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"Fusion Frames and Operators"

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At first I want to express my gratitude to my supervisor Peter Balazs. I first met Peter when he held a lecture course on Frame Theory at the University of Vienna, which very quickly picked up my interest on frames. After the exam for this course (which I remember to be the longest but also most funny oral exam I have ever had), I decided to attend the seminar on Frame Theory continuing the basic course, where I first came in touch with fusion frames. Peter gave me three papers about fusion frames, which I should later present via giving a talk at the ARI (Acoustics Research Institute) and he smilingly assigned to me the mission to find "the error" occurring in the first paper. Eagerly I worked through the notes and found a few minor errors, but I missed the essential one. Nevertheless, searching the paper for errors made my first dive into the theory of fusion frames a comparably extensive one, and in that process my interest for frames and fusion frames grew bigger and bigger. After giving my talk about the three papers on fusion frames at the ARI, I told Peter that I want to learn more about fusion frames and asked him, if I can write my master's thesis about this topic and if he would be my supervisor. As it appeared, Peter was prepared for my request and wrote down three operator identities for fusion frame systems on the blackboard, which he suspected to be true. These operator identities have been the starting point of this thesis and gave rise to many other ideas appearing in this work. In the process of writing this thesis, Peter always asked the right questions and gave the right clues, whenever I was stumped. Thank you for your questions, clues and also your patience, Peter!

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Abstract

Frames are families of vectors in a (separable) Hilbert space $\mathcal{H}$, which generalize the notion of an orthonormal basis and provide the possibility to represent and reconstruct vectors in $\mathcal{H}$ in a non-unique, redundant and stable way. These properties of frames are desired for a vast number of applications, e.g. signal and image processing, wireless communications, data compression or sampling theory. However, some applications (– for instance, to handle huge amounts of data coming from numerical computations; or the applications themselves –) additionally require distributed processing techniques. This naturally leads to the concept of fusion frames, which is the central topic of this work.

In this thesis, at first we motivate and recall the basic concepts of frame theory and mention some of its applications. Then we investigate Hilbert direct sums – since they are the representation spaces for fusion frames – and collect some results about these objects and component preserving operators between them, which – at least partly – have not been published in literature yet. After these two preparing chapters we present the basic notions and some of the fundamental results for the theory of fusion frames, before we prove some new results for fusion frames and fusion frame systems in terms of the above mentioned component preserving operators and operator identities involving these. After that we discuss the aspect of duality for fusion frames, which naturally leads to the study of general bounded operators between Hilbert direct sums. These operators will be represented by infinite matrices of operators, which not only helps to understand these objects better in an intuitive way, but also enables us to prove some convenient new results concerning boundedness and compactness in terms of the operators occurring in the matrix representations. Finally, we apply the latter results to fusion frames and thus are able to present the concept of fusion frame multipliers in a more general set up than it has been discussed in literature before. This not only allows for a richer theory, but also enables us to generalize more of the key results for frame multipliers.

Abstract (German)

Frames sind Familien von Vektoren in einem (separablen) Hilbertraum $\mathcal{H}$, die den Begriff der Orthonormalbasis verallgemeinern und es ermöglichen, Vektoren im Raum $\mathcal{H}$ auf uneindeutige, redundante und numerisch stabile Art und Weise darzustellen bzw. zu rekonstruieren. Die eben genannten Eigenschaften von Frames kommen bei einer Vielzahl von Anwendungen, wie zum Beispiel Bild- und Signalverarbeitung, drahtlose Kommunikation, Datenkomprimierung oder Abtastungs-Theorie, zu tragen. Zusätzlich benötigen manche Anwendungen (– etwa um große Datenmengen von numerischen Berechnungen verarbeiten zu können; oder die Anwendungen selbst –) eine verteilte Datenverarbeitung. Dies führt auf kanonische Art und Weise zum Konzept der Fusion Frames, welches das zentrale Thema dieser Arbeit ist.

In dieser Arbeit motivieren und wiederholen wir zuerst die Grundprinzipien der Frame Theorie und erwähnen einige ihrer Anwendungen. Anschließend untersuchen wir direkte Hilbertsummen – diese sind die Darstellungsräume für Fusion Frames – und sammeln einige Resultate über diese Objekte und komponententreue Operatoren zwischen ihnen, die – zumindest teilweise – noch nicht in der Fachliteratur veröffentlicht wurden. Nach diesen beiden vorbereitenden Kapiteln präsentieren wir die grundlegenden Begriffe der Theorie der Fusion Frames, bevor wir das eine oder andere neue Resultat im Zusammenhang mit Fusion Frames und Fusion Frame-Systemen mit Hilfe von Operatorenidentitäten, welche wir mit den oben genannten Resultaten über komponententreue Opertoren zwischen direkten Hilbertsummen herleiten werden, beweisen. Danach diskutieren wir den Aspekt der Dualität im Zusammenhang mit Fusion Frames, was auf natürliche Art und Weise zu Untersuchungen von beliebigen beschränkten Operatoren zwischen direkten Hilbertsummen führt. Wir stellen diese Operatoren durch unendliche Matrizen von Operatoren dar, was nicht nur hilfreich ist, um diese Objekte intuitiv besser verstehen zu können, sondern uns auch ermöglicht, einige nützliche und auch neue Resultate hinsichtlich Beschränktheit oder Kompaktheit in Zusammenhang mit den Operatoren in den Matrixdarstellungen zu beweisen. Schlussendlich wenden wir die letztgenannten Resultate auf Fusion Frames an und können dadurch das Konzept der Fusion Frame Multiplikatoren in einem Rahmen präsentieren, der allgemeiner ist als jener in der bisher veröffentlichten Fachliteratur zu diesem Thema. Dies gestattet nicht nur eine reichhaltigere Theorie, sondern ermöglicht uns auch, mehr der Schlüsselresultate aus der Theorie der Frame Multiplikatoren zu verallgemeinern.

Contents

1 Introduction

During the process of writing my master's thesis, I have very often been asked by various kind of people, including friends of mine, family members, or just random people I got to know recently, what my master's thesis is about and if there is any chance that I can explain it to them. Almost every time I have been asked this question, I tried to avoid giving a detailed answer, although I love explaining things, especially things which are related to mathematics!

The reason, why I have often avoided giving a detailed answer, is that compared to other mathematical topics or questions like *"Are there infinitely many twin prime numbers?"*, I find it very difficult to explain to a non-mathematician, what a *frame* is, not to mention what is meant by the title *Fusion Frames and Operators*. Therefore, in most cases I replied "Something about frames." and added with a smile *"Please don't ask me to explain what a frame is"* to laugh giving a proper answer off in that way.

However, while avoiding to give a proper answer to the question, what my master's thesis is about, over and over again, the idea to give such an explanation for non-mathematicians in the introduction of my thesis grew bigger and bigger in my head until I finally decided to do so. This is why – in what follows – I will motivate and explain the idea of *frames* (and thereafter *fusion frames* respectively) by starting with giving the (in my opinion) easiest possible example of a frame. I will explain everything on a very basic level, which should (as I hope!) be accessible for *every* person still reading. Bit by bit I will add more and more ideas and will slowly get more and more abstract, until we reach the point, where the concept of *fusion frames* is – at least roughly – motivated.

1.1 Example: Orthonormal bases

Consider any vector $x = (x_1, x_2)$ in the space $\mathbb{R}^2$, lets say we consider the vector $(3, 4)$, to have a concrete example. Using the vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, we can represent the vector $(3, 4)$ via

$$(3, 4) = 3e_1 + 4e_2. \tag{1.1}$$

The coefficients 3 and 4 are the only possible choice to write $(3, 4)$ as a linear combination of the vectors e_1 and e_2. The following figure illustrates this as well.

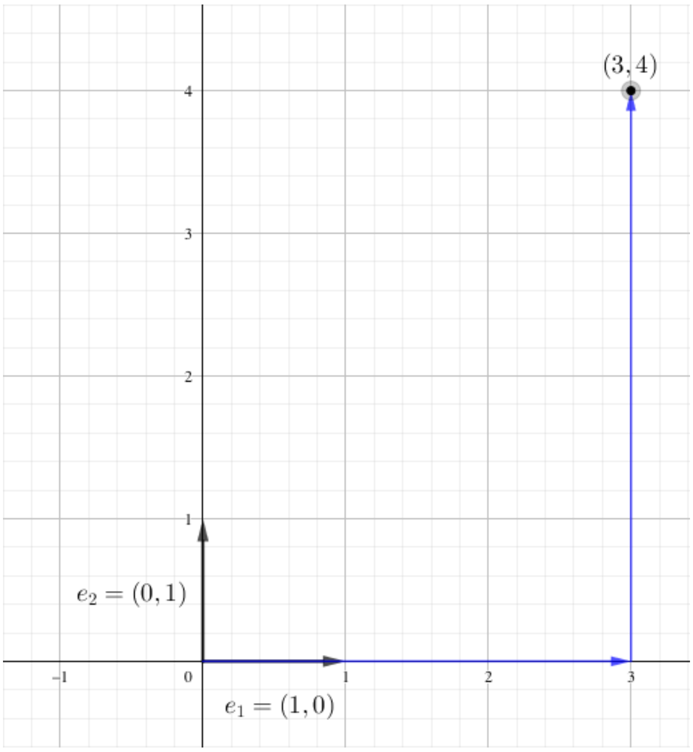

Figure 1: Illustration of $(3, 4) = 3e_1 + 4e_2$.

Of course we can do the same with the general vector $x = (x_1, x_2)$ and may write

$$x = (x_1, x_2) = x_1 e_1 + x_2 e_2. \tag{1.2}$$

Let us define the operation $\langle \cdot, \cdot \rangle$ (called an *inner product*), which takes two vectors $y = (y_1, y_2)$ and $z = (z_1, z_2)$ from $\mathbb{R}^2$ and *"multiplies"* them via $\langle y, z \rangle = y_1 z_1 + y_2 z_2$, which gives a real number as output. Then we have $\langle (3,4), (1,0) \rangle = 3 \cdot 1 + 4 \cdot 0 = 3$ and $\langle (3,4), (0,1) \rangle = 3 \cdot 0 + 4 \cdot 1 = 4$. More generally, we readily see that $x_1 = \langle x, e_1 \rangle$ and $x_2 = \langle x, e_2 \rangle$. Thus we may rewrite (1.2) as

$$x = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 = \sum_{i=1}^{2} \langle x, e_i \rangle e_i. \tag{1.3}$$

As above, the real numbers $\langle x, e_1 \rangle$ and $\langle x, e_2 \rangle$ are the unique choice of coefficients to write x as a linear combination of the vectors e_1 and e_2. In the language of mathematics we call the set $\{e_1, e_2\}$ an *orthonormal basis* for $\mathbb{R}^2$ and this orthonormal basis is also an example of a *frame* (for the space $\mathbb{R}^2$).

Until now, all we did was that we rewrote something very simple in a little bit more complicated way and it probably seems unclear to the reader with not much mathematical background, why we would do so. This will be made a little bit clearer later on.

If we leave the 2-dimensional space $\mathbb{R}^2$ and instead consider the 3-dimensional space $\mathbb{R}^3$, then we can make analogous considerations and rewrite an arbitrary vector

$x = (x_1, x_2, x_3)$ from the space $\mathbb{R}^3$ using the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ and the operation $\langle\cdot,\cdot\rangle$, this time defined by $\langle y,z\rangle = y_1z_1 + y_2z_2 + y_3z_3$ (where $y = (y_1,y_2,y_3)$ and $z = (z_1,z_2,z_3)$ again are arbitrary vectors in the space $\mathbb{R}^3$), as

$$x = \sum_{i=1}^{3} \langle x, e_i\rangle e_i. \tag{1.4}$$

Analogously to before, the set $\{e_1, e_2, e_3\}$ is an orthonormal basis for $\mathbb{R}^3$ and also an example of a frame for $\mathbb{R}^3$.

Nothing prevents us from doing the same procedure with vectors that have n components (n is an arbitrary natural number). This might seem a little bit weird to non-mathematicians, since we can only think and live in three dimensions, but doing things in a more general way than seemingly necessary can lead to great mathematical ideas and besides that, there exist plenty of applications for n-dimensional vectors. So, for any $x = (x_1, ..., x_n)$ from the n-dimensional space $\mathbb{R}^n$, we have

$$x = \sum_{i=1}^{n} \langle x, e_i\rangle e_i, \tag{1.5}$$

where the vectors $e_1, ..., e_n$ and the inner product $\langle\cdot,\cdot\rangle$ are defined analogously as above. The reader may be quickly convinced, that – as in the 2- and 3- dimensional case – the above representation of x is *unique*, as soon as we choose to use each of the vectors $e_1, ..., e_n$ precisely once. Hardly surprising, the set $\{e_1, ..., e_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and an example of a frame for $\mathbb{R}^n$.

Orthonormal bases can not only be used to rewrite any arbitrary vector x (as above) uniquely, but they also yield a possibility to reconstruct the vector x, if we know the scalars $\langle x, e_i\rangle$. The above examples are the most basic examples for an orthonormal basis for the space $\mathbb{R}^n$, a so-called *Hilbert space*, and in fact, any orthonormal basis for any Hilbert space is also a frame. However, orthonormal bases are not the best examples to describe the concept of a frame (– they are only the most simple ones). In the next section we will present other examples of frames, which also describe the intuition of what a frame is better.

1.2 Example: Spanning sets

Let us turn back to the beginning example from Section 1.1. There we wrote the vector $(3,4)$ as a linear combination of the vectors $(1,0)$ and $(0,1)$. We could also try to expand the vector $(3,4)$ as a linear combination of three (or even more) vectors. To give an example, consider the vectors $f_1 = (1,1)$, $f_2 = (1,0)$ and $f_3 = (0,2)$. Then we can expand the vector $(3,4)$ again as a linear combination of the vectors f_1, f_2 and f_3. However, as we immediately see (via direct calculation), we now have more than one possibility to do so. For instance,

$$(3,4) = 2f_1 + 1f_2 + 1f_3$$
$$(3,4) = 4f_1 - 1f_2 + 0f_3$$
$$(3,4) = 3f_1 + 0f_2 + \frac{1}{2}f_3$$
$$(3,4) = 0f_1 + 3f_2 + 2f_3.$$

In the first equation we represented $(3,4)$ as linear combination of all three of the vectors f_1, f_2 and f_3, while in the other three equations we essentially only used two vectors of the set $\{f_1, f_2, f_3\}$. In case we use all three vectors f_1, f_2 and f_3 to do so, there is an infinite amount of possibilities for appropriate coefficients to represent $(3,4)$. Intuitively explained, the reason for this phenomenon is that we work in a 2-dimensional space, but use 3 vectors to represent $(3,4)$, i.e. one more than necessary. The following figure illustrates this situation.

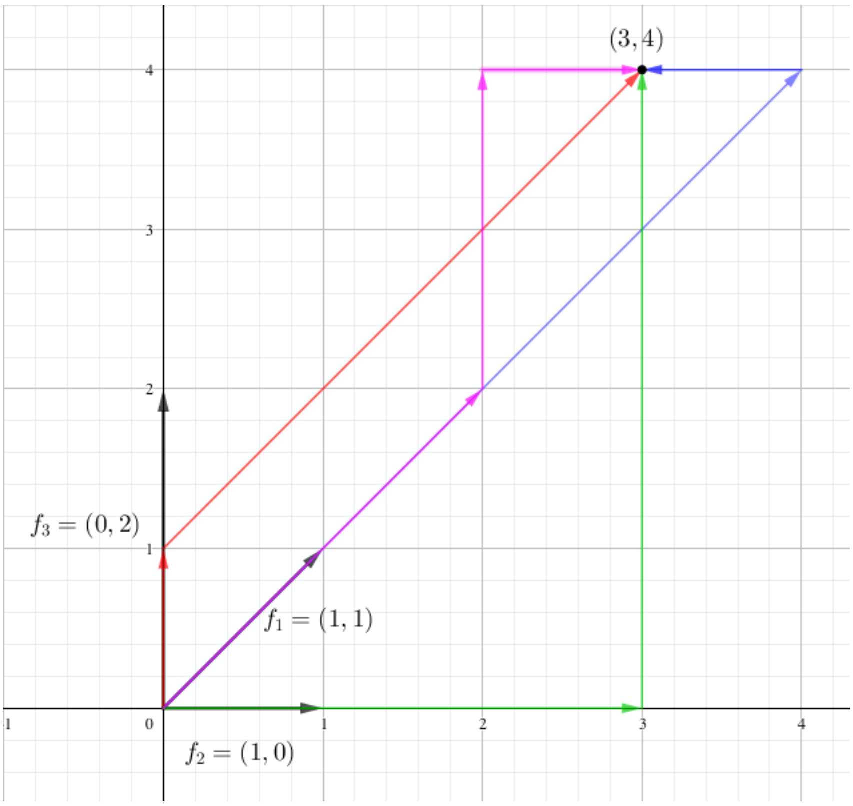

Figure 2: Various linear combinations for the vector $(3,4)$.

Intuitively spoken, the set $\{f_1, f_2, f_3\}$ is redundant in that way, that we could cancel one of the vectors from this set and would still be able to express $(3,4)$ as a *linear combination* of the remaining two vectors. This redundancy already describes one of the key properties of a frame (at least most frames have this property) and is desired for many applications.

The reader, who is familiar with linear algebra, will have already noticed, that what is described above is nothing else than an example of a spanning set for the space $\mathbb{R}^2$. In fact, the set $\{f_1, f_2, f_3\}$ also is an example for a frame for $\mathbb{R}^2$.

Of course the same procedure could be done for a general vector x from the space $\mathbb{R}^2$. Without further explanation (for now), let me tell you that there exists a specific choice of coefficients (called *frame coefficients*) to represent x using all three vectors f_1, f_2 and f_3, namely

$$x = \sum_{i=1}^{3} \langle x, S^{-1} f_i \rangle f_i, \tag{1.6}$$

where S^{-1} is the so-called *inverse frame operator* (see Chapter 2).

Again, analogous considerations can be made when working in the spaces $\mathbb{R}^3$ or $\mathbb{R}^n$ or other (Hilbert) spaces and in fact, any spanning set for the space $\mathbb{R}^n$ (n is an arbitrary natural number) is a frame.

1.3 A rough idea of frames

Without even having mentioned how *frames* (in Hilbert spaces) are defined, we already implicitly highlighted the spirit of what *frames* are. Roughly speaking, a *frame* is a set of vectors $\{f_i\}_{i \in I}$ (I is a countable index set) in a certain vector space, which enables us to not only represent any vector x of that space as a linear combination of those vectors – like in equation (1.6), but also to reconstruct x, if we know the numbers $\langle x, S^{-1} f_i \rangle$ (called *frame coefficients*). In addition to that, frames often (but not always – this depends on the frame) involve a redundancy, meaning that – as explained above – in some situations we might be able to omit some of the frame vectors f_i and still may be able to reconstruct x by using not all of the vectors f_i. This extra redundancy may seem like a harmless property; however, this property (which an orthonormal basis can never have) is desired for a vast number of applications. We will give further information on that aspect later on.

1.4 A rough idea of fusion frames

The concept of so-called *fusion frames* is based on the concept of frames. One possibility to motivate fusion frames is the following. Let us consider a frame (see Chapter 2 for an exact definition) that consists of a very large number of vectors, and imagine a computer calculating data corresponding to that frame. If the frame is simply too big to be handled numerically, it may be of advantage to split the frame into several components and compute the information at first locally (meaning each component separately) and then *fuse* the information back together (hence the name *fusion frame*). To be a tiny bit more precise, the above mentioned components will be modeled as subspaces of the space we work in. Those subspaces will have *"frame-like"* properties, which we will elaborate later. Moreover the process of *fusing* the information back together will mathematically remember us of the reconstruction via frames, which has been already hinted in (1.6) and will be presented in Chapter

2. Fusion frames can also be viewed as generalization of frames. Of course this will be discussed later on too.

1.5 What lies ahead

When we want to model a real life problem as a mathematical problem, the situation is often too complicated and difficult to work in the finite dimensional spaces $\mathbb{R}^n$ or $\mathbb{C}^n$. Therefore mathematicians consider more abstract spaces, like spaces that consist of functions and are not finite dimensional, for instance. This is the reason why I emphasized some superfluous seeming abstract notation from the very beginning, since then the connection to the abstract theory lying ahead will be more clear. The rest of this thesis is formulated in a very general and abstract set up and sadly will probably only be accessible to people with a solid mathematical background.

While this first chapter of this master's thesis mainly focused on introducing and motivating the content of the rest of this work, Chapter 2 recalls the basic definitions and properties of *frames* in *Hilbert spaces* and closely related notions, and also enlightens more of the vast number of applications for frames. Some of the results in Chapter 2 are only collected and cited (and later on used) without giving a detailed proof, since the focus of this thesis lies more on the study on *fusion frames* and other related notions. In Chapter 3 we focus on a certain class of Hilbert spaces, called *Hilbert direct sums*, and *operators* between them. Knowing about these spaces and operators provides us with great help for understanding some key ideas corresponding to fusion frames better. Fusion frames themselves are the content of Chapter 4. There, we do not only present the basic definitions and most fundamental concepts and results for fusion frames, but also view some aspects from a different perspective, using our theory from Chapter 3. Eventually, this does not only enable us to give some new proofs of already known results, but also yields proofs for some new results. Also the aspect of duality for fusion frames is discussed in Chapter 4. In Chapter 5 we consider general (bounded) operators between Hilbert direct sums and show that they can be represented by (infinite) matrices of operators. Furthermore, we prove some results involving compact operators and in particular Hilbert Schmidt and trace class operators. Finally, in Chapter 6 we present the notion of fusion frame multipliers in a more general way than it has been done in literature before, which allows for a larger variety of results generalizing the ones corresponding to frame multipliers. In particular, we will apply our operator theoretic results from Chapters 3 and 5 to prove some new properties of fusion frame multipliers.

2 Frames

In this chapter we will not only introduce the basic concepts of frame theory, which we will later refer to very often, but also discuss the usefulness of frames with respect to both abstract theory and applications. Before we get started, let us collect some preliminaries and notations, which we fix for this chapter and all later chapters ahead.

Preliminaries and notations.
In this thesis we set $\mathbb{N} = \{1, 2, 3, ...\}$, i.e. we do not include zero to the set of natural numbers. For $n \in \mathbb{N}$ we abbreviate $\langle n \rangle := \{1, \dots, n\}$. δ_{ij} denotes the Kronecker-delta, which is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Throughout this notes, $\mathcal{H}$ is always a separable Hilbert space. If not stated otherwise, index sets like I, J, K, L or J_i are always countable. $\mathcal{I}_X$ will denote the identity operator on a given space X. If V is a subspace of $\mathcal{H}$ then π_V denotes the orthogonal projection onto V.

Whenever we speak about an operator T, we mean that T is a linear map between two normed spaces. If T is some operator, then $\mathcal{R}(T)$ denotes the range of T, $\mathcal{N}(T)$ denotes the kernel of T and $\mathrm{dom}(T)$ denotes the domain of T. The set of bounded operators between two normed spaces X and Y is denoted by $\mathcal{B}(X, Y)$; in case $X = Y$, we set $\mathcal{B}(X) := \mathcal{B}(X, X)$. The norm of $T \in \mathcal{B}(X, Y)$ is defined by

$$\|T\|_{op} = \sup_{\|x\|_X = 1} \|Tx\|_Y.$$

If we want to emphasize the corresponding spaces X and Y, we write $\|T\|_{X \to Y}$ instead of $\|T\|_{op}$. We denote the set of compact operators $T : X \longrightarrow Y$ by $\mathcal{C}(X, Y)$ and write $\mathcal{C}(X) = \mathcal{C}(X, X)$. In case $X = \mathcal{H}_1$ and $Y = \mathcal{H}_2$ are Hilbert spaces, we denote the Schatten-p-classes of $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ by $\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ and their corresponding Schatten-p-norms by $\|.\|_p$. We denote the norm $\|.\|_1$ associated to the trace class $\mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2)$ by $\|.\|_{trace}$ and the norm $\|.\|_2$ associated to the class $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ of Hilbert Schmidt operators, called Hilbert Schmidt norm, by $\|.\|_{\mathcal{HS}}$. For more details about compact operators, Schatten-p-classes and in particular Hilbert Schmidt and trace class operators we refer to the Appendix.

For every positive and bounded operator $U \in \mathcal{B}(\mathcal{H})$, there exists a unique positive operator $U^{1/2}$, called the *square root* of U, such that $U^{1/2}U^{1/2} = U$. If U is self-adjoint, then so is $U^{1/2}$ (see [18] for more details). Since for any $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the operator $T^*T \in \mathcal{B}(\mathcal{H}_1)$ is positive (and self-adjoint) the operator $|T| := (T^*T)^{1/2}$, called the *absolute value* of T is well-defined.

Recall that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are arbitrary Hilbert spaces, then for any bounded operator $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ with closed range $\mathcal{R}(U)$, there exists a bounded operator $U^\dagger : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that

$$UU^\dagger x = x \quad (x \in \mathcal{R}(U)). \tag{2.1}$$

$U^\dagger$ is called the *pseudo inverse* of U. To be a little bit more precise, if U_0 is defined by $U_0 := U \restriction_{\mathcal{N}(U)^\perp} : \mathcal{N}(U)^\perp \longrightarrow \mathcal{H}_2$, then it can be shown [18] that U_0 is bounded and injective and that $\mathcal{R}(U_0) = \mathcal{R}(U)$. Therefore, by the Bounded Inverse Theorem (see Appendix), $U_0^{-1} : \mathcal{R}(U) \longrightarrow \mathcal{N}(U)^\perp$ exists and is bounded, and we define $U^\dagger$ as an extension of U_0^{-1} via

$$U^\dagger x = \begin{cases} U_0^{-1} x & \text{if } x \in \mathcal{R}(U) \\ 0 & \text{if } x \in \mathcal{R}(U)^\perp \end{cases}.$$

By construction, $U^\dagger$ has again closed range, namely

$$\mathcal{R}(U^\dagger) = \mathcal{N}(U)^\perp. \tag{2.2}$$

An alternative and equivalent way to define the pseudo inverse $U^\dagger$ of a given operator $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ with closed range $\mathcal{R}(U)$ is to define it as the unique operator $U^\dagger : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$, satisfying the three relations

$$\mathcal{N}(U^\dagger) = \mathcal{R}(U)^\perp, \quad \mathcal{R}(U^\dagger) = \mathcal{N}(U)^\perp, \quad UU^\dagger x = x \quad (x \in \mathcal{R}(U)). \tag{2.3}$$

See [18] for more details about the pseudo inverse of an operator.

2.1 Some basic principles of Frame Theory

As mentioned in Section 1.1, every orthonormal basis of a separable Hilbert space $\mathcal{H}$ is a *frame*. To see this, let us first consider Parseval's equation before giving an exact definition for a frame in $\mathcal{H}$. See [18] for a proof.

Theorem (Parseval's equation) [18] 2.1.1. *Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Then for every $f \in \mathcal{H}$*

$$\|f\|^2 = \sum_{i\in I} |\langle f, e_i\rangle|^2. \tag{2.4}$$

Next, we finally give a definition for a (discrete) frame in a separable Hilbert space. In literature, one can find several generalizations of the subsequent frame definition like the notion of *continuous frames* (see [18]) or *Banach frames* (see [5]).

Definition (Frame) 2.1.2. *A set of vectors $\psi = \{\psi_i\}_{i\in I}$ in $\mathcal{H}$ is called a* frame *for $\mathcal{H}$, if there exist constants $0 < A_\psi \leq B_\psi < \infty$, called* lower *and* upper frame bound *respectively, such that for every $f \in \mathcal{H}$*

$$A_\psi \|f\|^2 \leq \sum_{i\in I} |\langle f, \psi_i\rangle|^2 \leq B_\psi \|f\|^2. \tag{2.5}$$

In the introduction, we motivated the idea of a frame for $\mathbb{R}^n$ via a spanning set in $\mathbb{R}^n$. In fact, it is not difficult to show that every spanning set for $\mathbb{R}^n$ or $\mathbb{C}^n$ constitutes a frame for $\mathbb{R}^n$ or $\mathbb{C}^n$ respectively and conversely, that every frame for $\mathbb{R}^n$ or $\mathbb{C}^n$ is a spanning set for $\mathbb{R}^n$ or $\mathbb{C}^n$ respectively. For a proof we refer to [18].

Definition 2.1.2 not only immediately implies (via Parseval's equation (2.4)) that every orthonormal basis for $\mathcal{H}$ is a frame for $\mathcal{H}$ (with frame bounds $A_\psi = B_\psi = 1$), but also (at least formally) hints the flexibility of frames, which was mentioned in the introduction and will be discussed later on.

In general, we call a frame ψ an *A_ψ-tight frame* or simply *tight frame*, if the frame bounds A_ψ and B_ψ in (2.5) can be chosen to be equal. A 1-tight frame is also called *Parseval frame*. In this manner we see that every orthonormal basis for $\mathcal{H}$ is a Parseval frame for $\mathcal{H}$. However, the converse is not true. To give a counter example, let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and set $\{f_k\}_{k\in\mathbb{N}} := \left\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, ...\right\}$. Then, using Parseval's equation (2.4), one readily sees that $\{f_k\}_{k\in\mathbb{N}}$ is a Parseval frame, since for every $f \in \mathcal{H}$

$$\sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2 = \sum_{k\in\mathbb{N}} k\left|\left\langle f, k^{-1/2}e_k\right\rangle\right|^2 = \sum_{k\in\mathbb{N}} |\langle f, e_k\rangle|^2 = \|f\|^2,$$

but not an orthonormal basis for $\mathcal{H}$.

If in (2.5) only the upper inequality is considered, we call ψ a *Bessel sequence* and, in this case, call B_ψ a *Bessel bound*.

We say that a frame $\psi = \{\psi_i\}_{i\in I}$ is an *exact* frame, if it ceases to be a frame, when one of the frame vectors ψ_i is removed from the set ψ. A frame which is not an exact frame is called an *overcomplete* or *redundant* frame.

We say ψ is a *Riesz basis*, if $\overline{span}\{\psi_i\}_{i\in I} = \mathcal{H}$ (i.e. the sequence $\{\psi_i\}_{i\in I}$ is *complete*), and if there exist constants $0 < A_\psi \le B_\psi < \infty$, called *lower* and *upper Riesz bound* respectively, such that for any finite scalar sequence $\{c_j\}_{j\in J}$ we have

$$A_\psi \sum_{j\in J} |c_j|^2 \le \left\|\sum_{j\in J} c_j\psi_j\right\|^2 \le B_\psi \sum_{j\in J} |c_j|^2.$$

One can show [18] that every Riesz basis for $\mathcal{H}$ is a frame for $\mathcal{H}$. This fact also follows from Theorems 2.1.6 and 2.1.7, see below. In fact [18], one can even show that a frame is a Riesz basis if and only if it is an exact frame.

In the following, we define the *synthesis operator*, *analysis operator* and *frame operator* corresponding to a set of vectors $\psi = \{\psi_i\}_{i\in I}$. These operators are of fundamental importance in the field of frame theory. We will only consider the situation, where ψ is a Bessel sequence (e.g. a frame), in which case these operators are bounded. However, of course one may consider other situations, where the *synthesis operator*, *analysis operator* and *frame operator* respectively are possibly unbounded operators, see [8].

Given any Bessel sequence $\psi = \{\psi_i\}_{i\in I}$ with Bessel bound B_ψ, we define the operator T_ψ, called the *synthesis operator*, via

$$T_\psi : \ell^2(I) \longrightarrow \mathcal{H},$$

$$T_\psi\big(\{c_i\}_{i\in I}\big) = \sum_{i\in I} c_i\psi_i.$$

In [18] it is shown that $\|T_\psi\|_op \le \sqrt{B_\psi}$ and that the above series converges unconditionally. More precisely, one can show [18] that given an arbitrary sequence $\varphi = \{\varphi_i\}_{i\in I}$ of vectors in $\mathcal{H}$, its associated synthesis operator T_φ is well-defined on $\ell^2(I)$ and bounded with $\|T_\varphi\|_{op}^2 \le B$ $(B > 0)$ if and only if φ is a Bessel sequence with Bessel bound B.

If ψ is a Bessel sequence with Bessel bound B_ψ for $\mathcal{H}$ and T_ψ its corresponding synthesis operator, then its adjoint

$$T_\psi^* : \mathcal{H} \longrightarrow \ell^2(I)$$

is called *analysis operator* and is given by [18]

$$T_\psi^*(f) = \{\langle f, \psi_i \rangle\}_{i \in I}.$$

We have $\|T_\psi^*\|_{op} = \|T_\psi\|_{op} \leq \sqrt{B_\psi}$.

Composing the synthesis operator with the analysis operator yields the *frame operator*

$$S_\psi = T_\psi T_\psi^* : \mathcal{H} \longrightarrow \mathcal{H},$$

$$S_\psi(f) = \sum_{i \in I} \langle f, \psi_i \rangle \psi_i,$$

which, in case ψ is a Bessel sequence with Bessel bound B_ψ, is bounded by B_ψ, since then

$$\|S_\psi\|_{op} = \|T_\psi T_\psi^*\|_{op} = \|T_\psi\|_{op} \|T_\psi^*\|_{op} = \|T_\psi\|_{op}^2 \leq B_\psi.$$

Again, the above series converges unconditionally. Clearly, S_ψ is a self-adjoint operator, since

$$S_\psi^* = (T_\psi T_\psi^*)^* = T_\psi T_\psi^* = S_\psi.$$

In case ψ is a frame with frame bounds A_ψ and B_ψ, the frame inequalities (2.5) can be rewritten in terms of the frame operator S_ψ via

$$A_\psi \|f\|^2 \leq \langle S_\psi f, f \rangle \leq B_\psi \|f\|^2. \tag{2.6}$$

This implies that S_ψ is a positive operator. If we use the well known fact that

$$\|T\|_{op} = \sup_{\|f\|=1} |\langle Tf, f \rangle|$$

for any self-adjoint $T \in \mathcal{B}(\mathcal{H})$ (c.f. [18]), (2.6) then immediately implies

$$A_\psi \leq \|S_\psi\|_{op} \leq B_\psi. \tag{2.7}$$

After a small technical manipulation of (2.6) and an application of Neumann's Theorem (see Appendix) we can also conclude that S_ψ is invertible. See [18] for the details of the proof. We call S_ψ^{-1} the inverse frame operator.

Since for any frame ψ for $\mathcal{H}$, its corresponding frame operator S_ψ is invertible, we therefore may conclude that for any $f \in \mathcal{H}$ we have $f = S_\psi S_\psi^{-1} f = S_\psi^{-1} S_\psi f$. Applying the definition of S_ψ yields the *frame reconstruction* formulas

$$f = \sum_{i \in I} \langle f, S_\psi^{-1} \psi_i \rangle \psi_i = \sum_{i \in I} \langle f, \psi_i \rangle S_\psi^{-1} \psi_i. \tag{2.8}$$

We call the numbers $\langle f, S_\psi^{-1} \psi_i \rangle$ *frame coefficients*, as already mentioned in the introduction.

It can be shown [18] that $\{S_\psi^{-1}\psi_i\}_{i\in I}$ is also a frame for $\mathcal{H}$ with frame operator S_ψ^{-1}. Often the frame $\{S_\psi^{-1}\psi_i\}_{i\in I}$ is referred to as *canonical dual frame* of ψ or simply *canonical dual* of ψ. We will sometimes write $\{\tilde{\psi}_i\}_{i\in I} := \{S_\psi^{-1}\psi_i\}_{i\in I}$ for the canonical dual of $\psi = \{\psi_i\}_{i\in I}$ and sometimes will abbreviate $\tilde{\psi} = \{\tilde{\psi}_i\}_{i\in I}$.

If A_ψ and B_ψ are frame bounds for the frame ψ, then it can be shown [18] that its canonical dual $\tilde{\psi}$ has frame bounds B_ψ^{-1} and A_ψ^{-1}. Analogously to above we then see that

$$B_\psi^{-1} \leq \|S_\psi^{-1}\|_{op} \leq A_\psi^{-1}. \tag{2.9}$$

Obviously, the inverse frame operator of a frame is also bounded, positive, self-adjoint and invertible.

So far, we have elaborated, how the frame reconstruction for a given frame ψ for $\mathcal{H}$ works. If we know the frame vectors ψ_i and want to compute some $f \in \mathcal{H}$, all we need to know is the sequence $\langle f, S_\psi^{-1}\psi_i\rangle$ of frame coefficients. For this, knowledge of the inverse frame operator, or at least its action on the frame vectors, is crucial. However, when it comes to real life applications, computing the inverse frame operator generally is a major difficulty [18]. Luckily, the frame operator and inverse frame operator for tight frames can be easily computed, as we will show in the next Proposition. In case we work with frames that are not tight (i.e. the frame bounds cannot be chosen to be equal) then one may apply numerical algorithms to approximate the inverse frame operator, see [18] or [31] for more details.

In the following, we consider tight frames and compute their corresponding frame and inverse frame operators respectively. For the sake of completeness we also provide the short proofs.

Proposition 2.1.3. *A frame ψ for $\mathcal{H}$ is an A_ψ-tight frame if and only if $S_\psi = A_\psi \mathcal{I}_\mathcal{H}$.*

Proof. ψ is an A_ψ-tight frame if and only if

$$\langle S_\psi f, f\rangle = A_\psi \|f\|^2 = \langle A_\psi f, f\rangle \text{ for all } f \in \mathcal{H}. \tag{2.10}$$

(2.10) is equivalent to

$$\langle (S_\psi - A_\psi \mathcal{I}_\mathcal{H}) f, f\rangle = 0 \text{ for all } f \in \mathcal{H}. \tag{2.11}$$

Since $S_\psi - A_\psi \mathcal{I}_\mathcal{H}$ is self-adjoint, (2.11) is equivalent to

$$S_\psi - A_\psi \mathcal{I}_\mathcal{H} = 0.$$

□

Let us collect some easy consequences of the above.

Corollary 2.1.4. *A frame ψ is a Parseval frame if and only if $S_\psi = \mathcal{I}_\mathcal{H}$.*

Proof. This is the special case $A_\psi = 1$ in Proposition 2.1.3. □

Corollary 2.1.5. *For an A_ψ-tight frame ψ, the frame reconstruction formulas reduce to the reconstruction formula*

$$f = \frac{1}{A_\psi} \sum_{i\in I} \langle f, \psi_i\rangle \psi_i. \tag{2.12}$$

If ψ is a Parseval frame, then the reconstruction formulas reduce to

$$f = \sum_{i \in I} \langle f, \psi_i \rangle \psi_i. \tag{2.13}$$

Proof. The result follows immediately from (2.8), Proposition 2.1.3 and Corollary 2.1.5. □

Similarly to the notion of the canonical dual of a frame, we call a frame $\varphi = \{\varphi_i\}_{i \in I}$ a *dual frame* of ψ or simply a *dual* of ψ, if for all $f \in \mathcal{H}$

$$f = \sum_{i \in I} \langle f, \varphi_i \rangle \psi_i = \sum_{i \in I} \langle f, \psi_i \rangle \varphi_i. \tag{2.14}$$

In operator notation this reads

$$T_\psi T_\varphi^* = \mathcal{I}_{\mathcal{H}}, \tag{2.15}$$

which is equivalent to

$$T_\varphi T_\psi^* = \mathcal{I}_{\mathcal{H}}.$$

In other words, a frame φ is called a dual frame of ψ if and only if the so-called *mixed frame operator* $T_\psi T_\varphi^*$ equals the identity operator on $\mathcal{H}$. See [18] for more details.

The following result gives characterizations for a set of vectors being a frame in terms of its synthesis and analysis operator. For a proof and more details, we refer to [18] or [8].

Theorem [8] [18] 2.1.6. *Let $\psi = \{\psi_i\}_{i \in I}$ be a sequence in $\mathcal{H}$. Then the following are equivalent.*

(i) ψ is a frame for $\mathcal{H}$.

(ii) The synthesis operator T_ψ is bounded and surjective.

(iii) The analysis operator T_ψ^* is bounded and injective.

Similarly, Riesz bases can be characterized as follows, see [18] or [8].

Theorem [8] [18] 2.1.7. *Let $\psi = \{\psi_i\}_{i \in I}$ be a sequence in $\mathcal{H}$. Then the following are equivalent.*

(i) ψ is a Riesz basis for $\mathcal{H}$.

(ii) The synthesis operator T_ψ is bounded and bijective.

(iii) The analysis operator T_ψ^* is bounded and bijective.

The previous two results, where frames and Riesz bases (i.e. exact frames) are characterized in terms of their corresponding synthesis and analysis operators respectively, will be useful later on, when we elaborate some properties of *fusion frame systems*, see Chapter 4.

In case $\psi = \{\psi_i\}_{i\in I}$ is a frame but not a Riesz basis (i.e. ψ is an overcomplete frame), its associated synthesis operator T_ψ is not invertible, but has closed range $\mathcal{R}(T_\psi) = \mathcal{H}$, since T_ψ is surjective. Therefore we may consider its pseudo inverse operator. In [18] it is shown that the pseudo inverse

$$T_\psi^\dagger : \mathcal{H} \longrightarrow \ell^2(I)$$

of T_ψ is given by

$$T_\psi^\dagger f = \{\langle f, S_\psi^{-1}\psi_i\rangle\}_{i\in I}.$$

Since S_ψ^{-1} is self-adjoint, this is equivalent to $T_\psi^\dagger f = \{\langle S_\psi^{-1} f, \psi_i\rangle\}_{i\in I}$, i.e.

$$T_\psi^\dagger = T_\psi^* S_\psi^{-1}. \tag{2.16}$$

In Chapter 4, we will prove an analogous formula for the more general fusion frame setting.

2.2 Applications for frames

The purpose of this section is to enlighten the practicability of frames a little bit more, which will also help to explain the usefulness of the concept of *fusion frames* for real life applications. We will not go much into detail about the applications for frames themselves, but rather give an intuition and idea, why frames are used so frequently in real life situations. We will discuss this on the basis of *signal processing* and then mention some some other interesting aspects of the usage of frames.

Roughly speaking, *signal processing* is about *signals* (information, e.g. sounds, images or scientific measurements) being at first *analyzed*, then processed and then *synthesized*. The analysis stage usually consists of sampling the input signal before each sample is *quantized* to a finite number of bits. Then the quantized samples are processed and then synthesized back into a signal. See [39] for more details.

Signals often are modeled as elements f from a Hilbert space $\mathcal{H}$ (e.g. $\mathcal{H} = \mathbb{C}^n$ or $\mathcal{H} = L^2[\mathbb{R}^d]$). The canonical frame operators, namely the analysis operator, the synthesis operator and the frame operator, correspond to analyzing, synthesizing and analyzing and re-synthesizing such a signal [6]. More precisely, given a frame $\psi = \{\psi_i\}_{i\in I}$, its corresponding analysis operator T_ψ^* maps the signal f into the representation space $\ell^2(I)$ via $T_\psi^* f = \{\langle f, \psi_i\rangle\}_{i\in I}$. The synthesis operator T_ψ maps this sampled signal $T_\psi^* f$ back into the signal space via $T_\psi T_\psi^* f = \sum_{i\in I}\langle f, \psi_i\rangle\psi_i$. As we saw in the previous section, this process $S_\psi = T_\psi T_\psi^*$ of analyzing and re-synthesizing a signal f is even invertible. This property of *perfect reconstruction*, which is mirrored by the reconstruction formulas (2.8), basically means in this context, that any signal f can be reconstructed from its analyzed samples without losing any information. See [18] or [6] for more details.

Moreover, frames guarantee *stability*. More precisely, if two signals f_1 and f_2 are *close* to each other (i.e. $\|f_1 - f_2\|$ is small), then their analyzed samples are *close* to each other (meaning the ℓ^2 -norm of their difference is small) and vice versa. This can be checked immediately by rewriting the frame inequalities (2.1.2)

$$\sqrt{A_\psi}\,\|f_1 - f_2\|_{\mathcal{H}} \le \|\{\langle f_1, \psi_i\rangle\}_{i\in I} - \{\langle f_2, \psi_i\rangle\}_{i\in I}\|_{\ell^2(I)} \le \sqrt{B_\psi}\,\|f_1 - f_2\|_{\mathcal{H}}\,.$$

Furthermore, the representation (2.14) of a signal via dual frames ψ and φ is stable, which is also desirable. For instance, if a signal f is transmitted via the dual frame coefficients $\{\langle f, \psi_i\rangle\}_{i\in I}$ and if in the process of transmission some small (i.e. small ℓ^2-norm) perturbation error $\epsilon = \{\epsilon_i\}_{i\in I}$ occurs (which happens more or less always), then the reconstructed signal $\tilde{f} = \sum_{i\in I}(\langle f, \psi_i\rangle + \epsilon_i)\varphi_i$ will also be close to the original signal f, since

$$\|f - \tilde{f}\|_{\mathcal{H}} = \|\sum_{i\in I} \epsilon_i \varphi_i\|_{\mathcal{H}} \le \sqrt{B_\varphi}\|\epsilon\|_{\ell^2(I)},$$

where B_φ denotes the upper frame bound for φ. See [18] or [6] for more details.

To illustrate the advantage of *redundancy*, which overcomplete frames yield and which has already been mentioned in the introduction, let us consider a common situation in *signal transmission*, where packets of data (modeled as frame coefficients corresponding to a given frame) are sent from a transmitter to a receiver. Sometimes, one or more packets of data get lost during the transmission process, which corresponds to the removal of an element from the frame. If the frame is an exact frame (e.g. an orthonormal basis), then reconstruction of the data is not possible. However, if some redundancy is built into the system, i.e. the frame is overcomplete, reconstruction still might be possible, since the packets of data then contain information about each other and at least part of the original information can be reconstructed. See [18] for more details.

In the above we finally illustrated the usefulness of frames with respect to their properties perfect reconstruction, stability and redundancy. It is no surprise that these properties are not only desirable for signal processing, but also for many other fields of application. However, the flexibility and basis-like behavior of frames turns them also into a powerful tool for some purely mathematical questions, which makes Frame Theory interesting for both theory and application.

Originally, frames have first been introduced in 1952 by the authors Duffin and Schaeffer [27] in order to study non-harmonic Fourier series. Three decades later, the study of frames was continued by the authors Doubechies, Grossman and Meyer [21], before then being continued by many others, e.g. [18], [11], [29]. The study of frames led to many interesting theoretical results, such as versions of the *Balian-Low theorem*, see e.g. [18], or the *Feichtinger conjecture*, see e.g. [18], which was first proposed by the Austrian mathematician Hans Georg Feichtinger in 2003, was later found to be equivalent to the famous since 1953 open *Kadison-Singer problem* [16] and was finally proven in 2013 [38]. Certain constructions of a certain class of finite frames even involve group theoretic arguments [52]. Frame Theory is nowadays applied in many fields of application, like signal processing [39], because of their resilience to additive noise [22], or their resilience to quantization [30], or compressed sensing [12] [13] [14] [25] [26] [53], spherical codes [24] [50], LDPC codes [10], MIMO communications [34] [33], quantum measurements [28] [41] [44] and others. For the application of frames and *fusion frames* for distributed processing we refer to [15] and Chapter 4.

3 Hilbert direct sums and operators

Fusion frames are defined in a quite similar way as frames, see Chapter 4 for an exact definition. However, instead of families of vectors contained in $\mathcal{H}$, families of closed subspaces of $\mathcal{H}$ are considered. We will see, that fusion frames behave a lot like frames. In particular we will also define analogs of the canonical frame related operators, namely the *synthesis*, *analysis* and the *fusion frame operator* respectively.

In Chapter 2, we saw that in frame theory an input signal is mapped by the analysis operator into the representation space $\ell^2(I)$ (strictly speaking we first defined the synthesis operator with domain $\ell^2(I)$). However, in fusion frame theory we need another (and yet very similar) representation space, since we deal with families of subspaces instead of families of vectors. The representation space we will work in is the space

$$\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2} := \left\{ \{f_i\}_{i\in I} \,:\, f_i \in V_i \,(\forall i \in I), \sum_{i\in I} \|f_i\|_{V_i}^2 < \infty \right\}, \tag{3.1}$$

where each V_i is some Hilbert space (e.g. a closed subspace of $\mathcal{H}$, as in the fusion frame setting). As in [37] we call $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ a *Hilbert direct sum*. We remark that in [19] this space is referred to as *direct sum of the Hilbert spaces* V_i and denoted by $\bigoplus\{V_i : i \in I\}$. However, in this thesis we will stick to the notation as in (3.1), which is more common in fusion frame theory.

The purpose of this chapter is to investigate some properties of these Hilbert direct sums and a certain class of bounded operators between them. All results contained in this chapter (except Lemma 3.1.1) and their proofs have been found by the author. To the authors knowledge, these results have not been published in literature yet (except Proposition 3.2.1, which is formulated as an exercise in [19]).

3.1 Hilbert direct sums

As indicated above, Hilbert direct sums are vector spaces, which is easy to check. Moreover, for every Hilbert direct sum we may define an inner product as follows. For $f = \{f_i\}_{i\in I}, g = \{g_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ we define

$$\langle f, g\rangle_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} := \sum_{i\in I} \langle f_i, g_i\rangle_{V_i},$$

which is well-defined, since we have

$$\begin{aligned} |\langle f, g\rangle_{(\sum_{i\in I} \oplus V_i)_{\ell^2}}| &\leq \sum_{i\in I} |\langle f_i, g_i\rangle_{V_i}| \\ &\leq \sum_{i\in I} \|f_i\|_{V_i} \|g_i\|_{V_i} \\ &\leq \left(\sum_{i\in I} \|f_i\|_{V_i}^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} \|g_i\|_{V_i}^2\right)^{\frac{1}{2}} < \infty \end{aligned}$$

by the Cauchy-Schwartz inequality. It is clear that $\langle ., .\rangle_{(\sum_{i\in I} \oplus V_i)_{\ell^2}}$ inherits the three defining properties of an inner product from the inner products $\langle f_i, g_i\rangle_{V_i}$ corresponding to the Hilbert spaces V_i.

Hence, we may define a norm on $(\sum_{i\in I} \oplus V_i)_{\ell^2}$ via

$$\|.\|_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} := \langle .,. \rangle^{1/2}_{(\sum_{i\in I} \oplus V_i)_{\ell^2}}.$$

For $f = \{f_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ this implies

$$\|f\|^2_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} = \sum_{i\in I} \|f_i\|^2_{V_i}.$$

In the following lemma we show completeness. The proof works analogously to showing the completeness of $\ell^2(I)$. For convenience for the reader we present it in full detail.

Lemma 3.1.1. *Any Hilbert direct sum $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ is complete and thus a Hilbert space.*

Proof. Without loss of generality we assume that $I = \mathbb{N}$. Let $\{f^{(j)}\}_{j\in\mathbb{N}}$ be a Cauchy sequence in $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$, i.e. for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for all $m, n \in \mathbb{N}$ with $m, n \geq N$:

$$\|f^{(m)} - f^{(n)}\|_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} = \left(\sum_{i\in I} \|f_i^{(m)} - f_i^{(n)}\|^2\right)^{1/2} < \varepsilon. \tag{3.2}$$

This implies that for every $i \in \mathbb{N}$, the sequence $\{f_i^{(j)}\}_{j\in\mathbb{N}}$ is a Cauchy sequence in V_i. Since V_i is complete, $\{f_i^{(j)}\}_{j\in\mathbb{N}}$ converges to some $f_i \in V_i$. We show that $f := \{f_i\}_{i\in\mathbb{N}} \in \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ and that $\{f^{(j)}\}_{j\in\mathbb{N}}$ converges to f in $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$.

Observe that the sequence $\{\|f^{(m)}\|_{(\sum_{i\in I} \oplus V_i)_{\ell^2}}\}_{m\in\mathbb{N}}$ is bounded by some constant M, since it is a Cauchy sequence in $\mathbb{R}$, which follows from (3.2) after applying the inverse triangle inequality. Hence we see that for fixed $K < \infty$

$$\sum_{i=1}^{K} \|f_i\|^2_{V_i} = \lim_{j\longrightarrow\infty} \sum_{i=1}^{K} \|f_i^{(j)}\|^2_{V_i} \leq M^2.$$

Since this holds for any K, this implies

$$\|f\|_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} \leq M,$$

i.e. $f \in \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$.

To show that $f^{(j)} \longrightarrow f$ in $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$, choose again some fixed K and some $l \geq N$ to see that

$$\sum_{i=1}^{K} \|f_i - f_i^{(l)}\|^2_{V_i} = \lim_{j\longrightarrow\infty} \sum_{i=1}^{K} \|f_i^{(j)} - f_i^{(l)}\|^2_{V_i} \leq \varepsilon^2,$$

which implies that

$$\|f - f^{(l)}\|_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} \leq \varepsilon,$$

if $l \geq N$. Therefore $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ is complete and thus a Hilbert space. □

Observe that if $V_i = \mathbb{C}$ for all $i \in I$, then $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2} = \ell^2(I)$. Thus Hilbert direct sums are a generalization of the well-known Hilbert spaces $\ell^2(I)$.

The next result generalizes the well-known fact, that the space $c_{00}(I)$ of scalar sequences $\{x_i\}_{i\in I}$ with $x_n \neq 0$ for only finitely many n (which is sometimes also denoted by $\ell^{00}(I)$) is a dense subspace of $\ell^2(\mathbb{N})$.

Lemma 3.1.2. *Let* $\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}^{00}$ *be the set of elements* $f=\{f_i\}_{i\in I}\in\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$ *such that* $f_i\neq 0$ *for finitely many* $i\in I$. *Then* $\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}^{00}$ *is a dense subspace of* $\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$.

Proof. If I is a finite set, then there is nothing to show. We assume without loss of generality that $I=\mathbb{N}$. Let $f=\{f_i\}_{i=1}^{\infty}\in\left(\sum_{i\in\mathbb{N}}\oplus V_i\right)_{\ell^2}$. Then $\|f\|^2_{(\sum_{i\in\mathbb{N}}\oplus V_i)_{\ell^2}}=\sum_{i=1}^{\infty}\|f_i\|^2_{V_i}<\infty$ implies that for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $\sum_{i=N+1}^{\infty}\|f_i\|^2_{V_i}<\varepsilon$. Now observe that $\tilde{f}:=(f_1,\ldots f_N,0,0,\ldots)\in\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}^{00}$ and that the above means that $\|f-\tilde{f}\|^2_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}<\varepsilon$. □

Let us point out another similarity between Hilbert direct sums and the space $\ell^2(I)$. Recall, that the canonical orthonormal basis of $\ell^2(I)$ is $\{\delta_i\}_{i\in I}$, where $\delta_i=(\ldots,0,0,1,0,0,\ldots)$ (with 1 being in the i-th component). If we view the scalar 1 as orthonormal basis for the Hilbert space $\mathbb{C}$ (over the field $\mathbb{C}$), then this already hints a natural example of an orthonormal basis for the more general space $\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$.

The next result was proved here by the author, before a very similar special case of it was later found in [2].

Lemma 3.1.3. *For every* $i\in I$, *let* $\{e_{ij}\}_{j\in J_i}$ *be an orthonormal basis for* V_i. *Set*

$$\tilde{e}_{ij}=\{\delta_{ik}e_{kj}\}_{k\in I}=(\ldots,0,0,e_{ij},0,0,\ldots)\ (e_{ij}\text{ in the }i\text{-th component}).$$

Then $\{\tilde{e}_{ij}\}_{i\in I,j\in J_i}$ *is an orthonormal basis for* $\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$.

Proof. It suffices to show that $\{\tilde{e}_{ij}\}_{i\in I,j\in J_i}$ is a complete orthonormal system (see [18] for instance). Take two arbitrary elemts $\tilde{e}_{ij},\tilde{e}_{i'j'}$: In case $i\neq i'$ we clearly have $\langle\tilde{e}_{ij},\tilde{e}_{i'j'}\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}=0$ by the definition of $\langle.,.\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}$. In case $i=i'$ we have $\langle\tilde{e}_{ij},\tilde{e}_{ij'}\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}=\langle e_{ij},e_{ij'}\rangle_{V_i}=\delta_{jj'}$, since $\{e_{ij}\}_{j\in J_i}$ is an orthonormal basis for V_i. Thus the set $\{\tilde{e}_{ij}\}_{i\in I,j\in J_i}$ is an orthonormal system. To see that it is complete, observe that for any $\{f_i\}_{i\in I}\in\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$ we have

$$\begin{aligned}\{f_i\}_{i\in I}&=\sum_{i\in I}(\ldots,0,0,f_i,0,0,\ldots)\ (f_i\text{ in the }i\text{-th component})\\&=\sum_{i\in I}\Big(\ldots,0,0,\sum_{j\in J_i}\langle f_i,e_{ij}\rangle_{V_i}e_{ij},0,0,\ldots\Big)\ \Big(\sum_{j\in J_i}\langle f_i,e_{ij}\rangle_{V_i}e_{ij}\text{ in the }i\text{-th component}\Big)\\&=\sum_{i\in I}\sum_{j\in J_i}\langle f_i,e_{ij}\rangle_{V_i}\tilde{e}_{ij}.\end{aligned}$$

□

Recall, that the completeness-proof of the Hilbert direct sum $\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$ highly depends on the completeness of the Hilbert spaces V_i. If, for every $i\in I$, we consider some (not necessarily closed) subspace U_i of V_i, then we may define the normed space $\left(\sum_{i\in I}\oplus U_i\right)_{\ell^2}$ analogously to (3.1). Of course $\left(\sum_{i\in I}\oplus U_i\right)_{\ell^2}$ is a subspace of the Hilbert direct sum $\left(\sum_{i\in I}\oplus V_i\right)_{\ell^2}$, but not necessarily a closed subspace. In the following, we characterize all closed subspaces of the form $\left(\sum_{i\in I}\oplus U_i\right)_{\ell^2}$ in terms of the subspaces U_i.

Proposition 3.1.4. *Consider the Hilbert direct sum $\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ and for each $i\in I$, let U_i be a subspace of V_i. Then $\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$ is a closed subspace of $\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ if and only if U_i is a closed subspace of V_i for every $i\in I$.*

Proof. First, we prove the " $\Leftarrow$ "-part of the equivalence. Assume that U_i is a closed subspace of V_i for every $i\in I$ and let $\{f^{(n)}\}_{n\in\mathbb{N}}=\{f_i^{(n)}\}_{i\in I,n\in\mathbb{N}}$ be a Cauchy sequence in $\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$. Then $\{f^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ and thus has a limit $f=\{f_i\}_{i\in I}\in\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$. This means that for every $\varepsilon>0$, there exists $N\in\mathbb{N}$, such that for $n\geq N$

$$\|f^{(n)}-f\|_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}=\sum_{i\in I}\|f^{(n)}-f_i\|_{V_i}<\varepsilon.$$

This implies

$$\|f^{(n)}-f_i\|_{V_i}<\varepsilon$$

for every $i\in I$, i.e. $\{f_i^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in U_i with limit $f_i\in V_i$. However, since U_i is closed, $f_i\in U_i$ for every $i\in I$ and therefore $f=\{f_i\}_{i\in I}\in\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$.

To prove the " $\Rightarrow$ "-part of the equivalence, let $i\in I$ be arbitrary and assume that $\{g^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in U_i. Then $\{g^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in V_i as well and thus has a limit $g\in V_i$. Observe that $\{\tilde g^{(n)}\}_{n\in\mathbb{N}}=\{(\dots,0,0,g^{(n)},0,0,\dots)\}_{n\in\mathbb{N}}$ ($g^{(n)}$ in the i-th entry) is a Cauchy sequence in $\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$ and thus in $\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ and its limit is given by $\tilde g=(\dots,0,0,g,0,0,\dots)\in\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ (g in the i-th entry), since we have

$$\|\tilde g^{(n)}-\tilde g\|_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}=\|g^{(n)}-g\|_{V_i}.$$

Since $\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$ is closed, we have $\tilde g\in\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$ and thus $g\in U_i$. Hence U_i is closed. This finishes the proof. □

Let us consider orthogonal complements.

Proposition 3.1.5. *Consider the Hilbert direct sum $\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ and the (not necessarily closed) subspaces U_i of V_i ($i\in I$) and $\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$ of $\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$. Then*

$$\left(\Big(\sum_{i\in I}\oplus U_i\Big)_{\ell^2}\right)^{\perp}=\Big(\sum_{i\in I}\oplus U_i^{\perp}\Big)_{\ell^2}. \tag{3.3}$$

Proof. To show the "$\supseteq$"-part, choose some $g=\{g_i\}_{i\in I}\in\big(\sum_{i\in I}\oplus U_i^{\perp}\big)_{\ell^2}$ and observe that for any $f=\{f_i\}_{i\in I}\in\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$ we have

$$\langle f,g\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}=\sum_{i\in I}\langle f_i,g_i\rangle_{V_i}=0,$$

which implies that g is also an element of $\big(\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}\big)^{\perp}$.

To prove the "$\subseteq$"-part, let $h=\{h_i\}_{i\in I}\in\big(\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}\big)^{\perp}$ be arbitrary. It suffices to show that $h_i\in U_i^{\perp}$ for every $i\in I$. To this end, observe that for any $f\in\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$ we have

$$\langle f,h\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}=0.$$

Especially for arbitrary $i\in I$ and arbitrary $f_i\in U_i$ consider $\tilde f=(\dots,0,0,f_i,0,0,\dots)\in\big(\sum_{i\in I}\oplus U_i\big)_{\ell^2}$ (where $f_i\in U_i$ is in the i-th component). Then

$$0=\langle\tilde f,h\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}=\langle f_i,h_i\rangle_{V_i},$$

which implies $h_i\in U_i^{\perp}$. This finishes the proof. □

Recall that if V is some closed subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ can be decomposed as direct sum $\mathcal{H} = V \oplus V^{\perp}$. In case we deal with Hilbert direct sums, our previous results imply the following.

Corollary 3.1.6. *Consider the Hilbert direct sum $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$. If U_i is a closed subspace of V_i for every $i \in I$, then*

$$\Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2} = \Big(\sum_{i\in I} \oplus U_i\Big)_{\ell^2} \oplus \Big(\sum_{i\in I} \oplus U_i^{\perp}\Big)_{\ell^2}.$$

Proof. Since the spaces U_i are closed subspaces of V_i, $\big(\sum_{i\in I} \oplus U_i\big)_{\ell^2}$ is a closed subspace of $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ by Proposition 3.1.4. Using Proposition 3.1.5 then yields

$$\Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2} = \Big(\sum_{i\in I} \oplus U_i\Big)_{\ell^2} \oplus \left(\Big(\sum_{i\in I} \oplus U_i\Big)_{\ell^2}\right)^{\perp} = \Big(\sum_{i\in I} \oplus U_i\Big)_{\ell^2} \oplus \Big(\sum_{i\in I} \oplus U_i^{\perp}\Big)_{\ell^2},$$

which completes the proof. □

3.2 Component preserving operators

After describing some properties of Hilbert direct sums, we will now investigate some properties of a certain class of operators between those spaces. The results we present and prove in this section will be very useful for later purposes.

In the following we consider *component preserving* (meaning component-wise defined) operators between two Hilbert direct sums $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ and $\big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}$ with same index set I. Later we will consider more general operators between arbitrary Hilbert direct sums.

It will be convenient to adopt the notion of *completely bounded* families of operators from [40] and hence call a family $\{T_i\}_{i\in I}$ of operators $T_i : X_i \longrightarrow Y_i$, where X_i and Y_i are normed spaces for all $i \in I$, *completely bounded*, if there exists a constant $C > 0$ such that $\|T_i\| \leq C$ for all $i \in I$ and set $\|T\|_{cb} := \sup_{i\in I} \|T_i\|$. Obviously, this implies that all the operators $\mathcal{O}_i$ are bounded.

Proposition 3.2.1. *Consider the family $\{\mathcal{O}_i\}_{i\in I}$ of operators $\mathcal{O}_i : V_i \longrightarrow W_i$. If $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded, then*

$$\bigoplus_{i\in I} \mathcal{O}_i\big(\{f_i\}_{i\in I}\big) := \{\mathcal{O}_i f_i\}_{i\in I} \tag{3.4}$$

defines a well-defined and bounded operator from $(\sum_{i\in I} \oplus V_i)_{\ell^2}$ into $(\sum_{i\in I} \oplus W_i)_{\ell^2}$.

Conversely, if

$$\bigoplus_{i\in I} \mathcal{O}_i : \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2} \longrightarrow \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2},$$

$$\bigoplus_{i\in I} \mathcal{O}_i\big(\{f_i\}_{i\in I}\big) := \{\mathcal{O}_i f_i\}_{i\in I}$$

is well-defined and bounded, then the family $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded.

Proof. Assume that $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded. Then for any $f = \{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus V_i)_{\ell^2}$ we have

$$\begin{aligned} \|\bigoplus_{i\in I} \mathcal{O}_i(f)\|^2_{(\sum_{i\in I} \oplus W_i)_{\ell^2}} &= \sum_{i\in I} \|\mathcal{O}_i f_i\|^2_{W_i} \\ &\leq \sum_{i\in I} \|\mathcal{O}_i\|^2 \|f_i\|^2_{V_i} \\ &\leq \|\mathcal{O}\|^2_{cb} \|f\|^2_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} < \infty, \end{aligned}$$

hence $\bigoplus_{i\in I} \mathcal{O}_i$ not only maps into $\big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}$, but is also bounded by $\|\mathcal{O}\|_{cb}$.

Conversely, if we assume that $\bigoplus_{i\in I} \mathcal{O}_i$ is defined as in (3.4) and bounded by some constant $C > 0$, then this means that

$$\|\bigoplus_{i\in I} \mathcal{O}_i(\{f_i\}_{i\in I})\|_{(\sum_{i\in I} \oplus W_i)_{\ell^2}} \leq C \|\{f_i\}_{i\in I}\|_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} \tag{3.5}$$

for all $f = \{f_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$. For arbitrary $i \in I$ and $\tilde{f}_i \in V_i$ we may especially consider $\tilde{f} = (..., 0, 0, \tilde{f}_i, 0, 0, ...) \in \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ ($\tilde{f}_i$ in the i-th component) and observe that (3.5) reduces to

$$\|\mathcal{O}_i \tilde{f}_i\|_{W_i} \leq C \|\tilde{f}_i\|_{V_i},$$

which implies that $\|\mathcal{O}_i\| \leq C$. Since $i \in I$ was arbitrary, this implies $\|\mathcal{O}\|_{cb} \leq C$, which means that the family $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded by C. $\square$

We call operators $\bigoplus_{i\in I} \mathcal{O}_i$ (defined component wise as in (3.4)) *component preserving*. We remark that in [19], this class of operators is called *direct sum of the operators* $\mathcal{O}_i$.

Let us investigate further properties of bounded component preserving operators between two Hilbert direct sums. Recall that, by the above Proposition, whenever we write $\bigoplus_{i\in I} \mathcal{O}_i \in \mathcal{B}\Big(\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}, \big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}\Big)$, we always have that the family $\{\mathcal{O}_i\}_{i\in I}$ of operators $\mathcal{O}_i : V_i \longrightarrow W_i$ is completely bounded, which implies $\mathcal{O}_i \in \mathcal{B}(V_i, W_i)$ for every $i \in I$.

Proposition 3.2.2. *Let* $\bigoplus_{i\in I} \mathcal{O}_i \in \mathcal{B}\Big(\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}, \big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}\Big)$. *Then*

(a)
$$\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big)^* = \bigoplus_{i\in I} \mathcal{O}_i^*. \tag{3.6}$$

(b) $\bigoplus_{i\in I} \mathcal{O}_i$ *is self-adjoint if and only if* $\mathcal{O}_i$ *is self-adjoint for every* $i \in I$.

(c) If $\big(\sum_{i\in I} \oplus U_i\big)_{\ell^2}$ *is another Hilbert direct sum and*

$$\bigoplus_{i\in I} \mathcal{P}_i : \Big(\sum_{i\in I} \oplus U_i\Big)_{\ell^2} \longrightarrow \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2}$$

is defined analogously as in (3.4), then the composition of $\bigoplus_{i\in I} \mathcal{O}_i$ *with* $\bigoplus_{i\in I} \mathcal{P}_i$,

$$\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big)\Big(\bigoplus_{i\in I} \mathcal{P}_i\Big) : \Big(\sum_{i\in I} \oplus U_i\Big)_{\ell^2} \longrightarrow \Big(\sum_{i\in I} \oplus W_i\Big)_{\ell^2},$$

is given by

$$\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big)\Big(\bigoplus_{i\in I} \mathcal{P}_i\Big) = \bigoplus_{i\in I} (\mathcal{O}_i \mathcal{P}_i). \tag{3.7}$$

(d)

$$\Big|\bigoplus_{i\in I}\mathcal{O}_i\Big| = \bigoplus_{i\in I}|\mathcal{O}_i|.$$

Proof. (a) $(\bigoplus_{i\in I}\mathcal{O}_i)^*$ and $\mathcal{O}_i^*$ $(i\in I)$ are well-defined and bounded. For $f=\{f_i\}_{i\in I}\in\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ and $g=\{g_i\}_{i\in I}\in\big(\sum_{i\in I}\oplus W_i\big)_{\ell^2}$ we compute

$$\begin{aligned}\Big\langle f,(\bigoplus_{i\in I}\mathcal{O}_i)^*g\Big\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} &= \Big\langle(\bigoplus_{i\in I}\mathcal{O}_i)f,g\Big\rangle_{(\sum_{i\in I}\oplus W_i)_{\ell^2}}\\ &=\sum_{i\in I}\langle\mathcal{O}_i f_i,g_i\rangle_{W_i}\\ &=\sum_{i\in I}\langle f_i,\mathcal{O}_i^*g_i\rangle_{V_i}=\Big\langle f,\bigoplus_{i\in I}(\mathcal{O}_i^*)g\Big\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}},\end{aligned}$$

which implies (3.6).
(b) If all $\mathcal{O}_i$ are self-adjoint, then by (a) we have

$$\big(\bigoplus_{i\in I}\mathcal{O}_i\big)^*=\bigoplus_{i\in I}\mathcal{O}_i^*=\bigoplus_{i\in I}\mathcal{O}_i,$$

i.e. $\bigoplus_{i\in I}\mathcal{O}_i$ is self-adjoint. Conversely, if $\bigoplus_{i\in I}\mathcal{O}_i$ is self-adjoint, then for all $f=\{f_i\}_{i\in I}\in\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ and $g=\{g_i\}_{i\in I}\in\big(\sum_{i\in I}\oplus W_i\big)_{\ell^2}$ we have

$$\Big\langle(\bigoplus_{i\in I}\mathcal{O}_i)f,g\Big\rangle_{(\sum_{i\in I}\oplus W_i)_{\ell^2}}=\Big\langle f,(\bigoplus_{i\in I}\mathcal{O}_i)g\Big\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}.$$

For arbitrary $i\in I$ and $\tilde{f}_i\in V_i$ we may especially consider $\tilde{f}=(\ldots,0,0,\tilde{f}_i,0,0,\ldots)\in\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ ($\tilde{f}_i$ in the i-th component) and observe that

$$\begin{aligned}\langle\mathcal{O}_i\tilde{f}_i,g_i\rangle_{W_i}&=\Big\langle(\bigoplus_{i\in I}\mathcal{O}_i)\tilde{f},g\Big\rangle_{(\sum_{i\in I}\oplus W_i)_{\ell^2}}\\ &=\Big\langle\tilde{f},(\bigoplus_{i\in I}\mathcal{O}_i)g\Big\rangle_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}=\langle\tilde{f}_i,\mathcal{O}_i g_i\rangle_{V_i},\end{aligned}$$

i.e. $\mathcal{O}_i$ is self-adjoint (and i was arbitrary).
(c) Observe that for any $h=\{h_i\}_{i\in I}\in(\sum_{i\in I}\bigoplus U_i)_{\ell^2}$ we have

$$\begin{aligned}\big(\bigoplus_{i\in I}\mathcal{O}_i\big)\big(\bigoplus_{i\in I}\mathcal{P}_i\big)(\{h_i\}_{i\in I})&=\big(\bigoplus_{i\in I}\mathcal{O}_i\big)(\{\mathcal{P}_i h_i\}_{i\in I})\\ &=\{\mathcal{O}_i\mathcal{P}_i h_i\}_{i\in I}=\bigoplus_{i\in I}(\mathcal{O}_i\mathcal{P}_i)(\{h_i\}_{i\in I}).\end{aligned}$$

(d) By (a) and (c) we have $\big(\bigoplus_{i\in I}\mathcal{O}_i\big)^*\bigoplus_{i\in I}\mathcal{O}_i=\bigoplus_{i\in I}(\mathcal{O}_i^*\mathcal{O}_i)$. Moreover, by Proposition 3.2.1, the family $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded, which implies that the family $\{|\mathcal{O}_i|\}_{i\in I}$ is completely bounded, which implies that $\bigoplus_{i\in I}|\mathcal{O}_i|\in\mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2}\big)$. Since, by (c), $\big(\bigoplus_{i\in I}|\mathcal{O}_i|\big)^2=\bigoplus_{i\in I}(\mathcal{O}_i^*\mathcal{O}_i)$ we see that $\bigoplus_{i\in I}|\mathcal{O}_i|$ is a square root of $\big(\bigoplus_{i\in I}\mathcal{O}_i\big)^*\bigoplus_{i\in I}\mathcal{O}_i$. However, since the square root of an operator is unique, we conclude that $\big|\bigoplus_{i\in I}\mathcal{O}_i\big|=\bigoplus_{i\in I}|\mathcal{O}_i|$. □

Next we consider the kernel and the range of operators of the type $\bigoplus_{i\in I}\mathcal{O}_i$.

Proposition 3.2.3. *Let* $\bigoplus_{i\in I}\mathcal{O}_i\in\mathcal{B}\Big(\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2},\big(\sum_{i\in I}\oplus W_i\big)_{\ell^2}\Big)$. *Then*

(a)
$$\mathcal{N}\big(\bigoplus_{i\in I}\mathcal{O}_i\big) = \big(\sum_{i\in I}\oplus\mathcal{N}(\mathcal{O}_i)\big)_{\ell^2} \tag{3.8}$$

(b)
$$\mathcal{R}\big(\bigoplus_{i\in I}\mathcal{O}_i\big) \subseteq \big(\sum_{i\in I}\oplus\mathcal{R}(\mathcal{O}_i)\big)_{\ell^2} \tag{3.9}$$

Proof. (a) (3.8) follows from the component-wise definition of $\bigoplus_{i\in I}\mathcal{O}_i$:

$$\begin{aligned}\mathcal{N}\big(\bigoplus_{i\in I}\mathcal{O}_i\big) &= \Big\{\{f_i\}_{i\in I} \in \big(\sum_{i\in I}\bigoplus V_i\big)_{\ell^2} :\ \mathcal{O}_i f_i = 0\ \ (\forall i\in I)\Big\}\\ &= \Big\{\{f_i\}_{i\in I} : f_i \in \mathcal{N}(\mathcal{O}_i)\ \ (\forall i\in I),\ \sum_{i\in I}\|f_i\|^2 < \infty\Big\}\\ &= \big(\sum_{i\in I}\oplus\mathcal{N}(O_i)\big)_{\ell^2}.\end{aligned}$$

(b) (3.9) follows from the component-wise definition of $\bigoplus_{i\in I}\mathcal{O}_i$ as well:

$$\begin{aligned}\mathcal{R}\big(\bigoplus_{i\in I}\mathcal{O}_i\big) &= \Big\{\{g_i\}_{i\in I} \in \big(\sum_{i\in I}\oplus W_i\big)_{\ell^2} : \exists\{f_i\}_{i\in I} \in \big(\sum_{i\in I}\oplus V_i\big)_{\ell^2} : \mathcal{O}_i f_i = g_i\ (\forall i\in I)\Big\}\\ &= \Big\{\{g_i\}_{i\in I} \in \big(\sum_{i\in I}\oplus W_i\big)_{\ell^2} : \forall i\in I : \exists f_i \in V_i : \mathcal{O}_i f_i = g_i,\ \sum_{i\in I}\|f_i\|^2 < \infty\Big\}\\ &\subseteq \Big\{\{g_i\}_{i\in I} \in \big(\sum_{i\in I}\oplus W_i\big)_{\ell^2} : \forall i\in I : \exists f_i \in V_i : \mathcal{O}_i f_i = g_i\Big\} \qquad (3.10)\\ &= \Big\{\{g_i\}_{i\in I} \in \big(\sum_{i\in I}\oplus W_i\big)_{\ell^2} : g_i \in \mathcal{R}(\mathcal{O}_i)\ (\forall i\in I)\Big\}\\ &= \Big\{\{g_i\}_{i\in I} : g_i \in \mathcal{R}(\mathcal{O}_i),\ \sum_{i\in I}\|g_i\|^2 < \infty\Big\}\\ &= \big(\sum_{i\in I}\oplus\mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}.\end{aligned}$$

□

Let us reconsider part (b) of the above proposition. In general, we only have the relation "$\subseteq$" in (3.9) and cannot achieve equality without further assumptions. The reason for this is relation (3.10), since in general $\sum_{i\in I}\|\mathcal{O}_i f_i\|^2 < \infty$ does not imply $\sum_{i\in I}\|f_i\|^2 < \infty$ under the assumptions we have made. We refer to Example 3.2.7 for a concrete scenario, where this implication is not true.

However, if we assume I to be finite, then this convergence issue becomes trivial and we then have

$$\mathcal{R}\big(\bigoplus_{i\in I}\mathcal{O}_i\big) = \big(\sum_{i\in I}\oplus\mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}.$$

Another way to circumvent this obstacle and achieve equality in (3.9) is to assume that $\bigoplus_{i\in I}\mathcal{O}_i$ has closed range. This will be part of the next result.

Before we formulate the next proposition, let us recall the following two well-known facts [19]: If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is bounded, then we have the relation

$$\mathcal{N}(T^*) = \mathcal{R}(T)^\perp. \tag{3.11}$$

Moreover, if $\mathcal{M}$ is some subspace of a Hilbert space $\mathcal{H}$, then we have

$$\overline{\mathcal{M}} = (\mathcal{M}^\perp)^\perp. \tag{3.12}$$

Proposition 3.2.4. *Let* $\bigoplus_{i\in I} \mathcal{O}_i \in \mathcal{B}\Big(\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}, \big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}\Big)$ *and assume that* $\bigoplus_{i\in I} \mathcal{O}_i$ *has closed range. Then*

(a)

$$\mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big) = \big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}. \tag{3.13}$$

(b) $\mathcal{O}_i$ *has closed range for every* $i \in I$.

Proof. (a) We have

$$\begin{aligned}
\overline{\big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}} &= \left(\left(\big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}\right)^\perp\right)^\perp && \text{(by (3.12))} \\
&= \left(\big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)^\perp\big)_{\ell^2}\right)^\perp && \text{(by Proposition 3.1.5)} \\
&= \left(\big(\sum_{i\in I} \oplus \mathcal{N}(\mathcal{O}_i^*)\big)_{\ell^2}\right)^\perp && \text{(by (3.11))} \\
&= \mathcal{N}\big(\bigoplus_{i\in I} \mathcal{O}_i^*\big)^\perp && \text{(by Proposition 3.2.3 (a))} \\
&= \mathcal{N}\left(\big(\bigoplus_{i\in I} \mathcal{O}_i\big)^*\right)^\perp && \text{(by Proposition 3.2.2 (a))} \\
&= \left(\mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big)^\perp\right)^\perp && \text{(by (3.11))} \\
&= \overline{\mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big)} && \text{(by (3.12))} \\
&= \mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big) && \text{(by assumption)} \\
&\subseteq \big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2} && \text{(by Proposition 3.2.3 (b))} \\
&\subseteq \overline{\big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}},
\end{aligned}$$

which proves (a) and also implies that $\big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}$ is a closed subspace of $\big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}$. By Proposition 3.1.4 this implies that $\mathcal{R}(\mathcal{O}_i)$ is a closed subspace of W_i for every $i \in I$, i.e. that $\mathcal{O}_i$ has closed range for every $i \in I$, which proves (b). □

So, if $\bigoplus_{i\in I} \mathcal{O}_i$ has closed range, so do the operators $\mathcal{O}_i$ ($i \in I$). Therefore their corresponding pseudo inverses $\big(\bigoplus_{i\in I} \mathcal{O}_i\big)^\dagger$ and $\mathcal{O}_i^\dagger$ ($i \in I$) are well-defined. In the following result we prove a nice relation between the operators $\big(\bigoplus_{i\in I} \mathcal{O}_i\big)^\dagger$ and $\bigoplus_{i\in I} \mathcal{O}_i^\dagger$, assuming that $\bigoplus_{i\in I} \mathcal{O}_i^\dagger$ is well-defined and has closed range.

Proposition 3.2.5. *Let* $\bigoplus_{i\in I} \mathcal{O}_i \in \mathcal{B}\Big(\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}, \big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}\Big)$ *and assume that* $\bigoplus_{i\in I} \mathcal{O}_i$ *has closed range. If, in addition, the family* $\{\mathcal{O}_i^\dagger\}_{i\in I}$ *is completely bounded and if* $\bigoplus_{i\in I} \mathcal{O}_i^\dagger$ *has closed range, then*

$$\big(\bigoplus_{i\in I} \mathcal{O}_i\big)^\dagger = \bigoplus_{i\in I} \mathcal{O}_i^\dagger. \tag{3.14}$$

Proof. By assumption, the operators

$$\Big(\bigoplus_{i\in I}\mathcal{O}_i\Big)^\dagger : \big(\sum_{i\in I}\oplus W_i\big)_{\ell^2} \longrightarrow \big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$$

and

$$\bigoplus_{i\in I}\mathcal{O}_i^\dagger : \big(\sum_{i\in I}\oplus W_i\big)_{\ell^2} \longrightarrow \big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$$

are both well-defined and bounded (the latter follows from Proposition 3.2.1). By Proposition 3.2.2 (c) and (2.3), we have

$$\big(\bigoplus_{i\in I}\mathcal{O}_i\big)\big(\bigoplus_{i\in I}\mathcal{O}_i^\dagger\big)\big(\bigoplus_{i\in I}\mathcal{O}_i\big) = \bigoplus_{i\in I}(\mathcal{O}_i\mathcal{O}_i^\dagger\mathcal{O}_i) = \bigoplus_{i\in I}\mathcal{O}_i.$$

Furthermore, we have

$$\begin{aligned}
\mathcal{N}\big(\bigoplus_{i\in I}\mathcal{O}_i^\dagger\big) &= \big(\sum_{i\in I}\oplus\mathcal{N}(\mathcal{O}_i^\dagger)\big)_{\ell^2} && \text{(by Proposition 3.2.3 (a))}\\
&= \big(\sum_{i\in I}\oplus(\mathcal{R}(\mathcal{O}_i)^\perp\big)_{\ell^2} && \text{(by (2.3))}\\
&= \big(\sum_{i\in I}\oplus\mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}^\perp && \text{(by Proposition 3.1.5)}\\
&= \mathcal{R}\big(\bigoplus_{i\in I}\mathcal{O}_i\big)^\perp && \text{(by Proposition 3.2.4 (a)).}
\end{aligned}$$

The assumption, that $\bigoplus_{i\in I}\mathcal{O}_i^\dagger$ has closed range, guarantees via Proposition 3.2.4 (a) that

$$\mathcal{R}\big(\bigoplus_{i\in I}\mathcal{O}_i^\dagger\big) = \big(\sum_{i\in I}\oplus\mathcal{R}(\mathcal{O}_i^\dagger)\big)_{\ell^2}$$

and therefore we have

$$\begin{aligned}
\mathcal{R}\big(\bigoplus_{i\in I}\mathcal{O}_i^\dagger\big) = \big(\sum_{i\in I}\oplus\mathcal{R}(\mathcal{O}_i^\dagger)\big)_{\ell^2} &= \big(\sum_{i\in I}\oplus(\mathcal{N}(\mathcal{O}_i)^\perp\big)_{\ell^2} && \text{(by (2.3))}\\
&= \big(\sum_{i\in I}\oplus\mathcal{N}(\mathcal{O}_i)\big)_{\ell^2}^\perp && \text{(by Proposition 3.1.5)}\\
&= \mathcal{N}\big(\bigoplus_{i\in I}\mathcal{O}_i\big)^\perp && \text{(by Proposition 3.2.3 (a)).}
\end{aligned}$$

Thus the operator $\bigoplus_{i\in I}\mathcal{O}_i^\dagger$ fulfills all three characterizing relations in (2.3) for the pseudo inverse of $\bigoplus_{i\in I}\mathcal{O}_i$ and thus coincides with $\big(\bigoplus_{i\in I}\mathcal{O}_i\big)^\dagger$. □

We postpone the discussion about about pseudo inverses to Example 3.2.7 and continue with the properties injectivity, surjectivity and invertibility of operators of the type $\bigoplus_{i\in I}\mathcal{O}_i$.

Proposition 3.2.6. *Let* $\bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{B}\Big(\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}, \big(\sum_{i\in I}\oplus W_i\big)_{\ell^2}\Big)$*. Then*

(a) $\bigoplus_{i\in I}\mathcal{O}_i$ *is injective if and only if* $\mathcal{O}_i$ *is injective for all* $i\in I$*.*

(b) If $\bigoplus_{i\in I}\mathcal{O}_i$ *is surjective, then* $\mathcal{O}_i$ *is surjective for every* $i\in I$*.*

(c) If, for every $i\in I$*,* $\mathcal{O}_i$ *is surjective and the family* $\{\mathcal{O}_i^\dagger\}_{i\in I}$ *is completely bounded, then* $\bigoplus_{i\in I}\mathcal{O}_i$ *is surjective.*

(d) If for every $i \in I$, $\mathcal{O}_i$ is invertible with inverse operator $\mathcal{O}_i^{-1}$ and if $\{\mathcal{O}_i^{-1}\}_{i\in I}$ is completely bounded, then also $\bigoplus_{i\in I} \mathcal{O}_i$ is invertible with inverse

$$\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big)^{-1} = \bigoplus_{i\in I} \mathcal{O}_i^{-1}. \tag{3.15}$$

(e) If $\bigoplus_{i\in I} \mathcal{O}_i$ is invertible, then $\mathcal{O}_i$ is invertible for all $i \in I$ and (3.15) holds again.

Proof. (a) follows immediately from Proposition 3.2.3 (a).
(b) Assume that $\bigoplus_{i\in I} \mathcal{O}_i$ is surjective and choose $i \in I$ arbitrary. Take some $g_i \in W_i$. Then $(0, ..., 0, g_i, 0, 0, ...) \in \big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}$ and by assumption there exists some $\{f_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ such that $\big(\bigoplus_{i\in I} \mathcal{O}_i\big)(\{f_i\}_{i\in I}) = (0, ..., 0, g_i, 0, 0, ...)$. This implies $\mathcal{O}_i f_i = g_i$, hence $\mathcal{O}_i$ is surjective.
(c) Let $i \in I$ be arbitrary. Then, by assumption, $\mathcal{O}_i$ is bounded and has closed range $\mathcal{R}(\mathcal{O}_i) = W_i$. Hence its pseudo inverse $\mathcal{O}_i^\dagger : W_i \longrightarrow V_i$ is well-defined and has (among others) the property

$$\mathcal{O}_i \mathcal{O}_i^\dagger g_i = g_i \quad (g_i \in \mathcal{R}(\mathcal{O}_i) = W_i). \tag{3.16}$$

Since the familiy $\{\mathcal{O}_i^\dagger\}_{i\in I}$ is completely bounded, by Proposition 3.2.1 the operator

$$\bigoplus_{i\in I} \mathcal{O}_i^\dagger : \Big(\sum_{i\in I} \oplus W_i\Big)_{\ell^2} \longrightarrow \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2}$$

is well-defined and bounded. Moreover, (3.16) and Proposition 3.2.2 (c) imply that for any $g = \{g_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}$ we have

$$\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big)\Big(\bigoplus_{i\in I} \mathcal{O}_i^\dagger\Big)(\{g_i\}_{i\in I}) = \bigoplus_{i\in I} (\mathcal{O}_i \mathcal{O}_i^\dagger)(\{g_i\}_{i\in I}) = \{\mathcal{O}_i \mathcal{O}_i^\dagger g_i\}_{i\in I} = \{g_i\}_{i\in I}.$$

In other words, for any $g = \{g_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}$, we can find $f := \big(\bigoplus_{i\in I} \mathcal{O}_i^\dagger\big)(g)$ in $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ such that $\big(\bigoplus_{i\in I} \mathcal{O}_i\big)(f) = g$, i.e. $\bigoplus_{i\in I} \mathcal{O}_i$ is surjective.
(d) Let $i \in I$ be arbitrary. Since $\mathcal{O}_i$ is invertible, its pseudo inverse and its inverse coincide, i.e. $\mathcal{O}_i^\dagger = \mathcal{O}_i^{-1}$. Since the family $\{\mathcal{O}_i^{-1}\}_{i\in I}$ is completely bounded, we may use (a) and (c) to see that $\bigoplus_{i\in I} \mathcal{O}_i$ is bijective and thus (by the bounded inverse theorem, see Appendix) invertible with bounded inverse. Moreover $\bigoplus_{i\in I} \mathcal{O}_i^{-1}$ is well-defined and by using Proposition 3.2.2 (c), we see that

$$\Big(\bigoplus_{i\in I} \mathcal{O}_i^{-1}\Big)\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big) = \bigoplus_{i\in I} \big(\mathcal{O}_i^{-1} \mathcal{O}_i\big) = \bigoplus_{i\in I} \mathcal{I}_{W_i} = \mathcal{I}_{(\sum_{i\in I} \oplus W_i)_{\ell^2}}.$$

Analogously we have

$$\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big)\Big(\bigoplus_{i\in I} \mathcal{O}_i^{-1}\Big) = \mathcal{I}_{(\sum_{i\in I} \oplus V_i)_{\ell^2}}.$$

This proves (3.15).
(e) If $\bigoplus_{i\in I} \mathcal{O}_i$ is invertible and hence bijective, so is $\mathcal{O}_i$ for every $i \in I$ by (a) and (b). The proof of (3.15) follows as in (d). □

Example 3.2.7. *Let $I = \mathbb{N}$ and let $V_i = W_i \neq \{0\}$ for all $i \in I$. For every $i \in I$ we define*

$$\mathcal{O}_i : V_i \longrightarrow V_i, \ \ \mathcal{O}_i := \frac{1}{\sqrt{i}} \mathcal{I}_{V_i}.$$

For each $i \in I$ we have $\|\mathcal{O}_i\|_{op} \leq 1$, i.e. the family $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded by 1. Thus, by Proposition 3.2.1, $\bigoplus_{i\in I} \mathcal{O}_i : \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2} \longrightarrow \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ is a well-defined and bounded operator.

(a) *At first we reconsider Proposition 3.2.3 (b). We give an example of an element* $g \in \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$, *which is contained in* $\big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}$ *but not in* $\mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big)$: *For every* $i \in I$, *choose some normalized vector* $h_i \in V_i$ *and set* $f_i := \frac{h_i}{\sqrt{i}} \in V_i$ *and* $g_i := \frac{h_i}{i} \in V_i$. *Clearly we have* $\mathcal{O}_i f_i = g_i$ *for each* $i \in I$. *Observe that* $g = \{g_i\}_{i\in I} \in (\sum_{i\in I} \bigoplus V_i)_{\ell^2}$, *since*

$$\|g\|^2_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} = \sum_{i\in\mathbb{N}} \|g_i\|^2_{V_i} = \sum_{i\in\mathbb{N}} \frac{1}{i^2} = \frac{\pi^2}{6}. \tag{3.17}$$

On the other hand we have $f = \{f_i\}_{i\in I} \notin (\sum_{i\in I} \bigoplus V_i)_{\ell^2}$, *since*

$$\|f\|^2_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} = \sum_{i\in\mathbb{N}} \|f_i\|^2_{V_i} = \sum_{i\in\mathbb{N}} \frac{1}{i} = \infty. \tag{3.18}$$

(3.17) implies that $g \in \big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}$. *However,* $g \notin \mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big)$, *since, by component-wise definition of* $\bigoplus_{i\in I} \mathcal{O}_i$, f *is the only possible candidate to be mapped onto* g *by* $\bigoplus_{i\in I} \mathcal{O}_i$, *but (3.18) shows that* $f \notin \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$.

(b) *Next we reconsider Proposition 3.2.6 (c). At first glance, one might guess that if* $\{\mathcal{O}_i\}_{i\in I}$ *is a completely bounded family of surjective operators, then* $\bigoplus_{i\in I} \mathcal{O}_i$ *has to be surjective as well. However, the above example demonstrates that this is not the case: The operators* $O_i = \frac{1}{\sqrt{i}} \mathcal{I}_{V_i}$ *are not only surjective and completely bounded, but also injective. However,* $\bigoplus_{i\in I} \mathcal{O}_i$ *is not surjective, since (as shown above) there exists no element in* $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$, *which is mapped onto* g *by* $\bigoplus_{i\in I} \mathcal{O}_i$. *Moreover, we have* $\mathcal{O}_i^\dagger = \mathcal{O}_i^{-1} = \sqrt{i}\mathcal{I}_{V_i}$. *We immediately see that* $\|\mathcal{O}_i^\dagger\|_{op} = \|\mathcal{O}_i^{-1}\|_{op} = \sqrt{i}$, *i.e. that the family* $\{\mathcal{O}_i^\dagger\}_{i\in I} = \{\mathcal{O}_i^{-1}\}_{i\in I}$ *is not completely bounded. This emphasizes the importance of the condition that the operators* $\mathcal{O}_i^\dagger$ *are completely bounded in order to guarantee that the operator* $\bigoplus_{i\in I} \mathcal{O}_i$ *is surjective.*

(c) *Our observations from (b) imply that* $\bigoplus_{i\in I} \mathcal{O}_i$ *does not have an inverse, since* $\bigoplus_{i\in I} \mathcal{O}_i$ *is not surjective. However, on* $\mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big)$ *the operator* $\bigoplus_{i\in I} \mathcal{O}_i^{-1}$, *which is defined component-wise, i.e.* $\bigoplus_{i\in I} \mathcal{O}_i^{-1}\{g_i\}_{i\in I} = \{\sqrt{i} g_i\}_{i\in I}$, *is well-defined, since* $\bigoplus_{i\in I} \mathcal{O}_i$ *is injective by Proposition 3.2.6 (a).* $\bigoplus_{i\in I} \mathcal{O}_i^{-1}$ *is unbounded by Proposition 3.2.1. By component-wise definition,* $\bigoplus_{i\in I} \mathcal{O}_i^{-1}$ *is a right-inverse of* $\bigoplus_{i\in I} \mathcal{O}_i$ *on* $\mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big)$ *and the only possible candidate to be the pseudo inverse of* $\bigoplus_{i\in I} \mathcal{O}_i$. *However, the pseudo inverse of* $\bigoplus_{i\in I} \mathcal{O}_i$ *does not exist, since by (a) we have*

$$\mathcal{N}\Big(\bigoplus_{i\in I} \mathcal{O}_i^{-1}\Big) = \{0\}_{i\in I} = \Big(\sum_{i\in I} \oplus V_i\Big)^\perp_{\ell^2} \neq \mathcal{R}\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big)^\perp,$$

i.e. (2.3) is hurt. We further observe that $\bigoplus_{i\in I} \mathcal{O}_i$ *does not have closed range, since by Proposition 3.2.4 (a) this would imply* $\mathcal{R}\big(\bigoplus_{i\in I} \mathcal{O}_i\big) = \big(\sum_{i\in I} \oplus \mathcal{R}(\mathcal{O}_i)\big)_{\ell^2}$, *which we have disproved in (a) by giving a counter example.*

Remark 3.2.8. *In Chapter 5 we will continue our discussion about bounded and unbounded operators between Hilbert direct sums in a more general set up.*

4 Fusion frames and operators

In this chapter we finally present *fusion frames* and closely related notions. In order to motivate this concept, we will continue our discussion about frames and their applications in Section 4.1. In Section 4.2 we will present some of the basic concepts of the theory of fusion frames. In Section 4.3 we will consider fusion frame systems and will prove some new operator identities for fusion frame systems involving component preserving operators between Hilbert direct sums, which we have discussed in Chapter 3. We will investigate further properties of fusion frame systems and their connection to distributed processing in Section 4.4 by applying the operator theoretic results from Section 4.3 and our theory from Chapter 3. In Section 4.5 we will consider the aspect of duality for fusion frames and will be able to extend some results about the duality for fusion frames, that have already been published in literature.

4.1 Motivation for fusion frames

In Section 2.2 we have emphasized the usefulness of frames for real life applications in terms of their properties perfect reconstruction, stability and redundancy. However, as already mentioned in the introduction, sometimes we are overwhelmed by the big amount of data coming from one single frame, which has to be computed numerically. A natural idea is to split such a large frame system into several (*local*) components, which are processed separately from each other, before the information is fused together. Such a principle is called *distributed processing*.

Moreover, not only the numerical aspect of frame theory calls for a frame-like theory which models distributed processing, but also various applications themselves require distributed processing techniques. For instance, wireless sensor networks [36], geophones in geophysics measurements and studies [20], or the physiological structure of visual and hearing systems [42]. Fusion frames are also involved in solving operator equations numerically: Domain decomposition methods [55] solve a boundary value problem by splitting into smaller boundary value problems on subdomains. Fusion Frames are involved in combining the solution spaces in a natural way. See also [7] for more details.

To motivate the ingredients of the definition of a *fusion frame* a little bit further, let us continue with the discussion of wireless sensor networks. Roughly speaking, the principle of wireless sensor networks is that a number of sensors is located in a certain area to measure different things. Such a system of sensors is usually redundant. Each sensor is modeled as a frame element in a frame for a certain Hilbert space $\mathcal{H}$. Typically, such sensors have severe constraints in their processing power and transmission bandwith, – simply due to practical and financial reasons. Therefore, large sensor networks are often split into smaller redundant sensor networks (following the idea of "splitting up into local components" – as described above), which are modeled as sets of (closed) subspaces V_i of $\mathcal{H}$. Moreover, it is often advantageous to not only consider a family of subspaces, but also a family of weights assigned to it. Each subspace V_i is equipped with a weight v_i in order to assign its *"importance"* to it. For instance, consider the more concrete situation, where the sensors are located in a forest to measure attributes like temperature, sound, vibration or pressure. In some areas there might be more additional disturbing factors like wind or other natural forces than in other areas, which makes some of the sub-sensor networks less reliable than others and therefore demands a weighted structure of these sub-sensor networks. In addition to the latter, we don't restrict the family of subspaces to be orthogonal, but consider the general case, where the subspaces may have overlaps, since some applications require (or even enforce) non-orthogonal structures between the subspaces V_i. Of course, the procedure of splitting the large sensor network (i.e. the large frame) into smaller components which are processed at a *local* level first before being fused together to a *global* level, should yield the same output as processing the large family of sensors all at once at a *global* level.

We will see that *fusion frames* take all our demands into account. They not only fit all our needs explained above, but also generalize the concept of (classical) frames, which we introduced in Chapter 2. We will see that the concept of *fusion frames* indeed enables us to split a large frame into (possibly overlapping) smaller families of frames (with a weighted structure) and also allows the converse, – i.e. to fuse families of frames together to one large frame (if the subspaces spanned from these families of frames form a *fusion frame*). We will also see that *fusion frames* share a lot of properties with classical frames. However, some aspects, like the concept of duality, make *fusion frames* a non-trivial generalization of classical frames, which makes the theory more interesting from a mathematical point of view.

4.2 Some basic principles of Fusion Frame Theory

In the context of *fusion frames* (the exact definition follows) we always consider families $\{V_i\}_{i\in I}$ of *closed* subspaces V_i ($i \in I$, I countable) of $\mathcal{H}$. Since these subspaces V_i are closed, they are Hilbert spaces as well. As already revealed in Section 4.1, we assign to each family $\{V_i\}_{i\in I}$ of (closed) subspaces a family of *weights* $v = \{v_i\}_{i\in I}$, i.e. $v_i > 0$ for every $i \in I$.

In the following we give the basic definitions for a *fusion frame* and related notions. Originally, *fusion frames* were first introduced in [17] as *frames of subspaces*, however we will refer to them as *fusion frames*, following the labeling of later publications such as [15] or [45].

Definition (fusion frame) 4.2.1. *Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and let $\{v_i\}_{i\in I}$ be a family of weights. The sequence $V = \{(V_i, v_i)\}_{i\in I}$ is called a* fusion frame *for $\mathcal{H}$, if there exist constants $0 < C_V \leq D_V < \infty$, called* lower *and* upper fusion frame bound *respectively, such that for every $f \in \mathcal{H}$*

$$C_V\|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2 \leq D_V \|f\|^2. \tag{4.1}$$

As proclaimed before, every frame can be viewed as a fusion frame. Therefore, fusion frames may be viewed as a generalization of frames. For the proof to work, we assume without loss of generality that every frame element is non-zero. For the sake of completeness we present the proof (see also [15] for a similar proof).

Lemma 4.2.2. *Let $\{\psi_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with frame bounds A and B, such that $\psi_i \neq 0$ for all $i \in I$. For every $i \in I$ we set $V_i := \overline{span}\{\psi_i\}$ and $v_i := \|\psi_i\|^2$. Then $\{(V_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds A and B.*

Proof. Clearly, the singleton $\{\frac{\psi_i}{\|\psi_i\|}\}$ is an orthonormal basis for the one-dimensional closed subspace $\overline{span}\{\psi_i\}$ of $\mathcal{H}$. Therefore, for every $f \in \mathcal{H}$, we have

$$
\begin{aligned}
\sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2 &= \sum_{i\in I} \|\psi_i\|^2 \langle \pi_{\overline{span}\{\psi_i\}} f, f\rangle && (*) \\
&= \sum_{i\in I} \|\psi_i\|^2 \Big\langle \langle \pi_{\overline{span}\{\psi_i\}} f, \frac{\psi_i}{\|\psi_i\|}\rangle \frac{\psi_i}{\|\psi_i\|}, f\Big\rangle \\
&= \sum_{i\in I} \|\psi_i\|^2 \Big\langle \langle f, \frac{\psi_i}{\|\psi_i\|}\rangle \frac{\psi_i}{\|\psi_i\|}, f\Big\rangle && (**) \\
&= \sum_{i\in I} |\langle f, \psi_i\rangle|^2.
\end{aligned}
$$

This implies that for all $f \in \mathcal{H}$ we have

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, \psi_i\rangle|^2 = \sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2 \leq B\|f\|^2,$$

which finishes the proof. □

In the above proof in equation $(*)$ we used the fact that π_{V_i} is an orthogonal projection an therefore satisfies $\pi_{V_i}^* = \pi_{V_i}$ and $\pi_{V_i}^2 = \pi_{V_i}$ for every $i \in I$, which implies $\|\pi_{V_i} f\|^2 = \langle \pi_{V_i} f, \pi_{V_i} f\rangle = \langle \pi_{V_i} f, f\rangle$. In equation $(**)$ we implicitly used the orthogonal decomposition $\mathcal{H} = V_i \oplus V_i^\perp$ $(i \in I)$ which implies $\langle f, \psi_i\rangle = \langle \pi_{V_i} f, \psi_i\rangle$ for every $i \in I$, since $\psi_i \in V_i$ and thus $\langle f, \psi_i\rangle = \langle \pi_{V_i} f + \pi_{V_i^\perp} f, \psi_i\rangle = \langle \pi_{V_i} f, \psi_i\rangle + \langle \pi_{V_i^\perp} f, \psi_i\rangle = \langle \pi_{V_i} f, \psi_i\rangle$. We will implicitly use arguments like these several times in what lies ahead.

Since fusion frames can be viewed as generalization of classical frames, it is not surprising that a lot of frame related notions and properties also extend to the fusion frame setting. In the following we will give the relevant definitions and present some results for fusion frames, see [17], [15] or [45] for more details.

We call a fusion frame V a C_V*-tight* fusion frame or simply *tight* fusion frame, if the fusion frame bounds C_V and D_V in (4.1) can be chosen to be equal. A 1-tight fusion frame is also called *Parseval fusion frame* fusion frame. The above lemma and its proof imply that every A-tight frame defines an A-tight fusion frame and that every Parseval frame defines a Parseval fusion frame.

If in (4.1) only the upper inequality is considered, we call V a *Bessel fusion sequence* and in this case call D_V *Bessel fusion bound*.

A Bessel fusion sequence $V = \{(V_i, v_i)\}_{i\in I}$ is called v*-uniform* if $v_i = v > 0$ for all $i \in I$.

A fusion frame $V = \{(V_i, v_i)\}_{i\in I}$ is called a *Riesz decomposition*, if for every $f \in \mathcal{H}$ there is a unique choice of $f_i \in V_i$ such that $f = \sum_{i\in I} f_i$. The uniqueness of the vectors f_i implies that $V_i \cap V_j = \{0\}$ for all $i \neq j$.

A family $V = \{(V_i, v_i)\}_{i\in I}$ is called a *fusion Riesz basis*, if $\overline{span}\{V_i\}_{i\in I} = \mathcal{H}$ and if there exist constants $0 < C_V \le D_V < \infty$, called *lower* and *upper fusion Riesz basis constant* respectively, such that for any finite subset $J \subseteq I$ we have

$$C_V \sum_{j\in J} \|f_j\|^2 \le \left\| \sum_{j\in J} v_j f_j \right\|^2 \le D_V \sum_{j\in J} \|f_j\|^2, \tag{4.2}$$

for all sequences $\{f_j\}_{j\in J} \in \{V_j\}_{j\in J}$. It can be shown that every fusion Riesz basis is a fusion frame.

We call a family $\{V_i\}_{i\in I}$ of closed subspaces of $\mathcal{H}$ an orthonormal fusion basis, if $\mathcal{H} = \bigoplus_{i\in I} V_i$. Clearly, every orthonormal fusion basis is a Riesz decomposition.

Theorems 4.2.11 and 4.2.12, which we will formulate a little bit later, will make the relations between the above notions clearer.

The next lemma shows that the family of weights $\{v_i\}_{i\in I}$ corresponding to a Bessel fusion sequence (i.e. a fusion frame) $V = \{(V_i, v_i)\}_{i\in I}$ is in $\ell^\infty(I)$. We only have to assume without loss of generality that none of the spaces V_i is zero.

Lemma 4.2.3. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a Bessel fusion sequence with Bessel fusion bound D_V. Then for every $i \in I$ such that $V_i \neq 0$ we have $v_i \le \sqrt{D_V}$*

Proof. Let $i \in I$ be arbitrary such that $V_i \neq \{0\}$. Then we can choose some non-zero $f \in V_i$ and see that

$$v_i^2 \|f\|^2 = v_i^2 \|\pi_{V_i} f\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2 \le D_V \|f\|^2.$$

□

Corollary 4.2.4. *If $\psi = \{\psi_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with frame bounds A_ψ and B_ψ, then*

$$\|\psi_i\| \le \sqrt[4]{B_\psi} \text{ for all } i \in I.$$

Proof. Assume without loss of generality that $\psi_i \neq 0$ for all $i \in I$. By Lemma 4.2.2, $\{(\overline{span}\{\psi_i\}, \|\psi_i\|^2)\}_{i\in I}$ is a fusion frame with fusion frame bounds A_ψ and B_ψ. Thus by Lemma 4.2.3, $\|\psi_i\|^2 \le \sqrt{B_\psi}$. □

It is well-known that for any frame (or Bessel sequence) $\psi = \{\psi_i\}_{i\in I}$ for $\mathcal{H}$ with upper bound (Bessel bound) B_ψ we have $\|\psi_i\| \le \sqrt{B_\psi}$ for all $i \in I$. To see this, observe that for any $i \in I$ we have

$$\|\psi_i\|^4 = |\langle \psi_i, \psi_i \rangle|^2 \le \sum_{j\in I} |\langle \psi_i, \psi_j \rangle|^2 \le B_\psi \|\psi_i\|^2.$$

Thus, if $B_\psi > 1$, then Corollary 4.2.4 gives a tighter upper bound for the atoms of the frame (or Bessel sequence). Interestingly, this easy proof was done by viewing frames as fusion frames.

If $V = \{(V_i, v_i)\}_{i\in I}$ is a Bessel fusion sequence or a fusion frame, then in general the family of weights $\{v_i\}_{I\in I}$ is not necessarily bounded from below by some positive constant. However, in case $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, the associated family of weights $\{v_i\}_{i\in I}$ is indeed bounded from above and from below by a positive constant, where we again exclude trivial cases like $V_i = 0$ for some $i \in I$.

Lemma 4.2.5. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a fusion Riesz basis with fusion Riesz basis constants C and D. Then $\sqrt{C} \leq v_i \leq \sqrt{D}$ for all $i \in I$ such that $V_i \neq 0$.*

Proof. Recall that by definition of a fusion Riesz basis, for any finite subset $J \subseteq I$ and all families $\{f_j\}_{j\in J}$ we have

$$C \sum_{j\in J} \|f_j\|^2 \leq \left\| \sum_{j\in J} v_j f_j \right\|^2 \leq D \sum_{j\in J} \|f_j\|^2.$$

Now choose some arbitrary $i \in I$ with $V_i \neq \{0\}$, set $J = \{i\}$, choose some normalized $f \in V_i$ and apply the above. □

It is easy to see that every orthonormal fusion basis is a fusion Riesz basis:

Proposition 4.2.6. *Every orthonormal fusion basis $\{V_i\}_{i\in I}$ is a 1-uniform fusion Riesz basis with fusion Riesz basis constants $C = D = 1$.*

Proof. By assumption, we have $\mathcal{H} = \bigoplus_{i\in I} V_i$, which clearly implies $\overline{span}\{V_i\}_{i\in I} = \mathcal{H}$. Now choose some arbitrary finite subset $J \subseteq I$, and for any $j \in J$ choose some arbitrary $f_j \in V_j$. Then, since $V_k \perp V_l$ for all $k, l \in I$ with $k \neq l$, we have

$$\sum_{j\in J} \|f_j\|^2 = \left\| \sum_{j\in J} f_j \right\|^2 .$$

□

The above statement follows also directly from the following statement, which has been proved in [17].

Proposition [17] 4.2.7. *A family of subspaces $\{V_i\}_{i\in I}$ is an orthonormal fusion basis if and only if it is a 1-uniform Parseval fusion frame.*

Recall that for any Bessel sequence (hence also for any frame) ψ we defined its corresponding synthesis operator T_ψ, analysis operator T_ψ^* and frame operator $S_\psi = T_\psi T_\psi^*$. We can also define their analogs for fusion frames. As already mentioned in the beginning of Chapter 3, in the fusion frame setting we use the space $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ as representation space for our signals (and as domain for our new synthesis operator), instead of the representation space $\ell^2(I)$ as in the classical frame setting. Recall that in the fusion frame definition we demand the subspaces V_i to be *closed* subspaces of $\mathcal{H}$, which implies that they are Hilbert spaces themselves. This means that the Hilbert direct sum $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ is a Hilbert space and that we may apply all our results from Chapter 3. As in classical frame theory, the fusion frame related versions of the synthesis, analysis and frame operator (which we will define below) will be bounded operators between two Hilbert spaces.

For any Bessel fusion sequence $V = \{(V_i, v_i)\}_{i\in I}$ in $\mathcal{H}$ with Bessel fusion bound D_V we define the operator T_V, called *fusion synthesis operator*, via

$$T_V : \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2} \longrightarrow \mathcal{H},$$

$$T_V\big(\{f_i\}_{i\in I}\big) = \sum_{i\in I} v_i f_i.$$

In [17] it is shown that $\|T_V\|_{op} \leq \sqrt{D_V}$ and that the series above converges unconditionally. In fact, one can show [17] that given an arbitrary weighted sequence $W = \{(W_i, w_i)\}_{i\in I}$ of closed subspaces of $\mathcal{H}$, the fusion synthesis operator T_W is well-defined on $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ and bounded by the constant $D > 0$ if and only if W is a Bessel fusion sequence with Bessel fusion bound D.

If V is a Bessel fusion sequence for $\mathcal{H}$ with Bessel fusion bound D_V and T_V its associated (bounded) fusion synthesis operator, then its adjoint

$$T_V^* : \mathcal{H} \longrightarrow \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$$

is called *fusion analysis operator* and is given by [17]

$$T_V^*(f) = \{v_i \pi_{V_i} f\}_{i\in I}.$$

T_V^* then is of course also bounded, since we have $\|T_V^*\|_{op} = \|T_V\|_{op} \leq \sqrt{D_V}$.

Analogously to classical Frame Theory, we define the *fusion frame operator* as composition of the fusion synthesis operator with the fusion analysis operator

$$S_V = T_V T_V^* : \mathcal{H} \longrightarrow \mathcal{H},$$

$$S_V(f) = \sum_{i\in I} v_i^2 \pi_{V_i} f,$$

which, in case $V = \{(V_i, v_i)\}_{i\in I}$ is a Bessel fusion sequence with Bessel bound D_V, is bounded by D_V, since then

$$\|S_V\|_{op} = \|T_V T_V^*\|_{op} = \|T_V\|_{op}^2 \leq D_V.$$

Moreover, the above series converges unconditionally. Clearly, S_V is a self-adjoint operator, since

$$S_V^* = (T_V T_V^*)^* = T_V T_V^* = S_V.$$

In case V is a fusion frame with fusion frame bounds C_V and D_V, the fusion frame inequalities (4.1) can be rewritten in terms of the fusion frame operator S_V via

$$C_V \|f\|^2 \leq \langle S_V f, f\rangle \leq D_V \|f\|^2. \tag{4.3}$$

This implies that S_V is a positive operator. As in the classical frame situation this implies

$$C_V \leq \|S_V\|_{op} \leq D_V. \tag{4.4}$$

Completely analogously to the classical frame setting we can conclude that S_V is invertible. See [17] and [18] for more details. We call S_V^{-1} the inverse fusion frame operator, which is clearly also self-adjoint, invertible and bounded (by the Bounded Inverse Theorem, see Appendix). S_V^{-1} is also positive, which follows from the inequalities

$$D_V^{-1} \leq \|S_V^{-1}\|_{op} \leq C_V^{-1}. \tag{4.5}$$

To prove these inequalities, observe that $1 \leq \|S_V\|_{op}\|S_V^{-1}\|_{op}$ implies $D_V^{-1} \leq \|S_V\|_{op}^{-1} \leq \|S_V^{-1}\|_{op}$ and that by (4.3), for every $f \in \mathcal{H}$, we have

$$C_V\|S_V^{-1}f\|^2 \leq \langle S_V S_V^{-1}f, S_V^{-1}f\rangle = \langle f, S_V^{-1}f\rangle \leq \|S_V^{-1}\|\|f\|^2,$$

which implies

$$C_V\|S_V^{-1}\|_{op}^2 = \sup_{\|f\|=1} C_V\|S_V^{-1}f\|^2 \leq \sup_{\|f\|=1} \|S_V^{-1}\|\|f\|^2 = \|S_V^{-1}\|_{op},$$

which proves the right inequality from (4.5).

As in Frame Theory, the invertibility of the fusion frame operator yields reconstruction formulas. More precisely, for any $f \in \mathcal{H}$ we have $f = S_V S_V^{-1} f = S_V^{-1} S_V f$. Applying the definition of S_V yields the *fusion frame reconstruction* formulas

$$f = \sum_{i\in I} v_i^2 \pi_{V_i} S_V^{-1} f = \sum_{i\in I} v_i^2 S_V^{-1} \pi_{V_i} f. \tag{4.6}$$

Therefore, knowing the inverse fusion frame operator is of crucial importance. In general, as in the classical frame setting, this is not an easy task. However, if we are given a tight fusion frame, then the fusion frame operator and inverse fusion frame operator become very simple objects. For sake of completeness, we will formulate the results, but will not write down the proofs, since they coincide almost word by word with the proofs for the analogous frame theoretic results.

Proposition 4.2.8. *A fusion frame V for $\mathcal{H}$ is a C_V-tight fusion frame if and only if $S_V = C_V \mathcal{I}_{\mathcal{H}}$.*

Corollary 4.2.9. *A fusion frame V is a Parseval fusion frame if and only if $S_V = \mathcal{I}_{\mathcal{H}}$.*

Corollary 4.2.10. *For a C_V-tight fusion frame V, the fusion frame reconstruction formulas reduce to the fusion reconstruction formula*

$$f = \frac{1}{C_V}\sum_{i\in I} v_i^2 \pi_{V_i} f. \tag{4.7}$$

In case V is a Parseval fusion frame, then the reconstruction formulas reduce to

$$f = \sum_{i\in I} v_i^2 \pi_{V_i} f. \tag{4.8}$$

In Theorem 2.1.6 we gave characterizing conditions for a sequence $\{\psi_i\}_{i\in I} \subseteq \mathcal{H}$ being a frame in terms of its synthesis and analysis operator respectively. For fusion frames an analogous statement holds [17]:

Theorem [17] 4.2.11. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$ with weights $v_i > 0$. Then the following are equivalent.*

(i) V is a fusion frame for $\mathcal{H}$.

(ii) The synthesis operator T_V is bounded and surjective.

(iii) The analysis operator T_V^* is bounded and injective.

For fusion frames that are also fusion Riesz bases, the following characterizing conditions can be proven. For a proof all the subsequent characterizations we refer the reader to [17] and [45].

Theorem [17] [45] 4.2.12. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ and $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for V_i for each $i \in I$. Then the following are equivalent.*

(i) V is a fusion Riesz basis for $\mathcal{H}$.

(ii) The synthesis operator T_V is bounded and bijective.

(iii) The analysis operator T_V^* is bounded and bijective.

(iv) V is a Riesz decomposition for $\mathcal{H}$.

(v) $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a Riesz basis for $\mathcal{H}$.

(vi) $S_V^{-1} V_i \perp V_j$ for all $i, j \in I$ with $i \neq j$.

(vii) $v_i^2 \pi_{V_i} S_V^{-1} \pi_{V_j} = \delta_{ij} \pi_{V_j}$ for all $i, j \in I$.

The previous two results will be useful later on, when we prove some properties of so-called *fusion frame systems*.

Let $V = \{(V_i, v_i)\}_{i\in I}$ be a fusion Riesz basis with fusion Riesz basis constants $C \leq D$. Then V is also a fusion frame and we denote the fusion frame bounds by $C_V \leq D_V$. By Lemma 4.2.5, we know that

$$C \leq v_i \leq D \quad \text{for all } \ i \in I. \tag{4.9}$$

If we set $V^u = \{(V_i, 1)\}_{i\in I}$, then its corresponding analysis operator

$$T_{V^u}^* : \mathcal{H} \longrightarrow \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2}, \quad T_{V^u}^* f = \{\pi_{V_i} f\}_{i\in I}, \tag{4.10}$$

is well-defined and bounded, since for any $f \in \mathcal{H}$ we have

$$\begin{aligned} \|T_{V^u}^* f\|^2_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} &= \sum_{i\in I} \|\pi_{V_i} f\|^2 \\ &= \sum_{i\in I} v_i^2 v_i^{-2} \|\pi_{V_i} f\|^2 \\ &\leq C^{-2} \sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2 \leq C^{-2} D_V \|f\|^2 < \infty. \end{aligned}$$

Therefore T_{V^u} is also bounded by $C^{-2} D_V$. Now, recall that by Theorem 4.2.12, V is a fusion Riesz basis if and only if it is a Riesz decomposition. This means that for every $f \in \mathcal{H}$, there exists a unique sequence $\{f_i\}_{i\in I}$, where $f_i \in V_i$ (for all $i \in I$), such that $f = \sum_{i\in I} f_i$. This sequence is always an element in $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$: Since, by Theorem 4.2.12, T_V is bounded and bijective, for each $f \in \mathcal{H}$, there exists precisely one sequence $\{g_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$, such that $f = T_V \{g_i\}_{i\in I} = \sum_{i\in I} v_i g_i$. Thus $f_i = v_i g_i$ for every $i \in I$. However, observe that (4.9) implies that $\{g_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ if and only if $\{f_i\}_{i\in I} = \{v_i g_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$. Therefore, for any fusion Riesz basis V, the operator

$$\mathcal{Q} : \mathcal{H} \longrightarrow \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2}, \quad f \mapsto \{f_i\}_{i\in I}, \tag{4.11}$$

which assigns to any $f \in \mathcal{H}$ its corresponding sequence $\{f_i\}_{i\in I}$, such that $f = \sum_{i\in I} f_i$, is well-defined. Moreover, $\mathcal{Q}$ clearly is bijective and its inverse operator clearly is given by

$$\mathcal{Q}^{-1} : \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2} \longrightarrow \mathcal{H}, \quad \mathcal{Q}^{-1}\{f_i\}_{i\in I} = \sum_{i\in I} f_i.$$

In other words, this means that $\mathcal{Q}^{-1} = T_{V^u}$, i.e. $\mathcal{Q} = T_{V^u}^{-1}$ is bounded.

Now, we observe that since the operators $v_i^{-1}\mathcal{I}_{V_i} : V_i \longrightarrow V_i$ are completely bounded by $\frac{1}{C}$, by Proposition 3.2.1, the operator

$$\bigoplus_{i\in I}(v_i^{-1}\mathcal{I}_{V_i}) : \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2} \longrightarrow \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2}$$

is well-defined and bounded. We further observe that we have $T_V\{v_i^{-1}f_i\}_{i\in I} = \sum_{i\in I} f_i = f$. This means that $\bigoplus_{i\in I}(v_i^{-1}\mathcal{I}_{V_i})T_{V^u}^{-1}$ is a right-inverse of T_V and since, by Theorem 4.2.12, T_V is invertible, we obtain

$$T_V^{-1} = \bigoplus_{i\in I}(v_i^{-1}\mathcal{I}_{V_i})T_{V^u}^{-1} = \bigoplus_{i\in I}(v_i^{-1}\mathcal{I}_{V_i})\mathcal{Q}. \tag{4.12}$$

In other words this means

$$T_V^{-1} f = \{v_i^{-1} f_i\}_{i\in I}. \tag{4.13}$$

Note that analogously we see that

$$\begin{aligned}
T_V &= T_{V^u} \bigoplus_{i\in I}(v_i\mathcal{I}_{V_i}) \\
T_V^* &= \bigoplus_{i\in I}(v_i\mathcal{I}_{V_i})T_{V^u}^* \\
S_V &= T_{V^u} \bigoplus_{i\in I}(v_i^2\mathcal{I}_{V_i})T_{V^u}^* \\
(T_V^*)^{-1} &= (T_{V^u}^{-1})^* \bigoplus_{i\in I}(v_i^{-1}\mathcal{I}_{V_i}) \\
S_V^{-1} &= (T_V^*)^{-1}T_V^{-1} = (T_{V^u}^{-1})^* \bigoplus_{i\in I}(v_i^{-2}\mathcal{I}_{V_i})T_{V^u}^{-1}.
\end{aligned}$$

These formulas look nice, but neither the action of $\mathcal{Q}$ on $f \in \mathcal{H}$ (4.11), nor the action of T_V^{-1} on f (4.13) is given explicitly in terms of f. So the above observations don't seem to help us in making any progress. However, we can consider two special cases, where the above observations are indeed helpful.

Before we formulate the next result, we fix the following notation: For any $n \in I$, we set

$$\mathcal{V}_n := \ldots \times \{0\} \times \{0\} \times V_n \times \{0\} \times \{0\} \times \ldots \quad (V_n \text{ in the } n\text{-th component}). \tag{4.14}$$

Clearly, $\mathcal{V}_n$ is a closed subspace of $\left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ for every $n \in I$.

The following result will be applied in Section 4.4.

Propostion 4.2.13. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a fusion Riesz basis. Let $i \in I$ be arbitrary and $g_i \in V_i$. Then the action of $T_V^{-1} : \mathcal{H} \longrightarrow \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ on g_i is given by*

$$T_V^{-1} g_i = (\ldots, 0, 0, v_i^{-1} g_i, 0, 0, \ldots). \quad (v_i^{-1} g_i \text{ in the } i\text{-th component})$$

In other words, T_V^{-1} maps V_i into $\mathcal{V}_i$ for every $i \in I$.

Proof. Clearly $(\ldots, 0, 0, g_i, 0, 0, \ldots)$ is the unique sequence $\{f_i\}_{i\in I}$ with $f_i \in V_i$ $(i \in I)$, such that $g_i = \sum_{i\in I} f_i$. In other words, $\mathcal{Q} g_i = T_{V^u}^{-1} g_i = (\ldots, 0, 0, g_i, 0, 0, \ldots)$ (where g_i is in the i-th entry). Now (4.12) implies

$$T_V^{-1} g_i = \bigoplus_{i\in I} (v_i^{-1} \mathcal{I}_{V_i})(\ldots, 0, 0, g_i, 0, 0, \ldots) = (\ldots, 0, 0, v_i^{-1} g_i, 0, 0, \ldots).$$

□

In case V is an orthonormal fusion basis, we can give explicit formulas of the inverses of the fusion synthesis, fusion analysis and fusion frame operator respectively. Parts of the result have already been used in literature (see [15] for instance).

Propostion 4.2.14. *Assume that $V = \{(V_i, v_i)\}_{i\in I}$ is an orthonormal fusion basis, i.e. $\mathcal{H} = \bigoplus_{i\in I} V_i$. Then the inverse fusion synthesis operator*

$$T_V^{-1} : \mathcal{H} \longrightarrow \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$$

is given by

$$T_V^{-1} f = \{v_i^{-1} \pi_{V_i} f\}_{i\in I}, \tag{4.15}$$

the inverse fusion analysis operator

$$(T_V^*)^{-1} : \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2} \longrightarrow \mathcal{H}$$

is given by

$$(T_V^*)^{-1}(\{f_i\}_{i\in I}) = \sum_{i\in I} v_i^{-1} f_i, \tag{4.16}$$

and the inverse fusion frame operator

$$S_V^{-1} : \mathcal{H} \longrightarrow \mathcal{H}$$

is given by

$$S_V^{-1} f = \sum_{i\in I} v_i^{-2} \pi_{V_i} f. \tag{4.17}$$

Proof. $\mathcal{H} = \bigoplus_{i\in I} V_i$ implies that V is a Riesz decomposition and thus, by Theorem 4.2.12, a fusion Riesz basis. Moreover we immediately see that $\mathcal{Q} = T_{V^u}^{-1}$ is given by

$$\mathcal{Q} f = T_{V^u}^{-1} f = \{\pi_{V_i} f\}_{i\in I}.$$

Thus (4.12) implies (4.15). To prove (4.16), observe that since $(T_V^{-1})^* = (T_V^*)^{-1}$, it suffices to calculate the adjoint $(T_V^{-1})^*$ of T_V^{-1}. To this end, let $\{g_i\}_{i\in I} \in (\sum_{i\in I} \oplus V_i)_{\ell^2}$ and $f \in \mathcal{H}$ be arbitrary and observe that by the definition of an adjoint we have

$$\begin{aligned} \langle f, (T_V^{-1})^* \{g_i\}_{i\in I} \rangle_{\mathcal{H}} &= \langle T_V^{-1} f, \{g_i\}_{i\in I} \rangle_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} \\ &= \sum_{i\in I} \langle v_i^{-1} \pi_{V_i} f, g_i \rangle_{\mathcal{H}} \\ &= \sum_{i\in I} \langle f, v_i^{-1} \pi_{V_i} g_i \rangle_{\mathcal{H}} = \langle f, \sum_{i\in I} v_i^{-1} g_i \rangle_{\mathcal{H}}. \end{aligned}$$

This implies (4.16). Finally, (4.17) follows from observing that for any $f \in \mathcal{H}$ we have

$$S_V^{-1} f = (T_V^*)^{-1} T_V^{-1} f = (T_V^*)^{-1}(\{v_i^{-1}\pi_{V_i} f\}_{i\in I}) = \sum_{i\in I} v_i^{-2}\pi_{V_i} f.$$

This finishes the proof. □

In case $V = \{(V_i, v_i)\}_{i\in I}$ is a fusion frame but not a fusion Riesz basis, Theorems 4.2.11 and 4.2.12 state that its corresponding fusion synthesis operator T_V is not invertible, but has closed range $\mathcal{R}(T_V) = \mathcal{H}$, since T_V is surjective. Therefore we may consider its pseudo inverse operator $T_V^\dagger : \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2} \longrightarrow \mathcal{H}$. In Chapter 2, we saw that the pseudo inverse operator $T_\psi^\dagger$ of the synthesis operator T_ψ corresponding to a frame ψ is given by

$$T_\psi^\dagger = T_\psi^* S_\psi^{-1}.$$

A fair guess would be, that in the fusion frame setting we analogously have

$$T_V^\dagger = T_V^* S_V^{-1}.$$

In order to prove that this relation indeed is true, let us prove the following preparatory result.

Lemma 4.2.15. *If $V = \{(V_i, v_i)\}_{i\in I}$ is a fusion frame, then*

$$\mathcal{N}(T_V)^\perp = \mathcal{R}(T_V^*).$$

Proof. Since V is a fusion frame, T_V is bounded and surjective (by Theorem 4.2.11), i.e. has closed range, which implies that $\mathcal{R}(T_V^*)$ is closed. Together with (3.11) and (3.12) this implies

$$\mathcal{N}(T_V)^\perp = (\mathcal{R}(T_V^*)^\perp)^\perp = \overline{\mathcal{R}(T_V^*)} = \mathcal{R}(T_V^*).$$

□

We also note the following nice consequence of the previous result.

Corollary 4.2.16. *If $V = \{(V_i, v_i)\}_{i\in I}$ is a fusion frame, then*

$$\Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2} = \mathcal{N}(T_V) \oplus \mathcal{R}(T_V^*).$$

Proof. Since $\mathcal{N}(T_V)$ is a closed subspace of $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ we have $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^2} = \mathcal{N}(T_V) \oplus \mathcal{N}(T_V)^\perp$. Applying Corollary 4.2.15 finishes the proof. □

Corollary 4.2.15 enables us to prove the already mentioned explicit formula for the pseudo inverse of the fusion synthesis operator associated to a given fusion frame.

Proposition 4.2.17. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a fusion frame. Then the pseudo inverse $T_V^\dagger : \mathcal{H} \longrightarrow \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ of T_V is given by*

$$T_V^\dagger = T_V^* S_V^{-1}.$$

Proof. The pseudo inverse $T_V^\dagger$ of T_V is characterized by being the unique operator $T_V^\dagger : \mathcal{H} \longrightarrow \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$, which satisfies the three relations

$$\mathcal{N}(T_V^\dagger) = \mathcal{R}(T_V)^\perp, \;\; \mathcal{R}(T_V^\dagger) = \mathcal{N}(T_V)^\perp, \;\; T_V T_V^\dagger f = f \;\; (f \in \mathcal{R}(T_V)). \tag{4.18}$$

Since, by Theorem 4.2.11, T_V is surjective, we have $\mathcal{R}(T_V) = \mathcal{H}$ and $\mathcal{R}(T_V)^\perp = \{0\}$. This implies that the first relation in (4.18) is equivalent to $T_V^\dagger$ being injective and that the third relation in (4.18) is equivalent to $T_V^\dagger$ being a right-inverse of T_V. Now observe that $T_V^* S_V^{-1}$ is injective since both T_V^* and S_V^{-1} are injective. Moreover $T_V^* S_V^{-1}$ clearly is a right-inverse of T_V. Finally, Corollary 4.2.15 guarantees that $\mathcal{R}(T_V^* S_V^{-1}) = \mathcal{R}(T_V^*) = \mathcal{N}(T_V)^\perp$. □

In an analogous fashion we prove a formula for the pseudo inverse of the fusion analysis operator.

Proposition 4.2.18. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a fusion frame. Then the pseudo inverse $(T_V^*)^\dagger : \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2} \longrightarrow \mathcal{H}$ of T_V^* is given by*

$$(T_V^*)^\dagger = S_V^{-1} T_V.$$

In particular, we have

$$(T_V^*)^\dagger = (T_V^\dagger)^*.$$

Proof. This time, the three defining properties for the pseudo inverse $(T_V^*)^\dagger$ of T_V^* reduce to

$$\begin{aligned} \mathcal{N}((T_V^*)^\dagger) &= \mathcal{R}(T_V^*)^\perp = \mathcal{N}(T_V) && \text{(by Corollary 4.2.16)}\\ \mathcal{R}((T_V^*)^\dagger) &= \mathcal{N}(T_V^*)^\perp = \mathcal{H} && \text{(by Theorem 4.2.11)}\\ T_V^*(T_V^*)^\dagger \{f_i\}_{i\in I} &= \{f_i\}_{i\in I} \;\; (\{f_i\}_{i\in I} \in \mathcal{R}(T_V^*)). \end{aligned}$$

Now observe that $\mathcal{N}(S_V^{-1} T_V) = \mathcal{N}(T_V)$ since S_V^{-1} is bijective. Moreover $S_V^{-1} T_V$ is surjective since both S_V^{-1} and T_V are surjective for any fusion frame (by Theorem 4.2.11). Finally observe that for any $\{f_i\}_{i\in I} = T_V^* g \in \mathcal{R}(T_V^*))$ ($g \in \mathcal{H}$ suitable) we have

$$T_V^* S_V^{-1} T_V \{f_i\}_{i\in I} = T_V^* S_V^{-1} T_V T_V^* g = T_V^* S_V^{-1} S_V g = T_V^* g = \{f_i\}_{i\in I}.$$

□

4.3 Operator identities for fusion frame systems

In Section 4.1 we motivated the concept of fusion frames on the basis of the idea to split a large frame up into smaller components or to fuse several frame systems together to one large frame. However, in the previous section this idea did not reveal itself within the presented material. We merely gave the basic definitions for fusion frames and collected some results, which we have already encountered in a very similar fashion in Chapter 2, where we presented some of the basic concepts of classical Frame Theory.

Now, we will catch up to the idea of *"splitting up"* and *"fusing together"* by introducing the concept *fusion frame systems.* In this fashion, the following statement can be viewed as the starting point for the theory of fusion frames. It shows a connection between fusion frames and (classical) frames. For a better understanding of the situation and for sake of completeness we present the proof in full detail, see also [17].

Theorem 4.3.1. *Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{v_i\}_{i\in I}$ some family of weights. Furthermore, for every $i \in I$, let $\varphi^{(i)} := \{\varphi_{ij}\}_{j\in J_i}$ be a frame for V_i with frame bounds A_i and B_i, and suppose that there exist constants A and B such that $0 < A = \inf_{i\in I} A_i \leq \sup_{i\in I} B_i = B < \infty$. Then the following are equivalent:*

(i) *$\{(V_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$.*

(ii) *$v\varphi := \{v_i\varphi_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$.*

In particular, if $\{(V_i, v_i)\}_{i\in I}$ is a fusion frame with bounds $C \leq D$, then $v\varphi$ is a frame for $\mathcal{H}$ with bounds $AC \leq BD$. Conversely, if $v\varphi$ is a frame for $\mathcal{H}$ with bounds $C \leq D$, then $\{(V_i, v_i)\}_{i\in I}$ is a fusion frame with bounds $\frac{C}{B} \leq \frac{D}{A}$.

Proof. We use the frame reconstruction formulas (2.8) for the frames $\varphi^{(i)}$ to see that

$$\begin{aligned} A\sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2 &\leq \sum_{i\in I} A_i v_i^2 \|\pi_{V_i} f\|^2 \\ &\leq \sum_{i\in I}\sum_{j\in J_i} |\langle \pi_{V_i} f, v_i f_{ij}\rangle|^2 \\ &\leq \sum_{i\in I} B_i v_i^2 \|\pi_{V_i} f\|^2 \leq B\sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2. \end{aligned}$$

Moreover, we also have

$$\sum_{i\in I}\sum_{j\in J_i} |\langle \pi_{V_i} f, v_i f_{ij}\rangle|^2 = \sum_{i\in I}\sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2.$$

Thus, if we assume that $\{(V_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with bounds $C \leq D$, then we obtain

$$AC\|f\|^2 \leq \sum_{i\in I}\sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 \leq BD\|f\|^2,$$

i.e. $v\varphi$ is a frame for $\mathcal{H}$ with frame bounds $AC \leq BD$.

Conversely, if we assume that $v\varphi$ is a frame for $\mathcal{H}$ with frame bounds $C \leq D$, then the above implies

$$\frac{C}{B}\|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2 \leq \frac{D}{A}\|f\|^2,$$

i.e. $\{(V_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds $\frac{C}{B} \leq \frac{D}{A}$. □

As a consequence of the above we obtain the following (see also [15]).

Corollary 4.3.2. *Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{v_i\}_{i\in I}$ some family of weights. For every $i \in I$, let $\varphi^{(i)} := \{\varphi_{ij}\}_{j\in J_i}$ be a Parseval frame for V_i. Then the following are equivalent:*

(i) *$\{(V_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$.*

(ii) $v\varphi := \{v_i\varphi_{ij}\}_{i\in I, j\in J_i}$ *is a Parseval frame for* $\mathcal{H}$.

Proof. This is the special case $A = A_i = B_i = B = 1$ (for all $i \in I$) in Theorem 4.3.1. All occurring inequalities in the proof of Theorem 4.3.1 turn into equations and thus the result follows. □

Theorem 4.3.1 motivates the notion of a fusion frame system, see also [15].

Definition (fusion frame system) 4.3.3. *Let* $\{(V_i, v_i)\}_{i\in I}$ *be a fusion frame for* $\mathcal{H}$ *and let* $\varphi^{(i)} = \{\varphi_{ij}\}_{j\in J_i}$ *be a frame for* V_i *for every* $i \in I$*. If the frames* $\varphi^{(i)}$ *have common frame bounds as in Theorem 4.3.1, then we call* $\{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ *a* fusion frame system *for* $\mathcal{H}$*. Further we call the frames* $\varphi^{(i)}$ local frames*. By Theorem 4.3.1, in this situation* $v\varphi = \{v_i\varphi_{ij}\}_{i\in I, j\in J_i}$ *is a frame for* $\mathcal{H}$*, which we call the* global frame*. Whenever necessary, we call the frame bounds for the local frames* $\varphi^{(i)}$ local frame bounds*, and the frame bounds for global frame* $v\varphi$ global frame bounds*.*

For a given fusion frame system $\{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ Theorem 4.3.1 shows us how the fusion frame $\{(V_i, v_i)\}_{i\in I}$, the local frames $\varphi^{(i)}$ and its corresponding global frame $v\varphi$ are linked. Therefore it wouldn't be surprising, if also their associated (fusion) synthesis, (fusion) analysis and (fusion) frame operators respectively were linked in some way. We will prove some operator identities, which mirror their relation to each other.

Before doing so, we need to consider the concept of *multisets* and spaces of the type $\ell^2(\biguplus_{i\in I} J_i)$. This will be useful, when we consider the representation space for the global frame $v\varphi$. Therefore, in the following we introduce some of the basic definitions of the theory of *multisets*. For a similar introduction to this topic we refer to [51].

Roughly speaking, a *multiset* (or *bag*) is a set, in which the contained elements are allowed to occur more than once. Formally, a *multiset* is a set of the type $X \times \mathcal{N}$, where X is some arbitrary set and $\mathcal{N} \subseteq \mathbb{N}_0 \cup \{\infty\}$, i.e. it is a set of the form $\{(x, \alpha_x) : x \in X, \alpha_x \in \mathbb{N}_0 \cup \{\infty\}\}$. If we interpret multisets as sets that may contain some elements which occur more than once, then we interpret α_x as the number of occurrences of the element x in that (multi-)set, called the *multiplicities*. The case $\alpha_x = \infty$ is interpreted as *"x occurs infinitely often"*. If X is a non-empty set, then we interpret the multiset $X \times \{1\}$ as the set X itself. We included the case $\alpha_x = 0$ in order to make more sense of this interpretation in view of the *empty multiset* $\varnothing \times \{0\}$, which we interpret as the empty set $\varnothing$.

If $X = \{x_1, ..., x_n\}$ and $\{\alpha_x\}_{x\in X} \subseteq \mathbb{N}$ is some sequence of natural numbers, then we define

$$\{x_1, ..., x_1, x_2, ..., x_2, ..., x_n, ..., x_n\}_b := \{(x_i, \alpha_{x_i}) : x_i \in X, \alpha_{x_i} \in \{\alpha_x\}_{x\in X}\},$$

where in the above on the left side, each x_i occurs α_{x_i} times.

Further, if $\mathcal{A} = A \times \mathcal{N}_1 = \{(x, \alpha_x) : x \in A, \alpha_x \in \mathcal{N}_1\}$ and $\mathcal{B} = B \times \mathcal{N}_2 = \{(x, \beta_x) : x \in B, \beta_x \in \mathcal{N}_2\}$ are arbitrary multisets, then we define the multiset $\mathcal{A} \uplus \mathcal{B}$ via

$$\mathcal{A} \uplus \mathcal{B} := \{(x, \gamma_x) : x \in A \cup B, \gamma_x = \alpha_x + \beta_x\},$$

where we simply set $\alpha_x = 0$ if $x \notin A$ and $\beta_x = 0$ if $x \notin B$.

For sets C and D we define

$$C \uplus D := \mathcal{C} \uplus \mathcal{D}, \text{ where } \mathcal{C} := C \times \{1\} \text{ and } \mathcal{D} := D \times \{1\}.$$

Intuitively, we can think of the multiset $A \uplus B$ as the (multi-)set consisting of all the elements of the set $A \cup B$ and – in addition – all the elements of the set $A \cap B$. Note that we have $\mathcal{A} \uplus \varnothing = \mathcal{A}$ and $A \uplus \varnothing = A$ for any multiset $\mathcal{A}$ and any set A respectively (where we abused the notation and wrote $\varnothing$ for both the empty multiset and the empty set). Furthermore, note that if A and B are disjoint sets, then $A \uplus B$ coincindes with the multiset $(A \cup B) \times \{1\}$ which we interpret as the set $A \cup B$. Let us consider concrete examples in order to digest the above definitions a little bit quicker. We have $\{1,2,3\} \uplus \{3,4,5\} = \{1,2,3,3,4,5\}_b = \{(1,1),(2,1),(3,2),(4,1),(5,1)\}$ and interpret this multiset as the "set" which contains each of the numbers 1, 2, 4 and 5 once and the number 3 twice. The sets $\{e,\pi,\tau\}$ and $\{0,i,10^{80}\}$ are disjoint and we have $\{e,\pi,\tau\} \uplus \{0,i,10^{80}\} = \{e,\pi,\tau,0,i,10^{80}\}_b = \{e,\pi,\tau,0,i,10^{80}\} \times \{1\}$, which we simply interpret as the set $\{e,\pi,\tau,0,i,10^{80}\}$. We remark that the multiset operation $\uplus$ indeed is some kind of combination of the set operation $\cup$ and the operation $+$, which justifies its notation.

For multisets $\mathcal{A}_1, ..., \mathcal{A}_n$ and sets $C_1, ..., C_n$ we inductively define

$$\biguplus_{i=1}^{n} \mathcal{A}_i := \Big(\biguplus_{i=1}^{n-1} \mathcal{A}_i \Big) \uplus \mathcal{A}_n \quad \text{and} \quad \biguplus_{i=1}^{n} C_i := \Big(\biguplus_{i=1}^{n-1} C_i \Big) \uplus C_n.$$

We extend the above definitions to countable sequences $\{\mathcal{A}_i\}_{i\in I}$ of multisets $\mathcal{A}_i = \{(x, \alpha_x^{(i)}) : x \in A_i, \alpha_x^{(i)} \in \mathcal{N}_i\}$ $(i \in I)$, and define

$$\biguplus_{i\in\mathbb{N}} \mathcal{A}_i := \Big\{ (x, \alpha_x) : x \in \bigcup_{i\in\mathbb{N}} A_i, \alpha_x = \sum_{i\in\mathbb{N}} \alpha_x^{(i)} \Big\},$$

where we again simply set $\alpha_x^{(i)} = 0$ in case $x \notin A_i$.

Analogously to before, for countable sequences $\{C_i\}_{i\in I}$ of sets C_i we define

$$\biguplus_{i\in\mathbb{N}} C_i := \biguplus_{i\in\mathbb{N}} (C_i \times \{1\}).$$

Now, let $\{J_i\}_{i\in I}$ be a countable family of countable index sets J_i and consider the multiset $\biguplus_{i\in I} J_i = \{(k, \alpha_k) : k \in \bigcup_{i\in I} J_i\}$ (with a suitable family $\alpha = \{\alpha_k\}_{k\in\bigcup_{i\in I} J_i}$ of multiplicities α_k). Then we define

$$\ell^2\Big(\biguplus_{i\in I} J_i \Big) := \Big\{ \{c_k\}_{k\in\bigcup_{i\in I} J_i} \subseteq \mathbb{C} : \sum_{k\in\bigcup_{i\in I} J_i} \alpha_k |c_k|^2 = \sum_{i\in I} \sum_{j\in J_i} |c_k|^2 < \infty \Big\}.$$

Note that the sums in the above definition are indeed equal: In case one of them converges, it converges absolutely and thus unconditionally. Moreover, note that $\alpha = \{\alpha_k\}_{k\in\bigcup_{i\in I} J_i}$ always depends on the sets J_i, but is fixed as soon as we specified the sets J_i. Since $\alpha_k \geq 1$ for all k, the space $\ell^2(\biguplus_{i\in I} J_i)$ clearly is a subspace of the discrete Hilbert space $\ell^2(\bigcup_{i\in I} J_i)$. If we equip $\ell^2(\biguplus_{i\in I} J_i)$ with the inner product $\langle ., . \rangle_\alpha$ defined by

$$\Big\langle \{c_k\}_{k\in\bigcup_{i\in I} J_i}, \{d_k\}_{k\in\bigcup_{i\in I} J_i} \Big\rangle_\alpha := \sum_{k\in\bigcup_{i\in I} J_i} \alpha_k c_k \overline{d_k}$$

and as usual set $\|.\|_\alpha = \langle .,. \rangle_\alpha^{1/2}$, then we have

$$\|\{c_k\}_{k\in\bigcup_{i\in I} J_i}\|_\alpha = \sum_{k\in\bigcup_{i\in I} J_i} \alpha_k |c_k|^2 = \sum_{i\in I}\sum_{j\in J_i} |c_k|^2. \tag{4.19}$$

Moreover, the normed space $(\ell^2(\biguplus_{i\in I} J_i), \|.\|_\alpha)$ is easily seen to be complete (adapt any standard completeness proof of $\ell^2(\mathbb{N})$ or the proof of Lemma 3.1.1) and thus a Hilbert space as well. In case we have $\alpha_k \in \mathbb{N}$ for every k, we alternatively may describe $(\ell^2(\biguplus_{i\in I} J_i), \|.\|_\alpha)$ as the *weighted ℓ^2-space* $\ell^2_\alpha(\bigcup_{i\in I} J_i)$. For some more details about *weighted ℓ^p-spaces* $(1 \le p \le \infty)$, see [5] for instance.

The reason for this short voyage into the theory of multisets is the following. For a given fusion frame system $\{(V_i, v_i, \phi^{(i)})\}_{i\in I}$ in general the spaces V_i may have non-trivial intersection. Therefore, if we want to consider the representation space of the global frame $v\varphi = \{v_i\varphi_{ij}\}_{i\in I, j\in J_i}$, we not only have to take ℓ^2-sequences indexed by the index set $\bigcup_{i\in I} J_i$ into account, but also have to take the multiplicities corresponding to the intersections of the index sets J_i into account. This observation and equation (4.19) imply that $\ell^2(\biguplus_{i\in I} J_i)$ is precisely the representation space for the global frame $v\varphi$, i.e. the domain of the synthesis operator $T_{v\varphi}$.

The next result shows that we can identify the representation space $\ell^2(\biguplus_{i\in I} J_i)$ for the global frame $v\varphi$ with the Hilbert direct sum $\big(\sum_{i\in I} \oplus \ell^2(J_i)\big)_{\ell^2}$.

Proposition 4.3.4. *The Hilbert spaces $\ell^2\big(\biguplus_{i\in I} J_i\big)$ and $\big(\sum_{i\in I} \oplus \ell^2(J_i)\big)_{\ell^2}$ are isometrically isomorphic.*

Proof. Let $c = \{c_i\}_{i\in I} \in (\sum_{i\in I} \oplus \ell^2(J_i))_{\ell^2}$. Then, for every $i \in I$, $c_i = \{c_{ij}\}_{j\in J_i}$, i.e. $c = \{\{c_{ij}\}_{j\in J_i}\}_{i\in I}$. Define

$$U : \Big(\sum_{i\in I} \oplus \ell^2(J_i)\Big)_{\ell^2} \longrightarrow \ell^2\Big(\biguplus_{i\in I} J_i\Big),$$

$$Uc = U\{\{c_{ij}\}_{j\in J_i}\}_{i\in I} := \{c_{ij}\}_{i\in I, j\in J_i}.$$

Then

$$\|c\|^2_{(\sum_{i\in I} \oplus \ell^2(J_i))_{\ell^2}} = \sum_{i\in I} \|c_i\|^2_{\ell^2(J_i)} = \sum_{i\in I}\sum_{j\in J_i} |c_{ij}|^2 = \|Uc\|^2_{\ell^2(\biguplus_{i\in I} J_i)} \tag{4.20}$$

shows that U is norm-preserving and thus also bounded with $\|U\| = 1$. To show the surjectivity of U, choose some arbitrary $\{c_{ij}\}_{i\in I, j\in J_i} \in \ell^2(\biguplus_{i\in I} J_i)$ and observe that for any $i \in I$, $c_i := \{c_{ij}\}_{j\in J_i} \in \ell^2(J_i)$, since by (4.20) we have

$$\|c_i\|^2_{\ell^2(J_i)} \le \|\{c_{ij}\}_{i\in I, j\in J_i}\|^2_{\ell^2(\biguplus_{i\in I} J_i)} < \infty.$$

Again by (4.20), we see that $\{c_i\}_{i\in I} \in (\sum_{i\in I} \oplus \ell^2(J_i))_{\ell^2}$ and by definition of U we have $U\{c_i\}_{i\in I} = \{c_{ij}\}_{i\in I, j\in J_i}$. Hence, U is surjective and thus an isometric isomorphism. □

We are now finally ready to prove the announced operator identities. Let us consider a fusion frame system $\{(V_i, v_i, \phi^{(i)})\}_{i\in I}$ with corresponding global frame $v\varphi = \{v_i\varphi_{ij}\}_{i\in I, j\in J_i}$. Since the local frames $\phi^{(i)}$ have (by definition of a fusion frame system) a common upper frame bound B, their corresponding synthesis operators $T_{\phi^{(i)}} : \ell^2(J_i) \longrightarrow V_i$ are completely bounded by $\sqrt{B}$. This begs for an application of our results from Chapter 3 about component preserving operators between Hilbert direct sums. In fact, by Proposition 3.2.1,

$$\bigoplus_{i\in I} T_{\varphi^{(i)}} : \big(\sum_{i\in I} \oplus \ell^2(J_i)\big)_{\ell^2} \longrightarrow \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$$

is well-defined and bounded. Thus we may compose it with the fusion synthesis operator T_V which yields an operator from $(\sum_{i\in I} \oplus \ell^2(J_i))_{\ell^2}$ into $\mathcal{H}$. On the other hand, by our previous result, we may identify the domain of the synthesis operator $T_{v\varphi}$ associated to the global frame $v\varphi$ with the Hilbert direct sum $(\sum_{i\in I} \oplus \ell^2(J_i))_{\ell^2}$ and thus may consider $T_{v\varphi}$ as an operator from $(\sum_{i\in I} \oplus \ell^2(J_i))_{\ell^2}$ into $\mathcal{H}$. That the above composition of operators indeed coincides with the operator $T_{v\varphi}$ is proven in the following theorem.

Theorem 4.3.5. *Let $V = \{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system and $v\varphi$ be the corresponding global frame. Then*

$$T_{v\varphi} = T_V \bigoplus_{i\in I} T_{\varphi^{(i)}}. \tag{4.21}$$

Proof. If $c = \{c_{ij}\}_{i\in I, j\in J_i} \in \ell^2(\biguplus_{i\in I} J_i) \cong (\sum_{i\in I} \oplus \ell^2(J_i))_{l^2}$ then $c_i = \{c_{ij}\}_{j\in J_i} \in \ell^2(J_i)$ for every $i \in I$. Since the family $\{T_{\varphi^{(i)}}\}_{i\in I}$ is completely bounded by $\sqrt{B}$, where B is the common upper frame bound of the local frames $\varphi^{(i)}$, Proposition 3.2.1 guarantees that $\bigoplus_{i\in I} T_{\varphi^{(i)}}$ is well-defined and bounded, i.e. that

$$\{T_{\varphi^{(i)}}(c_i)\}_{i\in I} \in (\sum_{i\in I} \oplus V_i)_{l^2}.$$

This allows us to write

$$T_{v\varphi}(c) = \sum_{i\in I} \sum_{j\in J_i} c_{ij} v_i \varphi_{ij} = \sum_{i\in I} v_i T_{\varphi^{(i)}}(c_i) = T_V \bigoplus_{i\in I} T_{\varphi^{(i)}}(c).$$

□

Taking adjoints in (4.21) yields the following result.

Corollary 4.3.6. *Let $\{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system and $v\varphi$ be the corresponding global frame. Then*

$$T^*_{v\varphi} = \Big(\bigoplus_{i\in I} T^*_{\varphi^{(i)}}\Big) T^*_V.$$

Proof. Combining Proposition 3.2.2 (a) and Theorem 4.3.5 yields

$$T^*_{v\varphi} = \Big(T_V \bigoplus_{i\in I} T_{\varphi^{(i)}}\Big)^* = \Big(\bigoplus_{i\in I} T_{\varphi^{(i)}}\Big)^* T^*_V = \Big(\bigoplus_{i\in I} T^*_{\varphi^{(i)}}\Big) T^*_V.$$

□

Before we prove another consequence of the above, we introduce the notion of a *tensor product of operators* [23]. Let $\mathcal{H}_1$,$\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ be Hilbert spaces. For $S \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$ and $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the *tensor product* of two operators as an element of $\mathcal{B}\big(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_3), \mathcal{B}(\mathcal{H}_2, \mathcal{H}_4)\big)$ is defined by

$$(S \otimes T)(O) := SOT^*, \;\; (O \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_3)). \tag{4.22}$$

Furthermore [23]

$$(S \otimes T)^* = T^* \otimes S^*. \tag{4.23}$$

Corollary 4.3.7. *Let $\{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system and $v\varphi$ be the corresponding global frame. Then*

$$S_{v\varphi} = T_V\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\Big)T_V^* = (T_V \otimes T_V)\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\Big). \tag{4.24}$$

Proof. Applying Theorem 4.3.5, Corollary 4.3.6 and Proposition 3.2.2 (c) gives us

$$\begin{aligned} S_{v\varphi} = T_{v\varphi}T_{v\varphi}^* &= T_V\big(\bigoplus_{i\in I} T_{\varphi^{(i)}}\big)\big(\bigoplus_{i\in I} T_{\varphi^{(i)}}^*\big)T_V^* \\ &= T_V\big(\bigoplus_{i\in I}(T_{\varphi^{(i)}}T_{\varphi^{(i)}}^*)\big)T_V^* \\ &= T_V\big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\big)T_V^* = (T_V \otimes T_V)(\bigoplus_{i\in I} S_{\varphi^{(i)}}). && \text{(by (4.23))} \end{aligned}$$

□

In case $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, the operators T_V and T_V^* both are invertible, which enables us to invert (4.24).

Corollary 4.3.8. *Let $V = \{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system and $v\varphi$ be the corresponding global frame. If in addition $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, then*

$$S_{v\varphi}^{-1} = (T_V^{-1})^*\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big)T_V^{-1} = \big(T_V^{-1} \otimes T_V^{-1}\big)^*(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}). \tag{4.25}$$

Proof. By Proposition 4.2.12, T_V and T_V^* both are bounded and bijective, i.e. invertible. Since the local frames $\varphi^{(i)}$ have common frame bounds A and B, by (2.7) and (2.9) we have $\|S_{\varphi^{(i)}}\| \le B$ and $\|S_{\varphi^{(i)}}^{-1}\| \le A^{-1}$ for all $i \in I$. By Propositions 3.2.1 and 3.2.6, the operator $\bigoplus_{i\in I} S_{\varphi^{(i)}}$ is well defined, bounded and invertible with inverse $\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}$. Therefore, by Corollary 4.3.7 and (4.23), we have

$$\begin{aligned} S_{v\varphi}^{-1} = \Big(T_V\big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\big)T_V^*\Big)^{-1} &= (T_V^*)^{-1}\big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\big)^{-1}T_V^{-1} \\ &= (T_V^{-1})^*\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big)T_V^{-1} = \big(T_V^{-1} \otimes T_V^{-1}\big)^*(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}). \end{aligned}$$

□

The proof of the previous result strongly depends on the invertibility of the operator T_V, which is given if and only if $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis. In case the fusion frame $\{(V_i, v_i)\}_{i\in I}$ is not a fusion Riesz basis, we cannot consider the inverse T_V^{-1} of T_V, but we still may consider the pseudo inverse $T_V^{\dagger}$ of T_V. Thus, a fair guess would be, that (4.25) holds for the general case, if we substitute T_V^{-1} by $T_V^{\dagger}$. However, if $\{(V_i, v_i)\}_{i\in I}$ is not a fusion Riesz basis, then we don't necessarily have $V_i \cap V_j = \{0\}$ for $i \neq j$, i.e. the intersections of the subspaces V_i can be of much more complicated nature than in the latter case and thus we are only able to prove an analog of equation (4.25) for the following scenario:

Theorem 4.3.9. *Let $V = \{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system and $v\varphi$ be the corresponding global frame. If*

$$\pi_{V_i} S_V^{-1} S_{v\varphi} = S_{\varphi^{(i)}}\pi_{V_i} \quad \text{for all } i \in I,$$

then

$$S_{v\varphi}^{-1} = (T_V^{\dagger})^*\big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\big)T_V^{\dagger} = \big(T_V^{\dagger} \otimes T_V^{\dagger}\big)^*(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}). \tag{4.26}$$

Proof. Observe that for every $f \in \mathcal{H}$ we have

$$\begin{aligned}
(T_V^\dagger)^*\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big)T_V^\dagger S_{v\varphi} f &= (T_V^* S_V^{-1})^*\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big)T_V^* S_V^{-1} S_{v\varphi} f \qquad \text{(by Proposition 4.2.17)}\\
&= S_V^{-1} T_V\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big)T_V^* S_V^{-1} S_{v\varphi} f\\
&= S_V^{-1}\sum_{i\in I} S_{\varphi^{(i)}}^{-1} v_i^2 \pi_{V_i} S_V^{-1} S_{v\varphi} f\\
&= S_V^{-1}\sum_{i\in I} S_{\varphi^{(i)}}^{-1} v_i^2 S_{\varphi^{(i)}} \pi_{V_i} f \qquad \text{(by assumption)}\\
&= S_V^{-1}\sum_{i\in I} v_i^2 \pi_{V_i} f\\
&= S_V^{-1} S_V f = f.
\end{aligned}$$

This implies $(T_V^\dagger)^*\big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\big)T_V^\dagger = S_{v\varphi}^{-1}$. $\square$

On this note let us remark that if $\{(V_i, v_i)\}_{i\in I}$ is indeed a fusion Riesz basis, then by Theorem 4.2.12 this is equivalent to $v_i^2 \pi_{V_i} S_V^{-1} \pi_{V_j} = \delta_{ij}\pi_{V_j}$ for all $i, j \in I$ $(*)$. The latter implies that

$$\begin{aligned}
\pi_{V_i} S_V^{-1} S_{v\varphi} f &= \pi_{V_i} S_V^{-1} T_V\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\Big)T_V^* f \qquad \text{(by Theorem 4.3.7)}\\
&= v_i^2 \pi_{V_i} S_V^{-1}\Big(v_i^{-2}\sum_{j\in I} \pi_{V_j} v_j^2 S_{\varphi^{(j)}} \pi_{V_j} f\Big)\\
&= \pi_{V_i} S_{\varphi^{(i)}} \pi_{V_i} f \qquad \text{(by } (*)\text{)}\\
&= S_{\varphi^{(i)}} \pi_{V_i} f
\end{aligned}$$

for every $f \in \mathcal{H}$ and every $i \in I$, which means that the assumption from Theorem 4.3.9 is fulfilled. Thus the operator identity (4.26) holds. On the other hand, if $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, then $T_V^\dagger = T_V^*(T_V T_V^*)^{-1} = T_V^*(T_V^*)^{-1} T_V^{-1} = T_V^{-1}$ and thus (4.26) trivially reduces to the operator identity (4.25).

However, it might be possible, that there exist fusion frames, which are not a fusion Riesz basis but still fulfill the assumption (4.26) from Theorem 4.3.9.

Let us prove three more results, which we will apply in Section 4.4.

Lemma 4.3.10. *$V = \{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ is a fusion frame system with corresponding global frame $v\varphi$ if and only if $\{(V_i, v_i, \tilde{\varphi}^{(i)})\}_{i\in I}$ is a fusion frame system with corresponding global frame $v\tilde{\varphi} := \{v_i \tilde{\varphi}_{ij}\}_{i\in I, j\in J_i}$, where $\tilde{\varphi}^{(i)}$ denotes the canonical dual of $\varphi^{(i)}$.*

Proof. The local frames $\varphi^{(i)}$ of V have common frame bounds $A \le B$. Therefore $1/B \le \|S_{\varphi^{(i)}}^{-1}\| \le 1/A$ for all $i \in I$, i.e. the canonical duals $\tilde{\varphi}^{(i)}$ of the local frames $\varphi^{(i)}$ have common frame bounds $1/B \le 1/A$. As in Theorem 4.3.1 we then see that $\{(V_i, v_i, \tilde{\varphi}^{(i)})\}_{i\in I}$ is a fusion frame system with global frame $v\tilde{\varphi}$, since by assumption $\{(V_i, v_i)\}_{i\in I}$ is a fusion frame. We may prove the converse statement completely analogously, since the canonical dual of the canonical dual of a frame is the frame itself (c.f. Section 2.1 or [18]). $\square$

Proposition 4.3.11. *Consider the fusion frame system $\{(V_i, v_i, \tilde{\varphi}^{(i)})\}_{i\in I}$ with corresponding global frame $v\tilde{\varphi}$. Then*

$$S_{v\tilde{\varphi}} = T_V\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big)T_V^* = (T_V \otimes T_V)\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big). \tag{4.27}$$

Proof. By Corollary 4.3.7 we have

$$S_{v\tilde{\varphi}} = T_V\Big(\bigoplus_{i\in I} S_{\tilde{\varphi}^{(i)}}\Big)T_V^* = (T_V \otimes T_V)\Big(\bigoplus_{i\in I} S_{\tilde{\varphi}^{(i)}}\Big).$$

However, since $\tilde{\varphi}^{(i)}$ denotes the canonical dual of $\varphi^{(i)}$ and since the frame operator of $\tilde{\varphi}^{(i)}$ is given by $S^{-1}_{\varphi^{(i)}}$ (see Section 2.1), the result follows at once. □

In case $\{(V_i, v_i)\}_{i\in I}$ is also a fusion Riesz basis, we can invert the above identity.

Corollary 4.3.12. *Consider the fusion frame system $\{(V_i, v_i, \tilde{\varphi}^{(i)})\}_{i\in I}$ with corresponding global frame $v\tilde{\varphi}$. If in addition $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, then*

$$S^{-1}_{v\tilde{\varphi}} = (T_V^{-1})^*\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\Big)T_V^{-1} = \big(T_V^{-1} \otimes T_V^{-1}\big)^*\big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\big). \tag{4.28}$$

We remark that the global frame $v\tilde{\varphi}$ corresponding to the fusion frame system $\{(V_i, v_i, \tilde{\varphi}^{(i)})\}_{i\in I}$ does in general not coincide with the canonical dual $\widetilde{v\varphi}$ of the global frame $v\varphi$ corresponding to the fusion frame system $\{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$. We have $v\tilde{\varphi} = \{v_i S^{-1}_{\varphi^{(i)}}\varphi_{ij}\}_{i\in I, j\in J_i}$ and $\widetilde{v\varphi} = \{v_i S^{-1}_{v\varphi}\varphi_{ij}\}_{i\in I, j\in J_i}$. In the next section we will elaborate more on the similarities and differences between these two frames.

4.4 Fusion frame systems and distributed processing

Next we apply our operator identities from Section 4.3 to prove some properties about fusion frame systems. We will also refer to the concept of distributed processing once more, since the idea and applications of some of our results are closely connected to this topic.

Let us recall the philosophy of fusion frame systems. Suppose we are given some family of vectors $\{\varphi_k\}_{k\in K}$ (K countable) in some Hilbert space $\mathcal{H}$, which we choose to split up into possibly overlapping (i.e. there exist $i_1, i_2 \in I$ such that $V_{i_1} \cap V_{i_2} \neq \{0\}$) smaller families of vectors $\varphi^{(i)} := \{\varphi_{ij}\}_{j\in J_i}$ ($i \in I$), such that each family $\varphi^{(i)}$ constitutes a frame for its closed span $V_i := \overline{span}\{\varphi_{ij}\}_{j\in J_i}$ and such that all these frames $\varphi^{(i)}$ have common frame bounds. Then Theorem 4.3.1 guarantees us that the collection $v\varphi := \{v_i\varphi_{ij}\}_{i\in I, j\in J_i}$ of these (– if demanded – weighted and/or overlapping) sub-families is a frame for $\mathcal{H}$ if and only if $\{(V_i, v_i)\}_{i\in I}$ us a fusion frame for $\mathcal{H}$.

If one and therefore the other of the above conditions is true (in which case we call $\{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ a fusion frame system), then we have two possibilities of reconstructing a given signal $f \in \mathcal{H}$: We can perform frame reconstruction (compare to (2.8)) for the global frame $v\varphi$, i.e.

$$\begin{aligned} f &= \sum_{i\in I}\sum_{j\in J_i} \langle f, S^{-1}_{v\varphi} v_i\varphi_{ij}\rangle v_i\varphi_{ij} \\ &= \sum_{i\in I}\sum_{j\in J_i} \langle f, v_i\varphi_{ij}\rangle S^{-1}_{v\varphi} v_i\varphi_{ij}, \end{aligned} \tag{4.29}$$

or we combine frame reconstruction for $\pi_{V_i} f$ at a local level (for each $i \in I$) with fusion frame reconstruction (compare to (4.6)), i.e.

$$f = \sum_{i \in I} v_i^2 S_V^{-1} \sum_{j \in J_i} \langle f, \varphi_{ij} \rangle S_{\varphi^{(i)}}^{-1} \varphi_{ij} \tag{4.30}$$

$$= \sum_{i \in I} v_i^2 \sum_{j \in J_i} \langle f, \varphi_{ij} \rangle S_V^{-1} S_{\varphi^{(i)}}^{-1} \varphi_{ij} \tag{4.31}$$

$$= \sum_{i \in I} \sum_{j \in J_i} \langle f, v_i \varphi_{ij} \rangle S_V^{-1} S_{\varphi^{(i)}}^{-1} v_i \varphi_{ij}, \tag{4.32}$$

which yields different distributed fusion procedures. The following graphic illustrates our situation.

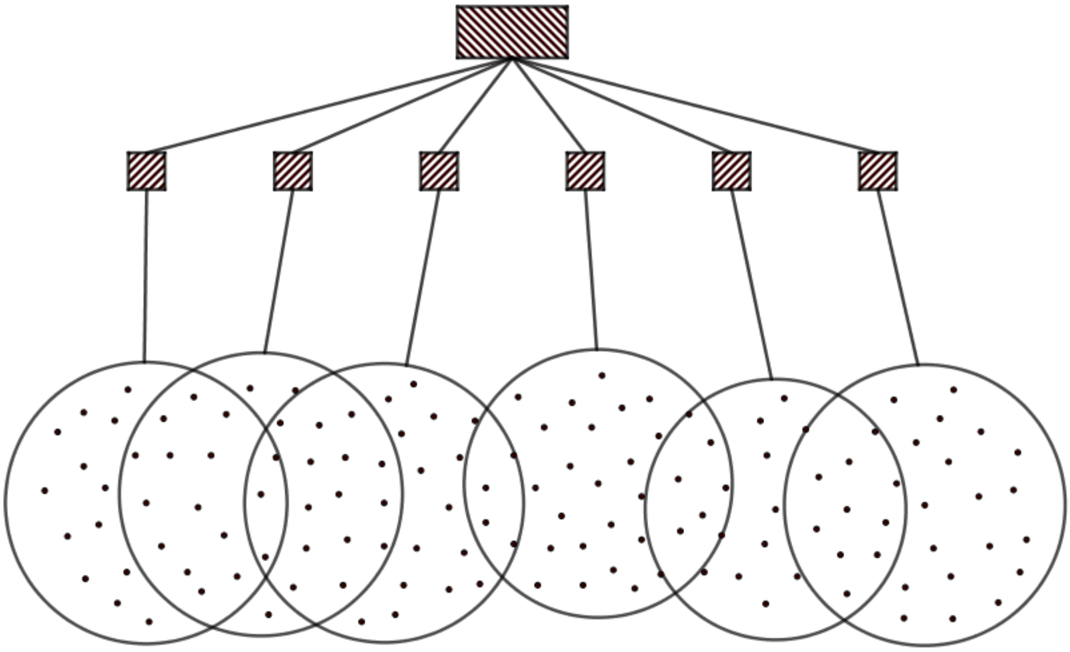

Figure 3: An illustration of distributed fusion processing for a fusion frame system.

The distributed fusion procedure (4.30), which first takes place in each subspace V_i before fusion frame reconstruction is performed, is required for sensor networks [36] or geophones in geophysics measurements [20], while procedure (4.31) – acting like a global reconstruction – is (according to [15]) applied for parallel processing of large frame systems, for instance. Especially (4.32) reminds us of the global frame reconstruction as in (4.29). Following the labelling from [15], we call the procedure (4.29) *centralized reconstruction* and the procedure (4.32) *distributed reconstruction*.

Let us elaborate more on the sequences $\{S_{v\varphi}^{-1} v_i \varphi_{ij}\}_{i \in I, j \in J_i}$ and $\{S_V^{-1} S_{\varphi^{(i)}}^{-1} v_i \varphi_{ij}\}_{i \in I, j \in J_i}$ corresponding to centralized and distributed reconstruction respectively. The sequence $\{S_{v\varphi}^{-1} v_i \varphi_{ij}\}_{i \in I, j \in J_i}$ is the canonical dual frame of the global frame $v\varphi$. By (4.32) we have that $\{S_V^{-1} S_{\varphi^{(i)}}^{-1} v_i \varphi_{ij}\}_{i \in I, j \in J_i}$ is always a dual frame of the global frame $v\varphi$.

In [15] the authors prove that in case $\{V_i\}_{i \in I}$ is an orthonormal fusion basis, $\{S_V^{-1} S_{\varphi^{(i)}}^{-1} v_i \varphi_{ij}\}_{i \in I, j \in J_i}$ is indeed the canonical dual frame of the global frame $v\varphi$:

Proposition [15] 4.4.1. *Let $V = \{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system and $v\varphi = \{v_i\varphi_{ij}\}_{i\in I, j\in J_i}$ be the corresponding global frame. If $\{V_i\}_{i\in I}$ is an orthonormal fusion basis, then $\{S_V^{-1}S_{\varphi^{(i)}}^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ is the canonical dual frame of the global frame $v\varphi$.*

In other words, if the family $\{V_i\}_{i\in I}$ associated to a given fusion frame system V is an orthonormal fusion basis, then centralized reconstruction and distributed reconstruction coincide.

Applying one of our operator identities from Section 4.3 and one of the results from Section 4.2 enables us to extend this result to the more general case, where we assume $\{(V_i, v_i)\}_{i\in I}$ to be a fusion Riesz basis:

Theorem 4.4.2. *Let $V = \{(V_i, v_i, \varphi^{(i)} = \{\varphi_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system and $v\varphi = \{v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ be the corresponding global frame. If $(\{V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, then $\{S_V^{-1}S_{\varphi^{(i)}}^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ is the canonical dual frame of the global frame $v\varphi$, i.e.*

$$\{S_{v\varphi}^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i} = \{S_V^{-1}S_{\varphi^{(i)}}^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}.$$

Proof. For each $i \in I$, $v_i\varphi_{ij} \in V_i$. By Proposition 4.2.13, we have $T_V^{-1}v_i\varphi_{ij} = (..., 0, 0, \varphi_{ij}, 0, 0, ...)$ (φ_{ij} in the i-th component). This yields

$$\begin{aligned}
\{S_{v\varphi}^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i} &= \left\{(T_V^*)^{-1}\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big)T_V^{-1}v_i\varphi_{ij}\right\}_{i\in I,j\in J_i} && \text{(Corollary 4.3.8)}\\
&= \left\{(T_V^*)^{-1}\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}^{-1}\Big)(..., 0, 0, \varphi_{ij}, 0, 0, ...)\right\}_{i\in I,j\in J_i} && \text{(Proposition 4.2.13)}\\
&= \left\{(T_V^*)^{-1}(..., 0, 0, S_{\varphi^{(i)}}^{-1}\varphi_{ij}, 0, 0, ...)\right\}_{i\in I,j\in J_i}\\
&= \left\{(T_V^*)^{-1}T_V^{-1}T_V(..., 0, 0, S_{\varphi^{(i)}}^{-1}\varphi_{ij}, 0, 0, ...)\right\}_{i\in I,j\in J_i}\\
&= \left\{(T_V^*)^{-1}T_V^{-1}S_{\varphi^{(i)}}^{-1}v_i\varphi_{ij}\right\}_{i\in I,j\in J_i}\\
&= \{S_V^{-1}S_{\varphi^{(i)}}^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}. && (4.33)
\end{aligned}$$

This completes the proof. □

We remark, that the above result does *not* state that "$S_{v\varphi}^{-1} = S_V^{-1}S_{\varphi^{(i)}}^{-1}$". It only states that the sequences $\{S_{v\varphi}^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ and $\{S_V^{-1}S_{\varphi^{(i)}}^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ coincide.

Let us note another nice duality relation, which was proven in [15], similar to the duality relation (4.33):

Proposition [15] 4.4.3. *Let $V = \{(V_i, v_i, \varphi^{(i)} = \{\varphi_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system. Then $v\tilde{\varphi} = \{v_iS_{\varphi^{(i)}}^{-1}\varphi_{ij}\}_{i\in I,j\in J_i}$ is a frame for $\mathcal{H}$ and $\{S_V^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ is a dual frame for it.*

Similarly to before, we can extend the previous result and show that if $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, then $\{S_V^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ is the canonical dual frame of $v\tilde{\varphi}$.

Proposition 4.4.4. *Let $V = \{(V_i, v_i, \varphi^{(i)} = \{\varphi_{ij}\}_{j\in J_i})\}_{i\in I}$ be a fusion frame system and $v\varphi = \{v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ be the corresponding global frame. If $(\{V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, then $\{S_V^{-1}v_i\varphi_{ij}\}_{i\in I,j\in J_i}$ is the canonical dual frame of $\{v_i\tilde{\varphi}_{ij}\}_{i\in I,j\in J_i}$.*

Proof. By Lemma 4.3.10 $\{(V_i, v_i, \tilde{\varphi}^{(i)})\}_{i\in I}$ is a fusion frame system with corresponding global frame $v\tilde{\varphi}$ and by Corollary 4.3.12 we have $S_{v\tilde{\varphi}}^{-1} = (T_V^*)^{-1}\big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\big)T_V^{-1}$. Now we may use precisely the same arguments as in the proof of Theorem 4.4.2 to see that

$$\begin{aligned}\{S_{v\tilde{\varphi}}^{-1} v_i \tilde{\varphi}_{ij}\}_{i\in I, j\in J_i} &= \{(T_V^*)^{-1}(\bigoplus_{i\in I} S_{\varphi^{(i)}})T_V^{-1} v_i S_{\varphi^{(i)}}^{-1}\varphi_{ij}\}_{i\in I, j\in J_i} \\ &= \{(T_V^*)^{-1}T_V^{-1} S_{\varphi^{(i)}} v_i S_{\varphi^{(i)}}^{-1}\varphi_{ij}\}_{i\in I, j\in J_i} = \{S_V^{-1} v_i \varphi_{ij}\}_{i\in I, j\in J_i}.\end{aligned}$$

□

In the following we will see that if we restrict some components of a fusion frame system to have certain structures or extra properties, then this may affect other components of the fusion frame system as well. For instance, recall that in Corollary 4.3.2 we have already presented one special case. There we proved that if all local frames of a given fusion frame system are Parseval frames (i.e. their associated frame operators take the simplest possible structure $S_{\varphi^{(i)}} = \mathcal{I}_{\mathcal{H}}$), then its corresponding global frame is a Parseval frame if and only if the corresponding fusion frame is a Parseval fusion frame.

In [17] a more general result is proven. There the authors show that if all the local frames of a fusion frame system are Parseval frames, then the fusion frame operator S_V equals the frame operator $S_{v\varphi}$ for the global frame (Note that Corollary 4.3.2 is the special case $S_V = S_{v\varphi} = \mathcal{I}_{\mathcal{H}}$.). However, our results from Section 4.3 also enable us to give a new and even shorter proof of this statement. Moreover, we will be able to prove the converse, if we additionally assume that the fusion frame $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis:

Proposition 4.4.5. *Let $V = \{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system and $v\varphi$ be the corresponding global frame. If $\varphi^{(i)}$ is a Parseval frame for every $i \in I$, then*

$$S_V = S_{v\varphi}. \tag{4.34}$$

Conversely, if (4.34) holds and if $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, then $\varphi^{(i)}$ is a Parseval frame for every $i \in I$.

Proof. Recall that by Corollary 2.1.4, $\varphi^{(i)}$ is a Parseval frame for every $i \in I$ if and only if $S_{\varphi^{(i)}} = \mathcal{I}_{V_i}$ for every $i \in I$. Applying Theorem 4.3.6 gives the first part of the statement, since we have

$$S_{v\varphi} = T_V\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\Big)T_V^* = T_V\Big(\bigoplus_{i\in I} \mathcal{I}_{V_i}\Big)T_V^* = T_V T_V^* = S_V.$$

Conversely, assume that (4.34) holds and that $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis. Applying Theorem 4.3.6 once again implies

$$0 = S_V - S_{v\varphi} = T_V T_V^* - T_V\Big(\bigoplus_{i\in I} S_{\varphi^{(i)}}\Big)T_V^* = T_V\Big(\mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - \bigoplus_{i\in I} S_{\varphi^{(i)}}\Big)T_V^*.$$

Since, by Theorem 4.2.12, T_V and T_V^* both are invertible, we may multiply the above equation with T_V^{-1} from the left and $(T_V^*)^{-1}$ from the right to see that

$$0 = \mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - \bigoplus_{i\in I} S_{\varphi^{(i)}} = \bigoplus_{i\in I} \mathcal{I}_{V_i} - \bigoplus_{i\in I} S_{\varphi^{(i)}},$$

i.e. $S_{\varphi^{(i)}} = \mathcal{I}_{V_i}$ for every $i \in I$. □

The following result shows, that if we consider the special case $S_V = S_{v\varphi} = \mathcal{I}_{\mathcal{H}}$, then all the local frames are Parseval frames, even in case $\{(V_i, v_i)\}_{i\in I}$ is not a fusion Riesz basis.

Proposition 4.4.6. *Let $V = \{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system, such that $\{(V_i, v_i)\}_{i\in I}$ is a Parseval fusion frame and $v\varphi$ a Parseval frame. Then all local frames $\varphi^{(i)}$ are Parseval frames.*

Proof. By the above assumptions, we have

$$\sum_{i\in I} v_i^2 \pi_{V_i} f = f = \sum_{i\in I}\sum_{j\in J_i} \langle \pi_{V_i} f, v_i\varphi_{ij}\rangle v_i\varphi_{ij}$$

for all $f \in \mathcal{H}$. Combining this with the frame reconstruction formulas (2.8) for the local frames $\varphi^{(i)}$ yields

$$f = \sum_{i\in I} v_i^2 \pi_{V_i} f = \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle \pi_{V_i} f, S_{\varphi^{(i)}}^{-1}\varphi_{ij}\rangle \varphi_{ij} = \sum_{i\in I}\sum_{j\in J_i} \langle S_{\varphi^{(i)}}^{-1}\pi_{V_i} f, v_i\varphi_{ij}\rangle v_i\varphi_{ij}$$

for all $f \in \mathcal{H}$. In particular, if we fix some $i \in I$, then for arbitrary $g \in V_i$ we have $S_{\varphi^{(i)}} g \in V_i$ and thus we obtain

$$S_{\varphi^{(i)}} g = \sum_{i\in I}\sum_{j\in J_i} \langle S_{\varphi^{(i)}}^{-1} S_{\varphi^{(i)}} g, v_i\varphi_{ij}\rangle v_i\varphi_{ij} = \sum_{i\in I}\sum_{j\in J_i} \langle g, v_i\varphi_{ij}\rangle v_i\varphi_{ij} = g.$$

This implies $S_{\varphi^{(i)}} = \mathcal{I}_{V_i}$ for every $i \in I$. □

The proof of the following theorem depicts pretty well, how useful our proven operator identities for fusion frame systems can be. We prove some properties for fusion frame systems by using operator theoretic arguments. In that process, our results about component preserving operators between Hilbert direct sums will be very convenient. Some parts of the subsequent statement can also be found in [45].

Theorem 4.4.7. *Let $V = \{(V_i, v_i, \varphi^{(i)})\}_{i\in I}$ be a fusion frame system and $v\varphi$ be the corresponding global frame. Then the following are equivalent:*

(i) $v\varphi$ is a Riesz basis

(ii) $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis and $\varphi^{(i)}$ is a Riesz basis for every $i \in I$.

Proof. Hereinafter, we will implicitly apply Theorems 2.1.6, 2.1.7, 4.2.11 and 4.2.12 several times. We will also make use of the well-known fact that if $f : Y \longrightarrow Z$ and $g : X \longrightarrow Y$ are some arbitrary functions, then $f \circ g$ being injective implies g being injective.

To show the implication (i) $\Rightarrow$ (ii), observe that $v\varphi$ being a Riesz basis implies that $T_{v\varphi}$ is bounded and bijective. Since, by Theorem 4.3.5, we have $T_{v\varphi} = T_V \bigoplus_{i\in I} T_{\varphi^{(i)}}$, this implies that $\bigoplus_{i\in I} T_{\varphi^{(i)}}$ is bounded and injective. By Proposition 3.2.6, this implies that $T_{\varphi^{(i)}}$ is injective for every $i \in I$ and therefore $T_{\varphi^{(i)}}$ is bounded and bijective for every $i \in I$, i.e. $\varphi^{(i)}$ is a Riesz basis for every $i \in I$. To show that $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis, consider the equation $T_{v\varphi} = T_V \bigoplus_{i\in I} T_{\varphi^{(i)}}$ once again. The idea is to multiply this equation from the right with the inverse of $\bigoplus_{i\in I} T_{\varphi^{(i)}}$ in order to write T_V as composition of bounded and bijective operators, which implies that T_V is bounded and bijective as well, i.e. $\{(V_i, v_i)\}_{i\in I}$ is a fusion Riesz basis. To this end, note that since for any fusion frame system, there exist common frame bounds $A \leq B$ for the local frames $\varphi^{(i)}$, we not only have that $\{T_{\varphi^{(i)}}\}_{i\in I}$ is completely bounded by $\sqrt{B}$, but also have that $\{T^{-1}_{\varphi^{(i)}}\}_{i\in I} = \{T^*_{\tilde{\varphi}^{(i)}}\}_{i\in I}$ is completely bounded by $1/\sqrt{A}$. Thus Proposition 3.2.6 guarantees that $\bigoplus_{i\in I} T_{\varphi^{(i)}}$ is indeed invertible.

To show the implication (ii) $\Rightarrow$ (i), observe that $\varphi^{(i)}$ being a Riesz basis for every $i \in I$ and $\{(V_i, v_i)\}_{i\in I}$ being a fusion Riesz basis implies that the operators T_V and $T_{\varphi^{(i)}}$ $(i \in I)$ are not only bounded and surjective, but also bijective. Using the same argumentation as above, we again conclude that $\bigoplus_{i\in I} T_{\varphi^{(i)}}$ is bounded and bijective. Thus the composition $T_{v\varphi} = T_V \bigoplus_{i\in I} T_{\varphi^{(i)}}$ is bounded and bijective, i.e. $v\varphi$ is a Riesz basis. □

4.5 Dual fusion frames

In the following we consider the concept of duality for fusion frames. For frames, we have already defined the notion of a dual frame in Chapter 2. There we called a frame $\varphi = \{\varphi_i\}_{i\in I}$ a dual frame of the frame $\psi = \{\psi_i\}_{i\in I}$ if $\sum_{i\in I}\langle f, \psi_i\rangle\varphi_i = f = \sum_{i\in I}\langle f, \varphi_i\rangle\psi_i$ for all $f \in \mathcal{H}$, or equivalently

$$T_\varphi T_\psi^* = \mathcal{I}_\mathcal{H} = T_\psi T_\varphi^*. \tag{4.35}$$

In order to define a notion of duality for fusion frames, this time – in contrast to the usual scenario – it is *not* possible to directly extend this definition to the fusion frame setting, since the following obstacle occurs. Assume that $V = \{(V_i, v_i)\}_{i\in I}$ and $W = \{(W_i, w_i)\}_{i\in I}$ are fusion frames for $\mathcal{H}$. Then in general a composition like $T_W T_V^*$ is *not* well-defined, since T_V^* maps into $(\sum_{i\in I} \oplus V_i)_{\ell^2}$ and we have $\operatorname{dom}(T_W) = (\sum_{i\in I} \oplus W_i)_{\ell^2}$, but in general $\mathcal{R}(T_V^*) \subseteq (\sum_{i\in I} \oplus V_i)_{\ell^2} \nsubseteq (\sum_{i\in I} \oplus W_i)_{\ell^2} = \operatorname{dom}(T_W)$. Therefore the concept of duality for fusion frames is non-trivial, which makes fusion frame theory more interesting.

In the following we present different approaches from literature to define the concept of duality for fusion frames and we will see that they lead to the same general definition of a *dual fusion frame.*

In [17] the authors call $\tilde{V} := \{S_V^{-1}V_i, v_i\}_{i\in I}$ a *dual fusion frame* of V and (after some error corrections) it is shown in [32] that $\{S_V^{-1}V_i, v_i\}_{i\in I}$ indeed is a fusion frame for $\mathcal{H}$. More precisely, the author P. Găvruţa rewrites the fusion frame reconstruction formula

$$f = S_V^{-1}S_V f = \sum_{i\in I} v_i^2 S_V^{-1}\pi_{V_i} f \tag{4.36}$$

using the formula

$$S_V^{-1}\pi_{V_i} = \pi_{S_V^{-1}V_i} S_V^{-1}\pi_{V_i} \tag{4.37}$$

and hence proves the reconstruction formula

$$f = \sum_{i\in I} v_i^2 \pi_{S_V^{-1}V_i} S_V^{-1}\pi_{V_i} f. \tag{4.38}$$

In contrast to frame theory, the fusion frame operator for the dual fusion frame $\tilde{V}$ is given by $S_{\tilde{V}} = S_V$ (which in general is *not* the inverse fusion frame operator S_V^{-1}, which would be the analog to frame theory), see [15].

Moreover, Găvruţa uses (4.38) to define the notion of an *alternate dual* of V. He defines a fusion frame W to be an *alternate dual* of the fusion frame V, if for all $f \in \mathcal{H}$

$$f = \sum_{i \in I} v_i w_i \pi_{W_i} S_V^{-1} \pi_{V_i} f.$$

Using our notation for component preserving operators from Chapter 3, we may rewrite this to

$$T_W \bigoplus_{i \in I} (\pi_{W_i} S_V^{-1}) T_V^* = \mathcal{I}_{\mathcal{H}}. \tag{4.39}$$

As in [45], where the notation $\phi_{WV} := \bigoplus_{i \in I} (\pi_{W_i} S_V^{-1})$ is used, we will call this type of dual a *Găvruţa dual*. Since the Găvruţa dual $\tilde{V} = \{(S_V^{-1} V_i, v_i)\}_{i \in I}$ is closely related to fusion frame reconstruction, we call $\tilde{V}$ the *canonical dual fusion frame* of V, analogously to frame theory.

Equation (4.39) already hints how the obstacle from before can be circumvented and how we generally may define duality for fusion frames. If we approach the definition for dual fusion frames from an operator theoretic point of view, and consider some $O \in \mathcal{B}\big((\sum_{i \in I} \oplus V_i)_{\ell^2}, (\sum_{i \in I} \oplus W_i)_{\ell^2} \big)$, then $T_W O T_V^*$ is indeed a well-defined bounded operator. Therefore, as in [35], we call a fusion frame W an *O-dual fusion frame* or simply *dual fusion frame* of the fusion frame V, if there exists some $O \in \mathcal{B}\big((\sum_{i \in I} \oplus V_i)_{\ell^2}, (\sum_{i \in I} \oplus W_i)_{\ell^2} \big)$ such that

$$T_W O T_V^* = \mathcal{I}_{\mathcal{H}}. \tag{4.40}$$

Observe that W is an O-dual fusion frame of V if and only if V is an O^*-dual fusion frame of W (see also [35]). Thus, the relation of duality for fusion frames is non-symmetric, in stark contrast to frame theory.

In [35], the authors call W a component preserving dual fusion frame of V, if O additionally satisfies

$$O P_i \Big(\sum_{i \in I} \oplus V_i \Big)_{\ell^2} = Q_i \Big(\sum_{i \in I} \oplus W_i \Big)_{\ell^2} \tag{4.41}$$

for all $i \in I$, where for any $n \in I$, $P_n := \pi_{\mathcal{V}_n}$ and $Q_n := \pi_{\mathcal{V}_n}$ ($\mathcal{W}_n$ is defined analogously as $\mathcal{V}_n$ in (4.14)), i.e.

$$P_n : \Big(\sum_{i \in I} \oplus V_i \Big)_{\ell^2} \longrightarrow \mathcal{V}_n,$$

$$P_n \{f_i\}_{i \in I} := \{\delta_{ni} f_i\}_{i \in I} \tag{4.42}$$

and analogously

$$Q_n : \Big(\sum_{i \in I} \oplus W_i \Big)_{\ell^2} \longrightarrow \mathcal{W}_n,$$

$$Q_n \{g_i\}_{i \in I} := \{\delta_{ni} g_i\}_{i \in I}. \tag{4.43}$$

Thus, (4.41) is equivalent to O being a component preserving operator $O = \bigoplus_{i \in I} \mathcal{O}_i$ such that in addition $\mathcal{O}_i$ is surjective for every $i \in I$.

However, in this thesis we will call W a *component preserving dual fusion frame* of V, if $T_W O T_V^* = \mathcal{I}_{\mathcal{H}}$ and if in addition we only have

$$OP_i\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2} \subseteq Q_i\big(\sum_{i\in I}\oplus W_i\big)_{\ell^2},$$

since in this case the latter is equivalent to O being a component preserving operator $O = \bigoplus_{i\in I}\mathcal{O}_i$ (without any extra conditions on the operators $\mathcal{O}_i$, because surjectivity of the operators $\mathcal{O}_i$ is not needed). Note that by this definition Gavruta duals are component preserving dual fusion frames.

If W is a component preserving dual fusion frame of V and if in addition we indeed have (4.41), i.e. $\mathcal{O}_i$ is surjective for every $i \in I$, then we call W a *perfect component preserving dual fusion frame*. The reason, why we distinguish between these two definitions will be clearer when considering the results ahead.

One of the main results from [35] is a characterization of all perfect component preserving dual fusion frames W of a given fusion frame V under the condition that $w_i \geq C > 0$ for all $i \in I$:

Proposition [35] 4.5.1. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ and let $w = \{w_i\}_{i\in I}$ be a family of weights such that $w_i \geq C > 0$ for all $i \in I$. Then the perfect component preserving dual fusion frames W of V are the Bessel fusion sequences $W = \{(W_i, w_i)\}_{i\in I}$, such that*

$$W_i = \Big[S_V^{-1}T_V + R\big(\mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - T_V^* S_V^{-1} T_V\big)\Big] P_i\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2} \quad \text{for all } i \in I, \tag{4.44}$$

with $R \in \mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2}, \mathcal{H}\big)$.

In case V is also a fusion Riesz basis, we will able to simplify condition (4.44) drastically. Before we formulate this result, let us formulate (and – for sake of completion – prove) a helpful intermediate lemma, which has also been used in [35] and is analogously proven for frames, see [18].

Lemma 4.5.2. *Let V be a Bessel fusion sequence. Then the set of all bounded left-inverses of T_V^* is precisely the set*

$$\Big\{S_V^{-1}T_V + R\big(\mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - T_V^* S_V^{-1} T_V\big) : R \in \mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2}, \mathcal{H}\big)\Big\}.$$

Proof. For any operator contained in the latter set we have

$$\Big[S_V^{-1}T_V + R\big(\mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - T_V^* S_V^{-1} T_V\big)\Big] T_V^* = \mathcal{I}_{\mathcal{H}} + RT_V^* - RT_V^* = \mathcal{I}_{\mathcal{H}}.$$

Conversely, assume that A is a bounded left-inverse of T_V*. Then we have $A \in \mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2}, \mathcal{H}\big)$ and taking $R = A$ yields

$$\begin{aligned} S_V^{-1}T_V + A\big(\mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - T_V^* S_V^{-1} T_V\big) &= S_V^{-1}T_V + A\mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - AT_V^* S_V^{-1} T_V \\ &= S_V^{-1}T_V + A - S_V^{-1}T_V = A. \end{aligned}$$

□

Proposition 4.5.3. *Let $V = \{(V_i, v_i)\}_{i\in}$ be a fusion Riesz basis for $\mathcal{H}$ and let $w = \{w_i\}_{i\in I}$ be a family of weights such that $w_i \geq C > 0$ for all $i \in I$. Then the perfect component preserving dual fusion frames W of V are the Bessel fusion sequences $W = \{(W_i, w_i)\}_{i\in I}$, where $W_i = S_V^{-1}V_i$ for all $i \in I$.*

Proof. First, assume that $W = \{(W_i, w_i)\}_{i\in I}$ is a Bessel fusion sequence with $w_i \geq C > 0$ and $W_i = S_V^{-1}V_i$ for all $i \in I$. Then, by Lemma 4.2.3, the family $\{\frac{v_i}{w_i}S_V^{-1}\}_{i\in I}$ of operators $\frac{v_i}{w_i}S_V^{-1} : V_i \longrightarrow S_V^{-1}V_i$ is completely bounded by $\frac{\sqrt{D_V}}{C_V C}$ (where C_V and D_V denote the lower and upper fusion frame bound for V respectively). Therefore, by Proposition 3.2.1, $\bigoplus_{i\in I}(\frac{v_i}{w_i}S_V^{-1}) \in \mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2}, (\sum_{i\in I}\oplus W_i)_{\ell^2}\big)$. Moreover, (compare to (4.42) and (4.43)) we have

$$\bigoplus_{i\in I}\Big(\frac{v_i}{w_i}S_V^{-1}\Big)P_i\Big(\sum_{i\in I}\oplus V_i\Big)_{\ell^2} = \bigtimes\Big\{\delta_{ij}\frac{v_j}{w_j}S_V^{-1}V_j\Big\}_{j\in I} = \bigtimes\{\delta_{ij}S_V^{-1}V_j\}_{j\in I} = Q_i\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}.$$

Now observe that for all $f \in \mathcal{H}$ we have

$$\begin{aligned} T_W\bigoplus_{i\in I}\Big(\frac{v_i}{w_i}S_V^{-1}\Big)T_V^* f &= \sum_{i\in I} w_i\pi_{W_i}\frac{v_i}{w_i}S_V^{-1}v_i\pi_{V_i}f \\ &= \sum_{i\in I} v_i^2\pi_{S_V^{-1}V_i}S_V^{-1}\pi_{V_i}f = f \qquad \text{(by (4.38))}. \end{aligned}$$

This means that W is a perfect component preserving dual fusion frame of V with respect to the operator $\bigoplus_{i\in I}\big(\frac{v_i}{w_i}S_V^{-1}\big)$.
Conversely, assume that $W = \{(W_i, w_i)\}_{i\in I}$ is a perfect component preserving dual fusion frame of V with respect to the operator $\bigoplus_{i\in I}\mathcal{O}_i$. Then, by definition, $T_W\bigoplus_{i\in I}\mathcal{O}_i$ is a bounded left-inverse of T_V^*. Applying Lemma 4.5.2 yields

$$\begin{aligned} W_i = w_i\pi_{W_i}W_i &= T_W Q_i\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2} \\ &= T_W\Big(\bigoplus_{i\in I}\mathcal{O}_i\Big)P_i\Big(\sum_{i\in I}\oplus V_i\Big)_{\ell^2} \\ &= \Big[S_V^{-1}T_V + R\big(\mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - T_V^*S_V^{-1}T_V\big)\Big]P_i\Big(\sum_{i\in I}\oplus V_i\Big)_{\ell^2} = (*), \end{aligned}$$

where $R \in \mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2}, \mathcal{H}\big)$. However, since V is a fusion Riesz basis, T_V is invertible, which implies that $T_V^*S_V^{-1}T_V = T_V^*(T_V^*)^{-1}T_V^{-1}T_V = \mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}}$ and thus $R(\mathcal{I}_{(\sum_{i\in I}\oplus V_i)_{\ell^2}} - T_V^*S_V^{-1}T_V) = 0$. Therefore, we obtain

$$\begin{aligned} W_i = (*) &= S_V^{-1}T_V P_i\Big(\sum_{i\in I}\oplus V_i\Big)_{\ell^2} \\ &= S_V^{-1}v_i\pi_{V_i}V_i \\ &= S_V^{-1}V_i \quad \text{for all } i \in I. \end{aligned}$$

□

We remark that we only used the condition $w_i \geq C > 0$ for all $i \in I$ to show that the family $\{\frac{v_i}{w_i}S_V^{-1}\}_{i\in I}$ is completely bounded. So, in the above we proved that any Bessel fusion sequence of the form $W = \{(S_V^{-1}V_i, w_i)\}_{i\in I}$ with $w_i \geq C > 0$ is a perfect component preserving dual fusion frame of the fusion frame $V = \{(V_i, v_i)\}_{i\in I}$ (where V is not necessarily a fusion Riesz basis, and the weights v_i are arbitrary) and conversely, we showed that any perfect component preserving dual fusion frame $W = \{(W_i, w_i)\}_{i\in I}$ (with arbitrary weights w_i) of a given fusion Riesz basis V must have $W_i = S_V^{-1} V_i$ for all $i \in I$. Note that in case $v_i = w_i$ for all $i \in I$ or in case $v_i \geq w_i$ for all except finitely many $i \in I$, the family $\{\frac{v_i}{w_i} S_V^{-1}\}_{i\in I}$ is still completely bounded and the proof works in these cases too. Thus we may note the following consequence of Theorem 4.5.3.

Corollary 4.5.4. *Let $V = \{(V_i, v_i)\}_{i\in I}$ be a fusion Riesz basis. The canonical dual fusion frame $\tilde{V} = \{(S_V^{-1} V_i, v_i)\}_{i\in I}$ is the unique perfect component preserving dual fusion frame of V having the same weights v_i.*

For (general) component preserving dual fusion frames we can prove the following result. We will keep the proof short, since it almost coincides word by word with the proof of Theorem 4.5.3.

Proposition 4.5.5. *Let $V = \{(V_i, v_i)\}_{i\in}$ be a fusion Riesz basis for $\mathcal{H}$ and let $w = \{w_i\}_{i\in I}$ be a family of weights such that $w_i \geq C > 0$ for all $i \in I$. Then the component preserving dual fusion frames W of V are the Bessel fusion sequences $W = \{(W_i, w_i)\}_{i\in I}$, where $S_V^{-1} V_i \subseteq W_i$ for all $i \in I$.*

Proof. If $W = \{(W_i, w_i)\}_{i\in I}$ is a Bessel fusion sequence with $w_i \geq C > 0$ and $S_V^{-1} V_i \subseteq W_i$ for all $i \in I$, then we see that $\bigoplus_{i\in I} (\frac{v_i}{w_i} S_V^{-1})$ is a well-defined and bounded component preserving operator from $(\sum_{i\in I} \oplus V_i)_{\ell^2})$ into $(\sum_{i\in I} \oplus S_V^{-1} V_i)_{\ell^2} \subseteq (\sum_{i\in I} \oplus W_i)_{\ell^2}$. Clearly, we have $T_W (\bigoplus_{i\in I} v_i w_i^{-1} S_V^{-1}) T_V^* = \mathcal{I}_{\mathcal{H}}$, as before.
Conversely, if $W = \{(W_i, w_i)\}_{i\in I}$ is a component preserving dual fusion frame of V then there exists a suitable operator $\bigoplus_{i\in I} \mathcal{O}_i$ such that $T_W \bigoplus_{i\in I} \mathcal{O}_i$ is a bounded left-inverse of T_V^*. As before, we obtain via Lemma 4.5.2

$$\begin{aligned}
W_i &= T_W Q_i \Big(\sum_{i\in I} \oplus W_i \Big)_{\ell^2} \\
&\supseteq T_W \Big(\bigoplus_{i\in I} \mathcal{O}_i \Big) P_i \Big(\sum_{i\in I} \oplus V_i \Big)_{\ell^2} \\
&= \Big[S_V^{-1} T_V + R \big(\mathcal{I}_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} - T_V^* S_V^{-1} T_V \big) \Big] P_i \Big(\sum_{i\in I} \oplus V_i \Big)_{\ell^2} \\
&= S_V^{-1} T_V P_i \Big(\sum_{i\in I} \oplus V_i \Big)_{\ell^2} = S_V^{-1} V_i,
\end{aligned}$$

where we again used that $\mathcal{I}_{(\sum_{i\in I} \oplus V_i)_{\ell^2}} - T_V^* S_V^{-1} T_V = 0$ for fusion Riesz bases, as in the proof of Proposition 4.5.3. □

Note that the above characterizing results strongly depend on the structure of component preserving operators. For general O-dual fusion frames, the above approach does do not apply, since the general case involves arbitrary bounded operators $O \in \mathcal{B}\big((\sum_{i\in I} \oplus V_i)_{\ell^2}, (\sum_{i\in I} \oplus W_i)_{\ell^2} \big)$, which can be of much more complicated nature (see Chapter 5). Thus proving results about general O-dual fusion frames might be much more difficult.

However, one particular special case is easy to show:

Corollary 4.5.6. *If W is an O-dual fusion frame of a fusion frame V and if both V and W are fusion Riesz bases, then $O = T_W^{-1} (T_V^*)^{-1}$.*

Proof. By assumption we have $T_W O T_V^* = \mathcal{I}_{\mathcal{H}}$. Since both T_W and T_V^* are invertible by Thoerem 4.2.12, $O = T_W^{-1} (T_V^*)^{-1}$ follows immediately. □

5 Infinite matrices of operators

So far, we have dealt with operators in $\mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2},(\sum_{i\in I}\oplus W_i)_{\ell^2}\big)$ several times. In Chapter 3 we have discussed bounded component preserving operators between two Hilbert direct sums (corresponding to the same index set I), which are a very special and simple class of operators from this space. We characterized this class of operators by giving a very simple equivalent condition and consequently derived some nice properties of bounded component preserving operators. In Chapter 4 we applied those results quite often in order to prove some fusion frame theoretic results. Since Hilbert direct sums are the natural representation spaces for fusion frames, it might be useful to also understand general bounded (not necessarily component preserving) operators between Hilbert direct sums. In particular, at least since Section 4.5, where we discussed the concept of duality for fusion frames, the question, how general operators $O\in\mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2},(\sum_{i\in I}\oplus W_i)_{\ell^2}\big)$ behave, arises.

In the following we will consider this general case. By doing so, we will not only generalize some results from Chapter 3, but we will also view component preserving operators in a different and probably more intuitive way than before.

In [37] operators from the space $\mathcal{B}\big((\sum_{i\in I}\oplus V_i)_{\ell^2},(\sum_{i\in I}\oplus V_i)_{\ell^2}\big)$ are represented by (possibly infinite) matrices of bounded operators and the author gives characterizing conditions for this being the case. In Sections 5.1 and 5.2 we will give definitions and results very similar to those in [37]. Nevertheless, we will prove the results in full detail, since in [37] many proof details are missing. Some of our proofs will be different from the ones in [37], while other proof ideas will be adapted from the ones in [37]. In addition to that we will give plenty of remarks and examples, which cannot be found there and which relate to previous contents of this thesis and other fusion frame related topics. Moreover, we will consider the slightly more general scenario, where $O\in\mathcal{B}\big((\sum_{j\in J}\oplus V_j)_{\ell^2},(\sum_{i\in I}\oplus W_i)_{\ell^2}\big)$, I and J are countable index sets of possibly different size (since we don't want to exclude cases like $J=\mathbb{N}, I=\{1,...,N\}$) and where for all $i\in I$ and all $j\in J$, V_j and W_i are arbitrary Hilbert spaces.

In the later sections of this chapter we consider classes of Banach spaces, which are similar to Hilbert direct sums. Also, compact operators between Hilbert direct sums will be investigated.

In the following we fix the notations $\mathcal{K}_V^2=(\sum_{j\in J}\oplus V_j)_{\ell^2}$ and $\mathcal{K}_W^2=(\sum_{i\in I}\oplus W_i)_{\ell^2}$, where I and J are fixed (but arbitrary countable) index sets. Moreover, in some situations it will be convenient to assume without loss of generality that $I=\mathbb{N}$ or $J=\mathbb{N}$, – in particular, when we consider *matrix representations* of operators (see ahead).

5.1 Matrix representations

Assume that $O_{ij}\in\mathcal{B}(V_j,W_i)$ for all $i\in I$, $j\in J$ and let $f=\{f_j\}_{j\in J}\in\mathcal{K}_V^2$. Then

$$Of := \Big\{\sum_{j\in J} O_{ij} f_j\Big\}_{i\in I} \tag{5.1}$$

defines a (possibly unbounded) operator

$$O : \mathrm{dom}(O) \longrightarrow \mathcal{K}^2_W,$$

where

$$\mathrm{dom}(O) = \left\{ f = \{f_j\}_{j\in J} \in \mathcal{K}^2_V : \|Of\|^2_{\mathcal{K}^2_W} = \sum_{i\in I} \Big\| \sum_{j\in J} O_{ij} f_j \Big\|^2_{W_i} < \infty \right\} \subseteq \mathcal{K}^2_V. \tag{5.2}$$

Note that $\mathrm{dom}(O) = \mathcal{K}^2_V$ means that

$$\|Of\|^2_{\mathcal{K}^2_W} = \sum_{i\in I} \Big\| \sum_{j\in J} O_{ij} f_j \Big\|^2_{W_i} < \infty \qquad \text{for all } f \in \mathcal{K}^2_V,$$

which in particular means that $\sum_{j\in J} O_{ij} f_j$ converges in the W_i-norm for every $i \in I$. We will often use the latter implicitly, when we consider operators from $\mathcal{K}^2_V$ into $\mathcal{K}^2_W$ (i.e. $\mathrm{dom}(O) = \mathcal{K}^2_V$) defined as in (5.1). If we view elements in $\mathcal{K}^2_V$ as column vectors of (in general) infinite size, then we can represent the operator O by the infinite matrix of operators

$$O = \begin{bmatrix} O_{11} & O_{12} & O_{13} & \dots \\ O_{21} & O_{22} & O_{23} & \dots \\ O_{31} & O_{32} & O_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{5.3}$$

since then, by the definition of O, Of corresponds to the formal matrix multiplication of O as in (5.3) to the column vector representing f.

In the following, we will show that if O is defined as in (5.1) and if $\mathrm{dom}(O) = \mathcal{K}^2_V$, i.e. if O maps the entire space $\mathcal{K}^2_V$ into $\mathcal{K}^2_W$, then this already implies that $O \in \mathcal{B}(\mathcal{K}^2_V, \mathcal{K}^2_W)$. We will show this by applying a variant of the Uniform Boundedness Principle (see Appendix). In order to write down the proof properly, we define the operators

$$P_{\langle n\rangle} : \mathcal{K}^2_V \longrightarrow \mathcal{K}^2_V,$$

$$P_{\langle n\rangle}\{f_j\}_{j\in J} = (f_1, ..., f_n, 0, 0, ...)$$

and

$$Q_{\langle n\rangle} : \mathcal{K}^2_W \longrightarrow \mathcal{K}^2_W,$$

$$Q_{\langle n\rangle}\{g_i\}_{i\in I} = (g_1, ..., g_n, 0, 0, ...),$$

where $n \in \mathbb{N}$. We immediately see that $P^2_{\langle n\rangle} = P_{\langle n\rangle}$ and $Q^2_{\langle n\rangle} = Q_{\langle n\rangle}$. Moreover, observe that for arbitrary $\{h_j\}_{j\in J} \in \mathcal{K}^2_V$ we have

$$\big\langle P_{\langle n\rangle}\{f_j\}_{j\in J}, \{h_j\}_{j\in J} \big\rangle_{\mathcal{K}^2_V} = \sum_{j=1}^n \langle f_j, h_j\rangle_{V_j} = \big\langle \{f_j\}_{j\in J}, P_{\langle n\rangle}\{h_j\}_{j\in J} \big\rangle_{\mathcal{K}^2_V},$$

i.e. $P_{\langle n\rangle}$ is self-adjoint. Analogously we see that $Q_{\langle n\rangle}$ is self-adoint. Thus, for every $n \in \mathbb{N}$, $P_{\langle n\rangle}$ and $Q_{\langle n\rangle}$ are orthogonal projections. Note that we have $P_{\langle n\rangle} = \sum_{i=1}^n P_i$ and $Q_{\langle n\rangle} = \sum_{i=1}^n Q_i$.

Proposition 5.1.1. *Let $O_{ij} \in \mathcal{B}(V_j, W_i)$ for all $j \in J$ and $i \in I$, and define O as in (5.1). If dom$(O) = \mathcal{K}_V^2$, then $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$.*

Proof. The assumption $\mathrm{dom}(O) = \mathcal{K}_V^2$ means that O is a well-defined (not necessarily bounded) operator from $\mathcal{K}_V^2$ into $\mathcal{K}_W^2$. Consider the operators $T_m = Q_{\langle m\rangle} O : \mathcal{K}_V^2 \longrightarrow \mathcal{K}_W^2$. If we can show that each T_m is bounded and that the operators T_m converge pointwise to O as $m \longrightarrow \infty$, then, by the Uniform Boundedness Principle (see Appendix), this implies $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$.

To see that the operators T_m converge pointwise to O, recall that our assumption $\mathrm{dom}(O) = \mathcal{K}_V^2$ implies that for all $f \in \mathcal{K}_V^2$ we have $\|Of\|^2_{\mathcal{K}_W^2} = \sum_{i=1}^\infty \| \sum_{j=1}^\infty O_{ij} f_j \|^2_{W_i} < \infty$, which in particular implies that

$$\lim_{m\longrightarrow\infty} \|(O - T_m)f\|^2_{\mathcal{K}_W^2} = \lim_{m\longrightarrow\infty} \|(\mathcal{I}_{\mathcal{K}_W^2} - Q_{\langle m\rangle})Of\|^2_{\mathcal{K}_W^2} = \lim_{m\longrightarrow\infty} \sum_{i=m+1}^\infty \left\| \sum_{j=1}^\infty O_{ij} f_j \right\|^2_{W_i} = 0.$$

It remains to show that $T_m \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ for each m. We will use the Uniform Boundedness Principle again to see that this is true. To this end, we fix an arbitrary $m \in \mathbb{N}$ and consider the operators $T_m^{(n)} = Q_{\langle m\rangle} O P_{\langle n\rangle} : \mathcal{K}_V^2 \longrightarrow \mathcal{K}_W^2$. To see that $T_m^{(n)}$ is bounded for each $n \in \mathbb{N}$, observe that we have

$$\begin{aligned}
\|Q_{\langle m\rangle} O P_{\langle n\rangle} f\|^2_{\mathcal{K}_W^2} &= \sum_{i=1}^m \left\| \sum_{j=1}^n O_{ij} f_j \right\|^2_{W_i} \\
&\le \sum_{i=1}^m \left(\sum_{j=1}^n \|O_{ij}\|_{V_j \to W_i} \|f_j\|_{V_j} \right)^2 \\
&\le \sum_{i=1}^m \left(\sum_{j=1}^n \|O_{ij}\|^2_{V_j \to W_i} \right) \left(\sum_{j=1}^n \|f_j\|^2_{V_j} \right),
\end{aligned}$$

where we used the Cauchy Schwartz inequality. This implies

$$\|Q_{\langle m\rangle} O P_{\langle n\rangle}\|^2_{\mathcal{K}_V^2 \to \mathcal{K}_W^2} \le mn \sup_{1\le i\le m, 1\le j\le n} \|O_{ij}\|^2_{V_j \to W_i} < \infty.$$

What's left to show is that $T_m^{(n)} \longrightarrow T_m$ pointwise (as $n \longrightarrow \infty$), i.e. that for any $f = \{f_j\}_{j\in J} \in \mathcal{K}_V^2$ we have

$$\lim_{n\longrightarrow\infty} \left\|(T_m - T_m^{(n)})f\right\|_{\mathcal{K}_W^2} = \lim_{n\longrightarrow\infty} \left\|Q_{\langle m\rangle} O(\mathcal{I}_{\mathcal{K}_V^2} - P_{\langle n\rangle})f\right\|_{\mathcal{K}_W^2} = 0. \tag{5.4}$$

To this end, fix an arbitrary $f = \{f_j\}_{j\in J} \in \mathcal{K}_V^2$ and define

$$g_n^{(i)} := \sum_{j=1}^n O_{ij} f_j.$$

Then clearly $g_n^{(i)} \in W_i$ for every $n \in \mathbb{N}$ and every i and by construction we have $\lim_{n\longrightarrow\infty} g_n^{(i)} = \sum_{j=1}^\infty O_{ij} f_j \in W_i$, since by assumption $Of \in \mathcal{K}_W^2$. This implies that for every i, the sequence $\{x_n^{(i)}\}_{n\in\mathbb{N}}$, defined by $x_n^{(i)} = \| \sum_{j=n}^\infty O_{ij} f_j \|^2_{W_i}$ is a convergent sequence in $\mathbb{R}$ with limit 0. This implies

$$\lim_{n\longrightarrow\infty} \left\|Q_{\langle m\rangle} O(\mathcal{I}_{\mathcal{K}_V^2} - P_{\langle n\rangle})f\right\|_{\mathcal{K}_W^2} = \lim_{n\to\infty} \sum_{i=1}^m \left\| \sum_{j=n+1}^\infty O_{ij} f_j \right\|^2_{W_i} = \sum_{i=1}^m \lim_{n\to\infty} \left\| \sum_{j=n+1}^\infty O_{ij} f_j \right\|^2_{W_i} = 0$$

and (5.4) is proven. □

In other words, any matrix (O_{ij}) of operators $O_{ij} \in \mathcal{B}(V_j, W_i)$, which maps the whole space $\mathcal{K}_V^2$ into $\mathcal{K}_W^2$, defines a bounded operator $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$. However, the next result shows that the converse is also true. In order to prove it, let us consider the following operators beforehand.

For any $k \in J$ we define the coordinate function

$$\Phi_k : \mathcal{K}_V^2 \longrightarrow V_k$$

via

$$\Phi_k \{f_j\}_{j\in J} := f_k.$$

Clearly, each of these operators Φ_k is bounded (with $\|\Phi_k\| \le 1$) and hence possesses a uniquely determined adjoint operator $\Phi_k^* : V_k \longrightarrow \mathcal{K}_V^2$. Φ_k^* maps $f_k \in V_k$ onto $(0, ..., 0, f_k, 0, 0, ...) \in \mathcal{K}_V^2$ (f_k in the k-th component), since by definition of Φ_k^* we have for any $\{g_j\}_{j\in J} \in \mathcal{K}_V^2$

$$\begin{aligned}
\left\langle \Phi_j^* f_j, \{g_j\}_{j\in J} \right\rangle_{\mathcal{K}_V^2} &= \langle f_j, \Phi_j \{g_j\}_{j\in J} \rangle_{V_j} \\
&= \langle f_j, g_j \rangle_{V_j} \\
&= \langle (0, ..., 0, f_j, 0, 0, ...), \{g_j\}_{j\in J} \rangle_{\mathcal{K}_V^2}.
\end{aligned}$$

Analogously, for every $l \in I$, we define the coordinate function

$$\Psi_l : \mathcal{K}_W^2 \longrightarrow W_l$$

via

$$\Psi_l \{h_i\}_{i\in I} := h_l$$

and see that $\|\Psi_l\| \le 1$ ($l \in I$) and that $\Psi_l^* : W_l \longrightarrow \mathcal{K}_W^2$ is given by $\Psi_l^* g_l = (0, ..., 0, g_l, 0, 0, ...)$ (g_l in the l-th component).

Proposition 5.1.2. *If $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$, then there exists a uniquely determined matrix (O_{ij}) of operators $O_{ij} \in \mathcal{B}(V_j, W_i)$, such that the action of O on any given $\{f_j\}_{j\in J} \in \mathcal{K}_V^2$ is given by the formal matrix multiplication of (O_{ij}) with $\{f_j\}_{j\in J}$, i.e.*

$$O\{f_j\}_{j\in J} = \left\{ \sum_{j\in J} O_{ij} f_j \right\}_{i\in I}.$$

Proof. For each $f = \{f_j\}_{j\in J} \in \mathcal{K}_V^2$ we have

$$Of = O \sum_{j\in J} P_j f = \sum_{j\in J} O P_j f = \sum_{j\in J} O \Phi_j^* f_j.$$

Since $Of \in \mathcal{K}_W^2$ we may write $Of = \{[Of]_i\}_{i\in I}$ where $[Of]_i \in W_i$ denotes the i-th component of Of. By definition of Ψ_i we therefore see that

$$\begin{aligned}
Of &= \{[Of]_i\}_{i\in I} \\
&= \{\Psi_i O f\}_{i\in I} \\
&= \left\{ \Psi_i \sum_{j\in J} O \Phi_j^* f_j \right\}_{i\in I} \\
&= \left\{ \sum_{j\in J} \Psi_i O \Phi_j^* f_j \right\}_{i\in I}.
\end{aligned}$$

We set $O_{ij} := \Psi_i O \Phi_j^* \in \mathcal{B}(V_j, W_i)$ and observe that O acts on $f = \{f_j\}_{j\in J} \in \mathcal{K}_V^2$ in the same way as the matrix (O_{ij}) does in (5.1), when considering f as column vector. The uniqueness of the matrix (O_{ij}) follows from its construction. $\square$

Due to Propositions 5.1.1 and 5.1.2 we may identify every operator $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ with a matrix (O_{ij}) of bounded operators, which (formally) acts on $f = \{f_j\}_{j\in J}$ via matrix multiplication (as in (5.1)), and which additionally satisfies $\operatorname{dom}(O) = \mathcal{K}_V^2$ (compare to (5.2)), and vice versa. We use the notation

$$\mathbb{M}(O) = (O_{ij})$$

and call $\mathbb{M}(O) = (O_{ij})$ the *matrix representation* of O.

In the following, we give some concrete examples of matrix representations of some bounded operators between Hilbert direct sums.

The bounded operators $P_{\langle n\rangle} : \mathcal{K}_V^2 \longrightarrow \mathcal{K}_V^2$ and $Q_{\langle n\rangle} : \mathcal{K}_W^2 \longrightarrow \mathcal{K}_W^2$ from above have the matrix representations

$$\mathbb{M}(P_{\langle n\rangle}) = \begin{bmatrix} \mathcal{I}_{V_1} & 0 & \dots & \dots & \dots \\ 0 & \ddots & 0 & \dots & \dots \\ \vdots & 0 & \mathcal{I}_{V_n} & 0 & \dots \\ \vdots & \vdots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\mathbb{M}(Q_{\langle n\rangle}) = \begin{bmatrix} \mathcal{I}_{W_1} & 0 & \dots & \dots & \dots \\ 0 & \ddots & 0 & \dots & \dots \\ \vdots & 0 & \mathcal{I}_{W_n} & 0 & \dots \\ \vdots & \vdots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The bounded operators $P_n : \mathcal{K}_V^2 \longrightarrow \mathcal{K}_V^2$ and $Q_n : \mathcal{K}_W^2 \longrightarrow \mathcal{K}_W^2$, defined as in (4.42) and (4.43) respectively, have the matrix representations

$$\mathbb{M}(P_n) = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & 0 & 0 & 0 & \dots \\ \dots & 0 & \mathcal{I}_{V_n} & 0 & \dots \\ \dots & 0 & 0 & 0 & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\mathbb{M}(Q_n) = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & 0 & 0 & 0 & \dots \\ \dots & 0 & \mathcal{I}_{W_n} & 0 & \dots \\ \dots & 0 & 0 & 0 & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the entries $\mathcal{I}_{V_n}$ and $\mathcal{I}_{W_n}$ both are in the (n,n)-th component of their respective matrices. Observe, that we again see that $P_{\langle n\rangle} = \sum_{i=1}^n P_i$ and $Q_{\langle n\rangle} = \sum_{i=1}^n Q_i$, – this time via matrix representations.

If we regard the Hilbert space V_k as the Hilbert direct sum $\left(\sum_{j\in J} \oplus V_j\right)_{\ell^2}$, where $I = \{k\}$, then we see that Φ_k has the matrix representation

$$\mathbb{M}(\Phi_k) = \begin{bmatrix} \dots & 0 & \mathcal{I}_{V_k} & 0 & \dots \end{bmatrix},$$

and analogously we have

$$\mathbb{M}(\Psi_l) = \begin{bmatrix} \dots & 0 & \mathcal{I}_{W_l} & 0 & \dots \end{bmatrix},$$

as well as

$$\mathbb{M}(\Phi_k^*) = \begin{bmatrix} \vdots \\ 0 \\ \mathcal{I}_{V_k} \\ 0 \\ \vdots \end{bmatrix}$$

and

$$\mathbb{M}(\Psi_l^*) = \begin{bmatrix} \vdots \\ 0 \\ \mathcal{I}_{W_l} \\ 0 \\ \vdots \end{bmatrix}.$$

The component preserving operators $\bigoplus_{i\in I} \mathcal{O}_i : \mathcal{K}_V^2 \longrightarrow \mathcal{K}_W^2$ (here we have $I = J$), which we have introduced in Chapter 3, have the matrix representation

$$\mathbb{M}\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big) = \begin{bmatrix} \mathcal{O}_1 & 0 & 0 & \dots \\ 0 & \mathcal{O}_2 & 0 & \dots \\ 0 & 0 & \mathcal{O}_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In Proposition 3.2.2 (a) we computed the adjoint of $\bigoplus_{i\in I} \mathcal{O}_i$, which has the matrix representation

$$\mathbb{M}\Big(\Big(\bigoplus_{i\in I} \mathcal{O}_i\Big)^*\Big) = \begin{bmatrix} \mathcal{O}_1^* & 0 & 0 & \dots \\ 0 & \mathcal{O}_2^* & 0 & \dots \\ 0 & 0 & \mathcal{O}_3^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We will now consider the general case and prove a generalized version of Proposition 3.2.2 (a).

Proposition 5.1.3. *Let $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ with matrix representation $\mathbb{M}(O) = (O_{ij})$. Then $O^* \in \mathcal{B}(\mathcal{K}_W^2, \mathcal{K}_V^2)$ has the matrix representation $\mathbb{M}(O^*) = ((O^*)_{ji}) = ((O_{ij})^*)$, i.e.*

$$\mathbb{M}(O^*) = \begin{bmatrix} O_{11}^* & O_{21}^* & O_{31}^* & \dots \\ O_{12}^* & O_{22}^* & O_{32}^* & \dots \\ O_{13}^* & O_{23}^* & O_{33}^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.5}$$

Proof. Since O is a bounded operator between two Hilbert spaces, its adjoint O^* is well-defined and bounded, i.e. $O^* \in \mathcal{B}(\mathcal{K}_W^2, \mathcal{K}_V^2)$. Thus for every $g = \{g_i\}_{i\in I} \in \mathcal{K}_W^2$, $O^* g$ is a sequence in $\mathcal{K}_V^2$. Hence we may write $O^* g = \{[O^* g]_j\}_{j\in J}$. As in the proof of Proposition 5.1.2 we use the operators Φ_j and Ψ_i and see that

$$\begin{aligned} O^* g &= \{[O^* g]_j\}_{j\in J} \\ &= \left\{ \left[O^* \sum_{i\in I} Q_i g \right]_j \right\}_{j\in J} \\ &= \left\{ \Phi_j \sum_{i\in I} O^* Q_i g \right\}_{j\in J} = \left\{ \sum_{i\in I} \Phi_j O^* \Psi_i^* g_i \right\}_{j\in J}. \end{aligned}$$

This implies $(O^*)_{ji} = \Phi_j O^* \Psi_i^* = (\Psi_i O \Phi_j^*)^* = (O_{ij})^* \in \mathcal{B}(W_i, V_j)$. □

Lets continue with some more relevant examples of matrix representations of a bounded operator from $\mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$.

In [45] the authors Balazs, Shamsabadi and Arefijamaal define the *U-fusion cross Gram matrix* as follows. If $W = \{(W_i, w_i)\}_{i\in I}$ is a Bessel fusion sequence for $\mathcal{H}$ and $V = \{(V_i, v_i)\}_{i\in I}$ a fusion frame for $\mathcal{H}$, then for any $U \in \mathcal{B}(\mathcal{H})$ the operator

$$\mathcal{G}_{U,W,V} : \mathcal{K}_W^2 \longrightarrow \mathcal{K}_W^2,$$

defined via

$$\mathcal{G}_{U,W,V}\{f_i\}_{i\in I} \coloneqq \phi_{WV} T_V^* U T_W = \bigoplus_{i\in I} (\pi_{W_i} S_V^{-1}) T_V^* U T_W,$$

is called *U-fusion cross Gram matrix*. This operator generalizes the *Gram matrix* associated to a frame (see [18] for more details). Since the action of $\mathcal{G}_{U,W,V}$ on any $\{f_i\}_{i\in I} \in \mathcal{K}_W^2$ is given by

$$\mathcal{G}_{U,W,V}\{f_i\}_{i\in I} = \Big\{\pi_{W_i} S_V^{-1} v_i \pi_{V_i} U \sum_{j\in I} w_j f_j\Big\}_{i\in I} = \Big\{\sum_{j\in I} \pi_{W_i} S_V^{-1} \pi_{V_i} w_j U f_j\Big\}_{i\in I},$$

the matrix representation of the U-fusion cross Gram matrix $\mathcal{G}_{U,W,V}$ is given by

$$\mathbb{M}(\mathcal{G}_{U,W,V}) = \begin{bmatrix} \pi_{W_1} S_V^{-1} \pi_{V_1} v_1 w_1 U & \pi_{W_1} S_V^{-1} \pi_{V_1} v_1 w_2 U & \pi_{W_1} S_V^{-1} \pi_{V_1} v_1 w_3 U & \dots \\ \pi_{W_2} S_V^{-1} \pi_{V_2} v_2 w_1 U & \pi_{W_2} S_V^{-1} \pi_{V_2} v_2 w_2 U & \pi_{W_2} S_V^{-1} \pi_{V_2} v_2 w_3 U & \dots \\ \pi_{W_3} S_V^{-1} \pi_{V_3} v_3 w_1 U & \pi_{W_3} S_V^{-1} \pi_{V_3} v_3 w_2 U & \pi_{W_3} S_V^{-1} \pi_{V_3} v_3 w_3 U & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This operator might take much simpler forms [45]: For instance, if $U = \mathcal{I}_{\mathcal{H}}$ and $W = V$ is an orthonormal fusion basis, then $\mathcal{G}_{U,W,W} = \mathcal{I}_{\mathcal{K}_W} = \operatorname{diag}[(\mathcal{I}_{W_i})_{i\in I}]$.

Moreover, in [45] it is shown that $\mathcal{G}_{U,W,W}$ is invertible if and only if U is invertible and W is a fusion Riesz basis, in which case we have $\mathcal{G}_{U,W,W}^{-1} = \mathcal{G}_{S_{W^u}^{-1} U^{-1} S_{W^u}^{-1}, W, W}$, i.e.

$$\mathbb{M}(\mathcal{G}_{U,W,W}^{-1}) = \begin{bmatrix} \pi_{W_1} S_W^{-1} \pi_{W_1} w_1^2 S_{W^u}^{-1} U^{-1} S_{W^u}^{-1} & \pi_{W_1} S_W^{-1} \pi_{W_1} w_1 w_2 S_{W^u}^{-1} U^{-1} S_{W^u}^{-1} & \dots \\ \pi_{W_2} S_W^{-1} \pi_{W_2} w_2 w_1 S_{W^u}^{-1} U^{-1} S_{W^u}^{-1} & \pi_{W_2} S_W^{-1} \pi_{W_2} w_2^2 S_{W^u}^{-1} U^{-1} S_{W^u}^{-1} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

We refer the interested reader to [45] for other results about U-fusion Cross Gram matrices.

In [7] the authors define the *matrix induced by the operator* U as follows. If V is a Bessel fusion sequence of a Hilbert space $\mathcal{H}_1$ with Bessel fusion bound D_V, W a Bessel fusion sequence of a Hilbert space $\mathcal{H}_2$ with Bessel fusion bound D_W, then for any $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the operator $\mathcal{M}^{(W,V)}(U)$, called the *matrix induced by the operator* U, is defined by the matrix representation

$$\mathcal{M}^{(W,V)}(U) \coloneqq \begin{bmatrix} w_1 \pi_{W_1} U \pi_{V_1} v_1 & w_2 \pi_{W_2} U \pi_{V_1} v_1 & w_3 \pi_{W_3} U \pi_{V_1} v_1 & \dots \\ w_1 \pi_{W_1} U \pi_{V_2} v_2 & w_2 \pi_{W_2} U \pi_{V_2} v_2 & w_3 \pi_{W_3} U \pi_{V_2} v_2 & \dots \\ w_1 \pi_{W_1} U \pi_{V_3} v_3 & w_2 \pi_{W_2} U \pi_{V_3} v_3 & w_3 \pi_{W_3} U \pi_{V_3} v_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In [7] it is shown that

$$\|\mathcal{M}^{(W,V)}(U)\|_{\mathcal{K}_V^2 \to \mathcal{K}_W^2} \leq D_V D_W \|U\|_{\mathcal{H}_1 \to \mathcal{H}_2},$$

i.e. $\mathcal{M}^{(W,V)}(U) \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$.

Moreover, the authors show that

$$\big(\mathcal{M}^{(W,V)}(U)\big)^* = \mathcal{M}^{(V,W)}(U^*),$$

which also follows immediately from Proposition 5.1.3.

We remark, that both types $\mathcal{M}^{(W,V)}(U)$ and $\mathcal{G}_{U,W,V}$ of operators are applied to obtain discretization schemes for operators, which are useful in order to find numerical solutions for physical equations [45], [7].

Even though the inverse of $\mathcal{G}_{U,W,V}$ under certain conditions on U, V and W is explicitly given, finding invertibility conditions for a general operator $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ in terms of the operators O_{ij} corresponding to the matrix representation $\mathbb{M}(O) = (O_{ij})$ of O is by far a non-trivial matter. In the next section we will see that even boundedness conditions of O in terms of the operators O_{ij} are in general much more complicated than in case O is a component preserving operator (i.e. $O = \bigoplus_{i \in I} \mathcal{O}_i$ is represented by a diagonal matrix $\mathbb{M}\big(\bigoplus_{i \in I} \mathcal{O}_i\big) = \operatorname{diag}\big(\{\mathcal{O}_i\}_{i \in I}\big)$ of operators) as in Chapter 3.

5.2 Bounded operators between Hilbert direct sums

The goal of this section is to prove an analog of Proposition 3.2.1 of general operators $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ (i.e. not necessarily diagonal matrices of operators). This needs some preparation. Again we will follow the ideas from [37].

Definition 5.2.1. *Assume that V_j ($j \in J$), W_i ($i \in I$) and X_l ($l \in L$) are Hilbert spaces, where J, I and L are (arbitrary but fixed) countable index sets. We set $\mathcal{K}_V^2 = (\sum_{j\in J} \oplus V_j)_{\ell^2}$, $\mathcal{K}_W^2 = (\sum_{i\in I} \oplus W_i)_{\ell^2}$ and $\mathcal{K}_X^2 = (\sum_{l\in L} \oplus X_l)_{\ell^2}$. Moreover:*

(a) We denote the the set of all (possibly infinite) matrices $A = (A_{ij})$ of operators $A_{ij} \in \mathcal{B}(V_j, W_i)$ by $M(V,W)$, i.e.

$$M(V,W) = \{A = (A_{ij}) : A_{ij} \in \mathcal{B}(V_j, W_i), i \in I, j \in J\}.$$

(b) For $A = (A_{ij}) \in M(V,W)$ we define the formal adjoint $A^* = (A_{ij})^* \in M(W,V)$ *of A by*

$$A^* = (A_{ij})^* = \begin{bmatrix} A_{11}^* & A_{21}^* & A_{31}^* & \dots \\ A_{12}^* & A_{22}^* & A_{32}^* & \dots \\ A_{13}^* & A_{23}^* & A_{33}^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.6}$$

(c) For $A = (A_{ij}) \in M(V,W)$ and $B = (B_{jl}) \in M(X,V)$ we say that AB exists*, if for all $i \in I$ and all $l \in L$ the series $\sum_{j\in J} A_{ij}B_{jl}$ converges in the strong operator topology, i.e. if for all $h_l \in X_l$ we have $\|\sum_{j\in J} A_{ij}B_{jl}h_l\|_{W_i} < \infty$.*

Recall that if $(O_{ij}) \in M(V,W)$ defines a bounded operator $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$, i.e. $\mathbb{M}(O) = (O_{ij})$ (which is the case if $\operatorname{dom}(O) = \mathcal{K}_V^2$, compare to (5.2) and Proposition 5.1.1), then, by Proposition 5.1.3, the formal adjoint $(O_{ij})^*$ of (O_{ij}) as in (5.6) coincides with the matrix representation of the adjoint operator $O^* \in \mathcal{B}(\mathcal{K}_W^2, \mathcal{K}_V^2)$ of O, i.e. $(O_{ij})^* = \mathbb{M}(O^*)$. However, in general the matrix $(O_{ij})^* \in M(W,V)$ might correspond to an unbounded operator.

Moreover, we remark that "AB exists" means that the entries of the formal matrix product AB are well-defined operators. In more formal terms, this can be explained as follows:

We define the *entry* $(AB)_{il}$ of AB to be $(AB)_{il} = \sum_{j\in J} A_{ij}B_{jl}$, which serves the intuition, since the formal matrix product AB takes the form

$$AB = ((AB)_{il}) = \begin{bmatrix} \sum_{j\in J} A_{1j}B_{j1} & \sum_{j\in J} A_{1j}B_{j2} & \sum_{j\in J} A_{1j}B_{j3} & \dots \\ \sum_{j\in J} A_{2j}B_{j1} & \sum_{j\in J} A_{2j}B_{j2} & \sum_{j\in J} A_{2j}B_{j3} & \dots \\ \sum_{j\in J} A_{3j}B_{j1} & \sum_{j\in J} A_{3j}B_{j2} & \sum_{j\in J} A_{1j}B_{j3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.7}$$

For any $i \in I$ and $l \in L$, let us define

$$\operatorname{dom}((AB)_{il}) = \Big\{h_l \in X_l : \Big\| \sum_{j\in J} A_{ij}B_{jl}h_l \Big\|_{W_i} < \infty \Big\} \subseteq X_l.$$

In general one might have $\operatorname{dom}((AB)_{il}) = \{0\}$. However, if AB exists then by definition we have $\operatorname{dom}((AB)_{il}) = X_l$ and thus

$$(AB)_{il}\{h_l\}_{l\in L} := \sum_{j\in J} A_{ij}B_{jl}h_l$$

defines a well-defined operator

$$(AB)_{il} : \mathrm{dom}((AB)_{il}) = X_l \longrightarrow W_i.$$

However, in this case an application of the Uniform Boundedness Principle (see Appendix) shows that $(AB)_{il}$ is even bounded: To see this, assume W.L.O.G. that $J = \mathbb{N}$ and observe that the operators $T_{il}^{(n)} : X_l \longrightarrow W_i$, defined by $T_{il}^{(n)} h_l := \sum_{j=1}^n A_{ij} B_{jl} h_l$ are in $\mathcal{B}(X_l, W_i)$ and converge pointwise to the operator $(AB)_{il}$ (for all $i \in I$, $l \in L$). Thus $(AB)_{il} \in \mathcal{B}(X_l, W_i)$. Since i and l were arbitrary, we see that if $A \in M(V, W)$, $B \in M(X, V)$ and if AB exists, then $AB \in M(X, W)$.

Conversely, if $A \in M(V, W)$, $B \in M(X, V)$ and $AB \in M(X, W)$ (AB is defined as in (5.7)), then by definition this means that $(AB)_{il} \in \mathcal{B}(X_l, W_i)$ for all $i \in I$ and all $l \in L$, which implies $\| \sum_{j\in J} A_{ij} B_{jl} h_l \|_{W_i} < \infty$ for all $h_l \in X_l$, $l \in L$ and all $i \in I$, i.e. that AB exists.

Thus we have proven the following:

Lemma 5.2.2. *Let $A \in M(V, W)$ and $B \in M(X, V)$. Then the following are equivalent.*

(i) AB exists.

(ii) $AB \in M(X, W)$.

Recall that if $O = (O_{ij}) \in M(V, W)$ then, by Propositions 5.1.1 and 5.1.2, $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ if and only if $\mathrm{dom}(O) = \mathcal{K}_V^2$, where $\mathrm{dom}(O) = \{f = \{f_j\}_{j\in J} \in \mathcal{K}_V^2 : \sum_{i\in I} \| \sum_{j\in J} O_{ij} f_j \|_{W_i}^2 < \infty\}$.

Now, if $A = (A_{ij}) \in M(V, W)$, $B = (B_{jl}) \in M(X, V)$ and if AB exists (or equivalently $AB \in M(X, W)$) we may define

$$\mathrm{dom}(AB) = \left\{ h = \{h_l\}_{l\in L} \in \mathcal{K}_X^2 : \sum_{i\in I} \Big\| \sum_{l\in L} (AB)_{il} h_l \Big\|_{W_i}^2 < \infty \right\}. \tag{5.8}$$

Note that by Propositions 5.1.1 and 5.1.2, and by the definition of $\mathrm{dom}(AB)$ we have that $\mathrm{dom}(AB) = \mathcal{K}_X^2$ if and only if $AB \in \mathcal{B}(\mathcal{K}_X^2, \mathcal{K}_W^2)$. In particular, the latter holds if $\mathrm{dom}(A) = \mathcal{K}_V^2$ and $\mathrm{dom}(B) = \mathcal{K}_X^2$, or – equivalently – if $A \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ and $B \in \mathcal{B}(\mathcal{K}_X^2, \mathcal{K}_V^2)$.

From now on we will sometimes slightly abuse the notation as follows: If a matrix $O = (O_{ij})$ in $M(V, W)$ defines a bounded operator in $\mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$, then we also write $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$, i.e. we will not distinguish between a matrix of of operators defining a bounded operator and the bounded operator itself.

Before we prove the next result, let us fix the following notation. We set $\mathcal{K}_V^{00} = \big(\sum_{j\in J} \oplus V_j\big)_{\ell^2}^{00}$. Recall (compare to Lemma 3.1.2) that we defined $\big(\sum_{j\in J} \oplus V_j\big)_{\ell^2}^{00}$ to be the set of elements $f = \{f_j\}_{j\in J} \in \mathcal{K}_V^2$ such that $f_j \neq 0$ for finitely many $j \in J$ and that $\mathcal{K}_V^{00} = \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}^{00}$ is a dense subspace of $\mathcal{K}_V^2$.

Lemma 5.2.3. *Let $O = (O_{ij}) \in M(V, W)$. Then the following are equivalent:*

(i) $dom(O) = \mathcal{K}_V^2$

(ii) O^*O exists and $dom(O^*O) = \mathcal{K}_V^2$.

Proof. Assume that (i) holds. Then by Proposition 5.1.1, O defines an operator in $\mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$. This implies (see Proposition 5.1.3) that the formal adjoint O^* is the matrix representation of the adjoint of $O \in \mathcal{B}(\mathcal{K}_W^2, \mathcal{K}_V^2)$. Hence the matrix O^*O defines an operator in $\mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_V^2)$. This implies $\operatorname{dom}(O^*O) = \mathcal{K}_V^2$. Moreover, Proposition 5.1.2 yields that $O^*O \in M(V,V)$ and by Lemma 5.2.2 this implies that O^*O exists.

Now assume that (ii) holds. Then by Lemma 5.2.2, $O^*O \in M(V,V)$. Therefore, by Proposition 5.1.1, the matrix O^*O defines a bounded operator $O^*O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_V^2)$. Let $f \in \mathcal{K}_V^2$ and observe that for arbitrary $m, n \in \mathbb{N}$ we have

$$\begin{aligned}\left\|Q_{\langle m\rangle}OP_{\langle n\rangle}f\right\|^2_{\mathcal{K}_W^2} &= \left\langle Q_{\langle m\rangle}OP_{\langle n\rangle}f, Q_{\langle m\rangle}OP_{\langle n\rangle}f\right\rangle_{\mathcal{K}_W^2}\\ &= \sum_{i=1}^{m}\left\langle \sum_{j=1}^{n} O_{ij}f_j, \sum_{k=1}^{n} O_{ik}f_k\right\rangle_{W_i}\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\left\langle f_j, \sum_{i=1}^{m}(O^*)_{ji}O_{ik}f_k\right\rangle_{V_j},\end{aligned}$$

where we used $(O_{ij})^* = (O^*)_{ji}$ (according to the definition of the formal adjoint) in the last equation. Since O^*O exists, we may let $m \longrightarrow \infty$ and obtain

$$\begin{aligned}\left\|OP_{\langle n\rangle}f\right\|^2_{\mathcal{K}_W^2} &= \sum_{j=1}^{n}\sum_{k=1}^{n}\left\langle f_j, \sum_{i=1}^{\infty}(O^*)_{ji}O_{ik}f_k\right\rangle_{V_j}\\ &= \sum_{j=1}^{n}\left\langle f_j, \sum_{k=1}^{n}(O^*O)_{jk}f_k\right\rangle_{V_j}\\ &= \left\langle P_{\langle n\rangle}f, O^*OP_{\langle n\rangle}f\right\rangle_{\mathcal{K}_V^2}\\ &\le \left\|O^*O\right\|_{\mathcal{K}_V^2\to\mathcal{K}_V^2}\left\|P_{\langle n\rangle}f\right\|^2_{\mathcal{K}_V^2} \qquad \text{(by the Cauchy Schwarz inequality).}\end{aligned}$$

This implies that $O \in \mathcal{B}\big(\mathcal{K}_V^{00}, \mathcal{K}_W^{00}\big)$ with $\|O\|_{\mathcal{K}_V^{00}\to\mathcal{K}_W^{00}} \le \|O^*O\|^{1/2}_{\mathcal{K}_V^2\to\mathcal{K}_V^2}$. Note that for every $f \in \mathcal{K}_V^2$, the sequence $\{P_{\langle n\rangle}f\}_{n\in\mathbb{N}} \subseteq \mathcal{K}_V^{00}$ converges to f in $\mathcal{K}_V^2$. Hence it is a Cauchy sequence in $\mathcal{K}_V^2$, i.e. for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > n \ge N$ we have

$$\|P_{\langle m\rangle}f - P_{\langle n\rangle}f\|_{\mathcal{K}_V^2} < \varepsilon.$$

For the same ε, N, m and n this implies

$$\|OP_{\langle m\rangle}f - OP_{\langle n\rangle}f\|_{\mathcal{K}_V^2} \le \|O^*O\|^{1/2}_{\mathcal{K}_V^2\to\mathcal{K}_V^2}\|P_{\langle m\rangle}f - P_{\langle n\rangle}f\|_{\mathcal{K}_V^2} < \|O^*O\|^{1/2}_{\mathcal{K}_V^2\to\mathcal{K}_V^2}\varepsilon,$$

i.e. $\{OP_{\langle n\rangle}f\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{K}_W^2$ and thus converges to some $g = \{g_i\}_{i\in I} \in \mathcal{K}_W^2$. However, we have

$$g_i = \lim_{n\longrightarrow\infty}\left[OP_{\langle n\rangle}f\right]_i = \lim_{n\longrightarrow\infty}\sum_{j=1}^{n}O_{ij}f_j = \sum_{j=1}^{\infty}O_{ij}f_j,$$

which means that $g = Of \in \mathcal{K}_W^2$ for all $f \in \mathcal{K}_V^2$. In other words, O is a well-defined operator from $\mathcal{K}_V^2$ into $\mathcal{K}_W^2$, i.e. $\operatorname{dom}(O) = \mathcal{K}_V^2$ and the proof is finished. □

Corollary 5.2.4. *Let Let $O = (O_{ij}) \in M(V,W)$. Then the following are equivalent:*

(i) O defines an operator in $\mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$.

(ii) O^*O defines an operator in $\mathcal{B}(\mathcal{K}_V^2)$.

Proof. If (i) holds then $\mathrm{dom}(O) = \mathcal{K}_V^2$ and by Lemma 5.2.3 this implies that O^*O exists and $\mathrm{dom}(O^*O) = \mathcal{K}_V^2$. By Lemma 5.2.2 we also have $O^*O \in M(V,V)$. Proposition 5.1.1 now implies (ii).

Conversely, if (ii) holds, then, by Proposition 5.1.2, we have $O^*O \in M(V,V)$ and $\mathrm{dom}(O^*O) = \mathcal{K}_V^2$. In particular, Lemma 5.2.2 gives us that O^*O exists. Finally, another application of Lemma 5.2.3 and Proposition 5.1.1 implies (i). □

The following technical result will be useful to prove the main result of this section.

Lemma 5.2.5. *Let $O \in M(V,W)$ and assume that O^*O exists. Then*

$$\|OP_{\langle n\rangle}\|^2_{\mathcal{K}_V^2 \to \mathcal{K}_W^2} \leq \|P_{\langle n\rangle} O^* O P_{\langle n\rangle}\|_{\mathcal{K}_V^2 \to \mathcal{K}_V^2}.$$

Proof. In the proof (ii) ⇒ (i) of Lemma 5.2.3 we showed (by only using the assumptions that $O \in M(V,W)$ and that O^*O exists) that for arbitrary $f = \{f_j\}_{j\in J} \in \mathcal{K}_V^2$ we have

$$\left\|OP_{\langle n\rangle} f\right\|^2_{\mathcal{K}_W^2} = \left\langle P_{\langle n\rangle} f, O^* O P_{\langle n\rangle} f\right\rangle_{\mathcal{K}_V^2}. \tag{5.9}$$

This implies

$$\begin{aligned}\left\|OP_{\langle n\rangle} f\right\|^2_{\mathcal{K}_W^2} &= \left\langle f, P_{\langle n\rangle} O^* O P_{\langle n\rangle} f\right\rangle_{\mathcal{K}_V^2} \\ &\leq \|f\|_{\mathcal{K}_V^2} \left\|P_{\langle n\rangle} O^* O P_{\langle n\rangle} f\right\|_{\mathcal{K}_V^2}\end{aligned}$$

and the result follows by taking suprema on both sides. □

Let us also note the following observation, which follows immediately by going through the same steps as in the first part of the proof of Proposition 5.1.1.

Lemma 5.2.6. *Let $O \in M(V,W)$ and $n \in \mathbb{N}$. Then*

$$\|Q_{\langle n\rangle} O P_{\langle n\rangle}\|_{\mathcal{K}_V^2 \to \mathcal{K}_W^2} \leq n \sup_{1\leq i,j\leq n} \|O_{ij}\|_{V_j \to W_i}.$$

We are now finally able to prove the main result of this section. We remark that in [37] the following result is stated.

Theorem [37] 5.2.7. *Let $O \in M(V,V)$. Then O defines an operator in $\mathcal{B}(\mathcal{K}_V^2)$ if and only if*

(i) $(O^*O)^n$ exists for all $n \in \mathbb{N}$

(ii) $\sup_{i,n\in\mathbb{N}} \left\|\left((O^*O)^n\right)_{ii}\right\|^{\frac{1}{n}}_{V_i \to V_i} =: K < \infty$.

Moreover, we have $K = \|O^*O\|_{\mathcal{K}_V^2 \to \mathcal{K}_V^2}$.

In contrast to [37] we will not only state an analogous result in a slightly more general context, where $O : \mathcal{K}_V^2 \longrightarrow \mathcal{K}_W^2$ (with possibly different corresponding index sets I and J), but also exchange one of the characterizing conditions for another, (– in our opinion –) more intuitive one.

Theorem 5.2.8. *Let $O \in M(V,W)$. Then O defines an operator in $\mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ if and only if*

(i) $(O^*O)^n \in M(V,V)$ *for all* $n \in \mathbb{N}$

(ii) $\sup_{i,n\in\mathbb{N}} \left\| \left((O^*O)^n\right)_{ii} \right\|_{V_i\to V_i}^{\frac{1}{n}} =: K < \infty$.

Moreover, we have $K = \|O^*O\|_{\mathcal{K}_V^2\to\mathcal{K}_V^2}$.

In case $\mathcal{K}_V^2 = \mathcal{K}_W^2$ we could immediately establish a proof of Theorem 5.2.8 by combining Theorem 5.2.7 with Lemma 5.2.2. However, for sake of completion, we provide the full proof for the slightly more general case $\mathcal{K}_V^2 \neq \mathcal{K}_W^2$:

Proof. Assume that $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$. Then $O^*O \in \mathcal{B}(\mathcal{K}_V^2)$ and thus $(O^*O)^n \in \mathcal{B}(\mathcal{K}_V^2)$ for every $n \in \mathbb{N}$, since $\|(O^*O)^n\|_{\mathcal{K}_V^2\to\mathcal{K}_V^2} \le \|O^*O\|^n_{\mathcal{K}_V^2\to\mathcal{K}_V^2}$. By Proposition 5.1.2 this implies (i). Moreover, for all $i \in I$ we have

$$\|[(O^*O)^n]_{ii}\|_{V_i\to V_i} = \|\Phi_i (O^*O)^n \Phi_i^*\|_{V_i\to V_i} \le \|(O^*O)^n\|_{\mathcal{K}_V^2\to\mathcal{K}_V^2}$$

(compare to the proof of Proposition 5.1.2) which implies (ii) and $K \le \|O^*O\|_{\mathcal{K}_V^2\to\mathcal{K}_V^2}$.

Conversely, suppose that (i) and (ii) hold. First, we prove that

$$\sup_{i,j,n\in\mathbb{N}} \|[(O^*O)^n]_{ij}\|^{1/n}_{V_j\to V_i} = K. \tag{5.10}$$

$K \le \sup_{i,j,n\in\mathbb{N}} \|[(O^*O)^n]_{ij}\|^{1/n}$ is clear. To see that the opposite inequality holds, we fix some arbitrary $n \in \mathbb{N}$ and set $A := (O^*O)^n \in M(V,V)$. Note that Lemma 5.2.2 implies that A exists. Moreover, we readily see that the formal adjoint $(O^*O)^*$ of the matrix (O^*O) equals the matrix (O^*O) itself. Thus $(O^*O)^*(O^*O) = (O^*O)^2$ and by iterating this we obtain $A^*A = A^2$. Using this observation, we see that for arbitrary $i \in I$, $j \in J$ and $f_j \in V_j$ we have

$$\begin{aligned}
\|A_{ij} f_j\|_{V_i}^2 &= \langle A_{ij} f_j, A_{ij} f_j\rangle_{V_i} \\
&\le \sum_{k=1}^{\infty} \langle A_{kj} f_j, A_{kj} f_j\rangle_{V_k} \\
&= \sum_{k=1}^{\infty} \langle (A^*)_{jk} A_{kj} f_j, f_j\rangle_{V_k} && ((A^*)_{ji} = (A_{ij})^* \text{ by definition}) \\
&= \left\langle \sum_{k=1}^{\infty} (A^*)_{jk} A_{kj} f_j, f_j \right\rangle_{V_j} && (A^*A \text{ exists}) \\
&= \langle (A^*A)_{jj} f_j, f_j\rangle_{V_j} \\
&= \langle (A^2)_{jj} f_j, f_j\rangle_{V_j} \\
&\le \|(A^2)_{jj}\|_{V_j\to V_j} \|f_j\|_{V_j}^2.
\end{aligned}$$

This implies $\|A_{ij}\|_{V_j\to V_i} \le \|(A^2)_{jj}\|^{1/2}_{V_j\to V_j}$ and thus $\|A_{ij}\|^{1/n}_{V_j\to V_i} \le \|(A^2)_{jj}\|^{1/2n}_{V_j\to V_j}$ for all $i \in I$, $j \in J$ and $n \in \mathbb{N}$. This means that

$$\begin{aligned}
\|[(O^*O)^n]_{ij}\|^{1/n}_{V_j\to V_i} &\le \|[(O^*O)^{2n}]_{jj}\|^{1/2n}_{V_j\to V_j} \\
&\le \sup_{i,n\in\mathbb{N}} \|[(O^*O)^n]_{ii}\|^{1/n}_{V_i\to V_i} = K.
\end{aligned}$$

Since this is true for all i and j, this implies

$$\sup_{i,j,n\in\mathbb{N}} \|[(O^*O)^n]_{ij}\|_{V_j\to V_i}^{1/n} \le K$$

for all $n\in\mathbb{N}$. Thus (5.10) is proved.

Now, since our assumptions imply by Lemma 5.2.2 that $(O^*O)^n$ exists for all $n\in\mathbb{N}$, we may apply Lemma 5.2.5 to the matrix O^*O and see that

$$\begin{aligned}
\|O^*OP_{\langle n\rangle}\|^4_{\mathcal{K}^2_V\to\mathcal{K}^2_V} &\le \|P_{\langle n\rangle}(O^*O)^2P_{\langle n\rangle}\|^2_{\mathcal{K}^2_V\to\mathcal{K}^2_V} && \text{(by Lemma 5.2.5)}\\
&\le \|(O^*O)^2P_{\langle n\rangle}\|^2_{\mathcal{K}^2_V\to\mathcal{K}^2_V}\\
&= \sup_{\|f\|_{\mathcal{K}^2_V}=1} \left\langle (O^*O)^2P_{\langle n\rangle}f, (O^*O)^2P_{\langle n\rangle}f\right\rangle_{\mathcal{K}^2_V}\\
&= \sup_{\|f\|_{\mathcal{K}^2_V}=1} \left\langle f, P_{\langle n\rangle}(O^*O)^4P_{\langle n\rangle}f\right\rangle_{\mathcal{K}^2_V}\\
&\le \|P_{\langle n\rangle}(O^*O)^4P_{\langle n\rangle}\|_{\mathcal{K}^2_V\to\mathcal{K}^2_V}. && \text{(by Cauchy Schwarz)}
\end{aligned}$$

However, $\|P_{\langle n\rangle}(O^*O)^4P_{\langle n\rangle}\|_{\mathcal{K}^2_V\to\mathcal{K}^2_V} \le \|(O^*O)^4P_{\langle n\rangle}\|_{\mathcal{K}^2_V\to\mathcal{K}^2_V}$ and assumption (i) allows us to repeat this argumentation successively. Hence we obtain

$$\|O^*OP_{\langle n\rangle}\|_{\mathcal{K}^2_V\to\mathcal{K}^2_V} \le \|P_{\langle n\rangle}(O^*O)^{2^k}P_{\langle n\rangle}\|^{2^{-k}}_{\mathcal{K}^2_V\to\mathcal{K}^2_V}$$

for all $k,n\in\mathbb{N}$. Now, an application of Lemma 5.2.6 to the matrix $(O^*O)^{2^k}\in M(V,V)$ yields

$$\begin{aligned}
\|O^*OP_{\langle n\rangle}\|_{\mathcal{K}^2_V\to\mathcal{K}^2_V} &\le \|P_{\langle n\rangle}(O^*O)^{2^k}P_{\langle n\rangle}\|^{2^{-k}}_{\mathcal{K}^2_V\to\mathcal{K}^2_V}\\
&\le n^{2^{-k}} \sup_{1\le i,j\le n} \|[(O^*O)^{2^k}]_{ij}\|^{2^{-k}}_{V_j\to V_i} && \text{(by Lemma 5.2.6)}\\
&\le n^{2^{-k}} \sup_{i,j\in\mathbb{N}} \|[(O^*O)^{2^k}]_{ij}\|^{2^{-k}}_{V_j\to V_i}\\
&\le n^{2^{-k}}K && \text{(by (5.10))}
\end{aligned}$$

for all $k,n\in\mathbb{N}$. Letting $k\longrightarrow\infty$ gives us

$$\|O^*OP_{\langle n\rangle}\|_{\mathcal{K}^2_V\to\mathcal{K}^2_V} \le K$$

for all $n\in\mathbb{N}$. This implies that the matrix O^*O defines a bounded operator in $\mathcal{B}(K^{00}_V)$ with $\|O^*O\|_{K^{00}_V\to K^{00}_V}\le K$. Now, using precisely the same completeness argument as in the proof of (ii) $\Longrightarrow$ (i) of Lemma 5.2.3, we may conclude that the matrix O^*O defines a well-defined operator from $\mathcal{K}^2_V$ into $\mathcal{K}^2_V$, which means that $\operatorname{dom}(O^*O)=\mathcal{K}^2_V$. Applying Proposition 5.1.1 yields that O^*O defines an operator from $\mathcal{B}(\mathcal{K}^2_V)$ and Corollary 5.2.4 implies that O defines an operator from $\mathcal{B}(\mathcal{K}^2_V,\mathcal{K}^2_W)$. This completes the proof. □

Theorem 5.2.8 is a direct generalization of Proposition 3.2.1. To see this, assume that $I=J$ and that $O\in M(V,W)$ defines a component-preserving operator $O=\bigoplus_{i\in I}\mathcal{O}_i$. Then $(O^*O)^n$ has the matrix representation

$$\mathbb{M}\big((O^*O)^n\big) = \begin{bmatrix} (\mathcal{O}_1^*\mathcal{O}_1)^n & 0 & 0 & \dots\\ 0 & (\mathcal{O}_2^*\mathcal{O}_2)^n & 0 & \dots\\ 0 & 0 & (\mathcal{O}_3^*\mathcal{O}_3)^n & \dots\\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus condition (i) from Theorem 5.2.8 is trivially fulfilled, since $O \in M(V,W)$ means $\mathcal{O}_i \in \mathcal{B}(V_i, V_i)$ for all $i \in I$, which implies $(\mathcal{O}_i^* \mathcal{O}_i)^n \in \mathcal{B}(V_i, V_i)$ for all $i \in I$, i.e. $(O^*O)^n \in M(V,V)$ (for every $n \in \mathbb{N}$). We show that condition (ii) from Theorem 5.2.8 is equivalent to $\{\mathcal{O}_i\}_{i\in I}$ being completely bounded:

First, assume that (ii) holds. Then

$$\sup_{i\in I} \|\mathcal{O}_i\|^2_{V_i \to V_i} = \sup_{i\in I} \|\mathcal{O}_i^* \mathcal{O}_i\|_{V_i\to V_i} \leq \sup_{i\in I, n\in\mathbb{N}} \|(\mathcal{O}_i^* \mathcal{O}_i)^n\|^{1/n}_{V_i\to V_i} = K.$$

Conversely, if $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded then

$$K = \sup_{i\in I, n\in\mathbb{N}} \|(\mathcal{O}_i^* \mathcal{O}_i)^n\|^{1/n}_{V_i\to V_i} \leq \sup_{i\in I} \|\mathcal{O}_i^* \mathcal{O}_i\|_{V_i\to V_i} = \|O\|^2_{cb}.$$

In particular, we have

$$K = \|O\|^2_{cb}.$$

We can also prove other sufficient conditions for an operator $O = (O_{ij}) \in M(V,W)$ being in $\mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_V^2)$ in terms of the operators O_{ij}. However, we will be able to prove the result not only for Hilbert direct sums, but also for other, more general sequence spaces. Among others, this result will be part of the next section.

5.3 Other norms

Until now, we considered Hilbert direct sums $K_V^2 = \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$ a lot, since they are the representation spaces for fusion frames (in case the Hilbert spaces V_i are closed subspaces of one single separable Hilbert space $\mathcal{H}$). However, Hilbert direct sums belong to a more general class of Banach spaces, whose definition is motivated by the well-known Banach spaces $\ell^p(I)$ (I countable, $1 \leq p \leq \infty$), which we sometimes simply abbreviate by ℓ^p and which are defined by

$$\ell^p(I) = \Big\{ c = \{c_i\}_{i\in I} \subseteq \mathbb{C} : \|c\|_{\ell^p} < \infty \Big\},$$

where

$$\|c\|_{\ell^p} = \Big(\sum_{i\in I} |c_i|^p \Big)^{1/p}$$

if $1 \leq p < \infty$ and

$$\|c\|_{\ell^\infty} = \sup_{i\in I} \{|c_i|\}$$

if $p = \infty$.

Let $\{V_i\}_{i\in I}$ be a collection of Banach spaces. Then, for $1 \leq p \leq \infty$, we define (see also [19])

$$\mathcal{K}_V^p = \Big(\sum_{i\in I} \oplus V_i \Big)_{\ell^p} = \Big\{ f = \{f_i\}_{i\in I} : f_i \in V_i, \|f\|_{\mathcal{K}_V^p} < \infty \Big\},$$

where

$$\|f\|_{\mathcal{K}_V^p} = \Big\| \{ \|f_i\|_{V_i} \}_{i\in I} \Big\|_{\ell^p},$$

i.e.

$$\|f\|_{\mathcal{K}_V^p} = \Big(\sum_{i\in I} \|f_i\|_{V_i}^p\Big)^{1/p}$$

if $1 \le p < \infty$ and

$$\|f\|_{\mathcal{K}_V^\infty} = \sup_{i\in I}\{\|f_i\|_{V_i}\}$$

if $p = \infty$.

Clearly $\|.\|_{\mathcal{K}_V^p}$ is a norm for $\mathcal{K}_V^p$, as indicated by the notation. Moreover, if we adapt the completeness-proof of $\mathcal{K}_V^2$ (see Lemma 3.1.1), then we see that $\mathcal{K}_V^p$ is a Banach space ($1 \le p \le \infty$), see also [19].

Moreover, the chain of subspaces

$$\ell^{00} \subseteq \ell^1 \subseteq \ell^p \subseteq \ell^2 \subseteq \ell^q \subseteq \ell^0 \subseteq \ell^\infty$$

(see [5] for instance), where $1 < p < 2 < q < \infty$ (and where ℓ^{00} is the space of complex scalar sequences with $c_i \neq 0$ for only finitely many i, and ℓ^0 is the space of complex scalar sequences with limit 0) implies the chain of subspaces

$$\mathcal{K}_V^{00} \subseteq \mathcal{K}_V^1 \subseteq \mathcal{K}_V^p \subseteq \mathcal{K}_V^2 \subseteq \mathcal{K}_V^q \subseteq \mathcal{K}_V^0 \subseteq \mathcal{K}_V^\infty, \tag{5.11}$$

where $\mathcal{K}_V^{00}$ and $\mathcal{K}_V^0$ are defined analogously as the set of all sequences $\{f_i\}_{i\in I}$ with $f_i \in V_i$ ($i \in I$) such that $f_i \neq 0$ for only finitely many i, or $\lim_{i\to\infty} \|f_i\|_{V_i=0}$ respectively. In case I is finite, we cancel ℓ^0 and $\mathcal{K}_V^0$ from their corresponding chains of subspaces. Note that we can easily adapt the density proof of of Lemma 3.1.2 to the cases $1 \le p < \infty$, which yields that $\mathcal{K}_V^{00}$ is a dense subspace of $\mathcal{K}_V^p$ ($1 \le p < \infty$).

The following result generalizes Schur's test, see [31] for instance. Our proof is adapted from there. Nonetheless, we will present it in full detail. We denote the *conjugate exponent* of p ($1 \le p \le \infty$) with p', i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, where we set $\frac{1}{\infty} = 0$.

Schur's test - general version 5.3.1. *Let $O = (O_{ij}) \in M(V, W)$. Suppose that there exist positive constants K_1 and K_2, such that*

$$\sup_{j\in J} \sum_{i\in I} \|O_{ij}\|_{V_j\to W_i} \le K_1 \tag{5.12}$$

$$\sup_{i\in I} \sum_{j\in J} \|O_{ij}\|_{V_j\to W_i} \le K_2. \tag{5.13}$$

Then, for $1 \le p \le \infty$, $O \in \mathcal{B}(\mathcal{K}_V^p, \mathcal{K}_W^p)$ with $\|O\|_{\mathcal{K}_V^p\to\mathcal{K}_W^p} \le K_1^{1/p'} K_2^{1/p}$.

Proof. First, observe that for any $f = \{f_j\}_{j\in J} \in \mathcal{K}_V^p$

$$\begin{aligned}\|[Of]_i\|_{W_i} &= \Big\|\sum_{j\in J} O_{ij} f_j\Big\|_{W_i} \\ &\le \sum_{j\in J} \|O_{ij}\|_{V_j\to W_i} \|f_j\|_{V_j} \\ &\le \Big(\sum_{j\in J} \|O_{ij}\|_{V_j\to W_i}\Big)^{1/p'} \Big(\sum_{j\in J} \|O_{ij}\|_{V_j\to W_i} \|f_j\|_{V_j}^p\Big)^{1/p} \quad \text{(by Hölder's inequality).}\end{aligned}$$

This implies

$$\begin{aligned}\|Of\|^p_{\mathcal{K}^p_W} = \sum_{i\in I}\big\|[Of]_i\big\|^p_{W_i} &\le K_2^{p/p'}\sum_{i\in I}\sum_{j\in J}\|O_{ij}\|_{V_j\to W_i}\|f_j\|^p_{V_j}\\ &= K_2^{p/p'}\sum_{j\in J}\Big(\sum_{i\in I}\|O_{ij}\|_{V_j\to W_i}\Big)\|f_j\|^p_{V_j} \qquad (*)\\ &\le K_2^{p/p'}K_1\|f\|^p_{\mathcal{K}^p_V}.\end{aligned}$$

□

We remark that the step $(*)$ in the above proof is either legitimized by Fubini's Theorem (see Appendix), or by a consequence of the Monotone Convergence Theorem (see [54]), since all summands in the double sum are non-negative.

In case $O = \bigoplus_{i\in I}\mathcal{O}_i$ is component preserving, then conditions (5.12) and (5.13) are equivalent to $\{\mathcal{O}_i\}_{i\in I}$ being completely bounded. This observation leads to the following result, which generalizes Proposition 3.2.1.

Proposition 5.3.2. *The component preserving operator $\bigoplus_{i\in I}\mathcal{O}_i$ is in $\mathcal{B}(\mathcal{K}^p_V,\mathcal{K}^p_W)$ if and only if $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded.*

Proof. If $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded, then the general version of Schur's test 5.3.1 yields $\bigoplus_{i\in I}\mathcal{O}_i\in\mathcal{B}(\mathcal{K}^p_V,\mathcal{K}^p_W)$. Conversely, if $\bigoplus_{i\in I}\mathcal{O}_i:\mathcal{K}^p_V\longrightarrow\mathcal{K}^p_W$ is bounded by some constant $C>0$, then this means that

$$\|\bigoplus_{i\in I}\mathcal{O}_i(\{f_i\}_{i\in I})\|_{\mathcal{K}^p_W}\le C\|\{f_i\}_{i\in I}\|_{\mathcal{K}^p_V} \tag{5.14}$$

for all $f=\{f_i\}_{i\in I}\in\mathcal{K}^p_V$. In particular, if we fix some arbitrary $i\in I$ and consider the sequence $\tilde f=\{\delta_{ik}f_k\}_{k\in I}\in\mathcal{K}^p_V$, then (5.14) reduces to

$$\|\mathcal{O}_if_i\|_{W_i}\le C\|f_i\|_{V_i},$$

which implies that $\|\mathcal{O}_i\|\le C$. Since $i\in I$ was arbitrary, the above holds for any $i\in I$, which implies $\|\mathcal{O}\|_{cb}\le C$, i.e. the family $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded by C. □

Let $O=(O_{ij})\in M(V,W)$. Then $\{\|O_{ij}\|_{V_j\to W_i}\}_{i\in I,j\in J}$ is a scalar sequence indexed by the index sets I and J, i.e. $\{\|O_{ij}\|\}_{i\in I,j\in J}\in\ell(I\times J)$. For $1\le p,q\le\infty$ we define the following class of linear spaces.

$$M(V,W)^{p,q} := \left\{O=(O_{ij})\in M(V,W): \|O\|_{p,q} = \Big\|\Big\{\big\|\{\|O_{ij}\|\}_{j\in J}\big\|_{\ell^q(J)}\Big\}_{i\in I}\Big\|_{\ell^p(I)}<\infty\right\}.$$

We call $\|.\|_{p,q}$ *mixed norm*. Note that if $p,q<\infty$ then

$$\|O\|_{p,q} = \Big(\sum_{i\in I}\Big(\sum_{j\in J}\|O_{ij}\|^q\Big)^{p/q}\Big)^{1/p}$$

and in case $p=\infty$, or $q=\infty$, or $p=q=\infty$ respectively we have

$$\|O\|_{\infty,q} = \sup_{i\in I}\Big(\sum_{j\in J}\|O_{ij}\|^q\Big)^{1/q}$$

$$\|O\|_{p,\infty} = \sup_{j\in J}\Big(\sum_{i\in I}\|O_{ij}\|^p\Big)^{1/p}$$

$$\|O\|_{\infty,\infty} = \sup_{i\in I,j\in J}\{\|O_{ij}\|\}.$$

It follows directly from the norm properties of the spaces ℓ^p $(1 \le p \le \infty)$ and the operator norm that the mixed norm $\|.\|_{p,q}$ is indeed a norm for $M(V,W)^{p,q}$ $(1 \le p,q \le \infty)$ which implies that $\big(M(V,W)^{p,q}, \|.\|_{p,q}\big)$ is indeed a normed space and thus a linear subspace of $M(V,W)$. Further important special cases are

$$\|O\|_{1,1} = \sum_{i\in I}\sum_{j\in J}\|O_{ij}\|$$
$$\|O\|_{1,\infty} = \sup_{j\in J}\sum_{i\in I}\|O_{ij}\|$$
$$\|O\|_{\infty,1} = \sup_{i\in I}\sum_{j\in J}\|O_{ij}\|$$
$$\|O\|_{2,2} = \Big(\sum_{i\in I}\sum_{j\in J}\|O_{ij}\|^2\Big)^{1/2}.$$

Note that we have already used some of these mixed norms implicitly. For instance, the conditions (5.12) and (5.13) of Schur's test 5.3.1 can be rewritten as

$$\|O\|_{1,\infty} < \infty \quad \text{and} \quad \|O\|_{\infty,1} < \infty.$$

Thus, the statement of Schur's test 5.3.1 may be rewritten as

$$M(V,W)^{1,\infty} \cap M(V,W)^{\infty,1} \subseteq \mathcal{B}(\mathcal{K}_V^p, \mathcal{K}_W^p) \qquad (1 \le p \le \infty).$$

Another example, where the mixed norms $\|.\|_{p,q}$ have been implicitly used, is Proposition 5.3.2 (or its special case Proposition 3.2.1). There we considered diagonal matrices of operators, i.e. component preserving operators $\bigoplus_{i\in I}\mathcal{O}_i$. Note that for these matrices (or operators respectively) we have

$$\|\bigoplus_{i\in I}\mathcal{O}_i\|_{1,\infty} = \|\bigoplus_{i\in I}\mathcal{O}_i\|_{\infty,1} = \|\bigoplus_{i\in I}\mathcal{O}_i\|_{\infty,\infty} = \|\mathcal{O}\|_{cb}.$$

Thus, we may rephrase Proposition 5.3.2 as follows.

$$\bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{B}(\mathcal{K}_V^p, \mathcal{K}_W^p) \Longleftrightarrow \|\bigoplus_{i\in I}\mathcal{O}_i\|_{1,\infty} = \|\bigoplus_{i\in I}\mathcal{O}_i\|_{\infty,1} = \|\bigoplus_{i\in I}\mathcal{O}_i\|_{\infty,\infty} = \|\mathcal{O}\|_{cb} < \infty.$$

Since the right-hand side of the above is independent of p, this yields the following corollary.

Corollary 5.3.3. *Let $1 \le p \le \infty$. Then $\bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{B}(\mathcal{K}_V^p, \mathcal{K}_W^p)$ if and only of $\bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{B}(\mathcal{K}_V^q, \mathcal{K}_W^q)$ for all q with $1 \le q \le \infty$.*

For general (i.e. not necessarily component preserving) operators $O \in \mathcal{B}(\mathcal{K}_V^p, \mathcal{K}_W^p)$ this is not necessarily true.

5.4 Compact operators between Hilbert direct sums

In the following, we consider compact, Hilbert Schmidt and trace class operators (see Appendix for more details about these classes of operators) between Hilbert direct sums. All results from this section are new. Once again, we remind the reader that we identify any bounded operator $O \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_\mathcal{W})$ with its matrix representation $\mathbb{M}(O) = (O_{ij})$ (see Sections 5.1 and 5.2).

Proposition 5.4.1. *Let $O \in \mathcal{C}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ with matrix representation $\mathbb{M}(O) = (O_{ij})$. Then $O_{ij} \in \mathcal{C}(V_j, W_i)$ for all $i \in I$ and $j \in J$.*

Proof. Since $O \in \mathcal{C}(\mathcal{K}_V^2, \mathcal{K}_W^2)$, (by the characterization result for compact operators, see Appendix) there exist finite rank operators $T^{(n)} \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ such that $\|O - T^{(n)}\|_{\mathcal{K}_V^2 \to \mathcal{K}_W^2} \longrightarrow 0$ (as $n \longrightarrow \infty$). For each n, let $(T_{ij}^{(n)}) \in M(V,W)$ be the matrix representation for $T^{(n)}$. For all $i \in I$, $j \in J$ and all n, $T_{ij}^{(n)}$ has finite rank, since

$$\begin{aligned} \operatorname{rank}(T_{ij}^{(n)}) &= \dim\{T_{ij}^{(n)} f_j : f_j \in V_j\} \\ &= \dim\Big\{\big\{\delta_{ik} \sum_{l \in J} T_{kl}^{(n)} \delta_{jl} f_l\big\}_{k \in I} : \{\delta_{jl} f_l\}_{l \in J} \in \mathcal{K}_V^2\Big\} \\ &\le \dim\Big\{\big\{\sum_{j \in J} T_{ij}^{(n)} f_j\big\}_{i \in I} : \{f_j\}_{j \in J} \in \mathcal{K}_V^2\Big\} \\ &= \operatorname{rank}(T^{(n)}) < \infty. \end{aligned}$$

Moreover, for all $i \in I$ and all $j \in J$, we have (compare to the proof of Proposition 5.1.2)

$$\begin{aligned} \|O_{ij} - T_{il}^{(n)}\|_{V_j \to W_i} &= \|\Psi_i O \Phi_j^* - \Psi_i T^{(n)} \Phi_j^*\|_{V_j \to W_i} \\ &= \|\Psi_i (O - T^{(n)}) \Phi_j^*\|_{V_j \to W_i} \\ &\le \|\Psi_i\|_{\mathcal{K}_W^2 \to W_i} \|O - T^{(n)}\|_{\mathcal{K}_V^2 \to \mathcal{K}_W^2} \|\Phi_j^*\|_{V_j \to \mathcal{K}_V^2} \\ &\le \|O - T^{(n)}\|_{\mathcal{K}_V^2 \to \mathcal{K}_W^2} \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \end{aligned}$$

Thus the finite rank operators $T_{il}^{(n)} \in \mathcal{B}(V_j, W_i)$ approximate O_{ij} in that sense, that $\|O_{ij} - T_{il}^{(n)}\|_{V_j \to W_i} \longrightarrow 0$ as $n \longrightarrow \infty$, which means that O_{ij} is compact for every $i \in I$ and every $j \in J$. □

For component preserving operators we can show a converse statement, if we assume one additional condition.

Proposition 5.4.2. *Let $\{\mathcal{O}_i\}_{i \in I}$ be a family of compact operators $\mathcal{O}_i \in \mathcal{C}(V_i, W_i)$ such that $\{\|\mathcal{O}_i\|_{V_i \to W_i}\}_{i \in I} \in \ell^0(I)$. Then $\bigoplus_{i \in I} \mathcal{O}_i \in \mathcal{C}(\mathcal{K}_V^2, \mathcal{K}_W^2)$.*

Proof. Without loss of generality we assume that $I = \mathbb{N}$. Since $\{\|\mathcal{O}_i\|_{V_i \to W_i}\}_{i \in I} \in \ell^0(I) \subseteq \ell^\infty(I)$, we obtain via Proposition 3.2.1 that $\bigoplus_{i \in I} \mathcal{O}_i \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$. We consider the operators $T_n \coloneqq Q_{\langle n \rangle}\big(\bigoplus_{i \in I} \mathcal{O}_i\big) P_{\langle n \rangle} \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ (for each $n \in \mathbb{N}$). We will make use of the fact that $\mathcal{C}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ is a closed (w.r.t. the topology induced by the operator-norm) subspace of $\mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ [19] and will show that the operators T_n are compact and that we have $\lim_{n \to \infty} \|\bigoplus_{i \in I} \mathcal{O}_i - T_n\|_{\mathcal{K}_V^2 \to \mathcal{K}_W^2} = 0$.

Since each operator $\mathcal{O}_i$ is compact, for every $i \in I$, there exists a sequence $\{S_i^{(k)}\}_{k \in \mathbb{N}}$ of finite rank operators $S_i^{(k)} : V_i \longrightarrow W_i$, such that $\lim_{k \to \infty} \|\mathcal{O}_i - S_i^{(k)}\|_{V_i \to W_i} = 0$. For fixed $n \in \mathbb{N}$, the operator $Q_{\langle n \rangle}\big(\bigoplus_{i \in I} S_i^{(k)}\big) P_{\langle n \rangle}$ has finite rank for each k, since

$$\begin{aligned} \operatorname{rank}\Big(Q_{\langle n \rangle}\big(\bigoplus_{i \in I} S_i^{(k)}\big) P_{\langle n \rangle}\Big) &= \dim\Big\{(S_1^{(k)} f_1, \ldots, S_n^{(k)} f_n, 0, 0, \ldots) : f_i \in V_i (1 \le i \le n)\Big\} \\ &= \sum_{i=1}^n \dim\Big\{S_i^{(k)} f_i : f_i \in V_i\Big\} \\ &= \sum_{i=1}^n \operatorname{rank}\big(S_i^{(k)}\big) < \infty. \end{aligned}$$

Now we see that the operators T_n are compact for each n, since

$$\begin{aligned}\lim_{k\to\infty}\Big\|T_n - Q_{\langle n\rangle}\big(\bigoplus_{i\in I} S_i^{(k)}\big)P_{\langle n\rangle}\Big\|^2_{\mathcal{K}_V^2\to\mathcal{K}_W^2} &= \lim_{k\to\infty}\sup_{\|f\|_{\mathcal{K}_V^2}=1}\sum_{i=1}^{n}\|(\mathcal{O}_i - S_i^{(k)})f_i\|^2_{W_i}\\ &\le \lim_{k\to\infty}\sup_{1\le i\le n}\|\mathcal{O}_i - S_i^{(k)}\|^2_{V_i\to W_i}\sup_{\|f\|_{\mathcal{K}_V^2}=1}\sum_{i=1}^{n}\|f_i\|^2_{V_i}\\ &\le \lim_{k\to\infty}\sup_{1\le i\le n}\|\mathcal{O}_i - S_i^{(k)}\|^2_{V_i\to W_i}\\ &\le \lim_{k\to\infty}\sum_{i=1}^{n}\|\mathcal{O}_i - S_i^{(k)}\|^2_{V_i\to W_i} = 0.\end{aligned}$$

Finally, we see that

$$\begin{aligned}\lim_{n\to\infty}\|\bigoplus_{i\in I}\mathcal{O}_i - T^{(n)}\|^2_{\mathcal{K}_V^2\to\mathcal{K}_W^2} &= \lim_{n\to\infty}\sup_{\|f\|_{\mathcal{K}_V^2}=1}\sum_{i=n+1}^{\infty}\|\mathcal{O}_i f_i\|^2_{W_i}\\ &\le \lim_{n\to\infty}\sup_{i\ge n+1}\|\mathcal{O}_i\|^2_{V_i\to W_i}\sup_{\|f\|_{\mathcal{K}_V^2}=1}\sum_{i=n+1}^{\infty}\|f_i\|^2_{W_i}\\ &\le \lim_{n\to\infty}\sup_{i\ge n+1}\|\mathcal{O}_i\|^2_{V_i\to W_i} = 0.\end{aligned}$$

and the proof is complete. □

Next, we consider Hilbert Schmidt operators.

Proposition 5.4.3. *Let $O\in\mathcal{HS}(\mathcal{K}_V^2,\mathcal{K}_W^2)$ with matrix representation $\mathbb{M}(O)=(O_{ij})$. Then $O_{ij}\in\mathcal{HS}(V_j,W_i)$ for all $i\in I$, $j\in J$.*

Proof. Assume that O is Hilbert Schmidt. Then (see Appendix)

$$\sum_k \|Oe_k\|^2_{\mathcal{K}_W^2} < \infty$$

for any orthonormal basis $\{e_k\}\subseteq\mathcal{K}_V^2$. Recall that if $\{e_{ij}\}_{j\in J_i}$ is an orthonormal basis for V_i (for each $i\in J$), then, by Lemma 3.1.3, $\{\tilde{e}_{ij}\}_{i\in J, j\in J_i}$, where $\tilde{e}_{ij} = (...,0,0,e_{ij},0,0,...)$ (e_{ij} in the i-th component), is an orthonormal basis for $\mathcal{K}_V^2$. For arbitrary $l\in I$ and $i\in J$, this implies

$$\begin{aligned}\sum_{j\in J_i}\|O_{li}e_{ij}\|^2_{W_l} &\le \sum_{l\in I}\sum_{i\in J}\sum_{j\in J_i}\|O_{li}e_{ij}\|^2_{W_l}\\ &= \sum_{i\in J}\sum_{j\in J_i}\sum_{l\in I}\|O_{li}e_{ij}\|^2_{W_l} \qquad \text{(by Fubini's Theorem)}\\ &= \sum_{i\in J}\sum_{j\in J_i}\|O\tilde{e}_{ij}\|^2_{\mathcal{K}_W^2} = \|O\|_{\mathcal{HS}} < \infty.\end{aligned}$$

Thus $O_{li}\in\mathcal{B}(V_i,W_l)$ is Hilbert Schmidt for every $l\in I$ and $i\in J$ by the characterization for Hilbert Schmidt operators (see Appendix). □

Proposition 5.4.4. *Let $O\in\mathcal{B}(\mathcal{K}_V^2,\mathcal{K}_W^2)$ with matrix representation $\mathbb{M}(O)=(O_{ij})$. If $O_{ij}\in\mathcal{HS}(V_j,W_i)$ for all $i\in I$, $j\in J$ and if $\sum_{i\in I}\sum_{j\in J}\|O_{ij}\|^2_{\mathcal{HS}}<\infty$, then O is a Hilbert Schmidt operator.*

Proof. As above, if $\{e_{ij}\}_{j\in J_i}$ is an orthonormal basis for V_i (for each $i\in J$), then, by Lemma 3.1.3, $\{\tilde{e}_{ij}\}_{i\in J, j\in J_i}$ is an orthonormal basis for $\mathcal{K}^2_V$. This implies

$$\begin{aligned}
\|O\|^2_{\mathcal{HS}} &= \sum_{i\in J}\sum_{j\in J_i}\|O\tilde{e}_{ij}\|^2_{\mathcal{K}^2_W} && \text{(see Appendix)}\\
&= \sum_{i\in J}\sum_{j\in J_i}\sum_{l\in I}\|O_{li}e_{ij}\|^2_{W_l}\\
&= \sum_{l\in I}\sum_{i\in J}\sum_{j\in J_i}\|O_{li}e_{ij}\|^2_{W_l} && \text{(by Fubini's Theorem)}\\
&= \sum_{l\in I}\sum_{i\in J}\|O_{li}\|^2_{\mathcal{HS}} < \infty.
\end{aligned}$$

□

This yields the following consequence.

Corollary 5.4.5. *If $\{\mathcal{O}_i\}_{i\in I}$ is a family of Hilbert Schmidt operators $\mathcal{O}_i \in \mathcal{HS}(V_i, W_i)$ such that $\{\|\mathcal{O}_i\|_{\mathcal{HS}}\}_{i\in I}\in \ell^2(I)$, then $\bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{HS}(\mathcal{K}^2_V, \mathcal{K}^2_W)$.*

Proof. Since $\|\mathcal{O}_i\|_{V_i\to W_i} \le \|\mathcal{O}_i\|_{\mathcal{HS}}$ for all i and j (see Appendix), we have

$$\|\bigoplus_{i\in I}\mathcal{O}_i\|_{2,2} = \Big\|\{\|\mathcal{O}_i\|_{V_i\to W_i}\}_{i\in I}\Big\|_{\ell^2} \le \Big\|\{\|\mathcal{O}_i\|_{\mathcal{HS}}\}_{i\in I}\Big\|_{\ell^2} < \infty.$$

Thus $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded, which implies $\bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{B}(\mathcal{K}^2_V, \mathcal{K}^2_W)$ by Proposition 3.2.1. Now apply Proposition 5.4.4. □

Finally, let us consider trace class operators between Hilbert direct sums.

Proposition 5.4.6. *Let $O\in \mathcal{S}_1(\mathcal{K}^2_V, \mathcal{K}^2_W)$ with matrix representation $\mathbb{M}(O) = (O_{ij})$. Then $O_{ij}\in \mathcal{S}_1(V_j, W_i)$ for all $i\in I$, $j\in J$.*

Proof. By the characterization of trace class operators (see Appendix), there exist sequences $\{f^{(n)}\}_{n\in\mathbb{N}}\subseteq \mathcal{K}^2_V$ and $\{g^{(n)}\}_{n\in\mathbb{N}}\subseteq \mathcal{K}^2_W$, such that the action of O on any $f = \{f_j\}_{j\in J}\in\mathcal{K}^2_V$ is given by

$$Of = \sum_{n\in\mathbb{N}}\langle f, f^{(n)}\rangle_{\mathcal{K}^2_V} g^{(n)}, \tag{5.15}$$

where we have

$$\sum_{n\in\mathbb{N}}\|f^{(n)}\|_{\mathcal{K}^2_V}\|g^{(n)}\|_{\mathcal{K}^2_W} < \infty. \tag{5.16}$$

Note that equation (5.15) reads

$$\Big\{\sum_{j\in J}O_{ij}f_j\Big\}_{i\in I} = \Big\{\sum_{n\in\mathbb{N}}\sum_{j\in J}\langle f_j, f_j^{(n)}\rangle_{V_j} g_i^{(n)}\Big\}_{i\in I} \tag{5.17}$$

and that equation (5.16) reads

$$\sum_{n\in\mathbb{N}}\Big(\sum_{j\in J}\|f_j^{(n)}\|^2_{V_j}\sum_{i\in I}\|g_i^{(n)}\|^2_{W_i}\Big)^{1/2} < \infty. \tag{5.18}$$

Now fix some arbitrary $i\in I$ and some arbitrary $j\in J$ and consider the sequences $\{f_j^{(n)}\}_{n\in\mathbb{N}}\subseteq V_j$ and $\{g_i^{(n)}\}_{n\in\mathbb{N}}\subseteq W_i$. Then (5.18) implies

$$\sum_{n\in\mathbb{N}}\|f_j^{(n)}\|_{V_j}\|g_i^{(n)}\|_{W_i} < \infty.$$

Moreover, if we apply (5.17) on $\{\delta_{jk} f_k\}_{k\in J}$ and only consider the i-th component, then we obtain

$$O_{ij} f_j = \sum_{n\in\mathbb{N}} \langle f_j, f_j^{(n)} \rangle_{V_j} g_i^{(n)}$$

(for any $f_j \in V_j$). Thus the characterization result for trass class operators (see Appendix) implies that O_{ij} is trace class. Since i and j were arbitrary, the proof is finished. □

We can show a converse result for component preserving operators $\bigoplus_{i\in I} \mathcal{O}_i$, if we make an extra assumption on the operators $\mathcal{O}_i$.

Proposition 5.4.7. *Assume that $\{\mathcal{O}_i\}_{i\in I}$ is a family of trace class operators $\mathcal{O}_i \in \mathcal{S}_1(V_i, W_i)$, such that $\{\|\mathcal{O}_i\|_{trace}\}_{i\in I} \in \ell^1(I)$. Then $\bigoplus_{i\in I} \mathcal{O}_i \in \mathcal{S}_1(\mathcal{K}_V^2, \mathcal{K}_W^2)$.*

Proof. Since $\|\mathcal{O}_i\|_{V_i\to W_i} \leq \|\mathcal{O}_i\|_{trace}$ for all i (see Appendix), we have

$$\|\bigoplus_{i\in I} \mathcal{O}_i\|_{1,1} = \left\|\{\|\mathcal{O}_i\|_{V_i\to W_i}\}_{i\in I}\right\|_{\ell^1} \leq \left\|\{\|\mathcal{O}_i\|_{trace}\}_{i\in I}\right\|_{\ell^1} < \infty.$$

This implies that $\{\mathcal{O}_i\}_{i\in I}$ is completely bounded, which implies $\bigoplus_{i\in I} \mathcal{O}_i \in \mathcal{B}(\mathcal{K}_V^2, \mathcal{K}_W^2)$ by Proposition 3.2.1. Now, as before we observe that if $\{e_{ij}\}_{j\in J_i}$ is an orthonormal basis for V_i (for each $i \in J$), then, by Lemma 3.1.3, $\{\tilde{e}_{ij}\}_{i\in J, j\in J_i}$ is an orthonormal basis for $\mathcal{K}_V^2$. Now observe that

$$\begin{aligned}
\left\|\bigoplus_{i\in I} \mathcal{O}_i\right\|_{trace} &= \sum_{i\in I}\sum_{j\in J_i} \left\langle \left|\bigoplus_{i\in I} \mathcal{O}_i\right| \tilde{e}_{ij}, \tilde{e}_{ij} \right\rangle_{\mathcal{K}_V^2} \\
&= \sum_{i\in I}\sum_{j\in J_i} \left\langle \bigoplus_{i\in I} |\mathcal{O}_i| \tilde{e}_{ij}, \tilde{e}_{ij} \right\rangle_{\mathcal{K}_V^2} && \text{(by Proposition 3.2.2 (d))} \\
&= \sum_{i\in I}\sum_{j\in J_i} \left\langle |\mathcal{O}_i| e_{ij}, e_{ij} \right\rangle_{V_i} \\
&= \sum_{i\in I} \|\mathcal{O}_i\|_{trace} = \left\|\{\|\mathcal{O}_i\|_{trace}\}_{i\in I}\right\|_{\ell^1} < \infty.
\end{aligned}$$

□

6 Fusion frame multipliers

The goal of this final chapter is to introduce the notion of *Bessel fusion multipliers*, *fusion frame multipliers* and *fusion Riesz multipliers* and prove some first results corresponding to these notions. Strictly speaking, these concepts are not new, since they have already been discussed in [46]. However, we will introduce these notions in a more general way, which enables us to prove a larger variety of results making the theory more interesting. The idea to consider these objects and many proof ideas of the results we will present are based on the already elaborated concepts *Bessel multipliers*, *frame multipliers* and *Riesz multipliers*. This chapter presents how the latter (frame related) concepts and some corresponding results can be generalized to the fusion frame setting in a satisfyingly general framework.

Bessel multipliers, *frame multipliers* and *Riesz multipliers* have been first introduced and studied by Peter Balazs in his PhD-thesis [3] in great detail. For a more compact publication we refer to [4]. Before generalizing them to the fusion frame setting, we will present the basic definitions and some results. We refer the interested reader to [9], [49], [48] and [47] for more details.

6.1 Frame multipliers

Frame multipliers are operators, which generalize the frame operator associated to a given frame. These operators are applied to implement time-variant filters, which are used in several fields in acoustics, such as psychoacoustical modeling or denoising (see [3] or [4] for more details). They are a special class of operators of so-called *Bessel multipliers*, which are defined as follows.

Assume that $\psi = \{\psi_i\}_{i\in I}$ is a Bessel sequence for some Hilbert space $\mathcal{H}_1$ with Bessel bound B_ψ and that $\varphi = \{\varphi_i\}_{i\in I}$ is a Bessel sequence for some Hilbert space $\mathcal{H}_2$ with Bessel bound B_φ. If $m = \{m_i\}_{i\in I} \in \ell^\infty(I)$ is a bounded sequence, then the operator

$$\mathbf{M}_{m,\varphi,\psi} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2,$$

defined by

$$\mathbf{M}_{m,\varphi,\psi} f = \sum_{i\in I} m_i \langle f, \psi_i \rangle \varphi_i, \tag{6.1}$$

is called *Bessel multiplier* and m is called the *symbol* corresponding to $\mathbf{M}_{m,\varphi,\psi}$. Note that the sum (6.1) converges unconditionally for all $f \in \mathcal{H}_1$, since φ is a Bessel sequence, which implies (compare to Section 2.1) that $\sum_{i\in I} c_i \varphi_i$ converges unconditionally for all sequences $\{c_i\}_{i\in I} \in \ell^2(I)$) and thus also for the ℓ^2-sequence $\{m_i \langle f, \psi_i \rangle\}_{i\in I}$.

A Bessel multiplier acts on a given signal $f \in \mathcal{H}_1$ by analysing, then multiplying by the symbol m, and then synthesising. In case ψ and φ both are frames or both are Riesz bases, we call $\mathbf{M}_{m,\varphi,\psi}$ *frame multiplier* or *Riesz multiplier* respectively.

To see that a Bessel multiplier is indeed a well-defined operator, observe that we even have $\mathbf{M}_{m,\varphi,\psi} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, due to

$$\begin{aligned}
\|\mathbf{M}_{m,\varphi,\psi}\| &= \sup_{\|f\|_{\mathcal{H}_1}=1} \|\mathbf{M}_{m,\varphi,\psi} f\|_{\mathcal{H}_2} \\
&= \sup_{\|f\|_{\mathcal{H}_1}=1} \Big\| \sum_{i\in I} m_i \langle f, \psi_i\rangle \varphi_i \Big\|_{\mathcal{H}_2} \\
&\le \sup_{\|f\|_{\mathcal{H}_1}=1} \|m\|_{\ell^\infty} \Big\| \sum_{i\in I} \langle f, \psi_i\rangle \varphi_i \Big\|_{\mathcal{H}_2} \\
&= \|m\|_{\ell^\infty} \|T_\varphi\| \|T_\psi^*\| \\
&\le \sqrt{B_\varphi B_\psi} \|m\|_{\ell^\infty}.
\end{aligned}$$

Note that the multiplication of the symbol m with the analysed signal $\{\langle f, \psi_i\rangle\}_{i\in I}$ is pointwise. Thus we may interpret the symbol m as a multiplication operator from $\ell^p(I)$ into $\ell^p(I)$, which is represented by the scalar matrix

$$m = \operatorname{diag}(\{m_i\}) = \begin{bmatrix} m_1 & 0 & 0 & \dots \\ 0 & m_2 & 0 & \dots \\ 0 & 0 & m_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{6.2}$$

Recall that m as multiplication operator from $\ell^p(I)$ into $\ell^p(I)$, represented by the scalar matrix (6.2), is indeed well-defined and bounded for every p with $1 \le p \le \infty$ by Corollary 5.3.3, since here we consider the special case $\big(\sum_{i\in I} \oplus V_i\big)_{\ell^p} = \big(\sum_{i\in I} \oplus W_i\big)_{\ell^p} = \big(\sum_{i\in I} \oplus \mathbb{C}\big)_{\ell^p} = \ell^p(I)$.

In particular, the case $p = 2$ is interesting. If we denote the multiplication operator from $\ell^2(I)$ into $\ell^2(I)$, defined by the symbol m as in (6.2), by $\mathcal{M}_m$, then

$$\mathcal{M}_m : \ell^2(I) \longrightarrow \ell^2(I)$$

is a well-defined and bounded operator between the representation spaces of the Bessel sequences φ and ψ. Therefore we may rewrite the definition (6.1) for the Bessel multiplier as

$$\mathbf{M}_{m,\varphi,\psi} = T_\varphi \mathcal{M}_m T_\psi^*. \tag{6.3}$$

Note that the constant symbol $1 = (1, 1, ...)$ defines the identity operator $\mathcal{I}_{\ell^p}$ (for all p with $1 \le p \le \infty$). Thus, if $\mathcal{H}_1 = \mathcal{H}_2$, then the Bessel multiplier

$$\mathbf{M}_{1,\varphi,\psi} = T_\varphi \mathcal{I}_{\ell^2} T_\psi^* = T_\varphi T_\psi^*$$

reduces to the mixed frame operator (compare to (2.15)). In case we also have $\varphi = \psi$, then the Bessel multiplier

$$\mathbf{M}_{1,\psi,\psi} = T_\psi T_\psi^* = S_\psi$$

reduces to the frame operator.

In the following we present some properties of the operator $\mathcal{M}_m : \ell^2(I) \longrightarrow \ell^2(I)$, which will help to understand the subsequent theorem a little bit better. For a proof we refer to [3] or [4]. However, note that most parts can also be followed from our results from Sections 3.2 and 5.4, since $\ell^p(I)$ can also be viewed as the Banach space $(\sum_{i\in I} \oplus \mathbb{C})_{\ell^p}$ $(1 \le p \le \infty)$.

Lemma [4] 6.1.1. *Consider the operator $\mathcal{M}_m : \ell^2(I) \longrightarrow \ell^2(I)$. Then the following properties hold.*

(a) $\|\mathcal{M}_m\|_{op} = \|m\|_{\ell^\infty}$.

(b) $\mathcal{M}_m^* = \mathcal{M}_{\overline{m}}$, *where* $\overline{m} = \{\overline{m_i}\}_{i \in I}$.

(c) If $m \in \ell^0(I)$, *then* $\mathcal{M}_m$ *is compact.*

(d) If $m \in \ell^2(I)$, *then* $\mathcal{M}_m$ *is a Hilbert Schmidt operator and* $\|\mathcal{M}_m\|_{\mathcal{HS}} = \|m\|_{\ell^2}$.

(e) If $m \in \ell^1(I)$, *then* $\mathcal{M}_m$ *is trace class with* $\|\mathcal{M}_m\|_{trace} = \|m\|_{\ell^1}$.

The following theorem is one of the main results from [3]. It can be proven by using the previous Lemma and other observations. Since we will prove a more general variant in Section 6.2, we refer the interested reader to [3] or [4] for a proof.

Theorem [4] 6.1.2. *Consider the Bessel multiplier* $\boldsymbol{M}_{m,\varphi,\psi} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *associated to the symbol* $m \in \ell^\infty$ *and the two Bessel sequences* φ *and* ψ *for* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *respectively and respective Bessel bounds* B_φ *and* B_ψ. *Then the following statements hold.*

(a) $\boldsymbol{M}_{m,\varphi,\psi}^* = \boldsymbol{M}_{\overline{m},\psi,\varphi}$. *Therefore, if* m *is real and if* $\varphi = \psi$ *(and* $\mathcal{H}_1 = \mathcal{H}_2$*), then* $\boldsymbol{M}_{m,\varphi,\psi}$ *is self-adjoint.*

(b) If $m \in \ell^0(I)$, *then* $\boldsymbol{M}_{m,\psi,\varphi}$ *is compact.*

(c) If $m \in \ell^2(I)$, *then* $\boldsymbol{M}_{m,\psi,\varphi}$ *is a Hilbert Schmidt operator with* $\|\boldsymbol{M}_{m,\psi,\varphi}\|_{\mathcal{HS}} \leq \sqrt{B_\varphi B_\psi}\|m\|_{\ell^2}$.

(d) If $m \in \ell^1(I)$, *then* $\boldsymbol{M}_{m,\psi,\varphi}$ *is trace class with* $\|\boldsymbol{M}_{m,\psi,\varphi}\|_{trace} \leq \sqrt{B_\varphi B_\psi}\|m\|_{\ell^1}$ *and* $tr(\boldsymbol{M}_{m,\psi,\varphi}) = \sum_{i \in I} m_i \langle \varphi_i, \psi_i \rangle$.

6.2 Fusion frame multipliers

The purpose of this section is to generalize the definitions and results of Section 6.1 to the fusion frame setting.

We assume that $V = \{(V_i, v_i)\}_{i \in I}$ is a Bessel fusion sequence for some Hilbert space $\mathcal{H}_1$ with Bessel fusion bound D_V and that $W = \{(W_i, w_i)\}_{i \in I}$ is a Bessel fusion sequence for some Hilbert space $\mathcal{H}_2$ with Bessel fusion bound D_W. If $\mathcal{O} = \{\mathcal{O}_i\}_{i \in I}$ is a completely bounded family of operators $\mathcal{O}_i \in \mathcal{B}(V_i, W_i)$, which we again call *symbol*, then the operator

$$\mathbf{M}_{\mathcal{O},W,V} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2,$$

defined by

$$\mathbf{M}_{\mathcal{O},W,V} f = \sum_{i \in I} v_i w_i \pi_{W_i} \mathcal{O}_i \pi_{V_i} f, \tag{6.4}$$

is called *Bessel fusion multiplier*. Any Bessel fusion multiplier $\mathbf{M}_{\mathcal{O},W,V}$ is in $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$, since

$$\begin{aligned}
\|\mathbf{M}_{\mathcal{O},W,V}\|_{op} &= \sup_{\|f\|_{\mathcal{H}_1}=1} \|\mathbf{M}_{\mathcal{O},W,V} f\|_{\mathcal{H}_2} \\
&= \sup_{\|f\|_{\mathcal{H}_1}=1} \Big\| \sum_{i\in I} v_i w_i \pi_{W_i} \mathcal{O}_i \pi_{V_i} f \Big\|_{\mathcal{H}_2} \\
&\leq \sup_{\|f\|_{\mathcal{H}_1}=1} \|\mathcal{O}\|_{cb} \Big\| \sum_{i\in I} v_i w_i \pi_{W_i} \pi_{V_i} f \Big\|_{\mathcal{H}_2} \\
&= \|\mathcal{O}\|_{cb} \sup_{\|f\|_{\mathcal{H}_1}=1} \Big\| \sum_{i\in I} w_i \big[(\bigoplus_{i\in I} \pi_{W_i}) T_V^* f \big]_i \Big\|_{\mathcal{H}_2} \\
&= \|\mathcal{O}\|_{cb} \sup_{\|f\|_{\mathcal{H}_1}=1} \Big\| T_W (\bigoplus_{i\in I} \pi_{W_i}) T_V^* f \Big\|_{\mathcal{H}_2} \\
&\leq \|\mathcal{O}\|_{cb} \sup_{\|f\|_{\mathcal{H}_1}=1} \|T_W\|_{\mathcal{K}_W^2\to\mathcal{H}_2} \Big\| (\bigoplus_{i\in I} \pi_{W_i}) T_V^* f \Big\|_{\mathcal{K}_W^2} \\
&\leq \|\mathcal{O}\|_{cb} \|T_W\|_{\mathcal{K}_W^2\to\mathcal{H}_2} \sup_{\|f\|_{\mathcal{H}_1}=1} \|T_V^* f\|_{\mathcal{K}_V^2} \\
&= \|\mathcal{O}\|_{cb} \|T_W\|_{\mathcal{K}_W^2\to\mathcal{H}_2} \|T_V^*\|_{\mathcal{H}_1\to\mathcal{K}_V^2} \\
&\leq \sqrt{D_V D_W} \|\mathcal{O}\|_{cb}.
\end{aligned}$$

Since the family $\mathcal{O} = \{\mathcal{O}_i\}_{i\in I}$ is completely bounded, by Proposition 5.3.2 the symbol $\mathcal{O}$ defines a bounded component preserving operator $\mathcal{O} \in \mathcal{B}(\mathcal{K}_V^p, \mathcal{K}_W^p)$ ($1 \leq p \leq \infty$). In case $p = 2$ we denote this operator by $\mathcal{O} = \bigoplus_{i\in I} \mathcal{O}_i$ and observe that $\bigoplus_{i\in I} \mathcal{O}_i$ maps from the signal representation space $\mathcal{K}_V^2$ of the Bessel fusion sequence V into the signal representation space $\mathcal{K}_W^2$ of the Bessel fusion sequence W. Therefore, we may rewrite the definition of the Bessel fusion multiplier $\mathbf{M}_{\mathcal{O},W,V}$ to

$$\mathbf{M}_{\mathcal{O},W,V} = T_W \Big(\bigoplus_{i\in I} \mathcal{O}_i \Big) T_V^*.$$

Thus we see that the sum (6.4) converges unconditionally for all $f \in \mathcal{H}_1$, since W is a Bessel fusion sequence, which implies (compare to Section 4.2) that $\sum_{i\in I} w_i g_i$ converges unconditionally for all sequences $\{g_i\}_{i\in I} \in \mathcal{K}_W^2$ and thus also for the $\mathcal{K}_W^2$-sequence $\{\mathcal{O}_i v_i \pi_{V_i} f\}_{i\in I} = \big(\bigoplus_{i\in I} \mathcal{O}_i \big) T_V^* f$.

In case V and W both are fusion frames or both are fusion Riesz bases, we call $\mathbf{M}_{\mathcal{O},W,V}$ *fusion frame multiplier* or *fusion Riesz multiplier* respectively.

Note that if $\mathcal{H}_1 = \mathcal{H}_2$ and $V = W$, then the symbol $\mathcal{I} = \{\mathcal{I}_{V_i}\}_{i\in I}$ defines the identity operator $\mathcal{I}_{\mathcal{K}_V^2}$. Thus, in this case, the Bessel fusion multiplier

$$\mathbf{M}_{\mathcal{I},V,V} = T_V \mathcal{I}_{\mathcal{K}_V^2} T_V^* = T_V T_V^* = S_V$$

reduces to the fusion frame operator.

We remark that our definition of Bessel fusion multipliers generalizes the following two approaches to define Bessel fusion multipliers:

In [1], the authors define Bessel fusion multipliers $S_{m,W,V} : \mathcal{H} \longrightarrow \mathcal{H}$ for Bessel fusion sequences V, W of the same Hilbert space $\mathcal{H}$ via

$$S_{m,W,V} f := \sum_{i\in I} m_i v_i w_i \pi_{W_i} \pi_{V_i} f,$$

where $m = \{m_i\}_{i\in I} \in \ell^\infty$ is a bounded sequence. In operator notation this reads

$$S_{m,W,V} = T_W\Big(\bigoplus_{i\in I}(m_i\pi_{W_i})\Big)T_V^*,$$

which is the special case $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $\bigoplus_{i\in I}\mathcal{O}_i = \bigoplus_{i\in I}(m_i\pi_{W_i})$ of our definition of a Bessel fusion multiplier.

Similarly, in [46], the authors defined Bessel fusion multipliers $\mathbf{M}_{m,W,V} : \mathcal{H} \longrightarrow \mathcal{H}$ for Bessel fusion sequences V, W of the same Hilbert space $\mathcal{H}$ via

$$\mathbf{M}_{m,W,V} f = \sum_{i\in I} m_i w_i \pi_{W_i} S_V^{-1} v_i \pi_{V_i} f,$$

where again $m = \{m_i\}_{i\in I} \in \ell^\infty$. In operator notation this reads

$$\mathbf{M}_{m,W,V} = T_W\Big(\bigoplus_{i\in I}(m_i\pi_{W_i}S_V^{-1})\Big)T_V^*,$$

which is the special case $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $\bigoplus_{i\in I}\mathcal{O}_i = \bigoplus_{i\in I}(m_i\pi_{W_i}S_V^{-1})$ of our definition of a Bessel fusion multiplier.

The following result generalizes Theorem 6.1.2 to the fusion frame setting. On this note, we remark that the proofs are short and easy only due to our effort in proving the corresponding results in Chapters 3 and 5.

Theorem 6.2.1. *Consider the Bessel fusion multiplier $\boldsymbol{M}_{\mathcal{O},W,V} \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ associated to the symbol $\mathcal{O}$ and the two Bessel fusion sequences V and W for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and let D_V and D_W denote their respective Bessel fusion bounds. Then the following statements hold.*

(a) $\boldsymbol{M}^*_{\mathcal{O},W,V} = \boldsymbol{M}_{\mathcal{O}^*,V,W}$, where $\mathcal{O}^* = \{\mathcal{O}_i^*\}_{i\in I}$. Therefore, if $\mathcal{O}_i$ is self-adjoint for every $i\in I$ and if $V = W$ (and $\mathcal{H}_1 = \mathcal{H}_2$), then $\boldsymbol{M}_{\mathcal{O},V,V}$ is self-adjoint.

(b) If $\mathcal{O}$ is a family of compact operators such that $\{\|\mathcal{O}_i\|_{op}\}_{i\in I} \in \ell^0(I)$, then $\boldsymbol{M}_{\mathcal{O},W,V}$ is compact.

(c) If $\mathcal{O}$ is a family of Hilbert Schmidt operators such that $\{\|\mathcal{O}_i\|_{\mathcal{HS}}\}_{i\in I} \in \ell^2(I)$, then $\boldsymbol{M}_{\mathcal{O},W,V}$ is a Hilbert Schmidt operator and $\|\boldsymbol{M}_{\mathcal{O},W,V}\|_{\mathcal{HS}} \leq \sqrt{D_V D_W}\|\{\|\mathcal{O}_i\|_{\mathcal{HS}}\}_{i\in I}\|_{\ell^2}$.

(d) If $\mathcal{O}$ is a family of trace class operators such that $\{\|\mathcal{O}_i\|_{trace}\}_{i\in I} \in \ell^1(I)$, then $\boldsymbol{M}_{\mathcal{O},W,V}$ is trace class and $\|\boldsymbol{M}_{\mathcal{O},W,V}\|_{trace} \leq \sqrt{D_V D_W}\Big\|\{\|\mathcal{O}_i\|_{trace}\}_{i\in I}\Big\|_{\ell^1}$.

Proof. (a) By Proposition 3.2.2 (a), the symbol $\mathcal{O}^* = \bigoplus_{i\in I}\mathcal{O}_i^*$ is the adjoint operator $(\bigoplus_{i\in I}\mathcal{O}_i)^* \in \mathcal{B}(\mathcal{K}_W^2,\mathcal{K}_V^2)$ of the symbol $\mathcal{O} = \bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{B}(\mathcal{K}_V^2,\mathcal{K}_W^2)$. Therefore

$$\mathbf{M}^*_{\mathcal{O},W,V} = \Big(T_W\Big(\bigoplus_{i\in I}\mathcal{O}_i\Big)T_V^*\Big)^* = T_V\Big(\bigoplus_{i\in I}\mathcal{O}_i^*\Big)T_W^* = \mathbf{M}_{\mathcal{O}^*,V,W}.$$

Moreover, if $\mathcal{O}_i$ is self-adjoint for every $i\in I$, then so is $\bigoplus_{i\in I}\mathcal{O}_i$ and thus we obtain

$$\mathbf{M}^*_{\mathcal{O},V,V} = \Big(T_V\Big(\bigoplus_{i\in I}\mathcal{O}_i\Big)T_V^*\Big)^* = T_V\Big(\bigoplus_{i\in I}\mathcal{O}_i\Big)T_V^* = \mathbf{M}_{\mathcal{O},V,V}.$$

(b) By Proposition 5.4.2, the assumptions imply that $\bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{C}(\mathcal{K}_V^2,\mathcal{K}_W^2)$. Therefore, by the ideal property for compact operators (see Appendix), $\mathbf{M}_{\mathcal{O},W,V}$ is compact.

(c) By Corollary 5.4.5, $\bigoplus_{i\in I}\mathcal{O}_i \in \mathcal{HS}(\mathcal{K}_V^2,\mathcal{K}_W^2)$ with $\|\bigoplus_{i\in I}\mathcal{O}_i\|_{\mathcal{HS}} \leq \|\{\|\mathcal{O}_i\|_{\mathcal{HS}}\}_{i\in I}\|_{\ell^2}$. Now, the ideal property for Hilbert Schmidt operators (see Appendix) implies that $\mathbf{M}_{\mathcal{O},W,V}$ is a Hilbert Schmidt operator and that

$$\begin{aligned}\|\mathbf{M}_{\mathcal{O},W,V}\|_{\mathcal{HS}} &= \|T_W\Big(\bigoplus_{i\in I}\mathcal{O}_i\Big)T_V^*\|_{\mathcal{HS}}\\ &\leq \|T_W\|_{\mathcal{K}_W^2\to\mathcal{H}_2}\|\bigoplus_{i\in I}\mathcal{O}_i\|_{\mathcal{HS}}\|T_V^*\|_{\mathcal{H}_1\to\mathcal{K}_V^2}\\ &\leq \sqrt{D_V D_W}\,\|\{\|\mathcal{O}_i\|_{\mathcal{HS}}\}_{i\in I}\|_{\ell^2}.\end{aligned}$$

(d) By Proposition 5.4.7, the assumptions imply that $\bigoplus_{i\in I}\mathcal{O}_i$ is trace class. Now, by the ideal property for trace class operators (see Appendix), we have

$$\begin{aligned}\|\mathbf{M}_{\mathcal{O},W,V}\|_{trace} &= \|T_W\Big(\bigoplus_{i\in I}\mathcal{O}_i\Big)T_V^*\|_{trace}\\ &\leq \|T_W\|_{\mathcal{K}_W^2\to\mathcal{H}_2}\|\bigoplus_{i\in I}\mathcal{O}_i\|_{trace}\|T_V^*\|_{\mathcal{H}_1\to\mathcal{K}_V^2}\\ &\leq \sqrt{D_V D_W}\,\|\{\|\mathcal{O}_i\|_{trace}\}_{i\in I}\|_{\ell^1}.\end{aligned}$$

□

7 Conclusio

In this thesis, Hilbert direct sums and bounded operators between them have been studied on a large scale. We first introduced a special case of this class of operators, namely bounded component preserving operators, in Chapter 3, after giving an introduction of the basic concepts of the theory of frames in Chapter 2. At first glance, it might not have been immediately clear to the reader, why such an extensive discussion about these kind of operators would be beneficial. However, throughout this thesis, we naturally encountered them quite frequently, while studying various kind of different concepts associated to the theory of fusion frames. For instance, in Chapter 4 we could give some new and shorter proofs of already known results for fusion frames and also could prove some new result here and there. Moreover, we could prove some new operator identities corresponding to fusion frame systems, which enabled us to relate extra properties of the corresponding global frame with extra properties of the corresponding fusion frame and local frames. This lead to new insights to fusion frame systems and distributed processing techniques. We also pointed out that bounded component preserving operators between Hilbert direct sums have been implicitly used in literature related to fusion frames several times. For instance, the concept of component preserving dual fusion frames occurring in [35] is based on this class of operators and in Section 4.5 we applied our theory from Chapter 3 to simplify one of the main results from there. In Chapter 5 we studied general bounded operators between Hilbert direct sums and showed some new results about compact operators and the sub-classes Hilbert Schmidt operators and trace class operators. Our theory of bounded component preserving operators enabled us to extend the definition of frame multipliers to the fusion frame setting in a nice and satisfyingly general way and – in addition to that – our results about compact component preserving operators between Hilbert direct sums from Chapter 5 allowed us to formulate and prove a fusion frame-theoretic analog of one of the main results about frame multipliers in Chapter 6.

In view of the above mentioned advantages, which component preserving operators yield, we therefore suggest to the mathematical community to view fusion frame theoretic problems or questions (– if possible –) in terms of these operators and we hope that applying the already elaborated results from this thesis yields some progress for the study fusion frames.

Appendix

Bounded Inverse Theorem [18]. *Any bounded and bijective operator between two Banach spaces has a bounded inverse.*

Theorem (Neumann) [18]. *Let X be a Banach space. If $T \in \mathcal{B}(X)$ and $\|\mathcal{I}_X - T\| < 1$, then T is invertible.*

Uniform Boundedness Principle [18]. *Let X be a Banach space and Y be a normed space. Suppose that $\{T_n\}_{n\in\mathbb{N}}$ is a family of bounded operators $T_n : X \longrightarrow Y$ and assume that $\{T_n\}_{n\in\mathbb{N}}$ converges pointwise (as $n \longrightarrow \infty$) to some operator $T : X \longrightarrow Y$, $Tx := \lim_{n\longrightarrow\infty} T_n x$. Then T defines a bounded linear operator and we have*

$$\|T\| \le \liminf_{n\in\mathbb{N}} \|T_n\| \le \sup_{n\in\mathbb{N}} \|T_n\| < \infty.$$

A proof of the following variant of Fubini's Theorem can be found in [54].

Fubini's Theorem [54]. *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be σ-finite measure spaces and let f be a $(\mathcal{S} \times \mathcal{T})$-measurable function on $X \times Y$.*

(a) If $0 \le f \le \infty$ and

$$\varphi(x) = \int_Y f_x d\lambda, \;\; \psi(y) = \int_X f_y d\mu \;\; (x \in X, y \in Y),$$

then φ is $\mathcal{S}$-measurable, ψ is $\mathcal{T}$-measurable and

$$\int_X \varphi d\mu = \int_{X\times Y} f d(\mu \times \lambda) = \int_Y \psi d\lambda.$$

b If f is complex and

$$\varphi^*(x) = \int_Y |f_x| d\lambda \;\; \textit{and} \;\; \int_X \varphi^* d\mu < \infty,$$

then $f \in L^1(\mu \times \lambda)$.

(c) If $f \in L^1(\mu \times \lambda)$, then $f_x \in L^1(\lambda)$ for almost every $x \in X$, $f_y \in L^1(\mu)$ for almost all $y \in Y$; the almost everywhere defined functions φ and ψ are in $L^1(\mu)$ and $L^1(\lambda)$ respectively, and

$$\int_X \varphi d\mu = \int_{X\times Y} f d(\mu \times \lambda) = \int_Y \psi d\lambda.$$

holds again.

Compact operators and Schatten-p-classes.
In the following we collect some results about compact operators and Schatten-p-classes of operators.

Recall that an operator $T : X \longrightarrow Y$, where X and Y are normed spaces, is called *compact*, if every sequence $\{Tf_n\}$ has a convergent subsequence, whenever $\{f_n\}$ is bounded. Equivalently [19], T is compact, if T maps bounded sets into relatively compact ones. The space $\mathcal{C}(X,Y)$ of all compact operators $T : X \longrightarrow Y$ is a subspace of $\mathcal{B}(X,Y)$. Moreover, if $S \in \mathcal{C}(X,Y)$ and $T \in \mathcal{B}(Y,Z)$ or if $S \in \mathcal{B}(X,Y)$ and $T \in \mathcal{C}(Y,Z)$, then in both cases $TS \in \mathcal{C}(X,Z)$. We call this property the *ideal property*, since, in particular, $\mathcal{C}(X) = \mathcal{C}(X,X)$ is an ideal of $\mathcal{B}(X)$.

Important is the following well-known Spectral Theorem for compact symmetric operators $T \in \mathcal{C}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, see [19] for more details.

Spectral Theorem for compact symmetric operators. *Let $T \in \mathcal{C}(\mathcal{H})$ be symmetric. Then there exists a sequence $\{\alpha_i\}_{i\in I}$ of real eigenvalues α_i converging to 0. The corresponding normalized eigenvectors u_i form an orthonormal set and every $f \in \mathcal{H}$ can be written as*

$$f = \sum_{i\in I} \langle u_i, f\rangle u_i + h,$$

where $h \in \mathcal{N}(T)$. In particular, this means, that the action of T on any $f \in \mathcal{H}$ is given by

$$Tf = \sum_{i\in I} \alpha_i \langle u_i, f\rangle u_i.$$

Moreover, if 0 *is not an eigenvalue, then the eigenvectors $\{u_i\}_{i\in I}$ form an orthonormal basis.*

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $T \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ then $T^*T \in \mathcal{C}(\mathcal{H}_1)$ is symmetric and the Spectral Theorem from above applies, i.e. there is a countable orthonormal set $\{u_i\}_{i\in I}$ and nonzero real numbers s_i^2 (the non-zero eigenvalues of T^*T) such that for all $f \in \mathcal{H}$

$$T^*Tf = \sum_{i\in I} s_i^2 \langle u_i, f\rangle u_i.$$

Since

$$\langle u_i, T^*Tu_i\rangle_{\mathcal{H}_1} = \|Tu_i\|^2_{\mathcal{H}_1} = s_i^2 > 0,$$

we can set

$$s_i := \|Tu_i\|_{\mathcal{H}_1}.$$

The numbers s_i are called *singular values* of T. This key-idea leads to the following theorem (see [19] for more details).

Singular value decomposition for compact operators. *Let $T \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and s_i be the singular values of T und u_i the corresponding orthonormal eigenvectors of T^*T. Then*

$$T = \sum_i s_i \langle u_i, \cdot\rangle v_i,$$

where $v_i = s^{-1}Tu_i$ and the operator norm of T is given by its largest singular value

$$\|T\| = \max_i s_i(T).$$

Moreover, the voctors v_i form an orthonormal set and are the eigenvectors of TT^* corresponding to the eigenvalues s_i^2.

For more details about the following characterization of compact operators we refer to [43].

Characterization of compact operators. *An operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is compact if and only if there exists a family of finite rank operators $T^{(n)} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\|T - T^{(n)}\|_{\mathcal{H}_1 \to \mathcal{H}_2} \longrightarrow 0$ (as $n \longrightarrow \infty$).*

There exist several sub-classes of $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$, called the *Schatten-p-classes*. A compact operator $T \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ belongs to the *Schatten-p-class* $\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ ($1 \leq p < \infty$), if the corresponding set of singular values is in ℓ^p, i.e. if

$$\|T\|_p := \Big(\sum_i s_i(T)^p \Big)^{1/p} < \infty.$$

As the notation indicates, $\|.\|_p$ is indeed a norm for $\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ and we have $\|T\|_{op} \leq \|T\|_p$ [43]. Moreover [43], we have $s_i(T) = s_i(T^*)$ for all i and thus $\|T\|_p = \|T^*\|_p$ for all p. This means that T is in the Schatten-p-class if and only if T^* is. The class $\mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2)$ is called the space of *trace class* operators, or simply *trace class*, and $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ is called the class of *Hilbert Schmidt* operators, which is often denoted by $\mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2)$.

For a proof of the following result we refer to [43].

Characterization of Hilbert Schmidt operators. *An operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is Hilbert Schmidt if and only if*

$$\sum_i \|Tw_i\|^2_{\mathcal{H}_2} < \infty$$

for some orthonormal basis $\{w_i\} \subseteq \mathcal{H}_1$. Moreover, if the above conditions hold, then

$$\|T\|_2 = \Big(\sum_i \|Te_i\|^2_{\mathcal{H}_2} \Big)^{1/2}$$

for every orthonormal basis $\{e_i\} \subseteq \mathcal{H}_1$.

Hilbert Schmidt operators also satisfy an *ideal property* [43]: If $K \in \mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2)$ and $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ ($\mathcal{H}_3$ is a Hilbert space) or if $K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $A \in \mathcal{HS}(\mathcal{H}_2, \mathcal{H}_3)$, then in both cases $AK \in \mathcal{HS}(\mathcal{H}_1, \mathcal{H}_3)$ and we have $\|AK\|_{\mathcal{HS}} \leq \|A\|_{op}\|K\|_{\mathcal{HS}}$ or $\|AK\|_{\mathcal{HS}} \leq \|A\|_{\mathcal{HS}}\|K\|_{op}$ respectively. In particular, $\mathcal{HS}(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$.

For more details about the following result, see [43].

Characterization of trace class operators. *An operator $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is trace class if and only if there exist sequences $\{f^{(n)}\} \subseteq \mathcal{H}_1$ and $\{g^{(n)}\} \subseteq \mathcal{H}_2$ such that the action of T on any $f \in \mathcal{H}_1$ is given by*

$$Tf = \sum_n \langle f, f^{(n)} \rangle_{\mathcal{H}_1} g^{(n)}$$

and such that

$$\sum_n \|f^{(n)}\|_{\mathcal{H}_1} \|g^{(n)}\|_{\mathcal{H}_2} < \infty.$$

Also trace class operators satisfy an *ideal property* [43]: If $K \in \mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2)$ and $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ or if $K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $A \in \mathcal{S}_1(\mathcal{H}_2, \mathcal{H}_3)$, then in both cases $AK \in \mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_3)$ and we have $\|AK\|_1 \leq \|A\|_{op}\|K\|_1$ or $\|AK\|_1 \leq \|A\|_1\|K\|_{op}$ respectively. In particular, $\mathcal{S}_1(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$. See [43] for more details. Moreover, it can be shown [43] that

$$\|T\|_{trace} := \|T\|_1 = \sum_k \langle |T| e_k, e_k \rangle_{\mathcal{H}_1},$$

where $\{e_k\} \subseteq \mathcal{H}_1$ is an orthonormal basis and $|T| = (T^*T)^{1/2}$ is the absolute value of T. The above sum is independent of the choice of the orthonormal basis.

References

[1] M. Laura Arias and Miriam Pacheco. Bessel fusion multipliers. *Journal of Mathematical Analysis and Applications*, 348(2):581–588, 2008.

[2] M. S. Asgari. New characterizations of fusion frames (frames of subspaces). *Proceedings of the Indian Academy of Sciences - Section A Part 3 Mathematical sciences*, 119:369–382, 07 2009.

[3] P. Balazs. *Regular and irregular Gabor multipliers with application to psychoacoustic masking.* PhD thesis, University of Vienna, 2005.

[4] P. Balazs. Basic definition and properties of Bessel multipliers. *J. Math. Anal. Appl.*, 2006.

[5] P. Balazs and K. Gröchenig. A Guide to Localized Frames and Applications to Galerkin-Like Representations of Operators. *Applied and Numerical Harmonic Analysis*, pages 47–79, 06 2017.

[6] P. Balazs, N. Holighaus, T. Necciari, and D. Stoeva. Frame Theory for Signal Processing in Psychoacoustics. *Balan R., Benedetto J., Czaja W., Dellatorre M., Okoudjou K. (eds) Excursions in Harmonic Analysis, Volume 5. Applied and Numerical Harmonic Analysis. Birkhäuser, Cham*, 2017.

[7] P. Balazs, M. Shamsabadi, A. A. Arefijamaal, and Gardon C. Representation of Operators Using Fusion Frames. *(pre-print)*, 07 2020.

[8] P. Balazs, D. Stoeva, and J. Antoine. Classification of General Sequences by Frame-Related Operators. *Sampling Theory in Signal and Image Processing*, 10, 01 2011.

[9] P. Balazs and D. T. Stoeva. Representation of the inverse of a frame multiplier. *Journal of Mathematical Analysis and Applications*, 422(2):981–994, 2015.

[10] M. Baldi, G. Cancellieri, and F. Chiaraluce. Array Convolutional Low-Density Parity-Check Codes. *Communications Letters, IEEE*, 18, 12 2013.

[11] J. J. Benedetto and P. J. S. G. Ferreira. *Modern sampling theory.* Mathematics and applications. Birkhäuser, 2001.

[12] E. J. Candes and T. Tao. Decoding by linear programming. *IEEE Transactions on Information Theory*, 51(12):4203–4215, 2005.

[13] E. J. Candès. The restricted isometry property and its implications for compressed sensing. *Compte Rendus de l'Academie des Sciences*, 346:589–592, 05 2008.

[14] E. J. Candès, J. Romberg, and T. Tao. Stable Signal Recovery from Incomplete and Inaccurate Measurements. *Communications on Pure and Applied Mathematics*, 59, 08 2006.

[15] P. G. Casazza, G. Kutyniok, and S. Li. Fusion Frames and Distributed Processing. *Applied and Computational Harmonic Analysis*, 25:114–132, 07 2008.

[16] P. G. Casazza and J. C. Tremain. The Kadison–Singer Problem in mathematics and engineering. *Proceedings of the National Academy of Sciences*, 103(7):2032–2039, Feb 2006.

[17] P.G. Casazza and G. Kutyniok. Frames of subspaces. *Contemporary Mathematics*, 2003.

[18] O. Christensen. *An Introduction to Frames and Riesz Bases.* Springer International Publishing, 2016.

[19] J. B. Conway. *A Course in Functional Analysis.* Springer New York, 1990.

[20] M. S. Craig and R. L. Genter. Geophone array formation and semblance evaluation. *Society of Exploration Geophysicists*, 71, 01 2006.

[21] I. Daubechies, A. Grossmann, and Y. Meyer. Painless nonorthogonal expansions. *J. Math. Phys. 27, 1271 (1986)*, 1986.

[22] I. Daubechies and C. Heil. Ten Lectures on Wavelets. *The Journal of the Acoustical Society of America 93*, 1993.

[23] A. Defant and K. Floret. *Tensor Norms and Operator Ideals.* North Holland, 1992.

[24] Ph. Delsarte, J. M. Goethals, and Seidel J. J. Spherical codes and designs. *Geometriae Dedicata, Springer*, 6(3):363–388, 1977.

[25] D. Donoho and M. Elad. Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ_1. *Proceedings of the National Academy of Sciences of the United States of America*, 100:2197–202, 04 2003.

[26] D. L. Donoho and X. Huo. Uncertainty principles and ideal atomic decomposition. *IEEE Transactions on Information Theory*, 47(7):2845–2862, 2001.

[27] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc. 72 (1952), 341-366*, 1952.

[28] Y. C. Eldar and G. David Forney Jr. Optimal Tight Frames and Quantum Measurement. *IEEE Trans. Inform. Theory*, 48:599–610, 03 2002.

[29] H.G. Feichtinger and T. Strohmer. *Gabor Analysis and Algorithms - Theory and Applications.* Birkhäuser Boston, 1998.

[30] V. Goyal, M. Vetterli, and N.T. Thao. Quantized Overcomplete Expansions in $\mathbb{R}^N$: Analysis, Synthesis, and Algorithms. *IEEE Trans. Inf. Theory*, 44:16–31, 1998.

[31] K. Gröchenig. *Foundations of Time–Frequency Analysis.* Springer-Verlag, 01 2001.

[32] P. Găvruţa. On the duality of fusion frames. *Journal of Mathematical Analysis and Applications*, 2007.

[33] R. Heath, H. Bolcskei, and A. J. Paulraj. Space-time signaling and frame theory. *Acoustics, Speech, and Signal Processing, 1988. ICASSP-88., 1988 International Conference on*, 4:2445 – 2448, 02 2001.

[34] R. W. Heath and A. J. Paulraj. Linear dispersion codes for MIMO systems based on frame theory. *IEEE Transactions on Signal Processing*, 50(10):2429–2441, 2002.

[35] S. Heineken, P. Morillas, A. Benavente, and M. Zakowicz. Dual Fusion Frames. *Archiv der Mathematik*, 2013.

[36] S. S. Iyengar and R. R. Brooks. *Distributed Sensor Networks*. CRC Press, 2016.

[37] I. J. Maddox. *Infinite Matrices of Operators*. Springer-Verlag, 1980.

[38] A. W. Marcus, D. A. Spielman, and N. Srivastava. Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem. *Annals of Mathematics, SECOND SERIES, Vol. 182, No. 1 , pp. 327-350*, 2015.

[39] S. J. Orfanidis. *Introduction to signal processing*. Pearson Education, Inc., 1996.

[40] V. Paulsen. *Completely Bounded Maps and Operator Algebras*. Cambridge University Press, 2003.

[41] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves. Symmetric informationally complete quantum measurements. *Journal of Mathematical Physics*, 45(6):2171–2180, Jun 2004.

[42] C.J. Rozell and D.H. Johnson. Analyzing the robustness of redundant population codes in sensory and feature extraction systems. *Neurocomputing*, 69(10–12):1215–1218, June 2006.

[43] R. Schatten. *Norm Ideals of Completely Continious Operators*. Springer Berlin, 1960.

[44] A. J. Scott. Tight informationally complete quantum measurements. *Journal of Physics A: Mathematical and General*, 39, 10 2006.

[45] M. Shamsabadi, A. Arefijamaal, and P. Balazs. The Invertibility of U-Fusion Cross Gram Matrices of Operators. *Mediterranean Journal of Mathematics*, 10 2017.

[46] M. Shamsabadi and A. A. Arefijamaal. The invertibility of fusion frame multipliers. *Linear and Multilinear Algebra*, 65(5):1062–1072, 2017.

[47] D. T. Stoeva and P. Balazs. Invertibility of multipliers. *Applied and Computational Harmonic Analysis*, 33(2):292–299, 2012.

[48] D. T. Stoeva and P. Balazs. Detailed characterization of unconditional convergence and invertibility of multipliers. *Sampling Theory in Signal and Image Processing (STSIP)*, 12(2-3):87–125, 2013.

[49] D. T. Stoeva and P. Balazs. The dual frame induced by an invertible frame multiplier. In *Proceedings of SampTA (2015)*, pages 101–104, Washington DC, 2015.

[50] T. Strohmer and R. W. Heath. Grassmannian frames with applications to coding and communication. *Applied and Computational Harmonic Analysis*, 14(3):257 – 275, 2003.

[51] A. Syropoulos. Mathematics of Multisets. *Springer-Verlag*, 12 2001.

[52] M. Thill and B. Hassibi. Low-Coherence Frames from Group Fourier Matrices. *IEEE Trans. Inf. Theory 63(6): 3386-3404*, 2017.

[53] J. A. Tropp and A. C. Gilbert. Signal Recovery From Random Measurements Via Orthogonal Matching Pursuit. *IEEE Transactions on Information Theory*, 53(12):4655–4666, 2007.

[54] Rudin W. *Reelle und Komplexe Analysis*. Oldenbourg, 2. Auflage, 2009.

[55] M. Werner. *Adaptive Wavelet Frame Domain Decomposition Methods for Elliptic Operator Equations*. Logos Verlag, Berlin, 2009.

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