general coordinate transformations with parameters $\xi^{\mu}(x)$ and local Lorentz transformations with parameter $\lambda^{A B}(x)$, where latin indices $A, B$ refer to the local frame and greek indices $\mu, \nu$ to the background spacetime.

The respective gauge fields are $e_{\mu}^{A}$ - the frame field - for local translations and $\omega_{\mu}{ }^{A B}$ - the spin connection - for local Lorentz transformations.

In more general terms, the symmetry group may contain additional generators with corresponding gauge fields. Since we only want to discard the local translations as part of the symmetry group, it is useful to define a set of 'standard gauge transformations' which contains the remaining symmetries. We take $B^{\hat{A}}$ to be their gauge fields with $\hat{A}, \hat{B}$ indices running over the restricted set of transformations. In the special case of the Poincaré algebra, the standard gauge transformations consist of only the local Lorentz transformations.

Finally, the last and most important modification for us is to go from the general coordinate transformations to covariant general coordinate transformations ('cgct') defined as

$$
\begin{equation*}
\delta_{\mathrm{cgct}}(\xi)=\delta_{\mathrm{gct}}(\xi)-\delta\left(\xi^{\mu} B_{\mu}\right) \tag{3.4}
\end{equation*}
$$

This is necessary since a general coordinate transformation of a field does not generally transform covariantly under the standard gauge transformations.

This has two interesting consequences. First, assuming that the structure of the gauge group is such that $f_{A \hat{B}} \hat{A}=0$ as is the case for the relativistic algebras we encounter, all the standard gauge fields transform under 'cgct' as

$$
\begin{equation*}
\delta_{\mathrm{cgct}}(\xi) B_{\mu}^{\hat{A}}=\xi^{\nu} \partial_{\nu} B_{\mu}^{\hat{A}}+B_{\nu}^{\hat{A}} \partial_{\mu} \xi^{\nu}-\partial_{\mu}\left(\xi^{\nu} B_{\nu}^{\hat{A}}\right)-\xi^{\nu} B_{\nu}^{\hat{C}} B_{\mu}^{\hat{B}} f_{\hat{B} \hat{C}}^{\hat{A}}=\xi^{\nu} R_{\nu \mu}^{\hat{A}}, \tag{3.5}
\end{equation*}
$$

which means that the gauge fields transform into their respective curvatures under covariant general coordinate transformations.

Secondly, when acting upon the frame field we find that its transformation

[^6]is
\[

$$
\begin{align*}
\delta_{\mathrm{cgct}}(\xi) e_{\mu}^{A}= & \xi^{\nu} \partial_{\nu} e_{\mu}^{A}+e_{\nu}^{A} \partial_{\mu} \xi^{\nu}-\xi^{\nu} B_{\nu}^{\hat{C}}\left(B_{\mu}^{\hat{B}} f_{\hat{B} \hat{C}}{ }^{A}+e_{\mu}^{B} f_{B \hat{C}}{ }^{A}\right) \\
= & \partial_{\mu} \xi^{A}+\xi^{\nu}\left(2 \partial_{[\nu} e_{\mu]}^{A}-B_{\nu}^{\hat{C}} B_{\mu}^{\hat{B}} f_{\hat{B} \hat{C}}{ }^{A}-B_{\nu}^{\hat{C}} e_{\mu}^{B} f_{B \hat{C}}{ }^{A}-e_{\nu}^{C} B_{\mu}^{\hat{B}} f_{\hat{B} C}{ }^{A}\right) \\
& +\xi^{C} B_{\mu}^{\hat{B}} f_{\hat{B} C}{ }^{A} \\
= & \partial_{\mu} \xi^{A}+\xi^{C} B_{\mu}^{\hat{B}} f_{\hat{B} C}{ }^{A}-\xi^{\nu} R_{\mu \nu}{ }^{A}, \tag{3.6}
\end{align*}
$$
\]

which we can now compare to the usual gauge transformation according to eq. 2.15. $\mathrm{sq}^{2}$

$$
\begin{equation*}
\delta e_{\mu}^{A}=\partial_{\mu} \xi^{A}+\xi^{C} B_{\mu}^{\hat{B}} f_{\hat{B} C}{ }^{A}+\epsilon^{\hat{C}}\left(B_{\mu}^{\hat{B}} f_{\hat{B} \hat{C}}{ }^{A}+e_{\mu}^{B} f_{B \hat{C}}{ }^{A}\right) . \tag{3.7}
\end{equation*}
$$

Comparing the first two terms in eq. 3.6 and eq. 3.7 , we see that the local translations are equivalent to covariant general coordinate transformations - up to standard gauge transformations - if and only if the curvature $R_{\mu \nu}{ }^{A}$ vanishes.

In practice, this means that we can start from global Poincaré symmetry, go to the local symmetry via gauging and set the curvature corresponding to the local translations to zero, to end up with a theory containing diffeomorphisms as a symmetry.

### 3.2 Gravity from Gauge Theory

In this section we set out to derive the kinematics of general relativity - Einstein gravity - building on the results from the previous section 3.1. The general derivation is well known (see [12 and [14]) but is certainly worth repeating in the broader context of this work.

For purely bosonic relativistic theories the dimension has no major impact on the calculations. This is starkly contrasted by the supersymmetric theories explored below, where the structure of the algebra is highly dependent on the spacetime dimensions due to the underlying Clifford algebra (see appendix A.

Additionally, we do not concern ourselves with the coupling to matter or the dynamics of the theory. In the relativistic theory these are well-known and easily implemented by imposing a certain action, once the structure via the kinematics is given.

We associate the parameter $\xi^{A}$ and the gauge field ${ }^{3} e^{A}$ with local translations and parameter $\lambda^{A B}$ and gauge field $\omega^{A B}$ with the local Lorentz transformations.

According to eq. 2.24 the transformations of the gauge fields are

$$
\begin{align*}
\delta e^{A} & =\mathrm{d} \xi^{A}+\omega^{A}{ }_{B} \xi^{B}-\lambda^{A}{ }_{B} e^{B}, \\
\delta \omega^{A B} & =\mathrm{d} \lambda^{A B}-2 \lambda^{[A}{ }_{C} \omega^{C \mid B]} . \tag{3.8}
\end{align*}
$$

Referring back to section 3.1 we can go from the local translations to 'cgct' by imposing the condition

$$
\begin{equation*}
R\left(P^{A}\right)=0 \tag{3.9}
\end{equation*}
$$

[^7]which is commonly referred to as a 'conventional constraint'. We then have
\[

$$
\begin{equation*}
\delta_{P} e^{A}=\delta_{\mathrm{cgct}} e^{A} \tag{3.10}
\end{equation*}
$$

\]

Thus, the theory now includes diffeomorphism invariance, which is exactly what was needed for gravity.

Writing out the constraint in full according to eq. 2.24 , we find

$$
\begin{equation*}
R\left(P^{A}\right)=\mathrm{d} e^{A}+\omega_{B}^{A} \wedge e^{B}=0 \tag{3.11}
\end{equation*}
$$

which are $\frac{1}{2} D^{2}(D-1)$ equations for the $\frac{1}{2} D^{2}(D-1)$ components of $\omega^{A B}$. This system can be solved by identifying the frame field $e^{A}$ with the Vielbein in general relativity and introducing the inverse Vielbein $e_{A}^{\mu}$ defined via

$$
\begin{equation*}
e_{\mu}^{A} e_{A}^{\nu}=\delta_{\mu}^{\nu}, \quad e_{A}^{\mu} e_{\mu}^{B}=\delta_{A}^{B} \tag{3.12}
\end{equation*}
$$

Thus, we find the solution for eq. 3.11 to be

$$
\begin{equation*}
\omega_{\mu}^{A B}(e)=2 e^{\rho[A} \partial_{[\rho} e_{\mu]}^{B]}+e_{\mu}^{C} e^{\rho[A} e^{B] \sigma} \partial_{[\rho} e_{\sigma] C} \tag{3.13}
\end{equation*}
$$

which takes us from the first order formulation of general relativity to the second order formulation with a dependent spin connection $\omega_{\mu}{ }^{A B}(e)$.

In fact, the constraint in eq. 3.11 also allows for solutions if we add a torsion term, according to

$$
\begin{equation*}
R\left(P^{A}\right)=\mathrm{d} e^{A}+\omega_{B}^{A} \wedge e^{B}=T^{A} . \tag{3.14}
\end{equation*}
$$

While this modifies the equality in eq. 3.10 , the concept of torsion extends the calculation to supergravity in the next section, where the fermionic gauge field contributes such a term. The full solution in the presence of torsion is then given by

$$
\begin{equation*}
\omega^{A B}=\omega^{A B}(e)+K^{A B} \tag{3.15}
\end{equation*}
$$

with the 'contorsion tensor'

$$
\begin{equation*}
K_{\mu[\nu \rho]}=-\frac{1}{2}\left(T_{[\mu \nu] \rho}-T_{[\nu \rho] \mu}+T_{[\rho \mu] \nu}\right) \tag{3.16}
\end{equation*}
$$

Note the relation of the 2-form torsion tensor and the 1-form contorsion tensor. In order to preserve readability we made use of the Vielbein to change the indices of the torsion tensor as $T_{\mu \nu}{ }^{A} e_{A \rho}=T_{[\mu \nu] \rho}$.

The remaining curvature of the spin connection is

$$
\begin{equation*}
R\left(M^{A B}\right)=\mathrm{d} \omega^{A B}+\omega_{C}^{A} \wedge \omega^{C B}, \tag{3.17}
\end{equation*}
$$

which upon insertion of the torsionless solution eq. 3.13 takes the familiar form of the spacetime curvature in Einsteinian gravity in the second order formalism.

The connection to gravity can also be made explicit by invoking the Vielbein postulate (see 21])

$$
\begin{equation*}
\partial_{\mu} e_{\nu}^{a}+\omega_{\mu}{ }^{a} e_{\nu}^{b}-\Gamma_{\mu \nu}^{\sigma} e_{\sigma}^{a}=0, \tag{3.18}
\end{equation*}
$$

which shows how the spin connection $\omega_{\mu}^{a b}$ relates to the affine connection $\Gamma_{\mu \nu}^{\sigma}$ of general relativity. Further, slightly rewriting eq. 3.12 illuminates how the metric $g_{\mu \nu}$ relates to the Vielbein $e_{\mu}^{A}$, since

$$
\begin{equation*}
g_{\mu \nu}=e_{\mu}^{A} \eta_{A B} e_{\nu}^{B} \tag{3.19}
\end{equation*}
$$

Combing eq. 3.17 through eq. 3.19 and the solution in eq. 3.13 we have the torsion-less connection

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\rho}(g)=\frac{1}{2} g^{\rho \sigma}\left(\partial_{\mu} g_{\sigma \nu}+\partial_{\nu} g_{\mu \sigma}-\partial_{\sigma} g_{\mu \nu}\right) \tag{3.20}
\end{equation*}
$$

and spacetime curvature

$$
\begin{equation*}
R_{\mu \nu}{ }^{\rho}{ }_{\sigma}=\partial_{[\mu} \Gamma_{\nu] \sigma}^{\rho}+\Gamma_{[\mu \mid \tau}^{\rho} \Gamma_{\nu] \sigma}^{\tau} . \tag{3.21}
\end{equation*}
$$

At this point we have all the elements needed for general relativity assembled and can include the dynamics via the Einstein equation, which may be derived from an action principle.

## $3.3 \mathcal{N}=1$ Supergravity from Gauge Theory in $D=3,4$

The step to describing the kinematics of supergravity comes in the form of one (or more) additional fermionic symmetry (supersymmetry) charges $Q_{\alpha}^{i}$ with $i=1, \ldots, \mathcal{N}$, which extend the symmetry algebra of ordinary Poincaré transformations. The index $i$ numerates the $\mathcal{N}$ distinct supercharges, with $\mathcal{N}=1$ 'simple supersymmetry' and $\mathcal{N}=2$ 'extended supersymmetry' being the only cases relevant to this work $4^{4}$. These charges satisfy anti-commutation relations and span a Lie superalgebra.

In order to describe the structure of a superalgebra we need the concept of a Clifford algebra. As many of the properties of the elements of a Clifford algebra depend on the dimension $D$ of spacetime we are working in, it requires additional care when trying to extend some results to different dimensions. For this reason we focus on $D=4$, since it serves as the immediate extension of Einstein gravity. However, since the symmetry properties of the elements of the Clifford algebra are the same for $D=3$, the results are also valid for three dimensions. A detailed derivation for these properties can be found in 21 with the relevant results compiled in appendix A.

Importantly, the gauge theory laid out in chapter 2 is already formulated, such that no further modification is necessary to accommodate the fermionic objects.

Thus, we can immediately proceed to the structure of the superalgebra by adding the fermionic sector

$$
\begin{align*}
{\left[P_{A}, Q^{i}\right] } & =0 \\
{\left[M_{A B}, Q^{i}\right] } & =-\frac{1}{2} \gamma_{A B} Q^{i},  \tag{3.22}\\
\left\{Q_{i}, Q^{j}\right\} & =-\delta_{i}^{j} \gamma^{A} C^{-1} P_{A},
\end{align*}
$$

to the Poincaré algebra given in eq. 3.1. For the rest of this section we limit the scope to $\mathcal{N}=1$, such that there is only one supercharge $Q_{\alpha}$.

Following the procedure laid out in section 3.2 we define a gauge field for the supersymmetry transformations $\bar{\psi}_{\mu}^{\alpha}$ and parameter $\bar{\epsilon}^{\alpha}$ and write down the transformations of all the gauge fields and corresponding curvatures.

[^8]Starting with the transformations we have according to eq. 2.24

$$
\begin{align*}
\delta e^{A} & =\mathrm{d} \xi^{A}+\omega_{B}^{A}{ }_{B}^{B}-\lambda_{B}^{A} e^{B}+\frac{1}{2} \bar{\epsilon} \gamma^{A} \psi, \\
\delta \omega^{A B} & =\mathrm{d} \lambda^{A B}-2 \lambda^{[A}{ }_{C} \omega^{C \mid B]}  \tag{3.23}\\
\delta \psi & =\mathrm{d} \epsilon+\frac{1}{4} \omega^{A B} \gamma_{A B} \epsilon-\frac{1}{4} \lambda^{A B} \gamma_{A B} \psi,
\end{align*}
$$

and we can again identify the local translations as covariant general coordinate transformations upon setting $R\left(P^{A}\right)=0$.

The next step is to write down the curvatures, which yields

$$
\begin{align*}
R\left(P^{A}\right) & =\mathrm{d} e^{A}+\omega_{B}^{A} \wedge e^{B}-\frac{1}{4} \bar{\psi} \gamma^{A} \wedge \psi, \\
R\left(M^{A B}\right) & =\mathrm{d} \omega^{A B}+\omega_{C}^{A} \wedge \omega^{C B},  \tag{3.24}\\
\Psi & =\mathrm{d} \psi+\frac{1}{4} \omega^{A B} \wedge \gamma_{A B} \psi .
\end{align*}
$$

For a theory of gravity this leads to the constraint

$$
\begin{equation*}
R\left(P^{A}\right)=\mathrm{d} e^{A}+\omega_{B}^{A} \wedge e^{B}-\frac{1}{4} \bar{\psi} \gamma^{A} \wedge \psi=0 \tag{3.25}
\end{equation*}
$$

which suggests the identification of the last term with the torsion $T^{A}$ introduced in eq. 3.14 .

Thus, this gives a straightforward solution for the spin connection $\omega^{A B}(e, \psi)$ according to eq. 3.15 . It depends on both the vielbein as well as a spin- $\frac{3}{2}$ field, which enters into the physical spectrum of the theory and is called the 'gravitino'.

### 3.4 Newton-Cartan Gravity from the Bargmann Algebra

Having considered some relativistic examples we can now turn to the nonrelativistic version of the gauging procedure. Just as gauging of the Poincaré algebra leads to the kinematics of Einstein gravity, we can gauge the nonrelativistic analog - the Bargmann algebra - to arrive at Newton gravity.

However, since the procedure is inherently formulated in a geometric language, we find a non-relativistic geometry, which can give a formulation called 'Newton-Cartan gravity'. This procedure is laid out in detail in 12 which we draw heavily upon.

We start by introducing the Bargmann algebra $\mathfrak{b}(D-1,1)$ in $D$ dimensions via the commutation relations

$$
\begin{array}{ll}
{\left[J_{i j}, J_{k l}\right]=4 \delta_{[i[k} J_{l] j]},} & {\left[J_{i j}, P_{k}\right]=-2 \delta_{k[i} P_{j]},} \\
{\left[J_{i j}, G_{k}\right]=-2 \delta_{k[i} G_{j]},} & {\left[G_{i}, H\right]=-P_{i},}  \tag{3.26}\\
{\left[G_{i}, P_{j}\right]=-\delta_{i j} M .} &
\end{array}
$$

The $J_{i j}$ are the generators of spatial rotations, $P_{i}$ the generators of spatial translations, $H$ the generator of time translations and $G_{i}$ the generators of
boosts. The central charge $M$ is the Noether charge of mass conservation and necessary for massive particle representations. A derivation of this algebra in the sense of a non-relativistic contraction of the Poincaré algebra is given in chapter 4 but for now we take it as given.

Taking the usual first step we introduce parameters and gauge fields

$$
\begin{array}{ll}
H \rightarrow\left\{\xi, \tau_{\mu}\right\}, & J_{a b} \rightarrow\left\{\lambda^{a b}, \omega_{\mu}^{a b}\right\}, \\
P_{a} \rightarrow\left\{\xi^{a}, e_{\mu}^{a}\right\}, & G_{a} \rightarrow\left\{\lambda^{a}, \omega_{\mu}^{a}\right\},  \tag{3.27}\\
M \rightarrow\left\{\sigma, m_{\mu}\right\} . &
\end{array}
$$

Using the relations in eq. 3.26 we can write down the symmetry transformations for the gauge fields

$$
\begin{align*}
\delta \tau & =\mathrm{d} \xi, \\
\delta e^{a} & =\mathrm{d} \xi^{a}+\lambda^{a} \tau-\xi \omega^{a}-\lambda^{a}{ }_{b} e^{b}-\xi^{b} \omega_{b}{ }^{a}, \\
\delta \omega^{a} & =\mathrm{d} \lambda^{a}-\lambda^{a}{ }_{b} \omega^{b}-\lambda^{b} \omega_{b}{ }^{a},  \tag{3.28}\\
\delta \omega^{a b} & =\mathrm{d} \lambda^{a b}-2 \lambda^{[a}{ }_{c} \omega^{c \mid b]}, \\
\delta m & =\mathrm{d} \sigma-\xi_{a} \omega^{a}+\lambda_{a} e^{a},
\end{align*}
$$

as well as the curvatures

$$
\begin{align*}
R(H) & =\mathrm{d} \tau \\
R\left(P^{a}\right) & =\mathrm{d} e^{a}+\omega^{a}{ }_{b} \wedge e^{b}+\tau \wedge \omega^{a} \\
R\left(J^{a b}\right) & =\mathrm{d} \omega^{a b}+\omega^{a}{ }_{c} \wedge \omega^{c b}  \tag{3.29}\\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\omega^{a}{ }_{b} \wedge \omega^{b} \\
R(M) & =\mathrm{d} m+e_{a} \wedge \omega^{a} .
\end{align*}
$$

As usual we want to replace the local translations with 'cgcts', which requires imposing the conventional constraint on our set of curvatures. Referring back to section 3.1 we see however, that we cannot include $M$ in the standard gauge transformations with the transformation rule for the gauge field $m_{\mu}$ as in eq. 3.5 , since it would violate the condition $f_{A \hat{B}}{ }^{\hat{A}}=0$. In fact, instead of taking $M$ to be part of the standard gauge transformations it is more useful to define an 'extended frame field ${ }^{5}$ which consists of $e_{\mu}^{a}, \tau_{\mu}$ and $m_{\mu}$, thereby preserving the algebra structure we used in section 3.1. Going from local translations to 'cgct' requires setting the curvatures corresponding to all these fields to zero as part of the conventional constraints, leading to the set,

$$
\begin{equation*}
R(H)=0, \quad R\left(P^{a}\right)=0, \quad R(M)=0 \tag{3.30}
\end{equation*}
$$

The first condition additionally lets us define a foliation of spacetime in form of $D$-1-dimensional 'constant time hypersurfaces ${ }^{6}$, since it implies via eq. 2.25 that there exists a function $t(x)$, such that

$$
\begin{equation*}
\tau=\mathrm{d} t \tag{3.31}
\end{equation*}
$$

[^9]which we identify with the Newtonian absolute time.
Next, we would like to solve for the dependent fields. Proceeding by analogy to the relativistic case, this means the spin connections $\omega^{a b}$ and $\omega^{a}$ should become functions of the independent fields.

The non-relativistic case introduces a complication, in that we cannot define the inverse vielbein in the usual way as in eq. 3.12, since the metric has become degenerat $\left.{ }^{7}\right]$. We thus introduce the respective inverses $\tau^{\mu}$ and $e_{a}^{\mu}$ via the 'projective inverse conditions'

$$
\begin{array}{ll}
e_{\mu}^{a} e_{b}^{\mu}=\delta_{b}^{a}, & \tau^{\mu} \tau_{\mu}=1 \\
\tau^{\mu} e_{\mu}^{a}=0, & \tau_{\mu} e_{a}^{\mu}=0  \tag{3.32}\\
e_{\mu}^{a} e_{a}^{\nu}=\delta_{\mu}^{\nu}-\tau_{\mu} \tau^{\nu}, &
\end{array}
$$

which let us solve the constraints for the dependent spin connections (for the detailed calculation see [12])

$$
\begin{align*}
\omega_{\mu}^{a b} & =2 e^{\rho[a} \partial_{[\rho} e_{\mu]}^{b]}+e_{\mu}^{c} e^{\rho a} e^{\nu b} \partial_{[\rho} e_{\nu]}^{c}-\tau_{\mu} e^{\rho a} e^{\nu b} \partial_{[\rho} m_{\nu]}  \tag{3.33}\\
\omega_{\mu}^{a} & =e^{\nu a} \partial_{[\mu} m_{\nu]}+e_{\mu}^{b} e^{\nu a} \tau^{\rho} \partial_{[\nu} e_{\rho]}^{b}+\tau^{\nu} \partial_{[\mu} e_{\nu]}^{a}+\tau_{\mu} \tau^{\nu} e^{\rho a} \partial_{[\nu} m_{\rho]} .
\end{align*}
$$

As a last step to Newton-Cartan gravity we can now set the spatial curvature to zero as a further constraint

$$
\begin{equation*}
R\left(J^{a b}\right)=0 \tag{3.34}
\end{equation*}
$$

to end up with flat space. This only leaves us with a non-vanishing curvature for boosts $R\left(G^{a}\right)$ which can be used to include the dynamics of Newton gravity via

$$
\begin{equation*}
R_{0 a}\left(G^{a}\right)=0 \tag{3.35}
\end{equation*}
$$

One can then continue to eliminate components of the remaining independent fields via partial gauge fixing as was carried out in 13, such that only $m_{0}$ is non-trivial. Identifying this remaining field with the Newton gravitational potential $\Phi$ and adding an appropriate source term to eq. 3.35 leads to the Poisson equation of Newton gravity

$$
\begin{equation*}
\Delta \Phi=4 \pi G \rho \tag{3.36}
\end{equation*}
$$

This concludes the derivation of Newton-Cartan gravity from symmetry principles. However, there is an interesting note to be made about the constraints we introduced in this section, especially in the light of a supersymmetric extension.

The constraints we used to solve for the dependent fields, so

$$
\begin{equation*}
R\left(P^{a}\right)=0, \quad R(M)=0 \tag{3.37}
\end{equation*}
$$

are no longer constraints in the second order formulation, since they are fulfilled identically upon inserting the spin connections. However, for

$$
\begin{equation*}
R(H)=0 \tag{3.38}
\end{equation*}
$$

[^10]this is not the case and it remains a true constraint, meaning that the variation of it may lead to further constraints on our set of curvatures.

From eq. 3.28 we can see that this is not the case here, however in the supersymmetric extension considered in [14, $\tau$ transforms into one of the fermionic gauge fields, which imposes a constraint on the corresponding curvature. This new constraint has to be varied as well leading to further constraints which might end up putting the theory on-shell by accident or imposing an unintended constraint on the spacetime curvature.

While only a curiosity here, this circumstance poses challenges for nonrelativistic supersymmetric theories.

## Chapter 4

## Lie Algebra Expansion

In the previous sections we introduced the Poincaré algebra and its supersymmetric extension and showed how these lead to the kinematics of theories of gravity. Thus, one would expect that modifications to these algebras would in turn lead to kinematics of novel theories showing characteristics of theories of gravity - which is to say diffeomorphism invariance and a dependent spinconnection.

In general, the process consists of finding consistent Lie algebras corresponding to spacetime symmetries ${ }^{11}$, gauging them and then setting the curvature of local translations to zero as a canonical constraint, which then causes diffeomorphisms to become part of the symmetries of the theory. Furthermore, it might be necessary to add additional constraints for the curvatures, to eliminate superfluous fields which should not be present as independent fields in the physical theory. The consequences of this have been discussed in section 3.4 in some detail.

In this chapter the focus lies on finding those consistent Lie algebras. First we quickly introduce the concept of Lie algebra contractions and derive the Bargmann algebra used in section 3.4 from an extended Poincaré algebra. Then we turn to the method of Lie algebra expansions as first laid out in [17. This technique has been recently used in $16,19,20$ to construct a number of nonrelativistic expansions for gravity and supergravity algebras.

### 4.1 Lie Algebra Contractions

Before moving to the somewhat novel method of Lie Algebra Expansions it is worthwhile to consider a different method of obtaining new algebras from known ones - namely Lie algebra contractions. These - specifically InönüWigner contractions (see [28]) - have been used extensively to construct nonrelativistic limits of (super-)Poincaré algebras as in $12-14$ and have served as the standard approach up to this point.

For the contraction we take an initial algebra $\mathcal{G}$ and re-define the generators $T_{A}$ by taking a linear combination involving a parameter $\lambda$, so

$$
\begin{equation*}
T_{A} \rightarrow T_{A}^{\prime}=c(\lambda)_{A}^{B} T_{B}, \tag{4.1}
\end{equation*}
$$

[^11]where $c(\lambda){ }_{A}^{B}$ are polynomial functions of $\lambda$. Doing so consistently and taking some finite value for $\lambda$ this gives an algebra equivalent to the initial one, since we are merely redefining the basis.

The next step for the procedure is to write out the commutation relations for the algebra and take a singular limit for the contraction parameter. This leads to a new set of commutation relations between the re-defined generators, which do not necessarily form a consistent algebra, since the Jacobi identities may not be fulfilled. By checking them explicitly it can be determined if the contraction procedure successfully yielded a new algebra.

In the special case of Inönü-Wigner ('IW') contractions - see 28] - the possible contractions as by eq. 4.1 are limited but consistency is guaranteed.

As an example for a Lie algebra contraction we can derive the Bargmann algebra in eq. 3.26 from an extended Poincaré algebra $\mathfrak{i s o}(D-1,1) \oplus \mathfrak{g}_{M}$ with $\mathfrak{g}_{M}$ a 1-parameter subalgebra spanned by a charge $M$. Since this additional charge is central, the commutation relations are equivalent to those of the Poincaré algebra in eq. 3.1. We choose the same re-definition for the generators as in 12, $\mathrm{sc}{ }^{2}$

$$
\begin{array}{ll}
P_{0} \rightarrow \lambda^{2} M+H, & P_{i} \rightarrow \lambda P_{i}  \tag{4.2}\\
M_{i 0} \rightarrow \lambda G_{i}, & M_{i j} \rightarrow J_{i j}
\end{array}
$$

The choice can be motivated physically, since it reflects the non-relativistic expansion of $P_{0}$ for a massive particle in terms of the speed of light $c$

$$
\begin{equation*}
P_{0} \approx M c^{2}+\frac{P^{i} P_{i}}{2 M} \tag{4.3}
\end{equation*}
$$

This gives some confidence that the results will indeed end up leading to a nonrelativistic analog of the Poincaré algebra. However, for the contraction procedure itself this is superfluous, since the resulting algebra justifies the choices post hoc.

We proceed by inserting the re-definitions in eq. 4.2 into the commutation relations 3.1, leading to

$$
\begin{array}{ll}
{\left[J_{i j}, J_{k l}\right]=4 \delta_{[i[k} J_{l] j]},} & {\left[\lambda P_{i}, J_{j k}\right]=2 \delta_{i[j} \lambda P_{k]},} \\
{\left[J_{i j}, \lambda G_{k}\right]=-2 \delta_{k[i} \lambda G_{j]},} & {\left[\lambda^{2} M+H, \lambda G_{i}\right]=\lambda P_{i},} \tag{4.4}
\end{array}
$$

for the non-vanishing expressions. Taking the singular limit $\lambda \rightarrow \infty$, only terms with the same order in $\lambda$ on both sides remain. Thus, we find the Bargmann algebra

$$
\begin{array}{ll}
{\left[J_{i j}, J_{k l}\right]=4 \delta_{[i[k} J_{l] j]},} & {\left[J_{i j}, P_{k}\right]=-2 \delta_{k[i} P_{j]},} \\
{\left[J_{i j}, G_{k}\right]=-2 \delta_{k[i} G_{j]},} & {\left[G_{i}, H\right]=-P_{i},}  \tag{4.5}\\
{\left[G_{i}, P_{j}\right]=-\delta_{i j} M .} &
\end{array}
$$

### 4.2 The Expansion Method

The idea of the Lie algebra expansion method was first laid out in 17 from which we will draw heavily in this section. Nonetheless, since all of the results obtained

[^12]in this section will be vital for the calculations in chapter 5 it is necessary to repeat the steps here, focusing only on the directly applicable aspects. The work in 17] treats additional structures for the initial algebra not relevant for the discussion of the non-relativistic expansion of the Poincaré algebra and we refer back to it for a broader approach to the subject.

We now want to find a new Lie algebra from a given one $\mathcal{G}$ by re-scaling some of the parameters as $g^{l} \rightarrow \lambda g^{l}$ and taking a singular limit for $\lambda$. This is conceptually similar to the IW contractions in 28 but instead of taking the contraction for the algebra, we want to use this re-scaling to induce an expansion for the Maurer-Cartan one-forms $\omega^{i}(g)$.

From there we can use the expanded one-forms in their corresponding MC equations and find a new set of MC equations by identifying terms in the expansion order by order. So far this is simply a rewriting of the MC equations.

The crucial step comes in the form of truncating the expansion of the oneforms $\omega^{i}(g)$ and requiring consistency for the MC equations by enforcing the Jacobi identity. This induces a constraint on the order in $\lambda$ up to which terms have to be included, which will in general be different for some subsets of MC one-forms.

Finally, we can read off the structure constants for the expanded algebra from the set of MC equations. In general, this new algebra will have more generators than $\mathcal{G}$ and will thus be of higher dimension, in contrast to the Lie algebra contractions considered above which retain the original dimensionality.

### 4.3 Derivation of the Lie Algebra Expansion

To make this approach systematic, we first introduce the Lie algebra $\mathcal{G}$ for the Lie group $G$ with parameters $\square^{3} g^{i}$ and left-invariant generators $X_{i}$. As explained in section 2.4 the MC one-forms $\omega^{i}(g)$ with MC equations

$$
\begin{equation*}
\mathrm{d} \omega^{k}(g)=-\frac{1}{2} c_{i j}^{k} \omega^{i}(g) \wedge \omega^{j}(g) \tag{4.6}
\end{equation*}
$$

with $c_{i j}{ }^{k}$ the structure constants of $\mathcal{G}$, give a dual description of the algebra.
We can then take the left-invariant one-form over $G$, which is defined as

$$
\begin{equation*}
\omega^{i} X_{i}=g^{-1} \mathrm{~d} g \tag{4.7}
\end{equation*}
$$

with $g \in G$, to find an expansion of $\omega^{i}$ in terms of $g^{i}$. The group elements $g$ are related to $\mathcal{G}$ via the exponential map, so

$$
\begin{equation*}
g=e^{g^{i} X_{i}} \tag{4.8}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\omega^{i} X_{i}=e^{-g^{i} X_{i}} \mathrm{~d} e^{g^{i} X_{i}} \tag{4.9}
\end{equation*}
$$

which yields a sum over nested commutators. Using the structure equations for the Lie algebra this can be worked out to (see [17])

$$
\begin{equation*}
\omega^{i}(g)=\left[\delta_{j}^{i}+\sum_{n=1}^{\infty} \frac{1}{(n+1)!} c_{j k_{1}}^{h_{1}} c_{h_{1} k_{2}}{ }^{h_{2}} \cdots c_{h_{n-1} k_{n}}{ }^{i} g^{k_{1}} g^{k_{2}} \cdots g^{k_{n}}\right] \mathrm{d} g^{j} \tag{4.10}
\end{equation*}
$$

[^13]The actual form of the expansion in eq. 4.10 is secondary to the fact that a re-scaling of certain parameters $g^{l} \rightarrow \lambda g^{l}$ will lead to a power series expansion of the MC one-forms as

$$
\begin{equation*}
\omega^{i}(g, \lambda)=\sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{i, \alpha}(g) \tag{4.11}
\end{equation*}
$$

which is the expansion most useful to us.
As noted above, the aim is to re-scale only a subset of the parameters, so we will have to make this concept more concrete. We split the algebra $\mathcal{G}$ into an arbitrary ${ }^{4}$ direct sum of vector-subspaces

$$
\begin{equation*}
\mathcal{G}=V_{0} \oplus V_{1}, \tag{4.12}
\end{equation*}
$$

leading to a split in the indices $i_{0}, i_{1}$, such that the basis one-forms split into sets $\left\{\omega^{i_{0}}(g)\right\}$ and $\left\{\omega^{i_{1}}(g)\right\}$. $V_{1}$ corresponds to the subspace with re-scaled parameters, so $g^{i_{1}} \rightarrow \lambda g^{i_{1}}$. Importantly, both $\omega^{i_{0}}(g)$ and $\omega^{i_{1}}(g)$ are expanded in a power series of $\lambda$, since the power series depends on all $g^{i}$, as can be seen in eq. 4.10. We have

$$
\begin{equation*}
\omega^{i_{p}}(g, \lambda)=\sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{i_{p}, \alpha}(g) \tag{4.13}
\end{equation*}
$$

with $p=0,1$.
The next step is to insert the expansions into the MC equation 4.6 and compare terms order by order. To read off the corresponding terms, the r.h.s requires some slight modification by shifting the sums, such that

$$
\begin{equation*}
\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \lambda^{\alpha+\beta} \omega^{i_{p}, \alpha} \wedge \omega^{j_{q}, \beta}=\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} \lambda^{\alpha} \omega^{i_{p}, \beta} \wedge \omega^{j_{q}, \alpha-\beta} \tag{4.14}
\end{equation*}
$$

and thus order by order

$$
\begin{equation*}
\mathrm{d} \omega^{k_{s}, \alpha}=-\frac{1}{2} c_{i_{p} j_{q}} \sum_{\beta=0}^{\alpha} \omega^{i_{p}, \beta} \wedge \omega^{j_{q}, \alpha-\beta} \tag{4.15}
\end{equation*}
$$

with $p, q, s=0,1$.
So far this was just an exercise in rewriting the MC one-forms and MC equations since we recover the original form by including all the terms in the expansion. The last step is now to make a truncation in eq. 4.13 (in principle independently for $p=0,1$ ) at some order $N_{0}$ and $N_{1}$, such that the resulting algebra is consistent and closed.

For the truncated set of MC one-forms $\left\{\omega^{i_{0}, 1}, \ldots, \omega^{i_{0}, N_{0}}, \omega^{i_{1}, 1}, \ldots, \omega^{i_{1}, N_{1}}\right\}$ we can check closure by requiring that $\mathrm{d} \omega^{k_{s}, \alpha}$ in eq. 4.15 does not contain any additional one-forms not already part of the set. In general this will put a constraint on possible choices of $N_{0}$ and $N_{1}$.

Secondly for consistency, we enforce the Jacobi identity by checking that

$$
\begin{equation*}
\mathrm{d} d \omega^{k_{s}, \alpha} \equiv 0 \tag{4.16}
\end{equation*}
$$

[^14]Starting from eq. 4.15 we can quickly work out that this condition holds if the structure constants satisfy

$$
\begin{equation*}
c_{i_{p}\left[j_{q}\right.}{ }^{k_{s}} c_{\left.l_{t} m_{u}\right]}{ }^{i_{p}}=0 \tag{4.17}
\end{equation*}
$$

which is the case if the original algebra $\mathcal{G}$ satisfies its Jacobi identities. Thus, consistency is a natural result of the formalism and we only need to work towards closure.

When both criteria are satisfied this yields a new algebra (or rather a class of algebras), which we call $\mathcal{G}\left(N_{0}, N_{1}\right)$.

From here, one can derive the constraint for the arbitrary split in eq. 4.12 or consider different structures along which the parameters can be split, like a subalgebra structure for $V_{0}$ or a symmetric coset structure for $V_{1}$. These cases are all discussed in detail in 17 . For this work however, we are mainly interested in the symmetric coset structure which is particularly useful in making non-relativistic expansion of theories containing a Poincaré sector.

### 4.3.1 $\quad V_{1}$ as a symmetric coset

For the structure of a symmetric coset we have the split

$$
\begin{equation*}
\left[V_{0}, V_{0}\right] \subset V_{0}, \quad\left[V_{0}, V_{1}\right] \subset V_{1}, \quad\left[V_{1}, V_{1}\right] \subset V_{0} \tag{4.18}
\end{equation*}
$$

This implies that a number of structure constants have to vanish, so

$$
\begin{equation*}
c_{i_{0} j_{0}}{ }^{k_{1}}=0, \quad c_{i_{0} j_{1}}{ }^{k_{0}}=0, \quad c_{i_{1} j_{1}}{ }^{k_{1}}=0 \tag{4.19}
\end{equation*}
$$

which by eq. 4.10 leads to an expansion in even (odd) powers ${ }^{5}$ of $\lambda$ for $\omega^{i_{0}}\left(\omega^{i_{1}}\right)$, so

$$
\begin{align*}
\omega^{i_{0}}(\lambda) & =\sum_{\sigma=0}^{\infty} \lambda^{2 \sigma} \omega^{i_{0}, 2 \sigma} \\
\omega^{i_{1}}(\lambda) & =\sum_{\sigma=0}^{\infty} \lambda^{2 \sigma+1} \omega^{i_{1}, 2 \sigma+1} \tag{4.20}
\end{align*}
$$

As explained above, we can now find $N_{0}$ and $N_{1}$ by checking that the exterior derivative does not lead to one-forms outside the truncated set. The MC equations read
$\mathrm{d} \omega^{k_{0}, 2 \sigma}=-\frac{1}{2} c_{i_{0} j_{0}}{ }^{k_{0}} \sum_{\rho=0}^{\sigma} \omega^{i_{0}, 2 \rho} \wedge \omega^{j_{0}, 2(\sigma-\rho)}-\frac{1}{2} c_{i_{1} j_{1}}{ }^{k_{0}} \sum_{\rho=1}^{\sigma} \omega^{i_{1}, 2 \rho-1} \wedge \omega^{j_{1}, 2(\sigma-\rho)+1}$,
$\mathrm{d} \omega^{k_{1}, 2 \sigma+1}=-c_{i_{0} j_{1}}{ }^{k_{1}} \sum_{\rho=0}^{\sigma} \omega^{i_{0}, 2 \rho} \wedge \omega^{j_{1}, 2(\sigma-\rho)+1}$,
where we made the even (odd) structure of the $\omega^{i_{p}}$ explicit. Making a choice for the highest order one-forms on the l.h.s, we find a closed truncation by checking that no higher order elements appear on the r.h.s, which is satisfied for

$$
\begin{equation*}
N_{1}=N_{0}-1 \quad \text { or } \quad N_{1}=N_{0}+1 \tag{4.22}
\end{equation*}
$$

[^15]This result naturally complies with the even (odd) constraint on the expansion and tells us how to consistently expand and truncate Lie algebras with symmetric coset structure.

The methods developed in this chapter serve as the theoretical framework for the concrete expansions in chapter 5 and especially the results in eq. 4.20 and eq. 4.22 are invaluable for our purposes.

## Chapter 5

## Applications of Lie Algebra Expansion

We are now ready to apply the results of the previous chapters to the Poincaré theory and its supersymmetric extensions in order to find a number of nonrelativistic expanded algebras.

First, we will treat the case without supersymmetry similar to the work in [16]. These calculations bear repeating here not just because they lead to interesting results and reproduce the algebras used in previous works in this area (see 12,28 as examples), but also as an exercise for the more cumbersome case of supersymmetry. Since the method of Lie algebra expansion reads much like a recipe, applying the same steps to different order to a number of initial algebras, the general layout repeats and will quickly be familiar. In addition, this section will allow us to adapt to the notation introduced in 16 and extend it for the case of SUSY.

Having dealt with ordinary gravity we can then finally turn to the algebra underlying supergravity and expand it. As the expansion method leads to an infinite number of possible algebras, we have to limit ourselves to a number of particularly interesting cases, only deriving the 3D and 4D extended Bargmann superalgebra and 4D extended string Bargmann superalgebra.

### 5.1 Expansion of Gravity

We focus on two cases which will be particularly useful once we turn to the SUSY extension and refer to 16 for a number of additional possible expansions. As a particularly straightforward calculation we take the limit of 4D Galilei gravity which gives an overview of all the necessary steps in practice. Afterwards we consider the expansion of 3D extended Bargmann gravity which will be broadly similar to the calculations necessary once we add the SUSY extension.

### 5.1.1 4D Galilei Gravity

We start the calculation by first replicating the 4D Poincaré algebra from eq. 3.1

$$
\begin{align*}
{\left[P_{A}, P_{B}\right] } & =0, \\
{\left[P_{A}, M_{B C}\right] } & =2 \eta_{A[B} P_{C]},  \tag{5.1}\\
{\left[M_{A B}, M_{C D}\right] } & =4 \eta_{[A[C} M_{D] B]},
\end{align*}
$$

with $A=0,1,2,3$. The expansion method tells us to first define the desired split for the algebra for which we want to choose the case with symmetric coset structure as developed in section 4.3.1. Physically motivated, we decompose the generators according to

$$
\begin{align*}
P_{A} & \rightarrow\left\{H, P_{a}\right\}  \tag{5.2}\\
M_{A B} & \rightarrow\left\{J_{a b}, G_{a}\right\},
\end{align*}
$$

with $a=1,2,3$ the spacelike directions and $G_{a}=M_{0 a}$ the generators for boosts. Inserting into the commutation relations of the Poincaré algebra we find the alternative formulation

$$
\begin{array}{ll}
{\left[G_{a}, H\right]=P_{a},} & {\left[G_{a}, P_{b}\right]=\delta_{a b} H,} \\
{\left[J_{a b}, P_{c}\right]=2 \delta_{c[b} P_{a]},} & {\left[J_{a b}, G_{c}\right]=2 \delta_{c[b} G_{a]},}  \tag{5.3}\\
{\left[G_{a}, G_{b}\right]=J_{a b},} & {\left[J_{a b}, J_{c d}\right]=4 \delta_{[a[c} J_{d] b]},}
\end{array}
$$

for the non-vanishing relations.
Since we merely re-labeled the generators without affecting the underlying structure this still describes the Poincaré algebra and leads to ordinary Einsteinian gravity via the methods laid out in chapter 3 .

We can now define the split as

$$
\begin{equation*}
V_{0}=\left\{J_{a b}, H\right\}, \quad V_{1}=\left\{P_{a}, G_{a}\right\} \tag{5.4}
\end{equation*}
$$

for which we can check compliance with the symmetric coset structure as in eq. 4.18 .

The next step is to introduce the associated gauge fields

$$
\begin{array}{ll}
H \rightarrow \tau_{\mu}, & J_{a b} \rightarrow \omega_{\mu}^{a b},  \tag{5.5}\\
P_{a} \rightarrow e_{\mu}^{a}, & G_{a} \rightarrow \omega_{\mu}^{a},
\end{array}
$$

where we included the spacetime index to stress that these are covariant vector fields, before going back to the more efficient one-form formalism.

With the gauge fields we can now write the curvatures according to eq. 2.24 such that we have

$$
\begin{align*}
R\left(J^{a b}\right) & =\mathrm{d} \omega^{a b}+\omega^{a}{ }_{c} \wedge \omega^{c b}+\omega^{a} \wedge \omega^{b}, \\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\omega^{a}{ }_{b} \wedge \omega^{b}, \\
R(H) & =\mathrm{d} \tau+\delta_{a b} \omega^{a} \wedge e^{b},  \tag{5.6}\\
R\left(P^{a}\right) & =\mathrm{d} e^{a}+\omega^{a} \wedge \tau+\omega^{a}{ }_{b} \wedge e^{b} .
\end{align*}
$$

It is important to stress again that we have yet to expand and truncate the algebra, so these curvatures are still just the unmodified curvatures for the ordinary Poincaré case.

At this point however, we can make the connection between the physically motivated approach using gauge fields and corresponding curvatures and the mathematically motivated approach in chapter 4 which uses the dual formulation of a coalgebra with one-forms and MC equations. We thus identify the gauge fields with the MC one-forms and consider the curvatures to quantify a failure of the MC equations, such that we can recover them by setting

$$
\begin{equation*}
R\left(B^{A}\right)=0 \tag{5.7}
\end{equation*}
$$

We can therefore expand the gauge fields and curvatures and find a consistent truncation of the corresponding MC equations upon setting the curvatures equal to zero. The new curvatures for the expanded algebra can then again be thought of a quantifying a failure of the MC equations and be introduced by hand.

We want to discuss the simplest expansion and thus consider $\mathcal{G}\left(N_{0}, N_{1}\right)=$ $\mathcal{G}(0,1)$. According to section 4.3 .1 the expansions for the one-forms can only contain even $\left(V_{0}\right)$ and odd $\left(V_{1}\right)$ terms respectively, such that we have for the expanded and truncated gauge fields

$$
\begin{array}{ll}
\tau \rightarrow \tau, & e^{a} \rightarrow \lambda e^{a},  \tag{5.8}\\
\omega^{a b} \rightarrow \omega^{a b}, & \omega^{a} \rightarrow \lambda \omega^{a},
\end{array}
$$

where we immediately renamed the terms in the expansions appropriately to cut down on cumbersome notation. For the curvatures we are then left with

$$
\begin{align*}
R\left(J^{a b}\right) & =\mathrm{d} \omega^{a b}+\omega^{a}{ }_{c} \wedge \omega^{c b}, \\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\omega^{a}{ }_{b} \wedge \omega^{b}, \\
R(H) & =\mathrm{d} \tau  \tag{5.9}\\
R\left(P^{a}\right) & =\mathrm{d} e^{a}+\omega^{a} \wedge \tau+\omega^{a}{ }_{b} \wedge e^{b},
\end{align*}
$$

where we truncated the $\lambda^{2}$ term in $R\left(J^{a b}\right)$ and $R(H)$ as part of the procedure. Consequently, we find the consistent algebra

$$
\begin{array}{ll}
{\left[G_{a}, H\right]=P_{a},} & {\left[J_{a b}, J_{c d}\right]=4 \delta_{[a[c} J_{d] b]},} \\
{\left[J_{a b}, P_{c}\right]=2 \delta_{c[b} P_{a]},} & {\left[J_{a b}, G_{c}\right]=2 \delta_{c[b} G_{a]} .} \tag{5.10}
\end{array}
$$

This algebra is known as the Galilean algebra and constitutes one possible non-relativistic limit of the Poincaré algebra. As discussed in 12 it is not quite the algebra needed for Newton-Cartan gravity, since it does not allow for massive representations. For these we require the Bargmann algebra which emerges as a central extension of the Galilean algebra.

Nevertheless, the methods employed here will translate cleanly to the rest of this chapter. In particular the split in eq. 5.4 will hold true and embodies a general feature of the non-relativistic expansion, since it corresponds to the limit $c \rightarrow \infty$ as can be seen by comparing to the derivation in 12 .

As an example for an alternative split one might consider we have the socalled 'Carroll gravity'. This is the spacetime symmetry algebra which underlies gravity in $\sqrt[29]{ }$ and can be considered an ultra-relativistic limit for the Poincaré algebra. The algebra and a physically motivated expansion are discussed in detail in [29, however we can also derive it using the above formalism by choosing a split

$$
\begin{equation*}
V_{0}=\left\{J_{a b}, P_{a}\right\}, \quad V_{1}=\left\{H, G_{a}\right\} . \tag{5.11}
\end{equation*}
$$

As can be checked by comparing to eq. 4.18 , this still respects the symmetric coset structure and leads to the Carroll algebra after the expansion. This illustrates rather well that we have some freedom in choosing the split, which leads to more possible algebras we can expand into.

### 5.1.2 3D Extended Bargmann gravity

Having dealt with a simple expansion one might make for the Poincaré algebra, we now want to go one step further and expand the algebra to $\mathcal{G}(2,1)^{11}$. Again, we know from section 4.3.1 that the order of the truncation has to comply with the consistency condition in eq. 4.22 for the symmetric coset structure we require.

We limit this section to the 3-dimensional case in part to keep the process streamlined, but also since this gives a direct link to the supersymmetry extension we want to derive below.

Starting from the Poincaré algebra in eq. 5.1 we again decompose the generators as above

$$
\begin{align*}
P_{A} & \rightarrow\left\{H, P_{a}\right\}, \\
M_{A B} & \rightarrow\left\{J, G_{a}\right\}, \tag{5.12}
\end{align*}
$$

where for three dimensions we $\operatorname{tak} \varepsilon^{2} G_{a}=\epsilon_{a}{ }^{b} M_{0 b}$ and $J=M_{12}$. This encompasses the main difference to higher dimensions, since only a single angular momentum charge exists in three dimensions and thus the structure simplifies as $[J, J]$ vanishes trivially.

The commutation relations for the algebra then take the form

$$
\begin{array}{ll}
{\left[H, G_{a}\right]=-\epsilon_{a}{ }^{b} P_{b},} & {\left[G_{a}, P_{b}\right]=\epsilon_{a b} H,} \\
{\left[J, P_{a}\right]=-\epsilon_{a}^{b} P_{b},} & {\left[J, G_{a}\right]=-\epsilon_{a}^{b} G_{b},}  \tag{5.13}\\
{\left[G_{a}, G_{b}\right]=\epsilon_{a b} J .} &
\end{array}
$$

We split the algebra in the same way as above into subspaces

$$
\begin{equation*}
V_{0}=\{J, H\}, \quad V_{1}=\left\{P_{a}, G_{a}\right\} \tag{5.14}
\end{equation*}
$$

and define gauge fields for the generators

$$
\begin{array}{ll}
H \rightarrow \tau_{\mu}, & J \rightarrow \omega_{\mu}  \tag{5.15}\\
P_{a} \rightarrow e_{\mu}^{a}, & G_{a} \rightarrow \omega_{\mu}^{a}
\end{array}
$$

with curvatures

$$
\begin{align*}
R(J) & =\mathrm{d} \omega+\frac{1}{2} \epsilon_{a b} \omega^{a} \wedge \omega^{b}, \\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\epsilon^{a}{ }_{b} \omega \wedge \omega^{b},  \tag{5.16}\\
R(H) & =\mathrm{d} \tau+\epsilon_{a b} \omega^{a} \wedge e^{b}, \\
R\left(P^{a}\right) & =\mathrm{d} e^{a}-\epsilon^{a}{ }_{b} \omega^{b} \wedge \tau+\epsilon^{a}{ }_{b} \omega \wedge e^{b} .
\end{align*}
$$

[^16]From here we can start the Lie algebra expansion to $\mathcal{G}(2,1)$ by truncating the infinite series for $V_{0}$ at quadratic order and introducing two additional one-forms $m$ and $s$ defined according to

$$
\begin{array}{ll}
\tau \rightarrow \tau+\lambda^{2} m, & e^{a} \rightarrow \lambda e^{a}  \tag{5.17}\\
\omega \rightarrow \omega+\lambda^{2} s, & \omega^{a} \rightarrow \lambda \omega^{a}
\end{array}
$$

where the expansion in $V_{1}$ is again to first order only.
Inserting the expansion into the MC equations and grouping terms by their order in $\lambda$ we can find the new set of MC equations. In order to make this step more explicit we will demonstrate it for the MC equation corresponding to $R(J)$. We have

$$
\begin{equation*}
0=\mathrm{d} \omega+\frac{1}{2} \epsilon_{a b} \omega^{a} \wedge \omega^{b} \tag{5.18}
\end{equation*}
$$

before expanding and redefining and

$$
\begin{align*}
0 & =\mathrm{d} \omega+\lambda^{2} \mathrm{~d} s+\frac{1}{2} \epsilon_{a b} \lambda^{2} \omega^{a} \wedge \omega^{b} \\
& =\mathrm{d} \omega+\lambda^{2}\left(\mathrm{~d} s+\frac{1}{2} \epsilon_{a b} \omega^{a} \wedge \omega^{b}\right) \tag{5.19}
\end{align*}
$$

afterwards. Thus, since the equation has to hold order by order, we end up with two MC equations

$$
\begin{align*}
& 0=\mathrm{d} \omega \\
& 0=\mathrm{d} s+\frac{1}{2} \epsilon_{a b} \omega^{a} \wedge \omega^{b} . \tag{5.20}
\end{align*}
$$

Going back to curvatures we observe that the MC one-form $s$ corresponds to a gauge field for some additional generator of the expanded algebra, which we call $S$. The same is true for $m$ and its corresponding generator, which we call $M$.

Finally, for the full set of curvatures we have

$$
\begin{align*}
R(J) & =\mathrm{d} \omega \\
R(S) & =\mathrm{d} s+\frac{1}{2} \epsilon_{a b} \omega^{a} \wedge \omega^{b}, \\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\epsilon^{a}{ }_{b} \omega \wedge \omega^{b},  \tag{5.21}\\
R(H) & =\mathrm{d} \tau \\
R(M) & =\mathrm{d} m+\epsilon_{a b} \omega^{a} \wedge e^{b}, \\
R\left(P^{a}\right) & =\mathrm{d} e^{a}-\epsilon^{a}{ }_{b} \omega^{b} \wedge \tau+\epsilon^{a}{ }_{b} \omega \wedge e^{b} .
\end{align*}
$$

and thus the algebra

$$
\begin{array}{ll}
{\left[H, G_{a}\right]=-\epsilon_{a}{ }^{b} P_{b},} & {\left[G_{a}, P_{b}\right]=\epsilon_{a b} M} \\
{\left[J, P_{a}\right]=-\epsilon_{a}^{b} P_{b},} & {\left[J, G_{a}\right]=-\epsilon_{a}^{b} G_{b}}  \tag{5.22}\\
{\left[G_{a}, G_{b}\right]=\epsilon_{a b} S .} &
\end{array}
$$

At this point we can observe how the extended Bargmann algebra relates to the Bargmann algebra. There are now two central charges that extend the

Galilean algebra. For one we have $M$ which extends to the Bargmann algebra and then $S$ which gives an additional central extension to the Bargmann algebra, thus justifying the name extended Bargmann algebra. While the expansion only leads to central extensions here, this will no longer be true in the supersymmetric and higher dimensional case.

As opposed to the 4D Galilei gravity considered above, where we started and ended with the same dimension for the algebra $\operatorname{dim} \mathcal{G}=\operatorname{dim} \mathcal{G}(0,1)$, this is no longer true here. We went from 6 generators for 3D Poincaré to 8 generators for the 3D extended Bargmann algebra, thus ending up with a higher dimensional algebra. This is not particularly surprising in light of the methods we used, but highlights a general property of Lie algebra expansion, namely

$$
\begin{equation*}
\operatorname{dim} \mathcal{G} \leq \operatorname{dim} \mathcal{G}\left(N_{0}, N_{1}\right) \tag{5.23}
\end{equation*}
$$

### 5.2 Expansion of Supergravity

After having laid the necessary ground work we are finally ready to approach the expansion of the $\mathcal{N}=2$ Poincaré superalgebra.

We omit the $\mathcal{N}=1$ case since then the SUSY generator anti-commutes only to a central charge after the expansion and not to spacetime translations as is necessary for a theory to truly contain supersymmetry. This is further discussed in 13. For $\mathcal{N}=2$ however, we can find a linear combination for the SUSY charges that leads to the desired structure after the expansion.

First, we aim to derive the 3D extended Bargmann superalgebra via the systematic approach offered by the Lie algebra expansion method, which leads to the algebra initially discovered in 15 . There, the method involved computationally checking the Jacobi identities for a certain set of commutation relations until a consistent set was found. The approach outlined below thus serves to illuminate the underlying structure that gives rise to this superalgebra, as was additionally shown in 19.

Secondly, we want to go a step further and derive two nove ${ }^{3}$ algebras which we will call 4D extended Bargmann superalgebra and 4D extended string Bargmann superalgebra. Going from $D=3$ to $D=4$ is rather simple using the expansion method, whereas the necessary modifications are not nearly as apparent using the computational technique. In the string case, we want to expand on the work in [18] which gives a starting point for non-relativistic string theory, by adding the fermionic sector to the algebra, using again the same systematic approach.

In all cases we start out from the Poincaré algebra as given in eq. 5.1 and now add the fermionic sector for $\mathcal{N}=2$ with charges $Q_{i}, i=1,2$ and structure relations ${ }^{4}$

$$
\begin{align*}
\left\{Q^{i}, Q^{j}\right\} & =-\frac{1}{2} \delta^{i j} \gamma^{A} C^{-1} P_{A}, \\
{\left[M_{A B}, Q^{i}\right] } & =-\frac{1}{2} \gamma_{A B} Q^{i},  \tag{5.24}\\
{\left[P_{A}, Q^{i}\right] } & =0 .
\end{align*}
$$

[^17]
### 5.2.1 3D Extended Bargmann Superalgebra

We want to apply the same coset split for the bosonic sector as in the case considered in section 5.1.2, since we know that this leads to a non-relativistic algebra for gravity. However, from eq. 5.24 it is clear that the second commutation relation contradicts the naive approach, because $M_{A B}$ contains generators pertaining to $V_{0}$ and $V_{1}$ and therefore the $Q^{i}$ fit neither.

This problem is solved by introducing rotated charges as in 13, 14, defined as

$$
\begin{equation*}
Q_{ \pm}=\frac{1}{\sqrt{2}}\left(Q^{1} \pm \gamma_{0} Q^{2}\right) \tag{5.25}
\end{equation*}
$$

which commute appropriately with spatial rotations and boosts. The commutation relations for the fermionic sector of the Poincaré superalgebra are then

$$
\begin{array}{lll}
{\left[M_{a b}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{a b} Q_{ \pm},} & \left\{Q_{+}, Q_{-}\right\}=-\gamma^{a} C^{-1} P_{a} \\
{\left[M_{0 a}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{0 a} Q_{\mp},} & \left\{Q_{ \pm}, Q_{ \pm}\right\}=-\gamma^{0} C^{-1} P_{0} . \tag{5.26}
\end{array}
$$

From here we decompose the bosonic charges as in eq. 5.12 and find the full set of commutation relations

$$
\begin{array}{lll}
{\left[H, G_{a}\right]=-\epsilon_{a}{ }^{b} P_{b},} & {\left[J, G_{a}\right]=-\epsilon_{a}^{b} G_{b},} & {\left[G_{a}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{a} Q_{\mp},} \\
{\left[G_{a}, P_{b}\right]=\epsilon_{a b} H,} & {\left[G_{a}, G_{b}\right]=\epsilon_{a b} J,} & \left\{Q_{+}, Q_{-}\right\}=-\gamma^{a} C^{-1} P_{a}, \\
{\left[J, P_{a}\right]=-\epsilon_{a}^{b} P_{b},} & {\left[J, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{0} Q_{ \pm},} & \left\{Q_{ \pm}, Q_{ \pm}\right\}=-\gamma^{0} C^{-1} H,
\end{array}
$$

which allow for the split

$$
\begin{equation*}
V_{0}=\left\{J, H, Q_{+}\right\}, \quad V_{1}=\left\{P_{a}, G_{a}, Q_{-}\right\} \tag{5.28}
\end{equation*}
$$

In addition to the gauge fields assigned in eq. 5.15 we further define gauge fields for the fermionic charges as

$$
\begin{equation*}
Q_{ \pm} \rightarrow \psi_{\mu}^{ \pm} \tag{5.29}
\end{equation*}
$$

leading to a set of curvatures

$$
\begin{align*}
R(J) & =\mathrm{d} \omega+\frac{1}{2} \epsilon_{a b} \omega^{a} \wedge \omega^{b}, \\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\epsilon^{a}{ }_{b} \omega \wedge \omega^{b}, \\
R(H) & =\mathrm{d} \tau+\epsilon_{a b} \omega^{a} \wedge e^{b}+\frac{1}{2} \bar{\psi}^{+} \gamma_{0} \wedge \psi^{+}+\frac{1}{2} \bar{\psi}^{-} \gamma_{0} \wedge \psi^{-},  \tag{5.30}\\
R\left(P^{a}\right) & =\mathrm{d} e^{a}-\epsilon^{a}{ }_{b} \omega^{b} \wedge \tau+\epsilon^{a}{ }_{b} \omega \wedge e^{b}-\bar{\psi}^{+} \gamma^{a} \wedge \psi^{-}, \\
R\left(Q_{ \pm}\right) & =\mathrm{d} \psi^{ \pm}+\frac{1}{2} \gamma_{0} \omega \wedge \psi^{ \pm}+\frac{1}{2} \gamma_{a} \omega^{a} \wedge \psi^{\mp} .
\end{align*}
$$

Since we aim for a supersymmetric extension of the extended Bargmann algebra, we again consider the algebra $\mathcal{G}(2,1)$ and thus expand the MC oneforms as

$$
\begin{array}{ll}
\tau \rightarrow \tau+\lambda^{2} m, & e^{a} \rightarrow \lambda e^{a}, \\
\omega \rightarrow \omega+\lambda^{2} s, & \omega^{a} \rightarrow \lambda \omega^{a},  \tag{5.31}\\
\psi^{+} \rightarrow \psi^{+}+\lambda^{2} \rho, & \psi^{-} \rightarrow \lambda \psi^{-} .
\end{array}
$$

Notably, the symmetric coset split also requires a non-trivial expansion of one of the fermionic one-forms, adding a MC one-form $\rho$ with associated charge $R$.

Inserting these expansions into the MC equations and then taking the step back to curvatures, we find the set

$$
\begin{align*}
R(J) & =\mathrm{d} \omega, \\
R(S) & =\mathrm{d} s+\frac{1}{2} \epsilon_{a b} \omega^{a} \wedge \omega^{b}, \\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\epsilon^{a}{ }_{b} \omega \wedge \omega^{b}, \\
R(H) & =\mathrm{d} \tau+\frac{1}{2} \bar{\psi}^{+} \gamma_{0} \wedge \psi^{+}, \\
R(M) & =\mathrm{d} m+\epsilon_{a b} \omega^{a} \wedge e^{b}+\bar{\psi}^{+} \gamma_{0} \wedge \rho+\frac{1}{2} \bar{\psi}^{-} \gamma_{0} \wedge \psi^{-},  \tag{5.32}\\
R\left(P^{a}\right) & =\mathrm{d} e^{a}-\epsilon^{a}{ }_{b} \omega^{b} \wedge \tau+\epsilon^{a}{ }_{b} \omega \wedge e^{b}-\bar{\psi}^{+} \gamma^{a} \wedge \psi^{-}, \\
R\left(Q_{+}\right) & =\mathrm{d} \psi^{+}+\frac{1}{2} \gamma_{0} \omega \wedge \psi^{+}, \\
R(R) & =\mathrm{d} \rho+\frac{1}{2} \gamma_{0} s \wedge \psi^{+}+\frac{1}{2} \gamma_{0} \omega \wedge \rho+\frac{1}{2} \gamma_{a} \omega^{a} \wedge \psi^{-}, \\
R\left(Q_{-}\right) & =\mathrm{d} \psi^{-}+\frac{1}{2} \gamma_{0} \omega \wedge \psi^{-}+\frac{1}{2} \gamma_{a} \omega^{a} \wedge \psi^{+},
\end{align*}
$$

and finally the 3D extended Bargmann superalgebra

$$
\begin{array}{ll}
{\left[H, G_{a}\right]=-\epsilon_{a}{ }^{b} P_{b},} & {\left[G_{a}, P_{b}\right]=\epsilon_{a b} M,} \\
{\left[J, P_{a}\right]=-\epsilon_{a}{ }^{b} P_{b},} & {\left[J, G_{a}\right]=-\epsilon_{a}^{b} G_{b},} \\
{\left[G_{a}, G_{b}\right]=\epsilon_{a b} S,} & {\left[J, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{0} Q_{ \pm},} \\
{[J, R]=-\frac{1}{2} \gamma_{0} R,} & {\left[S, Q_{+}\right]=-\frac{1}{2} \gamma_{0} R,} \\
{\left[G_{a}, Q_{+}\right]=-\frac{1}{2} \gamma_{a} Q_{-},} & {\left[G_{a}, Q_{-}\right]=-\frac{1}{2} \gamma_{a} R,} \\
\left\{Q_{+}, Q_{-}\right\}=-\gamma^{a} C^{-1} P_{a}, & \left\{Q_{+}, Q_{+}\right\}=-\gamma^{0} C^{-1} H, \\
\left\{Q_{-}, Q_{-}\right\}=-\gamma^{0} C^{-1} M, & \left\{Q_{+}, R\right\}=-\gamma^{0} C^{-1} M, \tag{5.33}
\end{array}
$$

which agrees with the algebra discovered in [15]. As mentioned above, the added charges no longer appear exclusively as central extensions to the algebra. Both $S$ and $R$ commute non-trivially with some of the generators, with only $M$ remaining central.

While the structure relations do not seem particularly clear ad hoc, they are illuminated by the derivation and follow a well-ordered system. This result is particularly important since it allows for a deeper understanding of the possible manipulations which lead to new and viable algebras.

As mentioned in 15 this theory allows for an action of Chern-Simons (CS) type given by

$$
\begin{align*}
S=\frac{k}{4 \pi} \int \mathrm{~d}^{3} x \epsilon^{\mu \nu \rho} & \left(e_{\mu}^{a} R_{\nu \rho}\left(G_{a}\right)-m_{\mu} R_{\nu \rho}(J)-\tau_{\mu} R_{\nu \rho}(S)\right.  \tag{5.34}\\
& \left.+\bar{\psi}_{\mu}^{+} R_{\nu \rho}(R)+\bar{\rho}_{\mu} R_{\nu \rho}\left(Q_{+}\right)+\bar{\psi}_{\mu}^{-} R_{\nu \rho}\left(Q_{-}\right)\right)
\end{align*}
$$

which unsurprisingly can be obtained by expanding the relativistic CS action. Not all terms in the expansion of the action will yield invariant actions (see 16]),
thus requiring some subtlety in the approach, as will be discussed for the $D=4$ case.

### 5.2.2 4D Extended Bargmann superalgebra

The calculations to obtain a 4D analog to the 3D extended Bargmann algebra are conveniently broadly similar to the $D=3$ case, since the $\gamma$-matrices have the same symmetry properties in both cases, as is outlined in appendix A. We therefore only have to include the now non-trivial commutator between two spatial rotations. Taking again the fermionic charges $Q_{ \pm}$as defined in eq. 5.25 we have the algebra given in eq. 5.1 and eq. 5.26 , adapting the range of the indices as appropriate.

We now decompose the charges as

$$
\begin{align*}
P_{A} & \rightarrow\left\{H, P_{a}\right\}, \\
M_{A B} & \rightarrow\left\{J_{a b}, G_{a}\right\}, \tag{5.35}
\end{align*}
$$

with $J_{a b}=M_{a b}, G_{a}=M_{0 a}$ and $H=P_{0}$, such that the commutation relations take the form

$$
\begin{array}{ll}
{\left[G_{a}, H\right]=P_{a},} & {\left[G_{a}, P_{b}\right]=\delta_{a b} H,} \\
{\left[J_{a b}, P_{c}\right]=2 \delta_{c[b} P_{a]},} & {\left[J_{a b}, G_{c}\right]=2 \delta_{c[b} G_{a]},} \\
{\left[G_{a}, G_{b}\right]=J_{a b},} & {\left[J_{a b}, J_{c d}\right]=4 \delta_{[a[c} J_{d] b]},} \\
{\left[J_{a b}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{a b} Q_{ \pm},} & {\left[G_{a}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{0 a} Q_{\mp},}  \tag{5.36}\\
\left\{Q_{+}, Q_{-}\right\}=-\gamma^{a} C^{-1} P_{a}, & \left\{Q_{ \pm}, Q_{ \pm}\right\}=-\gamma^{0} C^{-1} H .
\end{array}
$$

This allows for the same split as in section 5.2 .1 and the expansion to the algebra $\mathcal{G}(2,1)$. Using the same gauge fields assigned in eq. 5.15 and eq. 5.29 with the modification

$$
\begin{equation*}
J_{a b} \rightarrow \omega^{a b} \tag{5.37}
\end{equation*}
$$

we have the curvatures

$$
\begin{align*}
R\left(J^{a b}\right) & =\mathrm{d} \omega^{a b}+\omega^{a}{ }_{c} \wedge \omega^{c b}+\omega^{a} \wedge \omega^{b}, \\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\omega^{a}{ }_{b} \wedge \omega^{b}, \\
R(H) & =\mathrm{d} \tau+\delta_{a b} \omega^{a} \wedge e^{b}+\frac{1}{2} \bar{\psi}_{+} \gamma_{0} \wedge \psi_{+}+\frac{1}{2} \bar{\psi}_{-} \gamma_{0} \wedge \psi_{-},  \tag{5.38}\\
R\left(P^{a}\right) & =\mathrm{d} e^{a}+\omega^{a} \wedge \tau+\omega^{a}{ }_{b} \wedge e^{b}-\bar{\psi}_{+} \gamma^{a} \wedge \psi_{-}, \\
R\left(Q_{ \pm}\right) & =\mathrm{d} \psi_{ \pm}+\frac{1}{4} \gamma_{a b} \omega^{a b} \wedge \psi_{ \pm}+\frac{1}{2} \gamma_{0 a} \omega^{a} \wedge \psi_{\mp} .
\end{align*}
$$

Expanding and truncating the series as

$$
\begin{array}{ll}
\tau \rightarrow \tau+\lambda^{2} m, & e^{a} \rightarrow \lambda e^{a}, \\
\omega^{a b} \rightarrow \omega^{a b}+\lambda^{2} s^{a b}, & \omega^{a} \rightarrow \lambda \omega^{a}  \tag{5.39}\\
\psi^{+} \rightarrow \psi^{+}+\lambda^{2} \rho, & \psi^{-} \rightarrow \lambda \psi^{-},
\end{array}
$$

leads to the curvatures

$$
\begin{align*}
R\left(J^{a b}\right) & =\mathrm{d} \omega^{a b}+\omega^{a}{ }_{c} \wedge \omega^{c b} \\
R\left(S^{a b}\right) & =\mathrm{d} s^{a b}+2 \omega^{[a}{ }_{c} \wedge s^{c b]}+\omega^{a} \wedge \omega^{b} \\
R\left(G^{a}\right) & =\mathrm{d} \omega^{a}+\omega^{d}{ }_{c} \wedge \omega^{c}, \\
R(H) & =\mathrm{d} \tau+\frac{1}{2} \bar{\psi}_{+} \gamma_{0} \wedge \psi_{+}, \\
R(M) & =\mathrm{d} m+\delta_{a b} \omega^{a} \wedge e^{b}+\bar{\psi}_{+} \gamma_{0} \wedge \rho+\frac{1}{2} \bar{\psi}_{-} \gamma_{0} \wedge \psi_{-},  \tag{5.40}\\
R\left(P^{a}\right) & =\mathrm{d} e^{a}+\omega^{a} \wedge \tau+\omega^{a}{ }_{b} \wedge e^{b}-\bar{\psi}_{+} \gamma^{a} \wedge \psi_{-}, \\
R\left(Q_{+}\right) & =\mathrm{d} \psi_{+}+\frac{1}{4} \gamma_{a b} \omega^{a b} \wedge \psi_{+}, \\
R(R) & =\mathrm{d} \rho+\frac{1}{4} \gamma_{a b} \omega^{a b} \wedge \rho+\frac{1}{4} \gamma_{a b} s^{a b} \wedge \psi_{+}+\frac{1}{2} \gamma_{0 a} \omega^{a} \wedge \psi_{-}, \\
R\left(Q_{-}\right) & =\mathrm{d} \psi_{-}+\frac{1}{4} \gamma_{a b} \omega^{a b} \wedge \psi_{-}+\frac{1}{2} \gamma_{0 a} \omega^{a} \wedge \psi_{+} .
\end{align*}
$$

Thus, we can read off the algebra

$$
\begin{array}{ll}
{\left[G_{a}, H\right]=P_{a},} & {\left[G_{a}, P_{b}\right]=\delta_{a b} M,} \\
{\left[J_{a b}, P_{c}\right]=2 \delta_{c[b} P_{a]},} & {\left[J_{a b}, G_{c}\right]=2 \delta_{c[b} G_{a]},} \\
{\left[G_{a}, G_{b}\right]=S_{a b},} & {\left[J_{a b}, J_{c d}\right]=4 \delta_{[a[c} J_{d] b]},} \\
{\left[J_{a b}, S_{c d}\right]=4 \delta_{[a[c} S_{d] b]},} & {\left[J_{a b}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{a b} Q_{ \pm},} \\
{\left[J_{a b}, R\right]=-\frac{1}{2} \gamma_{a b} R,} & {\left[S_{a b}, Q_{+}\right]=-\frac{1}{2} \gamma_{a b} R,} \\
{\left[G_{a}, Q_{+}\right]=-\frac{1}{2} \gamma_{0 a} Q_{-},} & {\left[G_{a}, Q_{-}\right]=-\frac{1}{2} \gamma_{0 a} R,} \\
\left\{Q_{+}, Q_{-}\right\}=-\gamma^{a} C^{-1} P_{a}, & \left\{Q_{+}, Q_{+}\right\}=-\gamma^{0} C^{-1} H,  \tag{5.41}\\
\left\{Q_{-}, Q_{-}\right\}=-\gamma^{0} C^{-1} M, & \left\{Q_{+}, R\right\}=-\gamma^{0} C^{-1} M .
\end{array}
$$

In principle we would be done with the construction of the superalgebra, since this set of structure relations gives a consistent algebra $\mathcal{G}(2,1)$. However, this algebra does not in fact lead to an invariant action and can thus not be considered to be an algebra of 4D non-relativistic supergravity. This relates to the expansion of the Einstein Hilbert action and a consistent - i.e. invariant order-by-order - truncation thereof.

It suffices to consider the Einstein Hilbert Lagrangian density for 4D Poincaré in the form

$$
\begin{align*}
B & =\epsilon_{A B C D} R\left(M^{A B}\right) \wedge e^{C} \wedge e^{D} \\
& =-2 \epsilon_{0 a b c}\left(R\left(G^{a}\right) \wedge e^{b} \wedge e^{c}+R\left(J^{a b}\right) \wedge e^{c} \wedge \tau\right) \tag{5.42}
\end{align*}
$$

instead of the action itself to see where the $\mathcal{G}(2,1)$ expanded algebra fails. Expanding the terms in eq. 5.42 we observe that there are no even terms in the series. The order $\mathcal{O}(\lambda)$ term only receives a contribution from the second term and gives a consistent action for the $\mathcal{G}(0,1)$ expanded algebra (see 16 ).

This leads us to naively assume that the $\mathcal{O}\left(\lambda^{3}\right)$ terms give an action for $\mathcal{G}(2,1)$, but looking at the terms we find
$\stackrel{(3)}{B}=-2 \epsilon_{a b c}\left(R\left(G^{a}\right) \wedge e^{b} \wedge e^{c}+R\left(S^{a b}\right) \wedge e^{c} \wedge \tau+R\left(J^{a b}\right) \wedge e^{c} \wedge m+R\left(J^{a b}\right) \wedge{ }^{(3)} e^{c} \wedge \tau\right)$,
such that we need an additional gauge field $\stackrel{(3)}{e^{c}}$ to have an invariant action. Thus, we need to go one step further and consider the expanded algebra $\mathcal{G}(2,3)$ in order to have access to all the fields necessary.

The expansion only modifies the $V_{1}$ subspace and we add three additional MC one-forms defined by

$$
\begin{align*}
e^{a} & \rightarrow \lambda e^{a}+\lambda^{3} f^{a} \\
\omega^{a} & \rightarrow \lambda \omega^{a}+\lambda^{3} n^{a}  \tag{5.44}\\
\psi^{-} & \rightarrow \lambda \psi^{-}+\lambda^{3} \sigma
\end{align*}
$$

The curvatures in eq. 5.40 and the algebra in eq. 5.41 remain unchanged, since the third order one-forms cannot contribute to the lower order curvatures. We therefore merely add the corresponding curvatures

$$
\begin{align*}
R\left(N^{a}\right) & =\mathrm{d} n^{a}+\omega^{a}{ }_{b} \wedge n^{b}+s^{a}{ }_{b} \wedge \omega^{b}, \\
R\left(F^{a}\right) & =\mathrm{d} f^{a}+\omega^{a} \wedge m+n^{a} \wedge \tau+\omega^{a}{ }_{b} \wedge f^{b}+s^{a}{ }_{b} \wedge e^{b}-\bar{\psi}_{+} \gamma^{a} \wedge \sigma-\bar{\rho} \gamma^{a} \wedge \psi_{-}, \\
R(\Sigma) & =\mathrm{d} \sigma+\frac{1}{4} \gamma_{a b} \omega^{a b} \wedge \sigma+\frac{1}{4} \gamma_{a b} s^{a b} \wedge \psi_{-}+\frac{1}{2} \gamma_{0 a} \omega^{a} \wedge \rho+\frac{1}{2} \gamma_{0 a} n^{a} \wedge \psi_{+}, \tag{5.45}
\end{align*}
$$

to the ones appearing in the $\mathcal{G}(2,1)$ expansion, leading us to extend the algebra with the commutation relations

$$
\begin{array}{ll}
{\left[N_{a}, H\right]=F_{a},} & {\left[G_{a}, M\right]=F_{a},} \\
{\left[S_{a b}, P_{c}\right]=2 \delta_{c[b} F_{a]},} & {\left[J_{a b}, F_{c}\right]=2 \delta_{c[b} F_{a]},} \\
{\left[S_{a b}, G_{c}\right]=2 \delta_{c[b} N_{a]},} & {\left[J_{a b}, N_{c}\right]=2 \delta_{c[b} N_{a]},} \\
{\left[S_{a b}, Q_{-}\right]=-\frac{1}{2} \gamma_{a b} \Sigma,} & {\left[J_{a b}, \Sigma\right]=-\frac{1}{2} \gamma_{a b} \Sigma,} \\
{\left[N_{a}, Q_{+}\right]=-\frac{1}{2} \gamma_{0 a} \Sigma,} & {\left[G_{a}, R\right]=-\frac{1}{2} \gamma_{0 a} \Sigma,}  \tag{5.46}\\
\left\{R, Q_{-}\right\}=-\gamma^{a} C^{-1} F_{a}, & \left\{Q_{+}, \Sigma\right\}=-\gamma^{a} C^{-1} F_{a} .
\end{array}
$$

where we added new charges labeled as the capitalized version of their associated gauge fields.

Finally, we can write down the full Lagrangian density for the extended algebra $\mathcal{G}(2,3)$. The fermionic part derives from the relativistic Lagrange density given by (see 21])

$$
\begin{align*}
B^{\prime}= & 2 \mathrm{i} \bar{\psi}_{+} \gamma_{A} \gamma_{5} \wedge e^{A} \wedge R\left(Q_{+}\right)+2 \mathrm{i} \bar{\psi}_{-} \gamma_{A} \gamma_{5} \wedge e^{A} \wedge R\left(Q_{-}\right) \\
= & 2 \mathrm{i} \bar{\psi}+\gamma_{0} \gamma_{5} \wedge \tau \wedge R\left(Q_{+}\right)+2 \mathrm{i} \bar{\psi}_{+} \gamma_{a} \gamma_{5} \wedge e^{a} \wedge R\left(Q_{+}\right)  \tag{5.47}\\
& +2 \mathrm{i} \bar{\psi}_{-} \gamma_{0} \gamma_{5} \wedge \tau \wedge R\left(Q_{-}\right)+2 \mathrm{i} \bar{\psi}_{-} \gamma_{a} \gamma_{5} \wedge e^{a} \wedge R\left(Q_{-}\right),
\end{align*}
$$

which, combined with eq. 5.43 gives the Lagrangian density

$$
\begin{align*}
& \stackrel{(3)}{B}=-2 \epsilon_{a b c}\left(R\left(G^{a}\right) \wedge e^{b} \wedge e^{c}+R\left(S^{a b}\right) \wedge e^{c} \wedge \tau\right. \\
& \left.\quad+R\left(J^{a b}\right) \wedge e^{c} \wedge m+R\left(J^{a b}\right) \wedge f^{c} \wedge \tau\right) \\
& \quad \begin{array}{l}
\stackrel{(3)}{B^{\prime}}=2 \mathrm{i}\left(\bar{\rho} \gamma_{a} \gamma_{5} \wedge e^{a} \wedge R\left(Q_{+}\right)+\bar{\psi}_{+} \gamma_{a} \gamma_{5} \wedge e^{a} \wedge R(R)\right. \\
\left.\quad+\bar{\psi}_{+} \gamma_{a} \gamma_{5} \wedge f^{a} \wedge R\left(Q_{+}\right)+\bar{\psi}_{-} \gamma_{a} \gamma_{5} \wedge e^{a} \wedge R\left(Q_{-}\right)\right) .
\end{array} \tag{5.48}
\end{align*}
$$

The corresponding action is invariant under the whole symmetry group and constitutes a supergravity action in the first order formulation. Going to a second order formulation from here poses a non-trivial challenge due to the dependence of the spin connection on the fermionic gauge fields.

### 5.2.3 4D Extended String Bargmann Superalgebra

This section expands upon the work in 18 by adding the fermionic sector to the 4D Extended String Bargmann algebra constructed there. Non-relativistic string algebras give a starting point for the discussion of non-relativistic string theory.

The four-dimensional Poincaré algebra is split into a part longitudinal to the string with index $A=0,1$ and a transversal part $a=2,3$, which necessitates a modification to the fermionic generators $Q_{ \pm}$as defined in eq.5.25. Thus we take $Q_{ \pm}$to be

$$
\begin{equation*}
Q_{ \pm}=\frac{1}{\sqrt{2}}\left(Q_{1} \pm \gamma_{01} Q_{2}\right) \tag{5.49}
\end{equation*}
$$

which leads to the initial commutation relations

$$
\begin{array}{ll}
{\left[P_{\hat{A}}, P_{\hat{B}}\right]=0,} & {\left[M_{\hat{A} \hat{B}}, P_{\hat{C}}\right]=2 \eta_{\hat{C}[\hat{B}} P_{\hat{A}]},} \\
{\left[M_{\hat{A} \hat{B}}, M_{\hat{C} \hat{D}}\right]=4 \eta_{[\hat{A} \hat{C}} M_{\hat{D} \mid \hat{B}]},} & {\left[M_{A B}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{A B} Q_{ \pm},} \\
{\left[M_{a b}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{a b} Q_{ \pm},} & {\left[M_{A a}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{A A} Q_{\mp},}  \tag{5.50}\\
\left\{Q_{+}, Q_{-}\right\}=-\gamma^{a} C^{-1} P_{a}, & \left\{Q_{ \pm}, Q_{ \pm}\right\}=-\gamma^{A} C^{-1} P_{A},
\end{array}
$$

where we introduced an index $\hat{A}=0,1,2,3$ in order to differentiate from the longitudinal index. We now decompose the generators according to

$$
\begin{align*}
P_{A} & \rightarrow\left\{H_{A}, P_{a}\right\} \\
M_{\hat{A} \hat{B}} & \rightarrow\left\{J, G_{A a}, M\right\}, \tag{5.51}
\end{align*}
$$

with $J=M_{23}, M=M_{01}, G_{A a}=M_{A a}$ and $H_{A}=P_{A}$. The algebra then takes the form

$$
\begin{align*}
& {\left[J, P_{a}\right]=-\epsilon_{a}^{b} P_{b}} \\
& {\left[G_{A a}, H_{B}\right]=-\eta_{A B} P_{a}} \\
& {\left[J, G_{A a}\right]=-\epsilon_{a}^{b} G_{A b}} \\
& {\left[G_{A a}, G_{B b}\right]=-\eta_{A B} \epsilon_{a b} J-\delta_{a b} \epsilon_{A B} M} \\
& {\left[J, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{23} Q_{ \pm}} \\
& \left\{Q_{ \pm}, Q_{ \pm}\right\}=-\gamma^{A} C^{-1} H_{A} \tag{5.52}
\end{align*}
$$

$$
\begin{aligned}
& {\left[M, H_{A}\right]=\epsilon_{A}^{B} H_{B}} \\
& {\left[G_{A a}, P_{b}\right]=\delta_{a b} H_{A}} \\
& {\left[M, G_{A a}\right]=\epsilon_{A}^{B} G_{B a}} \\
& {\left[M, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{01} Q_{ \pm}} \\
& {\left[G_{A a}, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{A a} Q_{\mp}} \\
& \left\{Q_{+}, Q_{-}\right\}=-\gamma^{a} C^{-1} P_{a}
\end{aligned}
$$

which splits into $V_{0}$ and $V_{1}$ according to

$$
\begin{equation*}
V_{0}=\left\{J, H_{A}, M, Q_{+}\right\}, \quad V_{1}=\left\{P_{a}, G_{A a}, Q_{-}\right\} \tag{5.53}
\end{equation*}
$$

as laid out in chapter 4 .
Assigning gauge fields

$$
\begin{array}{ll}
J \rightarrow \omega, & G_{A a} \rightarrow \omega^{A a} \\
M \rightarrow m, & H_{A} \rightarrow \tau^{A}  \tag{5.54}\\
P_{a} \rightarrow e^{a}, & Q_{ \pm} \rightarrow \psi^{ \pm}
\end{array}
$$

we can write down the curvatures

$$
\begin{align*}
R(J) & =\mathrm{d} \omega-\frac{1}{2} \eta_{A B} \epsilon_{a b} \omega^{A a} \wedge \omega^{B b}, \\
R(M) & =\mathrm{d} m-\frac{1}{2} \delta_{a b} \epsilon_{A B} \omega^{A a} \wedge \omega^{B b}, \\
R\left(G^{A a}\right) & =\mathrm{d} \omega^{A a}+\epsilon^{a}{ }_{b} \omega \wedge \omega^{A b}-\epsilon^{A}{ }_{B} m \wedge \omega^{B a}, \\
R\left(H^{A}\right) & =\mathrm{d} \tau^{A}-\epsilon^{A}{ }_{B} m \wedge \tau^{B}+\delta_{a b} \omega^{A a} \wedge e^{b}-\frac{1}{2} \bar{\psi}^{+} \gamma^{A} \wedge \psi^{+}-\frac{1}{2} \bar{\psi}^{-} \gamma^{A} \wedge \psi_{-}, \\
R\left(P^{a}\right) & =\mathrm{d} e^{a}+\epsilon^{a}{ }_{b} \omega \wedge e^{b}-\eta_{A B} \omega^{A a} \wedge \tau_{B}-\bar{\psi}^{+} \gamma^{a} \wedge \psi^{-}, \\
R\left(Q_{ \pm}\right) & =\mathrm{d} \psi^{ \pm}+\frac{1}{2} \gamma_{01} m \wedge \psi^{ \pm}+\frac{1}{2} \gamma_{23} \omega \wedge \psi^{ \pm}+\frac{1}{2} \gamma_{A a} \omega^{A a} \wedge \psi^{\mp} . \tag{5.55}
\end{align*}
$$

The next step is now to expand the MC one-forms and truncate the series to end up with the expanded algebra $\mathcal{G}(2,1)$, which coincides with the expansion order in [18]. We have

$$
\begin{array}{ll}
\omega_{\mu} \rightarrow \omega_{\mu}+\lambda^{2} s_{\mu}, & e_{\mu}^{a} \rightarrow \lambda e_{\mu}^{a}, \\
m_{\mu} \rightarrow m_{\mu}+\lambda^{2} n_{\mu}, & \omega_{\mu}^{A a} \rightarrow \lambda \omega_{\mu}^{A a}, \\
\tau_{\mu}^{A} \rightarrow \tau_{\mu}^{A}+\lambda^{2} z_{\mu}^{A}, & \psi_{\mu}^{-} \rightarrow \lambda \psi_{\mu}^{-},  \tag{5.56}\\
\psi_{\mu}^{+} \rightarrow \psi_{\mu}^{+}+\lambda^{2} \rho_{\mu}, &
\end{array}
$$

where as usual a number of new one-forms was defined via the formalism.
Going through the effort of inserting the truncated one-forms into the MC equations and extracting the new set order by order yields for the curvatures

$$
\begin{align*}
R(J)= & \mathrm{d} \omega \\
R(S)= & \mathrm{d} s-\frac{1}{2} \eta_{A B} \epsilon_{a b} \omega^{A a} \wedge \omega^{B b}, \\
R(M)= & \mathrm{d} m, \\
R(N)= & \mathrm{d} n-\frac{1}{2} \delta_{a b} \epsilon_{A B} \omega^{A a} \wedge \omega^{B b}, \\
R\left(G^{A a}\right)= & \mathrm{d} \omega^{A a}+\epsilon^{a}{ }_{b} \omega \wedge \omega^{A b}-\epsilon^{A}{ }_{B} m \wedge \omega^{B a}, \\
R\left(H^{A}\right)= & \mathrm{d} \tau^{A}-\epsilon^{A}{ }_{B} m \wedge \tau^{B}-\frac{1}{2} \bar{\psi}^{+} \gamma^{A} \wedge \psi^{+}, \\
R\left(Z^{A}\right)= & \mathrm{d} z^{A}-\epsilon^{A}{ }_{B} n \wedge \tau^{B}-\epsilon^{A}{ }_{B} m \wedge z^{B}+\delta_{a b} \omega^{A a} \wedge e^{b}  \tag{5.57}\\
& -\bar{\psi}^{+} \gamma^{A} \wedge \rho-\frac{1}{2} \bar{\psi}^{-} \gamma^{A} \wedge \psi_{-}, \\
R\left(P^{a}\right)= & \mathrm{d} e^{a}+\epsilon^{a}{ }_{b} \omega \wedge e^{b}-\eta_{A B} \omega^{A a} \wedge \tau_{B}-\bar{\psi}^{+} \gamma^{a} \wedge \psi^{-}, \\
R\left(Q_{+}\right)= & \mathrm{d} \psi^{+}+\frac{1}{2} \gamma_{01} m \wedge \psi^{+}+\frac{1}{2} \gamma_{23} \omega \wedge \psi^{+}, \\
R(R)= & \mathrm{d} \rho+\frac{1}{2} \gamma_{01} n \wedge \psi^{+}+\frac{1}{2} \gamma_{01} m \wedge \rho+\frac{1}{2} \gamma_{23} s \wedge \psi^{+} \\
& +\frac{1}{2} \gamma_{23} \omega \wedge \rho+\frac{1}{2} \gamma_{A a} \omega^{A a} \wedge \psi^{-}, \\
R\left(Q_{-}\right)= & \mathrm{d} \psi^{-}+\frac{1}{2} \gamma_{01} m \wedge \psi^{-}+\frac{1}{2} \gamma_{23} \omega \wedge \psi^{-}+\frac{1}{2} \gamma_{A a} \omega^{A a} \wedge \psi^{-} .
\end{align*}
$$

Finally, we can read off the structure constants and write down the consistent algebra for 4D Extended String Bargmann Superalgebra as

$$
\begin{array}{ll}
{\left[J, P_{a}\right]=-\epsilon_{a}{ }^{b} P_{b},} & {\left[M, H_{A}\right]=\epsilon_{A}{ }^{B} H_{B},} \\
{\left[N, H_{A}\right]=\epsilon_{A}{ }^{B} Z_{B},} & {\left[M, Z_{A}\right]=\epsilon_{A}{ }^{B} Z_{B},} \\
{\left[G_{A a}, H_{B}\right]=-\eta_{A B} P_{a},} & {\left[G_{A a}, P_{b}\right]=\delta_{a b} Z_{A},} \\
{\left[J, G_{A a}\right]=-\epsilon_{a}^{b} G_{A b},} & {\left[M, G_{A a}\right]=\epsilon_{A}{ }^{B} G_{B a},} \\
{\left[G_{A a}, G_{B b}\right]=-\eta_{A B} \epsilon_{a b} S-\delta_{a b} \epsilon_{A B} N,} & {\left[M, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{01} Q_{ \pm},} \\
{[M, R]=-\frac{1}{2} \gamma_{01} R,} & {\left[N, Q_{+}\right]=-\frac{1}{2} \gamma_{01} R,} \\
{\left[J, Q_{ \pm}\right]=-\frac{1}{2} \gamma_{23} Q_{ \pm},} & {[J, R]=-\frac{1}{2} \gamma_{23} R,} \\
{\left[S, Q_{+}\right]=-\frac{1}{2} \gamma_{23} R,} & {\left[G_{A a}, Q_{+}\right]=-\frac{1}{2} \gamma_{A a} Q_{-},} \\
{\left[G_{A a}, Q_{-}\right]=-\frac{1}{2} \gamma_{A a} R,} & \left\{Q_{+}, Q_{+}\right\}=-\gamma^{A} C^{-1} H_{A}, \\
\left\{Q_{-}, Q_{-}\right\}=-\gamma^{A} C^{-1} Z_{A}, & \left\{Q_{+}, R\right\}=-\gamma^{A} C^{-1} Z_{A}, \\
\left\{Q_{+}, Q_{-}\right\}=-\gamma^{a} C^{-1} P_{a} . & \tag{5.58}
\end{array}
$$

This concludes our discussion of non-relativistic expansions for algebras of supergravity via the Lie algebra expansion method.

## Chapter 6

## Further Work

The formalism built up in chapters 2 2-4 gives a powerful and versatile tool to derive novel theories of gravity from some underlying, known algebra. Building up from the results derived in 16 and reproduced in section 5.1 we were able to derive the 3D extendend Bargmann superalgebra in 15 via the method of Lie algebra expansion, giving a better understanding of the structure of the algebra - especially the additional generators - and its role in the non-relativistic expansion. These results coincide with the independent derivation carried out in 19 .

Further, the theory allowed for the derivation of a novel $D=4$ algebra and its first order action for non-relativistic supergravity. Intriguingly, the lowest order algebra in the expansion $\mathcal{G}(2,1)$ does not have a corresponding consistent action, since higher order terms appear in its expansion. Thus, the simplest $D=4$ non-relativistic superalgebra is given by $\mathcal{G}(2,3)$, similar to the bosonic case noted in 16 .

A particular strength of the Lie algebra expansion - as compared to previous methods of algebra extension and contraction - is its ability to generalize results. Since any starting algebra leads to an infinite series of possible truncations following a simple pattern, we always find a whole family of resulting algebras and actions. These can then be classified according to their truncation order, as was done in 20 for non-relativistic 4D p-brane superalgebras, extending beyond the lowest order cases considered here. Additional contributions in this area were made more recently in [31] via a so-called 'semi-group expansion' procedure.

The possible applications of Lie algebra expansions in the study of gravity are not limited to the non-relativistic regime however. A different choice for the split of the algebra $\mathcal{G}$ as by eq. 4.12 changes the physical interpretation of the expansion. Splitting to a subalgebra

$$
\begin{equation*}
V_{0}=\left\{J_{a b}, H, \ldots\right\}, \tag{6.1}
\end{equation*}
$$

containing spatial rotations and time translations corresponds physically to carrying out a non-relativistic expansion. However, taking

$$
\begin{equation*}
V_{0}=\left\{J_{a b}, P_{a}, \ldots\right\} \tag{6.2}
\end{equation*}
$$

to be the subalgebra leads to an ultra-relativistic algebra, underlying the socalled 'Carroll gravity' (see $[29]$ ), which was done for the bosonic case in 16].


Table 6.1: A successful second order expansion would apply to the auxiliary fields of a gauged relativistic theory, leading to an off-shell formulation for nonrelativistic gravity which cannot be found by gauging a non-relativistic algebra.

This split cannot be straightforwardly extended to the supersymmetric sector however, since our definition for the supercharges in eq. 5.25 does not comply with the necessary symmetric coset structure as by eq. 4.18 . One approach for a Carroll superalgebra presented in 32 takes an algebra contraction with the same scaling for both $Q_{+}$and $Q_{-}$, which is similarly not compatible with our methods. It would be interesting to see, if the Lie algebra expansion can be applied in some way to derive ultra-relativistic superalgebras. A promising starting point is given in 33] which describes a link between $p$-brane Galilei algebras and ( $D-p-2$ )-brane Carroll algebras for the strictly bosonic case. This connection might allow for an extension to the supersymmetric case via the Galilei superalgebras considered in this work.

Thinking back to the diagram in table 1.1 there remains an open question, namely if the expansion could be applied to relativistic gravity directly, expanding the gauged fields in a second order formulation. The work in 14 successfully applied a non-relativistic limiting procedure via Lie algebra contraction for 3D $\mathcal{N}=2$ supergravity, which leads us to believe that a '2nd-Order Expansion' could also be possible. The advantage would be the extension of the Lie algebra expansion method to non-relativistic gravity theories which cannot be derived via a gauging process. In particular, off-shell formulations generally require auxiliary fields which need to be added by hand. For all but the simplest theories this is not a straightforward process as was argued in 13.

Expanding the known auxiliary fields of the relativistic theory directly would be immensely helpful. The main obstacle here is the expansion of the dependent gauge fields, which similar to 14 requires careful application of the relativistic constraints to avoid divergences upon taking a singular limit of the expansion parameter. The diagram in table 6.1 shows how the hypothetical ' 2 nd-Order Expansion' would fit into the work considered here.

Some recent work in this area 34,35 managed to construct non-relativistic $c^{-2}$ expansions of the metric in a second order formulation directly. However, their methods did not involve the Lie algebra expansion method as in 17 and thus cannot easily be generalized to higher orders and different expansions. Nevertheless, their results give new opportunities to study the structure of nonrelativistic gravity by comparing their brute force second order construction to the first order results derived via Lie algebra expansion.

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## Appendix A

## Clifford Algebra

Since this work is mainly concerned with supersymmetry, we naturally have to deal with a number of fermionic fields, which require the structure of a Clifford algebra to efficiently describe them. We are primarily interested in some properties of its elements - specifically their symmetries - and will not dwell on the practical manipulations. This chapter borrows heavily from 21 and aims to deliver the relevant results in a condensed manner.

## A. 1 Elements of the Clifford Algebra

We introduce $2^{[D / 2]}$-dimensiona ${ }^{1}$ matrices $\gamma^{\mu}$ as the generators of the $D$-dimensional Clifford algebra with the defining property

$$
\begin{equation*}
\gamma^{\mu} \gamma^{\nu}+\gamma^{\nu} \gamma^{\mu}=2 \eta^{\mu \nu} \mathbb{1} \tag{A.1}
\end{equation*}
$$

In general these matrices - and as a consequence the algebra as a whole are complex valued. However, there is a special case of particular interest to us, when the $\gamma$-matrices are real valued, since we can then take the spinors on which the algebra acts to be real as well, leading to Majonara spinors.

The equation A. 1 is sufficient (and necessary) to allow for a spinor field to satisfy a first order equation of motion and simultaneously the second order wave equation. However, for the theory to remain covariant under Lorentz transformations we additionally require one of the generators - we choose $\gamma^{0}-$ to be anti-hermitian. This property is best summarized as

$$
\begin{equation*}
\gamma^{\mu \dagger}=\gamma^{0} \gamma^{\mu} \gamma^{0} \tag{A.2}
\end{equation*}
$$

From this we can then build up the Clifford algebra via matrix multiplication of the generators, denoting a general element of the algebra as

$$
\begin{equation*}
\gamma^{\mu_{1} \ldots \mu_{r}}=\gamma^{\left[\mu_{1}\right.} \cdots \gamma^{\left.\mu_{r}\right]} \tag{A.3}
\end{equation*}
$$

with $r$ the rank of the element. There are only anti-symmetric elements for $r>1$, since any symmetric component simplifies to a lower rank element via eq. A.1.

[^18]| $D(\bmod 8)$ | $t_{r}=-1$ | $t_{r}=1$ |
| :---: | :---: | :---: |
| 0 | 0,1 | 2,3 |
| 1 | 0,1 | 2,3 |
| 2 | 1,2 | 0,3 |
| 3 | 1,2 | 0,3 |
| 4 | 1,2 | 0,3 |
| 5 | 2,3 | 0,1 |
| 6 | 0,3 | 1,2 |
| 7 | 0,3 | 1,2 |

Table A.1: Symmetries of the $\gamma$-matrices. The entries correspond to the rank $r$ for which $t_{r}= \pm 1$ for any given dimension $D$. The choice for even dimensions is ambiguous, however we only included the ones convenient for supersymmetry. For a full list see 21 .

## A. 2 Symmetries of $\gamma$-matrices

All elements of the Clifford algebra can be classified as either symmetric or antisymmetric with respect to a unitary matrix $C$, called the 'charge conjugation matrix', which makes this property manifest. The symmetry of any particular matrix depends solely on its rank, which allows us to discuss them as a class of Clifford elements with rank $r$ which we call $\Gamma^{(r)}$.

We write the (anti-)symmetry condition as

$$
\begin{equation*}
\left(C \Gamma^{(r)}\right)^{T}=-t_{r} C \Gamma^{(r)}, \quad t_{r}= \pm 1, \tag{A.4}
\end{equation*}
$$

which defines the coefficients $t_{r}$. They contain all the information about the symmetries of the $\gamma$-matrices for a given Clifford algebra and depend on the spacetime dimension $D$.

In fact, only two coefficients $-t_{0}$ and $t_{1}$ - are necessary to determine all the others, since they are related to the higher ranks via $t_{2}=-t_{0}, t_{3}=-t_{1}$ and then $t_{r+4}=t_{r}$. Since they are characteristic for any given spacetime dimension, we can list them, as is done in table A. 1

The main way in which theses symmetries - and thus a spacetime dependence - enter into our calculations are via the 'Majorana flip relations' given in the simplest form by

$$
\begin{equation*}
\bar{\lambda} \gamma_{\mu_{1} \ldots \mu_{r}} \chi=t_{r} \bar{\chi} \gamma_{\mu_{1} \ldots \mu_{r}} \lambda, \tag{A.5}
\end{equation*}
$$

and the more general case as

$$
\begin{equation*}
\bar{\lambda} \Gamma^{\left(r_{1}\right)} \Gamma^{\left(r_{2}\right)} \cdots \Gamma^{\left(r_{p}\right)} \chi=t_{0}^{p-1} t_{r_{1}} t_{r_{2}} \cdots t_{r_{p}} \bar{\chi} \Gamma^{\left(r_{p}\right)} \cdots \Gamma^{\left(r_{2}\right)} \Gamma^{\left(r_{1}\right)} \lambda, \tag{A.6}
\end{equation*}
$$

where barred spinors are defined via the 'Majorana conjugate' to be $\bar{\lambda}=\lambda^{T} C$.

## A. 3 Spinor Indices

The last concept we have to mention in this chapter are the spinor indices. While we try to avoid them in this work, it is often necessary to refer back to them in order to compare our equations with the literature.

For this purpose we define the position of the spinor indices first for the spinors and conjugate spinors themselves. We write $\lambda_{\alpha}$ for the components of $\lambda$ and $\lambda^{\alpha}$ for the components of $\bar{\lambda}$ and set a convention for the summation over the indices known as the 'northwest-southeast spinor convention'. This refers to the position the indices have to be placed in relative to each other, in order to be omitted.

As an example we have the bilinear

$$
\begin{equation*}
\bar{\lambda} \chi=\lambda^{\alpha} \chi_{\alpha} . \tag{A.7}
\end{equation*}
$$

The index placement for the various matrices follows naturally, since we have

$$
\begin{equation*}
\bar{\lambda} \gamma_{\mu} \chi=\lambda^{\alpha}\left(\gamma_{\mu}\right)_{\alpha}{ }^{\beta} \chi_{\beta}, \tag{A.8}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\bar{\lambda}^{T}=C^{T} \lambda \quad \Longleftrightarrow \quad \lambda^{\alpha}=\mathcal{C}^{\alpha \beta} \lambda_{\beta} \tag{A.9}
\end{equation*}
$$

The latter relation shows, that the charge conjugation matrix $\mathcal{C}^{\alpha \beta}$ can be thought of as a raising operator or correspondingly $\mathcal{C}_{\alpha \beta}$ as the lowering operator for the spinor indices. This leads to a property for summed over indices that involves the symmetry coefficients from above, namely

$$
\begin{equation*}
\lambda^{\alpha} \chi_{\alpha}=-t_{0} \lambda_{\alpha} \chi^{\alpha} \tag{A.10}
\end{equation*}
$$

Lastly, we have a relation that is particularly common in the context of the supercharge anti-commutator relations,

$$
\begin{equation*}
\left\{Q_{\alpha}, Q_{\beta}\right\}=-\frac{1}{2}\left(\gamma^{a}\right)_{\alpha \beta} P_{a} . \tag{A.11}
\end{equation*}
$$

It is clear that we cannot simply omit the indices, since the indices for the $\gamma$ matrix are not placed in accordance with our convention. However, we can use the lowering operator $\mathcal{C}_{\alpha \beta}$, which corresponds to the matrix $C^{-1}$ to write

$$
\begin{equation*}
\left(\gamma_{\mu}\right)_{\alpha \beta}=\left(\gamma_{\mu}\right)_{\alpha}^{\delta} \mathcal{C}_{\delta \beta} \tag{A.12}
\end{equation*}
$$

and since all indices are in their proper position as defined by our convention, we have for eq. A. 11 the form

$$
\begin{equation*}
\{Q, Q\}=-\frac{1}{2} \gamma^{a} C^{-1} P_{a} \tag{A.13}
\end{equation*}
$$

We will generally write relations such as this using the second version in order to avoid any cumbersome index manipulations.

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[^0]:    ${ }^{1}$ Due to the nature of the constraints we need to impose, the formalism might have unwanted constraints on the final theory of gravity we derive - in particular putting the theory on-shell or limiting the possible spacetime curvatures.

[^1]:    ${ }^{1}$ Global symmetries have no spacetime dependence in their parameters - i.e. they act the same in all points of spacetime.

[^2]:    ${ }^{2}$ It is implicitly assumed that repeated indices - one upper, one lower - are summed over

[^3]:    ${ }^{3}$ In the following this dependence will be implicitly assumed for the parameters.

[^4]:    ${ }^{4}$ Scalars are considered to be 0 -forms to complete the pattern.

[^5]:    ${ }^{5}$ For the r.h.s of eq. 2.31 this gives a non-trivial condition for the structure constants.

[^6]:    ${ }^{1}$ For general tensor fields there is an additional, matrix-valued contribution for the Lorentz transformations, which we omit in the following discussion since it remains unchanged in the local formalism.

[^7]:    ${ }^{2}$ Since translations do not commute to translations we take the corresponding structure constant to be zero, so $f_{B C}{ }^{A}=0$.
    ${ }^{3}$ Making use of the differential form formalism introduced in section 2.4

[^8]:    ${ }^{4}$ For generic $\mathcal{N}$ these theories are called ' $\mathcal{N}$-extended supersymmetry'.

[^9]:    ${ }^{5}$ The work in 22 expands further on this idea and the relation between massive matter and the Bargmann algebra.
    ${ }^{6}$ Also known as 'co-dimension 1 foliation'.

[^10]:    ${ }^{7}$ Taking the $c \rightarrow \infty$ limit for the Minkowski metric $\eta_{\mu \nu}$ and its inverse leads to two different degenerate metrics.

[^11]:    ${ }^{1}$ These types of algebras are grouped under the term 'kinematic Lie algebras' and extensively studied in 23.24 for the bosonic case and $25-27$ for the supersymmetric extensions.

[^12]:    ${ }^{2}$ Different re-definitions as in 14, taking $P_{0} \rightarrow \lambda M+\lambda^{-1} H, P_{i} \rightarrow P_{i}$ and $M \rightarrow-\lambda M+$ $\lambda^{-1} H$ are also available. They do not change the derivation of the algebra in the bosonic case considered here but can be useful for the supersymmetric extension.

[^13]:    ${ }^{3}$ We follow the notation laid out in 17 with group elements $g$ and parameters $g^{i}$, so some care has to be taken in properly distinguishing them. As a consequence the generators $T_{A}$ from previous chapters are now referred to as $X_{i}$.

[^14]:    ${ }^{4}$ The choice of this split significantly impacts the possible consistent truncations, but for the general outline we make no restrictions here.

[^15]:    ${ }^{5}$ In $V_{0}$ the only relevant structure constants are $c_{i_{0} j_{0}}{ }^{k_{0}}$ and $c_{i_{1} j_{1}}{ }^{k_{0}}$, such that the corresponding $g^{i_{p}}$ enter pairwise. Thus, only even power terms appear in the expansion. The same argument applies analogously for $V_{1}$.

[^16]:    ${ }^{1}$ This algebra was originally discoverd in 30 without the context of Lie algebra expansion.
    ${ }^{2}$ The first choice is motivated by simplifying the fermionic sector once we consider the supersymmetric extension.

[^17]:    ${ }^{3}$ Since the writing of this text, these algebras were independently discovered in 20 .
    ${ }^{4}$ We employ the spinor algebra notation defined in appendix A

[^18]:    ${ }^{1}[x]$ is the integer part of $x$.

